is surjective. Thus we can choose a map $\alpha : \mathcal{O}_{X'}^{\oplus r} \to \mathcal{G}$ which is compatible with the given trivialization of $\mathcal{G}|_E$. Thus α is an isomorphism over an open neighbourhood of E in X'. Thus every point of Z has an affine open neighbourhood where we can solve the problem. Since $X' \setminus E \to X \setminus Z$ is an isomorphism, the same holds for points of X not in Z. Thus another Zariski glueing argument finishes the proof. □

0EXE Proposition 38.39.4. Let p be a prime number. Let S be a scheme in characteristic p. Then the category fibred in groupoids

$$p : \mathcal{S} \longrightarrow (Sch/S)_h$$

whose fibre category over U is the category of finite locally free $\mathrm{colim}_F\, \mathcal{O}_U$-modules over U is a stack in groupoids. Moreover, if U is quasi-compact and quasi-separated, then $\mathcal{S}_U$ is $\mathrm{colim}_F\, \mathrm{Vect}(U)$.

Proof. The final assertion is the content of Lemma 38.39.1. To prove the proposition we will check conditions (1), (2), and (3) of Lemma 38.37.13.

Condition (1) holds because by definition we have glueing for the Zariski topology.

To see condition (2), suppose that $f : X \to Y$ is a surjective, flat, proper morphism of finite presentation over S with Y affine. Since $Y, X, X \times_Y X$ are quasi-compact and quasi-separated, we can use the description of fibre categories given in the statement of the proposition. Then it is clearly enough to show that

$$\mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X) \times_{\mathrm{Vect}(X \times_Y X)} \mathrm{Vect}(X)$$

is an equivalence (as this will imply the same for the colimits). This follows immediately from fppf descent of finite locally free modules, see Descent, Proposition 35.5.2 and Lemma 35.7.6.

Condition (3) is the content of Lemmas 38.39.2 and 38.39.3. □

0EXF Lemma 38.39.5. Let $f : X \to S$ be a proper morphism with geometrically connected fibres where S is the spectrum of a discrete valuation ring. Denote $\eta \in S$ the generic point and denote $X_n \subset X$ the closed subscheme cutout by the nth power of a uniformizer on S. Then there exists an integer n such that the following is true: any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ such that $\mathcal{E}|_{X_\eta}$ and $\mathcal{E}|_{X_n}$ are free, is free.

Proof. We first reduce to the case where $X \to S$ has a section. Say $S = \mathrm{Spec}(A)$. Choose a closed point ξ of X_η. Choose an extension of discrete valuation rings $A \subset B$ such that the fraction field of B is $\kappa(\xi)$. This is possible by Krull-Akizuki (Algebra, Lemma 10.120.18) and the fact that $\kappa(\xi)$ is a finite extension of the fraction field of A. By the valuative criterion of properness (Morphisms, Lemma 29.42.1) we get a B-valued point $\tau : \mathrm{Spec}(B) \to X$ which induces a section $\sigma : \mathrm{Spec}(B) \to X_B$. For a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let $\mathcal{E}_B$ be the pullback to the base change X_B. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we see that $H^0(X_B, \mathcal{E}_B) = H^0(X, \mathcal{E}) \otimes_A B$. Thus if $\mathcal{E}_B$ is free of rank r, then the sections in $H^0(X, \mathcal{E})$ generate the free B-module $\tau^*\mathcal{E} = \sigma^*\mathcal{E}_B$. In particular, we can find r global sections $s_1, \ldots, s_r$ of $\mathcal{E}$ which generate $\tau^*\mathcal{E}$. Then

$$s_1, \ldots, s_r : \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{E}$$

is a map of finite locally free $\mathcal{O}_X$-modules of rank r and the pullback to X_B is a map of free $\mathcal{O}_{X_B}$-modules which restricts to an isomorphism in one point of each fibre. Taking the determinant we get a function $g \in \Gamma(X_\eta, \mathcal{O}_{X_B})$ which is invertible in one point of each fibre. As the fibres are proper and connected, we see that g must be invertible (details omitted; hint: use Varieties, Lemma 33.9.3). Thus it suffices to prove the lemma for the base change $X_B \to \mathrm{Spec}(B)$.

Assume we have a section $\sigma : S \to X$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module which is assumed free on the generic fibre and on X_n (we will choose n later). Choose an isomorphism $\sigma^*\mathcal{E} = \mathcal{O}_S^{\oplus r}$. Consider the map

$$K = R\Gamma(X, \mathcal{E}) \longrightarrow R\Gamma(S, \sigma^*\mathcal{E}) = A^{\oplus r}$$

in $D(A)$. Arguing as above, we see $\mathcal{E}$ is free if (and only if) the induced map $H^0(K) = H^0(X, \mathcal{E}) \to A^{\oplus r}$ is surjective.

Set $L = R\Gamma(X, \mathcal{O}_X^{\oplus r})$ and observe that the corresponding map $L \to A^{\oplus r}$ has the desired property. Observe that $K \otimes_A Q(A) \cong L \otimes_A Q(A)$ by flat base change and the assumption that $\mathcal{E}$ is free on the generic fibre. Let $\pi \in A$ be a uniformizer. Observe that

$$K \otimes_A^{\mathbf{L}} A/\pi^m A = R\Gamma(X, \mathcal{E} \xrightarrow{\pi^m} \mathcal{E})$$

and similarly for L. Denote $\mathcal{E}_{tors} \subset \mathcal{E}$ the coherent subsheaf of sections supported on the special fibre and similarly for other $\mathcal{O}_X$-modules. Choose $k > 0$ such that $(\mathcal{O}_X)_{tors} \to \mathcal{O}_X/\pi^k\mathcal{O}_X$ is injective (Cohomology of Schemes, Lemma 30.10.3). Since $\mathcal{E}$ is locally free, we see that $\mathcal{E}_{tors} \subset \mathcal{E}/\pi^k\mathcal{E}$. Then for $n \geq m + k$ we have isomorphisms

$$\begin{aligned}
(\mathcal{E} \xrightarrow{\pi^m} \mathcal{E}) &\cong (\mathcal{E}/\pi^k\mathcal{E} \xrightarrow{\pi^m} \mathcal{E}/\pi^{k+m}\mathcal{E}) \\
&\cong (\mathcal{O}_X^{\oplus r}/\pi^k\mathcal{O}_X^{\oplus r} \xrightarrow{\pi^m} \mathcal{O}_X^{\oplus r}/\pi^{k+m}\mathcal{O}_X^{\oplus r}) \\
&\cong (\mathcal{O}_X^{\oplus r} \xrightarrow{\pi^m} \mathcal{O}_X^{\oplus r})
\end{aligned}$$

in $D(\mathcal{O}_X)$. This determines an isomorphism

$$K \otimes_A^{\mathbf{L}} A/\pi^m A \cong L \otimes_A^{\mathbf{L}} A/\pi^m A$$

in $D(A)$ (holds when $n \geq m + k$). Observe that these isomorphisms are compatible with pulling back by σ hence in particular we conclude that $K \otimes_A^{\mathbf{L}} A/\pi^m A \to (A/\pi^m A)^{\oplus r}$ defines an surjection on degree 0 cohomology modules (as this is true for L). Since A is a discrete valuation ring, we have

$$K \cong \bigoplus H^i(K)[-i] \quad \text{and} \quad L \cong \bigoplus H^i(L)[-i]$$

in $D(A)$. See More on Algebra, Example 15.69.3. The cohomology groups $H^i(K) = H^i(X, \mathcal{E})$ and $H^i(L) = H^i(X, \mathcal{O}_X)^{\oplus r}$ are finite A-modules by Cohomology of Schemes, Lemma 30.19.2. By More on Algebra, Lemma 15.124.3 these modules are direct sums of cyclic modules. We have seen above that the rank β_i of the free part of $H^i(K)$ and $H^i(L)$ are the same. Next, observe that

$$H^i(L \otimes_A^{\mathbf{L}} A/\pi^m A) = H^i(L)/\pi^m H^i(L) \oplus H^{i+1}(L)[\pi^m]$$

and similarly for K. Let e be the largest integer such that $A/\pi^e A$ occurs as a summand of $H^i(X, \mathcal{O}_X)$, or equivalently $H^i(L)$, for some i. Then taking $m = e + 1$ we see that $H^i(L \otimes_A^{\mathbf{L}} A/\pi^m A)$ is a direct sum of β_i copies of $A/\pi^m A$ and some other cyclic modules each annihilated by π^e. By the same reasoning for K and the isomorphism $K \otimes_A^{\mathbf{L}} A/\pi^m A \cong L \otimes_A^{\mathbf{L}} A/\pi^m A$ it follows that $H^i(K)$ cannot have any cyclic summands of the form $A/\pi^l A$ with $l > e$. (It also follows that K is isomorphic to L as an object of $D(A)$, but we won't need this.) Then the only way the map

$$H^0(K \otimes_A^{\mathbf{L}} A/\pi^{e+1}A) = H^0(K)/\pi^{e+1}H^0(K) \oplus H^1(K)[\pi^{e+1}] \longrightarrow (A/\pi^{e+1}A)^{\oplus r}$$

is surjective, is if it is surjective on the first summand. This is what we wanted to show. (To be precise, the integer n in the statement of the lemma, if there is a section σ, should be equal to $k+e+1$ where k and e are as above and depend only on X.) ☐

0EXG Lemma 38.39.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume

(1) f is flat and proper and $\mathcal{O}_S = f_*\mathcal{O}_X$,
(2) S is a normal Noetherian scheme,
(3) the pullback of $\mathcal{E}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is free for every codimension 1 point $s \in S$.

Then $\mathcal{E}$ is isomorphic to the pullback of a finite locally free $\mathcal{O}_S$-module.

Proof. We will prove the canonical map

$$\Phi : f^*f_*\mathcal{E} \longrightarrow \mathcal{E}$$

is an isomorphism. By flat base change (Cohomology of Schemes, Lemma 30.5.2) and assumptions (1) and (3) we see that the pullback of this to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is an isomorphism for every codimension 1 point $s \in S$. By Divisors, Lemma 31.2.11 it suffices to prove that $\mathrm{depth}((f^*f_*\mathcal{E})_x) \geq 2$ for any point $x \in X$ mapping to a point $s \in S$ of codimension ≥ 2. Since f is flat and $(f^*f_*\mathcal{E})_x = (f_*\mathcal{E})_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$, it suffices to prove that $\mathrm{depth}((f_*\mathcal{E})_s) \geq 2$, see Algebra, Lemma 10.163.2. Since S is a normal Noetherian scheme and $\dim(\mathcal{O}_{S,s}) \geq 2$ we have $\mathrm{depth}(\mathcal{O}_{S,s}) \geq 2$, see Properties, Lemma 28.12.5. Thus we get what we want from Divisors, Lemma 31.6.6. ☐

We can use the results above to prove the following miraculous statement.

0EXH Theorem 38.39.7. Let p be a prime number. Let Y be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. Let $f : X \to Y$ be a proper, surjective morphism of finite presentation with geometrically connected fibres. Then the functor

$$\mathrm{colim}_F \mathrm{Vect}(Y) \longrightarrow \mathrm{colim}_F \mathrm{Vect}(X)$$

is fully faithful with essential image described as follows. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume for all $y \in Y$ there exists integers $n_y, r_y \geq 0$ such that

$$F^{n_y,*}\mathcal{E}|_{X_{y,red}} \cong \mathcal{O}_{X_{y,red}}^{\oplus r_y}$$

Then for some $n \geq 0$ the nth Frobenius power pullback $F^{n,*}\mathcal{E}$ is the pullback of a finite locally free $\mathcal{O}_Y$-module.

Proof. Proof of fully faithfulness. Since vectorbundles on Y are locally trivial, this reduces to the statement that

$$\mathrm{colim}_F \Gamma(Y, \mathcal{O}_Y) \longrightarrow \mathrm{colim}_F \Gamma(X, \mathcal{O}_X)$$

is bijective. Since $\{X \to Y\}$ is an h covering, this will follow from Lemma 38.38.2 if we can show that the two maps

$$\mathrm{colim}_F \Gamma(X, \mathcal{O}_X) \longrightarrow \mathrm{colim}_F \Gamma(X \times_Y X, \mathcal{O}_{X \times_Y X})$$

are equal. Let $g \in \Gamma(X, \mathcal{O}_X)$ and denote g_1 and g_2 the two pullbacks of g to $X \times_Y X$. Since $X_{y,red}$ is geometrically connected, we see that $H^0(X_{y,red}, \mathcal{O}_{X_{y,red}})$ is a purely inseparable extension of $\kappa(y)$, see Varieties, Lemma 33.9.3. Thus $g^q|_{X_{y,red}}$ comes from an element of $\kappa(y)$ for some p-power q (which may depend on y). It follows that g_1^q and g_2^q map to the same element of the residue field at any point of $(X \times_Y X)_y = X_y \times_y X_y$. Hence $g_1 - g_2$ restricts to zero on $(X \times_Y X)_{red}$. Hence $(g_1 - g_2)^n = 0$ for some n which we may take to be a p-power as desired.

Description of essential image. Let $\mathcal{E}$ be as in the statement of the proposition. We first reduce to the Noetherian case.

Let $y \in Y$ be a point and view it as a morphism $y \to Y$ from the spectrum of the residue field into Y. We can write $y \to Y$ as a filtered limit of morphisms $Y_i \to Y$ of finite presentation with Y_i affine. (It is best to prove this yourself, but it also follows formally from Limits, Lemma 32.7.2 and 32.4.13.) For each i set $Z_i = Y_i \times_Y X$. Then $X_y = \lim Z_i$ and $X_{y,red} = \lim Z_{i,red}$. By Limits, Lemma 32.10.2 we can find an i such that $F^{n_y,*}\mathcal{E}|_{Z_{i,red}} \cong \mathcal{O}_{Z_{i,red}}^{\oplus r_y}$. Fix i. We have $Z_{i,red} = \lim Z_{i,j}$ where $Z_{i,j} \to Z_i$ is a thickening of finite presentation (Limits, Lemma 32.9.4). Using the same lemma as before we can find a j such that $F^{n_y,*}\mathcal{E}|_{Z_{i,j}} \cong \mathcal{O}_{Z_{i,j}}^{\oplus r_y}$. We conclude that for each $y \in Y$ there exists a morphism $Y_y \to Y$ of finite presentation whose image contains y and a thickening $Z_y \to Y_y \times_Y X$ such that $F^{n_y,*}\mathcal{E}|_{Z_y} \cong \mathcal{O}_{Z_y}^{\oplus r_y}$. Observe that the image of $Y_y \to Y$ is constructible (Morphisms, Lemma 29.22.2). Since Y is quasi-compact in the constructible topology (Topology, Lemma 5.23.2 and Properties, Lemma 28.2.4) we conclude that there are a finite number of morphisms

$$Y_1 \to Y,\ Y_2 \to Y,\ \ldots,\ Y_N \to Y$$

of finite presentation such that $Y = \bigcup \mathrm{Im}(Y_a \to Y)$ set theoretically and such that for each $a \in \{1, \ldots, N\}$ there exist integers $n_a, r_a \geq 0$ and there is a thickening $Z_a \subset Y_a \times_Y X$ of finite presentation such that $F^{n_a,*}\mathcal{E}|_{Z_a} \cong \mathcal{O}_{Z_a}^{\oplus r_a}$.

Formulated in this way, the condition descends to an absolute Noetherian approximation. We stronly urge the reader to skip this paragraph. First write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of schemes of finite type over $\mathbf{F}_p$ with affine transition morphisms (Limits, Lemma 32.7.2). Next, we can assume we have proper morphisms $f_i : X_i \to Y_i$ whose base change to Y recovers $f : X \to Y$, see Limits, Lemma 32.10.1. After increasing i we may assume there exists a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ whose pullback to X is isomorphic to $\mathcal{E}$, see Limits, Lemma 32.10.3. Pick $0 \in I$ and denote $E \subset Y_0$ the constructible subset where the geometric fibres of f_0 are connected, see More on Morphisms, Lemma 37.28.6. Then $Y \to Y_0$ maps into E, see More on Morphisms, Lemma 37.28.2. Thus $Y_i \to Y_0$ maps into E for $i \gg 0$, see Limits, Lemma 32.4.10. Hence we see that the fibres of f_i are geometrically connected for $i \gg 0$. By Limits, Lemma 32.10.1 for large enough i we can find morphisms $Y_{i,a} \to Y_i$ of finite type whose base change to Y recovers $Y_a \to Y$, $a \in \{1, \ldots, N\}$. After possibly increasing i we can find thickenings $Z_{i,a} \subset Y_{i,a} \times_{Y_i} X_i$ whose base change to $Y_a \times_Y X$ recovers Z_a (same reference as before combined with Limits, Lemmas 32.8.5 and 32.8.15). Since $Z_a = \lim Z_{i,a}$ we find that after increasing i we may assume $F^{n_a,*}\mathcal{E}_i|_{Z_{i,a}} \cong \mathcal{O}_{Z_{i,a}}^{\oplus r_a}$, see Limits, Lemma 32.10.2. Finally, after increasing i one more time we may assume $\coprod Y_{i,a} \to Y_i$ is surjective by Limits, Lemma 32.8.15. At this point all the assumptions hold for $X_i \to Y_i$ and $\mathcal{E}_i$ and we see that it suffices to prove result for $X_i \to Y_i$ and $\mathcal{E}_i$.

Assume Y is of finite type over $\mathbf{F}_p$. To prove the result we will use induction on $\dim(Y)$. We are trying to find an object of $\mathrm{colim}_F \mathrm{Vect}(Y)$ which pulls back to the object of $\mathrm{colim}_F \mathrm{Vect}(X)$ determined by $\mathcal{E}$. By the fully faithfulness already proven and because of Proposition 38.39.4 it suffices to construct a descent of $\mathcal{E}$ after replacing Y by the members of a h covering and X by the corresponding base change. This means that we may replace Y by a scheme proper and surjective over Y provided this does not increase the dimension of Y. If $T \subset T'$ is a thickening of schemes of finite type over $\mathbf{F}_p$ then $\mathrm{colim}_F \mathrm{Vect}(T) = \mathrm{colim}_F \mathrm{Vect}(T')$ as $\{T \to T'\}$ is a h covering such that $T \times_{T'} T = T$. If $T' \to T$ is a universal homeomorphism of schemes of finite type over $\mathbf{F}_p$, then $\mathrm{colim}_F \mathrm{Vect}(T) = \mathrm{colim}_F \mathrm{Vect}(T')$ as $\{T \to T'\}$ is a h covering such that the diagonal $T \subset T \times_{T'} T$ is a thickening.

Using the general remarks made above, we may and do replace X by its reduction and we may assume X is reduced. Consider the Stein factorization $X \to Y' \to Y$, see More on Morphisms, Theorem 37.53.4. Then $Y' \to Y$ is a universal homeomorphism of schemes of finite type over $\mathbf{F}_p$. By the above we may replace Y by Y'. Thus we may assume $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that Y is reduced. This reduces us to the case discussed in the next paragraph.

Assume Y is reduced and $f_*\mathcal{O}_X = \mathcal{O}_Y$ over a dense open subscheme of Y. Then $X \to Y$ is flat over a dense open subscheme $V \subset Y$, see Morphisms, Proposition 29.27.2. By Lemma 38.31.1 there is a V-admissible blowing up $Y' \to Y$ such that the strict transform X' of X is flat over Y'. Observe that $\dim(Y') = \dim(Y)$ as Y and Y' have a common dense open subscheme. By More on Morphisms, Lemma 37.53.7 and the fact that $V \subset Y'$ is dense all fibres of $f' : X' \to Y'$ are geometrically connected. We still have $(f'_*\mathcal{O}_{X'})|_V = \mathcal{O}_V$. Write

$$Y' \times_Y X = X' \cup E \times_Y X$$

where $E \subset Y'$ is the exceptional divisor of the blowing up. By the general remarks above, it suffices to prove existence for $Y' \times_Y X \to Y'$ and the restriction of $\mathcal{E}$ to $Y' \times_Y X$. Suppose that we find some object ξ' in $\mathrm{colim}_F \mathrm{Vect}(Y')$ pulling back to the restriction of $\mathcal{E}$ to X' (viewed as an object of the colimit category). By induction on $\dim(Y)$ we can find an object ξ'' in $\mathrm{colim}_F \mathrm{Vect}(E)$ pulling back to the restriction of $\mathcal{E}$ to $E \times_Y X$. Then the fully faithfullness determines a unique isomorphism $\xi'|_E \to \xi''$ compatible with the given identifications with the restriction of $\mathcal{E}$ to $E \times_{Y'} X'$. Since

$$\{E \times_Y X \to Y' \times_Y X, X' \to Y' \times_Y X\}$$

is a h covering given by a pair of closed immersions with

$$(E \times_Y X) \times_{(Y' \times_Y X)} X' = E \times_{Y'} X'$$

we conclude that ξ' pulls back to the restriction of $\mathcal{E}$ to $Y' \times_Y X$. Thus it suffices to find ξ' and we reduce to the case discussed in the next paragraph.

Assume Y is reduced, f is flat, and $f_*\mathcal{O}_X = \mathcal{O}_Y$ over a dense open subscheme of Y. In this case we consider the normalization $Y^\nu \to Y$ (Morphisms, Section 29.54). This is a finite surjective morphism (Morphisms, Lemma 29.54.10 and 29.18.2) which is an isomorphism over a dense open. Hence by our general remarks we may replace Y by Y^ν and X by $Y^\nu \times_Y X$. After this replacement we see that

$\mathcal{O}_Y = f_*\mathcal{O}_X$ (because the Stein factorization has to be an isomorphism in this case; small detail omitted).

Assume Y is a normal Noetherian scheme, that f is flat, and that $f_*\mathcal{O}_X = \mathcal{O}_Y$. After replacing $\mathcal{E}$ by a suitable Frobenius power pullback, we may assume $\mathcal{E}$ is trivial on the scheme theoretic fibres of f at the generic points of the irreducible components of Y (because $\mathrm{colim}_F \mathrm{Vect}(-)$ is an equivalence on universal homeomorphisms, see above). Similarly to the arguments above (in the reduction to the Noetherian case) we conclude there is a dense open subscheme $V \subset Y$ such that $\mathcal{E}|_{f^{-1}(V)}$ is free. Let $Z \subset Y$ be a closed subscheme such that $Y = V \amalg Z$ set theoretically. Let $z_1, \ldots, z_t \in Z$ be the generic points of the irreducible components of Z of codimension 1. Then $A_i = \mathcal{O}_{Y,z_i}$ is a discrete valuation ring. Let n_i be the integer found in Lemma 38.39.5 for the scheme X_{A_i} over A_i. After replacing $\mathcal{E}$ by a suitable Frobenius power pullback, we may assume $\mathcal{E}$ is free over $X_{A_i/\mathfrak{m}_i^{n_i}}$ (again because the colimit category is invariant under universal homeomorphisms, see above). Then Lemma 38.39.5 tells us that $\mathcal{E}$ is free on X_{A_i}. Thus finally we conclude by applying Lemma 38.39.6. $\square$

38.40. Blowing up complexes

0ESM This section finds normal forms for perfect objects of the derived category after blowups.

0ESP Lemma 38.40.1. Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. For every $p, k \in \mathbf{Z}$ there is an finite type quasi-coherent sheaf of ideals $\mathrm{Fit}_{p,k}(E) \subset \mathcal{O}_X$ with the following property: for $U \subset X$ open such that $E|_U$ is isomorphic to

$$\ldots \to \mathcal{O}_U^{\oplus n_{b-2}} \xrightarrow{d_{b-2}} \mathcal{O}_U^{\oplus n_{b-1}} \xrightarrow{d_{b-1}} \mathcal{O}_U^{\oplus n_b} \to 0 \to \ldots$$

the restriction $\mathrm{Fit}_{p,k}(E)|_U$ is generated by the minors of the matrix of d_p of size

$$-k + n_{p+1} - n_{p+2} + \ldots + (-1)^{b-p+1} n_b$$

Convention: the ideal generated by $r \times r$-minors is $\mathcal{O}_U$ if $r \leq 0$ and the ideal generated by $r \times r$-minors where $r > \min(n_p, n_{p+1})$ is zero.

Proof. Observe that E locally on X has the shape as stated in the lemma, see More on Algebra, Section 15.64, Cohomology, Section 20.47, and Derived Categories of Schemes, Section 36.10. Thus it suffices to prove that the ideal of minors is independent of the chosen representative. To do this, it suffices to check in local rings. Over a local ring $(R, \mathfrak{m}, \kappa)$ consider a bounded above complex

$$F^\bullet : \ldots \to R^{\oplus n_{b-2}} \xrightarrow{d_{b-2}} R^{\oplus n_{b-1}} \xrightarrow{d_{b-1}} R^{\oplus n_b} \to 0 \to \ldots$$

Denote $\mathrm{Fit}_{k,p}(F^\bullet) \subset R$ the ideal generated by the minors of size $k - n_{p+1} + n_{p+2} - \ldots + (-1)^{b-p} n_b$ in the matrix of d_p. Suppose some matrix coefficient of some differential of $F^\bullet$ is invertible. Then we pick a largest integer i such that d_i has an invertible matrix coefficient. By Algebra, Lemma 10.102.2 the complex $F^\bullet$ is isomorphic to a direct sum of a trivial complex $\ldots \to 0 \to R \to R \to 0 \to \ldots$ with nonzero terms in degrees i and $i+1$ and a complex $(F')^\bullet$. We leave it to the reader to see that $\mathrm{Fit}_{p,k}(F^\bullet) = \mathrm{Fit}_{p,k}((F')^\bullet)$; this is where the formula for the size of the minors is used. If $(F')^\bullet$ has another differential with an invertible matrix coefficient, we do it again, etc. Continuing in this manner, we eventually reach a complex $(F^\infty)^\bullet$ all of whose differentials have matrices with coefficients in

$\mathfrak{m}$. Here you may have to do an infinite number of steps, but for any cutoff only a finite number of these steps affect the complex in degrees $\geq$ the cutoff. Thus the "limit" $(F^\infty)^\bullet$ is a well defined bounded above complex of finite free modules, comes equipped with a quasi-isomorphism $(F^\infty)^\bullet \to F^\bullet$ into the complex we started with, and $\text{Fit}_{p,k}(F^\bullet) = \text{Fit}_{p,k}((F^\infty)^\bullet)$. Since the complex $(F^\infty)^\bullet$ is unique up to isomorphism by More on Algebra, Lemma 15.75.5 the proof is complete. □

0ESQ Lemma 38.40.2. Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $U \subset X$ be a scheme theoretically dense open subscheme such that $H^i(E|_U)$ is finite locally free of constant rank r_i for all $i \in \mathbf{Z}$. Then there exists a U-admissible blowup $b : X' \to X$ such that $H^i(Lb^*E)$ is a perfect $\mathcal{O}_{X'}$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$.

Proof. We will construct and study the blowup affine locally. Namely, suppose that $V \subset X$ is an affine open subscheme such that $E|_V$ can be represented by the complex

$$\mathcal{O}_V^{\oplus n_a} \xrightarrow{d_a} \ldots \xrightarrow{d_{b-1}} \mathcal{O}_V^{\oplus n_b}$$

Set $k_i = r_{i+1} - r_{i+2} + \ldots + (-1)^{b-i+1} r_b$. A computation which we omit show that over $U \cap V$ the rank of d_i is

$$\rho_i = -k_i + n_{i+1} - n_{i+2} + \ldots + (-1)^{b-i+1} n_b$$

in the sense that the cokernel of d_i is finite locally free of rank $n_{i+1} - \rho_i$. Let $\mathcal{I}_i \subset \mathcal{O}_V$ be the ideal generated by the minors of size $\rho_i \times \rho_i$ in the matrix of d_i.

On the one hand, comparing with Lemma 38.40.1 we see the ideal $\mathcal{I}_i$ corresponds to the global ideal $\text{Fit}_{i,k_i}(E)$ which was shown to be independent of the choice of the complex representing $E|_V$. On the other hand, $\mathcal{I}_i$ is the $(n_{i+1} - \rho_i)$th Fitting ideal of $\text{Coker}(d_i)$. Please keep this in mind.

We let $b : X' \to X$ be the blowing up in the product of the ideals $\text{Fit}_{i,k_i}(E)$; this makes sense as locally on X almost all of these ideals are equal to the unit ideal (see above). This blowup dominates the blowups $b_i : X'_i \to X$ in the ideals $\text{Fit}_{i,k_i}(E)$, see Divisors, Lemma 31.32.12. By Divisors, Lemma 31.35.3 each b_i is a U-admissible blowup. It follows that b is a U-admissible blowup (tiny detail omitted; compare with the proof of Divisors, Lemma 31.34.4). Finally, U is still a scheme theoretically dense open subscheme of X'. Thus after replacing X by X' we end up in the situation discussed in the next paragraph.

Assume $\text{Fit}_{i,k_i}(E)$ is an invertible ideal for all i. Choose an affine open V and a complex of finite free modules representing $E|_V$ as above. It follows from Divisors, Lemma 31.35.3 that $\text{Coker}(d_i)$ has tor dimension ≤ 1. Thus $\text{Im}(d_i)$ is finite locally free as the kernel of a map from a finite locally free module to a finitely presented module of tor dimension ≤ 1. Hence $\text{Ker}(d_i)$ is finite locally free as well (same argument). Thus the short exact sequence

$$0 \to \text{Im}(d_{i-1}) \to \text{Ker}(d_i) \to H^i(E)|_V \to 0$$

shows what we want and the proof is complete. □

0ESR Lemma 38.40.3. Let X be an integral scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Then there exists a nonempty open $U \subset X$ such that $H^i(E|_U)$ is finite locally free of constant rank r_i for all $i \in \mathbf{Z}$ and there exists a U-admissible blowup $b : X' \to X$ such that $H^i(Lb^*E)$ is a perfect $\mathcal{O}_{X'}$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$.

Proof. We strongly urge the reader to find their own proof of the existence of U. Let $\eta \in X$ be the generic point. The restriction of E to η is isomorphic in $D(\kappa(\eta))$ to a finite complex $V^\bullet$ of finite dimensional vector spaces with zero differentials. Set $r_i = \dim_{\kappa(\eta)} V^i$. Then the perfect object E' in $D(\mathcal{O}_X)$ represented by the complex with terms $\mathcal{O}_X^{\oplus r_i}$ and zero differentials becomes isomorphic to E after pulling back to η. Hence by Derived Categories of Schemes, Lemma 36.35.9 there is an open neighbourhood U of η such that $E|_U$ and $E'|_U$ are isomorphic. This proves the first assertion. The second follows from the first and Lemma 38.40.2 as any nonempty open is scheme theoretically dense in the integral scheme X. □

0F8J Remark 38.40.4. Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be a perfect object such that $H^i(E)$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. This property sometimes allows one to reduce questions about E to questions about $H^i(E)$. For example, suppose

$$\mathcal{E}^a \xrightarrow{d^a} \ldots \xrightarrow{d^{b-2}} \mathcal{E}^{b-1} \xrightarrow{d^{b-1}} \mathcal{E}^b$$

is a bounded complex of finite locally free $\mathcal{O}_X$-modules representing E. Then $\mathrm{Im}(d^i)$ and $\mathrm{Ker}(d^i)$ are finite locally free $\mathcal{O}_X$-modules for all i. Namely, suppose by induction we know this for all indices bigger than i. Then we can first use the short exact sequence

$$0 \to \mathrm{Im}(d^i) \to \mathrm{Ker}(d^{i+1}) \to H^{i+1}(E) \to 0$$

and the assumption that $H^{i+1}(E)$ is perfect of tor dimension ≤ 1 to conclude that $\mathrm{Im}(d^i)$ is finite locally free. The same argument used again for the short exact sequence

$$0 \to \mathrm{Ker}(d^i) \to \mathcal{E}^i \to \mathrm{Im}(d^i) \to 0$$

then gives that $\mathrm{Ker}(d^i)$ is finite locally free. It follows that the distinguished triangles

$$\tau_{\leq k-1}E \to \tau_{\leq k}E \to H^k(E)[-k] \to (\tau_{\leq k-1}E)[1]$$

are represented by the following short exact sequences of bounded complexes of finite locally free modules

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\mathcal{E}^a & \to & \ldots & \to & \mathcal{E}^{k-2} & \to & \mathrm{Ker}(d^{k-1}) & & \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{E}^a & \to & \ldots & \to & \mathcal{E}^{k-2} & \to & \mathcal{E}^{k-1} & \to & \mathrm{Ker}(d^k) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \mathrm{Im}(d^{k-1}) & \to & \mathrm{Ker}(d^k) \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Here the complexes are the rows and the "obvious" zeros are omitted from the display.

38.41. Blowing up perfect modules

0F8K This section tries to find normal forms for perfect modules of tor dimension ≤ 1 after blowups. We are only partially successful.

0ESS Lemma 38.41.1. Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. For any blowup $b : X' \to X$ we have $Lb^*\mathcal{F} = b^*\mathcal{F}$ and $b^*\mathcal{F}$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1.

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine and we may assume the A-module M corresponding to $\mathcal{F}$ has a presentation

$$0 \to A^{\oplus m} \to A^{\oplus n} \to M \to 0$$

Suppose $I \subset A$ is an ideal and $a \in I$. Recall that the affine blowup algebra $A[\frac{I}{a}]$ is a subring of A_a. Since localization is exact we see that $A_a^{\oplus m} \to A_a^{\oplus n}$ is injective. Hence $A[\frac{I}{a}]^{\oplus m} \to A[\frac{I}{a}]^{\oplus n}$ is injective too. This proves the lemma. $\square$

0EST Lemma 38.41.2. Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U$ is finite locally free of constant rank r. Then there exists a U-admissible blowup $b : X' \to X$ such that there is a canonical short exact sequence

$$0 \to \mathcal{K} \to b^*\mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is finite locally free of rank r and $\mathcal{K}$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 whose restriction to U is zero.

Proof. Combine Divisors, Lemma 31.35.3 and Lemma 38.41.1. $\square$

0ESU Lemma 38.41.3. Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. Let $U \subset X$ be an open such that $\mathcal{F}|_U = 0$. Then there is a U-admissible blowup

$$b : X' \to X$$

such that $\mathcal{F}' = b^*\mathcal{F}$ is equipped with two canonical locally finite filtrations

$$0 = F^0 \subset F^1 \subset F^2 \subset \ldots \subset \mathcal{F}' \quad \text{and} \quad \mathcal{F}' = F_1 \supset F_2 \supset F_3 \supset \ldots \supset 0$$

such that for each $n \geq 1$ there is an effective Cartier divisor $D_n \subset X'$ with the property that

$$F^i/F^{i-1} \quad \text{and} \quad F_i/F_{i+1}$$

are finite locally free of rank i on D_i.

Proof. Choose an affine open $V \subset X$ such that there exists a presentation

$$0 \to \mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus n} \to \mathcal{F} \to 0$$

for some n and some matrix A. The ideal we are going to blowup in is the product of the Fitting ideals $\mathrm{Fit}_k(\mathcal{F})$ for $k \geq 0$. This makes sense because in the affine situation above we see that $\mathrm{Fit}_k(\mathcal{F})|_V = \mathcal{O}_V$ for $k > n$. It is clear that this is a U-admissible blowing up. By Divisors, Lemma 31.32.12 we see that on X' the ideals $\mathrm{Fit}_k(\mathcal{F})$ are invertible. Thus we reduce to the case discussed in the next paragraph.

Assume $\mathrm{Fit}_k(\mathcal{F})$ is an invertible ideal for $k \geq 0$. If $E_k \subset X$ is the effective Cartier divisor defined by $\mathrm{Fit}_k(\mathcal{F})$ for $k \geq 0$, then the effective Cartier divisors D_k in the statement of the lemma will satisfy

$$E_k = D_{k+1} + 2D_{k+2} + 3D_{k+3} + \ldots$$

This makes sense as the collection D_k will be locally finite. Moreover, it uniquely determines the effective Cartier divisors D_k hence it suffices to construct D_k locally.

Choose an affine open $V \subset X$ and presentation of $\mathcal{F}|_V$ as above. We will construct the divisors and filtrations by induction on the integer n in the presentation. We set $D_k|_V = \emptyset$ for $k > n$ and we set $D_n|V = E_{n-1}|_V$. After shrinking V we may assume that $\text{Fit}_{n-1}(\mathcal{F})|_V$ is generated by a single nonzerodivisor $f \in \Gamma(V, \mathcal{O}_V)$. Since $\text{Fit}_{n-1}(\mathcal{F})|_V$ is the ideal generated by the entries of A, we see that there is a matrix A' in $\Gamma(V, \mathcal{O}_V)$ such that $A = fA'$. Define $\mathcal{F}'$ on V by the short exact sequence

$$0 \to \mathcal{O}_V^{\oplus n} \xrightarrow{A'} \mathcal{O}_V^{\oplus n} \to \mathcal{F}' \to 0$$

Since the entries of A' generate the unit ideal in $\Gamma(V, \mathcal{O}_V)$ we see that $\mathcal{F}'$ locally on V has a presentation with n decreased by 1, see Algebra, Lemma 10.102.2. Further note that $f^{n-k}\text{Fit}_k(\mathcal{F}') = \text{Fit}_k(\mathcal{F})|_V$ for $k = 0, \ldots, n$. Hence $\text{Fit}_k(\mathcal{F}')$ is an invertible ideal for all k. We conclude by induction that there exist effective Cartier divisors $D'_k \subset V$ such that $\mathcal{F}'$ has two canonical filtrations as in the statement of the lemma. Then we set $D_k|_V = D'_k$ for $k = 1, \ldots, n-1$. Observe that the equalities between effective Cartier divisors displayed above hold with these choices. Finally, we come to the construction of the filtrations. Namely, we have short exact sequences

$$0 \to \mathcal{O}_{D_n \cap V}^{\oplus n} \to \mathcal{F} \to \mathcal{F}' \to 0 \quad \text{and} \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{O}_{D_n \cap V}^{\oplus n} \to 0$$

coming from the two factorizations $A = A'f = fA'$ of A. These sequences are canonical because in the first one the submodule is $\text{Ker}(f : \mathcal{F} \to \mathcal{F})$ and in the second one the quotient module is $\text{Coker}(f : \mathcal{F} \to \mathcal{F})$. □

0ESV Lemma 38.41.4. Let X be a scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homorphism of perfect $\mathcal{O}_X$-modules of tor dimension ≤ 1. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U = 0$ and $\mathcal{G}|_U = 0$. Then there is a U-admissible blowup $b : X' \to X$ such that the kernel, image, and cokernel of $b^*\varphi$ are perfect $\mathcal{O}_{X'}$-modules of tor dimension ≤ 1.

Proof. The assumptions tell us that the object $(\mathcal{F} \to \mathcal{G})$ of $D(\mathcal{O}_X)$ is perfect. Thus we get a U-admissible blowup that works for the cokernel and kernel by Lemmas 38.40.2 and 38.41.1 (to see what the complex looks like after pullback). The image is the kernel of the cokernel and hence is going to be perfect of tor dimension ≤ 1 as well. □

38.42. An operator introduced by Berthelot and Ogus

0F8L Please read Cohomology, Section 20.55 first.

Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{I} = \mathcal{I}_D \subset \mathcal{O}_X$ be the ideal sheaf of D, see Divisors, Section 31.14. Clearly we can apply the discussion in Cohomology, Section 20.55 to X and $\mathcal{I}$.

0F8M Lemma 38.42.1. Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $\mathcal{I}$-torsion free for all i. Then $\eta_\mathcal{I}\mathcal{F}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules. Moreover, if $U = \text{Spec}(A) \subset X$ is affine open and $D \cap U = V(f)$, then $\eta_f(\mathcal{F}^\bullet(U))$ is canonically isomorphic to $(\eta_\mathcal{I}\mathcal{F}^\bullet)(U)$.

Proof. Omitted. □

0GTV Lemma 38.42.2. Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The functor $L\eta_\mathcal{I} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ of Cohomology, Lemma 20.55.7 sends $D_{QCoh}(\mathcal{O}_X)$ into itself. Moreover, if $X = \mathrm{Spec}(A)$ is affine and $D = V(f)$, then the functor $L\eta_f$ on $D(A)$ defined in More on Algebra, Lemma 15.95.4 and the functor $L\eta_\mathcal{I}$ on $D_{QCoh}(\mathcal{O}_X)$ correspond via the equivalence of Derived Categories of Schemes, Lemma 36.3.5.

Proof. Omitted. □

38.43. Blowing up complexes, II

0F8R The material in this section will be used to construct a version of Macpherson's graph construction in Section 38.44.

0F8S Situation 38.43.1. Here X is a scheme, $D \subset X$ is an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and M is a perfect object of $D(\mathcal{O}_X)$.

Let (X, D, M) be a triple as in Situation 38.43.1. Consider an affine open $U = \mathrm{Spec}(A) \subset X$ such that

(1) $D \cap U = V(f)$ for some nonzerodivisor $f \in A$, and
(2) there exists a bounded complex $M^\bullet$ of finite free A-modules representing $M|_U$ (via the equivalence of Derived Categories of Schemes, Lemma 36.3.5).

We will say that $(U, A, f, M^\bullet)$ is an affine chart for (X, D, M). Consider the ideals $I_i(M^\bullet, f) \subset A$ defined in More on Algebra, Section 15.96. Let us say (X, S, M) is a good triple if for every $x \in D$ there exists an affine chart $(U, A, f, M^\bullet)$ with $x \in U$ and $I_i(M^\bullet, f)$ principal ideals for all $i \in \mathbf{Z}$.

0F8T Lemma 38.43.2. In Situation 38.43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ of D is defined (Divisors, Definition 31.13.12). Let $(U, A, f, M^\bullet)$ is an affine chart for (X, D, M). Let $V = \mathrm{Spec}(B) \subset Y$ is an affine open with $h(V) \subset U$. Denote $g \in B$ the image of $f \in A$. Then

(1) $(V, B, g, M^\bullet \otimes_A B)$ is an affine chart for (Y, E, Lh^*M),
(2) $I_i(M^\bullet, f)B = I_i(M^\bullet \otimes_A B, g)$ in B, and
(3) if (X, D, M) is a good triple, then (Y, E, Lh^*M) is a good triple.

Proof. The first statement follows from the folowing observations: g is a nonzerodivisor in B which defines $E \cap V \subset V$ and $M^\bullet \otimes_A B$ represents $M^\bullet \otimes_A^{\mathbf{L}} B$ and hence represents the pullback of M to V by Derived Categories of Schemes, Lemma 36.3.8. Part (2) follows from part (1) and More on Algebra, Lemma 15.96.3. Combined with More on Algebra, Lemma 15.96.3 we conclude that the second statement of the lemma holds. □

0GTW Lemma 38.43.3. Let $X, D, \mathcal{I}, M$ be as in Situation 38.43.1. If (X, D, M) is a good triple, then $L\eta_\mathcal{I} M$ is a perfect object of $D(\mathcal{O}_X)$.

Proof. Translation of More on Algebra, Lemma 15.96.5. To do the translation use Lemma 38.42.2. □

0GTX Lemma 38.43.4. Let $X, D, \mathcal{I}, M$ be as in Situation 38.43.1. Assume (X, D, M) is a good triple. If there exists a locally bounded complex $\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing M, then there exists a locally bounded complex $\mathcal{Q}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules representing $L\eta_\mathcal{I} M$.

Proof. By Cohomology, Lemma 20.55.7 the complex $\mathcal{Q}^\bullet = \eta_\mathcal{I}\mathcal{M}^\bullet$ represents $L\eta_\mathcal{I}M$. To check that this complex is locally bounded and consists of finite locally free, we may work affine locally. Then the boundedness is clear. Choose an affine chart $(U, A, f, M^\bullet)$ for (X, D, M) such that the ideals $I_i(M^\bullet, f)$ are principal and such that $\mathcal{M}^i|_U$ is finite free for each i. By our assumption that (X, D, M) is a good triple we can do this. Writing $N^i = \Gamma(U, \mathcal{M}^i|_U)$ we get a bounded complex $N^\bullet$ of finite free A-modules representing the same object in $D(A)$ as the complex $M^\bullet$ (by Derived Categories of Schemes, Lemma 36.3.5). Then $I_i(N^\bullet, f)$ is a principal ideal for all i by More on Algebra, Lemma 15.96.1. Hence the complex $\eta_f N^\bullet$ is a bounded complex of finite locally free A-modules. Since $\mathcal{Q}^i|_U$ is the quasi-coherent $\mathcal{O}_U$-module corresponding to $\eta_f N^i$ by Lemma 38.42.1 we conclude. $\square$

0F9X **Lemma 38.43.5.** In Situation 38.43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ is defined. If (X, D, M) is a good triple, then

$$Lh^*(L\eta_\mathcal{I}M) = L\eta_\mathcal{J}(Lh^*M)$$

in $D(\mathcal{O}_Y)$ where $\mathcal{J}$ is the ideal sheaf of E.

Proof. Translation of More on Algebra, Lemma 15.96.6. Use Lemmas 38.42.1 and 38.42.2 to do the translation. $\square$

0GTY **Lemma 38.43.6.** In Situation 38.43.1 there is a unique morphism $b : X' \to X$ such that

(1) the pullback $D' = b^{-1}D$ is defined and (X', D', M') is a good triple where $M' = Lb^*M$, and
(2) for any morphism of schemes $h : Y \to X$ such that the pullback $E = h^{-1}D$ is defined and (Y, E, Lh^*M) is a good triple, there is a unique factorization of h through b.

Moreover, for any affine chart $(U, A, f, M^\bullet)$ the restriction $b^{-1}(U) \to U$ is the blowing up in the product of the ideals $I_i(M^\bullet, f)$ and for any quasi-compact open $W \subset X$ the restriction $b|_{b^{-1}(W)} : b^{-1}(W) \to W$ is a $W \setminus D$-admissible blowing up.

Proof. The proof is just that we will locally blow up X in the product ideals $I_i(M^\bullet, f)$ for any affine chart $(U, A, f, M^\bullet)$. The first few lemmas in More on Algebra, Section 15.96 show that this is well defined. The universal property (2) then follows from the universal property of blowing up. The details can be found below.

Let $U, A, f, M^\bullet$ be an affine chart for (X, D, M). All but a finite number of the ideals $I_i(M^\bullet, f)$ are equal to A hence it makes sense to look at

$$I = \prod\nolimits_i I_i(M^\bullet, f)$$

and this is a finitely generated ideal of A. Denote

$$b_U : U' \to U$$

the blowing up of U in I. Then $b_U^{-1}(U \cap D)$ is defined by Divisors, Lemma 31.32.11. Recall that $f^{r_i} \in I_i(M^\bullet, f)$ and hence b_U is a $(U \setminus D)$-admissible blowing up. By Divisors, Lemma 31.32.12 for each i the morphism b_U factors as $U' \to U'_i \to U$ where $U'_i \to U$ is the blowing up in $I_i(M^\bullet, f)$ and $U' \to U'_i$ is another blowing up. It follows that the pullback $I_i(M^\bullet, f)\mathcal{O}_{U'}$ of $I_i(M^\bullet, f)$ to U' is an invertible ideal sheaf, see Divisors, Lemmas 31.32.11 and 31.32.4. It follows that $(U', b^{-1}D, Lb^*M|_U)$ is a good triple, see Lemma 38.43.2 for the behaviour of the ideals $I_i(-, -)$ under pullback. Finally, we claim that $b_U : U' \to U$ has the universal property mentioned in part (2) of the statement of the lemma. Namely, suppose $h : Y \to U$ is a morphism of schemes such that the pullback $E = h^{-1}(D \cap U)$ is defined and (Y, E, Lh^*M) is a good triple. Then Y is covered by affine charts $(V, B, g, N^\bullet)$ such that $I_i(N^\bullet, g)$ is an invertible ideal for each i. Then g and the image of f in B differ by a unit as they both cut out the effective Cartier divisor $E \cap V$. Hence we may assume g is the image of f by More on Algebra, Lemma 15.96.2. Then $I_i(N^\bullet, g)$ is isomorphic to $I_i(M^\bullet \otimes_A B, g)$ as a B-module by More on Algebra, Lemma 15.96.1. Thus $I_i(M^\bullet \otimes_A B, g) = I_i(M^\bullet, f)B$ (Lemma 38.43.2) is an invertible B-module. Hence the ideal IB is invertible. It follows that $I\mathcal{O}_Y$ is invertible. Hence we obtain a unique factorization of h through b_U by Divisors, Lemma 31.32.5.

Let $\mathcal{B}$ be the set of affine opens $U \subset X$ such that there exists an affine chart $(U, A, f, M^\bullet)$ for (X, D, M). Then $\mathcal{B}$ is a basis for the topology on X; details omitted. For $U \in \mathcal{B}$ we have the morphism $b_U : U' \to U$ constructed above which satisfies the universal property over U. If $U_1 \subset U_2 \subset X$ are both in $\mathcal{B}$, then $b_{U_1} : U_1' \to U_1$ is canonically isomorphic to

$$b_{U_2}|_{b_{U_2}^{-1}(U_1)} : b_{U_2}^{-1}(U_1) \longrightarrow U_1$$

by the universal propery. In other words, we get an isomorphism $U_1' \to b_{U_2}^{-1}(U_1)$ over U_1. These isomorphisms satisfy the cocycle condition (again by the universal property) and hence by Constructions, Lemma 27.2.1 we get a morphism $b : X' \to X$ whose restriction to each U in $\mathcal{B}$ is isomorphic to $U' \to U$. Then the morphism $b : X' \to X$ satisfies properties (1) and (2) of the statement of the lemma as these properties may be checked locally (details omitted).

We still have to prove the final assertion of the lemma. Let $W \subset X$ be a quasi-compact open. Choose a finite covering $W = U_1 \cup \ldots \cup U_T$ such that for each $1 \leq t \leq T$ there exists an affine chart $(U_t, A_t, f_t, M_t^\bullet)$. We will use below that for any affine open $V = \text{Spec}(B) \subset U_t \cap U_{t'}$ we have (a) the images of f_t and $f_{t'}$ in B differ by a unit, and (b) the complexes $M_t^\bullet \otimes_A B$ and $M_{t'} \otimes_A B$ define isomorphic objects of $D(B)$. For $i \in \mathbf{Z}$, set

$$N_i = \max\nolimits_{t=1,\ldots,T} \left(\sum\nolimits_{j \geq i} (-1)^{j-i} rk(M_t^j)\right)$$

Then $N_t - \sum_{j \geq i} (-1)^{j-i} rk(M_t^j) \geq 0$ and we can consider the ideals

$$I_{t,i} = f_t^{N_i - \sum_{j \geq i} (-1)^{j-i} rk(M_t^j)} I_i(M_t^\bullet, f_t) \subset A_t$$

It follows from More on Algebra, Lemmas 15.96.2 and 15.96.1 that the ideals $I_{t,i}$ glue to a quasi-coherent, finite type ideal $\mathcal{I}_i \subset \mathcal{O}_W$. Moreover, all but a finite number of these ideals are equal to $\mathcal{O}_W$. Clearly, the morphism $X' \to X$ constructed above restricts to the blowing up of W in the product of the ideals $\mathcal{I}_i$. This finishes the proof. $\square$

0F8U Lemma 38.43.7. In Situation 38.43.1 let $b : X' \to X$ be the morphism of Lemma 38.43.6. Consider the effective Cartier divisor $D' = b^{-1}D$ with ideal sheaf $\mathcal{I}' \subset \mathcal{O}_{X'}$. Then $Q = L\eta_{\mathcal{I}'} Lb^*M$ is a perfect object of $D(\mathcal{O}_{X'})$.

Proof. Follows from Lemmas 38.43.6 and 38.43.3. $\square$

0F8V Lemma 38.43.8. In Situation 38.43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ is defined. Let $b : X' \to X$, resp. $c : Y' \to Y$ be as constructed in Lemma 38.43.6 for $D \subset X$ and M, resp. $E \subset Y$ and Lh^*M. Then Y' is the strict transform of Y with respect to $b : X' \to X$ (see proof for a precise formulation of this) and

$$L\eta_{\mathcal{J}'} L(h \circ c)^* M = L(Y' \to X')^* Q$$

where $Q = L\eta_{\mathcal{I}'} Lb^* M$ as in Lemma 38.43.7. In particular, if (Y, E, Lh^*M) is a good triple and $k : Y \to X'$ is the unique morphism such that $h = b \circ k$, then $L\eta_{\mathcal{J}} Lh^* M = Lk^* Q$.

Proof. Denote $E' = c^{-1}E$. Then $(Y', E', L(h \circ c)^* M)$ is a good triple. Hence by the universal property of Lemma 38.43.6 there is a unique morphism

$$h' : Y' \longrightarrow X'$$

such that $b \circ h' = h \circ c$. In particular, there is a morphism $(h', c) : Y' \to X' \times_X Y$. We claim that given $W \subset X$ quasi-compact open, such that $b^{-1}(W) \to W$ is a blowing up, this morphism identifies $Y'|_W$ with the strict transform of Y_W with respect to $b^{-1}(W) \to W$. In turn, to see this is true is a local question on W, and we may therefore prove the statement over an affine chart. We do this in the next paragraph.

Let $(U, A, f, M^\bullet)$ be an affine chart for (X, D, M). Recall from the proof of Lemma 38.43.7 that the restriction of $b : X' \to X$ to U is the blowing up of $U = \mathrm{Spec}(A)$ in the product of the ideals $I_i(M^\bullet, f)$. Now if $V = \mathrm{Spec}(B) \subset Y$ is any affine open with $h(V) \subset U$, then $(V, B, g, M^\bullet \otimes_A B)$ is an affine chart for (Y, E, Lh^*M) where $g \in B$ is the image of f, see Lemma 38.43.2. Hence the restriction of $c : Y' \to Y$ to V is the blowing up in the product of the ideals $I_i(M^\bullet, f)B$, i.e., the morphism $c : Y' \to Y$ over $h^{-1}(U)$ is the blowing up of $h^{-1}(U)$ in the ideal $\prod I_i(M^\bullet, f)\mathcal{O}_{h^{-1}(U)}$. Since this is also true for the strict transform, we see that our claim on strict transforms is true.

Having said this the equality $L\eta_{\mathcal{J}'} L(h \circ c)^* M = L(Y' \to X')^* Q$ follows from Lemma 38.43.5. The final statement is a special case of this (namely, the case where $c = \mathrm{id}_Y$ and $k = h'$). $\square$

0F8W Lemma 38.43.9. In Situation 38.43.1 let $W \subset X$ be the maximal open subscheme over which the cohomology sheaves of M are locally free. Then the morphism $b : X' \to X$ of Lemma 38.43.6 is an isomorphism over W.

Proof. This is true because for any affine chart $(U, A, f, M^\bullet)$ with $U \subset W$ we have that $I_i(M^\bullet, f)$ are locally generated by a power of f by More on Algebra, Lemma 15.96.4. Since f is a nonzerodivisor, the blowing up $b^{-1}(U) \to U$ is an isomorphism. $\square$

0GTZ Lemma 38.43.10. Let $X, D, \mathcal{I}, M$ be as in Situation 38.43.1. If (X, D, M) is a good triple, then there exists a closed immersion

$$i : T \longrightarrow D$$

of finite presentation with the following properties

(1) T scheme theoretically contains $D \cap W$ where $W \subset X$ is the maximal open over which the cohomology sheaves of M are locally free,

(2) the cohomology sheaves of $Li^*L\eta_\mathcal{I}M$ are locally free, and
(3) for any point $t \in T$ with image $x = i(t) \in W$ the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$ and the rank of $H^i(Li^*L\eta_\mathcal{I}M)_t$ over $\mathcal{O}_{T,t}$ agree.

Proof. Let $(U, A, f, M^\bullet)$ be an affine chart for (X, D, M) such that $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then we define $T \cap U \subset D \cap U$ as the closed subscheme defined by the ideal

$$J(M^\bullet, f) = \sum J_i(M^\bullet, f) \subset A/fA$$

studied in More on Algebra, Lemmas 15.96.8 and 15.96.9; in terms of the second lemma we see that $T \cap U \to D \cap U$ is given by the ring map $A/fA \to C$ studied there. Since (X, D, M) is a good triple we can cover X by affine charts of this form and by the first of the two lemmas, this construction glues. Hence we obtain a closed subscheme $T \subset D$ which on good affine charts as above is given by the ideal $J(M^\bullet, f)$. Then properties (1) and (2) follow from the second lemma. Details omitted. Small observation to help the reader: since $\eta_f M^\bullet$ is a complex of locally free modules by More on Algebra, Lemma 15.96.5 we see that $Li^*L\eta_\mathcal{I}M|_{T\cap U}$ is represented by the complex $\eta_f M^\bullet \otimes_A C$ of C-modules. The statement (3) on ranks follows from Cohomology, Lemma 20.55.10. □

0F8X Lemma 38.43.11. In Situation 38.43.1. Let $b : X' \to X$ and D' be as in Lemma 38.43.6. Let $Q = L\eta_{\mathcal{I}'}Lb^*M$ be as in Lemma 38.43.7. Let $W \subset X$ be the maximal open where M has locally free cohomology modules. Then there exists a closed immersion $i : T \to D'$ of finite presentation such that

(1) $D' \cap b^{-1}(W) \subset T$ scheme theoretically,
(2) Li^*Q has locally free cohomology sheaves, and
(3) for $t \in T$ mapping to $w \in W$ the rank of $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$.

Proof. Lemma 38.43.9 tells us that b is an isomorphism over W. Hence $b^{-1}(W) \subset X'$ is contained in the maximal open $W' \subset X'$ where Lb^*M has locally free cohomology sheaves. Then the actual statements in the lemma are an immediate consequence of Lemma 38.43.10 applied to (X', D', Lb^*M) and the other lemmas mentioned in the statement. □

0F8Y Lemma 38.43.12. In Situation 38.43.1. Let $b : X' \to X$, $D' \subset X'$, and Q be as in Lemma 38.43.7. Let $\rho = (\rho_i)_{i\in\mathbf{Z}}$ be integers. Let $W(\rho) \subset X$ be the maximal open subscheme where $H^i(M)$ is locally free of rank ρ_i for all i. Let $i : T \to D'$ be as in Lemma 38.43.11. Then there exists an open and closed subscheme $T(\rho) \subset T$ containing $D' \cap b^{-1}(W(\rho))$ scheme theoretically such that $H^i(Li^*Q|_{T(\rho)})$ is locally free of rank ρ_i for all i.

Proof. Let $T(\rho) \subset T$ be the open and closed subscheme where $H^i(Li^*Q)$ has rank ρ_i for all i. Then the statement is immediate from the assertion in Lemma 38.43.11 on ranks of the cohomology modules. □

0GU0 Lemma 38.43.13. In Situation 38.43.1. Let $b : X' \to X$, $D' \subset X'$, and Q be as in Lemma 38.43.7. If there exists a locally bounded complex $\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing M, then there exists a locally bounded complex $\mathcal{Q}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules representing Q.

Proof. Recall that $Q = L\eta_{\mathcal{I}'} Lb^*M$ where $\mathcal{I}'$ is the ideal sheaf of the effective Cartier divisor D'. The locally bounded complex $(\mathcal{M}')^\bullet = b^*\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules represents Lb^*M. Thus the lemma follows from Lemma 38.43.4. $\square$

0F9Y Lemma 38.43.14. Let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Let $M \in D(\mathcal{O}_X)$ be a perfect object. Let $W \subset X$ be the maximal open over which the cohomology sheaves $H^i(M)$ are locally free. There exists a proper morphism $b : X' \longrightarrow X$ and an object Q in $D(\mathcal{O}_{X'})$ with the following properties

1. $b : X' \to X$ is an isomorphism over $X \setminus D$,
2. $b : X' \to X$ is an isomorphism over W,
3. $D' = b^{-1}D$ is an effective Cartier divisor,
4. $Q = L\eta_{\mathcal{I}'} Lb^*M$ where $\mathcal{I}'$ is the ideal sheaf of D',
5. Q is a perfect object of $D(\mathcal{O}_{X'})$,
6. there exists a closed immersion $i : T \to D'$ of finite presentation such that
   (a) $D' \cap b^{-1}(W) \subset T$ scheme theoretically,
   (b) Li^*Q has finite locally free cohomology sheaves,
   (c) for $t \in T$ with image $w \in W$ the rank of $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$,
7. for any affine chart $(U, A, f, M^\bullet)$ for (X, D, M) the restriction of b to U is the blowing up of $U = \operatorname{Spec}(A)$ in the ideal $I = \prod I_i(M^\bullet, f)$, and
8. for any affine chart $(V, B, g, N^\bullet)$ for (X', D', Lb^*N) such that $I_i(N^\bullet, g)$ is principal, we have
   (a) $Q|_V$ corresponds to $\eta_g N^\bullet$,
   (b) $T \subset V \cap D'$ corresponds to the ideal $J(N^\bullet, g) = \sum J_i(N^\bullet, g) \subset B/gB$ studied in More on Algebra, Lemma 15.96.9.
9. If M can be represented by a locally bounded complex of finite locally free $\mathcal{O}_X$-modules, then Q can be represented by a bounded complex of finite locally free $\mathcal{O}_{X'}$-modules.

Proof. This statement collects the information obtained in Lemmas 38.43.2, 38.43.3, 38.43.5, 38.43.6, 38.43.7, 38.43.8, 38.43.9, 38.43.10, 38.43.11, and 38.43.13. $\square$

38.44. Blowing up complexes, III

0F8Z In this section we give an "algebra version" of the version of Macpherson's graph construction given in [Ful98, Section 18.1].

Let X be a scheme. Let E be a perfect object of $D(\mathcal{O}_X)$. Let $U \subset X$ be the maximal open subscheme such that $E|_U$ has locally free cohomology sheaves.

Consider the commutative diagram

$$\begin{tikzcd} \mathbf{A}^1_X \arrow[r] \arrow[dr] & \mathbf{P}^1_X \arrow[d, "p"] & (\mathbf{P}^1_X)_\infty \arrow[l] \arrow[dl] \\ & X \arrow[ur, bend right, "\infty"'] & \end{tikzcd}$$

Here we recall that $\mathbf{A}^1 = D_+(T_0)$ is the first standard affine open of $\mathbf{P}^1$ and that $\infty = V_+(T_0)$ is the complementary effective Cartier divisor and the diagram above is the pullback of these schemes to X. Observe that $\infty : X \to (\mathbf{P}^1_X)_\infty$ is an isomorphism. Then

$$(\mathbf{P}^1_X, (\mathbf{P}^1_X)_\infty, Lp^*E)$$

is a triple as in Situation 38.43.1 in Section 38.43. Let

$$b : W \longrightarrow \mathbf{P}^1_X \quad \text{and} \quad W_\infty = b^{-1}((\mathbf{P}^1_X)_\infty)$$

be the blowing up and effective Cartier divisor constructed starting with this triple in Lemma 38.43.6. We also denote

$$Q = L\eta_\mathcal{I} Lb^*M = L\eta_\mathcal{I} L(p \circ b)^*E$$

the perfect object of $D(\mathcal{O}_W)$ considered in Lemma 38.43.7. Here $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of W_∞.

0F90 **Lemma 38.44.1.** The construction above has the following properties:

(1) b is an isomorphism over $\mathbf{P}^1_U \cup \mathbf{A}^1_X$,
(2) the restriction of Q to $\mathbf{A}^1_X$ is equal to the pullback of E,
(3) there exists a closed immersion $i : T \to W_\infty$ of finite presentation such that $(W_\infty \to X)^{-1}U \subset T$ scheme theoretically and such that Li^*Q has locally free cohomology sheaves,
(4) for $t \in T$ with image $u \in U$ we have that the rank $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_u$ over $\mathcal{O}_{U,u}$,
(5) if E can be represented by a locally bounded complex of finite locally free $\mathcal{O}_X$-modules, then Q can be represented by a locally bounded complex of finite locally free $\mathcal{O}_W$-modules.

Proof. This follows immediately from the results in Section 38.43; for a statement collecting everything needed, see Lemma 38.43.14. $\square$

38.45. Other chapters

CHAPTER 39

Groupoid Schemes

022L
39.1. Introduction

022M This chapter is devoted to generalities concerning groupoid schemes. See for example the beautiful paper [KM97] by Keel and Mori.

39.2. Notation

022N Let S be a scheme. If U, T are schemes over S we denote $U(T)$ for the set of T-valued points of U over S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$. We try to reserve the letter T to denote a "test scheme" over S, as in the discussion that follows. Suppose we are given schemes X, Y over S and a morphism of schemes $f : X \to Y$ over S. For any scheme T over S we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which as indicated we denote by f also. In fact this construction is functorial in the scheme T/S. Yoneda's Lemma, see Categories, Lemma 4.3.5, says that f determines and is determined by this transformation of functors $f : h_X \to h_Y$. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are schemes over S, and if $m : X \times_S Y \to Z \times_S Z$ is a morphism of schemes over S, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

We continue our convention to label projection maps starting with index 0, so we have $\mathrm{pr}_0 : X \times_S Y \to X$ and $\mathrm{pr}_1 : X \times_S Y \to Y$.

39.3. Equivalence relations

022O Recall that a relation R on a set A is just a subset of $R \subset A \times A$. We usually write aRb to indicate $(a, b) \in R$. We say the relation is transitive if $aRb, bRc \Rightarrow aRc$. We say the relation is reflexive if aRa for all $a \in A$. We say the relation is symmetric if $aRb \Rightarrow bRa$. A relation is called an equivalence relation if it is transitive, reflexive and symmetric.

In the setting of schemes we are going to relax the notion of a relation a little bit and just require $R \to A \times A$ to be a map. Here is the definition.

022P Definition 39.3.1. Let S be a scheme. Let U be a scheme over S.

(1) A pre-relation on U over S is any morphism of schemes $j : R \to U \times_S U$. In this case we set $t = \mathrm{pr}_0 \circ j$ and $s = \mathrm{pr}_1 \circ j$, so that $j = (t, s)$.
(2) A relation on U over S is a monomorphism of schemes $j : R \to U \times_S U$.

(3) A pre-equivalence relation is a pre-relation $j : R \to U \times_S U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all T/S.
(4) We say a morphism $R \to U \times_S U$ of schemes is an equivalence relation on U over S if and only if for every scheme T over S the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

02V8 Lemma 39.3.2. Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. Finally, set

$$R' = (U' \times_S U') \times_{U \times_S U} R \xrightarrow{j'} U' \times_S U'$$

Then j' is a pre-relation on U' over S. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

02V9 Definition 39.3.3. Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. The pre-relation $j' : R' \to U' \times_S U'$ is called the restriction, or pullback of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

022Q Lemma 39.3.4. Let $j : R \to U \times_S U$ be a pre-relation. Consider the relation on points of the scheme U defined by the rule

$$x \sim y \Leftrightarrow \exists\, r \in R : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in R$ with $t(r) = x$, $s(r) = y$ and pick $r' \in R$ with $t(r') = y$, $s(r') = z$. Pick a field K fitting into the following commutative diagram

$$\begin{tikzcd} \kappa(r) \arrow[r] & K \\ \kappa(y) \arrow[u] \arrow[r] & \kappa(r') \arrow[u] \end{tikzcd}$$

Denote $x_K, y_K, z_K : \mathrm{Spec}(K) \to U$ the morphisms

$$\begin{matrix} \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r)) \to \mathrm{Spec}(\kappa(x)) \to U \\ \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r)) \to \mathrm{Spec}(\kappa(y)) \to U \\ \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r')) \to \mathrm{Spec}(\kappa(z)) \to U \end{matrix}$$

By construction $(x_K, y_K) \in j(R(K))$ and $(y_K, z_K) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x_K, z_K) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

0DT7 Lemma 39.3.5. Let $j : R \to U \times_S U$ be a pre-relation. Assume
(1) s, t are unramified,
(2) for any algebraically closed field k over S the map $R(k) \to U(k) \times U(k)$ is an equivalence relation,

(3) there are morphisms $e : U \to R$, $i : R \to R$, $c : R \times_{s,U,t} R \to R$ such that

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "\Delta"] & R \arrow[d, "j"] \\ U \times_S U \arrow[r] & U \times_S U \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "i"] \arrow[d, "j"] & R \arrow[d, "j"] \\ U \times_S U \arrow[r, "flip"] & U \times_S U \end{tikzcd} \quad \begin{tikzcd} R \times_{s,U,t} R \arrow[r, "c"] \arrow[d, "j \times j"] & R \arrow[d, "j"] \\ U \times_S U \times_S U \arrow[r, "\mathrm{pr}_{02}"] & U \times_S U \end{tikzcd}$$

are commutative.

Then j is an equivalence relation.

Proof. By condition (1) and Morphisms, Lemma 29.35.16 we see that j is a unramified. Then $\Delta_j : R \to R \times_{U \times_S U} R$ is an open immersion by Morphisms, Lemma 29.35.13. However, then condition (2) says Δ_j is bijective on k-valued points, hence Δ_j is an isomorphism, hence j is a monomorphism. Then it easily follows from the commutative diagrams that $R(T) \subset U(T) \times U(T)$ is an equivalence relation for all schemes T over S. $\square$

39.4. Group schemes

022R Let us recall that a group is a pair (G, m) where G is a set, and $m : G \times G \to G$ is a map of sets with the following properties:

1. (associativity) $m(g, m(g', g'')) = m(m(g, g'), g'')$ for all $g, g', g'' \in G$,
2. (identity) there exists a unique element $e \in G$ (called the identity, unit, or 1 of G) such that $m(g, e) = m(e, g) = g$ for all $g \in G$, and
3. (inverse) for all $g \in G$ there exists a $i(g) \in G$ such that $m(g, i(g)) = m(i(g), g) = e$, where e is the identity.

Thus we obtain a map $e : \{*\} \to G$ and a map $i : G \to G$ so that the quadruple (G, m, e, i) satisfies the axioms listed above.

A homomorphism of groups $\psi : (G, m) \to (G', m')$ is a map of sets $\psi : G \to G'$ such that $m'(\psi(g), \psi(g')) = \psi(m(g, g'))$. This automatically insures that $\psi(e) = e'$ and $i'(\psi(g)) = \psi(i(g))$. (Obvious notation.) We will use this below.

022S Definition 39.4.1. Let S be a scheme.

1. A group scheme over S is a pair (G, m), where G is a scheme over S and $m : G \times_S G \to G$ is a morphism of schemes over S with the following property: For every scheme T over S the pair $(G(T), m)$ is a group.
2. A morphism $\psi : (G, m) \to (G', m')$ of group schemes over S is a morphism $\psi : G \to G'$ of schemes over S such that for every T/S the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group scheme over the scheme S. By the discussion above (and the discussion in Section 39.2) we obtain morphisms of schemes over S: (identity) $e : S \to G$ and (inverse) $i : G \to G$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group listed above.

Let (G, m), (G', m') be group schemes over S. Let $f : G \to G'$ be a morphism of schemes over S. It follows from the definition that f is a morphism of group schemes over S if and only if the following diagram is commutative:

$$\begin{tikzcd} G \times_S G \arrow[r, "f \times f"'] \arrow[d, "m"'] & G' \times_S G' \arrow[d, "m'"] \\ G \arrow[r, "f"] & G' \end{tikzcd}$$

022T Lemma 39.4.2. Let (G, m) be a group scheme over S. Let $S' \to S$ be a morphism of schemes. The pullback $(G_{S'}, m_{S'})$ is a group scheme over S'.

Proof. Omitted. $\square$

047D Definition 39.4.3. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) A closed subgroup scheme of G is a closed subscheme $H \subset G$ such that $m|_{H \times_S H}$ factors through H and induces a group scheme structure on H over S.
(2) An open subgroup scheme of G is an open subscheme $G' \subset G$ such that $m|_{G' \times_S G'}$ factors through G' and induces a group scheme structure on G' over S.

Alternatively, we could say that H is a closed subgroup scheme of G if it is a group scheme over S endowed with a morphism of group schemes $i : H \to G$ over S which identifies H with a closed subscheme of G.

0G8L Lemma 39.4.4. Let S be a scheme. Let (G, m, e, i) be a group scheme over S.

(1) A closed subscheme $H \subset G$ is a closed subgroup scheme if and only if $e : S \to G$, $m|_{H \times_S H} : H \times_S H \to G$, and $i|_H : H \to G$ factor through H.
(2) An open subscheme $H \subset G$ is an open subgroup scheme if and only if $e : S \to G$, $m|_{H \times_S H} : H \times_S H \to G$, and $i|_H : H \to G$ factor through H.

Proof. Looking at T-valued points this translates into the well known conditions characterizing subsets of groups as subgroups. $\square$

047E Definition 39.4.5. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) We say G is a smooth group scheme if the structure morphism $G \to S$ is smooth.
(2) We say G is a flat group scheme if the structure morphism $G \to S$ is flat.
(3) We say G is a separated group scheme if the structure morphism $G \to S$ is separated.

Add more as needed.

39.5. Examples of group schemes

047F

022U Example 39.5.1 (Multiplicative group scheme). Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[x, x^{-1}])$$

The morphism giving the group structure is the morphism

$$\begin{array}{rcl} \mathbf{G}_m \times \mathbf{G}_m & \to & \mathbf{G}_m \\ \mathrm{Spec}(\mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}]) & \to & \mathrm{Spec}(\mathbf{Z}[x, x^{-1}]) \\ \mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}] & \leftarrow & \mathbf{Z}[x, x^{-1}] \\ x \otimes x & \leftarrow & x \end{array}$$

Hence we see that $\mathbf{G}_m$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{m,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{m,S}(T) = \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^*)$$

as before.

040M Example 39.5.2 (Roots of unity). Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the scheme

$$\mu_n = \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1)).$$

The morphism giving the group structure is the morphism

$$\begin{array}{rcl} \mu_n \times \mu_n & \to & \mu_n \\ \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1)) & \to & \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1)) \\ \mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1) & \leftarrow & \mathbf{Z}[x]/(x^n - 1) \\ x \otimes x & \leftarrow & x \end{array}$$

Hence we see that μ_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mu_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mu_{n,S}(T) = \mu_n(T) = \{f \in \Gamma(T, \mathcal{O}_T^*) \mid f^n = 1\}$$

as before.

022V Example 39.5.3 (Additive group scheme). Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_a = \mathrm{Spec}(\mathbf{Z}[x])$$

The morphism giving the group structure is the morphism

$$\begin{array}{rcl} \mathbf{G}_a \times \mathbf{G}_a & \to & \mathbf{G}_a \\ \mathrm{Spec}(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x]) & \to & \mathrm{Spec}(\mathbf{Z}[x]) \\ \mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x] & \leftarrow & \mathbf{Z}[x] \\ x \otimes 1 + 1 \otimes x & \leftarrow & x \end{array}$$

Hence we see that $\mathbf{G}_a$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{a,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{a,S}(T) = \mathbf{G}_a(T) = \Gamma(T, \mathcal{O}_T)$$

as before.

022W Example 39.5.4 (General linear group scheme). Let $n \geq 1$. Consider the functor which associates to any scheme T the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the scheme

$$\mathrm{GL}_n = \mathrm{Spec}(\mathbf{Z}[\{x_{ij}\}_{1 \leq i,j \leq n}][1/d])$$

where $d = \det((x_{ij}))$ with (x_{ij}) the $n \times n$ matrix with entry x_{ij} in the (i,j)-spot. The morphism giving the group structure is the morphism

$$\begin{array}{rcl} \mathrm{GL}_n \times \mathrm{GL}_n & \to & \mathrm{GL}_n \\ \mathrm{Spec}(\mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d]) & \to & \mathrm{Spec}(\mathbf{Z}[x_{ij}, 1/d]) \\ \mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d] & \leftarrow & \mathbf{Z}[x_{ij}, 1/d] \\ \sum x_{ik} \otimes x_{kj} & \leftarrow & x_{ij} \end{array}$$

Hence we see that GL_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathrm{GL}_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathrm{GL}_{n,S}(T) = \mathrm{GL}_n(T) = \mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

as before.

022X Example 39.5.5. The determinant defines a morphism of group schemes

$$\det : \mathrm{GL}_n \longrightarrow \mathbf{G}_m$$

over $\mathbf{Z}$. By base change it gives a morphism of group schemes $\mathrm{GL}_{n,S} \to \mathbf{G}_{m,S}$ over any base scheme S.

03YW Example 39.5.6 (Constant group). Let G be an abstract group. Consider the functor which associates to any scheme T the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the scheme

$$G_{\mathrm{Spec}(\mathbf{Z})} = \coprod\nolimits_{g \in G} \mathrm{Spec}(\mathbf{Z}).$$

The morphism giving the group structure is the morphism

$$G_{\mathrm{Spec}(\mathbf{Z})} \times_{\mathrm{Spec}(\mathbf{Z})} G_{\mathrm{Spec}(\mathbf{Z})} \longrightarrow G_{\mathrm{Spec}(\mathbf{Z})}$$

which maps the component corresponding to the pair (g, g') to the component corresponding to gg'. For any scheme S the base change G_S is a group scheme over S whose functor of points is

$$T/S \longmapsto G_S(T) = \{f : T \to G \text{ locally constant}\}$$

as before.

39.6. Properties of group schemes

045W In this section we collect some simple properties of group schemes which hold over any base.

047G Lemma 39.6.1. Let S be a scheme. Let G be a group scheme over S. Then $G \to S$ is separated (resp. quasi-separated) if and only if the identity morphism $e : S \to G$ is a closed immersion (resp. quasi-compact).

Proof. We recall that by Schemes, Lemma 26.21.11 we have that e is an immersion which is a closed immersion (resp. quasi-compact) if $G \to S$ is separated (resp. quasi-separated). For the converse, consider the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta_{G/S}"'] \arrow[d] & G \times_S G \arrow[d, "(g,g') \mapsto m(i(g),g')"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/S}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact) so is $\Delta_{G/S}$, see Schemes, Lemma 26.18.2 (resp. Schemes, Lemma 26.19.3). $\square$

047H Lemma 39.6.2. Let S be a scheme. Let G be a group scheme over S. Let T be a scheme over S and let $\psi : T \to G$ be a morphism over S. If T is flat over S, then the morphism

$$T \times_S G \longrightarrow G, \quad (t, g) \longmapsto m(\psi(t), g)$$

is flat. In particular, if G is flat over S, then $m : G \times_S G \to G$ is flat.

Proof. Consider the diagram

$$\begin{tikzcd} T \times_S G \arrow[r, "(t,g)\mapsto(t,m(\psi(t),g))"'] & T \times_S G \arrow[r, "\mathrm{pr}"'] \arrow[d] & G \arrow[d] \\ & T \arrow[r] & S \end{tikzcd}$$

The left top horizontal arrow is an isomorphism and the square is cartesian. Hence the lemma follows from Morphisms, Lemma 29.25.8. $\square$

047I Lemma 39.6.3. Let (G, m, e, i) be a group scheme over the scheme S. Denote $f : G \to S$ the structure morphism. Then there exist canonical isomorphisms [MvdGE, Proposition 3.15]

$$\Omega_{G/S} \cong f^*\mathcal{C}_{S/G} \cong f^*e^*\Omega_{G/S}$$

where $\mathcal{C}_{S/G}$ denotes the conormal sheaf of the immersion e. In particular, if S is the spectrum of a field, then $\Omega_{G/S}$ is a free $\mathcal{O}_G$-module.

Proof. By Morphisms, Lemma 29.32.10 we have

$$\Omega_{G \times_S G/G} = \mathrm{pr}_0^*\Omega_{G/S}$$

where on the left hand side we view $G \times_S G$ as a scheme over G using pr_1. Let $\tau : G \times_S G \to G \times_S G$ be the "shearing map" given by $(g, h) \mapsto (m(g, h), h)$ on points. This map is an automorphism of $G \times_S G$ viewed as a scheme over G via the projection pr_1. Combining these two remarks we obtain an isomorphism

$$\tau^*\mathrm{pr}_0^*\Omega_{G/S} \to \mathrm{pr}_0^*\Omega_{G/S}$$

Since $\mathrm{pr}_0 \circ \tau = m$ this can be rewritten as an isomorphism

$$m^*\Omega_{G/S} \to \mathrm{pr}_0^*\Omega_{G/S}$$

Pulling back this isomorphism by $(e \circ f, \mathrm{id}_G) : G \to G \times_S G$ and using that $m \circ (e \circ f, \mathrm{id}_G) = \mathrm{id}_G$ and $\mathrm{pr}_0 \circ (e \circ f, \mathrm{id}_G) = e \circ f$ we obtain an isomorphism

$$\Omega_{G/S} \to f^*e^*\Omega_{G/S}$$

as desired. By Morphisms, Lemma 29.32.16 we have $\mathcal{C}_{S/G} \cong e^*\Omega_{G/S}$. If S is the spectrum of a field, then any $\mathcal{O}_S$-module on S is free and the final statement follows. $\square$

0BF5 Lemma 39.6.4. Let S be a scheme. Let G be a group scheme over S. Let $s \in S$. Then the composition

$$T_{G/S,e(s)} \oplus T_{G/S,e(s)} = T_{G \times_S G/S,(e(s),e(s))} \to T_{G/S,e(s)}$$

is addition of tangent vectors. Here the $=$ comes from Varieties, Lemma 33.16.7 and the right arrow is induced from $m : G \times_S G \to G$ via Varieties, Lemma 33.16.6.

Proof. We will use Varieties, Equation (33.16.3.1) and work with tangent vectors in fibres. An element θ in the first factor $T_{G_s/s,e(s)}$ is the image of θ via the map $T_{G_s/s,e(s)} \to T_{G_s \times G_s/s,(e(s),e(s))}$ coming from $(1, e) : G_s \to G_s \times G_s$. Since $m \circ (1, e) = 1$ we see that θ maps to θ by functoriality. Since the map is linear we see that (θ_1, θ_2) maps to $\theta_1 + \theta_2$. $\square$

39.7. Properties of group schemes over a field

047J In this section we collect some properties of group schemes over a field. In the case of group schemes which are (locally) algebraic over a field we can say a lot more, see Section 39.8.

047K Lemma 39.7.1. If (G, m) is a group scheme over a field k, then the multiplication map $m : G \times_k G \to G$ is open.

Proof. The multiplication map is isomorphic to the projection map $\text{pr}_0 : G \times_k G \to G$ because the diagram

$$\begin{tikzcd} G \times_k G \arrow[r, "(g,g') \mapsto (m(g,g'),g')"'] \arrow[d, "m"] & G \times_k G \arrow[d, "(g,g') \mapsto g"] \\ G \arrow[r, "\text{id}"] & G \end{tikzcd}$$

is commutative with isomorphisms as horizontal arrows. The projection is open by Morphisms, Lemma 29.23.4. $\square$

0B7N Lemma 39.7.2. If (G, m) is a group scheme over a field k. Let $U \subset G$ open and $T \to G$ a morphism of schemes. Then the image of the composition $T \times_k U \to G \times_k G \to G$ is open.

Proof. For any field extension K/k the morphism $G_K \to G$ is open (Morphisms, Lemma 29.23.4). Every point ξ of $T \times_k U$ is the image of a morphism $(t, u) : \text{Spec}(K) \to T \times_k U$ for some K. Then the image of $T_K \times_K U_K = (T \times_k U)_K \to G_K$ contains the translate $t \cdot U_K$ which is open. Combining these facts we see that the image of $T \times_k U \to G$ contains an open neighbourhood of the image of ξ. Since ξ was arbitrary we win. $\square$

047L Lemma 39.7.3. Let G be a group scheme over a field. Then G is a separated scheme.

Proof. Say $S = \text{Spec}(k)$ with k a field, and let G be a group scheme over S. By Lemma 39.6.1 we have to show that $e : S \to G$ is a closed immersion. By Morphisms, Lemma 29.20.2 the image of $e : S \to G$ is a closed point of G. It is clear that $\mathcal{O}_G \to e_*\mathcal{O}_S$ is surjective, since $e_*\mathcal{O}_S$ is a skyscraper sheaf supported at the neutral element of G with value k. We conclude that e is a closed immersion by Schemes, Lemma 26.24.2. $\square$

047M Lemma 39.7.4. Let G be a group scheme over a field k. Then

(1) every local ring $\mathcal{O}_{G,g}$ of G has a unique minimal prime ideal,
(2) there is exactly one irreducible component Z of G passing through e, and
(3) Z is geometrically irreducible over k.

Proof. For any point $g \in G$ there exists a field extension K/k and a K-valued point $g' \in G(K)$ mapping to g. If we think of g' as a K-rational point of the group scheme G_K, then we see that $\mathcal{O}_{G,g} \to \mathcal{O}_{G_K,g'}$ is a faithfully flat local ring map (as $G_K \to G$ is flat, and a local flat ring map is faithfully flat, see Algebra, Lemma 10.39.17). The result for $\mathcal{O}_{G_K,g'}$ implies the result for $\mathcal{O}_{G,g}$, see Algebra, Lemma 10.30.5. Hence in order to prove (1) it suffices to prove it for k-rational points g of G. In this case translation by g defines an automorphism $G \to G$ which maps e to g. Hence $\mathcal{O}_{G,g} \cong \mathcal{O}_{G,e}$. In this way we see that (2) implies (1), since irreducible components passing through e correspond one to one with minimal prime ideals of $\mathcal{O}_{G,e}$.

In order to prove (2) and (3) it suffices to prove (2) when k is algebraically closed. In this case, let Z_1, Z_2 be two irreducible components of G passing through e. Since k is algebraically closed the closed subscheme $Z_1 \times_k Z_2 \subset G \times_k G$ is irreducible too, see Varieties, Lemma 33.8.4. Hence $m(Z_1 \times_k Z_2)$ is contained in an irreducible component of G. On the other hand it contains Z_1 and Z_2 since $m|_{e \times G} = \mathrm{id}_G$ and $m|_{G \times e} = \mathrm{id}_G$. We conclude $Z_1 = Z_2$ as desired. $\square$

04L9 Remark 39.7.5. Warning: The result of Lemma 39.7.4 does not mean that every irreducible component of G/k is geometrically irreducible. For example the group scheme $\mu_{3,\mathbf{Q}} = \mathrm{Spec}(\mathbf{Q}[x]/(x^3 - 1))$ over $\mathbf{Q}$ has two irreducible components corresponding to the factorization $x^3 - 1 = (x - 1)(x^2 + x + 1)$. The first factor corresponds to the irreducible component passing through the identity, and the second irreducible component is not geometrically irreducible over $\mathrm{Spec}(\mathbf{Q})$.

047R Lemma 39.7.6. Let G be a group scheme over a perfect field k. Then the reduction G_{red} of G is a closed subgroup scheme of G.

Proof. Omitted. Hint: Use that $G_{red} \times_k G_{red}$ is reduced by Varieties, Lemmas 33.6.3 and 33.6.7. $\square$

047S Lemma 39.7.7. Let k be a field. Let $\psi : G' \to G$ be a morphism of group schemes over k. If $\psi(G')$ is open in G, then $\psi(G')$ is closed in G.

Proof. Let $U = \psi(G') \subset G$. Let $Z = G \setminus \psi(G') = G \setminus U$ with the reduced induced closed subscheme structure. By Lemma 39.7.2 the image of

$$Z \times_k G' \longrightarrow Z \times_k U \longrightarrow G$$

is open (the first arrow is surjective). On the other hand, since ψ is a homomorphism of group schemes, the image of $Z \times_k G' \to G$ is contained in Z (because translation by $\psi(g')$ preserves U for all points g' of G'; small detail omitted). Hence $Z \subset G$ is an open subset (although not necessarily an open subscheme). Thus $U = \psi(G')$ is closed. $\square$

047T Lemma 39.7.8. Let $i : G' \to G$ be an immersion of group schemes over a field k. Then i is a closed immersion, i.e., $i(G')$ is a closed subgroup scheme of G.

Proof. To show that i is a closed immersion it suffices to show that $i(G')$ is a closed subset of G. Let $k \subset k'$ be a perfect extension of k. If $i(G'_{k'}) \subset G_{k'}$ is closed, then $i(G') \subset G$ is closed by Morphisms, Lemma 29.25.12 (as $G_{k'} \to G$ is flat, quasi-compact and surjective). Hence we may and do assume k is perfect. We will use without further mention that products of reduced schemes over k are reduced. We may replace G' and G by their reductions, see Lemma 39.7.6. Let $\overline{G'} \subset G$ be the closure of $i(G')$ viewed as a reduced closed subscheme. By Varieties, Lemma 33.24.1 we conclude that $\overline{G'} \times_k \overline{G'}$ is the closure of the image of $G' \times_k G' \to G \times_k G$. Hence

$$m\left(\overline{G'} \times_k \overline{G'}\right) \subset \overline{G'}$$

as m is continuous. It follows that $\overline{G'} \subset G$ is a (reduced) closed subgroup scheme. By Lemma 39.7.7 we see that $i(G') \subset \overline{G'}$ is also closed which implies that $i(G') = \overline{G'}$ as desired. □

0B7P Lemma 39.7.9. Let G be a group scheme over a field k. If G is irreducible, then G is quasi-compact.

Proof. Suppose that K/k is a field extension. If G_K is quasi-compact, then G is too as $G_K \to G$ is surjective. By Lemma 39.7.4 we see that G_K is irreducible. Hence it suffices to prove the lemma after replacing k by some extension. Choose K to be an algebraically closed field extension of very large cardinality. Then by Varieties, Lemma 33.14.2, we see that G_K is a Jacobson scheme all of whose closed points have residue field equal to K. In other words we may assume G is a Jacobson scheme all of whose closed points have residue field k.

Let $U \subset G$ be a nonempty affine open. Let $g \in G(k)$. Then $gU \cap U \neq \emptyset$. Hence we see that g is in the image of the morphism

$$U \times_{\mathrm{Spec}(k)} U \longrightarrow G, \quad (u_1, u_2) \longmapsto u_1 u_2^{-1}$$

Since the image of this morphism is open (Lemma 39.7.1) we see that the image is all of G (because G is Jacobson and closed points are k-rational). Since U is affine, so is $U \times_{\mathrm{Spec}(k)} U$. Hence G is the image of a quasi-compact scheme, hence quasi-compact. □

0B7Q Lemma 39.7.10. Let G be a group scheme over a field k. If G is connected, then G is irreducible.

Proof. By Varieties, Lemma 33.7.14 we see that G is geometrically connected. If we show that G_K is irreducible for some field extension K/k, then the lemma follows. Hence we may apply Varieties, Lemma 33.14.2 to reduce to the case where k is algebraically closed, G is a Jacobson scheme, and all the closed points are k-rational.

Let $Z \subset G$ be the unique irreducible component of G passing through the neutral element, see Lemma 39.7.4. Endowing Z with the reduced induced closed subscheme structure, we see that $Z \times_k Z$ is reduced and irreducible (Varieties, Lemmas 33.6.7 and 33.8.4). We conclude that $m|_{Z \times_k Z} : Z \times_k Z \to G$ factors through Z. Hence Z becomes a closed subgroup scheme of G.

To get a contradiction, assume there exists another irreducible component $Z' \subset G$. Then $Z \cap Z' = \emptyset$ by Lemma 39.7.4. By Lemma 39.7.9 we see that Z is quasi-compact. Thus we may choose a quasi-compact open $U \subset G$ with $Z \subset U$ and

$U \cap Z' = \emptyset$. The image W of $Z \times_k U \to G$ is open in G by Lemma 39.7.2. On the other hand, W is quasi-compact as the image of a quasi-compact space. We claim that W is closed.

Proof of the claim. Since W is quasi-compact, we see that points in the closure of W are specializations of points of W (Morphisms, Lemma 29.6.5). Thus we have to show that any irreducible component $Z'' \subset G$ of G which meets W is contained in W. As G is Jacobson and closed points are rational, $Z'' \cap W$ has a rational point $g \in Z''(k) \cap W(k)$ and hence $Z'' = Zg$. But $W = m(Z \times_k W)$ by construction, so $Z'' \cap W \neq \emptyset$ implies $Z'' \subset W$.

By the claim $W \subset G$ is an open and closed subset of G. Now $W \cap Z' = \emptyset$ since otherwise by the argument given in the precending paragraph we would get $Z' = Zg$ for some $g \in W(k)$. Then as Z is a subgroup we could even pick $g \in U(k)$ which would contradict $Z' \cap U = \emptyset$. Hence $W \subset G$ is a proper open and closed subset which contradicts the assumption that G is connected. $\square$

0B7R Proposition 39.7.11. Let G be a group scheme over a field k. There exists a canonical closed subgroup scheme $G^0 \subset G$ with the following properties

(1) $G^0 \to G$ is a flat closed immersion,
(2) $G^0 \subset G$ is the connected component of the identity,
(3) G^0 is geometrically irreducible, and
(4) G^0 is quasi-compact.

Proof. Let G^0 be the connected component of the identity with its canonical scheme structure (Morphisms, Definition 29.26.3). To show that G^0 is a closed subsgroup scheme we will use the criterion of Lemma 39.4.4. The morphism $e : \mathrm{Spec}(k) \to G$ factors through G^0 as we chose G^0 to be the connected component of G containing e. Since $i : G \to G$ is an automorphism fixing e, we see that i sends G^0 into itself. By Varieties, Lemma 33.7.13 the scheme G^0 is geometrically connected over k. Thus $G^0 \times_k G^0$ is connected (Varieties, Lemma 33.7.4). Thus $m(G^0 \times_k G^0) \subset G^0$ set theoretically. Thus $m|_{G^0 \times_k G^0} : G^0 \times_k G^0 \to G$ factors through G^0 by Morphisms, Lemma 29.26.1. Hence G^0 is a closed subgroup scheme of G. By Lemma 39.7.10 we see that G^0 is irreducible. By Lemma 39.7.4 we see that G^0 is geometrically irreducible. By Lemma 39.7.9 we see that G^0 is quasi-compact. $\square$

0B7T Lemma 39.7.12. Let k be a field. Let $T = \mathrm{Spec}(A)$ where A is a directed colimit of algebras which are finite products of copies of k. For any scheme X over k we have $|T \times_k X| = |T| \times |X|$ as topological spaces.

Proof. By taking an affine open covering we reduce to the case of an affine X. Say $X = \mathrm{Spec}(B)$. Write $A = \mathrm{colim}\, A_i$ with $A_i = \prod_{t \in T_i} k$ and T_i finite. Then $T_i = |\mathrm{Spec}(A_i)|$ with the discrete topology and the transition morphisms $A_i \to A_{i'}$ are given by set maps $T_{i'} \to T_i$. Thus $|T| = \lim T_i$ as a topological space, see

Limits, Lemma 32.4.6. Similarly we have

$$\begin{aligned}|T \times_k X| &= |\operatorname{Spec}(A \otimes_k B)| \\ &= |\operatorname{Spec}(\operatorname{colim} A_i \otimes_k B)| \\ &= \lim |\operatorname{Spec}(A_i \otimes_k B)| \\ &= \lim |\operatorname{Spec}(\prod\nolimits_{t \in T_i} B)| \\ &= \lim T_i \times |X| \\ &= (\lim T_i) \times |X| \\ &= |T| \times |X|\end{aligned}$$

by the lemma above and the fact that limits commute with limits. □

The following lemma says that in fact we can put a "algebraic profinite family of points" in an affine open. We urge the reader to read Lemma 39.8.6 first.

0B7U Lemma 39.7.13. Let k be an algebraically closed field. Let G be a group scheme over k. Assume that G is Jacobson and that all closed points are k-rational. Let $T = \operatorname{Spec}(A)$ where A is a directed colimit of algebras which are finite products of copies of k. For any morphism $f : T \to G$ there exists an affine open $U \subset G$ containing $f(T)$.

Proof. Let $G^0 \subset G$ be the closed subgroup scheme found in Proposition 39.7.11. The first two paragraphs serve to reduce to the case $G = G^0$.

Observe that T is a directed inverse limit of finite topological spaces (Limits, Lemma 32.4.6), hence profinite as a topological space (Topology, Definition 5.22.1). Let $W \subset G$ be a quasi-compact open containing the image of $T \to G$. After replacing W by the image of $G^0 \times W \to G \times G \to G$ we may assume that W is invariant under the action of left translation by G^0, see Lemma 39.7.2. Consider the composition

$$\psi = \pi \circ f : T \xrightarrow{f} W \xrightarrow{\pi} \pi_0(W)$$

The space $\pi_0(W)$ is profinite (Topology, Lemma 5.23.9 and Properties, Lemma 28.2.4). Let $F_\xi \subset T$ be the fibre of $T \to \pi_0(W)$ over $\xi \in \pi_0(W)$. Assume that for all ξ we can find an affine open $U_\xi \subset W$ with $F \subset U$. Since $\psi : T \to \pi_0(W)$ is universally closed as a map of topological spaces (Topology, Lemma 5.17.7), we can find a quasi-compact open $V_\xi \subset \pi_0(W)$ such that $\psi^{-1}(V_\xi) \subset f^{-1}(U_\xi)$ (easy topological argument omitted). After replacing U_ξ by $U_\xi \cap \pi^{-1}(V_\xi)$, which is open and closed in U_ξ hence affine, we see that $U_\xi \subset \pi^{-1}(V_\xi)$ and $U_\xi \cap T = \psi^{-1}(V_\xi)$. By Topology, Lemma 5.22.4 we can find a finite disjoint union decomposition $\pi_0(W) = \bigcup_{i=1,\ldots,n} V_i$ by quasi-compact opens such that $V_i \subset V_{\xi_i}$ for some i. Then we see that

$$f(T) \subset \bigcup\nolimits_{i=1,\ldots,n} U_{\xi_i} \cap \pi^{-1}(V_i)$$

the right hand side of which is a finite disjoint union of affines, therefore affine.

Let Z be a connected component of G which meets $f(T)$. Then Z has a k-rational point z (because all residue fields of the scheme T are isomorphic to k). Hence $Z = G^0 z$. By our choice of W, we see that $Z \subset W$. The argument in the preceding paragraph reduces us to the problem of finding an affine open neighbourhood of $f(T) \cap Z$ in W. After translation by a rational point we may assume that $Z = G^0$ (details omitted). Observe that the scheme theoretic inverse image $T' = f^{-1}(G^0) \subset$

T is a closed subscheme, which has the same type. After replacing T by T' we may assume that $f(T) \subset G^0$. Choose an affine open neighbourhood $U \subset G$ of $e \in G$, so that in particular $U \cap G^0$ is nonempty. We will show there exists a $g \in G^0(k)$ such that $f(T) \subset g^{-1}U$. This will finish the proof as $g^{-1}U \subset W$ by the left G^0-invariance of W.

The arguments in the preceding two paragraphs allow us to pass to G^0 and reduce the problem to the following: Assume G is irreducible and $U \subset G$ an affine open neighbourhood of e. Show that $f(T) \subset g^{-1}U$ for some $g \in G(k)$. Consider the morphism

$$U \times_k T \longrightarrow G \times_k T, \quad (t, u) \longrightarrow (uf(t)^{-1}, t)$$

which is an open immersion (because the extension of this morphism to $G \times_k T \to G \times_k T$ is an isomorphism). By our assumption on T we see that we have $|U \times_k T| = |U| \times |T|$ and similarly for $G \times_k T$, see Lemma 39.7.12. Hence the image of the displayed open immersion is a finite union of boxes $\bigcup_{i=1,\ldots,n} U_i \times V_i$ with $V_i \subset T$ and $U_i \subset G$ quasi-compact open. This means that the possible opens $Uf(t)^{-1}$, $t \in T$ are finite in number, say $Uf(t_1)^{-1}, \ldots, Uf(t_r)^{-1}$. Since G is irreducible the intersection

$$Uf(t_1)^{-1} \cap \ldots \cap Uf(t_r)^{-1}$$

is nonempty and since G is Jacobson with closed points k-rational, we can choose a k-valued point $g \in G(k)$ of this intersection. Then we see that $g \in Uf(t)^{-1}$ for all $t \in T$ which means that $f(t) \in g^{-1}U$ as desired. $\square$

047V Remark 39.7.14. If G is a group scheme over a field, is there always a quasi-compact open and closed subgroup scheme? By Proposition 39.7.11 this question is only interesting if G has infinitely many connected components (geometrically).

047U Lemma 39.7.15. Let G be a group scheme over a field. There exists an open and closed subscheme $G' \subset G$ which is a countable union of affines.

Proof. Let $e \in U(k)$ be a quasi-compact open neighbourhood of the identity element. By replacing U by $U \cap i(U)$ we may assume that U is invariant under the inverse map. As G is separated this is still a quasi-compact set. Set

$$G' = \bigcup\nolimits_{n \geq 1} m_n(U \times_k \ldots \times_k U)$$

where $m_n : G \times_k \ldots \times_k G \to G$ is the n-slot multiplication map $(g_1, \ldots, g_n) \mapsto m(m(\ldots(m(g_1, g_2), g_3), \ldots), g_n)$. Each of these maps are open (see Lemma 39.7.1) hence G' is an open subgroup scheme. By Lemma 39.7.7 it is also a closed subgroup scheme. $\square$

39.8. Properties of algebraic group schemes

0BF6 Recall that a scheme over a field k is (locally) algebraic if it is (locally) of finite type over $\mathrm{Spec}(k)$, see Varieties, Definition 33.20.1. This is the sense of algebraic we are using in the title of this section.

045X Lemma 39.8.1. Let k be a field. Let G be a locally algebraic group scheme over k. Then G is equidimensional and $\dim(G) = \dim_g(G)$ for all $g \in G$. For any closed point $g \in G$ we have $\dim(G) = \dim(\mathcal{O}_{G,g})$.

Proof. Let us first prove that $\dim_g(G) = \dim_{g'}(G)$ for any pair of points $g, g' \in G$. By Morphisms, Lemma 29.28.3 we may extend the ground field at will. Hence we may assume that both g and g' are defined over k. Hence there exists an automorphism of G mapping g to g', whence the equality. By Morphisms, Lemma 29.28.1 we have $\dim_g(G) = \dim(\mathcal{O}_{G,g}) + \text{trdeg}_k(\kappa(g))$. On the other hand, the dimension of G (or any open subset of G) is the supremum of the dimensions of the local rings of G, see Properties, Lemma 28.10.3. Clearly this is maximal for closed points g in which case $\text{trdeg}_k(\kappa(g)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 29.16). Hence the lemma follows. $\square$

The following result is sometimes referred to as Cartier's theorem.

047N Lemma 39.8.2. Let k be a field of characteristic 0. Let G be a locally algebraic group scheme over k. Then the structure morphism $G \to \text{Spec}(k)$ is smooth, i.e., G is a smooth group scheme.

Proof. By Lemma 39.6.3 the module of differentials of G over k is free. Hence smoothness follows from Varieties, Lemma 33.25.1. $\square$

047O Remark 39.8.3. Any group scheme over a field of characteristic 0 is reduced, see [Per75, I, Theorem 1.1 and I, Corollary 3.9, and II, Theorem 2.4] and also [Per76, Proposition 4.2.8]. This was a question raised in [Oor66, page 80]. We have seen in Lemma 39.8.2 that this holds when the group scheme is locally of finite type.

047P Lemma 39.8.4. Let k be a perfect field of characteristic $p > 0$ (see Lemma 39.8.2 for the characteristic zero case). Let G be a locally algebraic group scheme over k. If G is reduced then the structure morphism $G \to \text{Spec}(k)$ is smooth, i.e., G is a smooth group scheme.

Proof. By Lemma 39.6.3 the sheaf $\Omega_{G/k}$ is free. Hence the lemma follows from Varieties, Lemma 33.25.2. $\square$

047Q Remark 39.8.5. Let k be a field of characteristic $p > 0$. Let $\alpha \in k$ be an element which is not a pth power. The closed subgroup scheme

$$G = V(x^p + \alpha y^p) \subset \mathbf{G}_{a,k}^2$$

is reduced and irreducible but not smooth (not even normal).

The following lemma is a special case of Lemma 39.7.13 with a somewhat easier proof.

0B7S Lemma 39.8.6. Let k be an algebraically closed field. Let G be a locally algebraic group scheme over k. Let $g_1, \ldots, g_n \in G(k)$ be k-rational points. Then there exists an affine open $U \subset G$ containing $g_1, \ldots, g_n$.

Proof. We first argue by induction on n that we may assume all g_i are on the same connected component of G. Namely, if not, then we can find a decomposition $G = W_1 \amalg W_2$ with W_i open in G and (after possibly renumbering) $g_1, \ldots, g_r \in W_1$ and $g_{r+1}, \ldots, g_n \in W_2$ for some $0 < r < n$. By induction we can find affine opens U_1 and U_2 of G with $g_1, \ldots, g_r \in U_1$ and $g_{r+1}, \ldots, g_n \in U_2$. Then

$$g_1, \ldots, g_n \in (U_1 \cap W_1) \cup (U_2 \cap W_2)$$

is a solution to the problem. Thus we may assume $g_1, \ldots, g_n$ are all on the same connected component of G. Translating by g_1^{-1} we may assume $g_1, \ldots, g_n \in G^0$

where $G^0 \subset G$ is as in Proposition 39.7.11. Choose an affine open neighbourhood U of e, in particular $U \cap G^0$ is nonempty. Since G^0 is irreducible we see that

$$G^0 \cap (Ug_1^{-1} \cap \ldots \cap Ug_n^{-1})$$

is nonempty. Since $G \to \mathrm{Spec}(k)$ is locally of finite type, also $G^0 \to \mathrm{Spec}(k)$ is locally of finite type, hence any nonempty open has a k-rational point. Thus we can pick $g \in G^0(k)$ with $g \in Ug_i^{-1}$ for all i. Then $g_i \in g^{-1}U$ for all i and $g^{-1}U$ is the affine open we were looking for. $\square$

0BF7 Lemma 39.8.7. Let k be a field. Let G be an algebraic group scheme over k. Then G is quasi-projective over k.

Proof. By Varieties, Lemma 33.15.1 we may assume that k is algebraically closed. Let $G^0 \subset G$ be the connected component of G as in Proposition 39.7.11. Then every other connected component of G has a k-rational point and hence is isomorphic to G^0 as a scheme. Since G is quasi-compact and Noetherian, there are finitely many of these connected components. Thus we reduce to the case discussed in the next paragraph.

Let G be a connected algebraic group scheme over an algebraically closed field k. If the characteristic of k is zero, then G is smooth over k by Lemma 39.8.2. If the characteristic of k is $p > 0$, then we let $H = G_{red}$ be the reduction of G. By Divisors, Proposition 31.17.9 it suffices to show that H has an ample invertible sheaf. (For an algebraic scheme over k having an ample invertible sheaf is equivalent to being quasi-projective over k, see for example the very general More on Morphisms, Lemma 37.49.1.) By Lemma 39.7.6 we see that H is a group scheme over k. By Lemma 39.8.4 we see that H is smooth over k. This reduces us to the situation discussed in the next paragraph.

Let G be a quasi-compact irreducible smooth group scheme over an algebraically closed field k. Observe that the local rings of G are regular and hence UFDs (Varieties, Lemma 33.25.3 and More on Algebra, Lemma 15.121.2). The complement of a nonempty affine open of G is the support of an effective Cartier divisor D. This follows from Divisors, Lemma 31.16.6. (Observe that G is separated by Lemma 39.7.3.) We conclude there exists an effective Cartier divisor $D \subset G$ such that $G \setminus D$ is affine. We will use below that for any $n \geq 1$ and $g_1, \ldots, g_n \in G(k)$ the complement $G \setminus \bigcup Dg_i$ is affine. Namely, it is the intersection of the affine opens $G \setminus Dg_i \cong G \setminus D$ in the separated scheme G.

We may choose the top row of the diagram

$$\begin{tikzcd} G & U \arrow[l, "j"'] \arrow[r, "\pi"] & \mathbf{A}^d_k \\ & W \arrow[u] \arrow[r, "\pi'"] & V \arrow[u] \end{tikzcd}$$

such that $U \neq \emptyset$, $j : U \to G$ is an open immersion, and π is étale, see Morphisms, Lemma 29.36.20. There is a nonempty affine open $V \subset \mathbf{A}^d_k$ such that with $W = \pi^{-1}(V)$ the morphism $\pi' = \pi|_W : W \to V$ is finite étale. In particular π' is finite locally free, say of degree n. Consider the effective Cartier divisor

$$\mathcal{D} = \{(g, w) \mid m(g, j(w)) \in D\} \subset G \times W$$

(This is the restriction to $G \times W$ of the pullback of $D \subset G$ under the flat morphism $m : G \times G \to G$.) Consider the closed subset[1] $T = (1 \times \pi')(\mathcal{D}) \subset G \times V$. Since π' is finite locally free, every irreducible component of T has codimension 1 in $G \times V$. Since $G \times V$ is smooth over k we conclude these components are effective Cartier divisors (Divisors, Lemma 31.15.7 and lemmas cited above) and hence T is the support of an effective Cartier divisor E in $G \times V$. If $v \in V(k)$, then $(\pi')^{-1}(v) = \{w_1, \ldots, w_n\} \subset W(k)$ and we see that

$$E_v = \bigcup\nolimits_{i=1,\ldots,n} Dj(w_i)^{-1}$$

in G set theoretically. In particular we see that $G \setminus E_v$ is affine open (see above). Moreover, if $g \in G(k)$, then there exists a $v \in V$ such that $g \notin E_v$. Namely, the set W' of $w \in W$ such that $g \notin Dj(w)^{-1}$ is nonempty open and it suffices to pick v such that the fibre of $W' \to V$ over v has n elements.

Consider the invertible sheaf $\mathcal{M} = \mathcal{O}_{G\times V}(E)$ on $G \times V$. By Varieties, Lemma 33.30.5 the isomorphism class $\mathcal{L}$ of the restriction $\mathcal{M}_v = \mathcal{O}_G(E_v)$ is independent of $v \in V(k)$. On the other hand, for every $g \in G(k)$ we can find a v such that $g \notin E_v$ and such that $G \setminus E_v$ is affine. Thus the canonical section (Divisors, Definition 31.14.1) of $\mathcal{O}_G(E_v)$ corresponds to a section s_v of $\mathcal{L}$ which does not vanish at g and such that G_{s_v} is affine. This means that $\mathcal{L}$ is ample by definition (Properties, Definition 28.26.1). □

0BF8 Lemma 39.8.8. Let k be a field. Let G be a locally algebraic group scheme over k. Then the center of G is a closed subgroup scheme of G.

Proof. Let $\mathrm{Aut}(G)$ denote the contravariant functor on the category of schemes over k which associates to S/k the set of automorphisms of the base change G_S as a group scheme over S. There is a natural transformation

$$G \longrightarrow \mathrm{Aut}(G), \quad g \longmapsto \mathrm{inn}_g$$

sending an S-valued point g of G to the inner automorphism of G determined by g. The center C of G is by definition the kernel of this transformation, i.e., the functor which to S associates those $g \in G(S)$ whose associated inner automorphism is trivial. The statement of the lemma is that this functor is representable by a closed subgroup scheme of G.

Choose an integer $n \geq 1$. Let $G_n \subset G$ be the nth infinitesimal neighbourhood of the identity element e of G. For every scheme S/k the base change $G_{n,S}$ is the nth infinitesimal neighbourhood of $e_S : S \to G_S$. Thus we see that there is a natural transformation $\mathrm{Aut}(G) \to \mathrm{Aut}(G_n)$ where the right hand side is the functor of automorphisms of G_n as a scheme (G_n isn't in general a group scheme). Observe that G_n is the spectrum of an artinian local ring A_n with residue field k which has finite dimension as a k-vector space (Varieties, Lemma 33.20.2). Since every automorphism of G_n induces in particular an invertible linear map $A_n \to A_n$, we obtain transformations of functors

$$G \to \mathrm{Aut}(G) \to \mathrm{Aut}(G_n) \to \mathrm{GL}(A_n)$$

[1] Using the material in Divisors, Section 31.17 we could take as effective Cartier divisor E the norm of the effective Cartier divisor $\mathcal{D}$ along the finite locally free morphism $1 \times \pi'$ bypassing some of the arguments.

The final group valued functor is representable, see Example 39.5.4, and the last arrow is visibly injective. Thus for every n we obtain a closed subgroup scheme

$$H_n = \mathrm{Ker}(G \to \mathrm{Aut}(G_n)) = \mathrm{Ker}(G \to \mathrm{GL}(A_n)).$$

As a first approximation we set $H = \bigcap_{n \geq 1} H_n$ (scheme theoretic intersection). This is a closed subgroup scheme which contains the center C.

Let h be an S-valued point of H with S locally Noetherian. Then the automorphism inn_h induces the identity on all the closed subschemes $G_{n,S}$. Consider the kernel $K = \mathrm{Ker}(\mathrm{inn}_h : G_S \to G_S)$. This is a closed subgroup scheme of G_S over S containing the closed subschemes $G_{n,S}$ for $n \geq 1$. This implies that K contains an open neighbourhood of $e(S) \subset G_S$, see Algebra, Remark 10.51.6. Let $G^0 \subset G$ be as in Proposition 39.7.11. Since G^0 is geometrically irreducible, we conclude that K contains G^0_S (for any nonempty open $U \subset G^0_{k'}$ and any field extension k'/k we have $U \cdot U^{-1} = G^0_{k'}$, see proof of Lemma 39.7.9). Applying this with $S = H$ we find that G^0 and H are subgroup schemes of G whose points commute: for any scheme S and any S-valued points $g \in G^0(S)$, $h \in H(S)$ we have $gh = hg$ in $G(S)$.

Assume that k is algebraically closed. Then we can pick a k-valued point g_i in each irreducible component G_i of G. Observe that in this case the connected components of G are the irreducible components of G are the translates of G^0 by our g_i. We claim that

$$C = H \cap \bigcap\nolimits_i \mathrm{Ker}(\mathrm{inn}_{g_i} : G \to G) \quad \text{(scheme theoretic intersection)}$$

Namely, C is contained in the right hand side. On the other hand, every S-valued point h of the right hand side commutes with G^0 and with g_i hence with everything in $G = \bigcup G^0 g_i$.

The case of a general base field k follows from the result for the algebraic closure $\overline{k}$ by descent. Namely, let $A \subset G_{\overline{k}}$ the closed subgroup scheme representing the center of $G_{\overline{k}}$. Then we have

$$A \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} A$$

as closed subschemes of $G_{\overline{k} \otimes_k \overline{k}}$ by the functorial nature of the center. Hence we see that A descends to a closed subgroup scheme $Z \subset G$ by Descent, Lemma 35.37.2 (and Descent, Lemma 35.23.19). Then Z represents C (small argument omitted) and the proof is complete. $\square$

39.9. Abelian varieties

0BF9 An excellent reference for this material is Mumford's book on abelian varieties, see [Mum70]. We encourage the reader to look there. There are many equivalent definitions; here is one.

03RO **Definition 39.9.1.** Let k be a field. An *abelian variety* is a group scheme over k which is also a proper, geometrically integral variety over k[2].

We prove a few lemmas about this notion and then we collect all the results together in Proposition 39.9.11.

0BFA **Lemma 39.9.2.** *Let k be a field. Let A be an abelian variety over k. Then A is projective.*

[2]For equivalent definitions see Remark 39.9.12.

Proof. This follows from Lemma 39.8.7 and More on Morphisms, Lemma 37.50.1. ☐

0BFB Lemma 39.9.3. Let k be a field. Let A be an abelian variety over k. For any field extension K/k the base change A_K is an abelian variety over K.

Proof. Omitted. Note that this is why we insisted on A being geometrically integral; without that condition this lemma (and many others below) would be wrong. ☐

0BFC Lemma 39.9.4. Let k be a field. Let A be an abelian variety over k. Then A is smooth over k.

Proof. If k is perfect then this follows from Lemma 39.8.2 (characteristic zero) and Lemma 39.8.4 (positive characteristic). We can reduce the general case to this case by descent for smoothness (Descent, Lemma 35.23.27) and going to the perfect closure using Lemma 39.9.3. ☐

0BFD Lemma 39.9.5. An abelian variety is an abelian group scheme, i.e., the group law is commutative.

Proof. Let k be a field. Let A be an abelian variety over k. By Lemma 39.9.3 we may replace k by its algebraic closure. Consider the morphism

$$h : A \times_k A \longrightarrow A \times_k A, \quad (x, y) \longmapsto (x, xyx^{-1}y^{-1})$$

This is a morphism over A via the first projection on either side. Let $e \in A(k)$ be the unit. Then we see that $h|_{e \times A}$ is constant with value (e, e). By More on Morphisms, Lemma 37.44.3 there exists an open neighbourhood $U \subset A$ of e such that $h|_{U \times A}$ factors through some $Z \subset U \times A$ finite over U. This means that for $x \in U(k)$ the morphism $A \to A$, $y \mapsto xyx^{-1}y^{-1}$ takes finitely many values. Of course this means it is constant with value e. Thus $(x, y) \mapsto xyx^{-1}y^{-1}$ is constant with value e on $U \times A$ which implies that the group law on A is abelian. ☐

0BFE Lemma 39.9.6. Let k be a field. Let A be an abelian variety over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_A$-module. Then there is an isomorphism

$$m_{1,2,3}^*\mathcal{L} \otimes m_1^*\mathcal{L} \otimes m_2^*\mathcal{L} \otimes m_3^*\mathcal{L} \cong m_{1,2}^*\mathcal{L} \otimes m_{1,3}^*\mathcal{L} \otimes m_{2,3}^*\mathcal{L}$$

of invertible modules on $A \times_k A \times_k A$ where $m_{i_1,\ldots,i_t} : A \times_k A \times_k A \to A$ is the morphism $(x_1, x_2, x_3) \mapsto \sum x_{i_j}$.

Proof. Apply the theorem of the cube (More on Morphisms, Theorem 37.33.8) to the difference

$$\mathcal{M} = m_{1,2,3}^*\mathcal{L} \otimes m_1^*\mathcal{L} \otimes m_2^*\mathcal{L} \otimes m_3^*\mathcal{L} \otimes m_{1,2}^*\mathcal{L}^{\otimes -1} \otimes m_{1,3}^*\mathcal{L}^{\otimes -1} \otimes m_{2,3}^*\mathcal{L}^{\otimes -1}$$

This works because the restriction of $\mathcal{M}$ to $A \times A \times e = A \times A$ is equal to

$$n_{1,2}^*\mathcal{L} \otimes n_1^*\mathcal{L} \otimes n_2^*\mathcal{L} \otimes n_{1,2}^*\mathcal{L}^{\otimes -1} \otimes n_1^*\mathcal{L}^{\otimes -1} \otimes n_2^*\mathcal{L}^{\otimes -1} \cong \mathcal{O}_{A \times_k A}$$

where $n_{i_1,\ldots,i_t} : A \times_k A \to A$ is the morphism $(x_1, x_2) \mapsto \sum x_{i_j}$. Similarly for $A \times e \times A$ and $e \times A \times A$. ☐

0BFF Lemma 39.9.7. Let k be a field. Let A be an abelian variety over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_A$-module. Then

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n(n+1)/2} \otimes ([-1]^*\mathcal{L})^{\otimes n(n-1)/2}$$

where $[n] : A \to A$ sends x to $x + x + \ldots + x$ with n summands and where $[-1] : A \to A$ is the inverse of A.

Proof. Consider the morphism $A \to A \times_k A \times_k A$, $x \mapsto (x, x, -x)$ where $-x = [-1](x)$. Pulling back the relation of Lemma 39.9.6 we obtain

$$\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [-1]^*\mathcal{L} \cong [2]^*\mathcal{L}$$

which proves the result for $n = 2$. By induction assume the result holds for $1, 2, \ldots, n$. Then consider the morphism $A \to A \times_k A \times_k A$, $x \mapsto (x, x, [n-1]x)$. Pulling back the relation of Lemma 39.9.6 we obtain

$$[n+1]^*\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [n-1]^*\mathcal{L} \cong [2]^*\mathcal{L} \otimes [n]^*\mathcal{L} \otimes [n]^*\mathcal{L}$$

and the result follows by elementary arithmetic. □

0BFG Lemma 39.9.8. Let k be a field. Let A be an abelian variety over k. Let $[d] : A \to A$ be the multiplication by d. Then $[d]$ is finite locally free of degree $d^{2\dim(A)}$.

Proof. By Lemma 39.9.2 (and More on Morphisms, Lemma 37.50.1) we see that A has an ample invertible module $\mathcal{L}$. Since $[-1] : A \to A$ is an automorphism, we see that $[-1]^*\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module as well. Thus $\mathcal{N} = \mathcal{L} \otimes [-1]^*\mathcal{L}$ is ample, see Properties, Lemma 28.26.5. Since $\mathcal{N} \cong [-1]^*\mathcal{N}$ we see that $[d]^*\mathcal{N} \cong \mathcal{N}^{\otimes d^2}$ by Lemma 39.9.7.

To get a contradiction, let $C \subset X$ be a proper curve contained in a fibre of $[d]$. Then $\mathcal{N}^{\otimes d^2}|_C \cong \mathcal{O}_C$ is an ample invertible $\mathcal{O}_C$-module of degree 0 which contradicts Varieties, Lemma 33.44.14 for example. (You can also use Varieties, Lemma 33.45.9.) Thus every fibre of $[d]$ has dimension 0 and hence $[d]$ is finite for example by Cohomology of Schemes, Lemma 30.21.1. Moreover, since A is smooth over k by Lemma 39.9.4 we see that $[d] : A \to A$ is flat by Algebra, Lemma 10.128.1 (we also use that schemes smooth over fields are regular and that regular rings are Cohen-Macaulay, see Varieties, Lemma 33.25.3 and Algebra, Lemma 10.106.3). Thus $[d]$ is finite flat hence finite locally free by Morphisms, Lemma 29.48.2.

Finally, we come to the formula for the degree. By Varieties, Lemma 33.45.11 we see that

$$\deg_{\mathcal{N}^{\otimes d^2}}(A) = \deg([d]) \deg_{\mathcal{N}}(A)$$

Since the degree of A with respect to $\mathcal{N}^{\otimes d^2}$, respectively $\mathcal{N}$ is the coefficient of $n^{\dim(A)}$ in the polynomial

$$n \longmapsto \chi(A, \mathcal{N}^{\otimes nd^2}), \quad \text{respectively} \quad n \longmapsto \chi(A, \mathcal{N}^{\otimes n})$$

we see that $\deg([d]) = d^{2\dim(A)}$. □

0BFH Lemma 39.9.9. Let k be a field. Let A be a nonzero abelian variety over k. Then $[d] : A \to A$ is étale if and only if d is invertible in k.

Proof. Observe that $[d](x + y) = [d](x) + [d](y)$. Since translation by a point is an automorphism of A, we see that the set of points where $[d] : A \to A$ is étale is either empty or equal to A (some details omitted). Thus it suffices to check whether $[d]$ is étale at the unit $e \in A(k)$. Since we know that $[d]$ is finite locally free (Lemma 39.9.8) to see that it is étale at e is equivalent to proving that $\mathrm{d}[d] : T_{A/k,e} \to T_{A/k,e}$ is injective. See Varieties, Lemma 33.16.8 and Morphisms, Lemma 29.36.16. By Lemma 39.6.4 we see that $\mathrm{d}[d]$ is given by multiplication by d on $T_{A/k,e}$. □

0C0Y **Lemma 39.9.10.** Let k be a field of characteristic $p > 0$. Let A be an abelian variety of dimension g over k. The fibre of $[p] : A \to A$ over 0 has at most p^g distinct points.

Proof. To prove this, we may and do replace k by the algebraic closure. By Lemma 39.6.4 the derivative of $[p]$ is multiplication by p as a map $T_{A/k,e} \to T_{A/k,e}$ and hence is zero (compare with proof of Lemma 39.9.9). Since $[p]$ commutes with translation we conclude that the derivative of $[p]$ is everywhere zero, i.e., that the induced map $[p]^*\Omega_{A/k} \to \Omega_{A/k}$ is zero. Looking at generic points, we find that the corresponding map $[p]^* : k(A) \to k(A)$ of function fields induces the zero map on $\Omega_{k(A)/k}$. Let $t_1, \ldots, t_g$ be a p-basis of $k(A)$ over k (More on Algebra, Definition 15.46.1 and Lemma 15.46.2). Then $[p]^*(t_i)$ has a pth root by Algebra, Lemma 10.158.2. We conclude that $k(A)[x_1, \ldots, x_g]/(x_1^p - t_1, \ldots, x_g^p - t_g)$ is a subextension of $[p]^* : k(A) \to k(A)$. Thus we can find an affine open $U \subset A$ such that $t_i \in \mathcal{O}_A(U)$ and $x_i \in \mathcal{O}_A([p]^{-1}(U))$. We obtain a factorization

$$[p]^{-1}(U) \xrightarrow{\pi_1} \mathrm{Spec}(\mathcal{O}(U)[x_1, \ldots, x_g]/(x_1^p - t_1, \ldots, x_g^p - t_g)) \xrightarrow{\pi_2} U$$

of $[p]$ over U. After shrinking U we may assume that π_1 is finite locally free (for example by generic flatness – actually it is already finite locally free in our case). By Lemma 39.9.8 we see that $[p]$ has degree p^{2g}. Since π_2 has degree p^g we see that π_1 has degree p^g as well. The morphism π_2 is a universal homeomorphism hence the fibres are singletons. We conclude that the (set theoretic) fibres of $[p]^{-1}(U) \to U$ are the fibres of π_1. Hence they have at most p^g elements. Since $[p]$ is a homomorphism of group schemes over k, the fibre of $[p] : A(k) \to A(k)$ has the same cardinality for every $a \in A(k)$ and the proof is complete. $\square$

03RP **Proposition 39.9.11.** Let A be an abelian variety over a field k. Then Wonderfully explained in [Mum70].

(1) A is projective over k,
(2) A is a commutative group scheme,
(3) the morphism $[n] : A \to A$ is surjective for all $n \geq 1$,
(4) if k is algebraically closed, then $A(k)$ is a divisible abelian group,
(5) $A[n] = \mathrm{Ker}([n] : A \to A)$ is a finite group scheme of degree $n^{2\dim A}$ over k,
(6) $A[n]$ is étale over k if and only if $n \in k^*$,
(7) if $n \in k^*$ and k is algebraically closed, then $A(k)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2\dim(A)}$,
(8) if k is algebraically closed of characteristic $p > 0$, then there exists an integer $0 \leq f \leq \dim(A)$ such that $A(k)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$ for all $m \geq 1$.

Proof. Part (1) follows from Lemma 39.9.2. Part (2) follows from Lemma 39.9.5. Part (3) follows from Lemma 39.9.8. If k is algebraically closed then surjective morphisms of varieties over k induce surjective maps on k-rational points, hence (4) follows from (3). Part (5) follows from Lemma 39.9.8 and the fact that a base change of a finite locally free morphism of degree N is a finite locally free morphism of degree N. Part (6) follows from Lemma 39.9.9. Namely, if n is invertible in k, then $[n]$ is étale and hence $A[n]$ is étale over k. On the other hand, if n is not invertible in k, then $[n]$ is not étale at e and it follows that $A[n]$ is not étale over k at e (use Morphisms, Lemmas 29.36.16 and 29.35.15).

Assume k is algebraically closed. Set $g = \dim(A)$. Proof of (7). Let ℓ be a prime number which is invertible in k. Then we see that

$$A[\ell](k) = A(k)[\ell]$$

is a finite abelian group, annihilated by ℓ, of order ℓ^{2g}. It follows that it is isomorphic to $(\mathbf{Z}/\ell\mathbf{Z})^{2g}$ by the structure theory for finite abelian groups. Next, we consider the short exact sequence

$$0 \to A(k)[\ell] \to A(k)[\ell^2] \xrightarrow{\ell} A(k)[\ell] \to 0$$

Arguing similarly as above we conclude that $A(k)[\ell^2] \cong (\mathbf{Z}/\ell^2\mathbf{Z})^{2g}$. By induction on the exponent we find that $A(k)[\ell^m] \cong (\mathbf{Z}/\ell^m\mathbf{Z})^{2g}$. For composite integers n prime to the characteristic of k we take primary parts and we find the correct shape of the n-torsion in $A(k)$. The proof of (8) proceeds in exactly the same way, using that Lemma 39.9.10 gives $A(k)[p] \cong (\mathbf{Z}/p\mathbf{Z})^{\oplus f}$ for some $0 \leq f \leq g$. □

0H2U Remark 39.9.12. Let k be a field. There are $2 \times 4 \times 2 = 16$ equivalent definitions of abelian varieties. Let

- projective, proper,
- geometrically irreducible, irreducible, geometrically connected, connected,
- smooth, geometrically reduced

be three sets of properties, pick one from each of them, and let A be a group scheme over k with the chosen properties over k. Then A is an abelian variety. If we pick the options “proper, geometrically irreducible, geometrically reduced”, then we recover Definition 39.9.1 (use Varieties, Lemma 33.9.2). The weakest possible options would be “proper, connected, and geometrically reduced”, see for example Morphisms, Lemma 29.43.5 and Varieties, Lemma 33.25.4. So say A is a proper, connected, and geometrically reduced group scheme over k. Then A is geometrically irreducible by Lemmas 39.7.10 and 39.7.4 and hence an abelian variety. Finally, if A/k is an abelian variety, then it is projective and smooth over k (Proposition 39.9.11), whence satisfies the strongest possible options "projective, geometrically irreducible, smooth".

39.10. Actions of group schemes

022Y Let (G, m) be a group and let V be a set. Recall that a (left) action of G on V is given by a map $a : G \times V \to V$ such that

(1) (associativity) $a(m(g, g'), v) = a(g, a(g', v))$ for all $g, g' \in G$ and $v \in V$, and
(2) (identity) $a(e, v) = v$ for all $v \in V$.

We also say that V is a G-set (this usually means we drop the a from the notation – which is abuse of notation). A map of G-sets $\psi : V \to V'$ is any set map such that $\psi(a(g, v)) = a(g, \psi(v))$ for all $v \in V$.

022Z Definition 39.10.1. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) An action of G on the scheme X/S is a morphism $a : G \times_S X \to X$ over S such that for every T/S the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.

(2) Suppose that X, Y are schemes over S each endowed with an action of G. An equivariant or more precisely a G-equivariant morphism $\psi : X \to Y$ is a morphism of schemes over S such that for every T/S the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

03LD (39.10.1.1)
$$\begin{tikzcd} G \times_S G \times_S X \arrow[r, "1_G \times a"'] \arrow[d, "m \times 1_X"'] & G \times_S X \arrow[d, "a"] \\ G \times_S X \arrow[r, "a"] & X \end{tikzcd} \qquad \begin{tikzcd} G \times_S X \arrow[r, "a"'] & X \\ X \arrow[u, "e \times 1_X"] \arrow[ur, "1_X"'] & \end{tikzcd}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\mathrm{id} \times \psi"'] \arrow[d, "a"'] & G \times_S Y \arrow[d, "a"] \\ X \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes.

07S1 Definition 39.10.2. Let S, $G \to S$, and $X \to S$ as in Definition 39.10.1. Let $a : G \times_S X \to X$ be an action of G on X/S. We say the action is free if for every scheme T over S the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

07S2 Lemma 39.10.3. Situation as in Definition 39.10.2, The action a is free if and only if

$$G \times_S X \to X \times_S X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism.

Proof. Immediate from the definitions. $\square$

39.11. Principal homogeneous spaces

0497 In Cohomology on Sites, Definition 21.4.1 we have defined a torsor for a sheaf of groups on a site. Suppose $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ is a topology and (G, m) is a group scheme over S. Since τ is stronger than the canonical topology (see Descent, Lemma 35.13.7) we see that $\underline{G}$ (see Sites, Definition 7.12.3) is a sheaf of groups on $(Sch/S)_\tau$. Hence we already know what it means to have a torsor for $\underline{G}$ on $(Sch/S)_\tau$. A special situation arises if this sheaf is representable. In the following definitions we define directly what it means for the representing scheme to be a G-torsor.

0498 Definition 39.11.1. Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X.

(1) We say X is a pseudo G-torsor or that X is formally principally homogeneous under G if the induced morphism of schemes $G \times_S X \to X \times_S X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism of schemes over S.
(2) A pseudo G-torsor X is called trivial if there exists an G-equivariant isomorphism $G \to X$ over S where G acts on G by left multiplication.

It is clear that if $S' \to S$ is a morphism of schemes then the pullback $X_{S'}$ of a pseudo G-torsor over S is a pseudo $G_{S'}$-torsor over S'.

0499 **Lemma 39.11.2.** In the situation of Definition 39.11.1.

(1) The scheme X is a pseudo G-torsor if and only if for every scheme T over S the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.
(2) A pseudo G-torsor X is trivial if and only if the morphism $X \to S$ has a section.

Proof. Omitted. □

049A **Definition 39.11.3.** Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a pseudo G-torsor over S.

(1) We say X is a *principal homogeneous space* or a *G-torsor* if there exists a fpqc covering[3] $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section (i.e., is a trivial pseudo G_{S_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section.
(3) If X is a G-torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
(4) If X is a G-torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say "let X be a G-torsor over S" to indicate that X is a scheme over S equipped with an action of G which turns it into a principal homogeneous space over S. Next we show that this agrees with the notation introduced earlier when both apply.

049B **Lemma 39.11.4.** Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then X is a G-torsor in the τ-topology if and only if $\underline{X}$ is a $\underline{G}$-torsor on $(Sch/S)_\tau$.

Proof. Omitted. □

049C **Remark 39.11.5.** Let (G, m) be a group scheme over the scheme S. In this situation we have the following natural types of questions:

(1) If $X \to S$ is a pseudo G-torsor and $X \to S$ is surjective, then is X necessarily a G-torsor?
(2) Is every $\underline{G}$-torsor on $(Sch/S)_{fppf}$ representable? In other words, does every $\underline{G}$-torsor come from a fppf G-torsor?
(3) Is every G-torsor an fppf (resp. smooth, resp. étale, resp. Zariski) torsor?

In general the answers to these questions is no. To get a positive answer we need to impose additional conditions on $G \to S$. For example: If S is the spectrum of a field, then the answer to (1) is yes because then $\{X \to S\}$ is a fpqc covering trivializing X. If $G \to S$ is affine, then the answer to (2) is yes (this follows from Descent, Lemma 35.37.1). If $G = \mathrm{GL}_{n,S}$ then the answer to (3) is yes and in fact any $\mathrm{GL}_{n,S}$-torsor is locally trivial (this follows from Descent, Lemma 35.7.6).

[3] This means that the default type of torsor is a pseudo torsor which is trivial on an fpqc covering. This is the definition in [ABD+66, Exposé IV, 6.5]. It is a little bit inconvenient for us as we most often work in the fppf topology.

39.12. Equivariant quasi-coherent sheaves

03LE We think of "functions" as dual to "space". Thus for a morphism of spaces the map on functions goes the other way. Moreover, we think of the sections of a sheaf of modules as "functions". This leads us naturally to the direction of the arrows chosen in the following definition.

03LF **Definition 39.12.1.** Let S be a scheme, let (G, m) be a group scheme over S, and let $a : G \times_S X \to X$ be an action of the group scheme G on X/S. A *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply an *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G\times_S X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_S X \to X$ is the projection such that

(1) the diagram

$$\begin{tikzcd}
(1_G \times a)^*\mathrm{pr}_1^*\mathcal{F} \arrow[r, "\mathrm{pr}_{12}^*\alpha"'] & \mathrm{pr}_2^*\mathcal{F} \\
(1_G \times a)^*a^*\mathcal{F} \arrow[u, "(1_G\times a)^*\alpha"] \arrow[r, equal] & (m \times 1_X)^*a^*\mathcal{F} \arrow[u, "(m\times 1_X)^*\alpha"']
\end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{G\times_S G\times_S X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (39.10.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

03LG **Lemma 39.12.2.** *Let S be a scheme. Let G be a group scheme over S. Let $f : Y \to X$ be a G-equivariant morphism between S-schemes endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of G-equivariant quasi-coherent $\mathcal{O}_X$-modules to the category of G-equivariant quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Omitted. □

Let us give an example.

0EKJ **Example 39.12.3.** Let A be a $\mathbf{Z}$-graded ring, i.e., A comes with a direct sum decomposition $A = \bigoplus_{n\in\mathbf{Z}} A_n$ and $A_n \cdot A_m \subset A_{n+m}$. Set $X = \mathrm{Spec}(A)$. Then we obtain a $\mathbf{G}_m$-action

$$a : \mathbf{G}_m \times X \longrightarrow X$$

by the ring map $\mu : A \to A \otimes \mathbf{Z}[x, x^{-1}]$, $f \mapsto f \otimes x^{\deg(f)}$. Namely, to check this we have to verify that

$$\begin{tikzcd}
A \arrow[r, "\mu"'] \arrow[d, "\mu"'] & A \otimes \mathbf{Z}[x, x^{-1}] \arrow[d, "\mu\otimes 1"] \\
A \otimes \mathbf{Z}[x, x^{-1}] \arrow[r, "1\otimes m"] & A \otimes \mathbf{Z}[x, x^{-1}] \otimes \mathbf{Z}[x, x^{-1}]
\end{tikzcd}$$

where $m(x) = x \otimes x$, see Example 39.5.1. This is immediately clear when evaluating on a homogeneous element. Suppose that M is a graded A-module. Then we obtain a $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-module $\mathcal{F} = \widetilde{M}$ by using α as in Definition 39.12.1 corresponding to the $A \otimes \mathbf{Z}[x, x^{-1}]$-module map

$$M \otimes_{A,\mu} (A \otimes \mathbf{Z}[x, x^{-1}]) \longrightarrow M \otimes_{A,\mathrm{id}_A \otimes 1} (A \otimes \mathbf{Z}[x, x^{-1}])$$

sending $m \otimes 1 \otimes 1$ to $m \otimes 1 \otimes x^{\deg(m)}$ for $m \in M$ homogeneous.

0EKK Lemma 39.12.4. Let $a : \mathbf{G}_m \times X \to X$ be an action on an affine scheme. Then X is the spectrum of a $\mathbf{Z}$-graded ring and the action is as in Example 39.12.3.

Proof. Let $f \in A = \Gamma(X, \mathcal{O}_X)$. Then we can write

$$a^\sharp(f) = \sum\nolimits_{n \in \mathbf{Z}} f_n \otimes x^n \quad \text{in} \quad A \otimes \mathbf{Z}[x, x^{-1}] = \Gamma(\mathbf{G}_m \times X, \mathcal{O}_{\mathbf{G}_m \times X})$$

as a finite sum with f_n in A uniquely determined. Thus we obtain maps $A \to A$, $f \mapsto f_n$. Since a is an action, if we evaluate at $x = 1$, we see $f = \sum f_n$. Since a is an action we find that

$$\sum (f_n)_m \otimes x^m \otimes x^n = \sum f_n x^n \otimes x^n$$

(compare with computation in Example 39.12.3). Thus $(f_n)_m = 0$ if $n \neq m$ and $(f_n)_n = f_n$. Thus if we set

$$A_n = \{f \in A \mid f_n = f\}$$

then we get $A = \sum A_n$. On the other hand, the sum has to be direct since $f = 0$ implies $f_n = 0$ in the situation above. □

0EKL Lemma 39.12.5. Let A be a graded ring. Let $X = \mathrm{Spec}(A)$ with action $a : \mathbf{G}_m \times X \to X$ as in Example 39.12.3. Let $\mathcal{F}$ be a $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-module. Then $M = \Gamma(X, \mathcal{F})$ has a canonical grading such that it is a graded A-module and such that the isomorphism $\widetilde{M} \to \mathcal{F}$ (Schemes, Lemma 26.7.4) is an isomorphism of $\mathbf{G}_m$-equivariant modules where the $\mathbf{G}_m$-equivariant structure on $\widetilde{M}$ is the one from Example 39.12.3.

Proof. You can either prove this by repeating the arguments of Lemma 39.12.4 for the module M. Alternatively, you can consider the scheme $(X', \mathcal{O}_{X'}) = (X, \mathcal{O}_X \oplus \mathcal{F})$ where $\mathcal{F}$ is viewed as an ideal of square zero. There is a natural action $a' : \mathbf{G}_m \times X' \to X'$ defined using the action on X and on $\mathcal{F}$. Then apply Lemma 39.12.4 to X' and conclude. (The nice thing about this argument is that it immediately shows that the grading on A and M are compatible, i.e., that M is a graded A-module.) Details omitted. □

39.13. Groupoids

0230 Recall that a groupoid is a category in which every morphism is an isomorphism, see Categories, Definition 4.2.5. Hence a groupoid has a set of objects Ob, a set of arrows Arrows, a source and target map $s, t : \mathrm{Arrows} \to \mathrm{Ob}$, and a composition law $c : \mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \to \mathrm{Arrows}$. These maps satisfy exactly the following axioms

(1) (associativity) $c \circ (1, c) = c \circ (c, 1)$ as maps $\mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \to \mathrm{Arrows}$,
(2) (identity) there exists a map $e : \mathrm{Ob} \to \mathrm{Arrows}$ such that

   (a) $s \circ e = t \circ e = \text{id}$ as maps $\text{Ob} \to \text{Ob}$,
   (b) $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$ as maps $\text{Arrows} \to \text{Arrows}$,
(3) (inverse) there exists a map $i : \text{Arrows} \to \text{Arrows}$ such that
   (a) $s \circ i = t$, $t \circ i = s$ as maps $\text{Arrows} \to \text{Ob}$, and
   (b) $c \circ (1, i) = e \circ t$ and $c \circ (i, 1) = e \circ s$ as maps $\text{Arrows} \to \text{Arrows}$.

If this is the case the maps e and i are uniquely determined and i is a bijection. Note that if $(\text{Ob}', \text{Arrows}', s', t', c')$ is a second groupoid category, then a functor $f : (\text{Ob}, \text{Arrows}, s, t, c) \to (\text{Ob}', \text{Arrows}', s', t', c')$ is given by a pair of set maps $f : \text{Ob} \to \text{Ob}'$ and $f : \text{Arrows} \to \text{Arrows}'$ such that $s' \circ f = f \circ s$, $t' \circ f = f \circ t$, and $c' \circ (f, f) = f \circ c$. The compatibility with identity and inverse is automatic. We will use this below. (Warning: The compatibility with identity has to be imposed in the case of general categories.)

0231 Definition 39.13.1. Let S be a scheme.

(1) A groupoid scheme over S, or simply a groupoid over S is a quintuple (U, R, s, t, c) where U and R are schemes over S, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of schemes over S with the following property: For any scheme T over S the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category in the sense described above.
(2) A morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S is given by morphisms of schemes $f : U \to U'$ and $f : R \to R'$ with the following property: For any scheme T over S the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid over S. Note that, by the remarks preceding the definition and the Yoneda lemma, there are unique morphisms of schemes $e : U \to R$ and $i : R \to R$ over S such that for every scheme T over S the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) above, and conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid scheme. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid scheme over S we denote
$$j = (t, s) : R \longrightarrow U \times_S U$$
which is compatible with our conventions in Section 39.3 above. We sometimes say "let (U, R, s, t, c, e, i) be a groupoid over S" to stress the existence of identity and inverse.

0232 Lemma 39.13.2. Given a groupoid scheme (U, R, s, t, c) over S the morphism $j : R \to U \times_S U$ is a pre-equivalence relation.

Proof. Omitted. This is a nice exercise in the definitions. $\square$

0233 Lemma 39.13.3. Given an equivalence relation $j : R \to U \times_S U$ over S there is a unique way to extend it to a groupoid (U, R, s, t, c) over S.

Proof. Omitted. This is a nice exercise in the definitions. $\square$

02YE **Lemma 39.13.4.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. In the commutative diagram

$$\begin{tikzcd} & & U & & \\ R \arrow[urr, "t"] \arrow[d, "s"'] & & R \times_{s,U,t} R \arrow[ll, "\mathrm{pr}_0"'] \arrow[rr, "c"'] \arrow[d, "\mathrm{pr}_1"] & & R \arrow[ull, "t"'] \arrow[d, "s"] \\ U & & R \arrow[ll, "t"'] \arrow[rr, "s"] & & U \end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

03C6 **Lemma 39.13.5.** Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. The diagram

03C7 (39.13.5.1)
$$\begin{tikzcd} R \times_{t,U,t} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "{(\mathrm{pr}_0, c \circ (i,1))}"'] & R \arrow[r, "t"] \arrow[d, "\mathrm{id}_R"] & U \arrow[d, "\mathrm{id}_U"] \\ R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\ R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U & \end{tikzcd}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between $\text{Arrows} \times_{t,\text{Ob},t} \text{Arrows}$ and $\text{Arrows} \times_{s,\text{Ob},t} \text{Arrows}$. Hence the second assertion of the lemma. The last assertion follows from Lemma 39.13.4. □

0DT8 **Lemma 39.13.6.** Let (U, R, s, t, c) be a groupoid over a scheme S. Let $S' \to S$ be a morphism. Then the base changes $U' = S' \times_S U$, $R' = S' \times_S R$ endowed with the base changes s', t', c' of the morphisms s, t, c form a groupoid scheme (U', R', s', t', c') over S' and the projections determine a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes over S.

Proof. Omitted. Hint: $R' \times_{s',U',t'} R' = S' \times_S (R \times_{s,U,t} R)$. □

39.14. Quasi-coherent sheaves on groupoids

03LH See the introduction of Section 39.12 for our choices in direction of arrows.

03LI Definition 39.14.1. Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. A quasi-coherent module on (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

$$\begin{tikzcd} & \mathrm{pr}_1^*t^*\mathcal{F} \arrow[r, "\mathrm{pr}_1^*\alpha"'] \arrow[dl, equal] & \mathrm{pr}_1^*s^*\mathcal{F} \arrow[dr, equal] & \\ \mathrm{pr}_0^*s^*\mathcal{F} & & & c^*s^*\mathcal{F} \\ & \mathrm{pr}_0^*t^*\mathcal{F} \arrow[ul, "\mathrm{pr}_0^*\alpha"] \arrow[r, equal] & c^*t^*\mathcal{F} \arrow[ur, "c^*\alpha"'] & \end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{R\times_{s,U,t}R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 39.13.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

077Q Lemma 39.14.2. Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.

Proof. Pull back the commutative diagram of Definition 39.14.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. $\square$

03LJ Lemma 39.14.3. Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Then pullback f^* given by

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. $\square$

09VH Lemma 39.14.4. Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Assume that

(1) $f : U \to U'$ is quasi-compact and quasi-separated,

(2) the square

$$\begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "t"'] & R' \arrow[d, "t'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

is cartesian, and

(3) s' and t' are flat.

Then pushforward f_* given by

$$(\mathcal{F}, \alpha) \mapsto (f_*\mathcal{F}, f_*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U, R, s, t, c) to the category of quasi-coherent sheaves on (U', R', s', t', c') which is right adjoint to pullback as defined in Lemma 39.14.3.

Proof. Since $U \to U'$ is quasi-compact and quasi-separated we see that f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Schemes, Lemma 26.24.1). Moreover, since the squares

$$\begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "t"'] & R' \arrow[d, "t'"] \\ U \arrow[r, "f"] & U' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "s"'] & R' \arrow[d, "s'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

are cartesian we find that $(t')^*f_*\mathcal{F} = f_*t^*\mathcal{F}$ and $(s')^*f_*\mathcal{F} = f_*s^*\mathcal{F}$, see Cohomology of Schemes, Lemma 30.5.2. Thus it makes sense to think of $f_*\alpha$ as a map $(t')^*f_*\mathcal{F} \to (s')^*f_*\mathcal{F}$. A similar argument shows that $f_*\alpha$ satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and pushforward on modules on ringed spaces are adjoint. Some details omitted. □

077R Lemma 39.14.5. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Schemes, Section 26.24. Since colimits commute with pullback we see that $s^*\mathcal{F} = \operatorname{colim} s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = \operatorname{colim} t^*\mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

077S Lemma 39.14.6. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^* \operatorname{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^* \operatorname{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^* \operatorname{Ker}(\varphi) \to s^* \operatorname{Ker}(\varphi)$ and $\lambda : t^* \operatorname{Coker}(\varphi) \to s^* \operatorname{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\operatorname{Ker}(\varphi), \kappa)$ and $(\operatorname{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\operatorname{Coim}(\varphi) = \operatorname{Im}(\varphi)$ follows because it holds over U. □

39.15. Colimits of quasi-coherent modules

07TS In this section we prove some technical results saying that under suitable assumptions every quasi-coherent module on a groupoid is a filtered colimit of "small" quasi-coherent modules.

07TR Lemma 39.15.1. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are flat, quasi-compact, and quasi-separated. For any quasi-coherent module $\mathcal{G}$ on U, there exists a canonical isomorphism $\alpha : t^*s_*t^*\mathcal{G} \to s^*s_*t^*\mathcal{G}$ which turns $(s_*t^*\mathcal{G}, \alpha)$ into a quasi-coherent module on (U, R, s, t, c). This construction defines a functor

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$.

Proof. The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Schemes, Lemma 26.24.1. Hence $s_*t^*\mathcal{G}$ is quasi-coherent. With notation as in Lemma 39.13.4 we have

$$t^*s_*t^*\mathcal{G} = \mathrm{pr}_{1,*}\mathrm{pr}_0^*t^*\mathcal{G} = \mathrm{pr}_{1,*}c^*t^*\mathcal{G} = s^*s_*t^*\mathcal{G}$$

The middle equality because $t \circ c = t \circ \mathrm{pr}_0$ as morphisms $R \times_{s,U,t} R \to U$, and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Schemes, Lemma 30.5.2.

To verify the cocycle condition of Definition 39.14.1 for α and the adjointness property we describe the construction $\mathcal{G} \mapsto (s_*t^*\mathcal{G}, \alpha)$ in another way. Consider the groupoid scheme $(R, R \times_{t,U,t} R, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ associated to the equivalence relation $R \times_{t,U,t} R$ on R, see Lemma 39.13.3. There is a morphism

$$f : (R, R \times_{t,U,t} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by $s : R \to U$ and $R \times_{t,U,t} R \to R$ given by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$; we omit the verification of the commutativity of the required diagrams. Since $t, s : R \to U$ are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{tikzcd} R \times_{t,U,t} R \arrow[r, "(r_0,r_1)\mapsto r_0^{-1}\circ r_1"'] \arrow[d, "\mathrm{pr}_0"'] & R \arrow[d, "t"] \\ R \arrow[r, "s"] & U \end{tikzcd}$$

by Lemma 39.13.5 it follows that Lemma 39.14.4 applies to f. Thus pushforward and pullback of quasi-coherent modules along f are adjoint functors. To finish the proof we will identify these functors with the functors described above. To do this, note that

$$t^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(R, R \times_{t,U,t} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

is an equivalence by the theory of descent of quasi-coherent sheaves as $\{t : R \to U\}$ is an fpqc covering, see Descent, Proposition 35.5.2.

Pushforward along f precomposed with the equivalence t^* sends $\mathcal{G}$ to $(s_*t^*\mathcal{G}, \alpha)$; we omit the verification that the isomorphism α obtained in this fashion is the same as the one constructed above.

Pullback along f postcomposed with the inverse of the equivalence t^* sends $(\mathcal{F}, \beta)$ to the descent relative to $\{t : R \to U\}$ of the module $s^*\mathcal{F}$ endowed with the descent datum γ on $R \times_{t,U,t} R$ which is the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. Consider the isomorphism $\beta : t^*\mathcal{F} \to s^*\mathcal{F}$. The canonical descent datum (Descent, Definition 35.2.3) on $t^*\mathcal{F}$ relative to $\{t : R \to U\}$ translates via β into the map

$$\mathrm{pr}_0^*s^*\mathcal{F} \xrightarrow{\mathrm{pr}_0^*\beta^{-1}} \mathrm{pr}_0^*t^*\mathcal{F} \xrightarrow{can} \mathrm{pr}_1^*t^*\mathcal{F} \xrightarrow{\mathrm{pr}_1^*\beta} \mathrm{pr}_1^*s^*\mathcal{F}$$

Since β satisfies the cocycle condition, this is equal to the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. To see this take the actual cocycle relation in Definition 39.14.1 and pull it back by the morphism $(\text{pr}_0, c \circ (i, 1)) : R \times_{t,U,t} R \to R \times_{s,U,t} R$ which also plays a role in the commutative diagram of Lemma 39.13.5. It follows that $(s^*\mathcal{F}, \gamma)$ is isomorphic to $(t^*\mathcal{F}, can)$. All in all, we conclude that pullback by f postcomposed with the inverse of the equivalence t^* is isomorphic to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$. □

0GNF Remark 39.15.2. In the situation of Lemma 39.15.1 denote

$$F : QCoh(U, R, s, t, c) \to QCoh(\mathcal{O}_U), \quad (\mathcal{F}, \beta) \mapsto \mathcal{F}$$

the forgetful functor and denote

$$G : QCoh(\mathcal{O}_U) \to QCoh(U, R, s, t, c), \quad \mathcal{G} \mapsto (s_*t^*\mathcal{G}, \alpha)$$

the right adjoint constructed in the lemma. Then the unit $\eta : \text{id} \to G \circ F$ of the adjunction evaluated on $(\mathcal{F}, \beta)$ is given by the map

$$\mathcal{F} \to s_*s^*\mathcal{F} \xrightarrow{\beta^{-1}} s_*t^*\mathcal{F}$$

We omit the verification.

07TT Lemma 39.15.3. Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module, and let $\varphi : \mathcal{G} \to f^*\mathcal{F}$ be a module map. Assume

(1) φ is injective,
(2) f is quasi-compact, quasi-separated, flat, and surjective,
(3) X, Y are locally Noetherian, and
(4) $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module.

Then $\mathcal{F} \cap f_*\mathcal{G}$ defined as the pullback

$$\begin{tikzcd} \mathcal{F} \arrow[r] & f_*f^*\mathcal{F} \\ \mathcal{F} \cap f_*\mathcal{G} \arrow[r] \arrow[u] & f_*\mathcal{G} \arrow[u] \end{tikzcd}$$

is a coherent $\mathcal{O}_X$-module.

Proof. We will freely use the characterization of coherent modules of Cohomology of Schemes, Lemma 30.9.1 as well as the fact that coherent modules form a Serre subcategory of $QCoh(\mathcal{O}_X)$, see Cohomology of Schemes, Lemma 30.9.3. If f has a section σ, then we see that $\mathcal{F} \cap f_*\mathcal{G}$ is contained in the image of $\sigma^*\mathcal{G} \to \sigma^*f^*\mathcal{F} = \mathcal{F}$, hence coherent. In general, to show that $\mathcal{F} \cap f_*\mathcal{G}$ is coherent, it suffices the show that $f^*(\mathcal{F} \cap f_*\mathcal{G})$ is coherent (see Descent, Lemma 35.7.1). Since f is flat this is equal to $f^*\mathcal{F} \cap f^*f_*\mathcal{G}$. Since f is flat, quasi-compact, and quasi-separated we see $f^*f_*\mathcal{G} = p_*q^*\mathcal{G}$ where $p, q : Y \times_X Y \to Y$ are the projections, see Cohomology of Schemes, Lemma 30.5.2. Since p has a section we win. □

Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Assume that U is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is coherent if $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module.

07TU Lemma 39.15.4. Let (U, R, s, t, c) be a groupoid scheme over S. Assume that

(1) U, R are Noetherian,

(2) s, t are flat, quasi-compact, and quasi-separated.

Then every quasi-coherent module $(\mathcal{F}, \beta)$ on (U, R, s, t, c) is a filtered colimit of coherent modules.

Proof. We will use the characterization of Cohomology of Schemes, Lemma 30.9.1 of coherent modules on locally Noetherian scheme without further mention. We can write $\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ as the filtered colimit of coherent submodules $\mathcal{H}_i \subset \mathcal{F}$, see Cohomology of Schemes, Lemma 30.10.4. Given a quasi-coherent sheaf $\mathcal{H}$ on U we denote $(s_*t^*\mathcal{H}, \alpha)$ the quasi-coherent sheaf on (U, R, s, t, c) of Lemma 39.15.1. Consider the adjunction map $(\mathcal{F}, \beta) \to (s_*t^*\mathcal{F}, \alpha)$ in $QCoh(U, R, s, t, c)$, see Remark 39.15.2. Set

$$(\mathcal{F}_i, \beta_i) = (\mathcal{F}, \beta) \times_{(s_*t^*\mathcal{F}, \alpha)} (s_*t^*\mathcal{H}_i, \alpha)$$

in $QCoh(U, R, s, t, c)$. Since restriction to U is an exact functor on $QCoh(U, R, s, t, c)$ by the proof of Lemma 39.14.6 we obtain a pullback diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & s_*t^*\mathcal{F} \\ \mathcal{F}_i \arrow[u] \arrow[r] & s_*t^*\mathcal{H}_i \arrow[u] \end{tikzcd}$$

in other words $\mathcal{F}_i = \mathcal{F} \cap s_*t^*\mathcal{H}_i$. By the description of the adjunction map in Remark 39.15.2 this diagram is isomorphic to the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & s_*s^*\mathcal{F} \\ \mathcal{F}_i \arrow[u] \arrow[r] & s_*t^*\mathcal{H}_i \arrow[u] \end{tikzcd}$$

where the right vertical arrow is the result of appplying s_* to the map

$$t^*\mathcal{H}_i \to t^*\mathcal{F} \xrightarrow{\beta} s^*\mathcal{F}$$

This arrow is injective as t is a flat morphism. It follows that $\mathcal{F}_i$ is coherent by Lemma 39.15.3. Finally, because s is quasi-compact and quasi-separated we see that s_* commutes with colimits (see Cohomology of Schemes, Lemma 30.6.1). Hence $s_*t^*\mathcal{F} = \operatorname{colim} s_*t^*\mathcal{H}_i$ and hence $(\mathcal{F}, \beta) = \operatorname{colim}(\mathcal{F}_i, \beta_i)$ as desired. $\square$

Here is a curious lemma that is useful when working with groupoids on fields. In fact, this is the standard argument to prove that any representation of an algebraic group is a colimit of finite dimensional representations.

07TV Lemma 39.15.5. Let (U, R, s, t, c) be a groupoid scheme over S. Assume that

(1) U, R are affine,
(2) there exist $e_i \in \mathcal{O}_R(R)$ such that every element $g \in \mathcal{O}_R(R)$ can be uniquely written as $\sum s^*(f_i)e_i$ for some $f_i \in \mathcal{O}_U(U)$.

Then every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is a filtered colimit of finite type quasi-coherent modules.

Proof. The assumption means that $\mathcal{O}_R(R)$ is a free $\mathcal{O}_U(U)$-module via s with basis e_i. Hence for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$ we see that $s^*\mathcal{G}(R) = \bigoplus_i \mathcal{G}(U)e_i$. We will write $s(-)$ to indicate pullback of sections by s and similarly for other morphisms. Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Let $\sigma \in \mathcal{F}(U)$. By the above we can write

$$\alpha(t(\sigma)) = \sum s(\sigma_i)e_i$$

for some unique $\sigma_i \in \mathcal{F}(U)$ (all but finitely many are zero of course). We can also write

$$c(e_i) = \sum \mathrm{pr}_1(f_{ij})\mathrm{pr}_0(e_j)$$

as functions on $R \times_{s,U,t} R$. Then the commutativity of the diagram in Definition 39.14.1 means that

$$\sum \mathrm{pr}_1(\alpha(t(\sigma_i)))\mathrm{pr}_0(e_i) = \sum \mathrm{pr}_1(s(\sigma_i)f_{ij})\mathrm{pr}_0(e_j)$$

(calculation omitted). Picking off the coefficients of $\mathrm{pr}_0(e_l)$ we see that $\alpha(t(\sigma_l)) = \sum s(\sigma_i)f_{il}$. Hence the submodule $\mathcal{G} \subset \mathcal{F}$ generated by the elements σ_i defines a finite type quasi-coherent module preserved by α. Hence it is a subobject of $\mathcal{F}$ in $QCoh(U, R, s, t, c)$. This submodule contains σ (as one sees by pulling back the first relation by e). Hence we win. $\square$

We suggest the reader skip the rest of this section. Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Let κ be a cardinal. In the following we will say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is κ-generated if $\mathcal{F}$ is a κ-generated $\mathcal{O}_U$-module, see Properties, Definition 28.23.1.

077T Lemma 39.15.6. Let (U, R, s, t, c) be a groupoid scheme over S. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t, \alpha_t)_{t\in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) such that every κ-generated quasi-coherent module on (U, R, s, t, c) is isomorphic to one of the $(\mathcal{F}_t, \alpha_t)$.

Proof. For each quasi-coherent module $\mathcal{F}$ on U there is a (possibly empty) set of maps $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ such that $(\mathcal{F}, \alpha)$ is a quasi-coherent modules on (U, R, s, t, c). By Properties, Lemma 28.23.2 there exists a set of isomorphism classes of κ-generated quasi-coherent $\mathcal{O}_U$-modules. $\square$

077U Lemma 39.15.7. Let (U, R, s, t, c) be a groupoid scheme over S. Assume that s, t are flat. There exists a cardinal κ such that every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is the directed colimit of its κ-generated quasi-coherent submodules.

Proof. In the statement of the lemma and in this proof a submodule of a quasi-coherent module $(\mathcal{F}, \alpha)$ is a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as subsheaves of $s^*\mathcal{F}$. This makes sense because since s, t are flat the pullbacks s^* and t^* are exact, i.e., preserve subsheaves. The proof will be a repeat of the proof of Properties, Lemma 28.23.3. We urge the reader to read that proof first.

Choose an affine open covering $U = \bigcup_{i\in I} U_i$. For each pair i, j choose affine open coverings

$$U_i \cap U_j = \bigcup\nolimits_{k\in I_{ij}} U_{ijk} \quad \text{and} \quad s^{-1}(U_i) \cap t^{-1}(U_j) = \bigcup\nolimits_{k\in J_{ij}} W_{ijk}.$$

Write $U_i = \mathrm{Spec}(A_i)$, $U_{ijk} = \mathrm{Spec}(A_{ijk})$, $W_{ijk} = \mathrm{Spec}(B_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}, J_{ij}.

Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Set $M_i = \mathcal{F}(U_i)$, $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}$$

and that α gives isomorphisms

$$\alpha|_{W_{ijk}} : M_i \otimes_{A_i,t} B_{ijk} \longrightarrow M_j \otimes_{A_j,s} B_{ijk}$$

see Schemes, Lemma 26.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ or $k \in J_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'} \quad \text{or} \quad \alpha(m \otimes 1) = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes b_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$ or $b_{m'} \in B_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ when $k \in I_{ij}$. Fix such a collection $S(i, j, k, m)$

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S'_i$. Note that S'_i has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk} \quad \text{and} \quad \alpha(N_i \otimes_{A_i,t} B_{ijk}) = N_j \otimes_{A_j,s} B_{ijk}$$

as submodules of M_{ijk} or $M_j \otimes_{A_j,s} B_{ijk}$. Thus there exists a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as submodules of $s^*\mathcal{F}$. In other words, $(\mathcal{G}, \alpha|_{t^*\mathcal{G}})$ is a submodule of $(\mathcal{F}, \alpha)$. Moreover, by construction $\mathcal{G}$ is κ-generated.

Let $\{(\mathcal{G}_t, \alpha_t)\}_{t \in T}$ be the set of κ-generated quasi-coherent submodules of $(\mathcal{F}, \alpha)$. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent submodule as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\operatorname{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. ☐

39.16. Groupoids and group schemes

03LK There are many ways to construct a groupoid out of an action a of a group G on a set V. We choose the one where we think of an element $g \in G$ as an arrow with source v and target $a(g, v)$. This leads to the following construction for group actions of schemes.

0234 Lemma 39.16.1. Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y with identity e_G and inverse i_G. Let X/Y be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X/Y. Then we get a groupoid scheme (U, R, s, t, c, e, i) over S in the following manner:

(1) We set $U = X$, and $R = G \times_Y X$.

(2) We set $s : R \to U$ equal to $(g, x) \mapsto x$.
(3) We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.
(4) We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.
(5) We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.
(6) We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. $\square$

03LL Lemma 39.16.2. Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y. Let X be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X over Y. Let (U, R, s, t, c) be the groupoid scheme constructed in Lemma 39.16.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_Y X \to X$, see Definitions 39.12.1 and 39.14.1. Using the translation in Lemma 39.16.1 the commutativity requirements of the two definitions match up exactly. $\square$

39.17. The stabilizer group scheme

03LM Given a groupoid scheme we get a group scheme as follows.

0235 Lemma 39.17.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The scheme G defined by the cartesian square

$$\begin{tikzcd} G \arrow[r] \arrow[d] & R \arrow[d, "j=(t,s)"] \\ U \arrow[r, "\Delta"] & U \times_S U \end{tikzcd}$$

is a group scheme over U with composition law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. $\square$

Since Δ is an immersion we see that $G = j^{-1}(\Delta_{U/S})$ is a locally closed subscheme of R. Thinking of it in this way, the structure morphism $j^{-1}(\Delta_{U/S}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

0236 Definition 39.17.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The group scheme $j^{-1}(\Delta_{U/S}) \to U$ is called the stabilizer of the groupoid scheme (U, R, s, t, c).

In the literature the stabilizer group scheme is often denoted S (because the word stabilizer starts with an "s" presumably); we cannot do this since we have already used S for the base scheme.

0237 Lemma 39.17.3. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S, and let G/U be its stabilizer. Denote R_t/U the scheme R seen as a scheme over U via the morphism $t : R \to U$. There is a canonical left action

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/S we define $a(g, r) = c(g, r)$. $\square$

04Q2 **Lemma 39.17.4.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let G be the stabilizer group scheme of R. Let

$$G_0 = G \times_{U, \mathrm{pr}_0} (U \times_S U) = G \times_S U$$

as a group scheme over $U \times_S U$. The action of G on R of Lemma 39.17.3 induces an action of G_0 on R over $U \times_S U$ which turns R into a pseudo G_0-torsor over $U \times_S U$.

Proof. This is true because in a groupoid category $\mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is a principal homogeneous set under the group $\mathrm{Mor}_{\mathcal{C}}(y, y)$. $\square$

04Q3 **Lemma 39.17.5.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in U \times_S U$ be a point. Denote R_p the scheme theoretic fibre of $j = (t, s) : R \to U \times_S U$. If $R_p \neq \emptyset$, then the action

$$G_{0, \kappa(p)} \times_{\kappa(p)} R_p \longrightarrow R_p$$

(see Lemma 39.17.4) which turns R_p into a $G_{\kappa(p)}$-torsor over $\kappa(p)$.

Proof. The action is a pseudo-torsor by the lemma cited in the statement. And if R_p is not the empty scheme, then $\{R_p \to p\}$ is an fpqc covering which trivializes the pseudo-torsor. $\square$

39.18. Restricting groupoids

02VA Consider a (usual) groupoid $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$. Suppose we have a map of sets $g : \mathrm{Ob}' \to \mathrm{Ob}$. Then we can construct a groupoid $\mathcal{C}' = (\mathrm{Ob}', \mathrm{Arrows}', s', t', c')$ by thinking of a morphism between elements x', y' of Ob' as a morphism in $\mathcal{C}$ between $g(x'), g(y')$. In other words we set

$$\mathrm{Arrows}' = \mathrm{Ob}' \times_{g, \mathrm{Ob}, t} \mathrm{Arrows} \times_{s, \mathrm{Ob}, g} \mathrm{Ob}'.$$

with obvious choices for s', t', and c'. There is a canonical functor $\mathcal{C}' \to \mathcal{C}$ which is fully faithful, but not necessarily essentially surjective. This groupoid $\mathcal{C}'$ endowed with the functor $\mathcal{C}' \to \mathcal{C}$ is called the *restriction of the groupoid $\mathcal{C}$ to Ob'*.

02VB **Lemma 39.18.1.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Consider the following diagram

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right=60, "t'"'] & R \times_{s, U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U, t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s', U', t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid scheme over S and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes over S. Moreover, for any scheme T over S the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see above) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. $\square$

02VC Definition 39.18.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. The morphism of groupoids $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 39.18.1 is called the restriction of (U, R, s, t, c) to U'. We sometime use the notation $R' = R|_{U'}$ in this case.

02VD Lemma 39.18.3. The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 39.18.2 and 39.3.3 agree via the constructions of Lemmas 39.13.2 and 39.13.3.

Proof. What we are saying here is that R' of Lemma 39.18.1 is also equal to

$$R' = (U' \times_S U') \times_{U \times_S U} R \longrightarrow U' \times_S U'$$

In fact this might have been a clearer way to state that lemma. $\square$

04ML Lemma 39.18.4. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g. Let G be the stabilizer of (U, R, s, t, c) and let G' be the stabilizer of (U', R', s', t', c'). Then G' is the base change of G by g, i.e., there is a canonical identification $G' = U' \times_{g,U} G$.

Proof. Omitted. $\square$

39.19. Invariant subschemes

03LN In this section we discuss briefly the notion of an invariant subscheme.

03BC Definition 39.19.1. Let (U, R, s, t, c) be a groupoid scheme over the base scheme S.

(1) A subset $W \subset U$ is set-theoretically R-invariant if $t(s^{-1}(W)) \subset W$.
(2) An open $W \subset U$ is R-invariant if $t(s^{-1}(W)) \subset W$.
(3) A closed subscheme $Z \subset U$ is called R-invariant if $t^{-1}(Z) = s^{-1}(Z)$. Here we use the scheme theoretic inverse image, see Schemes, Definition 26.17.7.
(4) A monomorphism of schemes $T \to U$ is R-invariant if $T \times_{U,t} R = R \times_{s,U} T$ as schemes over R.

For subsets and open subschemes $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$ as subsets of R. If $W \subset U$ is an R-equivariant open subscheme then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant closed subscheme, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

03LO Lemma 39.19.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S.

(1) For any subset $W \subset U$ the subset $t(s^{-1}(W))$ is set-theoretically R-invariant.
(2) If s and t are open, then for every open $W \subset U$ the open $t(s^{-1}(W))$ is an R-invariant open subscheme.
(3) If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subschemes.

Proof. Part (1) follows from Lemmas 39.3.4 and 39.13.2, namely, $t(s^{-1}(W))$ is the set of points of U equivalent to a point of W. Next, assume s and t open and $W \subset U$ open. Since s is open the set $W' = t(s^{-1}(W))$ is an open subset of U. Finally, assume that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = t(s^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over all affine opens of U we see (3). □

0APA Lemma 39.19.3. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s and t quasi-compact and flat and U quasi-separated. Let $W \subset U$ be quasi-compact open. Then $t(s^{-1}(W))$ is an intersection of a nonempty family of quasi-compact open subsets of U.

Proof. Note that $s^{-1}(W)$ is quasi-compact open in R. As a continuous map t maps the quasi-compact subset $s^{-1}(W)$ to a quasi-compact subset $t(s^{-1}(W))$. As t is flat and $s^{-1}(W)$ is closed under generalization, so is $t(s^{-1}(W))$, see (Morphisms, Lemma 29.25.9 and Topology, Lemma 5.19.6). Pick a quasi-compact open $W' \subset U$ containing $t(s^{-1}(W))$. By Properties, Lemma 28.2.4 we see that W' is a spectral space (here we use that U is quasi-separated). Then the lemma follows from Topology, Lemma 5.24.7 applied to $t(s^{-1}(W)) \subset W'$. □

0APB Lemma 39.19.4. Assumptions and notation as in Lemma 39.19.3. There exists an R-invariant open $V \subset U$ and a quasi-compact open W' such that $W \subset V \subset W' \subset U$.

Proof. Set $E = t(s^{-1}(W))$. Recall that E is set-theoretically R-invariant (Lemma 39.19.2). By Lemma 39.19.3 there exists a quasi-compact open W' containing E. Let $Z = U \setminus W'$ and consider $T = t(s^{-1}(Z))$. Observe that $Z \subset T$ and that $E \cap T = \emptyset$ because $s^{-1}(E) = t^{-1}(E)$ is disjoint from $s^{-1}(Z)$. Since T is the image of the closed subset $s^{-1}(Z) \subset R$ under the quasi-compact morphism $t : R \to U$ we see that any point ξ in the closure $\overline{T}$ is the specialization of a point of T, see Morphisms, Lemma 29.6.5 (and Morphisms, Lemma 29.6.3 to see that the scheme theoretic image is the closure of the image). Say $\xi' \rightsquigarrow \xi$ with $\xi' \in T$. Suppose that $r \in R$ and $s(r) = \xi$. Since s is flat we can find a specialization $r' \rightsquigarrow r$ in R such that $s(r') = \xi'$ (Morphisms, Lemma 29.25.9). Then $t(r') \rightsquigarrow t(r)$. We conclude that $t(r') \in T$ as T is set-theoretically invariant by Lemma 39.19.2. Thus $\overline{T}$ is a set-theoretically R-invariant closed subset and $V = U \setminus \overline{T}$ is the open we are looking for. It is contained in W' which finishes the proof. □

39.20. Quotient sheaves

02VE Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-relation over S. Say U, R, S are objects of a τ-site Sch_τ (see Topologies, Section 34.2). Then we can consider the functors

$$h_U, h_R : (Sch/S)_\tau^{opp} \longrightarrow \text{Sets}.$$

These are sheaves, see Descent, Lemma 35.13.7. The morphism j induces a map $j : h_R \to h_U \times h_U$. For each object $T \in \text{Ob}((Sch/S)_\tau)$ we can take the equivalence relation $\sim_T$ generated by $j(T) : R(T) \to U(T) \times U(T)$ and consider the quotient. Hence we get a presheaf

02VF (39.20.0.1) $$(Sch/S)_\tau^{opp} \longrightarrow \text{Sets}, \quad T \longmapsto U(T)/\sim_T$$

02VG Definition 39.20.1. Let τ, S, and the pre-relation $j : R \to U \times_S U$ be as above. In this setting the quotient sheaf U/R associated to j is the sheafification of the presheaf (39.20.0.1) in the τ-topology. If $j : R \to U \times_S U$ comes from the action of a group scheme G/S on U as in Lemma 39.16.1 then we sometimes denote the quotient sheaf U/G.

This means exactly that the diagram

$$\begin{tikzcd} h_R \arrow[r, shift left] \arrow[r, shift right] & h_U \arrow[r] & U/R \end{tikzcd}$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_\tau$. Using the Yoneda embedding we may view $(Sch/S)_\tau$ as a full subcategory of sheaves on $(Sch/S)_\tau$ and hence identify schemes with representable functors. Using this abuse of notation we will often depict the diagram above simply

$$\begin{tikzcd} R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U \arrow[r] & U/R \end{tikzcd}$$

We will mostly work with the fppf topology when considering quotient sheaves of groupoids/equivalence relations.

03BD Definition 39.20.2. In the situation of Definition 39.20.1. We say that the pre-relation j has a representable quotient if the sheaf U/R is representable. We will say a groupoid (U, R, s, t, c) has a representable quotient if the quotient U/R with $j = (t, s)$ is representable.

The following lemma characterizes schemes M representing the quotient. It applies for example if $\tau = fppf$, $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

03C5 Lemma 39.20.3. In the situation of Definition 39.20.1. Assume there is a scheme M, and a morphism $U \to M$ such that

(1) the morphism $U \to M$ equalizes s, t,
(2) the morphism $U \to M$ induces a surjection of sheaves $h_U \to h_M$ in the τ-topology, and
(3) the induced map $(t, s) : R \to U \times_M U$ induces a surjection of sheaves $h_R \to h_{U \times_M U}$ in the τ-topology.

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $h_U \to h_M$ factors through U/R. Condition (2) says that $U/R \to h_M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to h_M$ is injective as a map of sheaves. Hence the lemma follows. $\square$

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

045Y Lemma 39.20.4. Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S. Assume U, R, S are objects of a τ-site Sch_τ. For $T \in \mathrm{Ob}((Sch/S)_\tau)$ and $a, b \in U(T)$ the following are equivalent:

(1) a and b map to the same element of $(U/R)(T)$, and
(2) there exists a τ-covering $\{f_i : T_i \to T\}$ of T and morphisms $r_i : T_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.

In other words, in this case the map of τ-sheaves

$$h_R \longrightarrow h_U \times_{U/R} h_U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (39.20.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 39.3.1. $\square$

045Z Lemma 39.20.5. Let $\tau \in \{Zariski, \acute{e}tale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S and $g : U' \to U$ a morphism of schemes over S. Let $j' : R' \to U' \times_S U'$ be the restriction of j to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If g defines a surjection $h_{U'} \to h_U$ of sheaves in the τ-topology (for example if $\{g : U' \to U\}$ is a τ-covering), then $U'/R' \to U/R$ is an isomorphism.

Proof. Suppose $\xi, \xi' \in (U'/R')(T)$ are sections which map to the same section of U/R. Then we can find a τ-covering $\mathcal{T} = \{T_i \to T\}$ of T such that $\xi|_{T_i}, \xi'|_{T_i}$ are given by $a_i, a_i' \in U'(T_i)$. By Lemma 39.20.4 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : T_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(T_i)$ and this shows that a_i and a_i' define the same section of U'/R' over T_i. By the sheaf condition this implies $\xi = \xi'$.

If $h_{U'} \to h_U$ is a surjection of sheaves, then of course $U'/R' \to U/R$ is surjective also. If $\{g : U' \to U\}$ is a τ-covering, then the map of sheaves $h_{U'} \to h_U$ is surjective, see Sites, Lemma 7.12.4. Hence $U'/R' \to U/R$ is surjective also in this case. $\square$

02VH Lemma 39.20.6. Let $\tau \in \{Zariski, \acute{e}tale, fppf, smooth, syntomic\}$. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ a morphism of schemes over S. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$\begin{tikzcd} U' \times_{g,U,t} R \arrow[r, "\mathrm{pr}_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

defines a surjection of sheaves in the τ-topology then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is a τ-covering, or if $U' \to U$ defines a surjection of sheaves in the τ-topology, or if $\{g : U' \to U\}$ is a covering in the τ-topology.

Proof. Injectivity follows on combining Lemmas 39.13.2 and 39.20.5. To see surjectivity (see Sites, Section 7.11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ is the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find a τ-covering $\{\varphi_i : T_i \to T\}$ and morphisms

$f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f'_i = \mathrm{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f'_i \in U'(T_i)$ maps to $g \circ f'_i \in U(T_i)$ and that $g \circ f'_i \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (39.20.0.1). Namely, the element of $R(T_i)$ giving the relation is $\mathrm{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is a τ-covering, then it induces a surjection of sheaves, see Sites, Lemma 7.12.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). □

07S3 Lemma 39.20.7. Let S be a scheme. Let $f : (U, R, j) \to (U', R', j')$ be a morphism between equivalence relations over S. Assume that

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "s"] & R' \arrow[d, "s'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

is cartesian. For any $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$ the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U/R \arrow[d, "f"] \\ U' \arrow[r] & U'/R' \end{tikzcd}$$

is a fibre product square of τ-sheaves.

Proof. By Lemma 39.20.4 the quotient sheaves have a simple description which we will use below without further mention. We first show that

$$U \longrightarrow U' \times_{U'/R'} U/R$$

is injective. Namely, assume $a, b \in U(T)$ map to the same element on the right hand side. Then $f(a) = f(b)$. After replacing T by the members of a τ-covering we may assume that there exists an $r \in R(T)$ such that $a = s(r)$ and $b = t(r)$. Then $r' = f(r)$ is a T-valued point of R' with $s'(r') = t'(r')$. Hence $r' = e'(f(a))$ (where e' is the identity of the groupoid scheme associated to j', see Lemma 39.13.3). Because the first diagram of the lemma is cartesian this implies that r has to equal $e(a)$. Thus $a = b$.

Finally, we show that the displayed arrow is surjective. Let T be a scheme over S and let $(a', \bar{b})$ be a section of the sheaf $U' \times_{U'/R'} U/R$ over T. After replacing T by the members of a τ-covering we may assume that $\bar{b}$ is the class of an element $b \in U(T)$. After replacing T by the members of a τ-covering we may assume that there exists an $r' \in R'(T)$ such that $a' = t(r')$ and $s'(r') = f(b)$. Because the first diagram of the lemma is cartesian we can find $r \in R(T)$ such that $s(r) = b$ and $f(r) = r'$. Then it is clear that $a = t(r) \in U(T)$ is a section which maps to $(a', \bar{b})$. □

39.21. Descent in terms of groupoids

0APC Cartesian morphisms are defined as follows.

0APD Definition 39.21.1. Let S be a scheme. Let $f : (U', R', s', t', c') \to (U, R, s, t, c)$ be a morphism of groupoid schemes over S. We say f is cartesian, or that (U', R', s', t', c') is cartesian over (U, R, s, t, c), if the diagram

$$\begin{tikzcd} R' \arrow[r, "f"] \arrow[d, "s'"'] & R \arrow[d, "s"] \\ U' \arrow[r, "f"] & U \end{tikzcd}$$

is a fibre square in the category of schemes. A morphism of groupoid schemes cartesian over (U, R, s, t, c) is a morphism of groupoid schemes compatible with the structure morphisms towards (U, R, s, t, c).

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian groupoid schemes over a fixed groupoid scheme.

0APE Lemma 39.21.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The category of groupoid schemes cartesian over (U, R, s, t, c) is equivalent to the category of pairs (V, φ) where V is a scheme over U and

$$\varphi : V \times_{U,t} R \longrightarrow R \times_{s,U} V$$

is an isomorphism over R such that $e^*\varphi = \text{id}_V$ and such that

$$c^*\varphi = \text{pr}_1^*\varphi \circ \text{pr}_0^*\varphi$$

as morphisms of schemes over $R \times_{s,U,t} R$.

Proof. The pullback notation in the lemma signifies base change. The displayed formula makes sense because

$$(R \times_{s,U,t} R) \times_{\text{pr}_1, R, \text{pr}_1} (V \times_{U,t} R) = (R \times_{s,U,t} R) \times_{\text{pr}_0, R, \text{pr}_0} (R \times_{s,U} V)$$

as schemes over $R \times_{s,U,t} R$.

Given (V, φ) we set $U' = V$ and $R' = V \times_{U,t} R$. We set $t' : R' \to U'$ equal to the projection $V \times_{U,t} R \to V$. We set s' equal to φ followed by the projection $R \times_{s,U} V \to V$. We set c' equal to the composition

$$\begin{aligned} R' \times_{s',U',t'} R' &\xrightarrow{\varphi, 1} (R \times_{s,U} V) \times_V (V \times_{U,t} R) \\ &\to R \times_{s,U} V \times_{U,t} R \\ &\xrightarrow{\varphi^{-1}, 1} V \times_{U,t} (R \times_{s,U,t} R) \\ &\xrightarrow{1, c} V \times_{U,t} R = R' \end{aligned}$$

A computation, which we omit shows that we obtain a groupoid scheme over (U, R, s, t, c). It is clear that this groupoid scheme is cartesian over (U, R, s, t, c).

Conversely, given $f : (U', R', s', t', c') \to (U, R, s, t, c)$ cartesian then the morphisms

$$U' \times_{U,t} R \xleftarrow{t', f} R' \xrightarrow{f, s'} R \times_{s,U} U'$$

are isomorphisms and we can set $V = U'$ and φ equal to the composition $(f, s') \circ (t', f)^{-1}$. We omit the proof that φ satisfies the conditions in the lemma. We omit the proof that these constructions are mutually inverse. $\square$

Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Then we obtain a groupoid scheme $(X, X \times_Y X, \text{pr}_1, \text{pr}_0, c)$ over S. Namely, $j : X \times_Y X \to X \times_S X$ is an equivalence relation and we can take the associated groupoid, see Lemma 39.13.3.

0APF Lemma 39.21.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. The construction of Lemma 39.21.2 determines an equivalence

$$\begin{matrix} \text{category of groupoid schemes} \\ \text{cartesian over } (X, X \times_Y X, \ldots) \end{matrix} \longrightarrow \begin{matrix} \text{category of descent data} \\ \text{relative to } X/Y \end{matrix}$$

Proof. This is clear from Lemma 39.21.2 and the definition of descent data on schemes in Descent, Definition 35.34.1. $\square$

39.22. Separation conditions

02YG This really means conditions on the morphism $j : R \to U \times_S U$ when given a groupoid (U, R, s, t, c) over S. As in the previous section we first formulate the corresponding diagram.

02YH Lemma 39.22.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme. The commutative diagram

$$\begin{tikzcd} R \arrow[rr, "f \mapsto (f, s(f))"'] \arrow[d, "\Delta_{R/U \times_S U}"] & & R \times_{s,U} U \arrow[r] \arrow[d] & U \arrow[d] \\ R \times_{(U \times_S U)} R \arrow[rr, "(f,g) \mapsto (f, f^{-1} \circ g)"] & & R \times_{s,U} G \arrow[r] & G \end{tikzcd}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. $\square$

02YI Lemma 39.22.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme.

(1) The following are equivalent
  (a) $j : R \to U \times_S U$ is separated,
  (b) $G \to U$ is separated, and
  (c) $e : U \to G$ is a closed immersion.
(2) The following are equivalent
  (a) $j : R \to U \times_S U$ is quasi-separated,
  (b) $G \to U$ is quasi-separated, and
  (c) $e : U \to G$ is quasi-compact.

Proof. The group scheme $G \to U$ is the base change of $R \to U \times_S U$ by the diagonal morphism $U \to U \times_S U$, see Lemma 39.17.1. Hence if j is separated (resp. quasi-separated), then $G \to U$ is separated (resp. quasi-separated). (See Schemes, Lemma 26.21.12). Thus (a) $\Rightarrow$ (b) in both (1) and (2).

If $G \to U$ is separated (resp. quasi-separated), then the morphism $U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. quasi-compact), see Schemes, Lemma 26.21.11. Thus (b) $\Rightarrow$ (a) in both (1) and (2).

By the result of Lemma 39.22.1 (and Schemes, Lemmas 26.18.2 and 26.19.3) we see that if e is a closed immersion (resp. quasi-compact) $\Delta_{R/U \times_S U}$ is a closed immersion (resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in both (1) and (2). $\square$

39.23. Finite flat groupoids, affine case

03BE Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine. In this case we get two ring maps $s^\sharp, t^\sharp : A \longrightarrow B$. Let C be the equalizer of $s^\sharp$ and $t^\sharp$. In a formula

03BF (39.23.0.1) $$C = \{a \in A \mid t^\sharp(a) = s^\sharp(a)\}.$$

We will sometimes call this the ring of R-invariant functions on U. What properties does $M = \mathrm{Spec}(C)$ have? The first observation is that the diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d] \\ U \arrow[r] & M \end{tikzcd}$$

is commutative, i.e., the morphism $U \to M$ equalizes s, t. Moreover, if T is any affine scheme, and if $U \to T$ is a morphism which equalizes s, t, then $U \to T$ factors through $U \to M$. In other words, $U \to M$ is a coequalizer in the category of affine schemes.

We would like to find conditions that guarantee the morphism $U \to M$ is really a "quotient" in the category of schemes. We will discuss this at length elsewhere (insert future reference here); here we just discuss some special cases. Namely, we will focus on the case where s, t are finite locally free.

03BG Example 39.23.1. Let k be a field. Let $U = \mathrm{GL}_{2,k}$. Let $B \subset \mathrm{GL}_2$ be the closed subgroup scheme of upper triangular matrices. Then the quotient sheaf $\mathrm{GL}_{2,k}/B$ (in the Zariski, étale or fppf topology, see Definition 39.20.1) is representable by the projective line: $\mathbf{P}^1 = \mathrm{GL}_{2,k}/B$. (Details omitted.) On the other hand, the ring of invariant functions in this case is just k. Note that in this case the morphisms $s, t : R = \mathrm{GL}_{2,k} \times_k B \to \mathrm{GL}_{2,k} = U$ are smooth of relative dimension 3.

Recall that in Exercises, Exercises 111.22.6 and 111.22.7 we have defined the determinant and the norm for finitely locally free modules and finite locally free ring extensions. If $\varphi : A \to B$ is a finite locally free ring map, then we will denote $\mathrm{Norm}_\varphi(b) \in A$ the norm of $b \in B$. In the case of a finite locally free morphism of schemes, the norm was constructed in Divisors, Lemma 31.17.6.

03BH Lemma 39.23.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let C be as in (39.23.0.1). Let $f \in A$. Then $\mathrm{Norm}_{s^\sharp}(t^\sharp(f)) \in C$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& U & \\
R \arrow[ur, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "\mathrm{pr}_0"'] \arrow[r, "c"] \arrow[d, "\mathrm{pr}_1"] & R \arrow[ul, "t"'] \arrow[d, "s"] \\
U & R \arrow[l, "t"'] \arrow[r, "s"] & U
\end{tikzcd}$$

of Lemma 39.13.4. Think of $f \in \Gamma(U, \mathcal{O}_U)$. The commutativity of the top part of the diagram shows that $\mathrm{pr}_0^\sharp(t^\sharp(f)) = c^\sharp(t^\sharp(f))$ as elements of $\Gamma(R \times_{S,U,t} R, \mathcal{O})$. Looking at the right lower cartesian square the compatibility of the norm construction with base change shows that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1^\sharp}(c^\sharp(t^\sharp(f)))$. Similarly we get $t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1^\sharp}(\mathrm{pr}_0^\sharp(t^\sharp(f)))$. Hence by the first equality of this proof we see that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f)))$ as desired. □

03BI **Lemma 39.23.3.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $s, t : R \to U$ finite locally free. Then

$$U = \coprod\nolimits_{r \geq 1} U_r$$

is a disjoint union of R-invariant opens such that the restriction R_r of R to U_r has the property that $s, t : R_r \to U_r$ are finite locally free of rank r.

Proof. By Morphisms, Lemma 29.48.5 there exists a decomposition $U = \coprod_{r \geq 0} U_r$ such that $s : s^{-1}(U_r) \to U_r$ is finite locally free of rank r. As s is surjective we see that $U_0 = \emptyset$. Note that $u \in U_r \Leftrightarrow$ if and only if the scheme theoretic fibre $s^{-1}(u)$ has degree r over $\kappa(u)$. Now, if $z \in R$ with $s(z) = u$ and $t(z) = u'$ then using notation as in Lemma 39.13.4

$$\mathrm{pr}_1^{-1}(z) \to \mathrm{Spec}(\kappa(z))$$

is the base change of both $s^{-1}(u) \to \mathrm{Spec}(\kappa(u))$ and $s^{-1}(u') \to \mathrm{Spec}(\kappa(u'))$ by the lemma cited. Hence $u \in U_r \Leftrightarrow u' \in U_r$, in other words, the open subsets U_r are R-invariant. In particular the restriction of R to U_r is just $s^{-1}(U_r)$ and $s : R_r \to U_r$ is finite locally free of rank r. As $t : R_r \to U_r$ is isomorphic to s by the inverse of R_r we see that it has also rank r. □

03BJ **Lemma 39.23.4.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (39.23.0.1). Then A is integral over C.

Proof. First, by Lemma 39.23.3 we know that (U, R, s, t, c) is a disjoint union of groupoid schemes (U_r, R_r, s, t, c) such that each $s, t : R_r \to U_r$ has constant rank r. As U is quasi-compact, we have $U_r = \emptyset$ for almost all r. It suffices to prove the lemma for each (U_r, R_r, s, t, c) and hence we may assume that s, t are finite locally free of rank r.

Assume that s, t are finite locally free of rank r. Let $f \in A$. Consider the element $x - f \in A[x]$, where we think of x as the coordinate on $\mathbf{A}^1$. Since

$$(U \times \mathbf{A}^1, R \times \mathbf{A}^1, s \times \mathrm{id}_{\mathbf{A}^1}, t \times \mathrm{id}_{\mathbf{A}^1}, c \times \mathrm{id}_{\mathbf{A}^1})$$

is also a groupoid scheme with finite source and target, we may apply Lemma 39.23.2 to it and we see that $P(x) = \text{Norm}_{s^\sharp}(t^\sharp(x - f))$ is an element of $C[x]$. Because $s^\sharp : A \to B$ is finite locally free of rank r we see that P is monic of degree r. Moreover $P(f) = 0$ by Cayley-Hamilton (Algebra, Lemma 10.16.1). □

03BK Lemma 39.23.5. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \text{Spec}(A)$ and $R = \text{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (39.23.0.1). Let $C \to C'$ be a ring map, and set $U' = \text{Spec}(A \otimes_C C')$, $R' = \text{Spec}(B \otimes_C C')$. Then

(1) The maps s, t, c induce maps s', t', c' such that (U', R', s', t', c') is a groupoid scheme. Let $C^1 \subset A'$ be the R'-invariant functions on U'.
(2) The canonical map $\varphi : C' \to C^1$ satisfies
    (a) for every $f \in C^1$ there exists an $n > 0$ and a polynomial $P \in C'[x]$ whose image in $C^1[x]$ is $(x - f)^n$, and
    (b) for every $f \in \text{Ker}(\varphi)$ there exists an $n > 0$ such that $f^n = 0$.
(3) If $C \to C'$ is flat then φ is an isomorphism.

Proof. The proof of part (1) is omitted. Let us denote $A' = A \otimes_C C'$ and $B' = B \otimes_C C'$. Then we have

$$C^1 = \{a \in A' \mid (t')^\sharp(a) = (s')^\sharp(a)\} = \{a \in A \otimes_C C' \mid t^\sharp \otimes 1(a) = s^\sharp \otimes 1(a)\}.$$

In other words, C^1 is the kernel of the difference map $(t^\sharp - s^\sharp) \otimes 1$ which is just the base change of the C-linear map $t^\sharp - s^\sharp : A \to B$ by $C \to C'$. Hence (3) follows.

Proof of part (2)(b). Since $C \to A$ is integral (Lemma 39.23.4) and injective we see that $\text{Spec}(A) \to \text{Spec}(C)$ is surjective, see Algebra, Lemma 10.36.17. Thus also $\text{Spec}(A') \to \text{Spec}(C')$ is surjective as a base change of a surjective morphism (Morphisms, Lemma 29.9.4). Hence $\text{Spec}(C^1) \to \text{Spec}(C')$ is surjective also. This implies (2)(b) holds for example by Algebra, Lemma 10.30.6.

Proof of part (2)(a). By Lemma 39.23.3 our groupoid scheme (U, R, s, t, c) decomposes as a finite disjoint union of groupoid schemes (U_r, R_r, s, t, c) such that $s, t : R_r \to U_r$ are finite locally free of rank r. Pulling back by $U' = \text{Spec}(C') \to U$ we obtain a similar decomposition of U' and $U^1 = \text{Spec}(C^1)$. We will show in the next paragraph that (2)(a) holds for the corresponding system of rings $A_r, B_r, C_r, C'_r, C^1_r$ with $n = r$. Then given $f \in C^1$ let $P_r \in C_r[x]$ be the polynomial whose image in $C^1_r[x]$ is the image of $(x - f)^r$. Choosing a sufficiently divisible integer n we see that there is a polynomial $P \in C'[x]$ whose image in $C^1[x]$ is $(x-f)^n$; namely, we take P to be the unique element of $C'[x]$ whose image in $C'_r[x]$ is $P_r^{n/r}$.

In this paragraph we prove (2)(a) in case the ring maps $s^\sharp, t^\sharp : A \to B$ are finite locally free of a fixed rank r. Let $f \in C^1 \subset A' = A \otimes_C C'$. Choose a flat C-algebra D and a surjection $D \to C'$. Choose a lift $g \in A \otimes_C D$ of f. Consider the polynomial

$$P = \text{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(x - g))$$

in $(A \otimes_C D)[x]$. By Lemma 39.23.2 and part (3) of the current lemma the coefficients of P are in D (compare with the proof of Lemma 39.23.4). On the other hand, the image of P in $(A \otimes_C C')[x]$ is $(x - f)^r$ because $t^\sharp \otimes 1(x - f) = s^\sharp(x - f)$ and $s^\sharp$ is finite locally free of rank r. This proves what we want with P as in the statement (2)(a) given by the image of our P under the map $D[x] \to C'[x]$. □

03BL **Lemma 39.23.6.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (39.23.0.1). Then $U \to M = \mathrm{Spec}(C)$ has the following properties:

(1) the map on points $|U| \to |M|$ is surjective and $u_0, u_1 \in |U|$ map to the same point if and only if there exists a $r \in |R|$ with $t(r) = u_0$ and $s(r) = u_1$, in a formula
$$|M| = |U|/|R|$$
(2) for any algebraically closed field k we have
$$M(k) = U(k)/R(k)$$

Proof. Since $C \to A$ is integral (Lemma 39.23.4) and injective we see that $\mathrm{Spec}(A) \to \mathrm{Spec}(C)$ is surjective, see Algebra, Lemma 10.36.17. Thus $|U| \to |M|$ is surjective.

Let k be an algebraically closed field and let $C \to k$ be a ring map. Since surjective morphisms are preserved under base change (Morphisms, Lemma 29.9.4) we see that $A \otimes_C k$ is not zero. Now $k \subset A \otimes_C k$ is a nonzero integral extension. Hence any residue field of $A \otimes_C k$ is an algebraic extension of k, hence equal to k. Thus we see that $U(k) \to M(k)$ is surjective.

Let $a_0, a_1 : A \to k$ be two ring maps. If there exists a ring map $b : B \to k$ such that $a_0 = b \circ t^\sharp$ and $a_1 = b \circ s^\sharp$ then we see that $a_0|_C = a_1|_C$ by definition. Thus the map $U(k) \to M(k)$ equalizes the two maps $R(k) \to U(k)$. Conversely, suppose that $a_0|_C = a_1|_C$. Let us name this algebra map $c : C \to k$. Consider the diagram

$$\begin{tikzcd}
 & B \arrow[dl, dashed] \\
k & A \arrow[l, shift right, "a_1"'] \arrow[l, shift left, "a_0"] \arrow[u, shift left] \arrow[u, shift right] \\
 & C \arrow[ul, "c"] \arrow[u]
\end{tikzcd}$$

If we can construct a dotted arrow making the diagram commute, then the proof of part (2) of the lemma is complete. Since $s : A \to B$ is finite there exist finitely many ring maps $b_1, \ldots, b_n : B \to k$ such that $b_i \circ s^\sharp = a_1$. If the dotted arrow does not exist, then we see that none of the $a_i' = b_i \circ t^\sharp$, $i = 1, \ldots, n$ is equal to a_0. Hence the maximal ideals
$$\mathfrak{m}_i' = \mathrm{Ker}(a_i' \otimes 1 : A \otimes_C k \to k)$$
of $A \otimes_C k$ are distinct from $\mathfrak{m} = \mathrm{Ker}(a_0 \otimes 1 : A \otimes_C k \to k)$. By Algebra, Lemma 10.15.2 we would get an element $f \in A \otimes_C k$ with $f \in \mathfrak{m}$, but $f \notin \mathfrak{m}_i'$ for $i = 1, \ldots, n$. Consider the norm
$$g = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A \otimes_C k$$
By Lemma 39.23.2 this lies in the invariants $C^1 \subset A \otimes_C k$ of the base change groupoid (base change via the map $c : C \to k$). On the one hand, $a_1(g) \in k^*$ since the value of $t^\sharp(f)$ at all the points (which correspond to $b_1, \ldots, b_n$) lying over a_1 is invertible (insert future reference on property determinant here). On the other hand, since $f \in \mathfrak{m}$, we see that f is not a unit, hence $t^\sharp(f)$ is not a unit (as $t^\sharp \otimes 1$ is faithfully flat), hence its norm is not a unit (insert future reference on property determinant here). We conclude that C^1 contains an element which is not nilpotent

and not a unit. We will now show that this leads to a contradiction. Namely, apply Lemma 39.23.5 to the map $c : C \to C' = k$, then we see that the map of k into the invariants C^1 is injective and moreover, that for any element $x \in C^1$ there exists an integer $n > 0$ such that $x^n \in k$. Hence every element of C^1 is either a unit or nilpotent.

We still have to finish the proof of (1). We already know that $|U| \to |M|$ is surjective. It is clear that $|U| \to |M|$ is $|R|$-invariant. Finally, suppose $u_0, u_1 \in U$ maps to the same point $m \in M$. Then the induced field extensions $\kappa(u_0)/\kappa(m)$ and $\kappa(u_1)/\kappa(m)$ are algebraic (as A is integral over C as used above). Hence if k is an algebraic closure of $\kappa(m)$ then we can find $\kappa(m)$-embeddings $\overline{u}_0 : \kappa(u_0) \to k$ and $\overline{u}_1 : \kappa(u_1) \to k$. These determine k-valued points $\overline{u}_0, \overline{u}_1 \in U(k)$ mapping to the same point of $M(k)$. By part (2) we see that there exists a point $\overline{r} \in R(k)$ with $s(\overline{r}) = \overline{u}_0$ and $t(\overline{r}) = \overline{u}_1$. The image $r \in R$ of $\overline{r}$ is a point with $s(r) = u_0$ and $t(r) = u_1$ as desired. □

0DT9 Lemma 39.23.7. Let S be a scheme. Let $f : (U', R', s', t') \to (U, R, s, t, c)$ be a morphism of groupoid schemes over S.

(1) U, R, U', R' are affine,
(2) s, t, s', t' are finite locally free,
(3) the diagrams

$$\begin{tikzcd} R' \arrow[r, "f"] \arrow[d, "s'"] & R \arrow[d, "s"] \\ U' \arrow[r, "f"] & U \end{tikzcd} \qquad \begin{tikzcd} R' \arrow[r, "f"] \arrow[d, "t'"] & R \arrow[d, "t"] \\ U' \arrow[r, "f"] & U \end{tikzcd} \qquad \begin{tikzcd} G' \arrow[r, "f"] \arrow[d] & G \arrow[d] \\ U' \arrow[r, "f"] & U \end{tikzcd}$$

are cartesian where G and G' are the stabilizer group schemes, and
(4) $f : U' \to U$ is étale.

Then the map $C \to C'$ from the R-invariant functions on U to the R'-invariant functions on U' is étale and $U' = \mathrm{Spec}(C') \times_{\mathrm{Spec}(C)} U$.

Proof. Set $M = \mathrm{Spec}(C)$ and $M' = \mathrm{Spec}(C')$. Write $U = \mathrm{Spec}(A)$, $U' = \mathrm{Spec}(A')$, $R = \mathrm{Spec}(B)$, and $R' = \mathrm{Spec}(B')$. We will use the results of Lemmas 39.23.4, 39.23.5, and 39.23.6 without further mention.

Assume C is a strictly henselian local ring. Let $p \in M$ be the closed point and let $p' \in M'$ map to p. Claim: in this case there is a disjoint union decomposition $(U', R', s', t', c') = (U, R, s, t, c) \amalg (U'', R'', s'', t'', c'')$ over (U, R, s, t, c) such that for the corresponding disjoint union decomposition $M' = M \amalg M''$ over M the point p' corresponds to $p \in M$.

The claim implies the lemma. Suppose that $M_1 \to M$ is a flat morphism of affine schemes. Then we can base change everything to M_1 without affecting the hypotheses (1) – (4). From Lemma 39.23.5 we see M_1, resp. M_1' is the spectrum of the R_1-invariant functions on U_1, resp. the R_1'-invariant functions on U_1'. Suppose that $p' \in M'$ maps to $p \in M$. Let M_1 be the spectrum of the strict henselization of $\mathcal{O}_{M,p}$ with closed point $p_1 \in M_1$. Choose a point $p_1' \in M_1'$ mapping to p_1 and p'. From the claim we get

$$(U_1', R_1', s_1', t_1', c_1') = (U_1, R_1, s_1, t_1, c_1) \amalg (U_1'', R_1'', s_1'', t_1'', c_1'')$$

and correspondingly $M'_1 = M_1 \amalg M''_1$ as a scheme over M_1. Write $M_1 = \mathrm{Spec}(C_1)$ and write $C_1 = \mathrm{colim}\, C_i$ as a filtered colimit of étale C-algebras. Set $M_i = \mathrm{Spec}(C_i)$. The $M_1 = \lim M_i$ and similarly for the other schemes. By Limits, Lemmas 32.4.11 and 32.8.11 we can find an i such that

$$(U'_i, R'_i, s'_i, t'_i, c'_i) = (U_i, R_i, s_i, t_i, c_i) \amalg (U''_i, R''_i, s''_i, t''_i, c''_i)$$

We conclude that $M'_i = M_i \amalg M''_i$. In particular $M' \to M$ becomes étale at a point over p' after an étale base change. This implies that $M' \to M$ is étale at p' (for example by Morphisms, Lemma 29.36.17). We will prove $U' \cong M' \times_M U$ after we prove the claim.

Proof of the claim. Observe that U_p and $U'_{p'}$ have finitely many points. For $u \in U_p$ we have $\kappa(u)/\kappa(p)$ is algebraic, hence $\kappa(u)$ is separably closed. As $U' \to U$ is étale, we conclude the morphism $U'_{p'} \to U_p$ induces isomorphisms on residue field extensions. Let $u' \in U'_{p'}$ with image $u \in U_p$. By assumption (3) the morphism of scheme theoretic fibres $(s')^{-1}(u') \to s^{-1}(u)$, $(t')^{-1}(u') \to t^{-1}(u)$, and $G'_{u'} \to G_u$ are isomorphisms. Observing that $U_p = t(s^{-1}(u))$ (set theoretically) we conclude that the points of $U'_{p'}$ surject onto the points of U_p. Suppose that u'_1 and u'_2 are points of $U'_{p'}$ mapping to the same point u of U_p. Then there exists a point $r' \in R'_{p'}$ with $s'(r') = u'_1$ and $t'(r') = u'_2$. Consider the two towers of fields

$$\kappa(r')/\kappa(u'_1)/\kappa(u)/\kappa(p) \quad \kappa(r')/\kappa(u'_2)/\kappa(u)/\kappa(p)$$

whose "ends" are the same as the two "ends" of the two towers

$$\kappa(r')/\kappa(u'_1)/\kappa(p')/\kappa(p) \quad \kappa(r')/\kappa(u'_2)/\kappa(p')/\kappa(p)$$

These two induce the same maps $\kappa(p') \to \kappa(r')$ as $(U'_{p'}, R'_{p'}, s', t', c')$ is a groupoid over p'. Since $\kappa(u)/\kappa(p)$ is purely inseparable, we conclude that the two induced maps $\kappa(u) \to \kappa(r')$ are the same. Therefore r' maps to a point of the fibre G_u. By assumption (3) we conclude that $r' \in (G')_{u'_1}$. Namely, we may think of G as a closed subscheme of R viewed as a scheme over U via s and use that the base change to U' gives $G' \subset R'$. In particular we have $u'_1 = u'_2$. We conclude that $U'_{p'} \to U_p$ is a bijective map on points inducing isomorphisms on residue fields. It follows that $U'_{p'}$ is a finite set of closed points (Algebra, Lemma 10.35.9) and hence $U'_{p'}$ is closed in U'. Let $J' \subset A'$ be the radical ideal cutting out $U'_{p'}$ set theoretically.

Second part proof of the claim. Let $\mathfrak{m} \subset C$ be the maximal ideal. Observe that $(A, \mathfrak{m}A)$ is a henselian pair by More on Algebra, Lemma 15.11.8. Let $J = \sqrt{\mathfrak{m}A}$. Then (A, J) is a henselian pair (More on Algebra, Lemma 15.11.7) and the étale ring map $A \to A'$ induces an isomorphism $A/J \to A'/J'$ by our deliberations above. We conclude that $A' = A \times A''$ by More on Algebra, Lemma 15.11.6. Consider the corresponding disjoint union decomposition $U' = U \amalg U''$. The open $(s')^{-1}(U)$ is the set of points of R' specializing to a point of $R'_{p'}$. Similarly for $(t')^{-1}(U)$. Similarly we have $(s')^{-1}(U'') = (t')^{-1}(U'')$ as this is the set of points which do not specialize to $R'_{p'}$. Hence we obtain a disjoint union decomposition

$$(U', R', s', t', c') = (U, R, s, t, c) \amalg (U'', R'', s'', t'', c'')$$

This immediately gives $M' = M \amalg M''$ and the proof of the claim is complete.

We still have to prove that the canonical map $U' \to M' \times_M U$ is an isomorphism. It is an étale morphism (Morphisms, Lemma 29.36.18). On the other hand, by base changing to strictly henselian local rings (as in the third paragraph of the proof) and using the bijectivity $U'_{p'} \to U_p$ established in the course of the proof of the claim, we see that $U' \to M' \times_M U$ is universally bijective (some details omitted). However, a universally bijective étale morphism is an isomorphism (Descent, Lemma 35.25.2) and the proof is complete. $\square$

03C8 **Lemma 39.23.8.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume

(1) $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and
(2) there exist elements $x_i \in A$, $i \in I$ such that $B = \bigoplus_{i \in I} s^\sharp(A)t^\sharp(x_i)$.

Then $A = \bigoplus_{i \in I} Cx_i$, and $B \cong A \otimes_C A$ where $C \subset A$ is the R-invariant functions on U as in (39.23.0.1).

Proof. During this proof we will write $s, t : A \to B$ instead of $s^\sharp, t^\sharp$, and similarly $c : B \to B \otimes_{s,A,t} B$. We write $p_0 : B \to B \otimes_{s,A,t} B$, $b \mapsto b \otimes 1$ and $p_1 : B \to B \otimes_{s,A,t} B$, $b \mapsto 1 \otimes b$. By Lemma 39.13.5 and the definition of C we have the following commutative diagram

$$\begin{tikzcd} B \otimes_{s,A,t} B & B \arrow[l, shift right, "c"'] \arrow[l, shift left, "p_0"] & A \arrow[l, "t"] \\ B \arrow[u, "p_1"] & A \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[u, "s"'] & C \arrow[l] \arrow[u] \end{tikzcd}$$

Moreover the tow left squares are cocartesian in the category of rings, and the top row is isomorphic to the diagram

$$\begin{tikzcd} B \otimes_{t,A,t} B & B \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_0"] & A \arrow[l, "t"] \end{tikzcd}$$

which is an equalizer diagram according to Descent, Lemma 35.3.6 because condition (2) implies in particular that s (and hence also then isomorphic arrow t) is faithfully flat. The lower row is an equalizer diagram by definition of C. We can use the x_i and get a commutative diagram

$$\begin{tikzcd} B \otimes_{s,A,t} B & B \arrow[l, shift right, "c"'] \arrow[l, shift left, "p_0"] & & A \arrow[ll, "t"] \\ \bigoplus_{i \in I} Bx_i \arrow[u, "p_1"] & \bigoplus_{i \in I} Ax_i \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[u, "s"'] & & \bigoplus_{i \in I} Cx_i \arrow[ll] \arrow[u] \end{tikzcd}$$

where in the right vertical arrow we map x_i to x_i, in the middle vertical arrow we map x_i to $t(x_i)$ and in the left vertical arrow we map x_i to $c(t(x_i)) = t(x_i) \otimes 1 = p_0(t(x_i))$ (equality by the commutativity of the top part of the diagram in Lemma 39.13.4). Then the diagram commutes. Moreover the middle vertical arrow is an isomorphism by assumption. Since the left two squares are cocartesian we conclude that also the left vertical arrow is an isomorphism. On the other hand, the horizontal rows are exact (i.e., they are equalizers). Hence we conclude that also the right vertical arrow is an isomorphism. $\square$

03BM **Proposition 39.23.9.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume

(1) $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine,

(2) $s, t : R \to U$ finite locally free, and
(3) $j = (t, s)$ is an equivalence.

In this case, let $C \subset A$ be as in (39.23.0.1). Then $U \to M = \mathrm{Spec}(C)$ is finite locally free and $R = U \times_M U$. Moreover, M represents the quotient sheaf U/R in the fppf topology (see Definition 39.20.1).

Proof. During this proof we use the notation $s, t : A \to B$ instead of the notation $s^\sharp, t^\sharp$. By Lemma 39.20.3 it suffices to show that $C \to A$ is finite locally free and that the map

$$t \otimes s : A \otimes_C A \longrightarrow B$$

is an isomorphism. First, note that j is a monomorphism, and also finite (since already s and t are finite). Hence we see that j is a closed immersion by Morphisms, Lemma 29.44.15. Hence $A \otimes_C A \to B$ is surjective.

We will perform base change by flat ring maps $C \to C'$ as in Lemma 39.23.5, and we will use that formation of invariants commutes with flat base change, see part (3) of the lemma cited. We will show below that for every prime $\mathfrak{p} \subset C$, there exists a local flat ring map $C_\mathfrak{p} \to C'_\mathfrak{p}$ such that the result holds after a base change to $C'_\mathfrak{p}$. This implies immediately that $A \otimes_C A \to B$ is injective (use Algebra, Lemma 10.23.1). It also implies that $C \to A$ is flat, by combining Algebra, Lemmas 10.39.17, 10.39.18, and 10.39.8. Then since $U \to \mathrm{Spec}(C)$ is surjective also (Lemma 39.23.6) we conclude that $C \to A$ is faithfully flat. Then the isomorphism $B \cong A \otimes_C A$ implies that A is a finitely presented C-module, see Algebra, Lemma 10.83.2. Hence A is finite locally free over C, see Algebra, Lemma 10.78.2.

By Lemma 39.23.3 we know that A is a finite product of rings A_r and B is a finite product of rings B_r such that the groupoid scheme decomposes accordingly (see the proof of Lemma 39.23.4). Then also C is a product of rings C_r and correspondingly C' decomposes as a product. Hence we may and do assume that the ring maps $s, t : A \to B$ are finite locally free of a fixed rank r.

The local ring maps $C_\mathfrak{p} \to C'_\mathfrak{p}$ we are going to use are any local flat ring maps such that the residue field of $C'_\mathfrak{p}$ is infinite. By Algebra, Lemma 10.159.1 such local ring maps exist.

Assume C is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field, and assume that $s, t : A \to B$ is finite locally free of constant rank $r > 0$. Since $C \subset A$ is integral (Lemma 39.23.4) all primes lying over $\mathfrak{m}$ are maximal, and all maximal ideals of A lie over $\mathfrak{m}$. Similarly for $C \subset B$. Pick a maximal ideal $\mathfrak{m}'$ of A lying over $\mathfrak{m}$ (exists by Lemma 39.23.6). Since $t : A \to B$ is finite locally free there exist at most finitely many maximal ideals of B lying over $\mathfrak{m}'$. Hence we conclude (by Lemma 39.23.6 again) that A has finitely many maximal ideals, i.e., A is semi-local. This in turn implies that B is semi-local as well. OK, and now, because $t \otimes s : A \otimes_C A \to B$ is surjective, we can apply Algebra, Lemma 10.78.8 to the ring map $C \to A$, the A-module $M = B$ (seen as an A-module via t) and the C-submodule $s(A) \subset B$. This lemma implies that there exist $x_1, \ldots, x_r \in A$ such that M is free over A on the basis $s(x_1), \ldots, s(x_r)$. Hence we conclude that $C \to A$ is finite free and $B \cong A \otimes_C A$ by applying Lemma 39.23.8. □

39.24. Finite flat groupoids

03JD In this section we prove a lemma that will help to show that the quotient of a scheme by a finite flat equivalence relation is a scheme, provided that each equivalence class is contained in an affine. See Properties of Spaces, Proposition 66.14.1.

03JE Lemma 39.24.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are finite locally free. Let $u \in U$ be a point such that $t(s^{-1}(\{u\}))$ is contained in an affine open of U. Then there exists an R-invariant affine open neighbourhood of u in U.

Proof. Since s is finite locally free it has finite fibres. Hence $t(s^{-1}(\{u\})) = \{u_1, \ldots, u_n\}$ is a finite set. Note that $u \in \{u_1, \ldots, u_n\}$. Let $W \subset U$ be an affine open containing $\{u_1, \ldots, u_n\}$, in particular $u \in W$. Consider $Z = R \setminus s^{-1}(W) \cap t^{-1}(W)$. This is a closed subset of R. The image $t(Z)$ is a closed subset of U which can be loosely described as the set of points of U which are R-equivalent to a point of $U \setminus W$. Hence $W' = U \setminus t(Z)$ is an R-invariant, open subscheme of U contained in W, and $\{u_1, \ldots, u_n\} \subset W'$. Picture

$$\{u_1, \ldots, u_n\} \subset W' \subset W \subset U.$$

Let $f \in \Gamma(W, \mathcal{O}_W)$ be an element such that $\{u_1, \ldots, u_n\} \subset D(f) \subset W'$. Such an f exists by Algebra, Lemma 10.15.2. By our choice of W' we have $s^{-1}(W') \subset t^{-1}(W)$, and hence we get a diagram

$$\begin{tikzcd} s^{-1}(W') \arrow[r, "t"'] \arrow[d, "s"'] & W \\ W' & \end{tikzcd}$$

The vertical arrow is finite locally free by assumption. Set

$$g = \mathrm{Norm}_s(t^\sharp f) \in \Gamma(W', \mathcal{O}_{W'})$$

By construction g is a function on W' which is nonzero in u, as $t^\sharp(f)$ is nonzero in each of the points of R lying over u, since f is nonzero in $u_1, \ldots, u_n$. Similarly, $D(g) \subset W'$ is equal to the set of points w such that f is not zero in any of the points equivalent to w. This means that $D(g)$ is an R-invariant affine open of W'. The final picture is

$$\{u_1, \ldots, u_n\} \subset D(g) \subset D(f) \subset W' \subset W \subset U$$

and hence we win. $\square$

39.25. Descending quasi-projective schemes

0CCH We can use Lemma 39.24.1 to show that a certain type of descent datum is effective.

0CCI Lemma 39.25.1. Let $X \to Y$ be a surjective finite locally free morphism. Let V be a scheme over X such that for all $(y, v_1, \ldots, v_d)$ where $y \in Y$ and $v_1, \ldots, v_d \in V_y$ there exists an affine open $U \subset V$ with $v_1, \ldots, v_d \in U$. Then any descent datum on $V/X/Y$ is effective.

Proof. Let φ be a descent datum as in Descent, Definition 35.34.1. Recall that the functor from schemes over Y to descent data relative to $\{X \to Y\}$ is fully faithful, see Descent, Lemma 35.35.11. Thus using Constructions, Lemma 27.2.1 it suffices to prove the lemma in the case that Y is affine. Some details omitted (this argument can be avoided if Y is separated or has affine diagonal, because then every morphism from an affine scheme to X is affine).

Assume Y is affine. If V is also affine, then we have effectivity by Descent, Lemma 35.37.1. Hence by Descent, Lemma 35.35.13 it suffices to prove that every point v of V has a φ-invariant affine open neighbourhood. Consider the groupoid $(X, X \times_Y X, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$. By Lemma 39.21.3 the descent datum φ determines and is determined by a cartesian morphism of groupoid schemes

$$(V, R, s, t, c) \longrightarrow (X, X \times_Y X, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

over $\mathrm{Spec}(\mathbf{Z})$. Since $X \to Y$ is finite locally free, we see that $\mathrm{pr}_i : X \times_Y X \to X$ and hence s and t are finite locally free. In particular the R-orbit $t(s^{-1}(\{v\}))$ of our point $v \in V$ is finite. Using the equivalence of categories of Lemma 39.21.3 once more we see that φ-invariant opens of V are the same thing as R-invariant opens of V. Our assumption shows there exists an affine open of V containing the orbit $t(s^{-1}(\{v\}))$ as all the points in this orbit map to the same point of Y. Thus Lemma 39.24.1 provides an R-invariant affine open containing v. □

0CCJ Lemma 39.25.2. Let $X \to Y$ be a surjective finite locally free morphism. Let V be a scheme over X such that one of the following holds

(1) $V \to X$ is projective,
(2) $V \to X$ is quasi-projective,
(3) there exists an ample invertible sheaf on V,
(4) there exists an X-ample invertible sheaf on V,
(5) there exists an X-very ample invertible sheaf on V.

Then any descent datum on $V/X/Y$ is effective.

Proof. We check the condition in Lemma 39.25.1. Let $y \in Y$ and $v_1, \ldots, v_d \in V$ points over y. Case (1) is a special case of (2), see Morphisms, Lemma 29.43.10. Case (2) is a special case of (4), see Morphisms, Definition 29.40.1. If there exists an ample invertible sheaf on V, then there exists an affine open containing $v_1, \ldots, v_d$ by Properties, Lemma 28.29.5. Thus (3) is true. In cases (4) and (5) it is harmless to replace Y by an affine open neighbourhood of y. Then X is affine too. In case (4) we see that V has an ample invertible sheaf by Morphisms, Definition 29.37.1 and the result follows from case (3). In case (5) we can replace V by a quasi-compact open containing $v_1, \ldots, v_d$ and we reduce to case (4) by Morphisms, Lemma 29.38.2. □

39.26. Other chapters

Preliminaries

CHAPTER 40

More on Groupoid Schemes

04LA 40.1. Introduction

04LB This chapter is devoted to advanced topics on groupoid schemes. Even though the results are stated in terms of groupoid schemes, the reader should keep in mind the 2-cartesian diagram

04LC (40.1.0.1)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & {[U/R]} \end{tikzcd}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 78.20.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [KM97] by Keel and Mori.

40.2. Notation

04LD We continue to abide by the conventions and notation introduced in Groupoids, Section 39.2.

40.3. Useful diagrams

04LE We briefly restate the results of Groupoids, Lemmas 39.13.4 and 39.13.5 for easy reference in this chapter. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. In the commutative diagram

04LF (40.3.0.1)
$$\begin{tikzcd} & U & \\ R \arrow[ur, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "\mathrm{pr}_0"] \arrow[r, "c"'] \arrow[d, "\mathrm{pr}_1"] & R \arrow[ul, "t"'] \arrow[d, "s"] \\ U & R \arrow[l, "t"'] \arrow[r, "s"] & U \end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

04LG (40.3.0.2)

$$\begin{tikzcd}
R \times_{t,U,t} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_0 \times c \circ (i,1)"'] & R \arrow[r, "t"] \arrow[d, "\mathrm{id}_R"] & U \arrow[d, "\mathrm{id}_U"] \\
R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

40.4. Sheaf of differentials

04R8 The following lemma is the analogue of Groupoids, Lemma 39.6.3.

04R9 **Lemma 40.4.1.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The sheaf of differentials of R seen as a scheme over U via t is a quotient of the pullback via t of the conormal sheaf of the immersion $e : U \to R$. In a formula: there is a canonical surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$. If s is flat, then this map is an isomorphism.

Proof. Note that $e : U \to R$ is an immersion as it is a section of the morphism s, see Schemes, Lemma 26.21.11. Consider the following diagram

$$\begin{tikzcd}
R \arrow[r, "(1,i)"'] \arrow[d, "t"'] & R \times_{s,U,t} R \arrow[r, "(\mathrm{pr}_0, i \circ \mathrm{pr}_1)"'] \arrow[d, "c"] & R \times_{t,U,t} R \\
U \arrow[r, "e"] & R &
\end{tikzcd}$$

The square on the left is cartesian, because if $a \circ b = e$, then $b = i(a)$. The composition of the horizontal maps is the diagonal morphism of $t : R \to U$. The right top horizontal arrow is an isomorphism. Hence since $\Omega_{R/U}$ is the conormal sheaf of the composition it is isomorphic to the conormal sheaf of $(1, i)$. By Morphisms, Lemma 29.31.4 we get the surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$ and if c is flat, then this is an isomorphism. Since c is a base change of s by the properties of Diagram (40.3.0.2) we conclude that if s is flat, then c is flat, see Morphisms, Lemma 29.25.8. $\square$

40.5. Local structure

0CK3 Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $u \in U$ be a point. In this section we explain what kind of structure we obtain on the local rings

$$A = \mathcal{O}_{U,u} \quad \text{and} \quad B = \mathcal{O}_{R,e(u)}$$

The convention we will use is to denote the local ring homomorphisms induced by the morphisms s, t, c, e, i by the corresponding letters. In particular we have a

commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "t"'] \arrow[drr, "1"] & & \\
& B \arrow[r, "e"] & A \\
A \arrow[ur, "s"] \arrow[urr, "1"'] & &
\end{tikzcd}$$

of local rings. Thus if $I \subset B$ denotes the kernel of $e : B \to A$, then $B = s(A) \oplus I = t(A) \oplus I$. Let us denote

$$C = \mathcal{O}_{R \times_{s,U,t} R, (e(u), e(u))}$$

Then we have

$$C = (B \otimes_{s,A,t} B)_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$$

Let $J \subset C$ be the ideal of C generated by $I \otimes B + B \otimes I$. Then J is also the kernel of the local ring homomorphism

$$(e, e) : C \longrightarrow A$$

The composition law $c : R \times_{s,U,t} R \to R$ corresponds to a ring map

$$c : B \longrightarrow C$$

sending I into J.

0CK4 Lemma 40.5.1. The map $I/I^2 \to J/J^2$ induced by c is the composition

$$I/I^2 \xrightarrow{(1,1)} I/I^2 \oplus I/I^2 \to J/J^2$$

where the second arrow comes from the equality $J = (I \otimes B + B \otimes I)C$. The map $i : B \to B$ induces the map $-1 : I/I^2 \to I/I^2$.

Proof. To describe a local homomorphism from C to another local ring it is enough to say what happens to elements of the form $b_1 \otimes b_2$. Keeping this in mind we have the two canonical maps

$$e_2 : C \to B,\ b_1 \otimes b_2 \mapsto b_1 s(e(b_2)), \quad e_1 : C \to B,\ b_1 \otimes b_2 \mapsto t(e(b_1)) b_2$$

corresponding to the embeddings $R \to R \times_{s,U,t} R$ given by $r \mapsto (r, e(s(r)))$ and $r \mapsto (e(t(r)), r)$. These maps define maps $J/J^2 \to I/I^2$ which jointly give an inverse to the map $I/I^2 \oplus I/I^2 \to J/J^2$ of the lemma. Thus to prove statement we only have to show that $e_1 \circ c : B \to B$ and $e_2 \circ c : B \to B$ are the identity maps. This follows from the fact that both compositions $R \to R \times_{s,U,t} R \to R$ are identities.

The statement on i follows from the statement on c and the fact that $c \circ (1, i) = e \circ t$. Some details omitted. $\square$

40.6. Properties of groupoids

02YD Let (U, R, s, t, c) be a groupoid scheme. The idea behind the results in this section is that $s : R \to U$ is a base change of the morphism $U \to [U/R]$ (see Diagram (40.1.0.1). Hence the local properties of $s : R \to U$ should reflect local properties of the morphism $U \to [U/R]$. This doesn't work, because $[U/R]$ is not always an algebraic stack, and hence we cannot speak of geometric or algebraic properties of $U \to [U/R]$. But it turns out that we can make some of it work without even referring to the quotient stack at all.

Here is a first example of such a result. The open $W \subset U'$ found in the lemma is roughly speaking the locus where the morphism $U' \to [U/R]$ has property $\mathcal{P}$.

04LH Lemma 40.6.1. Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. Let $g : U' \to U$ be a morphism of schemes. Denote h the composition

$$h : U' \times_{g,U,t} R \xrightarrow[\mathrm{pr}_1]{} R \xrightarrow[s]{} U.$$

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of schemes. Assume

(1) $\mathcal{R} \Rightarrow \mathcal{Q}$,
(2) $\mathcal{Q}$ is preserved under base change and composition,
(3) for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and
(4) for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.

If s, t have $\mathcal{R}$ and g has $\mathcal{Q}$, then there exists an open subscheme $W \subset U'$ such that $W \times_{g,U,t} R = W(\mathcal{P}, h)$.

Proof. Note that the following diagram is commutative

$$\begin{tikzcd} U' \times_{g,U,t} R \times_{t,U,t} R \arrow[rr, "\mathrm{pr}_{12}"'] \arrow[d, shift right, "\mathrm{pr}_{01}"'] \arrow[d, shift left, "\mathrm{pr}_{02}"] & & R \times_{t,U,t} R \arrow[d, shift right, "\mathrm{pr}_0"'] \arrow[d, shift left, "\mathrm{pr}_1"] \\ U' \times_{g,U,t} R \arrow[rr, "\mathrm{pr}_1"] & & R \end{tikzcd}$$

with both squares cartesian (this uses that the two maps $t \circ \mathrm{pr}_i : R \times_{t,U,t} R \to U$ are equal). Combining this with the properties of diagram (40.3.0.2) we get a commutative diagram

$$\begin{tikzcd} U' \times_{g,U,t} R \times_{t,U,t} R \arrow[rr, "c\circ(i,1)"'] \arrow[d, shift right, "\mathrm{pr}_{01}"'] \arrow[d, shift left, "\mathrm{pr}_{02}"] & & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ U' \times_{g,U,t} R \arrow[rr, "h"] & & U \end{tikzcd}$$

where both squares are cartesian.

Assume s, t have $\mathcal{R}$ and g has $\mathcal{Q}$. Then h has $\mathcal{Q}$ as a composition of s (which has $\mathcal{R}$ hence $\mathcal{Q}$) and a base change of g (which has $\mathcal{Q}$). Thus $W(\mathcal{P}, h) \subset U' \times_{g,U,t} R$ exists. By our assumptions we have $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ since both are the largest open on which $c \circ (i, 1)$ has $\mathcal{P}$. Note that the projection $U' \times_{g,U,t} R \to U'$ has a section, namely $\sigma : U' \to U' \times_{g,U,t} R$, $u' \mapsto (u', e(g(u')))$. Also via the isomorphism

$$(U' \times_{g,U,t} R) \times_{U'} (U' \times_{g,U,t} R) = U' \times_{g,U,t} R \times_{t,U,t} R$$

the two projections of the left hand side to $U' \times_{g,U,t} R$ agree with the morphisms pr_{01} and pr_{02} on the right hand side. Since $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ we conclude that $W(\mathcal{P}, h)$ is the inverse image of a subset of U, which is necessarily the open set $W = \sigma^{-1}(W(\mathcal{P}, h))$. □

04LI Remark 40.6.2. Warning: Lemma 40.6.1 should be used with care. For example, it applies to $\mathcal{P}$ =“flat”, $\mathcal{Q}$ =“empty”, and $\mathcal{R}$ =“flat and locally of finite presentation”. But given a morphism of schemes $f : X \to Y$ the largest open $W \subset X$ such that $f|_W$ is flat is not the set of points where f is flat!

047W Remark 40.6.3. Notwithstanding the warning in Remark 40.6.2 there are some cases where Lemma 40.6.1 can be used without causing too much ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q} = \mathcal{R}$ =“locally of finite type”, and $\mathcal{P}$ =“relative dimension $\leq d$”. See Morphisms, Definition 29.29.1 and Morphisms, Lemmas 29.28.4 and 29.28.3.
(2) $\mathcal{Q} = \mathcal{R}$ =“locally of finite type”, and $\mathcal{P}$ =“locally quasi-finite”. This is the case $d = 0$ of the previous item, see Morphisms, Lemma 29.29.5.
(3) $\mathcal{Q} = \mathcal{R}$ =“locally of finite type”, and $\mathcal{P}$ =“unramified”. See Morphisms, Lemmas 29.35.3 and 29.35.15.

What is interesting about the cases listed above is that we do not need to assume that s, t are flat to get a conclusion about the locus where the morphism h has property $\mathcal{P}$. We continue the list:

(4) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“flat”. See More on Morphisms, Theorem 37.15.1 and Lemma 37.15.2.
(5) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“Cohen-Macaulay”. See More on Morphisms, Definition 37.22.1 and More on Morphisms, Lemmas 37.22.6 and 37.22.7.
(6) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“syntomic” use Morphisms, Lemma 29.30.12 (the locus is automatically open).
(7) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“smooth”. See Morphisms, Lemma 29.34.15 (the locus is automatically open).
(8) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“étale”. See Morphisms, Lemma 29.36.17 (the locus is automatically open).

Here is the second result. The R-invariant open $W \subset U$ should be thought of as the inverse image of the largest open of $[U/R]$ over which the morphism $U \to [U/R]$ has property $\mathcal{P}$.

03JC Lemma 40.6.4. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$[1]. Let $\mathcal{P}$ be a property of morphisms of schemes which is τ-local on the target (Descent, Definition 35.22.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $s|_{s^{-1}(W)} : s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant, see Groupoids, Definition 39.19.1.

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 35.22.3. In Diagram (40.3.0.1) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\text{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from

[1]The fact that $fpqc$ is missing is not a typo.

the aforementioned Descent, Lemma 35.22.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. $\square$

06QQ Lemma 40.6.5. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be its stabilizer group scheme. Let $\tau \in \{fppf, \acute{e}tale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids, Definition 39.19.1).

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 35.22.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent, Lemma 35.22.3. $\square$

40.7. Comparing fibres

04LJ Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Diagram (40.3.0.1) gives us a way to compare the fibres of the map $s : R \to U$ in a groupoid. For a point $u \in U$ we will denote $F_u = s^{-1}(u)$ the scheme theoretic fibre of $s : R \to U$ over u. For example the diagram implies that if $u, u' \in U$ are points such that $s(r) = u$ and $t(r) = u'$, then $(F_u)_{\kappa(r)} \cong (F_{u'})_{\kappa(r)}$. This is a special case of the more general and more precise Lemma 40.7.1 below. To see this take $r' = i(r)$.

A pair (X, x) consisting of a scheme X and a point $x \in X$ is sometimes called the germ of X at x. A morphism of germs $f : (X, x) \to (S, s)$ is a morphism $f : U \to S$ defined on an open neighbourhood of x with $f(x) = s$. Two such f, f' are said to give the same morphism of germs if and only if f and f' agree in some open neighbourhood of x. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. We temporarily introduce the following concept: We say that two morphisms of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are isomorphic locally on the base in the τ-topology, if there exists a pointed scheme (S'', s'') and morphisms of germs $g : (S'', s'') \to (S, s)$, and $g' : (S'', s'') \to (S', s')$ such that

(1) g and g' are an open immersion (resp. étale, smooth, syntomic, flat and locally of finite presentation) at s'',
(2) there exists an isomorphism
$$(S'' \times_{g,S,f} X, \tilde{x}) \cong (S'' \times_{g',S',f'} X', \tilde{x}')$$
of germs over the germ (S'', s'') for some choice of points $\tilde{x}$ and $\tilde{x}'$ lying over (s'', x) and (s'', x').

Finally, we simply say that the maps of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are flat locally on the base isomorphic if there exist S'', s'', g, g' as above but with (1) replaced by the condition that g and g' are flat at s'' (this is much weaker than any of the τ conditions above as a flat morphism need not be open).

02YF Lemma 40.7.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $r, r' \in R$ with $t(r) = t(r')$ in U. Set $u = s(r)$, $u' = s(r')$. Denote $F_u = s^{-1}(u)$ and $F_{u'} = s^{-1}(u')$ the scheme theoretic fibres.

(1) There exists a common field extension $\kappa(u) \subset k$, $\kappa(u') \subset k$ and an isomorphism $(F_u)_k \cong (F_{u'})_k$.
(2) We may choose the isomorphism of (1) such that a point lying over r maps to a point lying over r'.
(3) If the morphisms s, t are flat then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are flat locally on the base isomorphic.
(4) If the morphisms s, t are étale (resp. smooth, syntomic, or flat and locally of finite presentation) then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally on the base isomorphic in the étale (resp. smooth, syntomic, or fppf) topology.

Proof. We repeatedly use the properties and the existence of diagram (40.3.0.1). By the properties of the diagram (and Schemes, Lemma 26.17.5) there exists a point ξ of $R \times_{s,U,t} R$ with $\text{pr}_0(\xi) = r$ and $c(\xi) = r'$. Let $\tilde{r} = \text{pr}_1(\xi) \in R$.

Proof of (1). Set $k = \kappa(\tilde{r})$. Since $t(\tilde{r}) = u$ and $s(\tilde{r}) = u'$ we see that k is a common extension of both $\kappa(u)$ and $\kappa(u')$ and in fact that both $(F_u)_k$ and $(F_{u'})_k$ are isomorphic to the fibre of $\text{pr}_1 : R \times_{s,U,t} R \to R$ over $\tilde{r}$. Hence (1) is proved.

Part (2) follows since the point ξ maps to r, resp. r'.

Part (3) is clear from the above (using the point ξ for $\tilde{u}$ and $\tilde{u}'$) and the definitions.

If s and t are flat and of finite presentation, then they are open morphisms (Morphisms, Lemma 29.25.10). Hence the image of some affine open neighbourhood V'' of $\tilde{r}$ will cover an open neighbourhood V of u, resp. V' of u'. These can be used to show that properties (1) and (2) of the definition of "locally on the base isomorphic in the τ-topology". □

40.8. Cohen-Macaulay presentations

04LK Given any groupoid (U, R, s, t, c) with s, t flat and locally of finite presentation there exists an "equivalent" groupoid (U', R', s', t', c') such that s' and t' are Cohen-Macaulay morphisms (and locally of finite presentation). See More on Morphisms, Section 37.22 for more information on Cohen-Macaulay morphisms. Here "equivalent" can be taken to mean that the quotient stacks $[U/R]$ and $[U'/R']$ are equivalent stacks, see Groupoids in Spaces, Section 78.20 and Section 78.25.

0460 Lemma 40.8.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Assume s and t are flat and locally of finite presentation. Then there exists an open $U' \subset U$ such that

(1) $t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay,
(2) $s^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism t is Cohen-Macaulay,
(3) the morphism $t|_{s^{-1}(U')} : s^{-1}(U') \to U$ is surjective,
(4) the morphism $s|_{t^{-1}(U')} : t^{-1}(U') \to U$ is surjective, and
(5) the restriction $R' = s^{-1}(U') \cap t^{-1}(U')$ of R to U' defines a groupoid (U', R', s', t', c') which has the property that the morphisms s' and t' are Cohen-Macaulay and locally of finite presentation.

Proof. Apply Lemma 40.6.1 with $g = \mathrm{id}$ and $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="Cohen-Macaulay", see Remark 40.6.3. This gives us an open $U' \subset U$ such that Let $t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay. This proves (1). Let $i : R \to R$ be the inverse of the groupoid. Since i is an isomorphism, and $s \circ i = t$ and $t \circ i = s$ we see that $s^{-1}(U')$ is also the largest open of R on which t is Cohen-Macaulay. This proves (2). By More on Morphisms, Lemma 37.22.7 the open subset $t^{-1}(U')$ is dense in every fibre of $s : R \to U$. This proves (3). Same argument for (4). Part (5) is a formal consequence of (1) and (2) and the discussion of restrictions in Groupoids, Section 39.18. $\square$

40.9. Restricting groupoids

04MM In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

04MN (40.9.0.1)
$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

of a restriction. See Groupoids, Lemma 39.18.1.

04MP **Lemma 40.9.1.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.

(1) If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.
(2) If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.
(3) If s, t are flat and g is flat, then s', t' are flat.
(4) Add more here.

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms, Lemmas 29.15.3 and 29.15.4. Hence (1) is clear from Diagram (40.9.0.1). For the other cases, see Morphisms, Lemmas 29.21.3, 29.21.4, 29.25.6, and 29.25.8. $\square$

The following lemma could have been used to prove the results of the preceding lemma in a more uniform way.

04MV **Lemma 40.9.2.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g, and let $h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \to U$. If $\mathcal{P}$ is a property of morphisms of schemes such that

(1) h has property $\mathcal{P}$, and

(2) $\mathcal{P}$ is preserved under base change,

then s', t' have property $\mathcal{P}$.

Proof. This is clear as s' is the base change of h by Diagram (40.9.0.1) and t' is isomorphic to s' as a morphism of schemes. $\square$

04MW Lemma 40.9.3. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ \text{pr}_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ \text{pr}_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ \text{pr}_1 : U'' \times_{g'',U,t} R \to U$. The following diagram is commutative

$$\begin{tikzcd} U'' \times_{g',U',t} R' \arrow[d, "h'"] & (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) \arrow[l] \arrow[r] \arrow[d] & U'' \times_{g'',U,t} R \arrow[d, "h''"'] \\ U' & U' \times_{g,U,t} R \arrow[l, "\text{pr}_0"'] \arrow[r, "h"] & U \end{tikzcd}$$

with both squares cartesian where the left upper horizontal arrow is given by the rule

$$\begin{matrix} (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g',U',t} R' \\ ((u', r_0), (u'', r_1)) & \longmapsto & (u'', (c(r_1, i(r_0)), (g'(u''), u'))) \end{matrix}$$

with notation as explained in the proof.

Proof. We work this out by exploiting the functorial point of view and reducing the lemma to a statement on arrows in restrictions of a groupoid category. In the last formula of the lemma the notation $((u', r_0), (u'', r_1))$ indicates a T-valued point of $(U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R)$. This means that u', u'', r_0, r_1 are T-valued points of U', U'', R, R and that $g(u') = t(r_0)$, $g(g'(u'')) = g''(u'') = t(r_1)$, and $s(r_0) = s(r_1)$. It would be more correct here to write $g \circ u' = t \circ r_0$ and so on but this makes the notation even more unreadable. If we think of r_1 and r_0 as arrows in a groupoid category then we can represent this by the picture

$$t(r_0) = g(u') \xleftarrow{r_0} s(r_0) = s(r_1) \xrightarrow{r_1} t(r_1) = g(g'(u''))$$

This diagram in particular demonstrates that the composition $c(r_1, i(r_0))$ makes sense. Recall that

$$R' = R \times_{(t,s),U \times_S U, g \times g} U' \times_S U'$$

hence a T-valued point of R' looks like $(r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. In particular given $((u', r_0), (u'', r_1))$ as above we get the T-valued point $(c(r_1, i(r_0)), (g'(u''), u'))$ of R' because we have $t(c(r_1, i(r_0))) = t(r_1) = g(g'(u''))$ and $s(c(r_1, i(r_0))) = s(i(r_0)) = t(r_0) = g(u')$. We leave it to the reader to show that the left square commutes with this definition.

To show that the left square is cartesian, suppose we are given (v'', p') and (v', p) which are T-valued points of $U'' \times_{g',U',t} R'$ and $U' \times_{g,U,t} R$ with $v' = s'(p')$. This also means that $g'(v'') = t'(p')$ and $g(v') = t(p)$. By the discussion above we know that we can write $p' = (r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. Using this notation we see that $v' = s'(p') = u'_1$ and $g'(v'') = t'(p') = u'_0$. Here is a picture

$$s(p) \xrightarrow{p} g(v') = g(u'_1) \xrightarrow{r} g(u'_0) = g(g'(v''))$$

What we have to show is that there exists a unique T-valued point $((u', r_0), (u'', r_1))$ as above such that $v' = u'$, $p = r_0$, $v'' = u''$ and $p' = (c(r_1, i(r_0)), (g'(u''), u'))$. Comparing the two diagrams above it is clear that we have no choice but to take

$$((u', r_0), (u'', r_1)) = ((v', p), (v'', c(r, p))$$

Some details omitted. □

04MX Lemma 40.9.4. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ \text{pr}_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ \text{pr}_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ \text{pr}_1 : U'' \times_{g'',U,t} R \to U$. Let $\tau \in \{\textit{Zariski}, \textit{étale}, \textit{smooth}, \textit{syntomic}, \textit{fppf}, \textit{fpqc}\}$. Let $\mathcal{P}$ be a property of morphisms of schemes which is preserved under base change, and which is local on the target for the τ-topology. If

(1) $h(U' \times_U R)$ is open in U,
(2) $\{h : U' \times_U R \to h(U' \times_U R)\}$ is a τ-covering,
(3) h' has property $\mathcal{P}$,

then h'' has property $\mathcal{P}$. Conversely, if

(a) $\{t : R \to U\}$ is a τ-covering,
(d) h'' has property $\mathcal{P}$,

then h' has property $\mathcal{P}$.

Proof. This follows formally from the properties of the diagram of Lemma 40.9.3. In the first case, note that the image of the morphism h'' is contained in the image of h, as $g'' = g \circ g'$. Hence we may replace the U in the lower right corner of the diagram by $h(U' \times_U R)$. This explains the significance of conditions (1) and (2) in the lemma. In the second case, note that $\{\text{pr}_0 : U' \times_{g,U,t} R \to U'\}$ is a τ-covering as a base change of τ and condition (a). □

40.10. Properties of groupoids on fields

04LL A "groupoid on a field" indicates a groupoid scheme (U, R, s, t, c) where U is the spectrum of a field. It does not mean that (U, R, s, t, c) is defined over a field, more precisely, it does not mean that the morphisms $s, t : R \to U$ are equal. Given any field k, an abstract group G and a group homomorphism $\varphi : G \to \text{Aut}(k)$ we obtain a groupoid scheme (U, R, s, t, c) over $\mathbf{Z}$ by setting

$$\begin{aligned} U &= \text{Spec}(k) \\ R &= \coprod\nolimits_{g \in G} \text{Spec}(k) \\ s &= \coprod\nolimits_{g \in G} \text{Spec}(\text{id}_k) \\ t &= \coprod\nolimits_{g \in G} \text{Spec}(\varphi(g)) \\ c &= \text{composition in } G \end{aligned}$$

This example still is a groupoid scheme over $\text{Spec}(k^G)$. Hence, if G is finite, then $U = \text{Spec}(k)$ is finite over $\text{Spec}(k^G)$. In some sense our goal in this section is to show that suitable finiteness conditions on s, t force any groupoid on a field to be defined over a finite index subfield $k' \subset k$.

If k is a field and (G, m) is a group scheme over k with structure morphism $p : G \to \mathrm{Spec}(k)$, then $(\mathrm{Spec}(k), G, p, p, m)$ is an example of a groupoid on a field (and in this case of course the whole structure is defined over a field). Hence this section can be viewed as the analogue of Groupoids, Section 39.7.

04LM Lemma 40.10.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then the composition morphism $c : R \times_{s,U,t} R \to R$ is open.

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (40.3.0.2). The projection is open by Morphisms, Lemma 29.23.4. $\square$

04LN Lemma 40.10.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then R is a separated scheme.

Proof. By Groupoids, Lemma 39.7.3 the stabilizer group scheme $G \to U$ is separated. By Groupoids, Lemma 39.22.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 26.17). Hence R is a separated scheme, see Schemes, Lemma 26.21.12. $\square$

04LP Lemma 40.10.3. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any points $r, r' \in R$ there exists a field extension k'/k and points $r_1, r_2 \in R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k')$ and a diagram

$$R \xleftarrow{\mathrm{pr}_0} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{\varphi} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{\mathrm{pr}_0} R$$

such that φ is an isomorphism of schemes over $\mathrm{Spec}(k')$, we have $\varphi(r_1) = r_2$, $\mathrm{pr}_0(r_1) = r$, and $\mathrm{pr}_0(r_2) = r'$.

Proof. This is a special case of Lemma 40.7.1 parts (1) and (2). $\square$

04LQ Lemma 40.10.4. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Let k'/k be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[dr, dotted] \arrow[rr, bend left, "s'"] \arrow[dd, bend right=60, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r] & U &
\end{tikzcd}$$

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 29.23.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms, Lemmas 29.9.4, 29.9.2, 29.23.3, 29.11.8, 29.11.7, 29.25.8, and 29.25.6. $\square$

04LR Lemma 40.10.5. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any point $r \in R$ there exist

(1) a field extension k'/k with k' algebraically closed,
(2) a point $r' \in R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$

such that

(1) the point r' maps to r under the morphism $R' \to R$, and
(2) the maps $s', t' : R' \to \mathrm{Spec}(k')$ induce isomorphisms $k' \to \kappa(r')$.

Proof. Translating the geometric statement into a statement on fields, this means that we can find a diagram

$$\begin{tikzcd} k' & k' \arrow[l, "1"'] & \\ k' \arrow[u, "\tau"] & \kappa(r) \arrow[ul, "\sigma"'] & k \arrow[l, "s"] \arrow[ul, "i"'] \\ & k \arrow[u, "t"'] \arrow[ul, "i"] & \end{tikzcd}$$

where $i : k \to k'$ is the embedding of k into k', the maps $s, t : k \to \kappa(r)$ are induced by $s, t : R \to U$, and the map $\tau : k' \to k'$ is an automorphism. To produce such a diagram we may proceed in the following way:

(1) Pick $i : k \to k'$ a field map with k' algebraically closed of very large transcendence degree over k.
(2) Pick an embedding $\sigma : \kappa(r) \to k'$ such that $\sigma \circ s = i$. Such a σ exists because we can just choose a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of $\kappa(r)$ over k and find $y_\alpha \in k'$, $\alpha \in A$ which are algebraically independent over $i(k)$, and map $s(k)(\{x_\alpha\})$ into k' by the rules $s(\lambda) \mapsto i(\lambda)$ for $\lambda \in k$ and $x_\alpha \mapsto y_\alpha$ for $\alpha \in A$. Then extend to $\tau : \kappa(\alpha) \to k'$ using that k' is algebraically closed.
(3) Pick an automorphism $\tau : k' \to k'$ such that $\tau \circ i = \sigma \circ t$. To do this pick a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of k over its prime field. On the one hand, extend $\{i(x_\alpha)\}$ to a transcendence basis of k' by adding $\{y_\beta\}_{\beta \in B}$ and extend $\{\sigma(t(x_\alpha))\}$ to a transcendence basis of k' by adding $\{z_\gamma\}_{\gamma \in C}$. As k' is algebraically closed we can extend the isomorphism $\sigma \circ t \circ i^{-1} : i(k) \to \sigma(t(k))$ to an isomorphism $\tau' : \overline{i(k)} \to \overline{\sigma(t(k))}$ of their algebraic closures in k'. As k' has large transcendence degree we see that the sets B and C have the same cardinality. Thus we can use a bijection $B \to C$ to extend τ' to an isomorphism
$$\overline{i(k)}(\{y_\beta\}) \longrightarrow \overline{\sigma(t(k))}(\{z_\gamma\})$$
and then since k' is the algebraic closure of both sides we see that this extends to an automorphism $\tau : k' \to k'$ as desired.

This proves the lemma. $\square$

04LS Lemma 40.10.6. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. If $r \in R$ is a point such that s, t induce isomorphisms $k \to \kappa(r)$, then the map

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

(see proof for precise notation) is an automorphism $R \to R$ which maps e to r.

Proof. This is completely obvious if you think about groupoids in a functorial way. But we will also spell it out completely. Denote $a : U \to R$ the morphism with image r such that $s \circ a = \mathrm{id}_U$ which exists by the hypothesis that $s : k \to \kappa(r)$ is an isomorphism. Similarly, denote $b : U \to R$ the morphism with image r such that $t \circ b = \mathrm{id}_U$. Note that $b = a \circ (t \circ a)^{-1}$, in particular $a \circ s \circ b = b$.

Consider the morphism $\Psi : R \to R$ given on T-valued points by

$$(f : T \to R) \longmapsto (c(a \circ t \circ f, f) : T \to R)$$

To see this is defined we have to check that $s \circ a \circ t \circ f = t \circ f$ which is obvious as $s \circ a = 1$. Note that $\Phi(e) = a$, so that in order to prove the lemma it suffices to show that Φ is an automorphism of R. Let $\Phi : R \to R$ be the morphism given on T-valued points by

$$(g : T \to R) \longmapsto (c(i \circ b \circ t \circ g, g) : T \to R).$$

This is defined because $s \circ i \circ b \circ t \circ g = t \circ b \circ t \circ g = t \circ g$. We claim that Φ and Ψ are inverse to each other. To see this we compute

$$\begin{aligned}
& c(a \circ t \circ c(i \circ b \circ t \circ g, g), c(i \circ b \circ t \circ g, g)) \\
& = c(a \circ t \circ i \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
& = c(a \circ s \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
& = c(b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
& = c(c(b \circ t \circ g, i \circ b \circ t \circ g), g)) \\
& = c(e, g) \\
& = g
\end{aligned}$$

where we have used the relation $a \circ s \circ b = b$ shown above. In the other direction we have

$$\begin{aligned}
& c(i \circ b \circ t \circ c(a \circ t \circ f, f), c(a \circ t \circ f, f)) \\
& = c(i \circ b \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
& = c(i \circ a \circ (t \circ a)^{-1} \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
& = c(i \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
& = c(c(i \circ a \circ t \circ f, a \circ t \circ f), f) \\
& = c(e, f) \\
& = f
\end{aligned}$$

The lemma is proved. $\square$

0B7V Lemma 40.10.7. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, $W \subset R$ is open, and $Z \to R$ is a morphism of schemes, then the image of the composition $Z \times_{s,U,t} W \to R \times_{s,U,t} R \to R$ is open.

Proof. Write $U = \mathrm{Spec}(k)$. Consider a field extension k'/k. Denote $U' = \mathrm{Spec}(k')$. Let R' be the restriction of R via $U' \to U$. Set $Z' = Z \times_R R'$ and $W' = R' \times_R W$. Consider a point $\xi = (z, w)$ of $Z \times_{s,U,t} W$. Let $r \in R$ be the image of z under $Z \to R$. Pick $k' \supset k$ and $r' \in R'$ as in Lemma 40.10.5. We can choose $z' \in Z'$ mapping to z and r'. Then we can find $\xi' \in Z' \times_{s',U',t'} W'$ mapping to z' and ξ. The open $c(r', W')$ (Lemma 40.10.6) is contained in the image of $Z' \times_{s',U',t'} W' \to R'$. Observe that $Z' \times_{s',U',t'} W' = (Z \times_{s,U,t} W) \times_{R \times_{s,U,t} R} (R' \times_{s',U',t'} R')$. Hence the image of $Z' \times_{s',U',t'} W' \to R' \to R$ is contained in the image of $Z \times_{s,U,t} W \to R$. As $R' \to R$ is open (Lemma 40.10.4) we conclude the image contains an open neighbourhood of the image of ξ as desired. □

04LT Lemma 40.10.8. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. By abuse of notation denote $e \in R$ the image of the identity morphism $e : U \to R$. Then

(1) every local ring $\mathcal{O}_{R,r}$ of R has a unique minimal prime ideal,
(2) there is exactly one irreducible component Z of R passing through e, and
(3) Z is geometrically irreducible over k via either s or t.

Proof. Let $r \in R$ be a point. In this proof we will use the correspondence between irreducible components of R passing through a point r and minimal primes of the local ring $\mathcal{O}_{R,r}$ without further mention. Choose $k \subset k'$ and $r' \in R'$ as in Lemma 40.10.5. Note that $\mathcal{O}_{R,r} \to \mathcal{O}_{R',r'}$ is faithfully flat and local, see Lemma 40.10.4. Hence the result for $r' \in R'$ implies the result for $r \in R$. In other words we may assume that $s, t : k \to \kappa(r)$ are isomorphisms. By Lemma 40.10.6 there exists an automorphism moving e to r. Hence we may assume $r = e$, i.e., part (1) follows from part (2).

We first prove (2) in case k is separably algebraically closed. Namely, let $X, Y \subset R$ be irreducible components passing through e. Then by Varieties, Lemma 33.8.4 and 33.8.3 the scheme $X \times_{s,U,t} Y$ is irreducible as well. Hence $c(X \times_{s,U,t} Y) \subset R$ is an irreducible subset. We claim it contains both X and Y (as subsets of R). Namely, let T be the spectrum of a field. If $x : T \to X$ is a T-valued point of X, then $c(x, e \circ s \circ x) = x$ and $e \circ s \circ x$ factors through Y as $e \in Y$. Similarly for points of Y. This clearly implies that $X = Y$, i.e., there is a unique irreducible component of R passing through e.

Proof of (2) and (3) in general. Let $k \subset k'$ be a separable algebraic closure, and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. By the previous paragraph there is exactly one irreducible component Z' of R' passing through e'. Denote $e'' \in R \times_{s,U} U'$ the base change of e. As $R' \to R \times_{s,U} U'$ is faithfully flat, see Lemma 40.10.4, and $e' \mapsto e''$ we see that there is exactly one irreducible component Z'' of $R \times_{s,k} k'$ passing through e''. This implies, as $R \times_k k' \to R$ is faithfully flat, that there is exactly one irreducible component Z of R passing through e. This proves (2).

To prove (3) let $Z''' \subset R \times_k k'$ be an arbitrary irreducible component of $Z \times_k k'$. By Varieties, Lemma 33.8.13 we see that $Z''' = \sigma(Z'')$ for some $\sigma \in \mathrm{Gal}(k'/k)$. Since $\sigma(e'') = e''$ we see that $e'' \in Z'''$ and hence $Z''' = Z''$. This means that Z is geometrically irreducible over $\mathrm{Spec}(k)$ via the morphism s. The same argument implies that Z is geometrically irreducible over $\mathrm{Spec}(k)$ via the morphism t. □

04LU Lemma 40.10.9. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then

(1) R is equidimensional,
(2) $\dim(R) = \dim_r(R)$ for all $r \in R$,
(3) for any $r \in R$ we have $\mathrm{trdeg}_{s(k)}(\kappa(r)) = \mathrm{trdeg}_{t(k)}(\kappa(r))$, and
(4) for any closed point $r \in R$ we have $\dim(R) = \dim(\mathcal{O}_{R,r})$.

Proof. Let $r, r' \in R$. Then $\dim_r(R) = \dim_{r'}(R)$ by Lemma 40.10.3 and Morphisms, Lemma 29.28.3. By Morphisms, Lemma 29.28.1 we have

$$\dim_r(R) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{s(k)}(\kappa(r)) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{t(k)}(\kappa(r)).$$

On the other hand, the dimension of R (or any open subset of R) is the supremum of the dimensions of the local rings of R, see Properties, Lemma 28.10.3. Clearly this is maximal for closed points r in which case $\mathrm{trdeg}_k(\kappa(r)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 29.16). Hence the lemma follows. ☐

04MQ Lemma 40.10.10. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then $\dim(R) = \dim(G)$ where G is the stabilizer group scheme of R.

Proof. Let $Z \subset R$ be the irreducible component passing through e (see Lemma 40.10.8) thought of as an integral closed subscheme of R. Let k'_s, resp. k'_t be the integral closure of $s(k)$, resp. $t(k)$ in $\Gamma(Z, \mathcal{O}_Z)$. Recall that k'_s and k'_t are fields, see Varieties, Lemma 33.28.4. By Varieties, Proposition 33.31.1 we have $k'_s = k'_t$ as subrings of $\Gamma(Z, \mathcal{O}_Z)$. As e factors through Z we obtain a commutative diagram

$$\begin{tikzcd}
k \arrow[rd, "t"] \arrow[rrd, "1"] & & \\
& \Gamma(Z, \mathcal{O}_Z) \arrow[r, "e"] & k \\
k \arrow[ru, "s"] \arrow[rru, "1"'] & &
\end{tikzcd}$$

This on the one hand shows that $k'_s = s(k)$, $k'_t = t(k)$, so $s(k) = t(k)$, which combined with the diagram above implies that $s = t$! In other words, we conclude that Z is a closed subscheme of $G = R \times_{(t,s), U \times_S U, \Delta} U$. The lemma follows as both G and R are equidimensional, see Lemma 40.10.9 and Groupoids, Lemma 39.8.1. ☐

04MR Remark 40.10.11. Warning: Lemma 40.10.10 is wrong without the condition that s and t are locally of finite type. An easy example is to start with the action

$$\mathbf{G}_{m,\mathbf{Q}} \times_{\mathbf{Q}} \mathbf{A}^1_{\mathbf{Q}} \to \mathbf{A}^1_{\mathbf{Q}}$$

and restrict the corresponding groupoid scheme to the generic point of $\mathbf{A}^1_{\mathbf{Q}}$. In other words restrict via the morphism $\mathrm{Spec}(\mathbf{Q}(x)) \to \mathrm{Spec}(\mathbf{Q}[x]) = \mathbf{A}^1_{\mathbf{Q}}$. Then you get a groupoid scheme (U, R, s, t, c) with $U = \mathrm{Spec}(\mathbf{Q}(x))$ and

$$R = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(xy)}, P \in \mathbf{Q}[T], P \neq 0\right]\right)$$

In this case $\dim(R) = 1$ and $\dim(G) = 0$.

04RA **Lemma 40.10.12.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume

(1) $U = \mathrm{Spec}(k)$ with k a field,
(2) s, t are locally of finite type, and
(3) the characteristic of k is zero.

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 40.4.1 the sheaf of differentials of $R \to U$ is free. Hence smoothness follows from Varieties, Lemma 33.25.1. $\square$

04RB **Lemma 40.10.13.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume

(1) $U = \mathrm{Spec}(k)$ with k a field,
(2) s, t are locally of finite type,
(3) R is reduced, and
(4) k is perfect.

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 40.4.1 the sheaf $\Omega_{R/U}$ is free. Hence the lemma follows from Varieties, Lemma 33.25.2. $\square$

40.11. Morphisms of groupoids on fields

04Q4 This section studies morphisms between groupoids on fields. This is slightly more general, but very akin to, studying morphisms of groupschemes over a field.

04Q5 **Situation 40.11.1.** Let S be a scheme. Let $U = \mathrm{Spec}(k)$ be a scheme over S with k a field. Let (U, R_1, s_1, t_1, c_1), (U, R_2, s_2, t_2, c_2) be groupoid schemes over S with identical first component. Let $a : R_1 \to R_2$ be a morphism such that (id_U, a) defines a morphism of groupoid schemes over S, see Groupoids, Definition 39.13.1. In particular, the following diagrams commute

$$\begin{tikzcd}
R_1 \arrow[rd, "a"] \arrow[rrd, "t_1"] \arrow[rdd, "s_1"'] & & \\
& R_2 \arrow[r, "s_2"'] \arrow[d, "t_2"] & U \\
& U &
\end{tikzcd}
\qquad
\begin{tikzcd}
R_1 \times_{s_1, U, t_1} R_1 \arrow[r, "c_1"'] \arrow[d, "a \times a"'] & R_1 \arrow[d, "a"] \\
R_2 \times_{s_2, U, t_2} R_2 \arrow[r, "c_2"] & R_2
\end{tikzcd}$$

The following lemma is a generalization of Groupoids, Lemma 39.7.7.

04Q6 **Lemma 40.11.2.** Notation and assumptions as in Situation 40.11.1. If $a(R_1)$ is open in R_2, then $a(R_1)$ is closed in R_2.

Proof. Let $r_2 \in R_2$ be a point in the closure of $a(R_1)$. We want to show $r_2 \in a(R_1)$. Pick $k \subset k'$ and $r_2' \in R_2'$ adapted to (U, R_2, s_2, t_2, c_2) and r_2 as in Lemma 40.10.5. Let R_i' be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to U = \mathrm{Spec}(k)$. Let $a' : R_1' \to R_2'$ be the base change of a. The diagram

$$\begin{tikzcd}
R_1' \arrow[r, "a'"'] \arrow[d, "p_1"'] & R_2' \arrow[d, "p_2"] \\
R_1 \arrow[r, "a"] & R_2
\end{tikzcd}$$

is a fibre square. Hence the image of a' is the inverse image of the image of a via the morphism $p_2 : R'_2 \to R_2$. By Lemma 40.10.4 the map p_2 is surjective and open. Hence by Topology, Lemma 5.6.4 we see that r'_2 is in the closure of $a'(R'_1)$. This means that we may assume that $r_2 \in R_2$ has the property that the maps $k \to \kappa(r_2)$ induced by s_2 and t_2 are isomorphisms.

In this case we can use Lemma 40.10.6. This lemma implies $c(r_2, a(R_1))$ is an open neighbourhood of r_2. Hence $a(R_1) \cap c(r_2, a(R_1)) \neq \emptyset$ as we assumed that r_2 was a point of the closure of $a(R_1)$. Using the inverse of R_2 and R_1 we see this means $c_2(a(R_1), a(R_1))$ contains r_2. As $c_2(a(R_1), a(R_1)) \subset a(c_1(R_1, R_1)) = a(R_1)$ we conclude $r_2 \in a(R_1)$ as desired. □

04Q7 **Lemma 40.11.3.** Notation and assumptions as in Situation 40.11.1. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then $c_2(Z \times_{s_2, U, t_2} Z) \subset Z$ set theoretically.

Proof. Consider the commutative diagram

$$\begin{tikzcd} R_1 \times_{s_1, U, t_1} R_1 \arrow[r] \arrow[d] & R_1 \arrow[d] \\ R_2 \times_{s_2, U, t_2} R_2 \arrow[r] & R_2 \end{tikzcd}$$

By Varieties, Lemma 33.24.2 the closure of the image of the left vertical arrow is (set theoretically) $Z \times_{s_2, U, t_2} Z$. Hence the result follows. □

04Q8 **Lemma 40.11.4.** Notation and assumptions as in Situation 40.11.1. Assume that k is perfect. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 33.6.3. Hence the scheme $Z \times_{s_2, U, t_2} Z \subset Z$ is reduced, see Varieties, Lemma 33.6.7. Hence by Lemma 40.11.3 we see that c induces a morphism $Z \times_{s_2, U, t_2} Z \to Z$. Finally, it is clear that e_2 factors through Z and that the map $i_2 : R_2 \to R_2$ preserves Z. Since the morphisms of the septuple $(U, R_2, s_2, t_2, c_2, e_2, i_2)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. □

04Q9 **Lemma 40.11.5.** Notation and assumptions as in Situation 40.11.1. If the image $a(R_1)$ is a locally closed subset of R_2 then it is a closed subset.

Proof. Let $k \subset k'$ be a perfect closure of the field k. Let R'_i be the restriction of R_i via the morphism $U' = \text{Spec}(k') \to \text{Spec}(k)$. Note that the morphisms $R'_i \to R_i$ are universal homeomorphisms as compositions of base changes of the universal homeomorphism $U' \to U$ (see diagram in statement of Lemma 40.10.4). Hence it suffices to prove that $a'(R'_1)$ is closed in R'_2. In other words, we may assume that k is perfect.

If k is perfect, then the closure of the image is a groupoid scheme $Z \subset R_2$, by Lemma 40.11.4. By the same lemma applied to $\text{id}_{R_1} : R_1 \to R_1$ we see that $(R_2)_{red}$ is a groupoid scheme. Thus we may apply Lemma 40.11.2 to the morphism $a|_{(R_2)_{red}} : (R_2)_{red} \to Z$ to conclude that Z equals the image of a. □

04QA Lemma 40.11.6. Notation and assumptions as in Situation 40.11.1. Assume that $a : R_1 \to R_2$ is a quasi-compact morphism. Let $Z \subset R_2$ be the scheme theoretic image (see Morphisms, Definition 29.6.2) of $a : R_1 \to R_2$. Then

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. The main difficulty is to show that $c_2|_{Z \times_{s_2,U,t_2} Z}$ maps into Z. Consider the commutative diagram

$$\begin{tikzcd} R_1 \times_{s_1,U,t_1} R_1 \arrow[r] \arrow[d, "a \times a"] & R_1 \arrow[d] \\ R_2 \times_{s_2,U,t_2} R_2 \arrow[r] & R_2 \end{tikzcd}$$

By Varieties, Lemma 33.24.3 we see that the scheme theoretic image of $a \times a$ is $Z \times_{s_2,U,t_2} Z$. By the commutativity of the diagram we conclude that $Z \times_{s_2,U,t_2} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 40.11.4 it is also true that $i_2(Z) \subset Z$ and that e_2 factors through Z. Hence we conclude as in the proof of that lemma. □

04QB Lemma 40.11.7. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then $\text{pr}_{02}(Z \times_{\text{pr}_1,U,\text{pr}_0} Z) \subset Z$ set theoretically.

Proof. As $(U, U \times_S U, \text{pr}_1, \text{pr}_0, \text{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 40.11.3. But we can also prove it directly as follows.

Write $U = \text{Spec}(k)$. Denote R_s (resp. Z_s, resp. U^2_s) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\text{pr}_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\text{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U^2_s$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{tikzcd} R_s \times_k {}_tR \arrow[r, "c"] \arrow[d, "j_s \times {}_tj"'] & R \arrow[d, "j"] \\ U^2_s \times_k {}_tU^2 \arrow[r] & U \times_S U \end{tikzcd}$$

By Varieties, Lemma 33.24.2 we see that the closure of the image of $j_s \times {}_tj$ is $Z_s \times_k {}_tZ$. By the commutativity of the diagram we conclude that $Z_s \times_k {}_tZ$ maps into Z by the bottom horizontal arrow. □

04QC Lemma 40.11.8. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a perfect field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then

$$(U, Z, \mathrm{pr}_0|_Z, \mathrm{pr}_1|_Z, \mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 40.11.4. But we can also prove it directly as follows.

We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 33.6.3. Hence the scheme $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \subset Z$ is reduced, see Varieties, Lemma 33.6.7. Hence by Lemma 40.11.7 we see that pr_{02} induces a morphism $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \to Z$. Finally, it is clear that $\Delta_{U/S}$ factors through Z and that the map $\sigma : U \times_S U \to U \times_S U$, $(x, y) \mapsto (y, x)$ preserves Z. Since $(U, U \times_S U, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02}, \Delta_{U/S}, \sigma)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. □

04QD Lemma 40.11.9. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field and assume R is quasi-compact (equivalently s, t are quasi-compact). Let $Z \subset U \times_S U$ be the scheme theoretic image (see Morphisms, Definition 29.6.2) of $j = (t, s) : R \to U \times_S U$. Then

$$(U, Z, \mathrm{pr}_0|_Z, \mathrm{pr}_1|_Z, \mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 40.11.6. But we can also prove it directly as follows.

The main difficulty is to show that $\mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z}$ maps into Z. Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{tikzcd} R_s \times_k {}_tR \arrow[r, "c"] \arrow[d, "j_s \times {}_tj"'] & R \arrow[d, "j"] \\ U_s^2 \times_k {}_tU^2 \arrow[r] & U \times_S U \end{tikzcd}$$

By Varieties, Lemma 33.24.3 we see that the scheme theoretic image of $j_s \times {}_tj$ is $Z_s \times_k {}_tZ$. By the commutativity of the diagram we conclude that $Z_s \times_k {}_tZ$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 40.11.8 it is also true that $\sigma(Z) \subset Z$ and that $\Delta_{U/S}$ factors through Z. Hence we conclude as in the proof of that lemma. □

40.12. Slicing groupoids

04LV The following lemma shows that we may slice a Cohen-Macaulay groupoid scheme in order to reduce the dimension of the fibres, provided that the dimension of the stabilizer is small. This is an essential step in the process of improving a given presentation of a quotient stack.

04MY Situation 40.12.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let $u \in U$ be a point, and let $u' \in U'$ be a point such that $g(u') = u$. Given these data, denote (U', R', s', t', c') the restriction of (U, R, s, t, c) via the morphism g. Denote $G \to U$ the stabilizer group scheme of R, which is a locally closed subscheme of R. Denote h the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U.$$

Denote $F_u = s^{-1}(u)$ (scheme theoretic fibre), and G_u the scheme theoretic fibre of G over u. Similarly for R' we denote $F'_{u'} = (s')^{-1}(u')$. Because $g(u') = u$ we have

$$F'_{u'} = h^{-1}(u) \times_{\mathrm{Spec}(\kappa(u))} \mathrm{Spec}(\kappa(u')).$$

The point $e(u) \in R$ may be viewed as a point on G_u and F_u also, and $e'(u')$ is a point of R' (resp. $G'_{u'}$, resp. $F'_{u'}$) which maps to $e(u)$ in R (resp. G_u, resp. F_u).

0461 Lemma 40.12.2. Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 29.16.3. With notation as in Situation 40.12.1, set

$$d_1 = \dim(G_u), \quad d_2 = \dim_{e(u)}(F_u).$$

If $d_2 > d_1$, then there exist an affine scheme U' and a morphism $g : U' \to U$ such that (with notation as in Situation 40.12.1)

(1) g is an immersion
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay at $(u, e(u))$, and
(5) we have $\dim_{e'(u)}(F'_u) = d_2 - 1$.

Proof. Let $\mathrm{Spec}(A) \subset U$ be an affine neighbourhood of u such that u corresponds to a closed point of U, see Morphisms, Lemma 29.16.4. Let $\mathrm{Spec}(B) \subset R$ be an affine neighbourhood of $e(u)$ which maps via j into the open $\mathrm{Spec}(A) \times_S \mathrm{Spec}(A) \subset U \times_S U$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to u. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $e(u)$. Pictures:

$$\begin{tikzcd} B & A \arrow[l, "s"'] \\ A \arrow[u, "t"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_{\mathfrak{q}} & A_{\mathfrak{m}} \arrow[l, "s"'] \\ A_{\mathfrak{m}} \arrow[u, "t"] & \end{tikzcd}$$

Note that the two induced maps $s, t : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{q})$ are equal and isomorphisms as $s \circ e = t \circ e = \mathrm{id}_U$. In particular we see that $\mathfrak{q}$ is a maximal ideal as well. The ring maps $s, t : A \to B$ are of finite presentation and flat. By assumption the ring

$$\mathcal{O}_{F_u, e(u)} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$$

is Cohen-Macaulay of dimension d_2. The equality of dimension holds by Morphisms, Lemma 29.28.1.

Let R'' be the restriction of R to $u = \mathrm{Spec}(\kappa(u))$ via the morphism $\mathrm{Spec}(\kappa(u)) \to U$. As $u \to U$ is locally of finite type, we see that $(\mathrm{Spec}(\kappa(u)), R'', s'', t'', c'')$ is a groupoid scheme with s'', t'' locally of finite type, see Lemma 40.9.1. By Lemma 40.10.10 this implies that $\dim(G'') = \dim(R'')$. We also have $\dim(R'') =$

$\dim_{e''}(R'') = \dim(\mathcal{O}_{R'',e''})$, see Lemma 40.10.9. By Groupoids, Lemma 39.18.4 we have $G'' = G_u$. Hence we conclude that $\dim(\mathcal{O}_{R'',e''}) = d_1$.

As a scheme R'' is

$$R'' = R \times_{(U \times_S U)} \Big(\mathrm{Spec}(\kappa(\mathfrak{m})) \times_S \mathrm{Spec}(\kappa(\mathfrak{m})) \Big)$$

Hence an affine open neighbourhood of e'' is the spectrum of the ring

$$B \otimes_{(A \otimes A)} (\kappa(\mathfrak{m}) \otimes \kappa(\mathfrak{m})) = B/s(\mathfrak{m})B + t(\mathfrak{m})B$$

We conclude that

$$\mathcal{O}_{R'',e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$$

and so now we know that this ring has dimension d_1.

We claim this implies we can find an element $f \in \mathfrak{m}$ such that

$$\dim(B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + fB_{\mathfrak{q}}) < d_2$$

Namely, suppose $\mathfrak{n}_j \supset s(\mathfrak{m})B_{\mathfrak{q}}$, $j = 1, \ldots, m$ correspond to the minimal primes of the local ring $B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$. There are finitely many as this ring is Noetherian (since it is essentially of finite type over a field – but also because a Cohen-Macaulay ring is Noetherian). By the Cohen-Macaulay condition we have $\dim(B_{\mathfrak{q}}/\mathfrak{n}_j) = d_2$, for example by Algebra, Lemma 10.104.4. Note that $\dim(B_{\mathfrak{q}}/(\mathfrak{n}_j + t(\mathfrak{m})B_{\mathfrak{q}})) \leq d_1$ as it is a quotient of the ring $\mathcal{O}_{R'',e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$ which has dimension d_1. As $d_1 < d_2$ this implies that $\mathfrak{m} \not\subset t^{-1}(\mathfrak{n}_i)$. By prime avoidance, see Algebra, Lemma 10.15.2, we can find $f \in \mathfrak{m}$ with $t(f) \notin \mathfrak{n}_j$ for $j = 1, \ldots, m$. For this choice of f we have the displayed inequality above, see Algebra, Lemma 10.60.13.

Set $A' = A/fA$ and $U' = \mathrm{Spec}(A')$. Then it is clear that $U' \to U$ is an immersion, locally of finite presentation and that $u \in U'$. Thus (1), (2) and (3) of the lemma hold. The morphism

$$U' \times_{g,U,t} R \longrightarrow U$$

factors through $\mathrm{Spec}(A)$ and corresponds to the ring map

$$B/t(f)B = A/(f) \otimes_{A,t} B \xleftarrow{s} A$$

Now, we see $t(f)$ is not a zerodivisor on $B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$ as this is a Cohen-Macaulay ring of positive dimension and f is not contained in any minimal prime, see for example Algebra, Lemma 10.104.2. Hence by Algebra, Lemma 10.128.5 we conclude that $s : A_{\mathfrak{m}} \to B_{\mathfrak{q}}/t(f)B_{\mathfrak{q}}$ is flat with fibre ring $B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + t(f)B_{\mathfrak{q}})$ which is Cohen-Macaulay by Algebra, Lemma 10.104.2 again. This implies part (4) of the lemma. To see part (5) note that by Diagram (40.9.0.1) the fibre F'_u is equal to the fibre of h over u. Hence $\dim_{e'(u)}(F'_u) = \dim(B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + t(f)B_{\mathfrak{q}}))$ by Morphisms, Lemma 29.28.1 and the dimension of this ring is $d_2 - 1$ by Algebra, Lemma 10.104.2 once more. This proves the final assertion of the lemma and we win. $\square$

Now that we know how to slice we can combine it with the preceding material to get the following "optimal" result. It is optimal in the sense that since G_u is a locally closed subscheme of F_u one always has the inequality $\dim(G_u) = \dim_{e(u)}(G_u) \leq \dim_{e(u)}(F_u)$ so it is not possible to slice more than in the lemma.

04MZ Lemma 40.12.3. Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 29.16.3. With notation as in Situation 40.12.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay and locally of finite presentation,
(5) the morphisms $s', t' : R' \to U'$ are Cohen-Macaulay and locally of finite presentation, and
(6) $\dim_{e(u)}(F'_u) = \dim(G'_u)$.

Proof. As s is locally of finite presentation the scheme F_u is locally of finite type over $\kappa(u)$. Hence $\dim_{e(u)}(F_u) < \infty$ and we may argue by induction on $\dim_{e(u)}(F_u)$.

If $\dim_{e(u)}(F_u) = \dim(G_u)$ there is nothing to prove. Assume $\dim_{e(u)}(F_u) > \dim(G_u)$. This means that Lemma 40.12.2 applies and we find a morphism $g : U' \to U$ which has properties (1), (2), (3), instead of (6) we have $\dim_{e(u)}(F'_u) < \dim_{e(u)}(F_u)$, and instead of (4) and (5) we have that the composition

$$h = s \circ \text{pr}_1 : U' \times_{g,U,t} R \longrightarrow U$$

is Cohen-Macaulay at the point $(u, e(u))$. We apply Remark 40.6.3 and we obtain an open subscheme $U'' \subset U'$ such that $U'' \times_{g,U,t} R \subset U' \times_{g,U,t} R$ is the largest open subscheme on which h is Cohen-Macaulay. Since $(u, e(u)) \in U'' \times_{g,U,t} R$ we see that $u \in U''$. Hence we may replace U' by U'' and assume that in fact h is Cohen-Macaulay everywhere! By Lemma 40.9.2 we conclude that s', t' are locally of finite presentation and Cohen-Macaulay (use Morphisms, Lemma 29.21.4 and More on Morphisms, Lemma 37.22.6).

By construction $\dim_{e'(u)}(F'_u) < \dim_{e(u)}(F_u)$, so we may apply the induction hypothesis to (U', R', s', t', c') and the point $u \in U'$. Note that u is also a finite type point of U' (for example you can see this using the characterization of finite type points from Morphisms, Lemma 29.16.4). Let $g' : U'' \to U'$ and $(U'', R'', s'', t'', c'')$ be the solution of the corresponding problem starting with (U', R', s', t', c') and the point $u \in U'$. We claim that the composition

$$g'' = g \circ g' : U'' \longrightarrow U$$

is a solution for the original problem. Properties (1), (2), (3), (5), and (6) are immediate. To see (4) note that the morphism

$$h'' = s \circ \text{pr}_1 : U'' \times_{g'',U,t} R \longrightarrow U$$

is locally of finite presentation and Cohen-Macaulay by an application of Lemma 40.9.4 (use More on Morphisms, Lemma 37.22.11 to see that Cohen-Macaulay morphisms are fppf local on the target). $\square$

In case the stabilizer group scheme has fibres of dimension 0 this leads to the following slicing lemma.

04N0 Lemma 40.12.4. Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 29.16.3. Assume that $G \to U$ is locally quasi-finite. With notation as in Situation 40.12.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is flat, locally of finite presentation, and locally quasi-finite, and
(5) the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation, and locally quasi-finite.

Proof. Take $g : U' \to U$ as in Lemma 40.12.3. Since $h^{-1}(u) = F'_u$ we see that h has relative dimension ≤ 0 at $(u, e(u))$. Hence, by Remark 40.6.3, we obtain an open subscheme $U'' \subset U'$ such that $u \in U''$ and $U'' \times_{g,U,t} R$ is the maximal open subscheme of $U' \times_{g,U,t} R$ on which h has relative dimension ≤ 0. After replacing U' by U'' we see that h has relative dimension ≤ 0. This implies that h is locally quasi-finite by Morphisms, Lemma 29.29.5. Since it is still locally of finite presentation and Cohen-Macaulay we see that it is flat, locally of finite presentation and locally quasi-finite, i.e., (4) above holds. This implies that s' is flat, locally of finite presentation and locally quasi-finite as a base change of h, see Lemma 40.9.2. $\square$

40.13. Étale localization of groupoids

03FK In this section we begin applying the étale localization techniques of More on Morphisms, Section 37.41 to groupoid schemes. More advanced material of this kind can be found in More on Groupoids in Spaces, Section 79.15. Lemma 40.13.2 will be used to prove results on algebraic spaces separated and quasi-finite over a scheme, namely Morphisms of Spaces, Proposition 67.50.2 and its corollary Morphisms of Spaces, Lemma 67.51.1.

03FL Lemma 40.13.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume that

(1) $U \to S$ is locally of finite type,
(2) $U \to S$ is quasi-finite at u,
(3) $U \to S$ is separated,
(4) $R \to S$ is separated,
(5) s, t are flat and locally of finite presentation, and
(6) $s^{-1}(\{u\})$ is finite.

Then there exists an étale neighbourhood $(S', p') \to (S, p)$ with $\kappa(p) = \kappa(p')$ and a base change diagram

$$
\begin{tikzcd}
R' \amalg W' \arrow[r, equal] & S' \times_S R \arrow[r] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
U' \amalg W \arrow[r, equal] & S' \times_S U \arrow[r] \arrow[d] & U \arrow[d] \\
& S' \arrow[r] & S
\end{tikzcd}
$$

where the equal signs are decompositions into open and closed subschemes such that

(a) there exists a point u' of U' mapping to u in U,
(b) the fibre $(U')_{p'}$ equals $t'\big((s')^{-1}(\{u'\})\big)$ set theoretically,
(c) the fibre $(R')_{p'}$ equals $(s')^{-1}\big((U')_{p'}\big)$ set theoretically,
(d) the schemes U' and R' are finite over S',
(e) we have $s'(R') \subset U'$ and $t'(R') \subset U'$,
(f) we have $c'(R' \times_{s', U', t'} R') \subset R'$ where c' is the base change of c, and
(g) the morphisms s', t', c' determine a groupoid structure by taking the system $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{s', U', t'} R'})$.

Proof. Let us denote $f : U \to S$ the structure morphism of U. By assumption (6) we can write $s^{-1}(\{u\}) = \{r_1, \ldots, r_n\}$. Since this set is finite, we see that s is quasi-finite at each of these finitely many inverse images, see Morphisms, Lemma 29.20.7. Hence we see that $f \circ s : R \to S$ is quasi-finite at each r_i (Morphisms, Lemma 29.20.12). Hence r_i is isolated in the fibre R_p, see Morphisms, Lemma 29.20.6. Write $t(\{r_1, \ldots, r_n\}) = \{u_1, \ldots, u_m\}$. Note that it may happen that $m < n$ and note that $u \in \{u_1, \ldots, u_m\}$. Since t is flat and locally of finite presentation, the morphism of fibres $t_p : R_p \to U_p$ is flat and locally of finite presentation (Morphisms, Lemmas 29.25.8 and 29.21.4), hence open (Morphisms, Lemma 29.25.10). The fact that each r_i is isolated in R_p implies that each $u_j = t(r_i)$ is isolated in U_p. Using Morphisms, Lemma 29.20.6 again, we see that f is quasi-finite at $u_1, \ldots, u_m$.

Denote $F_u = s^{-1}(u)$ and $F_{u_j} = s^{-1}(u_j)$ the scheme theoretic fibres. Note that F_u is finite over $\kappa(u)$ as it is locally of finite type over $\kappa(u)$ with finitely many points (for example it follows from the much more general Morphisms, Lemma 29.57.9). By Lemma 40.7.1 we see that F_u and F_{u_j} become isomorphic over a common field extension of $\kappa(u)$ and $\kappa(u_j)$. Hence we see that F_{u_j} is finite over $\kappa(u_j)$. In particular we see $s^{-1}(\{u_j\})$ is a finite set for each $j = 1, \ldots, m$. Thus we see that assumptions (2) and (6) hold for each u_j also (above we saw that $U \to S$ is quasi-finite at u_j). Hence the argument of the first paragraph applies to each u_j and we see that $R \to U$ is quasi-finite at each of the points of

$$\{r_1, \ldots, r_N\} = s^{-1}(\{u_1, \ldots, u_m\})$$

Note that $t(\{r_1, \ldots, r_N\}) = \{u_1, \ldots, u_m\}$ and $t^{-1}(\{u_1, \ldots, u_m\}) = \{r_1, \ldots, r_N\}$ since R is a groupoid[2]. Moreover, we have $\mathrm{pr}_0(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$

[2]Explanation in groupoid language: The original set $\{r_1, \ldots, r_n\}$ was the set of arrows with source u. The set $\{u_1, \ldots, u_m\}$ was the set of objects isomorphic to u. And $\{r_1, \ldots, r_N\}$ is the set of all arrows between all the objects equivalent to u.

and $\text{pr}_1(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$. Similarly we get $e(\{u_1, \ldots, u_m\}) \subset \{r_1, \ldots, r_N\}$ and $i(\{r_1, \ldots, r_N\}) = \{r_1, \ldots, r_N\}$.

We may apply More on Morphisms, Lemma 37.41.4 to the pairs $(U \to S, \{u_1, \ldots, u_m\})$ and $(R \to S, \{r_1, \ldots, r_N\})$ to get an étale neighbourhood $(S', p') \to (S, p)$ which induces an identification $\kappa(p) = \kappa(p')$ such that $S' \times_S U$ and $S' \times_S R$ decompose as

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

with $U' \to S'$ finite and $(U')_{p'}$ mapping bijectively to $\{u_1, \ldots, u_m\}$, and $R' \to S'$ finite and $(R')_{p'}$ mapping bijectively to $\{r_1, \ldots, r_N\}$. Moreover, no point of $W_{p'}$ (resp. $(W')_{p'}$) maps to any of the points u_j (resp. r_i). At this point (a), (b), (c), and (d) of the lemma are satisfied. Moreover, the inclusions of (e) and (f) hold on fibres over p', i.e., $s'((R')_{p'}) \subset (U')_{p'}$, $t'((R')_{p'}) \subset (U')_{p'}$, and $c'((R' \times_{s', U', t'} R')_{p'}) \subset (R')_{p'}$.

We claim that we can replace S' by a Zariski open neighbourhood of p' so that the inclusions of (e) and (f) hold. For example, consider the set $E = (s'|_{R'})^{-1}(W)$. This is open and closed in R' and does not contain any points of R' lying over p'. Since $R' \to S'$ is closed, after replacing S' by $S' \setminus (R' \to S')(E)$ we reach a situation where E is empty. In other words s' maps R' into U'. Note that this property is preserved under further shrinking S'. Similarly, we can arrange it so that t' maps R' into U'. At this point (e) holds. In the same manner, consider the set $E = (c'|_{R' \times_{s', U', t'} R'})^{-1}(W')$. It is open and closed in the scheme $R' \times_{s', U', t'} R'$ which is finite over S', and does not contain any points lying over p'. Hence after replacing S' by $S' \setminus (R' \times_{s', U', t'} R' \to S')(E)$ we reach a situation where E is empty. In other words we obtain the inclusion in (f). We may repeat the argument also with the identity $e' : S' \times_S U \to S' \times_S R$ and the inverse $i' : S' \times_S R \to S' \times_S R$ so that we may assume (after shrinking S' some more) that $(e'|_{U'})^{-1}(W') = \emptyset$ and $(i'|_{R'})^{-1}(W') = \emptyset$.

At this point we see that we may consider the structure

$$(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t', U', s'} R'}, e'|_{U'}, i'|_{R'}).$$

The axioms of a groupoid scheme over S' hold because they hold for the groupoid scheme $(S' \times_S U, S' \times_S R, s', t', c', e', i')$. $\square$

03X5 **Lemma 40.13.2.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume assumptions (1) – (6) of Lemma 40.13.1 hold as well as

(7) $j : R \to U \times_S U$ is universally closed[3].

Then we can choose $(S', p') \to (S, p)$ and decompositions $S' \times_S U = U' \amalg W$ and $S' \times_S R = R' \amalg W'$ and $u' \in U'$ such that (a) – (g) of Lemma 40.13.1 hold as well as

(h) R' is the restriction of $S' \times_S R$ to U'.

Proof. We apply Lemma 40.13.1 for the groupoid (U, R, s, t, c) over the scheme S with points p and u. Hence we get an étale neighbourhood $(S', p') \to (S, p)$ and disjoint union decompositions

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

[3] In view of the other conditions this is equivalent to requiring j to be proper.

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), and (g). We may shrink S' to a smaller neighbourhood of p' without affecting the conclusions (a) – (g). We will show that for a suitable shrinking conclusion (h) holds as well. Let us denote j' the base change of j to S'. By conclusion (e) it is clear that

$$j'^{-1}(U' \times_{S'} U') = R' \amalg Rest$$

for some open and closed $Rest$ piece. Since $U' \to S'$ is finite by conclusion (d) we see that $U' \times_{S'} U'$ is finite over S'. Since j is universally closed, also j' is universally closed, and hence $j'|_{Rest}$ is universally closed too. By conclusions (b) and (c) we see that the fibre of

$$(U' \times_{S'} U' \to S') \circ j'|_{Rest} : Rest \longrightarrow S'$$

over p' is empty. Hence, since $Rest \to S'$ is closed as a composition of closed morphisms, after replacing S' by $S' \setminus \mathrm{Im}(Rest \to S')$, we may assume that $Rest = \emptyset$. And this is exactly the condition that R' is the restriction of $S' \times_S R$ to the open subscheme $U' \subset S' \times_S U$, see Groupoids, Lemma 39.18.3 and its proof. $\square$

40.14. Finite groupoids

0AB8 A groupoid scheme (U, R, s, t, c) is sometimes called finite if the morphisms s and t are finite. This is potentially confusing as it doesn't imply that U or R or the quotient sheaf U/R are finite over anything.

0AB9 **Lemma 40.14.1.** Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite. There exists a sequence of R-invariant closed subschemes

$$U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$$

such that $\bigcap Z_r = \emptyset$ and such that $s^{-1}(Z_{r-1}) \setminus s^{-1}(Z_r) \to Z_{r-1} \setminus Z_r$ is finite locally free of rank r.

Proof. Let $\{Z_r\}$ be the stratification of U given by the Fitting ideals of the finite type quasi-coherent modules $s_*\mathcal{O}_R$. See Divisors, Lemma 31.9.6. Since the identity $e : U \to R$ is a section to s we see that $s_*\mathcal{O}_R$ contains $\mathcal{O}_S$ as a direct summand. Hence $U = Z_{-1} = Z_0$ (details omitted). Since formation of Fitting ideals commutes with base change (More on Algebra, Lemma 15.8.4) we find that $s^{-1}(Z_r)$ corresponds to the rth Fitting ideal of $\mathrm{pr}_{1,*}\mathcal{O}_{R \times_{s,U,t} R}$ because the lower right square of diagram (40.3.0.1) is cartesian. Using the fact that the lower left square is also cartesian we conclude that $s^{-1}(Z_r) = t^{-1}(Z_r)$, in other words Z_r is R-invariant. The morphism $s^{-1}(Z_{r-1}) \setminus s^{-1}(Z_r) \to Z_{r-1} \setminus Z_r$ is finite locally free of rank r because the module $s_*\mathcal{O}_R$ pulls back to a finite locally free module of rank r on $Z_{r-1} \setminus Z_r$ by Divisors, Lemma 31.9.6. $\square$

0ABA **Lemma 40.14.2.** Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite. There exists an open subscheme $W \subset U$ and a closed subscheme $W' \subset W$ such that

1. W and W' are R-invariant,
2. $U = t(s^{-1}(\overline{W}))$ set theoretically,
3. W is a thickening of W', and
4. the maps s', t' of the restriction (W', R', s', t', c') are finite locally free.

Proof. Consider the stratification $U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$ of Lemma 40.14.1.

We will construct disjoint unions $W = \coprod_{r \geq 1} W_r$ and $W' = \coprod_{r \geq 1} W'_r$ with each $W'_r \to W_r$ a thickening of R-invariant subschemes of U such that the morphisms s'_r, t'_r of the restrictions $(W'_r, R'_r, s'_r, t'_r, c'_r)$ are finite locally free of rank r. To begin we set $W_1 = W'_1 = U \setminus Z_1$. This is an R-invariant open subscheme of U, it is true that W_0 is a thickening of W'_0, and the maps s'_1, t'_1 of the restriction $(W'_1, R'_1, s'_1, t'_1, c'_1)$ are isomorphisms, i.e., finite locally free of rank 1. Moreover, every point of $U \setminus Z_1$ is in $t(s^{-1}(\overline{W_1}))$.

Assume we have found subschemes $W'_r \subset W_r \subset U$ for $r \leq n$ such that

(1) $W_1, \ldots, W_n$ are disjoint,
(2) W_r and W'_r are R-invariant,
(3) $U \setminus Z_n \subset \bigcup_{r \leq n} t(s^{-1}(\overline{W_r}))$ set theoretically,
(4) W_r is a thickening of W'_r,
(5) the maps s'_r, t'_r of the restriction $(W'_r, R'_r, s'_r, t'_r, c'_r)$ are finite locally free of rank r.

Then we set

$$W_{n+1} = Z_n \setminus \left(Z_{n+1} \cup \bigcup\nolimits_{r \leq n} t(s^{-1}(\overline{W_r}))\right)$$

set theoretically and

$$W'_{n+1} = Z_n \setminus \left(Z_{n+1} \cup \bigcup\nolimits_{r \leq n} t(s^{-1}(\overline{W_r}))\right)$$

scheme theoretically. Then W_{n+1} is an R-invariant open subscheme of U because $Z_{n+1} \setminus \overline{U \setminus Z_{n+1}}$ is open in U and $\overline{U \setminus Z_{n+1}}$ is contained in the closed subset $\bigcup_{r \leq n} t(s^{-1}(\overline{W_r}))$ we are removing by property (3) and the fact that t is a closed morphism. It is clear that W'_{n+1} is a closed subscheme of W_{n+1} with the same underlying topological space. Finally, properties (1), (2) and (3) are clear and property (5) follows from Lemma 40.14.1.

By Lemma 40.14.1 we have $\bigcap Z_r = \emptyset$. Hence every point of U is contained in $U \setminus Z_n$ for some n. Thus we see that $U = \bigcup_{r \geq 1} t(s^{-1}(\overline{W_r}))$ set theoretically and we see that (2) holds. Thus $W' \subset W$ satisfy (1), (2), (3), and (4). $\square$

Let (U, R, s, t, c) be a groupoid scheme. Given a point $u \in U$ the R-orbit of u is the subset $t(s^{-1}(\{u\}))$ of U.

0ABB Lemma 40.14.3. In Lemma 40.14.2 assume in addition that s and t are of finite presentation. Then

(1) the morphism $W' \to W$ is of finite presentation, and
(2) if $u \in U$ is a point whose R-orbit consists of generic points of irreducible components of U, then $u \in W$.

Proof. In this case the stratification $U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$ of Lemma 40.14.1 is given by closed immersions $Z_k \to U$ of finite presentation, see Divisors, Lemma 31.9.6. Part (1) follows immediately from this as $W' \to W$ is locally given by intersecting the open W by Z_r. To see part (2) let $\{u_1, \ldots, u_n\}$ be the orbit of u. Since the closed subschemes Z_k are R-invariant and $\bigcap Z_k = \emptyset$, we find an k such that $u_i \in Z_k$ and $u_i \notin Z_{k+1}$ for all i. The image of $Z_k \to U$ and $Z_{k+1} \to U$ is locally constructible (Morphisms, Theorem 29.22.3). Since $u_i \in U$ is a generic point of an irreducible component of U, there exists an open neighbourhood U_i of

u_i which is contained in $Z_k \setminus Z_{k+1}$ set theoretically (Properties, Lemma 28.2.2). In the proof of Lemma 40.14.2 we have constructed W as a disjoint union $\coprod W_r$ with $W_r \subset Z_{r-1} \setminus Z_r$ such that $U = \bigcup t(s^{-1}(\overline{W_r}))$. As $\{u_1, \ldots, u_n\}$ is an R-orbit we see that $u \in t(s^{-1}(\overline{W_r}))$ implies $u_i \in \overline{W_r}$ for some i which implies $U_i \cap W_r \neq \emptyset$ which implies $r = k$. Thus we conclude that u is in

$$W_{k+1} = Z_k \setminus \left(Z_{k+1} \cup \bigcup\nolimits_{r \leq k} t(s^{-1}(\overline{W_r}))\right)$$

as desired. □

0ABC Lemma 40.14.4. Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite and of finite presentation and U quasi-separated. Let $u_1, \ldots, u_m \in U$ be points whose orbits consist of generic points of irreducible components of U. Then there exist R-invariant subschemes $V' \subset V \subset U$ such that

(1) $u_1, \ldots, u_m \in V'$,
(2) V is open in U,
(3) V' and V are affine,
(4) $V' \subset V$ is a thickening of finite presentation,
(5) the morphisms s', t' of the restriction (V', R', s', t', c') are finite locally free.

Proof. Let $W' \subset W \subset U$ be as in Lemma 40.14.2. By Lemma 40.14.3 we get $u_j \in W$ and that $W' \to W$ is a thickening of finite presentation. By Limits, Lemma 32.11.3 it suffices to find an R-invariant affine open subscheme V' of W' containing u_j (because then we can let $V \subset W$ be the corresponding open subscheme which will be affine). Thus we may replace (U, R, s, t, c) by the restriction (W', R', s', t', c') to W'. In other words, we may assume we have a groupoid scheme (U, R, s, t, c) whose morphisms s and t are finite locally free. By Properties, Lemma 28.29.1 we can find an affine open containing the union of the orbits of $u_1, \ldots, u_m$. Finally, we can apply Groupoids, Lemma 39.24.1 to conclude. □

The following lemma is a special case of Lemma 40.14.4 but we redo the argument as it is slightly easier in this case (it avoids using Lemma 40.14.3).

0ABD Lemma 40.14.5. Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t finite, U is locally Noetherian, and $u_1, \ldots, u_m \in U$ points whose orbits consist of generic points of irreducible components of U. Then there exist R-invariant subschemes $V' \subset V \subset U$ such that

(1) $u_1, \ldots, u_m \in V'$,
(2) V is open in U,
(3) V' and V are affine,
(4) $V' \subset V$ is a thickening,
(5) the morphisms s', t' of the restriction (V', R', s', t', c') are finite locally free.

Proof. Let $\{u_{j1}, \ldots, u_{jn_j}\}$ be the orbit of u_j. Let $W' \subset W \subset U$ be as in Lemma 40.14.2. Since $U = t(s^{-1}(\overline{W}))$ we see that at least one $u_{ji} \in \overline{W}$. Since u_{ji} is a generic point of an irreducible component and U locally Noetherian, this implies that $u_{ji} \in W$. Since W is R-invariant, we conclude that $u_j \in W$ and in fact the whole orbit is contained in W. By Cohomology of Schemes, Lemma 30.13.3 it suffices to find an R-invariant affine open subscheme V' of W' containing $u_1, \ldots, u_m$

(because then we can let $V \subset W$ be the corresponding open subscheme which will be affine). Thus we may replace (U, R, s, t, c) by the restriction (W', R', s', t', c') to W'. In other words, we may assume we have a groupoid scheme (U, R, s, t, c) whose morphisms s and t are finite locally free. By Properties, Lemma 28.29.1 we can find an affine open containing $\{u_{ij}\}$ (a locally Noetherian scheme is quasi-separated by Properties, Lemma 28.5.4). Finally, we can apply Groupoids, Lemma 39.24.1 to conclude. $\square$

0ABE Lemma 40.14.6. Let (U, R, s, t, c) be a groupoid scheme over a scheme S with s, t integral. Let $g : U' \to U$ be an integral morphism such that every R-orbit in U meets $g(U')$. Let (U', R', s', t', c') be the restriction of R to U'. If $u' \in U'$ is contained in an R'-invariant affine open, then the image $u \in U$ is contained in an R-invariant affine open of U.

Proof. Let $W' \subset U'$ be an R'-invariant affine open. Set $\tilde{R} = U' \times_{g,U,t} R$ with maps $\text{pr}_0 : \tilde{R} \to U'$ and $h = s \circ \text{pr}_1 : \tilde{R} \to U$. Observe that pr_0 and h are integral. It follows that $\tilde{W} = \text{pr}_0^{-1}(W')$ is affine. Since W' is R'-invariant, the image $W = h(\tilde{W})$ is set theoretically R-invariant and $\tilde{W} = h^{-1}(W)$ set theoretically (details omitted). Thus, if we can show that W is open, then W is a scheme and the morphism $\tilde{W} \to W$ is integral surjective which implies that W is affine by Limits, Proposition 32.11.2. However, our assumption on orbits meeting U' implies that $h : \tilde{R} \to U$ is surjective. Since an integral surjective morphism is submersive (Topology, Lemma 5.6.5 and Morphisms, Lemma 29.44.7) it follows that W is open. $\square$

The following technical lemma produces "almost" invariant functions in the situation of a finite groupoid on a quasi-affine scheme.

0ABF Lemma 40.14.7. Let (U, R, s, t, c) be a groupoid scheme with s, t finite and of finite presentation. Let $u_1, \ldots, u_m \in U$ be points whose R-orbits consist of generic points of irreducible components of U. Let $j : U \to \text{Spec}(A)$ be an immersion. Let $I \subset A$ be an ideal such that $j(U) \cap V(I) = \emptyset$ and $V(I) \cup j(U)$ is closed in $\text{Spec}(A)$. Then there exists an $h \in I$ such that $j^{-1}D(h)$ is an R-invariant affine open subscheme of U containing $u_1, \ldots, u_m$.

Proof. Let $u_1, \ldots, u_m \in V' \subset V \subset U$ be as in Lemma 40.14.4. Since $U \setminus V$ is closed in U, j an immersion, and $V(I) \cup j(U)$ is closed in $\text{Spec}(A)$, we can find an ideal $J \subset I$ such that $V(J) = V(I) \cup j(U \setminus V)$. For example we can take the ideal of elements of I which vanish on $j(U \setminus V)$. Thus we can replace (U, R, s, t, c), $j : U \to \text{Spec}(A)$, and I by (V', R', s', t', c'), $j|_{V'} : V' \to \text{Spec}(A)$, and J. In other words, we may assume that U is affine and that s and t are finite locally free. Take any $f \in I$ which does not vanish at all the points in the R-orbits of $u_1, \ldots, u_m$ (Algebra, Lemma 10.15.2). Consider

$$g = \text{Norm}_s(t^\sharp(j^\sharp(f))) \in \Gamma(U, \mathcal{O}_U)$$

Since $f \in I$ and since $V(I) \cup j(U)$ is closed we see that $U \cap D(f) \to D(f)$ is a closed immersion. Hence $f^n g$ is the image of an element $h \in I$ for some $n > 0$. We claim that h works. Namely, we have seen in Groupoids, Lemma 39.23.2 that g is an R-invariant function, hence $D(g) \subset U$ is R-invariant. Since f does not vanish on the orbit of u_j, the function g does not vanish at u_j. Moreover, we have $V(g) \supset V(j^\sharp(f))$ and hence $j^{-1}D(h) = D(g)$. $\square$

0ABG Lemma 40.14.8. Let (U, R, s, t, c) be a groupoid scheme. If s, t are finite, and $u, u' \in R$ are distinct points in the same orbit, then u' is not a specialization of u.

Proof. Let $r \in R$ with $s(r) = u$ and $t(r) = u'$. If $u \rightsquigarrow u'$ then we can find a nontrivial specialization $r \rightsquigarrow r'$ with $s(r') = u'$, see Schemes, Lemma 26.19.8. Set $u'' = t(r')$. Note that $u'' \neq u'$ as there are no specializations in the fibres of a finite morphism. Hence we can continue and find a nontrivial specialization $r' \rightsquigarrow r''$ with $s(r'') = u''$, etc. This shows that the orbit of u contains an infinite sequence $u \rightsquigarrow u' \rightsquigarrow u'' \rightsquigarrow \ldots$ of specializations which is nonsense as the orbit $t(s^{-1}(\{u\}))$ is finite. $\square$

0ABH Lemma 40.14.9. Let $j : V \to \mathrm{Spec}(A)$ be a quasi-compact immersion of schemes. Let $f \in A$ be such that $j^{-1}D(f)$ is affine and $j(V) \cap V(f)$ is closed. Then V is affine.

Proof. This follows from Morphisms, Lemma 29.11.14 but we will also give a direct proof. Let $A' = \Gamma(V, \mathcal{O}_V)$. Then $j' : V \to \mathrm{Spec}(A')$ is a quasi-compact open immersion, see Properties, Lemma 28.18.4. Let $f' \in A'$ be the image of f. Then $(j')^{-1}D(f') = j^{-1}D(f)$ is affine. On the other hand, $j'(V) \cap V(f')$ is a subscheme of $\mathrm{Spec}(A')$ which maps isomorphically to the closed subscheme $j(V) \cap V(f)$ of $\mathrm{Spec}(A)$. Hence it is closed in $\mathrm{Spec}(A')$ for example by Schemes, Lemma 26.21.11. Thus we may replace A by A' and assume that j is an open immersion and $A = \Gamma(V, \mathcal{O}_V)$.

In this case we claim that $j(V) = \mathrm{Spec}(A)$ which finishes the proof. If not, then we can find a principal affine open $D(g) \subset \mathrm{Spec}(A)$ which meets the complement and avoids the closed subset $j(V) \cap V(f)$. Note that j maps $j^{-1}D(f)$ isomorphically onto $D(f)$, see Properties, Lemma 28.18.3. Hence $D(g)$ meets $V(f)$. On the other hand, $j^{-1}D(g)$ is a principal open of the affine open $j^{-1}D(f)$ hence affine. Hence by Properties, Lemma 28.18.3 again we see that $D(g)$ is isomorphic to $j^{-1}D(g) \subset j^{-1}D(f)$ which implies that $D(g) \subset D(f)$. This contradiction finishes the proof. $\square$

0ABI Lemma 40.14.10. Let (U, R, s, t, c) be a groupoid scheme. Let $u \in U$. Assume

(1) s, t are finite morphisms,
(2) U is separated and locally Noetherian,
(3) $\dim(\mathcal{O}_{U,u'}) \leq 1$ for every point u' in the orbit of u.

Then u is contained in an R-invariant affine open of U.

Proof. The R-orbit of u is finite. By conditions (2) and (3) it is contained in an affine open U' of U, see Varieties, Proposition 33.42.7. Then $t(s^{-1}(U \setminus U'))$ is an R-invariant closed subset of U which does not contain u. Thus $U \setminus t(s^{-1}(U \setminus U'))$ is an R-invariant open of U' containing u. Replacing U by this open we may assume U is quasi-affine.

By Lemma 40.14.6 we may replace U by its reduction and assume U is reduced. This means R-invariant subschemes $W' \subset W \subset U$ of Lemma 40.14.2 are equal $W' = W$. As $U = t(s^{-1}(\overline{W}))$ some point u' of the R-orbit of u is contained in $\overline{W}$ and by Lemma 40.14.6 we may replace U by $\overline{W}$ and u by u'. Hence we may assume there is a dense open R-invariant subscheme $W \subset U$ such that the morphisms s_W, t_W of the restriction (W, R_W, s_W, t_W, c_W) are finite locally free.

If $u \in W$ then we are done by Groupoids, Lemma 39.24.1 (because W is quasi-affine so any finite set of points of W is contained in an affine open, see Properties, Lemma 28.29.5). Thus we assume $u \notin W$ and hence none of the points of the orbit of u is in W. Let $\xi \in U$ be a point with a nontrivial specialization to a point u' in the orbit of u. Since there are no specializations among the points in the orbit of u (Lemma 40.14.8) we see that ξ is not in the orbit. By assumption (3) we see that ξ is a generic point of U and hence $\xi \in W$. As U is Noetherian there are finitely many of these points $\xi_1, \ldots, \xi_m \in W$. Because s_W, t_W are flat the orbit of each ξ_j consists of generic points of irreducible components of W (and hence U).

Let $j : U \to \text{Spec}(A)$ be an immersion of U into an affine scheme (this is possible as U is quasi-affine). Let $J \subset A$ be an ideal such that $V(J) \cap j(W) = \emptyset$ and $V(J) \cup j(W)$ is closed. Apply Lemma 40.14.7 to the groupoid scheme (W, R_W, s_W, t_W, c_W), the morphism $j|_W : W \to \text{Spec}(A)$, the points ξ_j, and the ideal J to find an $f \in J$ such that $(j|_W)^{-1}D(f)$ is an R_W-invariant affine open containing ξ_j for all j. Since $f \in J$ we see that $j^{-1}D(f) \subset W$, i.e., $j^{-1}D(f)$ is an R-invariant affine open of U contained in W containing all ξ_j.

Let Z be the reduced induced closed subscheme structure on

$$U \setminus j^{-1}D(f) = j^{-1}V(f).$$

Then Z is set theoretically R-invariant (but it may not be scheme theoretically R-invariant). Let (Z, R_Z, s_Z, t_Z, c_Z) be the restriction of R to Z. Since $Z \to U$ is finite, it follows that s_Z and t_Z are finite. Since $u \in Z$ the orbit of u is in Z and agrees with the R_Z-orbit of u viewed as a point of Z. Since $\dim(\mathcal{O}_{U,u'}) \leq 1$ and since $\xi_j \notin Z$ for all j, we see that $\dim(\mathcal{O}_{Z,u'}) \leq 0$ for all u' in the orbit of u. In other words, the R_Z-orbit of u consists of generic points of irreducible components of Z.

Let $I \subset A$ be an ideal such that $V(I) \cap j(U) = \emptyset$ and $V(I) \cup j(U)$ is closed. Apply Lemma 40.14.7 to the groupoid scheme (Z, R_Z, s_Z, t_Z, c_Z), the restriction $j|_Z$, the ideal I, and the point $u \in Z$ to obtain $h \in I$ such that $j^{-1}D(h) \cap Z$ is an R_Z-invariant open affine containing u.

Consider the R_W-invariant (Groupoids, Lemma 39.23.2) function

$$g = \text{Norm}_{s_W}(t_W^\sharp(j^\sharp(h)|_W)) \in \Gamma(W, \mathcal{O}_W)$$

(In the following we only need the restriction of g to $j^{-1}D(f)$ and in this case the norm is along a finite locally free morphism of affines.) We claim that

$$V = (W_g \cap j^{-1}D(f)) \cup (j^{-1}D(h) \cap Z)$$

is an R-invariant affine open of U which finishes the proof of the lemma. It is set theoretically R-invariant by construction. As V is a constuctible set, to see that it is open it suffices to show it is closed under generalization in U (Topology, Lemma 5.19.10 or the more general Topology, Lemma 5.23.6). Since $W_g \cap j^{-1}D(f)$ is open in U, it suffices to consider a specialization $u_1 \rightsquigarrow u_2$ of U with $u_2 \in j^{-1}D(h) \cap Z$. This means that h is nonzero in $j(u_2)$ and $u_2 \in Z$. If $u_1 \in Z$, then $j(u_1) \rightsquigarrow j(u_2)$ and since h is nonzero in $j(u_2)$ it is nonzero in $j(u_1)$ which implies $u_1 \in V$. If $u_1 \notin Z$ and also not in $W_g \cap j^{-1}D(f)$, then $u_1 \in W$, $u_1 \notin W_g$ because the complement of $Z = j^{-1}V(f)$ is contained in $W \cap j^{-1}D(f)$. Hence there exists a point $r_1 \in R$ with $s(r_1) = u_1$ such that h is zero in $t(r_1)$. Since s is finite we can find a specialization $r_1 \rightsquigarrow r_2$ with $s(r_2) = u_2$. However, then we conclude that h is zero in $u_2' = t(r_2)$ which contradicts the fact that $j^{-1}D(h) \cap Z$ is R-invariant and u_2 is in it. Thus V is open.

Observe that $V \subset j^{-1}D(h)$ for our function $h \in I$. Thus we obtain an immersion

$$j' : V \longrightarrow \mathrm{Spec}(A_h)$$

Let $f' \in A_h$ be the image of f. Then $(j')^{-1}D(f')$ is the principal open determined by g in the affine open $j^{-1}D(f)$ of U. Hence $(j')^{-1}D(f)$ is affine. Finally, $j'(V) \cap V(f') = j'(j^{-1}D(h) \cap Z)$ is closed in $\mathrm{Spec}(A_h/(f')) = \mathrm{Spec}((A/f)_h) = D(h) \cap V(f)$ by our choice of $h \in I$ and the ideal I. Hence we can apply Lemma 40.14.9 to conclude that V is affine as claimed above. □

40.15. Descending ind-quasi-affine morphisms

0APG Ind-quasi-affine morphisms were defined in More on Morphisms, Section 37.66. This section is the analogue of Descent, Section 35.38 for ind-quasi-affine-morphisms.

Let X be a quasi-separated scheme. Let $E \subset X$ be a subset which is an intersection of a nonempty family of quasi-compact opens of X. Say $E = \bigcap_{i \in I} U_i$ with $U_i \subset X$ quasi-compact open and I nonempty. By adding finite intersections we may assume that for $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$. In this situation we have

0APH (40.15.0.1)
$$\Gamma(E, \mathcal{F}|_E) = \mathrm{colim}\, \Gamma(U_i, \mathcal{F}|_{U_i})$$

for any sheaf $\mathcal{F}$ defined on X. Namely, fix $i_0 \in I$ and replace X by U_{i_0} and I by $\{i \in I \mid U_i \subset U_{i_0}\}$. Then X is quasi-compact and quasi-separated, hence a spectral space, see Properties, Lemma 28.2.4. Then we see the equality holds by Topology, Lemma 5.24.7 and Sheaves, Lemma 6.29.4. (In fact, the formula holds for higher cohomology groups as well if $\mathcal{F}$ is abelian, see Cohomology, Lemma 20.19.2.)

0API **Lemma 40.15.1.** Let X be an ind-quasi-affine scheme. Let $E \subset X$ be an intersection of a nonempty family of quasi-compact opens of X. Set $A = \Gamma(E, \mathcal{O}_X|_E)$ and $Y = \mathrm{Spec}(A)$. Then the canonical morphism

$$j : (E, \mathcal{O}_X|_E) \longrightarrow (Y, \mathcal{O}_Y)$$

of Schemes, Lemma 26.6.4 determines an isomorphism $(E, \mathcal{O}_X|_E) \to (E', \mathcal{O}_Y|_{E'})$ where $E' \subset Y$ is an intersection of quasi-compact opens. If $W \subset E$ is open in X, then $j(W)$ is open in Y.

Proof. Note that $(E, \mathcal{O}_X|_E)$ is a locally ringed space so that Schemes, Lemma 26.6.4 applies to $A \to \Gamma(E, \mathcal{O}_X|_E)$. Write $E = \bigcap_{i \in I} U_i$ with $I \neq \emptyset$ and $U_i \subset X$ quasi-compact open. We may and do assume that for $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$. Set $A_i = \Gamma(U_i, \mathcal{O}_{U_i})$. We obtain commutative diagrams

$$\begin{tikzcd} (E, \mathcal{O}_X|_E) \arrow[r] \arrow[d] & (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \arrow[d] \\ (U_i, \mathcal{O}_{U_i}) \arrow[r] & (\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)}) \end{tikzcd}$$

Since U_i is quasi-affine, we see that $U_i \to \mathrm{Spec}(A_i)$ is a quasi-compact open immersion. On the other hand $A = \mathrm{colim}\, A_i$. Hence $\mathrm{Spec}(A) = \lim \mathrm{Spec}(A_i)$ as topological spaces (Limits, Lemma 32.4.6). Since $E = \lim U_i$ (by Topology, Lemma 5.24.7) we see that $E \to \mathrm{Spec}(A)$ is a homeomorphism onto its image E' and that E' is the intersection of the inverse images of the opens $U_i \subset \mathrm{Spec}(A_i)$ in $\mathrm{Spec}(A)$. For any $e \in E$ the local ring $\mathcal{O}_{X,e}$ is the value of $\mathcal{O}_{U_i,e}$ which is the same as the value on $\mathrm{Spec}(A)$.

To prove the final assertion of the lemma we argue as follows. Pick $i, j \in I$ with $U_i \subset U_j$. Consider the following commutative diagrams

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & \mathrm{Spec}(A_i) \arrow[d] \\ U_i \arrow[r] & \mathrm{Spec}(A_j) \end{tikzcd} \quad \begin{tikzcd} W \arrow[r] \arrow[d] & \mathrm{Spec}(A_i) \arrow[d] \\ W \arrow[r] & \mathrm{Spec}(A_j) \end{tikzcd} \quad \begin{tikzcd} W \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ W \arrow[r] & \mathrm{Spec}(A_j) \end{tikzcd}$$

By Properties, Lemma 28.18.5 the first diagram is cartesian. Hence the second is cartesian as well. Passing to the limit we find that the third diagram is cartesian, so the top horizontal arrow of this diagram is an open immersion. □

0APJ Lemma 40.15.2. Suppose given a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[d] \\ Y \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

of schemes. Let $E \subset Y$ be an intersection of a nonempty family of quasi-compact opens of Y. Then

$$\Gamma(f^{-1}(E), \mathcal{O}_X|_{f^{-1}(E)}) = \Gamma(E, \mathcal{O}_Y|_E) \otimes_A B$$

provided Y is quasi-separated and $A \to B$ is flat.

Proof. Write $E = \bigcap_{i \in I} V_i$ with $V_i \subset Y$ quasi-compact open. We may and do assume that for $i, j \in I$ there exists a $k \in I$ with $V_k \subset V_i \cap V_j$. Then we have similarly that $f^{-1}(E) = \bigcap_{i \in I} f^{-1}(V_i)$ in X. Thus the result follows from equation (40.15.0.1) and the corresponding result for V_i and $f^{-1}(V_i)$ which is Cohomology of Schemes, Lemma 30.5.2. □

0APK Lemma 40.15.3 (Gabber). Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$, see Descent, Definition 35.34.3. If each morphism $V_i \to X_i$ is ind-quasi-affine, then the descent datum is effective.

Proof. Being ind-quasi-affine is a property of morphisms of schemes which is preserved under any base change, see More on Morphisms, Lemma 37.66.6. Hence Descent, Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is ind-quasi-affine, in other words, V is ind-quasi-affine.

Let (U, R, s, t, c) be the groupoid scheme over S with $U = X$ and $R = X \times_S X$ and s, t, c as usual. By Groupoids, Lemma 39.21.3 the pair (V, φ) corresponds to a cartesian morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes. Let $u' \in U'$ be any point. By Groupoids, Lemmas 39.19.2, 39.19.3, and 39.19.4 we can choose $u' \in W \subset E \subset U'$ where W is open and R'-invariant, and E is set-theoretically R'-invariant and an intersection of a nonempty family of quasi-compact opens.

Translating back to (V, φ), for any $v \in V$ we can find $v \in W \subset E \subset V$ with the following properties: (a) W is open and $\varphi(W \times_S X) = X \times_S W$ and (b) E an intersection of quasi-compact opens and $\varphi(E \times_S X) = X \times_S E$ set-theoretically. Here we use the notation $E \times_S X$ to mean the inverse image of E in $V \times_S X$ by the projection morphism and similarly for $X \times_S E$. By Lemma 40.15.2 this implies that φ defines an isomorphism

$$\begin{aligned}\Gamma(E, \mathcal{O}_V|_E) \otimes_R A &= \Gamma(E \times_S X, \mathcal{O}_{V \times_S X}|_{E \times_S X}) \\ &\to \Gamma(X \times_S E, \mathcal{O}_{X \times_S V}|_{X \times_S E}) \\ &= A \otimes_R \Gamma(E, \mathcal{O}_V|_E)\end{aligned}$$

of $A \otimes_R A$-algebras which we will call ψ. The cocycle condition for φ translates into the cocycle condition for ψ as in Descent, Definition 35.3.1 (details omitted). By Descent, Proposition 35.3.9 we find an R-algebra R' and an isomorphism $\chi : R' \otimes_R A \to \Gamma(E, \mathcal{O}_V|_E)$ of A-algebras, compatible with ψ and the canonical descent datum on $R' \otimes_R A$.

By Lemma 40.15.1 we obtain a canonical "embedding"

$$j : (E, \mathcal{O}_V|_E) \longrightarrow \mathrm{Spec}(\Gamma(E, \mathcal{O}_V|_E)) = \mathrm{Spec}(R' \otimes_R A)$$

of locally ringed spaces. The construction of this map is canonical and we get a commutative diagram

$$\begin{tikzcd}
 & E \times_S X \arrow[rr, "\varphi"] \arrow[dl] \arrow[dr, "j'"] & & X \times_S E \arrow[dl, "j''"'] \arrow[dr] & \\
E \arrow[dr, "j"] & & \mathrm{Spec}(R' \otimes_R A \otimes_R A) \arrow[dl] \arrow[dr] & & E \arrow[dl, "j"'] \\
 & \mathrm{Spec}(R' \otimes_R A) \arrow[dr] & & \mathrm{Spec}(R' \otimes_R A) \arrow[dl] & \\
 & & \mathrm{Spec}(R') & &
\end{tikzcd}$$

where j' and j'' come from the same construction applied to $E \times_S X \subset V \times_S X$ and $X \times_S E \subset X \times_S V$ via χ and the identifications used to construct ψ. It follows that $j(W)$ is an open subscheme of $\mathrm{Spec}(R' \otimes_R A)$ whose inverse image under the two projections $\mathrm{Spec}(R' \otimes_R A \otimes_R A) \to \mathrm{Spec}(R' \otimes_R A)$ are equal. By Descent, Lemma 35.13.6 we find an open $W_0 \subset \mathrm{Spec}(R')$ whose base change to $\mathrm{Spec}(A)$ is $j(W)$. Contemplating the diagram above we see that the descent datum $(W, \varphi|_{W \times_S X})$ is effective. By Descent, Lemma 35.35.13 we see that our descent datum is effective. $\square$

40.16. Other chapters

Preliminaries

CHAPTER 41

Étale Morphisms of Schemes

024J 41.1. Introduction

024K In this Chapter, we discuss étale morphisms of schemes. We illustrate some of the more important concepts by working with the Noetherian case. Our principal goal is to collect for the reader enough commutative algebra results to start reading a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase "the étale topology of schemes" an exercise in general nonsense, if (s)he does indulge in such blasphemy.

We will refer to the other chapters of the Stacks project for standard results in algebraic geometry (on schemes and commutative algebra). We will provide detailed proofs of the new results that we state here.

41.2. Conventions

039F In this chapter, frequently schemes will be assumed locally Noetherian and frequently rings will be assumed Noetherian. But in all the statements we will reiterate this when necessary, and make sure we list all the hypotheses! On the other hand, here are some general facts that we will use often and are useful to keep in mind:

(1) A ring homomorphism $A \to B$ of finite type with A Noetherian is of finite presentation. See Algebra, Lemma 10.31.4.
(2) A morphism (locally) of finite type between locally Noetherian schemes is automatically (locally) of finite presentation. See Morphisms, Lemma 29.21.9.
(3) Add more like this here.

41.3. Unramified morphisms

024L We first define "unramified homomorphisms of local rings" for Noetherian local rings. We cannot use the term "unramified" as there already is a notion of an unramified ring map (Algebra, Section 10.151) and it is different. After discussing the notion a bit we globalize it to describe unramified morphisms of locally Noetherian schemes.

024M Definition 41.3.1. Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be unramified homomorphism of local rings if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,
(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and
(3) B is essentially of finite type over A (this means that B is the localization of a finite type A-algebra at a prime).

This is the local version of the definition in Algebra, Section 10.151. In that section a ring map $R \to S$ is defined to be unramified if and only if it is of finite type, and $\Omega_{S/R} = 0$. We say $R \to S$ is unramified at a prime $\mathfrak q \subset S$ if there exists a $g \in S$, $g \not\in \mathfrak q$ such that $R \to S_g$ is an unramified ring map. It is shown in Algebra, Lemmas 10.151.5 and 10.151.7 that given a ring map $R \to S$ of finite type, and a prime $\mathfrak q$ of S lying over $\mathfrak p \subset R$, then we have

$$R \to S \text{ is unramified at } \mathfrak q \Leftrightarrow \mathfrak p S_{\mathfrak q} = \mathfrak q S_{\mathfrak q} \text{ and } \kappa(\mathfrak p) \subset \kappa(\mathfrak q) \text{ finite separable}$$

Thus we see that for a local homomorphism of local rings the properties of our definition above are closely related to the question of being unramified. In fact, we have proved the following lemma.

039G Lemma 41.3.2. Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak q$ be a prime of B lying over $\mathfrak p \subset A$. Then $A \to B$ is unramified at $\mathfrak q$ if and only if $A_{\mathfrak p} \to B_{\mathfrak q}$ is an unramified homomorphism of local rings.

Proof. See discussion above. □

We will characterize the property of being unramified in terms of completions. For a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to the maximal ideal. It is also a Noetherian local ring, see Algebra, Lemma 10.97.6.

039H Lemma 41.3.3. Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism.

(1) if $A \to B$ is an unramified homomorphism of local rings, then $B^\wedge$ is a finite $A^\wedge$ module,
(2) if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak m_A) = \kappa(\mathfrak m_B)$, then $A^\wedge \to B^\wedge$ is surjective,
(3) if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak m_A)$ is separably closed, then $A^\wedge \to B^\wedge$ is surjective,
(4) if A and B are complete discrete valuation rings, then $A \to B$ is an unramified homomorphism of local rings if and only if the uniformizer for A maps to a uniformizer for B, and the residue field extension is finite separable (and B is essentially of finite type over A).

Proof. Part (1) is a special case of Algebra, Lemma 10.97.7. For part (2), note that the $\kappa(\mathfrak m_A)$-vector space $B^\wedge/\mathfrak m_{A^\wedge}B^\wedge$ is generated by 1. Hence by Nakayama's lemma (Algebra, Lemma 10.20.1) the map $A^\wedge \to B^\wedge$ is surjective. Part (3) is a special case of part (2). Part (4) is immediate from the definitions. □

039I Lemma 41.3.4. Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent

(1) $A \to B$ is an unramified homomorphism of local rings
(2) $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, and
(3) $A^\wedge \to B^\wedge$ is unramified.

Proof. The equivalence of (1) and (2) follows from the fact that $\mathfrak m_A A^\wedge$ is the maximal ideal of $A^\wedge$ (and similarly for B) and faithful flatness of $B \to B^\wedge$. For example if $A^\wedge \to B^\wedge$ is unramified, then $\mathfrak m_A B^\wedge = (\mathfrak m_A B)B^\wedge = \mathfrak m_B B^\wedge$ and hence $\mathfrak m_A B = \mathfrak m_B$.

Assume the equivalent conditions (1) and (2). By Lemma 41.3.3 we see that $A^\wedge \to B^\wedge$ is finite. Hence $A^\wedge \to B^\wedge$ is of finite presentation, and by Algebra, Lemma

10.151.7 we conclude that $A^\wedge \to B^\wedge$ is unramified at $\mathfrak{m}_{B^\wedge}$. Since $B^\wedge$ is local we conclude that $A^\wedge \to B^\wedge$ is unramified.

Assume (3). By Algebra, Lemma 10.151.5 we conclude that $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, i.e., (2) holds. □

024N Definition 41.3.5. (See Morphisms, Definition 29.35.1 for the definition in the general case.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$.

(1) We say f is unramified at x if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.
(2) The morphism $f : X \to Y$ is said to be unramified if it is unramified at all points of X.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is unramified is open.

039J Lemma 41.3.6. Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is unramified at x in the sense of Definition 41.3.5 if and only if it is unramified in the sense of Morphisms, Definition 29.35.1.

Proof. This follows from Lemma 41.3.2 and the definitions. □

Here are some results on unramified morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) Unramifiedness is local on the source and the target in the Zariski topology.
(2) Unramified morphisms are stable under base change and composition. See Morphisms, Lemmas 29.35.5 and 29.35.4.
(3) Unramified morphisms of schemes are locally quasi-finite and quasi-compact unramified morphisms are quasi-finite. See Morphisms, Lemma 29.35.10
(4) Unramified morphisms have relative dimension 0. See Morphisms, Definition 29.29.1 and Morphisms, Lemma 29.29.5.
(5) A morphism is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the scheme theoretic fibres. See Morphisms, Lemma 29.35.12.
(6) Let X and Y be unramified over a base scheme S. Any S-morphism from X to Y is unramified. See Morphisms, Lemma 29.35.16.

41.4. Three other characterizations of unramified morphisms

024O The following theorem gives three equivalent notions of being unramified at a point. See Morphisms, Lemma 29.35.14 for (part of) the statement for general schemes.

024P Theorem 41.4.1. Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let x be a point of X. The following are equivalent

(1) f is unramified at x,

(2) the stalk $\Omega_{X/Y,x}$ of the module of relative differentials at x is trivial,
(3) there exist open neighbourhoods U of x and V of $f(x)$, and a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "i"'] \arrow[dr] & & \mathbf{A}^n_V \arrow[dl] \\ & V & \end{tikzcd}$$

where i is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ such that the differentials $\mathrm{d}g$ for $g \in \mathcal{I}_{i(x)}$ generate $\Omega_{\mathbf{A}^n_V/V,i(x)}$, and
(4) the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.

Proof. The equivalence of (1) and (2) is proved in Morphisms, Lemma 29.35.14.

If f is unramified at x, then f is unramified in an open neighbourhood of x; this does not follow immediately from Definition 41.3.5 of this chapter but it does follow from Morphisms, Definition 29.35.1 which we proved to be equivalent in Lemma 41.3.6. Choose affine opens $V \subset Y$, $U \subset X$ with $f(U) \subset V$ and $x \in U$, such that f is unramified on U, i.e., $f|_U : U \to V$ is unramified. By Morphisms, Lemma 29.35.13 the morphism $U \to U \times_V U$ is an open immersion. This proves that (1) implies (4).

If $\Delta_{X/Y}$ is a local isomorphism at x, then $\Omega_{X/Y,x} = 0$ by Morphisms, Lemma 29.32.7. Hence we see that (4) implies (2). At this point we know that (1), (2) and (4) are all equivalent.

Assume (3). The assumption on the diagram combined with Morphisms, Lemma 29.32.15 show that $\Omega_{U/V,x} = 0$. Since $\Omega_{U/V,x} = \Omega_{X/Y,x}$ we conclude (2) holds.

Finally, assume that (2) holds. To prove (3) we may localize on X and Y and assume that X and Y are affine. Say $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The point $x \in X$ corresponds to a prime $\mathfrak{q} \subset B$. Our assumption is that $\Omega_{B/A,\mathfrak{q}} = 0$ (see Morphisms, Lemma 29.32.5 for the relationship between differentials on schemes and modules of differentials in commutative algebra). Since Y is locally Noetherian and f locally of finite type we see that A is Noetherian and $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Properties, Lemma 28.5.2 and Morphisms, Lemma 29.15.2. In particular, $\Omega_{B/A}$ is a finite B-module. Hence we can find a single $g \in B$, $g \notin \mathfrak{q}$ such that the principal localization $(\Omega_{B/A})_g$ is zero. Hence after replacing B by B_g we see that $\Omega_{B/A} = 0$ (formation of modules of differentials commutes with localization, see Algebra, Lemma 10.131.8). This means that $\mathrm{d}(f_j)$ generate the kernel of the canonical map $\Omega_{A[x_1,\ldots,x_n]/A} \otimes_A B \to \Omega_{B/A}$. Thus the surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras gives the commutative diagram of (3), and the theorem is proved. $\square$

How can we use this theorem? Well, here are a few remarks:

(1) Suppose that $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type between locally Noetherian schemes. There is a canonical short exact sequence

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

see Morphisms, Lemma 29.32.9. The theorem therefore implies that if $g \circ f$ is unramified, then so is f. This is Morphisms, Lemma 29.35.16.

(2) Since $\Omega_{X/Y}$ is isomorphic to the conormal sheaf of the diagonal morphism (Morphisms, Lemma 29.32.7) we see that if $X \to Y$ is a monomorphism of locally Noetherian schemes and locally of finite type, then $X \to Y$ is unramified. In particular, open and closed immersions of locally Noetherian schemes are unramified. See Morphisms, Lemmas 29.35.7 and 29.35.8.
(3) The theorem also implies that the set of points where a morphism $f : X \to Y$ (locally of finite type of locally Noetherian schemes) is not unramified is the support of the coherent sheaf $\Omega_{X/Y}$. This allows one to give a scheme theoretic definition to the "ramification locus".

41.5. The functorial characterization of unramified morphisms

024Q In basic algebraic geometry we learn that some classes of morphisms can be characterized functorially, and that such descriptions are quite useful. Unramified morphisms too have such a characterization.

024R **Theorem 41.5.1.** Let $f : X \to S$ be a morphism of schemes. Assume S is a locally Noetherian scheme, and f is locally of finite type. Then the following are equivalent:

(1) f is unramified,
(2) the morphism f is formally unramified: for any affine S-scheme T and subscheme T_0 of T defined by a square-zero ideal, the natural map
$$\mathrm{Hom}_S(T, X) \longrightarrow \mathrm{Hom}_S(T_0, X)$$
is injective.

Proof. See More on Morphisms, Lemma 37.6.8 for a more general statement and proof. What follows is a sketch of the proof in the current case.

Firstly, one checks both properties are local on the source and the target. This we may assume that S and X are affine. Say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Say $T = \mathrm{Spec}(C)$. Let J be the square-zero ideal of C with $T_0 = \mathrm{Spec}(C/J)$. Assume that we are given the diagram

$$\begin{tikzcd} & B \arrow[d, "\phi"] \arrow[dr, "\bar{\phi}"] & \\ R \arrow[ur] \arrow[r] & C \arrow[r] & C/J \end{tikzcd}$$

Secondly, one checks that the association $\phi' \mapsto \phi' - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2) via the description of unramified morphisms having trivial module of differentials, see Theorem 41.4.1.

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega_{B/R}$. Thus, B/R is unramified. $\square$

41.6. Topological properties of unramified morphisms

024S The first topological result that will be of utility to us is one which says that unramified and separated morphisms have "nice" sections. The material in this section does not require any Noetherian hypotheses.

024T **Proposition 41.6.1.** Sections of unramified morphisms.

(1) Any section of an unramified morphism is an open immersion.
(2) Any section of a separated morphism is a closed immersion.
(3) Any section of an unramified separated morphism is open and closed.

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by Schemes, Definition 26.21.3 (resp. Morphisms, Lemma 29.35.13). Hence so is the graph as a base change (by Schemes, Lemma 26.18.2). In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). $\square$

We can now explicitly describe the sections of unramified morphisms.

024U **Theorem 41.6.2.** Let Y be a connected scheme. Let $f : X \to Y$ be unramified and separated. Every section of f is an isomorphism onto a connected component. There exists a bijective correspondence

$$\text{sections of } f \leftrightarrow \left\{ \begin{matrix} \text{connected components } X' \text{ of } X \text{ such that} \\ \text{the induced map } X' \to Y \text{ is an isomorphism} \end{matrix} \right\}$$

In particular, given $x \in X$ there is at most one section passing through x.

Proof. Direct from Proposition 41.6.1 part (3). $\square$

The preceding theorem gives us some idea of the "rigidity" of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also useful in the theory of the algebraic fundamental group (see [Gro71, Exposé V]). See also the more general Morphisms, Lemma 29.35.17.

024V **Proposition 41.6.3.** Let S is be a scheme. Let $\pi : X \to S$ be unramified and separated. Let Y be an S-scheme and $y \in Y$ a point. Let $f, g : Y \to X$ be two S-morphisms. Assume

(1) Y is connected
(2) $x = f(y) = g(y)$, and
(3) the induced maps $f^\sharp, g^\sharp : \kappa(x) \to \kappa(y)$ on residue fields are equal.

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated also (see Morphisms, Lemmas 29.35.5 and Schemes, Lemma 26.21.12). Thus according to Theorem 41.6.2 it suffices to prove that f' and g' pass through the same point of X_Y. And this is exactly what the hypotheses (2) and (3) guarantee, namely $f'(y) = g'(y) \in X_Y$. $\square$

0AKI Lemma 41.6.4. Let S be a Noetherian scheme. Let $X \to S$ be a quasi-compact unramified morphism. Let $Y \to S$ be a morphism with Y Noetherian. Then $\mathrm{Mor}_S(Y, X)$ is a finite set.

Proof. Assume first $X \to S$ is separated (which is often the case in practice). Since Y is Noetherian it has finitely many connected components. Thus we may assume Y is connected. Choose a point $y \in Y$ with image $s \in S$. Since $X \to S$ is unramified and quasi-compact then fibre X_s is finite, say $X_s = \{x_1, \ldots, x_n\}$ and $\kappa(x_i)/\kappa(s)$ is a finite field extension. See Morphisms, Lemma 29.35.10, 29.20.5, and 29.20.10. For each i there are at most finitely many $\kappa(s)$-algebra maps $\kappa(x_i) \to \kappa(y)$ (by elementary field theory). Thus $\mathrm{Mor}_S(Y, X)$ is finite by Proposition 41.6.3.

General case. There exists a nonempty open $U \subset S$ such that $X_U \to U$ is finite (in particular separated), see Morphisms, Lemma 29.51.1 (the lemma applies since we've already seen above that a quasi-compact unramified morphism is quasi-finite and since $X \to S$ is quasi-separated by Morphisms, Lemma 29.15.7). Let $Z \subset S$ be the reduced closed subscheme supported on the complement of U. By Noetherian induction, we see that $\mathrm{Mor}_Z(Y_Z, X_Z)$ is finite (details omitted). By the result of the first paragraph the set $\mathrm{Mor}_U(Y_U, X_U)$ is finite. Thus it suffices to show that

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_Z(Y_Z, X_Z) \times \mathrm{Mor}_U(Y_U, X_U)$$

is injective. This follows from the fact that the set of points where two morphisms $a, b : Y \to X$ agree is open in Y, due to the fact that $\Delta : X \to X \times_S X$ is open, see Morphisms, Lemma 29.35.13. □

41.7. Universally injective, unramified morphisms

06ND Recall that a morphism of schemes $f : X \to Y$ is universally injective if any base change of f is injective (on underlying topological spaces), see Morphisms, Definition 29.10.1. Universally injective and unramified morphisms can be characterized as follows.

05VH Lemma 41.7.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) f is unramified and a monomorphism,
(2) f is unramified and universally injective,
(3) f is locally of finite type and a monomorphism,
(4) f is universally injective, locally of finite type, and formally unramified,
(5) f is locally of finite type and X_s is either empty or $X_s \to s$ is an isomorphism for all $s \in S$.

Proof. We have seen in More on Morphisms, Lemma 37.6.8 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly universally injective and formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_S X$ is both an open immersion (Morphisms, Lemma 29.35.13) and surjective (Morphisms, Lemma 29.10.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1).

Condition (3) implies (5) because monomorphisms are preserved under base change (Schemes, Lemma 26.23.5) and because of the description of monomorphisms towards the spectra of fields in Schemes, Lemma 26.23.11. Condition (5) implies (4) by Morphisms, Lemmas 29.10.2 and 29.35.12. □

This leads to the following useful characterization of closed immersions.

04XV Lemma 41.7.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) f is a closed immersion,
(2) f is a proper monomorphism,
(3) f is proper, unramified, and universally injective,
(4) f is universally closed, unramified, and a monomorphism,
(5) f is universally closed, unramified, and universally injective,
(6) f is universally closed, locally of finite type, and a monomorphism,
(7) f is universally closed, universally injective, locally of finite type, and formally unramified.

Proof. The equivalence of (4) – (7) follows immediately from Lemma 41.7.1.

Let $f : X \to S$ satisfy (6). Then f is separated, see Schemes, Lemma 26.23.3 and has finite fibres. Hence More on Morphisms, Lemma 37.44.1 shows f is finite. Then Morphisms, Lemma 29.44.15 implies f is a closed immersion, i.e., (1) holds.

Note that (1) $\Rightarrow$ (2) because a closed immersion is proper and a monomorphism (Morphisms, Lemma 29.41.6 and Schemes, Lemma 26.23.8). By Lemma 41.7.1 we see that (2) implies (3). It is clear that (3) implies (5). $\square$

Here is another result of a similar flavor.

04DG Lemma 41.7.3. Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that

(1) π is finite,
(2) π is unramified,
(3) $\pi^{-1}(\{s\}) = \{x\}$, and
(4) $\kappa(s) \subset \kappa(x)$ is purely inseparable[1].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion.

Proof. The question is local on S. Hence we may assume that $S = \text{Spec}(A)$. By definition of a finite morphism this implies $X = \text{Spec}(B)$. Note that the ring map $\varphi : A \to B$ defining π is a finite unramified ring map. Let $\mathfrak{p} \subset A$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime corresponding to x. Conditions (2), (3) and (4) imply that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \kappa(\mathfrak{p})$. By Algebra, Lemma 10.41.11 we have $B_\mathfrak{q} = B_\mathfrak{p}$ (note that a finite ring map satisfies going up, see Algebra, Section 10.41.) Hence we see that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = \kappa(\mathfrak{p})$. As B is a finite A-module we see from Nakayama's lemma (see Algebra, Lemma 10.20.1) that $B_\mathfrak{p} = \varphi(A_\mathfrak{p})$. Hence (using the finiteness of B as an A-module again) there exists a $f \in A$, $f \notin \mathfrak{p}$ such that $B_f = \varphi(A_f)$ as desired. $\square$

The topological results presented above will be used to give a functorial characterization of étale morphisms similar to Theorem 41.5.1.

[1]In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

41.8. Examples of unramified morphisms

024W Here are a few examples.

024X Example 41.8.1. Let k be a field. Unramified quasi-compact morphisms $X \to \mathrm{Spec}(k)$ are affine. This is true because X has dimension 0 and is Noetherian, hence is a finite discrete set, and each point gives an affine open, so X is a finite disjoint union of affines hence affine. Noether normalization forces X to be the spectrum of a finite k-algebra A. This algebra is a product of finite separable field extensions of k. Thus, an unramified quasi-compact morphism to $\mathrm{Spec}(k)$ corresponds to a finite number of finite separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extension. As we will see later, $X \to \mathrm{Spec}(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

024Y Example 41.8.2. Property (3) in Theorem 41.4.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Let $I = (g_1, \ldots, g_m)$ be an ideal in $R[x_1, \ldots, x_n]$. Let $\mathfrak q \subset R[x_1, \ldots, x_n]$ be a prime. Assume $I \subset \mathfrak q$ and that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) \bmod \mathfrak q \quad \in \quad \mathrm{Mat}(n \times m, \kappa(\mathfrak q))$$

has rank n. Then the morphism $f : Z = \mathrm{Spec}(R[x_1, \ldots, x_n]/I) \to \mathrm{Spec}(R)$ is unramified at the point $x \in Z \subset \mathbf{A}^n_R$ corresponding to $\mathfrak q$. Clearly we must have $m \geq n$. In the extreme case $m = n$, i.e., the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$ defined by the g_i's is an isomorphism of the tangent spaces, then f is also flat x and, hence, is an étale map (see Algebra, Definition 10.137.6, Lemma 10.137.7 and Example 10.137.8).

024Z Example 41.8.3. Fix an extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $\mathrm{Spec}(\mathcal{O}_L)$ can be defined by vanishing set of the different; this is an ideal in $\mathcal{O}_L$. In fact, the different is nothing but the annihilator of the module $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Similarly, the discriminant is an ideal in $\mathcal{O}_K$, namely it is the norm of the different. The vanishing set of the discriminant is precisely the set of points of K which ramify in L. Thus, denoting by X the complement of the closed subset defined by the different in $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a morphism $X \to \mathrm{Spec}(\mathcal{O}_K)$ which is unramified. Furthermore, this morphism is also flat, as any local homomorphism of discrete valuation rings is flat, and hence this morphism is actually étale. If L/K is finite Galois, then denoting by Y the complement of the closed subset defined by the discriminant in $\mathrm{Spec}(\mathcal{O}_K)$, we see that we get even a finite étale morphism $X \to Y$. Thus, this is an example of a finite étale covering.

41.9. Flat morphisms

0250 This section simply exists to summarize the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for "hyperplane sections" of certain modules to be flat.

0251 Definition 41.9.1. Flatness of modules and rings.

(1) A module N over a ring A is said to be flat if the functor $M \mapsto M \otimes_A N$ is exact.
(2) If this functor is also faithful, we say that N is faithfully flat over A.
(3) A morphism of rings $f : A \to B$ is said to be flat (resp. faithfully flat) if the functor $M \mapsto M \otimes_A B$ is exact (resp. faithful and exact).

Here is a list of facts with references to the algebra chapter.

(1) Free and projective modules are flat. This is clear for free modules and follows for projective modules as they are direct summands of free modules and $\otimes$ commutes with direct sums.
(2) Flatness is a local property, that is, M is flat over A if and only if $M_{\mathfrak p}$ is flat over $A_{\mathfrak p}$ for all $\mathfrak p \in \text{Spec}(A)$. See Algebra, Lemma 10.39.18.
(3) If M is a flat A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a flat B-module. See Algebra, Lemma 10.39.7.
(4) Finite flat modules over local rings are free. See Algebra, Lemma 10.78.5.
(5) If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}(\mathfrak q)} \to B_{\mathfrak q}$ are flat for all $\mathfrak q \in \text{Spec}(B)$. See Algebra, Lemma 10.39.18
(6) If $f : A \to B$ is a local homomorphism of local rings, f is flat if and only if it is faithfully flat. See Algebra, Lemma 10.39.17.
(7) A map $A \to B$ of rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. See Algebra, Lemma 10.39.16.
(8) If A is a Noetherian local ring, the completion $A^\wedge$ is faithfully flat over A. See Algebra, Lemma 10.97.3.
(9) Let A be a Noetherian local ring and M an A-module. Then M is flat over A if and only if $M \otimes_A A^\wedge$ is flat over $A^\wedge$. (Combine the previous statement with Algebra, Lemma 10.39.8.)

Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules.

0252 Theorem 41.9.2. Let A, B be Noetherian local rings. Let $f : A \to B$ be a local homomorphism. If M is a finite B-module that is flat as an A-module, and $t \in \mathfrak m_B$ is an element such that multiplication by t is injective on $M/\mathfrak m_A M$, then M/tM is also A-flat.

Proof. See Algebra, Lemma 10.99.1. See also [Mat70a, Section 20]. □

0253 Definition 41.9.3. (See Morphisms, Definition 29.25.1). Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal F$ be a quasi-coherent $\mathcal O_X$-module.

(1) Let $x \in X$. We say $\mathcal F$ is flat over Y at $x \in X$ if $\mathcal F_x$ is a flat $\mathcal O_{Y,f(x)}$-module. This uses the map $\mathcal O_{Y,f(x)} \to \mathcal O_{X,x}$ to think of $\mathcal F_x$ as a $\mathcal O_{Y,f(x)}$-module.
(2) Let $x \in X$. We say f is flat at $x \in X$ if $\mathcal O_{Y,f(x)} \to \mathcal O_{X,x}$ is flat.
(3) We say f is flat if it is flat at all points of X.
(4) A morphism $f : X \to Y$ that is flat and surjective is sometimes said to be faithfully flat.

Once again, here is a list of results:

(1) The property (of a morphism) of being flat is, by fiat, local in the Zariski topology on the source and the target.
(2) Open immersions are flat. (This is clear because it induces isomorphisms on local rings.)
(3) Flat morphisms are stable under base change and composition. Morphisms, Lemmas 29.25.8 and 29.25.6.
(4) If $f : X \to Y$ is flat, then the pullback functor $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ is exact. This is immediate by looking at stalks.
(5) Let $f : X \to Y$ be a morphism of schemes, and assume Y is quasi-compact and quasi-separated. In this case if the functor f^* is exact then f is flat. (Proof omitted. Hint: Use Properties, Lemma 28.22.1 to see that Y has "enough" ideal sheaves and use the characterization of flatness in Algebra, Lemma 10.39.5.)

41.10. Topological properties of flat morphisms

0254 We "recall" below some openness properties that flat morphisms enjoy.

0255 Theorem 41.10.1. Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The set of points in X where $\mathcal{F}$ is flat over Y is an open set. In particular the set of points where f is flat is open in X.

Proof. See More on Morphisms, Theorem 37.15.1. □

039K Theorem 41.10.2. Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is flat and locally of finite type. Then f is (universally) open.

Proof. See Morphisms, Lemma 29.25.10. □

0256 Theorem 41.10.3. A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.

Proof. See Morphisms, Lemma 29.25.12. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrized by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrized by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some "continuity" to the fibres.

41.11. Étale morphisms

0257 In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterization presented in Theorem 41.16.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the basic tenet of étale cohomology – étale morphisms are the algebraic analogue of local isomorphisms.

As the title suggests, we will define the class of étale morphisms – the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site $S_{étale}$. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace "local isomorphism" with "formal local isomorphism" (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

We first define "étale homomorphisms of local rings" for Noetherian local rings. We cannot use the term "étale", as there already is a notion of an étale ring map (Algebra, Section 10.143) and it is different.

0258 Definition 41.11.1. Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be an étale homomorphism of local rings if it is flat and an unramified homomorphism of local rings (please see Definition 41.3.1).

This is the local version of the definition of an étale ring map in Algebra, Section 10.143. The exact definition given in that section is that it is a smooth ring map of relative dimension 0. It is shown (in Algebra, Lemma 10.143.2) that an étale R-algebra S always has a presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

such that

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in S. The following two lemmas link the two notions.

039L Lemma 41.11.2. Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is étale at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an étale homomorphism of local rings.

Proof. See Algebra, Lemmas 10.143.3 (flatness of étale maps), 10.143.5 (étale maps are unramified) and 10.143.7 (flat and unramified maps are étale). $\square$

039M Lemma 41.11.3. Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent

(1) $A \to B$ is an étale homomorphism of local rings
(2) $A^\wedge \to B^\wedge$ is an étale homomorphism of local rings, and
(3) $A^\wedge \to B^\wedge$ is étale.

Moreover, in this case $B^\wedge \cong (A^\wedge)^{\oplus n}$ as $A^\wedge$-modules for some $n \geq 1$.

Proof. To see the equivalences of (1), (2) and (3), as we have the corresponding results for unramified ring maps (Lemma 41.3.4) it suffices to prove that $A \to B$ is flat if and only if $A^\wedge \to B^\wedge$ is flat. This is clear from our lists of properties of flat maps since the ring maps $A \to A^\wedge$ and $B \to B^\wedge$ are faithfully flat. For the final statement, by Lemma 41.3.3 we see that $B^\wedge$ is a finite flat $A^\wedge$ module. Hence it is finite free by our list of properties on flat modules in Section 41.9. □

The integer n which occurs in the lemma above is nothing other than the degree $[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$ of the residue field extension. In particular, if $\kappa(\mathfrak{m}_A)$ is separably closed, we see that $A^\wedge \to B^\wedge$ is an isomorphism, which vindicates our earlier claims.

0259 Definition 41.11.4. (See Morphisms, Definition 29.36.1.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type.

(1) Let $x \in X$. We say f is étale at $x \in X$ if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an étale homomorphism of local rings.
(2) The morphism is said to be étale if it is étale at all its points.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is étale is open.

039N Lemma 41.11.5. Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is étale at x in the sense of Definition 41.11.4 if and only if it is étale at x in the sense of Morphisms, Definition 29.36.1.

Proof. This follows from Lemma 41.11.2 and the definitions. □

Here are some results on étale morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) An étale morphism is unramified. (Clear from our definitions.)
(2) Étaleness is local on the source and the target in the Zariski topology.
(3) Étale morphisms are stable under base change and composition. See Morphisms, Lemmas 29.36.4 and 29.36.3.
(4) Étale morphisms of schemes are locally quasi-finite and quasi-compact étale morphisms are quasi-finite. (This is true because it holds for unramified morphisms as seen earlier.)
(5) Étale morphisms have relative dimension 0. See Morphisms, Definition 29.29.1 and Morphisms, Lemma 29.29.5.
(6) A morphism is étale if and only if it is flat and all its fibres are étale. See Morphisms, Lemma 29.36.8.
(7) Étale morphisms are open. This is true because an étale morphism is flat, and Theorem 41.10.2.
(8) Let X and Y be étale over a base scheme S. Any S-morphism from X to Y is étale. See Morphisms, Lemma 29.36.18.

41.12. The structure theorem

025A

We present a theorem which describes the local structure of étale and unramified morphisms. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

To state it we need the notion of a standard étale ring map, see Algebra, Definition 10.144.1. Namely, suppose that R is a ring and $f, g \in R[t]$ are polynomials such that

(a) f is a monic polynomial, and
(b) $f' = \mathrm{d}f/\mathrm{d}t$ is invertible in the localization $R[t]_g/(f)$.

Then the map

$$R \longrightarrow R[t]_g/(f) = R[t, 1/g]/(f)$$

is a standard étale algebra, and any standard étale algebra is isomorphic to one of these. It is a pleasant exercise to prove that such a ring map is flat, and unramified and hence étale (as expected of course). A special case of a standard étale ring map is any ring map

$$R \longrightarrow R[t]_{f'}/(f) = R[t, 1/f']/(f)$$

with f a monic polynomial, and any standard étale algebra is (isomorphic to) a principal localization of one of these.

025B Theorem 41.12.1. Let $f : A \to B$ be an étale homomorphism of local rings. Then there exist $f, g \in A[t]$ such that

(1) $B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and
(2) B is isomorphic to a localization of B' at a prime.

Proof. Write $B = B'_{\mathfrak q}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 41.11.2 we see that $A \to B'$ is étale at $\mathfrak q$. Hence we may apply Algebra, Proposition 10.144.4 to see that a principal localization of B' is standard étale. □

Here is the version for unramified homomorphisms of local rings.

039O Theorem 41.12.2. Let $f : A \to B$ be an unramified morphism of local rings. Then there exist $f, g \in A[t]$ such that

(1) $B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and
(2) B is isomorphic to a quotient of a localization of B' at a prime.

Proof. Write $B = B'_{\mathfrak q}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 41.3.2 we see that $A \to B'$ is unramified at $\mathfrak q$. Hence we may apply Algebra, Proposition 10.152.1 to see that a principal localization of B' is a quotient of a standard étale A-algebra. □

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

025C Theorem 41.12.3. Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. Let $V \subset Y$ be an affine open neighbourhood of $\varphi(x)$. If φ is étale at x, then there exist exists an affine open $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that we have the following diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "j"'] \arrow[d] & \mathrm{Spec}(R[t]_{f'}/(f)) \arrow[d] \\ Y & V \arrow[l] \arrow[r, equal] & \mathrm{Spec}(R) \end{tikzcd}$$

where j is an open immersion, and $f \in R[t]$ is monic.

Proof. This is equivalent to Morphisms, Lemma 29.36.14 although the statements differ slightly. See also, Varieties, Lemma 33.18.3 for a variant for unramified morphisms. $\square$

41.13. Étale and smooth morphisms

039P An étale morphism is smooth of relative dimension zero. The projection $\mathbf{A}^n_S \to S$ is a standard example of a smooth morphism of relative dimension n. It turns out that any smooth morphism is étale locally of this form. Here is the precise statement.

039Q **Theorem 41.13.1.** Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist an integer $n \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & \mathbf{A}^n_R \arrow[r, equal] \arrow[d] & \mathrm{Spec}(R[x_1, \ldots, x_n]) \arrow[dl] \\ Y & V \arrow[l] \arrow[r, equal] & \mathrm{Spec}(R) & \end{tikzcd}$$

where π is étale.

Proof. See Morphisms, Lemma 29.36.20. $\square$

41.14. Topological properties of étale morphisms

025F We present a few of the topological properties of étale and unramified morphisms. First, we give what Grothendieck calls the fundamental property of étale morphisms, see [Gro71, Exposé I.5].

025G **Theorem 41.14.1.** Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

(1) f is an open immersion,
(2) f is universally injective and étale, and
(3) f is a flat monomorphism, locally of finite presentation.

Proof. An open immersion is universally injective since any base change of an open immersion is an open immersion. Moreover, it is étale by Morphisms, Lemma 29.36.9. Hence (1) implies (2).

Assume f is universally injective and étale. Since f is étale it is flat and locally of finite presentation, see Morphisms, Lemmas 29.36.12 and 29.36.11. By Lemma 41.7.1 we see that f is a monomorphism. Hence (2) implies (3).

Assume f is flat, locally of finite presentation, and a monomorphism. Then f is open, see Morphisms, Lemma 29.25.10. Thus we may replace Y by $f(X)$ and we may assume f is surjective. Then f is open and bijective hence a homeomorphism.

Hence f is quasi-compact. Hence Descent, Lemma 35.25.1 shows that f is an isomorphism and we win. □

Here is another result of a similar flavor.

04DH Lemma 41.14.2. Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that

(1) π is finite,
(2) π is étale,
(3) $\pi^{-1}(\{s\}) = \{x\}$, and
(4) $\kappa(s) \subset \kappa(x)$ is purely inseparable[2].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 41.7.3 there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion. But a morphism which is étale and a closed immersion is an open immersion (for example by Theorem 41.14.1). Hence after shrinking U we obtain an isomorphism. □

0EBS Lemma 41.14.3. Let $U \to X$ be an étale morphism of schemes where X is a scheme in characteristic p. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 33.36.6. The morphism $F_{U/X}$ is étale as a morphism between schemes étale over X (Morphisms, Lemma 29.36.18). Hence $F_{U/X}$ is an isomorphism by Theorem 41.14.1. □

41.15. Topological invariance of the étale topology

06NE Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

025H Theorem 41.15.1. Let X and Y be two schemes over a base scheme S. Let S_0 be a closed subscheme of S with the same underlying topological space (for example if the ideal sheaf of S_0 in S has square zero). Denote X_0 (resp. Y_0) the base change $S_0 \times_S X$ (resp. $S_0 \times_S Y$). If X is étale over S, then the map

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_{S_0}(Y_0, X_0)$$

is bijective.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. In this case the theorem states that any S-morphism $\sigma_0 : S_0 \to X$ actually factors uniquely through a section $S \to X$ of the étale structure morphism $f : X \to S$.

Uniqueness. Suppose we have two sections σ, σ' through which σ_0 factors. Because $X \to S$ is étale we see that $\Delta : X \to X \times_S X$ is an open immersion (Morphisms, Lemma 29.35.13). The morphism $(\sigma, \sigma') : S \to X \times_S X$ factors through this open because for any $s \in S$ we have $(\sigma, \sigma')(s) = (\sigma_0(s), \sigma_0(s))$. Thus $\sigma = \sigma'$.

To prove existence we first reduce to the affine case (we suggest the reader skip this step). Let $X = \bigcup X_i$ be an affine open covering such that each X_i maps into an affine open S_i of S. For every $s \in S$ we can choose an i such that $\sigma_0(s) \in X_i$.

[2]In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Choose an affine open neighbourhood $U \subset S_i$ of s such that $\sigma_0(U_0) \subset X_{i,0}$. Note that $X' = X_i \times_S U = X_i \times_{S_i} U$ is affine. If we can lift $\sigma_0|_{U_0} : U_0 \to X'_0$ to $U \to X'$, then by uniqueness these local lifts will glue to a global morphism $S \to X$. Thus we may assume S and X are affine.

Existence when S and X are affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Then $A \to B$ is étale and in particular smooth (of relative dimension 0). As $|S_0| = |S|$ we see that $S_0 = \mathrm{Spec}(A/I)$ with $I \subset A$ locally nilpotent. Thus existence follows from Algebra, Lemma 10.138.17. □

From the proof of preceeding theorem, we also obtain one direction of the promised functorial characterization of étale morphisms. The following theorem will be strengthened in Étale Cohomology, Theorem 59.45.2.

039R Theorem 41.15.2 (Une equivalence remarquable de catégories). Let S be a scheme. Let $S_0 \subset S$ be a closed subscheme with the same underlying topological space (for example if the ideal sheaf of S_0 in S has square zero). The functor [DG67, IV, Theorem 18.1.2]

$$X \longmapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{\text{schemes } X \text{ étale over } S\} \leftrightarrow \{\text{schemes } X_0 \text{ étale over } S_0\}$$

Proof. By Theorem 41.15.1 we see that this functor is fully faithful. It remains to show that the functor is essentially surjective. Let $Y \to S_0$ be an étale morphism of schemes.

Suppose that the result holds if S and Y are affine. In that case, we choose an affine open covering $Y = \bigcup V_j$ such that each V_j maps into an affine open of S. By assumption (affine case) we can find étale morphisms $W_j \to S$ such that $W_{j,0} \cong V_j$ (as schemes over S_0). Let $W_{j,j'} \subset W_j$ be the open subscheme whose underlying topological space corresponds to $V_j \cap V_{j'}$. Because we have isomorphisms

$$W_{j,j',0} \cong V_j \cap V_{j'} \cong W_{j',j,0}$$

as schemes over S_0 we see by fully faithfulness that we obtain isomorphisms $\theta_{j,j'} : W_{j,j'} \to W_{j',j}$ of schemes over S. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 26.14. Applying Schemes, Lemma 26.14.2 we obtain a scheme $X \to S$ by glueing the schemes W_j along the identifications $\theta_{j,j'}$. It is clear that $X \to S$ is étale and $X_0 \cong Y$ by construction.

Thus it suffices to show the lemma in case S and Y are affine. Say $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$ with I locally nilpotent. By Algebra, Lemma 10.143.2 we know that Y is the spectrum of a ring $\overline{A}$ with

$$\overline{A} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$$

such that

$$\overline{g} = \det \begin{pmatrix} \partial\overline{f}_1/\partial x_1 & \partial\overline{f}_2/\partial x_1 & \ldots & \partial\overline{f}_n/\partial x_1 \\ \partial\overline{f}_1/\partial x_2 & \partial\overline{f}_2/\partial x_2 & \ldots & \partial\overline{f}_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\overline{f}_1/\partial x_n & \partial\overline{f}_2/\partial x_n & \ldots & \partial\overline{f}_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in $\overline{A}$. Choose any lifts $f_i \in R[x_1, \ldots, x_n]$. Set

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Since I is locally nilpotent the ideal IA is locally nilpotent (Algebra, Lemma 10.32.3). Observe that $\overline{A} = A/IA$. It follows that the determinant of the matrix of partials of the f_i is invertible in the algebra A by Algebra, Lemma 10.32.4. Hence $R \to A$ is étale and the proof is complete. □

41.16. The functorial characterization

025J We finally present the promised functorial characterization. Thus there are four ways to think about étale morphisms of schemes:

(1) as a smooth morphism of relative dimension 0,
(2) as locally finitely presented, flat, and unramified morphisms,
(3) using the structure theorem, and
(4) using the functorial characterization.

025K Theorem 41.16.1. Let $f : X \to S$ be a morphism that is locally of finite presentation. The following are equivalent

(1) f is étale,
(2) for all affine S-schemes Y, and closed subschemes $Y_0 \subset Y$ defined by square-zero ideals, the natural map
$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_S(Y_0, X)$$
is bijective.

Proof. This is More on Morphisms, Lemma 37.8.9. □

This characterization says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

41.17. Étale local structure of unramified morphisms

04HG In the chapter More on Morphisms, Section 37.41 the reader can find some results on the étale local structure of quasi-finite morphisms. In this section we want to combine this with the topological properties of unramified morphisms we have seen in this chapter. The basic overall picture to keep in mind is

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & U \arrow[r] & S \end{tikzcd}$$

see More on Morphisms, Equation (37.41.0.1). We start with a very general case.

04HH Lemma 41.17.1. Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that

(1) $V_{i,j} \to U$ is a closed immersion passing through u,
(2) u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and
(3) any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.

Proof. By Morphisms, Definition 29.35.1 there exists an open neighbourhood of each x_i which is locally of finite type over S. Replacing X by an open neighbourhood of $\{x_1, \ldots, x_n\}$ we may assume f is locally of finite type. Apply More on Morphisms, Lemma 37.41.3 to get the étale neighbourhood (U, u) and the opens $V_{i,j}$ finite over U. By Lemma 41.7.3 after possibly shrinking U we get that $V_{i,j} \to U$ is a closed immersion. □

04HI Lemma 41.17.2. Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

(1) $V_{i,j} \to U$ is a closed immersion passing through u,
(2) the fibre W_u contains no point mapping to any x_i.

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. Apply Lemma 41.17.1. We may assume U is affine, so X_U is separated. Then $V_{i,j} \to X_U$ is a closed map, see Morphisms, Lemma 29.41.7. Suppose $(i, j) \neq (i', j')$. Then $V_{i,j} \cap V_{i',j'}$ is closed in $V_{i,j}$ and its image in U does not contain u. Hence after shrinking U we may assume that $V_{i,j} \cap V_{i',j'} = \emptyset$. Moreover, $\bigcup V_{i,j}$ is a closed and open subscheme of X_U and hence has an open and closed complement W. This finishes the proof. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite unramified morphism is étale locally on the base a closed immersion.

04HJ Lemma 41.17.3. Let $f : X \to S$ be a finite unramified morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is a closed immersion.

Proof. Since $X \to S$ is finite the fibre over s is a finite set $\{x_1, \ldots, x_n\}$ of points of X. Apply Lemma 41.17.2 to this set (a finite morphism is separated, see Morphisms, Section 29.44). The image of W in U is a closed subset (as $X_U \to U$ is finite, hence proper) which does not contain u. After removing this from U we see that $W = \emptyset$ as desired. □

41.18. Étale local structure of étale morphisms

04HK This is a bit silly, but perhaps helps form intuition about étale morphisms. We simply copy over the results of Section 41.17 and change "closed immersion" into "isomorphism".

04HL Lemma 41.18.1. Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that

(1) $V_{i,j} \to U$ is an isomorphism,
(2) u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and
(3) any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.

Proof. An étale morphism is unramified, hence we may apply Lemma 41.17.1. Now $V_{i,j} \to U$ is a closed immersion and étale. Hence it is an open immersion, for example by Theorem 41.14.1. Replace U by the intersection of the images of $V_{i,j} \to U$ to get the lemma. □

04HM Lemma 41.18.2. Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

of schemes such that

(1) $V_{i,j} \to U$ is an isomorphism,
(2) the fibre W_u contains no point mapping to any x_i.

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. An étale morphism is unramified, hence we may apply Lemma 41.17.2. As in the proof of Lemma 41.18.1 the morphisms $V_{i,j} \to U$ are open immersions and we win after replacing U by the intersection of their images. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite étale morphism is étale locally on the base a "topological covering space", i.e., a finite product of copies of the base.

04HN Lemma 41.18.3. Let $f : X \to S$ be a finite étale morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition

$$X_U = \coprod\nolimits_j V_j$$

of schemes such that each $V_j \to U$ is an isomorphism.

Proof. An étale morphism is unramified, hence we may apply Lemma 41.17.3. As in the proof of Lemma 41.18.1 we see that $V_{i,j} \to U$ is an open immersion and we win after replacing U by the intersection of their images. □

41.19. Permanence properties

025L In what follows, we present a few "permanence" properties of étale homomorphisms of Noetherian local rings (as defined in Definition 41.11.1). See More on Algebra, Sections 15.43 and 15.45 for the analogue of this material for the completion and henselization of a Noetherian local ring.

039S Lemma 41.19.1. Let A, B be Noetherian local rings. Let $A \to B$ be a étale homomorphism of local rings. Then $\dim(A) = \dim(B)$.

Proof. See for example Algebra, Lemma 10.112.7. □

039T Proposition 41.19.2. Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then $\text{depth}(A) = \text{depth}(B)$

Proof. See Algebra, Lemma 10.163.2. □

025Q Proposition 41.19.3. Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is Cohen-Macaulay if and only if B is so.

Proof. A local ring A is Cohen-Macaulay if and only if $\dim(A) = \text{depth}(A)$. As both of these invariants is preserved under an étale extension, the claim follows. $\square$

025N Proposition 41.19.4. Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is regular if and only if B is so.

Proof. If B is regular, then A is regular by Algebra, Lemma 10.110.9. Assume A is regular. Let $\mathfrak m$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak m)} \mathfrak m/\mathfrak m^2 = \dim(A) = \dim(B)$ (see Lemma 41.19.1). On the other hand, $\mathfrak mB$ is the maximal ideal of B and hence $\mathfrak m_B/\mathfrak m_B = \mathfrak mB/\mathfrak m^2B$ is generated by at most $\dim(B)$ elements. Thus B is regular. (You can also use the slightly more general Algebra, Lemma 10.112.8.) $\square$

025O Proposition 41.19.5. Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is reduced if and only if B is so.

Proof. It is clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. See also Algebra, Lemma 10.164.2. Conversely, assume A is reduced. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak q$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 41.11.2. Then we see that Algebra, Lemma 10.163.7 applies to $A \to B'$ and B' is reduced. Hence B is reduced. $\square$

039U Remark 41.19.6. The result on "reducedness" does not hold with a weaker definition of étale local ring maps $A \to B$ where one drops the assumption that B is essentially of finite type over A. Namely, it can happen that a Noetherian local domain A has nonreduced completion $A^\wedge$, see Examples, Section 110.16. But the ring map $A \to A^\wedge$ is flat, and $\mathfrak m_A A^\wedge$ is the maximal ideal of $A^\wedge$ and of course A and $A^\wedge$ have the same residue fields. This is why it is important to consider this notion only for ring extensions which are essentially of finite type (or essentially of finite presentation if A is not Noetherian).

025P Proposition 41.19.7. Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is a normal domain if and only if B is so. [Gro71, Expose I, Theorem 9.5 part (i)]

Proof. See Algebra, Lemma 10.164.3 for descending normality. Conversely, assume A is normal. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak q$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 41.11.2. Then we see that Algebra, Lemma 10.163.9 applies to $A \to B'$ and we conclude that B' is normal. Hence B is a normal domain. $\square$

The preceeding propositions give some indication as to why we'd like to think of étale maps as "local isomorphisms". Another property that gives an excellent indication that we have the "right" definition is the fact that for **C**-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the **C**-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings – the details are left to the reader.

41.20. Descending étale morphisms

0BTH In order to understand the language used in this section we encourage the reader to take a look at Descent, Section 35.34. Let $f : X \to S$ be a morphism of schemes. Consider the pullback functor

0BTI
$$\text{schemes } U \text{ étale over } S \longrightarrow \begin{matrix} \text{descent data } (V, \varphi) \text{ relative to } X/S \\ \text{with } V \text{ étale over } X \end{matrix} \tag{41.20.0.1}$$

sending U to the canonical descent datum $(X \times_S U, can)$.

0BTJ **Lemma 41.20.1.** If $f : X \to S$ is surjective, then the functor (41.20.0.1) is faithful.

Proof. Let $a, b : U_1 \to U_2$ be two morphisms between schemes étale over S. Assume the base changes of a and b to X agree. We have to show that $a = b$. By Proposition 41.6.3 it suffices to show that a and b agree on points and residue fields. This is clear because for every $u \in U_1$ we can find a point $v \in X \times_S U_1$ mapping to u. □

0BTK **Lemma 41.20.2.** Assume $f : X \to S$ is submersive and any étale base change of f is submersive. Then the functor (41.20.0.1) is fully faithful.

Proof. By Lemma 41.20.1 the functor is faithful. Let $U_1 \to S$ and $U_2 \to S$ be étale morphisms and let $a : X \times_S U_1 \to X \times_S U_2$ be a morphism compatible with canonical descent data. We will prove that a is the base change of a morphism $U_1 \to U_2$.

Let $U_2' \subset U_2$ be an open subscheme. Consider $W = a^{-1}(X \times_S U_2')$. This is an open subscheme of $X \times_S U_1$ which is compatible with the canonical descent datum on $V_1 = X \times_S U_1$. This means that the two inverse images of W by the projections $V_1 \times_{U_1} V_1 \to V_1$ agree. Since $V_1 \to U_1$ is surjective (as the base change of $X \to S$) we conclude that W is the inverse image of some subset $U_1' \subset U_1$. Since W is open, our assumption on f implies that $U_1' \subset U_1$ is open.

Let $U_2 = \bigcup U_{2,i}$ be an affine open covering. By the result of the preceding paragraph we obtain an open covering $U_1 = \bigcup U_{1,i}$ such that $X \times_S U_{1,i} = a^{-1}(X \times_S U_{2,i})$. If we can prove there exists a morphism $U_{1,i} \to U_{2,i}$ whose base change is the morphism $a_i : X \times_S U_{1,i} \to X \times_S U_{2,i}$ then we can glue these morphisms to a morphism $U_1 \to U_2$ (using faithfulness). In this way we reduce to the case that U_2 is affine. In particular $U_2 \to S$ is separated (Schemes, Lemma 26.21.13).

Assume $U_2 \to S$ is separated. Then the graph Γ_a of a is a closed subscheme of

$$V = (X \times_S U_1) \times_X (X \times_S U_2) = X \times_S U_1 \times_S U_2$$

by Schemes, Lemma 26.21.10. On the other hand the graph is open for example because it is a section of an étale morphism (Proposition 41.6.1). Since a is a morphism of descent data, the two inverse images of $\Gamma_a \subset V$ under the projections $V \times_{U_1 \times_S U_2} V \to V$ are the same. Hence arguing as in the second paragraph of the proof we find an open and closed subscheme $\Gamma \subset U_1 \times_S U_2$ whose base change to X gives Γ_a. Then $\Gamma \to U_1$ is an étale morphism whose base change to X is an isomorphism. This means that $\Gamma \to U_1$ is universally bijective, hence an isomorphism by Theorem 41.14.1. Thus Γ is the graph of a morphism $U_1 \to U_2$ and the base change of this morphism is a as desired. □

0BTL **Lemma 41.20.3.** Let $f : X \to S$ be a morphism of schemes. In the following cases the functor (41.20.0.1) is fully faithful:

(1) f is surjective and universally closed (e.g., finite, integral, or proper),
(2) f is surjective and universally open (e.g., locally of finite presentation and flat, smooth, or etale),
(3) f is surjective, quasi-compact, and flat.

Proof. This follows from Lemma 41.20.2. For example a closed surjective map of topological spaces is submersive (Topology, Lemma 5.6.5). Finite, integral, and proper morphisms are universally closed, see Morphisms, Lemmas 29.44.7 and 29.44.11 and Definition 29.41.1. On the other hand an open surjective map of topological spaces is submersive (Topology, Lemma 5.6.4). Flat locally finitely presented, smooth, and étale morphisms are universally open, see Morphisms, Lemmas 29.25.10, 29.34.10, and 29.36.13. The case of surjective, quasi-compact, flat morphisms follows from Morphisms, Lemma 29.25.12. □

0BTM Lemma 41.20.4. Let $f : X \to S$ be a morphism of schemes. Let (V, φ) be a descent datum relative to X/S with $V \to X$ étale. Let $S = \bigcup S_i$ be an open covering. Assume that

(1) the pullback of the descent datum (V, φ) to $X \times_S S_i/S_i$ is effective,
(2) the functor (41.20.0.1) for $X \times_S (S_i \cap S_j) \to (S_i \cap S_j)$ is fully faithful, and
(3) the functor (41.20.0.1) for $X \times_S (S_i \cap S_j \cap S_k) \to (S_i \cap S_j \cap S_k)$ is faithful.

Then (V, φ) is effective.

Proof. (Recall that pullbacks of descent data are defined in Descent, Definition 35.34.7.) Set $X_i = X \times_S S_i$. Denote (V_i, φ_i) the pullback of (V, φ) to X_i/S_i. By assumption (1) we can find an étale morphism $U_i \to S_i$ which comes with an isomorphism $X_i \times_{S_i} U_i \to V_i$ compatible with *can* and φ_i. By assumption (2) we obtain isomorphisms $\psi_{ij} : U_i \times_{S_i} (S_i \cap S_j) \to U_j \times_{S_j} (S_i \cap S_j)$. By assumption (3) these isomorphisms satisfy the cocycle condition so that (U_i, ψ_{ij}) is a descend datum for the Zariski covering $\{S_i \to S\}$. Then Descent, Lemma 35.35.10 (which is essentially just a reformulation of Schemes, Section 26.14) tells us that there exists a morphism of schemes $U \to S$ and isomorphisms $U \times_S S_i \to U_i$ compatible with ψ_{ij}. The isomorphisms $U \times_S S_i \to U_i$ determine corresponding isomorphisms $X_i \times_S U \to V_i$ which glue to a morphism $X \times_S U \to V$ compatible with the canonical descent datum and φ. □

0BTN Lemma 41.20.5. Let (A, I) be a henselian pair. Let $U \to \text{Spec}(A)$ be a quasi-compact, separated, étale morphism such that $U \times_{\text{Spec}(A)} \text{Spec}(A/I) \to \text{Spec}(A/I)$ is finite. Then

$$U = U_{fin} \amalg U_{away}$$

where $U_{fin} \to \text{Spec}(A)$ is finite and U_{away} has no points lying over Z.

Proof. By Zariski's main theorem, the scheme U is quasi-affine. In fact, we can find an open immersion $U \to T$ with T affine and $T \to \text{Spec}(A)$ finite, see More on Morphisms, Lemma 37.43.3. Write $Z = \text{Spec}(A/I)$ and denote $U_Z \to T_Z$ the base change. Since $U_Z \to Z$ is finite, we see that $U_Z \to T_Z$ is closed as well as open. Hence by More on Algebra, Lemma 15.11.6 we obtain a unique decomposition $T = T' \amalg T''$ with $T'_Z = U_Z$. Set $U_{fin} = U \cap T'$ and $U_{away} = U \cap T''$. Since $T'_Z \subset U_Z$ we see that all closed points of T' are in U hence $T' \subset U$, hence $U_{fin} = T'$, hence $U_{fin} \to \text{Spec}(A)$ is finite. We omit the proof of uniqueness of the decomposition. □

0BTP Proposition 41.20.6. Let $f : X \to S$ be a surjective integral morphism. The functor (41.20.0.1) induces an equivalence

$$\begin{matrix}\text{schemes quasi-compact,} \\ \text{separated, étale over } S\end{matrix} \longrightarrow \begin{matrix}\text{descent data } (V, \varphi) \text{ relative to } X/S \text{ with} \\ V \text{ quasi-compact, separated, étale over } X\end{matrix}$$

Proof. By Lemma 41.20.3 the functor (41.20.0.1) is fully faithful and the same remains the case after any base change $S \to S'$. Let (V, φ) be a descent data relative to X/S with $V \to X$ quasi-compact, separated, and étale. We can use Lemma 41.20.4 to see that it suffices to prove the effectivity Zariski locally on S. In particular we may and do assume that S is affine.

If S is affine we can find a directed set Λ and an inverse system $X_\lambda \to S_\lambda$ of finite morphisms of affine schemes of finite type over $\text{Spec}(\mathbf{Z})$ such that $(X \to S) = \lim(X_\lambda \to S_\lambda)$. See Algebra, Lemma 10.127.15. Since limits commute with limits we deduce that $X \times_S X = \lim X_\lambda \times_{S_\lambda} X_\lambda$ and $X \times_S X \times_S X = \lim X_\lambda \times_{S_\lambda} X_\lambda \times_{S_\lambda} X_\lambda$. Observe that $V \to X$ is a morphism of finite presentation. Using Limits, Lemmas 32.10.1 we can find an λ and a descent datum $(V_\lambda, \varphi_\lambda)$ relative to X_λ/S_λ whose pullback to X/S is (V, φ). Of course it is enough to show that $(V_\lambda, \varphi_\lambda)$ is effective. Note that V_λ is quasi-compact by construction. After possibly increasing λ we may assume that $V_\lambda \to X_\lambda$ is separated and étale, see Limits, Lemma 32.8.6 and 32.8.10. Thus we may assume that f is finite surjective and S affine of finite type over $\mathbf{Z}$.

Consider an open $S' \subset S$ such that the pullback (V', φ') of (V, φ) to $X' = X \times_S S'$ is effective. Below we will prove, that $S' \neq S$ implies there is a strictly larger open over which the descent datum is effective. Since S is Noetherian (and hence has a Noetherian underlying topological space) this will finish the proof. Let $\xi \in S$ be a generic point of an irreducible component of the closed subset $Z = S \setminus S'$. If $\xi \in S'' \subset S$ is an open over which the descent datum is effective, then the descent datum is effective over $S' \cup S''$ by the glueing argument of the first paragraph. Thus in the rest of the proof we may replace S by an affine open neighbourhood of ξ.

After a first such replacement we may assume that Z is irreducible with generic point Z. Let us endow Z with the reduced induced closed subscheme structure. After another shrinking we may assume $X_Z = X \times_S Z = f^{-1}(Z) \to Z$ is flat, see Morphisms, Proposition 29.27.1. Let (V_Z, φ_Z) be the pullback of the descent datum to X_Z/Z. By More on Morphisms, Lemma 37.57.1 this descent datum is effective and we obtain an étale morphism $U_Z \to Z$ whose base change is isomorphic to V_Z in a manner compatible with descent data. Of course $U_Z \to Z$ is quasi-compact and separated (Descent, Lemmas 35.23.1 and 35.23.6). Thus after shrinking once more we may assume that $U_Z \to Z$ is finite, see Morphisms, Lemma 29.51.1.

Let $S = \text{Spec}(A)$ and let $I \subset A$ be the prime ideal corresponding to $Z \subset S$. Let (A^h, IA^h) be the henselization of the pair (A, I). Denote $S^h = \text{Spec}(A^h)$ and $Z^h = V(IA^h) \cong Z$. We claim that it suffices to show effectivity after base change to S^h. Namely, $\{S^h \to S, S' \to S\}$ is an fpqc covering ($A \to A^h$ is flat by More on Algebra, Lemma 15.12.2) and by More on Morphisms, Lemma 37.57.1 we have fpqc descent for separated étale morphisms. Namely, if $U^h \to S^h$ and $U' \to S'$ are the objects corresponding to the pullbacks (V^h, φ^h) and (V', φ'), then the required isomorphisms

$$U^h \times_S S^h \to S^h \times_S V^h \quad \text{and} \quad U^h \times_S S' \to S^h \times_S U'$$

are obtained by the fully faithfulness pointed out in the first paragraph. In this way we reduce to the situation described in the next paragraph.

Here $S = \mathrm{Spec}(A)$, $Z = V(I)$, $S' = S \setminus Z$ where (A, I) is a henselian pair, we have $U' \to S'$ corresponding to the descent datum (V', φ') and we have a finite étale morphism $U_Z \to Z$ corresponding to the descent datum (V_Z, φ_Z). We no longer have that A is of finite type over $\mathbf{Z}$; but the rest of the argument will not even use that A is Noetherian. By More on Algebra, Lemma 15.13.2 we can find a finite étale morphism $U_{fin} \to S$ whose restriction to Z is isomorphic to $U_Z \to Z$. Write $X = \mathrm{Spec}(B)$ and $Y = V(IB)$. Since (B, IB) is a henselian pair (More on Algebra, Lemma 15.11.8) and since the restriction $V \to X$ to Y is finite (as base change of $U_Z \to Z$) we see that there is a canonical disjoint union decomposition

$$V = V_{fin} \amalg V_{away}$$

were $V_{fin} \to X$ is finite and where V_{away} has no points lying over Y. See Lemma 41.20.5. Using the uniqueness of this decomposition over $X \times_S X$ we see that φ preserves it and we obtain

$$(V, \varphi) = (V_{fin}, \varphi_{fin}) \amalg (V_{away}, \varphi_{away})$$

in the category of descent data. By More on Algebra, Lemma 15.13.2 there is a unique isomorphism

$$X \times_S U_{fin} \longrightarrow V_{fin}$$

compatible with the given isomorphism $Y \times_Z U_Z \to V \times_X Y$ over Y. By the uniqueness we see that this isomorphism is compatible with descent data, i.e., $(X \times_S U_{fin}, can) \cong (V_{fin}, \varphi_{fin})$. Denote $U'_{fin} = U_{fin} \times_S S'$. By fully faithfulness we obtain a morphism $U'_{fin} \to U'$ which is the inclusion of an open (and closed) subscheme. Then we set $U = U_{fin} \amalg_{U'_{fin}} U'$ (glueing of schemes as in Schemes, Section 26.14). The morphisms $X \times_S U_{fin} \to V$ and $X \times_S U' \to V$ glue to a morphism $X \times_S U \to V$ which is the desired isomorphism. $\square$

41.21. Normal crossings divisors

0CBN Here is the definition.

0BI9 Definition 41.21.1. Let X be a locally Noetherian scheme. A strict normal crossings divisor on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ the local ring $\mathcal{O}_{X,p}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that D is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{X,p}$.

We often encounter effective Cartier divisors E on locally Noetherian schemes X such that there exists a strict normal crossings divisor D with $E \subset D$ set theoretically. In this case we have $E = \sum a_i D_i$ with $a_i \geq 0$ where $D = \bigcup_{i \in I} D_i$ is the decomposition of D into its irreducible components. Observe that $D' = \bigcup_{a_i > 0} D_i$ is a strict normal crossings divisor with $E = D'$ set theoretically. When the above happens we will say that E is supported on a strict normal crossings divisor.

0BIA Lemma 41.21.2. Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Let $D_i \subset D$, $i \in I$ be its irreducible components viewed as reduced closed subschemes of X. The following are equivalent

(1) D is a strict normal crossings divisor, and

(2) D is reduced, each D_i is an effective Cartier divisor, and for $J \subset I$ finite the scheme theoretic intersection $D_J = \bigcap_{j \in J} D_j$ is a regular scheme each of whose irreducible components has codimension $|J|$ in X.

Proof. Assume D is a strict normal crossings divisor. Pick $p \in D$ and choose a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ as in Definition 41.21.1. Since $\mathcal{O}_{X,p}/(x_i)$ is a regular local ring (and in particular a domain) we see that the irreducible components $D_1, \ldots, D_r$ of D passing through p correspond 1-to-1 to the height one primes $(x_1), \ldots, (x_r)$ of $\mathcal{O}_{X,p}$. By Algebra, Lemma 10.106.3 we find that the intersections $D_{i_1} \cap \ldots \cap D_{i_s}$ have codimension s in an open neighbourhood of p and that this intersection has a regular local ring at p. Since this holds for all $p \in D$ we conclude that (2) holds.

Assume (2). Let $p \in D$. Since $\mathcal{O}_{X,p}$ is finite dimensional we see that p can be contained in at most $\dim(\mathcal{O}_{X,p})$ of the components D_i. Say $p \in D_1, \ldots, D_r$ for some $r \geq 1$. Let $x_1, \ldots, x_r \in \mathfrak{m}_p$ be local equations for $D_1, \ldots, D_r$. Then x_1 is a nonzerodivisor in $\mathcal{O}_{X,p}$ and $\mathcal{O}_{X,p}/(x_1) = \mathcal{O}_{D_1,p}$ is regular. Hence $\mathcal{O}_{X,p}$ is regular, see Algebra, Lemma 10.106.7. Since $D_1 \cap \ldots \cap D_r$ is a regular (hence normal) scheme it is a disjoint union of its irreducible components (Properties, Lemma 28.7.6). Let $Z \subset D_1 \cap \ldots \cap D_r$ be the irreducible component containing p. Then $\mathcal{O}_{Z,p} = \mathcal{O}_{X,p}/(x_1, \ldots, x_r)$ is regular of codimension r (note that since we already know that $\mathcal{O}_{X,p}$ is regular and hence Cohen-Macaulay, there is no ambiguity about codimension as the ring is catenary, see Algebra, Lemmas 10.106.3 and 10.104.4). Hence $\dim(\mathcal{O}_{Z,p}) = \dim(\mathcal{O}_{X,p}) - r$. Choose additional $x_{r+1}, \ldots, x_n \in \mathfrak{m}_p$ which map to a minimal system of generators of $\mathfrak{m}_{Z,p}$. Then $\mathfrak{m}_p = (x_1, \ldots, x_n)$ by Nakayama's lemma and we see that D is a normal crossings divisor. $\square$

0CBP Lemma 41.21.3. Let X be a locally Noetherian scheme. Let $D \subset X$ be a strict normal crossings divisor. If $f : Y \to X$ is a smooth morphism of schemes, then the pullback f^*D is a strict normal crossings divisor on Y.

Proof. As f is flat the pullback is defined by Divisors, Lemma 31.13.13 hence the statement makes sense. Let $q \in f^*D$ map to $p \in D$. Choose a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ as in Definition 41.21.1. Since f is smooth the local ring homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ is flat and the fibre ring

$$\mathcal{O}_{Y,q}/\mathfrak{m}_p\mathcal{O}_{Y,q} = \mathcal{O}_{Y_p,q}$$

is a regular local ring (see for example Algebra, Lemma 10.140.3). Pick $y_1, \ldots, y_n \in \mathfrak{m}_q$ which map to a regular system of parameters in $\mathcal{O}_{Y_p,q}$. Then $x_1, \ldots, x_d, y_1, \ldots, y_n$ generate the maximal ideal $\mathfrak{m}_q$. Hence $\mathcal{O}_{Y,q}$ is a regular local ring of dimension $d+n$ by Algebra, Lemma 10.112.7 and $x_1, \ldots, x_d, y_1, \ldots, y_n$ is a regular system of parameters. Since f^*D is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{Y,q}$ we conclude that the lemma is true. $\square$

Here is the definition of a normal crossings divisor.

0BSF Definition 41.21.4. Let X be a locally Noetherian scheme. A normal crossings divisor on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ there exists an étale morphism $U \to X$ with p in the image and $D \times_X U$ a strict normal crossings divisor on U.

For example $D = V(x^2 + y^2)$ is a normal crossings divisor (but not a strict one) on $\text{Spec}(\mathbf{R}[x, y])$ because after pulling back to the étale cover $\text{Spec}(\mathbf{C}[x, y])$ we obtain $(x - iy)(x + iy) = 0$.

0CBQ Lemma 41.21.5. Let X be a locally Noetherian scheme. Let $D \subset X$ be a normal crossings divisor. If $f : Y \to X$ is a smooth morphism of schemes, then the pullback f^*D is a normal crossings divisor on Y.

Proof. As f is flat the pullback is defined by Divisors, Lemma 31.13.13 hence the statement makes sense. Let $q \in f^*D$ map to $p \in D$. Choose an étale morphism $U \to X$ whose image contains p such that $D \times_X U \subset U$ is a strict normal crossings divisor as in Definition 41.21.4. Set $V = Y \times_X U$. Then $V \to Y$ is étale as a base change of $U \to X$ (Morphisms, Lemma 29.36.4) and the pullback $D \times_X V$ is a strict normal crossings divisor on V by Lemma 41.21.3. Thus we have checked the condition of Definition 41.21.4 for $q \in f^*D$ and we conclude. □

0CBR Lemma 41.21.6. Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme. The following are equivalent

(1) D is a normal crossings divisor in X,
(2) D is reduced, the normalization $\nu : D^\nu \to D$ is unramified, and for any $n \geq 1$ the scheme
$$Z_n = D^\nu \times_D \ldots \times_D D^\nu \setminus \{(p_1, \ldots, p_n) \mid p_i = p_j \text{ for some } i \neq j\}$$
is regular, the morphism $Z_n \to X$ is a local complete intersection morphism whose conormal sheaf is locally free of rank n.

Proof. First we explain how to think about condition (2). The diagonal of an unramified morphism is open (Morphisms, Lemma 29.35.13). On the other hand $D^\nu \to D$ is separated, hence the diagonal $D^\nu \to D^\nu \times_D D^\nu$ is closed. Thus Z_n is an open and closed subscheme of $D^\nu \times_D \ldots \times_D D^\nu$. On the other hand, $Z_n \to X$ is unramified as it is the composition
$$Z_n \to D^\nu \times_D \ldots \times_D D^\nu \to \ldots \to D^\nu \times_D D^\nu \to D^\nu \to D \to X$$
and each of the arrows is unramified. Since an unramified morphism is formally unramified (More on Morphisms, Lemma 37.6.8) we have a conormal sheaf $\mathcal{C}_n = \mathcal{C}_{Z_n/X}$ of $Z_n \to X$, see More on Morphisms, Definition 37.7.2.

Formation of normalization commutes with étale localization by More on Morphisms, Lemma 37.19.3. Checking that local rings are regular, or that a morphism is unramified, or that a morphism is a local complete intersection or that a morphism is unramified and has a conormal sheaf which is locally free of a given rank, may be done étale locally (see More on Algebra, Lemma 15.44.3, Descent, Lemma 35.23.28, More on Morphisms, Lemma 37.62.19 and Descent, Lemma 35.7.6).

By the remark of the preceding paragraph and the definition of normal crossings divisor it suffices to prove that a strict normal crossings divisor $D = \bigcup_{i \in I} D_i$ satisfies (2). In this case $D^\nu = \coprod D_i$ and $D^\nu \to D$ is unramified (being unramified is local on the source and $D_i \to D$ is a closed immersion which is unramified). Similarly, $Z_1 = D^\nu \to X$ is a local complete intersection morphism because we may check this locally on the source and each morphism $D_i \to X$ is a regular immersion as it is the inclusion of a Cartier divisor (see Lemma 41.21.2 and More on Morphisms, Lemma 37.62.9). Since an effective Cartier divisor has an invertible conormal sheaf, we conclude that the requirement on the conormal sheaf is satisfied. Similarly, the scheme Z_n for $n \geq 2$ is the disjoint union of the schemes $D_J = \bigcap_{j \in J} D_j$ where $J \subset I$ runs over the subsets of order n. Since $D_J \to X$ is a regular immersion of codimension n (by the definition of strict normal crossings and the fact that we may check this on stalks by Divisors, Lemma 31.20.8) it follows in the same manner that $Z_n \to X$ has the required properties. Some details omitted.

Assume (2). Let $p \in D$. Since $D^\nu \to D$ is unramified, it is finite (by Morphisms, Lemma 29.44.4). Hence $D^\nu \to X$ is finite unramified. By Lemma 41.17.3 and étale localization (permissible by the discussion in the second paragraph and the definition of normal crossings divisors) we reduce to the case where $D^\nu = \coprod_{i \in I} D_i$ with I finite and $D_i \to U$ a closed immersion. After shrinking X if necessary, we may assume $p \in D_i$ for all $i \in I$. The condition that $Z_1 = D^\nu \to X$ is an unramified local complete intersection morphism with conormal sheaf locally free of rank 1 implies that $D_i \subset X$ is an effective Cartier divisor, see More on Morphisms, Lemma 37.62.3 and Divisors, Lemma 31.21.3. To finish the proof we may assume $X = \mathrm{Spec}(A)$ is affine and $D_i = V(f_i)$ with $f_i \in A$ a nonzerodivisor. If $I = \{1, \ldots, r\}$, then $p \in Z_r = V(f_1, \ldots, f_r)$. The same reference as above implies that $(f_1, \ldots, f_r)$ is a Koszul regular ideal in A. Since the conormal sheaf has rank r, we see that $f_1, \ldots, f_r$ is a minimal set of generators of the ideal defining Z_r in $\mathcal{O}_{X,p}$. This implies that $f_1, \ldots, f_r$ is a regular sequence in $\mathcal{O}_{X,p}$ such that $\mathcal{O}_{X,p}/(f_1, \ldots, f_r)$ is regular. Thus we conclude by Algebra, Lemma 10.106.7 that $f_1, \ldots, f_r$ can be extended to a regular system of parameters in $\mathcal{O}_{X,p}$ and this finishes the proof. $\square$

0CBS Lemma 41.21.7. Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme. If X is J-2 or Nagata, then following are equivalent

(1) D is a normal crossings divisor in X,
(2) for every $p \in D$ the pullback of D to the spectrum of the strict henselization $\mathcal{O}_{X,p}^{sh}$ is a strict normal crossings divisor.

Proof. The implication (1) $\Rightarrow$ (2) is straightforward and does not need the assumption that X is J-2 or Nagata. Namely, let $p \in D$ and choose an étale neighbourhood $(U, u) \to (X, p)$ such that the pullback of D is a strict normal crossings divisor on U. Then $\mathcal{O}_{X,p}^{sh} = \mathcal{O}_{U,u}^{sh}$ and we see that the trace of D on $\mathrm{Spec}(\mathcal{O}_{U,u}^{sh})$ is cut out by part of a regular system of parameters as this is already the case in $\mathcal{O}_{U,u}$.

To prove the implication in the other direction we will use the criterion of Lemma 41.21.6. Observe that formation of the normalization $D^\nu \to D$ commutes with strict henselization, see More on Morphisms, Lemma 37.19.4. If we can show that $D^\nu \to D$ is finite, then we see that $D^\nu \to D$ and the schemes Z_n satisfy all desired properties because these can all be checked on the level of local rings (but the finiteness of the morphism $D^\nu \to D$ is not something we can check on local rings). We omit the detailed verifications.

If X is Nagata, then $D^\nu \to D$ is finite by Morphisms, Lemma 29.54.10.

Assume X is J-2. Choose a point $p \in D$. We will show that $D^\nu \to D$ is finite over a neighbourhood of p. By assumption there exists a regular system of parameters $f_1, \ldots, f_d$ of $\mathcal{O}_{X,p}^{sh}$ and $1 \leq r \leq d$ such that the trace of D on $\mathrm{Spec}(\mathcal{O}_{X,p}^{sh})$ is cut out by $f_1 \ldots f_r$. Then

$$D^\nu \times_X \mathrm{Spec}(\mathcal{O}_{X,p}^{sh}) = \coprod\nolimits_{i=1,\ldots,r} V(f_i)$$

Choose an affine étale neighbourhood $(U, u) \to (X, p)$ such that f_i comes from $f_i \in \mathcal{O}_U(U)$. Set $D_i = V(f_i) \subset U$. The strict henselization of $\mathcal{O}_{D_i,u}$ is $\mathcal{O}^{sh}_{X,p}/(f_i)$ which is regular. Hence $\mathcal{O}_{D_i,u}$ is regular (for example by More on Algebra, Lemma 15.45.10). Because X is J-2 the regular locus is open in D_i. Thus after replacing U by a Zariski open we may assume that D_i is regular for each i. It follows that

$$\coprod\nolimits_{i=1,\ldots,r} D_i = D^\nu \times_X U \longrightarrow D \times_X U$$

is the normalization morphism and it is clearly finite. In other words, we have found an étale neighbourhood (U, u) of (X, p) such that the base change of $D^\nu \to D$ to this neighbourhood is finite. This implies $D^\nu \to D$ is finite by descent (Descent, Lemma 35.23.23) and the proof is complete. □

41.22. Other chapters

Part 3

Topics in Scheme Theory

CHAPTER 42

Chow Homology and Chern Classes

02P3 42.1. Introduction

02P4 In this chapter we discuss Chow homology groups and the construction of Chern classes of vector bundles as elements of operational Chow cohomology groups (everything with $\mathbf{Z}$-coefficients).

We start this chapter by giving the shortest possible algebraic proof of the Key Lemma 42.6.3. We first define the Herbrand quotient (Section 42.2) and we compute it in some cases (Section 42.3). Next, we prove some simple algebra lemmas on existence of suitable factorizations after modifications (Section 42.4). Using these we construct/define the tame symbol in Section 42.5. Only the most basic properties of the tame symbol are needed to prove the Key Lemma, which we do in Section 42.6.

Next, we introduce the basic setup we work with in the rest of this chapter in Section 42.7. To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes X are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and is endowed with a dimension function. These assumptions suffice to be able to define the Chow homology groups $\mathrm{CH}_*(X)$ and the action of capping with Chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

Next, we follow the first few chapters of [Ful98] in order to define cycles, flat pullback, proper pushforward, and rational equivalence, except that we have been less precise about the supports of the cycles involved.

We diverge from the presentation given in [Ful98] by using the Key lemma mentioned above to prove a basic commutativity relation in Section 42.27. Using this we prove that the operation of intersecting with an invertible sheaf passes through rational equivalence and is commutative, see Section 42.28. One more application of the Key lemma proves that the Gysin map of an effective Cartier divisor passes through rational equivalence, see Section 42.30. Having proved this, it is straightforward to define Chern classes of vector bundles, prove additivity, prove the splitting principle, introduce Chern characters, Todd classes, and state the Grothendieck-Riemann-Roch theorem.

There are two appendices. In Appendix A (Section 42.68) we discuss an alternative (longer) construction of the tame symbol and corresponding proof of the Key Lemma. Finally, in Appendix B (Section 42.69) we briefly discuss the relationship with K-theory of coherent sheaves and we discuss some blowup lemmas. We suggest the reader look at their introductions for more information.

We will return to the Chow groups $\mathrm{CH}_*(X)$ for smooth projective varieties over algebraically closed fields in the next chapter. Using a moving lemma as in [Sam56], [Che58a], and [Che58b] and Serre's Tor-formula (see [Ser00] or [Ser65]) we will define a ring structure on $\mathrm{CH}_*(X)$. See Intersection Theory, Section 43.1 ff.

42.2. Periodic complexes and Herbrand quotients

02PF Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

02PG Definition 42.2.1. Let R be a ring.

(1) A 2-periodic complex over R is given by a quadruple (M, N, φ, ψ) consisting of R-modules M, N and R-module maps $\varphi : M \to N$, $\psi : N \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \ldots$$

is a complex. In this setting we define the cohomology modules of the complex to be the R-modules

$$H^0(M, N, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi) \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

We say the 2-periodic complex is exact if the cohomology groups are zero.

(2) A $(2, 1)$-periodic complex over R is given by a triple (M, φ, ψ) consisting of an R-module M and R-module maps $\varphi : M \to M$, $\psi : M \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

is a complex. Since this is a special case of a 2-periodic complex we have its cohomology modules $H^0(M, \varphi, \psi)$, $H^1(M, \varphi, \psi)$ and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for $(2, 1)$-periodic complexes. It is clear that the collection of 2-periodic complexes forms a category with morphisms $(f, g) : (M, N, \varphi, \psi) \to (M', N', \varphi', \psi')$ pairs of morphisms $f : M \to M'$ and $g : N \to N'$ such that $\varphi' \circ f = g \circ \varphi$ and $\psi' \circ g = f \circ \psi$. We obtain an abelian category, with kernels and cokernels as in Homology, Lemma 12.13.3.

02PH Definition 42.2.2. Let (M, N, φ, ψ) be a 2-periodic complex over a ring R whose cohomology modules have finite length. In this case we define the multiplicity of (M, N, φ, ψ) to be the integer

$$e_R(M, N, \varphi, \psi) = \mathrm{length}_R(H^0(M, N, \varphi, \psi)) - \mathrm{length}_R(H^1(M, N, \varphi, \psi))$$

In the case of a $(2, 1)$-periodic complex (M, φ, ψ), we denote this by $e_R(M, \varphi, \psi)$ and we will sometimes call this the (additive) Herbrand quotient.

If the cohomology groups of (M, φ, ψ) are finite abelian groups, then it is customary to call the (multiplicative) Herbrand quotient

$$q(M, \varphi, \psi) = \frac{\#H^0(M, \varphi, \psi)}{\#H^1(M, \varphi, \psi)}$$

In words: the multiplicative Herbrand quotient is the number of elements of H^0 divided by the number of elements of H^1. If R is local and if the residue field of R is finite with q elements, then we see that

$$q(M, \varphi, \psi) = q^{e_R(M,\varphi,\psi)}$$

An example of a $(2,1)$-periodic complex over a ring R is any triple of the form $(M, 0, \psi)$ where M is an R-module and ψ is an R-linear map. If the kernel and cokernel of ψ have finite length, then we obtain

0EA6 (42.2.2.1)
$$e_R(M, 0, \psi) = \text{length}_R(\text{Coker}(\psi)) - \text{length}_R(\text{Ker}(\psi))$$

We state and prove the obligatory lemmas on these notations.

0EA7 Lemma 42.2.3. Let R be a ring. Suppose that we have a short exact sequence of 2-periodic complexes

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. We abbreviate $A = (M_1, N_1, \varphi_1, \psi_1)$, $B = (M_2, N_2, \varphi_2, \psi_2)$ and $C = (M_3, N_3, \varphi_3, \psi_3)$. We have a long exact cohomology sequence

$$\ldots \to H^1(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to \ldots$$

This gives a finite exact sequence

$$0 \to I \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to K \to 0$$

with $0 \to K \to H^1(C) \to I \to 0$ a filtration. By additivity of the length function (Algebra, Lemma 10.52.3) we see the result. $\square$

0EA8 Lemma 42.2.4. Let R be a ring. If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length, then $e_R(M, N, \varphi, \psi) = \text{length}_R(M) - \text{length}_R(N)$. In particular, if (M, φ, ψ) is a $(2,1)$-periodic complex such that M has finite length, then $e_R(M, \varphi, \psi) = 0$.

Proof. This follows from the additity of Lemma 42.2.3 and the short exact sequence $0 \to (M, 0, 0, 0) \to (M, N, \varphi, \psi) \to (0, N, 0, 0) \to 0$. $\square$

0EA9 Lemma 42.2.5. Let R be a ring. Let $f : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a map of $(2,1)$-periodic complexes whose cohomology modules have finite length. If $\text{Ker}(f)$ and $\text{Coker}(f)$ have finite length, then $e_R(M, \varphi, \psi) = e_R(M', \varphi', \psi')$.

Proof. Apply the additivity of Lemma 42.2.3 and observe that $(\text{Ker}(f), \varphi, \psi)$ and $(\text{Coker}(f), \varphi', \psi')$ have vanishing multiplicity by Lemma 42.2.4. $\square$

42.3. Calculation of some multiplicities

0EAA To prove equality of certain cycles later on we need to compute some multiplicities. Our main tool, besides the elementary lemmas on multiplicities given in the previous section, will be Algebra, Lemma 10.121.7.

02QF Lemma 42.3.1. Let R be a Noetherian local ring. Let M be a finite R-module. Let $x \in R$. Assume that

(1) $\dim(\text{Supp}(M)) \leq 1$, and
(2) $\dim(\text{Supp}(M/xM)) \leq 0$.

Write $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Then

$$e_R(M, 0, x) = \sum\nolimits_{i=1,\ldots,t} \text{ord}_{R/\mathfrak{q}_i}(x)\text{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

Proof. We first make some preparatory remarks. The result of the lemma holds if M has finite length, i.e., if $t = 0$, because both the left hand side and the right hand side are zero in this case, see Lemma 42.2.4. Also, if we have a short exact sequence $0 \to M \to M' \to M'' \to 0$ of modules satisfying (1) and (2), then lemma for 2 out of 3 of these implies the lemma for the third by the additivity of length (Algebra, Lemma 10.52.3) and additivty of multiplicities (Lemma 42.2.3).

Denote M_i the image of M in $M_{\mathfrak{q}_i}$, so $\text{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The kernel and cokernel of the map $M \to \bigoplus M_i$ have support $\{\mathfrak{m}\}$ and hence have finite length. By our preparatory remarks, it follows that it suffices to prove the lemma for each M_i. Thus we may assume that $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$. In this case we have a finite filtration $M \supset \mathfrak{q}M \supset \mathfrak{q}^2M \supset \ldots \supset \mathfrak{q}^nM = 0$ by Algebra, Lemma 10.62.4. Again additivity shows that it suffices to prove the lemma in the case M is annihilated by $\mathfrak{q}$. In this case we can view M as a $R/\mathfrak{q}$-module, i.e., we may assume that R is a Noetherian local domain of dimension 1 with fraction field K. Dividing by the torsion submodule, i.e., by the kernel of $M \to M \otimes_R K = V$ (the torsion has finite length hence is handled by our preliminary remarks) we may assume that $M \subset V$ is a lattice (Algebra, Definition 10.121.3). Then $x : M \to M$ is injective and $\text{length}_R(M/xM) = d(M, xM)$ (Algebra, Definition 10.121.5). Since $\text{length}_K(V) = \dim_K(V)$ we see that $\det(x : V \to V) = x^{\dim_K(V)}$ and $\text{ord}_R(\det(x : V \to V)) = \dim_K(V)\text{ord}_R(x)$. Thus the desired equality follows from Algebra, Lemma 10.121.7 in this case. $\square$

02QG Lemma 42.3.2. Let R be a Noetherian local ring. Let $x \in R$. If M is a finite Cohen-Macaulay module over R with $\dim(\text{Supp}(M)) = 1$ and $\dim(\text{Supp}(M/xM)) = 0$, then

$$\text{length}_R(M/xM) = \sum\nolimits_i \text{length}_R(R/(x, \mathfrak{q}_i))\text{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the support of M. If $I \subset R$ is an ideal such that x is a nonzerodivisor on R/I and $\dim(R/I) = 1$, then

$$\text{length}_R(R/(x, I)) = \sum\nolimits_i \text{length}_R(R/(x, \mathfrak{q}_i))\text{length}_{R_{\mathfrak{q}_i}}((R/I)_{\mathfrak{q}_i})$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are the minimal primes over I.

Proof. These are special cases of Lemma 42.3.1. $\square$

Here is another case where we can determine the value of a multiplicity.

0EAB Lemma 42.3.3. Let R be a ring. Let M be an R-module. Let $\varphi : M \to M$ be an endomorphism and $n > 0$ such that $\varphi^n = 0$ and such that $\text{Ker}(\varphi)/\text{Im}(\varphi^{n-1})$ has finite length as an R-module. Then

$$e_R(M, \varphi^i, \varphi^{n-i}) = 0$$

for $i = 0, \ldots, n$.

Proof. The cases $i = 0, n$ are trivial as $\varphi^0 = \mathrm{id}_M$ by convention. Let us think of M as an $R[t]$-module where multiplication by t is given by φ. Let us write $K_i = \mathrm{Ker}(t^i : M \to M)$ and

$$a_i = \mathrm{length}_R(K_i/t^{n-i}M), \quad b_i = \mathrm{length}_R(K_i/tK_{i+1}), \quad c_i = \mathrm{length}_R(K_1/t^iK_{i+1})$$

Boundary values are $a_0 = a_n = b_0 = c_0 = 0$. The c_i are integers for $i < n$ as K_1/t^iK_{i+1} is a quotient of $K_1/t^{n-1}M$ which is assumed to have finite length. We will use frequently that $K_i \cap t^jM = t^jK_{i+j}$. For $0 < i < n-1$ we have an exact sequence

$$0 \to K_1/t^{n-i-1}K_{n-i} \to K_{i+1}/t^{n-i-1}M \xrightarrow{t} K_i/t^{n-i}M \to K_i/tK_{i+1} \to 0$$

By induction on i we conclude that a_i and b_i are integers for $i < n$ and that

$$c_{n-i-1} - a_{i+1} + a_i - b_i = 0$$

For $0 < i < n-1$ there is a short exact sequence

$$0 \to K_i/tK_{i+1} \to K_{i+1}/tK_{i+2} \xrightarrow{t^i} K_1/t^{i+1}K_{i+2} \to K_1/t^iK_{i+1} \to 0$$

which gives

$$b_i - b_{i+1} + c_{i+1} - c_i = 0$$

Since $b_0 = c_0$ we conclude that $b_i = c_i$ for $i < n$. Then we see that

$$a_2 = a_1 + b_{n-2} - b_1, \quad a_3 = a_2 + b_{n-3} - b_2, \quad \ldots$$

It is straighforward to see that this implies $a_i = a_{n-i}$ as desired. □

0EAC Lemma 42.3.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R with M finite and with cohomology groups of finite length over R. Let $x \in R$ be such that $\dim(\mathrm{Supp}(M/xM)) \leq 0$. Then

$$e_R(M, x\varphi, \psi) = e_R(M, \varphi, \psi) - e_R(\mathrm{Im}(\varphi), 0, x)$$

and

$$e_R(M, \varphi, x\psi) = e_R(M, \varphi, \psi) + e_R(\mathrm{Im}(\psi), 0, x)$$

Proof. We will only prove the first formula as the second is proved in exactly the same manner. Let $M' = M[x^\infty]$ be the x-power torsion submodule of M. Consider the short exact sequence $0 \to M' \to M \to M'' \to 0$. Then M'' is x-power torsion free (More on Algebra, Lemma 15.88.4). Since φ, ψ map M' into M' we obtain a short exact sequence

$$0 \to (M', \varphi', \psi') \to (M, \varphi, \psi) \to (M'', \varphi'', \psi'') \to 0$$

of $(2,1)$-periodic complexes. Also, we get a short exact sequence $0 \to M' \cap \mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi'') \to 0$. We have $e_R(M', \varphi, \psi) = e_R(M', x\varphi, \psi) = e_R(M' \cap \mathrm{Im}(\varphi), 0, x) = 0$ by Lemma 42.2.5. By additivity (Lemma 42.2.3) we see that it suffices to prove the lemma for (M'', φ'', ψ''). This reduces us to the case discussed in the next paragraph.

Assume $x : M \to M$ is injective. In this case $\mathrm{Ker}(x\varphi) = \mathrm{Ker}(\varphi)$. On the other hand we have a short exact sequence

$$0 \to \mathrm{Im}(\varphi)/x\,\mathrm{Im}(\varphi) \to \mathrm{Ker}(\psi)/\mathrm{Im}(x\varphi) \to \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi) \to 0$$

This together with (42.2.2.1) proves the formula. □

42.4. Preparation for tame symbols

0EAD In this section we put some lemma that will help us define the tame symbol in the next section.

0EAE **Lemma 42.4.1.** Let A be a Noetherian ring. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be pairwise distinct maximal ideals of A. For $i = 1, \ldots, r$ let $\varphi_i : A_{\mathfrak{m}_i} \to B_i$ be a ring map whose kernel and cokernel are annihilated by a power of $\mathfrak{m}_i$. Then there exists a ring map $\varphi : A \to B$ such that

1. the localization of φ at $\mathfrak{m}_i$ is isomorphic to φ_i, and
2. $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by a power of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$.

Moreover, if each φ_i is finite, injective, or surjective then so is φ.

Proof. Set $I = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$. Set $A_i = A_{\mathfrak{m}_i}$ and $A' = \prod A_i$. Then $IA' = \prod \mathfrak{m}_i A_i$ and $A \to A'$ is a flat ring map such that $A/I \cong A'/IA'$. Thus we may use More on Algebra, Lemma 15.89.16 to see that there exists an A-module map $\varphi : A \to B$ with φ_i isomorphic to the localization of φ at $\mathfrak{m}_i$. Then we can use the discussion in More on Algebra, Remark 15.89.19 to endow B with an A-algebra structure matching the given A-algebra structure on B_i. The final statement of the lemma follows easily from the fact that $\mathrm{Ker}(\varphi)_{\mathfrak{m}_i} \cong \mathrm{Ker}(\varphi_i)$ and $\mathrm{Coker}(\varphi)_{\mathfrak{m}_i} \cong \mathrm{Coker}(\varphi_i)$. $\square$

The following lemma is very similar to Algebra, Lemma 10.119.3.

02Q7 **Lemma 42.4.2.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $a, b \in R$ be nonzerodivisors. There exists a finite ring extension $R \subset R'$ with R'/R annihilated by a power of $\mathfrak{m}$ and nonzerodivisors $t, a', b' \in R'$ such that $a = ta'$ and $b = tb'$ and $R' = a'R' + b'R'$.

Proof. If a or b is a unit, then the lemma is true with $R = R'$. Thus we may assume $a, b \in \mathfrak{m}$. Set $I = (a, b)$. The idea is to blow up R in I. Instead of doing the algebraic argument we work geometrically. Let $X = \mathrm{Proj}(\bigoplus_{d \geq 0} I^d)$. By Divisors, Lemma 31.32.4 the morphism $X \to \mathrm{Spec}(R)$ is an isomorphism over the punctured spectrum $U = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. Thus we may and do view U as an open subscheme of X. The morphism $X \to \mathrm{Spec}(R)$ is projective by Divisors, Lemma 31.32.13. Also, every generic point of X lies in U, for example by Divisors, Lemma 31.32.10. It follows from Varieties, Lemma 33.17.2 that $X \to \mathrm{Spec}(R)$ is finite. Thus $X = \mathrm{Spec}(R')$ is affine and $R \to R'$ is finite. We have $R_a \cong R'_a$ as $U = D(a)$. Hence a power of a annihilates the finite R-module R'/R. As $\mathfrak{m} = \sqrt{(a)}$ we see that R'/R is annihilated by a power of $\mathfrak{m}$. By Divisors, Lemma 31.32.4 we see that IR' is a locally principal ideal. Since R' is semi-local we see that IR' is principal, see Algebra, Lemma 10.78.7, say $IR' = (t)$. Then we have $a = a't$ and $b = b't$ and everything is clear. $\square$

0EAF **Lemma 42.4.3.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $a, b \in R$ be nonzerodivisors with $a \in \mathfrak{m}$. There exists an integer $n = n(R, a, b)$ such that for a finite ring extension $R \subset R'$ if $b = a^m c$ for some $c \in R'$, then $m \leq n$.

Proof. Choose a minimal prime $\mathfrak{q} \subset R$. Observe that $\dim(R/\mathfrak{q}) = 1$, in particular $R/\mathfrak{q}$ is not a field. We can choose a discrete valuation ring A dominating $R/\mathfrak{q}$ with the same fraction field, see Algebra, Lemma 10.119.1. Observe that a and b map to nonzero elements of A as nonzerodivisors in R are not contained in $\mathfrak{q}$. Let v be the discrete valuation on A. Then $v(a) > 0$ as $a \in \mathfrak{m}$. We claim $n = v(b)/v(a)$ works.

Let $R \subset R'$ be given. Set $A' = A \otimes_R R'$. Since $\text{Spec}(R') \to \text{Spec}(R)$ is surjective (Algebra, Lemma 10.36.17) also $\text{Spec}(A') \to \text{Spec}(A)$ is surjective (Algebra, Lemma 10.30.3). Pick a prime $\mathfrak{q}' \subset A'$ lying over $(0) \subset A$. Then $A \subset A'' = A'/\mathfrak{q}'$ is a finite extension of rings (again inducing a surjection on spectra). Pick a maximal ideal $\mathfrak{m}'' \subset A''$ lying over the maximal ideal of A and a discrete valuation ring A''' dominating $A''_{\mathfrak{m}''}$ (see lemma cited above). Then $A \to A'''$ is an extension of discrete valuation rings and we have $b = a^m c$ in A'''. Thus $v'''(b) \geq mv'''(a)$. Since $v''' = ev$ where e is the ramification index of A'''/A, we find that $m \leq n$ as desired. □

0EAG Lemma 42.4.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $r \geq 2$ and let $a_1, \ldots, a_r \in A$ be nonzerodivisors not all units. Then there exist

1. a finite ring extension $A \subset B$ with B/A annihilated by a power of $\mathfrak{m}$,
2. for each maximal ideal $\mathfrak{m}_j \subset B$ a nonzerodivisor $\pi_j \in B_j = B_{\mathfrak{m}_j}$, and
3. factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ in B_j with $u_{i,j} \in B_j$ units and $e_{i,j} \geq 0$.

Proof. Since at least one a_i is not a unit we find that $\mathfrak{m}$ is not an associated prime of A. Moreover, for any $A \subset B$ as in the statement $\mathfrak{m}$ is not an associated prime of B and $\mathfrak{m}_j$ is not an associate prime of B_j. Keeping this in mind will help check the arguments below.

First, we claim that it suffices to prove the lemma for $r = 2$. We will argue this by induction on r; we suggest the reader skip the proof. Suppose we are given $A \subset B$ and π_j in $B_j = B_{\mathfrak{m}_j}$ and factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ for $i = 1, \ldots, r-1$ in B_j with $u_{i,j} \in B_j$ units and $e_{i,j} \geq 0$. Then by the case $r = 2$ for π_j and a_r in B_j we can find extensions $B_j \subset C_j$ and for every maximal ideal $\mathfrak{m}_{j,k}$ of C_j a nonzerodivisor $\pi_{j,k} \in C_{j,k} = (C_j)_{\mathfrak{m}_{j,k}}$ and factorizations

$$\pi_j = v_{j,k}\pi_{j,k}^{f_{j,k}} \quad \text{and} \quad a_r = w_{j,k}\pi_{j,k}^{g_{j,k}}$$

as in the lemma. There exists a unique finite extension $B \subset C$ with C/B annihilated by a power of $\mathfrak{m}$ such that $C_j \cong C_{\mathfrak{m}_j}$ for all j, see Lemma 42.4.1. The maximal ideals of C correspond 1-to-1 to the maximal ideals $\mathfrak{m}_{j,k}$ in the localizations and in these localizations we have

$$a_i = u_{i,j}\pi_j^{e_{i,j}} = u_{i,j}v_{j,k}^{e_{i,j}}\pi_{j,k}^{e_{i,j}f_{j,k}}$$

for $i \leq r-1$. Since a_r factors correctly too the proof of the induction step is complete.

Proof of the case $r = 2$. We will use induction on

$$\ell = \min(\text{length}_A(A/a_1A),\ \text{length}_A(A/a_2A)).$$

If $\ell = 0$, then either a_1 or a_2 is a unit and the lemma holds with $A = B$. Thus we may and do assume $\ell > 0$.

Suppose we have a finite extension of rings $A \subset A'$ such that A'/A is annihilated by a power of $\mathfrak{m}$ and such that $\mathfrak{m}$ is not an associated prime of A'. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \subset A'$ be the maximal ideals and set $A'_i = A'_{\mathfrak{m}_i}$. If we can solve the problem for a_1, a_2 in each A'_i, then we can apply Lemma 42.4.1 to produce a solution for a_1, a_2 in A. Choose $x \in \{a_1, a_2\}$ such that $\ell = \text{length}_A(A/xA)$. By Lemma 42.2.5 and (42.2.2.1) we have $\text{length}_A(A/xA) = \text{length}_A(A'/xA')$. On the other hand, we have

$$\text{length}_A(A'/xA') = \sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \text{length}_{A'_i}(A'_i/xA'_i)$$

by Algebra, Lemma 10.52.12. Since $x \in \mathfrak{m}$ we see that each term on the right hand side is positive. We conclude that the induction hypothesis applies to a_1, a_2 in each A'_i if $r > 1$ or if $r = 1$ and $[\kappa(\mathfrak{m}_1) : \kappa(\mathfrak{m})] > 1$. We conclude that we may assume each A' as above is local with the same residue field as A.

Applying the discussion of the previous paragraph, we may replace A by the ring constructed in Lemma 42.4.2 for $a_1, a_2 \in A$. Then since A is local we find, after possibly switching a_1 and a_2, that $a_2 \in (a_1)$. Write $a_2 = a_1^m c$ with $m > 0$ maximal. In fact, by Lemma 42.4.3 we may assume m is maximal even after replacing A by any finite extension $A \subset A'$ as in the previous paragraph. If c is a unit, then we are done. If not, then we replace A by the ring constructed in Lemma 42.4.2 for $a_1, c \in A$. Then either (1) $c = a_1 c'$ or (2) $a_1 = c a'_1$. The first case cannot happen since it would give $a_2 = a_1^{m+1} c'$ contradicting the maximality of m. In the second case we get $a_1 = c a'_1$ and $a_2 = c^{m+1}(a'_1)^m$. Then it suffices to prove the lemma for A and c, a'_1. If a'_1 is a unit we're done and if not, then $\text{length}_A(A/cA) < \ell$ because cA is a strictly bigger ideal than $a_1 A$. Thus we win by induction hypothesis. □

42.5. Tame symbols

0EAH Consider a Noetherian local ring $(A, \mathfrak{m})$ of dimension 1. We denote $Q(A)$ the total ring of fractions of A, see Algebra, Example 10.9.8. The tame symbol will be a map

$$\partial_A(-,-) : Q(A)^* \times Q(A)^* \longrightarrow \kappa(\mathfrak{m})^*$$

satisfying the following properties:

0EAI (1) $\partial_A(f, gh) = \partial_A(f, g)\partial_A(f, h)$ for $f, g, h \in Q(A)^*$,
0EAJ (2) $\partial_A(f, g)\partial_A(g, f) = 1$ for $f, g \in Q(A)^*$,
0EAK (3) $\partial_A(f, 1-f) = 1$ for $f \in Q(A)^*$ such that $1 - f \in Q(A)^*$,
0EAL (4) $\partial_A(aa', b) = \partial_A(a, b)\partial_A(a', b)$ and $\partial_A(a, bb') = \partial_A(a, b)\partial_A(a, b')$ for $a, a', b, b' \in A$ nonzerodivisors,
0EAM (5) $\partial_A(b, b) = (-1)^m$ with $m = \text{length}_A(A/bA)$ for $b \in A$ a nonzerodivisor,
0EAN (6) $\partial_A(u, b) = u^m \bmod \mathfrak{m}$ with $m = \text{length}_A(A/bA)$ for $u \in A$ a unit and $b \in A$ a nonzerodivisor, and
0EAP (7) $\partial_A(a, b-a)\partial_A(b, b) = \partial_A(b, b-a)\partial_A(a, b)$ for $a, b \in A$ such that $a, b, b-a$ are nonzerodivisors.

Since it is easier to work with elements of A we will often think of ∂_A as a map defined on pairs of nonzerodivisors of A satisfying (4), (5), (6), (7). It is an exercise to see that setting

$$\partial_A(\frac{a}{b}, \frac{c}{d}) = \partial_A(a, c)\partial_A(a, d)^{-1}\partial_A(b, c)^{-1}\partial_A(b, d)$$

we get a well defined map $Q(A)^* \times Q(A)^* \to \kappa(\mathfrak{m})^*$ satisfying (1), (2), (3) as well as the other properties.

We do not claim there is a unique map with these properties. Instead, we will give a recipe for constructing such a map. Namely, given $a_1, a_2 \in A$ nonzerodivisors, we choose a ring extension $A \subset B$ and local factorizations as in Lemma 42.4.4. Then we define

0EAQ
$$\partial_A(a_1, a_2) = \prod\nolimits_j \text{Norm}_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j} \quad (42.5.0.1)$$

where $m_j = \text{length}_{B_j}(B_j/\pi_j B_j)$ and the product is taken over the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of B.

0EAR Lemma 42.5.1. The formula (42.5.0.1) determines a well defined element of $\kappa(\mathfrak{m})^*$. In other words, the right hand side does not depend on the choice of the local factorizations or the choice of B.

Proof. Independence of choice of factorizations. Suppose we have a Noetherian 1-dimensional local ring B, elements $a_1, a_2 \in B$, and nonzerodivisors π, θ such that we can write

$$a_1 = u_1\pi^{e_1} = v_1\theta^{f_1}, \quad a_2 = u_2\pi^{e_2} = v_2\theta^{f_2}$$

with $e_i, f_i \geq 0$ integers and u_i, v_i units in B. Observe that this implies

$$a_1^{e_2} = u_1^{e_2}u_2^{-e_1}a_2^{e_1}, \quad a_1^{f_2} = v_1^{f_2}v_2^{-f_1}a_2^{f_1}$$

On the other hand, setting $m = \text{length}_B(B/\pi B)$ and $k = \text{length}_B(B/\theta B)$ we find $e_2 m = \text{length}_B(B/a_2B) = f_2 k$. Expanding $a_1^{e_2m} = a_1^{f_2k}$ using the above we find

$$(u_1^{e_2}u_2^{-e_1})^m = (v_1^{f_2}v_2^{-f_1})^k$$

This proves the desired equality up to signs. To see the signs work out we have to show me_1e_2 is even if and only if kf_1f_2 is even. This follows as both $me_2 = kf_2$ and $me_1 = kf_1$ (same argument as above).

Independence of choice of B. Suppose given two extensions $A \subset B$ and $A \subset B'$ as in Lemma 42.4.4. Then

$$C = (B \otimes_A B')/(\mathfrak{m}\text{-power torsion})$$

will be a third one. Thus we may assume we have $A \subset B \subset C$ and factorizations over the local rings of B and we have to show that using the same factorizations over the local rings of C gives the same element of $\kappa(\mathfrak{m})$. By transitivity of norms (Fields, Lemma 9.20.5) this comes down to the following problem: if B is a Noetherian local ring of dimension 1 and $\pi \in B$ is a nonzerodivisor, then

$$\lambda^m = \prod \text{Norm}_{\kappa_k/\kappa}(\lambda)^{m_k}$$

Here we have used the following notation: (1) κ is the residue field of B, (2) λ is an element of κ, (3) $\mathfrak{m}_k \subset C$ are the maximal ideals of C, (4) $\kappa_k = \kappa(\mathfrak{m}_k)$ is the residue field of $C_k = C_{\mathfrak{m}_k}$, (5) $m = \text{length}_B(B/\pi B)$, and (6) $m_k = \text{length}_{C_k}(C_k/\pi C_k)$. The displayed equality holds because $\text{Norm}_{\kappa_k/\kappa}(\lambda) = \lambda^{[\kappa_k:\kappa]}$ as $\lambda \in \kappa$ and because $m = \sum m_k[\kappa_k : \kappa]$. First, we have $m = \text{length}_B(B/xB) = \text{length}_B(C/\pi C)$ by Lemma 42.2.5 and (42.2.2.1). Finally, we have $\text{length}_B(C/\pi C) = \sum m_k[\kappa_k : \kappa]$ by Algebra, Lemma 10.52.12. □

0EAS Lemma 42.5.2. The tame symbol (42.5.0.1) satisfies (4), (5), (6), (7) and hence gives a map $\partial_A : Q(A)^* \times Q(A)^* \to \kappa(\mathfrak{m})^*$ satisfying (1), (2), (3).

Proof. Let us prove (4). Let $a_1, a_2, a_3 \in A$ be nonzerodivisors. Choose $A \subset B$ as in Lemma 42.4.4 for a_1, a_2, a_3. Then the equality

$$\partial_A(a_1a_2, a_3) = \partial_A(a_1, a_3)\partial_A(a_2, a_3)$$

follows from the equality

$$(-1)^{(e_{1,j}+e_{2,j})e_{3,j}}(u_{1,j}u_{2,j})^{e_{3,j}}u_{3,j}^{-e_{1,j}-e_{2,j}} = (-1)^{e_{1,j}e_{3,j}}u_{1,j}^{e_{3,j}}u_{3,j}^{-e_{1,j}}(-1)^{e_{2,j}e_{3,j}}u_{2,j}^{e_{3,j}}u_{3,j}^{-e_{2,j}}$$

in B_j. Properties (5) and (6) are equally immediate.

Let us prove (7). Let $a_1, a_2, a_1 - a_2 \in A$ be nonzerodivisors and set $a_3 = a_1 - a_2$. Choose $A \subset B$ as in Lemma 42.4.4 for a_1, a_2, a_3. Then it suffices to show

$$(-1)^{e_{1,j}e_{2,j}+e_{1,j}e_{3,j}+e_{2,j}e_{3,j}+e_{2,j}} u_{1,j}^{e_{2,j}-e_{3,j}} u_{2,j}^{e_{3,j}-e_{1,j}} u_{3,j}^{e_{1,j}-e_{2,j}} \bmod \mathfrak{m}_j = 1$$

This is clear if $e_{1,j} = e_{2,j} = e_{3,j}$. Say $e_{1,j} > e_{2,j}$. Then we see that $e_{3,j} = e_{2,j}$ because $a_3 = a_1 - a_2$ and we see that $u_{3,j}$ has the same residue class as $-u_{2,j}$. Hence the formula is true – the signs work out as well and this verification is the reason for the choice of signs in (42.5.0.1). The other cases are handled in exactly the same manner. □

0EAT Lemma 42.5.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $A \subset B$ be a finite ring extension with B/A annihilated by a power of $\mathfrak{m}$ and $\mathfrak{m}$ not an associated prime of B. For $a, b \in A$ nonzerodivisors we have

$$\partial_A(a, b) = \prod \mathrm{Norm}_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}(\partial_{B_j}(a, b))$$

where the product is over the maximal ideals $\mathfrak{m}_j$ of B and $B_j = B_{\mathfrak{m}_j}$.

Proof. Choose $B_j \subset C_j$ as in Lemma 42.4.4 for a, b. By Lemma 42.4.1 we can choose a finite ring extension $B \subset C$ with $C_j \cong C_{\mathfrak{m}_j}$ for all j. Let $\mathfrak{m}_{j,k} \subset C$ be the maximal ideals of C lying over $\mathfrak{m}_j$. Let

$$a = u_{j,k}\pi_{j,k}^{f_{j,k}}, \quad b = v_{j,k}\pi_{j,k}^{g_{j,k}}$$

be the local factorizations which exist by our choice of $C_j \cong C_{\mathfrak{m}_j}$. By definition we have

$$\partial_A(a, b) = \prod\nolimits_{j,k} \mathrm{Norm}_{\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m})}((-1)^{f_{j,k}g_{j,k}} u_{j,k}^{g_{j,k}} v_{j,k}^{-f_{j,k}} \bmod \mathfrak{m}_{j,k})^{m_{j,k}}$$

and

$$\partial_{B_j}(a, b) = \prod\nolimits_{k} \mathrm{Norm}_{\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m}_j)}((-1)^{f_{j,k}g_{j,k}} u_{j,k}^{g_{j,k}} v_{j,k}^{-f_{j,k}} \bmod \mathfrak{m}_{j,k})^{m_{j,k}}$$

The result follows by transitivity of norms for $\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})$, see Fields, Lemma 9.20.5. □

0EPG Lemma 42.5.4. Let $(A, \mathfrak{m}, \kappa) \to (A', \mathfrak{m}', \kappa')$ be a local homomorphism of Noetherian local rings. Assume $A \to A'$ is flat and $\dim(A) = \dim(A') = 1$. Set $m = \mathrm{length}_{A'}(A'/\mathfrak{m}A')$. For $a_1, a_2 \in A$ nonzerodivisors $\partial_A(a_1, a_2)^m$ maps to $\partial_{A'}(a_1, a_2)$ via $\kappa \to \kappa'$.

Proof. If a_1, a_2 are both units, then $\partial_A(a_1, a_2) = 1$ and $\partial_{A'}(a_1, a_2) = 1$ and the result is true. If not, then we can choose a ring extension $A \subset B$ and local factorizations as in Lemma 42.4.4. Denote $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ be the maximal ideals of B. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ be the maximal ideals of B with residue fields $\kappa_1, \ldots, \kappa_m$. For each $j \in \{1, \ldots, m\}$ denote $\pi_j \in B_j = B_{\mathfrak{m}_j}$ a nonzerodivisor such that we have factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ as in the lemma. By definition we have

$$\partial_A(a_1, a_2) = \prod\nolimits_j \mathrm{Norm}_{\kappa_j/\kappa}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j}$$

where $m_j = \mathrm{length}_{B_j}(B_j/\pi_j B_j)$.

Set $B' = A' \otimes_A B$. Since A' is flat over A we see that $A' \subset B'$ is a ring extension with B'/A' annihilated by a power of $\mathfrak{m}'$. Let

$$\mathfrak{m}'_{j,l}, \quad l = 1, \ldots, n_j$$

be the maximal ideals of B' lying over $\mathfrak m_j$. Denote $\kappa'_{j,l}$ the residue field of $\mathfrak m'_{j,l}$. Denote $B'_{j,l}$ the localization of B' at $\mathfrak m'_{j,l}$. As factorizations of a_1 and a_2 in $B'_{j,l}$ we use the image of the factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ given to us in B_j. By definition we have

$$\partial_{A'}(a_1, a_2) = \prod\nolimits_{j,l} \text{Norm}_{\kappa'_{j,l}/\kappa'}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak m'_{j,l})^{m'_{j,l}}$$

where $m'_{j,l} = \text{length}_{B'_{j,l}}(B'_{j,l}/\pi_j B'_{j,l})$.

Comparing the formulae above we see that it suffices to show that for each j and for any unit $u \in B_j$ we have

0GU1 (42.5.4.1) $$\left(\text{Norm}_{\kappa_j/\kappa}(u \bmod \mathfrak m_j)^{m_j}\right)^m = \prod\nolimits_l \text{Norm}_{\kappa'_{j,l}/\kappa'}(u \bmod \mathfrak m'_{j,l})^{m'_{j,l}}$$

in κ'. We are going to use the construction of determinants of endomorphisms of finite length modules in More on Algebra, Section 15.120 to prove this. Set $M = B_j/\pi_j B_j$. By More on Algebra, Lemma 15.120.2 we have

$$\text{Norm}_{\kappa_j/\kappa}(u \bmod \mathfrak m_j)^{m_j} = \det\nolimits_\kappa(u : M \to M)$$

Thus, by More on Algebra, Lemma 15.120.3, the left hand side of (42.5.4.1) is equal to $\det_{\kappa'}(u : M \otimes_A A' \to M \otimes_A A')$. We have an isomorphism

$$M \otimes_A A' = (B_j/\pi_j B_j) \otimes_A A' = \bigoplus\nolimits_l B'_{j,l}/\pi_j B'_{j,l}$$

of A'-modules. Setting $M'_l = B'_{j,l}/\pi_j B'_{j,l}$ we see that $\text{Norm}_{\kappa'_{j,l}/\kappa'}(u \bmod \mathfrak m'_{j,l})^{m'_{j,l}} = \det_{\kappa'}(u_j : M'_l \to M'_l)$ by More on Algebra, Lemma 15.120.2 again. Hence (42.5.4.1) holds by multiplicativity of the determinant construction, see More on Algebra, Lemma 15.120.1. □

42.6. A key lemma

0EAU In this section we apply the results above to prove Lemma 42.6.3. This lemma is a low degree case of the statement that there is a complex for Milnor K-theory similar to the Gersten-Quillen complex in Quillen's K-theory. See Remark 42.6.4.

0EAV **Lemma 42.6.1.** Let $(A, \mathfrak m)$ be a 2-dimensional Noetherian local ring. Let $t \in \mathfrak m$ be a nonzerodivisor. Say $V(t) = \{\mathfrak m, \mathfrak q_1, \ldots, \mathfrak q_r\}$. Let $A_{\mathfrak q_i} \subset B_i$ be a finite ring extension with $B_i/A_{\mathfrak q_i}$ annihilated by a power of t. Then there exists a finite extension $A \subset B$ of local rings identifying residue fields with $B_i \cong B_{\mathfrak q_i}$ and B/A annihilated by a power of t.

Proof. Choose $n > 0$ such that $B_i \subset t^{-n}A_{\mathfrak q_i}$. Let $M \subset t^{-n}A$, resp. $M' \subset t^{-2n}A$ be the A-submodule consisting of elements mapping to B_i in $t^{-n}A_{\mathfrak q_i}$, resp. $t^{-2n}A_{\mathfrak q_i}$. Then $M \subset M'$ are finite A-modules as A is Noetherian and $M_{\mathfrak q_i} = M'_{\mathfrak q_i} = B_i$ as localization is exact. Thus M'/M is annihilated by $\mathfrak m^c$ for some $c > 0$. Observe that $M \cdot M \subset M'$ under the multiplication $t^{-n}A \times t^{-n}A \to t^{-2n}A$. Hence $B = A + \mathfrak m^{c+1}M$ is a finite A-algebra with the correct localizations. We omit the verification that B is local with maximal ideal $\mathfrak m + \mathfrak m^{c+1}M$. □

0EAW **Lemma 42.6.2.** Let $(A, \mathfrak m)$ be a 2-dimensional Noetherian local ring. Let $a, b \in A$ be nonzerodivisors. Then we have

$$\sum \text{ord}_{A/\mathfrak q}(\partial_{A_{\mathfrak q}}(a, b)) = 0$$

where the sum is over the height 1 primes $\mathfrak q$ of A.

Proof. If $\mathfrak q$ is a height 1 prime of A such that a, b map to a unit of $A_{\mathfrak q}$, then $\partial_{A_{\mathfrak q}}(a, b) = 1$. Thus the sum is finite. In fact, if $V(ab) = \{\mathfrak m, \mathfrak q_1, \ldots, \mathfrak q_r\}$ then the sum is over $i = 1, \ldots, r$. For each i we pick an extension $A_{\mathfrak q_i} \subset B_i$ as in Lemma 42.4.4 for a, b. By Lemma 42.6.1 with $t = ab$ and the given list of primes we may assume we have a finite local extension $A \subset B$ with B/A annihilated by a power of ab and such that for each i the $B_{\mathfrak q_i} \cong B_i$. Observe that if $\mathfrak q_{i,j}$ are the primes of B lying over $\mathfrak q_i$ then we have

$$\text{ord}_{A/\mathfrak q_i}(\partial_{A_{\mathfrak q_i}}(a, b)) = \sum\nolimits_j \text{ord}_{B/\mathfrak q_{i,j}}(\partial_{B_{\mathfrak q_{i,j}}}(a, b))$$

by Lemma 42.5.3 and Algebra, Lemma 10.121.8. Thus we may replace A by B and reduce to the case discussed in the next paragraph.

Assume for each i there is a nonzerodivisor $\pi_i \in A_{\mathfrak q_i}$ and units $u_i, v_i \in A_{\mathfrak q_i}$ such that for some integers $e_i, f_i \geq 0$ we have

$$a = u_i \pi_i^{e_i}, \quad b = v_i \pi_i^{f_i}$$

in $A_{\mathfrak q_i}$. Setting $m_i = \text{length}_{A_{\mathfrak q_i}}(A_{\mathfrak q_i}/\pi_i)$ we have $\partial_{A_{\mathfrak q_i}}(a, b) = ((-1)^{e_i f_i} u_i^{f_i} v_i^{-e_i})^{m_i}$ by definition. Since a, b are nonzerodivisors the $(2, 1)$-periodic complex $(A/(ab), a, b)$ has vanishing cohomology. Denote M_i the image of $A/(ab)$ in $A_{\mathfrak q_i}/(ab)$. Then we have a map

$$A/(ab) \longrightarrow \bigoplus M_i$$

whose kernel and cokernel are supported in $\{\mathfrak m\}$ and hence have finite length. Thus we see that

$$\sum e_A(M_i, a, b) = 0$$

by Lemma 42.2.5. Hence it suffices to show $e_A(M_i, a, b) = -\text{ord}_{A/\mathfrak q_i}(\partial_{A_{\mathfrak q_i}}(a, b))$.

Let us prove this first, in case π_i, u_i, v_i are the images of elements $\pi_i, u_i, v_i \in A$ (using the same symbols should not cause any confusion). In this case we get

$$\begin{aligned} e_A(M_i, a, b) &= e_A(M_i, u_i \pi_i^{e_i}, v_i \pi_i^{f_i}) \\ &= e_A(M_i, \pi_i^{e_i}, \pi_i^{f_i}) - e_A(\pi_i^{e_i} M_i, 0, u_i) + e_A(\pi_i^{f_i} M_i, 0, v_i) \\ &= 0 - f_i m_i \text{ord}_{A/\mathfrak q_i}(u_i) + e_i m_i \text{ord}_{A/\mathfrak q_i}(v_i) \\ &= -m_i \text{ord}_{A/\mathfrak q_i}(u_i^{f_i} v_i^{-e_i}) = -\text{ord}_{A/\mathfrak q_i}(\partial_{A_{\mathfrak q_i}}(a, b)) \end{aligned}$$

The second equality holds by Lemma 42.3.4. Observe that $M_i \subset (M_i)_{\mathfrak q_i} = A_{\mathfrak q_i}/(\pi_i^{e_i + f_i})$ and $(\pi_i^{e_i} M_i)_{\mathfrak q_i} \cong A_{\mathfrak q_i}/\pi_i^{f_i}$ and $(\pi_i^{f_i} M_i)_{\mathfrak q_i} \cong A_{\mathfrak q_i}/\pi_i^{e_i}$. The 0 in the third equality comes from Lemma 42.3.3 and the other two terms come from Lemma 42.3.1. The last two equalities follow from multiplicativity of the order function and from the definition of our tame symbol.

In general, we may first choose $c \in A$, $c \notin \mathfrak q_i$ such that $c\pi_i \in A$. After replacing π_i by $c\pi_i$ and u_i by $c^{-e_i} u_i$ and v_i by $c^{-f_i} v_i$ we may and do assume π_i is in A. Next, choose an $c \in A$, $c \notin \mathfrak q_i$ with $cu_i, cv_i \in A$. Then we observe that

$$e_A(M_i, ca, cb) = e_A(M_i, a, b) - e_A(aM_i, 0, c) + e_A(bM_i, 0, c)$$

by Lemma 42.3.1. On the other hand, we have

$$\partial_{A_{\mathfrak q_i}}(ca, cb) = c^{m_i(f_i - e_i)} \partial_{A_{\mathfrak q_i}}(a, b)$$

in $\kappa(\mathfrak{q}_i)^*$ because c is a unit in $A_{\mathfrak{q}_i}$. The arguments in the previous paragraph show that $e_A(M_i, ca, cb) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(ca, cb))$. Thus it suffices to prove

$$e_A(aM_i, 0, c) = \text{ord}_{A/\mathfrak{q}_i}(c^{m_i f_i}) \quad \text{and} \quad e_A(bM_i, 0, c) = \text{ord}_{A/\mathfrak{q}_i}(c^{m_i e_i})$$

and this follows from Lemma 42.3.1 by the description (see above) of what happens when we localize at $\mathfrak{q}_i$. □

0EAX Lemma 42.6.3 (Key Lemma). Let A be a 2-dimensional Noetherian local domain with fraction field K. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A_{\mathfrak{q}}^*$. Then we have

When A is an excellent ring this is [Kat86, Proposition 1].

$$\sum\nolimits_{i=1,\ldots,t} \text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\partial_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height 1 prime $\mathfrak{q}$ of A where $f, g \in A_{\mathfrak{q}}^*$ we have $\partial_{A_{\mathfrak{q}}}(f, g) = 1$.

Proof. Since the tame symbols $\partial_{A_{\mathfrak{q}}}(f, g)$ are bilinear and the order functions $\text{ord}_{A/\mathfrak{q}}$ are additive it suffices to prove the formula when f and g are elements of A. This case is proven in Lemma 42.6.2. □

0EAY Remark 42.6.4 (Milnor K-theory). For a field k let us denote $K_*^M(k)$ the quotient of the tensor algebra on k^* divided by the two-sided ideal generated by the elements $x \otimes 1 - x$ for $x \in k \setminus \{0, 1\}$. Thus $K_0^M(k) = \mathbf{Z}$, $K_1^M(k) = k^*$, and

$$K_2^M(k) = k^* \otimes_{\mathbf{Z}} k^* / \langle x \otimes 1 - x \rangle$$

If A is a discrete valuation ring with fraction field $F = \text{Frac}(A)$ and residue field κ, there is a tame symbol

$$\partial_A : K_{i+1}^M(F) \to K_i^M(\kappa)$$

defined as in Section 42.5; see [Kat86]. More generally, this map can be extended to the case where A is an excellent local domain of dimension 1 using normalization and norm maps on K_i^M, see [Kat86]; presumably the method in Section 42.5 can be used to extend the construction of the tame symbol ∂_A to arbitrary Noetherian local domains A of dimension 1. Next, let X be a Noetherian scheme with a dimension function δ. Then we can use these tame symbols to get the arrows in the following:

$$\bigoplus\nolimits_{\delta(x)=j+1} K_{i+1}^M(\kappa(x)) \longrightarrow \bigoplus\nolimits_{\delta(x)=j} K_i^M(\kappa(x)) \longrightarrow \bigoplus\nolimits_{\delta(x)=j-1} K_{i-1}^M(\kappa(x))$$

However, it is not clear, that the composition is zero, i.e., that we obtain a complex of abelian groups. For excellent X this is shown in [Kat86]. When $i = 1$ and j arbitrary, this follows from Lemma 42.6.3.

42.7. Setup

02QK We will throughout work over a locally Noetherian universally catenary base S endowed with a dimension function δ. Although it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. It is exactly the generality discussed in [Tho90]. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

02QL Situation 42.7.1. Here S is a locally Noetherian, and universally catenary scheme. Moreover, we assume S is endowed with a dimension function $\delta : S \longrightarrow \mathbf{Z}$.

See Morphisms, Definition 29.17.1 for the notion of a universally catenary scheme, and see Topology, Definition 5.20.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 28.11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 29.17.5.

Let (S, δ) be as in Situation 42.7.1. Any scheme X locally of finite type over S is locally Noetherian and catenary. In fact, X has a canonical dimension function

$$\delta = \delta_{X/S} : X \longrightarrow \mathbf{Z}$$

associated to $(f : X \to S, \delta)$ given by the rule $\delta_{X/S}(x) = \delta(f(x)) + \text{trdeg}_{\kappa(f(x))}\kappa(x)$. See Morphisms, Lemma 29.52.3. Moreover, if $h : X \to Y$ is a morphism of schemes locally of finite type over S, and $x \in X$, $y = h(x)$, then obviously $\delta_{X/S}(x) = \delta_{Y/S}(y) + \text{trdeg}_{\kappa(y)}\kappa(x)$. We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g. $\mathbf{Z}$, or a discrete valuation ring).

02QM Example 42.7.2. Here $S = \text{Spec}(k)$ and k is a field. We set $\delta(pt) = 0$ where pt indicates the unique point of S. The pair (S, δ) is an example of a situation as in Situation 42.7.1 by Morphisms, Lemma 29.17.5.

02QN Example 42.7.3. Here $S = \text{Spec}(A)$, where A is a Noetherian domain of dimension 1. For example we could consider $A = \mathbf{Z}$. We set $\delta(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is a maximal ideal and $\delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = (0)$ corresponds to the generic point. This is an example of Situation 42.7.1 by Morphisms, Lemma 29.17.5.

0F91 Example 42.7.4. Here S is a Cohen-Macaulay scheme. Then S is universally catenary by Morphisms, Lemma 29.17.5. We set $\delta(s) = -\dim(\mathcal{O}_{S,s})$. If $s' \rightsquigarrow s$ is a nontrivial specialization of points of S, then $\mathcal{O}_{S,s'}$ is the localization of $\mathcal{O}_{S,s}$ at a nonmaximal prime ideal $\mathfrak{p} \subset \mathcal{O}_{S,s}$, see Schemes, Lemma 26.13.2. Thus $\dim(\mathcal{O}_{S,s}) = \dim(\mathcal{O}_{S,s'}) + \dim(\mathcal{O}_{S,s}/\mathfrak{p}) > \dim(\mathcal{O}_{S,s'})$ by Algebra, Lemma 10.104.4. Hence $\delta(s') > \delta(s)$. If $s' \rightsquigarrow s$ is an immediate specialization, then there is no prime ideal strictly between $\mathfrak{p}$ and $\mathfrak{m}_s$ and we find $\delta(s') = \delta(s) + 1$. Thus δ is a dimension function. In other words, the pair (S, δ) is an example of Situation 42.7.1.

If S is Jacobson and δ sends closed points to zero, then δ is the function sending a point to the dimension of its closure.

02QO Lemma 42.7.5. Let (S, δ) be as in Situation 42.7.1. Assume in addition S is a Jacobson scheme, and $\delta(s) = 0$ for every closed point s of S. Let X be locally of finite type over S. Let $Z \subset X$ be an integral closed subscheme and let $\xi \in Z$ be its generic point. The following integers are the same:

(1) $\delta_{X/S}(\xi)$,
(2) $\dim(Z)$, and
(3) $\dim(\mathcal{O}_{Z,z})$ where z is a closed point of Z.

Proof. Let $X \to S$, $\xi \in Z \subset X$ be as in the lemma. Since X is locally of finite type over S we see that X is Jacobson, see Morphisms, Lemma 29.16.9. Hence closed points of X are dense in every closed subset of Z and map to closed points of S. Hence given any chain of irreducible closed subsets of Z we can end it with a closed point of Z. It follows that $\dim(Z) = \sup_z(\dim(\mathcal{O}_{Z,z})$ (see Properties, Lemma 28.10.3) where $z \in Z$ runs over the closed points of Z. Note that $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z)$ by the properties of a dimension function. For each closed $z \in Z$ the field extension $\kappa(z)/\kappa(f(z))$ is finite, see Morphisms, Lemma 29.16.8. Hence $\delta_{X/S}(z) = \delta(f(z)) = 0$ for $z \in Z$ closed. It follows that all three integers are equal. ☐

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if $S = \text{Spec}(\mathbf{C}[[t]])$ and $X = \text{Spec}(\mathbf{C}((t)))$ then we would get $\delta(x) = 1$ for the unique point of X, but $\dim(X) = 0$. Still we want to think of $\delta_{X/S}$ as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

02QP Definition 42.7.6. Let (S, δ) as in Situation 42.7.1. For any scheme X locally of finite type over S and any irreducible closed subset $Z \subset X$ we define

$$\dim_\delta(Z) = \delta(\xi)$$

where $\xi \in Z$ is the generic point of Z. We will call this the δ-dimension of Z. If Z is a closed subscheme of X, then we define $\dim_\delta(Z)$ as the supremum of the δ-dimensions of its irreducible components.

42.8. Cycles

02QQ Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

02QR Definition 42.8.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) A cycle on X is a formal sum
$$\alpha = \sum n_Z[Z]$$
where the sum is over integral closed subschemes $Z \subset X$, each $n_Z \in \mathbf{Z}$, and the collection $\{Z; n_Z \neq 0\}$ is locally finite (Topology, Definition 5.28.4).

(2) A k-cycle on X is a cycle
$$\alpha = \sum n_Z[Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(3) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subschemes of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by

$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$

i.e., by adding the coefficients.

0GU2 Remark 42.8.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. Then we can write

$$Z_k(X) = \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x)) \quad \subset \quad \bigoplus\nolimits_{\delta(x)=k} K_0^M(\kappa(x))$$

with the following notation and conventions:

(1) $K_0^M(\kappa(x)) = \mathbf{Z}$ is the degree 0 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 42.6.4), and
(2) the direct sum on the right is over all points $x \in X$ with $\delta(x) = k$,
(3) the notation $\bigoplus'_x$ signifies that we consider the subgroup consisting of locally finite elements; namely, elements $\sum_x n_x$ such that for every quasi-compact open $U \subset X$ the set of $x \in U$ with $n_x \neq 0$ is finite.

0H46 Definition 42.8.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The support of a cycle $\alpha = \sum n_Z[Z]$ on X is

$$\text{Supp}(\alpha) = \bigcup\nolimits_{n_Z \neq 0} Z \subset X$$

Since the collection $\{Z; n_Z \neq 0\}$ is locally finite we see that $\text{Supp}(\alpha)$ is a closed subset of X. If α is a k-cycle, then every irreducible component Z of $\text{Supp}(\alpha)$ has δ-dimension k.

0H47 Definition 42.8.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. A cycle α on X is effective if it can be written as $\alpha = \sum n_Z[Z]$ with $n_Z \geq 0$ for all Z.

The set of all effective cycles is a monoid because the sum of two effective cycles is effective, but it is not a group (unless $X = \emptyset$).

42.9. Cycle associated to a closed subscheme

02QS
02QT Lemma 42.9.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

(1) Let $Z' \subset Z$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then
$$\text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi} < \infty$$
(2) If $\dim_\delta(Z) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of Z.

Proof. Let $Z' \subset Z$, $\xi \in Z'$ be as in (1). Then $\dim(\mathcal{O}_{Z,\xi}) = 0$ (for example by Properties, Lemma 28.10.3). Hence $\mathcal{O}_{Z,\xi}$ is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 10.60.7). Hence, it also has finite length over $\mathcal{O}_{X,\xi}$, see Algebra, Lemma 10.52.5.

Assume $\xi \in Z$ and $\delta(\xi) = k$. Consider the closure $Z' = \overline{\{\xi\}}$. It is an irreducible closed subscheme with $\dim_\delta(Z') = k$ by definition. Since $\dim_\delta(Z) = k$ it must be an irreducible component of Z. Hence we see (2) holds. □

02QU Definition 42.9.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

(1) For any irreducible component $Z' \subset Z$ with generic point ξ the integer $m_{Z',Z} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi}$ (Lemma 42.9.1) is called the multiplicity of Z' in Z.
(2) Assume $\dim_\delta(Z) \leq k$. The k-cycle associated to Z is

$$[Z]_k = \sum m_{Z',Z}[Z']$$

where the sum is over the irreducible components of Z of δ-dimension k. (This is a k-cycle by Divisors, Lemma 31.26.1.)

It is important to note that we only define $[Z]_k$ if the δ-dimension of Z does not exceed k. In other words, by convention, if we write $[Z]_k$ then this implies that $\dim_\delta(Z) \leq k$.

42.10. Cycle associated to a coherent sheaf

02QV

02QW Lemma 42.10.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) The collection of irreducible components of the support of $\mathcal{F}$ is locally finite.
(2) Let $Z' \subset \text{Supp}(\mathcal{F})$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then

$$\text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$

(3) If $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of $\text{Supp}(\mathcal{F})$.

Proof. By Cohomology of Schemes, Lemma 30.9.7 the support Z of $\mathcal{F}$ is a closed subset of X. We may think of Z as a reduced closed subscheme of X (Schemes, Lemma 26.12.4). Hence (1) follows from Divisors, Lemma 31.26.1 applied to Z and (3) follows from Lemma 42.9.1 applied to Z.

Let $\xi \in Z'$ be as in (2). In this case for any specialization $\xi' \rightsquigarrow \xi$ in X we have $\mathcal{F}_{\xi'} = 0$. Recall that the non-maximal primes of $\mathcal{O}_{X,\xi}$ correspond to the points of X specializing to ξ (Schemes, Lemma 26.13.2). Hence $\mathcal{F}_\xi$ is a finite $\mathcal{O}_{X,\xi}$-module whose support is $\{\mathfrak{m}_\xi\}$. Hence it has finite length by Algebra, Lemma 10.62.3. $\square$

02QX Definition 42.10.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) For any irreducible component $Z' \subset \text{Supp}(\mathcal{F})$ with generic point ξ the integer $m_{Z',\mathcal{F}} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$ (Lemma 42.10.1) is called the multiplicity of Z' in $\mathcal{F}$.
(2) Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The k-cycle associated to $\mathcal{F}$ is

$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$

where the sum is over the irreducible components of $\text{Supp}(\mathcal{F})$ of δ-dimension k. (This is a k-cycle by Lemma 42.10.1.)

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\text{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

02QY Lemma 42.10.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. If $\dim_\delta(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.

Proof. This is because in this case the multiplicities $m_{Z',Z}$ and $m_{Z',\mathcal{O}_Z}$ agree by definition. $\square$

02QZ Lemma 42.10.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.

Proof. Follows immediately from additivity of lengths, see Algebra, Lemma 10.52.3. $\square$

42.11. Preparation for proper pushforward

02R0

02R1 Lemma 42.11.1. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either $f(X)$ is contained in a proper closed subscheme of Y, or f is dominant and the extension of function fields $R(X)/R(Y)$ is finite.

Proof. The closure $\overline{f(X)} \subset Y$ is irreducible as X is irreducible (Topology, Lemmas 5.8.2 and 5.8.3). If $\overline{f(X)} \neq Y$, then we are done. If $\overline{f(X)} = Y$, then f is dominant and by Morphisms, Lemma 29.8.6 we see that the generic point η_Y of Y is in the image of f. Of course this implies that $f(\eta_X) = \eta_Y$, where $\eta_X \in X$ is the generic point of X. Since $\delta(\eta_X) = \delta(\eta_Y)$ we see that $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$ is an extension of transcendence degree 0. Hence $R(Y) \subset R(X)$ is a finite extension by Morphisms, Lemma 29.51.7 (which applies by Morphisms, Lemma 29.15.8). $\square$

02R2 Lemma 42.11.2. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is quasi-compact, and $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of X. Then $\{\overline{f(Z_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of Y.

Proof. Let $V \subset Y$ be a quasi-compact open subset. Since f is quasi-compact the open $f^{-1}(V)$ is quasi-compact. Hence the set $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$ is finite by a simple topological argument which we omit. Since this is the same as the set

$$\{i \in I \mid f(Z_i) \cap V \neq \emptyset\} = \{i \in I \mid \overline{f(Z_i)} \cap V \neq \emptyset\}$$

the lemma is proved. $\square$

42.12. Proper pushforward

02R3

02R4 Definition 42.12.1. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = k$. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if} \quad \dim_\delta(f(Z)) = k. \end{cases}$$

Here we think of $f(Z) \subset Y$ as an integral closed subscheme. The degree of Z over $f(Z)$ is finite if $\dim_\delta(f(Z)) = \dim_\delta(Z)$ by Lemma 42.11.1.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The pushforward of α as the sum
$$f_*\alpha = \sum n_Z f_*[Z]$$
where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 42.11.2 above.

By definition the proper pushforward of cycles
$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$
is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category of schemes locally of finite type over S with morphisms equal to proper morphisms.

02R5 Lemma 42.12.2. Let (S, δ) be as in Situation 42.7.1. Let X, Y, and Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.

Proof. Let $W \subset X$ be an integral closed subscheme of dimension k. Consider $W' = f(W) \subset Y$ and $W'' = g(f(W)) \subset Z$. Since f, g are proper we see that W' (resp. W'') is an integral closed subscheme of Y (resp. Z). We have to show that $g_*(f_*[W]) = (g \circ f)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see the induced morphisms
$$W \longrightarrow W' \longrightarrow W''$$
both satisfy the hypotheses of Lemma 42.11.1. Hence
$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (g \circ f)_*[W] = \deg(W/W'')[W''].$$
Then we can apply Morphisms, Lemma 29.51.9 to conclude. □

A closed immersion is proper. If $i : Z \to X$ is a closed immersion then the maps
$$i_* : Z_k(Z) \longrightarrow Z_k(X)$$
are all injective.

0F92 Lemma 42.12.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $X_1, X_2 \subset X$ be closed subschemes such that $X = X_1 \cup X_2$ set theoretically. For every $k \in \mathbf{Z}$ the sequence of abelian groups
$$Z_k(X_1 \cap X_2) \longrightarrow Z_k(X_1) \oplus Z_k(X_2) \longrightarrow Z_k(X) \longrightarrow 0$$
is exact. Here $X_1 \cap X_2$ is the scheme theoretic intersection and the maps are the pushforward maps with one multiplied by -1.

Proof. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. The groups $Z_k(X_1)$, $Z_k(X_2)$, $Z_k(X_1 \cap X_2)$ are free on the subset of these Z such that $Z \subset X_1$, $Z \subset X_2$, $Z \subset X_1 \cap X_2$. This immediately proves the lemma in this case. The general case is similar and the proof is omitted. □

02R6 Lemma 42.12.4. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a proper morphism of schemes which are locally of finite type over S.

(1) Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then
$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Then

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Schemes, Proposition 30.19.1.

Proof. Part (1) follows from (2) and Lemma 42.10.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. By Cohomology of Schemes, Lemma 30.9.7 there exists a closed subscheme $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$. Consider the commutative diagram of schemes

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f|_Z"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i'"] & Y \end{tikzcd}$$

We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion is straightforward (omitted). Note that $f|_Z : Z \to Z'$ is a dominant morphism (see Morphisms, Lemma 29.6.3). Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant. Since f is dominant, for every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subscheme Z is actually an irreducible component of Y of δ-dimension k. Pick such an integral closed subscheme $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 42.9.1). Since f is quasi-compact and X is locally Noetherian, there can be only finitely many of these and hence $f^{-1}(\{\eta\})$ is finite. By Morphisms, Lemma 29.51.1 there exists an open neighbourhood $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite.

Write $Y = \text{Spec}(R)$ and $X = \text{Spec}(A)$ (possible as a finite morphism is affine). Then R and A are Noetherian rings and A is finite over R. Moreover $\mathcal{F} = \widetilde{M}$ for some finite A-module M. Note that $f_*\mathcal{F}$ corresponds to M viewed as an R-module. Let $\mathfrak{p} \subset R$ be the minimal prime corresponding to $\eta \in Y$. The coefficient of Z in $[f_*\mathcal{F}]_k$ is clearly $\text{length}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the primes of A lying over $\mathfrak{p}$. Then $A_\mathfrak{p} = \prod A_{\mathfrak{q}_i}$ since $A_\mathfrak{p}$ is an Artinian ring being finite over the dimension zero local Noetherian ring $R_\mathfrak{p}$. Clearly the coefficient of Z in $f_*[\mathcal{F}]_k$ is

$$\sum\nolimits_{i=1,\ldots,t} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Hence the desired equality follows from Algebra, Lemma 10.52.12. □

42.13. Preparation for flat pullback

02R7 Recall that a morphism $f : X \to Y$ which is locally of finite type is said to have relative dimension r if every nonempty fibre is equidimensional of dimension r. See Morphisms, Definition 29.29.1.

02R8 Lemma 42.13.1. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r. For any closed subset $Z \subset Y$ we have

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

provided $f^{-1}(Z)$ is nonempty. If Z is irreducible and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. It suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by the scheme theoretic inverse image $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 29.25.10 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Note that $\dim_\xi(X_\eta) = r$ by assumption that f has relative dimension r. On the other hand, since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 29.28.1 we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

02R9 Lemma 42.13.2. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of Y. Then $\{f^{-1}(Z_i)\}_{i \in I}$ is a locally finite collection of closed subsets of X.

Proof. Let $U \subset X$ be a quasi-compact open subset. Since the image $f(U) \subset Y$ is a quasi-compact subset there exists a quasi-compact open $V \subset Y$ such that $f(U) \subset V$. Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

42.14. Flat pullback

02RA In the following we use $f^{-1}(Z)$ to denote the scheme theoretic inverse image of a closed subscheme $Z \subset Y$ for a morphism of schemes $f : X \to Y$. We recall that the scheme theoretic inverse image is the fibre product

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

and it is also the closed subscheme of X cut out by the quasi-coherent sheaf of ideals $f^{-1}(\mathcal{I})\mathcal{O}_X$, if $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y. (This is discussed in Schemes, Section 26.4 and Lemma 26.17.6 and Definition 26.17.7.)

02RB Definition 42.14.1. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. We define $f^*[Z]$ to be the $(k+r)$-cycle on X to the scheme theoretic inverse image
$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$
This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 42.13.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The flat pullback of α by f is the sum
$$f^*\alpha = \sum n_i f^*[Z_i]$$
where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 42.13.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the restriction of the cycle to U. Note that in this case the maps
$$j^* : Z_k(X) \longrightarrow Z_k(U)$$
are all surjective. The reason is that given any integral closed subscheme $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subscheme of X (see Schemes, Lemma 26.12.4). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

02RC Lemma 42.14.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. For every $k \in \mathbf{Z}$ the sequence
$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$
is an exact complex of abelian groups.

Proof. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. Such a basis element maps either to the basis element $[Z \cap U]$ or to zero if $Z \subset Y$. Hence the lemma is clear in this case. The general case is similar and the proof is omitted. $\square$

02RD Lemma 42.14.3. Let (S, δ) be as in Situation 42.7.1. Let X, Y, Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s. Then $g \circ f$ is flat of relative dimension $r + s$ and
$$f^* \circ g^* = (g \circ f)^*$$
as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms, Lemma 29.29.3. Suppose that

(1) $W \subset Z$ is a closed integral subscheme of δ-dimension k,
(2) $W' \subset Y$ is a closed integral subscheme of δ-dimension $k+s$ with $W' \subset g^{-1}(W)$, and
(3) $W'' \subset Y$ is a closed integral subscheme of δ-dimension $k+s+r$ with $W'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[W'']$ in $(g \circ f)^*[W]$ agrees with the coefficient m of $[W'']$ in $f^*(g^*[W])$. That it suffices to check the lemma in these cases follows from Lemma 42.13.1. Let $\xi'' \in W''$, $\xi' \in W'$ and $\xi \in W$ be the generic points. Consider the local rings $A = \mathcal{O}_{Z,\xi}$, $B = \mathcal{O}_{Y,\xi'}$ and $C = \mathcal{O}_{X,\xi''}$. Then we have local flat ring maps $A \to B$, $B \to C$ and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C)\text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 10.52.14. □

02RE Lemma 42.14.4. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r.

(1) Let $Z \subset Y$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.
(2) Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Then we have $\dim_\delta(\text{Supp}(f^*\mathcal{F})) \leq k + r$ and
$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$
in $Z_{k+r}(X)$.

Proof. The statements on dimensions follow immediately from Lemma 42.13.1. Part (1) follows from part (2) by Lemma 42.10.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Schemes, Lemma 30.9.1 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules, Lemma 17.9.2), hence $f^*\mathcal{F}$ is coherent (Cohomology of Schemes, Lemma 30.9.1 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X$ be an integral closed subscheme of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W']$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[f^*\mathcal{F}]_{k+r}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 42.10.1.) We have $f^*\mathcal{F}_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 10.52.13. □

42.15. Push and pull

02RF In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

02RG Lemma 42.15.1. Let (S, δ) be as in Situation 42.7.1. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of schemes locally of finite type over S. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_* (g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms, Lemma 29.41.5. The assertion that g' is flat of relative dimension r follows from Morphisms, Lemmas 29.29.2 and 29.25.8. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subscheme $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 42.10.3. By Lemmas 42.12.4 and 42.14.4 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^* f_* \mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Schemes, Lemma 30.5.2. □

02RH Lemma 42.15.2. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms, Definition 29.48.1). Then f is both proper and flat of relative dimension 0, and

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms, Lemma 29.48.2, and a finite morphism is proper by Morphisms, Lemma 29.44.11. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 29.48.4) we see that $f_* f^* \mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Hence

$$f_* f^* [Z] = f_* f^* [\mathcal{O}_Z]_k = [f_* f^* \mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 42.14.4 and 42.12.4. □

42.16. Preparation for principal divisors

02RI Some of the material in this section partially overlaps with the discussion in Divisors, Section 31.26.

02RK Lemma 42.16.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral.

(1) If $Z \subset X$ is an integral closed subscheme, then the following are equivalent:
  (a) Z is a prime divisor,
  (b) Z has codimension 1 in X, and
  (c) $\dim_\delta(Z) = \dim_\delta(X) - 1$.
(2) If Z is an irreducible component of an effective Cartier divisor on X, then $\dim_\delta(Z) = \dim_\delta(X) - 1$.

Proof. Part (1) follows from the definition of a prime divisor (Divisors, Definition 31.26.2) and the definition of a dimension function (Topology, Definition 5.20.1). Let $\xi \in Z$ be the generic point of an irreducible component Z of an effective Cartier divisor $D \subset X$. Then $\dim(\mathcal{O}_{D,\xi}) = 0$ and $\mathcal{O}_{D,\xi} = \mathcal{O}_{X,\xi}/(f)$ for some nonzerodivisor $f \in \mathcal{O}_{X,\xi}$ (Divisors, Lemma 31.15.2). Then $\dim(\mathcal{O}_{X,\xi}) = 1$ by Algebra, Lemma 10.60.13. Hence Z is as in (1) by Properties, Lemma 28.10.3 and the proof is complete. □

02RM **Lemma 42.16.2.** Let $f : X \to Y$ be a morphism of schemes. Let $\xi \in Y$ be a point. Assume that

(1) X, Y are integral,
(2) Y is locally Noetherian
(3) f is proper, dominant and $R(Y) \subset R(X)$ is finite, and
(4) $\dim(\mathcal{O}_{Y,\xi}) = 1$.

Then there exists an open neighbourhood $V \subset Y$ of ξ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. This lemma is a special case of Varieties, Lemma 33.17.2. Here is a direct argument in this case. By Cohomology of Schemes, Lemma 30.21.2 it suffices to prove that $f^{-1}(\{\xi\})$ is finite. We replace Y by an affine open, say $Y = \mathrm{Spec}(R)$. Note that R is Noetherian, as Y is assumed locally Noetherian. Since f is proper it is quasi-compact. Hence we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_n$ with each $U_i = \mathrm{Spec}(A_i)$. Note that $R \to A_i$ is a finite type injective homomorphism of domains such that the induced extension of fraction fields is finite. Thus the lemma follows from Algebra, Lemma 10.113.2. □

42.17. Principal divisors

02RN The following definition is the analogue of Divisors, Definition 31.26.5 in our current setup.

02RO **Definition 42.17.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to f* is the $(n-1)$-cycle

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

defined in Divisors, Definition 31.26.5. This makes sense because prime divisors have δ-dimension $n-1$ by Lemma 42.16.1.

In the situation of the definition for $f, g \in R(X)^*$ we have

$$\mathrm{div}_X(fg) = \mathrm{div}_X(f) + \mathrm{div}_X(g)$$

in $Z_{n-1}(X)$. See Divisors, Lemma 31.26.6. The following lemma will be superseded by the more general Lemma 42.20.2.

02RR **Lemma 42.17.2.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $g \in R(Y)^*$. Then

$$f^*(\mathrm{div}_Y(g)) = \mathrm{div}_X(g)$$

in $Z_{n+r-1}(X)$.

Proof. Note that since f is flat it is dominant so that f induces an embedding $R(Y) \subset R(X)$, and hence we may think of g as an element of $R(X)^*$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n + r - 1$. Let $\xi \in Z$ be its generic point. If $\dim_\delta(f(Z)) > n - 1$, then we see that the coefficient of $[Z]$ in the left and right hand side of the equation is zero. Hence we may assume that $Z' = \overline{f(Z)}$ is an integral closed subscheme of Y of δ-dimension $n - 1$. Let $\xi' = f(\xi)$. It is the generic point of Z'. Set $A = \mathcal{O}_{Y,\xi'}$, $B = \mathcal{O}_{X,\xi}$. The ring map $A \to B$ is a flat local homomorphism of Noetherian local domains of dimension 1. We have g in the fraction field of A. What we have to show is that

$$\text{ord}_A(g)\text{length}_B(B/\mathfrak{m}_A B) = \text{ord}_B(g).$$

This follows from Algebra, Lemma 10.52.13 (details omitted). □

42.18. Principal divisors and pushforward

02RS The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

02RT Lemma 42.18.1. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set

$$g = \text{Nm}_{R(X)/R(Y)}(f).$$

Then we have $p_*\text{div}(f) = \text{div}(g)$.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension $n - 1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. We may apply Lemma 42.16.2 to the morphism $p : X \to Y$ and the generic point $\xi \in Z$. Hence we may replace Y by an affine open neighbourhood of ξ and assume that $p : X \to Y$ is finite. Write $Y = \text{Spec}(R)$ and $X = \text{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an finite field extension L/K of fraction fields. Now we have $f \in L$, $g = \text{Nm}(f) \in K$, and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\text{div}_Y(g)$ is $\text{ord}_{R_\mathfrak{p}}(g)$. The coefficient of $[Z]$ in $p_*\text{div}_X(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\text{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 10.121.8. □

An important role in the discussion of principal divisors is played by the "universal" principal divisor $[0] - [\infty]$ on $\mathbf{P}^1_S$. To make this more precise, let us denote

0F93 (42.18.1.1) $$D_0, D_\infty \subset \mathbf{P}^1_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, T_1])$$

the closed subscheme cut out by the section T_1, resp. T_0 of $\mathcal{O}(1)$. These are effective Cartier divisors, see Divisors, Definition 31.13.1 and Lemma 31.14.10. The following lemma says that loosely speaking we have "$\text{div}(T_1/T_0) = [D_0] - [D_1]$" and that this is the universal principal divisor.

02RQ Lemma 42.18.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponds to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times_S \mathbf{P}^1_S$ be the closure of the graph of $f : U \to \mathbf{P}^1_S$. Then

(1) the projection morphism $p : Y \to X$ is proper,

(2) $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism,
(3) the pullbacks $Y_0 = q^{-1}D_0$ and $Y_\infty = q^{-1}D_\infty$ via the morphism $q : Y \to \mathbf{P}^1_S$ are defined (Divisors, Definition 31.13.12),
(4) we have
$$\mathrm{div}_Y(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$
(5) we have
$$\mathrm{div}_X(f) = p_*\mathrm{div}_Y(f)$$
(6) if we view Y_0 and Y_∞ as closed subschemes of X via the morphism p then we have
$$\mathrm{div}_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

Proof. Since X is integral, we see that U is integral. Hence Y is integral, and $(1, f)(U) \subset Y$ is an open dense subscheme. Also, note that the closed subscheme $Y \subset X \times_S \mathbf{P}^1_S$ does not depend on the choice of the open U, since after all it is the closure of the one point set $\{\eta'\} = \{(1, f)(\eta)\}$ where $\eta \in X$ is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that p is the composition of the closed immersion $Y \to X \times_S \mathbf{P}^1_S = \mathbf{P}^1_X$ with the proper morphism $\mathbf{P}^1_X \to X$. As a composition of proper morphisms is proper (Morphisms, Lemma 29.41.4) we conclude.

It is clear that $Y \cap U \times_S \mathbf{P}^1_S = (1, f)(U)$. Thus (2) follows. It also follows that $\dim_\delta(Y) = n$.

Note that $q(\eta') = f(\eta)$ is not contained in D_0 or D_∞ since $f \in R(X)^*$. Hence (3) by Divisors, Lemma 31.13.13. We obtain $\dim_\delta(Y_0) = n - 1$ and $\dim_\delta(Y_\infty) = n - 1$ from Lemma 42.16.1.

Consider the effective Cartier divisor Y_0. At every point $\xi \in Y_0$ we have $f \in \mathcal{O}_{Y,\xi}$ and the local equation for Y_0 is given by f. In particular, if $\delta(\xi) = n - 1$ so ξ is the generic point of a integral closed subscheme Z of δ-dimension $n - 1$, then we see that the coefficient of $[Z]$ in $\mathrm{div}_Y(f)$ is
$$\mathrm{ord}_Z(f) = \mathrm{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \mathrm{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y_0,\xi})$$
which is the coefficient of $[Z]$ in $[Y_0]_{n-1}$. A similar argument using the rational function $1/f$ shows that $-[Y_\infty]$ agrees with the terms with negative coefficients in the expression for $\mathrm{div}_Y(f)$. Hence (4) follows.

Note that $D_0 \to S$ is an isomorphism. Hence we see that $X \times_S D_0 \to X$ is an isomorphism as well. Clearly we have $Y_0 = Y \cap X \times_S D_0$ (scheme theoretic intersection) inside $X \times_S \mathbf{P}^1_S$. Hence it is really the case that $Y_0 \to X$ is a closed immersion. It follows that
$$p_*\mathcal{O}_{Y_0} = \mathcal{O}_{Y_0'}$$
where $Y_0' \subset X$ is the image of $Y_0 \to X$. By Lemma 42.12.4 we have $p_*[Y_0]_{n-1} = [Y_0']_{n-1}$. The same is true for D_∞ and Y_∞. Hence (6) is a consequence of (5). Finally, (5) follows immediately from Lemma 42.18.1. $\square$

The following lemma says that the degree of a principal divisor on a proper curve is zero.

02RU Lemma 42.18.3. Let K be any field. Let X be a 1-dimensional integral scheme endowed with a proper morphism $c : X \to \mathrm{Spec}(K)$. Let $f \in K(X)^*$ be an invertible rational function. Then

$$\sum\nolimits_{x \in X \text{ closed}} [\kappa(x) : K] \mathrm{ord}_{\mathcal{O}_{X,x}}(f) = 0$$

where ord is as in Algebra, Definition 10.121.2. In other words, $c_*\mathrm{div}(f) = 0$.

Proof. Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "p"'] \arrow[d, "q"'] & X \arrow[d, "c"] \\ \mathbf{P}^1_K \arrow[r, "c'"] & \mathrm{Spec}(K) \end{tikzcd}$$

that we constructed in Lemma 42.18.2 starting with X and the rational function f over $S = \mathrm{Spec}(K)$. We will use all the results of this lemma without further mention. We have to show that $c_*\mathrm{div}_X(f) = c_*p_*\mathrm{div}_Y(f) = 0$. This is the same as proving that $c'_*q_*\mathrm{div}_Y(f) = 0$. If $q(Y)$ is a closed point of $\mathbf{P}^1_K$ then we see that $\mathrm{div}_X(f) = 0$ and the lemma holds. Thus we may assume that q is dominant. Suppose we can show that $q : Y \to \mathbf{P}^1_K$ is finite locally free of degree d (see Morphisms, Definition 29.48.1). Since $\mathrm{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$ we see (by definition of flat pullback) that $\mathrm{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$. Then by Lemma 42.15.2 we get $q_*\mathrm{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$. Since clearly $c'_*[D_0]_0 = c'_*[D_\infty]_0$ we win.

It remains to show that q is finite locally free. (It will automatically have some given degree as $\mathbf{P}^1_K$ is connected.) Since $\dim(\mathbf{P}^1_K) = 1$ we see that q is finite for example by Lemma 42.16.2. All local rings of $\mathbf{P}^1_K$ at closed points are regular local rings of dimension 1 (in other words discrete valuation rings), since they are localizations of $K[T]$ (see Algebra, Lemma 10.114.1). Hence for $y \in Y$ closed the local ring $\mathcal{O}_{Y,y}$ will be flat over $\mathcal{O}_{\mathbf{P}^1_K, q(y)}$ as soon as it is torsion free (More on Algebra, Lemma 15.22.11). This is obviously the case as $\mathcal{O}_{Y,y}$ is a domain and q is dominant. Thus q is flat. Hence q is finite locally free by Morphisms, Lemma 29.48.2. □

42.19. Rational equivalence

02RV In this section we define rational equivalence on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 42.19.5. However, if we do not allow these then we do not know how to prove that capping with Chern classes of line bundles factors through rational equivalence.

02RW Definition 42.19.1. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider

$$\sum (i_j)_*\mathrm{div}(f_j) \in Z_k(X)$$

where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.

(2) We say that $\alpha \in Z_k(X)$ is rationally equivalent to zero if α is a cycle of the form displayed above.

(3) We say $\alpha, \beta \in Z_k(X)$ are rationally equivalent and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
(4) We define
$$\mathrm{CH}_k(X) = Z_k(X)/ \sim_{rat}$$
to be the Chow group of k-cycles on X. This is sometimes called the Chow group of k-cycles modulo rational equivalence on X.

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all.

0GU3 Remark 42.19.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. Let us show that we have a presentation
$$\bigoplus\nolimits'_{\delta(x)=k+1} K_1^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x)) \to \mathrm{CH}_k(X) \to 0$$
Here we use the notation and conventions introduced in Remark 42.8.2 and in addition
(1) $K_1^M(\kappa(x)) = \kappa(x)^*$ is the degree 1 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 42.6.4), and
(2) the differential ∂ is defined as follows: given an element $\xi = \sum_x f_x$ we denote $W_x = \overline{x}$ the integral closed subscheme of X with generic point x and we set
$$\partial(\xi) = \sum (W_x \to X)_*\mathrm{div}(f_x)$$
in $Z_k(X)$ which makes sense as we have seen that the second term of the complex is equal to $Z_k(X)$ by Remark 42.8.2.

The fact that we obtain a presentation of $\mathrm{CH}_k(X)$ follows immediately by comparing with Definition 42.19.1.

A very simple but important lemma is the following.

02RX Lemma 42.19.3. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that
$$\alpha \sim_{rat} \beta + i_*\gamma.$$
In other words, the sequence
$$\mathrm{CH}_k(Y) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \longrightarrow 0$$
is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j \in J}$ be a locally finite collection of integral closed subschemes of U of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_*\mathrm{div}(f_j)$ as in the definition. Set $W_j' \subset X$ equal to the closure of W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that
$$\alpha - \beta - \sum (i_j')_*\mathrm{div}(f_j)$$
is a cycle supported on Y and the lemma follows (see Lemma 42.14.2). $\square$

0F94 Lemma 42.19.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $X_1, X_2 \subset X$ be closed subschemes such that $X = X_1 \cup X_2$ set theoretically. For every $k \in \mathbf{Z}$ the sequence of abelian groups

$$\mathrm{CH}_k(X_1 \cap X_2) \longrightarrow \mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \longrightarrow \mathrm{CH}_k(X) \longrightarrow 0$$

is exact. Here $X_1 \cap X_2$ is the scheme theoretic intersection and the maps are the pushforward maps with one multiplied by -1.

Proof. By Lemma 42.12.3 the arrow $\mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \to \mathrm{CH}_k(X)$ is surjective. Suppose that (α_1, α_2) maps to zero under this map. Write $\alpha_1 = \sum n_{1,i}[W_{1,i}]$ and $\alpha_2 = \sum n_{2,i}[W_{2,i}]$. Then we obtain a locally finite collection $\{W_j\}_{j \in J}$ of integral closed subschemes of X of δ-dimension $k+1$ and $f_j \in R(W_j)^*$ such that

$$\sum n_{1,i}[W_{1,i}] + \sum n_{2,i}[W_{2,i}] = \sum (i_j)_*\mathrm{div}(f_j)$$

as cycles on X where $i_j : W_j \to X$ is the inclusion morphism. Choose a disjoint union decomposition $J = J_1 \amalg J_2$ such that $W_j \subset X_1$ if $j \in J_1$ and $W_j \subset X_2$ if $j \in J_2$. (This is possible because the W_j are integral.) Then we can write the equation above as

$$\sum n_{1,i}[W_{1,i}] - \sum\nolimits_{j \in J_1} (i_j)_*\mathrm{div}(f_j) = -\sum n_{2,i}[W_{2,i}] + \sum\nolimits_{j \in J_2} (i_j)_*\mathrm{div}(f_j)$$

Hence this expression is a cycle (!) on $X_1 \cap X_2$. In other words the element (α_1, α_2) is in the image of the first arrow and the proof is complete. $\square$

02RY Example 42.19.5. Here is a "strange" example. Suppose that S is the spectrum of a field k with δ as in Example 42.7.2. Suppose that $X = C_1 \cup C_2 \cup \ldots$ is an infinite union of curves $C_j \cong \mathbf{P}^1_k$ glued together in the following way: The point $\infty \in C_j$ is glued transversally to the point $0 \in C_{j+1}$ for $j = 1, 2, 3, \ldots$. Take the point $0 \in C_1$. This gives a zero cycle $[0] \in Z_0(X)$. The "strangeness" in this situation is that actually $[0] \sim_{rat} 0$! Namely we can choose the rational function $f_j \in R(C_j)$ to be the function which has a simple zero at 0 and a simple pole at ∞ and no other zeros or poles. Then we see that the sum $\sum (i_j)_*\mathrm{div}(f_j)$ is exactly the 0-cycle $[0]$. In fact it turns out that $\mathrm{CH}_0(X) = 0$ in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so X does not even exist for you).

02RZ Remark 42.19.6. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ such that α_i, β_i are supported on T_i and such that $\alpha_i = \beta_i$ in $\mathrm{CH}_k(T_i)$, in other words, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of subschemes $W_{i,j} \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

42.20. Rational equivalence and push and pull

02S0 In this section we show that flat pullback and proper pushforward commute with rational equivalence.

0EPH **Lemma 42.20.1.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be schemes locally of finite type over S. Assume Y integral with $\dim_\delta(Y) = k$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then for $g \in R(Y)^*$ we have

$$f^*\text{div}_Y(g) = \sum n_j i_{j,*}\text{div}_{X_j}(g \circ f|_{X_j})$$

as $(k + r - 1)$-cycles on X where the sum is over the irreducible components X_j of X and n_j is the multiplicity of X_j in X.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $k + r - 1$. We have to show that the coefficient n of $[Z]$ in $f^*\text{div}(g)$ is equal to the coefficient m of $[Z]$ in $\sum i_{j,*}\text{div}(g \circ f|_{X_j})$. Let Z' be the closure of $f(Z)$ which is an integral closed subscheme of Y. By Lemma 42.13.1 we have $\dim_\delta(Z') \geq k - 1$. Thus either $Z' = Y$ or Z' is a prime divisor on Y. If $Z' = Y$, then the coefficients n and m are both zero: this is clear for n by definition of f^* and follows for m because $g \circ f|_{X_j}$ is a unit in any point of X_j mapping to the generic point of Y. Hence we may assume that $Z' \subset Y$ is a prime divisor.

We are going to translate the equality of n and m into algebra. Namely, let $\xi' \in Z'$ and $\xi \in Z$ be the generic points. Set $A = \mathcal{O}_{Y,\xi'}$ and $B = \mathcal{O}_{X,\xi}$. Note that A, B are Noetherian, $A \to B$ is flat, local, A is a domain, and $\mathfrak{m}_A B$ is an ideal of definition of the local ring B. The rational function g is an element of the fraction field $Q(A)$ of A. By construction, the closed subschemes X_j which meet ξ correspond 1-to-1 with minimal primes

$$\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset B$$

The integers n_j are the corresponding lengths

$$n_i = \text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

The rational functions $g \circ f|_{X_j}$ correspond to the image $g_i \in \kappa(\mathfrak{q}_i)^*$ of $g \in Q(A)$. Putting everything together we see that

$$n = \text{ord}_A(g)\text{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \text{ord}_{B/\mathfrak{q}_i}(g_i)\text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

Writing $g = x/y$ for some nonzero $x, y \in A$ we see that it suffices to prove

$$\text{length}_A(A/(x))\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B/xB)$$

(equality uses Algebra, Lemma 10.52.13) equals

$$\sum\nolimits_{i=1,\ldots,s} \text{length}_{B/\mathfrak{q}_i}(B/(x, \mathfrak{q}_i))\text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

and similarly for y. As $A \to B$ is flat it follows that x is a nonzerodivisor in B. Hence the desired equality follows from Lemma 42.3.2. $\square$

02S1 **Lemma 42.20.2.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be schemes locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k + r)$-cycles on X.

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k + 1$ and rational functions $g_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on Y. Then we have to show that

$$f^*(\sum i_{j,*}\text{div}(g_j)) = \sum f^* i_{j,*}\text{div}(g_j)$$

is rationally equivalent to zero on X. The sum on the right makes sense as $\{W_j\}$ is locally finite in X by Lemma 42.13.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote $f_j : W'_j \to W_j$ the first projection. By Lemma 42.15.1 we can write the sum above as

$$\sum i'_{j,*}(f_j^*\text{div}(g_j))$$

By Lemma 42.20.1 we see that each $f_j^*\text{div}(g_j)$ is rationally equivalent to zero on W'_j. Hence each $i'_{j,*}(f_j^*\text{div}(g_j))$ is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 42.19.6. □

02S2 Lemma 42.20.3. Let (S, δ) be as in Situation 42.7.1. Let X, Y be schemes locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k + 1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\text{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_* i_{j,*}\text{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subscheme which is the image of $p \circ i_j$. The collection $\{W'_j\}$ is locally finite in Y by Lemma 42.11.2. Hence it suffices to show, for a given j, that either $p_* i_{j,*}\text{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\text{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a single integral closed subscheme $W \subset X$ of δ-dimension $k + 1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "i'"] & Y \end{tikzcd}$$

Note that $p_*i_*\text{div}(f) = i'_*(p')_*\text{div}(f)$ by Lemma 42.12.2. As explained above we have to show that $(p')_*\text{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_*\text{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_*\text{div}(f) = 0$ in this case. Let $\eta \in W'$ be the generic point. Note that $c : W_\eta \to \text{Spec}(K)$ is a proper integral curve over $K = \kappa(\eta)$ whose function field $K(W_\eta)$ is identified with $R(W)$. Here is a diagram

$$\begin{tikzcd} W_\eta \arrow[r] \arrow[d, "c"'] & W \arrow[d, "p'"] \\ \text{Spec}(K) \arrow[r] & W' \end{tikzcd}$$

Let us denote $f_\eta \in K(W_\eta)^*$ the rational function corresponding to $f \in R(W)^*$. Moreover, the closed points ξ of W_η correspond $1-1$ to the closed integral subschemes $Z = Z_\xi \subset W$ of δ-dimension k with $p'(Z) = W'$. Note that the multiplicity of Z_ξ in $\text{div}(f)$ is equal to $\text{ord}_{\mathcal{O}_{W_\eta,\xi}}(f_\eta)$ simply because the local rings $\mathcal{O}_{W_\eta,\xi}$ and $\mathcal{O}_{W,\xi}$ are identified (as subrings of their fraction fields). Hence we see that the multiplicity of $[W']$ in $(p')_*\text{div}(f)$ is equal to the multiplicity of $[\text{Spec}(K)]$ in $c_*\text{div}(f_\eta)$. By Lemma 42.18.3 this is zero.

The case $\dim_\delta(W') = k + 1$. In this case Lemma 42.18.1 applies, and we see that indeed $p'_*\text{div}(f) = \text{div}(g)$ for some $g \in R(W')^*$ as desired. $\square$

42.21. Rational equivalence and the projective line

02S3 Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Given any closed subscheme $Z \subset X \times_S \mathbf{P}^1_S = X \times \mathbf{P}^1$ we let Z_0, resp. Z_∞ be the scheme theoretic closed subscheme $Z_0 = \text{pr}_2^{-1}(D_0)$, resp. $Z_\infty = \text{pr}_2^{-1}(D_\infty)$. Here D_0, D_∞ are as in (42.18.1.1).

02S4 **Lemma 42.21.1.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $W \subset X \times_S \mathbf{P}^1_S$ be an integral closed subscheme of δ-dimension $k + 1$. Assume $W \neq W_0$, and $W \neq W_\infty$. Then

(1) W_0, W_∞ are effective Cartier divisors of W,
(2) W_0, W_∞ can be viewed as closed subschemes of X and
$$[W_0]_k \sim_{rat} [W_\infty]_k,$$
(3) for any locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k + 1$ with $W_i \neq (W_i)_0$ and $W_i \neq (W_i)_\infty$ we have $\sum([(W_i)_0]_k - [(W_i)_\infty]_k) \sim_{rat} 0$ on X, and
(4) for any $\alpha \in Z_k(X)$ with $\alpha \sim_{rat} 0$ there exists a locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ as above such that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$.

Proof. Part (1) follows from Divisors, Lemma 31.13.13 since the generic point of W is not mapped into D_0 or D_∞ under the projection $X \times_S \mathbf{P}^1_S \to \mathbf{P}^1_S$ by assumption.

Since $X \times_S D_0 \to X$ is a closed immersion, we see that W_0 is isomorphic to a closed subscheme of X. Similarly for W_∞. The morphism $p : W \to X$ is proper as a composition of the closed immersion $W \to X \times_S \mathbf{P}^1_S$ and the proper morphism $X \times_S \mathbf{P}^1_S \to X$. By Lemma 42.18.2 we have $[W_0]_k \sim_{rat} [W_\infty]_k$ as cycles on W. Hence part (2) follows from Lemma 42.20.3 as clearly $p_*[W_0]_k = [W_0]_k$ and similarly for W_∞.

The only content of statement (3) is, given parts (1) and (2), that the collection $\{(W_i)_0, (W_i)_\infty\}$ is a locally finite collection of closed subschemes of X. This is clear.

Suppose that $\alpha \sim_{rat} 0$. By definition this means there exist integral closed subschemes $V_i \subset X$ of δ-dimension $k+1$ and rational functions $f_i \in R(V_i)^*$ such that the family $\{V_i\}_{i \in I}$ is locally finite in X and such that $\alpha = \sum (V_i \to X)_* \text{div}(f_i)$. Let

$$W_i \subset V_i \times_S \mathbf{P}^1_S \subset X \times_S \mathbf{P}^1_S$$

be the closure of the graph of the rational map f_i as in Lemma 42.18.2. Then we have that $(V_i \to X)_* \text{div}(f_i)$ is equal to $[(W_i)_0]_k - [(W_i)_\infty]_k$ by that same lemma. Hence the result is clear. $\square$

02S5 Lemma 42.21.2. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let Z be a closed subscheme of $X \times \mathbf{P}^1$. Assume

(1) $\dim_\delta(Z) \leq k+1$,
(2) $\dim_\delta(Z_0) \leq k$, $\dim_\delta(Z_\infty) \leq k$, and
(3) for any embedded point ξ (Divisors, Definition 31.4.1) of Z either $\xi \not\in Z_0 \cup Z_\infty$ or $\delta(\xi) < k$.

Then $[Z_0]_k \sim_{rat} [Z_\infty]_k$ as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of Z which have δ-dimension $k+1$. Write

$$[Z]_{k+1} = \sum n_i [W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Divisors, Lemma 31.26.1. We claim that

$$[Z_0]_k = \sum n_i [(W_i)_0]_k$$

and similarly for $[Z_\infty]_k$. If we prove this then the lemma follows from Lemma 42.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[Z_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i [(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $I \subset A$ be the ideal cutting out Z, in other words so that $A/I = \mathcal{O}_{Z,\xi}$. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(A/I) = 1$. Since ξ is not an embedded point by assumption (3) we see that A/I is Cohen-Macaulay. Since $\dim_\delta(Z_0) = k$ we see that $\dim(A/(t, I)) = 0$ which implies that t is a nonzerodivisor on A/I. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $I \subset \mathfrak{q}_i$ over I. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 42.3.2. ☐

02S6 **Lemma 42.21.3.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbf{P}^1$. Let $i_0, i_\infty : X \to X \times \mathbf{P}^1$ be the closed immersion such that $i_t(x) = (x, t)$. Denote $\mathcal{F}_0 = i_0^*\mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^*\mathcal{F}$. Assume

(1) $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k + 1$,
(2) $\dim_\delta(\text{Supp}(\mathcal{F}_0)) \leq k$, $\dim_\delta(\text{Supp}(\mathcal{F}_\infty)) \leq k$, and
(3) for any embedded associated point ξ of $\mathcal{F}$ either $\xi \notin (X \times \mathbf{P}^1)_0 \cup (X \times \mathbf{P}^1)_\infty$ or $\delta(\xi) < k$.

Then $[\mathcal{F}_0]_k \sim_{rat} [\mathcal{F}_\infty]_k$ as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Write

$$[\mathcal{F}]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 42.10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[\mathcal{F}_\infty]_k$. If we prove this then the lemma follows from Lemma 42.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[\mathcal{F}_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(\text{Supp}(M)) = 1$. Since ξ is not an associated point of $\mathcal{F}$ by assumption (3) we see that M is a Cohen-Macaulay module. Since $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$ we see that $\dim(\text{Supp}(M/tM)) = 0$ which implies that t is a nonzerodivisor on M. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Ass}(M)$. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(M/tM)$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)A)\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 42.3.2. ☐

42.22. Chow groups and envelopes

0GU4 Here is the definition.

0GU5 **Definition 42.22.1.** Let X be a scheme. An *envelope* is a proper morphism $f : Y \to X$ which is completely decomposed (More on Morphisms, Definition 37.78.1). [Ful98, Definition 18.3]

The exact sequence of Lemma 42.22.4 is the main motivation for the definition.

0GU6 **Lemma 42.22.2.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. If $f : Y \to X$ and $g : Z \to Y$ are envelopes, then $f \circ g$ is an envelope.

Proof. Follows from Morphisms, Lemma 29.41.4 and More on Morphisms, Lemma 37.78.2. ☐

0GU7 Lemma 42.22.3. Let (S, δ) be as in Situation 42.7.1. Let $X' \to X$ be a morphism of schemes locally of finite type over S. If $f : Y \to X$ is an envelope, then the base change $f' : Y' \to X'$ of f is an envelope too.

Proof. Follows from Morphisms, Lemma 29.41.5 and More on Morphisms, Lemma 37.78.3. ☐

0GU8 Lemma 42.22.4. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $f : Y \to X$ be an envelope. Then we have an exact sequence

$$\mathrm{CH}_k(Y \times_X Y) \xrightarrow{p_* - q_*} \mathrm{CH}_k(Y) \xrightarrow{f_*} \mathrm{CH}_k(X) \to 0$$

for all $k \in \mathbf{Z}$. Here $p, q : Y \times_X Y \to Y$ are the projections.

Proof. Since f is an envelope, f is proper and hence pushforward on cycles and cycle classes is defined, see Sections 42.12 and 42.15. Similarly, the morphisms p and q are proper as base changes of f. The composition of the arrows is zero as $f_* \circ p_* = (p \circ f)_* = (q \circ f)_* = f_* \circ q_*$, see Lemma 42.12.2.

Let us show that $f_* : Z_k(Y) \to Z_k(X)$ is surjective. Namely, suppose that we have $\alpha = \sum n_i[Z_i] \in Z_k(X)$ where $Z_i \subset X$ is a locally finite family of integral closed subschemes. Let $x_i \in Z_i$ be the generic point. Since f is an envelope and hence completely decomposed, there exists a point $y_i \in Y$ with $f(y_i) = x_i$ and with $\kappa(y_i)/\kappa(x_i)$ trivial. Let $W_i \subset Y$ be the integral closed subscheme with generic point y_i. Since f is closed, we see that $f(W_i) = Z_i$. It follows that the family of closed subschemes W_i is locally finite on Y. Since $\kappa(y_i)/\kappa(x_i)$ is trivial we see that $\dim_\delta(W_i) = \dim_\delta(Z_i) = k$. Hence $\beta = \sum n_i[W_i]$ is in $Z_k(Y)$. Finally, since $\kappa(y_i)/\kappa(x_i)$ is trivial, the degree of the dominant morphism $f|_{W_i} : W_i \to Z_i$ is 1 and we conclude that $f_*\beta = \alpha$.

Since $f_* : Z_k(Y) \to Z_k(X)$ is surjective, a fortiori the map $f_* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X)$ is surjective.

Let $\beta \in Z_k(Y)$ be an element such that $f_*\beta$ is zero in $\mathrm{CH}_k(X)$. This means we can find a locally finite family of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k + 1$ and $f_j \in R(Z_j)^*$ such that

$$f_*\beta = \sum (Z_j \to X)_* \mathrm{div}(f_j)$$

as cycles where $i_j : Z_j \to X$ is the given closed immersion. Arguing exactly as above, we can find a locally finite family of integral closed subschemes $W_j \subset Y$ with $f(W_j) = Z_j$ and such that $W_j \to Z_j$ is birational, i.e., induces an isomorphism $R(Z_j) = R(W_j)$. Denote $g_j \in R(W_j)^*$ the element corresponding to f_j. Observe that $W_j \to Z_j$ is proper and that $(W_j \to Z_j)_*\mathrm{div}(g_j) = \mathrm{div}(f_j)$ as cycles on Z_j. It follows from this that if we replace β by the rationally equivalent cycle

$$\beta' = \beta - \sum (W_j \to Y)_* \mathrm{div}(g_j)$$

then we find that $f_*\beta' = 0$. (This uses Lemma 42.12.2.) Thus to finish the proof of the lemma it suffices to show the claim in the following paragraph.

Claim: if $\beta \in Z_k(Y)$ and $f_*\beta = 0$, then $\beta = \delta + p_*\gamma - q_*\gamma$ in $Z_k(Y)$ for some $\gamma \in Z_k(Y \times_X Y)$. Namely, write $\beta = \sum_{j \in J} n_j[W_j]$ with $\{W_j\}_{j \in J}$ a locally finite family of integral closed subschemes of Y with $\dim_\delta(W_j) = k$. Fix an integral closed subscheme $Z \subset X$. Consider the subset $J_Z = \{j \in J : f(W_j) = Z\}$. This is a finite set. There are three cases:

(1) $J_Z = \emptyset$. In this case we set $\gamma_Z = 0$.
(2) $J_Z \neq \emptyset$ and $\dim_\delta(Z) = k$. The condition $f_*\beta = 0$ implies by looking at the coefficient of Z that $\sum_{j \in J_Z} n_j \deg(W_j/Z) = 0$. In this case we choose an integral closed subscheme $W \subset Y$ which maps birationally onto Z (see above). Looking at generic points, we see that $W_j \times_Z W$ has a unique irreducible component $W'_j \subset W_j \times_Z W \subset Y \times_X Y$ mapping birationally to W_j. Then $W'_j \to W$ is dominant and $\deg(W'_j/W) = \deg(W_j/W)$. Thus if we set $\gamma_Z = \sum_{j \in J_Z} n_j[W'_j]$ then we see that $p_*\gamma_Z = \sum_{j \in J_Z} n_j[W_j]$ and $q_*\gamma_Z = \sum_{j \in J_Z} n_j \deg(W'_j/W)[W] = 0$.
(3) $J_Z \neq \emptyset$ and $\dim_\delta(Z) < k$. In this case we choose an integral closed subscheme $W \subset Y$ which maps birationally onto Z (see above). Looking at generic points, we see that $W_j \times_Z W$ has a unique irreducible component $W'_j \subset W_j \times_Z W \subset Y \times_X Y$ mapping birationally to W_j. Then $W'_j \to W$ is dominant and $k = \dim_\delta(W'_j) > \dim_\delta(W) = \dim_\delta(Z)$. Thus if we set $\gamma_Z = \sum_{j \in J_Z} n_j[W'_j]$ then we see that $p_*\gamma_Z = \sum_{j \in J_Z} n_j[W_j]$ and $q_*\gamma_Z = 0$.

Since the family of integral closed subschemes $\{f(W_j)\}$ is locally finite on X (Lemma 42.11.2) we see that the k-cycle

$$\gamma = \sum\nolimits_{Z \subset X \text{ integral closed}} \gamma_Z$$

on $Y \times_X Y$ is well defined. By our computations above it follows that $p_*\gamma_Z = \beta$ and $q_*\gamma_Z = 0$ which implies what we wanted to prove. □

42.23. Chow groups and K-groups

0FDQ In this section we are going to compare K_0 of the category of coherent sheaves to the chow groups.

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. We denote $\mathrm{Coh}(X) = \mathrm{Coh}(\mathcal{O}_X)$ the category of coherent sheaves on X. It is an abelian category, see Cohomology of Schemes, Lemma 30.9.2. For any $k \in \mathbf{Z}$ we let $\mathrm{Coh}_{\leq k}(X)$ be the full subcategory of $\mathrm{Coh}(X)$ consisting of those coherent sheaves $\mathcal{F}$ having $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$.

02S8 **Lemma 42.23.1.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. The categories $\mathrm{Coh}_{\leq k}(X)$ are Serre subcategories of the abelian category $\mathrm{Coh}(X)$.

Proof. The definition of a Serre subcategory is Homology, Definition 12.10.1. The proof of the lemma is straightforward and omitted. □

02S9 **Lemma 42.23.2.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. The maps

$$Z_k(X) \longrightarrow K_0(\mathrm{Coh}_{\leq k}(X)/\mathrm{Coh}_{\leq k-1}(X)), \quad \sum n_Z[Z] \mapsto \left[\bigoplus\nolimits_{n_Z > 0} \mathcal{O}_Z^{\oplus n_Z}\right] - \left[\bigoplus\nolimits_{n_Z < 0} \mathcal{O}_Z^{\oplus -n_Z}\right]$$

and

$$K_0(\mathrm{Coh}_{\leq k}(X)/\mathrm{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X), \quad \mathcal{F} \longmapsto [\mathcal{F}]_k$$

are mutually inverse isomorphisms.

Proof. Note that if $\sum n_Z[Z]$ is in $Z_k(X)$, then the direct sums $\bigoplus_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}$ and $\bigoplus_{n_Z<0} \mathcal{O}_Z^{\oplus -n_Z}$ are coherent sheaves on X since the family $\{Z \mid n_Z > 0\}$ is locally finite on X. The map $\mathcal{F} \to [\mathcal{F}]_k$ is additive on $\text{Coh}_{\leq k}(X)$, see Lemma 42.10.4. And $[\mathcal{F}]_k = 0$ if $\mathcal{F} \in \text{Coh}_{\leq k-1}(X)$. By part (1) of Homology, Lemma 12.11.3 this implies that the second map is well defined too. It is clear that the composition of the first map with the second map is the identity.

Conversely, say we start with a coherent sheaf $\mathcal{F}$ on X. Write $[\mathcal{F}]_k = \sum_{i\in I} n_i[Z_i]$ with $n_i > 0$ and $Z_i \subset X$, $i \in I$ pairwise distinct integral closed subschemes of δ-dimension k. We have to show that

$$[\mathcal{F}] = [\bigoplus\nolimits_{i\in I} \mathcal{O}_{Z_i}^{\oplus n_i}]$$

in $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$. Denote $\xi_i \in Z_i$ the generic point. If we set

$$\mathcal{F}' = \text{Ker}(\mathcal{F} \to \bigoplus \xi_{i,*}\mathcal{F}_{\xi_i})$$

then $\mathcal{F}'$ is the maximal coherent submodule of $\mathcal{F}$ whose support has dimension $\leq k-1$. In particular $\mathcal{F}$ and $\mathcal{F}/\mathcal{F}'$ have the same class in $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$. Thus after replacing $\mathcal{F}$ by $\mathcal{F}/\mathcal{F}'$ we may and do assume that the kernel $\mathcal{F}'$ displayed above is zero.

For each $i \in I$ we choose a filtration

$$\mathcal{F}_{\xi_i} = \mathcal{F}_i^0 \supset \mathcal{F}_i^1 \supset \ldots \supset \mathcal{F}_i^{n_i} = 0$$

such that the successive quotients are of dimension 1 over the residue field at ξ_i. This is possible as the length of $\mathcal{F}_{\xi_i}$ over $\mathcal{O}_{X,\xi_i}$ is n_i. For $p > n_i$ set $\mathcal{F}_i^p = 0$. For $p \geq 0$ we denote

$$\mathcal{F}^p = \text{Ker}\left(\mathcal{F} \longrightarrow \bigoplus \xi_{i,*}(\mathcal{F}_{\xi_i}/\mathcal{F}_i^p)\right)$$

Then $\mathcal{F}^p$ is coherent, $\mathcal{F}^0 = \mathcal{F}$, and $\mathcal{F}^p/\mathcal{F}^{p+1}$ is isomorphic to a free $\mathcal{O}_{Z_i}$-module of rank 1 (if $n_i > p$) or 0 (if $n_i \leq p$) in an open neighbourhood of ξ_i. Moreover, $\mathcal{F}' = \bigcap \mathcal{F}^p = 0$. Since every quasi-compact open $U \subset X$ contains only a finite number of ξ_i we conclude that $\mathcal{F}^p|_U$ is zero for $p \gg 0$. Hence $\bigoplus_{p\geq 0} \mathcal{F}^p$ is a coherent $\mathcal{O}_X$-module. Consider the short exact sequences

$$0 \to \bigoplus\nolimits_{p>0} \mathcal{F}^p \to \bigoplus\nolimits_{p\geq 0} \mathcal{F}^p \to \bigoplus\nolimits_{p>0} \mathcal{F}^p/\mathcal{F}^{p+1} \to 0$$

and

$$0 \to \bigoplus\nolimits_{p>0} \mathcal{F}^p \to \bigoplus\nolimits_{p\geq 0} \mathcal{F}^p \to \mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. This already shows that

$$[\mathcal{F}] = [\bigoplus \mathcal{F}^p/\mathcal{F}^{p+1}]$$

in $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$. Next, for every $p \geq 0$ and $i \in I$ such that $n_i > p$ we choose a nonzero ideal sheaf $\mathcal{I}_{i,p} \subset \mathcal{O}_{Z_i}$ and a map $\mathcal{I}_{i,p} \to \mathcal{F}^p/\mathcal{F}^{p+1}$ on X which is an isomorphism over the open neighbourhood of ξ_i mentioned above. This is possible by Cohomology of Schemes, Lemma 30.10.6. Then we consider the short exact sequence

$$0 \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{I}_{i,p} \to \bigoplus \mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{Q} \to 0$$

and the short exact sequence

$$0 \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{I}_{i,p} \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{O}_{Z_i} \to \mathcal{Q}' \to 0$$

Observe that both $\mathcal{Q}$ and $\mathcal{Q}'$ are zero in a neighbourhood of the points ξ_i and that they are supported on $\bigcup Z_i$. Hence $\mathcal{Q}$ and $\mathcal{Q}'$ are in $\text{Coh}_{\leq k-1}(X)$. Since

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_{Z_i}^{\oplus n_i} \cong \bigoplus\nolimits_{p \geq 0, i \in I, n_i > p} \mathcal{O}_{Z_i}$$

this concludes the proof. □

0FDR Lemma 42.23.3. Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over (S, δ) as in Situation 42.7.1. Then $\pi_* : \text{Coh}(X) \to \text{Coh}(Y)$ is an exact functor which sends $\text{Coh}_{\leq k}(X)$ into $\text{Coh}_{\leq k}(Y)$ and induces homomorphisms on K_0 of these categories and their quotients. The maps of Lemma 42.23.2 fit into a commutative diagram

$$\begin{tikzcd}
Z_k(X) \arrow[r] \arrow[d, "\pi_*"] & K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \arrow[r] \arrow[d, "\pi_*"] & Z_k(X) \arrow[d, "\pi_*"] \\
Z_k(Y) \arrow[r] & K_0(\text{Coh}_{\leq k}(Y)/\text{Coh}_{\leq k-1}(Y)) \arrow[r] & Z_k(Y)
\end{tikzcd}$$

Proof. A finite morphism is affine, hence pushforward of quasi-coherent modules along π is an exact functor by Cohomology of Schemes, Lemma 30.2.3. A finite morphism is proper, hence π_* sends coherent sheaves to coherent sheaves, see Cohomology of Schemes, Proposition 30.19.1. The statement on dimensions of supports is clear. Commutativity on the right follows immediately from Lemma 42.12.4. Since the horizontal arrows are bijections, we find that we have commutativity on the left as well. □

0FDS Lemma 42.23.4. Let X be a scheme locally of finite type over (S, δ) as in Situation 42.7.1. There is a canonical map

$$\text{CH}_k(X) \longrightarrow K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$$

induced by the map $Z_k(X) \to K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$ from Lemma 42.23.2.

Proof. We have to show that an element α of $Z_k(X)$ which is rationally equivalent to zero, is mapped to zero in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$. Write $\alpha = \sum (i_j)_* \text{div}(f_j)$ as in Definition 42.19.1. Observe that

$$\pi = \coprod i_j : W = \coprod W_j \longrightarrow X$$

is a finite morphism as each $i_j : W_j \to X$ is a closed immersion and the family of W_j is locally finite in X. Hence we may use Lemma 42.23.3 to reduce to the case of W. Since W is a disjoint union of integral scheme, we reduce to the case discussed in the next paragraph.

Assume X is integral of δ-dimension $k + 1$. Let f be a nonzero rational function on X. Let $\alpha = \text{div}(f)$. We have to show that α is mapped to zero in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of f, see Divisors, Definition 31.23.10. Then we have short exact sequences

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$$

and

$$0 \to \mathcal{I} \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_X/f\mathcal{I} \to 0$$

See Divisors, Lemma 31.23.9. We claim that

$$[\mathcal{O}_X/\mathcal{I}]_k - [\mathcal{O}_X/f\mathcal{I}]_k = \text{div}(f)$$

The claim implies the element $\alpha = \text{div}(f)$ is represented by $[\mathcal{O}_X/\mathcal{I}] - [\mathcal{O}_X/f\mathcal{I}]$ in $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$. Then the short exact sequences show that this element maps to zero in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$.

To prove the claim, let $Z \subset X$ be an integral closed subscheme of δ-dimension k and let $\xi \in Z$ be its generic point. Then $I = \mathcal{I}_\xi \subset A = \mathcal{O}_{X,\xi}$ is an ideal such that $fI \subset A$. Now the coefficient of $[Z]$ in $\text{div}(f)$ is $\text{ord}_A(f)$. (Of course as usual we identify the function field of X with the fraction field of A.) On the other hand, the coefficient of $[Z]$ in $[\mathcal{O}_X/\mathcal{I}] - [\mathcal{O}_X/f\mathcal{I}]$ is

$$\text{length}_A(A/I) - \text{length}_A(A/fI)$$

Using the distance fuction of Algebra, Definition 10.121.5 we can rewrite this as

$$d(A, I) - d(A, fI) = d(I, fI) = \text{ord}_A(f)$$

The equalities hold by Algebra, Lemmas 10.121.6 and 10.121.7. (Using these lemmas isn't necessary, but convenient.) □

02SD Remark 42.23.5. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. We will see later (in Lemma 42.69.3) that the map

$$\text{CH}_k(X) \longrightarrow K_0(\text{Coh}_{k+1}(X)/\text{Coh}_{\leq k-1}(X))$$

of Lemma 42.23.4 is injective. Composing with the canonical map

$$K_0(\text{Coh}_{k+1}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow K_0(\text{Coh}(X)/\text{Coh}_{\leq k-1}(X))$$

we obtain a canonical map

$$\text{CH}_k(X) \longrightarrow K_0(\text{Coh}(X)/\text{Coh}_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map should be injective or not. It seems reasonable to expect the kernel of this map to be torsion. We will return to this question (insert future reference).

0FDT Lemma 42.23.6. Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Denote $\text{Coh}_Z(X) \subset \text{Coh}(X)$ the Serre subcategory of coherent $\mathcal{O}_X$-modules whose set theoretic support is contained in Z. Then the exact inclusion functor $\text{Coh}(Z) \to \text{Coh}_Z(X)$ induces an isomorphism

$$K'_0(Z) = K_0(\text{Coh}(Z)) \longrightarrow K_0(\text{Coh}_Z(X))$$

Proof. Let $\mathcal{F}$ be an object of $\text{Coh}_Z(X)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of Z. Consider the descending filtration

$$\ldots \subset \mathcal{F}^p = \mathcal{I}^p\mathcal{F} \subset \mathcal{F}^{p-1} \subset \ldots \subset \mathcal{F}^0 = \mathcal{F}$$

Exactly as in the proof of Lemma 42.23.4 this filtration is locally finite and hence $\bigoplus_{p \geq 0} \mathcal{F}^p$, $\bigoplus_{p \geq 1} \mathcal{F}^p$, and $\bigoplus_{p \geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}$ are coherent $\mathcal{O}_X$-modules supported on Z. Hence we get

$$[\mathcal{F}] = [\bigoplus\nolimits_{p \geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}]$$

in $K_0(\text{Coh}_Z(X))$ exactly as in the proof of Lemma 42.23.4. Since the coherent module $\bigoplus_{p \geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}$ is annihilated by $\mathcal{I}$ we conclude that $[\mathcal{F}]$ is in the image. Actually, we claim that the map

$$\mathcal{F} \longmapsto c(\mathcal{F}) = [\bigoplus\nolimits_{p \geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}]$$

factors through $K_0(\mathrm{Coh}_Z(X))$ and is an inverse to the map in the statement of the lemma. To see this all we have to show is that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence in $\mathrm{Coh}_Z(X)$, then we get $c(\mathcal{G}) = c(\mathcal{F}) + c(\mathcal{H})$. Observe that for all $q \geq 0$ we have a short exact sequence

$$0 \to (\mathcal{F} \cap \mathcal{I}^q\mathcal{G})/(\mathcal{F} \cap \mathcal{I}^{q+1}\mathcal{G}) \to \mathcal{G}^q/\mathcal{G}^{q+1} \to \mathcal{H}^q/\mathcal{H}^{q+1} \to 0$$

For $p, q \geq 0$ consider the coherent submodule

$$\mathcal{F}^{p,q} = \mathcal{I}^p\mathcal{F} \cap \mathcal{I}^q\mathcal{G}$$

Arguing exactly as above and using that the filtrations $\mathcal{F}^p = \mathcal{I}^p\mathcal{F}$ and $\mathcal{F} \cap \mathcal{I}^q\mathcal{G}$ are locally finite, we find that

$$[\bigoplus\nolimits_{p \geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}] = [\bigoplus\nolimits_{p,q \geq 0} \mathcal{F}^{p,q}/(\mathcal{F}^{p+1,q} + \mathcal{F}^{p,q+1})] = [\bigoplus\nolimits_{q \geq 0} (\mathcal{F} \cap \mathcal{I}^q\mathcal{G})/(\mathcal{F} \cap \mathcal{I}^{q+1}\mathcal{G})]$$

in $K_0(\mathrm{Coh}(Z))$. Combined with the exact sequences above we obtain the desired result. Some details omitted. $\square$

42.24. The divisor associated to an invertible sheaf

02SI The following definition is the analogue of Divisors, Definition 31.27.4 in our current setup.

02SJ Definition 42.24.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the Weil divisor associated to s is the $(n-1)$-cycle

$$\mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z]$$

defined in Divisors, Definition 31.27.4. This makes sense because Weil divisors have δ-dimension $n-1$ by Lemma 42.16.1.

(2) We define Weil divisor associated to $\mathcal{L}$ as

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \mathrm{div}_\mathcal{L}(s) \in \mathrm{CH}_{n-1}(X)$$

where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Divisors, Lemma 31.27.3.

Let X and S be as in Definition 42.24.1 above. Set $n = \dim_\delta(X)$. It is clear from the definitions that $Cl(X) = \mathrm{CH}_{n-1}(X)$ where $Cl(X)$ is the Weil divisor class group of X as defined in Divisors, Definition 31.26.7. The map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{CH}_{n-1}(X), \quad \mathcal{L} \longmapsto c_1(\mathcal{L}) \cap [X]$$

is the same as the map $\mathrm{Pic}(X) \to Cl(X)$ constructed in Divisors, Equation (31.27.5.1) for arbitrary locally Noetherian integral schemes. In particular, this map is a homomorphism of abelian groups, it is injective if X is a normal scheme, and an isomorphism if all local rings of X are UFDs. See Divisors, Lemmas 31.27.6 and 31.27.7. There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

02SK **Lemma 42.24.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then

$$\mathrm{div}_\mathcal{L}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $\mathrm{CH}_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $s = fs_\xi$ for some $f \in \mathcal{O}_{X,\xi}$. By definition of $Z(s)$, see Divisors, Definition 31.14.8 we see that $Z(s)$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = (f)$. Hence $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \mathrm{ord}_{\mathcal{O}_{X,x}}(f)$ as desired. ☐

The following lemma will be superseded by the more general Lemma 42.26.2.

02SM **Lemma 42.24.3.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then

$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$

in $\mathrm{CH}_{n+r-1}(X)$.

Proof. Let s be a nonzero meromorphic section of $\mathcal{L}$. We will show that actually $f^*\mathrm{div}_\mathcal{L}(s) = \mathrm{div}_{f^*\mathcal{L}}(f^*s)$ and hence the lemma holds. To see this let $\xi \in Y$ be a point and let $s_\xi \in \mathcal{L}_\xi$ be a generator. Write $s = gs_\xi$ with $g \in R(Y)^*$. Then there is an open neighbourhood $V \subset Y$ of ξ such that $s_\xi \in \mathcal{L}(V)$ and such that s_ξ generates $\mathcal{L}|_V$. Hence we see that

$$\mathrm{div}_\mathcal{L}(s)|_V = \mathrm{div}_Y(g)|_V.$$

In exactly the same way, since f^*s_ξ generates $f^*\mathcal{L}$ over $f^{-1}(V)$ and since $f^*s = gf^*s_\xi$ we also have

$$\mathrm{div}_\mathcal{L}(f^*s)|_{f^{-1}(V)} = \mathrm{div}_X(g)|_{f^{-1}(V)}.$$

Thus the desired equality of cycles over $f^{-1}(V)$ follows from the corresponding result for pullbacks of principal divisors, see Lemma 42.17.2. ☐

42.25. Intersecting with an invertible sheaf

02SN In this section we study the following construction.

02SO **Definition 42.25.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to \mathrm{CH}_k(X)$$

called intersection with the first Chern class of $\mathcal{L}$.

1. Given an integral closed subscheme $i : W \to X$ with $\dim_\delta(W) = k+1$ we define
$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$
where the right hand side is defined in Definition 42.24.1.
2. For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subschemes of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subschemes on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subschemes of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $\mathrm{CH}_k(\coprod W_i) = \prod \mathrm{CH}_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence. This is not a triviality.

02SP Lemma 42.25.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $\mathrm{CH}_k(X)$ for every $\alpha \in Z_{k+1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Divisors, Lemma 31.27.5 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subscheme $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. $\square$

Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 31.14.8.

0EPI Lemma 42.25.3. Let (S, δ) be as in Situation 42.7.1. Let Y be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $s \in \Gamma(Y, \mathcal{L})$. Assume

(1) $\dim_\delta(Y) \leq k + 1$,
(2) $\dim_\delta(Z(s)) \leq k$, and
(3) for every generic point ξ of an irreducible component of $Z(s)$ of δ-dimension k the multiplication by s induces an injection $\mathcal{O}_{Y,\xi} \to \mathcal{L}_\xi$.

Write $[Y]_{k+1} = \sum n_i[Y_i]$ where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k + 1$. Set $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$. Then

02SR (42.25.3.1)
$$[Z(s)]_k = \sum n_i[Z(s_i)]_k$$

as k-cycles on Y.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. We want to compare the coefficient n of $[Z]$ in the expression $\sum n_i[Z(s_i)]_k$ with the coefficient m of $[Z]$ in the expression $[Z(s)]_k$. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $A = \mathcal{O}_{Y,\xi}$, $L = \mathcal{L}_\xi$. Then $L = As_\xi$. Write $s = fs_\xi$ for some (unique) $f \in A$. Hypothesis (3) means that $f : A \to A$ is injective. Since $\dim_\delta(Y) \leq k + 1$ and $\dim_\delta(Z) = k$ we have $\dim(A) = 0$ or 1. We have

$$m = \mathrm{length}_A(A/(f))$$

which is finite in either case.

If $\dim(A) = 0$, then $f : A \to A$ being injective implies that $f \in A^*$. Hence in this case m is zero. Moreover, the condition $\dim(A) = 0$ means that ξ does not lie on any irreducible component of δ-dimension $k + 1$, i.e., $n = 0$ as well.

Now, let $\dim(A) = 1$. Since A is a Noetherian local ring it has finitely many minimal primes $\mathfrak q_1, \ldots, \mathfrak q_t$. These correspond 1-1 with the Y_i passing through ξ'. Moreover $n_i = \text{length}_{A_{\mathfrak q_i}}(A_{\mathfrak q_i})$. Also, the multiplicity of $[Z]$ in $[Z(s_i)]_k$ is $\text{length}_A(A/(f, \mathfrak q_i))$. Hence the equation to prove in this case is

$$\text{length}_A(A/(f)) = \sum \text{length}_{A_{\mathfrak q_i}}(A_{\mathfrak q_i})\text{length}_A(A/(f, \mathfrak q_i))$$

which follows from Lemma 42.3.2. □

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 31.14.8.

02SQ Lemma 42.25.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y \subset X$ be a closed subscheme. Let $s \in \Gamma(Y, \mathcal{L}|_Y)$. Assume

(1) $\dim_\delta(Y) \leq k+1$,
(2) $\dim_\delta(Z(s)) \leq k$, and
(3) for every generic point ξ of an irreducible component of $Z(s)$ of δ-dimension k the multiplication by s induces an injection $\mathcal{O}_{Y,\xi} \to (\mathcal{L}|_Y)_\xi$[1].

Then

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in $\text{CH}_k(X)$.

Proof. Write

$$[Y]_{k+1} = \sum n_i [Y_i]$$

where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ is not zero, and hence is a regular section. By Lemma 42.24.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{Y_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i [Z(s_i)]_k$$

Thus the result follows from Lemma 42.25.3. □

42.26. Intersecting with an invertible sheaf and push and pull

0AYA In this section we prove that the operation $c_1(\mathcal{L}) \cap -$ commutes with flat pullback and proper pushforward.

0EPJ Lemma 42.26.1. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume Y is integral and $n = \dim_\delta(Y)$. Let s be a nonzero meromorphic section of $\mathcal{L}$. Then we have

$$f^*\text{div}_\mathcal{L}(s) = \sum n_i \text{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

in $Z_{n+r-1}(X)$. Here the sum is over the irreducible components $X_i \subset X$ of δ-dimension $n + r$, the section $s_i = f|_{X_i}^*(s)$ is the pullback of s, and $n_i = m_{X_i,X}$ is the multiplicity of X_i in X.

[1]For example, this holds if s is a regular section of $\mathcal{L}|_Y$.

Proof. To prove this equality of cycles, we may work locally on Y. Hence we may assume Y is affine and $s = p/q$ for some nonzero sections $p \in \Gamma(Y, \mathcal{L})$ and $q \in \Gamma(Y, \mathcal{O})$. If we can show both

$$f^*\mathrm{div}_{\mathcal{L}}(p) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(p_i) \quad \text{and} \quad f^*\mathrm{div}_{\mathcal{O}}(q) = \sum n_i \mathrm{div}_{\mathcal{O}_{X_i}}(q_i)$$

(with obvious notations) then we win by the additivity, see Divisors, Lemma 31.27.5. Thus we may assume that $s \in \Gamma(Y, \mathcal{L})$. In this case we may apply the equality (42.25.3.1) to see that

$$[Z(f^*(s))]_{k+r-1} = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

where $f^*(s) \in f^*\mathcal{L}$ denotes the pullback of s to X. On the other hand we have

$$f^*\mathrm{div}_{\mathcal{L}}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 42.24.2 and 42.14.4. Since $Z(f^*(s)) = f^{-1}(Z(s))$ we win. □

02SS Lemma 42.26.2. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $\mathrm{CH}_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i[W_i]$. We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$$

in $\mathrm{CH}_{k+r-1}(X)$ by producing a rational equivalence on the closed subscheme $f^{-1}(W_i)$ of X. By the discussion in Remark 42.19.6 this will prove the equality of the lemma is true.

Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. Consider the closed subscheme $W' = f^{-1}(W) = W \times_Y X$ so that we have the fibre product diagram

$$\begin{tikzcd} W' \arrow[r] \arrow[d, "h"'] & X \arrow[d, "f"] \\ W \arrow[r] & Y \end{tikzcd}$$

We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Let $W'_i \subset W'$ be the irreducible components of δ-dimension $k + r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with n_i the multiplicity of W'_i in W' as per definition. So $f^*[W] = \sum n_i[W'_i]$ in $Z_{k+r}(X)$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. By Lemma 42.26.1 we have the following equality of cycles

$$h^*\mathrm{div}_{\mathcal{L}|_W}(s) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{W'_i}}(s_i)$$

in $Z_{k+r-1}(W')$. This finishes the proof since the left hand side is a cycle on W' which pushes to $f^*(c_1(\mathcal{L}) \cap [W])$ in $\mathrm{CH}_{k+r-1}(X)$ and the right hand side is a cycle on W' which pushes to $c_1(f^*\mathcal{L}) \cap f^*[W]$ in $\mathrm{CH}_{k+r-1}(X)$. □

02ST Lemma 42.26.3. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Then

$$f_* \left(\text{div}_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)]\text{div}_{\mathcal{L}}(s).$$

as cycles on Y. In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = [R(X) : R(Y)]c_1(\mathcal{L}) \cap [Y] = c_1(\mathcal{L}) \cap f_*[X]$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. It turns out that we can re-use Lemma 42.18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on Y. Hence we may assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$, and we are simply trying to prove

$$f_* \left(\text{div}_X(g)\right) = [R(X) : R(Y)]\text{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 42.18.1 we see this true since $\text{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. □

02SU Lemma 42.26.4. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $\text{CH}_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subscheme $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the image of W_i (as an integral closed subscheme). Consider the diagram

$$\begin{tikzcd} X' = \coprod W_i \arrow[r, "q"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' = \coprod W_i' \arrow[r, "q'"] & Y. \end{tikzcd}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 42.26.3. □

42.27. The key formula

0AYB Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible sheaves on X. Let s be a nonzero meromorphic section of $\mathcal{L}$ and let t be a nonzero meromorphic section of $\mathcal{N}$. Let $Z_i \subset X$, $i \in I$ be a locally finite set of irreducible closed subsets of codimension 1 with the following property: If $Z \not\in \{Z_i\}$ with generic point ξ, then s is a generator for $\mathcal{L}_\xi$ and t is a generator for $\mathcal{N}_\xi$. Such a set exists by Divisors, Lemma 31.27.2. Then

$$\mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{Z_i, \mathcal{L}}(s)[Z_i]$$

and similarly

$$\mathrm{div}_\mathcal{N}(t) = \sum \mathrm{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each i generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ where ξ_i is the generic point of Z_i. Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

Set $B_i = \mathcal{O}_{X, \xi_i}$. Then by definition

$$\mathrm{ord}_{Z_i, \mathcal{L}}(s) = \mathrm{ord}_{B_i}(f_i) \quad \text{and} \quad \mathrm{ord}_{Z_i, \mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$$

Since t_i is a generator of $\mathcal{N}_{\xi_i}$ we see that its image in the fibre $\mathcal{N}_{\xi_i} \otimes \kappa(\xi_i)$ is a nonzero meromorphic section of $\mathcal{N}|_{Z_i}$. We will denote this image $t_i|_{Z_i}$. From our definitions it follows that

$$c_1(\mathcal{N}) \cap \mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{B_i}(f_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \mathrm{div}_\mathcal{N}(t) = \sum \mathrm{ord}_{B_i}(g_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in $\mathrm{CH}_{n-2}(X)$. We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^*$$

see Section 42.5.

0AYC **Lemma 42.27.1** (Key formula). *In the situation above the cycle*

$$\sum (Z_i \to X)_* \left(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \mathrm{ord}_{B_i}(g_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})\right)$$

is equal to the cycle

$$\sum (Z_i \to X)_*\mathrm{div}(\partial_{B_i}(f_i, g_i))$$

Proof. First, let us examine what happens if we replace s_i by us_i for some unit u in B_i. Then f_i gets replaced by $u^{-1}f_i$. Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$-\mathrm{ord}_{B_i}(g_i)\mathrm{div}(u|_{Z_i})$$

(where $u|_{Z_i}$ is the image of u in the residue field) by Divisors, Lemma 31.27.3 and in the second expression we add

$$\mathrm{div}(\partial_{B_i}(u^{-1}, g_i))$$

by bi-linearity of the tame symbol. These terms agree by property (6) of the tame symbol.

Let $Z \subset X$ be an irreducible closed with $\dim_\delta(Z) = n - 2$. To show that the coefficients of Z of the two cycles of the lemma is the same, we may do a replacement $s_i \mapsto us_i$ as in the previous paragraph. In exactly the same way one shows that we may do a replacement $t_i \mapsto vt_i$ for some unit v of B_i.

Since we are proving the equality of cycles we may argue one coefficient at a time. Thus we choose an irreducible closed $Z \subset X$ with $\dim_\delta(Z) = n - 2$ and compare coefficients. Let $\xi \in Z$ be the generic point and set $A = \mathcal{O}_{X,\xi}$. This is a Noetherian local domain of dimension 2. Choose generators σ and τ for $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$. After shrinking X, we may and do assume σ and τ define trivializations of the invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ over all of X. Because Z_i is locally finite after shrinking X we may assume $Z \subset Z_i$ for all $i \in I$ and that I is finite. Then ξ_i corresponds to a prime $\mathfrak{q}_i \subset A$ of height 1. We may write $s_i = a_i\sigma$ and $t_i = b_i\tau$ for some a_i and b_i units in $A_{\mathfrak{q}_i}$. By the remarks above, it suffices to prove the lemma when $a_i = b_i = 1$ for all i.

Assume $a_i = b_i = 1$ for all i. Then the first expression of the lemma is zero, because we choose σ and τ to be trivializing sections. Write $s = f\sigma$ and $t = g\tau$ with f and g in the fraction field of A. By the previous paragraph we have reduced to the case $f_i = f$ and $g_i = g$ for all i. Moreover, for a height 1 prime $\mathfrak{q}$ of A which is not in $\{\mathfrak{q}_i\}$ we have that both f and g are units in $A_\mathfrak{q}$ (by our choice of the family $\{Z_i\}$ in the discussion preceding the lemma). Thus the coefficient of Z in the second expression of the lemma is

$$\sum\nolimits_i \text{ord}_{A/\mathfrak{q}_i}(\partial_{B_i}(f, g))$$

which is zero by the key Lemma 42.6.3. $\square$

0GU9 Remark 42.27.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. We claim that there is a complex

$$\bigoplus\nolimits'_{\delta(x)=k+2} K_2^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k+1} K_1^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x))$$

Here we use notation and conventions introduced in Remark 42.19.2 and in addition

(1) $K_2^M(\kappa(x))$ is the degree 2 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 42.6.4) which is the quotient of $\kappa(x)^* \otimes_\mathbf{Z} \kappa(x)^*$ by the subgroup generated by elements of the form $\lambda \otimes (1 - \lambda)$ for $\lambda \in \kappa(x) \setminus \{0, 1\}$, and
(2) the first differential ∂ is defined as follows: given an element $\xi = \sum_x \alpha_x$ in the first term we set

$$\partial(\xi) = \sum\nolimits_{x \leadsto x',\ \delta(x')=k+1} \partial_{\mathcal{O}_{W_x,x'}}(\alpha_x)$$

where $\partial_{\mathcal{O}_{W_x,x'}} : K_2^M(\kappa(x)) \to K_1^M(\kappa(x))$ is the tame symbol constructed in Section 42.5.

We claim that we get a complex, i.e., that $\partial \circ \partial = 0$. To see this it suffices to take an element ξ as above and a point $x'' \in X$ with $\delta(x'') = k$ and check that the coefficient of x'' in the element $\partial(\partial(\xi))$ is zero. Because $\xi = \sum \alpha_x$ is a locally finite sum, we may in fact assume by additivity that $\xi = \alpha_x$ for some $x \in X$ with $\delta(x) = k + 2$ and $\alpha_x \in K_2^M(\kappa(x))$. By linearity again we may assume that $\alpha_x = f \otimes g$ for some $f, g \in \kappa(x)^*$. Denote $W \subset X$ the integral closed subscheme with generic point x.

If $x'' \not\in W$, then it is immediately clear that the coefficient of x in $\partial(\partial(\xi))$ is zero. If $x'' \in W$, then we see that the coefficient of x'' in $\partial(\partial(x))$ is equal to

$$\sum\nolimits_{x \leadsto x' \leadsto x'',\ \delta(x') = k+1} \text{ord}_{\mathcal{O}_{\overline{\{x'\}}, x''}}(\partial_{\mathcal{O}_{W, x'}}(f, g))$$

The key algebraic Lemma 42.6.3 says exactly that this is zero.

0GUA Remark 42.27.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. The complex in Remark 42.27.2 and the presentation of $\text{CH}_k(X)$ in Remark 42.19.2 suggests that we can define a first higher Chow group

$$\text{CH}_k^M(X, 1) = H_1(\text{the complex of Remark 42.27.2})$$

We use the supscript ^{M} to distinguish our notation from the higher chow groups defined in the literature, e.g., in the papers by Spencer Bloch ([Blo86] and [Blo94]). Let $U \subset X$ be open with complement $Y \subset X$ (viewed as reduced closed subscheme). Then we find a split short exact sequence

$$0 \to \bigoplus\nolimits'_{y \in Y, \delta(y) = k+i} K_i^M(\kappa(y)) \to \bigoplus\nolimits'_{x \in X, \delta(x) = k+i} K_i^M(\kappa(x)) \to \bigoplus\nolimits'_{u \in U, \delta(u) = k+i} K_i^M(\kappa(u)) \to 0$$

for $i = 2, 1, 0$ compatible with the boundary maps in the complexes of Remark 42.27.2. Applying the snake lemma (see Homology, Lemma 12.13.6) we obtain a six term exact sequence

$$\text{CH}_k^M(Y, 1) \to \text{CH}_k^M(X, 1) \to \text{CH}_k^M(U, 1) \to \text{CH}_k(Y) \to \text{CH}_k(X) \to \text{CH}_k(U) \to 0$$

extending the canonical exact sequence of Lemma 42.19.3. With some work, one may also define flat pullback and proper pushforward for the first higher chow group $\text{CH}_k^M(X, 1)$. We will return to this later (insert future reference here).

42.28. Intersecting with an invertible sheaf and rational equivalence

02TG Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that $c_1(\mathcal{L}) \cap -$ factors through rational equivalence and that these operations for different invertible sheaves commute.

02TH Lemma 42.28.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = \text{div}_\mathcal{L}(s)$ and $\beta = \text{div}_\mathcal{N}(t)$. Then

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $\text{CH}_{n-2}(X)$.

Proof. Immediate from the key Lemma 42.27.1 and the discussion preceding it. □

02TI Lemma 42.28.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation

$$c_1(\mathcal{L}) \cap - : \text{CH}_{k+1}(X) \to \text{CH}_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 42.25.1 is zero. By Definition 42.19.1 there exists a locally finite family $\{W_j\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \text{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\text{div}_{W_j}(f_j)$. By Lemma 42.26.4 we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \text{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \text{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \text{div}_\mathcal{L}(s)$. By Lemma 42.28.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 42.25.2 we see that the right hand side is zero in $\text{CH}_k(X)$ as desired. □

Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. We will denote

$$c_1(\mathcal{L}) \cap - : \text{CH}_{k+1}(X) \to \text{CH}_k(X)$$

the operation $c_1(\mathcal{L}) \cap -$. This makes sense by Lemma 42.28.2. We will denote $c_1(\mathcal{L})^s \cap -$ the s-fold iterate of this operation for all $s \geq 0$.

02TJ Lemma 42.28.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in \text{CH}_{k+2}(X)$ we have

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $\text{CH}_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 42.26.4 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 42.28.1 and the definitions. □

42.29. Gysin homomorphisms

02T7 In this section we define the gysin map for the zero locus D of a section of an invertible sheaf. An interesting case occurs when D is an effective Cartier divisor, but the generalization to arbitrary D allows us a flexibility to formulate various compatibilities, see Remark 42.29.7 and Lemmas 42.29.8, 42.29.9, and 42.30.5. These results can be generalized to locally principal closed subschemes endowed with a virtual normal bundle (Remark 42.29.2) or to pseudo-divisors (Remark 42.29.3).

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and s is a regular global section, see Divisors, Lemma 31.14.10. If D corresponds to $(\mathcal{L}, s)$, then $\mathcal{L} = \mathcal{O}_X(D)$. Please keep this in mind while reading this section.

02T8 Definition 42.29.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s)$ be a pair consisting of an invertible sheaf and a global section $s \in \Gamma(X, \mathcal{L})$. Let $D = Z(s)$ be the zero scheme of s, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a Gysin homomorphism

$$i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D).$$

by the following rules:

(1) Given a integral closed subscheme $W \subset X$ with $\dim_\delta(W) = k + 1$ we define
   (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
   (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.
(2) For a general $(k+1)$-cycle $\alpha = \sum n_j[W_j]$ we set
$$i^*\alpha = \sum n_j i^*[W_j]$$
(3) If D is an effective Cartier divisor, then we denote $D \cdot \alpha = i_* i^* \alpha$ the pushforward of the class $i^*\alpha$ to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the pullback of the class α.

0B70 Remark 42.29.2. Let X be a scheme locally of finite type over S as in Situation 42.7.1. Let $(D, \mathcal{N}, \sigma)$ be a triple consisting of a locally principal (Divisors, Definition 31.13.1) closed subscheme $i : D \to X$, an invertible $\mathcal{O}_D$-module $\mathcal{N}$, and a surjection $\sigma : \mathcal{N}^{\otimes -1} \to i^*\mathcal{I}_D$ of $\mathcal{O}_D$-modules[2]. Here $\mathcal{N}$ should be thought of as a virtual normal bundle of D in X. The construction of $i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D)$ in Definition 42.29.1 generalizes to such triples, see Section 42.54.

0B7D Remark 42.29.3. Let X be a scheme locally of finite type over S as in Situation 42.7.1. In [Ful98] a pseudo-divisor on X is defined as a triple $D = (\mathcal{L}, Z, s)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, $Z \subset X$ is a closed subset, and $s \in \Gamma(X \setminus Z, \mathcal{L})$ is a nowhere vanishing section. Similarly to the above, one can define for every α in $\mathrm{CH}_{k+1}(X)$ a product $D \cdot \alpha$ in $\mathrm{CH}_k(Z \cap |\alpha|)$ where $|\alpha|$ is the support of α.

02T9 Lemma 42.29.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Let α be a $(k+1)$-cycle on X. Then $i_* i^* \alpha = c_1(\mathcal{L}) \cap \alpha$ in $\mathrm{CH}_k(X)$. In particular, if D is an effective Cartier divisor, then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.

Proof. Write $\alpha = \sum n_j[W_j]$ where $i_j : W_j \to X$ are integral closed subschemes with $\dim_\delta(W_j) = k$. Since D is the zero scheme of s we see that $D \cap W_j$ is the zero scheme of the restriction $s|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$ by Lemma 42.25.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{L})|_{W_j}) \cap [W_j])$$

in $\mathrm{CH}_k(X)$ by Definition 42.25.1. The right hand side matches (termwise) the pushforward of the class $i^*\alpha$ on D from Definition 42.29.1. Hence we win. $\square$

02TB Lemma 42.29.5. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1.

[2] This condition assures us that if D is an effective Cartier divisor, then $\mathcal{N} = \mathcal{O}_X(D)|_D$.

(1) Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*[Z]_{k+1} = [D \cap Z]_k$.
(2) Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k+1$ and $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Then

$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$

in $\text{CH}_k(D)$.

Proof. Assume $Z \subset X$ as in (1). Then set $\mathcal{F} = \mathcal{O}_Z$. The assumption that $D \cap Z$ is an effective Cartier divisor is equivalent to the assumption that $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Moreover $[Z]_{k+1} = [\mathcal{F}]_{k+1}$ and $[D \cap Z]_k = [\mathcal{O}_{D\cap Z}]_k = [i^*\mathcal{F}]_k$. See Lemma 42.10.3. Hence part (1) follows from part (2).

Write $[\mathcal{F}]_{k+1} = \sum m_j[W_j]$ with $m_j > 0$ and pairwise distinct integral closed subschemes $W_j \subset X$ of δ-dimension $k+1$. The assumption that $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective implies that $W_j \not\subset D$ for all j. By definition we see that

$$i^*[\mathcal{F}]_{k+1} = \sum m_j[D \cap W_j]_k.$$

We claim that

$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$

as cycles. Let $Z \subset D$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. Let $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$. Let $f \in A$ be an element generating the ideal of D, i.e., such that $\mathcal{O}_{D,\xi} = A/fA$. By assumption $\dim(\text{Supp}(M)) = 1$, the map $f : M \to M$ is injective, and $\text{length}_A(M/fM) < \infty$. Moreover, $\text{length}_A(M/fM)$ is the coefficient of $[Z]$ in $[i^*\mathcal{F}]_k$. On the other hand, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then

$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \text{ord}_{A/\mathfrak{q}_i}(f)$$

is the coefficient of $[Z]$ in $\sum [D \cap W_j]_k$. Hence we see the equality by Lemma 42.3.2. □

0B6Z Remark 42.29.6. Let $X \to S$, $\mathcal{L}$, s, $i : D \to X$ be as in Definition 42.29.1 and assume that $\mathcal{L}|_D \cong \mathcal{O}_D$. In this case we can define a canonical map $i^* : Z_{k+1}(X) \to Z_k(D)$ on cycles, by requiring that $i^*[W] = 0$ whenever $W \subset D$ is an integral closed subscheme. The possibility to do this will be useful later on.

0B6Y Remark 42.29.7. Let $f : X' \to X$ be a morphism of schemes locally of finite type over S as in Situation 42.7.1. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. Then we can set $\mathcal{L}' = f^*\mathcal{L}$, $s' = f^*s$, and $D' = X' \times_X D = Z(s')$. This gives a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

and we can ask for various compatibilities between i^* and $(i')^*$.

02TA Lemma 42.29.8. Let (S, δ) be as in Situation 42.7.1. Let $f : X' \to X$ be a proper morphism of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in

Definition 42.29.1. Form the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Remark 42.29.7. For any $(k+1)$-cycle α' on X' we have $i^* f_* \alpha' = g_*(i')^*\alpha'$ in $\mathrm{CH}_k(D)$ (this makes sense as f_* is defined on the level of cycles).

Proof. Suppose $\alpha = [W']$ for some integral closed subscheme $W' \subset X'$. Let $W = f(W') \subset X$. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \mathrm{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \mathrm{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Lemma 42.26.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$ is equal to $c_1(\mathcal{L}|_W) \cap [W]$ in $\mathrm{CH}_k(W)$ by the second assertion of Lemma 42.26.3. By Remark 42.19.6 the result follows for general α'. $\square$

0B71 Lemma 42.29.9. Let (S, δ) be as in Situation 42.7.1. Let $f : X' \to X$ be a flat morphism of relative dimension r of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Form the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Remark 42.29.7. For any $(k+1)$-cycle α on X we have $(i')^* f^* \alpha = g^* i^* \alpha$ in $\mathrm{CH}_{k+r}(D')$ (this makes sense as f^* is defined on the level of cycles).

Proof. Suppose $\alpha = [W]$ for some integral closed subscheme $W \subset X$. Let $W' = f^{-1}(W) \subset X'$. In case $W \not\subset D$, then $W' \not\subset D'$ and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subschemes of D'. Hence the equality holds as cycles, see Lemma 42.14.4. In case $W \subset D$, then $W' \subset D'$ and $W' = g^{-1}(W)$ with $[W']_{k+1+r} = g^*[W]$ and equality holds in $\mathrm{CH}_{k+r}(D')$ by Lemma 42.26.2. By Remark 42.19.6 the result follows for general α'. $\square$

42.30. Gysin homomorphisms and rational equivalence

02TK In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence. We also prove an important commutativity property.

02TM Lemma 42.30.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module and let t be a nonzero meromorphic section of $\mathcal{N}$. Then $i^*\mathrm{div}_{\mathcal{N}}(t) = c_1(\mathcal{N}|_D) \cap [D]_{n-1}$ in $\mathrm{CH}_{n-2}(D)$.

Proof. Write $\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$ for some integral closed subschemes $Z_i \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_i\}$ is locally finite, that $t \in \Gamma(U, \mathcal{N}|_U)$ is a generator where $U = X \setminus \bigcup Z_i$, and that every irreducible component of D is one of the Z_i, see Divisors, Lemmas 31.26.1, 31.26.4, and 31.27.2.

Set $\mathcal{L} = \mathcal{O}_X(D)$. Denote $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$ the canonical section. We will apply the discussion of Section 42.27 to our current situation. For each i let $\xi_i \in Z_i$ be its generic point. Let $B_i = \mathcal{O}_{X,\xi_i}$. For each i we pick generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ over B_i but we insist that we pick $s_i = s$ if $Z_i \not\subset D$. Write $s = f_i s_i$ and $t = g_i t_i$ with $f_i, g_i \in B_i$. Then $\text{ord}_{Z_i,\mathcal{N}}(t) = \text{ord}_{B_i}(g_i)$. On the other hand, we have $f_i \in B_i$ and

$$[D]_{n-1} = \sum \text{ord}_{B_i}(f_i)[Z_i]$$

because of our choices of s_i. We claim that

$$i^*\text{div}_{\mathcal{N}}(t) = \sum \text{ord}_{B_i}(g_i)\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for $\text{div}_{\mathcal{N}}(t)$ and the fact that $\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$ when $Z_i \not\subset D$ because in that case $s_i|_{Z_i} = s|_{Z_i}$ is a regular section, see Lemma 42.24.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \text{ord}_{B_i}(f_i)\text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 42.27.1) gives the equality

$$\sum \Big(\text{ord}_{B_i}(f_i)\text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \text{ord}_{B_i}(g_i)\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})\Big) = \sum \text{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If $Z_i \not\subset D$, then $f_i = 1$ and hence $\text{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$. Thus we get a rational equivalence between our specific cycles representing $i^*\text{div}_{\mathcal{N}}(t)$ and $c_1(\mathcal{N}) \cap [D]_{n-1}$ on D. This finishes the proof. $\square$

02TO Lemma 42.30.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. The Gysin homomorphism factors through rational equivalence to give a map $i^* : \text{CH}_{k+1}(X) \to \text{CH}_k(D)$.

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subschemes $W_j \subset X$ of δ-dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\text{div}_{W_j}(f_j)$. Set $X' = \coprod W_i$ and consider the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

of Remark 42.29.7. Since $X' \to X$ is proper we see that $i^*p_* = q_*(i')^*$ by Lemma 42.29.8. As we know that q_* factors through rational equivalence (Lemma 42.20.3), it suffices to prove the result for $\alpha' = \sum \text{div}_{W_j}(f_j)$ on X'. Clearly this reduces us to the case where X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$. If $X = D$, then we see that $i^*\alpha$ is equal to $c_1(\mathcal{L}) \cap \alpha$. This is rationally equivalent to zero by Lemma 42.28.2. If $D \neq X$, then we see that $i^*\text{div}_X(f)$ is equal to $c_1(\mathcal{O}_D) \cap [D]_{n-1}$ in $\text{CH}_{n-2}(D)$ by Lemma 42.30.1. Of course capping with $c_1(\mathcal{O}_D)$ is the zero map (Lemma 42.25.2). $\square$

0F95 Lemma 42.30.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Then $i^*i_* : \text{CH}_k(D) \to \text{CH}_{k-1}(D)$ sends α to $c_1(\mathcal{L}|_D) \cap \alpha$.

Proof. This is immediate from the definition of i_* on cycles and the definition of i^* given in Definition 42.29.1. □

0B72 Lemma 42.30.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module. Then $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^*\mathcal{N}) \cap i^*\alpha$ in $\mathrm{CH}_{k-2}(D)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. With exactly the same proof as in Lemma 42.30.2 this follows from Lemmas 42.26.4, 42.28.3, and 42.30.1. □

0B73 Lemma 42.30.5. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ and $(\mathcal{L}', s', i' : D' \to X)$ be two triples as in Definition 42.29.1. Then the diagram

$$\begin{tikzcd} \mathrm{CH}_k(X) \arrow[r, "i^*"'] \arrow[d, "(i')^*"'] & \mathrm{CH}_{k-1}(D) \arrow[d, "j^*"] \\ \mathrm{CH}_{k-1}(D') \arrow[r, "(j')^*"] & \mathrm{CH}_{k-2}(D \cap D') \end{tikzcd}$$

commutes where each of the maps is a gysin map.

Proof. Denote $j : D \cap D' \to D$ and $j' : D \cap D' \to D'$ the closed immersions corresponding to $(\mathcal{L}|_{D'}, s|_{D'}$ and $(\mathcal{L}'_D, s|_D)$. We have to show that $(j')^*i^*\alpha = j^*(i')^*\alpha$ for all $\alpha \in \mathrm{CH}_k(X)$. Let $W \subset X$ be an integral closed subscheme of dimension k. Let us prove the equality in case $\alpha = [W]$. We will deduce it from the key formula.

We let σ be a nonzero meromorphic section of $\mathcal{L}|_W$ which we require to be equal to $s|_W$ if $W \not\subset D$. We let σ' be a nonzero meromorphic section of $\mathcal{L}'|_W$ which we require to be equal to $s'|_W$ if $W \not\subset D'$. Write

$$\mathrm{div}_{\mathcal{L}|_W}(\sigma) = \sum \mathrm{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\mathrm{div}_{\mathcal{L}'|_W}(\sigma') = \sum \mathrm{ord}_{Z_i, \mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 42.27. Then we see that $Z_i \subset D$ if $n_i \neq 0$ and $Z'_i \subset D'$ if $n'_i \neq 0$. For each i, let $\xi_i \in Z_i$ be the generic point. As in Section 42.27 we choose for each i an element $\sigma_i \in \mathcal{L}_{\xi_i}$, resp. $\sigma'_i \in \mathcal{L}'_{\xi_i}$ which generates over $B_i = \mathcal{O}_{W, \xi_i}$ and which is equal to the image of s, resp. s' if $Z_i \not\subset D$, resp. $Z_i \not\subset D'$. Write $\sigma = f_i\sigma_i$ and $\sigma' = f'_i\sigma'_i$ so that $n_i = \mathrm{ord}_{B_i}(f_i)$ and $n'_i = \mathrm{ord}_{B_i}(f'_i)$. From our definitions it follows that

$$(j')^*i^*[W] = \sum \mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^*[W] = \sum \mathrm{ord}_{B_i}(f'_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 42.27.1) now gives the equality

$$\sum \left(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \mathrm{ord}_{B_i}(f'_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})\right) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that $\mathrm{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$ if $Z_i \not\subset D \cap D'$ because in this case either $f_i = 1$ or $f'_i = 1$. Thus we get a rational equivalence between our specific cycles representing $(j')^*i^*[W]$ and $j^*(i')^*[W]$ on $D \cap D' \cap W$. By Remark 42.19.6 the result follows for general α. □

42.31. Relative effective Cartier divisors

02TP Relative effective Cartier divisors are defined and studied in Divisors, Section 31.18. To develop the basic results on Chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

02TR **Lemma 42.31.1.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be a relative effective Cartier divisor (Divisors, Definition 31.18.2). Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in \mathrm{CH}_{k+1}(Y)$ we have

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $\mathrm{CH}_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $\mathrm{CH}_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subscheme of δ-dimension $k+1$. By Divisors, Lemma 31.18.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 42.29.5 we get the first equality in

$$i^*[p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$ as schemes. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ we see that $i^*p^*[W] = (p|_D)^*[W]$ as cycles. If $\alpha = \sum m_j[W_j]$ is a general $k+1$ cycle, then we get $i^*\alpha = \sum m_j i^*p^*[W_j] = \sum m_j(p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 42.29.4. $\square$

42.32. Affine bundles

02TS For an affine bundle the pullback map is surjective on Chow groups.

02TT **Lemma 42.32.1.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Assume that for every $y \in Y$, there exists an open neighbourhood $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is identified with the morphism $U \times \mathbf{A}^r \to U$. Then $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.

Proof. Let $\alpha \in \mathrm{CH}_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subschemes of δ-dimension $k+r$. Then the family $\{W_j\}$ is locally finite in X. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection $Z_j = \overline{f(W_j)}$ of closures of images is a locally finite collection of integral closed subschemes of Y.

Consider the fibre product diagrams

$$\begin{tikzcd} f^{-1}(Z_j) \arrow[r] \arrow[d, "f_j"'] & X \arrow[d, "f"] \\ Z_j \arrow[r] & Y \end{tikzcd}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in \mathrm{CH}_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$

will be rationally equivalent to α (see Remark 42.19.6). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $\mathrm{CH}_{k+r}(X) = 0$ and the lemma holds. By assumption there exists a dense open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 42.14.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $\mathrm{CH}_k(f^{-1}(Y \setminus V)) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(f^{-1}(V))$ of Lemma 42.19.3 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in \mathrm{CH}_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

Thus we may assume that $X = Y \times \mathbf{A}^r$. In this case we can factor f as

$$X = Y \times \mathbf{A}^r \to Y \times \mathbf{A}^{r-1} \to \ldots \to Y \times \mathbf{A}^1 \to Y.$$

Hence it suffices to do the case $r = 1$. By the argument in the second paragraph of the proof we are reduced to the case $\alpha = [W]$, Y integral, and $W \to Y$ dominant. Again we can do induction on $d = \dim_\delta(Y)$. If $W = Y \times \mathbf{A}^1$, then $[W] = f^*[Y]$. Lastly, $W \subset Y \times \mathbf{A}^1$ is a proper inclusion, then $W \to Y$ induces a finite field extension $R(W)/R(Y)$. Let $P(T) \in R(Y)[T]$ be the monic irreducible polynomial such that the generic fibre of $W \to Y$ is cut out by P in $\mathbf{A}^1_{R(Y)}$. Let $V \subset Y$ be a nonempty open such that $P \in \Gamma(V, \mathcal{O}_Y)[T]$, and such that $W \cap f^{-1}(V)$ is still cut out by P. Then we see that $\alpha|_{f^{-1}(V)} \sim_{rat} 0$ and hence $\alpha \sim_{rat} \alpha'$ for some cycle α' on $(Y \setminus V) \times \mathbf{A}^1$. By induction on the dimension we win. □

0B74 Lemma 42.32.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let

$$p : L = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{L})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+1}(L)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 42.32.1. Let $o : X \to L$ be the zero section of $L \to X$, i.e., the morphism corresponding to the surjection $\mathrm{Sym}^*(\mathcal{L}) \to \mathcal{O}_X$ which maps $\mathcal{L}^{\otimes n}$ to zero for all $n > 0$. Then $p \circ o = \mathrm{id}_X$ and $o(X)$ is an effective Cartier divisor on L. Hence by Lemma 42.31.1 we see that $o^* \circ p^* = \mathrm{id}$ and we conclude that p^* is injective too. □

02TU Remark 42.32.3. We will see later (Lemma 42.36.3) that if X is a vector bundle of rank r over Y then the pullback map $\mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is an isomorphism. This is true whenever $X \to Y$ satisfies the assumptions of Lemma 42.32.1, see [Tot14, Lemma 2.2]. We will sketch a proof in Remark 42.32.8 using higher chow groups.

0F96 Lemma 42.32.4. In the situation of Lemma 42.32.2 denote $o : X \to L$ the zero section (see proof of the lemma). Then we have

(1) $o(X)$ is the zero scheme of a regular global section of $p^*\mathcal{L}^{\otimes -1}$,
(2) $o_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(L)$ as o is a closed immersion,
(3) $o^* : \mathrm{CH}_{k+1}(L) \to \mathrm{CH}_k(X)$ as $o(X)$ is an effective Cartier divisor,
(4) $o^*p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(X)$ is the identity map,
(5) $o_*\alpha = -p^*(c_1(\mathcal{L}) \cap \alpha)$ for any $\alpha \in \mathrm{CH}_k(X)$, and
(6) $o^*o_* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(X)$ is equal to the map $\alpha \mapsto -c_1(\mathcal{L}) \cap \alpha$.

Proof. Since $p_*\mathcal{O}_L = \text{Sym}^*(\mathcal{L})$ we have $p_*(p^*\mathcal{L}^{\otimes -1}) = \text{Sym}^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$ by the projection formula (Cohomology, Lemma 20.54.2) and the section mentioned in (1) is the canonical trivialization $\mathcal{O}_X \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$. We omit the proof that the vanishing locus of this section is precisely $o(X)$. This proves (1).

Parts (2), (3), and (4) we've seen in the course of the proof of Lemma 42.32.2. Of course (4) is the first formula in Lemma 42.31.1.

Part (5) follows from the second formula in Lemma 42.31.1, additivity of capping with c_1 (Lemma 42.25.2), and the fact that capping with c_1 commutes with flat pullback (Lemma 42.26.2).

Part (6) follows from Lemma 42.30.3 and the fact that $o^*p^*\mathcal{L} = \mathcal{L}$. □

0F97 **Lemma 42.32.5.** Let Y be a scheme. Let $\mathcal{L}_i$, $i = 1, 2$ be invertible $\mathcal{O}_Y$-modules. Let s be a global section of $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Denote $i : D \to X$ the zero scheme of s. Then there exists a commutative diagram

$$\begin{tikzcd} D_1 \arrow[r, "i_1"'] \arrow[d, "p_1"'] & L \arrow[d, "p"] & D_2 \arrow[l, "i_2"] \arrow[d, "p_2"] \\ D \arrow[r, "i"] & Y & D \arrow[l, "i"'] \end{tikzcd}$$

and sections s_i of $p^*\mathcal{L}_i$ such that the following hold:

(1) $p^*s = s_1 \otimes s_2$,
(2) p is of finite type and flat of relative dimension 1,
(3) D_i is the zero scheme of s_i,
(4) $D_i \cong \underline{\text{Spec}}(\text{Sym}^*(\mathcal{L}_{1-i}^{\otimes -1})|_D))$ over D for $i = 1, 2$,
(5) $p^{-1}D = D_1 \cup D_2$ (scheme theoretic union),
(6) $D_1 \cap D_2$ (scheme theoretic intersection) maps isomorphically to D, and
(7) $D_1 \cap D_2 \to D_i$ is the zero section of the line bundle $D_i \to D$ for $i = 1, 2$.

Moreover, the formation of this diagram and the sections s_i commutes with arbitrary base change.

Proof. Let $p : X \to Y$ be the relative spectrum of the quasi-coherent sheaf of $\mathcal{O}_Y$-algebras

$$\mathcal{A} = \left(\bigoplus\nolimits_{a_1, a_2 \geq 0} \mathcal{L}_1^{\otimes -a_1} \otimes_{\mathcal{O}_Y} \mathcal{L}_2^{\otimes -a_2}\right)/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by local sections of the form $st - t$ for t a local section of any summand $\mathcal{L}_1^{\otimes -a_1} \otimes \mathcal{L}_2^{\otimes -a_2}$ with $a_1, a_2 > 0$. The sections s_i viewed as maps $p^*\mathcal{L}_i^{\otimes -1} \to \mathcal{O}_X$ are defined as the adjoints of the maps $\mathcal{L}_i^{\otimes -1} \to \mathcal{A} = p_*\mathcal{O}_X$. For any $y \in Y$ we can choose an affine open $V \subset Y$, say $V = \text{Spec}(B)$, containing y and trivializations $z_i : \mathcal{O}_V \to \mathcal{L}_i^{\otimes -1}|_V$. Observe that $f = s(z_1 z_2) \in A$ cuts out the closed subscheme D. Then clearly

$$p^{-1}(V) = \text{Spec}(B[z_1, z_2]/(z_1 z_2 - f))$$

Since D_i is cut out by z_i everything is clear. □

0F98 **Lemma 42.32.6.** In the situation of Lemma 42.32.5 assume Y is locally of finite type over (S, δ) as in Situation 42.7.1. Then we have $i_1^*p^*\alpha = p_1^*i^*\alpha$ in $\text{CH}_k(D_1)$ for all $\alpha \in \text{CH}_k(Y)$.

Proof. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. We distinguish two cases.

Assume $W \subset D$. Then $i^*[W] = c_1(\mathcal{L}_1) \cap [W] + c_1(\mathcal{L}_2) \cap [W]$ in $\mathrm{CH}_{k-1}(D)$ by our definition of gysin homomorphisms and the additivity of Lemma 42.25.2. Hence $p_1^* i^*[W] = p_1^*(c_1(\mathcal{L}_1) \cap [W]) + p_1^*(c_1(\mathcal{L}_2) \cap [W])$. On the other hand, we have $p^*[W] = [p^{-1}(W)]_{k+1}$ by construction of flat pullback. And $p^{-1}(W) = W_1 \cup W_2$ (scheme theoretically) where $W_i = p_i^{-1}(W)$ is a line bundle over W by the lemma (since formation of the diagram commutes with base change). Then $[p^{-1}(W)]_{k+1} = [W_1] + [W_2]$ as W_i are integral closed subschemes of L of δ-dimension $k+1$. Hence

$$\begin{aligned} i_1^* p^*[W] &= i_1^*[p^{-1}(W)]_{k+1} \\ &= i_1^*([W_1] + [W_2]) \\ &= c_1(p_1^*\mathcal{L}_1) \cap [W_1] + [W_1 \cap W_2]_k \\ &= c_1(p_1^*\mathcal{L}_1) \cap p_1^*[W] + [W_1 \cap W_2]_k \\ &= p_1^*(c_1(\mathcal{L}_1) \cap [W]) + [W_1 \cap W_2]_k \end{aligned}$$

by construction of gysin homomorphisms, the definition of flat pullback (for the second equality), and compatibility of $c_1 \cap -$ with flat pullback (Lemma 42.26.2). Since $W_1 \cap W_2$ is the zero section of the line bundle $W_1 \to W$ we see from Lemma 42.32.4 that $[W_1 \cap W_2]_k = p_1^*(c_1(\mathcal{L}_2) \cap [W])$. Note that here we use the fact that D_1 is the line bundle which is the relative spectrum of the inverse of $\mathcal{L}_2$. Thus we get the same thing as before.

Assume $W \not\subset D$. In this case, both $i_1^* p^*[W]$ and $p_1^* i^*[W]$ are represented by the $k-1$ cycle associated to the scheme theoretic inverse image of W in D_1. $\square$

0F99 Lemma 42.32.7. In Situation 42.7.1 let X be a scheme locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. There exists a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "p"] & X' \arrow[d, "g"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

such that

(1) p and g are of finite type and flat of relative dimension 1,
(2) $p^* : \mathrm{CH}_k(D) \to \mathrm{CH}_{k+1}(D')$ is injective for all k,
(3) $D' \subset X'$ is the zero scheme of a global section $s' \in \Gamma(X', \mathcal{O}_{X'})$,
(4) $p^* i^* = (i')^* g^*$ as maps $\mathrm{CH}_k(X) \to \mathrm{CH}_k(D')$.

Moreover, these properties remain true after arbitrary base change by morphisms $Y \to X$ which are locally of finite type.

Proof. Observe that $(i')^*$ is defined because we have the triple $(\mathcal{O}_{X'}, s', i' : D' \to X')$ as in Definition 42.29.1. Thus the statement makes sense.

Set $\mathcal{L}_1 = \mathcal{O}_X$, $\mathcal{L}_2 = \mathcal{L}$ and apply Lemma 42.32.5 with the section s of $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Take $D' = D_1$. The results now follow from the lemma, from Lemma 42.32.6 and injectivity by Lemma 42.32.2. $\square$

0GUB Remark 42.32.8. Let (S, δ) be as in Situation 42.7.1. Let Y be locally of finite type over S. Let $r \geq 0$. Let $f : X \to Y$ be a morphism of schemes. Assume every

$y \in Y$ is contained in an open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. In this remark we sketch a proof of the fact that $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is an isomorphism. First, by Lemma 42.32.1 the map is surjective. Let $\alpha \in \mathrm{CH}_k(Y)$ with $f^*\alpha = 0$. We will prove that $\alpha = 0$.

Step 1. We may assume that $\dim_\delta(Y) < \infty$. (This is immediate in all cases in practice so we suggest the reader skip this step.) Namely, any rational equivalence witnessing that $f^*\alpha = 0$ on X, will use a locally finite collection of integral closed subschemes of dimension $k + r + 1$. Taking the union of the closures of the images of these in Y we get a closed subscheme $Y' \subset Y$ of $\dim_\delta(Y') \leq k + r + 1$ such that α is the image of some $\alpha' \in \mathrm{CH}_k(Y')$ and such that $(f')^*\alpha = 0$ where f' is the base change of f to Y'.

Step 2. Assume $d = \dim_\delta(Y) < \infty$. Then we can use induction on d. If $d < k$, then $\alpha = 0$ and we are done; this is the base case of the induction. In general, our assumption on f shows we can choose a dense open $V \subset Y$ such that $U = f^{-1}(V) = \mathbf{A}^r_V$. Denote $Y' \subset Y$ the complement of V as a reduced closed subscheme and set $X' = f^{-1}(Y')$. Consider

$$\begin{tikzcd} \mathrm{CH}^M_{k+r}(U,1) \arrow[r] & \mathrm{CH}_{k+r}(X') \arrow[r] & \mathrm{CH}_{k+r}(X) \arrow[r] & \mathrm{CH}_{k+r}(U) \arrow[r] & 0 \\ \mathrm{CH}^M_k(V,1) \arrow[r] \arrow[u] & \mathrm{CH}_k(Y') \arrow[r] \arrow[u] & \mathrm{CH}_k(Y) \arrow[r] \arrow[u] & \mathrm{CH}_k(V) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Here we use the first higher Chow groups of V and U and the six term exact sequences constructed in Remark 42.27.3, as well as flat pullback for these higher chow groups and compatibility of flat pullback with these six term exact sequences. Since $U = \mathbf{A}^r_V$ the vertical map on the right is an isomorphism. The map $\mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ is bijective by induction on d. Hence to finish the argument is suffices to show that

$$\mathrm{CH}^M_k(V,1) \longrightarrow \mathrm{CH}^M_{k+r}(U,1)$$

is surjective. Arguing as in the proof of Lemma 42.32.1 this reduces to Step 3 below.

Step 3. Let F be a field. Then $\mathrm{CH}^M_0(\mathbf{A}^1_F, 1) = 0$. (In the proof of the lemma cited above we proved analogously that $\mathrm{CH}_0(\mathbf{A}^1_F) = 0$.) We have

$$\mathrm{CH}^M_0(\mathbf{A}^1_F, 1) = \mathrm{Coker}\left(\partial : K^M_2(F(T)) \longrightarrow \bigoplus\nolimits_{\mathfrak p \subset F[T] \text{ maximal}} \kappa(\mathfrak p)^*\right)$$

The classical argument for the vanishing of the cokernel is to show by induction on the degree of $\kappa(\mathfrak p)/F$ that the summand corresponding to $\mathfrak p$ is in the image. If $\mathfrak p$ is generated by the irreducible monic polynomial $P(T) \in F[T]$ and if $u \in \kappa(x)^*$ is the residue class of some $Q(T) \in F[T]$ with $\deg(Q) < \deg(P)$ then one shows that $\partial(Q, P)$ produces the element u at $\mathfrak p$ and perhaps some other units at primes dividing Q which have lower degree. This finishes the sketch of the proof.

42.33. Bivariant intersection theory

0B75 In order to intelligently talk about higher Chern classes of vector bundles we introduce bivariant chow classes as in [Ful98]. Our definition differs from [Ful98] in two respects: (1) we work in a different setting, and (2) we only require our bivariant classes commute with the gysin homomorphisms for zero schemes of sections of invertible modules (Section 42.29). We will see later, in Lemma 42.54.8, that our bivariant classes commute with all higher codimension gysin homomorphisms and hence satisfy all properties required of them in [Ful98]; see also [Ful98, Theorem 17.1].

0B76 Definition 42.33.1. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. A bivariant class c of degree p for f is given by a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map Similar to [Ful98, Definition 17.1]

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $X' = Y' \times_Y X$, satisfying the following conditions

(1) if $Y'' \to Y'$ is a proper, then $c \cap (Y'' \to Y')_*\alpha'' = (X'' \to X')_*(c \cap \alpha'')$ for all α'' on Y'' where $X'' = Y'' \times_Y X$,
(2) if $Y'' \to Y'$ is flat locally of finite type of fixed relative dimension, then $c \cap (Y'' \to Y')^*\alpha' = (X'' \to X')^*(c \cap \alpha')$ for all α' on Y', and
(3) if $(\mathcal{L}', s', i' : D' \to Y')$ is as in Definition 42.29.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X', then we have $c \cap (i')^*\alpha' = (j')^*(c \cap \alpha')$ for all α' on Y'.

The collection of all bivariant classes of degree p for f is denoted $A^p(X \to Y)$.

Let (S, δ) be as in Situation 42.7.1. Let $X \to Y$ and $Y \to Z$ be morphisms of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. It is clear that $A^p(X \to Y)$ is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \to Y) \times A^q(Y \to Z) \to A^{p+q}(X \to Z)$$

which is associative.

0B78 Lemma 42.33.2. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a flat morphism of relative dimension r between schemes locally of finite type over S. Then the rule that to $Y' \to Y$ assigns $(f')^* : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ where $X' = X \times_Y Y'$ is a bivariant class of degree $-r$.

Proof. This follows from Lemmas 42.20.2, 42.14.3, 42.15.1, and 42.29.9. □

0B79 Lemma 42.33.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. Then the rule that to $f : X' \to X$ assigns $(i')^* : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(D')$ where $D' = D \times_X X'$ is a bivariant class of degree 1.

Proof. This follows from Lemmas 42.30.2, 42.29.8, 42.29.9, and 42.30.5. □

0EPK Lemma 42.33.4. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes locally of finite type over S. Let $c \in A^p(X \to Z)$ and assume f is proper. Then the rule that to $Z' \to Z$ assigns $\alpha \longmapsto f'_*(c \cap \alpha)$ is a bivariant class denoted $f_* \circ c \in A^p(Y \to Z)$.

Proof. This follows from Lemmas 42.12.2, 42.15.1, and 42.29.8. □

0F9Z Remark 42.33.5. Let (S, δ) be as in Situation 42.7.1. Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes locally of finite type over S. Let $X' = Y' \times_Y X$. Then there is an obvious restriction map

$$A^p(X \to Y) \longrightarrow A^p(X' \to Y'), \quad c \longmapsto res(c)$$

obtained by viewing a scheme Y'' locally of finite type over Y' as a scheme locally of finite type over Y and settting $res(c) \cap \alpha'' = c \cap \alpha''$ for any $\alpha'' \in \mathrm{CH}_k(Y'')$. This restriction operation is compatible with compositions in an obvious manner.

0FA2 Remark 42.33.6. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. For $i = 1, 2$ let $Z_i \to X$ be a morphism of schemes locally of finite type. Let $c_i \in A^{p_i}(Z_i \to X)$, $i = 1, 2$ be bivariant classes. For any $\alpha \in \mathrm{CH}_k(X)$ we can ask whether

$$c_1 \cap c_2 \cap \alpha = c_2 \cap c_1 \cap \alpha$$

in $\mathrm{CH}_{k-p_1-p_2}(Z_1 \times_X Z_2)$. If this is true and if it holds after any base change by $X' \to X$ locally of finite type, then we say c_1 and c_2 commute. Of course this is the same thing as saying that

$$res(c_1) \circ c_2 = res(c_2) \circ c_1$$

in $A^{p_1+p_2}(Z_1 \times_X Z_2 \to X)$. Here $res(c_1) \in A^{p_1}(Z_1 \times_X Z_2 \to Z_2)$ is the restriction of c_1 as in Remark 42.33.5; similarly for $res(c_2)$.

0FA3 Example 42.33.7. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ a triple as in Definition 42.29.1. Let $Z \to X$ be a morphism of schemes locally of finite type and let $c \in A^p(Z \to X)$ be a bivariant class. Then the bivariant gysin class $c' \in A^1(D \to X)$ of Lemma 42.33.3 commutes with c in the sense of Remark 42.33.6. Namely, this is a restatement of condition (3) of Definition 42.33.1.

0FDU Remark 42.33.8. There is a more general type of bivariant class that doesn't seem to be considered in the literature. Namely, suppose we are given a diagram

$$X \longrightarrow Z \longleftarrow Y$$

of schemes locally of finite type over (S, δ) as in Situation 42.7.1. Let $p \in \mathbf{Z}$. Then we can consider a rule c which assigns to every $Z' \to Z$ locally of finite type maps

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

for all $k \in \mathbf{Z}$ where $X' = X \times_Z Z'$ and $Y' = Z' \times_Z Y$ compatible with

(1) proper pushforward if given $Z'' \to Z'$ proper,
(2) flat pullback if given $Z'' \to Z'$ flat of fixed relative dimension, and
(3) gysin maps if given $D' \subset Z'$ as in Definition 42.29.1.

We omit the detailed formulations. Suppose we denote the collection of all such operations $A^p(X \to Z \leftarrow Y)$. A simple example of the utility of this concept is when we have a proper morphism $f : X_2 \to X_1$. Then f_* isn't a bivariant operation in the sense of Definition 42.33.1 but it is in the above generalized sense, namely, $f_* \in A^0(X_1 \to X_1 \leftarrow X_2)$.

42.34. Chow cohomology and the first Chern class

0FDV We will be most interested in $A^p(X) = A^p(X \to X)$, which will always mean the bivariant cohomology classes for id_X. Namely, that is where Chern classes will live.

0B7E Definition 42.34.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The Chow cohomology of X is the graded $\mathbf{Z}$-algebra $A^*(X)$ whose degree p component is $A^p(X \to X)$.

Warning: It is not clear that the $\mathbf{Z}$-algebra structure on $A^*(X)$ is commutative, but we will see that Chern classes live in its center.

0B7F Remark 42.34.2. Let (S, δ) be as in Situation 42.7.1. Let $f : Y' \to Y$ be a morphism of schemes locally of finite type over S. As a special case of Remark 42.33.5 there is a canonical $\mathbf{Z}$-algebra map $res : A^*(Y) \to A^*(Y')$. This map is often denoted f^* in the literature.

0B77 Lemma 42.34.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the rule that to $f : X' \to X$ assigns $c_1(f^*\mathcal{L}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(X')$ is a bivariant class of degree 1.

Proof. This follows from Lemmas 42.28.2, 42.26.4, 42.26.2, and 42.30.4. $\square$

The lemma above finally allows us to make the following definition.

0FDW Definition 42.34.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first Chern class $c_1(\mathcal{L}) \in A^1(X)$ of $\mathcal{L}$ is the bivariant class of Lemma 42.34.3.

For finite locally free modules we construct the Chern classes in Section 42.38. Let us prove that $c_1(\mathcal{L})$ is in the center of $A^*(X)$.

0B7B Lemma 42.34.5. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

1. $c_1(\mathcal{L}) \in A^1(X)$ is in the center of $A^*(X)$ and
2. if $f : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_1(\mathcal{L}) = c_1(f^*\mathcal{L}) \circ c$.

Proof. Of course (2) implies (1). Let $p : L \to X$ be as in Lemma 42.32.2 and let $o : X \to L$ be the zero section. Denote $p' : L' \to X'$ and $o' : X' \to L'$ their base changes. By Lemma 42.32.4 we have

$$p^*(c_1(\mathcal{L}) \cap \alpha) = -o_*\alpha \quad \text{and} \quad (p')^*(c_1(f^*\mathcal{L}) \cap \alpha') = -o'_*\alpha'$$

Since c is a bivariant class we have

$$\begin{aligned} (p')^*(c \cap c_1(\mathcal{L}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}) \cap \alpha) \\ &= -c \cap o_*\alpha \\ &= -o'_*(c \cap \alpha) \\ &= (p')^*(c_1(f^*\mathcal{L}) \cap c \cap \alpha) \end{aligned}$$

Since $(p')^*$ is injective by one of the lemmas cited above we obtain $c \cap c_1(\mathcal{L}) \cap \alpha = c_1(f^*\mathcal{L}) \cap c \cap \alpha$. The same is true after any base change by $Y \to X$ locally of finite type and hence we have the equality of bivariant classes stated in (2). $\square$

0FDX Lemma 42.34.6. Let (S, δ) be as in Situation 42.7.1. Let X be a finite type scheme over S which has an ample invertible sheaf. Assume $d = \dim(X) < \infty$ (here we really mean dimension and not δ-dimension). Then for any invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ on X we have $c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_{d+1}) = 0$ in $A^{d+1}(X)$.

Proof. We prove this by induction on d. The base case $d = 0$ is true because in this case X is a finite set of closed points and hence every invertible module is trivial. Assume $d > 0$. By Divisors, Lemma 31.15.12 we can write $\mathcal{L}_{d+1} \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors $D, D' \subset X$. Then $c_1(\mathcal{L}_{d+1})$ is the difference of $c_1(\mathcal{O}_X(D))$ and $c_1(\mathcal{O}_X(D'))$ and hence we may assume $\mathcal{L}_{d+1} = \mathcal{O}_X(D)$ for some effective Cartier divisor.

Denote $i : D \to X$ the inclusion morphism and denote $i^* \in A^1(D \to X)$ the bivariant class given by the gysin hommomorphism as in Lemma 42.33.3. We have $i_* \circ i^* = c_1(\mathcal{L}_{d+1})$ in $A^1(X)$ by Lemma 42.29.4 (and Lemma 42.33.4 to make sense of the left hand side). Since $c_1(\mathcal{L}_i)$ commutes with both i_* and i^* (by definition of bivariant classes) we conclude that

$$c_1(\mathcal{L}_1)\circ\ldots\circ c_1(\mathcal{L}_{d+1}) = i_*\circ c_1(\mathcal{L}_1)\circ\ldots\circ c_1(\mathcal{L}_d)\circ i^* = i_*\circ c_1(\mathcal{L}_1|_D)\circ\ldots\circ c_1(\mathcal{L}_d|_D)\circ i^*$$

Thus we conclude by induction on d. Namely, we have $\dim(D) < d$ as none of the generic points of X are in D. □

0FA0 Remark 42.34.7. Let (S, δ) be as in Situation 42.7.1. Let $Z \to X$ be a closed immersion of schemes locally of finite type over S and let $p \geq 0$. In this setting we define

$$A^{(p)}(Z \to X) = \prod\nolimits_{i \leq p-1} A^i(X) \times \prod\nolimits_{i \geq p} A^i(Z \to X).$$

Then $A^{(p)}(Z \to X)$ canonically comes equipped with the structure of a graded algebra. In fact, more generally there is a multiplication

$$A^{(p)}(Z \to X) \times A^{(q)}(Z \to X) \longrightarrow A^{(\max(p,q))}(Z \to X)$$

In order to define these we define maps

$$A^i(Z \to X) \times A^j(X) \to A^{i+j}(Z \to X)$$
$$A^i(X) \times A^j(Z \to X) \to A^{i+j}(Z \to X)$$
$$A^i(Z \to X) \times A^j(Z \to X) \to A^{i+j}(Z \to X)$$

For the first we use composition of bivariant classes. For the second we use restriction $A^i(X) \to A^i(Z)$ (Remark 42.33.5) and composition $A^i(Z) \times A^j(Z \to X) \to A^{i+j}(Z \to X)$. For the third, we send (c, c') to $res(c) \circ c'$ where $res : A^i(Z \to X) \to A^i(Z)$ is the restriction map (see Remark 42.33.5). We omit the verification that these multiplications are associative in a suitable sense.

0FA1 Remark 42.34.8. Let (S, δ) be as in Situation 42.7.1. Let $Z \to X$ be a closed immersion of schemes locally of finite type over S. Denote $res : A^p(Z \to X) \to A^p(Z)$ the restriction map of Remark 42.33.5. For $c \in A^p(Z \to X)$ we have $res(c) \cap \alpha = c \cap i_*\alpha$ for $\alpha \in \mathrm{CH}_*(Z)$. Namely $res(c) \cap \alpha = c \cap \alpha$ and compatibility of c with proper pushforward gives $(Z \to Z)_*(c \cap \alpha) = c \cap (Z \to X)_*\alpha$.

42.35. Lemmas on bivariant classes

0FDY In this section we prove some elementary results on bivariant classes. Here is a criterion to see that an operation passes through rational equivalence.

0B7A Lemma 42.35.1. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. Suppose given a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map

Very weak form of [Ful98, Theorem 17.1]

$$c \cap - : Z_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $Y' = X' \times_X Y$, satisfying condition (3) of Definition 42.33.1 whenever $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$. Then $c \cap -$ factors through rational equivalence.

Proof. The statement makes sense because given a triple $(\mathcal{L}, s, i : D \to X)$ as in Definition 42.29.1 such that $\mathcal{L}|_D \cong \mathcal{O}_D$, then the operation i^* is defined on the level of cycles, see Remark 42.29.6. Let $\alpha \in Z_k(X')$ be a cycle which is rationally equivalent to zero. We have to show that $c \cap \alpha = 0$. By Lemma 42.21.1 there exists a cycle $\beta \in Z_{k+1}(X' \times \mathbf{P}^1)$ such that $\alpha = i_0^*\beta - i_\infty^*\beta$ where $i_0, i_\infty : X' \to X' \times \mathbf{P}^1$ are the closed immersions of X' over $0, \infty$. Since these are examples of effective Cartier divisors with trivial normal bundles, we see that $c \cap i_0^*\beta = j_0^*(c \cap \beta)$ and $c \cap i_\infty^*\beta = j_\infty^*(c \cap \beta)$ where $j_0, j_\infty : Y' \to Y' \times \mathbf{P}^1$ are closed immersions as before. Since $j_0^*(c \cap \beta) \sim_{rat} j_\infty^*(c \cap \beta)$ (follows from Lemma 42.21.1) we conclude. $\square$

0F9A **Lemma 42.35.2.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. Suppose given a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map Weak form of [Ful98, Theorem 17.1]

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $Y' = X' \times_X Y$, satisfying conditions (1), (2) of Definition 42.33.1 and condition (3) whenever $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$. Then $c \cap -$ is a bivariant class.

Proof. Let $Y' \to Y$ be a morphism of schemes which is locally of finite type. Let $(\mathcal{L}', s', i' : D' \to Y')$ be as in Definition 42.29.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X'. We have to show that $c \cap (i')^*\alpha' = (j')^*(c \cap \alpha')$ for all $\alpha' \in \mathrm{CH}_k(Y')$.

Denote $g : Y'' \to Y'$ the smooth morphism of relative dimension 1 with $i'' : D'' \to Y''$ and $p : D'' \to D'$ constructed in Lemma 42.32.7. (Warning: D'' isn't the full inverse image of D'.) Denote $f : X'' \to X'$ and $E'' \subset X''$ their base changes by $X' \to Y'$. Picture

$$\begin{tikzcd}
& X'' \arrow[rr] \arrow[dd, "h"] & & Y'' \arrow[dd, "g"] \\
E'' \arrow[ur, "j''"] \arrow[rr] \arrow[dd, "q"'] & & D'' \arrow[ur, "i''"] \arrow[dd, "p"] & \\
& X' \arrow[rr] & & Y' \\
E' \arrow[ur, "j'"] \arrow[rr] & & D' \arrow[ur, "i'"] &
\end{tikzcd}$$

By the properties given in the lemma we know that $\beta' = (i')^*\alpha'$ is the unique element of $\mathrm{CH}_{k-1}(D')$ such that $p^*\beta' = (i'')^*g^*\alpha'$. Similarly, we know that $\gamma' = (j')^*(c \cap \alpha')$ is the unique element of $\mathrm{CH}_{k-1-p}(E')$ such that $q^*\gamma' = (j'')^*h^*(c \cap \alpha')$. Since we know that

$$(j'')^*h^*(c \cap \alpha') = (j'')^*(c \cap g^*\alpha') = c \cap (i'')^*g^*\alpha'$$

by our assuptions on c; note that the modified version of (3) assumed in the statement of the lemma applies to i'' and its base change j''. We similarly know that

$$q^*(c \cap \beta') = c \cap p^*\beta'$$

We conclude that $\gamma' = c \cap \beta'$ by the uniqueness pointed out above. $\square$

Here a criterion for when a bivariant class is zero.

02UC **Lemma 42.35.3.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $c \in A^p(X \to Y)$. For $Y'' \to Y' \to Y$ set $X'' = Y'' \times_Y X$ and $X' = Y' \times_Y X$. The following are equivalent

(1) c is zero,
(2) $c \cap [Y'] = 0$ in $\mathrm{CH}_*(X')$ for every integral scheme Y' locally of finite type over Y, and
(3) for every integral scheme Y' locally of finite type over Y, there exists a proper birational morphism $Y'' \to Y'$ such that $c \cap [Y''] = 0$ in $\mathrm{CH}_*(X'')$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. Assumption (3) implies (2) because $(Y'' \to Y')_*[Y''] = [Y']$ and hence $c \cap [Y'] = (X'' \to X')_*(c \cap [Y''])$ as c is a bivariant class. Assume (2). Let $Y' \to Y$ be locally of finite type. Let $\alpha \in \mathrm{CH}_k(Y')$. Write $\alpha = \sum n_i[Y'_i]$ with $Y'_i \subset Y'$ a locally finite collection of integral closed subschemes of δ-dimension k. Then we see that α is pushforward of the cycle $\alpha' = \sum n_i[Y'_i]$ on $Y'' = \coprod Y'_i$ under the proper morphism $Y'' \to Y'$. By the properties of bivariant classes it suffices to prove that $c \cap \alpha' = 0$ in $\mathrm{CH}_{k-p}(X'')$. We have $\mathrm{CH}_{k-p}(X'') = \prod \mathrm{CH}_{k-p}(X'_i)$ where $X'_i = Y'_i \times_Y X$. This follows immediately from the definitions. The projection maps $\mathrm{CH}_{k-p}(X'') \to \mathrm{CH}_{k-p}(X'_i)$ are given by flat pullback. Since capping with c commutes with flat pullback, we see that it suffices to show that $c \cap [Y'_i]$ is zero in $\mathrm{CH}_{k-p}(X'_i)$ which is true by assumption. $\square$

0FDZ **Lemma 42.35.4.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Assume we have disjoint union decompositions $X = \coprod_{i \in I} X_i$ and $Y = \coprod_{j \in J} Y_j$ by open and closed subschemes and a map $a : I \to J$ of sets such that $f(X_i) \subset Y_{a(i)}$. Then

$$A^p(X \to Y) = \prod\nolimits_{i \in I} A^p(X_i \to Y_{a(i)})$$

Proof. Suppose given an element $(c_i) \in \prod_i A^p(X_i \to Y_{a(i)})$. Then given $\beta \in \mathrm{CH}_k(Y)$ we can map this to the element of $\mathrm{CH}_{k-p}(X)$ whose restriction to X_i is $c_i \cap \beta|_{Y_{a(i)}}$. This works because $\mathrm{CH}_{k-p}(X) = \prod_i \mathrm{CH}_{k-p}(X_i)$. The same construction works after base change by any $Y' \to Y$ locally of finite type and we get $c \in A^p(X \to Y)$. Thus we obtain a map Ψ from the right hand side of the formula to the left hand side of the formula. Conversely, given $c \in A^p(X \to Y)$ and an element $\beta_i \in \mathrm{CH}_k(Y_{a(i)})$ we can consider the element $(c \cap (Y_{a(i)} \to Y)_*\beta_i)|_{X_i}$ in $\mathrm{CH}_{k-p}(X_i)$. The same thing works after base change by any $Y' \to Y$ locally of finite type and we get $c_i \in A^p(X_i \to Y_{a(i)})$. Thus we obtain a map Φ from the left hand side of the formula to the right hand side of the formula. It is immediate that $\Phi \circ \Psi = \mathrm{id}$. For the converse, suppose that $c \in A^p(X \to Y)$ and $\beta \in \mathrm{CH}_k(Y)$. Say $\Phi(c) = (c_i)$. Let $j \in J$. Because c commutes with flat pullback we get

$$(c \cap \beta)|_{\coprod_{a(i)=j} X_i} = c \cap \beta|_{Y_j}$$

Because c commutes with proper pushforward we get

$$(\coprod\nolimits_{a(i)=j} X_i \to X)_*((c \cap \beta)|_{\coprod_{a(i)=j} X_i}) = c \cap (Y_j \to Y)_*\beta|_{Y_j}$$

The left hand side is the cycle on X restricting to $(c \cap \beta)|_{X_i}$ on X_i for $i \in I$ with $a(i) = j$ and 0 else. The right hand side is a cycle on X whose restriction to X_i is $c_i \cap \beta|_{Y_j}$ for $i \in I$ with $a(i) = j$. Thus $c \cap \beta = \Psi((c_i))$ as desired. $\square$

0FE0 Remark 42.35.5. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $X = \coprod_{i \in I} X_i$ and $Y = \coprod_{j \in J} Y_j$ be the decomposition of X and Y into their connected components (the connected components are open as X and Y are locally Noetherian, see Topology, Lemma 5.9.6 and Properties, Lemma 28.5.5). Let $a(i) \in J$ be the index such that $f(X_i) \subset Y_{a(i)}$. Then $A^p(X \to Y) = \prod A^p(X_i \to Y_{a(i)})$ by Lemma 42.35.4. In this setting it is convenient to set

$$A^*(X \to Y)^\wedge = \prod\nolimits_i A^*(X_i \to Y_{a(i)})$$

This "completed" bivariant group is the subset

$$A^*(X \to Y)^\wedge \quad \subset \quad \prod\nolimits_{p \geq 0} A^p(X)$$

consisting of elements $c = (c_0, c_1, c_2, \ldots)$ such that for each connected component X_i the image of c_p in $A^p(X_i \to Y_{a(i)})$ is zero for almost all p. If $Y \to Z$ is a second morphism, then the composition $A^*(X \to Y) \times A^*(Y \to Z) \to A^*(X \to Z)$ extends to a composition $A^*(X \to Y)^\wedge \times A^*(Y \to Z)^\wedge \to A^*(X \to Z)^\wedge$ of completions. We sometimes call $A^*(X)^\wedge = A^*(X \to X)^\wedge$ the completed bivariant cohomology ring of X.

0GUC Lemma 42.35.6. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $g : Y' \to Y$ be an envelope (Definition 42.22.1) and denote $X' = Y' \times_Y X$. Let $p \in \mathbf{Z}$ and let $c' \in A^p(X' \to Y')$. If the two restrictions

$$res_1(c') = res_2(c') \in A^p(X' \times_X X' \to Y' \times_Y Y')$$

are equal (see proof), then there exists a unique $c \in A^p(X \to Y)$ whose restriction $res(c) = c'$ in $A^p(X' \to Y')$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
X' \times_X X' \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "f''"] & X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \times_Y Y' \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

The element $res_1(c')$ is the restriction (see Remark 42.33.5) of c' for the cartesian square with morphisms a, f', p, f'' and the element $res_2(c')$ is the restriction of c' for the cartesian square with morphisms b, f', q, f''. Assume $res_1(c') = res_2(c')$ and let $\beta \in \mathrm{CH}_k(Y)$. By Lemma 42.22.4 we can find a $\beta' \in \mathrm{CH}_k(Y')$ with $g_*\beta' = \beta$. Then we set

$$c \cap \beta = h_*(c' \cap \beta')$$

To see that this is independent of the choice of β' it suffices to show that $h_*(c' \cap (p_*\gamma - q_*\gamma))$ is zero for $\gamma \in \mathrm{CH}_k(Y' \times_Y Y')$. Since c' is a bivariant class we have

$$h_*(c' \cap (p_*\gamma - q_*\gamma)) = h_*(a_*(c' \cap \gamma) - b_*(c' \cap \gamma)) = 0$$

the last equality since $h_* \circ a_* = h_* \circ b_*$ as $h \circ a = h \circ b$.

Observe that our choice for $c \cap \beta$ is forced by the requirement that $res(c) = c'$ and the compatibility of bivariant classes with proper pushforward.

Of course, in order to define the bivariant class c we need to construct maps $c \cap - : \mathrm{CH}_k(Y_1) \to \mathrm{CH}_{k+p}(Y_1 \times_Y X)$ for any morphism $Y_1 \to Y$ locally of finite type satisfying the conditions listed in Definition 42.33.1. Denote $Y_1' = Y' \times_Y Y_1$, $X_1 = X \times_Y Y_1$. The morphism $Y_1' \to Y_1$ is an envelope by Lemma 42.22.3. Hence we can use the base changed diagram

$$\begin{tikzcd} X_1' \times_{X_1} X_1' \arrow[r, shift left, "a_1"] \arrow[r, shift right, "b_1"'] \arrow[d, "f_1''"] & X_1' \arrow[r, "h_1"'] \arrow[d, "f_1'"] & X_1 \arrow[d, "f_1"] \\ Y_1' \times_{Y_1} Y_1' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "q_1"'] & Y_1' \arrow[r, "g_1"] & Y_1 \end{tikzcd}$$

and the same arguments to get a well defined map $c \cap - : \mathrm{CH}_k(Y_1) \to \mathrm{CH}_{k+p}(X_1)$ as before.

Next, we have to check conditions (1), (2), and (3) of Definition 42.33.1 for c. For example, suppose that $t : Y_2 \to Y_1$ is a proper morphism of schemes locally of finite type over Y. Denote as above the base changes of the first diagram to Y_1, resp. Y_2, by subscripts $_1$, resp. $_2$. Denote $t' : Y_2' \to Y_1'$, $s : X_2 \to X_1$, and $s' : X_2' \to X_1'$ the base changes of t to Y', X, and X'. We have to show that

$$s_*(c \cap \beta_2) = c \cap t_*\beta_2$$

for $\beta_2 \in \mathrm{CH}_k(Y_2)$. Choose $\beta_2' \in \mathrm{CH}_k(Y_2')$ with $g_{2,*}\beta_2' = \beta_2$. Since c' is a bivariant class and the diagrams

$$\begin{tikzcd} X_2' \arrow[r, "h_2"'] \arrow[d, "s'"'] & X_2 \arrow[d, "s"] \\ X_1' \arrow[r, "h_1"] & X_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2' \arrow[r, "f_2'"'] \arrow[d, "s'"'] & Y_2' \arrow[d, "t'"] \\ X_2' \arrow[r, "f_1'"] & Y_1' \end{tikzcd}$$

are cartesian we have

$$s_*(c \cap \beta_2) = s_*(h_{2,*}(c' \cap \beta_2')) = h_{1,*}s'_*(c' \cap \beta_2') = h_{1,*}(c' \cap (t'_*\beta_2'))$$

and the final expression computes $c \cap t_*\beta_2$ by construction: $t'_*\beta_2' \in \mathrm{CH}_k(Y_1')$ is a class whose image by $g_{1,*}$ is $t_*\beta_2$. This proves condition (1). The other conditions are proved in the same manner and we omit the detailed arguments. $\square$

42.36. Projective space bundle formula

02TV Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the projective bundle associated to $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

02TW **Lemma 42.36.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in \mathrm{CH}_k(X)$ the element

$$\pi_* \left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha\right) \in \mathrm{CH}_{k+r-1-s}(X)$$

is 0 if $s < r - 1$ and is equal to α when $s = r - 1$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Note that $\pi^*[Z] = [\pi^{-1}(Z)]$ as $\pi^{-1}(Z)$ is integral of δ-dimension $r - 1$. If $s < r - 1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$ is represented by a $(k + r - 1 - s)$-cycle supported on $\pi^{-1}(Z)$. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by $X \setminus T$ where $T \subset Z$ is a proper closed subset. Let ξ be the generic point of Z. We can choose elements $e_1, \ldots, e_{r-1} \in \mathcal{E}_\xi$ which form part of a basis of $\mathcal{E}_\xi$. These give rational sections $s_1, \ldots, s_{r-1}$ of $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$ whose common zero set is the closure of the image a rational section of $\mathbf{P}(\mathcal{E}|_Z) \to Z$ union a closed subset whose support maps to a proper closed subset T of Z. After removing T from X (and correspondingly $\pi^{-1}(T)$ from P), we see that $s_1, \ldots, s_n$ form a sequence of global sections $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$ whose common zero set is the image of a section $Z \to \pi^{-1}(Z)$. Hence we see successively that

$$\begin{aligned} \pi^*[Z] &= [\pi^{-1}(Z)] \\ c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] &= [Z(s_1)] \\ c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] &= [Z(s_1) \cap Z(s_2)] \\ \ldots &= \ldots \\ c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] &= [Z(s_1) \cap \ldots \cap Z(s_{r-1})] \end{aligned}$$

by repeated applications of Lemma 42.25.4. Since the pushforward by π of the image of a section of π over Z is clearly $[Z]$ we see the result when $\alpha = [Z]$. We omit the verification that these arguments imply the result for a general cycle $\alpha = \sum n_j[Z_j]$. $\square$

02TX **Lemma 42.36.2** (Projective space bundle formula)**.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map

$$\bigoplus\nolimits_{i=0}^{r-1} \mathrm{CH}_{k+i}(X) \longrightarrow \mathrm{CH}_{k+r-1}(P),$$
$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 42.36.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let X_i, $i \in I$ be the irreducible components of X. Then $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$, $i \in I$ are the irreducible components of P. Consider the commutative diagram

$$\begin{tikzcd} \coprod P_i \arrow[r, "p"'] \arrow[d, "\coprod \pi_i"'] & P \arrow[d, "\pi"] \\ \coprod X_i \arrow[r, "q"] & X \end{tikzcd}$$

Observe that p_* is surjective. If $\beta \in \mathrm{CH}_k(\coprod X_i)$ then $\pi^* q_* \beta = p_*(\coprod \pi_i)^*\beta$, see Lemma 42.15.1. Similarly for capping with $c_1(\mathcal{O}(1))$ by Lemma 42.26.4. Hence, if the map of the lemma is surjective for each of the morphisms $\pi_i : P_i \to X_i$, then the map is surjective for $\pi : P \to X$. Hence we may assume X is irreducible. Thus $\dim_\delta(X) < \infty$ and in particular we may use induction on $\dim_\delta(X)$.

The result is clear if $\dim_\delta(X) < k$. Let $\alpha \in \mathrm{CH}_{k+r-1}(P)$. For any locally closed subscheme $T \subset X$ denote $\gamma_T : \bigoplus \mathrm{CH}_{k+i}(T) \to \mathrm{CH}_{k+r-1}(\pi^{-1}(T))$ the map

$$\gamma_T(\alpha_0, \ldots, \alpha_{r-1}) = \pi^*\alpha_0 + \ldots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^*\alpha_{r-1}.$$

Suppose for some nonempty open $U \subset X$ we have $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \ldots, \alpha_{r-1})$. Then we may choose lifts $\alpha'_i \in \mathrm{CH}_{k+i}(X)$ and we see that $\alpha - \gamma_X(\alpha'_0, \ldots, \alpha'_{r-1})$ is by Lemma 42.19.3 rationally equivalent to a k-cycle on $P_Y = \mathbf{P}(\mathcal{E}|_Y)$ where $Y = X \setminus U$ as a reduced closed subscheme. Note that $\dim_\delta(Y) < \dim_\delta(X)$. By induction the result holds for $P_Y \to Y$ and hence the result holds for α. Hence we may replace X by any nonempty open of X.

In particular we may assume that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. In this case $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$. Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \amalg \mathbf{A}^{r-2} \amalg \ldots \amalg \mathbf{A}^0$$

The closure of each stratum is a $\mathbf{P}^{r-1-i}$ which is a representative of $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$. Hence P has a similar stratification

$$P = U^{r-1} \amalg U^{r-2} \amalg \ldots \amalg U^0$$

Let P^i be the closure of U^i. Let $\pi^i : P^i \to X$ be the restriction of π to P^i. Let $\alpha \in \mathrm{CH}_{k+r-1}(P)$. By Lemma 42.32.1 we can write $\alpha|_{U^{r-1}} = \pi^*\alpha_0|_{U^{r-1}}$ for some $\alpha_0 \in \mathrm{CH}_k(X)$. Hence the difference $\alpha - \pi^*\alpha_0$ is the image of some $\alpha' \in \mathrm{CH}_{k+r-1}(P^{r-2})$. By Lemma 42.32.1 again we can write $\alpha'|_{U^{r-2}} = (\pi^{r-2})^*\alpha_1|_{U^{r-2}}$ for some $\alpha_1 \in \mathrm{CH}_{k+1}(X)$. By Lemma 42.31.1 we see that the image of $(\pi^{r-2})^*\alpha_1$ represents $c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$. We also see that $\alpha - \pi^*\alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$ is the image of some $\alpha'' \in \mathrm{CH}_{k+r-1}(P^{r-3})$. And so on. □

02TY Lemma 42.36.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let

$$p : E = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$ is an isomorphism for all k.

Proof. (For the case of linebundles, see Lemma 42.32.2.) For surjectivity see Lemma 42.32.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors, Lemma 31.14.10). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : \mathrm{CH}_{k+r}(P) \longrightarrow \mathrm{CH}_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 42.19.3. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in \mathrm{CH}_{k+r}(D)$. By Lemma 42.36.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in \mathrm{CH}_{k+i}(X)$. By Lemmas 42.31.1 and 42.26.4 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 42.26.4 unless all α and all β_i are zero. ☐

42.37. The Chern classes of a vector bundle

02TZ We can use the projective space bundle formula to define the Chern classes of a rank r vector bundle in terms of the expansion of $c_1(\mathcal{O}(1))^r$ in terms of the lower powers, see formula (42.37.1.1). The reason for the signs will be explained later.

02U0 Definition 42.37.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$.

(1) By Lemma 42.36.2 there are elements $c_i \in \mathrm{CH}_{n-i}(X)$, $i = 0, \ldots, r$ such that $c_0 = [X]$, and

02U1 (42.37.1.1) $$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0.$$

(2) With notation as above we set $c_i(\mathcal{E}) \cap [X] = c_i$ as an element of $\mathrm{CH}_{n-i}(X)$. We call these the Chern classes of $\mathcal{E}$ on X.
(3) The total Chern class of $\mathcal{E}$ on X is the combination

$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \ldots + c_r(\mathcal{E}) \cap [X]$$

which is an element of $\mathrm{CH}_*(X) = \bigoplus_{k \in \mathbf{Z}} \mathrm{CH}_k(X)$.

Let us check that this does not give a new notion in case the vector bundle has rank 1.

02U2 Lemma 42.37.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first Chern class of $\mathcal{L}$ on X of Definition 42.37.1 is equal to the Weil divisor associated to $\mathcal{L}$ by Definition 42.24.1.

Proof. In this proof we use $c_1(\mathcal{L}) \cap [X]$ to denote the construction of Definition 42.24.1. Since $\mathcal{L}$ has rank 1 we have $\mathbf{P}(\mathcal{L}) = X$ and $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$ by our normalizations. Hence (42.37.1.1) reads

$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$

Since $c_0 = [X]$, we conclude $c_1 = c_1(\mathcal{L}) \cap [X]$ as desired. □

02U3 Remark 42.37.3. We could also rewrite equation 42.37.1.1 as

05M8 (42.37.3.1)
$$\sum\nolimits_{i=0}^{r} c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0.$$

but we find it easier to work with the tautological quotient sheaf $\mathcal{O}_P(1)$ instead of its dual.

42.38. Intersecting with Chern classes

02U4 In this section we define Chern classes of vector bundles on X as bivariant classes on X, see Lemma 42.38.7 and the discussion following this lemma. Our construction follows the familiar pattern of first defining the operation on prime cycles and then summing. In Lemma 42.38.2 we show that the result is determined by the usual formula on the associated projective bundle. Next, we show that capping with Chern classes passes through rational equivalence, commutes with proper pushforward, commutes with flat pullback, and commutes with the gysin maps for inclusions of effective Cartier divisors. These lemmas could have been avoided by directly using the characterization in Lemma 42.38.2 and using Lemma 42.33.4; the reader who wishes to see this worked out should consult Chow Groups of Spaces, Lemma 82.28.1.

02U5 Definition 42.38.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. We define, for every integer k and any $0 \leq j \leq r$, an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \to \mathrm{CH}_{k-j}(X)$$

called intersection with the jth Chern class of $\mathcal{E}$.

(1) Given an integral closed subscheme $i : W \to X$ of δ-dimension k we define
$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in \mathrm{CH}_{k-j}(X)$$
where $c_j(i^*\mathcal{E}) \cap [W]$ is as defined in Definition 42.37.1.

(2) For a general k-cycle $\alpha = \sum n_i [W_i]$ we set
$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

If $\mathcal{E}$ has rank 1 then this agrees with our previous definition (Definition 42.25.1) by Lemma 42.37.2.

02U6 Lemma 42.38.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For $\alpha \in Z_k(X)$ the elements $c_j(\mathcal{E}) \cap \alpha$ are the unique elements α_j of $\mathrm{CH}_{k-j}(X)$ such that $\alpha_0 = \alpha$ and

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

holds in the Chow group of P.

Proof. The uniqueness of $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$ and such that the displayed equation holds follows from the projective space bundle formula Lemma 42.36.2. The identity holds by definition for $\alpha = [W]$ where W is an integral closed subscheme of X. For a general k-cycle α on X write $\alpha = \sum n_a[W_a]$ with $n_a \neq 0$, and $i_a : W_a \to X$ pairwise distinct integral closed subschemes. Then the family $\{W_a\}$ is locally finite on X. Set $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$. Denote $i'_a : P_a \to P$ the corresponding closed immersions. Consider the fibre product diagram

$$\begin{tikzcd} P' \arrow[r, equal] \arrow[d, "\pi'"'] & \coprod P_a \arrow[r, "\coprod i'_a"] \arrow[d, "\coprod \pi_a"'] & P \arrow[d, "\pi"] \\ X' \arrow[r, equal] & \coprod W_a \arrow[r, "\coprod i_a"] & X \end{tikzcd}$$

The morphism $p : X' \to X$ is proper. Moreover $\pi' : P' \to X'$ together with the invertible sheaf $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$ which is also the pullback of $\mathcal{O}_P(1)$ is the projective bundle associated to $\mathcal{E}' = p^*\mathcal{E}$. By definition

$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$

Write $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$ which is an element of $\mathrm{CH}_{k-j}(W_a)$. We have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$

for each a by definition. Thus clearly we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$

with $\beta_j = \sum n_a \beta_{a,j} \in \mathrm{CH}_{k-j}(X')$. Denote $p' : P' \to P$ the morphism $\coprod i'_a$. We have $\pi^* p_* \beta_j = p'_*(\pi')^*\beta_j$ by Lemma 42.15.1. By the projection formula of Lemma 42.26.4 we conclude that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\beta_j) = 0$$

Since $p_*\beta_j$ is a representative of $c_j(\mathcal{E}) \cap \alpha$ we win. ☐

We will consistently use this characterization of Chern classes to prove many more properties.

02U7 **Lemma 42.38.3.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. If $\alpha \sim_{rat} \beta$ are rationally equivalent k-cycles on X then $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$ in $\mathrm{CH}_{k-j}(X)$.

Proof. By Lemma 42.38.2 the elements $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, $j \geq 1$ and $\beta_j = c_j(\mathcal{E}) \cap \beta$, $j \geq 1$ are uniquely determined by the same equation in the chow group of the projective bundle associated to $\mathcal{E}$. (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 42.20.2.) Hence they are equal. ☐

In other words capping with Chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-j}(X).$$

Our next task is to show that Chern classes are bivariant classes, see Definition 42.33.1.

02U9 Lemma 42.38.4. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $p : X \to Y$ be a proper morphism. Let α be a k-cycle on X. Let $\mathcal{E}$ be a finite locally free sheaf on Y. Then

$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_*\alpha$$

Proof. Let $(\pi : P \to Y, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. Then $P_X = X \times_Y P$ is the projective bundle associated to $p^*\mathcal{E}$ and $\mathcal{O}_{P_X}(1)$ is the pullback of $\mathcal{O}_P(1)$. Write $\alpha_j = c_j(p^*\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of P_X. Consider the fibre product diagram

$$\begin{tikzcd} P_X \arrow[r, "p'"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

Apply proper pushforward p'_* (Lemma 42.20.3) to the displayed equality above. Using Lemmas 42.26.4 and 42.15.1 we obtain

$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\alpha_{r-i}) = 0$$

in the chow group of P. By the characterization of Lemma 42.38.2 we conclude. □

02U8 Lemma 42.38.5. Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on Y. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let α be a k-cycle on Y. Then

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^*\alpha$$

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd} P_X = \mathbf{P}(f^*\mathcal{E}) \arrow[r, "f'"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Note that $\mathcal{O}_{P_X}(1)$ is the pullback of $\mathcal{O}_P(1)$. Apply flat pullback $(f')^*$ (Lemma 42.20.2) to the displayed equation above. By Lemmas 42.26.2 and 42.14.3 we see that

$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of P_X. By the characterization of Lemma 42.38.2 we conclude. □

0B7G Lemma 42.38.6. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Then $c_j(\mathcal{E}|_D) \cap i^*\alpha = i^*(c_j(\mathcal{E}) \cap \alpha)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to X, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd} P_D = \mathbf{P}(\mathcal{E}|_D) \arrow[r, "i'"'] \arrow[d, "\pi_D"'] & P \arrow[d, "\pi"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

Note that $\mathcal{O}_{P_D}(1)$ is the pullback of $\mathcal{O}_P(1)$. Apply the gysin map $(i')^*$ (Lemma 42.30.2) to the displayed equation above. Applying Lemmas 42.30.4 and 42.29.9 we obtain

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_D}(1))^i \cap \pi_D^*(i^*\alpha_{r-i}) = 0$$

in the chow group of P_D. By the characterization of Lemma 42.38.2 we conclude. $\square$

At this point we have enough material to be able to prove that capping with Chern classes defines a bivariant class.

0B7H Lemma 42.38.7. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Let $0 \leq p \leq r$. Then the rule that to $f : X' \to X$ assigns $c_p(f^*\mathcal{E}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-p}(X')$ is a bivariant class of degree p.

Proof. Immediate from Lemmas 42.38.3, 42.38.4, 42.38.5, and 42.38.6 and Definition 42.33.1. $\square$

This lemma allows us to define the Chern classes of a finite locally free module as follows.

0FE1 Definition 42.38.8. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. For $i = 0, \ldots, r$ the ith Chern class of $\mathcal{E}$ is the bivariant class $c_i(\mathcal{E}) \in A^i(X)$ of degree i constructed in Lemma 42.38.7. The total Chern class of $\mathcal{E}$ is the formal sum

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots + c_r(\mathcal{E})$$

which is viewed as a nonhomogeneous bivariant class on X.

By the remark following Definition 42.38.1 if $\mathcal{E}$ is invertible, then this definition agrees with Definition 42.34.4. Next we see that Chern classes are in the center of the bivariant Chow cohomology ring $A^*(X)$.

02UA Lemma 42.38.9. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then

(1) $c_j(\mathcal{E}) \in A^j(X)$ is in the center of $A^*(X)$ and
(2) if $f : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_j(\mathcal{E}) = c_j(f^*\mathcal{E}) \circ c$.

In particular, if $\mathcal{F}$ is a second locally free $\mathcal{O}_X$-module on X of rank s, then

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $\mathrm{CH}_{k-i-j}(X)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. It is immediate that (2) implies (1). Let $\alpha \in \mathrm{CH}_k(X)$. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Denote $\pi' : P' \to X'$ the base change of π by f. Using Lemma 42.34.5 and the properties of bivariant classes we obtain

$$\begin{aligned} 0 &= c \cap \left(\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i})\right) \\ &= \sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(c \cap \alpha_{r-i}) \end{aligned}$$

in the Chow group of P' (calculation omitted). Hence we see that $c \cap \alpha_j$ is equal to $c_j(f^*\mathcal{E}) \cap (c \cap \alpha)$ by the characterization of Lemma 42.38.2. This proves the lemma. $\square$

0ESW Remark 42.38.10. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. If the rank of $\mathcal{E}$ is not constant then we can still define the Chern classes of $\mathcal{E}$. Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

where $X_r \subset X$ is the open and closed subspace where the rank of $\mathcal{E}$ is r. By Lemma 42.35.4 we have $A^p(X) = \prod A^p(X_r)$. Hence we can define $c_p(\mathcal{E})$ to be the product of the classes $c_p(\mathcal{E}|_{X_r})$ in $A^p(X_r)$. Explicitly, if $X' \to X$ is a morphism locally of finite type, then we obtain by pullback a corresponding decomposition of X' and we find that

$$\mathrm{CH}_*(X') = \prod\nolimits_{r \geq 0} \mathrm{CH}_*(X'_r)$$

by our definitions. Then $c_p(\mathcal{E}) \in A^p(X)$ is the bivariant class which preserves these direct product decompositions and acts by the already defined operations $c_i(\mathcal{E}|_{X_r}) \cap -$ on the factors. Observe that in this setting it may happen that $c_p(\mathcal{E})$ is nonzero for infinitely many p. It follows that the total chern class is an element

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots \in A^*(X)^\wedge$$

of the completed bivariant cohomology ring, see Remark 42.35.5. In this setting we define the "rank" of $\mathcal{E}$ to be the element $r(\mathcal{E}) \in A^0(X)$ as the bivariant operation which sends $(\alpha_r) \in \prod \mathrm{CH}_*(X'_r)$ to $(r\alpha_r) \in \prod \mathrm{CH}_*(X'_r)$. Note that it is still true that $c_p(\mathcal{E})$ and $r(\mathcal{E})$ are in the center of $A^*(X)$.

0FA4 Remark 42.38.11. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. In general we write $X = \coprod X_r$ as in Remark 42.38.10. If only a finite number of the X_r are nonempty, then we can set

$$c_{top}(\mathcal{E}) = \sum\nolimits_r c_r(\mathcal{E}|_{X_r}) \in A^*(X) = \bigoplus A^*(X_r)$$

where the equality is Lemma 42.35.4. If infinitely many X_r are nonempty, we will use the same notation to denote

$$c_{top}(\mathcal{E}) = \prod c_r(\mathcal{E}|_{X_r}) \in \prod A^r(X_r) \subset A^*(X)^\wedge$$

see Remark 42.35.5 for notation.

42.39. Polynomial relations among Chern classes

02UB Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of finite locally free sheaves on X. By Lemma 42.38.9 we see that the Chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central) $\mathbf{Z}$-subalgebra of the Chow cohomology algebra $A^*(X)$. Thus we can say what it means for a polynomial in these Chern classes to be zero, or for two polynomials to be the same. As an example, saying that $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$ means that the operations

$$\mathrm{CH}_k(Y) \longrightarrow \mathrm{CH}_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms $f : Y \to X$ which are locally of finite type. By Lemma 42.35.3 this is equivalent to the requirement that given any morphism $f : Y \to X$ where Y is an integral scheme locally of finite type over S the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in $\mathrm{CH}_{\dim(Y)-5}(Y)$.

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 42.25.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

02UD **Lemma 42.39.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have

02UE (42.39.1.1)
$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$

in $A^*(X)$.

Proof. This should hold for any triple $(X, \mathcal{E}, \mathcal{L})$. In particular it should hold when X is integral and by Lemma 42.35.3 it is enough to prove it holds when capping with $[X]$ for such X. Thus assume that X is integral. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ be the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{E} \otimes \mathcal{L}$. Consider the canonical morphism

$$\begin{tikzcd} P \arrow[rr, "g"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

see Constructions, Lemma 27.20.1. It has the property that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}$. This means that we have

$$\sum\nolimits_{i=0}^{r} (-1)^i (\xi + x)^i \cap \pi^*(c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in $\mathrm{CH}_*(P)$, where ξ represents $c_1(\mathcal{O}_P(1))$ and x represents $c_1(\pi^*\mathcal{L})$. By simple algebra this is equivalent to

$$\sum\nolimits_{i=0}^{r} (-1)^i \xi^i \left(\sum\nolimits_{j=i}^{r} (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^*(c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (42.37.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum\nolimits_{j=i}^{r} \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation. $\square$

Some example cases of (42.39.1.1) are

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + rc_1(\mathcal{L})$$
$$c_2(\mathcal{E} \otimes \mathcal{L}) = c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2$$
$$c_3(\mathcal{E} \otimes \mathcal{L}) = c_3(\mathcal{E}) + (r-2)c_2(\mathcal{E})c_1(\mathcal{L}) + \binom{r-1}{2} c_1(\mathcal{E})c_1(\mathcal{L})^2 + \binom{r}{3} c_1(\mathcal{L})^3$$

42.40. Additivity of Chern classes

02UF All of the preliminary lemmas follow trivially from the final result.

02UG **Lemma 42.40.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

in $A^*(X)$.

Proof. By Lemma 42.35.3 it suffices to show that if X is integral then $c_j(\mathcal{E}) \cap [X] = c_j(\mathcal{F}) \cap [X]$. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ denote the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{F}$. The surjection $\mathcal{E} \to \mathcal{F}$ gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over X. Moreover, the element $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$ gives rise to a global section $s \in \Gamma(P, \mathcal{O}_P(1))$ whose zero set is exactly P'. Hence P' is an effective Cartier divisor on P such that $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$. Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha = i_*((\pi')^*\alpha)$$

for any cycle class α on X by Lemma 42.31.1. By Lemma 42.38.2 we see that $\alpha_j = c_j(\mathcal{F}) \cap [X]$, $j = 0, \ldots, r-1$ satisfy

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^*\alpha_j = 0$$

Pushing this to P and using the remark above as well as Lemma 42.26.4 we get

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^*\alpha_j = 0$$

By the uniqueness of Lemma 42.38.2 we conclude that $c_r(\mathcal{E}) \cap [X] = 0$ and $c_j(\mathcal{E}) \cap [X] = \alpha_j = c_j(\mathcal{F}) \cap [X]$ for $j = 0, \ldots, r-1$. Hence the lemma holds. $\square$

02UH Lemma 42.40.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in $A^*(X)$.

Proof. This relation really just says that $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$. By Lemma 42.40.1 we have $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})$ for $j = 0, \ldots, r$ were we set $c_r(\mathcal{F} \otimes \mathcal{L}^{-1}) = 0$ by convention. Applying Lemma 42.39.1 we deduce

$$\sum_{j=0}^{i} \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^{i} \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ gives a "solution" of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification. □

02UI Lemma 42.40.3. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Suppose that $\mathcal{E}$ sits in an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. The total Chern classes satisfy

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

in $A^*(X)$.

Proof. By Lemma 42.35.3 we may assume that X is integral and we have to show the identity when capping against $[X]$. By induction on r_1. The case $r_1 = 1$ is Lemma 42.40.2. Assume $r_1 > 1$. Let $(\pi : P \to X, \mathcal{O}_P(1))$ denote the projective space bundle associated to $\mathcal{E}_1$. Note that

(1) $\pi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(P)$ is injective, and
(2) $\pi^*\mathcal{E}_1$ sits in a short exact sequence $0 \to \mathcal{F} \to \pi^*\mathcal{E}_1 \to \mathcal{L} \to 0$ where $\mathcal{L}$ is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact $\mathcal{L} = \mathcal{O}_P(1)$.) Let $Q = \pi^*\mathcal{E}/\mathcal{F}$, which sits in an exact sequence $0 \to \mathcal{L} \to Q \to \pi^*\mathcal{E}_2 \to 0$. By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since $[P] = \pi^*[X]$ we win by Lemma 42.38.5. □

02UJ Lemma 42.40.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be a locally free rank $\mathcal{O}_X$-module endowed with a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r}(1 + x_i)$$

in $A^*(X)$.

Proof. Apply Lemma 42.40.2 and induction. $\square$

42.41. Degrees of zero cycles

0AZ0 We start defining the degree of a zero cycle on a proper scheme over a field. One approach is to define it directly as in Lemma 42.41.2 and then show it is well defined by Lemma 42.18.3. Instead we define it as follows.

0AZ1 **Definition 42.41.1.** Let k be a field (Example 42.7.2). Let $p : X \to \mathrm{Spec}(k)$ be proper. The *degree of a zero cycle* on X is given by proper pushforward

$$p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}(k))$$

(Lemma 42.20.3) combined with the natural isomorphism $\mathrm{CH}_0(\mathrm{Spec}(k)) = \mathbf{Z}$ which maps $[\mathrm{Spec}(k)]$ to 1. Notation: $\deg(\alpha)$.

Let us spell this out further.

0AZ2 **Lemma 42.41.2.** *Let k be a field. Let X be proper over k. Let $\alpha = \sum n_i[Z_i]$ be in $Z_0(X)$. Then*

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where $\deg(Z_i)$ is the degree of $Z_i \to \mathrm{Spec}(k)$, i.e., $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$.

Proof. This is the definition of proper pushforward (Definition 42.12.1). $\square$

Next, we make the connection with degrees of vector bundles over 1-dimensional proper schemes over fields as defined in Varieties, Section 33.44.

0AZ3 **Lemma 42.41.3.** *Let k be a field. Let X be a proper scheme over k of dimension ≤ 1. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of constant rank. Then*

$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

where the left hand side is defined in Varieties, Definition 33.44.1.

Proof. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1 with reduced induced scheme structure and let m_i be the multiplicity of C_i in X. Then $[X]_1 = \sum m_i[C_i]$ and $c_1(\mathcal{E}) \cap [X]_1$ is the sum of the pushforwards of the cycles $m_i c_1(\mathcal{E}|_{C_i}) \cap [C_i]$. Since we have a similar decomposition of the degree of $\mathcal{E}$ by Varieties, Lemma 33.44.6 it suffices to prove the lemma in case X is a proper curve over k.

Assume X is a proper curve over k. By Divisors, Lemma 31.36.1 there exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_{X'}$-modules. Since $f_*[X']_1 = [X]_1$ we conclude from Lemma 42.38.4 that

$$\deg(c_1(\mathcal{E}) \cap [X]_1) = \deg(c_1(f^*\mathcal{E}) \cap [X']_1)$$

Since we have a similar relationship for the degree by Varieties, Lemma 33.44.4 we reduce to the case where $\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_X$-modules. In this case, we may use additivity of the degree (Varieties, Lemma

33.44.3) and of first Chern classes (Lemma 42.40.3) to reduce to the case discussed in the next paragraph.

Assume X is a proper curve over k and $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module. By Divisors, Lemma 31.15.12 we see that $\mathcal{E}$ is isomorphic to $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors D, D' on X (this also uses that X is projective, see Varieties, Lemma 33.43.4 for example). By additivity of degree under tensor product of invertible sheaves (Varieties, Lemma 33.44.7) and additivity of c_1 under tensor product of invertible sheaves (Lemma 42.25.2 or 42.39.1) we reduce to the case $\mathcal{E} = \mathcal{O}_X(D)$. In this case the left hand side gives $\deg(D)$ (Varieties, Lemma 33.44.9) and the right hand side gives $\deg([D]_0)$ by Lemma 42.25.4. Since

$$[D]_0 = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x})[x] = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[x]$$

by definition, we see

$$\deg([D]_0) = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[\kappa(x) : k] = \dim_k \Gamma(D, \mathcal{O}_D) = \deg(D)$$

The penultimate equality by Algebra, Lemma 10.52.12 using that D is affine. $\square$

Finally, we can tie everything up with the numerical intersections defined in Varieties, Section 33.45.

0BFI Lemma 42.41.4. Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d)$$

where the left hand side is defined in Varieties, Definition 33.45.3. In particular,

$$\deg_{\mathcal{L}}(Z) = \deg(c_1(\mathcal{L})^d \cap [Z]_d)$$

if $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module.

Proof. We will prove this by induction on d. If $d = 0$, then the result is true by Varieties, Lemma 33.33.3. Assume $d > 0$.

Let $Z_i \subset Z$, $i = 1, \ldots, t$ be the irreducible components of dimension d with reduced induced scheme structure and let m_i be the multiplicity of Z_i in Z. Then $[Z]_d = \sum m_i[Z_i]$ and $c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d$ is the sum of the cycles $m_i c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z_i]$. Since we have a similar decomposition for $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ by Varieties, Lemma 33.45.2 it suffices to prove the lemma in case $Z = X$ is a proper variety of dimension d over k.

By Chow's lemma there exists a birational proper morphism $f : Y \to X$ with Y H-projective over k. See Cohomology of Schemes, Lemma 30.18.1 and Remark 30.18.2. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Y) = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Varieties, Lemma 33.45.7 and we have

$$f_*(c_1(f^*\mathcal{L}_1) \cap \ldots \cap c_1(f^*\mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 42.26.4. Thus we may replace X by Y and assume that X is projective over k.

If X is a proper d-dimensional projective variety, then we can write $\mathcal{L}_1 = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors $D, D' \subset X$ by Divisors, Lemma 31.15.12. By additivity for both sides of the equation (Varieties, Lemma 33.45.5

and Lemma 42.25.2) we reduce to the case $\mathcal{L}_1 = \mathcal{O}_X(D)$ for some effective Cartier divisor D. By Varieties, Lemma 33.45.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

and by Lemma 42.25.4 we have

$$c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X] = c_1(\mathcal{L}_2) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [D]_{d-1}$$

Thus we obtain the result from our induction hypothesis. □

42.42. Cycles of given codimension

0FE2 In some cases there is a second grading on the abelian group of all cycles given by codimension.

0FE3 **Lemma 42.42.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Write $\delta = \delta_{X/S}$ as in Section 42.7. The following are equivalent

(1) There exists a decomposition $X = \coprod_{n \in \mathbf{Z}} X_n$ into open and closed subschemes such that $\delta(\xi) = n$ whenever $\xi \in X_n$ is a generic point of an irreducible component of X_n.
(2) For all $x \in X$ there exists an open neighbourhood $U \subset X$ of x and an integer n such that $\delta(\xi) = n$ whenever $\xi \in U$ is a generic point of an irreducible component of U.
(3) For all $x \in X$ there exists an integer n_x such that $\delta(\xi) = n_x$ for any generic point ξ of an irreducible component of X containing x.

The conditions are satisfied if X is either normal or Cohen-Macaulay[3].

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Conversely, if (3) holds, then we set $X_n = \{x \in X \mid n_x = n\}$ and we get a decomposition as in (1). Namely, X_n is open because given x the union of the irreducible components of X passing through x minus the union of the irreducible components of X not passing through x is an open neighbourhood of x. If X is normal, then X is a disjoint union of integral schemes (Properties, Lemma 28.7.7) and hence the properties hold. If X is Cohen-Macaulay, then $\delta' : X \to \mathbf{Z}$, $x \mapsto -\dim(\mathcal{O}_{X,x})$ is a dimension function on X (see Example 42.7.4). Since $\delta - \delta'$ is locally constant (Topology, Lemma 5.20.3) and since $\delta'(\xi) = 0$ for every generic point ξ of X we see that (2) holds. □

Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1. For an integral closed subscheme $Z \subset X$ we have the codimension $\text{codim}(Z, X)$ of Z in X, see Topology, Definition 5.11.1. We define a codimension p-cycle to be a cycle $\alpha = \sum n_Z[Z]$ on X such that $n_Z \neq 0 \Rightarrow \text{codim}(Z, X) = p$. The abelian group of all codimension p-cycles is denoted $Z^p(X)$. Let $X = \coprod X_n$ be the decomposition given in Lemma 42.42.1 part (1). Recalling that our cycles are defined as locally finite sums, it is clear that

$$Z^p(X) = \prod\nolimits_n Z_{n-p}(X_n)$$

Moreover, we see that $\prod_p Z^p(X) = \prod_k Z_k(X)$. We could now define rational equivalence of codimension p cycles on X in exactly the same manner as before and in fact we could redevelop the whole theory from scratch for cycles of a given

[3]In fact, it suffices if X is (S_2). Compare with Local Cohomology, Lemma 51.3.2.

codimension for X as in Lemma 42.42.1. However, instead we simply define the Chow group of codimension p-cycles as

$$\mathrm{CH}^p(X) = \prod\nolimits_n \mathrm{CH}_{n-p}(X_n)$$

As before we have $\prod_p \mathrm{CH}^p(X) = \prod_k \mathrm{CH}_k(X)$. If X is quasi-compact, then the product in the formula is finite (and hence is a direct sum) and we have $\bigoplus_p \mathrm{CH}^p(X) = \bigoplus_k \mathrm{CH}_k(X)$. If X is quasi-compact and finite dimensional, then only a finite number of these groups is nonzero.

Many of the constructions and results for Chow groups proved above have natural counterparts for the Chow groups $\mathrm{CH}^*(X)$. Each of these is shown by decomposing the relevant schemes into "equidimensional" pieces as in Lemma 42.42.1 and applying the results already proved for the factors in the product decomposition given above. Let us list some of them.

(1) If $f : X \to Y$ is a flat morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 then flat pullback determines a map

$$f^* : \mathrm{CH}^p(Y) \to \mathrm{CH}^p(X)$$

(2) If $f : X \to Y$ is a morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 let us say f has codimension $r \in \mathbf{Z}$ if for all pairs of irreducible components $Z \subset X$, $W \subset Y$ with $f(Z) \subset W$ we have $\dim_\delta(W) - \dim_\delta(Z) = r$.
(3) If $f : X \to Y$ is a proper morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 and f has codimension r, then proper pushforward is a map

$$f_* : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+r}(Y)$$

(4) If $f : X \to Y$ is a morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 and f has codimension r and $c \in A^q(X \to Y)$, then c induces maps

$$c \cap - : \mathrm{CH}^p(Y) \to \mathrm{CH}^{p+q-r}(X)$$

(5) If X is a scheme locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1 and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+1}(X)$$

(6) If X is a scheme locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1 and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module, then

$$c_i(\mathcal{E}) \cap - : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+i}(X)$$

Warning: the property for a morphism to have codimension r is not preserved by base change.

0FE4 Remark 42.42.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1. Let $X = \coprod X_n$ be the decomposition into open and closed subschemes such that every irreducible component of X_n has δ-dimension n. In this situation we sometimes set

$$[X] = \sum\nolimits_n [X_n]_n \in \mathrm{CH}^0(X)$$

This class is a kind of "fundamental class" of X in Chow theory.

42.43. The splitting principle

02UK In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

02UL **Lemma 42.43.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a projective flat morphism $\pi : P \to X$ of relative dimension d such that*

1. *for any morphism $f : Y \to X$ the map $\pi_Y^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(Y \times_X P)$ is injective, and*
2. *each $\pi^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.*

Moreover, when (1) holds the restriction map $A^*(X) \to A^*(P)$ (Remark 42.34.2) is injective.

Proof. We may assume $r_i \geq 1$ for all i. We will prove the lemma by induction on $\sum(r_i - 1)$. If this integer is 0, then $\mathcal{E}_i$ is invertible for all i and we conclude by taking $\pi = \mathrm{id}_X$. If not, then we can pick an i such that $r_i > 1$ and consider the morphism $\pi_i : P_i = \mathbf{P}(\mathcal{E}_i) \to X$. We have a short exact sequence

$$0 \to \mathcal{F} \to \pi_i^*\mathcal{E}_i \to \mathcal{O}_{P_i}(1) \to 0$$

of finite locally free $\mathcal{O}_{P_i}$-modules of ranks $r_i - 1$, r_i, and 1. Observe that π_i^* is injective on chow groups after any base change by the projective bundle formula (Lemma 42.36.2). By the induction hypothesis applied to the finite locally free $\mathcal{O}_{P_i}$-modules $\mathcal{F}$ and $\pi_{i'}^*\mathcal{E}_{i'}$ for $i' \neq i$, we find a morphism $\pi : P \to P_i$ with properties stated as in the lemma. Then the composition $\pi_i \circ \pi : P \to X$ does the job. Some details omitted. $\square$

0FVE **Remark 42.43.2.** The proof of Lemma 42.43.1 shows that the morphism $\pi : P \to X$ has the following additional properties:

1. π is a finite composition of projective space bundles associated to locally free modules of finite constant rank, and
2. for every $\alpha \in \mathrm{CH}_k(X)$ we have $\alpha = \pi_*(\xi_1 \cap \ldots \cap \xi_d \cap \pi^*\alpha)$ where ξ_i is the first Chern class of some invertible $\mathcal{O}_P$-module.

The second observation follows from the first and Lemma 42.36.1. We will add more observations here as needed.

Let (S, δ), X, and $\mathcal{E}_i$ be as in Lemma 42.43.1. The splitting principle refers to the practice of symbolically writing

$$c(\mathcal{E}_i) = \prod(1 + x_{i,j})$$

The symbols $x_{i,1}, \ldots, x_{i,r_i}$ are called the *Chern roots* of $\mathcal{E}_i$. In other words, the pth Chern class of $\mathcal{E}_i$ is the pth elementary symmetric function in the Chern roots. The usefulness of the splitting principle comes from the assertion that in order to prove a polynomial relation among Chern classes of the $\mathcal{E}_i$ it is enough to prove the corresponding relation among the Chern roots.

Namely, let $\pi : P \to X$ be as in Lemma 42.43.1. Recall that there is a canonical $\mathbf{Z}$-algebra map $\pi^* : A^*(X) \to A^*(P)$, see Remark 42.34.2. The injectivity of π_Y^*

on Chow groups for every Y over X, implies that the map $\pi^* : A^*(X) \to A^*(P)$ is injective (details omitted). We have

$$\pi^* c(\mathcal{E}_i) = \prod (1 + c_1(\mathcal{L}_{i,j}))$$

by Lemma 42.40.4. Thus we may think of the Chern roots $x_{i,j}$ as the elements $c_1(\mathcal{L}_{i,j}) \in A^*(P)$ and the displayed equation as taking place in $A^*(P)$ after applying the injective map $\pi^* : A^*(X) \to A^*(P)$ to the left hand side of the equation.

To see how this works, it is best to give some examples.

0FA5 Lemma 42.43.3. In Situation 42.7.1 let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$$

in $A^i(X)$.

Proof. Choose a morphism $\pi : P \to X$ as in Lemma 42.43.1. By the injectivity of π^* (after any base change) it suffices to prove the relation between the Chern classes of $\mathcal{E}$ and $\mathcal{E}^\vee$ after pulling back to P. Thus we may assume there exist invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ and a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Then we obtain the dual filtration

$$0 = \mathcal{E}_r^\perp \subset \mathcal{E}_1^\perp \subset \mathcal{E}_2^\perp \subset \ldots \subset \mathcal{E}_0^\perp = \mathcal{E}^\vee$$

such that $\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp \cong \mathcal{L}_i^{\otimes -1}$. Set $x_i = c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{L}_i^{\otimes -1}) = -x_i$ by Lemma 42.25.2. By Lemma 42.40.4 we have

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^r (1 + x_i) \quad \text{and} \quad c(\mathcal{E}^\vee) = \prod\nolimits_{i=1}^r (1 - x_i)$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

0FA6 Lemma 42.43.4. In Situation 42.7.1 let X be locally of finite type over S. Let $\mathcal{E}$ and $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-modules of ranks r and s. Then we have

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$

$$c_2(\mathcal{E} \otimes \mathcal{F}) = rc_2(\mathcal{F}) + sc_2(\mathcal{E}) + \binom{r}{2} c_1(\mathcal{F})^2 + (rs - 1)c_1(\mathcal{F})c_1(\mathcal{E}) + \binom{s}{2} c_1(\mathcal{E})^2$$

and so on in $A^*(X)$.

Proof. Arguing exactly as in the proof of Lemma 42.43.3 we may assume we have invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ $\mathcal{N}_i$, $i = 1, \ldots, s$ filtrations

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E} \quad \text{and} \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ and such that $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{N}_j$. Ordering pairs (i, j) lexicographically we obtain a filtration

$$0 \subset \ldots \subset \mathcal{E}_i \otimes \mathcal{F}_j + \mathcal{E}_{i-1} \otimes \mathcal{F} \subset \ldots \subset \mathcal{E} \otimes \mathcal{F}$$

with successive quotients

$$\mathcal{L}_1 \otimes \mathcal{N}_1, \mathcal{L}_1 \otimes \mathcal{N}_2, \ldots, \mathcal{L}_1 \otimes \mathcal{N}_s, \mathcal{L}_2 \otimes \mathcal{N}_1, \ldots, \mathcal{L}_r \otimes \mathcal{N}_s$$

By Lemma 42.40.4 we have

$$c(\mathcal{E}) = \prod (1 + x_i), \quad c(\mathcal{F}) = \prod (1 + y_j), \quad \text{and} \quad c(\mathcal{E} \otimes \mathcal{F}) = \prod (1 + x_i + y_j),$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

0FA7 Remark 42.43.5. The equalities proven above remain true even when we work with finite locally free $\mathcal{O}_X$-modules whose rank is allowed to be nonconstant. In fact, we can work with polynomials in the rank and the Chern classes as follows. Consider the graded polynomial ring $\mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ where r has degree 0 and c_i has degree i. Let

$$P \in \mathbf{Z}[r, c_1, c_2, c_3, \ldots]$$

be a homogeneous polynomial of degree p. Then for any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ on X we can consider

$$P(\mathcal{E}) = P(r(\mathcal{E}), c_1(\mathcal{E}), c_2(\mathcal{E}), c_3(\mathcal{E}), \ldots) \in A^p(X)$$

see Remark 42.38.10 for notation and conventions. To prove relations among these polynomials (for multiple finite locally free modules) we can work locally on X and use the splitting principle as above. For example, we claim that

$$c_2(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = P(\mathcal{E})$$

where $P = 2rc_2 - (r-1)c_1^2$. Namely, since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^\vee$ this follows easily from Lemmas 42.43.3 and 42.43.4 above by decomposing X into parts where the rank of $\mathcal{E}$ is constant as in Remark 42.38.10.

0F9B Example 42.43.6. For every $p \geq 1$ there is a unique homogeneous polynomial $P_p \in \mathbf{Z}[c_1, c_2, c_3, \ldots]$ of degree p such that, for any $n \geq p$ we have

$$P_p(s_1, s_2, \ldots, s_p) = \sum x_i^p$$

in $\mathbf{Z}[x_1, \ldots, x_n]$ where $s_1, \ldots, s_p$ are the elementary symmetric polynomials in $x_1, \ldots, x_n$, so

$$s_i = \sum\nolimits_{1 \leq j_1 < \ldots < j_i \leq n} x_{j_1} x_{j_2} \ldots x_{j_i}$$

The existence of P_p comes from the well known fact that the elementary symmetric functions generate the ring of all symmetric functions over the integers. Another way to characterize $P_p \in \mathbf{Z}[c_1, c_2, c_3, \ldots]$ is that we have

$$\log(1 + c_1 + c_2 + c_3 + \ldots) = \sum\nolimits_{p \geq 1} (-1)^{p-1} \frac{P_p}{p}$$

as formal power series. This is clear by writing $1 + c_1 + c_2 + \ldots = \prod(1 + x_i)$ and applying the power series for the logarithm function. Expanding the left hand side we get

$$\begin{aligned}
&(c_1 + c_2 + \ldots) - (1/2)(c_1 + c_2 + \ldots)^2 + (1/3)(c_1 + c_2 + \ldots)^3 - \ldots \\
&= c_1 + (c_2 - (1/2)c_1^2) + (c_3 - c_1c_2 + (1/3)c_1^3) + \ldots
\end{aligned}$$

In this way we find that

$$\begin{aligned}
P_1 &= c_1, \\
P_2 &= c_1^2 - 2c_2, \\
P_3 &= c_1^3 - 3c_1c_2 + 3c_3, \\
P_4 &= c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4,
\end{aligned}$$

and so on. Since the Chern classes of a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ are the elementary symmetric polynomials in the Chern roots x_i, we see that

$$P_p(\mathcal{E}) = \sum x_i^p$$

For convenience we set $P_0 = r$ in $\mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ so that $P_0(\mathcal{E}) = r(\mathcal{E})$ as a bivariant class (as in Remarks 42.38.10 and 42.43.5).

42.44. Chern classes and sections

0FA8 A brief section whose main result is that we may compute the top Chern class of a finite locally free module using the vanishing locus of a "regular section.

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Let $f : X' \to X$ be locally of finite type. Let

$$s \in \Gamma(X', f^*\mathcal{E})$$

be a global section of the pullback of $\mathcal{E}$ to X'. Let $Z(s) \subset X'$ be the zero scheme of s. More precisely, we define $Z(s)$ to be the closed subscheme whose quasi-coherent sheaf of ideals is the image of the map $s : f^*\mathcal{E}^\vee \to \mathcal{O}_{X'}$.

0FA9 Lemma 42.44.1. In the situation described just above assume $\dim_\delta(X') = n$, that $f^*\mathcal{E}$ has constant rank r, that $\dim_\delta(Z(s)) \leq n - r$, and that for every generic point $\xi \in Z(s)$ with $\delta(\xi) = n - r$ the ideal of $Z(s)$ in $\mathcal{O}_{X',\xi}$ is generated by a regular sequence of length r. Then

$$c_r(\mathcal{E}) \cap [X']_n = [Z(s)]_{n-r}$$

in $\mathrm{CH}_*(X')$.

Proof. Since $c_r(\mathcal{E})$ is a bivariant class (Lemma 42.38.7) we may assume $X = X'$ and we have to show that $c_r(\mathcal{E}) \cap [X]_n = [Z(s)]_{n-r}$ in $\mathrm{CH}_{n-r}(X)$. We will prove the lemma by induction on $r \geq 0$. (The case $r = 0$ is trivial.) The case $r = 1$ is handled by Lemma 42.25.4. Assume $r > 1$.

Let $\pi : P \to X$ be the projective space bundle associated to $\mathcal{E}$ and consider the short exact sequence

$$0 \to \mathcal{E}' \to \pi^*\mathcal{E} \to \mathcal{O}_P(1) \to 0$$

By the projective space bundle formula (Lemma 42.36.2) it suffices to prove the equality after pulling back by π. Observe that $\pi^{-1}Z(s) = Z(\pi^*s)$ has δ-dimension $\leq n - 1$ and that the assumption on regular sequences at generic points of δ-dimension $n - 1$ holds by flat pullback, see Algebra, Lemma 10.68.5. Let $t \in \Gamma(P, \mathcal{O}_P(1))$ be the image of π^*s. We claim

$$[Z(t)]_{n+r-2} = c_1(\mathcal{O}_P(1)) \cap [P]_{n+r-1}$$

Assuming the claim we finish the proof as follows. The restriction $\pi^*s|_{Z(t)}$ maps to zero in $\mathcal{O}_P(1)|_{Z(t)}$ hence comes from a unique element $s' \in \Gamma(Z(t), \mathcal{E}'|_{Z(t)})$. Note that $Z(s') = Z(\pi^*s)$ as closed subschemes of P. If $\xi \in Z(s')$ is a generic point with $\delta(\xi) = n - 1$, then the ideal of $Z(s')$ in $\mathcal{O}_{Z(t),\xi}$ can be generated by a regular sequence of length $r - 1$: it is generated by $r - 1$ elements which are the images of $r - 1$ elements in $\mathcal{O}_{P,\xi}$ which together with a generator of the ideal of $Z(t)$ in $\mathcal{O}_{P,\xi}$ form a regular sequence of length r in $\mathcal{O}_{P,\xi}$. Hence we can apply the induction hypothesis to s' on $Z(t)$ to get $c_{r-1}(\mathcal{E}') \cap [Z(t)]_{n+r-2} = [Z(s')]_{n-1}$. Combining all of the above we obtain

$$\begin{aligned} c_r(\pi^*\mathcal{E}) \cap [P]_{n+r-1} &= c_{r-1}(\mathcal{E}') \cap c_1(\mathcal{O}_P(1)) \cap [P]_{n+r-1} \\ &= c_{r-1}(\mathcal{E}') \cap [Z(t)]_{n+r-2} \\ &= [Z(s')]_{n-1} \\ &= [Z(\pi^*s)]_{n-1} \end{aligned}$$

which is what we had to show.

Proof of the claim. This will follow from an application of the already used Lemma 42.25.4. We have $\pi^{-1}(Z(s)) = Z(\pi^*s) \subset Z(t)$. On the other hand, for $x \in X$ if $P_x \subset Z(t)$, then $t|_{P_x} = 0$ which implies that s is zero in the fibre $\mathcal{E} \otimes \kappa(x)$, which implies $x \in Z(s)$. It follows that $\dim_\delta(Z(t)) \leq n + (r-1) - 1$. Finally, let $\xi \in Z(t)$ be a generic point with $\delta(\xi) = n + r - 2$. If ξ is not the generic point of the fibre of $P \to X$ it is immediate that a local equation of $Z(t)$ is a nonzerodivisor in $\mathcal{O}_{P,\xi}$ (because we can check this on the fibre by Algebra, Lemma 10.99.2). If ξ is the generic point of a fibre, then $x = \pi(\xi) \in Z(s)$ and $\delta(x) = n + r - 2 - (r-1) = n - 1$. This is a contradiction with $\dim_\delta(Z(s)) \leq n - r$ because $r > 1$ so this case doesn't happen. $\square$

0FAA Lemma 42.44.2. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E} \to 0$$

be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Consider the closed embedding

$$i : N' = \underline{\mathrm{Spec}}_X(\mathrm{Sym}((\mathcal{N}')^\vee)) \longrightarrow N = \underline{\mathrm{Spec}}_X(\mathrm{Sym}(\mathcal{N}^\vee))$$

For $\alpha \in \mathrm{CH}_k(X)$ we have

$$i_*(p')^*\alpha = p^*(c_{top}(\mathcal{E}) \cap \alpha)$$

where $p' : N' \to X$ and $p : N \to X$ are the structure morphisms.

Proof. Here $c_{top}(\mathcal{E})$ is the bivariant class defined in Remark 42.38.11. By its very definition, in order to verify the formula, we may assume that $\mathcal{E}$ has constant rank. We may similarly assume $\mathcal{N}'$ and $\mathcal{N}$ have constant ranks, say r' and r, so $\mathcal{E}$ has rank $r - r'$ and $c_{top}(\mathcal{E}) = c_{r-r'}(\mathcal{E})$. Observe that $p^*\mathcal{E}$ has a canonical section

$$s \in \Gamma(N, p^*\mathcal{E}) = \Gamma(X, p_*p^*\mathcal{E}) = \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Sym}(\mathcal{N}^\vee) \supset \Gamma(X, \mathcal{H}om(\mathcal{N}, \mathcal{E}))$$

corresponding to the surjection $\mathcal{N} \to \mathcal{E}$ given in the statement of the lemma. The vanishing scheme of this section is exactly $N' \subset N$. Let $Y \subset X$ be an integral closed subscheme of δ-dimension n. Then we have

1. $p^*[Y] = [p^{-1}(Y)]$ since $p^{-1}(Y)$ is integral of δ-dimension $n + r$,
2. $(p')^*[Y] = [(p')^{-1}(Y)]$ since $(p')^{-1}(Y)$ is integral of δ-dimension $n + r'$,
3. the restriction of s to $p^{-1}Y$ has vanishing scheme $(p')^{-1}Y$ and the closed immersion $(p')^{-1}Y \to p^{-1}Y$ is a regular immersion (locally cut out by a regular sequence).

We conclude that

$$(p')^*[Y] = c_{r-r'}(p^*\mathcal{E}) \cap p^*[Y] \quad \text{in} \quad \mathrm{CH}_*(N)$$

by Lemma 42.44.1. This proves the lemma. $\square$

42.45. The Chern character and tensor products

02UM Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. We define the Chern character of a finite locally free $\mathcal{O}_X$-module to be the formal expression

$$ch(\mathcal{E}) = \sum\nolimits_{i=1}^{r} e^{x_i}$$

if the x_i are the Chern roots of $\mathcal{E}$. Writing this as a polynomial in the Chern classes we obtain

$$\begin{aligned} ch(\mathcal{E}) &= r(\mathcal{E}) + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \frac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})) \\ &\quad + \frac{1}{24}(c_1(\mathcal{E})^4 - 4c_1(\mathcal{E})^2c_2(\mathcal{E}) + 4c_1(\mathcal{E})c_3(\mathcal{E}) + 2c_2(\mathcal{E})^2 - 4c_4(\mathcal{E})) + \ldots \\ &= \sum\nolimits_{p=0,1,2,\ldots} \frac{P_p(\mathcal{E})}{p!} \end{aligned}$$

with P_p polynomials in the Chern classes as in Example 42.43.6. The degree p component of the above is

$$ch_p(\mathcal{E}) = \frac{P_p(\mathcal{E})}{p!} \in A^p(X) \otimes \mathbf{Q}$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of $ch_p(\mathcal{E})$ as an element of $A^p(X) \otimes \mathbf{Q}$.

0ESX Remark 42.45.1. In the discussion above we have defined the components of the Chern character $ch_p(\mathcal{E}) \in A^p(X) \otimes \mathbf{Q}$ of $\mathcal{E}$ even if the rank of $\mathcal{E}$ is not constant. See Remarks 42.38.10 and 42.43.5. Thus the full Chern character of $\mathcal{E}$ is an element of $\prod_{p \geq 0}(A^p(X) \otimes \mathbf{Q})$. If X is quasi-compact and $\dim(X) < \infty$ (usual dimension), then one can show using Lemma 42.34.6 and the splitting principle that $ch(\mathcal{E}) \in A^*(X) \otimes \mathbf{Q}$.

0F9C Lemma 42.45.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Then we have the equality

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

More precisely, we have $P_p(\mathcal{E}) = P_p(\mathcal{E}_1) + P_p(\mathcal{E}_2)$ in $A^p(X)$ where P_p is as in Example 42.43.6.

Proof. It suffices to prove the more precise statement. By Section 42.43 this follows because if $x_{1,i}$, $i = 1, \ldots, r_1$ and $x_{2,i}$, $i = 1, \ldots, r_2$ are the Chern roots of $\mathcal{E}_1$ and $\mathcal{E}_2$, then $x_{1,1}, \ldots, x_{1,r_1}, x_{2,1}, \ldots, x_{2,r_2}$ are the Chern roots of $\mathcal{E}$. Hence we get the result from our choice of P_p in Example 42.43.6. □

0F9D Lemma 42.45.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be finite locally free $\mathcal{O}_X$-modules. Then we have the equality

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

More precisely, we have

$$P_p(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(\mathcal{E}_1)P_{p_2}(\mathcal{E}_2)$$

in $A^p(X)$ where P_p is as in Example 42.43.6.

Proof. It suffices to prove the more precise statement. By Section 42.43 this follows because if $x_{1,i}$, $i = 1, \ldots, r_1$ and $x_{2,i}$, $i = 1, \ldots, r_2$ are the Chern roots of $\mathcal{E}_1$ and $\mathcal{E}_2$, then $x_{1,i} + x_{2,j}$, $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ are the Chern roots of $\mathcal{E}_1 \otimes \mathcal{E}_2$. Hence we get the result from the binomial formula for $(x_{1,i} + x_{2,j})^p$ and the shape of our polynomials P_p in Example 42.43.6. $\square$

0FAB Lemma 42.45.4. In Situation 42.7.1 let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then $ch_i(\mathcal{E}^\vee) = (-1)^i ch_i(\mathcal{E})$ in $A^i(X) \otimes \mathbf{Q}$.

Proof. Follows from the corresponding result for Chern classes (Lemma 42.43.3). $\square$

42.46. Chern classes and the derived category

0ESY In this section we define the total Chern class of a perfect object E of the derived category of a scheme X, under the assumption that E may be represented by a finite complex of finite locally free modules on an envelope of X.

Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let

$$\mathcal{E}^a \to \mathcal{E}^{a+1} \to \ldots \to \mathcal{E}^b$$

be a bounded complex of finite locally free $\mathcal{O}_X$-modules of constant rank. Then we define the total Chern class of the complex by the formula

$$c(\mathcal{E}^\bullet) = \prod\nolimits_{n=a,\ldots,b} c(\mathcal{E}^n)^{(-1)^n} \in \prod\nolimits_{p \geq 0} A^p(X)$$

Here the inverse is the formal inverse, so

$$(1 + c_1 + c_2 + c_3 + \ldots)^{-1} = 1 - c_1 + c_1^2 - c_2 - c_1^3 + 2c_1c_2 - c_3 + \ldots$$

We will denote $c_p(\mathcal{E}^\bullet) \in A^p(X)$ the degree p part of $c(\mathcal{E}^\bullet)$. We similarly define the Chern character of the complex by the formula

$$ch(\mathcal{E}^\bullet) = \sum\nolimits_{n=a,\ldots,b} (-1)^n ch(\mathcal{E}^n) \in \prod\nolimits_{p \geq 0} (A^p(X) \otimes \mathbf{Q})$$

We will denote $ch_p(\mathcal{E}^\bullet) \in A^p(X) \otimes \mathbf{Q}$ the degree p part of $ch(\mathcal{E}^\bullet)$. Finally, for $P_p \in \mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ as in Example 42.43.6 we define

$$P_p(\mathcal{E}^\bullet) = \sum\nolimits_{n=a,\ldots,b} (-1)^n P_p(\mathcal{E}^n)$$

in $A^p(X)$. Then we have $ch_p(\mathcal{E}^\bullet) = (1/p!)P_p(\mathcal{E}^\bullet)$ as usual. The next lemma shows that these constructions only depends on the image of the complex in the derived category.

0ESZ Lemma 42.46.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be an object such that there exists a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. Then a slight generalization of the above constructions

$$c(\mathcal{E}^\bullet) \in \prod\nolimits_{p \geq 0} A^p(X), \quad ch(\mathcal{E}^\bullet) \in \prod\nolimits_{p \geq 0} A^p(X) \otimes \mathbf{Q}, \quad P_p(\mathcal{E}^\bullet) \in A^p(X)$$

are independent of the choice of the complex $\mathcal{E}^\bullet$.

Proof. We prove this for the total Chern class; the other two cases follow by the same arguments using Lemma 42.45.2 instead of Lemma 42.40.3.

As in Remark 42.38.10 in order to define the total chern class $c(\mathcal{E}^\bullet)$ we decompose X into open and closed subschemes

$$X = \coprod\nolimits_{i \in I} X_i$$

such that the rank $\mathcal{E}^n$ is constant on X_i for all n and i. (Since these ranks are locally constant functions on X we can do this.) Since $\mathcal{E}^\bullet$ is locally bounded, we see that only a finite number of the sheaves $\mathcal{E}^n|_{X_i}$ are nonzero for a fixed i. Hence we can define

$$c(\mathcal{E}^\bullet|_{X_i}) = \prod\nolimits_n c(\mathcal{E}^n|_{X_i})^{(-1)^n} \in \prod\nolimits_{p \geq 0} A^p(X_i)$$

as above. By Lemma 42.35.4 we have $A^p(X) = \prod_i A^p(X_i)$. Hence for each $p \in \mathbf{Z}$ we have a unique element $c_p(\mathcal{E}^\bullet) \in A^p(X)$ restricting to $c_p(\mathcal{E}^\bullet|_{X_i})$ on X_i for all i.

Suppose we have a second locally bounded complex $\mathcal{F}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. Let $g : Y \to X$ be a morphism locally of finite type with Y integral. By Lemma 42.35.3 it suffices to show that with $c(g^*\mathcal{E}^\bullet) \cap [Y]$ is the same as $c(g^*\mathcal{F}^\bullet) \cap [Y]$ and it even suffices to prove this after replacing Y by an integral scheme proper and birational over Y. Then first we conclude that $g^*\mathcal{E}^\bullet$ and $g^*\mathcal{F}^\bullet$ are bounded complexes of finite locally free $\mathcal{O}_Y$-modules of constant rank. Next, by More on Flatness, Lemma 38.40.3 we may assume that $H^i(Lg^*E)$ is perfect of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. This reduces us to the case discussed in the next paragraph.

Assume X is integral, $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are bounded complexes of finite locally free modules of constant rank, and $H^i(E)$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. We have to show that $c(\mathcal{E}^\bullet) \cap [X]$ is the same as $c(\mathcal{F}^\bullet) \cap [X]$. Denote $d^i_\mathcal{E} : \mathcal{E}^i \to \mathcal{E}^{i+1}$ and $d^i_\mathcal{F} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ the differentials of our complexes. By More on Flatness, Remark 38.40.4 we know that $\text{Im}(d^i_\mathcal{E})$, $\text{Ker}(d^i_\mathcal{E})$, $\text{Im}(d^i_\mathcal{F})$, and $\text{Ker}(d^i_\mathcal{F})$ are finite locally free $\mathcal{O}_X$-modules for all i. By additivity (Lemma 42.40.3) we see that

$$c(\mathcal{E}^\bullet) = \prod\nolimits_i c(\text{Ker}(d^i_\mathcal{E}))^{(-1)^i} c(\text{Im}(d^i_\mathcal{E}))^{(-1)^i}$$

and similarly for $\mathcal{F}^\bullet$. Since we have the short exact sequences

$$0 \to \text{Im}(d^i_\mathcal{E}) \to \text{Ker}(d^i_\mathcal{E}) \to H^i(E) \to 0 \quad \text{and} \quad 0 \to \text{Im}(d^i_\mathcal{F}) \to \text{Ker}(d^i_\mathcal{F}) \to H^i(E) \to 0$$

we reduce to the problem stated and solved in the next paragraph.

Assume X is integral and we have two short exact sequences

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{Q} \to 0 \quad \text{and} \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q} \to 0$$

with $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{F}$, $\mathcal{F}'$ finite locally free. Problem: show that $c(\mathcal{E})c(\mathcal{E}')^{-1} \cap [X] = c(\mathcal{F})c(\mathcal{F}')^{-1} \cap [X]$. To do this, consider the short exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \oplus \mathcal{F} \to \mathcal{Q} \to 0$$

defining $\mathcal{G}$. Since $\mathcal{Q}$ has tor dimension ≤ 1 we see that $\mathcal{G}$ is finite locally free. A diagram chase shows that the kernel of the surjection $\mathcal{G} \to \mathcal{F}$ maps isomorphically to $\mathcal{E}'$ in $\mathcal{E}$ and the kernel of the surjection $\mathcal{G} \to \mathcal{E}$ maps isomorphically to $\mathcal{F}'$ in $\mathcal{F}$. (Working affine locally this follows from or is equivalent to Schanuel's lemma, see Algebra, Lemma 10.109.1.) We conclude that

$$c(\mathcal{E})c(\mathcal{F}') = c(\mathcal{G}) = c(\mathcal{F})c(\mathcal{E}')$$

as desired. ☐

0GUD Lemma 42.46.2. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Assume there exists an envelope $f : Y \to X$ (Definition 42.22.1) such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_Y$-modules. Then there exists unique bivariant classes $c(E) \in \prod_{p \geq 0} A^p(X)$, $ch(E) \in \prod_{p \geq 0} A^p(X) \otimes \mathbf{Q}$, and $P_p(E) \in A^p(X)$, independent of the choice of $f : Y \to X$ and $\mathcal{E}^\bullet$, such that the restriction of these classes to Y are equal to $c(\mathcal{E}^\bullet)$, $ch(\mathcal{E}^\bullet)$, and $P_p(\mathcal{E}^\bullet)$.

Proof. Fix $p \in \mathbf{Z}$. We will prove the lemma for the chern class $c_p(E) \in A^p(X)$ and omit the arguments for the other cases.

Let $g : T \to X$ be a morphism locally of finite type such that there exists a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_T$-modules representing Lg^*E in $D(\mathcal{O}_T)$. The bivariant class $c_p(\mathcal{E}^\bullet) \in A^p(T)$ is independent of the choice of $\mathcal{E}^\bullet$ by Lemma 42.46.1. Let us write $c_p(Lg^*E) \in A^p(T)$ for this class. For any further morphism $h : T' \to T$ which is locally of finite type, setting $g' = g \circ h$ we see that $L(g')^*E = L(g \circ h)^*E = Lh^*Lg^*E$ is represented by $h^*\mathcal{E}^\bullet$ in $D(\mathcal{O}_{T'})$. We conclude that $c_p(L(g')^*E)$ makes sense and is equal to the restriction (Remark 42.33.5) of $c_p(Lg^*E)$ to T' (strictly speaking this requires an application of Lemma 42.38.7).

Let $f : Y \to X$ and $\mathcal{E}^\bullet$ be as in the statement of the lemma. We obtain a bivariant class $c_p(E) \in A^p(X)$ from an application of Lemma 42.35.6 to $f : Y \to X$ and the class $c' = c_p(Lf^*E)$ we constructed in the previous paragraph. The assumption in the lemma is satisfied because by the discussion in the previous paragraph we have $res_1(c') = c_p(Lg^*E) = res_2(c')$ where $g = f \circ p = f \circ q : Y \times_X Y \to X$.

Finally, suppose that $f' : Y' \to X$ is a second envelope such that $L(f')^*E$ is represented by a bounded complex of finite locally free $\mathcal{O}_{Y'}$-modules. Then it follows that the restrictions of $c_p(Lf^*E)$ and $c_p(L(f')^*E)$ to $Y \times_X Y'$ are equal. Since $Y \times_X Y' \to X$ is an envelope (Lemmas 42.22.3 and 42.22.2), we see that our two candidates for $c_p(E)$ agree by the unicity in Lemma 42.35.6. ☐

0F9E Definition 42.46.3. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object.

(1) We say the Chern classes of E are defined[4] if there exists an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules.
(2) If the Chern classes of E are defined, then we define
$$c(E) \in \prod\nolimits_{p \geq 0} A^p(X), \quad ch(E) \in \prod\nolimits_{p \geq 0} A^p(X) \otimes \mathbf{Q}, \quad P_p(E) \in A^p(X)$$
by an application of Lemma 42.46.2.

This definition applies in many but not all situations envisioned in this chapter, see Lemma 42.46.4. Perhaps an elementary construction of these bivariant classes for general $E/X/(S, \delta)$ as in the definition exists; we don't know.

0GUE Lemma 42.46.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. If one of the following conditions hold, then the Chern classes of E are defined:

[4]See Lemma 42.46.4 for some criteria.

(1) there exists an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules,
(2) E can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules,
(3) the irreducible components of X are quasi-compact,
(4) X is quasi-compact,
(5) there exists a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object E' on X' whose chern classes are defined, or
(6) add more here.

Proof. Condition (1) is just Definition 42.46.3 part (1). Condition (2) implies (1).

As in (3) assume the irreducible components X_i of X are quasi-compact. We view X_i as a reduced integral closed subscheme over X. The morphism $\coprod X_i \to X$ is an envelope. For each i there exists an envelope $X'_i \to X_i$ such that X'_i has an ample family of invertible modules, see More on Morphisms, Proposition 37.80.3. Observe that $f : Y = \coprod X'_i \to X$ is an envelope; small detail omitted. By Derived Categories of Schemes, Lemma 36.36.7 each X'_i has the resolution property. Thus the perfect object $L(f|_{X'_i})^*E$ of $D(\mathcal{O}_{X'_i})$ can be represented by a bounded complex of finite locally free $\mathcal{O}_{X'_i}$-modules, see Derived Categories of Schemes, Lemma 36.37.2. This proves (3) implies (1).

Part (4) implies (3).

Let $g : X \to X'$ and E' be as in part (5). Then there exists an envelope $f' : Y' \to X'$ such that $L(f')^*E'$ is represented by a locally bounded complex $(\mathcal{E}')^\bullet$ of $\mathcal{O}_{Y'}$-modules. Then the base change $f : Y \to X$ is an envelope by Lemma 42.22.3. Moreover, the pullback $\mathcal{E}^\bullet = g^*(\mathcal{E}')^\bullet$ represents Lf^*E and we see that the chern classes of E are defined. $\square$

0GUF Lemma 42.46.5. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Assume the Chern classes of E are defined. For $g : W \to X$ locally of finite type with W integral, there exists a commutative diagram

$$\begin{tikzcd} W' \arrow[rr, "b"] \arrow[dr, "g'"'] & & W \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

with W' integral and $b : W' \to W$ proper birational such that $L(g')^*E$ is represented by a bounded complex $\mathcal{E}^\bullet$ of locally free $\mathcal{O}_{W'}$-modules of constant rank and we have $res(c_p(E)) = c_p(\mathcal{E}^\bullet)$ in $A^p(W')$.

Proof. Choose an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_Y$-modules. The base change $Y \times_X W \to W$ of f is an envelope by Lemma 42.22.3. Choose a point $\xi \in Y \times_X W$ mapping to the generic point of W with the same residue field. Consider the integral closed subscheme $W' \subset Y \times_X W$ with generic point ξ. The restriction of the projection $Y \times_X W \to W$ to W' is a proper birational morphism $b : W' \to W$. Set $g' = g \circ b$. Finally, consider the pullback $(W' \to Y)^*\mathcal{E}^\bullet$. This is a locally bounded complex of finite locally free modules on W'. Since W' is integral it follows that it is bounded and that the terms have constant rank. Finally, by construction $(W' \to Y)^*\mathcal{E}^\bullet$ represents $L(g')^*E$ and by construction its pth chern class gives the restriction of $c_p(E)$ by $W' \to X$. This finishes the proof. □

0FAC Lemma 42.46.6. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be perfect. If the Chern classes of E are defined then

(1) $c_p(E)$ is in the center of the algebra $A^*(X)$, and
(2) if $g : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_p(E) = c_p(Lg^*E) \circ c$.

Proof. Part (1) follows immediately from part (2). Let $g : X' \to X$ and $c \in A^*(X' \to X)$ be as in (2). To show that $c \circ c_p(E) - c_p(Lg^*E) \circ c = 0$ we use the criterion of Lemma 42.35.3. Thus we may assume that X is integral and by Lemma 42.46.5 we may even assume that E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules of constant rank. Then we have to show that

$$c \cap c_p(\mathcal{E}^\bullet) \cap [X] = c_p(\mathcal{E}^\bullet) \cap c \cap [X]$$

in $\mathrm{CH}_*(X')$. This is immediate from Lemma 42.38.9 and the construction of $c_p(\mathcal{E}^\bullet)$ as a polynomial in the chern classes of the locally free modules $\mathcal{E}^n$. □

0F9F Lemma 42.46.7. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. If one of the following conditions holds

(1) there exists an envelope $f : Y \to X$ such that $Lf^*E_1 \to Lf^*E_2$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules,
(2) $E_1 \to E_2$ can be represented be a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules,
(3) the irreducible components of X are quasi-compact,
(4) X is quasi-compact, or
(5) add more here,

then the Chern classes of E_1, E_2, E_3 are defined and we have $c(E_2) = c(E_1)c(E_3)$, $ch(E_2) = ch(E_1) + ch(E_3)$, and $P_p(E_2) = P_p(E_1) + P_p(E_3)$.

Proof. Let $f : Y \to X$ be an envelope and let $\alpha^\bullet : \mathcal{E}_1^\bullet \to \mathcal{E}_2^\bullet$ be a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules representing $Lf^*E_1 \to Lf^*E_2$. Then the cone $C(\alpha)^\bullet$ represents Lf^*E_3. Since $C(\alpha)^n = \mathcal{E}_2^n \oplus \mathcal{E}_1^{n+1}$ we see that $C(\alpha)^\bullet$ is a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules. We conclude that the Chern classes of E_1, E_2, E_3 are defined. Moreover, recall that $c_p(E_1)$ is defined as the unique element of $A^p(X)$ which restricts to $c_p(\mathcal{E}_1^\bullet)$ in $A^p(Y)$. Similarly for E_2 and E_3. Hence it suffices to prove $c(\mathcal{E}_2^\bullet) = c(\mathcal{E}_1^\bullet)c(C(\alpha)^\bullet)$ in $\prod_{p \geq 0} A^p(Y)$. In turn, it suffices to prove this after restricting to a connected component of Y. Hence we may assume the complexes $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ are bounded complexes of finite locally free $\mathcal{O}_Y$-modules of fixed rank. In this case the desired equality follows from the multiplicativity of Lemma 42.40.3. In the case of ch or P_p we use Lemmas 42.45.2.

In the previous paragraph we have seen that the lemma holds if condition (1) is satisfied. Since (2) implies (1) this deals with the second case. Assume (3). Arguing exactly as in the proof of Lemma 42.46.4 we find an envelope $f : Y \to X$ such that

Y is a disjoint union $Y = \coprod Y_i$ of quasi-compact (and quasi-separated) schemes each having the resolution property. Then we may represent the restriction of $Lf^*E_1 \to Lf^*E_2$ to Y_i by a map of bounded complexes of finite locally free modules, see Derived Categories of Schemes, Proposition 36.37.5. In this way we see that condition (3) implies condition (1). Of course condition (4) implies condition (3) and the proof is complete. □

0FAD Remark 42.46.8. The Chern classes of a perfect complex, when defined, satisfy a kind of splitting principle. Namely, suppose that $(S, \delta), X, E$ are as in Definition 42.46.3 such that the Chern classes of E are defined. Say we want to prove a relation between the bivariant classes $c_p(E)$, $P_p(E)$, and $ch_p(E)$. To do this, we may choose an envelope $f : Y \to X$ and a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. By the uniqueness in Lemma 42.46.2 it suffices to prove the desired relation between the bivariant classes $c_p(\mathcal{E}^\bullet)$, $P_p(\mathcal{E}^\bullet)$, and $ch_p(\mathcal{E}^\bullet)$. Thus we may replace X by a connected component of Y and assume that E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free modules of fixed rank. Using the splitting principle (Lemma 42.43.1) we may assume each $\mathcal{E}^i$ has a filtration whose successive quotients $\mathcal{L}_{i,j}$ are invertible modules. Settting $x_{i,j} = c_1(\mathcal{L}_{i,j})$ we see that

$$c(E) = \prod\nolimits_{i \text{ even}} (1 + x_{i,j}) \prod\nolimits_{i \text{ odd}} (1 + x_{i,j})^{-1}$$

and

$$P_p(E) = \sum\nolimits_{i \text{ even}} (x_{i,j})^p - \sum\nolimits_{i \text{ odd}} (x_{i,j})^p$$

Formally taking the logarithm for the expression for $c(E)$ above we find that

$$\log(c(E)) = \sum (-1)^{p-1} \frac{P_p(E)}{p}$$

Looking at the construction of the polynomials P_p in Example 42.43.6 it follows that $P_p(E)$ is the exact same expression in the Chern classes of E as in the case of vector bundles, in other words, we have

$$\begin{aligned}
P_1(E) &= c_1(E), \\
P_2(E) &= c_1(E)^2 - 2c_2(E), \\
P_3(E) &= c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E), \\
P_4(E) &= c_1(E)^4 - 4c_1(E)^2c_2(E) + 4c_1(E)c_3(E) + 2c_2(E)^2 - 4c_4(E),
\end{aligned}$$

and so on. On the other hand, the bivariant class $P_0(E) = r(E) = ch_0(E)$ cannot be recovered from the Chern class $c(E)$ of E; the chern class doesn't know about the rank of the complex.

0FAE Lemma 42.46.9. In Situation 42.7.1 let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object whose Chern classes are defined. Then $c_i(E^\vee) = (-1)^i c_i(E)$, $P_i(E^\vee) = (-1)^i P_i(E)$, and $ch_i(E^\vee) = (-1)^i ch_i(E)$ in $A^i(X)$.

Proof. First proof: argue as in the proof of Lemma 42.46.6 to reduce to the case where E is represented by a bounded complex of finite locally free modules of fixed rank and apply Lemma 42.43.3. Second proof: use the splitting principle discussed in Remark 42.46.8 and use that the chern roots of $E^\vee$ are the negatives of the chern roots of E. □

0FAF Lemma 42.46.10. In Situation 42.7.1 let X be locally of finite type over S. Let E be a perfect object of $D(\mathcal{O}_X)$ whose Chern classes are defined. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

$$c_i(E \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(E) c_1(\mathcal{L})^j$$

provided E has constant rank $r \in \mathbf{Z}$.

Proof. In the case where E is locally free of rank r this is Lemma 42.39.1. The reader can deduce the lemma from this special case by a formal computation. An alternative is to use the splitting principle of Remark 42.46.8. In this case one ends up having to prove the following algebra fact: if we write formally

$$\frac{\prod_{a=1,\ldots,n}(1+x_a)}{\prod_{n=1,\ldots,m}(1+y_b)} = 1 + c_1 + c_2 + c_3 + \ldots$$

with c_i homogeneous of degree i in $\mathbf{Z}[x_i, y_j]$ then we have

$$\frac{\prod_{a=1,\ldots,n}(1+x_a+t)}{\prod_{b=1,\ldots,m}(1+y_b+t)} = \sum\nolimits_{i \geq 0} \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j} t^j$$

where $r = n - m$. We omit the details. $\square$

0FAG Lemma 42.46.11. In Situation 42.7.1 let X be locally of finite type over S. Let E and F be perfect objects of $D(\mathcal{O}_X)$ whose Chern classes are defined. Then we have

$$c_1(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = r(E)c_1(\mathcal{F}) + r(F)c_1(\mathcal{E})$$

and for $c_2(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ we have the expression

$$r(E)c_2(F) + r(F)c_2(E) + \binom{r(E)}{2} c_1(F)^2 + (r(E)r(F)-1)c_1(F)c_1(E) + \binom{r(F)}{2} c_1(E)^2$$

and so on for higher Chern classes in $A^*(X)$. Similarly, we have $ch(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = ch(E)ch(F)$ in $A^*(X) \otimes \mathbf{Q}$. More precisely, we have

$$P_p(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(E) P_{p_2}(F)$$

in $A^p(X)$.

Proof. After choosing an envelope $f : Y \to X$ such that Lf^*E and Lf^*F can be represented by locally bounded complexes of finite locally free $\mathcal{O}_X$-modules this follows by a compuation from the corresponding result for vector bundles in Lemmas 42.43.4 and 42.45.3. A better proof is probably to use the splitting principle as in Remark 42.46.8 and reduce the lemma to computations in polynomial rings which we describe in the next paragraph.

Let A be a commutative ring (for us this will be the subring of the bivariant chow ring of X generated by Chern classes). Let S be a finite set together with maps $\epsilon : S \to \{\pm 1\}$ and $f : S \to A$. Define

$$P_p(S, f, \epsilon) = \sum\nolimits_{s \in S} \epsilon(s) f(s)^p$$

in A. Given a second triple (S', ϵ', f') the equality that has to be shown for P_p is the equality

$$P_p(S \times S', f + f', \epsilon\epsilon') = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(S, f, \epsilon) P_{p_2}(S', f', \epsilon')$$

To see this is true, one reduces to the polynomial ring on variables $S \amalg S'$ and one shows that each term $f(s)^i f'(s')^j$ occurs on the left and right hand side with the same coefficient. To verify the formulas for $c_1(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ and $c_2(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ we use the splitting principle to reduce to checking these formulae in a torsion free ring. Then we use the relationship between $P_j(E)$ and $c_i(E)$ proved in Remark 42.46.8. For example

$$c_1(E \otimes F) = P_1(E \otimes F) = r(F)P_1(E) + r(E)P_1(F) = r(F)c_1(E) + r(E)c_1(F)$$

the middle equation because $r(E) = P_0(E)$ by definition. Similarly, we have

$$\begin{aligned}
&2c_2(E \otimes F) \\
&= c_1(E \otimes F)^2 - P_2(E \otimes F) \\
&= (r(F)c_1(E) + r(E)c_1(F))^2 - r(F)P_2(E) - P_1(E)P_1(F) - r(E)P_2(F) \\
&= (r(F)c_1(E) + r(E)c_1(F))^2 - r(F)(c_1(E)^2 - 2c_2(E)) - c_1(E)c_1(F) - \\
&\quad r(E)(c_1(F)^2 - 2c_2(F))
\end{aligned}$$

which the reader can verify agrees with the formula in the statement of the lemma up to a factor of 2. ☐

42.47. A baby case of localized Chern classes

0F9G In this section we discuss some properties of the bivariant classes constructed in the following lemma; most of these properties follow immediately from the characterization given in the lemma. We urge the reader to skip the rest of the section.

0F9H Lemma 42.47.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E_2 \in D(\mathcal{O}_{X_2})$ be a perfect object. Assume

1. Chern classes of E_2 are defined,
2. the restriction $E_2|_{X_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0.

Then there is a canonical bivariant class

$$P'_p(E_2),\ \text{resp. } c'_p(E_2) \in A^p(X_2 \to X)$$

characterized by the property

$$P'_p(E_2) \cap i_{2,*}\alpha_2 = P_p(E_2) \cap \alpha_2 \quad \text{and} \quad P'_p(E_2) \cap i_{1,*}\alpha_1 = 0,$$

respectively

$$c'_p(E_2) \cap i_{2,*}\alpha_2 = c_p(E_2) \cap \alpha_2 \quad \text{and} \quad c'_p(E_2) \cap i_{1,*}\alpha_1 = 0$$

for $\alpha_i \in \mathrm{CH}_k(X_i)$ and similarly after any base change $X' \to X$ locally of finite type.

Proof. We are going to use the material of Section 42.46 without further mention.

Assume $E_2|_{X_1 \cap X_2}$ is zero. Consider a morphism of schemes $X' \to X$ which is locally of finite type and denote $i'_j : X'_j \to X'$ the base change of i_j. By Lemma 42.19.4 we can write any element $\alpha' \in \mathrm{CH}_k(X')$ as $i'_{1,*}\alpha'_1 + i'_{2,*}\alpha'_2$ where $\alpha'_2 \in \mathrm{CH}_k(X'_2)$ is well defined up to an element in the image of pushforward by $X'_1 \cap X'_2 \to X'_2$. Then we can set $P'_p(E_2) \cap \alpha' = P_p(E_2) \cap \alpha'_2 \in \mathrm{CH}_{k-p}(X'_2)$. This is well defined by our assumption that E_2 restricts to zero on $X_1 \cap X_2$.

If $E_2|_{X_1 \cap X_2}$ is isomorphic to a finite locally free $\mathcal{O}_{X_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0, then $c_p(E_2|_{X_1 \cap X_2}) = 0$ by rank considerations and we can argue in exactly the same manner. □

0FAH Lemma 42.47.2. In Lemma 42.47.1 the bivariant class $P'_p(E_2)$, resp. $c'_p(E_2)$ in $A^p(X_2 \to X)$ does not depend on the choice of X_1.

Proof. Suppose that $X'_1 \subset X$ is another closed subscheme such that $X = X'_1 \cup X_2$ set theoretically and the restriction $E_2|_{X'_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X'_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0. Then $X = (X_1 \cap X'_1) \cup X_2$. Hence we can write any element $\alpha \in \mathrm{CH}_k(X)$ as $i_*\beta + i_{2,*}\alpha_2$ with $\alpha_2 \in \mathrm{CH}_k(X'_2)$ and $\beta \in \mathrm{CH}_k(X_1 \cap X'_1)$. Thus it is clear that $P'_p(E_2) \cap \alpha = P_p(E_2) \cap \alpha_2 \in \mathrm{CH}_{k-p}(X_2)$, resp. $c'_p(E_2) \cap \alpha = c_p(E_2) \cap \alpha_2 \in \mathrm{CH}_{k-p}(X_2)$, is independent of whether we use X_1 or X'_1. Similarly after any base change. □

0GUG Lemma 42.47.3. In Lemma 42.47.1 let $X' \to X$ be a morphism which is locally of finite type. Denote $X' = X'_1 \cup X'_2$ and $E'_2 \in D(\mathcal{O}_{X'_2})$ the pullbacks to X'. Then the class $P'_p(E'_2)$, resp. $c'_p(E'_2)$ in $A^p(X'_2 \to X')$ constructed in Lemma 42.47.1 using $X' = X'_1 \cup X'_2$ and E'_2 is the restriction (Remark 42.33.5) of the class $P'_p(E_2)$, resp. $c'_p(E_2)$ in $A^p(X_2 \to X)$.

Proof. Immediate from the characterization of these classes in Lemma 42.47.1. □

0F9I Lemma 42.47.4. In Lemma 42.47.1 say E_2 is the restriction of a perfect $E \in D(\mathcal{O}_X)$ such that $E|_{X_1}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0. If Chern classes of E are defined, then $i_{2,*} \circ P'_p(E_2) = P_p(E)$, resp. $i_{2,*} \circ c'_p(E_2) = c_p(E)$ (with $\circ$ as in Lemma 42.33.4).

Proof. First, assume $E|_{X_1}$ is zero. With notations as in the proof of Lemma 42.47.1 the lemma in this case follows from

$$\begin{aligned} P_p(E) \cap \alpha' &= i'_{1,*}(P_p(E) \cap \alpha'_1) + i'_{2,*}(P_p(E) \cap \alpha'_2) \\ &= i'_{1,*}(P_p(E|_{X_1}) \cap \alpha'_1) + i'_{2,*}(P'_p(E_2) \cap \alpha') \\ &= i'_{2,*}(P'_p(E_2) \cap \alpha') \end{aligned}$$

The case where $E|_{X_1}$ is isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0 is similar. □

0FAI Lemma 42.47.5. In Lemma 42.47.1 suppose we have closed subschemes $X'_2 \subset X_2$ and $X_1 \subset X'_1 \subset X$ such that $X = X'_1 \cup X'_2$ set theoretically. Assume $E_2|_{X'_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free module of rank $< p$ placed in degree 0. Then we have $(X'_2 \to X_2)_* \circ P'_p(E_2|_{X'_2}) = P'_p(E_2)$, resp. $(X'_2 \to X_2)_* \circ c'_p(E_2|_{X'_2}) = c_p(E_2)$ (with $\circ$ as in Lemma 42.33.4).

Proof. This follows immediately from the characterization of these classes in Lemma 42.47.1. □

0FAJ Lemma 42.47.6. In Lemma 42.47.1 let $f : Y \to X$ be locally of finite type and say $c \in A^*(Y \to X)$. Then

$$c \circ P'_p(E_2) = P'_p(Lf_2^* E_2) \circ c \quad \text{resp.} \quad c \circ c'_p(E_2) = c'_p(Lf_2^* E_2) \circ c$$

in $A^*(Y_2 \to Y)$ where $f_2 : Y_2 \to X_2$ is the base change of f.

Proof. Let $\alpha \in \mathrm{CH}_k(X)$. We may write

$$\alpha = \alpha_1 + \alpha_2$$

with $\alpha_i \in \mathrm{CH}_k(X_i)$; we are omitting the pushforwards by the closed immersions $X_i \to X$. The reader then checks that $c'_p(E_2) \cap \alpha = c_p(E_2) \cap \alpha_2$, $c \cap c'_p(E_2) \cap \alpha = c \cap c_p(E_2) \cap \alpha_2$, $c \cap \alpha = c \cap \alpha_1 + c \cap \alpha_2$, and $c'_p(Lf_2^* E_2) \cap c \cap \alpha = c_p(Lf_2^* E_2) \cap c \cap \alpha_2$. We conclude by Lemma 42.46.6. □

0FAK Lemma 42.47.7. In Lemma 42.47.1 assume $E_2|_{X_1 \cap X_2}$ is zero. Then

$$P'_1(E_2) = c'_1(E_2),$$
$$P'_2(E_2) = c'_1(E_2)^2 - 2c'_2(E_2),$$
$$P'_3(E_2) = c'_1(E_2)^3 - 3c'_1(E_2)c'_2(E_2) + 3c'_3(E_2),$$
$$P'_4(E_2) = c'_1(E_2)^4 - 4c'_1(E_2)^2 c'_2(E_2) + 4c'_1(E_2)c'_3(E_2) + 2c'_2(E_2)^2 - 4c'_4(E_2),$$

and so on with multiplication as in Remark 42.34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c'_p(E_2)$ are defined for all $p \geq 1$. The equalities follow immediately from the characterization of the classes produced by Lemma 42.47.1 and the corresponding result for capping with the Chern classes of E_2 given in Remark 42.46.8. □

0FAL Lemma 42.47.8. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E, F \in D(\mathcal{O}_X)$ be perfect objects. Assume

(1) Chern classes of E and F are defined,
(2) the restrictions $E|_{X_1 \cap X_2}$ and $F|_{X_1 \cap X_2}$ are isomorphic to a finite locally free $\mathcal{O}_{X_1}$-modules of rank $< p$ and $< q$ sitting in cohomological degree 0.

With notation as in Remark 42.34.7 set

$$c^{(p)}(E) = 1 + c_1(E) + \ldots + c_{p-1}(E) + c'_p(E|_{X_2}) + c'_{p+1}(E|_{X_2}) + \ldots \in A^{(p)}(X_2 \to X)$$

with $c'_p(E|_{X_2})$ as in Lemma 42.47.1. Similarly for $c^{(q)}(F)$ and $c^{(p+q)}(E \oplus F)$. Then $c^{(p+q)}(E \oplus F) = c^{(p)}(E) c^{(q)}(F)$ in $A^{(p+q)}(X_2 \to X)$.

Proof. Immediate from the characterization of the classes in Lemma 42.47.1 and the additivity in Lemma 42.46.7. □

0FAM Lemma 42.47.9. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E, F \in D(\mathcal{O}_{X_2})$ be perfect objects. Assume

(1) Chern classes of E and F are defined,
(2) the restrictions $E|_{X_1 \cap X_2}$ and $F|_{X_1 \cap X_2}$ are zero,

Denote $P'_p(E), P'_p(F), P'_p(E \oplus F) \in A^p(X_2 \to X)$ for $p \geq 0$ the classes constructed in Lemma 42.47.1. Then $P'_p(E \oplus F) = P'_p(E) + P'_p(F)$.

Proof. Immediate from the characterization of the classes in Lemma 42.47.1 and the additivity in Lemma 42.46.7. □

0FAN Lemma 42.47.10. In Lemma 42.47.1 assume E_2 has constant rank 0. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

$$c'_i(E_2 \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{-i+j}{j} c'_{i-j}(E_2) c_1(\mathcal{L})^j$$

Proof. The assumption on rank implies that $E_2|_{X_1 \cap X_2}$ is zero. Hence $c'_i(E_2)$ is defined for all $i \geq 1$ and the statement makes sense. The actual equality follows immediately from Lemma 42.46.10 and the characterization of c'_i in Lemma 42.47.1. □

0FE5 Lemma 42.47.11. In Situation 42.7.1 let X be locally of finite type over S. Let

$$X = X_1 \cup X_2 = X'_1 \cup X'_2$$

be two ways of writing X as a set theoretic union of closed subschemes. Let E, E' be perfect objects of $D(\mathcal{O}_X)$ whose Chern classes are defined. Assume that $E|_{X_1}$ and $E'|_{X'_1}$ are zero[5] for $i = 1, 2$. Denote

(1) $r = P'_0(E) \in A^0(X_2 \to X)$ and $r' = P'_0(E') \in A^0(X'_2 \to X)$,
(2) $\gamma_p = c'_p(E|_{X_2}) \in A^p(X_2 \to X)$ and $\gamma'_p = c'_p(E'|_{X'_2}) \in A^p(X'_2 \to X)$,
(3) $\chi_p = P'_p(E|_{X_2}) \in A^p(X_2 \to X)$ and $\chi'_p = P'_p(E'|_{X'_2}) \in A^p(X'_2 \to X)$

the classes constructed in Lemma 42.47.1. Then we have

$$c'_1((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = r\gamma'_1 + r'\gamma_1$$

in $A^1(X_2 \cap X'_2 \to X)$ and

$$c'_2((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = r\gamma'_2 + r'\gamma_2 + \binom{r}{2}(\gamma'_1)^2 + (rr' - 1)\gamma'_1\gamma_1 + \binom{r'}{2}\gamma_1^2$$

in $A^2(X_2 \cap X'_2 \to X)$ and so on for higher Chern classes. Similarly, we have

$$P'_p((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = \sum\nolimits_{p_1 + p_2 = p} \binom{p}{p_1} \chi_{p_1}\chi'_{p_2}$$

in $A^p(X_2 \cap X'_2 \to X)$.

Proof. First we observe that the statement makes sense. Namely, we have $X = (X_2 \cap X'_2) \cup Y$ where $Y = (X_1 \cap X'_1) \cup (X_1 \cap X'_2) \cup (X_2 \cap X'_1)$ and the object $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E'$ restricts to zero on Y. The actual equalities follow from the characterization of our classes in Lemma 42.47.1 and the equalities of Lemma 42.46.11. We omit the details. □

42.48. Gysin at infinity

0FAP This section is about the bivariant class constructed in the next lemma. We urge the reader to skip the rest of the section.

0F9J Lemma 42.48.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $b : W \to \mathbf{P}^1_X$ be a proper morphism of schemes which is an isomorphism over $\mathbf{A}^1_X$. Denote $i_\infty : W_\infty \to W$ the inverse image of the divisor $D_\infty \subset \mathbf{P}^1_X$ with complement $\mathbf{A}^1_X$. Then there is a canonical bivariant class

$$C \in A^0(W_\infty \to X)$$

with the property that $i_{\infty,*}(C \cap \alpha) = i_{0,*}\alpha$ for $\alpha \in \mathrm{CH}_k(X)$ and similarly after any base change by $X' \to X$ locally of finite type.

[5] Presumably there is a variant of this lemma where we only assume these restrictions are isomorphic to a finite locally free modules of rank $< p$ and $< p'$.

Proof. Given $\alpha \in \mathrm{CH}_k(X)$ there exists a $\beta \in \mathrm{CH}_{k+1}(W)$ restricting to the flat pullback of α on $b^{-1}(\mathbf{A}^1_X)$, see Lemma 42.14.2. A second choice of β differs from β by a cycle supported on W_∞, see Lemma 42.19.3. Since the normal bundle of the effective Cartier divisor $D_\infty \subset \mathbf{P}^1_X$ of (42.18.1.1) is trivial, the gysin homomorphism i_∞^* kills cycle classes supported on W_∞, see Remark 42.29.6. Hence setting $C \cap \alpha = i_\infty^* \beta$ is well defined.

Since W_∞ and $W_0 = X \times \{0\}$ are the pullbacks of the rationally equivalent effective Cartier divisors D_0, D_∞ in $\mathbf{P}^1_X$, we see that $i_\infty^*\beta$ and $i_0^*\beta$ map to the same cycle class on W; namely, both represent the class $c_1(\mathcal{O}_{\mathbf{P}^1_X}(1)) \cap \beta$ by Lemma 42.29.4. By our choice of β we have $i_0^*\beta = \alpha$ as cycles on $W_0 = X \times \{0\}$, see for example Lemma 42.31.1. Thus we see that $i_{\infty,*}(C \cap \alpha) = i_{0,*}\alpha$ as stated in the lemma.

Observe that the assumptions on b are preserved by any base change by $X' \to X$ locally of finite type. Hence we get an operation $C \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_k(W'_\infty)$ by the same construction as above. To see that this family of operations defines a bivariant class, we consider the diagram

$$
\begin{tikzcd}
 & & \mathrm{CH}_*(X) \arrow[d, "\text{flat pullback}"] & \\
\mathrm{CH}_{*+1}(W_\infty) \arrow[r] \arrow[dr, "0"] & \mathrm{CH}_{*+1}(W) \arrow[r, "\text{flat pullback}"] \arrow[d, "i_\infty^*"] & \mathrm{CH}_{*+1}(\mathbf{A}^1_X) \arrow[r] \arrow[dl, dotted, "C\cap-"] & 0 \\
 & \mathrm{CH}_*(W_\infty) & &
\end{tikzcd}
$$

for X as indicated and the base change of this diagram for any $X' \to X$. We know that flat pullback and i_∞^* are bivariant operations, see Lemmas 42.33.2 and 42.33.3. Then a formal argument (involving huge diagrams of schemes and their chow groups) shows that the dotted arrow is a bivariant operation. □

0GUH **Lemma 42.48.2.** In Lemma 42.48.1 let $X' \to X$ be a morphism which is locally of finite type. Denote $b' : W' \to \mathbf{P}^1_{X'}$ and $i'_\infty : W'_\infty \to W'$ the base changes of b and i_∞. Then the class $C' \in A^0(W'_\infty \to X')$ constructed as in Lemma 42.48.1 using b' is the restriction (Remark 42.33.5) of C.

Proof. Immediate from the construction and the fact that a similar statement holds for flat pullback and i_∞^*. □

0FAQ **Lemma 42.48.3.** In Lemma 42.48.1 let $g : W' \to W$ be a proper morphism which is an isomorphism over $\mathbf{A}^1_X$. Let $C' \in A^0(W'_\infty \to X)$ and $C \in A^0(W_\infty \to X)$ be the classes constructed in Lemma 42.48.1. Then $g_{\infty,*} \circ C' = C$ in $A^0(W_\infty \to X)$.

Proof. Set $b' = b \circ g : W' \to \mathbf{P}^1_X$. Denote $i'_\infty : W'_\infty \to W'$ the inclusion morphism. Denote $g_\infty : W'_\infty \to W_\infty$ the restriction of g. Given $\alpha \in \mathrm{CH}_k(X)$ choose $\beta' \in \mathrm{CH}_{k+1}(W')$ restricting to the flat pullback of α on $(b')^{-1}\mathbf{A}^1_X$. Then $\beta = g_*\beta' \in \mathrm{CH}_{k+1}(W)$ restricts to the flat pullback of α on $b^{-1}\mathbf{A}^1_X$. Then $i_\infty^*\beta = g_{\infty,*}(i'_\infty)^*\beta'$ by Lemma 42.29.8. This and the corresponding fact after base change by morphisms $X' \to X$ locally of finite type, corresponds to the assertion made in the lemma. □

0FAR **Lemma 42.48.4.** In Lemma 42.48.1 we have $C \circ (W_\infty \to X)_* \circ i_\infty^* = i_\infty^*$.

Proof. Let $\beta \in \mathrm{CH}_{k+1}(W)$. Denote $i_0 : X = X \times \{0\} \to W$ the closed immersion of the fibre over 0 in $\mathbf{P}^1$. Then $(W_\infty \to X)_* i_\infty^* \beta = i_0^* \beta$ in $\mathrm{CH}_k(X)$ because $i_{\infty,*} i_\infty^* \beta$ and $i_{0,*} i_0^* \beta$ represent the same class on W (for example by Lemma 42.29.4) and hence pushforward to the same class on X. The restriction of β to $b^{-1}(\mathbf{A}^1_X)$ restricts to the flat pullback of $i_0^* \beta = (W_\infty \to X)_* i_\infty^* \beta$ because we can check this after pullback by i_0, see Lemmas 42.32.2 and 42.32.4. Hence we may use β when computing the image of $(W_\infty \to X)_* i_\infty^* \beta$ under C and we get the desired result. $\square$

0FAS Lemma 42.48.5. In Lemma 42.48.1 let $f : Y \to X$ be a morphism locally of finite type and $c \in A^*(Y \to X)$. Then $C \circ c = c \circ C$ in $A^*(W_\infty \times_X Y \to X)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
W_\infty \times_X Y \arrow[r, equal] & W_{Y,\infty} \arrow[r, "i_{Y,\infty}"'] \arrow[d] & W_Y \arrow[r, "b_Y"'] \arrow[d] & \mathbf{P}^1_Y \arrow[r, "p_Y"'] \arrow[d] & Y \arrow[d, "f"] \\
 & W_\infty \arrow[r, "i_\infty"] & W \arrow[r, "b"] & \mathbf{P}^1_X \arrow[r, "p"] & X
\end{tikzcd}$$

with cartesian squares. For an elemnent $\alpha \in \mathrm{CH}_k(X)$ choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is the flat pullback of α. Then $c \cap \beta$ is a class in $\mathrm{CH}_*(W_Y)$ whose restriction to $b_Y^{-1}(\mathbf{A}^1_Y)$ is the flat pullback of $c \cap \alpha$. Next, we have

$$i_{Y,\infty}^*(c \cap \beta) = c \cap i_\infty^* \beta$$

because c is a bivariant class. This exactly says that $C \cap c \cap \alpha = c \cap C \cap \alpha$. The same argument works after any base change by $X' \to X$ locally of finite type. This proves the lemma. $\square$

42.49. Preparation for localized Chern classes

0FAT In this section we discuss some properties of the bivariant classes constructed in the following lemma. We urge the reader to skip the rest of the section.

0F9K Lemma 42.49.1. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. Let

$$b : W \longrightarrow \mathbf{P}^1_X$$

be a proper morphism of schemes. Let $Q \in D(\mathcal{O}_W)$ be a perfect object. Denote $W_\infty \subset W$ the inverse image of the divisor $D_\infty \subset \mathbf{P}^1_X$ with complement $\mathbf{A}^1_X$. We assume

(A0) Chern classes of Q are defined (Section 42.46),
(A1) b is an isomorphism over $\mathbf{A}^1_X$,
(A2) there exists a closed subscheme $T \subset W_\infty$ containing all points of W_∞ lying over $X \setminus Z$ such that $Q|_T$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$ sitting in cohomological degree 0.

Then there exists a canonical bivariant class

$$P'_p(Q), \text{ resp. } c'_p(Q) \in A^p(Z \to X)$$

with $(Z \to X)_* \circ P'_p(Q) = P_p(Q|_{X \times \{0\}})$, resp. $(Z \to X)_* \circ c'_p(Q) = c_p(Q|_{X \times \{0\}})$.

Proof. Denote $E \subset W_\infty$ the inverse image of Z. Then $W_\infty = T \cup E$ and b induces a proper morphism $E \to Z$. Denote $C \in A^0(W_\infty \to X)$ the bivariant class constructed in Lemma 42.48.1. Denote $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ in $A^p(E \to W_\infty)$ the bivariant class constructed in Lemma 42.47.1. This makes sense because $(Q|_E)|_{T \cap E}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{E \cap T}$-module of rank $< p$ sitting in cohomological degree 0 by assumption (A2). Then we define

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C, \text{ resp. } c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

This is a bivariant class, see Lemma 42.33.4. Since $E \to Z \to X$ is equal to $E \to W_\infty \to W \to X$ we see that

$$\begin{aligned} (Z \to X)_* \circ c'_p(Q) &= (W \to X)_* \circ i_{\infty,*} \circ (E \to W_\infty)_* \circ c'_p(Q|_E) \circ C \\ &= (W \to X)_* \circ i_{\infty,*} \circ c_p(Q|_{W_\infty}) \circ C \\ &= (W \to X)_* \circ c_p(Q) \circ i_{\infty,*} \circ C \\ &= (W \to X)_* \circ c_p(Q) \circ i_{0,*} \\ &= (W \to X)_* \circ i_{0,*} \circ c_p(Q|_{X \times \{0\}}) \\ &= c_p(Q|_{X \times \{0\}}) \end{aligned}$$

The second equality holds by Lemma 42.47.4. The third equality because $c_p(Q)$ is a bivariant class. The fourth equality by Lemma 42.48.1. The fifth equality because $c_p(Q)$ is a bivariant class. The final equality because $(W_0 \to W) \circ (W \to X)$ is the identity on X if we identify W_0 with X as we've done above. The exact same sequence of equations works to prove the property for $P'_p(Q)$. ☐

0GUI Lemma 42.49.2. In Lemma 42.49.1 let $X' \to X$ be a morphism which is locally of finite type. Denote Z', $b' : W' \to \mathbf{P}^1_{X'}$, and $T' \subset W'_\infty$ the base changes of Z, $b : W \to \mathbf{P}^1_X$, and $T \subset W_\infty$. Set $Q' = (W' \to W)^*Q$. Then the class $P'_p(Q')$, resp. $c'_p(Q')$ in $A^p(Z' \to X')$ constructed as in Lemma 42.49.1 using b', Q', and T' is the restriction (Remark 42.33.5) of the class $P'_p(Q)$, resp. $c'_p(Q)$ in $A^p(Z \to X)$.

Proof. Recall that the construction is as follows

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C, \text{ resp. } c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

Thus the lemma follows from the corresponding base change property for C (Lemma 42.48.2) and the fact that the same base change property holds for the classes constructed in Lemma 42.47.1 (small detail omitted). ☐

0FAU Lemma 42.49.3. In Lemma 42.49.1 the bivariant class $P'_p(Q)$, resp. $c'_p(Q)$ is independent of the choice of the closed subscheme T. Moreover, given a proper morphism $g : W' \to W$ which is an isomorphism over $\mathbf{A}^1_X$, then setting $Q' = g^*Q$ we have $P'_p(Q) = P'_p(Q')$, resp. $c'_p(Q) = c'_p(Q')$.

Proof. The independence of T follows immediately from Lemma 42.47.2.

Let $g : W' \to W$ be a proper morphism which is an isomorphism over $\mathbf{A}^1_X$. Observe that taking $T' = g^{-1}(T) \subset W'_\infty$ is a closed subscheme satisfying (A2) hence the operator $P'_p(Q')$, resp. $c'_p(Q')$ in $A^p(Z \to X)$ corresponding to $b' = b \circ g : W' \to \mathbf{P}^1_X$ and Q' is defined. Denote $E' \subset W'_\infty$ the inverse image of Z in W'_∞. Recall that

$$c'_p(Q') = (E' \to Z)_* \circ c'_p(Q'|_{E'}) \circ C'$$

with $C' \in A^0(W'_\infty \to X)$ and $c'_p(Q'|_{E'}) \in A^p(E' \to W'_\infty)$. By Lemma 42.48.3 we have $g_{\infty,*} \circ C' = C$. Observe that E' is also the inverse image of E in W'_∞ by g_∞.

Since moreover $Q' = g^*Q$ we find that $c'_p(Q'|_{E'})$ is simply the restriction of $c'_p(Q|_E)$ to schemes lying over W'_∞, see Remark 42.33.5. Thus we obtain

$$\begin{aligned} c'_p(Q') &= (E' \to Z)_* \circ c'_p(Q'|_{E'}) \circ C' \\ &= (E \to Z)_* \circ (E' \to E)_* \circ c'_p(Q|_E) \circ C' \\ &= (E \to Z)_* \circ c'_p(Q|_E) \circ g_{\infty,*} \circ C' \\ &= (E \to Z)_* \circ c'_p(Q|_E) \circ C \\ &= c'_p(Q) \end{aligned}$$

In the third equality we used that $c'_p(Q|_E)$ commutes with proper pushforward as it is a bivariant class. The equality $P'_p(Q) = P'_p(Q')$ is proved in exactly the same way. □

0FAV Lemma 42.49.4. In Lemma 42.49.1 assume $Q|_T$ is isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$. Denote $C \in A^0(W_\infty \to X)$ the class of Lemma 42.48.1. Then

$$C \circ c_p(Q|_{X \times \{0\}}) = C \circ (Z \to X)_* \circ c'_p(Q) = c_p(Q|_{W_\infty}) \circ C$$

Proof. The first equality holds because $c_p(Q|_{X\times\{0\}}) = (Z \to X)_* \circ c'_p(Q)$ by Lemma 42.49.1. We may prove the second equality one cycle class at a time (see Lemma 42.35.3). Since the construction of the bivariant classes in the lemma is compatible with base change, we may assume we have some $\alpha \in \mathrm{CH}_k(X)$ and we have to show that $C \cap (Z \to X)_*(c'_p(Q) \cap \alpha) = c_p(Q|_{W_\infty}) \cap C \cap \alpha$. Observe that

$$\begin{aligned} C \cap (Z \to X)_*(c'_p(Q) \cap \alpha) &= C \cap (Z \to X)_*(E \to Z)_*(c'_p(Q|_E) \cap C \cap \alpha) \\ &= C \cap (W_\infty \to X)_*(E \to W_\infty)_*(c'_p(Q|_E) \cap C \cap \alpha) \\ &= C \cap (W_\infty \to X)_*(E \to W_\infty)_*(c'_p(Q|_E) \cap i_\infty^*\beta) \\ &= C \cap (W_\infty \to X)_*(c_p(Q|_{W_\infty}) \cap i_\infty^*\beta) \\ &= C \cap (W_\infty \to X)_* i_\infty^*(c_p(Q) \cap \beta) \\ &= i_\infty^*(c_p(Q) \cap \beta) \\ &= c_p(Q|_{W_\infty}) \cap i_\infty^*\beta \\ &= c_p(Q|_{W_\infty}) \cap C \cap \alpha \end{aligned}$$

as desired. For the first equality we used that $c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$ where $E \subset W_\infty$ is the inverse image of Z and $c'_p(Q|_E)$ is the class constructed in Lemma 42.47.1. The second equality is just the statement that $E \to Z \to X$ is equal to $E \to W_\infty \to X$. For the third equality we choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is the flat pullback of α so that $C \cap \alpha = i_\infty^*\beta$ by construction. The fourth equality is Lemma 42.47.4. The fifth equality is the fact that $c_p(Q)$ is a bivariant class and hence commutes with i_∞^*. The sixth equality is Lemma 42.48.4. The seventh uses again that $c_p(Q)$ is a bivariant class. The final holds as $C \cap \alpha = i_\infty^*\beta$. □

0FAW Lemma 42.49.5. In Lemma 42.49.1 let $Y \to X$ be a morphism locally of finite type and let $c \in A^*(Y \to X)$ be a bivariant class. Then

$$P'_p(Q) \circ c = c \circ P'_p(Q) \quad \text{resp.} \quad c'_p(Q) \circ c = c \circ c'_p(Q)$$

in $A^*(Y \times_X Z \to X)$.

Proof. Let $E \subset W_\infty$ be the inverse image of Z. Recall that $P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C$, resp. $c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$ where C is as in Lemma 42.48.1 and $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ are as in Lemma 42.47.1. By Lemma 42.48.5 we see that C commutes with c and by Lemma 42.47.6 we see that $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ commutes with c. Since c is a bivariant class it commutes with proper pushforward by $E \to Z$ by definition. This finishes the proof. □

0FAX Lemma 42.49.6. In Lemma 42.49.1 assume $Q|_T$ is zero. In $A^*(Z \to X)$ we have

$$\begin{aligned}
P'_1(Q) &= c'_1(Q), \\
P'_2(Q) &= c'_1(Q)^2 - 2c'_2(Q), \\
P'_3(Q) &= c'_1(Q)^3 - 3c'_1(Q)c'_2(Q) + 3c'_3(Q), \\
P'_4(Q) &= c'_1(Q)^4 - 4c'_1(Q)^2c'_2(Q) + 4c'_1(Q)c'_3(Q) + 2c'_2(Q)^2 - 4c'_4(Q),
\end{aligned}$$

and so on with multiplication as in Remark 42.34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c'_p(Q)$ are defined for all $p \geq 1$. In the proof of Lemma 42.49.1 we have constructed the classes $P'_p(Q)$ and $c'_p(Q)$ using the bivariant class $C \in A^0(W_\infty \to X)$ of Lemma 42.48.1 and the bivariant classes $P'_p(Q|_E)$ and $c'_p(Q|_E)$ of Lemma 42.47.1 for the restriction $Q|_E$ of Q to the inverse image E of Z in W_∞. Observe that by Lemma 42.47.7 we have the desired relationship between $P'_p(Q|_E)$ and $c'_p(Q|_E)$. Recall that

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C \quad \text{and} \quad c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

To finish the proof it suffices to show the multiplications defined in Remark 42.34.7 on the classes $a_p = c'_p(Q)$ and on the classes $b_p = c'_p(Q|_E)$ agree:

$$a_{p_1}a_{p_2}\ldots a_{p_r} = (E \to Z)_* \circ b_{p_1}b_{p_2}\ldots b_{p_r} \circ C$$

Some details omitted. If $r = 1$, then this is true. For $r > 1$ note that by Remark 42.34.8 the multiplication in Remark 42.34.7 proceeds by inserting $(Z \to X)_*$, resp. $(E \to W_\infty)_*$ in between the factors of the product $a_{p_1}a_{p_2}\ldots a_{p_r}$, resp. $b_{p_1}b_{p_2}\ldots b_{p_r}$ and taking compositions as bivariant classes. Now by Lemma 42.47.1 we have

$$(E \to W_\infty)_* \circ b_{p_i} = c_{p_i}(Q|_{W_\infty})$$

and by Lemma 42.49.4 we have

$$C \circ (Z \to X)_* \circ a_{p_i} = c_{p_i}(Q|_{W_\infty}) \circ C$$

for $i = 2, \ldots, r$. A calculation shows that the left and right hand side of the desired equality both simplify to

$$(E \to Z)_* \circ c'_{p_1}(Q|_E) \circ c_{p_2}(Q|_{W_\infty}) \circ \ldots \circ c_{p_r}(Q|_{W_\infty}) \circ C$$

and the proof is complete. □

0FAY Lemma 42.49.7. In Lemma 42.49.1 assume $Q|_T$ is isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$. Assume we have another perfect object $Q' \in D(\mathcal{O}_W)$ whose Chern classes are defined with $Q'|_T$ isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p'$ placed in cohomological degree 0. With notation as in Remark 42.34.7 set

$$c^{(p)}(Q) = 1 + c_1(Q|_{X \times \{0\}}) + \ldots + c_{p-1}(Q|_{X \times \{0\}}) + c'_p(Q) + c'_{p+1}(Q) + \ldots$$

in $A^{(p)}(Z \to X)$ with $c'_i(Q)$ for $i \geq p$ as in Lemma 42.49.1. Similarly for $c^{(p')}(Q')$ and $c^{(p+p')}(Q \oplus Q')$. Then $c^{(p+p')}(Q \oplus Q') = c^{(p)}(Q)c^{(p')}(Q')$ in $A^{(p+p')}(Z \to X)$.

Proof. Recall that the image of $c'_i(Q)$ in $A^p(X)$ is equal to $c_i(Q|_{X \times \{0\}})$ for $i \geq p$ and similarly for Q' and $Q \oplus Q'$, see Lemma 42.49.1. Hence the equality in degrees $< p + p'$ follows from the additivity of Lemma 42.46.7.

Let's take $n \geq p + p'$. As in the proof of Lemma 42.49.1 let $E \subset W_\infty$ denote the inverse image of Z. Observe that we have the equality

$$c^{(p+p')}(Q|_E \oplus Q'|_E) = c^{(p)}(Q|_E)c^{(p')}(Q'|_E)$$

in $A^{(p+p')}(E \to W_\infty)$ by Lemma 42.47.8. Since by construction

$$c'_p(Q \oplus Q') = (E \to Z)_* \circ c'_p(Q|_E \oplus Q'|_E) \circ C$$

we conclude that suffices to show for all $i + j = n$ we have

$$(E \to Z)_* \circ c_i^{(p)}(Q|_E)c_j^{(p')}(Q'|_E) \circ C = c_i^{(p)}(Q)c_j^{(p')}(Q')$$

in $A^n(Z \to X)$ where the multiplication is the one from Remark 42.34.7 on both sides. There are three cases, depending on whether $i \geq p$, $j \geq p'$, or both.

Assume $i \geq p$ and $j \geq p'$. In this case the products are defined by inserting $(E \to W_\infty)_*$, resp. $(Z \to X)_*$ in between the two factors and taking compositions as bivariant classes, see Remark 42.34.8. In other words, we have to show

$$(E \to Z)_* \circ c'_i(Q|_E) \circ (E \to W_\infty)_* \circ c'_j(Q'|_E) \circ C = c'_i(Q) \circ (Z \to X)_* \circ c'_j(Q')$$

By Lemma 42.47.1 the left hand side is equal to

$$(E \to Z)_* \circ c'_i(Q|_E) \circ c_j(Q'|_{W_\infty}) \circ C$$

Since $c'_i(Q) = (E \to Z)_* \circ c'_i(Q|_E) \circ C$ the right hand side is equal to

$$(E \to Z)_* \circ c'_i(Q|_E) \circ C \circ (Z \to X)_* \circ c'_j(Q')$$

which is immediately seen to be equal to the above by Lemma 42.49.4.

Assume $i \geq p$ and $j < p$. Unwinding the products in this case we have to show

$$(E \to Z)_* \circ c'_i(Q|_E) \circ c_j(Q'|_{W_\infty}) \circ C = c'_i(Q) \circ c_j(Q'|_{X \times \{0\}})$$

Again using that $c'_i(Q) = (E \to Z)_* \circ c'_i(Q|_E) \circ C$ we see that it suffices to show $c_j(Q'|_{W_\infty}) \circ C = C \circ c_j(Q'|_{X \times \{0\}})$ which is part of Lemma 42.49.4.

Assume $i < p$ and $j \geq p'$. Unwinding the products in this case we have to show

$$(E \to Z)_* \circ c_i(Q|_E) \circ c'_j(Q'|_E) \circ C = c_i(Q|_{Z \times \{0\}}) \circ c'_j(Q')$$

However, since $c'_j(Q|_E)$ and $c'_j(Q')$ are bivariant classes, they commute with capping with Chern classes (Lemma 42.38.9). Hence it suffices to prove

$$(E \to Z)_* \circ c'_j(Q'|_E) \circ c_i(Q|_{W_\infty}) \circ C = c'_j(Q') \circ c_i(Q|_{X \times \{0\}})$$

which we reduces us to the case discussed in the preceding paragraph. □

0FAZ Lemma 42.49.8. In Lemma 42.49.1 assume $Q|_T$ is zero. Assume we have another perfect object $Q' \in D(\mathcal{O}_W)$ whose Chern classes are defined such that the restriction $Q'|_T$ is zero. In this case the classes $P'_p(Q), P'_p(Q'), P'_p(Q \oplus Q') \in A^p(Z \to X)$ constructed in Lemma 42.49.1 satisfy $P'_p(Q \oplus Q') = P'_p(Q) + P'_p(Q')$.

Proof. This follows immediately from the construction of these classes and Lemma 42.47.9. □

42.50. Localized Chern classes

0FB0 Outline of the construction. Let F be a field, let X be a variety over F, let E be a perfect object of $D(\mathcal{O}_X)$, and let $Z \subset X$ be a closed subscheme such that $E|_{X \setminus Z} = 0$. Then we want to construct elements

$$c_p(Z \to X, E) \in A^p(Z \to X)$$

We will do this by constructing a diagram

$$\begin{tikzcd} W \arrow[r, "q"'] \arrow[d, "f"'] & X \\ \mathbf{P}^1_F \end{tikzcd}$$

and a perfect object Q of $D(\mathcal{O}_W)$ such that

(1) f is flat, and f, q are proper; for $t \in \mathbf{P}^1_F$ denote W_t the fibre of f, $q_t : W_t \to X$ the restriction of q, and $Q_t = Q|_{W_t}$,
(2) $q_t : W_t \to X$ is an isomorphism and $Q_t = q_t^*E$ for $t \in \mathbf{A}^1_F$,
(3) $q_\infty : W_\infty \to X$ is an isomorphism over $X \setminus Z$,
(4) if $T \subset W_\infty$ is the closure of $q_\infty^{-1}(X \setminus Z)$ then $Q_\infty|_T$ is zero.

The idea is to think of this as a family $\{(W_t, Q_t)\}$ parametrized by $t \in \mathbf{P}^1$. For $t \neq \infty$ we see that $c_p(Q_t)$ is just $c_p(E)$ on the chow groups of $Q_t = X$. But for $t = \infty$ we see that $c_p(Q_\infty)$ sends classes on Q_∞ to classes supported on $E = q_\infty^{-1}(Z)$ since $Q_\infty|_T = 0$. We think of E as the exceptional locus of $q_\infty : W_\infty \to X$. Since any $\alpha \in \mathrm{CH}_*(X)$ gives rise to a "family" of cycles $\alpha_t \in \mathrm{CH}_*(W_t)$ it makes sense to define $c_p(Z \to X, E) \cap \alpha$ as the pushforward $(E \to Z)_*(c_p(Q_\infty) \cap \alpha_\infty)$.

To make this work there are two main ingredients: (1) the construction of W and Q is a sort of algebraic Macpherson's graph construction; it is done in More on Flatness, Section 38.44. (2) the construction of the actual class given W and Q is done in Section 42.49 relying on Sections 42.48 and 42.47.

0GUJ Situation 42.50.1. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $i : Z \to X$ be a closed immersion. Let $E \in D(\mathcal{O}_X)$ be an object. Let $p \geq 0$. Assume

(1) E is a perfect object of $D(\mathcal{O}_X)$,
(2) the restriction $E|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, and
(3) at least one[6] of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) there exists a locally bounded complex of finite locally free $\mathcal{O}_X$-modules representing E, or (d) there exists a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object on X' and the irreducible components of X' are quasi-compact.

0FB2 Lemma 42.50.2. In Situation 42.50.1 there exists a canonical bivariant class

$$P_p(Z \to X, E) \in A^p(Z \to X), \quad \text{resp.} \quad c_p(Z \to X, E) \in A^p(Z \to X)$$

with the property that

0FB1 (42.50.2.1) $$i_* \circ P_p(Z \to X, E) = P_p(E), \quad \text{resp.} \quad i_* \circ c_p(Z \to X, E) = c_p(E)$$

[6] Please ignore this technical condition on a first reading; see discussion in Remark 42.50.5.

as bivariant classes on X (with $\circ$ as in Lemma 42.33.4).

Proof. The construction of these bivariant classes is as follows. Let

$$b : W \longrightarrow \mathbf{P}^1_X \quad \text{and} \quad T \longrightarrow W_\infty \quad \text{and} \quad Q$$

be the blowing up, the perfect object Q in $D(\mathcal{O}_W)$, and the closed immersion constructed in More on Flatness, Section 38.44 and Lemma 38.44.1. Let $T' \subset T$ be the open and closed subscheme such that $Q|_{T'}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{T'}$-module of rank $< p$ sitting in cohomological degree 0. By condition (2) of Situation 42.50.1 the morphisms

$$T' \to T \to W_\infty \to X$$

are all isomorphisms of schemes over the open subscheme $X \setminus Z$ of X. Below we check the chern classes of Q are defined. Recalling that $Q|_{X \times \{0\}} \cong E$ by construction, we conclude that the bivariant class constructed in Lemma 42.49.1 using W, b, Q, T' gives us classes

$$P_p(Z \to X, E) = P'_p(Q) \in A^p(Z \to X)$$

and

$$c_p(Z \to X, E) = c'_p(Q) \in A^p(Z \to X)$$

satisfying (42.50.2.1).

In this paragraph we prove that the chern classes of Q are defined (Definition 42.46.3); we suggest the reader skip this. If assumption (3)(a) or (3)(b) of Situation 42.50.1 holds, i.e., if X has quasi-compact irreducible components, then the same is true for W (because $W \to X$ is proper). Hence we conclude that the chern classes of any perfect object of $D(\mathcal{O}_W)$ are defined by Lemma 42.46.4. If (3)(c) hold, i.e., if E can be represented by a locally bounded complex of finite locally free modules, then the object Q can be represented by a locally bounded complex of finite locally free $\mathcal{O}_W$-modules by part (5) of More on Flatness, Lemma 38.44.1. Hence the chern classes of Q are defined. Finally, assume (3)(d) holds, i.e., assume we have a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object E' on X' and the irreducible components of X' are quasi-compact. Let $b' : W' \to \mathbf{P}^1_{X'}$ and $Q' \in D(\mathcal{O}_{W'})$ be the morphism and perfect object constructed as in More on Flatness, Section 38.44 starting with the triple $(\mathbf{P}^1_{X'}, (\mathbf{P}^1_{X'})_\infty, L(p')^*E')$. By the discussion above we see that the chern classes of Q' are defined. Since b and b' were constructed via an application of More on Flatness, Lemma 38.43.6 it follows from More on Flatness, Lemma 38.43.8 that there exists a morphism $W \to W'$ such that $Q = L(W \to W')^*Q'$. Then it follows from Lemma 42.46.4 that the chern classes of Q are defined. $\square$

0FB5 Definition 42.50.3. With (S, δ), X, $E \in D(\mathcal{O}_X)$, and $i : Z \to X$ as in Situation 42.50.1.

(1) If the restriction $E|_{X \setminus Z}$ is zero, then for all $p \geq 0$ we define

$$P_p(Z \to X, E) \in A^p(Z \to X)$$

by the construction in Lemma 42.50.2 and we define the localized Chern character by the formula

$$ch(Z \to X, E) = \sum\nolimits_{p=0,1,2,\ldots} \frac{P_p(Z \to X, E)}{p!} \quad \text{in} \quad \prod\nolimits_{p \geq 0} A^p(Z \to X) \otimes \mathbf{Q}$$

(2) If the restriction $E|_{X \setminus Z}$ is isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, then we define the localized pth Chern class $c_p(Z \to X, E)$ by the construction in Lemma 42.50.2.

In the situation of the definition assume $E|_{X \setminus Z}$ is zero. Then, to be sure, we have the equality

$$i_* \circ ch(Z \to X, E) = ch(E)$$

in $A^*(X) \otimes \mathbf{Q}$ because we have shown the equality (42.50.2.1) above.

Here is an important sanity check.

0FB3 Lemma 42.50.4. In Situation 42.50.1 let $f : X' \to X$ be a morphism of schemes which is locally of finite type. Denote $E' = f^*E$ and $Z' = f^{-1}(Z)$. Then the bivariant class of Definition 42.50.3

$$P_p(Z' \to X', E') \in A^p(Z' \to X'), \quad \text{resp.} \quad c_p(Z' \to X', E') \in A^p(Z' \to X')$$

constructed as in Lemma 42.50.2 using X', Z', E' is the restriction (Remark 42.33.5) of the bivariant class $P_p(Z \to X, E) \in A^p(Z \to X)$, resp. $c_p(Z \to X, E) \in A^p(Z \to X)$.

Proof. Denote $p : \mathbf{P}^1_X \to X$ and $p' : \mathbf{P}^1_{X'} \to X'$ the structure morphisms. Recall that $b : W \to \mathbf{P}^1_X$ and $b' : W' \to \mathbf{P}^1_{X'}$ are the morphism constructed from the triples $(\mathbf{P}^1_X, (\mathbf{P}^1_X)\infty, p^*E)$ and $(\mathbf{P}^1_{X'}, (\mathbf{P}^1_{X'})\infty, (p')^*E')$ in More on Flatness, Lemma 38.43.6. Furthermore $Q = L\eta_{\mathcal{I}_\infty} p^*E$ and $Q = L\eta_{\mathcal{I}'_\infty}(p')^*E'$ where $\mathcal{I}_\infty \subset \mathcal{O}_W$ is the ideal sheaf of W_∞ and $\mathcal{I}'_\infty \subset \mathcal{O}_{W'}$ is the ideal sheaf of W'_∞. Next, $h : \mathbf{P}^1_{X'} \to \mathbf{P}^1_X$ is a morphism of schemes such that the pullback of the effective Cartier divisor $(\mathbf{P}^1_X)_\infty$ is the effective Cartier divisor $(\mathbf{P}^1_{X'})_\infty$ and such that $h^*p^*E = (p')^*E'$. By More on Flatness, Lemma 38.43.8 we obtain a commutative diagram

$$\begin{tikzcd} W' \arrow[r, "g"'] \arrow[dr, "b'"'] & \mathbf{P}^1_{X'} \times_{\mathbf{P}^1_X} W \arrow[r, "q"'] \arrow[d, "r"'] & W \arrow[d, "b"] \\ & \mathbf{P}^1_{X'} \arrow[r] & \mathbf{P}^1_X \end{tikzcd}$$

such that W' is the "strict transform" of $\mathbf{P}^1_{X'}$ with respect to b and such that $Q' = (q \circ g)^*Q$. Now recall that $P_p(Z \to X, E) = P'_p(Q)$, resp. $c_p(Z \to X, E) = c'_p(Q)$ where $P'_p(Q)$, resp. $c'_p(Q)$ are constructed in Lemma 42.49.1 using b, Q, T' where T' is a closed subscheme $T' \subset W_\infty$ with the following two properties: (a) T' contains all points of W_∞ lying over $X \setminus Z$, and (b) $Q|_{T'}$ is zero, resp. isomorphic to a finite locally free module of rank $< p$ placed in degree 0. In the construction of Lemma 42.49.1 we chose a particular closed subscheme T' with properties (a) and (b) but the precise choice of T' is immaterial, see Lemma 42.49.3.

Next, by Lemma 42.49.2 the restriction of the bivariant class $P_p(Z \to X, E) = P'_p(Q)$, resp. $c_p(Z \to X, E) = c_p(Q')$ to X' corresponds to the class $P'_p(q^*Q)$, resp. $c'_p(q^*Q)$ constructed as in Lemma 42.49.1 using $r : \mathbf{P}^1_{X'} \times_{\mathbf{P}^1_X} W \to \mathbf{P}^1_{X'}$, the complex q^*Q, and the inverse image $q^{-1}(T')$.

Now by the second statement of Lemma 42.49.3 we have $P'_p(Q') = P'_p(q^*Q)$, resp. $c'_p(q^*Q) = c'_p(Q')$. Since $P_p(Z' \to X', E') = P'_p(Q')$, resp. $c_p(Z' \to X', E') = c'_p(Q')$ we conclude that the lemma is true. $\square$

0GUK Remark 42.50.5. In Situation 42.50.1 it would have been more natural to replace assumption (3) with the assumption: “the chern classes of E are defined”. In fact, combining Lemmas 42.50.2 and 42.50.4 with Lemma 42.35.6 it is easy to extend the definition to this (slightly) more general case. If we ever need this we will do so here.

0FB4 Lemma 42.50.6. In Situation 42.50.1 we have

$$P_p(Z \to X, E) \cap i_*\alpha = P_p(E|_Z) \cap \alpha, \quad \text{resp.} \quad c_p(Z \to X, E) \cap i_*\alpha = c_p(E|_Z) \cap \alpha$$

in $\text{CH}_*(Z)$ for any $\alpha \in \text{CH}_*(Z)$.

Proof. We only prove the second equality and we omit the proof of the first. Since $c_p(Z \to X, E)$ is a bivariant class and since the base change of $Z \to X$ by $Z \to X$ is $\text{id} : Z \to Z$ we have $c_p(Z \to X, E) \cap i_*\alpha = c_p(Z \to X, E) \cap \alpha$. By Lemma 42.50.4 the restriction of $c_p(Z \to X, E)$ to Z (!) is the localized Chern class for $\text{id} : Z \to Z$ and $E|_Z$. Thus the result follows from (42.50.2.1) with $X = Z$. □

0FB6 Lemma 42.50.7. In Situation 42.50.1 if $\alpha \in \text{CH}_k(X)$ has support disjoint from Z, then $P_p(Z \to X, E) \cap \alpha = 0$, resp. $c_p(Z \to X, E) \cap \alpha = 0$.

Proof. This is immediate from the construction of the localized Chern classes. It also follows from the fact that we can compute $c_p(Z \to X, E) \cap \alpha$ by first restricting $c_p(Z \to X, E)$ to the support of α, and then using Lemma 42.50.4 to see that this restriction is zero. □

0FB7 Lemma 42.50.8. In Situation 42.50.1 assume $Z \subset Z' \subset X$ where Z' is a closed subscheme of X. Then $P_p(Z' \to X, E) = (Z \to Z')_* \circ P_p(Z \to X, E)$, resp. $c_p(Z' \to X, E) = (Z \to Z')_* \circ c_p(Z \to X, E)$ (with $\circ$ as in Lemma 42.33.4).

Proof. The construction of $P_p(Z' \to X, E)$, resp. $c_p(Z' \to X, E)$ in Lemma 42.50.2 uses the exact same morphism $b : W \to \mathbf{P}^1_X$ and perfect object Q of $D(\mathcal{O}_W)$. Then we can use Lemma 42.47.5 to conclude. Some details omitted. □

0FB8 Lemma 42.50.9. In Lemma 42.47.1 say E_2 is the restriction of a perfect $E \in D(\mathcal{O}_X)$ whose restriction to X_1 is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0. Then the class $P'_p(E_2)$, resp. $c'_p(E_2)$ of Lemma 42.47.1 agrees with $P_p(X_2 \to X, E)$, resp. $c_p(X_2 \to X, E)$ of Definition 42.50.3 provided E satisfies assumption (3) of Situation 42.50.1.

Proof. The assumptions on E imply that there is an open $U \subset X$ containing X_1 such that $E|_U$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_U$-module of rank $< p$. See More on Algebra, Lemma 15.75.6. Let $Z \subset X$ be the complement of U in X endowed with the reduced induced closed subscheme structure. Then $P_p(X_2 \to X, E) = (Z \to X_2)_* \circ P_p(Z \to X, E)$, resp. $c_p(X_2 \to X, E) = (Z \to X_2)_* \circ c_p(Z \to X, E)$ by Lemma 42.50.8. Now we can prove that $P_p(X_2 \to X, E)$, resp. $c_p(X_2 \to X, E)$ satisfies the characterization of $P'_p(E_2)$, resp. $c'_p(E_2)$ given in Lemma 42.47.1. Namely, by the relation $P_p(X_2 \to X, E) = (Z \to X_2)_* \circ P_p(Z \to X, E)$, resp. $c_p(X_2 \to X, E) = (Z \to X_2)_* \circ c_p(Z \to X, E)$ just proven and the fact that $X_1 \cap Z = \emptyset$, the composition $P_p(X_2 \to X, E) \circ i_{1,*}$, resp. $c_p(X_2 \to X, E) \circ i_{1,*}$ is zero by Lemma 42.50.7. On the other hand, $P_p(X_2 \to X, E) \circ i_{2,*} = P_p(E_2)$, resp. $c_p(X_2 \to X, E) \circ i_{2,*} = c_p(E_2)$ by Lemma 42.50.6. □

42.51. Two technical lemmas

0FE6 In this section we develop some additional tools to allow us to work more comfortably with localized Chern classes. The following lemma is a more precise version of something we've already encountered in the proofs of Lemmas 42.49.6 and 42.49.7.

0FE7 **Lemma 42.51.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $b : W \longrightarrow \mathbf{P}^1_X$ be a proper morphism of schemes. Let $n \geq 1$. For $i = 1, \ldots, n$ let $Z_i \subset X$ be a closed subscheme, let $Q_i \in D(\mathcal{O}_W)$ be a perfect object, let $p_i \geq 0$ be an integer, and let $T_i \subset W_\infty$, $i = 1, \ldots, n$ be closed. Denote $W_i = b^{-1}(\mathbf{P}^1_{Z_i})$. Assume

(1) for $i = 1, \ldots, n$ the assumption of Lemma 42.49.1 hold for b, Z_i, Q_i, T_i, p_i,
(2) $Q_i|_{W \setminus W_i}$ is zero, resp. isomorphic to a finite locally free module of rank $< p_i$ placed in cohomological degree 0,
(3) Q_i on W satisfies assumption (3) of Situation 42.50.1.

Then $P'_{p_n}(Q_n) \circ \ldots \circ P'_{p_1}(Q_1)$ is equal to

$$(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ P'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \ldots \circ P'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

in $A^{p_n + \ldots + p_1}(Z_n \cap \ldots \cap Z_1 \to X)$, resp. $c'_{p_n}(Q_n) \circ \ldots \circ c'_{p_1}(Q_1)$ is equal to

$$(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

in $A^{p_n + \ldots + p_1}(Z_n \cap \ldots \cap Z_1 \to X)$.

Proof. Let us prove the statement on Chern classes by induction on n; the statement on $P_p(-)$ is proved in the exact same manner. The case $n = 1$ is the construction of $c'_{p_1}(Q_1)$ because $W_{1,\infty}$ is the inverse image of Z_1 in W_∞. For $n > 1$ we have by induction that $c'_{p_n}(Q_n) \circ \ldots \circ c'_{p_1}(Q_1)$ is equal to

$$c'_{p_n}(Q_n) \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_* \circ c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

By Lemma 42.49.2 the restriction of $c'_{p_n}(Q_n)$ to $Z_{n-1} \cap \ldots \cap Z_1$ is computed by the closed subset $Z_n \cap \ldots \cap Z_1$, the morphism $b' : W_{n-1} \cap \ldots \cap W_1 \to \mathbf{P}^1_{Z_{n-1} \cap \ldots \cap Z_1}$ and the restriction of Q_n to $W_{n-1} \cap \ldots \cap W_1$. Observe that $(b')^{-1}(Z_n) = W_n \cap \ldots \cap W_1$ and that $(W_n \cap \ldots \cap W_1)_\infty = W_{n,\infty} \cap \ldots \cap W_{1,\infty}$. Denote $C_{n-1} \in A^0(W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)$ the class of Lemma 42.48.1. We conclude the restriction of $c'_{p_n}(Q_n)$ to $Z_{n-1} \cap \ldots \cap Z_1$ is

$$\begin{aligned}
&(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{(W_n \cap \ldots \cap W_1)_\infty}) \circ C_{n-1} \\
&= (W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ C_{n-1}
\end{aligned}$$

where the equality follows from Lemma 42.47.3 (we omit writing the restriction on the right). Hence the above becomes

$$\begin{aligned}
(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \\
C_{n-1} \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_* \\
\circ c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C
\end{aligned}$$

By Lemma 42.48.4 we know that the composition $C_{n-1} \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_*$ is the identity on elements in the image of the gysin map

$$(W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to W_{n-1} \cap \ldots \cap W_1)^*$$

Thus it suffices to show that any element in the image of $c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$ is in the image of the gysin map. We may write

$$c'_{p_i}(Q_i|_{W_{i,\infty}}) = \text{restriction of } c_{p_i}(W_i \to W, Q_i) \text{ to } W_{i,\infty}$$

by Lemma 42.50.9 and assumptions (2) and (3) on Q_i in the statement of the lemma. Thus, if $\beta \in \mathrm{CH}_{k+1}(W)$ restricts to the flat pullback of α on $b^{-1}(\mathbf{A}^1_X)$, then

$$\begin{aligned}
& c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \cap \ldots \cap c'_{p_1}(Q_1|_{W_{1,\infty}}) \cap C \cap \alpha \\
& = c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \cap \ldots \cap c'_{p_1}(Q_1|_{W_{1,\infty}}) \cap i_\infty^*\beta \\
& = c_{p_{n-1}}(W_{n-1} \to W, Q_{n-1}) \cap \ldots \cap c_{p_{n-1}}(W_1 \to W, Q_1) \cap i_\infty^*\beta \\
& = (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to W_{n-1} \cap \ldots \cap W_1)^* \left(c_{p_{n-1}}(W_{n-1} \to W, Q_{n-1}) \cap \ldots \cap c_{p_1}(W_1 \to W, Q_1) \cap \beta\right)
\end{aligned}$$

as desired. Namely, for the last equality we use that $c_{p_i}(W_i \to W, Q_i)$ is a bivariant class and hence commutes with i_∞^* by definition. $\square$

The following lemma gives us a tremendous amount of flexibility if we want to compute the localized Chern classes of a complex.

0FE8 Lemma 42.51.2. Assume $(S, \delta), X, Z, b : W \to \mathbf{P}^1_X, Q, T, p$ satisfy the assumptions of Lemma 42.49.1. Let $F \in D(\mathcal{O}_X)$ be a perfect object such that

1. the restriction of Q to $b^{-1}(\mathbf{A}^1_X)$ is isomorphic to the pullback of F,
2. $F|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, and
3. Q on W and F on X satisfy assumption (3) of Situation 42.50.1.

Then the class $P'_p(Q)$, resp. $c'_p(Q)$ in $A^p(Z \to X)$ constructed in Lemma 42.49.1 is equal to $P_p(Z \to X, F)$, resp. $c_p(Z \to X, F)$ from Definition 42.50.3.

Proof. The assumptions are preserved by base change with a morphism $X' \to X$ locally of finite type. Hence it suffices to show that $P_p(Z \to X, F) \cap \alpha = P'_p(Q) \cap \alpha$, resp. $c_p(Z \to X, F) \cap \alpha = c'_p(Q) \cap \alpha$ for any $\alpha \in \mathrm{CH}_k(X)$. Choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is equal to the flat pullback of α as in the construction of C in Lemma 42.48.1. Denote $W' = b^{-1}(Z)$ and denote $E = W'_\infty \subset W_\infty$ the inverse image of Z by $W_\infty \to X$. The lemma follows from the following sequence of equalities (the case of P_p is similar)

$$\begin{aligned}
c'_p(Q) \cap \alpha &= (E \to Z)_*(c'_p(Q|_E) \cap i_\infty^*\beta) \\
&= (E \to Z)_*(c_p(E \to W_\infty, Q|_{W_\infty}) \cap i_\infty^*\beta) \\
&= (W'_\infty \to Z)_*(c_p(W' \to W, Q) \cap i_\infty^*\beta) \\
&= (W'_\infty \to Z)_*((i'_\infty)^*(c_p(W' \to W, Q) \cap \beta)) \\
&= (W'_\infty \to Z)_*((i'_\infty)^*(c_p(Z' \to X, F) \cap \beta)) \\
&= (W'_0 \to Z)_*((i'_0)^*(c_p(Z' \to X, F) \cap \beta)) \\
&= (W'_0 \to Z)_*(c_p(Z' \to X, F) \cap i_0^*\beta)) \\
&= c_p(Z \to X, F) \cap \alpha
\end{aligned}$$

The first equality is the construction of $c'_p(Q)$ in Lemma 42.49.1. The second is Lemma 42.50.9. The base change of $W' \to W$ by $W_\infty \to W$ is the morphism $E = W'_\infty \to W_\infty$. Hence the third equality holds by Lemma 42.50.4. The fourth equality, in which $i'_\infty : W'_\infty \to W'$ is the inclusion morphism, follows from the fact that $c_p(W' \to W, Q)$ is a bivariant class. For the fith equality, observe that $c_p(W' \to W, Q)$ and $c_p(Z' \to X, F)$ restrict to the same bivariant class in $A^p((b')^{-1} \to b^{-1}(\mathbf{A}^1_X))$ by assumption (1) of the lemma which says that Q and F restrict to the same object of $D(\mathcal{O}_{b^{-1}(\mathbf{A}^1_X)})$; use Lemma 42.50.4. Since $(i'_\infty)^*$ annihilates cycles supported on W'_∞ (see Remark 42.29.6) we conclude the fifth equality is true. The sixth equality holds because W'_∞ and W'_0 are the pullbacks of the rationally equivalent effective Cartier divisors D_0, D_∞ in $\mathbf{P}^1_Z$ and hence $i_\infty^*\beta$ and $i_0^*\beta$ map to the same cycle class on W'; namely, both represent the class $c_1(\mathcal{O}_{\mathbf{P}^1_Z}(1)) \cap c_p(Z \to X, F) \cap \beta$ by Lemma 42.29.4. The seventh equality holds because $c_p(Z \to X, F)$ is a bivariant class. By construction $W'_0 = Z$ and $i_0^*\beta = \alpha$ which explains why the final equality holds. $\square$

42.52. Properties of localized Chern classes

0FB9 The main results in this section are additivity and multiplicativity for localized Chern classes.

0FBA **Lemma 42.52.1.** In Situation 42.50.1 assume $E|_{X \setminus Z}$ is zero. Then

$$P_1(Z \to X, E) = c_1(Z \to X, E),$$
$$P_2(Z \to X, E) = c_1(Z \to X, E)^2 - 2c_2(Z \to X, E),$$
$$P_3(Z \to X, E) = c_1(Z \to X, E)^3 - 3c_1(Z \to X, E)c_2(Z \to X, E) + 3c_3(Z \to X, E),$$

and so on where the products are taken in the algebra $A^{(1)}(Z \to X)$ of Remark 42.34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c_p(Z \to X, E)$ are defined for all $p \geq 1$. The result itself follows immediately from the more general Lemma 42.49.6 as the localized Chern classes where defined using the procedure of Lemma 42.49.1 in Section 42.50. $\square$

0FBB **Lemma 42.52.2.** In Situation 42.50.1 let $Y \to X$ be locally of finite type and $c \in A^*(Y \to X)$. Then

$$P_p(Z \to X, E) \circ c = c \circ P_p(Z \to X, E),$$

respectively

$$c_p(Z \to X, E) \circ c = c \circ c_p(Z \to X, E)$$

in $A^*(Y \times_X Z \to X)$.

Proof. This follows from Lemma 42.49.5. More precisely, let

$$b : W \to \mathbf{P}^1_X \quad \text{and} \quad Q \quad \text{and} \quad T' \subset T \subset W_\infty$$

be as in the proof of Lemma 42.50.2. By definition $c_p(Z \to X, E) = c'_p(Q)$ as bivariant operations where the right hand side is the bivariant class constructed in Lemma 42.49.1 using W, b, Q, T'. By Lemma 42.49.5 we have $P'_p(Q) \circ c = c \circ P'_p(Q)$, resp. $c'_p(Q) \circ c = c \circ c'_p(Q)$ in $A^*(Y \times_X Z \to X)$ and we conclude. $\square$

0FBC **Remark 42.52.3.** In Situation 42.50.1 it is convenient to define

$$c^{(p)}(Z \to X, E) = 1 + c_1(E) + \ldots + c_{p-1}(E) + c_p(Z \to X, E) + c_{p+1}(Z \to X, E) + \ldots$$

as an element of the algebra $A^{(p)}(Z \to X)$ considered in Remark 42.34.7.

0FBD Lemma 42.52.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \to X$ be a closed immersion. Let

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. Assume

(1) the restrictions $E_1|_{X \setminus Z}$ and $E_3|_{X \setminus Z}$ are isomorphic to finite locally free $\mathcal{O}_{X \setminus Z}$-modules of rank $< p_1$ and $< p_3$ placed in degree 0, and
(2) at least one of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) $E_3 \to E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules, or (d) there exists an envelope $f : Y \to X$ such that $Lf^*E_3 \to Lf^*E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules.

With notation as in Remark 42.52.3 we have

$$c^{(p_1+p_3)}(Z \to X, E_2) = c^{(p_1)}(Z \to X, E_1)c^{(p_3)}(Z \to X, E_3)$$

in $A^{(p_1+p_3)}(Z \to X)$.

Proof. Observe that the assumptions imply that $E_2|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p_1 + p_3$. Thus the statement makes sense.

Let $f : Y \to X$ be an envelope. Expanding the left and right hand sides of the formula in the statement of the lemma we see that we have to prove some equalities of classes in $A^*(X)$ and in $A^*(Z \to X)$. By the uniqueness in Lemma 42.35.6 it suffices to prove the corresponding relations in $A^*(Y)$ and $A^*(Z \to Y)$. Since moreover the construction of the classes involved is compatible with base change (Lemma 42.50.4) we may replace X by Y and the distinguished triangle by its pullback.

In the proof of Lemma 42.46.7 we have seen that conditions (2)(a), (2)(b), and (2)(c) imply condition (2)(d). Combined with the discussion in the previous paragraph we reduce to the case discussed in the next paragraph.

Let $\varphi^\bullet : \mathcal{E}_3^\bullet[-1] \to \mathcal{E}_1^\bullet$ be a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules representing the map $E_3[-1] \to E_1$ in the derived category. Consider the scheme $X' = \mathbf{A}^1 \times X$ with projection $g : X' \to X$. Let $Z' = g^{-1}(Z) = \mathbf{A}^1 \times Z$. Denote t the coordinate on $\mathbf{A}^1$. Consider the cone $\mathcal{C}^\bullet$ of the map of complexes

$$tg^*\varphi^\bullet : g^*\mathcal{E}_3^\bullet[-1] \longrightarrow g^*\mathcal{E}_1^\bullet$$

over X'. We obtain a distinguished triangle

$$g^*\mathcal{E}_1^\bullet \to \mathcal{C}^\bullet \to g^*\mathcal{E}_3^\bullet \to g^*\mathcal{E}_1^\bullet[1]$$

where the first three terms form a termwise split short exact sequence of complexes. Clearly $\mathcal{C}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_{X'}$-modules whose restriction to $X' \setminus Z'$ is isomorphic to a finite locally free $\mathcal{O}_{X' \setminus Z'}$-module of rank $< p_1 + p_3$ placed in degree 0. Thus we have the localized Chern classes

$$c_p(Z' \to X', \mathcal{C}^\bullet) \in A^p(Z' \to X')$$

for $p \geq p_1 + p_3$. For any $\alpha \in \mathrm{CH}_k(X)$ consider

$$c_p(Z' \to X', \mathcal{C}^\bullet) \cap g^*\alpha \in \mathrm{CH}_{k+1-p}(\mathbf{A}^1 \times X)$$

If we restrict to $t = 0$, then the map $tg^*\varphi^\bullet$ restricts to zero and $\mathcal{C}^\bullet|_{t=0}$ is the direct sum of $\mathcal{E}_1^\bullet$ and $\mathcal{E}_3^\bullet$. By compatibility of localized Chern classes with base change (Lemma 42.50.4) we conclude that

$$i_0^* \circ c^{(p_1+p_3)}(Z' \to X', \mathcal{C}^\bullet) \circ g^* = c^{(p_1+p_2)}(Z \to X, E_1 \oplus E_3)$$

in $A^{(p_1+p_3)}(Z \to X)$. On the other hand, if we restrict to $t = 1$, then the map $tg^*\varphi^\bullet$ restricts to φ and $\mathcal{C}^\bullet|_{t=1}$ is a bounded complex of finite locally free modules representing E_2. We conclude that

$$i_1^* \circ c^{(p_1+p_3)}(Z' \to X', \mathcal{C}^\bullet) \circ g^* = c^{(p_1+p_2)}(Z \to X, E_2)$$

in $A^{(p_1+p_3)}(Z \to X)$. Since $i_0^* = i_1^*$ by definition of rational equivalence (more precisely this follows from the formulae in Lemma 42.32.4) we conclude that

$$c^{(p_1+p_2)}(Z \to X, E_2) = c^{(p_1+p_2)}(Z \to X, E_1 \oplus E_3)$$

This reduces us to the case discussed in the next paragraph.

Assume $E_2 = E_1 \oplus E_3$ and the triples (X, Z, E_i) are as in Situation 42.50.1. For $i = 1, 3$ let

$$b_i : W_i \to \mathbf{P}^1_X \quad \text{and} \quad Q_i \quad \text{and} \quad T'_i \subset T_i \subset W_{i,\infty}$$

be as in the proof of Lemma 42.50.2. By definition

$$c_p(Z \to X, E_i) = c'_p(Q_i)$$

where the right hand side is the bivariant class constructed in Lemma 42.49.1 using W_i, b_i, Q_i, T'_i. Set $W = W_1 \times_{b_1, \mathbf{P}^1_X, b_2} W_2$ and consider the cartesian diagram

$$\begin{tikzcd} W \arrow[r, "g_3"] \arrow[d, "g_1"'] \arrow[dr, "b"] & W_3 \arrow[d, "b_3"] \\ W_1 \arrow[r, "b_1"] & \mathbf{P}^1_X \end{tikzcd}$$

Of course $b^{-1}(\mathbf{A}^1)$ maps isomorphically to $\mathbf{A}^1_X$. Observe that $T' = g_1^{-1}(T'_1) \cap g_2^{-1}(T'_2)$ still contains all the points of W_∞ lying over $X \setminus Z$. By Lemma 42.49.3 we may use W, b, $g_i^*\mathcal{Q}_i$, and T' to construct $c_p(Z \to X, E_i)$ for $i = 1, 3$. Also, by the stronger independence given in Lemma 42.51.2 we may use W, b, $g_1^*Q_1 \oplus g_3^*Q_3$, and T' to compute the classes $c_p(Z \to X, E_2)$. Thus the desired equality follows from Lemma 42.49.7. $\square$

0FBE Lemma 42.52.5. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \to X$ be a closed immersion. Let

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. Assume

(1) the restrictions $E_1|_{X \setminus Z}$ and $E_3|_{X \setminus Z}$ are zero, and
(2) at least one of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) $E_3 \to E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules, or (d) there exists an envelope $f : Y \to X$ such that $Lf^*E_3 \to Lf^*E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules.

Then we have

$$P_p(Z \to X, E_2) = P_p(Z \to X, E_1) + P_p(Z \to X, E_3)$$

for all $p \in \mathbf{Z}$ and consequently $ch(Z \to X, E_2) = ch(Z \to X, E_1) + ch(Z \to X, E_3)$.

Proof. The proof is exactly the same as the proof of Lemma 42.52.4 except it uses Lemma 42.49.8 at the very end. For $p > 0$ we can deduce this lemma from Lemma 42.52.4 with $p_1 = p_3 = 1$ and the relationship between $P_p(Z \to X, E)$ and $c_p(Z \to X, E)$ given in Lemma 42.52.1. The case $p = 0$ can be shown directly (it is only interesting if X has a connected component entirely contained in Z). $\square$

0FBF Lemma 42.52.6. In Situation 42.7.1 let X be locally of finite type over S. Let $Z_i \subset X$, $i = 1, 2$ be closed subschemes. Let F_i, $i = 1, 2$ be perfect objects of $D(\mathcal{O}_X)$. Assume for $i = 1, 2$ that $F_i|_{X \setminus Z_i}$ is zero[7] and that F_i on X satisfies assumption (3) of Situation 42.50.1. Denote $r_i = P_0(Z_i \to X, F_i) \in A^0(Z_i \to X)$. Then we have

$$c_1(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = r_1 c_1(Z_2 \to X, F_2) + r_2 c_1(Z_1 \to X, F_1)$$

in $A^1(Z_1 \cap Z_2 \to X)$ and

$$\begin{aligned} c_2(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = {} & r_1 c_2(Z_2 \to X, F_2) + r_2 c_2(Z_1 \to X, F_1) + \\ & \binom{r_1}{2} c_1(Z_2 \to X, F_2)^2 + \\ & (r_1 r_2 - 1) c_1(Z_2 \to X, F_2) c_1(Z_1 \to X, F_1) + \\ & \binom{r_2}{2} c_1(Z_1 \to X, F_1)^2 \end{aligned}$$

in $A^2(Z_1 \cap Z_2 \to X)$ and so on for higher Chern classes. Similarly, we have

$$ch(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = ch(Z_1 \to X, F_1) ch(Z_2 \to X, F_2)$$

in $\prod_{p \geq 0} A^p(Z_1 \cap Z_2 \to X) \otimes \mathbf{Q}$. More precisely, we have

$$P_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = \sum\nolimits_{p_1 + p_2 = p} \binom{p}{p_1} P_{p_1}(Z_1 \to X, F_1) P_{p_2}(Z_2 \to X, F_2)$$

in $A^p(Z_1 \cap Z_2 \to X)$.

Proof. Choose proper morphisms $b_i : W_i \to \mathbf{P}^1_X$ and $Q_i \in D(\mathcal{O}_{W_i})$ as well as closed subschemes $T_i \subset W_{i,\infty}$ as in the construction of the localized Chern classes for F_i or more generally as in Lemma 42.51.2. Choose a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g_2"] \arrow[d, "g_1"] \arrow[dr, "b"] & W_2 \arrow[d, "b_2"] \\ W_1 \arrow[r, "b_1"] & \mathbf{P}^1_X \end{tikzcd}$$

where all morphisms are proper and isomorphisms over $\mathbf{A}^1_X$. For example, we can take W to be the closure of the graph of the isomorphism between $b_1^{-1}(\mathbf{A}^1_X)$ and $b_2^{-1}(\mathbf{A}^1_X)$. By Lemma 42.51.2 we may work with W, $b = b_i \circ g_i$, $Lg_i^* Q_i$, and $g_i^{-1}(T_i)$

[7] Presumably there is a variant of this lemma where we only assume $F_i|_{X \setminus Z_i}$ is isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z_i}$-module of rank $< p_i$.

to construct the localized Chern classes $c_p(Z_i \to X, F_i)$. Thus we reduce to the situation described in the next paragraph.

Assume we have

(1) a proper morphism $b : W \to \mathbf{P}^1_X$ which is an isomorphism over $\mathbf{A}^1_X$,
(2) $E_i \subset W_\infty$ is the inverse image of Z_i,
(3) perfect objects $Q_i \in D(\mathcal{O}_W)$ whose Chern classes are defined, such that
   (a) the restriction of Q_i to $b^{-1}(\mathbf{A}^1_X)$ is the pullback of F_i, and
   (b) there exists a closed subscheme $T_i \subset W_\infty$ containing all points of W_∞ lying over $X \setminus Z_i$ such that $Q_i|_{T_i}$ is zero.

By Lemma 42.51.2 we have

$$c_p(Z_i \to X, F_i) = c'_p(Q_i) = (E_i \to Z_i)_* \circ c'_p(Q_i|_{E_i}) \circ C$$

and

$$P_p(Z_i \to X, F_i) = P'_p(Q_i) = (E_i \to Z_i)_* \circ P'_p(Q_i|_{E_i}) \circ C$$

for $i = 1, 2$. Next, we observe that $Q = Q_1 \otimes^{\mathbf{L}}_{\mathcal{O}_W} Q_2$ satisfies (3)(a) and (3)(b) for $F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2$ and $T_1 \cup T_2$. Hence we see that

$$c_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = (E_1 \cap E_2 \to Z_1 \cap Z_2)_* \circ c'_p(Q|_{E_1 \cap E_2}) \circ C$$

and

$$P_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = (E_1 \cap E_2 \to Z_1 \cap Z_2)_* \circ P'_p(Q|_{E_1 \cap E_2}) \circ C$$

by the same lemma. By Lemma 42.47.11 the classes $c'_p(Q|_{E_1 \cap E_2})$ and $P'_p(Q|_{E_1 \cap E_2})$ can be expanded in the correct manner in terms of the classes $c'_p(Q_i|_{E_i})$ and $P'_p(Q_i|_{E_i})$. Then finally Lemma 42.51.1 tells us that polynomials in $c'_p(Q_i|_{E_i})$ and $P'_p(Q_i|_{E_i})$ agree with the corresponding polynomials in $c'_p(Q_i)$ and $P'_p(Q_i)$ as desired. $\square$

42.53. Blowing up at infinity

0FBG Let X be a scheme. Let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Denote $X' \to X$ the blowing up with center Z. Let $b : W \to \mathbf{P}^1_X$ be the blowing up with center $\infty(Z)$. Denote $E \subset W$ the exceptional divisor. There is a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "b"] \\ X \arrow[r, "\infty"] & \mathbf{P}^1_X \end{tikzcd}$$

whose horizontal arrows are closed immersion (Divisors, Lemma 31.33.2). Denote $E \subset W$ the exceptional divisor and $W_\infty \subset W$ the inverse image of $(\mathbf{P}^1_X)_\infty$. Then the following are true

(1) b is an isomorphism over $\mathbf{A}^1_X \cup \mathbf{P}^1_{X \setminus Z}$,
(2) X' is an effective Cartier divisor on W,
(3) $X' \cap E$ is the exceptional divisor of $X' \to X$,
(4) $W_\infty = X' + E$ as effective Cartier divisors on W,
(5) $E = \underline{\text{Proj}}_Z(\mathcal{C}_{Z/X,*}[S])$ where S is a variable placed in degree 1,
(6) $X' \cap E = \underline{\text{Proj}}_Z(\mathcal{C}_{Z/X,*})$,

0FBH (7) $E \setminus X' = E \setminus (X' \cap E) = \underline{\text{Spec}}_Z(\mathcal{C}_{Z/X,*}) = C_Z X$, 0FE9 (8) there is a closed immersion $\mathbf{P}^1_Z \to W$ whose composition with b is the inclusion morphism $\mathbf{P}^1_Z \to \mathbf{P}^1_X$ and whose base change by ∞ is the composition $Z \to C_Z X \to E \to W_\infty$ where the first arrow is the vertex of the cone.

We recall that $\mathcal{C}_{Z/X,*}$ is the conormal algebra of Z in X, see Divisors, Definition 31.19.1 and that $C_Z X$ is the normal cone of Z in X, see Divisors, Definition 31.19.5.

We now give the proof of the numbered assertions above. We strongly urge the reader to work through some examples instead of reading the proofs.

Part (1) follows from the corresponding assertion of Divisors, Lemma 31.32.4. Observe that $E \subset W$ is an effective Cartier divisor by the same lemma.

Observe that W_∞ is an effective Cartier divisor by Divisors, Lemma 31.32.11. Since $E \subset W_\infty$ we can write $W_\infty = D + E$ for some effective Cartier divisor D, see Divisors, Lemma 31.13.8. We will see below that $D = X'$ which will prove (2) and (4).

Since X' is the strict transform of the closed immersion $\infty : X \to \mathbf{P}^1_X$ (see above) it follows that the exceptional divisor of $X' \to X$ is equal to the intersection $X' \cap E$ (for example because both are cut out by the pullback of the ideal sheaf of Z to X'). This proves (3).

The intersection of $\infty(Z)$ with $\mathbf{P}^1_Z$ is the effective Cartier divisor $(\mathbf{P}^1_Z)_\infty$ hence the strict transform of $\mathbf{P}^1_Z$ by the blowing up b maps isomorphically to $\mathbf{P}^1_Z$ (see Divisors, Lemmas 31.33.2 and 31.32.7). This gives us the morphism $\mathbf{P}^1_Z \to W$ mentioned in (8). It is a closed immersion as b is separated, see Schemes, Lemma 26.21.11.

Suppose that $\text{Spec}(A) \subset X$ is an affine open and that $Z \cap \text{Spec}(A)$ corresponds to the finitely generated ideal $I \subset A$. An affine neighbourhood of $\infty(Z \cap \text{Spec}(A))$ is the affine space over A with coordinate $s = T_0/T_1$. Denote $J = (I, s) \subset A[s]$ the ideal generated by I and s. Let $B = A[s] \oplus J \oplus J^2 \oplus \ldots$ be the Rees algebra of $(A[s], J)$. Observe that

$$J^n = I^n \oplus sI^{n-1} \oplus s^2 I^{n-2} \ldots \oplus s^n A \oplus s^{n+1} A \oplus \ldots$$

as an A-submodule of $A[s]$ for all $n \geq 0$. Consider the open subscheme

$$\text{Proj}(B) = \text{Proj}(A[s] \oplus J \oplus J^2 \oplus \ldots) \subset W$$

Finally, denote S the element $s \in J$ viewed as a degree 1 element of B.

Since formation of Proj commutes with base change (Constructions, Lemma 27.11.6) we see that

$$E = \text{Proj}(B \otimes_{A[s]} A/I) = \text{Proj}((A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots)[S])$$

The verification that $B \otimes_{A[s]} A/I = \bigoplus J^n/J^{n+1}$ is as given follows immediately from our description of the powers J^n above. This proves (5) because the conormal algebra of $Z \cap \text{Spec}(A)$ in $\text{Spec}(A)$ corresponds to the graded A-algebra $A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$ by Divisors, Lemma 31.19.2.

Recall that $\text{Proj}(B)$ is covered by the affine opens $D_+(S)$ and $D_+(f^{(1)})$ for $f \in I$ which are the spectra of affine blowup algebras $A[s][\frac{J}{s}]$ and $A[s][\frac{J}{f}]$, see Divisors, Lemma 31.32.2 and Algebra, Definition 10.70.1. We will describe each of these affine opens and this will finish the proof.

The open $D_+(S)$, i.e., the spectrum of $A[s][\frac{J}{s}]$. It follows from the description of the powers of J above that

$$A[s][\tfrac{J}{s}] = \sum s^{-n} I^n[s] \subset A[s, s^{-1}]$$

The element s is a nonzerodivisor in this ring, defines the exceptional divisor E as well as W_∞. Hence $D \cap D_+(S) = \emptyset$. Finally, the quotient of $A[s][\frac{J}{s}]$ by s is the conormal algebra

$$A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

This proves (7).

The open $D_+(f^{(1)})$, i.e., the spectrum of $A[s][\frac{J}{f}]$. It follows from the description of the powers of J above that

$$A[s][\tfrac{J}{f}] = A[\tfrac{I}{f}][\tfrac{s}{f}]$$

where $\frac{s}{f}$ is a variable. The element f is a nonzerodivisor in this ring whose zero scheme defines the exceptional divisor E. Since s defines W_∞ and $s = f \cdot \frac{s}{f}$ we conclude that $\frac{s}{f}$ defines the divisor D constructed above. Then we see that

$$D \cap D_+(f^{(1)}) = \text{Spec}(A[\tfrac{I}{f}])$$

which is the corresponding open of the blowup X' over $\text{Spec}(A)$. Namely, the surjective graded $A[s]$-algebra map $B \to A \oplus I \oplus I^2 \oplus \ldots$ to the Rees algebra of (A, I) corresponds to the closed immersion $X' \to W$ over $\text{Spec}(A[s])$. This proves $D = X'$ as desired.

Let us prove (6). Observe that the zero scheme of $\frac{s}{f}$ in the previous paragraph is the restriction of the zero scheme of S on the affine open $D_+(f^{(1)})$. Hence we see that $S = 0$ defines $X' \cap E$ on E. Thus (6) follows from (5).

Finally, we have to prove the last part of (8). This is clear because the map $\mathbf{P}^1_Z \to W$ is affine locally given by the surjection

$$B \to B \otimes_{A[s]} A/I = (A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots)[S] \to A/I[S]$$

and the identification $\text{Proj}(A/I[S]) = \text{Spec}(A/I)$. Some details omitted.

42.54. Higher codimension gysin homomorphisms

0FBI Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. In this section we are going to consider triples

$$(Z \to X, \mathcal{N}, \sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X})$$

consisting of a closed immersion $Z \to X$ and a locally free $\mathcal{O}_Z$-module $\mathcal{N}$ and a surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ from the dual of $\mathcal{N}$ to the conormal sheaf of Z in X, see Morphisms, Section 29.31. We will say $\mathcal{N}$ is a virtual normal sheaf for Z in X.

0FBJ **Lemma 42.54.1.** Let (S, δ) be as in Situation 42.7.1. Let

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

be a cartesian diagram of schemes locally of finite type over S whose horizontal arrows are closed immersions. If $\mathcal{N}$ is a virtual normal sheaf for Z in X, then $\mathcal{N}' = g^*\mathcal{N}$ is a virtual normal sheaf for Z' in X'.

Proof. This follows from the surjectivity of the map $g^*\mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'}$ proved in Morphisms, Lemma 29.31.4. $\square$

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal bundle for a closed immersion $Z \to X$. In this situation we set

$$p : N = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{N}^\vee)) \longrightarrow Z$$

equal to the vector bundle over Z whose sections correspond to sections of $\mathcal{N}$. In this situation we have canonical closed immersions

$$C_Z X \longrightarrow N_Z X \longrightarrow N$$

The first closed immersion is Divisors, Equation (31.19.5.1) and the second closed immersion corresponds to the surjection $\mathrm{Sym}(\mathcal{N}^\vee) \to \mathrm{Sym}(\mathcal{C}_{Z/X})$ induced by σ. Let

$$b : W \longrightarrow \mathbf{P}^1_X$$

be the blowing up in $\infty(Z)$ constructed in Section 42.53. By Lemma 42.48.1 we have a canonical bivariant class in

$$C \in A^0(W_\infty \to X)$$

Consider the open immersion $j : C_Z X \to W_\infty$ of (7) and the closed immersion $i : C_Z X \to N$ constructed above. By Lemma 42.36.3 for every $\alpha \in \mathrm{CH}_k(X)$ there exists a unique $\beta \in \mathrm{CH}_*(Z)$ such that

$$i_* j^*(C \cap \alpha) = p^*\beta$$

We set $c(Z \to X, \mathcal{N}) \cap \alpha = \beta$.

0FBK Lemma 42.54.2. The construction above defines a bivariant class[8]

$$c(Z \to X, \mathcal{N}) \in A^*(Z \to X)^\wedge$$

and moreover the construction is compatible with base change as in Lemma 42.54.1. If $\mathcal{N}$ has constant rank r, then $c(Z \to X, \mathcal{N}) \in A^r(Z \to X)$.

Proof. Since both $i_* \circ j^* \circ C$ and p^* are bivariant classes (see Lemmas 42.33.2 and 42.33.4) we can use the equation

$$i_* \circ j^* \circ C = p^* \circ c(Z \to X, \mathcal{N})$$

(suitably interpreted) to define $c(Z \to X, \mathcal{N})$ as a bivariant class. This works because p^* is always bijective on chow groups by Lemma 42.36.3.

Let $X' \to X$, $Z' \to X'$, and $\mathcal{N}'$ be as in Lemma 42.54.1. Write $c = c(Z \to X, \mathcal{N})$ and $c' = c(Z' \to X', \mathcal{N}')$. The second statement of the lemma means that c' is the restriction of c as in Remark 42.33.5. Since we claim this is true for all X'/X locally of finite type, a formal argument shows that it suffices to check that $c' \cap \alpha' = c \cap \alpha'$ for $\alpha' \in \mathrm{CH}_k(X')$. To see this, note that we have a commutative diagram

$$\begin{tikzcd} C_{Z'}X' \arrow[r] \arrow[d] & W'_\infty \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & \mathbf{P}^1_{X'} \arrow[d] \\ C_Z X \arrow[r] & W_\infty \arrow[r] & W \arrow[r] & \mathbf{P}^1_X \end{tikzcd}$$

[8]The notation $A^*(Z \to X)^\wedge$ is discussed in Remark 42.35.5. If X is quasi-compact, then $A^*(Z \to X)^\wedge = A^*(Z \to X)$.

which induces closed immersions:

$$W' \to W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'}, \quad W'_\infty \to W_\infty \times_X X', \quad C_{Z'}X' \to C_ZX \times_Z Z'$$

To get $c \cap \alpha'$ we use the class $C \cap \alpha'$ defined using the morphism $W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'} \to \mathbf{P}^1_{X'}$ in Lemma 42.48.1. To get $c' \cap \alpha'$ on the other hand, we use the class $C' \cap \alpha'$ defined using the morphism $W' \to \mathbf{P}^1_{X'}$. By Lemma 42.48.3 the pushforward of $C' \cap \alpha'$ by the closed immersion $W'_\infty \to (W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'})_\infty$, is equal to $C \cap \alpha'$. Hence the same is true for the pullbacks to the opens

$$C_{Z'}X' \subset W'_\infty, \quad C_ZX \times_Z Z' \subset (W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'})_\infty$$

by Lemma 42.15.1. Since we have a commutative diagram

$$\begin{tikzcd} C_{Z'}X' \arrow[r] \arrow[d] & N' \arrow[d, equal] \\ C_ZX \times_Z Z' \arrow[r] & N \times_Z Z' \end{tikzcd}$$

these classes pushforward to the same class on N' which proves that we obtain the same element $c \cap \alpha' = c' \cap \alpha'$ in $\mathrm{CH}_*(Z')$. ☐

0FBL Lemma 42.54.3. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Suppose that we have a short exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E} \to 0$ of finite locally free $\mathcal{O}_Z$-modules such that the given surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ factors through a map $\sigma' : (\mathcal{N}')^\vee \to \mathcal{C}_{Z/X}$. Then

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}) \circ c(Z \to X, \mathcal{N}')$$

as bivariant classes.

Proof. Denote $N' \to N$ the closed immersion of vector bundles corresponding to the surjection $\mathcal{N}^\vee \to (\mathcal{N}')^\vee$. Then we have closed immersions

$$C_ZX \to N' \to N$$

Thus the desired relationship between the bivariant classes follows immediately from Lemma 42.44.2. ☐

0FV7 Lemma 42.54.4. Let (S, δ) be as in Situation 42.7.1. Consider a cartesian diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

of schemes locally of finite type over S whose horizontal arrows are closed immersions. Let $\mathcal{N}$, resp. $\mathcal{N}'$ be a virtual normal sheaf for $Z \subset X$, resp. $Z' \to X'$. Assume given a short exact sequence $0 \to \mathcal{N}' \to g^*\mathcal{N} \to \mathcal{E} \to 0$ of finite locally free modules on Z' such that the diagram

$$\begin{tikzcd} g^*\mathcal{N}^\vee \arrow[r] \arrow[d] & (\mathcal{N}')^\vee \arrow[d] \\ g^*\mathcal{C}_{Z/X} \arrow[r] & \mathcal{C}_{Z'/X'} \end{tikzcd}$$

commutes. Then we have

$$res(c(Z \to X, \mathcal{N})) = c_{top}(\mathcal{E}) \circ c(Z' \to X', \mathcal{N}')$$

in $A^*(Z' \to X')^\wedge$.

Proof. By Lemma 42.54.2 we have $res(c(Z \to X, \mathcal{N})) = c(Z' \to X', g^*\mathcal{N})$ and the equality follows from Lemma 42.54.3. $\square$

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Let $Y \to X$ be a morphism which is locally of finite type. Assume $Z \times_X Y \to Y$ is a regular closed immersion, see Divisors, Section 31.21. In this case the conormal sheaf $\mathcal{C}_{Z \times_X Y/Y}$ is a finite locally free $\mathcal{O}_{Z \times_X Y}$-module and we obtain a short exact sequence

$$0 \to \mathcal{E}^\vee \to \mathcal{N}^\vee|_{Z \times_X Y} \to \mathcal{C}_{Z \times_X Y/Y} \to 0$$

The quotient $\mathcal{N}|_{Y \times_X Z} \to \mathcal{E}$ is called the excess normal sheaf of the situation.

0FBM Lemma 42.54.5. In the situation described just above assume $\dim_\delta(Y) = n$ and that $\mathcal{C}_{Y \times_X Z/Z}$ has constant rank r. Then

$$c(Z \to X, \mathcal{N}) \cap [Y]_n = c_{top}(\mathcal{E}) \cap [Z \times_X Y]_{n-r}$$

in $\mathrm{CH}_*(Z \times_X Y)$.

Proof. The bivariant class $c_{top}(\mathcal{E}) \in A^*(Z \times_X Y)$ was defined in Remark 42.38.11. By Lemma 42.54.2 we may replace X by Y. Thus we may assume $Z \to X$ is a regular closed immersion of codimension r, we have $\dim_\delta(X) = n$, and we have to show that $c(Z \to X, \mathcal{N}) \cap [X]_n = c_{top}(\mathcal{E}) \cap [Z]_{n-r}$ in $\mathrm{CH}_*(Z)$. By Lemma 42.54.3 we may even assume $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is an isomorphism. In other words, we have to show $c(Z \to X, \mathcal{C}_{Z/X}^\vee) \cap [X]_n = [Z]_{n-r}$ in $\mathrm{CH}_*(Z)$.

Let us trace through the steps in the definition of $c(Z \to X, \mathcal{C}_{Z/X}^\vee) \cap [X]_n$. Let $b : W \to \mathbf{P}^1_X$ be the blowing up of $\infty(Z)$. We first have to compute $C \cap [X]_n$ where $C \in A^0(W_\infty \to X)$ is the class of Lemma 42.48.1. To do this, note that $[W]_{n+1}$ is a cycle on W whose restriction to $\mathbf{A}^1_X$ is equal to the flat pullback of $[X]_n$. Hence $C \cap [X]_n$ is equal to $i_\infty^*[W]_{n+1}$. Since W_∞ is an effective Cartier divisor on W we have $i_\infty^*[W]_{n+1} = [W_\infty]_n$, see Lemma 42.29.5. The restriction of this class to the open $C_Z X \subset W_\infty$ is of course just $[C_Z X]_n$. Because $Z \subset X$ is regularly embedded we have

$$\mathcal{C}_{Z/X,*} = \mathrm{Sym}(\mathcal{C}_{Z/X})$$

as graded $\mathcal{O}_Z$-algebras, see Divisors, Lemma 31.21.5. Hence $p : N = C_Z X \to Z$ is the structure morphism of the vector bundle associated to the finite locally free module $\mathcal{C}_{Z/X}$ of rank r. Then it is clear that $p^*[Z]_{n-r} = [C_Z X]_n$ and the proof is complete. $\square$

0FEA Lemma 42.54.6. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Let $Y \to X$ be a morphism which is locally of finite type. Given integers r, n assume

(1) $\mathcal{N}$ is locally free of rank r,
(2) every irreducible component of Y has δ-dimension n,
(3) $\dim_\delta(Z \times_X Y) \leq n - r$, and
(4) for $\xi \in Z \times_X Y$ with $\delta(\xi) = n - r$ the local ring $\mathcal{O}_{Y,\xi}$ is Cohen-Macaulay.

Then $c(Z \to X, \mathcal{N}) \cap [Y]_n = [Z \times_X Y]_{n-r}$ in $\mathrm{CH}_{n-r}(Z \times_X Y)$.

Proof. The statement makes sense as $Z \times_X Y$ is a closed subscheme of Y. Because $\mathcal{N}$ has rank r we know that $c(Z \to X, \mathcal{N}) \cap [Y]_n$ is in $\mathrm{CH}_{n-r}(Z \times_X Y)$. Since $\dim_\delta(Z \cap Y) \leq n - r$ the chow group $\mathrm{CH}_{n-r}(Z \times_X Y)$ is freely generated by the cycle classes of the irreducible components $W \subset Z \times_X Y$ of δ-dimension $n - r$. Let $\xi \in W$ be the generic point. By assumption (2) we see that $\dim(\mathcal{O}_{Y,\xi}) = r$. On the other hand, since $\mathcal{N}$ has rank r and since $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is surjective, we see that the ideal sheaf of Z is locally cut out by r equations. Hence the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ of $Z \times_X Y$ in Y is locally generated by r elements. Since $\mathcal{O}_{Y,\xi}$ is Cohen-Macaulay of dimension r and since $\mathcal{I}_\xi$ is an ideal of definition (as ξ is a generic point of $Z \times_X Y$) it follows that $\mathcal{I}_\xi$ is generated by a regular sequence (Algebra, Lemma 10.104.2). By Divisors, Lemma 31.20.8 we see that $\mathcal{I}$ is generated by a regular sequence over an open neighbourhood $V \subset Y$ of ξ. By our description of $\mathrm{CH}_{n-r}(Z \times_X Y)$ it suffices to show that $c(Z \to X, \mathcal{N}) \cap [V]_n = [Z \times_X V]_{n-r}$ in $\mathrm{CH}_{n-r}(Z \times_X V)$. This follows from Lemma 42.54.5 because the excess normal sheaf is 0 over V. $\square$

0FBN Lemma 42.54.7. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. The gysin homomorphism i^* viewed as an element of $A^1(D \to X)$ (see Lemma 42.33.3) is the same as the bivariant class $c(D \to X, \mathcal{N}) \in A^1(D \to X)$ constructed using $\mathcal{N} = i^*\mathcal{L}$ viewed as a virtual normal sheaf for D in X.

Proof. We will use the criterion of Lemma 42.35.3. Thus we may assume that X is an integral scheme and we have to show that $i^*[X]$ is equal to $c \cap [X]$. Let $n = \dim_\delta(X)$. As usual, there are two cases.

If $X = D$, then we see that both classes are represented by $c_1(\mathcal{N}) \cap [X]_n$. See Lemma 42.54.5 and Definition 42.29.1.

If $D \neq X$, then $D \to X$ is an effective Cartier divisor and in particular a regular closed immersion of codimension 1. Again by Lemma 42.54.5 we conclude $c(D \to X, \mathcal{N}) \cap [X]_n = [D]_{n-1}$. The same is true by definition for the gysin homomorphism and we conclude once again. $\square$

0FBP Lemma 42.54.8. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $Z \subset X$ be a closed subscheme with virtual normal sheaf $\mathcal{N}$. Let $Y \to X$ be locally of finite type and $c \in A^*(Y \to X)$. Then c and $c(Z \to X, \mathcal{N})$ commute (Remark 42.33.6).

Proof. To check this we may use Lemma 42.35.3. Thus we may assume X is an integral scheme and we have to show $c \cap c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to X, \mathcal{N}) \cap c \cap [X]$ in $\mathrm{CH}_*(Z \times_X Y)$.

If $Z = X$, then $c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{N})$ by Lemma 42.54.5 which commutes with the bivariant class c, see Lemma 42.38.9.

Assume that Z is not equal to X. By Lemma 42.35.3 it even suffices to prove the result after blowing up X (in a nonzero ideal). Let us blowup X in the ideal sheaf of Z. This reduces us to the case where Z is an effective Cartier divisor, see Divisors, Lemma 31.32.4, If Z is an effective Cartier divisor, then we have

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}) \circ i^*$$

where $i^* \in A^1(Z \to X)$ is the gysin homomorphism associated to $i : Z \to X$ (Lemma 42.33.3) and $\mathcal{E}$ is the dual of the kernel of $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$, see Lemmas 42.54.3 and 42.54.7. Then we conclude because Chern classes are in the center of the bivariant ring (in the strong sense formulated in Lemma 42.38.9) and c commutes with the gysin homomorphism i^* by definition of bivariant classes. $\square$

Let (S, δ) be as in Situation 42.7.1. Let X be an integral scheme locally of finite type over S of δ-dimension n. Let $Z \subset Y \subset X$ be closed subschemes which are both effective Cartier divisors in X. Denote $o : Y \to C_Y X$ the zero section of the normal line cone of Y in X. As $C_Y X$ is a line bundle over Y we obtain a bivariant class $o^* \in A^1(Y \to C_Y X)$, see Lemma 42.33.3.

0FEB Lemma 42.54.9. With notation as above we have

$$o^*[C_Z X]_n = [C_Z Y]_{n-1}$$

in $\mathrm{CH}_{n-1}(Y \times_{o, C_Y X} C_Z X)$.

Proof. Denote $W \to \mathbf{P}^1_X$ the blowing up of $\infty(Z)$ as in Section 42.53. Similarly, denote $W' \to \mathbf{P}^1_X$ the blowing up of $\infty(Y)$. Since $\infty(Z) \subset \infty(Y)$ we get an opposite inclusion of ideal sheaves and hence a map of the graded algebras defining these blowups. This produces a rational morphism from W to W' which in fact has a canonical representative

$$W \supset U \longrightarrow W'$$

See Constructions, Lemma 27.18.1. A local calculation (omitted) shows that U contains at least all points of W not lying over ∞ and the open subscheme $C_Z X$ of the special fibre. After shrinking U we may assume $U_\infty = C_Z X$ and $\mathbf{A}^1_X \subset U$. Another local calculation (omitted) shows that the morphism $U_\infty \to W'_\infty$ induces the canonical morphism $C_Z X \to C_Y X \subset W'_\infty$ of normal cones induced by the inclusion of ideals sheaves coming from $Z \subset Y$. Denote $W'' \subset W$ the strict transform of $\mathbf{P}^1_Y \subset \mathbf{P}^1_X$ in W. Then W'' is the blowing up of $\mathbf{P}^1_Y$ in $\infty(Z)$ by Divisors, Lemma 31.33.2 and hence $(W'' \cap U)_\infty = C_Z Y$.

Consider the effective Cartier divisor $i : \mathbf{P}^1_Y \to W'$ from (8) and its associated bivariant class $i^* \in A^1(\mathbf{P}^1_Y \to W')$ from Lemma 42.33.3. We similarly denote $(i'_\infty)^* \in A^1(W'_\infty \to W')$ the gysin map at infinity. Observe that the restriction of i'_∞ (Remark 42.33.5) to U is the restriction of $i^*_\infty \in A^1(W_\infty \to W)$ to U. On the one hand we have

$$(i'_\infty)^* i^*[U]_{n+1} = i^*_\infty i^*[U]_{n+1} = i^*_\infty[(W'' \cap U)_\infty]_{n+1} = [C_Z Y]_n$$

because i^*_∞ kills all classes supported over ∞, because $i^*[U]$ and $[W'']$ agree as cycles over $\mathbf{A}^1$, and because $C_Z Y$ is the fibre of $W'' \cap U$ over ∞. On the other hand, we have

$$(i'_\infty)^* i^*[U]_{n+1} = i^* i^*_\infty[U]_{n+1} = i^*[U_\infty] = o^*[C_Y X]_n$$

because $(i'_\infty)^*$ and i^* commute (Lemma 42.30.5) and because the fibre of $i : \mathbf{P}^1_Y \to W'$ over ∞ factors as $o : Y \to C_Y X$ and the open immersion $C_Y X \to W'_\infty$. The lemma follows. $\square$

0FEC Lemma 42.54.10. Let (S, δ) be as in Situation 42.7.1. Let $Z \subset Y \subset X$ be closed subschemes of a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for $Z \subset X$. Let $\mathcal{N}'$ be a virtual normal sheaf for $Z \subset Y$. Let $\mathcal{N}''$ be a virtual normal sheaf for $Y \subset X$. Assume there is a commutative diagram

$$\begin{tikzcd} (\mathcal{N}'')^\vee|_Z \arrow[r] \arrow[d] & \mathcal{N}^\vee \arrow[r] \arrow[d] & (\mathcal{N}')^\vee \arrow[d] \\ \mathcal{C}_{Y/X}|_Z \arrow[r] & \mathcal{C}_{Z/X} \arrow[r] & \mathcal{C}_{Z/Y} \end{tikzcd}$$

where the sequence at the bottom is from More on Morphisms, Lemma 37.7.12 and the top sequence is a short exact sequence. Then

$$c(Z \to X, \mathcal{N}) = c(Z \to Y, \mathcal{N}') \circ c(Y \to X, \mathcal{N}'')$$

in $A^*(Z \to X)^\wedge$.

Proof. Observe that the assumptions remain satisfied after any base change by a morphism $X' \to X$ which is locally of finite type (the short exact sequence of virtual normal sheaves is locally split hence remains exact after any base change). Thus to check the equality of bivariant classes we may use Lemma 42.35.3. Thus we may assume X is an integral scheme and we have to show $c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to Y, \mathcal{N}') \cap c(Y \to X, \mathcal{N}'') \cap [X]$.

If $Y = X$, then we have

$$\begin{aligned} c(Z \to Y, \mathcal{N}') \cap c(Y \to X, \mathcal{N}'') \cap [X] &= c(Z \to Y, \mathcal{N}') \cap c_{top}(\mathcal{N}'') \cap [Y] \\ &= c_{top}(\mathcal{N}''|_Z) \cap c(Z \to Y, \mathcal{N}') \cap [Y] \\ &= c(Z \to X, \mathcal{N}) \cap [X] \end{aligned}$$

The first equality by Lemma 42.54.3. The second because Chern classes commute with bivariant classes (Lemma 42.38.9). The third equality by Lemma 42.54.3.

Assume $Y \neq X$. By Lemma 42.35.3 it even suffices to prove the result after blowing up X in a nonzero ideal. Let us blowup X in the product of the ideal sheaf of Y and the ideal sheaf of Z. This reduces us to the case where both Y and Z are effective Cartier divisors on X, see Divisors, Lemmas 31.32.4 and 31.32.12.

Denote $\mathcal{N}'' \to \mathcal{E}$ the surjection of finite locally free $\mathcal{O}_Z$-modules such that $0 \to \mathcal{E}^\vee \to (\mathcal{N}'')^\vee \to \mathcal{C}_{Y/X} \to 0$ is a short exact sequence. Then $\mathcal{N} \to \mathcal{E}|_Z$ is a surjection as well. Denote $\mathcal{N}_1$ the finite locally free kernel of this map and observe that $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ factors through $\mathcal{N}_1$. By Lemma 42.54.3 we have

$$c(Y \to X, \mathcal{N}'') = c_{top}(\mathcal{E}) \circ c(Y \to X, \mathcal{C}_{Y/X}^\vee)$$

and

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}|_Z) \circ c(Z \to X, \mathcal{N}_1)$$

Since Chern classes of bundles commute with bivariant classes (Lemma 42.38.9) it suffices to prove

$$c(Z \to X, \mathcal{N}_1) = c(Z \to Y, \mathcal{N}') \circ c(Y \to X, \mathcal{C}_{Y/X}^\vee)$$

in $A^*(Z \to X)$. This we may assume that $\mathcal{N}'' = \mathcal{C}_{Y/X}$. This reduces us to the case discussed in the next paragraph.

In this paragraph Z and Y are effective Cartier divisors on X integral of dimension n, we have $\mathcal{N}'' = \mathcal{C}_{Y/X}$. In this case $c(Y \to X, \mathcal{C}_{Y/X}^\vee) \cap [X] = [Y]_{n-1}$ by Lemma 42.54.5. Thus we have to prove that $c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to Y, \mathcal{N}') \cap [Y]_{n-1}$. Denote N and N' the vector bundles over Z associated to $\mathcal{N}$ and $\mathcal{N}'$. Consider the commutative diagram

$$\begin{tikzcd} N' \arrow[r, "i"'] & N \arrow[r] & (C_Y X) \times_Y Z \\ C_Z Y \arrow[u] \arrow[r] & C_Z X \arrow[u] & \end{tikzcd}$$

of cones and vector bundles over Z. Observe that N' is a relative effective Cartier divisor in N over Z and that

$$\begin{tikzcd} N' \arrow[r, "i"'] \arrow[d] & N \arrow[d] \\ Z \arrow[r, "o"] & (C_Y X) \times_Y Z \end{tikzcd}$$

is cartesian where o is the zero section of the line bundle $C_Y X$ over Y. By Lemma 42.54.9 we have $o^*[C_Z X]_n = [C_Z Y]_{n-1}$ in

$$\mathrm{CH}_{n-1}(Y \times_{o, C_Y X} C_Z X) = \mathrm{CH}_{n-1}(Z \times_{o, (C_Y X) \times_Y Z} C_Z X)$$

By the cartesian property of the square above this implies that

$$i^*[C_Z X]_n = [C_Z Y]_{n-1}$$

in $\mathrm{CH}_{n-1}(N')$. Now observe that $\gamma = c(Z \to X, \mathcal{N}) \cap [X]$ and $\gamma' = c(Z \to Y, \mathcal{N}') \cap [Y]_{n-1}$ are characterized by $p^*\gamma = [C_Z X]_n$ in $\mathrm{CH}_n(N)$ and by $(p')^*\gamma' = [C_Z Y]_{n-1}$ in $\mathrm{CH}_{n-1}(N')$. Hence the proof is finished as $i^* \circ p^* = (p')^*$ by Lemma 42.31.1. $\square$

0FBQ Remark 42.54.11 (Variant for immersions). Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $i : Z \to X$ be an immersion of schemes. In this situation

(1) the conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X is defined (Morphisms, Definition 29.31.1),
(2) we say a pair consisting of a finite locally free $\mathcal{O}_Z$-module $\mathcal{N}$ and a surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is a virtual normal bundle for the immersion $Z \to X$,
(3) choose an open subscheme $U \subset X$ such that $Z \to X$ factors through a closed immersion $Z \to U$ and set $c(Z \to X, \mathcal{N}) = c(Z \to U, \mathcal{N}) \circ (U \to X)^*$.

The bivariant class $c(Z \to X, \mathcal{N})$ does not depend on the choice of the open subscheme U. All of the lemmas have immediate counterparts for this slightly more general construction. We omit the details.

42.55. Calculating some classes

0FED To get further we need to compute the values of some of the classes we've constructed above.

0FEE **Lemma 42.55.1.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then

$$\prod\nolimits_{n=0,\ldots,r} c(\wedge^n \mathcal{E})^{(-1)^n} = 1 - (r-1)! c_r(\mathcal{E}) + \ldots$$

Proof. By the splitting principle we can turn this into a calculation in the polynomial ring on the Chern roots $x_1, \ldots, x_r$ of $\mathcal{E}$. See Section 42.43. Observe that

$$c(\wedge^n \mathcal{E}) = \prod\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} (1 + x_{i_1} + \ldots + x_{i_n})$$

Thus the logarithm of the left hand side of the equation in the lemma is

$$-\sum\nolimits_{p \geq 1} \sum\nolimits_{n=0}^{r} \sum\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} \frac{(-1)^{p+n}}{p} (x_{i_1} + \ldots + x_{i_n})^p$$

Please notice the minus sign in front. However, we have

$$\sum\nolimits_{p \geq 0} \sum\nolimits_{n=0}^{r} \sum\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} \frac{(-1)^{p+n}}{p!} (x_{i_1} + \ldots + x_{i_n})^p = \prod (1 - e^{-x_i})$$

Hence we see that the first nonzero term in our Chern class is in degree r and equal to the predicted value. $\square$

0FEF **Lemma 42.55.2.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{C}$ be a locally free $\mathcal{O}_X$-module of rank r. Consider the morphisms

$$X = \underline{\mathrm{Proj}}_X(\mathcal{O}_X[T]) \xrightarrow{i} E = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{C})[T]) \xrightarrow{\pi} X$$

Then $c_t(i_* \mathcal{O}_X) = 0$ for $t = 1, \ldots, r-1$ and in $A^0(C \to E)$ we have

$$p^* \circ \pi_* \circ c_r(i_* \mathcal{O}_X) = (-1)^{r-1} (r-1)! j^*$$

where $j : C \to E$ and $p : C \to X$ are the inclusion and structure morphism of the vector bundle $C = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{C}))$.

Proof. The canonical map $\pi^* \mathcal{C} \to \mathcal{O}_E(1)$ vanishes exactly along $i(X)$. Hence the Koszul complex on the map

$$\pi^* \mathcal{C} \otimes \mathcal{O}_E(-1) \to \mathcal{O}_E$$

is a resolution of $i_* \mathcal{O}_X$. In particular we see that $i_* \mathcal{O}_X$ is a perfect object of $D(\mathcal{O}_E)$ whose Chern classes are defined. The vanishing of $c_t(i_* \mathcal{O}_X)$ for $t = 1, \ldots, t-1$ follows from Lemma 42.55.1. This lemma also gives

$$c_r(i_* \mathcal{O}_X) = -(r-1)! c_r(\pi^* \mathcal{C} \otimes \mathcal{O}_E(-1))$$

On the other hand, by Lemma 42.43.3 we have

$$c_r(\pi^* \mathcal{C} \otimes \mathcal{O}_E(-1)) = (-1)^r c_r(\pi^* \mathcal{C}^\vee \otimes \mathcal{O}_E(1))$$

and $\pi^* \mathcal{C}^\vee \otimes \mathcal{O}_E(1)$ has a section s vanishing exactly along $i(X)$.

After replacing X by a scheme locally of finite type over X, it suffices to prove that both sides of the equality have the same effect on an element $\alpha \in \mathrm{CH}_*(E)$. Since $C \to X$ is a vector bundle, every cycle class on C is of the form $p^* \beta$ for some $\beta \in \mathrm{CH}_*(X)$ (Lemma 42.36.3). Hence by Lemma 42.19.3 we can write $\alpha = \pi^* \beta + \gamma$ where γ is supported on $E \setminus C$. Using the equalities above it suffices to show that

$$p^*(\pi_*(c_r(\pi^* \mathcal{C}^\vee \otimes \mathcal{O}_E(1)) \cap [W])) = j^*[W]$$

when $W \subset E$ is an integral closed subscheme which is either (a) disjoint from C or (b) is of the form $W = \pi^{-1}Y$ for some integral closed subscheme $Y \subset X$. Using the section s and Lemma 42.44.1 we find in case (a) $c_r(\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1)) \cap [W] = 0$ and in case (b) $c_r(\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1)) \cap [W] = [i(Y)]$. The result follows easily from this; details omitted. □

0FEG Lemma 42.55.3. Let (S, δ) be as in Situation 42.7.1. Let $i : Z \to X$ be a regular closed immersion of codimension r between schemes locally of finite type over S. Let $\mathcal{N} = \mathcal{C}_{Z/X}^\vee$ be the normal sheaf. If X is quasi-compact (or has quasi-compact irreducible components), then $c_t(Z \to X, i_*\mathcal{O}_Z) = 0$ for $t = 1, \ldots, r-1$ and

$$c_r(Z \to X, i_*\mathcal{O}_Z) = (-1)^{r-1}(r-1)!c(Z \to X, \mathcal{N}) \quad \text{in} \quad A^r(Z \to X)$$

where $c_t(Z \to X, i_*\mathcal{O}_Z)$ is the localized Chern class of Definition 42.50.3.

Proof. For any $x \in Z$ we can choose an affine open neighbourhood $\text{Spec}(A) \subset X$ such that $Z \cap \text{Spec}(A) = V(f_1, \ldots, f_r)$ where $f_1, \ldots, f_r \in A$ is a regular sequence. See Divisors, Definition 31.21.1 and Lemma 31.20.8. Then we see that the Koszul complex on $f_1, \ldots, f_r$ is a resolution of $A/(f_1, \ldots, f_r)$ for example by More on Algebra, Lemma 15.30.2. Hence $A/(f_1, \ldots, f_r)$ is perfect as an A-module. It follows that $F = i_*\mathcal{O}_Z$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to $X \setminus Z$ is zero. The assumption that X is quasi-compact (or has quasi-compact irreducible components) means that the localized Chern classes $c_t(Z \to X, i_*\mathcal{O}_Z)$ are defined, see Situation 42.50.1 and Definition 42.50.3. All in all we conclude that the statement makes sense.

Denote $b : W \to \mathbf{P}^1_X$ the blowing up in $\infty(Z)$ as in Section 42.53. By (8) we have a closed immersion

$$i' : \mathbf{P}^1_Z \longrightarrow W$$

We claim that $Q = i'_*\mathcal{O}_{\mathbf{P}^1_Z}$ is a perfect object of $D(\mathcal{O}_W)$ and that F and Q satisfy the assumptions of Lemma 42.51.2.

Assume the claim. The output of Lemma 42.51.2 is that we have

$$c_p(Z \to X, F) = c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

for all $p \geq 1$. Observe that $Q|_E$ is equal to the pushforward of the structure sheaf of Z via the morphism $Z \to E$ which is the base change of i' by ∞. Thus the vanishing of $c_t(Z \to X, F)$ for $1 \leq t \leq r-1$ by Lemma 42.55.2 applied to $E \to Z$. Because $\mathcal{C}_{Z/X} = \mathcal{N}^\vee$ is locally free the bivariant class $c(Z \to X, \mathcal{N})$ is characterized by the relation

$$j^* \circ C = p^* \circ c(Z \to X, \mathcal{N})$$

where $j : C_ZX \to W_\infty$ and $p : C_ZX \to Z$ are the given maps. (Recall $C \in A^0(W_\infty \to X)$ is the class of Lemma 42.48.1.) Thus the displayed equation in the statement of the lemma follows from the corresponding equation in Lemma 42.55.2.

Proof of the claim. Let A and $f_1, \ldots, f_r$ be as above. Consider the affine open $\text{Spec}(A[s]) \subset \mathbf{P}^1_X$ as in Section 42.53. Recall that $s = 0$ defines $(\mathbf{P}^1_X)_\infty$ over this open. Hence over $\text{Spec}(A[s])$ we are blowing up in the ideal generated by the regular sequence $s, f_1, \ldots, f_r$. By More on Algebra, Lemma 15.31.2 the $r+1$ affine charts are global complete intersections over $A[s]$. The chart corresponding to the affine blowup algebra

$$A[s][f_1/s, \ldots, f_r/s] = A[s, y_1, \ldots, y_r]/(sy_i - f_i)$$

contains $i'(Z \cap \mathrm{Spec}(A))$ as the closed subscheme cut out by $y_1, \ldots, y_r$. Since $y_1, \ldots, y_r, sy_1 - f_1, \ldots, sy_r - f_r$ is a regular sequence in the polynomial ring $A[s, y_1, \ldots, y_r]$ we find that i' is a regular immersion. Some details omitted. As above we conclude that $Q = i'_*\mathcal{O}_{\mathbf{P}^1_Z}$ is a perfect object of $D(\mathcal{O}_W)$. All the other assumptions on F and Q in Lemma 42.51.2 (and Lemma 42.49.1) are immediately verified. $\square$

0FEH Lemma 42.55.4. In the situation of Lemma 42.55.3 say $\dim_\delta(X) = n$. Then we have

(1) $c_t(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = 0$ for $t = 1, \ldots, r-1$,
(2) $c_r(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = (-1)^{r-1}(r-1)![Z]_{n-r}$,
(3) $ch_t(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = 0$ for $t = 0, \ldots, r-1$, and
(4) $ch_r(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = [Z]_{n-r}$.

Proof. Parts (1) and (2) follow immediately from Lemma 42.55.3 combined with Lemma 42.54.5. Then we deduce parts (3) and (4) using the relationship between $ch_p = (1/p!)P_p$ and c_p given in Lemma 42.52.1. (Namely, $(-1)^{r-1}(r-1)!ch_r = c_r$ provided $c_1 = c_2 = \ldots = c_{r-1} = 0$.) $\square$

42.56. An Adams operator

0FEI We do the minimal amount of work to define the second adams operator. Let X be a scheme. Recall that $\mathrm{Vect}(X)$ denotes the category of finite locally free $\mathcal{O}_X$-modules. Moreover, recall that we have constructed a zeroth K-group $K_0(\mathrm{Vect}(X))$ associated to this category in Derived Categories of Schemes, Section 36.38. Finally, $K_0(\mathrm{Vect}(X))$ is a ring, see Derived Categories of Schemes, Remark 36.38.6.

0FEJ Lemma 42.56.1. Let X be a scheme. There is a ring map

$$\psi^2 : K_0(\mathrm{Vect}(X)) \longrightarrow K_0(\mathrm{Vect}(X))$$

which sends $[\mathcal{L}]$ to $[\mathcal{L}^{\otimes 2}]$ when $\mathcal{L}$ is invertible and is compatible with pullbacks.

Proof. Let X be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We will consider the element

$$\psi^2(\mathcal{E}) = [\mathrm{Sym}^2(\mathcal{E})] - [\wedge^2(\mathcal{E})]$$

of $K_0(\mathrm{Vect}(X))$.

Let X be a scheme and consider a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of finite locally free $\mathcal{O}_X$-modules. Let us think of this as a filtration on $\mathcal{F}$ with 2 steps. The induced filtration on $\mathrm{Sym}^2(\mathcal{F})$ has 3 steps with graded pieces $\mathrm{Sym}^2(\mathcal{E})$, $\mathcal{E} \otimes \mathcal{F}$, and $\mathrm{Sym}^2(\mathcal{G})$. Hence

$$[\mathrm{Sym}^2(\mathcal{F})] = [\mathrm{Sym}^2(\mathcal{E})] + [\mathcal{E} \otimes \mathcal{F}] + [\mathrm{Sym}^2(\mathcal{G})]$$

In exactly the same manner one shows that

$$[\wedge^2(\mathcal{F})] = [\wedge^2(\mathcal{E})] + [\mathcal{E} \otimes \mathcal{F}] + [\wedge^2(\mathcal{G})]$$

Thus we see that $\psi^2(\mathcal{F}) = \psi^2(\mathcal{E}) + \psi^2(\mathcal{G})$. We conclude that we obtain a well defined additive map $\psi^2 : K_0(\mathrm{Vect}(X)) \to K_0(\mathrm{Vect}(X))$.

It is clear that this map commutes with pullbacks.

We still have to show that ψ^2 is a ring map. Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be finite locally free $\mathcal{O}_X$-modules. Observe that there is a short exact sequence

$$0 \to \wedge^2(\mathcal{E}) \otimes \wedge^2(\mathcal{F}) \to \text{Sym}^2(\mathcal{E} \otimes \mathcal{F}) \to \text{Sym}^2(\mathcal{E}) \otimes \text{Sym}^2(\mathcal{F}) \to 0$$

where the first map sends $(e \wedge e') \otimes (f \wedge f')$ to $(e \otimes f)(e' \otimes f') - (e' \otimes f)(e \otimes f')$ and the second map sends $(e \otimes f)(e' \otimes f')$ to $ee' \otimes ff'$. Similarly, there is a short exact sequence

$$0 \to \text{Sym}^2(\mathcal{E}) \otimes \wedge^2(\mathcal{F}) \to \wedge^2(\mathcal{E} \otimes \mathcal{F}) \to \wedge^2(\mathcal{E}) \otimes \text{Sym}^2(\mathcal{F}) \to 0$$

where the first map sends $ee' \otimes f \wedge f'$ to $(e \otimes f) \wedge (e' \otimes f') + (e' \otimes f) \wedge (e \otimes f')$ and the second map sends $(e \otimes f) \wedge (e' \otimes f')$ to $(e \wedge e') \otimes (ff')$. As above this proves the map ψ^2 is multiplicative. Since it is clear that $\psi^2(1) = 1$ this concludes the proof. □

0FEK Remark 42.56.2. Let X be a scheme such that 2 is invertible on X. Then the Adams operator ψ^2 can be defined on the K-group $K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$ (Derived Categories of Schemes, Definition 36.38.2) in a straightforward manner. Namely, given a perfect complex L on X we get an action of the group $\{\pm 1\}$ on $L \otimes^{\mathbf{L}} L$ by switching the factors. Then we can set

$$\psi^2(L) = [(L \otimes^{\mathbf{L}} L)^+] - [(L \otimes^{\mathbf{L}} L)^-]$$

where $(-)^+$ denotes taking invariants and $(-)^-$ denotes taking anti-invariants (suitably defined). Using exactness of taking invariants and anti-invariants one can argue similarly to the proof of Lemma 42.56.1 to show that this is well defined. When 2 is not invertible on X the situation is a good deal more complicated and another approach has to be used.

0FV8 Lemma 42.56.3. Let X be a scheme. There is a ring map $\psi^{-1} : K_0(\text{Vect}(X)) \to K_0(\text{Vect}(X))$ which sends $[\mathcal{E}]$ to $[\mathcal{E}^\vee]$ when $\mathcal{E}$ is finite locally free and is compatible with pullbacks.

Proof. The only thing to check is that taking duals is compatible with short exact sequences and with pullbacks. This is clear. □

0FEL Remark 42.56.4. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The Chern class map defines a canonical map

$$c : K_0(\text{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

by sending a generator $[\mathcal{E}]$ on the left hand side to $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots$ and extending multiplicatively. Thus $-[\mathcal{E}]$ is sent to the formal inverse $c(\mathcal{E})^{-1}$ which is why we have the infinite product on the right hand side. This is well defined by Lemma 42.40.3.

0FEM Remark 42.56.5. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The Chern character map defines a canonical ring map

$$ch : K_0(\text{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X) \otimes \mathbf{Q}$$

by sending a generator $[\mathcal{E}]$ on the left hand side to $ch(\mathcal{E})$ and extending additively. This is well defined by Lemma 42.45.2 and a ring homomorphism by Lemma 42.45.3.

0FEN Lemma 42.56.6. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. If ψ^2 is as in Lemma 42.56.1 and c and ch are as in Remarks 42.56.4 and 42.56.5 then we have $c_i(\psi^2(\alpha)) = 2^i c_i(\alpha)$ and $ch_i(\psi^2(\alpha)) = 2^i ch_i(\alpha)$ for all $\alpha \in K_0(\text{Vect}(X))$.

Proof. Observe that the map $\prod_{i \geq 0} A^i(X) \to \prod_{i \geq 0} A^i(X)$ multiplying by 2^i on $A^i(X)$ is a ring map. Hence, since ψ^2 is also a ring map, it suffices to prove the formulas for additive generators of $K_0(\text{Vect}(X))$. Thus we may assume $\alpha = [\mathcal{E}]$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. By construction of the Chern classes of $\mathcal{E}$ we immediately reduce to the case where $\mathcal{E}$ has constant rank r, see Remark 42.38.10. In this case, we can choose a projective smooth morphism $p : P \to X$ such that restriction $A^*(X) \to A^*(P)$ is injective and such that $p^*\mathcal{E}$ has a finite filtration whose graded parts are invertible $\mathcal{O}_P$-modules $\mathcal{L}_j$, see Lemma 42.43.1. Then $[p^*\mathcal{E}] = \sum [\mathcal{L}_j]$ and hence $\psi^2([p^{\mathcal{E}}]) = \sum [\mathcal{L}_j^{\otimes 2}]$ by definition of ψ^2. Setting $x_j = c_1(\mathcal{L}_j)$ we have

$$c(\alpha) = \prod (1 + x_j) \quad \text{and} \quad c(\psi^2(\alpha)) = \prod (1 + 2x_j)$$

in $\prod A^i(P)$ and we have

$$ch(\alpha) = \sum \exp(x_j) \quad \text{and} \quad ch(\psi^2(\alpha)) = \sum \exp(2x_j)$$

in $\prod A^i(P)$. From these formulas the desired result follows. □

0FEP Remark 42.56.7. Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Consider the strictly full, saturated, triangulated subcategory

$$D_{Z,perf}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are set-theoretically supported on Z. Denote $\text{Coh}_Z(X) \subset \text{Coh}(X)$ the Serre subcategory of coherent $\mathcal{O}_X$-modules whose set theoretic support is contained in Z. Observe that given $E \in D_{Z,perf}(\mathcal{O}_X)$ Zariski locally on X only a finite number of the cohomology sheaves $H^i(E)$ are nonzero (and they are all settheoretically supported on Z). Hence we can define

$$K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow K_0(\text{Coh}_Z(X)) = K'_0(Z)$$

(equality by Lemma 42.23.6) by the rule

$$E \longmapsto [\bigoplus\nolimits_{i \in \mathbf{Z}} H^{2i}(E)] - [\bigoplus\nolimits_{i \in \mathbf{Z}} H^{2i+1}(E)]$$

This works because given a distinguished triangle in $D_{Z,perf}(\mathcal{O}_X)$ we have a long exact sequence of cohomology sheaves.

0FEQ Remark 42.56.8. Let X, Z, $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 42.56.7. Assume X is regular. Then there is a canonical map

$$K_0(\text{Coh}(Z)) \longrightarrow K_0(D_{Z,perf}(\mathcal{O}_X))$$

defined as follows. For any coherent $\mathcal{O}_Z$-module $\mathcal{F}$ denote $\mathcal{F}[0]$ the object of $D(\mathcal{O}_X)$ which has $\mathcal{F}$ in degree 0 and is zero in other degrees. Then $\mathcal{F}[0]$ is a perfect complex on X by Derived Categories of Schemes, Lemma 36.11.8. Hence $\mathcal{F}[0]$ is an object of $D_{Z,perf}(\mathcal{O}_X)$. On the other hand, given a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of coherent $\mathcal{O}_Z$-modules we obtain a distinguished triangle $\mathcal{F}[0] \to \mathcal{F}'[0] \to \mathcal{F}''[0] \to \mathcal{F}[1]$, see Derived Categories, Section 13.12. This shows that we obtain a map $K_0(\text{Coh}(Z)) \to K_0(D_{Z,perf}(\mathcal{O}_X))$ by sending $[\mathcal{F}]$ to $[\mathcal{F}[0]]$ with apologies for the horrendous notation.

0FER Lemma 42.56.9. Let X be a Noetherian regular scheme. Let $Z \subset X$ be a closed subscheme. The maps constructed in Remarks 42.56.7 and 42.56.8 are mutually inverse and we get $K'_0(Z) = K_0(D_{Z,perf}(\mathcal{O}_X))$.

Proof. Clearly the composition

$$K_0(\text{Coh}(Z)) \longrightarrow K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow K_0(\text{Coh}(Z))$$

is the identity map. Thus it suffices to show the first arrow is surjective. Let E be an object of $D_{Z,perf}(\mathcal{O}_X)$. Recall that $D_{perf}(\mathcal{O}_X) = D^b_{\text{Coh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.8. Hence the cohomologies $H^i(E)$ are coherent, can be viewed as objects of $D_{Z,perf}(\mathcal{O}_X)$, and only a finite number are nonzero. Using the distinguished triangles of canonical truncations the reader sees that

$$[E] = \sum (-1)^i [H^i(E)[0]]$$

in $K_0(D_{Z,perf}(\mathcal{O}_X))$. Then it suffices to show that $[\mathcal{F}[0]]$ is in the image of the map for any coherent $\mathcal{O}_X$-module set theoretically supported on Z. Since we can find a finite filtration on $\mathcal{F}$ whose subquotients are $\mathcal{O}_Z$-modules, the proof is complete. $\square$

0FES Remark 42.56.10. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme and let $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 42.56.7. If X is quasi-compact (or more generally the irreducible components of X are quasi-compact), then the localized Chern classes define a canonical map

$$c(Z \to X, -) : K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow A^0(X) \times \prod\nolimits_{i \geq 1} A^i(Z \to X)$$

by sending a generator $[E]$ on the left hand side to

$$c(Z \to X, E) = 1 + c_1(Z \to X, E) + c_2(Z \to X, E) + \ldots$$

and extending multiplicatively (with product on the right hand side as in Remark 42.34.7). The quasi-compactness condition on X guarantees that the localized chern classes are defined (Situation 42.50.1 and Definition 42.50.3) and that these localized chern classes convert distinguished triangles into the corresponding products in the bivariant chow rings (Lemma 42.52.4).

0FET Remark 42.56.11. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme and let $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 42.56.7. If the irreducible components of X are quasi-compact, then the localized Chern character defines a canonical additive and multiplicative map

$$ch(Z \to X, -) : K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(Z \to X) \otimes \mathbf{Q}$$

by sending a generator $[E]$ on the left hand side to $ch(Z \to X, E)$ and extending additively. Namely, the condition on the irreducible components of X guarantees that the localized chern character is defined (Situation 42.50.1 and Definition 42.50.3) and that these localized chern characters convert distinguished triangles into the corresponding sums in the bivariant chow rings (Lemma 42.52.5). The multiplication on $K_0(D_{Z,perf}(X))$ is defined using derived tensor product (Derived Categories of Schemes, Remark 36.38.9) hence $ch(Z \to X, \alpha\beta) = ch(Z \to X, \alpha) ch(Z \to X, \beta)$ by Lemma 42.52.6. If X is quasi-compact, then the map $ch(Z \to X, -)$ has image contained in $A^*(Z \to X) \otimes \mathbf{Q}$; we omit the details.

0FEU Remark 42.56.12. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S and assume X is quasi-compact (or more generally the irreducible components of X are quasi-compact). With $Z = X$ and notation as in Remarks 42.56.10 and 42.56.11 we have $D_{Z,perf}(\mathcal{O}_X) = D_{perf}(\mathcal{O}_X)$ and we see that

$$K_0(D_{Z,perf}(\mathcal{O}_X)) = K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$$

see Derived Categories of Schemes, Definition 36.38.2. Hence we get

$$c : K_0(X) \to \prod A^i(X) \quad \text{and} \quad ch : K_0(X) \to \prod A^i(X) \otimes \mathbf{Q}$$

as a special case of Remarks 42.56.10 and 42.56.11. Of course, instead we could have just directly used Definition 42.46.3 and Lemmas 42.46.7 and 42.46.11 to construct these maps (as this immediately seen to produce the same classes). Recall that there is a canonical map $K_0(\text{Vect}(X)) \to K_0(X)$ which sends a finite locally free module to itself viewed as a perfect complex (placed in degree 0), see Derived Categories of Schemes, Section 36.38. Then the diagram

$$\begin{tikzcd} K_0((\text{Vect}(X)) \arrow[rr] \arrow[dr, "c"'] & & K_0(D_{perf}(\mathcal{O}_X)) = K_0(X) \arrow[dl, "c"] \\ & \prod A^i(X) & \end{tikzcd}$$

commutes where the south-east arrow is the one constructed in Remark 42.56.4. Similarly, the diagram

$$\begin{tikzcd} K_0((\text{Vect}(X)) \arrow[rr] \arrow[dr, "ch"'] & & K_0(D_{perf}(\mathcal{O}_X)) = K_0(X) \arrow[dl, "ch"] \\ & \prod A^i(X) \otimes \mathbf{Q} & \end{tikzcd}$$

commutes where the south-east arrow is the one constructed in Remark 42.56.5.

42.57. Chow groups and K-groups revisited

0FEV This section is the continuation of Section 42.23. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The K-group $K'_0(X) = K_0(\text{Coh}(X))$ of coherent sheaves on X has a canonical increasing filtration

$$F_k K'_0(X) = \text{Im}\Big(K_0(\text{Coh}_{\leq k}(X)) \to K_0(\text{Coh}(X)\Big)$$

This is called the filtration by dimension of supports. Observe that

$$\text{gr}_k K'_0(X) \subset K'_0(X)/F_{k-1}K'_0(X) = K_0(\text{Coh}(X)/\text{Coh}_{\leq k-1}(X))$$

where the equality holds by Homology, Lemma 12.11.3. The discussion in Remark 42.23.5 shows that there are canonical maps

$$\text{CH}_k(X) \longrightarrow \text{gr}_k K'_0(X)$$

defined by sending the class of an integral closed subscheme $Z \subset X$ of δ-dimension k to the class of $[\mathcal{O}_Z]$ on the right hand side.

0FEW Proposition 42.57.1. Let (S, δ) be as in Situation 42.7.1. Assume given a closed immersion $X \to Y$ of schemes locally of finite type over S with Y regular and quasi-compact. Then the composition

$$K'_0(X) \to K_0(D_{X,perf}(\mathcal{O}_Y)) \to A^*(X \to Y) \otimes \mathbf{Q} \to \text{CH}_*(X) \otimes \mathbf{Q}$$

of the map $\mathcal{F} \mapsto \mathcal{F}[0]$ from Remark 42.56.8, the map $ch(X \to Y, -)$ from Remark 42.56.11, and the map $c \mapsto c \cap [Y]$ induces an isomorphism

$$K'_0(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

which depends on the choice of Y. Moreover, the canonical map

$$\mathrm{CH}_k(X) \otimes \mathbf{Q} \longrightarrow \mathrm{gr}_k K'_0(X) \otimes \mathbf{Q}$$

(see above) is an isomorphism of $\mathbf{Q}$-vector spaces for all $k \in \mathbf{Z}$.

Proof. Since Y is regular, the construction in Remark 42.56.8 applies. Since Y is quasi-compact, the construction in Remark 42.56.11 applies. We have that Y is locally equidimensional (Lemma 42.42.1) and thus the "fundamental cycle" $[Y]$ is defined as an element of $\mathrm{CH}_*(Y)$, see Remark 42.42.2. Combining this with the map $\mathrm{CH}_k(X) \to \mathrm{gr}_k K'_0(X)$ constructed above we see that it suffices to prove

(1) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module whose support has δ-dimension $\leq k$, then the composition above sends $[\mathcal{F}]$ into $\bigoplus_{k' \leq k} \mathrm{CH}_{k'}(X) \otimes \mathbf{Q}$.
(2) If $Z \subset X$ is an integral closed subscheme of δ-dimension k, then the composition above sends $[\mathcal{O}_Z]$ to an element whose degree k part is the class of $[Z]$ in $\mathrm{CH}_k(X) \otimes \mathbf{Q}$.

Namely, if this holds, then our maps induce maps $\mathrm{gr}_k K'_0(X) \otimes \mathbf{Q} \to CH_k(X) \otimes \mathbf{Q}$ which are inverse to the canonical maps $\mathrm{CH}_k(X) \otimes \mathbf{Q} \to \mathrm{gr}_k K'_0(X) \otimes \mathbf{Q}$ given above the proposition.

Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$ the composition above sends $[\mathcal{F}]$ to

$$ch(X \to Y, \mathcal{F}[0]) \cap [Y] \in \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

If $\mathcal{F}$ is (set theoretically) supported on a closed subscheme $Z \subset X$, then we have

$$ch(X \to Y, \mathcal{F}[0]) = (Z \to X)_* \circ ch(Z \to Y, \mathcal{F}[0])$$

by Lemma 42.50.8. We conclude that in this case we end up in the image of $\mathrm{CH}_*(Z) \to \mathrm{CH}_*(X)$. Hence we get condition (1).

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. The composition above sends $[\mathcal{O}_Z]$ to the element

$$ch(X \to Y, \mathcal{O}_Z[0]) \cap [Y] = (Z \to X)_* ch(Z \to Y, \mathcal{O}_Z[0]) \cap [Y]$$

by the same argument as above. Thus it suffices to prove that the degree k part of $ch(Z \to Y, \mathcal{O}_Z[0]) \cap [Y] \in \mathrm{CH}_*(Z) \otimes \mathbf{Q}$ is $[Z]$. Since $\mathrm{CH}_k(Z) = \mathbf{Z}$, in order to prove this we may replace Y by an open neighbourhood of the generic point ξ of Z. Since the maximal ideal of the regular local ring $\mathcal{O}_{X,\xi}$ is generated by a regular sequence (Algebra, Lemma 10.106.3) we may assume the ideal of Z is generated by a regular sequence, see Divisors, Lemma 31.20.8. Thus we deduce the result from Lemma 42.55.4. □

42.58. Rational intersection products on regular schemes

0FEX We will show that $\mathrm{CH}_*(X) \otimes \mathbf{Q}$ has an intersection product if X is Noetherian, regular, finite dimensional, with affine diagonal. The basis for the construction is the following result (which is a corollary of the proposition in the previous section).

0FEY **Lemma 42.58.1.** Let (S, δ) be as in Situation 42.7.1. Let X be a quasi-compact regular scheme of finite type over S with affine diagonal and $\delta_{X/S} : X \to \mathbf{Z}$ bounded. Then the composition

$$K_0(\text{Vect}(X)) \otimes \mathbf{Q} \longrightarrow A^*(X) \otimes \mathbf{Q} \longrightarrow \text{CH}_*(X) \otimes \mathbf{Q}$$

of the map ch from Remark 42.56.5 and the map $c \mapsto c \cap [X]$ is an isomorphism.

Proof. We have $K'_0(X) = K_0(X) = K_0(\text{Vect}(X))$ by Derived Categories of Schemes, Lemmas 36.38.4, 36.36.8, and 36.38.5. By Remark 42.56.12 the composition given agrees with the map of Proposition 42.57.1 for $X = Y$. Thus the result follows from the proposition. $\square$

Let X, S, δ be as in Lemma 42.58.1. For simplicity let us work with cycles of a given codimension, see Section 42.42. Let $[X]$ be the fundamental cycle of X, see Remark 42.42.2. Pick $\alpha \in CH^i(X)$ and $\beta \in CH^j(X)$. By the lemma we can find a unique $\alpha' \in K_0(\text{Vect}(X)) \otimes \mathbf{Q}$ with $ch(\alpha') \cap [X] = \alpha$. Of course this means that $ch_{i'}(\alpha') \cap [X] = 0$ if $i' \neq i$ and $ch_i(\alpha') \cap [X] = \alpha$. By Lemma 42.56.6 we see that $\alpha'' = 2^{-i}\psi^2(\alpha')$ is another solution. By uniqueness we get $\alpha'' = \alpha'$ and we conclude that $ch_{i'}(\alpha) = 0$ in $A^{i'}(X) \otimes \mathbf{Q}$ for $i' \neq i$. Then we can define

$$\alpha \cdot \beta = ch(\alpha') \cap \beta = ch_i(\alpha') \cap \beta$$

in $\text{CH}^{i+j}(X) \otimes \mathbf{Q}$ by the property of α' we observed above. This is a symmetric pairing: namely, if we pick $\beta' \in K_0(\text{Vect}(X)) \otimes \mathbf{Q}$ lifting β, then we get

$$\alpha \cdot \beta = ch(\alpha') \cap \beta = ch(\alpha') \cap ch(\beta') \cap [X]$$

and we know that Chern classes commute. The intersection product is associative for the same reason

$$(\alpha \cdot \beta) \cdot \gamma = ch(\alpha') \cap ch(\beta') \cap ch(\gamma') \cap [X]$$

because we know composition of bivariant classes is associative. Perhaps a better way to formulate this is as follows: there is a unique commutative, associative intersection product on $\text{CH}^*(X) \otimes \mathbf{Q}$ compatible with grading such that the isomorphism $K_0(\text{Vect}(X)) \otimes \mathbf{Q} \to \text{CH}^*(X) \otimes \mathbf{Q}$ is an isomorphism of rings.

42.59. Gysin maps for local complete intersection morphisms

0FEZ Before reading this section, we suggest the reader read up on regular immersions (Divisors, Section 31.21) and local complete intersection morphisms (More on Morphisms, Section 37.62).

Let (S, δ) be as in Situation 42.7.1. Let $i : X \to Y$ be a regular immersion[9] of schemes locally of finite type over S. In particular, the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free (see Divisors, Lemma 31.21.5). Hence the normal sheaf

$$\mathcal{N}_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$$

is finite locally free as well and we have a surjection $\mathcal{N}_{X/Y}^\vee \to \mathcal{C}_{X/Y}$ (because an isomorphism is also a surjection). The construction in Section 42.54 gives us a canonical bivariant class

$$i^! = c(X \to Y, \mathcal{N}_{X/Y}) \in A^*(X \to Y)^\wedge$$

[9]See Divisors, Definition 31.21.1. Observe that regular immersions are the same thing as Koszul-regular immersions or quasi-regular immersions for locally Noetherian schemes, see Divisors, Lemma 31.21.3. We will use this without further mention in this section.

We need a couple of lemmas about this notion.

0FF0 **Lemma 42.59.1.** Let (S, δ) be as in Situation 42.7.1. Let $i : X \to Y$ and $j : Y \to Z$ be regular immersions of schemes locally of finite type over S. Then $j \circ i$ is a regular immersion and $(j \circ i)^! = i^! \circ j^!$.

Proof. The first statement is Divisors, Lemma 31.21.7. By Divisors, Lemma 31.21.6 there is a short exact sequence

$$0 \to i^*(\mathcal{C}_{Y/Z}) \to \mathcal{C}_{X/Z} \to \mathcal{C}_{X/Y} \to 0$$

Thus the result by the more general Lemma 42.54.10. $\square$

0FF1 **Lemma 42.59.2.** Let (S, δ) be as in Situation 42.7.1. Let $p : P \to X$ be a smooth morphism of schemes locally of finite type over S and let $s : X \to P$ be a section. Then s is a regular immersion and $1 = s^! \circ p^*$ in $A^*(X)^\wedge$ where $p^* \in A^*(P \to X)^\wedge$ is the bivariant class of Lemma 42.33.2.

Proof. The first statement is Divisors, Lemma 31.22.8. It suffices to show that $s^! \cap p^*[Z] = [Z]$ in $\mathrm{CH}_*(X)$ for any integral closed subscheme $Z \subset X$ as the assumptions are preserved by base change by $X' \to X$ locally of finite type. After replacing P by an open neighbourhood of $s(Z)$ we may assume $P \to X$ is smooth of fixed relative dimension r. Say $\dim_\delta(Z) = n$. Then every irreducible component of $p^{-1}(Z)$ has dimension $r+n$ and $p^*[Z]$ is given by $[p^{-1}(Z)]_{n+r}$. Observe that $s(X) \cap p^{-1}(Z) = s(Z)$ scheme theoretically. Hence by the same reference as used above $s(X) \cap p^{-1}(Z)$ is a closed subscheme regularly embedded in $\overline{p}^{-1}(Z)$ of codimension r. We conclude by Lemma 42.54.5. $\square$

Let (S, δ) be as in Situation 42.7.1. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & P \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

of schemes locally of finite type over S such that g is smooth and i is a regular immersion. Combining the bivariant class $i^!$ discussed above with the bivariant class $g^* \in A^*(P \to Y)^\wedge$ of Lemma 42.33.2 we obtain

$$f^! = i^! \circ g^* \in A^*(X \to Y)$$

Observe that the morphism f is a local complete intersection morphism, see More on Morphisms, Definition 37.62.2. Conversely, if $f : X \to Y$ is a local complete intersection morphism of locally Noetherian schemes and $f = g \circ i$ with g smooth, then i is a regular immersion. We claim that our construction of $f^!$ only depends on the morphism f and not on the choice of factorization $f = g \circ i$.

0FF2 **Lemma 42.59.3.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. The bivariant class $f^!$ is independent of the choice of the factorization $f = g \circ i$ with g smooth (provided one exists).

Proof. Given a second such factorization $f = g' \circ i'$ we can consider the smooth morphism $g'' : P \times_Y P' \to Y$, the immersion $i'' : X \to P \times_Y P'$ and the factorization

$f = g'' \circ i''$. Thus we may assume that we have a diagram

$$\begin{tikzcd} & P' \arrow[d, "p"] \arrow[rd, "g'"] & \\ X \arrow[ru, "i'"] \arrow[r, "i"] & P \arrow[r, "g"] & Y \end{tikzcd}$$

where p is a smooth morphism. Then $(g')^* = p^* \circ g^*$ (Lemma 42.14.3) and hence it suffices to show that $i^! = (i')^! \circ p^*$ in $A^*(X \to P)$. Consider the commutative diagram

$$\begin{tikzcd} & X \times_P P' \arrow[r, "j"'] \arrow[d, "\overline{p}"] & P' \arrow[d, "p"] \\ X \arrow[ru, "s"] \arrow[r, "1"] & X \arrow[r, "i"] & P \end{tikzcd}$$

where $s = (1, i')$. Then s and j are regular immersions (by Divisors, Lemma 31.22.8 and Divisors, Lemma 31.21.4) and $i' = j \circ s$. By Lemma 42.59.1 we have $(i')^! = s^! \circ j^!$. Since the square is cartesian, the bivariant class $j^!$ is the restriction (Remark 42.33.5) of $i^!$ to P', see Lemma 42.54.2. Since bivariant classes commute with flat pullbacks we find $j^! \circ p^* = \overline{p}^* \circ i^!$. Thus it suffices to show that $s^! \circ \overline{p}^* = \mathrm{id}$ which is done in Lemma 42.59.2. □

0FF3 Definition 42.59.4. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. We say the gysin map for f exists if we can write $f = g \circ i$ with g smooth and i an immersion. In this case we define the gysin map $f^! = i^! \circ g^* \in A^*(X \to Y)$ as above.

It follows from the definition that for a regular immersion this agrees with the construction earlier and for a smooth morphism this agrees with flat pullback. In fact, this agreement holds for all syntomic morphisms.

0FF4 Lemma 42.59.5. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. If the gysin map exists for f and f is flat, then $f^!$ is equal to the bivariant class of Lemma 42.33.2.

Proof. Choose a factorization $f = g \circ i$ with $i : X \to P$ an immersion and $g : P \to Y$ smooth. Observe that for any morphism $Y' \to Y$ which is locally of finite type, the base changes of f', g', i' satisfy the same assumptions (see Morphisms, Lemmas 29.34.5 and 29.30.4 and More on Morphisms, Lemma 37.62.8). Thus we reduce to proving that $f^*[Y] = i^!(g^*[Y])$ in case Y is integral, see Lemma 42.35.3. Set $n = \dim_\delta(Y)$. After decomposing X and P into connected components we may assume f is flat of relative dimension r and g is smooth of relative dimension t. Then $f^*[Y] = [X]_{n+s}$ and $g^*[Y] = [P]_{n+t}$. On the other hand i is a regular immersion of codimension $t - s$. Thus $i^![P]_{n+t} = [X]_{n+s}$ (Lemma 42.54.5) and the proof is complete. □

0FF5 Lemma 42.59.6. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ and $g : Y \to Z$ be local complete intersection morphisms of schemes locally of finite type over S. Assume the gysin map exists for $g \circ f$ and g. Then the gysin map exists for f and $(g \circ f)^! = f^! \circ g^!$.

Proof. Observe that $g \circ f$ is a local complete intersection morphism by More on Morphisms, Lemma 37.62.7 and hence the statement of the lemma makes sense. If $X \to P$ is an immersion of X into a scheme P smooth over Z then $X \to P \times_Z Y$ is an immersion of X into a scheme smooth over Y. This prove the first assertion of the lemma. Let $Y \to P'$ be an immersion of Y into a scheme P' smooth over Z. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & P \times_Z Y \arrow[r, "a"] \arrow[dl, "p"] & P \times_Z P' \arrow[dl, "q"] \\
Y \arrow[r, "b"] \arrow[d] & P' \arrow[dl] & \\
Z & &
\end{tikzcd}$$

Here the horizontal arrows are regular immersions, the south-west arrows are smooth, and the square is cartesian. Whence $a^! \circ q^* = p^* \circ b^!$ as bivariant classes commute with flat pullback. Combining this fact with Lemmas 42.59.1 and 42.14.3 the reader finds the statement of the lemma holds true. Small detail omitted. □

0FF6 Lemma 42.59.7. Let (S, δ) be as in Situation 42.7.1. Consider a commutative diagram

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y'' \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

of schemes locally of finite type over S with both square cartesian. Assume $f : X \to Y$ is a local complete intersection morphism such that the gysin map exists for f. Let $c \in A^*(Y'' \to Y')$. Denote $res(f^!) \in A^*(X' \to Y')$ the restriction of $f^!$ to Y' (Remark 42.33.5). Then c and $res(f^!)$ commute (Remark 42.33.6).

Proof. Choose a factorization $f = g \circ i$ with g smooth and i an immersion. Since $f^! = i^! \circ g^!$ it suffices to prove the lemma for $g^!$ (which is given by flat pullback) and for $i^!$. The result for flat pullback is part of the definition of a bivariant class. The case of $i^!$ follows immediately from Lemma 42.54.8. □

0FF7 Lemma 42.59.8. Let (S, δ) be as in Situation 42.7.1. Consider a cartesian diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

of schemes locally of finite type over S. Assume

1. (1) f is a local complete intersection morphism and the gysin map exists for f,
2. (2) X, X', Y, Y' satisfy the equivalent conditions of Lemma 42.42.1,
3. (3) for $x' \in X'$ with images x, y', and y in X, Y', and Y we have $n_{x'} - n_{y'} = n_x - n_y$ where $n_{x'}$, n_x, $n_{y'}$, and n_y are as in the lemma, and
4. (4) for every generic point $\xi \in X'$ the local ring $\mathcal{O}_{Y', f'(\xi)}$ is Cohen-Macaulay.

Then $f^![Y'] = [X']$ where $[Y']$ and $[X']$ are as in Remark 42.42.2.

Proof. Recall that $n_{x'}$ is the common value of $\delta(\xi)$ where ξ is the generic point of an irreducible component passing through x'. Moreover, the functions $x' \mapsto n_{x'}$, $x \mapsto n_x$, $y' \mapsto n_{y'}$, and $y \mapsto n_y$ are locally constant. Let X'_n, X_n, Y'_n, and Y_n be the open and closed subscheme of X', X, Y', and Y where the function has value n. Recall that $[X'] = \sum [X'_n]_n$ and $[Y'] = \sum [Y'_n]_n$. Having said this, it is clear that to prove the lemma we may replace X' by one of its connected components and X, Y', Y by the connected component that it maps into. Then we know that X', X, Y', and Y are δ-equidimensional in the sense that each irreducible component has the same δ-dimension. Say n', n, m', and m is this common value for X', X, Y', and Y. The last assumption means that $n' - m' = n - m$.

Choose a factorization $f = g \circ i$ where $i : X \to P$ is an immersion and $g : P \to Y$ is smooth. As X is connected, we see that the relative dimension of $P \to Y$ at points of $i(X)$ is constant. Hence after replacing P by an open neighbourhood of $i(X)$, we may assume that $P \to Y$ has constant relative dimension and $i : X \to P$ is a closed immersion. Denote $g' : Y' \times_Y P \to Y'$ the base change of g and denote $i' : X' \to Y' \times_Y P$ the base change of i. It is clear that $g^*[Y] = [P]$ and $(g')^*[Y'] = [Y' \times_Y P]$. Finally, if $\xi' \in X'$ is a generic point, then $\mathcal{O}_{Y' \times_Y P, i'(\xi)}$ is Cohen-Macaulay. Namely, the local ring map $\mathcal{O}_{Y', f'(\xi)} \to \mathcal{O}_{Y' \times_Y P, i'(\xi)}$ is flat with regular fibre (see Algebra, Section 10.142), a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.106.3), $\mathcal{O}_{Y', f'(\xi)}$ is Cohen-Macaulay by assumption (4) and we get what we want from Algebra, Lemma 10.163.3. Thus we reduce to the case discussed in the next paragraph.

Assume f is a regular closed immersion and X', X, Y', and Y are δ-equidimensional of δ-dimensions n', n, m', and m and $m' - n' = m - n$. In this case we obtain the result immediately from Lemma 42.54.6. $\square$

0FF8 Remark 42.59.9. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. Assume the gysin map exists for f. Then $f^! \circ c_i(\mathcal{E}) = c_i(f^*\mathcal{E}) \circ f^!$ and similarly for the Chern character, see Lemma 42.59.7. If X and Y satisfy the equivalent conditions of Lemma 42.42.1 and Y is Cohen-Macaulay (for example), then $f^![Y] = [X]$ by Lemma 42.59.8. In this case we also get $f^!(c_i(\mathcal{E}) \cap [Y]) = c_i(f^*\mathcal{E}) \cap [X]$ and similarly for the Chern character.

0FV9 Lemma 42.59.10. Let (S, δ) be as in Situation 42.7.1. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes locally of finite type over S. Assume

(1) both f and f' are local complete intersection morphisms, and
(2) the gysin map exists for f

Then $\mathcal{C} = \mathrm{Ker}(H^{-1}((g')^* NL_{X/Y}) \to H^{-1}(NL_{X'/Y'}))$ is a finite locally free $\mathcal{O}_{X'}$-module, the gysin map exists for f', and we have

$$res(f^!) = c_{top}(\mathcal{C}^\vee) \circ (f')^!$$

in $A^*(X' \to Y')$.

Proof. The fact that $\mathcal{C}$ is finite locally free follows immediately from More on Algebra, Lemma 15.85.5. Choose a factorization $f = g \circ i$ with $g : P \to Y$ smooth and i an immersion. Then we can factor $f' = g' \circ i'$ where $g' : P' \to Y'$ and $i' : X' \to P'$ the base changes. Picture

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & P \arrow[r] & Y \end{tikzcd}$$

In particular, we see that the gysin map exists for f'. By More on Morphisms, Lemmas 37.13.13 we have

$$NL_{X/Y} = \left(\mathcal{C}_{X/P} \to i^*\Omega_{P/Y}\right)$$

where $\mathcal{C}_{X/P}$ is the conormal sheaf of the embedding i. Similarly for the primed version. We have $(g')^*i^*\Omega_{P/Y} = (i')^*\Omega_{P'/Y'}$ because $\Omega_{P/Y}$ pulls back to $\Omega_{P'/Y'}$ by Morphisms, Lemma 29.32.10. Also, recall that $(g')^*\mathcal{C}_{X/P} \to \mathcal{C}_{X'/P'}$ is surjective, see Morphisms, Lemma 29.31.4. We deduce that the sheaf $\mathcal{C}$ is canonicallly isomorphic to the kernel of the map $(g')^*\mathcal{C}_{X/P} \to \mathcal{C}_{X'/P'}$ of finite locally free modules. Recall that $i^!$ is defined using $\mathcal{N} = \mathcal{C}_{Z/X}^\vee$ and similarly for $(i')^!$. Thus we have

$$res(i^!) = c_{top}(\mathcal{C}^\vee) \circ (i')^!$$

in $A^*(X' \to P')$ by an application of Lemma 42.54.4. Since finally we have $f^! = i^! \circ g^*$, $(f')^! = (i')^! \circ (g')^*$, and $(g')^* = res(g^*)$ we conclude. $\square$

0FVA Lemma 42.59.11 (Blow up formula). Let (S, δ) be as in Situation 42.7.1. Let $i : Z \to X$ be a regular closed immersion of schemes locally of finite type over S. Let $b : X' \to X$ be the blowing up with center Z. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Assume that the gysin map exists for b. Then we have

$$res(b^!) = c_{top}(\mathcal{F}^\vee) \circ \pi^*$$

in $A^*(E \to Z)$ where $\mathcal{F}$ is the kernel of the canonical map $\pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'}$.

Proof. Observe that the morphism b is a local complete intersection morphism by More on Algebra, Lemma 15.31.2 and hence the statement makes sense. Since $Z \to X$ is a regular immersion (and hence a fortiori quasi-regular) we see that $\mathcal{C}_{Z/X}$ is finite locally free and the map $\text{Sym}^*(\mathcal{C}_{Z/X}) \to \mathcal{C}_{Z/X,*}$ is an isomorphism, see Divisors, Lemma 31.21.5. Since $E = \text{Proj}(\mathcal{C}_{Z/X,*})$ we conclude that $E = \mathbf{P}(\mathcal{C}_{Z/X})$ is a projective space bundle over Z. Thus $E \to Z$ is smooth and certainly a local complete intersection morphism. Thus Lemma 42.59.10 applies and we see that

$$res(b^!) = c_{top}(\mathcal{C}^\vee) \circ \pi^!$$

with $\mathcal{C}$ as in the statement there. Of course $\pi^* = \pi^!$ by Lemma 42.59.5. It remains to show that $\mathcal{F}$ is equal to the kernel $\mathcal{C}$ of the map $H^{-1}(j^*\, NL_{X'/X}) \to H^{-1}(NL_{E/Z})$.

Since $E \to Z$ is smooth we have $H^{-1}(NL_{E/Z}) = 0$, see More on Morphisms, Lemma 37.13.7. Hence it suffices to show that $\mathcal{F}$ can be identified with $H^{-1}(j^* NL_{X'/X})$. By More on Morphisms, Lemmas 37.13.11 and 37.13.9 we have an exact sequence

$$0 \to H^{-1}(j^* NL_{X'/X}) \to H^{-1}(NL_{E/X}) \to \mathcal{C}_{E/X'} \to \ldots$$

By the same lemmas applied to $E \to Z \to X$ we obtain an isomorphism $\pi^*\mathcal{C}_{Z/X} = H^{-1}(\pi^* NL_{Z/X}) \to H^{-1}(NL_{E/X})$. Thus we conclude. □

0FF9 Lemma 42.59.12. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S such that both X and Y are quasi-compact, regular, have affine diagonal, and finite dimension. Then f is a local complete intersection morphism. Assume moreover the gysin map exists for f. Then

$$f^!(\alpha \cdot \beta) = f^!\alpha \cdot f^!\beta$$

in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$ where the intersection product is as in Section 42.58.

Proof. The first statement follows from More on Morphisms, Lemma 37.62.11. Observe that $f^![Y] = [X]$, see Lemma 42.59.8. Write $\alpha = ch(\alpha') \cap [Y]$ and $\beta = ch(\beta') \cap [Y]$ where $\alpha', \beta' \in K_0(\mathrm{Vect}(X)) \otimes \mathbf{Q}$ as in Section 42.58. Setting $c = ch(\alpha')$ and $c' = ch(\beta')$ we find $\alpha \cdot \beta = c \cap c' \cap [Y]$ by construction. By Lemma 42.59.7 we know that $f^!$ commutes with both c and c'. Hence

$$\begin{aligned} f^!(\alpha \cdot \beta) &= f^!(c \cap c' \cap [Y]) \\ &= c \cap c' \cap f^![Y] \\ &= c \cap c' \cap [X] \\ &= (c \cap [X]) \cdot (c' \cap [X]) \\ &= (c \cap f^![Y]) \cdot (c' \cap f^![Y]) \\ &= f^!(\alpha) \cdot f^!(\beta) \end{aligned}$$

as desired. □

0FFA Lemma 42.59.13. Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S such that both X and Y are quasi-compact, regular, have affine diagonal, and finite dimension. Then f is a local complete intersection morphism. Assume moreover the gysin map exists for f and that f is proper. Then

$$f_*(\alpha \cdot f^!\beta) = f_*\alpha \cdot \beta$$

in $\mathrm{CH}^*(Y) \otimes \mathbf{Q}$ where the intersection product is as in Section 42.58.

Proof. The first statement follows from More on Morphisms, Lemma 37.62.11. Observe that $f^![Y] = [X]$, see Lemma 42.59.8. Write $\alpha = ch(\alpha') \cap [X]$ and

$\beta = ch(\beta') \cap [Y]$ $\alpha' \in K_0(\text{Vect}(X)) \otimes \mathbf{Q}$ and $\beta' \in K_0(\text{Vect}(Y)) \otimes \mathbf{Q}$ as in Section 42.58. Set $c = ch(\alpha')$ and $c' = ch(\beta')$. We have

$$\begin{aligned} f_*(\alpha \cdot f^!\beta) &= f_*(c \cap f^!(c' \cap [Y]_e)) \\ &= f_*(c \cap c' \cap f^![Y]_e) \\ &= f_*(c \cap c' \cap [X]_d) \\ &= f_*(c' \cap c \cap [X]_d) \\ &= c' \cap f_*(c \cap [X]_d) \\ &= \beta \cdot f_*(\alpha) \end{aligned}$$

The first equality by the construction of the intersection product. By Lemma 42.59.7 we know that $f^!$ commutes with c'. The fact that Chern classes are in the center of the bivariant ring justifies switching the order of capping $[X]$ with c and c'. Commuting c' with f_* is allowed as c' is a bivariant class. The final equality is again the construction of the intersection product. $\square$

42.60. Gysin maps for diagonals

0FBR Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a smooth morphism of schemes locally of finite type over S. Then the diagonal morphism $\Delta : X \longrightarrow X \times_Y X$ is a regular immersion, see More on Morphisms, Lemma 37.62.18. Thus we have the gysin map

$$\Delta^! \in A^*(X \to X \times_Y X)^\wedge$$

constructed in Section 42.59. If $X \to Y$ has constant relative dimension d, then $\Delta^! \in A^d(X \to X \times_Y X)$.

0FBS **Lemma 42.60.1.** In the situation above we have $\Delta^! \circ \text{pr}_i^! = 1$ in $A^0(X)$.

Proof. Observe that the projections $\text{pr}_i : X \times_Y X \to X$ are smooth and hence we have gysin maps for these projections as well. Thus the lemma makes sense and is a special case of Lemma 42.59.6. $\square$

0FBT **Proposition 42.60.2.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes locally of finite type over S. If g is smooth of relative dimension d, then $A^p(X \to Y) = A^{p-d}(X \to Z)$. [Ful98, Proposition 17.4.2]

Proof. We will use that smooth morphisms are local complete intersection morphisms whose gysin maps exist (see Section 42.59). In particular we have $g^! \in A^{-d}(Y \to Z)$. Then we can send $c \in A^p(X \to Y)$ to $c \circ g^! \in A^{p-d}(X \to Z)$.

Conversely, let $c' \in A^{p-d}(X \to Z)$. Denote $res(c')$ the restriction (Remark 42.33.5) of c' by the morphism $Y \to Z$. Since the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\text{pr}_2"'] \arrow[d, "\text{pr}_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is cartesian we find $res(c') \in A^{p-d}(X \times_Z Y \to Y)$. Let $\Delta : Y \to Y \times_Z Y$ be the diagonal and denote $res(\Delta^!)$ the restriction of $\Delta^!$ to $X \times_Z Y$ by the morphism

$X \times_Z Y \to Y \times_Z Y$. Since the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_Z Y \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

is cartesian we see that $res(\Delta^!) \in A^d(X \to X \times_Z Y)$. Combining these two restrictions we obtain

$$res(\Delta^!) \circ res(c') \in A^p(X \to Y)$$

Thus we have produced maps $A^p(X \to Y) \to A^{p-d}(X \to Z)$ and $A^{p-d}(X \to Z) \to A^p(X \to Y)$. To finish the proof we will show these maps are mutually inverse.

Let us start with $c \in A^p(X \to Y)$. Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] & \\ X \times_Z Y \arrow[r] \arrow[d, "\mathrm{pr}_1"] & Y \times_Z Y \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

whose squares are carteisan. The lower two square of this diagram show that $res(c \circ g^!) = res(c) \cap p_2^!$ where in this formula $res(c)$ means the restriction of c via p_1. Looking at the upper square of the diagram and using Lemma 42.59.7 we get $c \circ \Delta^! = res(\Delta^!) \circ res(c)$. We compute

$$\begin{aligned} res(\Delta^!) \circ res(c \circ g^!) &= res(\Delta^!) \circ res(c) \circ p_2^! \\ &= c \circ \Delta^! \circ p_2^! \\ &= c \end{aligned}$$

The final equality by Lemma 42.60.1.

Conversely, let us start with $c' \in A^{p-d}(X \to Z)$. Looking at the lower rectangle of the diagram above we find $res(c') \circ g^! = \mathrm{pr}_1^! \circ c'$. We compute

$$\begin{aligned} res(\Delta^!) \circ res(c') \circ g^! &= res(\Delta^!) \circ \mathrm{pr}_1^! \circ c' \\ &= c' \end{aligned}$$

The final equality holds because the left two squares of the diagram show that $\mathrm{id} = res(\Delta^! \circ p_1^!) = res(\Delta^!) \circ \mathrm{pr}_1^!$. This finishes the proof. □

42.61. Exterior product

0FBU Let k be a field. In this section we work over $S = \mathrm{Spec}(k)$ with $\delta : S \to \mathbf{Z}$ defined by sending the unique point to 0, see Example 42.7.2.

Consider a cartesian square

$$\begin{tikzcd} X \times_k Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k) = S \end{tikzcd}$$

of schemes locally of finite type over k. Then there is a canonical map

$$\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{n+m}(X \times_k Y)$$

which is uniquely determined by the following rule: given integral closed subschemes $X' \subset X$ and $Y' \subset Y$ of dimensions n and m we have

$$[X'] \times [Y'] = [X' \times_k Y']_{n+m}$$

in $\mathrm{CH}_{n+m}(X \times_k Y)$.

0FBV Lemma 42.61.1. The map $\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y)$ is well defined.

Proof. A first remark is that if $\alpha = \sum n_i[X_i]$ and $\beta = \sum m_j[Y_j]$ with $X_i \subset X$ and $Y_j \subset Y$ locally finite families of integral closed subschemes of dimensions n and m, then $X_i \times_k Y_j$ is a locally finite collection of closed subschemes of $X \times_k Y$ of dimensions $n + m$ and we can indeed consider

$$\alpha \times \beta = \sum n_i m_j [X_i \times_k Y_j]_{n+m}$$

as a $(n+m)$-cycle on $X \times_k Y$. In this way we obtain an additive map $\times : Z_n(X) \otimes_{\mathbf{Z}} Z_m(Y) \to Z_{n+m}(X \times_k Y)$. The problem is to show that this procedure is compatible with rational equivalence.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of dimension n. Then flat pullback along the morphism $p' : X' \to \mathrm{Spec}(k)$ is an element $(p')^* \in A^{-n}(X' \to \mathrm{Spec}(k))$ by Lemma 42.33.2 and hence $c' = i_* \circ (p')^* \in A^{-n}(X \to \mathrm{Spec}(k))$ by Lemma 42.33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n}(X \times_k Y)$$

which the reader easily sends $[Y']$ to $[X' \times_k Y']_{n+m}$ for any integral closed subscheme $Y' \subset Y$ of dimension m. Hence the construction $([X'], [Y']) \mapsto [X' \times_k Y']_{n+m}$ factors through rational equivalence in the second variable, i.e., gives a well defined map $Z_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y)$. By symmetry the same is true for the other variable and we conclude. ☐

0FBW Lemma 42.61.2. Let k be a field. Let X be a scheme locally of finite type over k. Then we have a canonical identification

$$A^p(X \to \mathrm{Spec}(k)) = \mathrm{CH}_{-p}(X)$$

for all $p \in \mathbf{Z}$.

Proof. Consider the element $[\mathrm{Spec}(k)] \in \mathrm{CH}_0(\mathrm{Spec}(k))$. We get a map $A^p(X \to \mathrm{Spec}(k)) \to \mathrm{CH}_{-p}(X)$ by sending c to $c \cap [\mathrm{Spec}(k)]$.

Conversely, suppose we have $\alpha \in \mathrm{CH}_{-p}(X)$. Then we can define $c_\alpha \in A^p(X \to \mathrm{Spec}(k))$ as follows: given $X' \to \mathrm{Spec}(k)$ and $\alpha' \in \mathrm{CH}_n(X')$ we let

$$c_\alpha \cap \alpha' = \alpha \times \alpha'$$

in $\mathrm{CH}_{n-p}(X \times_k X')$. To show that this is a bivariant class we write $\alpha = \sum n_i[X_i]$ as in Definition 42.8.1. Consider the composition

$$\coprod X_i \xrightarrow{g} X \to \mathrm{Spec}(k)$$

and denote $f : \coprod X_i \to \mathrm{Spec}(k)$ the composition. Then g is proper and f is flat of relative dimension $-p$. Pullback along f is a bivariant class $f^* \in A^p(\coprod X_i \to$

$\mathrm{Spec}(k))$ by Lemma 42.33.2. Denote $\nu \in A^0(\coprod X_i)$ the bivariant class which multiplies a cycle by n_i on the ith component. Thus $\nu \circ f^* \in A^p(\coprod X_i \to X)$. Finally, we have a bivariant class

$$g_* \circ \nu \circ f^*$$

by Lemma 42.33.4. The reader easily verifies that c_α is equal to this class and hence is itself a bivariant class.

To finish the proof we have to show that the two constructions are mutually inverse. Since $c_\alpha \cap [\mathrm{Spec}(k)] = \alpha$ this is clear for one of the two directions. For the other, let $c \in A^p(X \to \mathrm{Spec}(k))$ and set $\alpha = c \cap [\mathrm{Spec}(k)]$. It suffices to prove that

$$c \cap [X'] = c_\alpha \cap [X']$$

when X' is an integral scheme locally of finite type over $\mathrm{Spec}(k)$, see Lemma 42.35.3. However, then $p' : X' \to \mathrm{Spec}(k)$ is flat of relative dimension $\dim(X')$ and hence $[X'] = (p')^*[\mathrm{Spec}(k)]$. Thus the fact that the bivariant classes c and c_α agree on $[\mathrm{Spec}(k)]$ implies they agree when capped against $[X']$ and the proof is complete. □

0FBX Lemma 42.61.3. Let k be a field. Let X be a scheme locally of finite type over k. Let $c \in A^p(X \to \mathrm{Spec}(k))$. Let $Y \to Z$ be a morphism of schemes locally of finite type over k. Let $c' \in A^q(Y \to Z)$. Then $c \circ c' = c' \circ c$ in $A^{p+q}(X \times_k Y \to Z)$.

Proof. In the proof of Lemma 42.61.2 we have seen that c is given by a combination of proper pushforward, multiplying by integers over connected components, and flat pullback. Since c' commutes with each of these operations by definition of bivariant classes, we conclude. Some details omitted. □

0FBY Remark 42.61.4. The upshot of Lemmas 42.61.2 and 42.61.3 is the following. Let k be a field. Let X be a scheme locally of finite type over k. Let $\alpha \in \mathrm{CH}_*(X)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over k. Let $c' \in A^q(Y \to Z)$. Then

$$\alpha \times (c' \cap \beta) = c' \cap (\alpha \times \beta)$$

in $\mathrm{CH}_*(X \times_k Y)$ for any $\beta \in \mathrm{CH}_*(Z)$. Namely, this follows by taking $c = c_\alpha \in A^*(X \to \mathrm{Spec}(k))$ the bivariant class corresponding to α, see proof of Lemma 42.61.2.

0FBZ Lemma 42.61.5. Exterior product is associative. More precisely, let k be a field, let X, Y, Z be schemes locally of finite type over k, let $\alpha \in \mathrm{CH}_*(X)$, $\beta \in \mathrm{CH}_*(Y)$, $\gamma \in \mathrm{CH}_*(Z)$. Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(X \times_k Y \times_k Z)$.

Proof. Omitted. Hint: associativity of fibre product of schemes. □

42.62. Intersection products

0FC0 Let k be a field. In this section we work over $S = \mathrm{Spec}(k)$ with $\delta : S \to \mathbf{Z}$ defined by sending the unique point to 0, see Example 42.7.2.

Let X be a smooth scheme over k. The bivariant class $\Delta^!$ of Section 42.60 allows us to define a kind of intersection product on chow groups of schemes locally of finite type over X. Namely, suppose that $Y \to X$ and $Z \to X$ are morphisms of schemes which are locally of finite type. Then observe that

$$Y \times_X Z = (Y \times_k Z) \times_{X \times_k X, \Delta} X$$

Hence we can consider the following sequence of maps

$$\mathrm{CH}_n(Y) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Z) \xrightarrow{\times} \mathrm{CH}_{n+m}(Y \times_k Z) \xrightarrow{\Delta^!} \mathrm{CH}_{n+m-*}(Y \times_X Z)$$

Here the first arrow is the exterior product constructed in Section 42.61 and the second arrow is the gysin map for the diagonal studied in Section 42.60. If X is equidimensional of dimension d, then we end up in $\mathrm{CH}_{n+m-d}(Y \times_X Z)$ and in general we can decompose into the parts lying over the open and closed subschemes of X where X has a given dimension. Given $\alpha \in \mathrm{CH}_*(Y)$ and $\beta \in \mathrm{CH}_*(Z)$ we will denote

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in \mathrm{CH}_*(Y \times_X Z)$$

In the special case where $X = Y = Z$ we obtain a multiplication

$$\mathrm{CH}_*(X) \times \mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is called the intersection product. We observe that this product is clearly symmetric. Associativity follows from the next lemma.

0FC1 Lemma 42.62.1. The product defined above is associative. More precisely, let k be a field, let X be smooth over k, let Y, Z, W be schemes locally of finite type over X, let $\alpha \in \mathrm{CH}_*(Y)$, $\beta \in \mathrm{CH}_*(Z)$, $\gamma \in \mathrm{CH}_*(W)$. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ in $\mathrm{CH}_*(Y \times_X Z \times_X W)$.

Proof. By Lemma 42.61.5 we have $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(Y \times_k Z \times_k W)$. Consider the closed immersions

$$\Delta_{12} : X \times_k X \longrightarrow X \times_k X \times_k X, \quad (x, x') \mapsto (x, x, x')$$

and

$$\Delta_{23} : X \times_k X \longrightarrow X \times_k X \times_k X, \quad (x, x') \mapsto (x, x', x')$$

Denote $\Delta_{12}^!$ and $\Delta_{23}^!$ the corresponding bivariant classes; observe that $\Delta_{12}^!$ is the restriction (Remark 42.33.5) of $\Delta^!$ to $X \times_k X \times_k X$ by the map pr_{12} and that $\Delta_{23}^!$ is the restriction of $\Delta^!$ to $X \times_k X \times_k X$ by the map pr_{23}. Thus clearly the restriction of $\Delta_{12}^!$ by Δ_{23} is $\Delta^!$ and the restriction of $\Delta_{23}^!$ by Δ_{12} is $\Delta^!$ too. Thus by Lemma 42.54.8 we have

$$\Delta^! \circ \Delta_{12}^! = \Delta^! \circ \Delta_{23}^!$$

Now we can prove the lemma by the following sequence of equalities:

$$\begin{aligned}
(\alpha \cdot \beta) \cdot \gamma &= \Delta^!(\Delta^!(\alpha \times \beta) \times \gamma) \\
&= \Delta^!(\Delta_{12}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!(\alpha \times (\beta \times \gamma)) \\
&= \Delta^!(\alpha \times \Delta^!(\beta \times \gamma)) \\
&= \alpha \cdot (\beta \cdot \gamma)
\end{aligned}$$

All equalities are clear from the above except perhaps for the second and penultimate one. The equation $\Delta_{23}^!(\alpha \times (\beta \times \gamma)) = \alpha \times \Delta^!(\beta \times \gamma)$ holds by Remark 42.61.4. Similarly for the second equation. □

0FC2 Lemma 42.62.2. Let k be a field. Let X be a smooth scheme over k, equidimensional of dimension d. The map

$$A^p(X) \longrightarrow \mathrm{CH}_{d-p}(X), \quad c \longmapsto c \cap [X]_d$$

is an isomorphism. Via this isomorphism composition of bivariant classes turns into the intersection product defined above.

Proof. Denote $g : X \to \mathrm{Spec}(k)$ the structure morphism. The map is the composition of the isomorphisms

$$A^p(X) \to A^{p-d}(X \to \mathrm{Spec}(k)) \to \mathrm{CH}_{d-p}(X)$$

The first is the isomorphism $c \mapsto c \circ g^*$ of Proposition 42.60.2 and the second is the isomorphism $c \mapsto c \cap [\mathrm{Spec}(k)]$ of Lemma 42.61.2. From the proof of Lemma 42.61.2 we see that the inverse to the second arrow sends $\alpha \in \mathrm{CH}_{d-p}(X)$ to the bivariant class c_α which sends $\beta \in \mathrm{CH}_*(Y)$ for Y locally of finite type over k to $\alpha \times \beta$ in $\mathrm{CH}_*(X \times_k Y)$. From the proof of Proposition 42.60.2 we see the inverse to the first arrow in turn sends c_α to the bivariant class which sends $\beta \in \mathrm{CH}_*(Y)$ for $Y \to X$ locally of finite type to $\Delta^!(\alpha \times \beta) = \alpha \cdot \beta$. From this the final result of the lemma follows. $\square$

0FFB Lemma 42.62.3. Let k be a field. Let $f : X \to Y$ be a morphism of schemes smooth over k. Then the gysin map exists for f and $f^!(\alpha \cdot \beta) = f^!\alpha \cdot f^!\beta$.

Proof. Observe that $X \to X \times_k Y$ is an immersion of X into a scheme smooth over Y. Hence the gysin map exists for f (Definition 42.59.4). To prove the formula we may decompose X and Y into their connected components, hence we may assume X is smooth over k and equidimensional of dimension d and Y is smooth over k and equidimensional of dimension e. Observe that $f^![Y]_e = [X]_d$ (see for example Lemma 42.59.8). Write $\alpha = c \cap [Y]_e$ and $\beta = c' \cap [Y]_e$ and hence $\alpha \cdot \beta = c \cap c' \cap [Y]_e$, see Lemma 42.62.2. By Lemma 42.59.7 we know that $f^!$ commutes with both c and c'. Hence

$$\begin{aligned} f^!(\alpha \cdot \beta) &= f^!(c \cap c' \cap [Y]_e) \\ &= c \cap c' \cap f^![Y]_e \\ &= c \cap c' \cap [X]_d \\ &= (c \cap [X]_d) \cdot (c' \cap [X]_d) \\ &= (c \cap f^![Y]_e) \cdot (c' \cap f^![Y]_e) \\ &= f^!(\alpha) \cdot f^!(\beta) \end{aligned}$$

as desired where we have used Lemma 42.62.2 for X as well.

An alternative proof can be given by proving that $(f \times f)^!(\alpha \times \beta) = f^!\alpha \times f^!\beta$ and using Lemma 42.59.6. $\square$

0FFC Lemma 42.62.4. Let k be a field. Let $f : X \to Y$ be a proper morphism of schemes smooth over k. Then the gysin map exists for f and $f_*(\alpha \cdot f^!\beta) = f_*\alpha \cdot \beta$.

Proof. Observe that $X \to X \times_k Y$ is an immersion of X into a scheme smooth over Y. Hence the gysin map exists for f (Definition 42.59.4). To prove the formula we may decompose X and Y into their connected components, hence we may assume X is smooth over k and equidimensional of dimension d and Y is smooth over k

and equidimensional of dimension e. Observe that $f^![Y]_e = [X]_d$ (see for example Lemma 42.59.8). Write $\alpha = c \cap [X]_d$ and $\beta = c' \cap [Y]_e$, see Lemma 42.62.2. We have

$$\begin{aligned} f_*(\alpha \cdot f^!\beta) &= f_*(c \cap f^!(c' \cap [Y]_e)) \\ &= f_*(c \cap c' \cap f^![Y]_e) \\ &= f_*(c \cap c' \cap [X]_d) \\ &= f_*(c' \cap c \cap [X]_d) \\ &= c' \cap f_*(c \cap [X]_d) \\ &= \beta \cdot f_*(\alpha) \end{aligned}$$

The first equality by the result of Lemma 42.62.2 for X. By Lemma 42.59.7 we know that $f^!$ commutes with c'. The commutativity of the intersection product justifies switching the order of capping $[X]_d$ with c and c' (via the lemma). Commuting c' with f_* is allowed as c' is a bivariant class. The final equality is again the lemma. □

0FFD Lemma 42.62.5. Let k be a field. Let X be an integral scheme smooth over k. Let $Y, Z \subset X$ be integral closed subschemes. Set $d = \dim(Y) + \dim(Z) - \dim(X)$. Assume

(1) $\dim(Y \cap Z) \leq d$, and
(2) $\mathcal{O}_{Y,\xi}$ and $\mathcal{O}_{Z,\xi}$ are Cohen-Macaulay for every $\xi \in Y \cap Z$ with $\delta(\xi) = d$.

Then $[Y] \cdot [Z] = [Y \cap Z]_d$ in $\mathrm{CH}_d(X)$.

Proof. Recall that $[Y] \cdot [Z] = \Delta^!([Y \times Z])$ where $\Delta^! = c(\Delta : X \to X \times X, \mathcal{T}_{X/k})$ is a higher codimension gysin map (Section 42.54) with $\mathcal{T}_{X/k} = \mathcal{H}om(\Omega_{X/k}, \mathcal{O}_X)$ locally free of rank $\dim(X)$. We have the equality of schemes

$$Y \cap Z = X \times_{\Delta, (X \times X)} (Y \times Z)$$

and $\dim(Y \times Z) = \dim(Y) + \dim(Z)$ and hence conditions (1), (2), and (3) of Lemma 42.54.6 hold. Finally, if $\xi \in Y \cap Z$, then we have a flat local homomorphism

$$\mathcal{O}_{Y,\xi} \longrightarrow \mathcal{O}_{Y \times Z,\xi}$$

whose "fibre" is $\mathcal{O}_{Z,\xi}$. It follows that if both $\mathcal{O}_{Y,\xi}$ and $\mathcal{O}_{Z,\xi}$ are Cohen-Macaulay, then so is $\mathcal{O}_{Y \times Z,\xi}$, see Algebra, Lemma 10.163.3. In this way we see that all the hypotheses of Lemma 42.54.6 are satisfied and we conclude. □

0FFE Lemma 42.62.6. Let k be a field. Let X be a scheme smooth over k. Let $i : Y \to X$ be a regular closed immersion. Let $\alpha \in \mathrm{CH}_*(X)$. If Y is equidimensional of dimension e, then $\alpha \cdot [Y]_e = i_*(i^!(\alpha))$ in $\mathrm{CH}_*(X)$.

Proof. After decomposing X into connected components we may and do assume X is equidimensional of dimension d. Write $\alpha = c \cap [X]_n$ with $x \in A^*(X)$, see Lemma 42.62.2. Then

$$i_*(i^!(\alpha)) = i_*(i^!(c \cap [X]_n)) = i_*(c \cap i^![X]_n) = i_*(c \cap [Y]_e) = c \cap i_*[Y]_e = \alpha \cdot [Y]_e$$

The first equality by choice of c. Then second equality by Lemma 42.59.7. The third because $i^![X]_d = [Y]_e$ in $\mathrm{CH}_*(Y)$ (Lemma 42.59.8). The fourth because bivariant classes commute with proper pushforward. The last equality by Lemma 42.62.2. □

0FFF Lemma 42.62.7. Let k be a field. Let X be a smooth scheme over k which is quasi-compact and has affine diagonal. Then the intersection product on $\mathrm{CH}^*(X)$ constructed in this section agrees after tensoring with $\mathbf{Q}$ with the intersection product constructed in Section 42.58.

Proof. Let $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(X)$. Write $\alpha = ch(\alpha') \cap [X]$ and $\beta = ch(\beta') \cap [X]$ $\alpha', \beta' \in K_0(\mathrm{Vect}(X)) \otimes \mathbf{Q}$ as in Section 42.58. Set $c = ch(\alpha')$ and $c' = ch(\beta')$. Then the intersection product in Section 42.58 produces $c \cap c' \cap [X]$. This is the same as $\alpha \cdot \beta$ by Lemma 42.62.2 (or rather the generalization that $A^i(X) \to \mathrm{CH}^i(X)$, $c \mapsto c \cap [X]$ is an isomorphism for any smooth scheme X over k). $\square$

42.63. Exterior product over Dedekind domains

0FC3 Let S be a locally Noetherian scheme which has an open covering by spectra of Dedekind domains. Set $\delta(s) = 0$ for $s \in S$ closed and $\delta(s) = 1$ otherwise. Then (S, δ) is a special case of our general Situation 42.7.1; see Example 42.7.3. Observe that S is normal (Algebra, Lemma 10.120.17) and hence a disjoint union of normal integral schemes (Properties, Lemma 28.7.7). Thus all of the arguments below reduce to the case where S is irreducible. On the other hand, we allow S to be nonseparated (so S could be the affine line with 0 doubled for example).

Consider a cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

of schemes locally of finite type over S. We claim there is a canonical map

$$\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{n+m-1}(X \times_S Y)$$

which is uniquely determined by the following rule: given integral closed subschemes $X' \subset X$ and $Y' \subset Y$ of δ-dimensions n and m we set

(1) $[X'] \times [Y'] = [X' \times_S Y']_{n+m-1}$ if X' or Y' dominates an irreducible component of S,
(2) $[X'] \times [Y'] = 0$ if neither X' nor Y' dominates an irreducible component of S.

0FC4 Lemma 42.63.1. The map $\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m-1}(X \times_S Y)$ is well defined.

Proof. Consider n and m cycles $\alpha = \sum_{i \in I} n_i[X_i]$ and $\beta = \sum_{j \in J} m_j[Y_j]$ with $X_i \subset X$ and $Y_j \subset Y$ locally finite families of integral closed subschemes of δ-dimensions n and m. Let $K \subset I \times J$ be the set of pairs $(i,j) \in I \times J$ such that X_i or Y_j dominates an irreducible component of S. Then $\{X_i \times_S Y_j\}_{(i,j) \in K}$ is a locally finite collection of closed subschemes of $X \times_S Y$ of δ-dimension $n + m - 1$. This means we can indeed consider

$$\alpha \times \beta = \sum\nolimits_{(i,j) \in K} n_i m_j [X_i \times_S Y_j]_{n+m-1}$$

as a $(n + m - 1)$-cycle on $X \times_S Y$. In this way we obtain an additive map $\times : Z_n(X) \otimes_{\mathbf{Z}} Z_m(Y) \to Z_{n+m}(X \times_S Y)$. The problem is to show that this procedure is compatible with rational equivalence.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of δ-dimension n which dominates an irreducible component of S. Then $p' : X' \to S$ is flat of relative dimension $n - 1$, see More on Algebra, Lemma 15.22.11. Hence flat pullback along p' is an element $(p')^* \in A^{-n+1}(X' \to S)$ by Lemma 42.33.2 and hence $c' = i_* \circ (p')^* \in A^{-n+1}(X \to S)$ by Lemma 42.33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n-1}(X \times_S Y)$$

which sends $[Y']$ to $[X' \times_S Y']_{n+m-1}$ for any integral closed subscheme $Y' \subset Y$ of δ-dimension m.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of δ-dimension n such that the composition $X' \to X \to S$ factors through a closed point $s \in S$. Since s is a closed point of the spectrum of a Dedekind domain, we see that s is an effective Cartier divisor on S whose normal bundle is trivial. Denote $c \in A^1(s \to S)$ the gysin homomorphism, see Lemma 42.33.3. The morphism $p' : X' \to s$ is flat of relative dimension n. Hence flat pullback along p' is an element $(p')^* \in A^{-n}(X' \to S)$ by Lemma 42.33.2. Thus

$$c' = i_* \circ (p')^* \circ c \in A^{-n}(X \to S)$$

by Lemma 42.33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n-1}(X \times_S Y)$$

which for any integral closed subscheme $Y' \subset Y$ of δ-dimension m sends $[Y']$ to either $[X' \times_S Y']_{n+m-1}$ if Y' dominates an irreducible component of S or to 0 if not.

From the previous two paragraphs we conclude the construction $([X'], [Y']) \mapsto [X' \times_S Y']_{n+m-1}$ factors through rational equivalence in the second variable, i.e., gives a well defined map $Z_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m-1}(X \times_S Y)$. By symmetry the same is true for the other variable and we conclude. $\square$

0FC5 Lemma 42.63.2. Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Then we have a canonical identification

$$A^p(X \to S) = \mathrm{CH}_{1-p}(X)$$

for all $p \in \mathbf{Z}$.

Proof. Consider the element $[S]_1 \in \mathrm{CH}_1(S)$. We get a map $A^p(X \to S) \to \mathrm{CH}_{1-p}(X)$ by sending c to $c \cap [S]_1$.

Conversely, suppose we have $\alpha \in \mathrm{CH}_{1-p}(X)$. Then we can define $c_\alpha \in A^p(X \to S)$ as follows: given $X' \to S$ and $\alpha' \in \mathrm{CH}_n(X')$ we let

$$c_\alpha \cap \alpha' = \alpha \times \alpha'$$

in $\mathrm{CH}_{n-p}(X \times_S X')$. To show that this is a bivariant class we write $\alpha = \sum_{i \in I} n_i [X_i]$ as in Definition 42.8.1. In particular the morphism

$$g : \coprod\nolimits_{i \in I} X_i \longrightarrow X$$

is proper. Pick $i \in I$. If X_i dominates an irreducible component of S, then the structure morphism $p_i : X_i \to S$ is flat and we have $\xi_i = p_i^* \in A^p(X_i \to S)$. On the other hand, if p_i factors as $p'_i : X_i \to s_i$ followed by the inclusion $s_i \to S$ of a closed point, then we have $\xi_i = (p'_i)^* \circ c_i \in A^p(X_i \to S)$ where $c_i \in A^1(s_i \to S)$ is the gysin homomorphism and $(p'_i)^*$ is flat pullback. Observe that

$$A^p(\coprod\nolimits_{i \in I} X_i \to S) = \prod\nolimits_{i \in I} A^p(X_i \to S)$$

Thus we have

$$\xi = \sum n_i \xi_i \in A^p(\coprod\nolimits_{i \in I} X_i \to S)$$

Finally, since g is proper we have a bivariant class

$$g_* \circ \xi \in A^p(X \to S)$$

by Lemma 42.33.4. The reader easily verifies that c_α is equal to this class (please compare with the proof of Lemma 42.63.1) and hence is itself a bivariant class.

To finish the proof we have to show that the two constructions are mutually inverse. Since $c_\alpha \cap [S]_1 = \alpha$ this is clear for one of the two directions. For the other, let $c \in A^p(X \to S)$ and set $\alpha = c \cap [S]_1$. It suffices to prove that

$$c \cap [X'] = c_\alpha \cap [X']$$

when X' is an integral scheme locally of finite type over S, see Lemma 42.35.3. However, either $p' : X' \to S$ is flat of relative dimension $\dim_\delta(X') - 1$ and hence $[X'] = (p')^*[S]_1$ or $X' \to S$ factors as $X' \to s \to S$ and hence $[X'] = (p')^*(s \to S)^*[S]_1$. Thus the fact that the bivariant classes c and c_α agree on $[S]_1$ implies they agree when capped against $[X']$ (since bivariant classes commute with flat pullback and gysin maps) and the proof is complete. ☐

0FC6 Lemma 42.63.3. Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Let $c \in A^p(X \to S)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over S. Let $c' \in A^q(Y \to Z)$. Then $c \circ c' = c' \circ c$ in $A^{p+q}(X \times_S Y \to X \times_S Z)$.

Proof. In the proof of Lemma 42.63.2 we have seen that c is given by a combination of proper pushforward, multiplying by integers over connected components, flat pullback, and gysin maps. Since c' commutes with each of these operations by definition of bivariant classes, we conclude. Some details omitted. ☐

0FC7 Remark 42.63.4. The upshot of Lemmas 42.63.2 and 42.63.3 is the following. Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Let $\alpha \in \mathrm{CH}_*(X)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over S. Let $c' \in A^q(Y \to Z)$. Then

$$\alpha \times (c' \cap \beta) = c' \cap (\alpha \times \beta)$$

in $\mathrm{CH}_*(X \times_S Y)$ for any $\beta \in \mathrm{CH}_*(Z)$. Namely, this follows by taking $c = c_\alpha \in A^*(X \to S)$ the bivariant class corresponding to α, see proof of Lemma 42.63.2.

0FC8 Lemma 42.63.5. Exterior product is associative. More precisely, let (S, δ) be as above, let X, Y, Z be schemes locally of finite type over S, let $\alpha \in \mathrm{CH}_*(X)$, $\beta \in \mathrm{CH}_*(Y)$, $\gamma \in \mathrm{CH}_*(Z)$. Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(X \times_S Y \times_S Z)$.

Proof. Omitted. Hint: associativity of fibre product of schemes. ☐

42.64. Intersection products over Dedekind domains

0FC9 Let S be a locally Noetherian scheme which has an open covering by spectra of Dedekind domains. Set $\delta(s) = 0$ for $s \in S$ closed and $\delta(s) = 1$ otherwise. Then (S, δ) is a special case of our general Situation 42.7.1; see Example 42.7.3 and discussion in Section 42.63.

Let X be a smooth scheme over S. The bivariant class $\Delta^!$ of Section 42.60 allows us to define a kind of intersection product on chow groups of schemes locally of finite type over X. Namely, suppose that $Y \to X$ and $Z \to X$ are morphisms of schemes which are locally of finite type. Then observe that

$$Y \times_X Z = (Y \times_S Z) \times_{X \times_S X, \Delta} X$$

Hence we can consider the following sequence of maps

$$\mathrm{CH}_n(Y) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \xrightarrow{\times} \mathrm{CH}_{n+m-1}(Y \times_S Z) \xrightarrow{\Delta^!} \mathrm{CH}_{n+m-*}(Y \times_X Z)$$

Here the first arrow is the exterior product constructed in Section 42.63 and the second arrow is the gysin map for the diagonal studied in Section 42.60. If X is equidimensional of dimension d, then $X \to S$ is smooth of relative dimension $d-1$ and hence we end up in $\mathrm{CH}_{n+m-d}(Y \times_X Z)$. In general we can decompose into the parts lying over the open and closed subschemes of X where X has a given dimension. Given $\alpha \in \mathrm{CH}_*(Y)$ and $\beta \in \mathrm{CH}_*(Z)$ we will denote

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in \mathrm{CH}_*(Y \times_X Z)$$

In the special case where $X = Y = Z$ we obtain a multiplication

$$\mathrm{CH}_*(X) \times \mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is called the intersection product. We observe that this product is clearly symmetric. Associativity follows from the next lemma.

0FCA **Lemma 42.64.1.** The product defined above is associative. More precisely, with (S, δ) as above, let X be smooth over S, let Y, Z, W be schemes locally of finite type over X, let $\alpha \in \mathrm{CH}_*(Y)$, $\beta \in \mathrm{CH}_*(Z)$, $\gamma \in \mathrm{CH}_*(W)$. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ in $\mathrm{CH}_*(Y \times_X Z \times_X W)$.

Proof. By Lemma 42.63.5 we have $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(Y \times_S Z \times_S W)$. Consider the closed immersions

$$\Delta_{12} : X \times_S X \longrightarrow X \times_S X \times_S X, \quad (x, x') \mapsto (x, x, x')$$

and

$$\Delta_{23} : X \times_S X \longrightarrow X \times_S X \times_S X, \quad (x, x') \mapsto (x, x', x')$$

Denote $\Delta_{12}^!$ and $\Delta_{23}^!$ the corresponding bivariant classes; observe that $\Delta_{12}^!$ is the restriction (Remark 42.33.5) of $\Delta^!$ to $X \times_S X \times_S X$ by the map pr_{12} and that $\Delta_{23}^!$ is the restriction of $\Delta^!$ to $X \times_S X \times_S X$ by the map pr_{23}. Thus clearly the restriction of $\Delta_{12}^!$ by Δ_{23} is $\Delta^!$ and the restriction of $\Delta_{23}^!$ by Δ_{12} is $\Delta^!$ too. Thus by Lemma 42.54.8 we have

$$\Delta^! \circ \Delta_{12}^! = \Delta^! \circ \Delta_{23}^!$$

Now we can prove the lemma by the following sequence of equalities:

$$\begin{aligned}(\alpha \cdot \beta) \cdot \gamma &= \Delta^!(\Delta^!(\alpha \times \beta) \times \gamma) \\ &= \Delta^!(\Delta^!_{12}((\alpha \times \beta) \times \gamma)) \\ &= \Delta^!(\Delta^!_{23}((\alpha \times \beta) \times \gamma)) \\ &= \Delta^!(\Delta^!_{23}(\alpha \times (\beta \times \gamma)) \\ &= \Delta^!(\alpha \times \Delta^!(\beta \times \gamma)) \\ &= \alpha \cdot (\beta \cdot \gamma)\end{aligned}$$

All equalities are clear from the above except perhaps for the second and penultimate one. The equation $\Delta^!_{23}(\alpha \times (\beta \times \gamma)) = \alpha \times \Delta^!(\beta \times \gamma)$ holds by Remark 42.61.4. Similarly for the second equation. ☐

0FCB Lemma 42.64.2. Let (S, δ) be as above. Let X be a smooth scheme over S, equidimensional of dimension d. The map

$$A^p(X) \longrightarrow \mathrm{CH}_{d-p}(X), \quad c \longmapsto c \cap [X]_d$$

is an isomorphism. Via this isomorphism composition of bivariant classes turns into the intersection product defined above.

Proof. Denote $g : X \to S$ the structure morphism. The map is the composition of the isomorphisms

$$A^p(X) \to A^{p-d+1}(X \to S) \to \mathrm{CH}_{d-p}(X)$$

The first is the isomorphism $c \mapsto c \circ g^*$ of Proposition 42.60.2 and the second is the isomorphism $c \mapsto c \cap [S]_1$ of Lemma 42.63.2. From the proof of Lemma 42.63.2 we see that the inverse to the second arrow sends $\alpha \in \mathrm{CH}_{d-p}(X)$ to the bivariant class c_α which sends $\beta \in \mathrm{CH}_*(Y)$ for Y locally of finite type over k to $\alpha \times \beta$ in $\mathrm{CH}_*(X \times_k Y)$. From the proof of Proposition 42.60.2 we see the inverse to the first arrow in turn sends c_α to the bivariant class which sends $\beta \in \mathrm{CH}_*(Y)$ for $Y \to X$ locally of finite type to $\Delta^!(\alpha \times \beta) = \alpha \cdot \beta$. From this the final result of the lemma follows. ☐

42.65. Todd classes

02UN A final class associated to a vector bundle $\mathcal{E}$ of rank r is its Todd class $Todd(\mathcal{E})$. In terms of the Chern roots $x_1, \ldots, x_r$ it is defined as

$$Todd(\mathcal{E}) = \prod\nolimits_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

In terms of the Chern classes $c_i = c_i(\mathcal{E})$ we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots$$

We have made the appropriate remarks about denominators in the previous section. It is the case that given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

42.66. Grothendieck-Riemann-Roch

02UO Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. Let $f : X \to Y$ be a proper smooth morphism. Assume that $R^i f_*\mathcal{E}$ are locally free sheaves on Y of finite rank. The Grothendieck-Riemann-Roch theorem say in this case that

$$f_*(Todd(T_{X/Y})ch(\mathcal{E})) = \sum(-1)^i ch(R^i f_*\mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of X over Y. If $Y = \mathrm{Spec}(k)$ where k is a field, then we can restate this as

$$\chi(X, \mathcal{E}) = \deg(Todd(T_{X/k})ch(\mathcal{E}))$$

The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

42.67. Change of base scheme

0FVF In this section we explain how to compare theories for different base schemes.

0FVG Situation 42.67.1. Here (S, δ) and (S', δ') are as in Situation 42.7.1. Furthermore $g : S' \to S$ is a flat morphism of schemes and $c \in \mathbf{Z}$ is an integer such that: for all $s \in S$ and $s' \in S'$ a generic point of an irreducible component of $g^{-1}(\{s\})$ we have $\delta(s') = \delta(s) + c$.

We will see that for a scheme X locally of finite type over S there is a well defined map $\mathrm{CH}_k(X) \to \mathrm{CH}_{k+c}(X \times_S S')$ of Chow groups which (by and large) commutes with the operations we have defined in this chapter.

0FVH Lemma 42.67.2. In Situation 42.67.1 let $X \to S$ be locally of finite type. Denote $X' \to S'$ the base change by $S' \to S$. If X is integral with $\dim_\delta(X) = k$, then every irreducible component Z' of X' has $\dim_{\delta'}(Z') = k + c$,

Proof. The projection $X' \to X$ is flat as a base change of the flat morphism $S' \to S$ (Morphisms, Lemma 29.25.8). Hence every generic point x' of an irreducible component of X' maps to the generic point $x \in X$ (because generalizations lift along $X' \to X$ by Morphisms, Lemma 29.25.9). Let $s \in S$ be the image of x. Recall that the scheme $S'_s = S' \times_S s$ has the same underlying topological space as $g^{-1}(\{s\})$ (Schemes, Lemma 26.18.5). We may view x' as a point of the scheme $S'_s \times_s x$ which comes equipped with a monomorphism $S'_s \times_s x \to S' \times_S X$. Of course, x' is a generic point of an irreducible component of $S'_s \times_s x$ as well. Using the flatness of $\mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\kappa(s)) = s$ and arguing as above, we see that x' maps to a generic point s' of an irreducible component of $g^{-1}(\{s\})$. Hence $\delta'(s') = \delta(s) + c$ by assumption. We have $\dim_x(X_s) = \dim_{x'}(X_{s'})$ by Morphisms, Lemma 29.28.3. Since x is a generic point of an irreducible component X_s (this is an irreducible scheme but we don't need this) and x' is a generic point of an irreducible component of $X'_{s'}$ we conclude that $\mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \mathrm{trdeg}_{\kappa(s')}(\kappa(x'))$ by Morphisms, Lemma 29.28.1. Then

$$\delta_{X'/S'}(x') = \delta(s') + \mathrm{trdeg}_{\kappa(s')}(\kappa(x')) = \delta(s) + c + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \delta_{X/S}(x) + c$$

This proves what we want by Definition 42.7.6. □

In Situation 42.67.1 let $X \to S$ be locally of finite type. Denote $X' \to S'$ the base change by $g : S' \to S$. There is a unique homomorphism

$$g^* : Z_k(X) \longrightarrow Z_{k+c}(X')$$

which given an integral closed subscheme $Z \subset X$ of δ-dimension k sends $[Z]$ to $[Z \times_S S']_{k+c}$. This makes sense by Lemma 42.67.2.

0FVI **Lemma 42.67.3.** In Situation 42.67.1 let $X \to S$ locally of finite type and let $X' \to S$ be the base change by $S' \to S$.

(1) Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$ and base change $Z' \subset X'$. Then we have $\dim_{\delta'}(Z')) \leq k + c$ and $[Z']_{k+c} = g^*[Z]_k$ in $Z_{k+c}(X')$.
(2) Let $\mathcal{F}$ be a coherent sheaf on X with $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ and base change $\mathcal{F}'$ on X'. Then we have $\dim_\delta(\text{Supp}(\mathcal{F}')) \leq k + c$ and $g^*[\mathcal{F}]_k = [\mathcal{F}']_{k+c}$ in $Z_{k+c}(X')$.

Proof. The proof is exactly the same is the proof of Lemma 42.14.4 and we suggest the reader skip it.

The statements on dimensions follow from Lemma 42.67.2. Part (1) follows from part (2) by Lemma 42.10.3 and the fact that the base change of the coherent module $\mathcal{O}_Z$ is $\mathcal{O}_{Z'}$.

Proof of (2). As X, X' are locally Noetherian we may apply Cohomology of Schemes, Lemma 30.9.1 to see that $\mathcal{F}$ is of finite type, hence $\mathcal{F}'$ is of finite type (Modules, Lemma 17.9.2), hence $\mathcal{F}'$ is coherent (Cohomology of Schemes, Lemma 30.9.1 again). Thus the lemma makes sense. Let $W \subset X$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X'$ be an integral closed subscheme of δ'-dimension $k + c$ mapping into W under $X' \to X$. We have to show that the coefficient n of $[W']$ in $g^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[\mathcal{F}']_{k+c}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{X',\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 42.10.1.) We have $\mathcal{F}'_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 10.52.13. $\square$

0FVJ **Lemma 42.67.4.** In Situation 42.67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. The map $g^* : Z_k(X) \to Z_{k+c}(X')$ above factors through rational equivalence to give a map

$$g^* : \text{CH}_k(X) \longrightarrow \text{CH}_{k+c}(X')$$

of chow groups.

Proof. Suppose that $\alpha \in Z_k(X)$ is a k-cycle which is rationally equivalent to zero. By Lemma 42.21.1 there exists a locally finite family of integral closed subschemes $W_i \subset X \times \mathbf{P}^1$ of δ-dimension k not contained in the divisors $(X \times \mathbf{P}^1)_0$ or $(X \times \mathbf{P}^1)_\infty$ of $X \times \mathbf{P}^1$ such that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$. Thus it suffices to prove for $W \subset X \times \mathbf{P}^1$ integral closed of δ-dimension k not contained in the divisors $(X \times \mathbf{P}^1)_0$ or $(X \times \mathbf{P}^1)_\infty$ of $X \times \mathbf{P}^1$ we have

(1) the base change $W' \subset X' \times \mathbf{P}^1$ satisfies the assumptions of Lemma 42.21.2 with k replaced by $k + c$, and
(2) $g^*[W_0]_k = [(W')_0]_{k+c}$ and $g^*[W_\infty]_k = [(W')_\infty]_{k+c}$.

Part (2) follows immediately from Lemma 42.67.3 and the fact that $(W')_0$ is the base change of W_0 (by associativity of fibre products). For part (1), first the statement on dimensions follows from Lemma 42.67.2. Then let $w' \in (W')_0$ with image $w \in W_0$ and $z \in \mathbf{P}^1_S$. Denote $t \in \mathcal{O}_{\mathbf{P}^1_S, z}$ the usual equation for $0 : S \to \mathbf{P}^1_S$. Since $\mathcal{O}_{W,w} \to \mathcal{O}_{W',w'}$ is flat and since t is a nonzerodivisor on $\mathcal{O}_{W,w}$ (as W is integral and $W \neq W_0$) we see that also t is a nonzerodivisor in $\mathcal{O}_{W',w'}$. Hence W' has no associated points lying on W'_0. $\square$

0FVK Lemma 42.67.5. In Situation 42.67.1 let $Y \to X \to S$ be locally of finite type and let $Y' \to X' \to S'$ be the base change by $S' \to S$. Assume $f : Y \to X$ is flat of relative dimension r. Then $f' : Y' \to X'$ is flat of relative dimension r and the diagrams

$$\begin{tikzcd} Z_{k+r}(Y) \arrow[r, "g^*"'] & Z_{k+c+r}(Y') \\ Z_k(X) \arrow[u, "(f')^*"] \arrow[r, "g^*"] & Z_{k+c}(X') \arrow[u, "f^*"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{CH}_{k+r}(Y) \arrow[r, "g^*"'] & \mathrm{CH}_{k+c+r}(Y') \\ \mathrm{CH}_k(X) \arrow[u, "(f')^*"] \arrow[r, "g^*"] & \mathrm{CH}_{k+c}(X') \arrow[u, "f^*"'] \end{tikzcd}$$

of cycle and chow groups commutes.

Proof. It suffices to show the first diagram commutes. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k and denote $Z' \subset X'$ its base change. By construction we have $g^*[Z] = [Z']_{k+c}$. By Lemma 42.14.4 we have $(f')^*g^*[Z] = [Z' \times_{X'} Y']_{k+c+r}$. Conversely, we have $f^*[Z] = [Z \times_X Y]_{k+r}$ by Definition 42.14.1. By Lemma 42.67.3 we have $g^*f^*[Z] = [(Z \times_X Y)']_{k+r+c}$. Since $(Z \times_X Y)' = Z' \times_{X'} Y'$ by associativity of fibre product we conclude. $\square$

0FVL Lemma 42.67.6. In Situation 42.67.1 let $Y \to X \to S$ be locally of finite type and let $Y' \to X' \to S'$ be the base change by $S' \to S$. Assume $f : Y \to X$ is proper. Then $f' : Y' \to X'$ is proper and the diagram

$$\begin{tikzcd} Z_k(Y) \arrow[r, "g^*"'] \arrow[d, "f_*"'] & Z_{k+c}(Y') \arrow[d, "f'_*"] \\ Z_k(X) \arrow[r, "g^*"] & Z_{k+c}(X') \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{CH}_k(Y) \arrow[r, "g^*"'] \arrow[d, "f_*"'] & \mathrm{CH}_{k+c}(Y') \arrow[d, "f'_*"] \\ \mathrm{CH}_k(X) \arrow[r, "g^*"] & \mathrm{CH}_{k+c}(X') \end{tikzcd}$$

of cycle and chow groups commutes.

Proof. It suffices to show the first diagram commutes. To see this, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k and denote $Z' \subset X'$ its base change. By construction we have $g^*[Z] = [Z']_{k+c}$. By Lemma 42.12.4 we have $(f')_*g^*[Z] = [f'_*\mathcal{O}_{Z'}]_{k+c}$. By the same lemma we have $f_*[Z] = [f_*\mathcal{O}_Z]_k$. By Lemma 42.67.3 we have $g^*f_*[Z] = [(X' \to X)^*f_*\mathcal{O}_Z]_{k+r}$. Thus it suffices to show that

$$(X' \to X)^*f_*\mathcal{O}_Z \cong f'_*\mathcal{O}_{Z'}$$

as coherent modules on X'. As $X' \to X$ is flat and as $\mathcal{O}_{Z'} = (Y' \to Y)^*\mathcal{O}_Z$, this follows from flat base change, see Cohomology of Schemes, Lemma 30.5.2. $\square$

0FVM Lemma 42.67.7. In Situation 42.67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with base change $\mathcal{L}'$ on X'. Then the diagram

$$\begin{tikzcd} \mathrm{CH}_k(X) \arrow[r, "g^*"'] \arrow[d, "c_1(\mathcal{L})\cap -"'] & \mathrm{CH}_{k+c}(X') \arrow[d, "c_1(\mathcal{L}')\cap -"] \\ \mathrm{CH}_{k-1}(X) \arrow[r, "g^*"] & \mathrm{CH}_{k+c-1}(X') \end{tikzcd}$$

of chow groups commutes.

Proof. Let $p : L \to X$ be the line bundle associated to $\mathcal{L}$ with zero section $o : X \to L$. For $\alpha \in CH_k(X)$ we know that $\beta = c_1(\mathcal{L})\cap\alpha$ is the unique element of $\mathrm{CH}_{k-1}(X)$ such that $o_*\alpha = -p^*\beta$, see Lemmas 42.32.2 and 42.32.4. The same characterization holds after pullback. Hence the lemma follows from Lemmas 42.67.5 and 42.67.6. □

0FVN Lemma 42.67.8. In Situation 42.67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of rank r with base change $\mathcal{E}'$ on X'. Then the diagram

$$\begin{tikzcd} \mathrm{CH}_k(X) \arrow[r, "g^*"'] \arrow[d, "c_i(\mathcal{E})\cap -"'] & \mathrm{CH}_{k+c}(X') \arrow[d, "c_i(\mathcal{E}')\cap -"] \\ \mathrm{CH}_{k-i}(X) \arrow[r, "g^*"] & \mathrm{CH}_{k+c-i}(X') \end{tikzcd}$$

of chow groups commutes for all i.

Proof. Set $P = \mathbf{P}(\mathcal{E})$. The base change P' of P is equal to $\mathbf{P}(\mathcal{E}')$. Since we already know that flat pullback and cupping with c_1 of an invertible module commute with base change (Lemmas 42.67.5 and 42.67.7) the lemma follows from the characterization of capping with $c_i(\mathcal{E})$ given in Lemma 42.38.2. □

0FVP Lemma 42.67.9. Let (S, δ), (S', δ'), (S'', δ'') be as in Situation 42.7.1. Let $g : S' \to S$ and $g' : S'' \to S'$ be flat morphisms of schemes and let $c, c' \in \mathbf{Z}$ be integers such that $S, \delta, S', \delta', g, c$ and S', δ', S'', g', c' are as in Situation 42.67.1. Let $X \to S$ be locally of finite type and denote $X' \to S'$ and $X'' \to S''$ the base changes by $S' \to S$ and $S'' \to S$. Then

(1) $S, \delta, S'', \delta'', g \circ g', c + c'$ is as in Situation 42.67.1,
(2) the maps $g^* : Z_k(X) \to Z_{k+c}(X')$ and $(g')^* : Z_{k+c}(X') \to Z_{k+c+c'}(X'')$ of compose to give the map $(g \circ g')^* : Z_k(X) \to Z_{k+c+c'}(X'')$, and
(3) the maps $g^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+c}(X')$ and $(g')^* : \mathrm{CH}_{k+c}(X') \to \mathrm{CH}_{k+c+c'}(X'')$ of Lemma 42.67.4 compose to give the map $(g \circ g')^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+c+c'}(X'')$ of Lemma 42.67.4.

Proof. Let $s \in S$ and let $s'' \in S''$ be a generic point of an irreducible component of $(g \circ g')^{-1}(\{s\})$. Set $s' = g'(s'')$. Clearly, s'' is a generic point of an irreducible component of $(g')^{-1}(\{s'\})$. Moreover, since g' is flat and hence generalizations lift along g' (Morphisms, Lemma 29.25.8) we see that also s' is a generic point of an irreducible component of $g^{-1}(\{s\})$. Thus by assumption $\delta'(s') = \delta(s) + c$ and $\delta''(s'') = \delta'(s') + c'$. We conclude $\delta''(s'') = \delta(s) + c + c'$ and the first part of the statement is true.

For the second part, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Denote $Z' \subset X'$ and $Z'' \subset X''$ the base changes. By definition we have $g^*[Z] = [Z']_{k+c}$. By Lemma 42.67.3 we have $(g')^*[Z']_{k+c} = [Z'']_{k+c+c'}$. This proves the final statement. $\square$

0FVQ Lemma 42.67.10. In Situation 42.67.1 assume $c = 0$ and assume that $S' = \lim_{i \in I} S_i$ is a filtered limit of schemes S_i affine over S such that

(1) with δ_i equal to $S_i \to S \xrightarrow{\delta} \mathbf{Z}$ the pair (S_i, δ_i) is as in Situation 42.7.1,
(2) $S_i, \delta_i, S, \delta, S \to S_i, c = 0$ is as in Situation 42.67.1,
(3) $S_i, \delta_i, S_{i'}, \delta_{i'}, S_i \to S_{i'}, c = 0$ for $i \geq i'$ is as in Situation 42.67.1.

Then for a quasi-compact scheme X of finite type over S with base change X' and X_i by $S' \to S$ and $S_i \to S$ we have $Z_k(X') = \operatorname{colim} Z_k(X_i)$ and $\mathrm{CH}_k(X') = \operatorname{colim} \mathrm{CH}_k(X_i)$.

Proof. By the result of Lemma 42.67.9 we obtain a system of cycle groups $Z_k(X_i)$ and a system of chow groups $\mathrm{CH}_k(X_i)$ as well as maps $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ and $\operatorname{colim} \mathrm{CH}_i(X_i) \to \mathrm{CH}_k(X')$. We may replace S by a quasi-compact open through which $X \to S$ factors, hence we may and do assume all the schemes occuring in this proof are Noetherian (and hence quasi-compact and quasi-separated).

Let us show that the map $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ is surjective. Namely, let $Z' \subset X'$ be an integral closed subscheme of δ'-dimension k. By Limits, Lemma 32.10.1 we can find an i and a morphism $Z_i \to X_i$ of finite presentation whose base change is Z'. Afer increasing i we may assume Z_i is a closed subscheme of X_i, see Limits, Lemma 32.8.5. Then $Z' \to X_i$ factors through Z_i and we may replace Z_i by the scheme theoretic image of $Z' \to X_i$. In this way we see that we may assume Z_i is an integral closed subscheme of X_i. By Lemma 42.67.2 we conclude that $\dim_{\delta_i}(Z_i) = \dim_{\delta'}(Z') = k$. Thus $Z_k(X_i) \to Z_k(X')$ maps $[Z_i]$ to $[Z']$ and we conclude surjectivity holds.

Let us show that the map $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ is injective. Let $\alpha_i = \sum n_j[Z_j] \in Z_k(X_i)$ be a cycle whose image in $Z_k(X')$ is zero. We may and do assume $Z_j \neq Z_{j'}$ if $j \neq j'$ and $n_j \neq 0$ for all j. Denote $Z'_j \subset X'$ the base change of Z_j. By Lemma 42.67.2 each irreducible component of Z'_j has δ'-dimension k. Moreover, as Z_j is irreducible and $Z'_j \to Z_j$ is flat (as the base change of $S' \to S$) we see that $Z'_j \to Z_j$ is dominant. Hence if Z'_j is nonempty, then some irreducible component, say Z', of Z'_j dominates Z_j. It follows that Z' cannot be an irreducible component of $Z'_{j'}$ for $j' \neq j$. Hence if Z'_j is nonempty, then we see that $(S' \to S_i)^*\alpha_i = \sum [Z'_j]_r$ is nonzero (as the coefficient of Z' would be nonzero). Thus we see that $Z'_j = \emptyset$ for all j. However, this means that the base change of Z_j by some transition map $S_{i'} \to S_i$ is empty by Limits, Lemma 32.4.3. Thus α_i dies in the colimit as desired.

The surjectivity of $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ implies that $\operatorname{colim} \mathrm{CH}_k(X_i) \to \mathrm{CH}_k(X')$ is surjective. To finish the proof we show that this map is injective. Let $\alpha_i \in \mathrm{CH}_k(X_i)$ be a cycle whose image $\alpha' \in \mathrm{CH}_k(X')$ is zero. Then there exist integral closed subschemes $W'_l \subset X'$, $l = 1, \ldots, r$ of δ"-dimension $k+1$ and nonzero rational functions f'_l on W'_l such that $\alpha' = \sum_{l=1,\ldots,r} \operatorname{div}_{W'_l}(f'_l)$. Arguing as above we can find an i and integral closed subschemes $W_{i,l} \subset X_i$ of δ_i-dimension $k+1$ whose base change is W'_l. After increasin i we may assume we have rational functions $f_{i,l}$ on $W_{i,l}$. Namely, we may think of f'_l as a section of the structure sheaf over a nonempty open $U'_l \subset W'_l$, we can descend these opens by Limits, Lemma 32.4.11 and after increasing i we may descend f'_l by Limits, Lemma 32.4.7. We claim that

$$\alpha_i = \sum\nolimits_{l=1,\ldots,r} \text{div}_{W_{i,l}}(f_{i,l})$$

after possibly increasing i.

To prove the claim, let $Z'_{l,j} \subset W'_l$ be a finite collection of integral closed subschemes of δ'-dimension k such that f'_l is an invertible regular function outside $\bigcup_j Y'_{l,j}$. After increasing i (by the arguments above) we may assume there exist integral closed subschemes $Z_{i,l,j} \subset W_i$ of δ_i-dimension k such that $f_{i,l}$ is an invertible regular function outside $\bigcup_j Z_{i,l,j}$. Then we may write

$$\text{div}_{W'_l}(f'_l) = \sum n_{l,j}[Z'_{l,j}]$$

and

$$\text{div}_{W_{i,l}}(f_{i,l}) = \sum n_{i,l,j}[Z_{i,l,j}]$$

To prove the claim it suffices to show that $n_{l,i} = n_{i,l,j}$. Namely, this will imply that $\beta_i = \alpha_i - \sum_{l=1,\ldots,r} \text{div}_{W_{i,l}}(f_{i,l})$ is a cycle on X_i whose pullback to X' is zero as a cycle! It follows that β_i pulls back to zero as a cycle on $X_{i'}$ for some $i' \geq i$ by an easy argument we omit.

To prove the equality $n_{l,i} = n_{i,l,j}$ we choose a generic point $\xi' \in Z'_{l,j}$ and we denote $\xi \in Z_{i,l,j}$ the image which is a generic point also. Then the local ring map

$$\mathcal{O}_{W_{i,l},\xi} \longrightarrow \mathcal{O}_{W'_l,\xi'}$$

is flat as $W'_l \to W_{i,l}$ is the base change of the flat morphism $S' \to S_i$. We also have $\mathfrak{m}_\xi \mathcal{O}_{W'_l,\xi'} = \mathfrak{m}_{\xi'}$ because $Z_{i,l,j}$ pulls back to $Z'_{l,j}$! Thus the equality of

$$n_{l,j} = \text{ord}_{Z'_{l,j}}(f'_l) = \text{ord}_{\mathcal{O}_{W'_l,\xi'}}(f'_l) \quad \text{and} \quad n_{i,l,j} = \text{ord}_{Z_{i,l,j}}(f_{i,l}) = \text{ord}_{\mathcal{O}_{W_{i,l},\xi}}(f_{i,l})$$

follows from Algebra, Lemma 10.52.13 and the construction of ord in Algebra, Section 10.121. □

42.68. Appendix A: Alternative approach to key lemma

0EAZ In this appendix we first define determinants $\det_\kappa(M)$ of finite length modules M over local rings $(R, \mathfrak{m}, \kappa)$, see Subsection 42.68.1. The determinant $\det_\kappa(M)$ is a 1-dimensional κ-vector space. We use this in Subsection 42.68.12 to define the determinant $\det_\kappa(M, \varphi, \psi) \in \kappa^*$ of an exact $(2,1)$-periodic complex (M, φ, ψ) with M of finite length. In Subsection 42.68.26 we use these determinants to construct a tame symbol $d_R(a, b) = \det_\kappa(R/ab, a, b)$ for a pair of nonzerodivisors $a, b \in R$ when R is Noetherian of dimension 1. Although there is no doubt that

$$d_R(a, b) = \partial_R(a, b)$$

where ∂_R is as in Section 42.5, we have not (yet) added the verification. The advantage of the tame symbol as constructed in this appendix is that it extends (for example) to pairs of injective endomorphisms φ, ψ of a finite R-module M of dimension 1 such that $\varphi(\psi(M)) = \psi(\varphi(M))$. In Subsection 42.68.40 we relate Herbrand quotients and determinants. An easy to state version of the main result (Proposition 42.68.43) is the formula

$$-e_R(M, \varphi, \psi) = \text{ord}_R(\det_K(M_K, \varphi, \psi))$$

when (M, φ, ψ) is a $(2,1)$-periodic complex whose Herbrand quotient e_R (Definition 42.2.2) is defined over a 1-dimensonal Noetherian local domain R with fraction field K. We use this proposition to give an alternative proof of the key lemma (Lemma 42.6.3) for the tame symbol constructed in this appendix, see Lemma 42.68.46.

02P5 42.68.1. Determinants of finite length modules. The material in this section is related to the material in the paper [KM76] and to the material in the thesis [Ros09].

Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\varphi : M \to M$ be an R-linear endomorphism of a finite length R-module M. In More on Algebra, Section 15.120 we have already defined the determinant $\det_\kappa(\varphi)$ (and the trace and the characteristic polynomial) of φ relative to κ. In this section, we will construct a canonical 1-dimensional κ-vector space $\det_\kappa(M)$ such that $\det_\kappa(\varphi : M \to M) : \det_\kappa(M) \to \det_\kappa(M)$ is equal to multiplication by $\det_\kappa(\varphi)$. If M is annihilated by $\mathfrak{m}$, then M can be viewed as a finite dimension κ-vector space and then we have $\det_\kappa(M) = \wedge^n_\kappa(M)$ where $n = \dim_\kappa(M)$. Our construction will generalize this to all finite length modules over R and if R contains its residue field, then the determinant $\det_\kappa(M)$ will be given by the usual determinant in a suitable sense, see Remark 42.68.9.

02P6 Definition 42.68.2. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. Say $l = \text{length}_R(M)$.

(1) Given elements $x_1, \ldots, x_r \in M$ we denote $\langle x_1, \ldots, x_r\rangle = Rx_1 + \ldots + Rx_r$ the R-submodule of M generated by $x_1, \ldots, x_r$.
(2) We will say an l-tuple of elements $(e_1, \ldots, e_l)$ of M is admissible if $\mathfrak{m}e_i \subset \langle e_1, \ldots, e_{i-1}\rangle$ for $i = 1, \ldots, l$.
(3) A symbol $[e_1, \ldots, e_l]$ will mean $(e_1, \ldots, e_l)$ is an admissible l-tuple.
(4) An admissible relation between symbols is one of the following:
   (a) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$, then $[e_1, \ldots, e_l] = 0$,
   (b) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$, then
   $$[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l]$$
   where $\overline{\lambda} \in \kappa^*$ is the image of λ in the residue field, and
   (c) if $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$ then
   $$[e_1, \ldots, e_l] = -[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l].$$
(5) We define the determinant of the finite length R-module M to be
$$\det_\kappa(M) = \left\{ \frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}} \right\}$$

We stress that always $l = \text{length}_R(M)$. We also stress that it does not follow that the symbol $[e_1, \ldots, e_l]$ is additive in the entries (this will typically not be the case). Before we can show that the determinant $\det_\kappa(M)$ actually has dimension 1 we have to show that it has dimension at most 1.

02P7 Lemma 42.68.3. With notations as above we have $\dim_\kappa(\det_\kappa(M)) \leq 1$.

Proof. Fix an admissible sequence $(f_1, \ldots, f_l)$ of M such that
$$\text{length}_R(\langle f_1, \ldots, f_i\rangle) = i$$

for $i = 1, \ldots, l$. Such an admissible sequence exists exactly because M has length l. We will show that any element of $\det_\kappa(M)$ is a κ-multiple of the symbol $[f_1, \ldots, f_l]$. This will prove the lemma.

Let $(e_1, \ldots, e_l)$ be an admissible sequence of M. It suffices to show that $[e_1, \ldots, e_l]$ is a multiple of $[f_1, \ldots, f_l]$. First assume that $\langle e_1, \ldots, e_l\rangle \neq M$. Then there exists an $i \in [1, \ldots, l]$ such that $e_i \in \langle e_1, \ldots, e_{i-1}\rangle$. It immediately follows from the first admissible relation that $[e_1, \ldots, e_n] = 0$ in $\det_\kappa(M)$. Hence we may assume that $\langle e_1, \ldots, e_l\rangle = M$. In particular there exists a smallest index $i \in \{1, \ldots, l\}$ such that $f_1 \in \langle e_1, \ldots, e_i\rangle$. This means that $e_i = \lambda f_1 + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$ and $\lambda \in R^*$. By the second admissible relation this means that $[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_l]$. Note that $\mathfrak m f_1 = 0$. Hence by applying the third admissible relation $i-1$ times we see that

$$[e_1, \ldots, e_l] = (-1)^{i-1}\overline{\lambda}[f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l].$$

Note that it is also the case that $\langle f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l\rangle = M$. By induction suppose we have proven that our original symbol is equal to a scalar times

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l]$$

for some admissible sequence $(f_1, \ldots, f_j, e_{j+1}, \ldots, e_l)$ whose elements generate M, i.e., with $\langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_l\rangle = M$. Then we find the smallest i such that $f_{j+1} \in \langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_i\rangle$ and we go through the same process as above to see that

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l] = (\text{scalar})[f_1, \ldots, f_j, f_{j+1}, e_{j+1}, \ldots, \hat{e_i}, \ldots, e_l]$$

Continuing in this vein we obtain the desired result. ☐

Before we show that $\det_\kappa(M)$ always has dimension 1, let us show that it agrees with the usual top exterior power in the case the module is a vector space over κ.

02P8 Lemma 42.68.4. Let R be a local ring with maximal ideal $\mathfrak m$ and residue field κ. Let M be a finite length R-module which is annihilated by $\mathfrak m$. Let $l = \dim_\kappa(M)$. Then the map

$$\det_\kappa(M) \longrightarrow \wedge^l_\kappa(M), \quad [e_1, \ldots, e_l] \longmapsto e_1 \wedge \ldots \wedge e_l$$

is an isomorphism.

Proof. It is clear that the rule described in the lemma gives a κ-linear map since all of the admissible relations are satisfied by the usual symbols $e_1 \wedge \ldots \wedge e_l$. It is also clearly a surjective map. Since by Lemma 42.68.3 the left hand side has dimension at most one we see that the map is an isomorphism. ☐

02P9 Lemma 42.68.5. Let R be a local ring with maximal ideal $\mathfrak m$ and residue field κ. Let M be a finite length R-module. The determinant $\det_\kappa(M)$ defined above is a κ-vector space of dimension 1. It is generated by the symbol $[f_1, \ldots, f_l]$ for any admissible sequence such that $\langle f_1, \ldots f_l\rangle = M$.

Proof. We know $\det_\kappa(M)$ has dimension at most 1, and in fact that it is generated by $[f_1, \ldots, f_l]$, by Lemma 42.68.3 and its proof. We will show by induction on $l = \text{length}(M)$ that it is nonzero. For $l = 1$ it follows from Lemma 42.68.4. Choose a nonzero element $f \in M$ with $\mathfrak m f = 0$. Set $\overline{M} = M/\langle f\rangle$, and denote the quotient map $x \mapsto \overline{x}$. We will define a surjective map

$$\psi : \det_k(M) \to \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of $\overline{M}$ is nonzero.

We define ψ on symbols as follows. Let $(e_1, \ldots, e_l)$ be an admissible sequence. If $f \not\in \langle e_1, \ldots, e_l\rangle$ then we simply set $\psi([e_1, \ldots, e_l]) = 0$. If $f \in \langle e_1, \ldots, e_l\rangle$ then we choose an i minimal such that $f \in \langle e_1, \ldots, e_i\rangle$. We may write $e_i = \lambda f + x$ for some unit $\lambda \in R$ and $x \in \langle e_1, \ldots, e_{i-1}\rangle$. In this case we set

$$\psi([e_1, \ldots, e_l]) = (-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l].$$

Note that it is indeed the case that $(\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l)$ is an admissible sequence in $\overline{M}$, so this makes sense. Let us show that extending this rule κ-linearly to linear combinations of symbols does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Then $i \neq a$ and $\overline{e}_a \in \langle \overline{e}_1, \ldots, \hat{\overline{e}}_i, \ldots, \overline{e}_{a-1}\rangle$ if $i < a$ or $\overline{e}_a \in \langle \overline{e}_1, \ldots, \overline{e}_{a-1}\rangle$ if $i > a$. Thus the same admissible relation for $\det_\kappa(\overline{M})$ forces the symbol $[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$ to be zero as desired.

Type (b) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \mu f + y$ with $y \in \langle e_1, \ldots, e_{i-1}\rangle$. If $i < a$ then the desired equality is

$$(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = (-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. If $i > a$ then the desired equality is

$$(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = (-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. The interesting case is when $i = a$. In this case we have $e_a = \lambda e'_a + x = \mu f + y$. Hence also $e'_a = \lambda^{-1}(\mu f + y - x)$. Thus we see that

$$\psi([e_1, \ldots, e_l]) = (-1)^i\overline{\mu}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \psi(\overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l])$$

as desired.

Type (c) relations. Suppose that $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$. We distinguish 4 cases:

Case 1: $i < a - 1$. The desired equality is

$$\begin{aligned}&(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]\\ &= (-1)^{i+1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]\end{aligned}$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$.

Case 2: $i > a$. The desired equality is

$$\begin{aligned}&(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]\\ &= (-1)^{i+1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]\end{aligned}$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$.

Case 3: $i = a$. We write $e_a = \lambda f + \mu e_{a-1} + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$. Then

$$\psi([e_1, \ldots, e_l]) = (-1)^a \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

by definition. If $\overline{\mu}$ is nonzero, then we have $e_{a-1} = -\mu^{-1}\lambda f + \mu^{-1}e_a - \mu^{-1}y$ and we obtain

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^a \overline{\mu^{-1}\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

by definition. Since in $\overline{M}$ we have $\overline{e}_a = \mu\overline{e}_{a-1} + \overline{y}$ we see the two outcomes are equal by relation (a) for $\det_\kappa(\overline{M})$. If on the other hand $\overline{\mu}$ is zero, then we can write $e_a = \lambda f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$ and we have

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^a \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is equal to $\psi([e_1, \ldots, e_l])$.

Case 4: $i = a - 1$. Here we have

$$\psi([e_1, \ldots, e_l]) = (-1)^{a-1} \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l]$$

by definition. If $f \not\in \langle e_1, \ldots, e_{a-2}, e_a\rangle$ then

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^{a+1} \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l]$$

Since $(-1)^{a-1} = (-1)^{a+1}$ the two expressions are the same. Finally, assume $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case we see that $e_{a-1} = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{a-2}\rangle$ and $e_a = \mu f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$ for units $\lambda, \mu \in R$. We conclude that both $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$ and $e_{a-1} \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case a relation of type (a) applies to both $[e_1, \ldots, e_l]$ and $[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]$ and the compatibility of ψ with these shown above to see that both

$$\psi([e_1, \ldots, e_l]) \quad \text{and} \quad \psi([e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l])$$

are zero, as desired.

At this point we have shown that ψ is well defined, and all that remains is to show that it is surjective. To see this let $(\overline{f}_2, \ldots, \overline{f}_l)$ be an admissible sequence in $\overline{M}$. We can choose lifts $f_2, \ldots, f_l \in M$, and then $(f, f_2, \ldots, f_l)$ is an admissible sequence in M. Since $\psi([f, f_2, \ldots, f_l]) = [f_2, \ldots, f_l]$ we win. $\square$

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Note that if $\varphi : M \to N$ is an isomorphism of finite length R-modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \to \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \ldots, e_l]) = [\varphi(e_1), \ldots, \varphi(e_l)]$$

for any symbol $[e_1, \ldots, e_l]$ for M. Hence we see that $\det_\kappa$ is a functor

05M7 (42.68.5.1)
$$\left\{\begin{matrix} \text{finite length } R\text{-modules} \\ \text{with isomorphisms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{1-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms}\end{matrix}\right\}$$

This is typical for a "determinant functor" (see [Knu02]), as is the following additivity property.

02PA **Lemma 42.68.6.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For every short exact sequence

$$0 \to K \to L \to M \to 0$$

of finite length R-modules there exists a canonical isomorphism

$$\gamma_{K \to L \to M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

defined by the rule on nonzero symbols

$$[e_1, \ldots, e_k] \otimes [\overline{f}_1, \ldots, \overline{f}_m] \longrightarrow [e_1, \ldots, e_k, f_1, \ldots, f_m]$$

with the following properties:

(1) For every isomorphism of short exact sequences, i.e., for every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "u"] & L \arrow[r] \arrow[d, "v"] & M \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

with short exact rows and isomorphisms u, v, w we have

$$\gamma_{K' \to L' \to M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K \to L \to M},$$

(2) for every commutative square of finite length R-modules with exact rows and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

the following diagram is commutative

$$\begin{tikzcd} \det_\kappa(A) \otimes \det_\kappa(C) \otimes \det_\kappa(G) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A \to B \to C} \otimes \gamma_{G \to H \to I}"'] \arrow[dd, "\epsilon"'] & \det_\kappa(B) \otimes \det_\kappa(H) \arrow[d, "\gamma_{B \to E \to H}"] \\ & \det_\kappa(E) \\ \det_\kappa(A) \otimes \det_\kappa(G) \otimes \det_\kappa(C) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A \to D \to G} \otimes \gamma_{C \to F \to I}"] & \det_\kappa(D) \otimes \det_\kappa(F) \arrow[u, "\gamma_{D \to E \to F}"'] \end{tikzcd}$$

where ϵ is the switch of the factors in the tensor product times $(-1)^{cg}$ with $c = \text{length}_R(C)$ and $g = \text{length}_R(G)$, and

(3) the map $\gamma_{K \to L \to M}$ agrees with the usual isomorphism if $0 \to K \to L \to M \to 0$ is actually a short exact sequence of κ-vector spaces.

Proof. The significance of taking nonzero symbols in the explicit description of the map $\gamma_{K\to L\to M}$ is simply that if $(e_1, \ldots, e_l)$ is an admissible sequence in K, and $(\overline{f}_1, \ldots, \overline{f}_m)$ is an admissible sequence in M, then it is not guaranteed that $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ is an admissible sequence in L (where of course $f_i \in L$ signifies a lift of $\overline{f}_i$). However, if the symbol $[e_1, \ldots, e_l]$ is nonzero in $\det_\kappa(K)$, then necessarily $K = \langle e_1, \ldots, e_k\rangle$ (see proof of Lemma 42.68.3), and in this case it is true that $(e_1, \ldots, e_k, f_1, \ldots, f_m)$ is an admissible sequence. Moreover, by the admissible relations of type (b) for $\det_\kappa(L)$ we see that the value of $[e_1, \ldots, e_k, f_1, \ldots, f_m]$ in $\det_\kappa(L)$ is independent of the choice of the lifts f_i in this case also. Given this remark, it is clear that an admissible relation for $e_1, \ldots, e_k$ in K translates into an admissible relation among $e_1, \ldots, e_k, f_1, \ldots, f_m$ in L, and similarly for an admissible relation among the $\overline{f}_1, \ldots, \overline{f}_m$. Thus γ defines a linear map of vector spaces as claimed in the lemma.

By Lemma 42.68.5 we know $\det_\kappa(L)$ is generated by any single symbol $[x_1, \ldots, x_{k+m}]$ such that $(x_1, \ldots, x_{k+m})$ is an admissible sequence with $L = \langle x_1, \ldots, x_{k+m}\rangle$. Hence it is clear that the map $\gamma_{K\to L\to M}$ is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_\kappa(v)([e_1, \ldots, e_k, f_1, \ldots, f_m]) \\ = \quad & [v(e_1), \ldots, v(e_k), v(f_1), \ldots, v(f_m)] \\ = \quad & \gamma_{K'\to L'\to M'}([u(e_1), \ldots, u(e_k)] \otimes [w(f_1), \ldots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol $[\alpha_1, \ldots, \alpha_a]$ generating $\det_\kappa(A)$, a symbol $[\gamma_1, \ldots, \gamma_c]$ generating $\det_\kappa(C)$, a symbol $[\zeta_1, \ldots, \zeta_g]$ generating $\det_\kappa(G)$, and a symbol $[\iota_1, \ldots, \iota_i]$ generating $\det_\kappa(I)$ we have

$$\begin{aligned} & [\alpha_1, \ldots, \alpha_a, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \\ = \quad & (-1)^{cg}[\alpha_1, \ldots, \alpha_a, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \end{aligned}$$

(for suitable lifts $\tilde{x}$ in E) in $\det_\kappa(E)$. This holds because we may use the admissible relations of type (c) cg times in the following order: move the $\tilde{\zeta}_1$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_1 \subset A$), then move $\tilde{\zeta}_2$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_2 \subset A + R\tilde{\zeta}_1$), and so on.

Part (3) of the lemma is obvious. This finishes the proof. $\square$

We can use the maps γ of the lemma to define more general maps γ as follows. Suppose that $(R, \mathfrak{m}, \kappa)$ is a local ring. Let M be a finite length R-module and suppose we are given a finite filtration (see Homology, Definition 12.19.1)

$$0 = F^m \subset F^{m-1} \subset \ldots \subset F^{n+1} \subset F^n = M$$

then there is a well defined and canonical isomorphism

$$\gamma_{(M,F)} : \det_\kappa(F^{m-1}/F^m) \otimes_\kappa \ldots \otimes_k \det_\kappa(F^n/F^{n+1}) \longrightarrow \det_\kappa(M)$$

To construct it we use isomorphisms of Lemma 42.68.6 coming from the short exact sequences $0 \to F^{i-1}/F^i \to M/F^i \to M/F^{i-1} \to 0$. Part (2) of Lemma 42.68.6 with $G = 0$ shows we obtain the same isomorphism if we use the short exact sequences $0 \to F^i \to F^{i-1} \to F^{i-1}/F^i \to 0$.

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

02PB Lemma 42.68.7. Let $(R, \mathfrak{m}, \kappa)$ be any local ring. The functor

$$\det_\kappa : \left\{ \begin{matrix} \text{finite length } R\text{-modules} \\ \text{with isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} 1\text{-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms} \end{matrix} \right\}$$

endowed with the maps $\gamma_{K \to L \to M}$ is characterized by the following properties

(1) its restriction to the subcategory of modules annihilated by $\mathfrak{m}$ is isomorphic to the usual determinant functor (see Lemma 42.68.4), and
(2) (1), (2) and (3) of Lemma 42.68.6 hold.

Proof. Omitted. □

02PC Lemma 42.68.8. Let $(R', \mathfrak{m}') \to (R, \mathfrak{m})$ be a local ring homomorphism which induces an isomorphism on residue fields κ. Then for every finite length R-module the restriction $M_{R'}$ is a finite length R'-module and there is a canonical isomorphism

$$\det_{R,\kappa}(M) \longrightarrow \det_{R',\kappa}(M_{R'})$$

This isomorphism is functorial in M and compatible with the isomorphisms $\gamma_{K \to L \to M}$ of Lemma 42.68.6 defined for $\det_{R,\kappa}$ and $\det_{R',\kappa}$.

Proof. If the length of M as an R-module is l, then the length of M as an R'-module (i.e., $M_{R'}$) is l as well, see Algebra, Lemma 10.52.12. Note that an admissible sequence $x_1, \ldots, x_l$ of M over R is an admissible sequence of M over R' as $\mathfrak{m}'$ maps into $\mathfrak{m}$. The isomorphism is obtained by mapping the symbol $[x_1, \ldots, x_l] \in \det_{R,\kappa}(M)$ to the corresponding symbol $[x_1, \ldots, x_l] \in \det_{R',\kappa}(M)$. It is immediate to verify that this is functorial for isomorphisms and compatible with the isomorphisms γ of Lemma 42.68.6. □

0BDQ Remark 42.68.9. Let $(R, \mathfrak{m}, \kappa)$ be a local ring and assume either the characteristic of κ is zero or it is p and $pR = 0$. Let $M_1, \ldots, M_n$ be finite length R-modules. We will show below that there exists an ideal $I \subset \mathfrak{m}$ annihilating M_i for $i = 1, \ldots, n$ and a section $\sigma : \kappa \to R/I$ of the canonical surjection $R/I \to \kappa$. The restriction $M_{i,\kappa}$ of M_i via σ is a κ-vector space of dimension $l_i = \text{length}_R(M_i)$ and using Lemma 42.68.8 we see that

$$\det_\kappa(M_i) = \wedge_\kappa^{l_i}(M_{i,\kappa})$$

These isomorphisms are compatible with the isomorphisms $\gamma_{K \to M \to L}$ of Lemma 42.68.6 for short exact sequences of finite length R-modules annihilated by I. The conclusion is that verifying a property of $\det_\kappa$ often reduces to verifying corresponding properties of the usual determinant on the category finite dimensional vector spaces.

For I we can take the annihilator (Algebra, Definition 10.40.3) of the module $M = \bigoplus M_i$. In this case we see that $R/I \subset \text{End}_R(M)$ hence has finite length. Thus R/I is an Artinian local ring with residue field κ. Since an Artinian local ring is complete we see that R/I has a coefficient ring by the Cohen structure theorem (Algebra, Theorem 10.160.8) which is a field by our assumption on R.

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 42.68.11 for a random example.

02PD Lemma 42.68.10. Let R be a local ring with residue field κ. Let $u \in R^*$ be a unit. Let M be a module of finite length over R. Denote $u_M : M \to M$ the map multiplication by u. Then

$$\det_\kappa(u_M) : \det_\kappa(M) \longrightarrow \det_\kappa(M)$$

is multiplication by $\overline{u}^l$ where $l = \text{length}_R(M)$ and $\overline{u} \in \kappa^*$ is the image of u.

Proof. Denote $f_M \in \kappa^*$ the element such that $\det_\kappa(u_M) = f_M \text{id}_{\det_\kappa(M)}$. Suppose that $0 \to K \to L \to M \to 0$ is a short exact sequence of finite R-modules. Then we see that u_k, u_L, u_M give an isomorphism of short exact sequences. Hence by Lemma 42.68.6 (1) we conclude that $f_K f_M = f_L$. This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial. □

02PE Example 42.68.11. Consider the local ring $R = \mathbf{Z}_p$. Set $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$. Let $u : M \to M$ be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Z}_p$, and $a, d \in \mathbf{Z}_p^*$. In this case $\det_\kappa(u)$ equals multiplication by $a^2d^3 \bmod p \in \mathbf{F}_p^*$. This can easily be seen by consider the effect of u on the symbol $[p^2e, pe, pf, e, f]$ where $e = (0, 1) \in M$ and $f = (1, 0) \in M$.

0BDR 42.68.12. Periodic complexes and determinants. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2, 1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. We are going to use the determinant construction to define an invariant of this situation. See Subsection 42.68.1. Let us abbreviate $K_\varphi = \text{Ker}(\varphi)$, $I_\varphi = \text{Im}(\varphi)$, $K_\psi = \text{Ker}(\psi)$, and $I_\psi = \text{Im}(\psi)$. The short exact sequences

$$0 \to K_\varphi \to M \to I_\varphi \to 0, \quad 0 \to K_\psi \to M \to I_\psi \to 0$$

give isomorphisms

$$\gamma_\varphi : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M), \quad \gamma_\psi : \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi) \longrightarrow \det_\kappa(M),$$

see Lemma 42.68.6. On the other hand the exactness of the complex gives equalities $K_\varphi = I_\psi$, and $K_\psi = I_\varphi$ and hence an isomorphism

$$\sigma : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

02PJ Definition 42.68.13. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2, 1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. The determinant of (M, φ, ψ) is the element

$$\det_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det_\kappa(M)$$

is multiplication by $(-1)^{\text{length}_R(I_\varphi)\text{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$.

02PK Remark 42.68.14. Here is a more down to earth description of the determinant introduced above. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Let us abbreviate $I_\varphi = \mathrm{Im}(\varphi)$, $I_\psi = \mathrm{Im}(\psi)$ as above. Assume that $\text{length}_R(I_\varphi) = a$ and $\text{length}_R(I_\psi) = b$, so that $a + b = \text{length}_R(M)$ by exactness. Choose admissible sequences $x_1, \ldots, x_a \in I_\varphi$ and $y_1, \ldots, y_b \in I_\psi$ such that the symbol $[x_1, \ldots, x_a]$ generates $\det_\kappa(I_\varphi)$ and the symbol $[x_1, \ldots, x_b]$ generates $\det_\kappa(I_\psi)$. Choose $\tilde{x}_i \in M$ such that $\varphi(\tilde{x}_i) = x_i$. Choose $\tilde{y}_j \in M$ such that $\psi(\tilde{y}_j) = y_j$. Then $\det_\kappa(M, \varphi, \psi)$ is characterized by the equality

$$[x_1, \ldots, x_a, \tilde{y}_1, \ldots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi)[y_1, \ldots, y_b, \tilde{x}_1, \ldots, \tilde{x}_a]$$

in $\det_\kappa(M)$. This also explains the sign.

02PL Lemma 42.68.15. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

Proof. Omitted. □

02PM Lemma 42.68.16. Let R be a local ring with residue field κ. Let (M, φ, φ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, φ) is exact. Then $\text{length}_R(M) = 2\text{length}_R(\mathrm{Im}(\varphi))$ and

$$\det_\kappa(M, \varphi, \varphi) = (-1)^{\text{length}_R(\mathrm{Im}(\varphi))} = (-1)^{\frac{1}{2}\text{length}_R(M)}$$

Proof. Follows directly from the sign rule in the definitions. □

02PN Lemma 42.68.17. Let R be a local ring with residue field κ. Let M be a finite length R-module.

(1) if $\varphi : M \to M$ is an isomorphism then $\det_\kappa(M, \varphi, 0) = \det_\kappa(\varphi)$.
(2) if $\psi : M \to M$ is an isomorphism then $\det_\kappa(M, 0, \psi) = \det_\kappa(\psi)^{-1}$.

Proof. Let us prove (1). Set $\psi = 0$. Then we may, with notation as above Definition 42.68.13, identify $K_\varphi = I_\psi = 0$, $I_\varphi = K_\psi = M$. With these identifications, the map

$$\gamma_\varphi : \kappa \otimes \det_\kappa(M) = \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M)$$

is identified with $\det_\kappa(\varphi^{-1})$. On the other hand the map γ_ψ is identified with the identity map. Hence $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ is equal to $\det_\kappa(\varphi)$ in this case. Whence the result. We omit the proof of (2). □

02PO Lemma 42.68.18. Let R be a local ring with residue field κ. Suppose that we have a short exact sequence of $(2,1)$-periodic complexes

$$0 \to (M_1, \varphi_1, \psi_1) \to (M_2, \varphi_2, \psi_2) \to (M_3, \varphi_3, \psi_3) \to 0$$

with all M_i of finite length, and each (M_1, φ_1, ψ_1) exact. Then

$$\det_\kappa(M_2, \varphi_2, \psi_2) = \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3).$$

in κ^*.

Proof. Let us abbreviate $I_{\varphi,i} = \mathrm{Im}(\varphi_i)$, $K_{\varphi,i} = \mathrm{Ker}(\varphi_i)$, $I_{\psi,i} = \mathrm{Im}(\psi_i)$, and $K_{\psi,i} = \mathrm{Ker}(\psi_i)$. Observe that we have a commutative square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_{\varphi,1} \arrow[r] \arrow[d] & K_{\varphi,2} \arrow[r] \arrow[d] & K_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{\varphi,1} \arrow[r] \arrow[d] & I_{\varphi,2} \arrow[r] \arrow[d] & I_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of finite length R-modules with exact rows and columns. The top row is exact since it can be identified with the sequence $I_{\psi,1} \to I_{\psi,2} \to I_{\psi,3} \to 0$ of images, and similarly for the bottom row. There is a similar diagram involving the modules $I_{\psi,i}$ and $K_{\psi,i}$. By definition $\det_\kappa(M_2, \varphi_2, \psi_2)$ corresponds, up to a sign, to the composition of the left vertical maps in the following diagram

$$\begin{tikzcd}
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] \arrow[d, "\gamma^{-1}\otimes\gamma^{-1}"] & \det_\kappa(M_2) \arrow[d, "\gamma^{-1}"] \\
\det_\kappa(K_{\varphi,1}) \otimes \det_\kappa(I_{\varphi,1}) \otimes \det_\kappa(K_{\varphi,3}) \otimes \det_\kappa(I_{\varphi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\sigma\otimes\sigma"] & \det_\kappa(K_{\varphi,2}) \otimes \det_\kappa(I_{\varphi,2}) \arrow[d, "\sigma"] \\
\det_\kappa(K_{\psi,1}) \otimes \det_\kappa(I_{\psi,1}) \otimes \det_\kappa(K_{\psi,3}) \otimes \det_\kappa(I_{\psi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma\otimes\gamma"] & \det_\kappa(K_{\psi,2}) \otimes \det_\kappa(I_{\psi,2}) \arrow[d, "\gamma"] \\
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] & \det_\kappa(M_2)
\end{tikzcd}$$

The top and bottom squares are commutative up to sign by applying Lemma 42.68.6 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that $\det_\kappa(M_2, \varphi_2, \psi_2) = \epsilon \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3)$ for some sign ϵ. And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
\epsilon &= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(K_{\varphi,3}) + \mathrm{length}(I_{\psi,1})\mathrm{length}(K_{\psi,3})} \\
&= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(I_{\psi,3}) + \mathrm{length}(I_{\psi,1})\mathrm{length}(I_{\varphi,3})}
\end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well. $\square$

02PP Example 42.68.19. Let k be a field. Consider the ring $R = k[T]/(T^2)$ of dual numbers over k. Denote t the class of T in R. Let $M = R$ and $\varphi = ut$, $\psi = vt$ with $u, v \in k^*$. In this case $\det_k(M)$ has generator $e = [t, 1]$. We identify $I_\varphi = K_\varphi = I_\psi = K_\psi = (t)$. Then $\gamma_\varphi(t \otimes t) = u^{-1}[t, 1]$ (since $u^{-1} \in M$ is a lift of $t \in I_\varphi$) and $\gamma_\psi(t \otimes t) = v^{-1}[t, 1]$ (same reason). Hence we see that $\det_k(M, \varphi, \psi) = -u/v \in k^*$.

02PQ Example 42.68.20. Let $R = \mathbf{Z}_p$ and let $M = \mathbf{Z}_p/(p^l)$. Let $\varphi = p^b u$ and $\varphi = p^a v$ with $a, b \geq 0$, $a + b = l$ and $u, v \in \mathbf{Z}_p^*$. Then a computation as in Example 42.68.19 shows that

$$\begin{aligned} \det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a / v^b \bmod p \\ &= (-1)^{\text{ord}_p(\alpha)\text{ord}_p(\beta)} \frac{\alpha^{\text{ord}_p(\beta)}}{\beta^{\text{ord}_p(\alpha)}} \bmod p \end{aligned}$$

with $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$. See Lemma 42.68.37 for a more general case (and a proof).

02PR Example 42.68.21. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus b}$ be $l = a + b$ dimensional. Let φ and ψ be the following diagonal matrices

$$\varphi = \text{diag}(u_1, \ldots, u_a, 0, \ldots, 0), \quad \psi = \text{diag}(0, \ldots, 0, v_1, \ldots, v_b)$$

with $u_i, v_j \in k^*$. In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \ldots u_a}{v_1 \ldots v_b}.$$

This can be seen by a direct computation or by computing in case $l = 1$ and using the additivity of Lemma 42.68.18.

02PS Example 42.68.22. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus a}$ be $l = 2a$ dimensional. Let φ and ψ be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with $U, V \in \text{Mat}(a \times a, k)$ invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case $a = 1$ is similar to the computation in Example 42.68.19.

02PT Example 42.68.23. Let $R = k$ be a field. Let $M = k^{\oplus 4}$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in k^*$. Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write $I_\varphi = \text{Im}(\varphi)$, and $K_\varphi = \text{Ker}(\varphi)$ for any R-module map $\varphi : M \to M$. We also denote $\varphi\psi = \varphi \circ \psi$ for a pair of morphisms $\varphi, \psi : M \to M$.

02PU Lemma 42.68.24. Let R be a local ring with residue field κ. Let M be a finite length R-module. Let α, β, γ be endomorphisms of M. Assume that

(1) $I_\alpha = K_{\beta\gamma}$, and similarly for any permutation of α, β, γ,
(2) $K_\alpha = I_{\beta\gamma}$, and similarly for any permutation of α, β, γ.

Then

(1) The triple $(M, \alpha, \beta\gamma)$ is an exact $(2,1)$-periodic complex.
(2) The triple $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex.
(3) The triple $(M/K_\beta, \alpha, \gamma)$ is an exact $(2,1)$-periodic complex.
(4) We have

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta) \det_\kappa(M/K_\beta, \alpha, \gamma).$$

Proof. It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that $I_{\gamma\alpha} = I_{\alpha\gamma}$, and similarly for kernels and any other pair of morphisms. Moreover, we see that $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$ and similarly for any other pair. In particular we get a short exact sequence

$$0 \to I_{\beta\gamma} \to I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \to 0$$

and similarly we get a short exact sequence

$$0 \to I_{\alpha\gamma} \to I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \to 0.$$

This proves $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex. Hence part (2) of the lemma holds.

To see that α, γ give well defined endomorphisms of M/K_β we have to check that $\alpha(K_\beta) \subset K_\beta$ and $\gamma(K_\beta) \subset K_\beta$. This is true because $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$, and similarly in the other case. The kernel of the map $\alpha : M/K_\beta \to M/K_\beta$ is $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$. Similarly, the kernel of $\gamma : M/K_\beta \to M/K_\beta$ is equal to I_α/K_β. Hence we conclude that (3) holds.

We introduce $r = \text{length}_R(K_\alpha)$, $s = \text{length}_R(K_\beta)$ and $t = \text{length}_R(K_\gamma)$. By the exact sequences above and our hypotheses we have $\text{length}_R(I_\alpha) = s + t$, $\text{length}_R(I_\beta) = r + t$, $\text{length}_R(I_\gamma) = r + s$, and $\text{length}(M) = r + s + t$. Choose

(1) an admissible sequence $x_1, \ldots, x_r \in K_\alpha$ generating K_α
(2) an admissible sequence $y_1, \ldots, y_s \in K_\beta$ generating K_β,
(3) an admissible sequence $z_1, \ldots, z_t \in K_\gamma$ generating K_γ,
(4) elements $\tilde{x}_i \in M$ such that $\beta\gamma\tilde{x}_i = x_i$,
(5) elements $\tilde{y}_i \in M$ such that $\alpha\gamma\tilde{y}_i = y_i$,
(6) elements $\tilde{z}_i \in M$ such that $\beta\alpha\tilde{z}_i = z_i$.

With these choices the sequence $y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t$ is an admissible sequence in I_α generating it. Hence, by Remark 42.68.14 the determinant $D = \det_\kappa(M, \alpha, \beta\gamma)$ is the unique element of κ^* such that

$$\begin{aligned} & [y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_s, \tilde{x}_1, \ldots, \tilde{x}_r] \\ &= (-1)^{r(s+t)} D[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t] \end{aligned}$$

By the same remark, we see that $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$ is characterized by

$$[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t, \tilde{x}_1, \ldots, \tilde{x}_r] = (-1)^{rt} D_1[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t]$$

By the same remark, we see that $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$ is characterized by

$$[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t] = (-1)^{rs} D_2[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]$$

Combining the formulas above we see that $D = D_1 D_2$ as desired. □

02PV Lemma 42.68.25. Let R be a local ring with residue field κ. Let $\alpha : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a morphism of $(2,1)$-periodic complexes over R. Assume

(1) M, M' have finite length,

(2) (M, φ, ψ), (M', φ', ψ') are exact,
(3) the maps φ, ψ induce the zero map on $K = \mathrm{Ker}(\alpha)$, and
(4) the maps φ, ψ induce the zero map on $Q = \mathrm{Coker}(\alpha)$.

Denote $N = \alpha(M) \subset M'$. We obtain two short exact sequences of $(2, 1)$-periodic complexes

$$0 \to (N, \varphi', \psi') \to (M', \varphi', \psi') \to (Q, 0, 0) \to 0$$
$$0 \to (K, 0, 0) \to (M, \varphi, \psi) \to (N, \varphi', \psi') \to 0$$

which induce two isomorphisms $\alpha_i : Q \to K$, $i = 0, 1$. Then

$$\det\nolimits_\kappa(M, \varphi, \psi) = \det\nolimits_\kappa(\alpha_0^{-1} \circ \alpha_1) \det\nolimits_\kappa(M', \varphi', \psi')$$

In particular, if $\alpha_0 = \alpha_1$, then $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$.

Proof. There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps α_i are. Namely, α_0 is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \to H^1(N, \varphi', \psi') \to H^2(K, 0, 0) = K$$

and α_1 is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \to H^2(N, \varphi', \psi') \to H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes (M, φ, ψ), (M', φ', ψ') are exact implies these maps are isomorphisms.

We will use the notation $I_\varphi = \mathrm{Im}(\varphi)$, $K_\varphi = \mathrm{Ker}(\varphi)$ and similarly for the other maps. Exactness for M and M' means that $K_\varphi = I_\psi$ and three similar equalities. We introduce $k = \mathrm{length}_R(K)$, $a = \mathrm{length}_R(I_\varphi)$, $b = \mathrm{length}_R(I_\psi)$. Then we see that $\mathrm{length}_R(M) = a + b$, and $\mathrm{length}_R(N) = a + b - k$, $\mathrm{length}_R(Q) = k$ and $\mathrm{length}_R(M') = a + b$. The exact sequences below will show that also $\mathrm{length}_R(I_{\varphi'}) = a$ and $\mathrm{length}_R(I_{\psi'}) = b$.

The assumption that $K \subset K_\varphi = I_\psi$ means that φ factors through N to give an exact sequence

$$0 \to \alpha(I_\psi) \to N \xrightarrow{\varphi\alpha^{-1}} I_\psi \to 0.$$

Here $\varphi\alpha^{-1}(x') = y$ means $x' = \alpha(x)$ and $y = \varphi(x)$. Similarly, we have

$$0 \to \alpha(I_\varphi) \to N \xrightarrow{\psi\alpha^{-1}} I_\varphi \to 0.$$

The assumption that ψ' induces the zero map on Q means that $I_{\psi'} = K_{\varphi'} \subset N$. This means the quotient $\varphi'(N) \subset I_{\varphi'}$ is identified with Q. Note that $\varphi'(N) = \alpha(I_\varphi)$. Hence we conclude there is an isomorphism

$$\varphi' : Q \to I_{\varphi'}/\alpha(I_\varphi)$$

simply described by $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$. In exactly the same way we get

$$\psi' : Q \to I_{\psi'}/\alpha(I_\psi)$$

Finally, note that α_0 is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$.

To shorten the formulas below we are going to write αx instead of $\alpha(x)$ in the following. No confusion should result since all maps are indicated by Greek letters and elements by Roman letters. We are going to choose

(1) an admissible sequence $z_1, \ldots, z_k \in K$ generating K,
(2) elements $z_i' \in M$ such that $\varphi z_i' = z_i$,
(3) elements $z_i'' \in M$ such that $\psi z_i'' = z_i$,
(4) elements $x_{k+1}, \ldots, x_a \in I_\varphi$ such that $z_1, \ldots, z_k, x_{k+1}, \ldots, x_a$ is an admissible sequence generating I_φ,
(5) elements $\tilde{x}_i \in M$ such that $\varphi\tilde{x}_i = x_i$,
(6) elements $y_{k+1}, \ldots, y_b \in I_\psi$ such that $z_1, \ldots, z_k, y_{k+1}, \ldots, y_b$ is an admissible sequence generating I_ψ,
(7) elements $\tilde{y}_i \in M$ such that $\psi\tilde{y}_i = y_i$, and
(8) elements $w_1, \ldots, w_k \in M'$ such that $w_1 \bmod N, \ldots, w_k \bmod N$ are an admissible sequence in Q generating Q.

By Remark 42.68.14 the element $D = \det_\kappa(M, \varphi, \psi) \in \kappa^*$ is characterized by

$$\begin{aligned} & [z_1, \ldots, z_k, x_{k+1}, \ldots, x_a, z_1'', \ldots, z_k'', \tilde{y}_{k+1}, \ldots, \tilde{y}_b] \\ = \quad & (-1)^{ab} D[z_1, \ldots, z_k, y_{k+1}, \ldots, y_b, z_1', \ldots, z_k', \tilde{x}_{k+1}, \ldots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above $\alpha x_{k+1}, \ldots, \alpha x_a, \varphi w_1, \ldots, \varphi w_k$ is an admissible sequence generating $I_{\varphi'}$ and $\alpha y_{k+1}, \ldots, \alpha y_b, \psi w_1, \ldots, \psi w_k$ is an admissible sequence generating $I_{\psi'}$. Hence by Remark 42.68.14 the element $D' = \det_\kappa(M', \varphi', \psi') \in \kappa^*$ is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b, w_1, \ldots, w_k] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a, w_1, \ldots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the first, resp. last k entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

in $\det_\kappa(N)$. Note that $\varphi' w_1, \ldots, \varphi' w_k$ and $\alpha z_1'', \ldots, z_k''$ are admissible sequences generating the module $I_{\varphi'}/\alpha(I_\varphi)$. Write

$$[\varphi' w_1, \ldots, \varphi' w_k] = \lambda_0[\alpha z_1'', \ldots, \alpha z_k'']$$

in $\det_\kappa(I_{\varphi'}/\alpha(I_\varphi))$ for some $\lambda_0 \in \kappa^*$. Similarly, write

$$[\psi' w_1, \ldots, \psi' w_k] = \lambda_1[\alpha z_1', \ldots, \alpha z_k']$$

in $\det_\kappa(I_{\psi'}/\alpha(I_\psi))$ for some $\lambda_1 \in \kappa^*$. On the one hand it is clear that

$$\alpha_i([w_1, \ldots, w_k]) = \lambda_i[z_1, \ldots, z_k]$$

for $i = 0, 1$ by our description of α_i above, which means that

$$\det_\kappa(\alpha_0^{-1} \circ \alpha_1) = \lambda_1/\lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0[\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z''_1, \ldots, \alpha z''_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z'_1, \ldots, \alpha z'_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \\ = \quad & [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

which imply $\lambda_0 D = \lambda_1 D'$. The lemma follows. ☐

02PW 42.68.26. Symbols. The correct generality for this construction is perhaps the situation of the following lemma.

02PX Lemma 42.68.27. Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Assume $\varphi, \psi : M \to M$ are two injective A-module maps, and assume $\varphi(\psi(M)) = \psi(\varphi(M))$, for example if φ and ψ commute. Then $\text{length}_R(M/\varphi\psi M) < \infty$ and $(M/\varphi\psi M, \varphi, \psi)$ is an exact $(2, 1)$-periodic complex.

Proof. Let $\mathfrak{q}$ be a minimal prime of the support of M. Then $M_{\mathfrak{q}}$ is a finite length $A_{\mathfrak{q}}$-module, see Algebra, Lemma 10.62.3. Hence both φ and ψ induce isomorphisms $M_{\mathfrak{q}} \to M_{\mathfrak{q}}$. Thus the support of $M/\varphi\psi M$ is $\{\mathfrak{m}_A\}$ and hence it has finite length (see lemma cited above). Finally, the kernel of φ on $M/\varphi\psi M$ is clearly $\psi M/\varphi\psi M$, and hence the kernel of φ is the image of ψ on $M/\varphi\psi M$. Similarly the other way since $M/\varphi\psi M = M/\psi\varphi M$ by assumption. ☐

02PY Lemma 42.68.28. Let A be a Noetherian local ring. Let $a, b \in A$.

(1) If M is a finite A-module of dimension 1 such that a, b are nonzerodivisors on M, then $\text{length}_A(M/abM) < \infty$ and $(M/abM, a, b)$ is a $(2, 1)$-periodic exact complex.
(2) If a, b are nonzerodivisors and $\dim(A) = 1$ then $\text{length}_A(A/(ab)) < \infty$ and $(A/(ab), a, b)$ is a $(2, 1)$-periodic exact complex.

In particular, in these cases $\det_\kappa(M/abM, a, b) \in \kappa^*$, resp. $\det_\kappa(A/(ab), a, b) \in \kappa^*$ are defined.

Proof. Follows from Lemma 42.68.27. ☐

02PZ Definition 42.68.29. Let A be a Noetherian local ring with residue field κ. Let $a, b \in A$. Let M be a finite A-module of dimension 1 such that a, b are nonzerodivisors on M. We define the symbol associated to M, a, b to be the element

$$d_M(a, b) = \det_\kappa(M/abM, a, b) \in \kappa^*$$

02Q0 Lemma 42.68.30. Let A be a Noetherian local ring. Let $a, b, c \in A$. Let M be a finite A-module with $\dim(\text{Supp}(M)) = 1$. Assume a, b, c are nonzerodivisors on M. Then

$$d_M(a, bc) = d_M(a, b) d_M(a, c)$$

and $d_M(a, b) d_M(b, a) = 1$.

Proof. The first statement follows from Lemma 42.68.24 applied to $M/abcM$ and endomorphisms α, β, γ given by multiplication by a, b, c. The second comes from Lemma 42.68.15. ☐

02Q1 Definition 42.68.31. Let A be a Noetherian local domain of dimension 1 with residue field κ. Let K be the fraction field of A. We define the tame symbol of A to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where $d_A(x, y)$ is extended to $K^* \times K^*$ by the multiplicativity of Lemma 42.68.30.

It is clear that we may extend more generally $d_M(-,-)$ to certain rings of fractions of A (even if A is not a domain).

0AY9 Lemma 42.68.32. Let A be a Noetherian local ring and M a finite A-module of dimension 1. Let $a \in A$ be a nonzerodivisor on M. Then $d_M(a, a) = (-1)^{\text{length}_A(M/aM)}$.

Proof. Immediate from Lemma 42.68.16. □

02Q2 Lemma 42.68.33. Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Let $b \in A$ be a nonzerodivisor on M, and let $u \in A^*$. Then

$$d_M(u, b) = u^{\text{length}_A(M/bM)} \bmod \mathfrak{m}_A.$$

In particular, if $M = A$, then $d_A(u, b) = u^{\text{ord}_A(b)} \bmod \mathfrak{m}_A$.

Proof. Note that in this case $M/ubM = M/bM$ on which multiplication by b is zero. Hence $d_M(u, b) = \det_\kappa(u|_{M/bM})$ by Lemma 42.68.17. The lemma then follows from Lemma 42.68.10. □

02Q3 Lemma 42.68.34. Let A be a Noetherian local ring. Let $a, b \in A$. Let

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of A-modules of dimension 1 such that a, b are nonzerodivisors on all three A-modules. Then

$$d_{M'}(a, b) = d_M(a, b) d_{M''}(a, b)$$

in κ^*.

Proof. It is easy to see that this leads to a short exact sequence of exact $(2, 1)$-periodic complexes

$$0 \to (M/abM, a, b) \to (M'/abM', a, b) \to (M''/abM'', a, b) \to 0$$

Hence the lemma follows from Lemma 42.68.18. □

02Q4 Lemma 42.68.35. Let A be a Noetherian local ring. Let $\alpha : M \to M'$ be a homomorphism of finite A-modules of dimension 1. Let $a, b \in A$. Assume

(1) a, b are nonzerodivisors on both M and M', and
(2) $\dim(\text{Ker}(\alpha)), \dim(\text{Coker}(\alpha)) \leq 0$.

Then $d_M(a, b) = d_{M'}(a, b)$.

Proof. If $a \in A^*$, then the equality follows from the equality $\text{length}(M/bM) = \text{length}(M'/bM')$ and Lemma 42.68.33. Similarly if b is a unit the lemma holds as well (by the symmetry of Lemma 42.68.30). Hence we may assume that $a, b \in \mathfrak{m}_A$. This in particular implies that $\mathfrak{m}$ is not an associated prime of M, and hence $\alpha : M \to M'$ is injective. This permits us to think of M as a submodule of M'. By assumption M'/M is a finite A-module with support $\{\mathfrak{m}_A\}$ and hence has finite length. Note that for any third module M'' with $M \subset M'' \subset M'$ the maps $M \to M''$ and $M'' \to M'$ satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of M'/M, to the case where $\text{length}_A(M'/M) = 1$. Finally, in this case consider the map

$$\overline{\alpha} : M/abM \longrightarrow M'/abM'.$$

By construction the cokernel Q of $\overline{\alpha}$ has length 1. Since $a, b \in \mathfrak{m}_A$, they act trivially on Q. It also follows that the kernel K of $\overline{\alpha}$ has length 1 and hence also a, b act trivially on K. Hence we may apply Lemma 42.68.25. Thus it suffices to see that the two maps $\alpha_i : Q \to K$ are the same. In fact, both maps are equal to the map $q = x' \bmod \text{Im}(\overline{\alpha}) \mapsto abx' \in K$. We omit the verification. □

02Q5 Lemma 42.68.36. Let A be a Noetherian local ring. Let M be a finite A-module with $\dim(\text{Supp}(M)) = 1$. Let $a, b \in A$ nonzerodivisors on M. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then

$$d_M(a, b) = \prod\nolimits_{i=1,\ldots,t} d_{A/\mathfrak{q}_i}(a, b)^{\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$$

as elements of κ^*.

Proof. Choose a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_j/M_{j-1} is isomorphic to $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of A. See Algebra, Lemma 10.62.1. For each j we have either $\mathfrak{p}_j = \mathfrak{q}_i$ for some i, or $\mathfrak{p}_j = \mathfrak{m}_A$. Moreover, for a fixed i, the number of j such that $\mathfrak{p}_j = \mathfrak{q}_i$ is equal to $\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$ by Algebra, Lemma 10.62.5. Hence $d_{M_j}(a, b)$ is defined for each j and

$$d_{M_j}(a, b) = \begin{cases} d_{M_{j-1}}(a, b) d_{A/\mathfrak{q}_i}(a, b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a, b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 42.68.34 in the first instance and Lemma 42.68.35 in the second. Hence the lemma. □

02Q6 Lemma 42.68.37. Let A be a discrete valuation ring with fraction field K. For nonzero $x, y \in K$ we have

$$d_A(x, y) = (-1)^{\text{ord}_A(x)\text{ord}_A(y)} \frac{x^{\text{ord}_A(y)}}{y^{\text{ord}_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

Proof. By multiplicativity it suffices to prove this when $x, y \in A$. Let $t \in A$ be a uniformizer. Write $x = t^b u$ and $y = t^b v$ for some $a, b \geq 0$ and $u, v \in A^*$. Set $l = a + b$. Then $t^{l-1}, \ldots, t^b$ is an admissible sequence in $(x)/(xy)$ and $t^{l-1}, \ldots, t^a$ is an admissible sequence in $(y)/(xy)$. Hence by Remark 42.68.14 we see that $d_A(x, y)$ is characterized by the equation

$$[t^{l-1}, \ldots, t^b, v^{-1}t^{b-1}, \ldots, v^{-1}] = (-1)^{ab} d_A(x, y) [t^{l-1}, \ldots, t^a, u^{-1}t^{a-1}, \ldots, u^{-1}].$$

Hence by the admissible relations for the symbols $[x_1, \ldots, x_l]$ we see that

$$d_A(x, y) = (-1)^{ab} u^a / v^b \bmod \mathfrak{m}_A$$

as desired. □

02Q8 Lemma 42.68.38. Let A be a Noetherian local ring. Let $a, b \in A$. Let M be a finite A-module of dimension 1 on which each of a, b, $b-a$ are nonzerodivisors. Then

$$d_M(a, b-a)d_M(b,b) = d_M(b, b-a)d_M(a,b)$$

in κ^*.

Proof. By Lemma 42.68.36 it suffices to show the relation when $M = A/\mathfrak q$ for some prime $\mathfrak q \subset A$ with $\dim(A/\mathfrak q) = 1$.

In case $M = A/\mathfrak q$ we may replace A by $A/\mathfrak q$ and a, b by their images in $A/\mathfrak q$. Hence we may assume $A = M$ and A a local Noetherian domain of dimension 1. The reason is that the residue field κ of A and $A/\mathfrak q$ are the same and that for any $A/\mathfrak q$-module M the determinant taken over A or over $A/\mathfrak q$ are canonically identified. See Lemma 42.68.8.

It suffices to show the relation when both a, b are in the maximal ideal. Namely, the case where one or both are units follows from Lemmas 42.68.33 and 42.68.32.

Choose an extension $A \subset A'$ and factorizations $a = ta'$, $b = tb'$ as in Lemma 42.4.2. Note that also $b - a = t(b' - a')$ and that $A' = (a', b') = (a', b' - a') = (b' - a', b')$. Here and in the following we think of A' as an A-module and a, b, a', b', t as A-module endomorphisms of A'. We will use the notation $d^A_{A'}(a', b')$ and so on to indicate

$$d^A_{A'}(a', b') = \det\nolimits_\kappa(A'/a'b'A', a', b')$$

which is defined by Lemma 42.68.27. The upper index ^{A} is used to distinguish this from the already defined symbol $d_{A'}(a', b')$ which is different (for example because it has values in the residue field of A' which may be different from κ). By Lemma 42.68.35 we see that $d_A(a, b) = d^A_{A'}(a, b)$, and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d^A_{A'}(a', b' - a')d^A_{A'}(b', b') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

Now, since $(a', b') = A'$ and so on we have

$$\begin{array}{rcl} A'/(a'(b'-a')) & \cong & A'/(a') \oplus A'/(b'-a') \\ A'/(b'(b'-a')) & \cong & A'/(b') \oplus A'/(b'-a') \\ A'/(a'b') & \cong & A'/(a') \oplus A'/(b') \end{array}$$

Moreover, note that multiplication by $b' - a'$ on $A/(a')$ is equal to multiplication by b', and that multiplication by $b' - a'$ on $A/(b')$ is equal to multiplication by $-a'$. Using Lemmas 42.68.17 and 42.68.18 we conclude

$$\begin{array}{rcl} d^A_{A'}(a', b'-a') & = & \det_\kappa(b'|_{A'/(a')})^{-1}\det_\kappa(a'|_{A'/(b'-a')}) \\ d^A_{A'}(b', b'-a') & = & \det_\kappa(-a'|_{A'/(b')})^{-1}\det_\kappa(b'|_{A'/(b'-a')}) \\ d^A_{A'}(a', b') & = & \det_\kappa(b'|_{A'/(a')})^{-1}\det_\kappa(a'|_{A'/(b')}) \end{array}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))}d^A_{A'}(a', b' - a') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

the sign coming from the $-a'$ in the second equality above. On the other hand, by Lemma 42.68.16 we have $d^A_{A'}(b', b') = (-1)^{\text{length}_A(A'/(b'))}$ and the lemma is proved. $\square$

The tame symbol is a Steinberg symbol.

02Q9 Lemma 42.68.39. Let A be a Noetherian local domain of dimension 1 with fraction field K. For $x \in K \setminus \{0, 1\}$ we have

$$d_A(x, 1-x) = 1$$

Proof. Write $x = a/b$ with $a, b \in A$. The hypothesis implies, since $1-x = (b-a)/b$, that also $b - a \neq 0$. Hence we compute

$$d_A(x, 1-x) = d_A(a, b-a) d_A(a, b)^{-1} d_A(b, b-a)^{-1} d_A(b, b)$$

Thus we have to show that $d_A(a, b-a) d_A(b, b) = d_A(b, b-a) d_A(a, b)$. This is Lemma 42.68.38. □

02QA 42.68.40. Lengths and determinants. In this section we use the determinant to compare lattices. The key lemma is the following.

02QB Lemma 42.68.41. Let R be a Noetherian local ring. Let $\mathfrak q \subset R$ be a prime with $\dim(R/\mathfrak q) = 1$. Let $\varphi : M \to N$ be a homomorphism of finite R-modules. Assume there exist $x_1, \ldots, x_l \in M$ and $y_1, \ldots, y_l \in M$ with the following properties

(1) $M = \langle x_1, \ldots, x_l \rangle$,
(2) $\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \cong R/\mathfrak q$ for $i = 1, \ldots, l$,
(3) $N = \langle y_1, \ldots, y_l \rangle$, and
(4) $\langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle \cong R/\mathfrak q$ for $i = 1, \ldots, l$.

Then φ is injective if and only if $\varphi_{\mathfrak q}$ is an isomorphism, and in this case we have

$$\text{length}_R(\text{Coker}(\varphi)) = \text{ord}_{R/\mathfrak q}(f)$$

where $f \in \kappa(\mathfrak q)$ is the element such that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l]$$

in $\det_{\kappa(\mathfrak q)}(N_{\mathfrak q})$.

Proof. First, note that the lemma holds in case $l = 1$. Namely, in this case x_1 is a basis of M over $R/\mathfrak q$ and y_1 is a basis of N over $R/\mathfrak q$ and we have $\varphi(x_1) = f y_1$ for some $f \in R$. Thus φ is injective if and only if $f \notin \mathfrak q$. Moreover, $\text{Coker}(\varphi) = R/(f, \mathfrak q)$ and hence the lemma holds by definition of $\text{ord}_{R/q}(f)$ (see Algebra, Definition 10.121.2).

In fact, suppose more generally that $\varphi(x_i) = f_i y_i$ for some $f_i \in R$, $f_i \notin \mathfrak q$. Then the induced maps

$$\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \longrightarrow \langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle$$

are all injective and have cokernels isomorphic to $R/(f_i, \mathfrak q)$. Hence we see that

$$\text{length}_R(\text{Coker}(\varphi)) = \sum \text{ord}_{R/\mathfrak q}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f_1 \ldots f_l [y_1, \ldots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on l. Assume $l > 1$. Let $i \in \{1, \ldots, l\}$ be minimal such that $\varphi(x_1) \in \langle y_1, \ldots, y_i\rangle$. We will argue by induction on i. If $i = 1$, then we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \langle x_1\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle/\langle x_1\rangle \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \langle y_1\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle/\langle y_1\rangle \arrow[r] & 0 \end{tikzcd}$$

and the lemma follows from the snake lemma and induction on l. Assume now that $i > 1$. Write $\varphi(x_1) = a_1y_1 + \ldots + a_{i-1}y_{i-1} + ay_i$ with $a_j, a \in R$ and $a \notin \mathfrak{q}$ (since otherwise i was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if} \quad j = 1 \\ ax_j & \text{if} \quad j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if} \quad j < i \\ ay_j & \text{if} \quad j \geq i \end{cases}$$

Let $M' = \langle x'_1, \ldots, x'_l\rangle$ and $N' = \langle y'_1, \ldots, y'_l\rangle$. Since $\varphi(x'_1) = a_1y'_1 + \ldots + a_{i-1}y'_{i-1} + y'_i$ by construction and since for $j > 1$ we have $\varphi(x'_j) = a\varphi(x_i) \in \langle y'_1, \ldots, y'_l\rangle$ we get a commutative diagram of R-modules and maps

$$\begin{tikzcd} M' \arrow[r, "\varphi'"'] \arrow[d] & N' \arrow[d] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

By the result of the second paragraph of the proof we know that $\text{length}_R(M/M') = (l-1)\text{ord}_{R/\mathfrak{q}}(a)$ and similarly $\text{length}_R(M/M') = (l-i+1)\text{ord}_{R/\mathfrak{q}}(a)$. By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i \ \text{ord}_{R/\mathfrak{q}}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l], \quad [\varphi'(x'_1), \ldots, \varphi(x'_l)] = f'[y'_1, \ldots, y'_l]$$

we have $f' = a^if$. Hence it suffices to prove the lemma for the case that $\varphi(x_1) = a_1y_1 + \ldots a_{i-1}y_{i-1} + y_i$, i.e., in the case that $a = 1$. Next, recall that

$$[y_1, \ldots, y_l] = [y_1, \ldots, y_{i-1}, a_1y_1 + \ldots a_{i-1}y_{i-1} + y_i, y_{i+1}, \ldots, y_l]$$

by the admissible relations for symbols. The sequence $y_1, \ldots, y_{i-1}, a_1y_1 + \ldots + a_{i-1}y_{i-1} + y_i, y_{i+1}, \ldots, y_l$ satisfies the conditions (3), (4) of the lemma also. Hence, we may actually assume that $\varphi(x_1) = y_i$. In this case, note that we have $\mathfrak{q}x_1 = 0$ which implies also $\mathfrak{q}y_i = 0$. We have

$$[y_1, \ldots, y_l] = -[y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l]$$

by the third of the admissible relations defining $\det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$. Hence we may replace $y_1, \ldots, y_l$ by the sequence $y'_1, \ldots, y'_l = y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l$ (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant i by 1 and we win by induction on i. □

To use the previous lemma we show that often sequences of elements with the required properties exist.

02QC Lemma 42.68.42. Let R be a local Noetherian ring. Let $\mathfrak{q} \subset R$ be a prime ideal. Let M be a finite R-module such that $\mathfrak{q}$ is one of the minimal primes of the support of M. Then there exist $x_1, \ldots, x_l \in M$ such that

(1) the support of $M/\langle x_1, \ldots, x_l\rangle$ does not contain $\mathfrak q$, and
(2) $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak q$ for $i = 1, \ldots, l$.

Moreover, in this case $l = \text{length}_{R_\mathfrak q}(M_\mathfrak q)$.

Proof. The condition that $\mathfrak q$ is a minimal prime in the support of M implies that $l = \text{length}_{R_\mathfrak q}(M_\mathfrak q)$ is finite (see Algebra, Lemma 10.62.3). Hence we can find $y_1, \ldots, y_l \in M_\mathfrak q$ such that $\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong \kappa(\mathfrak q)$ for $i = 1, \ldots, l$. We can find $f_i \in R$, $f_i \notin \mathfrak q$ such that $f_i y_i$ is the image of some element $z_i \in M$. Moreover, as R is Noetherian we can write $\mathfrak q = (g_1, \ldots, g_t)$ for some $g_j \in R$. By assumption $g_j y_i \in \langle y_1, \ldots, y_{i-1}\rangle$ inside the module $M_\mathfrak q$. By our choice of z_i we can find some further elements $f_{ji} \in R$, $f_{ij} \notin \mathfrak q$ such that $f_{ij} g_j z_i \in \langle z_1, \ldots, z_{i-1}\rangle$ (equality in the module M). The lemma follows by taking

$$x_1 = f_{11} f_{12} \ldots f_{1t} z_1, \quad x_2 = f_{11} f_{12} \ldots f_{1t} f_{21} f_{22} \ldots f_{2t} z_2,$$

and so on. Namely, since all the elements f_i, f_{ij} are invertible in $R_\mathfrak q$ we still have that $R_\mathfrak q x_1 + \ldots + R_\mathfrak q x_i / R_\mathfrak q x_1 + \ldots + R_\mathfrak q x_{i-1} \cong \kappa(\mathfrak q)$ for $i = 1, \ldots, l$. By construction, $\mathfrak q x_i \in \langle x_1, \ldots, x_{i-1}\rangle$. Thus $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle$ is an R-module generated by one element, annihilated $\mathfrak q$ such that localizing at $\mathfrak q$ gives a q-dimensional vector space over $\kappa(\mathfrak q)$. Hence it is isomorphic to $R/\mathfrak q$. □

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 42.68.44 his/herself.

02QD Proposition 42.68.43. Let R be a local Noetherian ring with residue field κ. Suppose that (M, φ, ψ) is a $(2, 1)$-periodic complex over R. Assume

(1) M is a finite R-module,
(2) the cohomology modules of (M, φ, ψ) are of finite length, and
(3) $\dim(\text{Supp}(M)) = 1$.

Let $\mathfrak q_i$, $i = 1, \ldots, t$ be the minimal primes of the support of M. Then we have[10]

$$-e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} \text{ord}_{R/\mathfrak q_i}\left(\det\nolimits_{\kappa(\mathfrak q_i)}(M_{\mathfrak q_i}, \varphi_{\mathfrak q_i}, \psi_{\mathfrak q_i})\right)$$

Proof. We first reduce to the case $t = 1$ in the following way. Note that $\text{Supp}(M) = \{\mathfrak m, \mathfrak q_1, \ldots, \mathfrak q_t\}$, where $\mathfrak m \subset R$ is the maximal ideal. Let M_i denote the image of $M \to M_{\mathfrak q_i}$, so $\text{Supp}(M_i) = \{\mathfrak m, \mathfrak q_i\}$. The map φ (resp. ψ) induces an R-module map $\varphi_i : M_i \to M_i$ (resp. $\psi_i : M_i \to M_i$). Thus we get a morphism of $(2, 1)$-periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,t} (M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support contained in $\{\mathfrak m\}$. Hence by Lemma 42.2.5 we have

$$e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have $M_{\mathfrak q_i} = M_{i,\mathfrak q_i}$, and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\text{ord}_{R/\mathfrak q_i}\left(\det\nolimits_{\kappa(\mathfrak q_i)}(M_{i,\mathfrak q_i}, \varphi_{i,\mathfrak q_i}, \psi_{i,\mathfrak q_i})\right)$$

[10]Obviously we could get rid of the minus sign by redefining $\det_\kappa(M, \varphi, \psi)$ as the inverse of its current value, see Definition 42.68.13.

In other words, if we can prove the lemma for each of the modules M_i, then the lemma holds. This reduces us to the case $t = 1$.

Assume we have a $(2,1)$-periodic complex (M, φ, ψ) over a Noetherian local ring with M a finite R-module, $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$, and finite length cohomology modules. The proof in this case follows from Lemma 42.68.41 and careful bookkeeping. Denote $K_\varphi = \text{Ker}(\varphi)$, $I_\varphi = \text{Im}(\varphi)$, $K_\psi = \text{Ker}(\psi)$, and $I_\psi = \text{Im}(\psi)$. Since R is Noetherian these are all finite R-modules. Set

$$a = \text{length}_{R_\mathfrak{q}}(I_{\varphi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\psi,\mathfrak{q}}), \quad b = \text{length}_{R_\mathfrak{q}}(I_{\psi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\varphi,\mathfrak{q}}).$$

Equalities because the complex becomes exact after localizing at $\mathfrak{q}$. Note that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is equal to $l = a + b$.

We are going to use Lemma 42.68.42 to choose sequences of elements in finite R-modules N with support contained in $\{\mathfrak{m}, \mathfrak{q}\}$. In this case $N_\mathfrak{q}$ has finite length, say $n \in \mathbf{N}$. Let us call a sequence $w_1, \ldots, w_n \in N$ with properties (1) and (2) of Lemma 42.68.42 a "good sequence". Note that the quotient $N/\langle w_1, \ldots, w_n\rangle$ of N by the submodule generated by a good sequence has support (contained in) $\{\mathfrak{m}\}$ and hence has finite length (Algebra, Lemma 10.62.3). Moreover, the symbol $[w_1, \ldots, w_n] \in \det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$ is a generator, see Lemma 42.68.5.

Having said this we choose good sequences

$$\begin{array}{llllll} x_1, \ldots, x_b & \text{in} & K_\varphi, & t_1, \ldots, t_a & \text{in} & K_\psi, \\ y_1, \ldots, y_a & \text{in} & I_\varphi \cap \langle t_1, \ldots t_a\rangle, & s_1, \ldots, s_b & \text{in} & I_\psi \cap \langle x_1, \ldots, x_b\rangle. \end{array}$$

We will adjust our choices a little bit as follows. Choose lifts $\tilde{y}_i \in M$ of $y_i \in I_\varphi$ and $\tilde{s}_i \in M$ of $s_i \in I_\psi$. It may not be the case that $\mathfrak{q}\tilde{y}_1 \subset \langle x_1, \ldots, x_b\rangle$ and it may not be the case that $\mathfrak{q}\tilde{s}_1 \subset \langle t_1, \ldots, t_a\rangle$. However, using that $\mathfrak{q}$ is finitely generated (as in the proof of Lemma 42.68.42) we can find a $d \in R$, $d \notin \mathfrak{q}$ such that $\mathfrak{q}d\tilde{y}_1 \subset \langle x_1, \ldots, x_b\rangle$ and $\mathfrak{q}d\tilde{s}_1 \subset \langle t_1, \ldots, t_a\rangle$. Thus after replacing y_i by dy_i, $\tilde{y}_i$ by $d\tilde{y}_i$, s_i by ds_i and $\tilde{s}_i$ by $d\tilde{s}_i$ we see that we may assume also that $x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_b$ and $t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b$ are good sequences in M.

Finally, we choose a good sequence $z_1, \ldots, z_l$ in the finite R-module

$$\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle \cap \langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle.$$

Note that this is also a good sequence in M.

Since $I_{\varphi,\mathfrak{q}} = K_{\psi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[y_1, \ldots, y_a] = h[t_1, \ldots, t_a]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\psi,\mathfrak{q}})$. Similarly, as $I_{\psi,\mathfrak{q}} = K_{\varphi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[s_1, \ldots, s_b] = g[x_1, \ldots, x_b]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\varphi,\mathfrak{q}})$. We can also do this with the three good sequences we have in M. All in all we get the following identities

$$\begin{aligned} [y_1, \ldots, y_a] &= h[t_1, \ldots, t_a] \\ [s_1, \ldots, s_b] &= g[x_1, \ldots, x_b] \\ [z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\ [z_1, \ldots, z_l] &= f_\psi[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \end{aligned}$$

for some $g, h, f_\varphi, f_\psi \in \kappa(\mathfrak{q})$.

Having set up all this notation let us compute $\det_{\kappa(\mathfrak q)}(M, \varphi, \psi)$. Namely, consider the element $[z_1, \ldots, z_l]$. Under the map $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ of Definition 42.68.13 we have

$$\begin{array}{rcl} [z_1, \ldots, z_l] & = & f_\varphi[x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a] \\ & \mapsto & f_\varphi[x_1, \ldots, x_b] \otimes [y_1, \ldots, y_a] \\ & \mapsto & f_\varphi h/g[t_1, \ldots, t_a] \otimes [s_1, \ldots, s_b] \\ & \mapsto & f_\varphi h/g[t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b] \\ & = & f_\varphi h/f_\psi g[z_1, \ldots, z_l] \end{array}$$

This means that $\det_{\kappa(\mathfrak q)}(M_{\mathfrak q}, \varphi_{\mathfrak q}, \psi_{\mathfrak q})$ is equal to $f_\varphi h/f_\psi g$ up to a sign.

We abbreviate the following quantities

$$\begin{array}{rcl} k_\varphi & = & \text{length}_R(K_\varphi/\langle x_1, \ldots, x_b\rangle) \\ k_\psi & = & \text{length}_R(K_\psi/\langle t_1, \ldots, t_a\rangle) \\ i_\varphi & = & \text{length}_R(I_\varphi/\langle y_1, \ldots, y_a\rangle) \\ i_\psi & = & \text{length}_R(I_\psi/\langle s_1, \ldots, s_a\rangle) \\ m_\varphi & = & \text{length}_R(M/\langle x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a\rangle) \\ m_\psi & = & \text{length}_R(M/\langle t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b\rangle) \\ \delta_\varphi & = & \text{length}_R(\langle x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a\rangle\langle z_1, \ldots, z_l\rangle) \\ \delta_\psi & = & \text{length}_R(\langle t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b\rangle\langle z_1, \ldots, z_l\rangle) \end{array}$$

Using the exact sequences $0 \to K_\varphi \to M \to I_\varphi \to 0$ we get $m_\varphi = k_\varphi + i_\varphi$. Similarly we have $m_\psi = k_\psi + i_\psi$. We have $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$ since this is equal to the colength of $\langle z_1, \ldots, z_l\rangle$ in M. Finally, we have

$$\delta_\varphi = \text{ord}_{R/\mathfrak q}(f_\varphi), \quad \delta_\psi = \text{ord}_{R/\mathfrak q}(f_\psi)$$

by our first application of the key Lemma 42.68.41.

Next, let us compute the multiplicity of the periodic complex

$$\begin{array}{rcl} e_R(M, \varphi, \psi) & = & \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\ & = & \text{length}_R(\langle x_1, \ldots, x_b\rangle/\langle s_1, \ldots, s_b\rangle) + k_\varphi - i_\psi \\ & & -\text{length}_R(\langle t_1, \ldots, t_a\rangle/\langle y_1, \ldots, y_a\rangle) - k_\psi + i_\varphi \\ & = & \text{ord}_{R/\mathfrak q}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\ & = & \text{ord}_{R/\mathfrak q}(g/h) + m_\varphi - m_\psi \\ & = & \text{ord}_{R/\mathfrak q}(g/h) + \delta_\psi - \delta_\varphi \\ & = & \text{ord}_{R/\mathfrak q}(f_\psi g/f_\varphi h) \end{array}$$

where we used the key Lemma 42.68.41 twice in the third equality. By our computation of $\det_{\kappa(\mathfrak q)}(M_{\mathfrak q}, \varphi_{\mathfrak q}, \psi_{\mathfrak q})$ this proves the proposition. □

In most applications the following lemma suffices.

02QE **Lemma 42.68.44.** Let R be a Noetherian local ring with maximal ideal $\mathfrak m$. Let M be a finite R-module, and let $\psi : M \to M$ be an R-module map. Assume that

(1) $\text{Ker}(\psi)$ and $\text{Coker}(\psi)$ have finite length, and
(2) $\dim(\text{Supp}(M)) \leq 1$.

Write $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ and denote $f_i \in \kappa(\mathfrak{q}_i)^*$ the element such that $\det_{\kappa(\mathfrak{q}_i)}(\psi_{\mathfrak{q}_i}) : \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}) \to \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i})$ is multiplication by f_i. Then we have

$$\text{length}_R(\text{Coker}(\psi)) - \text{length}_R(\text{Ker}(\psi)) = \sum\nolimits_{i=1,\ldots,t} \text{ord}_{R/\mathfrak{q}_i}(f_i).$$

Proof. Recall that $H^0(M, 0, \psi) = \text{Coker}(\psi)$ and $H^1(M, 0, \psi) = \text{Ker}(\psi)$, see remarks above Definition 42.2.2. The lemma follows by combining Proposition 42.68.43 with Lemma 42.68.17.

Alternative proof. Reduce to the case $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$ as in the proof of Proposition 42.68.43. Then directly combine Lemmas 42.68.41 and 42.68.42 to prove this specific case of Proposition 42.68.43. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted. □

02QI 42.68.45. Application to the key lemma. In this section we apply the results above to show the analogue of the key lemma (Lemma 42.6.3) with the tame symbol d_A constructed above. Please see Remark 42.6.4 for the relationship with Milnor K-theory.

02QJ Lemma 42.68.46 (Key Lemma). Let A be a 2-dimensional Noetherian local domain with fraction field K. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A^*_{\mathfrak{q}}$. Then we have

When A is an excellent ring this is [Kat86, Proposition 1].

$$\sum\nolimits_{i=1,\ldots,t} \text{ord}_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height one prime $\mathfrak{q}$ of A where $f, g \in A^*_{\mathfrak{q}}$ we have $d_{A_{\mathfrak{q}}}(f, g) = 1$ by Lemma 42.68.33.

Proof. Since the tame symbols $d_{A_{\mathfrak{q}}}(f, g)$ are additive (Lemma 42.68.30) and the order functions $\text{ord}_{A/\mathfrak{q}}$ are additive (Algebra, Lemma 10.121.1) it suffices to prove the formula when $f = a \in A$ and $g = b \in A$. In this case we see that we have to show

$$\sum\nolimits_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\det_\kappa(A_{\mathfrak{q}}/(ab), a, b)) = 0$$

By Proposition 42.68.43 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$ is exact we win. □

42.69. Appendix B: Alternative approaches

0AYD In this appendix we first briefly try to connect the material in the main text with K-theory of coherent sheaves. In particular we describe how cupping with c_1 of an invertible module is related to tensoring by this invertible module, see Lemma 42.69.7. This material is obviously very interesting and deserves a much more detailed and expansive exposition.

02S7 42.69.1. Rational equivalence and K-groups. This section is a continuation of Section 42.23. The motivation for the following lemma is Homology, Lemma 12.11.3.

02SB Lemma 42.69.2. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let

$$\ldots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \ldots$$

be a complex as in Homology, Equation (12.11.2.1). Assume that

(1) $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k+1$.
(2) $\dim_\delta(\text{Supp}(H^i(\mathcal{F}, \varphi, \psi))) \leq k$ for $i = 0, 1$.

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{rat} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as k-cycles on X.

Proof. Let $\{W_j\}_{j \in J}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k+1$. Note that $\{W_j\}$ is a locally finite collection of closed subsets of X by Lemma 42.10.1. For every j, let $\xi_j \in W_j$ be the generic point. Set

$$f_j = \det{}_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 42.68.13 for notation. We claim that

$$-[H^0(\mathcal{F}, \varphi, \psi)]_k + [H^1(\mathcal{F}, \varphi, \psi)]_k = \sum (W_j \to X)_* \text{div}(f_j)$$

If we prove this then the lemma follows.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z]$ in $[H^0(\mathcal{F}, \varphi, \psi)]_k - [H^1(\mathcal{F}, \varphi, \psi)]_k$ is the same as the coefficient m of $[Z]$ in $\sum (W_j \to X)_* \text{div}(f_j)$. Let $\xi \in Z$ be the generic point. Consider the local ring $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Denote $\varphi, \psi : M \to M$ the action of φ, ψ on the stalk. By our choice of $\xi \in Z$ we have $\delta(\xi) = k$ and hence $\dim(\text{Supp}(M)) = 1$. Finally, the integral closed subschemes W_j passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Supp}(M)$. In each case the element $f_j \in R(W_j)^*$ corresponds to the element $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$ in $\kappa(\mathfrak{q}_i)^*$. Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \text{ord}_{A/\mathfrak{q}_i}(\det{}_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 42.68.43. □

02SC Lemma 42.69.3. Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. The map

$$\text{CH}_k(X) \longrightarrow K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$$

from Lemma 42.23.4 induces a bijection from $\text{CH}_k(X)$ onto the image $B_k(X)$ of the map

$$K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X)).$$

Proof. By Lemma 42.23.2 we have $Z_k(X) = K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$ compatible with the map of Lemma 42.23.4. Thus, suppose we have an element $[A] - [B]$ of $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$ which maps to zero in $B_k(X)$, i.e., maps to zero in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$. We have to show that $[A] - [B]$ corresponds to a cycle rationally equivalent to zero on X. Suppose $[A] = [\mathcal{A}]$ and $[B] = [\mathcal{B}]$ for some coherent sheaves $\mathcal{A}, \mathcal{B}$ on X supported in δ-dimension $\leq k$. The assumption that $[A] - [B]$ maps to zero in the group $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ means that there exists coherent sheaves $\mathcal{A}', \mathcal{B}'$ on X supported in δ-dimension $\leq k-1$ such that $[\mathcal{A} \oplus \mathcal{A}'] - [\mathcal{B} \oplus \mathcal{B}']$ is zero in $K_0(\text{Coh}_{k+1}(X))$ (use part (1) of Homology, Lemma 12.11.3). By part (2) of Homology, Lemma 12.11.3 this means there exists a $(2,1)$-periodic complex $(\mathcal{F}, \varphi, \psi)$ in the category $\text{Coh}_{\leq k+1}(X)$ such that $\mathcal{A} \oplus \mathcal{A}' = H^0(\mathcal{F}, \varphi, \psi)$ and $\mathcal{B} \oplus \mathcal{B}' = H^1(\mathcal{F}, \varphi, \psi)$. By Lemma 42.69.2 this implies that

$$[\mathcal{A} \oplus \mathcal{A}']_k \sim_{rat} [\mathcal{B} \oplus \mathcal{B}']_k$$

This proves that $[A] - [B]$ maps to a cycle rationally equivalent to zero by the map

$$K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

of Lemma 42.23.2. This is what we had to prove and the proof is complete. $\square$

02SV **42.69.4. Cartier divisors and K-groups.** In this section we describe how the intersection with the first Chern class of an invertible sheaf $\mathcal{L}$ corresponds to tensoring with $\mathcal{L} - \mathcal{O}$ in K-groups.

02QH **Lemma 42.69.5.** Let A be a Noetherian local ring. Let M be a finite A-module. Let $a, b \in A$. Assume

(1) $\dim(A) = 1$,
(2) both a and b are nonzerodivisors in A,
(3) A has no embedded primes,
(4) M has no embedded associated primes,
(5) $\text{Supp}(M) = \text{Spec}(A)$.

Let $I = \{x \in A \mid x(a/b) \in A\}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of A. Then $(a/b)IM \subset M$ and

$$\text{length}_A(M/(a/b)IM) - \text{length}_A(M/IM) = \sum\nolimits_i \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})\text{ord}_{A/\mathfrak{q}_i}(a/b)$$

Proof. Since M has no embedded associated primes, and since the support of M is $\text{Spec}(A)$ we see that $\text{Ass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Hence a, b are nonzerodivisors on M. Note that

$$\begin{aligned} &\text{length}_A(M/(a/b)IM) \\ &= \text{length}_A(bM/aIM) \\ &= \text{length}_A(M/aIM) - \text{length}_A(M/bM) \\ &= \text{length}_A(M/aM) + \text{length}_A(aM/aIM) - \text{length}_A(M/bM) \\ &= \text{length}_A(M/aM) + \text{length}_A(M/IM) - \text{length}_A(M/bM) \end{aligned}$$

as the injective map $b : M \to bM$ maps $(a/b)IM$ to aIM and the injective map $a : M \to aM$ maps IM to aIM. Hence the left hand side of the equation of the lemma is equal to

$$\text{length}_A(M/aM) - \text{length}_A(M/bM).$$

Applying the second formula of Lemma 42.3.2 with $x = a, b$ respectively and using Algebra, Definition 10.121.2 of the ord-functions we get the result. $\square$

02SW Lemma 42.69.6. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a meromorphic section of $\mathcal{L}$. Assume

(1) $\dim_\delta(X) \leq k + 1$,
(2) X has no embedded points,
(3) $\mathcal{F}$ has no embedded associated points,
(4) the support of $\mathcal{F}$ is X, and
(5) the section s is regular meromorphic.

In this situation let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s, see Divisors, Definition 31.23.10. Then we have the following:

(1) there are short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

(2) the coherent sheaves $\mathcal{Q}_1$, $\mathcal{Q}_2$ are supported in δ-dimension $\leq k$,
(3) the section s restricts to a regular meromorphic section s_i on every irreducible component X_i of X of δ-dimension $k + 1$, and
(4) writing $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$ we have

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* \text{div}_{\mathcal{L}|_{X_i}}(s_i)$$

in $Z_k(X)$, in particular

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$

in $\text{CH}_k(X)$.

Proof. Recall from Divisors, Lemma 31.24.5 the existence of injective maps $1 : \mathcal{I}\mathcal{F} \to \mathcal{F}$ and $s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ whose cokernels are supported on a closed nowhere dense subsets T. Denote $\mathcal{Q}_i$ there cokernels as in the lemma. We conclude that $\dim_\delta(\text{Supp}(\mathcal{Q}_i)) \leq k$. By Divisors, Lemmas 31.23.5 and 31.23.8 the pullbacks s_i are defined and are regular meromorphic sections for $\mathcal{L}|_{X_i}$. The equality of cycles in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* \text{div}_{\mathcal{L}|_{X_i}}(s_i)$$

holds in $Z_k(X)$. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be the generic point. Let $A = \mathcal{O}_{X,\xi}$ and $M = \mathcal{F}_\xi$. Moreover, choose a generator $s_\xi \in \mathcal{L}_\xi$. Then we can write $s = (a/b)s_\xi$ where $a, b \in A$ are nonzerodivisors. In this case $I = \mathcal{I}_\xi = \{x \in A \mid x(a/b) \in A\}$. In this case the coefficient of $[Z]$ in the left hand side is

$$\text{length}_A(M/(a/b)IM) - \text{length}_A(M/IM)$$

and the coefficient of $[Z]$ in the right hand side is

$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \text{ord}_{A/\mathfrak{q}_i}(a/b)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the 1-dimensional local ring A. Hence the result follows from Lemma 42.69.5. $\square$

02SX Lemma 42.69.7. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k+1$. Then the element

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$$

lies in the subgroup $B_k(X)$ of Lemma 42.69.3 and maps to the element $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$ via the map $B_k(X) \to \text{CH}_k(X)$.

Proof. Let

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$$

be the short exact sequence constructed in Divisors, Lemma 31.4.6. This in particular means that $\mathcal{F}'$ has no embedded associated points. Since the support of $\mathcal{K}$ is nowhere dense in the support of $\mathcal{F}$ we see that $\dim_\delta(\text{Supp}(\mathcal{K})) \leq k$. We may re-apply Divisors, Lemma 31.4.6 starting with $\mathcal{K}$ to get a short exact sequence

$$0 \to \mathcal{K}'' \to \mathcal{K} \to \mathcal{K}' \to 0$$

where now $\dim_\delta(\text{Supp}(\mathcal{K}'')) < k$ and $\mathcal{K}'$ has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves $\mathcal{F}'$ and $\mathcal{K}'$. Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}'']_{k+1}$$

(use Lemma 42.10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$$

(use the $\otimes \mathcal{L}$ is exact) and the trivial vanishing of $[\mathcal{K}'']_{k+1}$ and $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$ in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ that the result holds for $\mathcal{F}$. What this means is that we may assume that the sheaf $\mathcal{F}$ has no embedded associated points.

Assume X, $\mathcal{F}$ as in the lemma, and assume in addition that $\mathcal{F}$ has no embedded associated points. Consider the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, the corresponding closed subscheme $i : Z \to X$ and the coherent $\mathcal{O}_Z$-module $\mathcal{G}$ constructed in Divisors, Lemma 31.4.7. Recall that Z is a locally Noetherian scheme without embedded points, $\mathcal{G}$ is a coherent sheaf without embedded associated points, with $\text{Supp}(\mathcal{G}) = Z$ and such that $i_*\mathcal{G} = \mathcal{F}$. Moreover, set $\mathcal{N} = \mathcal{L}|_Z$.

By Divisors, Lemma 31.25.4 the invertible sheaf $\mathcal{N}$ has a regular meromorphic section s over Z. Let us denote $\mathcal{J} \subset \mathcal{O}_Z$ the sheaf of denominators of s. By Lemma 42.69.6 there exist short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

such that $\dim_\delta(\text{Supp}(\mathcal{Q}_i)) \leq k$ and such that the cycle $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$ is a representative of $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$. We see (using the fact that $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$ by the projection formula, see Cohomology, Lemma 20.54.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$. This already shows that $[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}]$ is an element of $B_k(X)$. Moreover we have

$$\begin{aligned} [i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*\left([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k\right) \\ &= i_*\left(c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}\right) \\ &= c_1(\mathcal{L}) \cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1} \end{aligned}$$

by the above and Lemmas 42.26.4 and 42.12.4. And this agree with the image of the element under $B_k(X) \to \text{CH}_k(X)$ by definition. Hence the lemma is proved. □

42.70. Other chapters

CHAPTER 43

Intersection Theory

0AZ6

43.1. Introduction

0AZ7 In this chapter we construct the intersection product on the Chow groups modulo rational equivalence on a nonsingular projective variety over an algebraically closed field. Our tools are Serre's Tor formula (see [Ser65, Chapter V]), reduction to the diagonal, and the moving lemma.

We first recall cycles and how to construct proper pushforward and flat pullback of cycles. Next, we introduce rational equivalence of cycles which gives us the Chow groups $\mathrm{CH}_*(X)$. Proper pushforward and flat pullback factor through rational equivalence to give operations on Chow groups. This takes up Sections 43.3, 43.4, 43.5, 43.6, 43.7, 43.8, 43.9, 43.10, and 43.11. For proofs we mostly refer to the chapter on Chow homology where these results have been proven in the setting of schemes locally of finite type over a universally catenary Noetherian base, see Chow Homology, Section 42.7 ff.

Since we work on a nonsingular projective X any irreducible component of the intersection $V \cap W$ of two irreducible closed subvarieties has dimension at least $\dim(V)+\dim(W)-\dim(X)$. We say V and W intersect properly if equality holds for every irreducible component Z. In this case we define the intersection multiplicity $e_Z = e(X, V \cdot W, Z)$ by the formula

$$e_Z = \sum\nolimits_i (-1)^i \text{length}_{\mathcal{O}_{X,Z}} \text{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})$$

We need to do a little bit of commutative algebra to show that these intersection multiplicities agree with intuition in simple cases, namely, that sometimes

$$e_Z = \text{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{V \cap W, Z},$$

in other words, only Tor_0 contributes. This happens when V and W are Cohen-Macaulay in the generic point of Z or when W is cut out by a regular sequence in $\mathcal{O}_{X,Z}$ which also defines a regular sequence on $\mathcal{O}_{V,Z}$. However, Example 43.14.4 shows that higher tors are necessary in general. Moreover, there is a relationship with the Samuel multiplicity. These matters are discussed in Sections 43.13, 43.14, 43.15, 43.16, and 43.17.

Reduction to the diagonal is the statement that we can intersect V and W by intersecting $V \times W$ with the diagonal in $X \times X$. This innocuous statement, which is clear on the level of scheme theoretic intersections, reduces an intersection of a general pair of closed subschemes, to the case where one of the two is locally cut out by a regular sequence. We use this, following Serre, to obtain positivity of intersection multiplicities. Moreover, reduction to the diagonal leads to additivity of intersection multiplicities, associativity, and a projection formula. This can be found in Sections 43.18, 43.19, 43.20, 43.21, and 43.22.

Finally, we come to the moving lemmas and applications. There are two parts to the moving lemma. The first is that given closed subvarieties

$$Z \subset X \subset \mathbf{P}^N$$

with X nonsingular, we can find a subvariety $C \subset \mathbf{P}^N$ intersecting X properly such that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

and such that the other components Z_j are "more general" than Z. The second part is that one can move $C \subset \mathbf{P}^N$ over a rational curve to a subvariety in general position with respect to any given list of subvarieties. Combined these results imply that it suffices to define the intersection product of cycles on X which intersect properly which was done above. Of course this only leads to an intersection product on $\mathrm{CH}_*(X)$ if one can show, as we do in the text, that these products pass through rational equivalence. This and some applications are discussed in Sections 43.23, 43.24, 43.25, 43.26, 43.27, and 43.28.

43.2. Conventions

0AZ8 We fix an algebraically closed ground field $\mathbf{C}$ of any characteristic. All schemes and varieties are over $\mathbf{C}$ and all morphisms are over $\mathbf{C}$. A variety X is nonsingular if X is a regular scheme (see Properties, Definition 28.9.1). In our case this means that the morphism $X \to \mathrm{Spec}(\mathbf{C})$ is smooth (see Varieties, Lemma 33.12.6).

43.3. Cycles

0AZ9 Let X be a variety. A closed subvariety of X is an integral closed subscheme $Z \subset X$. A k-cycle on X is a finite formal sum $\sum n_i[Z_i]$ where each Z_i is a closed subvariety of dimension k. Whenever we use the notation $\alpha = \sum n_i[Z_i]$ for a k-cycle we always assume the subvarieties Z_i are pairwise distinct and $n_i \neq 0$ for all i. In this case the support of α is the closed subset

$$\mathrm{Supp}(\alpha) = \bigcup Z_i \subset X$$

of dimension k. The group of k-cycles is denoted $Z_k(X)$. See Chow Homology, Section 42.8.

43.4. Cycle associated to closed subscheme

0AZA Suppose that X is a variety and that $Z \subset X$ be a closed subscheme with $\dim(Z) \leq k$. Let Z_i be the irreducible components of Z of dimension k and let n_i be the multiplicity of Z_i in Z defined as

$$n_i = \mathrm{length}_{\mathcal{O}_{X,Z_i}} \mathcal{O}_{Z,Z_i}$$

where $\mathcal{O}_{X,Z_i}$, resp. $\mathcal{O}_{Z,Z_i}$ is the local ring of X, resp. Z at the generic point of Z_i. We define the k-cycle associated to Z to be the k-cycle

$$[Z]_k = \sum n_i [Z_i].$$

See Chow Homology, Section 42.9.

43.5. Cycle associated to a coherent sheaf

0AZB Suppose that X is a variety and that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module with $\dim(\text{Supp}(\mathcal{F})) \leq k$. Let Z_i be the irreducible components of $\text{Supp}(\mathcal{F})$ of dimension k and let n_i be the multiplicity of Z_i in $\mathcal{F}$ defined as

$$n_i = \text{length}_{\mathcal{O}_{X,Z_i}} \mathcal{F}_{\xi_i}$$

where $\mathcal{O}_{X,Z_i}$ is the local ring of X at the generic point ξ_i of Z_i and $\mathcal{F}_{\xi_i}$ is the stalk of $\mathcal{F}$ at this point. We define the k-cycle associated to $\mathcal{F}$ to be the k-cycle

$$[\mathcal{F}]_k = \sum n_i [Z_i].$$

See Chow Homology, Section 42.10. Note that, if $Z \subset X$ is a closed subscheme with $\dim(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$ by definition.

43.6. Proper pushforward

0AZC Suppose that $f : X \to Y$ is a proper morphism of varieties. Let $Z \subset X$ be a k-dimensional closed subvariety. We define $f_*[Z]$ to be 0 if $\dim(f(Z)) < k$ and $d \cdot [f(Z)]$ if $\dim(f(Z)) = k$ where

$$d = [\mathbf{C}(Z) : \mathbf{C}(f(Z))] = \deg(Z/f(Z))$$

is the degree of the dominant morphism $Z \to f(Z)$, see Morphisms, Definition 29.51.8. Let $\alpha = \sum n_i [Z_i]$ be a k-cycle on X. The pushforward of α is the sum $f_*\alpha = \sum n_i f_*[Z_i]$ where each $f_*[Z_i]$ is defined as above. This defines a homomorphism

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

See Chow Homology, Section 42.12.

0AZD **Lemma 43.6.1.** Suppose that $f : X \to Y$ is a proper morphism of varieties. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\text{Supp}(\mathcal{F})) \leq k$, then $f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k$. In particular, if $Z \subset X$ is a closed subscheme of dimension $\leq k$, then $f_*[Z]_k = [f_*\mathcal{O}_Z]_k$. See [Ser65, Chapter V].

Proof. See Chow Homology, Lemma 42.12.4. $\square$

0B0N **Lemma 43.6.2.** Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of varieties. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.

Proof. Special case of Chow Homology, Lemma 42.12.2. $\square$

43.7. Flat pullback

0AZE Suppose that $f : X \to Y$ is a flat morphism of varieties. By Morphisms, Lemma 29.28.2 every fibre of f has dimension $r = \dim(X) - \dim(Y)$[1]. Let $Z \subset X$ be a k-dimensional closed subvariety. We define $f^*[Z]$ to be the $(k+r)$-cycle associated to the scheme theoretic inverse image: $f^*[Z] = [f^{-1}(Z)]_{k+r}$. Let $\alpha = \sum n_i [Z_i]$ be a k-cycle on Y. The pullback of α is the sum $f_*\alpha = \sum n_i f^*[Z_i]$ where each $f^*[Z_i]$ is defined as above. This defines a homomorphism

$$f^* : Z_k(Y) \longrightarrow Z_{k+r}(X)$$

See Chow Homology, Section 42.14.

[1]Conversely, if $f : X \to Y$ is a dominant morphism of varieties, X is Cohen-Macaulay, Y is nonsingular, and all fibres have the same dimension r, then f is flat. This follows from Algebra, Lemma 10.128.1 and Varieties, Lemma 33.20.4 showing $\dim(X) = \dim(Y) + r$.

0AZF **Lemma 43.7.1.** Let $f : X \to Y$ be a flat morphism of varieties. Set $r = \dim(X) - \dim(Y)$. Then $f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$ if $\mathcal{F}$ is a coherent sheaf on Y and the dimension of the support of $\mathcal{F}$ is at most k.

Proof. See Chow Homology, Lemma 42.14.4. □

0B0P **Lemma 43.7.2.** Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of varieties. Then $g \circ f$ is flat and $f^* \circ g^* = (g \circ f)^*$ as maps $Z_k(Z) \to Z_{k+\dim(X)-\dim(Z)}(X)$.

Proof. Special case of Chow Homology, Lemma 42.14.3. □

43.8. Rational Equivalence

0AZG We are going to define rational equivalence in a way which at first glance may seem different from what you are used to, or from what is in [Ful98, Chapter I] or Chow Homology, Section 42.19. However, in Section 43.9 we will show that the two notions agree.

Let X be a variety. Let $W \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $k + 1$. Let a, b be distinct closed points of $\mathbf{P}^1$. Assume that $X \times a$, $X \times b$ and W intersect properly:

$$\dim(W \cap X \times a) \leq k, \quad \dim(W \cap X \times b) \leq k.$$

This is true as soon as $W \to \mathbf{P}^1$ is dominant or if W is contained in a fibre of the projection over a closed point different from a or b (this is an uninteresting case which we will discard). In this situation the scheme theoretic fibre W_a of the morphism $W \to \mathbf{P}^1$ is equal to the scheme theoretic intersection $W \cap X \times a$ in $X \times \mathbf{P}^1$. Identifying $X \times a$ and $X \times b$ with X we may think of the fibres W_a and W_b as closed subschemes of X of dimension $\leq k$[2]. A basic example of a rational equivalence is

$$[W_a]_k \sim_{rat} [W_b]_k$$

The cycles $[W_a]_k$ and $[W_b]_k$ are easy to compute in practice (given W) because they are obtained by proper intersection with a Cartier divisor (we will see this in Section 43.17). Since the automorphism group of $\mathbf{P}^1$ is 2-transitive we may move the pair of closed points a, b to any pair we like. A traditional choice is to choose $a = 0$ and $b = \infty$.

More generally, let $\alpha = \sum n_i[W_i]$ be a $(k + 1)$-cycle on $X \times \mathbf{P}^1$. Let a_i, b_i be pairs of distinct closed points of $\mathbf{P}^1$. Assume that $X \times a_i$, $X \times b_i$ and W_i intersect properly, in other words, each W_i, a_i, b_i satisfies the condition discussed above. A cycle rationally equivalent to zero is any cycle of the form

$$\sum n_i([W_{i,a_i}]_k - [W_{i,b_i}]_k).$$

This is indeed a k-cycle. The collection of k-cycles rationally equivalent to zero is an additive subgroup of the group of k-cycles. We say two k-cycles are rationally equivalent, notation $\alpha \sim_{rat} \alpha'$, if $\alpha - \alpha'$ is a cycle rationally equivalent to zero.

We define

$$\mathrm{CH}_k(X) = Z_k(X)/ \sim_{rat}$$

to be the Chow group of k-cycles on X. We will see in Lemma 43.9.1 that this agrees with the Chow group as defined in Chow Homology, Definition 42.19.1.

[2]We will sometimes think of W_a as a closed subscheme of $X \times \mathbf{P}^1$ and sometimes as a closed subscheme of X. It should always be clear from context which point of view is taken.

43.9. Rational equivalence and rational functions

0AZH Let X be a variety. Let $W \subset X$ be a subvariety of dimension $k+1$. Let $f \in \mathbf{C}(W)^*$ be a nonzero rational function on W. For every subvariety $Z \subset W$ of dimension k one can define the order of vanishing $\text{ord}_{W,Z}(f)$ of f at Z. If f is an element of the local ring $\mathcal{O}_{W,Z}$, then one has

$$\text{ord}_{W,Z}(f) = \text{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{W,Z}/f\mathcal{O}_{W,Z}$$

where $\mathcal{O}_{X,Z}$, resp. $\mathcal{O}_{W,Z}$ is the local ring of X, resp. W at the generic point of Z. In general one extends the definition by multiplicativity. The principal divisor associated to f is

$$\text{div}_W(f) = \sum \text{ord}_{W,Z}(f)[Z]$$

in $Z_k(W)$. Since $W \subset X$ is a closed subvariety we may think of $\text{div}_W(f)$ as a cycle on X. See Chow Homology, Section 42.17.

0AZI **Lemma 43.9.1.** Let X be a variety. Let $W \subset X$ be a subvariety of dimension $k+1$. Let $f \in \mathbf{C}(W)^*$ be a nonzero rational function on W. Then $\text{div}_W(f)$ is rationally equivalent to zero on X. Conversely, these principal divisors generate the abelian group of cycles rationally equivalent to zero on X.

Proof. The first assertion follows from Chow Homology, Lemma 42.18.2. More precisely, let $W' \subset X \times \mathbf{P}^1$ be the closure of the graph of f. Then $\text{div}_W(f) = [W'_0]_k - [W'_\infty]$ in $Z_k(W) \subset Z_k(X)$, see part (6) of Chow Homology, Lemma 42.18.2.

For the second, let $W' \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $k+1$ which dominates $\mathbf{P}^1$. We will show that $[W'_0]_k - [W'_\infty]_k$ is a principal divisor which will finish the proof. Let $W \subset X$ be the image of W' under the projection to X. Then $W \subset X$ is a closed subvariety and $W' \to W$ is proper and dominant with fibres of dimension 0 or 1. If $\dim(W) = k$, then $W' = W \times \mathbf{P}^1$ and we see that $[W'_0]_k - [W'_\infty]_k = [W] - [W] = 0$. If $\dim(W) = k+1$, then $W' \to W$ is generically finite[3]. Let f denote the projection $W' \to \mathbf{P}^1$ viewed as an element of $\mathbf{C}(W')^*$. Let $g = \text{Nm}(f) \in \mathbf{C}(W)^*$ be the norm. By Chow Homology, Lemma 42.18.1 we have

$$\text{div}_W(g) = \text{pr}_{X,*}\text{div}_{W'}(f)$$

Since $\text{div}_{W'}(f) = [W'_0]_k - [W'_\infty]_k$ by Chow Homology, Lemma 42.18.2 the proof is complete. □

43.10. Proper pushforward and rational equivalence

0AZJ Suppose that $f : X \to Y$ is a proper morphism of varieties. Let $\alpha \sim_{rat} 0$ be a k-cycle on X rationally equivalent to 0. Then the pushforward of α is rationally equivalent to zero: $f_*\alpha \sim_{rat} 0$. See Chapter I of [Ful98] or Chow Homology, Lemma 42.20.3.

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[3]If $W' \to W$ is birational, then the result follows from Chow Homology, Lemma 42.18.2. Our task is to show that even if $W' \to W$ has degree > 1 the basic rational equivalence $[W'_0]_k \sim_{rat} [W'_\infty]_k$ comes from a principal divisor on a subvariety of X.

Therefore we obtain a commutative diagram

$$\begin{tikzcd} Z_k(X) \arrow[r] \arrow[d, "f_*"'] & \mathrm{CH}_k(X) \arrow[d, "f_*"] \\ Z_k(Y) \arrow[r] & \mathrm{CH}_k(Y) \end{tikzcd}$$

of groups of k-cycles.

43.11. Flat pullback and rational equivalence

0AZK Suppose that $f : X \to Y$ is a flat morphism of varieties. Set $r = \dim(X) - \dim(Y)$. Let $\alpha \sim_{rat} 0$ be a k-cycle on Y rationally equivalent to 0. Then the pullback of α is rationally equivalent to zero: $f^*\alpha \sim_{rat} 0$. See Chapter I of [Ful98] or Chow Homology, Lemma 42.20.2.

Therefore we obtain a commutative diagram

$$\begin{tikzcd} Z_{k+r}(X) \arrow[r] & \mathrm{CH}_{k+r}(X) \\ Z_k(Y) \arrow[r] \arrow[u, "f^*"] & \mathrm{CH}_k(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

of groups of k-cycles.

43.12. The short exact sequence for an open

0B5Z Let X be a variety and let $U \subset X$ be an open subvariety. Let $X \setminus U = \bigcup Z_i$ be the decomposition into irreducible components[4]. Then for each $k \geq 0$ there exists a commutative diagram

$$\begin{tikzcd} \bigoplus Z_k(Z_i) \arrow[r] \arrow[d] & Z_k(X) \arrow[r] \arrow[d] & Z_k(U) \arrow[r] \arrow[d] & 0 \\ \bigoplus \mathrm{CH}_k(Z_i) \arrow[r] & \mathrm{CH}_k(X) \arrow[r] & \mathrm{CH}_k(U) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Here the vertical arrows are the canonical quotient maps. The left horizontal arrows are given by proper pushforward along the closed immersions $Z_i \to X$. The right horizontal arrows are given by flat pullback along the open immersion $j : U \to X$. Since we have seen that these maps factor through rational equivalence we obtain the commutativity of the squares. The top row is exact simply because every subvariety of X is either contained in some Z_i or has irreducible intersection with U. The bottom row is exact because every principal divisor $\mathrm{div}_W(f)$ on U is the restriction of a principal divisor on X. More precisely, if $W \subset U$ is a $(k+1)$-dimensional closed subvariety and $f \in \mathbf{C}(W)^*$, then denote $\overline{W}$ the closure of W in X. Then $W \subset \overline{W}$ is an open immersion, so $\mathbf{C}(W) = \mathbf{C}(\overline{W})$ and we may think of f as a nonconstant rational function on $\overline{W}$. Then clearly

$$j^*\mathrm{div}_{\overline{W}}(f) = \mathrm{div}_W(f)$$

in $Z_k(X)$. The exactness of the lower row follows easily from this. For details see Chow Homology, Lemma 42.19.3.

[4]Since in this chapter we only consider Chow groups of varieties, we are prohibited from taking $Z_k(X \setminus U)$ and $\mathrm{CH}_k(X \setminus U)$, hence the approach using the varieties Z_i.

43.13. Proper intersections

0AZL First a few lemmas to get dimension estimates.

0AZM **Lemma 43.13.1.** Let X and Y be varieties. Then $X \times Y$ is a variety and $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Proof. The scheme $X \times Y = X \times_{\mathrm{Spec}(\mathbf{C})} Y$ is a variety by Varieties, Lemma 33.3.3. The statement on dimension is Varieties, Lemma 33.20.5. $\square$

Recall that a regular immersion $i : X \to Y$ of schemes is a closed immersion whose corresponding sheaf of ideals is locally generated by a regular sequence, see Divisors, Section 31.21. Moreover, the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free of rank equal to the length of the regular sequence. Let us say i is a regular immersion of codimension c if $\mathcal{C}_{X/Y}$ is locally free of rank c.

More generally, recall (More on Morphisms, Section 37.62) that $f : X \to Y$ is a local complete intersection morphism if we can cover X by opens U such that we can factor $f|_U$ as

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr] & & \mathbf{A}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where i is a Koszul regular immersion (if Y is locally Noetherian this is the same as asking i to be a regular immersion, see Divisors, Lemma 31.21.3). Let us say that f is a local complete intersection morphism of relative dimension r if for any factorization as above, the closed immersion i has conormal sheaf of rank $n - r$ (in other words if i is a Koszul-regular immersion of codimension $n - r$ which in the Noetherian case just means it is regular immersion of codimension $n - r$).

0AZN **Lemma 43.13.2.** Let $f : X \to Y$ be a morphism of varieties.

(1) If $Z \subset Y$ is a subvariety dimension d and f is a regular immersion of codimension c, then every irreducible component of $f^{-1}(Z)$ has dimension $\geq d - c$.
(2) If $Z \subset Y$ is a subvariety of dimension d and f is a local complete intersection morphism of relative dimension r, then every irreducible component of $f^{-1}(Z)$ has dimension $\geq d + r$.

Proof. Proof of (1). We may work locally, hence we may assume that $Y = \mathrm{Spec}(A)$ and $X = V(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in A. If $Z = \mathrm{Spec}(A/\mathfrak{p})$, then we see that $f^{-1}(Z) = \mathrm{Spec}(A/\mathfrak{p} + (f_1, \ldots, f_c))$. If V is an irreducible component of $f^{-1}(Z)$, then we can choose a closed point $v \in V$ not contained in any other irreducible component of $f^{-1}(Z)$. Then

$$\dim(Z) = \dim \mathcal{O}_{Z,v} \quad \text{and} \quad \dim(V) = \dim \mathcal{O}_{V,v} = \dim \mathcal{O}_{Z,v}/(f_1, \ldots, f_c)$$

The first equality for example by Algebra, Lemma 10.116.1 and the second equality by our choice of closed point. The result now follows from the fact that dividing by one element in the maximal ideal decreases the dimension by at most 1, see Algebra, Lemma 10.60.13.

Proof of (2). Choose a factorization as in the definition of a local complete intersection and apply (1). Some details omitted. $\square$

0B0Q **Lemma 43.13.3.** Let X be a nonsingular variety. Then the diagonal $\Delta : X \to X \times X$ is a regular immersion of codimension $\dim(X)$.

Proof. In fact, any closed immersion between nonsingular projective varieties is a regular immersion, see Divisors, Lemma 31.22.11. $\square$

The following lemma demonstrates how reduction to the diagonal works.

0AZP **Lemma 43.13.4.** Let X be a nonsingular variety and let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Then every irreducible component Z of $V \cap W$ has dimension $\geq r + s - \dim(X)$.

Proof. Since $V \cap W = \Delta^{-1}(V \times W)$ (scheme theoretically) we conclude by Lemmas 43.13.3 and 43.13.2. $\square$

This lemma suggests the following definition.

0AZQ **Definition 43.13.5.** Let X be a nonsingular variety.

(1) Let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. We say that W and V *intersect properly* if $\dim(V \cap W) \leq r + s - \dim(X)$.
(2) Let $\alpha = \sum n_i[W_i]$ be an s-cycle, and $\beta = \sum_j m_j[V_j]$ be an r-cycle on X. We say that α and β *intersect properly* if W_i and V_j intersect properly for all i and j.

43.14. Intersection multiplicities using Tor formula

0AZR A basic fact we will use frequently is that given sheaves of modules $\mathcal{F}, \mathcal{G}$ on a ringed space $(X, \mathcal{O}_X)$ and a point $x \in X$ we have

$$\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x = \mathrm{Tor}_p^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

as $\mathcal{O}_{X,x}$-modules. This can be seen in several ways from our construction of derived tensor products in Cohomology, Section 20.26, for example it follows from Cohomology, Lemma 20.26.4. Moreover, if X is a scheme and $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, then the modules $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are quasi-coherent too, see Derived Categories of Schemes, Lemma 36.3.9. More important for our purposes is the following result.

0AZS **Lemma 43.14.1.** Let X be a locally Noetherian scheme.

(1) If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is too.
(2) If L and K are in $D^-_{\mathit{Coh}}(\mathcal{O}_X)$, then so is $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$.

Proof. Let us explain how to prove (1) in a more elementary way and part (2) using previously developed general theory.

Proof of (1). Since formation of Tor commutes with localization we may assume X is affine. Hence $X = \mathrm{Spec}(A)$ for some Noetherian ring A and $\mathcal{F}$, $\mathcal{G}$ correspond to finite A-modules M and N (Cohomology of Schemes, Lemma 30.9.1). By Derived Categories of Schemes, Lemma 36.3.9 we may compute the Tor's by first computing the Tor's of M and N over A, and then taking the associated $\mathcal{O}_X$-module. Since the modules $\mathrm{Tor}_p^A(M, N)$ are finite by Algebra, Lemma 10.75.7 we conclude.

By Derived Categories of Schemes, Lemma 36.10.3 the assumption is equivalent to asking L and K to be (locally) pseudo-coherent. Then $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$ is pseudo-coherent by Cohomology, Lemma 20.47.5. $\square$

0AZT Lemma 43.14.2. Let X be a nonsingular variety. Let $\mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-module $\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent, has stalk at x equal to $\text{Tor}_p^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$, is supported on $\text{Supp}(\mathcal{F}) \cap \text{Supp}(\mathcal{G})$, and is nonzero only for $p \in \{0, \ldots, \dim(X)\}$.

Proof. The result on stalks was discussed above and it implies the support condition. The Tor's are coherent by Lemma 43.14.1. The vanishing of negative Tor's is immediate from the construction. The vanishing of Tor_p for $p > \dim(X)$ can be seen as follows: the local rings $\mathcal{O}_{X,x}$ are regular (as X is nonsingular) of dimension $\leq \dim(X)$ (Algebra, Lemma 10.116.1), hence $\mathcal{O}_{X,x}$ has finite global dimension $\leq \dim(X)$ (Algebra, Lemma 10.110.8) which implies that Tor-groups of modules vanish beyond the dimension (More on Algebra, Lemma 15.66.19). □

Let X be a nonsingular variety and $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Assume V and W intersect properly. In this case Lemma 43.13.4 tells us all irreducible components of $V \cap W$ have dimension equal to $r + s - \dim(X)$. The sheaves $\text{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)$ are coherent, supported on $V \cap W$, and zero if $j < 0$ or $j > \dim(X)$ (Lemma 43.14.2). We define the intersection product as

$$W \cdot V = \sum\nolimits_i (-1)^i [\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)]_{r+s-\dim(X)}.$$

We stress that this makes sense only because of our assumption that V and W intersect properly. This fact will necessitate a moving lemma in order to define the intersection product in general.

With this notation, the cycle $V \cdot W$ is a formal linear combination $\sum e_Z Z$ of the irreducible components Z of the intersection $V \cap W$. The integers e_Z are called the intersection multiplicities

$$e_Z = e(X, V \cdot W, Z) = \sum\nolimits_i (-1)^i \text{length}_{\mathcal{O}_{X,Z}} \text{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})$$

where $\mathcal{O}_{X,Z}$, resp. $\mathcal{O}_{W,Z}$, resp. $\mathcal{O}_{V,Z}$ denotes the local ring of X, resp. W, resp. V at the generic point of Z. These alternating sums of lengths of Tor's satisfy many good properties, as we will see later on.

In the case of transversal intersections, the intersection number is 1.

0B1I Lemma 43.14.3. Let X be a nonsingular variety. Let $V, W \subset X$ be closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ and assume that the multiplicity (in the sense of Section 43.4) of Z in the closed subscheme $V \cap W$ is 1. Then $e(X, V \cdot W, Z) = 1$ and V and W are smooth in a general point of Z.

Proof. Let $(A, \mathfrak{m}, \kappa) = (\mathcal{O}_{X,\xi}, \mathfrak{m}_\xi, \kappa(\xi))$ where $\xi \in Z$ is the generic point. Then $\dim(A) = \dim(X) - \dim(Z)$, see Varieties, Lemma 33.20.3. Let $I, J \subset A$ cut out the trace of V and W in $\text{Spec}(A)$. Set $\overline{I} = I + \mathfrak{m}^2/\mathfrak{m}^2$. Then $\dim_\kappa \overline{I} \leq \dim(X) - \dim(V)$ with equality if and only if A/I is regular (this follows from the lemma cited above and the definition of regular rings, see Algebra, Definition 10.60.10 and the discussion preceding it). Similarly for $\overline{J}$. If the multiplicity is 1, then $\text{length}_A(A/I + J) = 1$, hence $I + J = \mathfrak{m}$, hence $\overline{I} + \overline{J} = \mathfrak{m}/\mathfrak{m}^2$. Then we get equality everywhere (because the intersection is proper). Hence we find $f_1, \ldots, f_a \in I$ and $g_1, \ldots g_b \in J$ such that $\overline{f}_1, \ldots, \overline{g}_b$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$. Then $f_1, \ldots, g_b$ is a regular system of parameters and a regular sequence (Algebra, Lemma 10.106.3). The same lemma shows $A/(f_1, \ldots, f_a)$ is a regular local ring of dimension

$\dim(X) - \dim(V)$, hence $A/(f_1, \ldots, f_a) \to A/I$ is an isomorphism (if the kernel is nonzero, then the dimension of A/I is strictly less, see Algebra, Lemmas 10.106.2 and 10.60.13). We conclude $I = (f_1, \ldots, f_a)$ and $J = (g_1, \ldots, g_b)$ by symmetry. Thus the Koszul complex $K_\bullet(A, f_1, \ldots, f_a)$ on $f_1, \ldots, f_a$ is a resolution of A/I, see More on Algebra, Lemma 15.30.2. Hence

$$\begin{aligned} \text{Tor}_p^A(A/I, A/J) &= H_p(K_\bullet(A, f_1, \ldots, f_a) \otimes_A A/J) \\ &= H_p(K_\bullet(A/J, f_1 \bmod J, \ldots, f_a \bmod J)) \end{aligned}$$

Since we've seen above that $f_1 \bmod J, \ldots, f_a \bmod J$ is a regular system of parameters in the regular local ring A/J we conclude that there is only one cohomology group, namely $H_0 = A/(I + J) = \kappa$. This finishes the proof. □

0B2S Example 43.14.4. In this example we show that it is necessary to use the higher tors in the formula for the intersection multiplicities above. Let X be a nonsingular variety of dimension 4. Let $p \in X$ be a closed point. Let $V, W \subset X$ be closed subvarieties in X. Assume that there is an isomorphism

$$\mathcal{O}_{X,p}^\wedge \cong \mathbf{C}[[x, y, z, w]]$$

such that the ideal of V is (xz, xw, yz, yw) and the ideal of W is $(x - z, y - w)$. Then a computation shows that

$$\text{length } \mathbf{C}[[x, y, z, w]]/(xz, xw, yz, yw, x - z, y - w) = 3$$

On the other hand, the multiplicity $e(X, V \cdot W, p) = 2$ as can be seen from the fact that formal locally V is the union of two smooth planes $x = y = 0$ and $z = w = 0$ at p, each of which has intersection multiplicity 1 with the plane $x - z = y - w = 0$ (Lemma 43.14.3). To make an actual example, take a general morphism $f : \mathbf{P}^2 \to \mathbf{P}^4$ given by 5 homogeneous polynomials of degree > 1. The image $V \subset \mathbf{P}^4 = X$ will have singularities of the type described above, because there will be $p_1, p_2 \in \mathbf{P}^2$ with $f(p_1) = f(p_2)$. To find W take a general plane passing through such a point.

43.15. Algebraic multiplicities

0AZU Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let M be a finite A-module and let $I \subset A$ be an ideal of definition (Algebra, Definition 10.59.1). Recall that the function

$$\chi_{I,M}(n) = \text{length}_A(M/I^nM) = \sum\nolimits_{p=0,\ldots,n-1} \text{length}_A(I^pM/I^{p+1}M)$$

is a numerical polynomial (Algebra, Proposition 10.59.5). The degree of this polynomial is equal to $\dim(\text{Supp}(M))$ by Algebra, Lemma 10.62.6.

0AZV Definition 43.15.1. In the situation above, if $d \geq \dim(\text{Supp}(M))$, then we set $e_I(M, d)$ equal to 0 if $d > \dim(\text{Supp}(M))$ and equal to $d!$ times the leading coefficient of the numerical polynomial $\chi_{I,M}$ so that

$$\chi_{I,M}(n) \sim e_I(M, d)\frac{n^d}{d!} + \text{lower order terms}$$

The multiplicity of M for the ideal of definition I is $e_I(M) = e_I(M, \dim(\text{Supp}(M)))$.

We have the following properties of these multiplicities.

0AZW **Lemma 43.15.2.** Let A be a Noetherian local ring. Let $I \subset A$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite A-modules. Let $d \geq \dim(\text{Supp}(M))$. Then

$$e_I(M, d) = e_I(M', d) + e_I(M'', d)$$

Proof. Immediate from the definitions and Algebra, Lemma 10.59.10. □

0AZX **Lemma 43.15.3.** Let A be a Noetherian local ring. Let $I \subset A$ be an ideal of definition. Let M be a finite A-module. Let $d \geq \dim(\text{Supp}(M))$. Then

$$e_I(M, d) = \sum \text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) e_I(A/\mathfrak{p}, d)$$

where the sum is over primes $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p}) = d$.

Proof. Both the left and side and the right hand side are additive in short exact sequences of modules of dimension $\leq d$, see Lemma 43.15.2 and Algebra, Lemma 10.52.3. Hence by Algebra, Lemma 10.62.1 it suffices to prove this when $M = A/\mathfrak{q}$ for some prime $\mathfrak{q}$ of A with $\dim(A/\mathfrak{q}) \leq d$. This case is obvious. □

0AZY **Lemma 43.15.4.** Let P be a polynomial of degree r with leading coefficient a. Then

$$r!a = \sum\nolimits_{i=0,\ldots,r} (-1)^i \binom{r}{i} P(t-i)$$

for any t.

Proof. Let us write Δ the operator which to a polynomial P associates the polynomial $\Delta(P) = P(t) - P(t-1)$. We claim that

$$\Delta^r(P) = \sum\nolimits_{i=0,\ldots,r} (-1)^i \binom{r}{i} P(t-i)$$

This is true for $r = 0, 1$ by inspection. Assume it is true for r. Then we compute

$$\begin{aligned} \Delta^{r+1}(P) &= \sum\nolimits_{i=0,\ldots,r} (-1)^i \binom{r}{i} \Delta(P)(t-i) \\ &= \sum\nolimits_{n=-r,\ldots,0} (-1)^i \binom{r}{i} (P(t-i) - P(t-i-1)) \end{aligned}$$

Thus the claim follows from the equality

$$\binom{r+1}{i} = \binom{r}{i} + \binom{r}{i-1}$$

The lemma follows from the fact that $\Delta(P)$ is of degree $r-1$ with leading coefficient ra if the degree of P is r. □

An important fact is that one can compute the multiplicity in terms of the Koszul complex. Recall that if R is a ring and $f_1, \ldots, f_r \in R$, then $K_\bullet(f_1, \ldots, f_r)$ denotes the Koszul complex, see More on Algebra, Section 15.28.

0AZZ **Theorem 43.15.5.** Let A be a Noetherian local ring. Let $I = (f_1, \ldots, f_r) \subset A$ be an ideal of definition. Let M be a finite A-module. Then [Ser65, Theorem 1 in part B of Chapter IV]

$$e_I(M, r) = \sum (-1)^i \text{length}_A H_i(K_\bullet(f_1, \ldots, f_r) \otimes_A M)$$

Proof. Let us change the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ into a cochain complex $K^\bullet$ by setting $K^n = K_{-n}(f_1, \ldots, f_r)$. Then $K^\bullet$ is sitting in degrees $-r, \ldots, 0$ and $H^i(K^\bullet \otimes_A M) = H_{-i}(K_\bullet(f_1, \ldots, f_r) \otimes_A M)$. The statement of the theorem makes sense as the modules $H^i(K^\bullet \otimes M)$ are annihilated by $f_1, \ldots, f_r$ (More on Algebra, Lemma 15.28.6) hence have finite length. Define a filtration on the complex $K^\bullet$ by setting

$$F^p(K^n \otimes_A M) = I^{\max(0,p+n)}(K^n \otimes_A M), \quad p \in \mathbf{Z}$$

Since $f_i I^p \subset I^{p+1}$ this is a filtration by subcomplexes. Thus we have a filtered complex and we obtain a spectral sequence, see Homology, Section 12.24. We have

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q} = \bigoplus\nolimits_{p,q} \text{gr}^p(K^{p+q} \otimes_A M) = \text{Gr}_I(K^\bullet \otimes_A M)$$

Since K^n is finite free we have

$$\text{Gr}_I(K^\bullet \otimes_A M) = \text{Gr}_I(K^\bullet) \otimes_{\text{Gr}_I(A)} \text{Gr}_I(M)$$

Note that $\text{Gr}_I(K^\bullet)$ is the Koszul complex over $\text{Gr}_I(A)$ on the elements $\overline{f}_1, \ldots, \overline{f}_r \in I/I^2$. A simple calculation (omitted) shows that the differential d_0 on E_0 agrees with the differential coming from the Koszul complex. Since $\text{Gr}_I(M)$ is a finite $\text{Gr}_I(A)$-module and since $\text{Gr}_I(A)$ is Noetherian (as a quotient of $A/I[x_1, \ldots, x_r]$ with $x_i \mapsto \overline{f}_i$), the cohomology module $E_1 = \bigoplus E_1^{p,q}$ is a finite $\text{Gr}_I(A)$-module. However, as above E_1 is annihilated by $\overline{f}_1, \ldots, \overline{f}_r$. We conclude E_1 has finite length. In particular we find that $\text{Gr}^p_F(K^\bullet \otimes M)$ is acyclic for $p \gg 0$.

Next, we check that the spectral sequence above converges using Homology, Lemma 12.24.10. The required equalities follow easily from the Artin-Rees lemma in the form stated in Algebra, Lemma 10.51.3. Thus we see that

$$\begin{aligned} \sum (-1)^i \text{length}_A(H^i(K^\bullet \otimes_A M)) &= \sum (-1)^{p+q} \text{length}_A(E_\infty^{p,q}) \\ &= \sum (-1)^{p+q} \text{length}_A(E_1^{p,q}) \end{aligned}$$

because as we've seen above the length of E_1 is finite (of course this uses additivity of lengths). Pick t so large that $\text{Gr}^p_F(K^\bullet \otimes M)$ is acyclic for $p \geq t$ (see above). Using additivity again we see that

$$\sum (-1)^{p+q} \text{length}_A(E_1^{p,q}) = \sum\nolimits_n \sum\nolimits_{p \leq t} (-1)^n \text{length}_A(\text{gr}^p(K^n \otimes_A M))$$

This is equal to

$$\sum\nolimits_{n=-r,\ldots,0} (-1)^n \binom{r}{|n|} \chi_{I,M}(t+n)$$

by our choice of filtration above and the definition of $\chi_{I,M}$ in Algebra, Section 10.59. The lemma follows from Lemma 43.15.4 and the definition of $e_I(M, r)$. □

0B00 Remark 43.15.6 (Trivial generalization). Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let M be a finite A-module. Let $I \subset A$ be an ideal. The following are equivalent

(1) $I' = I + \text{Ann}(M)$ is an ideal of definition (Algebra, Definition 10.59.1),
(2) the image $\overline{I}$ of I in $\overline{A} = A/\text{Ann}(M)$ is an ideal of definition,
(3) $\text{Supp}(M/IM) \subset \{\mathfrak{m}\}$,
(4) $\dim(\text{Supp}(M/IM)) \leq 0$, and
(5) $\text{length}_A(M/IM) < \infty$.

This follows from Algebra, Lemma 10.62.3 (details omitted). If this is the case we have $M/I^nM = M/(I')^nM$ for all n and $M/I^nM = M/\overline{I}^nM$ for all n if M is viewed as an $\overline{A}$-module. Thus we can define

$$\chi_{I,M}(n) = \text{length}_A(M/I^nM) = \sum\nolimits_{p=0,\ldots,n-1} \text{length}_A(I^pM/I^{p+1}M)$$

and we get

$$\chi_{I,M}(n) = \chi_{I',M}(n) = \chi_{\overline{I},M}(n)$$

for all n by the equalities above. All the results of Algebra, Section 10.59 and all the results in this section, have analogues in this setting. In particular we can define multiplicities $e_I(M, d)$ for $d \geq \dim(\text{Supp}(M))$ and we have

$$\chi_{I,M}(n) \sim e_I(M, d)\frac{n^d}{d!} + \text{lower order terms}$$

as in the case where I is an ideal of definition.

43.16. Computing intersection multiplicities

0B01 In this section we discuss some cases where the intersection multiplicities can be computed by different means. Here is a first example.

0B02 Lemma 43.16.1. Let X be a nonsingular variety and $W, V \subset X$ closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ with generic point ξ. Assume that $\mathcal{O}_{W,\xi}$ and $\mathcal{O}_{V,\xi}$ are Cohen-Macaulay. Then

$$e(X, V \cdot W, Z) = \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{V\cap W,\xi})$$

where $V \cap W$ is the scheme theoretic intersection. In particular, if both V and W are Cohen-Macaulay, then $V \cdot W = [V \cap W]_{\dim(V)+\dim(W)-\dim(X)}$.

Proof. Set $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{V,\xi}$, and $C = \mathcal{O}_{W,\xi}$. By Auslander-Buchsbaum (Algebra, Proposition 10.111.1) we can find a finite free resolution $F_\bullet \to B$ of length

$$\text{depth}(A) - \text{depth}(B) = \dim(A) - \dim(B) = \dim(C)$$

First equality as A and B are Cohen-Macaulay and the second as V and W intersect properly. Then $F_\bullet \otimes_A C$ is a complex of finite free modules representing $B \otimes_A^{\mathbf{L}} C$ hence has cohomology modules with support in $\{\mathfrak{m}_A\}$. By the Acyclicity lemma (Algebra, Lemma 10.102.8) which applies as C is Cohen-Macaulay we conclude that $F_\bullet \otimes_A C$ has nonzero cohomology only in degree 0. This finishes the proof. □

0B03 Lemma 43.16.2. Let A be a Noetherian local ring. Let $I = (f_1, \ldots, f_r)$ be an ideal generated by a regular sequence. Let M be a finite A-module. Assume that $\dim(\text{Supp}(M/IM)) = 0$. Then

$$e_I(M, r) = \sum (-1)^i \text{length}_A(\text{Tor}_i^A(A/I, M))$$

Here $e_I(M, r)$ is as in Remark 43.15.6.

Proof. Since $f_1, \ldots, f_r$ is a regular sequence the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ is a resolution of A/I over A, see More on Algebra, Lemma 15.30.7. Thus the right hand side is equal to

$$\sum (-1)^i \text{length}_A H_i(K_\bullet(f_1, \ldots, f_r) \otimes_A M)$$

Now the result follows immediately from Theorem 43.15.5 if I is an ideal of definition. In general, we replace A by $\overline{A} = A/\text{Ann}(M)$ and $f_1, \ldots, f_r$ by $\overline{f}_1, \ldots, \overline{f}_r$ which is allowed because

$$K_\bullet(f_1, \ldots, f_r) \otimes_A M = K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) \otimes_{\overline{A}} M$$

Since $e_I(M, r) = e_{\overline{I}}(M, r)$ where $\overline{I} = (\overline{f}_1, \ldots, \overline{f}_r) \subset \overline{A}$ is an ideal of definition the result follows from Theorem 43.15.5 in this case as well. □

0B04 Lemma 43.16.3. Let X be a nonsingular variety. Let $W, V \subset X$ be closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ with generic point ξ. Suppose the ideal of V in $\mathcal{O}_{X,\xi}$ is cut out by a regular sequence $f_1, \ldots, f_c \in \mathcal{O}_{X,\xi}$. Then $e(X, V \cdot W, Z)$ is equal to $c!$ times the leading coefficient in the Hilbert polynomial

$$t \mapsto \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{W,\xi}/(f_1, \ldots, f_c)^t, \quad t \gg 0.$$

In particular, this coefficient is > 0.

Proof. The equality

$$e(X, V \cdot W, Z) = e_{(f_1, \ldots, f_c)}(\mathcal{O}_{W,\xi}, c)$$

follows from the more general Lemma 43.16.2. To see that $e_{(f_1, \ldots, f_c)}(\mathcal{O}_{W,\xi}, c)$ is > 0 or equivalently that $e_{(f_1, \ldots, f_c)}(\mathcal{O}_{W,\xi}, c)$ is the leading coefficient of the Hilbert polynomial it suffices to show that the dimension of $\mathcal{O}_{W,\xi}$ is c, because the degree of the Hilbert polynomial is equal to the dimension by Algebra, Proposition 10.60.9. Say $\dim(V) = r$, $\dim(W) = s$, and $\dim(X) = n$. Then $\dim(Z) = r + s - n$ as the intersection is proper. Thus the transcendence degree of $\kappa(\xi)$ over $\mathbf{C}$ is $r + s - n$, see Algebra, Lemma 10.116.1. We have $r + c = n$ because V is cut out by a regular sequence in a neighbourhood of ξ, see Divisors, Lemma 31.20.8 and then Lemma 43.13.2 applies (for example). Thus

$$\dim(\mathcal{O}_{W,\xi}) = s - (r + s - n) = s - ((n - c) + s - n) = c$$

the first equality by Algebra, Lemma 10.116.3. □

0B05 Lemma 43.16.4. In Lemma 43.16.3 assume that $c = 1$, i.e., V is an effective Cartier divisor. Then

$$e(X, V \cdot W, Z) = \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi}).$$

Proof. In this case the image of f_1 in $\mathcal{O}_{W,\xi}$ is nonzero by properness of intersection, hence a nonzerodivisor divisor. Moreover, $\mathcal{O}_{W,\xi}$ is a Noetherian local domain of dimension 1. Thus

$$\text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1^t\mathcal{O}_{W,\xi}) = t\text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi})$$

for all $t \geq 1$, see Algebra, Lemma 10.121.1. This proves the lemma. □

0B06 Lemma 43.16.5. In Lemma 43.16.3 assume that the local ring $\mathcal{O}_{W,\xi}$ is Cohen-Macaulay. Then we have

$$e(X, V \cdot W, Z) = \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi} + \ldots + f_c\mathcal{O}_{W,\xi}).$$

Proof. This follows immediately from Lemma 43.16.1. Alternatively, we can deduce it from Lemma 43.16.3. Namely, by Algebra, Lemma 10.104.2 we see that $f_1, \ldots, f_c$

is a regular sequence in $\mathcal{O}_{W,\xi}$. Then Algebra, Lemma 10.69.2 shows that $f_1, \ldots, f_c$ is a quasi-regular sequence. This easily implies the length of $\mathcal{O}_{W,\xi}/(f_1, \ldots, f_c)^t$ is

$$\binom{c+t}{c} \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi} + \ldots + f_c\mathcal{O}_{W,\xi}).$$

Looking at the leading coefficient we conclude. □

43.17. Intersection product using Tor formula

0B08 Let X be a nonsingular variety. Let $\alpha = \sum n_i[W_i]$ be an r-cycle and $\beta = \sum_j m_j[V_j]$ be an s-cycle on X. Assume that α and β intersect properly, see Definition 43.13.5. In this case we define

$$\alpha \cdot \beta = \sum\nolimits_{i,j} n_i m_j W_i \cdot V_j.$$

where $W_i \cdot V_j$ is as defined in Section 43.14. If $\beta = [V]$ where V is a closed subvariety of dimension s, then we sometimes write $\alpha \cdot \beta = \alpha \cdot V$.

0B07 **Lemma 43.17.1.** Let X be a nonsingular variety. Let $a, b \in \mathbf{P}^1$ be distinct closed points. Let $k \geq 0$.

(1) If $W \subset X \times \mathbf{P}^1$ is a closed subvariety of dimension $k+1$ which intersects $X \times a$ properly, then
  (a) $[W_a]_k = W \cdot X \times a$ as cycles on $X \times \mathbf{P}^1$, and
  (b) $[W_a]_k = \text{pr}_{X,*}(W \cdot X \times a)$ as cycles on X.
(2) Let α be a $(k+1)$-cycle on $X \times \mathbf{P}^1$ which intersects $X \times a$ and $X \times b$ properly. Then $pr_{X,*}(\alpha \cdot X \times a - \alpha \cdot X \times b)$ is rationally equivalent to zero.
(3) Conversely, any k-cycle which is rationally equivalent to 0 is of this form.

Proof. First we observe that $X \times a$ is an effective Cartier divisor in $X \times \mathbf{P}^1$ and that W_a is the scheme theoretic intersection of W with $X \times a$. Hence the equality in (1)(a) is immediate from the definitions and the calculation of intersection multiplicity in case of a Cartier divisor given in Lemma 43.16.4. Part (1)(b) holds because $W_a \to X \times \mathbf{P}^1 \to X$ maps isomorphically onto its image which is how we viewed W_a as a closed subscheme of X in Section 43.8. Parts (2) and (3) are formal consequences of part (1) and the definitions. □

For transversal intersections of closed subschemes the intersection multiplicity is 1.

0B1J **Lemma 43.17.2.** Let X be a nonsingular variety. Let $r, s \geq 0$ and let $Y, Z \subset X$ be closed subschemes with $\dim(Y) \leq r$ and $\dim(Z) \leq s$. Assume $[Y]_r = \sum n_i[Y_i]$ and $[Z]_s = \sum m_j[Z_j]$ intersect properly. Let T be an irreducible component of $Y_{i_0} \cap Z_{j_0}$ for some i_0 and j_0 and assume that the multiplicity (in the sense of Section 43.4) of T in the closed subscheme $Y \cap Z$ is 1. Then

(1) the coefficient of T in $[Y]_r \cdot [Z]_s$ is 1,
(2) Y and Z are nonsingular at the generic point of Z,
(3) $n_{i_0} = 1$, $m_{j_0} = 1$, and
(4) T is not contained in Y_i or Z_j for $i \neq i_0$ and $j \neq j_0$.

Proof. Set $n = \dim(X)$, $a = n - r$, $b = n - s$. Observe that $\dim(T) = r + s - n = n - a - b$ by the assumption that the intersections are transversal. Let $(A, \mathfrak{m}, \kappa) = (\mathcal{O}_{X,\xi}, \mathfrak{m}_\xi, \kappa(\xi))$ where $\xi \in T$ is the generic point. Then $\dim(A) = a + b$, see Varieties, Lemma 33.20.3. Let $I_0, I, J_0, J \subset A$ cut out the trace of Y_{i_0}, Y, Z_{j_0}, Z in $\text{Spec}(A)$. Then $\dim(A/I) = \dim(A/I_0) = b$ and $\dim(A/J) = \dim(A/J_0) = a$

by the same reference. Set $\overline{I} = I + \mathfrak{m}^2/\mathfrak{m}^2$. Then $I \subset I_0 \subset \mathfrak{m}$ and $J \subset J_0 \subset \mathfrak{m}$ and $I + J = \mathfrak{m}$. By Lemma 43.14.3 and its proof we see that $I_0 = (f_1, \ldots, f_a)$ and $J_0 = (g_1, \ldots, g_b)$ where $f_1, \ldots, g_b$ is a regular system of parameters for the regular local ring A. Since $I + J = \mathfrak{m}$, the map

$$I \oplus J \to \mathfrak{m}/\mathfrak{m}^2 = \kappa f_1 \oplus \ldots \oplus \kappa f_a \oplus \kappa g_1 \oplus \ldots \oplus \kappa g_b$$

is surjective. We conclude that we can find $f'_1, \ldots, f'_a \in I$ and $g'_1, \ldots, g'_b \in J$ whose residue classes in $\mathfrak{m}/\mathfrak{m}^2$ are equal to the residue classes of $f_1, \ldots, f_a$ and $g_1, \ldots, g_b$. Then $f'_1, \ldots, g'_b$ is a regular system of parameters of A. By Algebra, Lemma 10.106.3 we find that $A/(f'_1, \ldots, f'_a)$ is a regular local ring of dimension b. Thus any nontrivial quotient of $A/(f'_1, \ldots, f'_a)$ has strictly smaller dimension (Algebra, Lemmas 10.106.2 and 10.60.13). Hence $I = (f'_1, \ldots, f'_a) = I_0$. By symmetry $J = J_0$. This proves (2), (3), and (4). Finally, the coefficient of T in $[Y]_r \cdot [Z]_s$ is the coefficient of T in $Y_{i_0} \cdot Z_{j_0}$ which is 1 by Lemma 43.14.3. □

43.18. Exterior product

0B09 Let X and Y be varieties. Let V, resp. W be a closed subvariety of X, resp. Y. The product $V \times W$ is a closed subvariety of $X \times Y$ (Lemma 43.13.1). For a k-cycle $\alpha = \sum n_i [V_i]$ and a l-cycle $\beta = \sum m_j [V_j]$ on Y we define the exterior product of α and β to be the cycle $\alpha \times \beta = \sum n_i m_j [V_i \times W_j]$. Exterior product defines a $\mathbf{Z}$-linear map

$$Z_r(X) \otimes_{\mathbf{Z}} Z_s(Y) \longrightarrow Z_{r+s}(X \times Y)$$

Let us prove that exterior product factors through rational equivalence.

0B0S Lemma 43.18.1. Let X and Y be varieties. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(Y)$. If $\alpha \sim_{rat} 0$ or $\beta \sim_{rat} 0$, then $\alpha \times \beta \sim_{rat} 0$.

Proof. By linearity and symmetry in X and Y, it suffices to prove this when $\alpha = [V]$ for some subvariety $V \subset X$ of dimension s and $\beta = [W_a]_s - [W_b]_s$ for some closed subvariety $W \subset Y \times \mathbf{P}^1$ of dimension $s+1$ which intersects $Y \times a$ and $Y \times b$ properly. In this case the lemma follows if we can prove

$$[(V \times W)_a]_{r+s} = [V] \times [W_a]_s$$

and similarly with a replaced by b. Namely, then we see that $\alpha \times \beta = [(V \times W)_a]_{r+s} - [(V \times W)_b]_{r+s}$ as desired. To see the displayed equality we note the equality

$$V \times W_a = (V \times W)_a$$

of schemes. The projection $V \times W_a \to W_a$ induces a bijection of irreducible components (see for example Varieties, Lemma 33.8.4). Let $W' \subset W_a$ be an irreducible component with generic point ζ. Then $V \times W'$ is the corresponding irreducible component of $V \times W_a$ (see Lemma 43.13.1). Let ξ be the generic point of $V \times W'$. We have to show that

$$\text{length}_{\mathcal{O}_{Y,\zeta}}(\mathcal{O}_{W_a,\zeta}) = \text{length}_{\mathcal{O}_{X \times Y,\xi}}(\mathcal{O}_{V \times W_a,\xi})$$

In this formula we may replace $\mathcal{O}_{Y,\zeta}$ by $\mathcal{O}_{W_a,\zeta}$ and we may replace $\mathcal{O}_{X \times Y,\zeta}$ by $\mathcal{O}_{V \times W_a,\zeta}$ (see Algebra, Lemma 10.52.5). As $\mathcal{O}_{W_a,\zeta} \to \mathcal{O}_{V \times W_a,\xi}$ is flat, by Algebra, Lemma 10.52.13 it suffices to show that

$$\text{length}_{\mathcal{O}_{V \times W_a,\xi}}(\mathcal{O}_{V \times W_a,\xi}/\mathfrak{m}_\zeta \mathcal{O}_{V \times W_a,\xi}) = 1$$

This is true because the quotient on the right is the local ring $\mathcal{O}_{V\times W',\xi}$ of a variety at a generic point hence equal to $\kappa(\xi)$. $\square$

We conclude that exterior product defines a commutative diagram

$$\begin{tikzcd} Z_r(X) \otimes_{\mathbf{Z}} Z_s(Y) \arrow[r] \arrow[d] & Z_{r+s}(X \times Y) \arrow[d] \\ \mathrm{CH}_r(X) \otimes_{\mathbf{Z}} \mathrm{CH}_s(Y) \arrow[r] & \mathrm{CH}_{r+s}(X \times Y) \end{tikzcd}$$

for any pair of varieties X and Y. For nonsingular varieties we can think of the exterior product as an intersection product of pullbacks.

0B0R Lemma 43.18.2. Let X and Y be nonsingular varieties. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(Y)$. Then

(1) $\mathrm{pr}_Y^*(\beta) = [X] \times \beta$ and $\mathrm{pr}_X^*(\alpha) = \alpha \times [Y]$,
(2) $\alpha \times [Y]$ and $[X] \times \beta$ intersect properly on $X \times Y$, and
(3) we have $\alpha \times \beta = (\alpha \times [Y]) \cdot ([X] \times \beta) = pr_Y^*(\alpha) \cdot pr_X^*(\beta)$ in $Z_{r+s}(X \times Y)$.

Proof. By linearity we may assume $\alpha = [V]$ and $\beta = [W]$. Then (1) says that $\mathrm{pr}_Y^{-1}(W) = X \times W$ and $\mathrm{pr}_X^{-1}(V) = V \times Y$. This is clear. Part (2) holds because $X \times W \cap V \times Y = V \times W$ and $\dim(V \times W) = r + s$ by Lemma 43.13.1.

Proof of (3). Let ξ be the generic point of $V \times W$. Since the projections $X \times W \to W$ is smooth as a base change of $X \to \mathrm{Spec}(\mathbf{C})$, we see that $X \times W$ is nonsingular at every point lying over the generic point of W, in particular at ξ. Similarly for $V \times Y$. Hence $\mathcal{O}_{X\times W,\xi}$ and $\mathcal{O}_{V\times Y,\xi}$ are Cohen-Macaulay local rings and Lemma 43.16.1 applies. Since $V \times Y \cap X \times W = V \times W$ scheme theoretically the proof is complete. $\square$

43.19. Reduction to the diagonal

0B0A Let X be a nonsingular variety. We will use Δ to denote either the diagonal morphism $\Delta : X \to X \times X$ or the image $\Delta \subset X \times X$. Reduction to the diagonal is the statement that intersection products on X can be reduced to intersection products of exterior products with the diagonal on $X \times X$.

0B0T Lemma 43.19.1. Let X be a nonsingular variety.

(1) If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then there are canonical isomorphisms
$$\mathrm{Tor}_i^{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times X}} \mathrm{pr}_2^*\mathcal{G}) = \Delta_*\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$
(2) If K and M are in $D_{QCoh}(\mathcal{O}_X)$, then there is a canonical isomorphism
$$L\Delta^*\left(L\mathrm{pr}_1^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} L\mathrm{pr}_2^*M\right) = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$
in $D_{QCoh}(\mathcal{O}_X)$ and a canonical isomorphism
$$\mathcal{O}_\Delta \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} L\mathrm{pr}_1^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} L\mathrm{pr}_2^*M = \Delta_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$
in $D_{QCoh}(X \times X)$.

Proof. Let us explain how to prove (1) in a more elementary way and part (2) using more general theory. As (2) implies (1) the reader can skip the proof of (1).

Proof of (1). Choose an affine open $\mathrm{Spec}(A) \subset X$. Then A is a Noetherian $\mathbf{C}$-algebra and $\mathcal{F}$, $\mathcal{G}$ correspond to finite A-modules M and N (Cohomology of Schemes, Lemma 30.9.1). By Derived Categories of Schemes, Lemma 36.3.9 we may compute Tor_i over $\mathcal{O}_X$ by first computing the Tor's of M and N over A, and then taking the associated $\mathcal{O}_X$-module. For the Tor_i over $\mathcal{O}_{X \times X}$ we compute the tor of A and $M \otimes_{\mathbf{C}} N$ over $A \otimes_{\mathbf{C}} A$ and then take the associated $\mathcal{O}_{X \times X}$-module. Hence on this affine patch we have to prove that

$$\mathrm{Tor}_i^{A \otimes_{\mathbf{C}} A}(A, M \otimes_{\mathbf{C}} N) = \mathrm{Tor}_i^A(M, N)$$

To see this choose resolutions $F_\bullet \to M$ and $G_\bullet \to M$ by finite free A-modules (Algebra, Lemma 10.71.1). Note that $\mathrm{Tot}(F_\bullet \otimes_{\mathbf{C}} G_\bullet)$ is a resolution of $M \otimes_{\mathbf{C}} N$ as it computes Tor groups over $\mathbf{C}$! Of course the terms of $F_\bullet \otimes_{\mathbf{C}} G_\bullet$ are finite free $A \otimes_{\mathbf{C}} A$-modules. Hence the left hand side of the displayed equation is the module

$$H_i(A \otimes_{A \otimes_{\mathbf{C}} A} \mathrm{Tot}(F_\bullet \otimes_{\mathbf{C}} G_\bullet))$$

and the right hand side is the module

$$H_i(\mathrm{Tot}(F_\bullet \otimes_A G_\bullet))$$

Since $A \otimes_{A \otimes_{\mathbf{C}} A} (F_p \otimes_{\mathbf{C}} G_q) = F_p \otimes_A G_q$ we see that these modules are equal. This defines an isomorphism over the affine open $\mathrm{Spec}(A) \times \mathrm{Spec}(A)$ (which is good enough for the application to equality of intersection numbers). We omit the proof that these isomorphisms glue.

Proof of (2). The second statement follows from the first by the projection formula as stated in Derived Categories of Schemes, Lemma 36.22.1. To see the first, represent K and M by K-flat complexes $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$. Since pullback and tensor product preserve K-flat complexes (Cohomology, Lemmas 20.26.5 and 20.26.8) we see that it suffices to show

$$\Delta^* \mathrm{Tot}(\mathrm{pr}_1^* \mathcal{K}^\bullet \otimes_{\mathcal{O}_{X \times X}} \mathrm{pr}_2^* \mathcal{M}^\bullet) = \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

Thus it suffices to see that there are canonical isomorphisms

$$\Delta^*(\mathrm{pr}_1^* \mathcal{K} \otimes_{\mathcal{O}_{X \times X}} \mathrm{pr}_2^* \mathcal{M}) \longrightarrow \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}$$

whenever $\mathcal{K}$ and $\mathcal{M}$ are $\mathcal{O}_X$-modules (not necessarily quasi-coherent or flat). We omit the details. ☐

0B0U Lemma 43.19.2. Let X be a nonsingular variety. Let α, resp. β be an r-cycle, resp. s-cycle on X. Assume α and β intersect properly. Then

(1) $\alpha \times \beta$ and $[\Delta]$ intersect properly
(2) we have $\Delta_*(\alpha \cdot \beta) = [\Delta] \cdot \alpha \times \beta$ as cycles on $X \times X$,
(3) if X is proper, then $\mathrm{pr}_{1,*}([\Delta] \cdot \alpha \times \beta) = \alpha \cdot \beta$, where $pr_1 : X \times X \to X$ is the projection.

Proof. By linearity it suffices to prove this when $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V \subset X$ and $W \subset Y$ which intersect properly. Recall that $V \times W$ is a closed subvariety of dimension $r + s$. Observe that scheme theoretically we have $V \cap W = \Delta^{-1}(V \times W)$ as well as $\Delta(V \cap W) = \Delta \cap V \times W$. This proves (1).

Proof of (2). Let $Z \subset V \cap W$ be an irreducible component with generic point ξ. We have to show that the coefficient of Z in $\alpha \cdot \beta$ is the same as the coefficient of $\Delta(Z)$ in $[\Delta] \cdot \alpha \times \beta$. The first is given by the integer

$$\sum (-1)^i \text{length}_{\mathcal{O}_{X,\xi}} \text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_\xi$$

and the second by the integer

$$\sum (-1)^i \text{length}_{\mathcal{O}_{X \times Y, \Delta(\xi)}} \text{Tor}_i^{\mathcal{O}_{X \times Y}}(\mathcal{O}_\Delta, \mathcal{O}_{V \times W})_{\Delta(\xi)}$$

However, by Lemma 43.19.1 we have

$$\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_\xi \cong \text{Tor}_i^{\mathcal{O}_{X \times Y}}(\mathcal{O}_\Delta, \mathcal{O}_{V \times W})_{\Delta(\xi)}$$

as $\mathcal{O}_{X \times X, \Delta(\xi)}$-modules. Thus equality of lengths (by Algebra, Lemma 10.52.5 to be precise).

Part (2) implies (3) because $\text{pr}_{1,*} \circ \Delta_* = \text{id}$ by Lemma 43.6.2. □

0B0V **Proposition 43.19.3.** Let X be a nonsingular variety. Let $V \subset X$ and $W \subset Y$ be closed subvarieties which intersect properly. Let $Z \subset V \cap W$ be an irreducible component. Then $e(X, V \cdot W, Z) > 0$. This is one of the main results of [Ser65].

Proof. By Lemma 43.19.2 we have

$$e(X, V \cdot W, Z) = e(X \times X, \Delta \cdot V \times W, \Delta(Z))$$

Since $\Delta : X \to X \times X$ is a regular immersion (see Lemma 43.13.3), we see that $e(X \times X, \Delta \cdot V \times W, \Delta(Z))$ is a positive integer by Lemma 43.16.3. □

The following is a key lemma in the development of the theory as is done in this chapter. Essentially, this lemma tells us that the intersection numbers have a suitable additivity property.

0B0W **Lemma 43.19.4.** Let X be a nonsingular variety. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X with $\dim(\text{Supp}(\mathcal{F})) \leq r$, $\dim(\text{Supp}(\mathcal{G})) \leq s$, and $\dim(\text{Supp}(\mathcal{F}) \cap \text{Supp}(\mathcal{G})) \leq r + s - \dim X$. In this case $[\mathcal{F}]_r$ and $[\mathcal{G}]_s$ intersect properly and [Ser65, Chapter V]

$$[\mathcal{F}]_r \cdot [\mathcal{G}]_s = \sum (-1)^p [\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim(X)}.$$

Proof. The statement that $[\mathcal{F}]_r$ and $[\mathcal{G}]_s$ intersect properly is immediate. Since we are proving an equality of cycles we may work locally on X. (Observe that the formation of the intersection product of cycles, the formation of Tor-sheaves, and forming the cycle associated to a coherent sheaf, each commute with restriction to open subschemes.) Thus we may and do assume that X is affine.

Denote

$$RHS(\mathcal{F}, \mathcal{G}) = [\mathcal{F}]_r \cdot [\mathcal{G}]_s \quad \text{and} \quad LHS(\mathcal{F}, \mathcal{G}) = \sum (-1)^p [\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim(X)}$$

Consider a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

of coherent sheaves on X with $\text{Supp}(\mathcal{F}_i) \subset \text{Supp}(\mathcal{F})$, then both $LHS(\mathcal{F}_i, \mathcal{G})$ and $RHS(\mathcal{F}_i, \mathcal{G})$ are defined for $i = 1, 2, 3$ and we have

$$RHS(\mathcal{F}_2, \mathcal{G}) = RHS(\mathcal{F}_1, \mathcal{G}) + RHS(\mathcal{F}_3, \mathcal{G})$$

and similarly for LHS. Namely, the support condition guarantees that everything is defined, the short exact sequence and additivity of lengths gives

$$[\mathcal{F}_2]_r = [\mathcal{F}_1]_r + [\mathcal{F}_3]_r$$

(Chow Homology, Lemma 42.10.4) which implies additivity for RHS. The long exact sequence of Tors

$$\ldots \to \text{Tor}_1(\mathcal{F}_3, \mathcal{G}) \to \text{Tor}_0(\mathcal{F}_1, \mathcal{G}) \to \text{Tor}_0(\mathcal{F}_2, \mathcal{G}) \to \text{Tor}_0(\mathcal{F}_3, \mathcal{G}) \to 0$$

and additivity of lengths as before implies additivity for LHS.

By Algebra, Lemma 10.62.1 and the fact that X is affine, we can find a filtration of $\mathcal{F}$ whose graded pieces are structure sheaves of closed subvarieties of $\text{Supp}(\mathcal{F})$. The additivity shown in the previous paragraph, implies that it suffices to prove $LHS = RHS$ with $\mathcal{F}$ replaced by $\mathcal{O}_V$ where $V \subset \text{Supp}(\mathcal{F})$. By symmetry we can do the same for $\mathcal{G}$. This reduces us to proving that

$$LHS(\mathcal{O}_V, \mathcal{O}_W) = RHS(\mathcal{O}_V, \mathcal{O}_W)$$

where $W \subset \text{Supp}(\mathcal{G})$ is a closed subvariety. If $\dim(V) = r$ and $\dim(W) = s$, then this equality is the definition of $V \cdot W$. On the other hand, if $\dim(V) < r$ or $\dim(W) < s$, i.e., $[V]_r = 0$ or $[W]_s = 0$, then we have to prove that $RHS(\mathcal{O}_V, \mathcal{O}_W) = 0$ [5].

Let $Z \subset V \cap W$ be an irreducible component of dimension $r+s-\dim(X)$. This is the maximal dimension of a component and it suffices to show that the coefficient of Z in RHS is zero. Let $\xi \in Z$ be the generic point. Write $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{X \times X, \Delta(\xi)}$, and $C = \mathcal{O}_{V \times W, \Delta(\xi)}$. By Lemma 43.19.1 we have

$$\text{coeff of } Z \text{ in } RHS(\mathcal{O}_V, \mathcal{O}_W) = \sum (-1)^i \text{length}_B \text{Tor}_i^B(A, C)$$

Since $\dim(V) < r$ or $\dim(W) < s$ we have $\dim(V \times W) < r + s$ which implies $\dim(C) < \dim(X)$ (small detail omitted). Moreover, the kernel I of $B \to A$ is generated by a regular sequence of length $\dim(X)$ (Lemma 43.13.3). Hence vanishing by Lemma 43.16.2 because the Hilbert function of C with respect to I has degree $\dim(C) < n$ by Algebra, Proposition 10.60.9. □

0B0X Remark 43.19.5. Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring. Let M and N be nonzero finite A-modules such that $M \otimes_A N$ is supported in $\{\mathfrak{m}\}$. Then

$$\chi(M, N) = \sum (-1)^i \text{length}_A \text{Tor}_i^A(M, N)$$

is finite. Let $r = \dim(\text{Supp}(M))$ and $s = \dim(\text{Supp}(N))$. In [Ser65] it is shown that $r + s \leq \dim(A)$ and the following conjectures are made:

(1) if $r + s < \dim(A)$, then $\chi(M, N) = 0$, and
(2) if $r + s = \dim(A)$, then $\chi(M, N) > 0$.

The arguments that prove Lemma 43.19.4 and Proposition 43.19.3 can be leveraged (as is done in Serre's text) to show that (1) and (2) are true if A contains a field. Currently, conjecture (1) is known in general and it is known that $\chi(M, N) \geq 0$ in general (Gabber). Positivity is, as far as we know, still an open problem.

[5] The reader can see that this is not a triviality by taking $r = s = 1$ and X a nonsingular surface and $V = W$ a closed point x of X. In this case there are 3 nonzero Tors of lengths $1, 2, 1$ at x.

43.20. Associativity of intersections

0B1K It is clear that proper intersections as defined above are commutative. Using the key Lemma 43.19.4 we can prove that (proper) intersection products are associative.

0B1L Lemma 43.20.1. Let X be a nonsingular variety. Let U, V, W be closed subvarieties. Assume that U, V, W intersect properly pairwise and that $\dim(U \cap V \cap W) \leq \dim(U) + \dim(V) + \dim(W) - 2\dim(X)$. Then

$$U \cdot (V \cdot W) = (U \cdot V) \cdot W$$

as cycles on X.

Proof. We are going to use Lemma 43.19.4 without further mention. This implies that

$$\begin{aligned} V \cdot W &= \sum (-1)^i [\mathrm{Tor}_i(\mathcal{O}_V, \mathcal{O}_W)]_{b+c-n} \\ U \cdot (V \cdot W) &= \sum (-1)^{i+j} [\mathrm{Tor}_j(\mathcal{O}_U, \mathrm{Tor}_i(\mathcal{O}_V, \mathcal{O}_W))]_{a+b+c-2n} \\ U \cdot V &= \sum (-1)^i [\mathrm{Tor}_i(\mathcal{O}_U, \mathcal{O}_V)]_{a+b-n} \\ (U \cdot V) \cdot W &= \sum (-1)^{i+j} [\mathrm{Tor}_j(\mathrm{Tor}_i(\mathcal{O}_U, \mathcal{O}_V), \mathcal{O}_W))]_{a+b+c-2n} \end{aligned}$$

where $\dim(U) = a$, $\dim(V) = b$, $\dim(W) = c$, $\dim(X) = n$. The assumptions in the lemma guarantee that the coherent sheaves in the formulae above satisfy the required bounds on dimensions of supports in order to make sense of these. Now consider the object

$$K = \mathcal{O}_U \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_V \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_W$$

of the derived category $D_{\mathrm{Coh}}(\mathcal{O}_X)$. We claim that the expressions obtained above for $U \cdot (V \cdot W)$ and $(U \cdot V) \cdot W$ are equal to

$$\sum (-1)^k [H^k(K)]_{a+b+c-2n}$$

This will prove the lemma. By symmetry it suffices to prove one of these equalities. To do this we represent $\mathcal{O}_U$ and $\mathcal{O}_V \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_W$ by K-flat complexes $M^\bullet$ and $L^\bullet$ and use the spectral sequence associated to the double complex $M^\bullet \otimes_{\mathcal{O}_X} L^\bullet$ in Homology, Section 12.25. This is a spectral sequence with E_2 page

$$E_2^{p,q} = \mathrm{Tor}_{-p}(\mathcal{O}_U, \mathrm{Tor}_{-q}(\mathcal{O}_V, \mathcal{O}_W))$$

converging to $H^{p+q}(K)$ (details omitted; compare with More on Algebra, Example 15.62.4). Since lengths are additive in short exact sequences we see that the result is true. □

43.21. Flat pullback and intersection products

0B0B Short discussion of the interaction between intersections and flat pullback.

0B0Y Lemma 43.21.1. Let $f : X \to Y$ be a flat morphism of nonsingular varieties. Set $e = \dim(X) - \dim(Y)$. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on Y with $\dim(\mathrm{Supp}(\mathcal{F})) \leq r$, $\dim(\mathrm{Supp}(\mathcal{G})) \leq s$, and $\dim(\mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(\mathcal{G})) \leq r + s - \dim(Y)$. In this case the cycles $[f^*\mathcal{F}]_{r+e}$ and $[f^*\mathcal{G}]_{s+e}$ intersect properly and

$$f^*([\mathcal{F}]_r \cdot [\mathcal{G}]_s) = [f^*\mathcal{F}]_{r+e} \cdot [f^*\mathcal{G}]_{s+e}$$

Proof. The statement that $[f^*\mathcal{F}]_{r+e}$ and $[f^*\mathcal{G}]_{s+e}$ intersect properly is immediate from the assumption that f has relative dimension e. By Lemmas 43.19.4 and 43.7.1 it suffices to show that

$$f^*\text{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) = \text{Tor}_i^{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

as $\mathcal{O}_X$-modules. This follows from Cohomology, Lemma 20.27.3 and the fact that f^* is exact, so $Lf^*\mathcal{F} = f^*\mathcal{F}$ and similarly for $\mathcal{G}$. □

0B0Z Lemma 43.21.2. Let $f : X \to Y$ be a flat morphism of nonsingular varieties. Let α be a r-cycle on Y and β an s-cycle on Y. Assume that α and β intersect properly. Then $f^*\alpha$ and $f^*\beta$ intersect properly and $f^*(\alpha \cdot \beta) = f^*\alpha \cdot f^*\beta$.

Proof. By linearity we may assume that $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V, W \subset Y$ of dimension r, s. Say f has relative dimension e. Then the lemma is a special case of Lemma 43.21.1 because $[V] = [\mathcal{O}_V]_r$, $[W] = [\mathcal{O}_W]_r$, $f^*[V] = [f^{-1}(V)]_{r+e} = [f^*\mathcal{O}_V]_{r+e}$, and $f^*[W] = [f^{-1}(W)]_{s+e} = [f^*\mathcal{O}_W]_{s+e}$. □

43.22. Projection formula for flat proper morphisms

0B0C Short discussion of the projection formula for flat proper morphisms.

0B10 Lemma 43.22.1. Let $f : X \to Y$ be a flat proper morphism of nonsingular varieties. Set $e = \dim(X) - \dim(Y)$. Let α be an r-cycle on X and let β be a s-cycle on Y. Assume that α and $f^*(\beta)$ intersect properly. Then $f_*(\alpha)$ and β intersect properly and

$$f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$$

See [Ser65, Chapter V, C), Section 7, formula (10)] for a more general formula.

Proof. By linearity we reduce to the case where $\alpha = [V]$ and $\beta = [W]$ for some closed subvariety $V \subset X$ and $W \subset Y$ of dimension r and s. Then $f^{-1}(W)$ has pure dimension $s + e$. We assume the cycles $[V]$ and $f^*[W]$ intersect properly. We will use without further mention the fact that $V \cap f^{-1}(W) \to f(V) \cap W$ is surjective.

Let a be the dimension of the generic fibre of $V \to f(V)$. If $a > 0$, then $f_*[V] = 0$. In particular $f_*\alpha$ and β intersect properly. To finish this case we have to show that $f_*([V] \cdot f^*[W]) = 0$. However, since every fibre of $V \to f(V)$ has dimension $\geq a$ (see Morphisms, Lemma 29.28.4) we conclude that every irreducible component Z of $V \cap f^{-1}(W)$ has fibres of dimension $\geq a$ over $f(Z)$. This certainly implies what we want.

Assume that $V \to f(V)$ is generically finite. Let $Z \subset f(V) \cap W$ be an irreducible component. Let $Z_i \subset V \cap f^{-1}(W)$, $i = 1, \ldots, t$ be the irreducible components of $V \cap f^{-1}(W)$ dominating Z. By assumption each Z_i has dimension $r + s + e - \dim(X) = r + s - \dim(Y)$. Hence $\dim(Z) \leq r + s - \dim(Y)$. Thus we see that $f(V)$ and W intersect properly, $\dim(Z) = r + s - \dim(Y)$, and each $Z_i \to Z$ is generically finite. In particular, it follows that $V \to f(V)$ has finite fibre over the generic point ξ of Z. Thus $V \to Y$ is finite in an open neighbourhood of ξ, see Cohomology of Schemes, Lemma 30.21.2. Using a very general projection formula for derived tensor products, we get

$$Rf_*(\mathcal{O}_V \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*\mathcal{O}_W) = Rf_*\mathcal{O}_V \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{O}_W$$

see Derived Categories of Schemes, Lemma 36.22.1. Since f is flat, we see that $Lf^*\mathcal{O}_W = f^*\mathcal{O}_W$. Since $f|_V$ is finite in an open neighbourhood of ξ we have

$$(Rf_*\mathcal{F})_\xi = (f_*\mathcal{F})_\xi$$

for any coherent sheaf on X whose support is contained in V (see Cohomology of Schemes, Lemma 30.20.8). Thus we conclude that

0B11 (43.22.1.1)
$$\left(f_*\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)\right)_\xi = \left(\text{Tor}_i^{\mathcal{O}_Y}(f_*\mathcal{O}_V, \mathcal{O}_W)\right)_\xi$$

for all i. Since $f^*[W] = [f^*\mathcal{O}_W]_{s+e}$ by Lemma 43.7.1 we have

$$[V] \cdot f^*[W] = \sum (-1)^i [\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]_{r+s-\dim(Y)}$$

by Lemma 43.19.4. Applying Lemma 43.6.1 we find

$$f_*([V] \cdot f^*[W]) = \sum (-1)^i [f_*\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]_{r+s-\dim(Y)}$$

Since $f_*[V] = [f_*\mathcal{O}_V]_r$ by Lemma 43.6.1 we have

$$[f_*V] \cdot [W] = \sum (-1)^i [\text{Tor}_i^{\mathcal{O}_X}(f_*\mathcal{O}_V, \mathcal{O}_W)]_{r+s-\dim(Y)}$$

again by Lemma 43.19.4. Comparing the formula for $f_*([V] \cdot f^*[W])$ with the formula for $f_*[V] \cdot [W]$ and looking at the coefficient of Z by taking lengths of stalks at ξ, we see that (43.22.1.1) finishes the proof. ☐

0B1M **Lemma 43.22.2.** Let $X \to P$ be a closed immersion of nonsingular varieties. Let $C' \subset P \times \mathbf{P}^1$ be a closed subvariety of dimension $r+1$. Assume

(1) the fibre $C = C'_0$ has dimension r, i.e., $C' \to \mathbf{P}^1$ is dominant,
(2) C' intersects $X \times \mathbf{P}^1$ properly,
(3) $[C]_r$ intersects X properly.

Then setting $\alpha = [C]_r \cdot X$ viewed as cycle on X and $\beta = C' \cdot X \times \mathbf{P}^1$ viewed as cycle on $X \times \mathbf{P}^1$, we have

$$\alpha = \text{pr}_{X,*}(\beta \cdot X \times 0)$$

as cycles on X where $\text{pr}_X : X \times \mathbf{P}^1 \to X$ is the projection.

Proof. Let $\text{pr} : P \times \mathbf{P}^1 \to P$ be the projection. Since we are proving an equality of cycles it suffices to think of α, resp. β as a cycle on P, resp. $P \times \mathbf{P}^1$ and prove the result for pushing forward by pr. Because $\text{pr}^*X = X \times \mathbf{P}^1$ and pr defines an isomorphism of C'_0 onto C the projection formula (Lemma 43.22.1) gives

$$\text{pr}_*([C'_0]_r \cdot X \times \mathbf{P}^1) = [C]_r \cdot X = \alpha$$

On the other hand, we have $[C'_0]_r = C' \cdot P \times 0$ as cycles on $P \times \mathbf{P}^1$ by Lemma 43.17.1. Hence

$$[C'_0]_r \cdot X \times \mathbf{P}^1 = (C' \cdot P \times 0) \cdot X \times \mathbf{P}^1 = (C' \cdot X \times \mathbf{P}^1) \cdot P \times 0$$

by associativity (Lemma 43.20.1) and commutativity of the intersection product. It remains to show that the intersection product of $C' \cdot X \times \mathbf{P}^1$ with $P \times 0$ on $P \times \mathbf{P}^1$ is equal (as a cycle) to the intersection product of β with $X \times 0$ on $X \times \mathbf{P}^1$. Write $C' \cdot X \times \mathbf{P}^1 = \sum n_k [E_k]$ and hence $\beta = \sum n_k [E_k]$ for some subvarieties $E_k \subset X \times \mathbf{P}^1 \subset P \times \mathbf{P}^1$. Then both intersections are equal to $\sum m_k [E_{k,0}]$ by Lemma 43.17.1 applied twice. This finishes the proof. ☐

43.23. Projections

0B1N Recall that we are working over a fixed algebraically closed ground field $\mathbf{C}$. If V is a finite dimensional vector space over $\mathbf{C}$ then we set

$$\mathbf{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V))$$

where $\mathrm{Sym}(V)$ is the symmetric algebra on V over $\mathbf{C}$. See Constructions, Example 27.21.2. The normalization is chosen such that $V = \Gamma(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1))$. Of course we have $\mathbf{P}(V) \cong \mathbf{P}^n_{\mathbf{C}}$ if $\dim(V) = n + 1$. We note that $\mathbf{P}(V)$ is a nonsingular projective variety.

Let $p \in \mathbf{P}(V)$ be a closed point. The point p corresponds to a surjection $V \to L_p$ of vector spaces where $\dim(L_p) = 1$, see Constructions, Lemma 27.12.3. Let us denote $W_p = \mathrm{Ker}(V \to L_p)$. Projection from p is the morphism

$$r_p : \mathbf{P}(V) \setminus \{p\} \longrightarrow \mathbf{P}(W_p)$$

of Constructions, Lemma 27.11.1. Here is a lemma to warm up.

0B1P Lemma 43.23.1. Let V be a vector space of dimension $n + 1$. Let $X \subset \mathbf{P}(V)$ be a closed subscheme. If $X \neq \mathbf{P}(V)$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the restriction of the projection r_p defines a finite morphism $r_p|_X : X \to \mathbf{P}(W_p)$.

Proof. We claim the lemma holds with $U = \mathbf{P}(V) \setminus X$. For a closed point p of U we indeed obtain a morphism $r_p|_X : X \to \mathbf{P}(W_p)$. This morphism is proper because X is a proper scheme (Morphisms, Lemmas 29.43.5 and 29.41.7). On the other hand, the fibres of r_p are affine lines as can be seen by a direct calculation. Hence the fibres of $r_p|X$ are proper and affine, whence finite (Morphisms, Lemma 29.44.11). Finally, a proper morphism with finite fibres is finite (Cohomology of Schemes, Lemma 30.21.1). □

0B1Q Lemma 43.23.2. Let V be a vector space of dimension $n + 1$. Let $X \subset \mathbf{P}(V)$ be a closed subvariety. Let $x \in X$ be a nonsingular point.

1. If $\dim(X) < n-1$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_X : X \to r_p(X)$ is an isomorphism over an open neighbourhood of $r_p(x)$.
2. If $\dim(X) = n-1$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_X : X \to \mathbf{P}(W_p)$ is étale at x.

Proof. Proof of (1). Note that if $x, y \in X$ have the same image under r_p then p is on the line $\overline{xy}$. Consider the finite type scheme

$$T = \{(y, p) \mid y \in X \setminus \{x\},\ p \in \mathbf{P}(V),\ p \in \overline{xy}\}$$

and the morphisms $T \to X$ and $T \to \mathbf{P}(V)$ given by $(y, p) \mapsto y$ and $(y, p) \mapsto p$. Since each fibre of $T \to X$ is a line, we see that the dimension of T is $\dim(X) + 1 < \dim(\mathbf{P}(V))$. Hence $T \to \mathbf{P}(V)$ is not surjective. On the other hand, consider the finite type scheme

$$T' = \{p \mid p \in \mathbf{P}(V) \setminus \{x\},\ \overline{xp} \text{ tangent to } X \text{ at } x\}$$

Then the dimension of T' is $\dim(X) < \dim(\mathbf{P}(V))$. Thus the morphism $T' \to \mathbf{P}(V)$ is not surjective either. Let $U \subset \mathbf{P}(V) \setminus X$ be nonempty open and disjoint from these images; such a U exists because the images of T and T' in $\mathbf{P}(V)$ are constructible by Morphisms, Lemma 29.22.2. Then for $p \in U$ closed the projection $r_p|_X : X \to \mathbf{P}(W_p)$ is injective on the tangent space at x and $r_p^{-1}(\{r_p(x)\}) = \{x\}$. This means that r_p is unramified at x (Varieties, Lemma 33.16.8), finite by Lemma 43.23.1, and $r_p^{-1}(\{r_p(x)\}) = \{x\}$ thus Étale Morphisms, Lemma 41.7.3 applies and there is an open neighbourhood R of $r_p(x)$ in $\mathbf{P}(W_p)$ such that $(r_p|_X)^{-1}(R) \to R$ is a closed immersion which proves (1).

Proof of (2). In this case we still conclude that the morphism $T' \to \mathbf{P}(V)$ is not surjective. Arguing as above we conclude that for $U \subset \mathbf{P}(V)$ avoiding X and the image of T', the projection $r_p|_X : X \to \mathbf{P}(W_p)$ is étale at x and finite. $\square$

0B1R Lemma 43.23.3. Let V be a vector space of dimension $n+1$. Let $Y, Z \subset \mathbf{P}(V)$ be closed subvarieties. There is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ we have

$$Y \cap r_p^{-1}(r_p(Z)) = (Y \cap Z) \cup E$$

with $E \subset Y$ closed and $\dim(E) \leq \dim(Y) + \dim(Z) + 1 - n$.

Proof. Set $Y' = Y \setminus Y \cap Z$. Let $y \in Y'$, $z \in Z$ be closed points with $r_p(y) = r_p(z)$. Then p is on the line $\overline{yz}$ passing through y and z. Consider the finite type scheme

$$T = \{(y, z, p) \mid y \in Y', z \in Z, p \in \overline{yz}\}$$

and the morphism $T \to \mathbf{P}(V)$ given by $(y, z, p) \mapsto p$. Observe that T is irreducible and that $\dim(T) = \dim(Y) + \dim(Z) + 1$. Hence the general fibre of $T \to \mathbf{P}(V)$ has dimension at most $\dim(Y) + \dim(Z) + 1 - n$, more precisely, there exists a nonempty open $U \subset \mathbf{P}(V) \setminus (Y \cup Z)$ over which the fibre has dimension at most $\dim(Y) + \dim(Z) + 1 - n$ (Varieties, Lemma 33.20.4). Let $p \in U$ be a closed point and let $F \subset T$ be the fibre of $T \to \mathbf{P}(V)$ over p. Then

$$(Y \cap r_p^{-1}(r_p(Z))) \setminus (Y \cap Z)$$

is the image of $F \to Y$, $(y, z, p) \mapsto y$. Again by Varieties, Lemma 33.20.4 the closure of the image of $F \to Y$ has dimension at most $\dim(Y) + \dim(Z) + 1 - n$. $\square$

0B2T Lemma 43.23.4. Let V be a vector space. Let $B \subset \mathbf{P}(V)$ be a closed subvariety of codimension ≥ 2. Let $p \in \mathbf{P}(V)$ be a closed point, $p \notin B$. Then there exists a line $\ell \subset \mathbf{P}(V)$ with $\ell \cap B = \emptyset$. Moreover, these lines sweep out an open subset of $\mathbf{P}(V)$.

Proof. Consider the image of B under the projection $r_p : \mathbf{P}(V) \to \mathbf{P}(W_p)$. Since $\dim(W_p) = \dim(V) - 1$, we see that $r_p(B)$ has codimension ≥ 1 in $\mathbf{P}(W_p)$. For any $q \in \mathbf{P}(V)$ with $r_p(q) \notin r_p(B)$ we see that the line $\ell = \overline{pq}$ connecting p and q works. $\square$

0B2U Lemma 43.23.5. Let V be a vector space. Let $G = \mathrm{PGL}(V)$. Then $G \times \mathbf{P}(V) \to \mathbf{P}(V)$ is doubly transitive.

Proof. Omitted. Hint: This follows from the fact that $\mathrm{GL}(V)$ acts doubly transitive on pairs of linearly independent vectors. $\square$

0B2V Lemma 43.23.6. Let k be a field. Let $n \geq 1$ be an integer and let $x_{ij}, 1 \leq i, j \leq n$ be variables. Then

$$\det \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1} & \ldots & \ldots & x_{nn} \end{pmatrix}$$

is an irreducible element of the polynomial ring $k[x_{ij}]$.

Proof. Let V be an n dimensional vector space. Translating into geometry the lemma signifies that the variety C of non-invertible linear maps $V \to V$ is irreducible. Let W be a vector space of dimension $n-1$. By elementary linear algebra, the morphism

$$\mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, V), \quad (\psi, \varphi) \longmapsto \psi \circ \varphi$$

has image C. Since the source is irreducible, so is the image. $\square$

Let V be a vector space of dimension $n+1$. Set $E = \mathrm{End}(V)$. Let $E^\vee = \mathrm{Hom}(E, \mathbf{C})$ be the dual vector space. Write $\mathbf{P} = \mathbf{P}(E^\vee)$. There is a canonical linear map

$$V \longrightarrow V \otimes_{\mathbf{C}} E^\vee = \mathrm{Hom}(E, V)$$

sending $v \in V$ to the map $g \mapsto g(v)$ in $\mathrm{Hom}(E, V)$. Recall that we have a canonical map $E^\vee \to \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ which is an isomorphism. Hence we obtain a canonical map

$$\psi : V \otimes \mathcal{O}_{\mathbf{P}} \to V \otimes \mathcal{O}_{\mathbf{P}}(1)$$

of sheaves of modules on $\mathbf{P}$ which on global sections recovers the given map. Recall that a projective bundle $\mathbf{P}(\mathcal{E})$ is defined as the relative Proj of the symmetric algebra on $\mathcal{E}$, see Constructions, Definition 27.21.1. We are going to study the rational map between $\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}(1))$ and $\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}})$ associated to ψ. By Constructions, Lemma 27.16.10 we have a canonical isomorphism

$$\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}) = \mathbf{P} \times \mathbf{P}(V)$$

By Constructions, Lemma 27.20.1 we see that

$$\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}(1)) = \mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}) = \mathbf{P} \times \mathbf{P}(V)$$

Combining this with Constructions, Lemma 27.18.1 we obtain

0B2W (43.23.6.1)
$$\mathbf{P} \times \mathbf{P}(V) \supset U(\psi) \xrightarrow{r_\psi} \mathbf{P} \times \mathbf{P}(V)$$

To understand this better we work out what happens on fibres over $\mathbf{P}$. Let $g \in E$ be nonzero. This defines a nonzero map $E^\vee \to \mathbf{C}$, hence a point $[g] \in \mathbf{P}$. On the other hand, g defines a $\mathbf{C}$-linear map $g : V \to V$. Hence we obtain, by Constructions, Lemma 27.11.1 a map

$$\mathbf{P}(V) \supset U(g) \xrightarrow{r_g} \mathbf{P}(V)$$

What we will use below is that $U(g)$ is the fibre $U(\psi)_{[g]}$ and that r_g is the fibre of r_ψ over the point $[g]$. Another observation we will use is that the complement of $U(g)$ in $\mathbf{P}(V)$ is the image of the closed immersion

$$\mathbf{P}(\mathrm{Coker}(g)) \longrightarrow \mathbf{P}(V)$$

and the image of r_g is the image of the closed immersion

$$\mathbf{P}(\mathrm{Im}(g)) \longrightarrow \mathbf{P}(V)$$

0B1S Lemma 43.23.7. With notation as above. Let X, Y be closed subvarieties of $\mathbf{P}(V)$ which intersect properly such that $X \neq \mathbf{P}(V)$ and $X \cap Y \neq \emptyset$. For a general line $\ell \subset \mathbf{P}$ with $[\mathrm{id}_V] \in \ell$ we have

(1) $X \subset U_g$ for all $[g] \in \ell$,
(2) $g(X)$ intersects Y properly for all $[g] \in \ell$.

Proof. Let $B \subset \mathbf{P}$ be the set of "bad" points, i.e., those points $[g]$ that violate either (1) or (2). Note that $[\mathrm{id}_V] \not\in B$ by assumption. Moreover, B is closed. Hence it suffices to prove that $\dim(B) \leq \dim(\mathbf{P}) - 2$ (Lemma 43.23.4).

First, consider the open $G = \mathrm{PGL}(V) \subset \mathbf{P}$ consisting of points $[g]$ such that $g : V \to V$ is invertible. Since G acts doubly transitively on $\mathbf{P}(V)$ (Lemma 43.23.5) we see that

$$T = \{(x, y, [g]) \mid x \in X, y \in Y, [g] \in G, r_g(x) = y\}$$

is a locally trivial fibration over $X \times Y$ with fibre equal to the stabilizer of a point in G. Hence T is a variety. Observe that the fibre of $T \to G$ over $[g]$ is $r_g(X) \cap Y$. The morphism $T \to G$ is surjective, because any translate of X intersects Y (note that by the assumption that X and Y intersect properly and that $X \cap Y \neq \emptyset$ we see that $\dim(X) + \dim(Y) \geq \dim(\mathbf{P}(V))$ and then Varieties, Lemma 33.34.3 implies all translates of X intersect Y). Since the dimensions of fibres of a dominant morphism of varieties do not jump in codimension 1 (Varieties, Lemma 33.20.4) we conclude that $B \cap G$ has codimension ≥ 2.

Next we look at the complement $Z = \mathbf{P} \setminus G$. This is an irreducible variety because the determinant is an irreducible polynomial (Lemma 43.23.6). Thus it suffices to prove that B does not contain the generic point of Z. For a general point $[g] \in Z$ the cokernel $V \to \mathrm{Coker}(g)$ has dimension 1, hence $U(g)$ is the complement of a point. Since $X \neq \mathbf{P}(V)$ we see that for a general $[g] \in Z$ we have $X \subset U(g)$. Moreover, the morphism $r_g|_X : X \to r_g(X)$ is finite, hence $\dim(r_g(X)) = \dim(X)$. On the other hand, for such a g the image of r_g is the closed subspace $H = \mathbf{P}(\mathrm{Im}(g)) \subset \mathbf{P}(V)$ which has codimension 1. For general point of Z we see that $H \cap Y$ has dimension 1 less than Y (compare with Varieties, Lemma 33.35.3). Thus we see that we have to show that $r_g(X)$ and $H \cap Y$ intersect properly in H. For a fixed choice of H, we can by postcomposing g by an automorphism, move $r_g(X)$ by an arbitrary automorphism of $H = \mathbf{P}(\mathrm{Im}(g))$. Thus we can argue as above to conclude that the intersection of $H \cap Y$ with $r_g(X)$ is proper for general g with given $H = \mathbf{P}(\mathrm{Im}(g))$. Some details omitted. □

43.24. Moving Lemma

0B0D The moving lemma states that given an r-cycle α and an s-cycle β there exists α', $\alpha' \sim_{rat} \alpha$ such that α' and β intersect properly (Lemma 43.24.3). See [Sam56], [Che58a], [Che58b]. The key to this is Lemma 43.24.1; the reader may find this lemma in the form stated in [Ful98, Example 11.4.1] and find a proof in [Rob72].

0B0E **Lemma 43.24.1.** Let $X \subset \mathbf{P}^N$ be a nonsingular closed subvariety. Let $n = \dim(X)$ and $0 \leq d, d' < n$. Let $Z \subset X$ be a closed subvariety of dimension d and $T_i \subset X$, $i \in I$ be a finite collection of closed subvarieties of dimension d'. Then there exists a subvariety $C \subset \mathbf{P}^N$ such that C intersects X properly and such that See [Rob72].

$$C \cdot X = Z + \sum\nolimits_{j \in J} m_j Z_j$$

where $Z_j \subset X$ are irreducible of dimension d, distinct from Z, and

$$\dim(Z_j \cap T_i) \leq \dim(Z \cap T_i)$$

with strict inequality if Z does not intersect T_i properly in X.

Proof. Write $\mathbf{P}^N = \mathbf{P}(V_N)$ so $\dim(V_N) = N + 1$ and set $X_N = X$. We are going to choose a sequence of projections from points

$$\begin{aligned} &r_N : \mathbf{P}(V_N) \setminus \{p_N\} \to \mathbf{P}(V_{N-1}),\\ &r_{N-1} : \mathbf{P}(V_{N-1}) \setminus \{p_{N-1}\} \to \mathbf{P}(V_{N-2}),\\ &\ldots,\\ &r_{n+1} : \mathbf{P}(V_{n+1}) \setminus \{p_{n+1}\} \to \mathbf{P}(V_n) \end{aligned}$$

as in Section 43.23. At each step we will choose $p_N, p_{N-1}, \ldots, p_{n+1}$ in a suitable Zariski open set. Pick a closed point $x \in Z \subset X$. For every i pick closed points $x_{it} \in T_i \cap Z$, at least one in each irreducible component of $T_i \cap Z$. Taking the composition we obtain a morphism

$$\pi = (r_{n+1} \circ \ldots \circ r_N)|_X : X \longrightarrow \mathbf{P}(V_n)$$

which has the following properties

(1) π is finite,
(2) π is étale at x and all x_{it},
(3) $\pi|_Z : Z \to \pi(Z)$ is an isomorphism over an open neighbourhood of $\pi(x_{it})$,
(4) $T_i \cap \pi^{-1}(\pi(Z)) = (T_i \cap Z) \cup E_i$ with $E_i \subset T_i$ closed and $\dim(E_i) \leq d + d' + 1 - (n + 1) = d + d' - n$.

It follows in a straightforward manner from Lemmas 43.23.1, 43.23.2, and 43.23.3 and induction that we can do this; observe that the last projection is from $\mathbf{P}(V_{n+1})$ and that $\dim(V_{n+1}) = n + 2$ which explains the inequality in (4).

Let $C \subset \mathbf{P}(V_N)$ be the scheme theoretic closure of $(r_{n+1}\circ\ldots\circ r_N)^{-1}(\pi(Z))$. Because π is étale at the point x of Z, we see that the closed subscheme $C \cap X$ contains Z with multiplicity 1 (local calculation omitted). Hence by Lemma 43.17.2 we conclude that

$$C \cdot X = [Z] + \sum m_j[Z_j]$$

for some subvarieties $Z_j \subset X$ of dimension d. Note that

$$C \cap X = \pi^{-1}(\pi(Z))$$

set theoretically. Hence $T_i \cap Z_j \subset T_i \cap \pi^{-1}(\pi(Z)) \subset T_i \cap Z \cup E_i$. For any irreducible component of $T_i \cap Z$ contained in E_i we have the desired dimension bound. Finally, let V be an irreducible component of $T_i \cap Z_j$ which is contained in $T_i \cap Z$. To finish the proof it suffices to show that V does not contain any of the points x_{it}, because then $\dim(V) < \dim(Z \cap T_i)$. To show this it suffices to show that $x_{it} \notin Z_j$ for all i, t, j.

Set $Z' = \pi(Z)$ and $Z'' = \pi^{-1}(Z')$, scheme theoretically. By condition (3) we can find an open $U \subset \mathbf{P}(V_n)$ containing $\pi(x_{it})$ such that $\pi^{-1}(U) \cap Z \to U \cap Z'$ is an isomorphism. In particular, $Z \to Z'$ is a local isomorphism at x_{it}. On the other hand, $Z'' \to Z'$ is étale at x_{it} by condition (2). Hence the closed immersion $Z \to Z''$ is étale at x_{it} (Morphisms, Lemma 29.36.18). Thus $Z = Z''$ in a Zariski neighbourhood of x_{it} which proves the assertion. □

The actual moving is done using the following lemma.

0B1T Lemma 43.24.2. Let $C \subset \mathbf{P}^N$ be a closed subvariety. Let $X \subset \mathbf{P}^N$ be subvariety and let $T_i \subset X$ be a finite collection of closed subvarieties. Assume that C and X intersect properly. Then there exists a closed subvariety $C' \subset \mathbf{P}^N \times \mathbf{P}^1$ such that

(1) $C' \to \mathbf{P}^1$ is dominant,
(2) $C'_0 = C$ scheme theoretically,
(3) C' and $X \times \mathbf{P}^1$ intersect properly,
(4) C'_∞ properly intersects each of the given T_i.

Proof. If $C \cap X = \emptyset$, then we take the constant family $C' = C \times \mathbf{P}^1$. Thus we may and do assume $C \cap X \neq \emptyset$.

Write $\mathbf{P}^N = \mathbf{P}(V)$ so $\dim(V) = N + 1$. Let $E = \mathrm{End}(V)$. Let $E^\vee = \mathrm{Hom}(E, \mathbf{C})$. Set $\mathbf{P} = \mathbf{P}(E^\vee)$ as in Lemma 43.23.7. Choose a general line $\ell \subset \mathbf{P}$ passing through id_V. Set $C' \subset \ell \times \mathbf{P}(V)$ equal to the closed subscheme having fibre $r_g(C)$ over $[g] \in \ell$. More precisely, C' is the image of

$$\ell \times C \subset \mathbf{P} \times \mathbf{P}(V)$$

under the morphism (43.23.6.1). By Lemma 43.23.7 this makes sense, i.e., $\ell \times C \subset U(\psi)$. The morphism $\ell \times C \to C'$ is finite and $C'_{[g]} = r_g(C)$ set theoretically for all $[g] \in \ell$. Parts (1) and (2) are clear with $0 = [\mathrm{id}_V] \in \ell$. Part (3) follows from the fact that $r_g(C)$ and X intersect properly for all $[g] \in \ell$. Part (4) follows from the fact that a general point $\infty = [g] \in \ell$ is a general point of $\mathbf{P}$ and for such as point $r_g(C) \cap T$ is proper for any closed subvariety T of $\mathbf{P}(V)$. Details omitted. $\square$

0B1U Lemma 43.24.3. Let X be a nonsingular projective variety. Let α be an r-cycle and β be an s-cycle on X. Then there exists an r-cycle α' such that $\alpha' \sim_{rat} \alpha$ and such that α' and β intersect properly.

Proof. Write $\beta = \sum n_i[T_i]$ for some subvarieties $T_i \subset X$ of dimension s. By linearity we may assume that $\alpha = [Z]$ for some irreducible closed subvariety $Z \subset X$ of dimension r. We will prove the lemma by induction on the maximum e of the integers

$$\dim(Z \cap T_i)$$

The base case is $e = r + s - \dim(X)$. In this case Z intersects β properly and the lemma is trivial.

Induction step. Assume that $e > r + s - \dim(X)$. Choose an embedding $X \subset \mathbf{P}^N$ and apply Lemma 43.24.1 to find a closed subvariety $C \subset \mathbf{P}^N$ such that $C \cdot X = [Z] + \sum m_j[Z_j]$ and such that the induction hypothesis applies to each Z_j. Next, apply Lemma 43.24.2 to C, X, T_i to find $C' \subset \mathbf{P}^N \times \mathbf{P}^1$. Let $\gamma = C' \cdot X \times \mathbf{P}^1$ viewed as a cycle on $X \times \mathbf{P}^1$. By Lemma 43.22.2 we have

$$[Z] + \sum m_j[Z_j] = \mathrm{pr}_{X,*}(\gamma \cdot X \times 0)$$

On the other hand the cycle $\gamma_\infty = \mathrm{pr}_{X,*}(\gamma \cdot X \times \infty)$ is supported on $C'_\infty \cap X$ hence intersects β transversally. Thus we see that $[Z] \sim_{rat} -\sum m_j[Z_j] + \gamma_\infty$ by Lemma 43.17.1. Since by induction each $[Z_j]$ is rationally equivalent to a cycle which properly intersects β this finishes the proof. $\square$

43.25. Intersection products and rational equivalence

0B0F With definitions as above we show that the intersection product is well defined modulo rational equivalence. We first deal with a special case.

0B60 Lemma 43.25.1. Let X be a nonsingular variety. Let $W \subset X \times \mathbf{P}^1$ be an $(s+1)$-dimensional subvariety dominating $\mathbf{P}^1$. Let W_a, resp. W_b be the fibre of $W \to \mathbf{P}^1$ over a, resp. b. Let V be a r-dimensional subvariety of X such that V intersects both W_a and W_b properly. Then $[V] \cdot [W_a]_r \sim_{rat} [V] \cdot [W_b]_r$.

Proof. We have $[W_a]_r = \text{pr}_{X,*}(W \cdot X \times a)$ and similarly for $[W_b]_r$, see Lemma 43.17.1. Thus we reduce to showing

$$V \cdot \text{pr}_{X,*}(W \cdot X \times a) \sim_{rat} V \cdot \text{pr}_{X,*}(W \cdot X \times b).$$

Applying the projection formula Lemma 43.22.1 we get

$$V \cdot \text{pr}_{X,*}(W \cdot X \times a) = \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a))$$

and similarly for b. Thus we reduce to showing

$$\text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \sim_{rat} \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b))$$

If $V \times \mathbf{P}^1$ intersects W properly, then associativity for the intersection multiplicities (Lemma 43.20.1) gives $V \times \mathbf{P}^1 \cdot (W \cdot X \times a) = (V \times \mathbf{P}^1 \cdot W) \cdot X \times a$ and similarly for b. Thus we reduce to showing

$$\text{pr}_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times a) \sim_{rat} \text{pr}_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times b)$$

which is true by Lemma 43.17.1.

The argument above does not quite work. The obstruction is that we do not know that $V \times \mathbf{P}^1$ and W intersect properly. We only know that V and W_a and V and W_b intersect properly. Let Z_i, $i \in I$ be the irreducible components of $V \times \mathbf{P}^1 \cap W$. Then we know that $\dim(Z_i) \geq r+1+s+1-n-1 = r+s+1-n$ where $n = \dim(X)$, see Lemma 43.13.4. Since we have assumed that V and W_a intersect properly, we see that $\dim(Z_{i,a}) = r+s-n$ or $Z_{i,a} = \emptyset$. On the other hand, if $Z_{i,a} \neq \emptyset$, then $\dim(Z_{i,a}) \geq \dim(Z_i) - 1 = r+s-n$. It follows that $\dim(Z_i) = r+s+1-n$ if Z_i meets $X \times a$ and in this case $Z_i \to \mathbf{P}^1$ is surjective. Thus we may write $I = I' \amalg I''$ where I' is the set of $i \in I$ such that $Z_i \to \mathbf{P}^1$ is surjective and I'' is the set of $i \in I$ such that Z_i lies over a closed point $t_i \in \mathbf{P}^1$ with $t_i \neq a$ and $t_i \neq b$. Consider the cycle

$$\gamma = \sum\nolimits_{i \in I'} e_i [Z_i]$$

where we take

$$e_i = \sum\nolimits_p (-1)^p \text{length}_{\mathcal{O}_{X \times \mathbf{P}^1, Z_i}} \text{Tor}_p^{\mathcal{O}_{X \times \mathbf{P}^1, Z_i}}(\mathcal{O}_{V \times \mathbf{P}^1, Z_i}, \mathcal{O}_{W, Z_i})$$

We will show that γ can be used as a replacement for the intersection product of $V \times \mathbf{P}^1$ and W.

We will show this using associativity of intersection products in exactly the same way as above. Let $U = \mathbf{P}^1 \setminus \{t_i, i \in I''\}$. Note that $X \times a$ and $X \times b$ are contained in $X \times U$. The subvarieties

$$V \times U, \quad W_U, \quad X \times a \quad \text{of} \quad X \times U$$

intersect transversally pairwise by our choice of U and moreover $\dim(V \times U \cap W_U \cap X \times a) = \dim(V \cap W_a)$ has the expected dimension. Thus we see that

$$V \times U \cdot (W_U \cdot X \times a) = (V \times U \cdot W_U) \cdot X \times a$$

as cycles on $X \times U$ by Lemma 43.20.1. By construction γ restricts to the cycle $V \times U \cdot W_U$ on $X \times U$. Trivially, $V \times \mathbf{P}^1 \cdot (W \times X \times a)$ restricts to $V \times U \cdot (W_U \cdot X \times a)$ on $X \times U$. Hence

$$V \times \mathbf{P}^1 \cdot (W \cdot X \times a) = \gamma \cdot X \times a$$

as cycles on $X \times \mathbf{P}^1$ (because both sides are contained in $X \times U$ and are equal after restricting to $X \times U$ by what was said before). Since we have the same for b we conclude

$$\begin{aligned} V \cdot [W_a] &= \mathrm{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \\ &= \mathrm{pr}_{X,*}(\gamma \cdot X \times a) \\ &\sim_{rat} \mathrm{pr}_{X,*}(\gamma \cdot X \times b) \\ &= \mathrm{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b)) \\ &= V \cdot [W_b] \end{aligned}$$

The first and the last equality by the first paragraph of the proof, the second and penultimate equalities were shown in this paragraph, and the middle equivalence is Lemma 43.17.1. □

0B1V Theorem 43.25.2. Let X be a nonsingular projective variety. Let α, resp. β be an r, resp. s cycle on X. Assume that α and β intersect properly so that $\alpha \cdot \beta$ is defined. Finally, assume that $\alpha \sim_{rat} 0$. Then $\alpha \cdot \beta \sim_{rat} 0$.

Proof. Pick a closed immersion $X \subset \mathbf{P}^N$. By linearity it suffices to prove the result when $\beta = [Z]$ for some s-dimensional closed subvariety $Z \subset X$ which intersects α properly. The condition $\alpha \sim_{rat} 0$ means there are finitely many $(r+1)$-dimensional closed subvarieties $W_i \subset X \times \mathbf{P}^1$ such that

$$\alpha = \sum [W_{i,a_i}]_r - [W_{i,b_i}]_r$$

for some pairs of points a_i, b_i of $\mathbf{P}^1$. Let W^t_{i,a_i} and W^t_{i,b_i} be the irreducible components of W_{i,a_i} and W_{i,b_i}. We will use induction on the maximum d of the integers

$$\dim(Z \cap W^t_{i,a_i}), \quad \dim(Z \cap W^t_{i,b_i})$$

The main problem in the rest of the proof is that although we know that Z intersects α properly, it may not be the case that Z intersects the "intermediate" varieties W^t_{i,a_i} and W^t_{i,b_i} properly, i.e., it may happen that $d > r + s - \dim(X)$.

Base case: $d = r+s-\dim(X)$. In this case all the intersections of Z with the W^t_{i,a_i} and W^t_{i,b_i} are proper and the desired result follows from Lemma 43.25.1, because it applies to show that $[Z] \cdot [W_{i,a_i}]_r \sim_{rat} [Z] \cdot [W_{i,b_i}]_r$ for each i.

Induction step: $d > r + s - \dim(X)$. Apply Lemma 43.24.1 to $Z \subset X$ and the family of subvarieties $\{W^t_{i,a_i}, W^t_{i,b_i}\}$. Then we find a closed subvariety $C \subset \mathbf{P}^N$ intersecting X properly such that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

and such that

$$\dim(Z_j \cap W^t_{i,a_i}) \leq \dim(Z \cap W^t_{i,a_i}), \quad \dim(Z_j \cap W^t_{i,b_i}) \leq \dim(Z \cap W^t_{i,b_i})$$

with strict inequality if the right hand side is $> r + s - \dim(X)$. This implies two things: (a) the induction hypothesis applies to each Z_j, and (b) $C \cdot X$ and α intersect properly (because α is a linear combination of those $[W^t_{i,a_i}]$ and $[W^t_{i,a_i}]$

which intersect Z properly). Next, pick $C' \subset \mathbf{P}^N \times \mathbf{P}^1$ as in Lemma 43.24.2 with respect to C, X, and W^t_{i,a_i}, W^t_{i,b_i}. Write $C' \cdot X \times \mathbf{P}^1 = \sum n_k[E_k]$ for some subvarieties $E_k \subset X \times \mathbf{P}^1$ of dimension $s+1$. Note that $n_k > 0$ for all k by Proposition 43.19.3. By Lemma 43.22.2 we have

$$[Z] + \sum m_j[Z_j] = \sum n_k[E_{k,0}]_s$$

Since $E_{k,0} \subset C \cap X$ we see that $[E_{k,0}]_s$ and α intersect properly. On the other hand, the cycle

$$\gamma = \sum n_k[E_{k,\infty}]_s$$

is supported on $C'_\infty \cap X$ and hence properly intersects each W^t_{i,a_i}, W^t_{i,b_i}. Thus by the base case and linearity, we see that

$$\gamma \cdot \alpha \sim_{rat} 0$$

As we have seen that $E_{k,0}$ and $E_{k,\infty}$ intersect α properly Lemma 43.25.1 applied to $E_k \subset X \times \mathbf{P}^1$ and α gives

$$[E_{k,0}] \cdot \alpha \sim_{rat} [E_{k,\infty}] \cdot \alpha$$

Putting everything together we have

$$\begin{aligned}
[Z] \cdot \alpha &= (\sum n_k[E_{k,0}]_r - \sum m_j[Z_j]) \cdot \alpha \\
&\sim_{rat} \sum n_k[E_{k,0}] \cdot \alpha \quad \text{(by induction hypothesis)} \\
&\sim_{rat} \sum n_k[E_{k,\infty}] \cdot \alpha \quad \text{(by the lemma)} \\
&= \gamma \cdot \alpha \\
&\sim_{rat} 0 \quad \text{(by base case)}
\end{aligned}$$

This finishes the proof. □

0B61 Remark 43.25.3. Lemma 43.24.3 and Theorem 43.25.2 also hold for nonsingular quasi-projective varieties with the same proof. The only change is that one needs to prove the following version of the moving Lemma 43.24.1: Let $X \subset \mathbf{P}^N$ be a closed subvariety. Let $n = \dim(X)$ and $0 \leq d, d' < n$. Let $X^{reg} \subset X$ be the open subset of nonsingular points. Let $Z \subset X^{reg}$ be a closed subvariety of dimension d and $T_i \subset X^{reg}$, $i \in I$ be a finite collection of closed subvarieties of dimension d'. Then there exists a subvariety $C \subset \mathbf{P}^N$ such that C intersects X properly and such that

$$(C \cdot X)|_{X^{reg}} = Z + \sum\nolimits_{j \in J} m_j Z_j$$

where $Z_j \subset X^{reg}$ are irreducible of dimension d, distinct from Z, and

$$\dim(Z_j \cap T_i) \leq \dim(Z \cap T_i)$$

with strict inequality if Z does not intersect T_i properly in X^{reg}.

43.26. Chow rings

0B0G Let X be a nonsingular projective variety. We define the intersection product

$$\mathrm{CH}_r(X) \times \mathrm{CH}_s(X) \longrightarrow \mathrm{CH}_{r+s-\dim(X)}(X), \quad (\alpha, \beta) \longmapsto \alpha \cdot \beta$$

as follows. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(X)$. If α and β intersect properly, we use the definition given in Section 43.17. If not, then we choose $\alpha \sim_{rat} \alpha'$ as in Lemma 43.24.3 and we set

$$\alpha \cdot \beta = \text{class of } \alpha' \cdot \beta \in \mathrm{CH}_{r+s-\dim(X)}(X)$$

This is well defined and passes through rational equivalence by Theorem 43.25.2. The intersection product on $\mathrm{CH}_*(X)$ is commutative (this is clear), associative (Lemma 43.20.1) and has a unit $[X] \in \mathrm{CH}_{\dim(X)}(X)$.

We often use $\mathrm{CH}^c(X) = \mathrm{CH}_{\dim X - c}(X)$ to denote the Chow group of cycles of codimension c, see Chow Homology, Section 42.42. The intersection product defines a product

$$\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \longrightarrow \mathrm{CH}^{k+l}(X)$$

which is commutative, associative, and has a unit $1 = [X] \in \mathrm{CH}^0(X)$.

43.27. Pullback for a general morphism

0B0H Let $f : X \to Y$ be a morphism of nonsingular projective varieties. We define

$$f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+\dim X - \dim Y}(X)$$

by the rule

$$f^*(\alpha) = pr_{X,*}(\Gamma_f \cdot pr_Y^*(\alpha))$$

where $\Gamma_f \subset X \times Y$ is the graph of f. Note that in this generality, it is defined only on cycle classes and not on cycles. With the notation CH^* introduced in Section 43.26 we may think of pullback as a map

$$f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$$

in other words, it is a map of graded abelian groups.

0B2X **Lemma 43.27.1.** Let $f : X \to Y$ be a morphism of nonsingular projective varieties. The pullback map on chow groups satisfies:

1. $f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ is a ring map,
2. $(g \circ f)^* = f^* \circ g^*$ for a composable pair f, g,
3. the projection formula holds: $f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$, and
4. if f is flat then it agrees with the previous definition.

Proof. All of these follow readily from the results above.

For (1) it suffices to show that $\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) = \mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha) \cdot \mathrm{pr}_{X,*}(\Gamma_f \cdot \beta)$ for cycles α, β on $X \times Y$. If α is a cycle on $X \times Y$ which intersects Γ_f properly, then it is easy to see that

$$\Gamma_f \cdot \alpha = \Gamma_f \cdot \mathrm{pr}_X^*(\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha))$$

as cycles because Γ_f is a graph. Thus we get the first equality in

$$\begin{aligned} \mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) &= \mathrm{pr}_{X,*}(\Gamma_f \cdot \mathrm{pr}_X^*(\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha)) \cdot \beta) \\ &= \mathrm{pr}_{X,*}(\mathrm{pr}_X^*(\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha)) \cdot (\Gamma_f \cdot \beta)) \\ &= \mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha) \cdot \mathrm{pr}_{X,*}(\Gamma_f \cdot \beta) \end{aligned}$$

the last step by the projection formula in the flat case (Lemma 43.22.1).

If $g : Y \to Z$ then property (2) follows formally from the observation that

$$\Gamma = \mathrm{pr}_{X \times Y}^* \Gamma_f \cdot \mathrm{pr}_{Y \times Z}^* \Gamma_g$$

in $Z_*(X \times Y \times Z)$ where $\Gamma = \{(x, f(x), g(f(x))\}$ and maps isomorphically to $\Gamma_{g \circ f}$ in $X \times Z$. The equality follows from the scheme theoretic equality and Lemma 43.14.3.

For (3) we use the projection formula for flat maps twice

$$\begin{aligned} f_*(\alpha \cdot pr_{X,*}(\Gamma_f \cdot pr_Y^*(\beta))) &= f_*(pr_{X,*}(pr_X^*\alpha \cdot \Gamma_f \cdot pr_Y^*(\beta))) \\ &= pr_{Y,*}(pr_X^*\alpha \cdot \Gamma_f \cdot pr_Y^*(\beta))) \\ &= pt_{Y,*}(pr_X^*\alpha \cdot \Gamma_f) \cdot \beta \\ &= f_*(\alpha) \cdot \beta \end{aligned}$$

where in the last equality we use the remark on graphs made above. This proves (3).

Property (4) rests on identifying the intersection product $\Gamma_f \cdot pr_Y^*\alpha$ in the case f is flat. Namely, in this case if $V \subset Y$ is a closed subvariety, then every generic point ξ of the scheme $f^{-1}(V) \cong \Gamma_f \cap pr_Y^{-1}(V)$ lies over the generic point of V. Hence the local ring of $pr_Y^{-1}(V) = X \times V$ at ξ is Cohen-Macaulay. Since $\Gamma_f \subset X \times Y$ is a regular immersion (as a morphism of smooth projective varieties) we find that

$$\Gamma_f \cdot pr_Y^*[V] = [\Gamma_f \cap pr_Y^{-1}(V)]_d$$

with d the dimension of $\Gamma_f \cap pr_Y^{-1}(V)$, see Lemma 43.16.5. Since $\Gamma_f \cap pr_Y^{-1}(V)$ maps isomorphically to $f^{-1}(V)$ we conclude. $\square$

43.28. Pullback of cycles

0B0I Suppose that X and Y be nonsingular projective varieties, and let $f : X \to Y$ be a morphism. Suppose that $Z \subset Y$ is a closed subvariety. Let $f^{-1}(Z)$ be the scheme theoretic inverse image:

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a fibre product diagram of schemes. In particular $f^{-1}(Z) \subset X$ is a closed subscheme of X. In this case we always have

$$\dim f^{-1}(Z) \geq \dim Z + \dim X - \dim Y.$$

If equality holds in the formula above, then $f^*[Z] = [f^{-1}(Z)]_{\dim Z + \dim X - \dim Y}$ provided that the scheme Z is Cohen-Macaulay at the images of the generic points of $f^{-1}(Z)$. This follows by identifying $f^{-1}(Z)$ with the scheme theoretic intersection of Γ_f and $X \times Z$ and using Lemma 43.16.5. Details are similar to the proof of part (4) of Lemma 43.27.1 above.

43.29. Other chapters

Preliminaries

CHAPTER 44

Picard Schemes of Curves

0B92 44.1. Introduction

0B93 In this chapter we do just enough work to construct the Picard scheme of a projective nonsingular curve over an algebraically closed field. See [Kle05] for a more thorough discussion as well as historical background.

Later in the Stacks project we will discuss Hilbert and Quot functors in much greater generality.

44.2. Hilbert scheme of points

0B94 Let $X \to S$ be a morphism of schemes. Let $d \geq 0$ be an integer. For a scheme T over S we let

$$\mathrm{Hilb}^d_{X/S}(T) = \left\{ \begin{matrix} Z \subset X_T \text{ closed subscheme such that} \\ Z \to T \text{ is finite locally free of degree } d \end{matrix} \right\}$$

If $T' \to T$ is a morphism of schemes over S and if $Z \in \mathrm{Hilb}^d_{X/S}(T)$, then the base change $Z_{T'} \subset X_{T'}$ is an element of $\mathrm{Hilb}^d_{X/S}(T')$. In this way we obtain a functor

$$\mathrm{Hilb}^d_{X/S} : (Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \mathrm{Hilb}^d_{X/S}(T)$$

In general $\mathrm{Hilb}^d_{X/S}$ is an algebraic space (insert future reference here). In this section we will show that $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme if any finite number of points in a fibre of $X \to S$ are contained in an affine open. If $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme, we often denote this scheme by $\underline{\mathrm{Hilb}}^d_{X/S}$.

0B95 **Lemma 44.2.1.** *Let $X \to S$ be a morphism of schemes. The functor $\mathrm{Hilb}^d_{X/S}$ satisfies the sheaf property for the fpqc topology (Topologies, Definition 34.9.12).*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T (Topologies, Lemma 34.9.7) and that $X_{T_i \times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $Z_i \in \mathrm{Hilb}^d_{X/S}(T_i)$ is a collection of elements such that Z_i and $Z_{i'}$ map to the same element of $\mathrm{Hilb}^d_{X/S}(T_i \times_T T_{i'})$. By effective descent for closed immersions (Descent, Lemma 35.37.2) there is a closed immersion $Z \to X_T$ whose base change by $X_i \to X_T$ is equal to $Z_i \to X_i$. The morphism $Z \to T$ then has the property that its base change to T_i is the morphism $Z_i \to T_i$. Hence $Z \to T$ is finite locally free of degree d by Descent, Lemma 35.23.30. $\square$

0B96 **Lemma 44.2.2.** *Let $X \to S$ be a morphism of schemes. If $X \to S$ is of finite presentation, then the functor $\mathrm{Hilb}^d_{X/S}$ is limit preserving (Limits, Remark 32.6.2).*

Proof. Let $T = \lim T_i$ be a limit of affine schemes over S. We have to show that $\mathrm{Hilb}^d_{X/S}(T) = \mathrm{colim}\,\mathrm{Hilb}^d_{X/S}(T_i)$. Observe that if $Z \to X_T$ is an element of $\mathrm{Hilb}^d_{X/S}(T)$, then $Z \to T$ is of finite presentation. Hence by Limits, Lemma 32.10.1 there exists an i, a scheme Z_i of finite presentation over T_i, and a morphism $Z_i \to X_{T_i}$ over T_i whose base change to T gives $Z \to X_T$. We apply Limits, Lemma 32.8.5 to see that we may assume $Z_i \to X_{T_i}$ is a closed immersion after increasing i. We apply Limits, Lemma 32.8.8 to see that $Z_i \to T_i$ is finite locally free of degree d after possibly increasing i. Then $Z_i \in \mathrm{Hilb}^d_{X/S}(T_i)$ as desired. $\square$

Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes over S. Then there is a transformation of functors

$$\mathrm{Hilb}^d_{X/S} \longrightarrow \mathrm{Hilb}^d_{Y/S}$$

which maps an element $Z \in \mathrm{Hilb}^d_{X/S}(T)$ to $i_T(Z) \subset Y_T$ in $\mathrm{Hilb}^d_{Y/S}$. Here $i_T : X_T \to Y_T$ is the base change of i.

0B97 Lemma 44.2.3. Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes. If $\mathrm{Hilb}^d_{Y/S}$ is representable by a scheme, so is $\mathrm{Hilb}^d_{X/S}$ and the corresponding morphism of schemes $\underline{\mathrm{Hilb}}^d_{X/S} \to \underline{\mathrm{Hilb}}^d_{Y/S}$ is a closed immersion.

Proof. Let T be a scheme over S and let $Z \in \mathrm{Hilb}^d_{Y/S}(T)$. Claim: there is a closed subscheme $T_X \subset T$ such that a morphism of schemes $T' \to T$ factors through T_X if and only if $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$. Applying this to a scheme T_{univ} representing $\mathrm{Hilb}^d_{Y/S}$ and the universal object[1] $Z_{univ} \in \mathrm{Hilb}^d_{Y/S}(T_{univ})$ we get a closed subscheme $T_{univ,X} \subset T_{univ}$ such that $Z_{univ,X} = Z_{univ} \times_{T_{univ}} T_{univ,X}$ is a closed subscheme of $X \times_S T_{univ,X}$ and hence defines an element of $\mathrm{Hilb}^d_{X/S}(T_{univ,X})$. A formal argument then shows that $T_{univ,X}$ is a scheme representing $\mathrm{Hilb}^d_{X/S}$ with universal object $Z_{univ,X}$.

Proof of the claim. Consider $Z' = X_T \times_{Y_T} Z$. Given $T' \to T$ we see that $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$ if and only if $Z'_{T'} \to Z_{T'}$ is an isomorphism. Thus the claim follows from the very general More on Flatness, Lemma 38.23.4. However, in this special case one can prove the statement directly as follows: first reduce to the case $T = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$. After shrinking T further we may assume there is an isomorphism $\varphi : B \to A^{\oplus d}$ as A-modules. Then $Z' = \mathrm{Spec}(B/J)$ for some ideal $J \subset B$. Let $g_\beta \in J$ be a collection of generators and write $\varphi(g_\beta) = (g^1_\beta, \ldots, g^d_\beta)$. Then it is clear that T_X is given by $\mathrm{Spec}(A/(g^j_\beta))$. $\square$

0B98 Lemma 44.2.4. Let $X \to S$ be a morphism of schemes. If $X \to S$ is separated and $\mathrm{Hilb}^d_{X/S}$ is representable, then $\underline{\mathrm{Hilb}}^d_{X/S} \to S$ is separated.

Proof. In this proof all unadorned products are over S. Let $H = \underline{\mathrm{Hilb}}^d_{X/S}$ and let $Z \in \mathrm{Hilb}^d_{X/S}(H)$ be the universal object. Consider the two objects $Z_1, Z_2 \in \mathrm{Hilb}^d_{X/S}(H \times H)$ we get by pulling back Z by the two projections $H \times H \to H$. Then $Z_1 = Z \times H \subset X_{H \times H}$ and $Z_2 = H \times Z \subset X_{H \times H}$. Since H represents the functor $\mathrm{Hilb}^d_{X/S}$, the diagonal morphism $\Delta : H \to H \times H$ has the following universal property: A morphism of schemes $T \to H \times H$ factors through Δ if and only if $Z_{1,T} = Z_{2,T}$ as elements of $\mathrm{Hilb}^d_{X/S}(T)$. Set $Z = Z_1 \times_{X_{H \times H}} Z_2$. Then we see

[1] See Categories, Section 4.3

that $T \to H \times H$ factors through Δ if and only if the morphisms $Z_T \to Z_{1,T}$ and $Z_T \to Z_{2,T}$ are isomorphisms. It follows from the very general More on Flatness, Lemma 38.23.4 that Δ is a closed immersion. In the proof of Lemma 44.2.3 the reader finds an alternative easier proof of the needed result in our special case. $\square$

0B99 Lemma 44.2.5. Let $X \to S$ be a morphism of affine schemes. Let $d \geq 0$. Then $\mathrm{Hilb}^d_{X/S}$ is representable.

Proof. Say $S = \mathrm{Spec}(R)$. Then we can choose a closed immersion of X into the spectrum of $R[x_i; i \in I]$ for some set I (of sufficiently large cardinality. Hence by Lemma 44.2.3 we may assume that $X = \mathrm{Spec}(A)$ where $A = R[x_i; i \in I]$. We will use Schemes, Lemma 26.15.4 to prove the lemma in this case.

Condition (1) of the lemma follows from Lemma 44.2.1.

For every subset $W \subset A$ of cardinality d we will construct a subfunctor F_W of $\mathrm{Hilb}^d_{X/S}$. (It would be enough to consider the case where W consists of a collection of monomials in the x_i but we do not need this.) Namely, we will say that $Z \in \mathrm{Hilb}^d_{X/S}(T)$ is in $F_W(T)$ if and only if the $\mathcal{O}_T$-linear map

$$\bigoplus\nolimits_{f \in W} \mathcal{O}_T \longrightarrow (Z \to T)_*\mathcal{O}_Z, \quad (g_f) \longmapsto \sum g_f f|_Z$$

is surjective (equivalently an isomorphism). Here for $f \in A$ and $Z \in \mathrm{Hilb}^d_{X/S}(T)$ we denote $f|_Z$ the pullback of f by the morphism $Z \to X_T \to X$.

Openness, i.e., condition (2)(b) of the lemma. This follows from Algebra, Lemma 10.79.4.

Covering, i.e., condition (2)(c) of the lemma. Since

$$A \otimes_R \mathcal{O}_T = (X_T \to T)_*\mathcal{O}_{X_T} \to (Z \to T)_*\mathcal{O}_Z$$

is surjective and since $(Z \to T)_*\mathcal{O}_Z$ is finite locally free of rank d, for every point $t \in T$ we can find a finite subset $W \subset A$ of cardinality d whose images form a basis of the d-dimensional $\kappa(t)$-vector space $((Z \to T)_*\mathcal{O}_Z)_t \otimes_{\mathcal{O}_{T,t}} \kappa(t)$. By Nakayama's lemma there is an open neighbourhood $V \subset T$ of t such that $Z_V \in F_W(V)$.

Representable, i.e., condition (2)(a) of the lemma. Let $W \subset A$ have cardinality d. We claim that F_W is representable by an affine scheme over R. We will construct this affine scheme here, but we encourage the reader to think it trough for themselves. Choose a numbering $f_1, \ldots, f_d$ of the elements of W. We will construct a universal element $Z_{univ} = \mathrm{Spec}(B_{univ})$ of F_W over $T_{univ} = \mathrm{Spec}(R_{univ})$ which will be the spectrum of

$$B_{univ} = R_{univ}[e_1, \ldots, e_d]/(e_k e_l - \sum c_{kl}^m e_m)$$

where the e_l will be the images of the f_l and where the closed immersion $Z_{univ} \to X_{T_{univ}}$ is given by the ring map

$$A \otimes_R R_{univ} \longrightarrow B_{univ}$$

mapping $1 \otimes 1$ to $\sum b^l e_l$ and x_i to $\sum b^l_i e_l$. In fact, we claim that F_W is represented by the spectrum of the ring

$$R_{univ} = R[c^m_{kl}, b^l, b^l_i]/\mathfrak{a}_{univ}$$

where the ideal $\mathfrak{a}_{univ}$ is generated by the following elements:

(1) multiplication on B_{univ} is commutative, i.e., $c^m_{lk} - c^m_{kl} \in \mathfrak{a}_{univ}$,

(2) multiplication on B_{univ} is associative, i.e., $c_{lk}^m c_{mn}^p - c_{lq}^p c_{kn}^q \in \mathfrak{a}_{univ}$,
(3) $\sum b^l e_l$ is a multiplicative 1 in B_{univ}, in other words, we should have $(\sum b^l e_l)e_k = e_k$ for all k, which means $\sum b^l c_{lk}^m - \delta_{km} \in \mathfrak{a}_{univ}$ (Kronecker delta).

After dividing out by the ideal $\mathfrak{a}'_{univ}$ of the elements listed sofar we obtain a well defined ring map

$$\Psi : A \otimes_R R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{univ} \longrightarrow \left(R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{univ}\right)[e_1, \ldots, e_d]/(e_k e_l - \sum c_{kl}^m e_m)$$

sending $1 \otimes 1$ to $\sum b^l e_l$ and $x_i \otimes 1$ to $\sum b_i^l e_l$. We need to add some more elements to our ideal because we need

(5) f_l to map to e_l in B_{univ}. Write $\Psi(f_l) - e_l = \sum h_l^m e_m$ with $h_l^m \in R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{univ}$ then we need to set h_l^m equal to zero.

Thus setting $\mathfrak{a}_{univ} \subset R[c_{kl}^m, b^l, b_i^l]$ equal to $\mathfrak{a}'_{univ}+$ ideal generated by lifts of h_l^m to $R[c_{kl}^m, b^l, b_i^l]$, then it is clear that F_W is represented by $\mathrm{Spec}(R_{univ})$. □

0B9A Proposition 44.2.6. Let $X \to S$ be a morphism of schemes. Let $d \geq 0$. Assume for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$. Then $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme.

Proof. Either using relative glueing (Constructions, Section 27.2) or using the functorial point of view (Schemes, Lemma 26.15.4) we reduce to the case where S is affine. Details omitted.

Assume S is affine. For $U \subset X$ affine open, denote $F_U \subset \mathrm{Hilb}^d_{X/S}$ the subfunctor such that for a scheme T/S an element $Z \in \mathrm{Hilb}^d_{X/S}(T)$ is in $F_U(T)$ if and only if $Z \subset U_T$. We will use Schemes, Lemma 26.15.4 and the subfunctors F_U to conclude.

Condition (1) is Lemma 44.2.1.

Condition (2)(a) follows from the fact that $F_U = \mathrm{Hilb}^d_{U/S}$ and that this is representable by Lemma 44.2.5. Namely, if $Z \in F_U(T)$, then Z can be viewed as a closed subscheme of U_T which is finite locally free of degree d over T and hence $Z \in \mathrm{Hilb}^d_{U/S}(T)$. Conversely, if $Z \in \mathrm{Hilb}^d_{U/S}(T)$ then $Z \to U_T \to X_T$ is a closed immersion[2] and we may view Z as an element of $F_U(T)$.

Let $Z \in \mathrm{Hilb}^d_{X/S}(T)$ for some scheme T over S. Let

$$B = (Z \to T)\left((Z \to X_T \to X)^{-1}(X \setminus U)\right)$$

This is a closed subset of T and it is clear that over the open $T_{Z,U} = T \setminus B$ the restriction $Z_{t'}$ maps into $U_{T'}$. On the other hand, for any $b \in B$ the fibre Z_b does not map into U. Thus we see that given a morphism $T' \to T$ we have $Z_{T'} \in F_U(T')$ $\Leftrightarrow$ $T' \to T$ factors through the open $T_{Z,U}$. This proves condition (2)(b).

[2] This is clear if $X \to S$ is separated as in this case Morphisms, Lemma 29.41.7 tells us that the immersion $\varphi : Z \to X_T$ has closed image and hence is a closed immersion by Schemes, Lemma 26.10.4. We suggest the reader skip the rest of this footnote as we don't know of any instance where the assumptions on $X \to S$ hold but $X \to S$ is not separated. In the general case, let $x \in X_T$ be a point in the closure of $\varphi(Z)$. We have to show that $x \in \varphi(Z)$. Let $t \in T$ be the image of x. By assumption on $X \to S$ we can choose an affine open $W \subset X_T$ containing x and $\varphi(Z_t)$. Then $\varphi^{-1}(W)$ is an open containing the whole fibre Z_t and since $Z \to T$ is closed, we may after replacing T by an open neighbourhood of t assume that $Z = \varphi^{-1}(W)$. Then $\varphi(Z) \subset W$ is closed by the separated case (as $W \to T$ is separated) and we conclude $x \in \varphi(Z)$.

Condition (2)(c) follows from our assumption on X/S. All we have to do is show the following: If T is the spectrum of a field and $Z \subset X_T$ is a closed subscheme, finite flat of degree d over T, then $Z \to X_T \to X$ factors through an affine open U of X. This is clear because Z will have at most d points and these will all map into the fibre of X over the image point of $T \to S$. □

0B9B Remark 44.2.7. Let $f : X \to S$ be a morphism of schemes. The assumption of Proposition 44.2.6 and hence the conclusion holds in each of the following cases:

(1) X is quasi-affine,
(2) f is quasi-affine,
(3) f is quasi-projective,
(4) f is locally projective,
(5) there exists an ample invertible sheaf on X,
(6) there exists an f-ample invertible sheaf on X, and
(7) there exists an f-very ample invertible sheaf on X.

Namely, in each of these cases, every finite set of points of a fibre X_s is contained in a quasi-compact open U of X which comes with an ample invertible sheaf, is isomorphic to an open of an affine scheme, or is isomorphic to an open of Proj of a graded ring (in each case this follows by unwinding the definitions). Thus the existence of suitable affine opens by Properties, Lemma 28.29.5.

44.3. Moduli of divisors on smooth curves

0B9C For a smooth morphism $X \to S$ of relative dimension 1 the functor $\mathrm{Hilb}^d_{X/S}$ parametrizes relative effective Cartier divisors as defined in Divisors, Section 31.18.

0B9D Lemma 44.3.1. Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D \subset X$ be a closed subscheme. Consider the following conditions

(1) $D \to S$ is finite locally free,
(2) D is a relative effective Cartier divisor on X/S,
(3) $D \to S$ is locally quasi-finite, flat, and locally of finite presentation, and
(4) $D \to S$ is locally quasi-finite and flat.

We always have the implications

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$$

If S is locally Noetherian, then the last arrow is an if and only if. If $X \to S$ is proper (and S arbitrary), then the first arrow is an if and only if.

Proof. Equivalence of (2) and (3). This follows from Divisors, Lemma 31.18.9 if we can show the equivalence of (2) and (3) when S is the spectrum of a field k. Let $x \in X$ be a closed point. As X is smooth of relative dimension 1 over k and we see that $\mathcal{O}_{X,x}$ is a regular local ring of dimension 1 (see Varieties, Lemma 33.25.3). Thus $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 10.119.7) and hence a PID. It follows that every sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is nonvanishing at all the generic points of X is invertible (Divisors, Lemma 31.15.2). In other words, every closed subscheme of X which does not contain a generic point is an effective Cartier divisor. It follows that (2) and (3) are equivalent.

If S is Noetherian, then any locally quasi-finite morphism $D \to S$ is locally of finite presentation (Morphisms, Lemma 29.21.9), whence (3) is equivalent to (4).

If $X \to S$ is proper (and S is arbitrary), then $D \to S$ is proper as well. Since a proper locally quasi-finite morphism is finite (More on Morphisms, Lemma 37.44.1) and a finite, flat, and finitely presented morphism is finite locally free (Morphisms, Lemma 29.48.2), we see that (1) is equivalent to (2). $\square$

0B9E Lemma 44.3.2. Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D_1, D_2 \subset X$ be closed subschemes finite locally free of degrees d_1, d_2 over S. Then $D_1 + D_2$ is finite locally free of degree $d_1 + d_2$ over S.

Proof. By Lemma 44.3.1 we see that D_1 and D_2 are relative effective Cartier divisors on X/S. Thus $D = D_1 + D_2$ is a relative effective Cartier divisor on X/S by Divisors, Lemma 31.18.3. Hence $D \to S$ is locally quasi-finite, flat, and locally of finite presentation by Lemma 44.3.1. Applying Morphisms, Lemma 29.41.11 the surjective integral morphism $D_1 \amalg D_2 \to D$ we find that $D \to S$ is separated. Then Morphisms, Lemma 29.41.9 implies that $D \to S$ is proper. This implies that $D \to S$ is finite (More on Morphisms, Lemma 37.44.1) and in turn we see that $D \to S$ is finite locally free (Morphisms, Lemma 29.48.2). Thus it suffice to show that the degree of $D \to S$ is $d_1 + d_2$. To do this we may base change to a fibre of $X \to S$, hence we may assume that $S = \mathrm{Spec}(k)$ for some field k. In this case, there exists a finite set of closed points $x_1, \ldots, x_n \in X$ such that D_1 and D_2 are supported on $\{x_1, \ldots, x_n\}$. In fact, there are nonzerodivisors $f_{i,j} \in \mathcal{O}_{X,x_i}$ such that

$$D_1 = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,1})) \quad \text{and} \quad D_2 = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,2}))$$

Then we see that

$$D = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,1}f_{i,2}))$$

From this one sees easily that D has degree $d_1 + d_2$ over k (if need be, use Algebra, Lemma 10.121.1). $\square$

0B9F Lemma 44.3.3. Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D_1, D_2 \subset X$ be closed subschemes finite locally free of degrees d_1, d_2 over S. If $D_1 \subset D_2$ (as closed subschemes) then there is a closed subscheme $D \subset X$ finite locally free of degree $d_2 - d_1$ over S such that $D_2 = D_1 + D$.

Proof. This proof is almost exactly the same as the proof of Lemma 44.3.2. By Lemma 44.3.1 we see that D_1 and D_2 are relative effective Cartier divisors on X/S. By Divisors, Lemma 31.18.4 there is a relative effective Cartier divisor $D \subset X$ such that $D_2 = D_1 + D$. Hence $D \to S$ is locally quasi-finite, flat, and locally of finite presentation by Lemma 44.3.1. Since D is a closed subscheme of D_2, we see that $D \to S$ is finite. It follows that $D \to S$ is finite locally free (Morphisms, Lemma 29.48.2). Thus it suffice to show that the degree of $D \to S$ is $d_2 - d_1$. This follows from Lemma 44.3.2. $\square$

Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. By Lemma 44.3.1 for a scheme T over S and $D \in \mathrm{Hilb}^d_{X/S}(T)$, we can view D as a relative effective Cartier divisor on X_T/T such that $D \to T$ is finite locally free of degree d. Hence, by Lemma 44.3.2 we obtain a transformation of functors

$$\mathrm{Hilb}^{d_1}_{X/S} \times \mathrm{Hilb}^{d_2}_{X/S} \longrightarrow \mathrm{Hilb}^{d_1+d_2}_{X/S}, \quad (D_1, D_2) \longmapsto D_1 + D_2$$

If $\mathrm{Hilb}^d_{X/S}$ is representable for all degrees d, then this transformation of functors corresponds to a morphism of schemes

$$\underline{\mathrm{Hilb}}^{d_1}_{X/S} \times_S \underline{\mathrm{Hilb}}^{d_2}_{X/S} \longrightarrow \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/S}$$

over S. Observe that $\underline{\mathrm{Hilb}}^0_{X/S} = S$ and $\underline{\mathrm{Hilb}}^1_{X/S} = X$. A special case of the morphism above is the morphism

$$\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \longrightarrow \underline{\mathrm{Hilb}}^{d+1}_{X/S}, \quad (D, x) \longmapsto D + x$$

0B9G Lemma 44.3.4. Let $X \to S$ be a smooth morphism of schemes of relative dimension 1 such that the functors $\mathrm{Hilb}^d_{X/S}$ are representable. The morphism $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \to \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ is finite locally free of degree $d + 1$.

Proof. Let $D_{univ} \subset X \times_S \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ be the universal object. There is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \arrow[rr] \arrow[dr] & & D_{univ} \arrow[r, hook] \arrow[dl] & \underline{\mathrm{Hilb}}^{d+1}_{X/S} \times_S X \\
& \underline{\mathrm{Hilb}}^{d+1}_{X/S} & &
\end{tikzcd}$$

where the top horizontal arrow maps (D', x) to $(D' + x, x)$. We claim this morphism is an isomorphism which certainly proves the lemma. Namely, given a scheme T over S, a T-valued point ξ of D_{univ} is given by a pair $\xi = (D, x)$ where $D \subset X_T$ is a closed subscheme finite locally free of degree $d + 1$ over T and $x : T \to X$ is a morphism whose graph $x : T \to X_T$ factors through D. Then by Lemma 44.3.3 we can write $D = D' + x$ for some $D' \subset X_T$ finite locally free of degree d over T. Sending $\xi = (D, x)$ to the pair (D', x) is the desired inverse. □

0B9H Lemma 44.3.5. Let $X \to S$ be a smooth morphism of schemes of relative dimension 1 such that the functors $\mathrm{Hilb}^d_{X/S}$ are representable. The schemes $\underline{\mathrm{Hilb}}^d_{X/S}$ are smooth over S of relative dimension d.

Proof. We have $\underline{\mathrm{Hilb}}^0_{X/S} = S$ and $\underline{\mathrm{Hilb}}^1_{X/S} = X$ thus the result is true for $d = 0, 1$. Assuming the result for d, we see that $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X$ is smooth over S (Morphisms, Lemma 29.34.5 and 29.34.4). Since $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \to \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ is finite locally free of degree $d + 1$ by Lemma 44.3.4 the result follows from Descent, Lemma 35.14.5. We omit the verification that the relative dimension is as claimed (you can do this by looking at fibres, or by keeping track of the dimensions in the argument above). □

We collect all the information obtained sofar in the case of a proper smooth curve over a field.

0B9I Proposition 44.3.6. Let X be a geometrically irreducible smooth proper curve over a field k.

(1) The functors $\mathrm{Hilb}^d_{X/k}$ are representable by smooth proper varieties $\underline{\mathrm{Hilb}}^d_{X/k}$ of dimension d over k.
(2) For a field extension k'/k the k'-rational points of $\underline{\mathrm{Hilb}}^d_{X/k}$ are in 1-to-1 bijection with effective Cartier divisors of degree d on $X_{k'}$.

(3) For $d_1, d_2 \geq 0$ there is a morphism

$$\underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_k \underline{\mathrm{Hilb}}^{d_2}_{X/k} \longrightarrow \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k}$$

which is finite locally free of degree $\binom{d_1+d_2}{d_1}$.

Proof. The functors $\mathrm{Hilb}^d_{X/k}$ are representable by Proposition 44.2.6 (see also Remark 44.2.7) and the fact that X is projective (Varieties, Lemma 33.43.4). The schemes $\underline{\mathrm{Hilb}}^d_{X/k}$ are separated over k by Lemma 44.2.4. The schemes $\underline{\mathrm{Hilb}}^d_{X/k}$ are smooth over k by Lemma 44.3.5. Starting with $X = \underline{\mathrm{Hilb}}^1_{X/k}$, the morphisms of Lemma 44.3.4, and induction we find a morphism

$$X^d = X \times_k X \times_k \ldots \times_k X \longrightarrow \underline{\mathrm{Hilb}}^d_{X/k}, \quad (x_1, \ldots, x_d) \longrightarrow x_1 + \ldots + x_d$$

which is finite locally free of degree $d!$. Since X is proper over k, so is X^d, hence $\underline{\mathrm{Hilb}}^d_{X/k}$ is proper over k by Morphisms, Lemma 29.41.9. Since X is geometrically irreducible over k, the product X^d is irreducible (Varieties, Lemma 33.8.4) hence the image is irreducible (in fact geometrically irreducible). This proves (1). Part (2) follows from the definitions. Part (3) follows from the commutative diagram

$$\begin{tikzcd} X^{d_1} \times_k X^{d_2} \arrow[r, equal] \arrow[d] & X^{d_1+d_2} \arrow[d] \\ \underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_k \underline{\mathrm{Hilb}}^{d_2}_{X/k} \arrow[r] & \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k} \end{tikzcd}$$

and multiplicativity of degrees of finite locally free morphisms. □

0B9J Remark 44.3.7. Let X be a geometrically irreducible smooth proper curve over a field k as in Proposition 44.3.6. Let $d \geq 0$. The universal closed object is a relatively effective divisor

$$D_{univ} \subset \underline{\mathrm{Hilb}}^{d+1}_{X/k} \times_k X$$

over $\underline{\mathrm{Hilb}}^{d+1}_{X/k}$ by Lemma 44.3.1. In fact, D_{univ} is isomorphic as a scheme to $\underline{\mathrm{Hilb}}^d_{X/k} \times_k X$, see proof of Lemma 44.3.4. In particular, D_{univ} is an effective Cartier divisor and we obtain an invertible module $\mathcal{O}(D_{univ})$. If $[D] \in \underline{\mathrm{Hilb}}^{d+1}_{X/k}$ denotes the k-rational point corresponding to the effective Cartier divisor $D \subset X$ of degree $d+1$, then the restriction of $\mathcal{O}(D_{univ})$ to the fibre $[D] \times X$ is $\mathcal{O}_X(D)$.

44.4. The Picard functor

0B9K Given any scheme X we denote $\mathrm{Pic}(X)$ the set of isomorphism classes of invertible $\mathcal{O}_X$-modules. See Modules, Definition 17.25.9. Given a morphism $f : X \to Y$ of schemes, pullback defines a group homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. The assignment $X \leadsto \mathrm{Pic}(X)$ is a contravariant functor from the category of schemes to the category of abelian groups. This functor is not representable, but it turns out that a relative variant of this construction sometimes is representable.

Let us define the Picard functor for a morphism of schemes $f : X \to S$. The idea behind our construction is that we'll take it to be the sheaf $R^1 f_* \mathbf{G}_m$ where we use the fppf topology to compute the higher direct image. Unwinding the definitions this leads to the following more direct definition.

0B9L Definition 44.4.1. Let Sch_{fppf} be a big site as in Topologies, Definition 34.7.8. Let $f : X \to S$ be a morphism of this site. The Picard functor $\mathrm{Pic}_{X/S}$ is the fppf sheafification of the functor

$$(Sch/S)_{fppf} \longrightarrow \text{Sets}, \quad T \longmapsto \mathrm{Pic}(X_T)$$

If this functor is representable, then we denote $\underline{\mathrm{Pic}}_{X/S}$ a scheme representing it.

An often used remark is that if $T \in \mathrm{Ob}((Sch/S)_{fppf})$, then $\mathrm{Pic}_{X_T/T}$ is the restriction of $\mathrm{Pic}_{X/S}$ to $(Sch/T)_{fppf}$. It turns out to be nontrivial to see what the value of $\mathrm{Pic}_{X/S}$ is on schemes T over S. Here is a lemma that helps with this task.

0B9M Lemma 44.4.2. Let $f : X \to S$ be as in Definition 44.4.1. If $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all $T \in \mathrm{Ob}((Sch/S)_{fppf})$, then

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S}(T)$$

is an exact sequence for all T.

Proof. We may replace S by T and X by X_T and assume that $S = T$ to simplify the notation. Let $\mathcal{N}$ be an invertible $\mathcal{O}_S$-module. If $f^*\mathcal{N} \cong \mathcal{O}_X$, then we see that $f_*f^*\mathcal{N} \cong f_*\mathcal{O}_X \cong \mathcal{O}_S$ by assumption. Since $\mathcal{N}$ is locally trivial, we see that the canonical map $\mathcal{N} \to f_*f^*\mathcal{N}$ is locally an isomorphism (because $\mathcal{O}_S \to f_*f^*\mathcal{O}_S$ is an isomorphism by assumption). Hence we conclude that $\mathcal{N} \to f_*f^*\mathcal{N} \to \mathcal{O}_S$ is an isomorphism and we see that $\mathcal{N}$ is trivial. This proves the first arrow is injective.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is in the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/S}(S)$. Then there exists an fppf covering $\{S_i \to S\}$ such that $\mathcal{L}$ pulls back to the trivial invertible sheaf on X_{S_i}. Choose a trivializing section s_i. Then $\mathrm{pr}_0^*s_i$ and $\mathrm{pr}_1^*s_j$ are both trivialising sections of $\mathcal{L}$ over $X_{S_i \times_S S_j}$ and hence differ by a multiplicative unit

$$f_{ij} \in \Gamma(X_{S_i \times_S S_j}, \mathcal{O}^*_{X_{S_i \times_S S_j}}) = \Gamma(S_i \times_S S_j, \mathcal{O}^*_{S_i \times_S S_j})$$

(equality by our assumption on pushforward of structure sheaves). Of course these elements satisfy the cocycle condition on $S_i \times_S S_j \times_S S_k$, hence they define a descent datum on invertible sheaves for the fppf covering $\{S_i \to S\}$. By Descent, Proposition 35.5.2 there is an invertible $\mathcal{O}_S$-module $\mathcal{N}$ with trivializations over S_i whose associated descent datum is $\{f_{ij}\}$. Then $f^*\mathcal{N} \cong \mathcal{L}$ as the functor from descent data to modules is fully faithful (see proposition cited above). $\square$

0B9N Lemma 44.4.3. Let $f : X \to S$ be as in Definition 44.4.1. Assume f has a section σ and that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Then

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S}(T) \to 0$$

is a split exact sequence with splitting given by $\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T)$.

Proof. Denote $K(T) = \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$. Since σ is a section of f we see that $\mathrm{Pic}(X_T)$ is the direct sum of $\mathrm{Pic}(T)$ and $K(T)$. Thus by Lemma 44.4.2 we see that $K(T) \subset \mathrm{Pic}_{X/S}(T)$ for all T. Moreover, it is clear from the construction that $\mathrm{Pic}_{X/S}$ is the sheafification of the presheaf K. To finish the proof it suffices to show that K satisfies the sheaf condition for fppf coverings which we do in the next paragraph.

Let $\{T_i \to T\}$ be an fppf covering. Let $\mathcal{L}_i$ be elements of $K(T_i)$ which map to the same elements of $K(T_i \times_T T_j)$ for all i and j. Choose an isomorphism $\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^*\mathcal{L}_i$ for all i. Choose an isomorphism

$$\varphi_{ij} : \mathcal{L}_i|_{X_{T_i \times_T T_j}} \longrightarrow \mathcal{L}_j|_{X_{T_i \times_T T_j}}$$

If the map

$$\alpha_j|_{T_i \times_T T_j} \circ \sigma_{T_i \times_T T_j}^* \varphi_{ij} \circ \alpha_i|_{T_i \times_T T_j} : \mathcal{O}_{T_i \times_T T_j} \to \mathcal{O}_{T_i \times_T T_j}$$

is not equal to multiplication by 1 but some u_{ij}, then we can scale φ_{ij} by u_{ij}^{-1} to correct this. Having done this, consider the self map

$$\varphi_{ki}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{jk}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{ij}|_{X_{T_i \times_T T_j \times_T T_k}} \quad \text{on} \quad \mathcal{L}_i|_{X_{T_i \times_T T_j \times_T T_k}}$$

which is given by multiplication by some regular function f_{ijk} on the scheme $X_{T_i \times_T T_j \times_T T_k}$. By our choice of φ_{ij} we see that the pullback of this map by σ is equal to multiplication by 1. By our assumption on functions on X, we see that $f_{ijk} = 1$. Thus we obtain a descent datum for the fppf covering $\{X_{T_i} \to X\}$. By Descent, Proposition 35.5.2 there is an invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ and an isomorphism $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ whose pullback to X_{T_i} recovers $(\mathcal{L}_i, \alpha_i)$ (small detail omitted). Thus $\mathcal{L}$ defines an object of $K(T)$ as desired. □

44.5. A representability criterion

0B9P To prove the Picard functor is representable we will use the following criterion.

0B9Q **Lemma 44.5.1.** Let k be a field. Let $G : (Sch/k)^{opp} \to \textit{Groups}$ be a functor. With terminology as in Schemes, Definition 26.15.3, assume that

(1) G satisfies the sheaf property for the Zariski topology,
(2) there exists a subfunctor $F \subset G$ such that
   (a) F is representable,
   (b) $F \subset G$ is representable by open immersion,
   (c) for every field extension K of k and $g \in G(K)$ there exists a $g' \in G(k)$ such that $g'g \in F(K)$.

Then G is representable by a group scheme over k.

Proof. This follows from Schemes, Lemma 26.15.4. Namely, take $I = G(k)$ and for $i = g' \in I$ take $F_i \subset G$ the subfunctor which associates to T over k the set of elements $g \in G(T)$ with $g'g \in F(T)$. Then $F_i \cong F$ by multiplication by g'. The map $F_i \to G$ is isomorphic to the map $F \to G$ by multiplication by g', hence is representable by open immersions. Finally, the collection $(F_i)_{i \in I}$ covers G by assumption (2)(c). Thus the lemma mentioned above applies and the proof is complete. □

44.6. The Picard scheme of a curve

0B9R In this section we will apply Lemma 44.5.1 to show that $\mathrm{Pic}_{X/k}$ is representable, when k is an algebraically closed field and X is a smooth projective curve over k. To make this work we use a bit of cohomology and base change developed in the chapter on derived categories of schemes.

0B9U **Lemma 44.6.1.** Let k be a field. Let X be a smooth projective curve over k which has a k-rational point. Then the hypotheses of Lemma 44.4.3 are satisfied.

Proof. The meaning of the phrase "has a k-rational point" is exactly that the structure morphism $f : X \to \mathrm{Spec}(k)$ has a section, which verifies the first condition. By Varieties, Lemma 33.26.2 we see that $k' = H^0(X, \mathcal{O}_X)$ is a field extension of k. Since X has a k-rational point there is a k-algebra homomorphism $k' \to k$ and we conclude $k' = k$. Since k is a field, any morphism $T \to \mathrm{Spec}(k)$ is flat. Hence we see by cohomology and base change (Cohomology of Schemes, Lemma 30.5.2) that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. This finishes the proof. □

Let X be a smooth projective curve over a field k with a k-rational point σ. Then the functor

$$\mathrm{Pic}_{X/k,\sigma} : (Sch/k)^{opp} \longrightarrow \mathrm{Ab}, \quad T \longmapsto \mathrm{Ker}(\mathrm{Pic}(X_T) \xrightarrow{\sigma_T^*} \mathrm{Pic}(T))$$

is isomorphic to $\mathrm{Pic}_{X/k}$ on $(Sch/k)_{fppf}$ by Lemmas 44.6.1 and 44.4.3. Hence it will suffice to prove that $\mathrm{Pic}_{X/k,\sigma}$ is representable. We will use the notation "$\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$" to signify that T is a scheme over k and $\mathcal{L}$ is an invertible $\mathcal{O}_{X_T}$-module whose restriction to T via σ_T is isomorphic to $\mathcal{O}_T$.

0B9V Lemma 44.6.2. Let k be a field. Let X be a smooth projective curve over k with a k-rational point σ. For a scheme T over k, consider the subset $F(T) \subset \mathrm{Pic}_{X/k,\sigma}(T)$ consisting of $\mathcal{L}$ such that $Rf_{T,*}\mathcal{L}$ is isomorphic to an invertible $\mathcal{O}_T$-module placed in degree 0. Then $F \subset \mathrm{Pic}_{X/k,\sigma}$ is a subfunctor and the inclusion is representable by open immersions.

Proof. Immediate from Derived Categories of Schemes, Lemma 36.32.3 applied with $i = 0$ and $r = 1$ and Schemes, Definition 26.15.3. □

To continue it is convenient to make the following definition.

0B9W Definition 44.6.3. Let k be a field. Let X be a smooth projective geometrically irreducible curve over k. The genus of X is $g = \dim_k H^1(X, \mathcal{O}_X)$.

0B9X Lemma 44.6.4. Let k be a field. Let X be a smooth projective curve of genus g over k with a k-rational point σ. The open subfunctor F defined in Lemma 44.6.2 is representable by an open subscheme of $\underline{\mathrm{Hilb}}^g_{X/k}$.

Proof. In this proof unadorned products are over $\mathrm{Spec}(k)$. By Proposition 44.3.6 the scheme $H = \underline{\mathrm{Hilb}}^g_{X/k}$ exists. Consider the universal divisor $D_{univ} \subset H \times X$ and the associated invertible sheaf $\mathcal{O}(D_{univ})$, see Remark 44.3.7. We adjust by tensoring with the pullback via $\sigma_H : H \to H \times X$ to get

$$\mathcal{L}_H = \mathcal{O}(D_{univ}) \otimes_{\mathcal{O}_{H\times X}} \mathrm{pr}_H^*\sigma_H^*\mathcal{O}(D_{univ})^{\otimes -1} \in \mathrm{Pic}_{X/k,\sigma}(H)$$

By the Yoneda lemma (Categories, Lemma 4.3.5) the invertible sheaf $\mathcal{L}_H$ defines a natural transformation

$$h_H \longrightarrow \mathrm{Pic}_{X/k,\sigma}$$

Because F is an open subfactor, there exists a maximal open $W \subset H$ such that $\mathcal{L}_H|_{W\times X}$ is in $F(W)$. Of course, this open is nothing else than the open subscheme constructed in Derived Categories of Schemes, Lemma 36.32.3 with $i = 0$ and $r = 1$ for the morphism $H \times X \to H$ and the sheaf $\mathcal{F} = \mathcal{O}(D_{univ})$. Applying the Yoneda lemma again we obtain a commutative diagram

$$\begin{tikzcd} h_W \arrow[r] \arrow[d] & F \arrow[d] \\ h_H \arrow[r] & \mathrm{Pic}_{X/k,\sigma} \end{tikzcd}$$

To finish the proof we will show that the top horizontal arrow is an isomorphism.

Let $\mathcal{L} \in F(T) \subset \mathrm{Pic}_{X/k,\sigma}(T)$. Let $\mathcal{N}$ be the invertible $\mathcal{O}_T$-module such that $Rf_{T,*}\mathcal{L} \cong \mathcal{N}[0]$. The adjunction map

$$f_T^*\mathcal{N} \longrightarrow \mathcal{L} \quad \text{corresponds to a section } s \text{ of} \quad \mathcal{L} \otimes f_T^*\mathcal{N}^{\otimes -1}$$

on X_T. Claim: The zero scheme of s is a relative effective Cartier divisor D on $(T \times X)/T$ finite locally free of degree g over T.

Let us finish the proof of the lemma admitting the claim. Namely, D defines a morphism $m : T \to H$ such that D is the pullback of D_{univ}. Then

$$(m \times \mathrm{id}_X)^*\mathcal{O}(D_{univ}) \cong \mathcal{O}_{T\times X}(D)$$

Hence $(m \times \mathrm{id}_X)^*\mathcal{L}_H$ and $\mathcal{O}(D)$ differ by the pullback of an invertible sheaf on H. This in particular shows that $m : T \to H$ factors through the open $W \subset H$ above. Moreover, it follows that these invertible modules define, after adjusting by pullback via σ_T as above, the same element of $\mathrm{Pic}_{X/k,\sigma}(T)$. Chasing diagrams using Yoneda's lemma we see that $m \in h_W(T)$ maps to $\mathcal{L} \in F(T)$. We omit the verification that the rule $F(T) \to h_W(T)$, $\mathcal{L} \mapsto m$ defines an inverse of the transformation of functors above.

Proof of the claim. Since D is a locally principal closed subscheme of $T \times X$, it suffices to show that the fibres of D over T are effective Cartier divisors, see Lemma 44.3.1 and Divisors, Lemma 31.18.9. Because taking cohomology of $\mathcal{L}$ commutes with base change (Derived Categories of Schemes, Lemma 36.30.4) we reduce to $T = \mathrm{Spec}(K)$ where K/k is a field extension. Then $\mathcal{L}$ is an invertible sheaf on X_K with $H^0(X_K, \mathcal{L}) = K$ and $H^1(X_K, \mathcal{L}) = 0$. Thus

$$\deg(\mathcal{L}) = \chi(X_K, \mathcal{L}) - \chi(X_K, \mathcal{O}_{X_K}) = 1 - (1 - g) = g$$

See Varieties, Definition 33.44.1. To finish the proof we have to show a nonzero section of $\mathcal{L}$ defines an effective Cartier divisor on X_K. This is clear. $\square$

0B9Y Lemma 44.6.5. Let k be a separably closed field. Let X be a smooth projective curve of genus g over k. Let K/k be a field extension and let $\mathcal{L}$ be an invertible sheaf on X_K. Then there exists an invertible sheaf $\mathcal{L}_0$ on X such that $\dim_K H^0(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 1$ and $\dim_K H^1(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 0$.

Proof. This proof is a variant of the proof of Varieties, Lemma 33.44.16. We encourage the reader to read that proof first.

First we pick an ample invertible sheaf $\mathcal{L}_0$ and we replace $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0^{\otimes n}|_{X_K}$ for some $n \gg 0$. The result will be that we may assume that $H^0(X_K, \mathcal{L}) \neq 0$ and $H^1(X_K, \mathcal{L}) = 0$. Namely, we will get the vanishing by Cohomology of Schemes, Lemma 30.17.1 and the nonvanishing because the degree of the tensor product is $\gg 0$. We will finish the proof by descending induction on $t = \dim_K H^0(X_K, \mathcal{L})$. The base case $t = 1$ is trivial. Assume $t > 1$.

Observe that for a k-rational point x of X, the inverse image x_K is a K-rational point of X_K. Moreover, there are infinitely many k-rational points by Varieties, Lemma 33.25.6. Therefore the points x_K form a Zariski dense collection of points of X_K.

Let $s \in H^0(X_K, \mathcal{L})$ be nonzero. From the previous paragraph we deduce there exists a k-rational point x such that s does not vanish in x_K. Let $\mathcal{I}$ be the ideal sheaf of $i : x_K \to X_K$ as in Varieties, Lemma 33.43.8. Look at the short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L} \to \mathcal{L} \to i_*i^*\mathcal{L} \to 0$$

Observe that $H^0(X_K, i_*i^*\mathcal{L}) = H^0(x_K, i^*\mathcal{L})$ has dimension 1 over K. Since s does not vanish at x we conclude that

$$H^0(X_K, \mathcal{L}) \longrightarrow H^0(X, i_*i^*\mathcal{L})$$

is surjective. Hence $\dim_K H^0(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = t - 1$. Finally, the long exact sequence of cohomology also shows that $H^1(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = 0$ thereby finishing the proof of the induction step. □

0B9Z Proposition 44.6.6. Let k be a separably closed field. Let X be a smooth projective curve over k. The Picard functor $\mathrm{Pic}_{X/k}$ is representable.

Proof. Since k is separably closed there exists a k-rational point σ of X, see Varieties, Lemma 33.25.6. As discussed above, it suffices to show that the functor $\mathrm{Pic}_{X/k,\sigma}$ classifying invertible modules trivial along σ is representable. To do this we will check conditions (1), (2)(a), (2)(b), and (2)(c) of Lemma 44.5.1.

The functor $\mathrm{Pic}_{X/k,\sigma}$ satisfies the sheaf condition for the fppf topology because it is isomorphic to $\mathrm{Pic}_{X/k}$. It would be more correct to say that we've shown the sheaf condition for $\mathrm{Pic}_{X/k,\sigma}$ in the proof of Lemma 44.4.3 which applies by Lemma 44.6.1. This proves condition (1)

As our subfunctor we use F as defined in Lemma 44.6.2. Condition (2)(b) follows. Condition (2)(a) is Lemma 44.6.4. Condition (2)(c) is Lemma 44.6.5. □

In fact, the proof given above produces more information which we collect here.

0BA0 Lemma 44.6.7. Let k be a separably closed field. Let X be a smooth projective curve of genus g over k.

(1) $\underline{\mathrm{Pic}}_{X/k}$ is a disjoint union of g-dimensional smooth proper varieties $\underline{\mathrm{Pic}}^d_{X/k}$,
(2) k-points of $\underline{\mathrm{Pic}}^d_{X/k}$ correspond to invertible $\mathcal{O}_X$-modules of degree d,
(3) $\underline{\mathrm{Pic}}^0_{X/k}$ is an open and closed subgroup scheme,
(4) for $d \geq 0$ there is a canonical morphism $\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$
(5) the morphisms γ_d are surjective for $d \geq g$ and smooth for $d \geq 2g - 1$,
(6) the morphism $\underline{\mathrm{Hilb}}^g_{X/k} \to \underline{\mathrm{Pic}}^g_{X/k}$ is birational.

Proof. Pick a k-rational point σ of X. Recall that $\mathrm{Pic}_{X/k}$ is isomorphic to the functor $\mathrm{Pic}_{X/k,\sigma}$. By Derived Categories of Schemes, Lemma 36.32.2 for every $d \in \mathbf{Z}$ there is an open subfunctor

$$\mathrm{Pic}^d_{X/k,\sigma} \subset \mathrm{Pic}_{X/k,\sigma}$$

whose value on a scheme T over k consists of those $\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$ such that $\chi(X_t, \mathcal{L}_t) = d + 1 - g$ and moreover we have

$$\mathrm{Pic}_{X/k,\sigma} = \coprod\nolimits_{d \in \mathbf{Z}} \mathrm{Pic}^d_{X/k,\sigma}$$

as fppf sheaves. It follows that the scheme $\underline{\mathrm{Pic}}_{X/k}$ (which exists by Proposition 44.6.6) has a corresponding decomposition

$$\underline{\mathrm{Pic}}_{X/k,\sigma} = \coprod\nolimits_{d \in \mathbf{Z}} \underline{\mathrm{Pic}}^d_{X/k,\sigma}$$

where the points of $\underline{\mathrm{Pic}}^d_{X/k,\sigma}$ correspond to isomorphism classes of invertible modules of degree d on X.

Fix $d \geq 0$. There is a morphism

$$\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \longrightarrow \underline{\mathrm{Pic}}^d_{X/k}$$

coming from the invertible sheaf $\mathcal{O}(D_{univ})$ on $\underline{\mathrm{Hilb}}^d_{X/k} \times_k X$ (Remark 44.3.7) by the Yoneda lemma (Categories, Lemma 4.3.5). Our proof of the representability of the Picard functor of X/k in Proposition 44.6.6 and Lemma 44.6.4 shows that γ_g induces an open immersion on a nonempty open of $\underline{\mathrm{Hilb}}^g_{X/k}$. Moreover, the proof shows that the translates of this open by k-rational points of the group scheme $\underline{\mathrm{Pic}}_{X/k}$ define an open covering. Since $\underline{\mathrm{Hilb}}^g_{X/K}$ is smooth of dimension g (Proposition 44.3.6) over k, we conclude that the group scheme $\underline{\mathrm{Pic}}_{X/k}$ is smooth of dimension g over k.

By Groupoids, Lemma 39.7.3 we see that $\underline{\mathrm{Pic}}_{X/k}$ is separated. Hence, for every $d \geq 0$, the image of γ_d is a proper variety over k (Morphisms, Lemma 29.41.10).

Let $d \geq g$. Then for any field extension K/k and any invertible $\mathcal{O}_{X_K}$-module $\mathcal{L}$ of degree d, we see that $\chi(X_K, \mathcal{L}) = d + 1 - g > 0$. Hence $\mathcal{L}$ has a nonzero section and we conclude that $\mathcal{L} = \mathcal{O}_{X_K}(D)$ for some divisor $D \subset X_K$ of degree d. It follows that γ_d is surjective.

Combining the facts mentioned above we see that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper for $d \geq g$. This finishes the proof of (2) because now we see that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper for $d \geq g$ but then all $\underline{\mathrm{Pic}}^d_{X/k}$ are proper by translation.

It remains to prove that γ_d is smooth for $d \geq 2g - 1$. Consider an invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree d. Then the fibre of the point corresponding to $\mathcal{L}$ is

$$Z = \{D \subset X \mid \mathcal{O}_X(D) \cong \mathcal{L}\} \subset \underline{\mathrm{Hilb}}^d_{X/k}$$

with its natural scheme structure. Since any isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ is well defined up to multiplying by a nonzero scalar, we see that the canonical section $1 \in \mathcal{O}_X(D)$ is mapped to a section $s \in \Gamma(X, \mathcal{L})$ well defined up to multiplication by a nonzero scalar. In this way we obtain a morphism

$$Z \longrightarrow \mathrm{Proj}(\mathrm{Sym}(\Gamma(X, \mathcal{L})^*))$$

(dual because of our conventions). This morphism is an isomorphism, because given an section of $\mathcal{L}$ we can take the associated effective Cartier divisor, in other words we can construct an inverse of the displayed morphism; we omit the precise formulation and proof. Since $\dim H^0(X, \mathcal{L}) = d + 1 - g$ for every $\mathcal{L}$ of degree $d \geq 2g - 1$ by Varieties, Lemma 33.44.17 we see that $\mathrm{Proj}(\mathrm{Sym}(\Gamma(X, \mathcal{L})^*)) \cong \mathbf{P}^{d-g}_k$. We conclude that $\dim(Z) = \dim(\mathbf{P}^{d-g}_k) = d - g$. We conclude that the fibres of the morphism γ_d

all have dimension equal to the difference of the dimensions of $\underline{\mathrm{Hilb}}^d_{X/k}$ and $\underline{\mathrm{Pic}}^d_{X/k}$. It follows that γ_d is flat, see Algebra, Lemma 10.128.1. As moreover the fibres are smooth, we conclude that γ_d is smooth by Morphisms, Lemma 29.34.3. □

44.7. Some remarks on Picard groups

0CDS This section continues the discussion in Varieties, Section 33.30 and will be continued in Algebraic Curves, Section 53.17.

0CDT Lemma 44.7.1. Let k be a field. Let X be a quasi-compact and quasi-separated scheme over k with $H^0(X, \mathcal{O}_X) = k$. If X has a k-rational point, then for any Galois extension k'/k we have

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{k'})^{\mathrm{Gal}(k'/k)}$$

Moreover the action of $\mathrm{Gal}(k'/k)$ on $\mathrm{Pic}(X_{k'})$ is continuous.

Proof. Since $\mathrm{Gal}(k'/k) = \mathrm{Aut}(k'/k)$ it acts (from the right) on $\mathrm{Spec}(k')$, hence it acts (from the right) on $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$, and since $\mathrm{Pic}(-)$ is a contravariant functor, it acts (from the left) on $\mathrm{Pic}(X_{k'})$. If k'/k is an infinite Galois extension, then we write $k' = \mathrm{colim}\, k'_\lambda$ as a filtered colimit of finite Galois extensions, see Fields, Lemma 9.22.3. Then $X_{k'} = \lim X_{k_\lambda}$ (as in Limits, Section 32.2) and we obtain

$$\mathrm{Pic}(X_{k'}) = \mathrm{colim}\, \mathrm{Pic}(X_{k_\lambda})$$

by Limits, Lemma 32.10.3. Moreover, the transition maps in this system of abelian groups are injective by Varieties, Lemma 33.30.3. It follows that every element of $\mathrm{Pic}(X_{k'})$ is fixed by one of the open subgroups $\mathrm{Gal}(k'/k_\lambda)$, which exactly means that the action is continuous. Injectivity of the transition maps implies that it suffices to prove the statement on fixed points in the case that k'/k is finite Galois.

Assume k'/k is finite Galois with Galois group $G = \mathrm{Gal}(k'/k)$. Let $\mathcal{L}$ be an element of $\mathrm{Pic}(X_{k'})$ fixed by G. We will use Galois descent (Descent, Lemma 35.6.1) to prove that $\mathcal{L}$ is the pullback of an invertible sheaf on X. Recall that $f_\sigma = \mathrm{id}_X \times \mathrm{Spec}(\sigma) : X_{k'} \to X_{k'}$ and that σ acts on $\mathrm{Pic}(X_{k'})$ by pulling back by f_σ. Hence for each $\sigma \in G$ we can choose an isomorphism $\varphi_\sigma : \mathcal{L} \to f_\sigma^*\mathcal{L}$ because $\mathcal{L}$ is a fixed by the G-action. The trouble is that we don't know if we can choose φ_σ such that the cocycle condition $\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ holds. To see that this is possible we use that X has a k-rational point $x \in X(k)$. Of course, x similarly determines a k'-rational point $x' \in X_{k'}$ which is fixed by f_σ for all σ. Pick a nonzero element s in the fibre of $\mathcal{L}$ at x'; the fibre is the 1-dimensional $k' = \kappa(x')$-vector space

$$\mathcal{L}_{x'} \otimes_{\mathcal{O}_{X_{k'},x'}} \kappa(x').$$

Then f_σ^*s is a nonzero element of the fibre of $f_\sigma^*\mathcal{L}$ at x'. Since we can multiply φ_σ by an element of $(k')^*$ we may assume that φ_σ sends s to f_σ^*s. Then we see that both $\varphi_{\sigma\tau}$ and $f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ send s to $f_{\sigma\tau}^*s = f_\tau^* f_\sigma^* s$. Since $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ these two isomorphisms have to be the same (as one is a global unit times the other and they agree in x') and the proof is complete. □

0CD5 Lemma 44.7.2. Let k be a field of characteristic $p > 0$. Let X be a quasi-compact and quasi-separated scheme over k with $H^0(X, \mathcal{O}_X) = k$. Let n be an integer prime to p. Then the map

$$\mathrm{Pic}(X)[n] \longrightarrow \mathrm{Pic}(X_{k'})[n]$$

is bijective for any purely inseparable extension k'/k.

Proof. First we observe that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k'})$ is injective by Varieties, Lemma 33.30.3. Hence we have to show the map in the lemma is surjective. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{X_{k'}}$-module which has order dividing n in $\mathrm{Pic}(X_{k'})$. Choose an isomorphism $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_{X_{k'}}$ of invertible modules. We will prove that we can descend the pair $(\mathcal{L}, \alpha)$ to X.

Set $A = k' \otimes_k k'$. Since k'/k is purely inseparable, the kernel of the multiplication map $A \to k'$ is a locally nilpotent ideal I of A. Observe that

$$X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A) = X_{k'} \times_X X_{k'}$$

comes with two projections $\mathrm{pr}_i : X_A \to X_{k'}$, $i = 0, 1$ which agree over A/I. Hence the invertible modules $\mathcal{L}_i = \mathrm{pr}_i^*\mathcal{L}$ agree over the closed subscheme $X_{A/I} = X_{k'}$. Since $X_{A/I} \to X_A$ is a thickening and since $\mathcal{L}_i$ are n-torsion, we see that there exists an isomorphism $\varphi : \mathcal{L}_0 \to \mathcal{L}_1$ by More on Morphisms, Lemma 37.4.2. We may pick φ to reduce to the identity modulo I. Namely, $H^0(X, \mathcal{O}_X) = k$ implies $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ by Cohomology of Schemes, Lemma 30.5.2 and $A \to k'$ is surjective hence we can adjust φ by multiplying by a suitable element of A. Consider the map

$$\lambda : \mathcal{O}_{X_A} \xrightarrow{\mathrm{pr}_0^*\alpha^{-1}} \mathcal{L}_0^{\otimes n} \xrightarrow{\varphi^{\otimes n}} \mathcal{L}_1^{\otimes n} \xrightarrow{\mathrm{pr}_0^*\alpha} \mathcal{O}_{X_A}$$

We can view λ as an element of A because $H^0(X_A, \mathcal{O}_{X_A}) = A$ (same reference as above). Since φ reduces to the identity modulo I we see that $\lambda = 1 \bmod I$. Then there is a unique nth root of λ in $1 + I$ (Algebra, Lemma 10.32.8) and after multiplying φ by its inverse we get $\lambda = 1$. We claim that $(\mathcal{L}, \varphi)$ is a descent datum for the fpqc covering $\{X_{k'} \to X\}$ (Descent, Definition 35.2.1). If true, then $\mathcal{L}$ is the pullback of an invertible $\mathcal{O}_X$-module $\mathcal{N}$ by Descent, Proposition 35.5.2. Injectivity of the map on Picard groups shows that $\mathcal{N}$ is a torsion element of $\mathrm{Pic}(X)$ of the same order as $\mathcal{L}$.

Proof of the claim. To see this we have to verify that

$$\mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi = \mathrm{pr}_{02}^*\varphi \quad \text{on} \quad X_{k'} \times_X X_{k'} \times_X X_{k'} = X_{k' \otimes_k k' \otimes_k k'}$$

As before the diagonal morphism $\Delta : X_{k'} \to X_{k' \otimes_k k' \otimes_k k'}$ is a thickening. The left and right hand sides of the equality signs are maps $a, b : p_0^*\mathcal{L} \to p_2^*\mathcal{L}$ compatible with $p_0^*\alpha$ and $p_2^*\alpha$ where $p_i : X_{k' \otimes_k k' \otimes_k k'} \to X_{k'}$ are the projection morphisms. Finally, a, b pull back to the same map under Δ. Affine locally (in local trivializations) this means that a, b are given by multiplication by invertible functions which reduce to the same function modulo a locally nilpotent ideal and which have the same nth powers. Then it follows from Algebra, Lemma 10.32.8 that these functions are the same. $\square$

44.8. Other chapters

Preliminaries

CHAPTER 45

Weil Cohomology Theories

0FFG 45.1. Introduction

0FFH In this chapter we discuss Weil cohomology theories for smooth projective schemes over a base field. Briefly, for us such a cohomology theory H^* is one which has Künneth, Poincaré duality, and cycle classes (with suitable compatibilities). We warn the reader that there is no universal agreement in the literature as to what constitutes a "Weil cohomology theory".

Before reading this chapter the reader should take a look at Categories, Section 4.43 and Homology, Section 12.17 where we define (symmetric) monoidal categories and we develop just enough basic language concerning these categories for the needs of this chapter. Equipped with this language we construct in Section 45.3 the symmetric monoidal graded category whose objects are smooth projective schemes and whose morphisms are correspondences. In Section 45.4 we add images of projectors and invert the Lefschetz motive in order to obtain the symmetric monoidal Karoubian category M_k of Chow motives. This category comes equipped with a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

As we will see below, a key property of a Weil cohomology theory is that it factors over h.

First, in the case of an algebraically closed base field, we define what we call a "classical Weil cohomology theory", see Section 45.7. This notion is the same as the notion introduced in [Kle68, Section 1.2] and agrees with the notion introduced in [Kle72, page 65]. However, our notion does not a priori agree with the notion introduced in [Kle94, page 10] because there the author adds two Lefschetz type axioms and it isn't known whether any classical Weil cohomology theory as defined in this chapter satisfies those axioms. At the end of Section 45.7 we show that a classical Weil cohomology theory is of the form $H^* = G \circ h$ where G is a symmetric monoidal functor from M_k to the category of graded vector spaces over the coefficient field of H^*.

In Section 45.8 we prove a couple of lemmas on cycle groups over non-closed fields which will be used in discussing Weil cohomology theories on smooth projective schemes over arbitrary fields.

Our motivation for our axioms of a Weil cohomology theory H^* over a general base field k are the following

(1) $H^* = G \circ h$ for a symmetric monoidal functor G from M_k to the category of graded vector spaces over the coefficient field F of H^*,

(2) G should send the Tate motive (inverse of the Lefschetz motive) to a 1-dimensional vector space $F(1)$ sitting in degree -2,
(3) when k is algebraically closed we should recover the notion discussion in Section 45.7 up to choosing a basis element of $F(1)$.

First, in Section 45.9 we analyze the first two conditions. After developing a few more results in Section 45.10 in Section 45.11 we add the necessary axioms to obtain property (3).

In the final Section 45.14 we detail an alternative approach to Weil cohomology theories, using a first Chern class map instead of cycle classes. It is this approach that will be most suited for proving that certain cohomology theories are Weil cohomology theories in later chapters, see de Rham Cohomology, Section 50.22.

45.2. Conventions and notation

0FFI Let F be a field. In this chapter we view the category of F-graded vector spaces as an F-linear symmetric monoidal category with associativity constraint as usual and with commutativity constraint involving signs. See Homology, Example 12.17.4.

Let R be a ring. In this chapter a graded commutative R-algebra A is a commutative differential graded R-algebra (Differential Graded Algebra, Definitions 22.3.1 and 22.3.3) whose differential is zero. Thus A is an R-module endowed with a grading $A = \bigoplus_{n \in \mathbf{Z}} A^n$ by R-submodules. The R-bilinear multiplication

$$A^n \times A^m \longrightarrow A^{n+m}, \quad \alpha \times \beta \longmapsto \alpha \cup \beta$$

will be called the cup product in this chapter. The commutativity constraint is $\alpha \cup \beta = (-1)^{nm} \beta \cup \alpha$ if $\alpha \in A^n$ and $\beta \in A^m$. Finally, there is a multiplicative unit $1 \in A^0$, or equivalently, there is an additive and multiplicative map $R \to A^0$ which is compatible the R-module structure on A.

Let k be a field. Let X be a scheme of finite type over k. The Chow groups $\mathrm{CH}_k(X)$ of X of cycles of dimension k modulo rational equivalence have been defined in Chow Homology, Definition 42.19.1. If X is normal or Cohen-Macaulay, then we can also consider the Chow groups $\mathrm{CH}^p(X)$ of cycles of codimension p (Chow Homology, Section 42.42) and then $[X] \in \mathrm{CH}^0(X)$ denotes the "fundamental class" of X, see Chow Homology, Remark 42.42.2. If X is smooth and α and β are cycles on X, then $\alpha \cdot \beta$ denotes the intersection product of α and β, see Chow Homology, Section 42.62.

45.3. Correspondences

0FFZ Let k be a field. For schemes X and Y over k we denote $X \times Y$ the product of X and Y in the category of schemes over k. In this section we construct the graded category over $\mathbf{Q}$ whose objects are smooth projective schemes over k and whose morphisms are correspondences.

Let X and Y be smooth projective schemes over k. Let $X = \coprod X_d$ be the decomposition of X into the open and closed subschemes which are equidimensional with $\dim(X_d) = d$. We define the $\mathbf{Q}$-vector space of correspondences of degree r from X to Y by the formula:

$$\mathrm{Corr}^r(X, Y) = \bigoplus\nolimits_d \mathrm{CH}^{d+r}(X_d \times Y) \otimes \mathbf{Q} \subset \mathrm{CH}^*(X \times Y) \otimes \mathbf{Q}$$

Given $c \in \mathrm{Corr}^r(X, Y)$ and $\beta \in \mathrm{CH}_k(Y) \otimes \mathbf{Q}$ we can define the pullback of β by c using the formula

$$c^*(\beta) = \mathrm{pr}_{1,*}(c \cdot \mathrm{pr}_2^*\beta) \quad \text{in} \quad \mathrm{CH}_{k-r}(X) \otimes \mathbf{Q}$$

This makes sense because pr_2 is flat of relative dimension d on $X_d \times Y$, hence $\mathrm{pr}_2^*\beta$ is a cycle of dimension $d + k$ on $X_d \times Y$, hence $c \cdot \mathrm{pr}_2^*\alpha$ is a cycle of dimension $k - r$ on $X_d \times Y$ whose pushforward by the proper morphism pr_1 is a cycle of the same dimension. Similarly, switching to grading by codimension, given $\alpha \in \mathrm{CH}^i(X) \otimes \mathbf{Q}$ we can define the pushforward of α by c using the formula

$$c_*(\alpha) = \mathrm{pr}_{2,*}(c \cdot \mathrm{pr}_1^*\alpha) \quad \text{in} \quad \mathrm{CH}^{i+r}(Y) \otimes \mathbf{Q}$$

This makes sense because $\mathrm{pr}_1^*\alpha$ is a cycle of codimension i on $X \times Y$, hence $c \cdot \mathrm{pr}_1^*\alpha$ is a cycle of codimension $i + d + r$ on $X_d \times Y$, which pushes forward to a cycle of codimension $i + r$ on Y.

Given a three smooth projective schemes X, Y, Z over k we define a composition of correspondences

$$\mathrm{Corr}^s(Y, Z) \times \mathrm{Corr}^r(X, Y) \longrightarrow \mathrm{Corr}^{r+s}(X, Z)$$

by the rule

$$(c', c) \longmapsto c' \circ c = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*c \cdot \mathrm{pr}_{23}^*c')$$

where $\mathrm{pr}_{12} : X \times Y \times Z \to X \times Y$ is the projection and similarly for pr_{13} and pr_{23}.

0FG0 Lemma 45.3.1. We have the following for correspondences:

(1) composition of correspondences is $\mathbf{Q}$-bilinear and associative,
(2) there is a canonical isomorphism

$$\mathrm{CH}_{-r}(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(X, \mathrm{Spec}(k))$$

such that pullback by correspondences corresponds to composition,
(3) there is a canonical isomorphism

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(\mathrm{Spec}(k), X)$$

such that pushforward by correspondences corresponds to composition,
(4) composition of correspondences is compatible with pushforward and pullback of cycles.

Proof. Bilinearity follows immediately from the linearity of pushforward and pullback and the bilinearity of the intersection product. To prove associativity, say we have X, Y, Z, W and $c \in \mathrm{Corr}(X, Y)$, $c' \in \mathrm{Corr}(Y, Z)$, and $c'' \in \mathrm{Corr}(Z, W)$. Then we have

$$\begin{aligned}
c'' \circ (c' \circ c) &= \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{134,*}_{13}\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{12}c \cdot \mathrm{pr}^{123,*}_{23}c') \cdot \mathrm{pr}^{134,*}_{34}c'') \\
&= \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}\mathrm{pr}^{1234,*}_{123}(\mathrm{pr}^{123,*}_{12}c \cdot \mathrm{pr}^{123,*}_{23}c') \cdot \mathrm{pr}^{134,*}_{34}c'') \\
&= \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}(\mathrm{pr}^{1234,*}_{12}c \cdot \mathrm{pr}^{1234,*}_{23}c') \cdot \mathrm{pr}^{134,*}_{34}c'') \\
&= \mathrm{pr}^{134}_{14,*}\mathrm{pr}^{1234}_{134,*}((\mathrm{pr}^{1234,*}_{12}c \cdot \mathrm{pr}^{1234,*}_{23}c') \cdot \mathrm{pr}^{1234,*}_{34}c'') \\
&= \mathrm{pr}^{1234}_{14,*}((\mathrm{pr}^{1234,*}_{12}c \cdot \mathrm{pr}^{1234,*}_{23}c') \cdot \mathrm{pr}^{1234,*}_{34}c'')
\end{aligned}$$

Here we use the notation

$$p^{1234}_{134} : X \times Y \times Z \times W \to X \times Z \times W \quad \text{and} \quad p^{134}_{14} : X \times Z \times W \to X \times W$$

the projections and similarly for other indices. The first equality is the definition of the composition. The second equality holds because $\mathrm{pr}_{13}^{134,*}\mathrm{pr}_{13,*}^{123} = \mathrm{pr}_{134,*}^{1234}\mathrm{pr}_{123}^{1234,*}$ by Chow Homology, Lemma 42.15.1. The third equality holds because intersection product commutes with the gysin map for p_{123}^{1234} (which is given by flat pullback), see Chow Homology, Lemma 42.62.3. The fourth equality follows from the projection formula for p_{134}^{1234}, see Chow Homology, Lemma 42.62.4. The fourth equality is that proper pushforward is compatible with composition, see Chow Homology, Lemma 42.12.2. Since intersection product is associative by Chow Homology, Lemma 42.62.1 this concludes the proof of associativity of composition of correspondences.

We omit the proofs of (2) and (3) as these are essentially proved by carefully bookkeeping where various cycles live and in what (co)dimension.

The statement on pushforward and pullback of cycles means that $(c' \circ c)^*(\alpha) = c^*((c')^*(\alpha))$ and $(c' \circ c)_*(\alpha) = (c')_*(c_*(\alpha))$. This follows on combining (1), (2), and (3). □

0FG1 Example 45.3.2. Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Denote $\Gamma_f \subset X \times Y$ the graph of f. More precisely, Γ_f is the image of the closed immersion

$$(f, \mathrm{id}_Y) : Y \longrightarrow X \times Y$$

Let $X = \coprod X_d$ be the decomposition of X into its open and closed parts X_d which are equidimensional of dimension d. Then $\Gamma_f \cap (X_d \times Y)$ has pure codimension d. Hence $[\Gamma_f] \in \mathrm{CH}^*(X \times Y) \otimes \mathbf{Q}$ is contained in $\mathrm{Corr}^0(X \times Y)$, i.e., $[\Gamma_f]$ is a correspondence of degree 0 from X to Y.

0FG2 Lemma 45.3.3. Smooth projective schemes over k with correspondences and composition of correspondences as defined above form a graded category over $\mathbf{Q}$ (Differential Graded Algebra, Definition 22.25.1).

Proof. Everything is clear from the construction and Lemma 45.3.1 except for the existence of identity morphisms. Given a smooth projective scheme X consider the class $[\Delta]$ of the diagonal $\Delta \subset X \times X$ in $\mathrm{Corr}^0(X, X)$. We note that Δ is equal to the graph of the identity $\mathrm{id}_X : X \to X$ which is a fact we will use below.

To prove that $[\Delta]$ can serve as an identity we have to show that $[\Delta] \circ c = c$ and $c' \circ [\Delta] = c'$ for any correspondences $c \in \mathrm{Corr}^r(Y, X)$ and $c' \in \mathrm{Corr}^s(X, Y)$. For the second case we have to show that

$$c' = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*[\Delta] \cdot \mathrm{pr}_{23}^*c')$$

where $\mathrm{pr}_{12} : X \times X \times Y \to X \times X$ is the projection and simlarly for pr_{13} and pr_{23}. We may write $c' = \sum a_i[Z_i]$ for some integral closed subschemes $Z_i \subset X \times Y$ and rational numers a_i. Thus it clearly suffices to show that

$$[Z] = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*[\Delta] \cdot \mathrm{pr}_{23}^*[Z])$$

in the chow group of $X \times Y$ for any integral closed subscheme Z of $X \times Y$. After replacing X and Y by the irreducible component containing the image of Z under the two projections we may assume X and Y are integral as well. Then we have to show

$$[Z] = \mathrm{pr}_{13,*}([\Delta \times Y] \cdot [X \times Z])$$

Denote $Z' \subset X \times X \times Y$ the image of Z by the morphism $(\Delta, 1) : X \times Y \to X \times X \times Y$. Then Z' is a closed subscheme of $X \times X \times Y$ isomorphic to Z and $Z' = \Delta \times Y \cap X \times Z$ scheme theoretically. By Chow Homology, Lemma 42.62.5[1] we conclude that

$$[Z'] = [\Delta \times Y] \cdot [X \times Z]$$

Since Z' maps isomorphically to Z by pr_{13} also we conclude. The verification that $[\Delta] \circ c = c$ is similar and we omit it. □

0FG3 Lemma 45.3.4. There is a contravariant functor from the category of smooth projective schemes over k to the category of correspondences which is the identity on objects and sends $f : Y \to X$ to the element $[\Gamma_f] \in \text{Corr}^0(X, Y)$.

Proof. In the proof of Lemma 45.3.3 we have seen that this construction sends identities to identities. To finish the proof we have to show if $g : Z \to Y$ is another morphism of smooth projective schemes over k, then we have $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{f \circ g}]$ in $\text{Corr}^0(X, Z)$. Arguing as in the proof of Lemma 45.3.3 we see that it suffices to show

$$[\Gamma_{f \circ g}] = \text{pr}_{13,*}([\Gamma_f \times Z] \cdot [X \times \Gamma_g])$$

in $\text{CH}^*(X \times Z)$ when X, Y, Z are integral. Denote $Z' \subset X \times Y \times Z$ the image of the closed immersion $(f \circ g, g, 1) : Z \to X \times Y \times Z$. Then $Z' = \Gamma_f \times Z \cap X \times \Gamma_g$ scheme theoretically and we conclude using Chow Homology, Lemma 42.62.5 that

$$[Z'] = [\Gamma_f \times Z] \cdot [X \times \Gamma_g]$$

Since it is clear that $\text{pr}_{13,*}([Z']) = [\Gamma_{f \circ g}]$ the proof is complete. □

0FG4 Remark 45.3.5. Let X and Y be smooth projective schemes over k. Assume X is equidimensional of dimension d and Y is equidimensional of dimension e. Then the isomorphism $X \times Y \to Y \times X$ switching the factors determines an isomorphism

$$\text{Corr}^r(X, Y) \longrightarrow \text{Corr}^{d-e+r}(Y, X), \quad c \longmapsto c^t$$

called the transpose. It acts on cycles as well as cycle classes. An example which is sometimes useful, is the transpose $[\Gamma_f]^t = [\Gamma_f^t]$ of the graph of a morphism $f : Y \to X$.

0FG5 Lemma 45.3.6. Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Let $[\Gamma_f] \in \text{Corr}^0(X, Y)$ be as in Example 45.3.2. Then

(1) pushforward of cycles by the correspondence $[\Gamma_f]$ agrees with the gysin map $f^! : \text{CH}^*(X) \to \text{CH}^*(Y)$,
(2) pullback of cycles by the correspondence $[\Gamma_f]$ agrees with the pushforward map $f_* : \text{CH}_*(Y) \to \text{CH}_*(X)$,
(3) if X and Y are equidimensional of dimensions d and e, then
  (a) pushforward of cycles by the correspondence $[\Gamma_f^t]$ of Remark 45.3.5 corresponds to pushforward of cycles by f, and
  (b) pullback of cycles by the correspondence $[\Gamma_f^t]$ of Remark 45.3.5 corresponds to the gysin map $f^!$.

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[1] The reader verifies that $\dim(Z') = \dim(\Delta \times Y) + \dim(X \times Z) - \dim(X \times X \times Y)$ and that Z' has a unique generic point mapping to the generic point of Z (where the local ring is CM) and to some point of X (where the local ring is CM). Thus all the hypothese of the lemma are indeed verified.

Proof. Proof of (1). Recall that $[\Gamma_f]_*(\alpha) = \text{pr}_{2,*}([\Gamma_f] \cdot \text{pr}_1^*\alpha)$. We have

$$[\Gamma_f] \cdot \text{pr}_1^*\alpha = (f,1)_*((f,1)^!\text{pr}_1^*\alpha) = (f,1)_*((f,1)^!\text{pr}_1^!\alpha) = (f,1)_*(f^!\alpha)$$

The first equality by Chow Homology, Lemma 42.62.6. The second by Chow Homology, Lemma 42.59.5. The third because $\text{pr}_1 \circ (f,1) = f$ and Chow Homology, Lemma 42.59.6. Then we coclude because $\text{pr}_{2,*} \circ (f,1)_* = 1_*$ by Chow Homology, Lemma 42.12.2.

Proof of (2). Recall that $[\Gamma_f]_*(\beta) = \text{pr}_{1,*}([\Gamma_f] \cdot \text{pr}_2^*\beta)$. Arguing exactly as above we have

$$[\Gamma_f] \cdot \text{pr}_2^*\beta = (f,1)_*\beta$$

Thus the result follows as before.

Proof of (3). Proved in exactly the same manner as above. □

0FG6 **Example 45.3.7.** Let $X = \mathbf{P}^1_k$. Then we have

$$\text{Corr}^0(X,X) = \text{CH}^1(X \times X) \otimes \mathbf{Q} = \text{CH}_1(X \times X) \otimes \mathbf{Q}$$

Choose a k-rational point $x \in X$ and consider the cycles $c_0 = [x \times X]$ and $c_2 = [X \times x]$. A computation shows that $1 = [\Delta] = c_0 + c_2$ in $\text{Corr}^0(X,X)$ and that we have the following rules for composition $c_0 \circ c_0 = c_0$, $c_0 \circ c_2 = 0$, $c_2 \circ c_0 = 0$, and $c_2 \circ c_2 = c_2$. In other words, c_0 and c_2 are orthogonal idempotents in the algebra $\text{Corr}^0(X,X)$ and in fact we get

$$\text{Corr}^0(X,X) = \mathbf{Q} \times \mathbf{Q}$$

as a $\mathbf{Q}$-algebra.

The category of correspondences is a symmetric monoidal category. Given smooth projective schemes X and Y over k, we define $X \otimes Y = X \times Y$. Given four smooth projective schemes X, X', Y, Y' over k we define a tensor product

$$\otimes : \text{Corr}^r(X,Y) \times \text{Corr}^{r'}(X',Y') \longrightarrow \text{Corr}^{r+r'}(X \times X', Y \times Y')$$

by the rule

$$(c,c') \longmapsto c \otimes c' = \text{pr}_{13}^*c \cdot \text{pr}_{24}^*c'$$

where $\text{pr}_{13} : X \times X' \times Y \times Y' \to X \times Y$ and $\text{pr}_{24} : X \times X' \times Y \times Y' \to X' \times Y'$ are the projections. As associativity constraint

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

we use the usual associativity constraint on products of schemes. The commutativity constraint will be given by the isomorphism $X \times Y \to Y \times X$ switching the factors.

0FG7 **Lemma 45.3.8.** *The tensor product of correspondences defined above turns the category of correspondences into a symmetric monoidal category with unit* $\text{Spec}(k)$.

Proof. Omitted. □

0FG8 **Lemma 45.3.9.** *Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Assume X and Y equidimensional of dimensions d and e. Denote $a = [\Gamma_f] \in \text{Corr}^0(X,Y)$ and $a^t = [\Gamma_f^t] \in \text{Corr}^{d-e}(Y,X)$. Set $\eta_X = [\Gamma_{X \to X \times X}] \in$*

$\mathrm{Corr}^0(X \times X, X)$, $\eta_Y = [\Gamma_{Y \to Y \times Y}] \in \mathrm{Corr}^0(Y \times Y, Y)$, $[X] \in \mathrm{Corr}^{-d}(X, \mathrm{Spec}(k))$, and $[Y] \in \mathrm{Corr}^{-e}(Y, \mathrm{Spec}(k))$. The diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "a\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes a^t"'] & Y \otimes Y \arrow[r, "\eta_Y"'] & Y \arrow[d, "{[Y]}"] \\ X \otimes X \arrow[r, "\eta_X"] & X \arrow[r, "{[X]}"] & \mathrm{Spec}(k) \end{tikzcd}$$

is commutative in the category of correspondences.

Proof. Recall that $\mathrm{Corr}^r(W, \mathrm{Spec}(k)) = \mathrm{CH}_{-r}(W)$ for any smooth projective scheme W over k and given $c \in \mathrm{Corr}^s(W', W)$ the composition with c agrees with pullback by c as a map $\mathrm{CH}_{-r}(W) \to \mathrm{CH}_{-r-s}(W')$ (Lemma 45.3.1). Finally, we have Lemma 45.3.6 which tells us how to convert this into usual pushforward and pullback of cycles. We have

$$(a \otimes \mathrm{id})^*\eta_Y^*[Y] = (a \otimes \mathrm{id})^*[\Delta_Y] = (f \times \mathrm{id})_*\Delta_Y = [\Gamma_f]$$

and the other way around we get

$$(\mathrm{id} \otimes a^t)^*\eta_X^*[X] = (\mathrm{id} \otimes a^t)^*[\Delta_X] = (\mathrm{id} \times f)^![\Delta_X] = [\Gamma_f]$$

The last equality follows from Chow Homology, Lemma 42.59.8. In other words, going either way around the diagram we obtain the element of $\mathrm{Corr}^d(X \times Y, \mathrm{Spec}(k))$ corresponding to the cycle $\Gamma_f \subset X \times Y$. $\square$

45.4. Chow motives

0FG9 We fix a base field k. In this section we construct an additive Karoubian $\mathbf{Q}$-linear category M_k endowed with a symmetric monoidal structure and a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

which maps products to tensor products and disjoint unions to direct sums. Our construction will be characterized by the fact that h factors through the symmetric monoidal category whose objects are smooth projective varieties and whose morphisms are correspondences of degree 0 such that the image of the projector c_2 on $h(\mathbf{P}^1_k)$ from Example 45.3.7 is invertible in M_k, see Lemma 45.4.8. At the end of the section we will show that every motive, i.e., every object of M_k to has a (left) dual, see Lemma 45.4.10.

A motive or a Chow motive over k will be a triple (X, p, m) where

(1) X is a smooth projective scheme over k,
(2) $p \in \mathrm{Corr}^0(X, X)$ satisfies $p \circ p = p$,
(3) $m \in \mathbf{Z}$.

Given a second motive (Y, q, n) we define a morphism of motives or a morphism of Chow motives to be an element of

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q \circ \mathrm{Corr}^{n-m}(X, Y) \circ p \subset \mathrm{Corr}^{n-m}(X, Y)$$

Composition of morphisms of motives is defined using the composition of correspondences defined above.

0FGA Lemma 45.4.1. The category M_k whose objects are motives over k and morphisms are morphisms of motives over k is a $\mathbf{Q}$-linear category. There is a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

defined by $h(X) = (X, 1, 0)$ and $h(f) = [\Gamma_f]$.

Proof. Follows immediately from Lemma 45.3.4. □

0FGB Lemma 45.4.2. The category M_k is Karoubian.

Proof. Let $M = (X, p, m)$ be a motive and let $a \in \text{Mor}(M, M)$ be a projector. Then $a = a \circ a$ both in $\text{Mor}(M, M)$ as well as in $\text{Corr}^0(X, X)$. Set $N = (X, a, m)$. Since we have $a = p \circ a \circ a$ in $\text{Corr}^0(X, X)$ we see that $a : N \to M$ is a morphism of M_k. Next, suppose that $b : (Y, q, n) \to M$ is a morphism such that $(1 - a) \circ b = 0$. Then $b = a \circ b$ as well as $b = b \circ q$. Hence b is a morphism $b : (Y, q, n) \to N$. Thus we see that the projector $1 - a$ has a kernel, namely N and we find that M_k is Karoubian, see Homology, Definition 12.4.1. □

We define a functor

$$\otimes : M_k \times M_k \longrightarrow M_k$$

On objects we use the formula

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

On morphisms, we use

$$\begin{array}{c} \text{Mor}((X, p, m), (Y, q, n)) \times \text{Mor}((X', p', m'), (Y', q', n')) \\ \downarrow \\ \text{Mor}((X \times X', p \otimes p', m + m'), (Y \times Y', q \otimes q', n + n')) \end{array}$$

given by the rule $(a, a') \longmapsto a \otimes a'$ where $\otimes$ on correspondences is as in Section 45.3. This makes sense: by definition of morphisms of motives we can write $a = q \circ c \circ p$ and $a' = q' \circ c' \circ p'$ with $c \in \text{Corr}^{n-m}(X, Y)$ and $c' \in \text{Corr}^{n'-m'}(X', Y')$ and then we obtain

$$a \otimes a' = (q \circ c \circ p) \otimes (q' \circ c' \circ p') = (q \otimes q') \circ (c \otimes c') \circ (p \otimes p')$$

which is indeed a morphism of motives from $(X \times X', p \otimes p', m + m')$ to $(Y \times Y', q \otimes q', n + n')$.

0FGC Lemma 45.4.3. The category M_k with tensor product defined as above is symmetric monoidal with the obvious associativity and commutativity constraints and with unit $\mathbf{1} = (\text{Spec}(k), 1, 0)$.

Proof. Follows readily from Lemma 45.3.8. Details omitted. □

The motives $\mathbf{1}(n) = (\text{Spec}(k), 1, n)$ are useful. Observe that

$$\mathbf{1} = \mathbf{1}(0) \quad \text{and} \quad \mathbf{1}(n + m) = \mathbf{1}(n) \otimes \mathbf{1}(m)$$

Thus tensoring with $\mathbf{1}(1)$ is an autoequivalence of the category of motives. Given a motive M we sometimes write $M(n) = M \otimes \mathbf{1}(n)$. Observe that if $M = (X, p, m)$, then $M(n) = (X, p, m + n)$.

0FGD Lemma 45.4.4. With notation as in Example 45.3.7

(1) the motive $(X, c_0, 0)$ is isomorphic to the motive $\mathbf{1} = (\mathrm{Spec}(k), 1, 0)$.
(2) the motive $(X, c_2, 0)$ is isomorphic to the motive $\mathbf{1}(-1) = (\mathrm{Spec}(k), 1, -1)$.

Proof. We will use Lemma 45.3.4 without further mention. The structure morphism $X \to \mathrm{Spec}(k)$ gives a correspondence $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), X)$. On the other hand, the rational point x is a morphism $\mathrm{Spec}(k) \to X$ which gives a correspondence $b \in \mathrm{Corr}^0(X, \mathrm{Spec}(k))$. We have $b \circ a = 1$ as a correspondence on $\mathrm{Spec}(k)$. The composition $a \circ b$ corresponds to the graph of the composition $X \to x \to X$ which is $c_0 = [x \times X]$. Thus $a = a \circ b \circ a = c_0 \circ a$ and $b = a \circ b \circ a = b \circ c_0$. Hence, unwinding the definitions, we see that a and b are mutually inverse morphisms $a : (\mathrm{Spec}(k), 1, 0) \to (X, c_0, 0)$ and $b : (X, c_0, 0) \to (\mathrm{Spec}(k), 1, 0)$.

We will proceed exactly as above to prove the second statement. Denote

$$a' \in \mathrm{Corr}^1(\mathrm{Spec}(k), X) = \mathrm{CH}^1(X)$$

the class of the point x. Denote

$$b' \in \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k)) = \mathrm{CH}_1(X)$$

the class of $[X]$. We have $b' \circ a' = 1$ as a correspondence on $\mathrm{Spec}(k)$ because $[x] \cdot [X] = [x]$ on $X = \mathrm{Spec}(k) \times X \times \mathrm{Spec}(k)$. Computing the intersection product $\mathrm{pr}_{12}^*b' \cdot \mathrm{pr}_{23}^*a'$ on $X \times \mathrm{Spec}(k) \times X$ gives the cycle $X \times \mathrm{Spec}(k) \times x$. Hence the composition $a' \circ b'$ is equal to c_2 as a correspondence on X. Thus $a' = a' \circ b \circ a' = c_2 \circ a'$ and $b' = b' \circ a' \circ b' = b' \circ c_2$. Recall that

$$\mathrm{Mor}((\mathrm{Spec}(k), 1, -1), (X, c_2, 0)) = c_2 \circ \mathrm{Corr}^1(\mathrm{Spec}(k), X) \subset \mathrm{Corr}^1(\mathrm{Spec}(k), X)$$

and

$$\mathrm{Mor}((X, c_2, 0), (\mathrm{Spec}(k), 1, -1)) = \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k)) \circ c_2 \subset \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k))$$

Hence, we see that a' and b' are mutually inverse morphisms $a' : (\mathrm{Spec}(k), 1, -1) \to (X, c_0, 0)$ and $b' : (X, c_0, 0) \to (\mathrm{Spec}(k), 1, -1)$. $\square$

0FGE Remark 45.4.5 (Lefschetz and Tate motive). Let $X = \mathbf{P}^1_k$ and c_2 be as in Example 45.3.7. In the literature the motive $(X, c_2, 0)$ is sometimes called the Lefschetz motive and depending on the reference the notation L, $\mathbf{L}$, $\mathbf{Q}(-1)$, or $h^2(\mathbf{P}^1_k)$ may be used to denote it. By Lemma 45.4.4 the Lefschetz motive is isomorphic to $\mathbf{1}(-1)$. Hence the Lefschetz motive is invertible (Categories, Definition 4.43.4) with inverse $\mathbf{1}(1)$. The motive $\mathbf{1}(1)$ is sometimes called the Tate motive and depending on the reference the notation L^{-1}, $\mathbf{L}^{-1}$, $\mathbf{T}$, or $\mathbf{Q}(1)$ may be used to denote it.

0FGF Lemma 45.4.6. The category M_k is additive.

Proof. Let (Y, p, m) and (Z, q, n) be motives. If $n = m$, then a direct sum is given by $(Y \amalg Z, p + q, m)$, with obvious notation. Details omitted.

Suppose that $n < m$. Let X, c_2 be as in Example 45.3.7. Then we consider

$$\begin{aligned} (Z, q, n) &= (Z, q, m) \otimes (\mathrm{Spec}(k), 1, -1) \otimes \ldots \otimes (\mathrm{Spec}(k), 1, -1) \\ &\cong (Z, q, m) \otimes (X, c_2, 0) \otimes \ldots \otimes (X, c_2, 0) \\ &\cong (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m) \end{aligned}$$

where we have used Lemma 45.4.4. This reduces us to the case discussed in the first paragraph. $\square$

0FGG Lemma 45.4.7. In M_k we have $h(\mathbf{P}^1_k) \cong \mathbf{1} \oplus \mathbf{1}(-1)$.

Proof. This follows from Example 45.3.7 and Lemma 45.4.4. □

0FGH **Lemma 45.4.8.** Let X, c_2 be as in Example 45.3.7. Let $\mathcal{C}$ be a $\mathbf{Q}$-linear Karoubian symmetric monoidal category. Any $\mathbf{Q}$-linear functor

$$F : \left\{ \begin{matrix} \text{smooth projective schemes over } k \\ \text{morphisms are correspondences of degree } 0 \end{matrix} \right\} \longrightarrow \mathcal{C}$$

of symmetric monoidal categories such that the image of $F(c_2)$ on $F(X)$ is an invertible object, factors uniquely through a functor $F : M_k \to \mathcal{C}$ of symmetric monoidal categories.

Proof. Denote U in $\mathcal{C}$ the invertible object which is assumed to exist in the statement of the lemma. We extend F to motives by setting

$$F(X, p, m) = (\text{the image of the projector } F(p) \text{ in } F(X)) \otimes U^{\otimes -m}$$

which makes sense because U is invertible and because $\mathcal{C}$ is Karoubian. An important feature of this choice is that $F(X, c_2, 0) = U$. Observe that

$$\begin{aligned} F((X, p, m) \otimes (Y, q, n)) &= F(X \times Y, p \otimes q, m + n) \\ &= (\text{the image of } F(p \otimes q) \text{ in } F(X \times Y)) \otimes U^{\otimes -m-n} \\ &= F(X, p, m) \otimes F(Y, q, n) \end{aligned}$$

Thus we see that our rule is compatible with tensor products on the level of objects (details omitted).

Next, we extend F to morphisms of motives. Suppose that

$$a \in \mathrm{Hom}((Y, p, m), (Z, q, n)) = q \circ \mathrm{Corr}^{n-m}(Y, Z) \circ p \subset \mathrm{Corr}^{n-m}(Y, Z)$$

is a morphism. If $n = m$, then a is a correspondence of degree 0 and we can use $F(a) : F(Y) \to F(Z)$ to get the desired map $F(Y, p, m) \to F(Z, q, n)$. If $n < m$ we get canonical identifications

$$\begin{aligned} s : F((Z, q, n)) &\to F(Z, q, m) \otimes U^{m-n} \\ &\to F(Z, q, m) \otimes F(X, c_2, 0) \otimes \ldots \otimes F(X, c_2, 0) \\ &\to F((Z, q, m) \otimes (X, c_2, 0) \otimes \ldots \otimes (X, c_2, 0)) \\ &\to F((Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)) \end{aligned}$$

Namely, for the first isomorphism we use the definition of F on motives above. For the second, we use the choice of U. For the third we use the compatibility of F on tensor products of motives. The fourth is the definition of tensor products on motives. On the other hand, since we similarly have an isomorphism

$$\sigma : (Z, q, n) \to (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)$$

(see proof of Lemma 45.4.6). Composing a with this isomorphism gives

$$\sigma \circ a \in \mathrm{Hom}((Y, p, m), (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m))$$

Putting everything together we obtain

$$s^{-1} \circ F(\sigma \circ a) : F(Y, p, m) \to F(Z, q, n)$$

If $n > m$ we similarly define isomorphisms

$$t : F((Y, p, m)) \to F((Y \times X^{n-m}, p \otimes c_2 \otimes \ldots \otimes c_2, n))$$

and

$$\tau : (Y, p, m)) \to (Y \times X^{n-m}, p \otimes c_2 \otimes \ldots \otimes c_2, n)$$

and we set $F(a) = F(a \circ \tau^{-1}) \circ t$. We omit the verification that this construction defines a functor of symmetric monoidal categories. □

0FGI Lemma 45.4.9. Let X be a smooth projective scheme over k which is equidimensional of dimension d. Then $h(X)(d)$ is a left dual to $h(X)$ in M_k.

Proof. We will use Lemma 45.3.1 without further mention. We compute

$$\mathrm{Hom}(\mathbf{1}, h(X) \otimes h(X)(d)) = \mathrm{Corr}^d(\mathrm{Spec}(k), X \times X) = \mathrm{CH}^d(X \times X)$$

Here we have $\eta = [\Delta]$. On the other hand, we have

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

and here we have the class $\epsilon = [\Delta]$ of the diagonal as well. The composition of the correspondence $[\Delta] \otimes 1$ with $1 \otimes [\Delta]$ either way is the correspondence $[\Delta] = 1$ in $\mathrm{Corr}^0(X, X)$ which proves the required diagrams of Categories, Definition 4.43.5 commute. Namely, observe that

$$[\Delta] \otimes 1 \in \mathrm{Corr}^d(X, X \times X \times X) = \mathrm{CH}^{2d}(X \times X \times X \times X)$$

is given by the class of the cycle $\mathrm{pr}_{23}^{1234,-1}(\Delta) \cap \mathrm{pr}_{14}^{1234,-1}(\Delta)$ with obvious notation. Similarly, the class

$$1 \otimes [\Delta] \in \mathrm{Corr}^{-d}(X \times X \times X, X) = \mathrm{CH}^{2d}(X \times X \times X \times X)$$

is given by the class of the cycle $\mathrm{pr}_{23}^{1234,-1}(\Delta) \cap \mathrm{pr}_{14}^{1234,-1}(\Delta)$. The composition $(1 \otimes [\Delta]) \circ ([\Delta] \otimes 1)$ is by definition the pushforward $\mathrm{pr}_{15,*}^{12345}$ of the intersection product

$$[\mathrm{pr}_{23}^{12345,-1}(\Delta) \cap \mathrm{pr}_{14}^{12345,-1}(\Delta)] \cdot [\mathrm{pr}_{34}^{12345,-1}(\Delta) \cap \mathrm{pr}_{15}^{12345,-1}(\Delta)] = [\text{small diagonal in } X^5]$$

which is equal to Δ as desired. We omit the proof of the formula for the composition in the other order. □

0FGJ Lemma 45.4.10. Every object of M_k has a left dual.

Proof. Let $M = (X, p, m)$ be an object of M_k. Then M is a summand of $(X, 0, m) = h(X)(m)$. By Homology, Lemma 12.17.3 it suffices to show that $h(X)(m) = h(X) \otimes \mathbf{1}(m)$ has a dual. By construction $\mathbf{1}(-m)$ is a left dual of $\mathbf{1}(m)$. Hence it suffices to show that $h(X)$ has a left dual, see Categories, Lemma 4.43.8. Let $X = \coprod X_i$ be the decomposition of X into irreducible components. Then $h(X) = \bigoplus h(X_i)$ and it suffices to show that $h(X_i)$ has a left dual, see Homology, Lemma 12.17.2. This follows from Lemma 45.4.9. □

45.5. Chow groups of motives

0FGK We define the Chow groups of a motive as follows.

0FGL Definition 45.5.1. Let k be a base field. Let $M = (X, p, m)$ be a Chow motive over k. For $i \in \mathbf{Z}$ we define the ith Chow group of M by the formula

$$\mathrm{CH}^i(M) = p\left(\mathrm{CH}^{i+m}(X) \otimes \mathbf{Q}\right)$$

We have $\mathrm{CH}^i(h(X)) = \mathrm{CH}^i(X) \otimes \mathbf{Q}$ if X is a smooth projective scheme over k.

Observe that $\mathrm{CH}^i(-)$ is a functor from M_k to $\mathbf{Q}$-vector spaces. Indeed, if $c : M \to N$ is a morphism of motives $M = (X, p, m)$ and $N = (Y, q, n)$, then c is a correspondence of degree $n - m$ from X to Y and hence pushforward along c (Section 45.3) is a family of maps

$$c_* : \mathrm{CH}^{i+m}(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}^{i+n}(Y) \otimes \mathbf{Q}$$

Since $c = q \circ c \circ p$ by definition of morphisms of motives, we see that indeed we obtain

$$c_* : \mathrm{CH}^i(M) \to \mathrm{CH}^i(N)$$

for all $i \in \mathbf{Z}$. This is compatible with compositions of morphisms of motives by Lemma 45.3.1. This functoriality of Chow groups can also be deduced from the following lemma.

0FGM Lemma 45.5.2. Let k be a base field. The functor $\mathrm{CH}^i(-)$ on the category of motives M_k is representable by $\mathbf{1}(-i)$, i.e., we have

$$\mathrm{CH}^i(M) = \mathrm{Hom}_{M_k}(\mathbf{1}(-i), M)$$

functorially in M in M_k.

Proof. Immediate from the definitions and Lemma 45.3.1. □

The reader can imagine that we can use Lemma 45.5.2, the Yoneda lemma, and the duality in Lemma 45.4.9 to obtain the following.

0FGN Lemma 45.5.3 (Manin). Let k be a base field. Let $c : M \to N$ be a morphism of motives. If for every smooth projective scheme X over k the map $c \otimes 1 : M \otimes h(X) \to N \otimes h(X)$ induces an isomorphism on Chow groups, then c is an isomorphism.

Proof. Any object L of M_k is a summand of $h(X)(m)$ for some smooth projective scheme X over k and some $m \in \mathbf{Z}$. Observe that the Chow groups of $M \otimes h(X)(m)$ are the same as the Chow groups of of $M \otimes h(X)$ up to a shift in degrees. Hence our assumption implies that $c \otimes 1 : M \otimes L \to N \otimes L$ induces an isomorphism on Chow groups for every object L of M_k. By Lemma 45.5.2 we see that

$$\mathrm{Hom}_{M_k}(\mathbf{1}, M \otimes L) \to \mathrm{Hom}_{M_k}(\mathbf{1}, N \otimes L)$$

is an isomorphism for every L. Since every object of M_k has a left dual (Lemma 45.4.10) we conclude that

$$\mathrm{Hom}_{M_k}(K, M) \to \mathrm{Hom}_{M_k}(K, N)$$

is an isomorphism for every object K of M_k, see Categories, Lemma 4.43.6. We conclude by the Yoneda lemma (Categories, Lemma 4.3.5). □

45.6. Projective space bundle formula

0FGP Let k be a base field. Let X be a smooth projective scheme over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Our convention is that the projective bundle associated to $\mathcal{E}$ is the morphism

$$P = \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{p} X$$

over X with $\mathcal{O}_P(1)$ normalized so that $p_*(\mathcal{O}_P(1)) = \mathcal{E}$. Recall that

$$[\Gamma_p] \in \mathrm{Corr}^0(X, P) \subset \mathrm{CH}^*(X \times P) \otimes \mathbf{Q}$$

See Example 45.3.2. For $i = 0, \ldots, r-1$ consider the correspondences

$$c_i = c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^i \cap [\Gamma_p] \in \mathrm{Corr}^i(X, P)$$

We may and do think of c_i as a morphism $h(X)(-i) \to h(P)$.

0FGQ Lemma 45.6.1 (Projective space bundle formula). In the situation above, the map

$$\sum\nolimits_{i=0,\ldots,r-1} c_i : \bigoplus\nolimits_{i=0,\ldots,r-1} h(X)(-i) \longrightarrow h(P)$$

is an isomorphism in the category of motives.

Proof. By Lemma 45.5.3 it suffices to show that our map defines an isomorphism on Chow groups of motives after taking the product with any smooth projective scheme Z. Observe that $P \times Z \to X \times Z$ is the projective bundle associated to the pullback of $\mathcal{E}$ to $X \times Z$. Hence the statement on Chow groups is true by the projective space bundle formula given in Chow Homology, Lemma 42.36.2. Namely, pushforward of cycles along $[\Gamma_p]$ is given by pullback of cycles by p according to Lemma 45.3.6 and Chow Homology, Lemma 42.59.5. Hence pushforward along c_i sends α to $c_1(\mathcal{O}_P(1))^i \cap p^*\alpha$. Some details omitted. $\square$

In the situation above, for $j = 0, \ldots, r-1$ consider the correspondences

$$c'_j = c_1(\mathrm{pr}_1^*\mathcal{O}_P(1))^{r-1-j} \cap [\Gamma_p^t] \in \mathrm{Corr}^{-j}(P, X)$$

For $i, j \in \{0, \ldots, r-1\}$ we have

$$c'_j \circ c_i = \mathrm{pr}_{13,*}\left(c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^{i+r-1-j} \cap (\mathrm{pr}_{12}^*[\Gamma_p] \cdot \mathrm{pr}_{23}^*[\Gamma_p^t])\right)$$

The cycles $\mathrm{pr}_{12}^{-1}\Gamma_p$ and $\mathrm{pr}_{23}^{-1}\Gamma_p^t$ intersect transversally and with intersection equal to the image of $(p, 1, p) : P \to X \times P \times X$. Observe that the fibres of $(p, p) = \mathrm{pr}_{13} \circ (p, 1, p) : P \to X \times X$ have dimension $r - 1$. We immediately conclude $c'_j \circ c_i = 0$ for $i + r - 1 - j < r - 1$, in other words when $i < j$. On the other hand, by the projective space bundle formula (Chow Homology, Lemma 42.36.2) the cycle $c_1(\mathcal{O}_P(1))^{r-1} \cap [P]$ maps to $[X]$ in X. Hence for $i = j$ the pushforward above gives the class of the diagonal and hence we see that

$$c'_i \circ c_i = 1 \in \mathrm{Corr}^0(X, X)$$

for all $i \in \{0, \ldots, r-1\}$. Thus we see that the matrix of the composition

$$\bigoplus h(X)(-i) \xrightarrow{\bigoplus c_i} h(P) \xrightarrow{\bigoplus c'_j} \bigoplus h(X)(-j)$$

is invertible (upper triangular with 1s on the diagonal). We conclude from the projective space bundle formula (Lemma 45.6.1) that also the composition the other way around is invertible, but it seems a bit harder to prove this directly.

0FGR Lemma 45.6.2. Let $p : P \to X$ be as in Lemma 45.6.1. The class $[\Delta_P]$ of the diagonal of P in $\mathrm{CH}^*(P \times P)$ can be written as

$$[\Delta_P] = \left(\sum\nolimits_{i=0,\ldots,r-1} \binom{r-1}{i} c_{r-1-i}(\mathrm{pr}_1^*\mathcal{S}^\vee) \cap c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^i\right) \cap (p \times p)^*[\Delta_X]$$

where $\mathcal{S}$ is the kernel of the canonical surjection $p^*\mathcal{E} \to \mathcal{O}_P(1)$.

Proof. Observe that $(p \times p)^*[\Delta_X] = [P \times_X P]$. Since $\Delta_P \subset P \times_X P \subset P \times P$ and since capping with Chern classes commutes with proper pushforward (Chow Homology, Lemma 42.38.4) it suffices to show that the class of $\Delta_P \subset P \times_X P$ in $\mathrm{CH}^*(P \times_X P)$ is equal to

$$\left(\sum\nolimits_{i=0,\ldots,r-1} \binom{r-1}{i} c_{r-1-i}(q_1^*\mathcal{S}^\vee) \cap c_1(q_2^*\mathcal{O}_P(1))^i\right) \cap [P \times_X P]$$

where $q_i : P \times_X P \to P$, $i = 1, 2$ are the projections. Set $q = p \circ q_1 = p \circ q_2 : P \times_X P \to X$. Consider the maps

$$q_1^*\mathcal{S} \otimes q_2^*\mathcal{O}_P(-1) \to q^*\mathcal{E} \otimes q^*\mathcal{E}^\vee \to \mathcal{O}_{P \times_X P}$$

where the final arrow is the pullback by q of the evaluation map $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathcal{O}_X$. The source of the composition is a module locally free of rank $r-1$ and a local calculation shows that this map vanishes exactly along Δ_P. By Chow Homology, Lemma 42.44.1 the class $[\Delta_P]$ is the top Chern class of the dual

$$q_1^*\mathcal{S}^\vee \otimes q_2^*\mathcal{O}_P(1)$$

The desired result follows from Chow Homology, Lemma 42.39.1. □

45.7. Classical Weil cohomology theories

0FGS In this section we define what we will call a classical Weil cohomology theory. This is exactly what is called a Weil cohomology theory in [Kle68, Section 1.2].

We fix an algebraically closed field k (the base field). In this section variety will mean a variety over k, see Varieties, Section 33.3. We fix a field F of characteristic 0 (the coefficient field). A Weil cohomology theory is given by data (D1), (D2), and (D3) subject to axioms (A), (B), and (C).

The data is given by:

- (D1) A contravariant functor H^* from the category of smooth projective varieties to the category of graded commutative F-algebras.
- (D2) For every smooth projective variety X a group homomorphism $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X)$.
- (D3) For every smooth projective variety X of dimension d a map $\int_X : H^{2d}(X) \to F$.

We make some remarks to explain what this means and to introduce some terminology associated with this.

Remarks on (D1). Given a smooth projective variety X we say that $H^*(X)$ is the cohomology of X. Given a morphism $f : X \to Y$ of smooth projective varieties we denote $f^* : H^*(Y) \to H^*(X)$ the map $H^*(f)$ and we call it the pullback map.

Remarks on (D2). The map γ is called the cycle class map. We say that $\gamma(\alpha)$ is the cohomology class of α. If $Z \subset Y \subset X$ are closed subschemes with Y and X smooth projective varieties and Z integral, then $[Z]$ could mean the class of the cycle $[Z]$ in $\mathrm{CH}^*(Y)$ or in $\mathrm{CH}^*(X)$. In this case the notation $\gamma([Z])$ is ambiguous and the intended meaning has to be deduced from context.

Remarks on (D3). The map $\int_X$ is sometimes called the trace map and is sometimes denoted Tr_X.

The first axiom is often called Poincaré duality

(A) Let X be a smooth projective variety of dimension d. Then
  (a) $\dim_F H^i(X) < \infty$ for all i,
  (b) $H^i(X) \times H^{2d-i}(X) \to H^{2d}(X) \to F$ is a perfect pairing for all i where the final map is the trace map $\int_X$,
  (c) $H^i(X) = 0$ unless $i \in [0, 2d]$, and
  (d) $\int_X : H^{2d}(X) \to F$ is an isomorphism.

Let $f : X \to Y$ be a morphism of smooth projective varieties with $\dim(X) = d$ and $\dim(Y) = e$. Using Poincaré duality we can define a pushforward

$$f_* : H^{2d-i}(X) \longrightarrow H^{2e-i}(Y)$$

as the contragredient of the linear map $f^* : H^i(Y) \to H^i(X)$. In a formula, for $a \in H^{2d-i}(X)$, the element $f_*a \in H^{2e-i}(Y)$ is characterized by

$$\int_X f^*b \cup a = \int_Y b \cup f_*a$$

for all $b \in H^i(Y)$.

0FGT Lemma 45.7.1. Assume given (D1) and (D3) satisfying (A). For $f : X \to Y$ a morphism of smooth projective varieties we have $f_*(f^*b \cup a) = b \cup f_*a$. If $g : Y \to Z$ is a second morphism of smooth projective varieties, then $g_* \circ f_* = (g \circ f)_*$.

Proof. The first equality holds because

$$\int_Y c \cup b \cup f_*a = \int_X f^*c \cup f^*b \cup a = \int_Y c \cup f_*(f^*b \cup a).$$

The second equality holds because

$$\int_Z c \cup (g \circ f)_*a = \int_X (g \circ f)^*c \cup a = \int_X f^*g^*c \cup a = \int_Y g^*c \cup f_*a = \int_Z c \cup g_*f_*a$$

This ends the proof. □

The second axiom says that H^* respects the monoidal structure given by products via the Künneth formula

(B) Let X and Y be smooth projective varieties. The map

$$H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y), \quad a \otimes b \mapsto \text{pr}_1^*a \cup \text{pr}_2^*b$$

is an isomorphism.

The third axiom concerns the cycle class maps

(C) The cycle class maps satisfy the following rules
  (a) for a morphism $f : X \to Y$ of smooth projective varieties we have $\gamma(f^!\beta) = f^*\gamma(\beta)$ for $\beta \in \text{CH}^*(Y)$,
  (b) for a morphism $f : X \to Y$ of smooth projective varieties we have $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ for $\alpha \in \text{CH}^*(X)$,
  (c) for any smooth projective variety X we have $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cup \gamma(\beta)$ for $\alpha, \beta \in \text{CH}^*(X)$, and
  (d) $\int_{\text{Spec}(k)} \gamma([\text{Spec}(k)]) = 1$.

0FGU Remark 45.7.2. Let X be a smooth projective variety. We obtain maps

$$H^*(X) \otimes_F H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where the first arrow is as in axiom (B) and Δ^* is pullback along the diagonal morphism $\Delta : X \to X \times X$. The composition is the cup product as pullback is an algebra homomorphism and $\text{pr}_i \circ \Delta = \text{id}$. On the other hand, given cycles α, β on X the intersection product is defined by the formula

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta)$$

In other words, $\alpha \cdot \beta$ is the pullback of the exterior product $\alpha \times \beta$ on $X \times X$ by the diagonal. Note also that $\alpha \times \beta = \text{pr}_1^*\alpha \cdot \text{pr}_2^*\beta$ in $\text{CH}^*(X \times X)$ (we omit the proof). Hence, given axiom (C)(a), axiom (C)(c) is equivalent to the statement that γ is compatible with exterior product in the sense that $\gamma(\alpha \times \beta)$ is equal to $\text{pr}_1^*\gamma(\alpha) \cup \text{pr}_2^*\gamma(\beta)$. This is how axiom (C)(c) is formulated in [Kle68].

0FGV Definition 45.7.3. Let k be an algebraically closed field. Let F be a field of characteristic 0. A classical Weil cohomology theory over k with coefficients in F is given by data (D1), (D2), and (D3) satisfying Poincaré duality, the Künneth formula, and compatibility with cycle classes, more precisely, satisfying (A), (B), and (C).

We do a tiny bit of work.

0FGW Lemma 45.7.4. Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X be a smooth projective variety of dimension d. The diagram

$$\begin{tikzcd} \text{CH}^d(X) \arrow[r, "\gamma"'] \arrow[d, equal] & H^{2d}(X) \arrow[d, "\int_X"] \\ \text{CH}_0(X) \arrow[r, "\deg"] & F \end{tikzcd}$$

commutes where $\deg : \text{CH}_0(X) \to \mathbf{Z}$ is the degree of zero cycles discussed in Chow Homology, Section 42.41.

Proof. The result holds for $\text{Spec}(k)$ by axiom (C)(d). Let $x : \text{Spec}(k) \to X$ be a closed point of X. Then we have $\gamma([x]) = x_*\gamma([\text{Spec}(k)])$ in $H^{2d}(X)$ by axiom (C)(b). Hence $\int_X \gamma([x]) = 1$ by the definition of x_*. □

0FGX Lemma 45.7.5. Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X and Y be smooth projective varieties. Then $\int_{X \times Y} = \int_X \otimes \int_Y$.

Proof. Say $\dim(X) = d$ and $\dim(Y) = e$. By axiom (B) we have $H^{2d+2e}(X \times Y) = H^{2d}(X) \otimes H^{2e}(Y)$ and by axiom (A)(d) this is 1-dimensional. By Lemma 45.7.4 this 1-dimensional vector space generated by the class $\gamma([x \times y])$ of a closed point (x, y) and $\int_{X \times Y} \gamma([x \times y]) = 1$. Since $\gamma([x \times y]) = \gamma([x]) \otimes \gamma([y])$ by axioms (C)(a) and (C)(c) and since $\int_X \gamma([x]) = 1$ and $\int_Y \gamma([y]) = 1$ we conclude. □

0FGY Lemma 45.7.6. Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X and Y be smooth projective varieties. Then $\text{pr}_{2,*} : H^*(X \times Y) \to H^*(Y)$ sends $a \otimes b$ to $(\int_X a)b$.

Proof. This is equivalent to the result of Lemma 45.7.5. □

0FGZ Lemma 45.7.7. Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X be a smooth projective variety of dimension d. Choose a basis $e_{i,j}, j = 1, \ldots, \beta_i$ of $H^i(X)$ over F. Using Künneth write

$$\gamma([\Delta]) = \sum\nolimits_{i=0,\ldots,2d} \sum\nolimits_j e_{i,j} \otimes e'_{2d-i,j} \quad \text{in} \quad \bigoplus\nolimits_i H^i(X) \otimes_F H^{2d-i}(X)$$

with $e'_{2d-i,j} \in H^{2d-i}(X)$. Then $\int_X e_{i,j} \cup e'_{2d-i,j'} = (-1)^i \delta_{jj'}$.

Proof. Recall that $\Delta^* : H^*(X \times X) \to H^*(X)$ is equal to the cup product map $H^*(X) \otimes_F H^*(X) \to H^*(X)$, see Remark 45.7.2. On the other hand we have $\gamma([\Delta]) = \Delta_*\gamma([X]) = \Delta_* 1$ by axiom (C)(b) and the fact that $\gamma([X]) = 1$. Namely, $[X] \cdot [X] = [X]$ hence by axiom (C)(c) the cohomology class $\gamma([X])$ is 0 or 1 in the 1-dimensional F-algebra $H^0(X)$; here we have also used axioms (A)(d) and (A)(b). But $\gamma([X])$ cannot be zero as $[X] \cdot [x] = [x]$ for a closed point x of X and we have the nonvanishing of $\gamma([x])$ by Lemma 45.7.4. Hence

$$\int_{X \times X} \gamma([\Delta]) \cup a \otimes b = \int_{X \times X} \Delta_* 1 \cup a \otimes b = \int_X a \cup b$$

by the definition of Δ_*. On the other hand, we have

$$\int_{X \times X} (\sum e_{i,j} \otimes e'_{2d-i,j}) \cup a \otimes b = \sum (\int_X a \cup e_{i,j})(\int_X e'_{2d-i,j} \cup b)$$

by Lemma 45.7.5; note that we made two switches of order so that the sign is 1. Thus if we choose a such that $\int_X a \cup e_{i,j} = 1$ and all other pairings equal to zero, then we conclude that $\int_X e'_{2d-i,j} \cup b = \int_X a \cup b$ for all b, i.e., $e'_{2d-i,j} = a$. This proves the lemma. □

0FH0 Lemma 45.7.8. Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X be a smooth projective variety. We have

$$\sum\nolimits_{i=0,\ldots,2\dim(X)} (-1)^i \dim_F H^i(X) = \deg([\Delta] \cdot [\Delta]) = \deg(c_d(\mathcal{T}_X) \cap [X])$$

Proof. Equality on the right. We have $[\Delta] \cdot [\Delta] = \Delta_*(\Delta^![\Delta])$ (Chow Homology, Lemma 42.62.6). Since Δ_* preserves degrees of 0-cycles it suffices to compute the degree of $\Delta^![\Delta]$. The class $\Delta^![\Delta]$ is given by capping $[\Delta]$ with the top Chern class of the normal sheaf of $\Delta \subset X \times X$ (Chow Homology, Lemma 42.54.5). Since the conormal sheaf of Δ is $\Omega_{X/k}$ (Morphisms, Lemma 29.32.7) we see that the normal sheaf is equal to the tangent sheaf $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ as desired.

Equality on the left. By Lemma 45.7.4 we have

$$\begin{aligned} \deg([\Delta] \cdot [\Delta]) &= \int_{X \times X} \gamma([\Delta]) \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_* 1 \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_*(\Delta^*\gamma([\Delta])) \\ &= \int_X \Delta^*\gamma([\Delta]) \end{aligned}$$

Write $\gamma([\Delta]) = \sum e_{i,j} \otimes e'_{2d-i,j}$ as in Lemma 45.7.7. Recalling that Δ^* is given by cup product we obtain

$$\int_X \sum\nolimits_{i,j} e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} \int_X e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} (-1)^i = \sum (-1)^i \beta_i$$

as desired. □

We will now tie classical Weil cohomology theories in with motives as follows.

0FH1 Lemma 45.7.9. Let k be an algebraically closed field. Let F be a field of characteristic 0. Consider a $\mathbf{Q}$-linear functor

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $G(\mathbf{1}(1))$ is nonzero only in degree -2. Then we obtain data (D1), (D2), (D3) satisfying all of (A), (B), (C) except for possibly (A)(c) and (A)(d).

Proof. We obtain a contravariant functor from the category of smooth projective varieties to the category of graded F-vector spaces by setting $H^*(X) = G(h(X))$. By assumption we have a canonical isomorphism

$$H^*(X \times Y) = G(h(X \times Y)) = G(h(X) \otimes h(Y)) = G(h(X)) \otimes G(h(Y)) = H^*(X) \otimes H^*(Y)$$

compatible with pullbacks. Using pullback along the diagonal $\Delta : X \to X \times X$ we obtain a canonical map

$$H^*(X) \otimes H^*(X) = H^*(X \times X) \to H^*(X)$$

of graded vector spaces compatible with pullbacks. This defines a functorial graded F-algebra structure on $H^*(X)$. Since Δ commutes with the commutativity constraint $h(X) \otimes h(X) \to h(X) \otimes h(X)$ (switching the factors) and since G is a functor of symmetric monoidal categories (so compatible with commutativity constraints), and by our convention in Homology, Example 12.17.4 we conclude that $H^*(X)$ is a graded commutative algebra! Hence we get our datum (D1).

Since $\mathbf{1}(1)$ is invertible in the category of motives we see that $G(\mathbf{1}(1))$ is invertible in the category of graded F-vector spaces. Thus $\sum_i \dim_F G^i(\mathbf{1}(1)) = 1$. By assumption we only get something nonzero in degree -2 and we may choose an isomorphism $F[2] \to G(\mathbf{1}(1))$ of graded F-vector spaces. Here and below $F[n]$ means the graded F-vector space which has F in degree $-n$ and zero elsewhere. Using compatibility with tensor products, we find for all $n \in \mathbf{Z}$ an isomorphism $F[2n] \to G(\mathbf{1}(n))$ compatible with tensor products.

Let X be a smooth projective variety. By Lemma 45.3.1 we have

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(\mathrm{Spec}(k), X) = \mathrm{Hom}(\mathbf{1}(-r), h(X))$$

Applying the functor G we obtain

$$\gamma : \mathrm{CH}^r(X) \otimes \mathbf{Q} \longrightarrow \mathrm{Hom}(G(\mathbf{1}(-r)), H^*(X)) = H^{2r}(X)$$

This is the datum (D2).

Let X be a smooth projective variety of dimension d. By Lemma 45.3.1 we have

$$\mathrm{Mor}(h(X)(d), \mathbf{1}) = \mathrm{Mor}((X, 1, d), (\mathrm{Spec}(k), 1, 0)) = \mathrm{Corr}^{-d}(X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X)$$

Thus the class of the cycle $[X]$ in $\mathrm{CH}_d(X)$ defines a morphism $h(X)(d) \to \mathbf{1}$. Applying G we obtain

$$H^*(X) \otimes F[-2d] = G(h(X)(d)) \longrightarrow G(\mathbf{1}) = F$$

This map is zero except in degree 0 where we obtain $\int_X : H^{2d}(X) \to F$. This is the datum (D3).

Let X be a smooth projective variety of dimension d. By Lemma 45.4.9 we know that $h(X)(d)$ is a left dual to $h(X)$. Hence $G(h(X)(d)) = H^*(X) \otimes F[-2d]$ is a left dual to $H^*(X)$ in the category of graded F-vector spaces. By Homology, Lemma 12.17.5 we find that $\sum_i \dim_F H^i(X) < \infty$ and that $\epsilon : h(X)(d) \otimes h(X) \to \mathbf{1}$

produces nondegenerate pairings $H^{2d-i}(X) \otimes_F H^i(X) \to F$. In the proof of Lemma 45.4.9 we have seen that ϵ is given by $[\Delta]$ via the identifications

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

Thus ϵ is the composition of $[X] : h(X)(d) \to \mathbf{1}$ and $h(\Delta)(d) : h(X)(d) \otimes h(X) \to h(X)(d)$. It follows that the pairings above are given by cup product followed by $\int_X$. This proves axiom (A) parts (a) and (b).

Axiom (B) follows from the assumption that G is compatible with tensor structures and our construction of the cup product above.

Axiom (C). Our construction of γ takes a cycle α on X, interprets it as a correspondence a from $\mathrm{Spec}(k)$ to X of some degree, and then applies G. If $f : Y \to X$ is a morphism of smooth projective varieties, then $f^!\alpha$ is the pushforward (!) of α by the correspondence $[\Gamma_f]$ from X to Y, see Lemma 45.3.6. Hence $f^!\alpha$ viewed as a correspondence from $\mathrm{Spec}(k)$ to Y is equal to $a \circ [\Gamma_f]$, see Lemma 45.3.1. Since G is a functor, we conclude γ is compatible with pullbacks, i.e., axiom (C)(a) holds.

Let $f : Y \to X$ be a morphism of smooth projective varieties and let $\beta \in \mathrm{CH}^r(Y)$ be a cycle on Y. We have to show that

$$\int_Y \gamma(\beta) \cup f^*c = \int_X \gamma(f_*\beta) \cup c$$

for all $c \in H^*(X)$. Let $a, a^t, \eta_X, \eta_Y, [X], [Y]$ be as in Lemma 45.3.9. Let b be β viewed as a correspondence from $\mathrm{Spec}(k)$ to Y of degree r. Then $f_*\beta$ viewed as a correspondence from $\mathrm{Spec}(k)$ to X is equal to $a^t \circ b$, see Lemmas 45.3.6 and 45.3.1. The displayed equality above holds if we can show that

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{b \otimes 1} h(Y)(r) \otimes h(X) \xrightarrow{1 \otimes a} h(Y)(r) \otimes h(Y) \xrightarrow{\eta_Y} h(Y)(r) \xrightarrow{[Y]} \mathbf{1}(r-e)$$

is equal to

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{a^t \circ b \otimes 1} h(X)(r+d-e) \otimes h(X) \xrightarrow{\eta_X} h(X)(r+d-e) \xrightarrow{[X]} \mathbf{1}(r-e)$$

This follows immediately from Lemma 45.3.9. Thus we have axiom (C)(b).

To prove axiom (C)(c) we use the discussion in Remark 45.7.2. Hence it suffices to prove that γ is compatible with exterior products. Let X, Y be smooth projective varieties and let α, β be cycles on them. Denote a, b the corresponding correspondences from $\mathrm{Spec}(k)$ to X, Y. Then $\alpha \times \beta$ corresponds to the correspondence $a \otimes b$ from $\mathrm{Spec}(k)$ to $X \otimes Y = X \times Y$. Hence the requirement follows from the fact that G is compatible with the tensor structures on both sides.

Axiom (C)(d) follows because the cycle $[\mathrm{Spec}(k)]$ corresponds to the identity morphism on $h(\mathrm{Spec}(k))$. This finishes the proof of the lemma. □

0FH2 **Lemma 45.7.10.** Let k be an algebraically closed field. Let F be a field of characteristic 0. Let H^* be a classical Weil cohomology theory. Then we can construct a $\mathbf{Q}$-linear functor

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $H^*(X) = G(h(X))$.

Proof. By Lemma 45.4.8 it suffices to construct a functor G on the category of smooth projective schemes over k with morphisms given by correspondences of degree 0 such that the image of $G(c_2)$ on $G(\mathbf{P}^1)$ is an invertible graded F-vector space. Since every smooth projective scheme is canonically a disjoint union of smooth projective varieties, it suffices to construct G on the category whose objects are smooth projective varieties and whose morphisms are correspondences of degree 0. (Some details omitted.)

Given a smooth projective variety X we set $G(X) = H^*(X)$.

Given a correspondence $c \in \text{Corr}^0(X, Y)$ between smooth projective varieties we consider the map $G(c) : G(X) = H^*(X) \to G(Y) = H^*(Y)$ given by the rule

$$a \longmapsto G(c)(a) = \text{pr}_{2,*}(\gamma(c) \cup \text{pr}_1^* a)$$

It is clear that $G(c)$ is additive in c and hence $\mathbf{Q}$-linear. Compatibility of γ with pullbacks, pushforwards, and intersection products given by axioms (C)(a), (C)(b), and (C)(c) shows that we have $G(c' \circ c) = G(c') \circ G(c)$ if $c' \in \text{Corr}^0(Y, Z)$. Namely, for $a \in H^*(X)$ we have

$$\begin{aligned}
(G(c') \circ G(c))(a) &= \text{pr}^{23}_{3,*}(\gamma(c') \cup \text{pr}_2^{23,*}(\text{pr}^{12}_{2,*}(\gamma(c) \cup \text{pr}_1^{12,*} a))) \\
&= \text{pr}^{23}_{3,*}(\gamma(c') \cup \text{pr}^{123}_{23,*}(\text{pr}_{12}^{123,*}(\gamma(c) \cup \text{pr}_1^{12,*} a))) \\
&= \text{pr}^{23}_{3,*}\text{pr}^{123}_{23,*}(\text{pr}_{23}^{123,*}\gamma(c') \cup \text{pr}_{12}^{123,*}\gamma(c) \cup \text{pr}_1^{123,*} a) \\
&= \text{pr}^{23}_{3,*}\text{pr}^{123}_{23,*}(\gamma(\text{pr}_{23}^{123,*}c') \cup \gamma(\text{pr}_{12}^{123,*}c) \cup \text{pr}_1^{123,*} a) \\
&= \text{pr}^{13}_{3,*}\text{pr}^{123}_{13,*}(\gamma(\text{pr}_{23}^{123,*}c' \cdot \text{pr}_{12}^{123,*}c) \cup \text{pr}_1^{123,*} a) \\
&= \text{pr}^{13}_{3,*}(\gamma(\text{pr}^{123}_{13,*}(\text{pr}_{23}^{123,*}c' \cdot \text{pr}_{12}^{123,*}c)) \cup \text{pr}_1^{13,*} a) \\
&= G(c' \circ c)(a)
\end{aligned}$$

with obvious notation. The first equality follows from the definitions. The second equality holds because $\text{pr}_2^{23,*} \circ \text{pr}^{12}_{2,*} = \text{pr}^{123}_{23,*} \circ \text{pr}_{12}^{123,*}$ as follows immediately from the description of pushforward along projections given in Lemma 45.7.6. The third equality holds by Lemma 45.7.1 and the fact that H^* is a functor. The fourth equalith holds by axiom (C)(a) and the fact that the gysin map agrees with flat pullback for flat morphisms (Chow Homology, Lemma 42.59.5). The fifth equality uses axiom (C)(c) as well as Lemma 45.7.1 to see that $\text{pr}^{23}_{3,*} \circ \text{pr}^{123}_{23,*} = \text{pr}^{13}_{3,*} \circ \text{pr}^{123}_{13,*}$. The sixth equality uses the projection formula from Lemma 45.7.1 as well as axiom (C)(b) to see that $\text{pr}^{123}_{13,*}\gamma(\text{pr}_{23}^{123,*}c' \cdot \text{pr}_{12}^{123,*}c) = \gamma(\text{pr}^{123}_{13,*}(\text{pr}_{23}^{123,*}c' \cdot \text{pr}_{12}^{123,*}c))$. Finally, the last equality is the definition.

To finish the proof that G is a functor, we have to show identities are preserved. In other words, if $1 = [\Delta] \in \text{Corr}^0(X, X)$ is the identity in the category of correspondences (see Lemma 45.3.3 and its proof), then we have to show that $G([\Delta]) = \text{id}$. This follows from the determination of $\gamma([\Delta])$ in Lemma 45.7.7 and Lemma 45.7.6. This finishes the construction of G as a functor on smooth projective varieties and correspondences of degree 0.

It follows from axioms (A)(c) and (A)(d) that $G(\text{Spec}(k)) = H^*(\text{Spec}(k))$ is canonically isomorphic to F as an F-algebra. The Künneth axiom (B) shows our functor is compatible with tensor products. Thus our functor is a functor of symmetric monoidal categories.

We still have to check that the image of $G(c_2)$ on $G(\mathbf{P}^1)$ is an invertible graded F-vector space (in particular we don't know yet that G extends to M_k). By axiom (A)(d) the map $\int_{\mathbf{P}^1} : H^2(\mathbf{P}^1) \to F$ is an isomorphism. By axiom (A)(b) we see that $\dim_F H^0(\mathbf{P}^1) = 1$. By Lemma 45.7.8 and axiom (A)(c) we obtain $2 - \dim_F H^1(\mathbf{P}^1) = c_1(T_{\mathbf{P}^1}) = 2$. Hence $H^1(\mathbf{P}^1) = 0$. Thus

$$G(\mathbf{P}^1) = H^0(\mathbf{P}^1) \oplus H^2(\mathbf{P}^1)$$

Recall that $1 = c_0 + c_2$ is a decomposition of the identity into a sum of orthogonal idempotents in $\mathrm{Corr}^0(\mathbf{P}^1, \mathbf{P}^1)$, see Example 45.3.7. We have $c_0 = a \circ b$ where $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), \mathbf{P}^1)$ and $b \in \mathrm{Corr}^0(\mathbf{P}^1, \mathrm{Spec}(k))$ and where $b \circ a = 1$ in $\mathrm{Corr}^0(\mathrm{Spec}(k), \mathrm{Spec}(k))$, see proof of Lemma 45.4.4. Since $F = G(\mathrm{Spec}(k))$, it follows from functoriality that $G(c_0)$ is the projector onto the summand $H^0(\mathbf{P}^1) \subset G(\mathbf{P}^1)$. Hence $G(c_2)$ must necessarily be the projection onto $H^2(\mathbf{P}^1)$ and the proof is complete. □

0FH3 Proposition 45.7.11. Let k be an algebraically closed field. Let F be a field of characteristic 0. A classical Weil cohomology theory is the same thing as a $\mathbf{Q}$-linear functor

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories together with an isomorphism $F[2] \to G(\mathbf{1}(1))$ of graded F-vector spaces such that in addition

(1) $G(h(X))$ lives in nonnegative degrees, and
(2) $\dim_F G^0(h(X)) = 1$

for any smooth projective variety X.

Proof. Given G and $F[2] \to G(\mathbf{1}(1))$ by setting $H^*(X) = G(h(X))$ we obtain data (D1), (D2), and (D3) satisfying all of (A), (B), and (C) except for possibly (A)(c) and (A)(d), see Lemma 45.7.9 and its proof. Observe that assumptions (1) and (2) imply axioms (A)(c) and (A)(d) in the presence of the known axioms (A)(a) and (A)(b).

Conversely, given H^* we get a functor G by the construction of Lemma 45.7.10. Let $X = \mathbf{P}^1, c_0, c_2$ be as in Example 45.3.7. We have constructed an isomorphism $1(-1) \to (X, c_2, 0)$ of motives in Lemma 45.4.4. In the proof of Lemma 45.7.10 we have seen that $G(1(-1)) = G(X, c_2, 0) = H^2(\mathbf{P}^1)[-2]$. Hence the isomorphism $\int_{\mathbf{P}^1} : H^2(\mathbf{P}^1) \to F$ of axiom (A)(d) gives an isomorphism $G(1(-1)) \to F[-2]$ which determines an isomorphism $F[2] \to G(\mathbf{1}(1))$. Finally, since $G(h(X)) = H^*(X)$ assumptions (1) and (2) follow from axiom (A). □

45.8. Cycles over non-closed fields

0FH4 Some lemmas which will help us in our study of motives over base fields which are not algebraically closed.

0FH5 Lemma 45.8.1. Let k be a field. Let X be a smooth projective scheme over k. Then $\mathrm{CH}_0(X)$ is generated by classes of closed points whose residue fields are separable over k.

Proof. The lemma is immediate if k has characteristic 0 or is perfect. Thus we may assume k is an infinite field of characteristic $p > 0$.

We may assume X is irreducible of dimension d. Then $k' = H^0(X, \mathcal{O}_X)$ is a finite separable field extension of k and that X is geometrically integral over k'. See Varieties, Lemmas 33.25.4, 33.9.3, and 33.9.4. We may and do replace k by k' and assume that X is geometrically integral.

Let $x \in X$ be a closed point. To prove the lemma we are going to show that $[x] \in \mathrm{CH}_0(X)$ is rationally equivalent to an integer linear combination of classes of closed points whose residue fields are separable over k. Choose an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. Set

$$V = \{s \in H^0(X, \mathcal{L}) \mid s(x) = 0\}$$

After replacing $\mathcal{L}$ by a power we may assume (a) $\mathcal{L}$ is very ample, (b) V generates $\mathcal{L}$ over $X \setminus x$, (c) the morphism $X \setminus x \to \mathbf{P}(V)$ is an immersion, (d) the map $V \to \mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$ is surjective, see Morphisms, Lemma 29.39.5, Varieties, Lemma 33.47.1, and Properties, Proposition 28.26.13. Consider the set

$$V^d \supset U = \{(s_1, \ldots, s_d) \in V^d \mid s_1, \ldots, s_d \text{ generate } \mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x \text{ over } \kappa(x)\}$$

Since $\mathcal{O}_{X,x}$ is a regular local ring of dimension d we have $\dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2) = d$ and hence we see that U is a nonempty (Zariski) open of V^d. For $(s_1, \ldots, s_d) \in U$ set $H_i = Z(s_i)$. Since $s_1, \ldots, s_d$ generate $\mathfrak{m}_x\mathcal{L}_x$ we see that

$$H_1 \cap \ldots \cap H_d = x \amalg Z$$

scheme theoretically for some closed subscheme $Z \subset X$. By Bertini (in the form of Varieties, Lemma 33.47.3) for a general element $s_1 \in V$ the scheme $H_1 \cap (X \setminus x)$ is smooth over k of dimension $d-1$. Having chosen s_1, for a general element $s_2 \in V$ the scheme $H_1 \cap H_2 \cap (X \setminus x)$ is smooth over k of dimension $d-2$. And so on. We conclude that for sufficiently general $(s_1, \ldots, s_d) \in U$ the scheme Z is étale over $\mathrm{Spec}(k)$. In particular $H_1 \cap \ldots \cap H_d$ has dimension 0 and hence

$$[H_1] \cdot \ldots \cdot [H_d] = [x] + [Z]$$

in $\mathrm{CH}_0(X)$ by repeated application of Chow Homology, Lemma 42.62.5 (details omitted). This finishes the proof as it shows that $[x] \sim_{rat} -[Z] + [Z']$ where $Z' = H'_1 \cap \ldots \cap H'_d$ is a general complete intersection of vanishing loci of sufficiently general sections of $\mathcal{L}$ which will be étale over k by the same argument as before. $\square$

0FH6 Lemma 45.8.2. Let K/k be an algebraic field extension. Let X be a finite type scheme over k. Then $\mathrm{CH}_i(X_K) = \operatorname{colim} \mathrm{CH}_i(X_{k'})$ where the colimit is over the subextensions $K/k'/k$ with k'/k finite.

Proof. This is a special case of Chow Homology, Lemma 42.67.10. $\square$

0FH7 Lemma 45.8.3. Let k be a field. Let X be a geometrically irreducible smooth projective scheme over k. Let $x, x' \in X$ be k-rational points. Let n be an integer invertible in k. Then there exists a finite separable extension k'/k such that the pullback of $[x] - [x']$ to $X_{k'}$ is divisible by n in $\mathrm{CH}_0(X_{k'})$.

Proof. Let k' be a separable algebraic closure of k. Suppose that we can show the the pullback of $[x] - [x']$ to $X_{k'}$ is divisible by n in $\mathrm{CH}_0(X_{k'})$. Then we conclude by Lemma 45.8.2. Thus we may and do assume k is separably algebraically closed.

Suppose $\dim(X) > 1$. Let $\mathcal{L}$ be an ample invertible sheaf on X. Set

$$V = \{s \in H^0(X, \mathcal{L}) \mid s(x) = 0 \text{ and } s(x') = 0\}$$

After replacing $\mathcal{L}$ by a power we see that for a general $v \in V$ the corresponding divisor $H_v \subset X$ is smooth away from x and x', see Varieties, Lemmas 33.47.1 and 33.47.3. To find v we use that k is infinite (being separably algebraically closed). If we choose s general, then the image of s in $\mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$ will be nonzero, which implies that H_v is smooth at x (details omitted). Similarly for x'. Thus H_v is smooth. By Varieties, Lemma 33.48.3 (applied to the base change of everything to the algebraic closure of k) we see that H_v is geometrically connected. It suffices to prove the result for $[x] - [x']$ seen as an element of $\mathrm{CH}_0(H_v)$. In this way we reduce to the case of a curve.

Assume X is a curve. Then we see that $\mathcal{O}_X(x - x')$ defines a k-rational point g of $J = \underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 44.6.7. Recall that J is a proper smooth variety over k which is also a group scheme over k (same reference). Hence J is geometrically integral (see Varieties, Lemma 33.7.13 and 33.25.4). In other words, J is an abelian variety, see Groupoids, Definition 39.9.1. Thus $[n] : J \to J$ is finite étale by Groupoids, Proposition 39.9.11 (this is where we use n is invertible in k). Since k is separably closed we conclude that $g = [n](g')$ for some $g' \in J(k)$. If $\mathcal{L}$ is the degree 0 invertible module on X corresponding to g', then we conclude that $\mathcal{O}_X(x - x') \cong \mathcal{L}^{\otimes n}$ as desired. □

0FH8 Lemma 45.8.4. Let K/k be an algebraic extension of fields. Let X be a finite type scheme over k. The kernel of the map $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_K)$ constructed in Lemma 45.8.2 is torsion.

Proof. It clearly suffices to show that the kernel of flat pullback $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_{k'})$ by $\pi : X_{k'} \to X$ is torsion for any finite extension k'/k. This is clear because $\pi_*\pi^*\alpha = [k' : k]\alpha$ by Chow Homology, Lemma 42.15.2. □

0FH9 Lemma 45.8.5 (Voevodsky). Let k be a field. Let X be a geometrically irreducible smooth projective scheme over k. Let $x, x' \in X$ be k-rational points. For n large enough the class of the zero cycle [Voe95]

$$([x] - [x']) \times \ldots \times ([x] - [x']) \in \mathrm{CH}_0(X^n)$$

is torsion.

Proof. If we can show this after base change to the algebraic closure of k, then the result follows over k because the kernel of pullback is torsion by Lemma 45.8.4. Hence we may and do assume k is algebraically closed.

Using Bertini we can choose a smooth curve $C \subset X$ passing through x and x'. See proof of Lemma 45.8.3. Hence we may assume X is a curve.

Assume X is a curve and k is algebraically closed. Write $S^n(X) = \underline{\mathrm{Hilb}}^n_{X/k}$ with notation as in Picard Schemes of Curves, Sections 44.2 and 44.3. There is a canonical morphism

$$\pi : X^n \longrightarrow S^n(X)$$

which sends the k-rational point $(x_1, \ldots, x_n)$ to the k-rational point corresponding to the divisor $[x_1] + \ldots + [x_n]$ on X. There is a faithful action of the symmetric group S_n on X^n. The morphism π is S_n-invariant and the fibres of π are S_n-orbits (set theoretically). Finally, π is finite flat of degree $n!$, see Picard Schemes of Curves, Lemma 44.3.4.

Let α_n be the zero cycle on X^n given by the formula in the statement of the lemma. Let $\mathcal{L} = \mathcal{O}_X(x - x')$. Then $c_1(\mathcal{L}) \cap [X] = [x] - [x']$. Thus

$$\alpha_n = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_n) \cap [X^n]$$

where $\mathcal{L}_i = \text{pr}_i^*\mathcal{L}$ and $\text{pr}_i : X^n \to X$ is the ith projection. By either Divisors, Lemma 31.17.6 or Divisors, Lemma 31.17.7 there is a norm for π. Set $\mathcal{N} = \text{Norm}_\pi(\mathcal{L}_1)$, see Divisors, Lemma 31.17.2. We have

$$\pi^*\mathcal{N} = (\mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n)^{\otimes(n-1)!}$$

in $\text{Pic}(X^n)$ by a calculation. Deails omitted; hint: this follows from the fact that $\text{Norm}_\pi : \pi_*\mathcal{O}_{X^n} \to \mathcal{O}_{S^n(X)}$ composed with the natural map $\pi_*\mathcal{O}_{S^n(X)} \to \mathcal{O}_{X^n}$ is equal to the product over all $\sigma \in S_n$ of the action of σ on $\pi_*\mathcal{O}_{X^n}$. Consider

$$\beta_n = c_1(\mathcal{N})^n \cap [S^n(X)]$$

in $\text{CH}_0(S^n(X))$. Observe that $c_1(\mathcal{L}_i) \cap c_1(\mathcal{L}_i) = 0$ because $\mathcal{L}_i$ is pulled back from a curve, see Chow Homology, Lemma 42.34.6. Thus we see that

$$\begin{aligned}
\pi^*\beta_n &= ((n-1)!)^n (\sum\nolimits_{i=1,\ldots,n} c_1(\mathcal{L}_i))^n \cap [X^n] \\
&= ((n-1)!)^n n^n c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_n) \cap [X^n] \\
&= (n!)^n \alpha_n
\end{aligned}$$

Thus it suffices to show that β_n is torsion.

There is a canonical morphism

$$f : S^n(X) \longrightarrow \underline{\text{Pic}}^n_{X/k}$$

See Picard Schemes of Curves, Lemma 44.6.7. For $n \geq 2g - 1$ this morphism is a projective space bundle (details omitted; compare with the proof of Picard Schemes of Curves, Lemma 44.6.7). The invertible sheaf $\mathcal{N}$ is trivial on the fibres of f, see below. Thus by the projective space bundle formula (Chow Homology, Lemma 42.36.2) we see that $\mathcal{N} = f^*\mathcal{M}$ for some invertible module $\mathcal{M}$ on $\underline{\text{Pic}}^n_{X/k}$. Of course, then we see that

$$c_1(\mathcal{N})^n = f^*(c_1(\mathcal{M})^n)$$

is zero because $n > g = \dim(\underline{\text{Pic}}^n_{X/k})$ and we can use Chow Homology, Lemma 42.34.6 as before.

We still have to show that $\mathcal{N}$ is trivial on a fibre F of f. Since the fibres of f are projective spaces and since $\text{Pic}(\mathbf{P}^m_k) = \mathbf{Z}$ (Divisors, Lemma 31.28.5), this can be shown by computing the degree of $\mathcal{N}$ on a line contained in the fibre. Instead we will prove it by proving that $\mathcal{N}$ is algebraically equivalent to zero. First we claim there is a connected finite type scheme T over k, an invertible module $\mathcal{L}'$ on $T \times X$ and k-rational points $p, q \in T$ such that $\mathcal{M}_p \cong \mathcal{O}_X$ and $\mathcal{M}_q = \mathcal{L}$. Namely, since $\mathcal{L} = \mathcal{O}_X(x - x')$ we can take $T = X$, $p = x'$, $q = x$, and $\mathcal{L}' = \mathcal{O}_{X \times X}(\Delta) \otimes \text{pr}_2^*\mathcal{O}_X(-x')$. Then we let $\mathcal{L}'_i$ on $T \times X^n$ for $i = 1, \ldots, n$ be the pullback of $\mathcal{L}'$ by $\text{id}_T \times \text{pr}_i : T \times X^n \to T \times X$. Finally, we let $\mathcal{N}' = \text{Norm}_{\text{id}_T \times \pi}(\mathcal{L}'_1)$ on $T \times S^n(X)$. By construction we have $\mathcal{N}'_p = \mathcal{O}_{S^n(X)}$ and $\mathcal{N}'_q = \mathcal{N}$. We conclude that

$$\mathcal{N}'|_{T \times F}$$

is an invertible module on $T \times F \cong T \times \mathbf{P}^m_k$ whose fibre over p is the trivial invertible module and whose fibre over q is $\mathcal{N}|_F$. Since the euler characteristic of the trivial bundle is 1 and since this euler characteristic is locally constant in families (Derived Categories of Schemes, Lemma 36.32.2) we conclude $\chi(F, \mathcal{N}^{\otimes s}|_F) = 1$ for all $s \in \mathbf{Z}$. This can happen only if $\mathcal{N}|_F \cong \mathcal{O}_F$ (see Cohomology of Schemes, Lemma 30.8.1) and the proof is complete. Some details omitted. □

45.9. Weil cohomology theories, I

0FHA This section is the analogue of Section 45.7 over arbitrary fields. In other words, we work out what data and axioms correspond to functors G of symmetric monoidal categories from the category of motives to the category of graded vector spaces such that $G(\mathbf{1}(1))$ sits in degree -2. In Section 45.11 we will define a Weil cohomology theory by adding a single suplementary condition.

We fix a field k (the base field). We fix a field F of characteristic 0 (the coefficient field). The data is given by:

- (D0) A 1-dimensional F-vector space $F(1)$.
- (D1) A contravariant functor H^* from the category of smooth projective schemes over k to the category of graded commutative F-algebras.
- (D2) For every smooth projective scheme X over k a group homomorphism $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X)(i)$.
- (D3) For every nonempty smooth projective scheme X over k which is equidimensional of dimension d a map $\int_X : H^{2d}(X)(d) \to F$.

We make some remarks to explain what this means and to introduce some terminology associated with this.

Remarks on (D0). The vector space $F(1)$ gives rise to Tate twists on the category of F-vector spaces. Namely, for $n \in \mathbf{Z}$ we set $F(n) = F(1)^{\otimes n}$ if $n \geq 0$, we set $F(-1) = \mathrm{Hom}_F(F(1), F)$, and we set $F(n) = F(-1)^{\otimes -n}$ if $n < 0$. Please compare with More on Algebra, Section 15.117. For an F-vector space V we define the nth Tate twist

$$V(n) = V \otimes_F F(n)$$

We will use obvious notation, e.g., given F-vector spaces U, V and W and a linear map $U \otimes_F V \to W$ we obtain a linear map $U(n) \otimes_F V(m) \to W(n+m)$ for $n, m \in \mathbf{Z}$.

Remarks on (D1). Given a smooth projective scheme X over k we say that $H^*(X)$ is the cohomology of X. Given a morphism $f : X \to Y$ of smooth projective schemes over k we denote $f^* : H^*(Y) \to H^*(X)$ the map $H^*(f)$ and we call it the pullback map.

Remarks on (D2). The map γ is called the cycle class map. We say that $\gamma(\alpha)$ is the cohomology class of α. If $Z \subset Y \subset X$ are closed subschemes with Y and X smooth projective over k and Z integral, then $[Z]$ could mean the class of the cycle $[Z]$ in $\mathrm{CH}^*(Y)$ or in $\mathrm{CH}^*(X)$. In this case the notation $\gamma([Z])$ is ambiguous and the intended meaning has to be deduced from context.

Remarks on (D3). The map $\int_X$ is sometimes called the trace map and is sometimes denoted Tr_X.

The first axiom is often called Poincaré duality

- (A) Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. Then
  - (a) $\dim_F H^i(X) < \infty$ for all i,

(b) $H^i(X) \times H^{2d-i}(X)(d) \to H^{2d}(X)(d) \to F$ is a perfect pairing for all i where the final map is the trace map $\int_X$.

Let $f : X \to Y$ be a morphism of nonempty smooth projective schemes with X equidimensional of dimension d and Y is equidimensional of dimension e. Using Poincaré duality we can define a pushforward

$$f_* : H^{2d-i}(X)(d) \longrightarrow H^{2e-i}(Y)(e)$$

as the contragredient of the linear map $f^* : H^i(Y) \to H^i(X)$. In a formula, for $a \in H^{2d-i}(X)(d)$, the element $f_*a \in H^{2e-i}(Y)(e)$ is characterized by

$$\int_X f^*b \cup a = \int_Y b \cup f_*a$$

for all $b \in H^i(Y)$.

0FHB Lemma 45.9.1. Assume given (D0), (D1), and (D3) satisfying (A). For $f : X \to Y$ a morphism of nonempty equidimensional smooth projective schemes over k we have $f_*(f^*b \cup a) = b \cup f_*a$. If $g : Y \to Z$ is a second morphism with Z nonempty smooth projective and equidimensional, then $g_* \circ f_* = (g \circ f)_*$.

Proof. The first equality holds because

$$\int_Y c \cup b \cup f_*a = \int_X f^*c \cup f^*b \cup a = \int_Y c \cup f_*(f^*b \cup a).$$

The second equality holds because

$$\int_Z c \cup (g \circ f)_*a = \int_X (g \circ f)^*c \cup a = \int_X f^*g^*c \cup a = \int_Y g^*c \cup f_*a = \int_Z c \cup g_*f_*a$$

This ends the proof. □

The second axiom says that H^* respects the monoidal structure given by products via the Künneth formula

(B) Let X and Y be smooth projective schemes over k.
- (a) $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $\alpha \otimes \beta \mapsto \text{pr}_1^*\alpha \cup \text{pr}_2^*\beta$ is an isomorphism,
- (b) if X and Y are nonempty and equidimensional, then $\int_{X \times Y} = \int_X \otimes \int_Y$ via (a).

Using axiom (B)(b) we can compute pushforwards along projections.

0FHC Lemma 45.9.2. Assume given (D0), (D1), and (D3) satisfying (A) and (B). Let X and Y be nonempty smooth projective schemes over k equidimensional of dimensions d and e. Then $\text{pr}_{2,*} : H^*(X \times Y)(d+e) \to H^*(Y)(e)$ sends $a \otimes b$ to $(\int_X a)b$.

Proof. This follows from axioms (B)(a) and (B)(b). □

The third axiom concerns the cycle class maps

(C) The cycle class maps satisfy the following rules
- (a) for a morphism $f : X \to Y$ of smooth projective schemes over k we have $\gamma(f^!\beta) = f^*\gamma(\beta)$ for $\beta \in \text{CH}^*(Y)$,
- (b) for a morphism $f : X \to Y$ of nonempty equidimensional smooth projective schemes over k we have $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ for $\alpha \in \text{CH}^*(X)$,

(c) for any smooth projective scheme X over k we have $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cup \gamma(\beta)$ for $\alpha, \beta \in \mathrm{CH}^*(X)$, and
(d) $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$.

Let us elucidate axiom (C)(b). Namely, say $f : X \to Y$ is as in (C)(b) with $\dim(X) = d$ and $\dim(Y) = e$. Then we see that pushforward on Chow groups gives

$$f_* : \mathrm{CH}^{d-i}(X) = \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y) = \mathrm{CH}^{e-i}(Y)$$

Say $\alpha \in \mathrm{CH}^{d-i}(X)$. On the one hand, we have $f_*\alpha \in \mathrm{CH}^{e-i}(Y)$ and hence $\gamma(f_*\alpha) \in H^{2e-2i}(Y)(e-i)$. On the other hand, we have $\gamma(\alpha) \in H^{2d-2i}(X)(d-i)$ and hence $f_*\gamma(\alpha) \in H^{2e-2i}(Y)(e-i)$ as well. Thus the condition $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ makes sense.

0FHD Remark 45.9.3. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C)(a). Let X be a smooth projective scheme over k. We obtain maps

$$H^*(X) \otimes_F H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where the first arrow is as in axiom (B) and Δ^* is pullback along the diagonal morphism $\Delta : X \to X \times X$. The composition is the cup product as pullback is an algebra homomorphism and $\mathrm{pr}_i \circ \Delta = \mathrm{id}$. On the other hand, given cycles α, β on X the intersection product is defined by the formula

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta)$$

In other words, $\alpha \cdot \beta$ is the pullback of the exterior product $\alpha \times \beta$ on $X \times X$ by the diagonal. Note also that $\alpha \times \beta = \mathrm{pr}_1^*\alpha \cdot \mathrm{pr}_2^*\beta$ in $\mathrm{CH}^*(X \times X)$ (we omit the proof). Hence, given axiom (C)(a), axiom (C)(c) is equivalent to the statement that γ is compatible with exterior product in the sense that $\gamma(\alpha \times \beta)$ is equal to $\mathrm{pr}_1^*\gamma(\alpha) \cup \mathrm{pr}_2^*\gamma(\beta)$.

0FHE Lemma 45.9.4. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Then $H^i(\mathrm{Spec}(k)) = 0$ for $i \neq 0$ and there is a unique F-algebra isomorphism $F = H^0(\mathrm{Spec}(k))$. We have $\gamma([\mathrm{Spec}(k)]) = 1$ and $\int_{\mathrm{Spec}(k)} 1 = 1$.

Proof. By axiom (C)(d) we see that $H^0(\mathrm{Spec}(k))$ is nonzero and even $\gamma([\mathrm{Spec}(k)])$ is nonzero. Since $\mathrm{Spec}(k) \times \mathrm{Spec}(k) = \mathrm{Spec}(k)$ we get

$$H^*(\mathrm{Spec}(k)) \otimes_F H^*(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$$

by axiom (B)(a) which implies (look at dimensions) that only H^0 is nonzero and moreover has dimension 1. Thus $F = H^0(\mathrm{Spec}(k))$ via the unique F-algebra isomorphism given by mapping $1 \in F$ to $1 \in H^0(\mathrm{Spec}(k))$. Since $[\mathrm{Spec}(k)] \cdot [\mathrm{Spec}(k)] = [\mathrm{Spec}(k)]$ in the Chow ring of $\mathrm{Spec}(k)$ we conclude that $\gamma([\mathrm{Spec}(k)) \cup \gamma([\mathrm{Spec}(k)]) = \gamma([\mathrm{Spec}(k)])$ by axiom (C)(c). Since we already know that $\gamma([\mathrm{Spec}(k)])$ is nonzero we conclude that it has to be equal to 1. Finally, we have $\int_{\mathrm{Spec}(k)} 1 = 1$ since $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$ by axiom (C)(d). $\square$

0FHF Lemma 45.9.5. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme over k. If $X = \emptyset$, then $H^*(X) = 0$. If X is nonempty, then $\gamma([X]) = 1$ and $1 \neq 0$ in $H^0(X)$.

Proof. First assume X is nonempty. Observe that $[X]$ is the pullback of $[\mathrm{Spec}(k)]$ by the structure morphism $p : X \to \mathrm{Spec}(k)$. Hence we get $\gamma([X]) = 1$ by axiom (C)(a) and Lemma 45.9.4. Let $X' \subset X$ be an irreducible component. By functoriality it suffices to show $1 \neq 0$ in $H^0(X')$. Thus we may and do assume X is irreducible, and in particular nonempty and equidimensional, say of dimension d. To see that $1 \neq 0$ it suffices to show that $H^*(X)$ is nonzero.

Let $x \in X$ be a closed point whose residue field k' is separable over k, see Varieties, Lemma 33.25.6. Let $i : \mathrm{Spec}(k') \to X$ be the inclusion morphism. Denote $p : X \to \mathrm{Spec}(k)$ is the structure morphism. Observe that $p_*i_*[\mathrm{Spec}(k')] = [k' : k][\mathrm{Spec}(k)]$ in $\mathrm{CH}_0(\mathrm{Spec}(k))$. Using axiom (C)(b) twice and Lemma 45.9.4 we conclude that

$$p_*i_*\gamma([\mathrm{Spec}(k')]) = \gamma([k' : k][\mathrm{Spec}(k)]) = [k' : k] \in F = H^0(\mathrm{Spec}(k))$$

is nonzero. Thus $i_*\gamma([\mathrm{Spec}(k)]) \in H^{2d}(X)(d)$ is nonzero (because it maps to something nonzero via p_*). This concludes the proof in case X is nonempty.

Finally, we consider the case of the empty scheme. Axiom (B)(a) gives $H^*(\emptyset) \otimes H^*(\emptyset) = H^*(\emptyset)$ and we get that $H^*(\emptyset)$ is either zero or 1-dimensional in degree 0. Then axiom (B)(a) again shows that $H^*(\emptyset) \otimes H^*(X) = H^*(\emptyset)$ for all smooth projective schemes X over k. Using axiom (A)(b) and the nonvanishing of $H^0(X)$ we've seen above we find that $H^*(X)$ is nonzero in at least two degrees if $\dim(X) > 0$. This then forces $H^*(\emptyset)$ to be zero. ☐

0FHG Lemma 45.9.6. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $i : X \to Y$ be a closed immersion of nonempty smooth projective equidimensional schemes over k. Then $\gamma([X]) = i_*1$ in $H^{2c}(Y)(c)$ where $c = \dim(Y) - \dim(X)$.

Proof. This is true because $1 = \gamma([X])$ in $H^0(X)$ by Lemma 45.9.5 and then we can apply axiom (C)(b). ☐

0FHH Lemma 45.9.7. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Choose a basis $e_{i,j}, j = 1, \ldots, \beta_i$ of $H^i(X)$ over F. Using Künneth write

$$\gamma([\Delta]) = \sum\nolimits_i \sum\nolimits_j e_{i,j} \otimes e'_{2d-i,j} \quad \text{in} \quad \bigoplus\nolimits_i H^i(X) \otimes_F H^{2d-i}(X)(d)$$

with $e'_{2d-i,j} \in H^{2d-i}(X)(d)$. Then $\int_X e_{i,j} \cup e'_{2d-i,j'} = (-1)^i\delta_{jj'}$.

Proof. Recall that $\Delta^* : H^*(X \times X) \to H^*(X)$ is equal to the cup product map $H^*(X) \otimes_F H^*(X) \to H^*(X)$, see Remark 45.9.3. On the other hand, recall that $\gamma([\Delta]) = \Delta_*1$ (Lemma 45.9.6) and hence

$$\int_{X \times X} \gamma([\Delta]) \cup a \otimes b = \int_{X \times X} \Delta_*1 \cup a \otimes b = \int_X a \cup b$$

by Lemma 45.9.1. On the other hand, we have

$$\int_{X \times X} (\sum e_{i,j} \otimes e'_{2d-i,j}) \cup a \otimes b = \sum (\int_X a \cup e_{i,j})(\int_X e'_{2d-i,j} \cup b)$$

by axiom (B)(b); note that we made two switches of order so that the sign for each term is 1. Thus if we choose a such that $\int_X a \cup e_{i,j} = 1$ and all other pairings equal to zero, then we conclude that $\int_X e'_{2d-i,j} \cup b = \int_X a \cup b$ for all b, i.e., $e'_{2d-i,j} = a$. This proves the lemma. ☐

0FHI Lemma 45.9.8. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Then $H^*(\mathbf{P}^1_k)$ is 1-dimensional in dimensions 0 and 2 and zero in other degrees.

Proof. Let $x \in \mathbf{P}^1_k$ be a k-rational point. Observe that $\Delta = \text{pr}_1^*x + \text{pr}_2^*x$ as divisors on $\mathbf{P}^1_k \times \mathbf{P}^1_k$. Using axiom (C)(a) and additivity of γ we see that

$$\gamma([\Delta]) = \text{pr}_1^*\gamma([x]) + \text{pr}_2^*\gamma([x]) = \gamma([x]) \otimes 1 + 1 \otimes \gamma([x])$$

in $H^*(\mathbf{P}^1_k \times \mathbf{P}^1_k) = H^*(\mathbf{P}^1_k) \otimes_F H^*(\mathbf{P}^1_k)$. However, by Lemma 45.9.7 we know that $\gamma([\Delta])$ cannot be written as a sum of fewer than $\sum \beta_i$ pure tensors where $\beta_i = \dim_F H^i(\mathbf{P}^1_k)$. Thus we see that $\sum \beta_i \leq 2$. By Lemma 45.9.5 we have $H^0(\mathbf{P}^1_k) \neq 0$. By Poincaré duality, more precisely axiom (A)(b), we have $\beta_0 = \beta_2$. Therefore the lemma holds. $\square$

0FHJ Lemma 45.9.9. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). If X and Y are smooth projective schemes over k, then $H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$, $a \mapsto (i^*a, j^*a)$ is an isomorphism where i, j are the coprojections.

Proof. If X or Y is empty, then this is true because $H^*(\emptyset) = 0$ by Lemma 45.9.5. Thus we may assume both X and Y are nonempty.

We first show that the map is injective. First, observe that we can find morphisms $X' \to X$ and $Y' \to Y$ of smooth projective schemes so that X' and Y' are equidimensional of the same dimension and such that $X' \to X$ and $Y' \to Y$ each have a section. Namely, decompose $X = \coprod X_d$ and $Y = \coprod Y_e$ into open and closed subschemes equidimensional of dimension d and e. Then take $X' = \coprod X_d \times \mathbf{P}^{n-d}$ and $Y' = \coprod Y_e \times \mathbf{P}^{n-e}$ for some n sufficiently large. Thus pullback by $X' \amalg Y' \to X \amalg Y$ is injective (because there is a section) and it suffices to show the injectivity for X', Y' as we do in the next parapgrah.

Let us show the map is injective when X and Y are equidimensional of the same dimension d. Observe that $[X \amalg Y] = [X] + [Y]$ in $\text{CH}^0(X \amalg Y)$ and that $[X]$ and $[Y]$ are orthogonal idempotents in $\text{CH}^0(X \amalg Y)$. Thus

$$1 = \gamma([X \amalg Y] = \gamma([X]) + \gamma([Y]) = i_*1 + j_*1$$

is a decomposition into orthogonal idempotents. Here we have used Lemmas 45.9.5 and 45.9.6 and axiom (C)(c). Then we see that

$$a = a \cup 1 = a \cup i_*1 + a \cup j_*1 = i_*(i^*a) + j_*(j^*a)$$

by the projection formula (Lemma 45.9.1) and hence the map is injective.

We show the map is surjective. Write $e = \gamma([X])$ and $f = \gamma([Y])$ viewed as elements in $H^0(X \amalg Y)$. We have $i^*e = 1$, $i^*f = 0$, $j^*e = 0$, and $j^*f = 1$ by axiom (C)(a). Hence if $i^* : H^*(X \amalg Y) \to H^*(X)$ and $j^* : H^*(X \amalg Y) \to H^*(Y)$ are surjective, then so is (i^*, j^*). Namely, for $a, a' \in H^*(X \amalg Y)$ we have

$$(i^*a, j^*a') = (i^*(a \cup e + a' \cup f), j^*(a \cup e + a' \cup f))$$

By symmetry it suffices to show $i^* : H^*(X \amalg Y) \to H^*(X)$ is surjective. If there is a morphism $Y \to X$, then there is a morphism $g : X \amalg Y \to X$ with $g \circ i = \text{id}_X$ and we conclude. To finish the proof, observe that in order to prove i^* is surjective, it suffices to do so after tensoring by a nonzero graded F-vector space. Hence by axiom (B)(b) and nonvanishing of cohomology (Lemma 45.9.5) it suffices to prove i^* is surjective after replacing X and Y by $X \times \text{Spec}(k')$ and $Y \times \text{Spec}(k')$ for some finite separable extension k'/k. If we choose k' such that there exists a closed point $x \in X$ with $\kappa(x) = k'$ (and this is possible by Varieties, Lemma 33.25.6) then there is a morphism $Y \times \text{Spec}(k') \to X \times \text{Spec}(k')$ and we find that the proof is complete. $\square$

0FHK Lemma 45.9.10. Let k be a field. Let F be a field of characteristic 0. Assume given a $\mathbf{Q}$-linear functor

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $G(\mathbf{1}(1))$ is nonzero only in degree -2. Then we obtain data (D0), (D1), (D2), and (D3) satisfying all of (A), (B), and (C) above.

Proof. This proof is the same as the proof of Lemma 45.7.9; we urge the reader to read the proof of that lemma instead.

We obtain a contravariant functor from the category of smooth projective schemes over k to the category of graded F-vector spaces by setting $H^*(X) = G(h(X))$. By assumption we have a canonical isomorphism

$$H^*(X\times Y) = G(h(X\times Y)) = G(h(X)\otimes h(Y)) = G(h(X))\otimes G(h(Y)) = H^*(X)\otimes H^*(Y)$$

compatible with pullbacks. Using pullback along the diagonal $\Delta : X \to X \times X$ we obtain a canonical map

$$H^*(X) \otimes H^*(X) = H^*(X \times X) \to H^*(X)$$

of graded vector spaces compatible with pullbacks. This defines a functorial graded F-algebra structure on $H^*(X)$. Since Δ commutes with the commutativity constraint $h(X)\otimes h(X) \to h(X)\otimes h(X)$ (switching the factors) and since G is a functor of symmetric monoidal categories (so compatible with commutativity constraints), and by our convention in Homology, Example 12.17.4 we conclude that $H^*(X)$ is a graded commutative algebra! Hence we get our datum (D1).

Since $\mathbf{1}(1)$ is invertible in the category of motives we see that $G(\mathbf{1}(1))$ is invertible in the category of graded F-vector spaces. Thus $\sum_i \dim_F G^i(\mathbf{1}(1)) = 1$. By assumption we only get something nonzero in degree -2. Our datum (D0) is the vector space $F(1) = G^{-2}(\mathbf{1}(1))$. Since G is a symmetric monoidal functor we see that $F(n) = G^{-2n}(\mathbf{1}(n))$ for all $n \in \mathbf{Z}$. It follows that

$$H^{2r}(X)(r) = G^{2r}(h(X)) \otimes G^{-2r}(\mathbf{1}(r)) = G^0(h(X)(r))$$

a formula we will frequently use below.

Let X be a smooth projective scheme over k. By Lemma 45.3.1 we have

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(\mathrm{Spec}(k), X) = \mathrm{Hom}(\mathbf{1}(-r), h(X)) = \mathrm{Hom}(\mathbf{1}, h(X)(r))$$

Applying the functor G this maps into $\mathrm{Hom}(G(\mathbf{1}), G(h(X)(r)))$. By taking the image of 1 in $G^0(\mathbf{1}) = F$ into $G^0(h(X)(r)) = H^{2r}(X)(r)$ we obtain

$$\gamma : \mathrm{CH}^r(X) \otimes \mathbf{Q} \longrightarrow H^{2r}(X)(r)$$

This is the datum (D2).

Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. By Lemma 45.3.1 we have

$$\mathrm{Mor}(h(X)(d), \mathbf{1}) = \mathrm{Mor}((X, 1, d), (\mathrm{Spec}(k), 1, 0)) = \mathrm{Corr}^{-d}(X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X)$$

Thus the class of the cycle $[X]$ in $\mathrm{CH}_d(X)$ defines a morphism $h(X)(d) \to \mathbf{1}$. Applying G and taking degree 0 parts we obtain

$$H^{2d}(X)(d) = G^0(h(X)(d)) \longrightarrow G^0(\mathbf{1}) = F$$

This map $\int_X : H^{2d}(X)(d) \to F$ is the datum (D3).

Let X be a smooth projective scheme over k which is nonempty and equidimensional of dimension d. By Lemma 45.4.9 we know that $h(X)(d)$ is a left dual to $h(X)$. Hence $G(h(X)(d)) = H^*(X) \otimes_F F(d)[2d]$ is a left dual to $H^*(X)$ in the category of graded F-vector spaces. Here $[n]$ is the shift functor on graded vector spaces. By Homology, Lemma 12.17.5 we find that $\sum_i \dim_F H^i(X) < \infty$ and that $\epsilon : h(X)(d) \otimes h(X) \to \mathbf{1}$ produces nondegenerate pairings $H^{2d-i}(X)(d) \otimes_F H^i(X) \to F$. In the proof of Lemma 45.4.9 we have seen that ϵ is given by $[\Delta]$ via the identifications

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

Thus ϵ is the composition of $[X] : h(X)(d) \to \mathbf{1}$ and $h(\Delta)(d) : h(X)(d) \otimes h(X) \to h(X)(d)$. It follows that the pairings above are given by cup product followed by $\int_X$. This proves axiom (A).

Axiom (B) follows from the assumption that G is compatible with tensor structures and our construction of the cup product above.

Axiom (C). Our construction of γ takes a cycle α on X, interprets it a correspondence a from $\mathrm{Spec}(k)$ to X of some degree, and then applies G. If $f : Y \to X$ is a morphism of nonempty equidimensional smooth projective schemes over k, then $f^!\alpha$ is the pushforward (!) of α by the correspondence $[\Gamma_f]$ from X to Y, see Lemma 45.3.6. Hence $f^!\alpha$ viewed as a correspondence from $\mathrm{Spec}(k)$ to Y is equal to $a \circ [\Gamma_f]$, see Lemma 45.3.1. Since G is a functor, we conclude γ is compatible with pullbacks, i.e., axiom (C)(a) holds.

Let $f : Y \to X$ be a morphism of nonempty equidimensional smooth projective schemes over k and let $\beta \in \mathrm{CH}^r(Y)$ be a cycle on Y. We have to show that

$$\int_Y \gamma(\beta) \cup f^*c = \int_X \gamma(f_*\beta) \cup c$$

for all $c \in H^*(X)$. Let $a, a^t, \eta_X, \eta_Y, [X], [Y]$ be as in Lemma 45.3.9. Let b be β viewed as a correspondence from $\mathrm{Spec}(k)$ to Y of degree r. Then $f_*\beta$ viewed as a correspondence from $\mathrm{Spec}(k)$ to X is equal to $a^t \circ b$, see Lemmas 45.3.6 and 45.3.1. The displayed equality above holds if we can show that

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{b \otimes 1} h(Y)(r) \otimes h(X) \xrightarrow{1 \otimes a} h(Y)(r) \otimes h(Y) \xrightarrow{\eta_Y} h(Y)(r) \xrightarrow{[Y]} \mathbf{1}(r-e)$$

is equal to

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{a^t \circ b \otimes 1} h(X)(r+d-e) \otimes h(X) \xrightarrow{\eta_X} h(X)(r+d-e) \xrightarrow{[X]} \mathbf{1}(r-e)$$

This follows immediately from Lemma 45.3.9. Thus we have axiom (C)(b).

To prove axiom (C)(c) we use the discussion in Remark 45.7.2. Hence it suffices to prove that γ is compatible with exterior products. Let X, Y be nonempty smooth projective schemes over k and let α, β be cycles on them. Denote a, b the corresponding correspondences from $\mathrm{Spec}(k)$ to X, Y. Then $\alpha \times \beta$ corresponds to the correspondence $a \otimes b$ from $\mathrm{Spec}(k)$ to $X \otimes Y = X \times Y$. Hence the requirement follows from the fact that G is compatible with the tensor structures on both sides.

Axiom (C)(d) follows because the cycle $[\mathrm{Spec}(k)]$ corresponds to the identity morphism on $h(\mathrm{Spec}(k))$. This finishes the proof of the lemma. $\square$

0FHL Lemma 45.9.11. Let k be a field. Let F be a field of characteristic 0. Given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) we can construct a $\mathbf{Q}$-linear functor

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $H^*(X) = G(h(X))$.

Proof. The proof of this lemma is the same as the proof of Lemma 45.7.10; we urge the reader to read the proof of that lemma instead.

By Lemma 45.4.8 it suffices to construct a functor G on the category of smooth projective schemes over k with morphisms given by correspondences of degree 0 such that the image of $G(c_2)$ on $G(\mathbf{P}^1_k)$ is an invertible graded F-vector space.

Let X be a smooth projective scheme over k. There is a canonical decomposition

$$X = \coprod\nolimits_{0 \leq d \leq \dim(X)} X_d$$

into open and closed subschemes such that X_d is equidimensional of dimension d. By Lemma 45.9.9 we have correspondingly

$$H^*(X) \longrightarrow \prod\nolimits_{0 \leq d \leq \dim(X)} H^*(X_d)$$

If Y is a second smooth projective scheme over k and we similarly decompose $Y = \coprod Y_e$, then

$$\text{Corr}^0(X, Y) = \bigoplus \text{Corr}^0(X_d, Y_e)$$

As well we have $X \otimes Y = \coprod X_d \otimes Y_e$ in the category of correspondences. From these observations it follows that it suffices to construct G on the category whose objects are equidimensional smooth projective schemes over k and whose morphisms are correspondences of degree 0. (Some details omitted.)

Given an equdimensional smooth projective scheme X over k we set $G(X) = H^*(X)$. Observe that $G(X) = 0$ if $X = \emptyset$ (Lemma 45.9.5). Thus maps from and to $G(\emptyset)$ are zero and we may and do assume our schemes are nonempty in the discussions below.

Given a correspondence $c \in \text{Corr}^0(X, Y)$ between nonempty equidmensional smooth projective schemes over k we consider the map $G(c) : G(X) = H^*(X) \to G(Y) = H^*(Y)$ given by the rule

$$a \longmapsto G(c)(a) = \text{pr}_{2,*}(\gamma(c) \cup \text{pr}_1^* a)$$

It is clear that $G(c)$ is additive in c and hence $\mathbf{Q}$-linear. Compatibility of γ with pullbacks, pushforwards, and intersection products given by axioms (C)(a), (C)(b), and (C)(c) shows that we have $G(c' \circ c) = G(c') \circ G(c)$ if $c' \in \text{Corr}^0(Y, Z)$. Namely, for $a \in H^*(X)$ we have

$$\begin{aligned}
(G(c') \circ G(c))(a) &= \mathrm{pr}^{23}_{3,*}(\gamma(c') \cup \mathrm{pr}^{23,*}_2(\mathrm{pr}^{12}_{2,*}(\gamma(c) \cup \mathrm{pr}^{12,*}_1 a))) \\
&= \mathrm{pr}^{23}_{3,*}(\gamma(c') \cup \mathrm{pr}^{123}_{23,*}(\mathrm{pr}^{123,*}_{12}(\gamma(c) \cup \mathrm{pr}^{12,*}_1 a))) \\
&= \mathrm{pr}^{23}_{3,*}\mathrm{pr}^{123}_{23,*}(\mathrm{pr}^{123,*}_{23}\gamma(c') \cup \mathrm{pr}^{123,*}_{12}\gamma(c) \cup \mathrm{pr}^{123,*}_1 a) \\
&= \mathrm{pr}^{23}_{3,*}\mathrm{pr}^{123}_{23,*}(\gamma(\mathrm{pr}^{123,*}_{23}c') \cup \gamma(\mathrm{pr}^{123,*}_{12}c) \cup \mathrm{pr}^{123,*}_1 a) \\
&= \mathrm{pr}^{13}_{3,*}\mathrm{pr}^{123}_{13,*}(\gamma(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c) \cup \mathrm{pr}^{123,*}_1 a) \\
&= \mathrm{pr}^{13}_{3,*}(\gamma(\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c)) \cup \mathrm{pr}^{13,*}_1 a) \\
&= G(c' \circ c)(a)
\end{aligned}$$

with obvious notation. The first equality follows from the definitions. The second equality holds because $\mathrm{pr}^{23,*}_2 \circ \mathrm{pr}^{12}_{2,*} = \mathrm{pr}^{123}_{23,*} \circ \mathrm{pr}^{123,*}_{12}$ as follows immediately from the description of pushforward along projections given in Lemma 45.9.2. The third equality holds by Lemma 45.9.1 and the fact that H^* is a functor. The fourth equalith holds by axiom (C)(a) and the fact that the gysin map agrees with flat pullback for flat morphisms (Chow Homology, Lemma 42.59.5). The fifth equality uses axiom (C)(c) as well as Lemma 45.9.1 to see that $\mathrm{pr}^{23}_{3,*} \circ \mathrm{pr}^{123}_{23,*} = \mathrm{pr}^{13}_{3,*} \circ \mathrm{pr}^{123}_{13,*}$. The sixth equality uses the projection formula from Lemma 45.9.1 as well as axiom (C)(b) to see that $\mathrm{pr}^{123}_{13,*}\gamma(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c) = \gamma(\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c))$. Finally, the last equality is the definition.

To finish the proof that G is a functor, we have to show identities are preserved. In other words, if $1 = [\Delta] \in \mathrm{Corr}^0(X, X)$ is the identity in the category of correspondences (Lemma 45.3.3), then we have to show that $G([\Delta]) = \mathrm{id}$. This follows from the determination of $\gamma([\Delta])$ in Lemma 45.9.7 and Lemma 45.9.2. This finishes the construction of G as a functor on smooth projective schemes over k and correspondences of degree 0.

By Lemma 45.9.4 we have that $G(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$ is canonically isomorphic to F as an F-algebra. The Künneth axiom (B)(a) shows our functor is compatible with tensor products. Thus our functor is a functor of symmetric monoidal categories.

We still have to check that the image of $G(c_2)$ on $G(\mathbf{P}^1_k) = H^*(\mathbf{P}^1_k)$ is an invertible graded F-vector space (in particular we don't know yet that G extends to M_k). By Lemma 45.9.8 we only have nonzero cohomology in degrees 0 and 2 both of dimension 1. We have $1 = c_0 + c_2$ is a decomposition of the identity into a sum of orthogonal idempotents in $\mathrm{Corr}^0(\mathbf{P}^1_k, \mathbf{P}^1_k)$, see Example 45.3.7. Further we have $c_0 = a \circ b$ where $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), \mathbf{P}^1_k)$ and $b \in \mathrm{Corr}^0(\mathbf{P}^1_k, \mathrm{Spec}(k))$ and where $b \circ a = 1$ in $\mathrm{Corr}^0(\mathrm{Spec}(k), \mathrm{Spec}(k))$, see proof of Lemma 45.4.4. Thus $G(c_0)$ is the projector onto the degree 0 part. It follows that $G(c_2)$ must be the projector onto the degree 2 part and the proof is complete. □

0FHM Proposition 45.9.12. Let k be a field. Let F be a field of characteristic 0. There is a 1-to-1 correspondence between the following

(1) data (D0), (D1), (D2), and (D3) satisfying (A), (B), and(C), and
(2) $\mathbf{Q}$-linear symmetric monoidal functors

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

such that $G(\mathbf{1}(1))$ is nonzero only in degree -2.

Proof. Given G as in (2) by setting $H^*(X) = G(h(X))$ we obtain data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C), see Lemma 45.9.10 and its proof.

Conversely, given data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) we get a functor G as in (2) by the construction of the proof of Lemma 45.9.11.

We omit the detailed proof that these constructions are inverse to each other. $\square$

45.10. Further properties

0FHN In this section we prove a few more results one obtains if given data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) as in Section 45.9.

0FHP Lemma 45.10.1. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X, Y be nonempty smooth projective schemes both equidimensional of dimension d over k. Then $\int_{X \amalg Y} = \int_X + \int_Y$.

Proof. Denote $i : X \to X \amalg Y$ and $j : Y \to X \amalg Y$ be the coprojections. By Lemma 45.9.9 the map $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism. The statement of the lemma means that under the isomorphism $(i^*, j^*) : H^{2d}(X \amalg Y)(d) \to H^{2d}(X)(d) \oplus H^{2d}(Y)(d)$ the map $\int_X + \int_Y$ is tranformed into $\int_{X \amalg Y}$. This is true because

$$\int_{X \amalg Y} a = \int_{X \amalg Y} i_*(i^*a) + j_*(j^*a) = \int_X i^*a + \int_Y j^*a$$

where the equality $a = i_*(i^*a) + j_*(j^*a)$ was shown in the proof of Lemma 45.9.9. $\square$

0FHQ Lemma 45.10.2. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme of dimension zero over k. Then

(1) $H^i(X) = 0$ for $i \neq 0$,
(2) $H^0(X)$ is a finite separable algebra over F,
(3) $\dim_F H^0(X) = \deg(X \to \mathrm{Spec}(F))$,
(4) $\int_X : H^0(X) \to F$ is the trace map,
(5) $\gamma([X]) = 1$, and
(6) $\int_X \gamma([X]) = \deg(X \to \mathrm{Spec}(k))$.

Proof. We can write $X = \mathrm{Spec}(k')$ where k' is a finite separable algebra over k. Observe that $\deg(X \to \mathrm{Spec}(k)) = [k' : k]$. Choose a finite Galois extension k''/k containing each of the factors of k'. (Recall that a finite separable k-algebra is a product of finite separable field extension of k.) Set $\Sigma = \mathrm{Hom}_k(k', k'')$. Then we get

$$k' \otimes_k k'' = \prod\nolimits_{\sigma \in \Sigma} k''$$

Setting $Y = \mathrm{Spec}(k'')$ axioms (B)(a) and Lemma 45.9.9 give

$$H^*(X) \otimes_F H^*(Y) = \prod\nolimits_{\sigma \in \Sigma} H^*(Y)$$

as graded commutative F-algebras. By Lemma 45.9.5 the F-algebra $H^*(Y)$ is nonzero. Comparing dimensions on either side of the displayed equation we conclude that $H^*(X)$ sits only in degree 0 and $\dim_F H^0(X) = [k' : k]$. Applying this to Y we get $H^*(Y) = H^0(Y)$. Since

$$H^0(X) \otimes_F H^0(Y) = H^0(Y) \times \ldots \times H^0(Y)$$

as F-algebras, it follows that $H^0(X)$ is a separable F-algebra because we may check this after the faithfully flat base change $F \to H^0(Y)$.

The displayed isomorphism above is given by the map

$$H^0(X) \otimes_F H^0(Y) \longrightarrow \prod\nolimits_{\sigma \in \Sigma} H^0(Y), \quad a \otimes b \longmapsto \prod\nolimits_\sigma \mathrm{Spec}(\sigma)^* a \cup b$$

Via this isomorphism we have $\int_{X \times Y} = \sum_\sigma \int_Y$ by Lemma 45.10.1. Thus

$$\int_X a = \mathrm{pr}_{1,*}(a \otimes 1) = \sum \mathrm{Spec}(\sigma)^* a$$

in $H^0(Y)$; the first equality by Lemma 45.9.2 and the second by the observation we just made. Choose an algebraic closure $\overline{F}$ and a F-algebra map $\tau : H^0(Y) \to \overline{F}$. The isomorphism above base changes to the isomorphism

$$H^0(X) \otimes_F \overline{F} \longrightarrow \prod\nolimits_{\sigma \in \Sigma} \overline{F}, \quad a \otimes b \longmapsto \prod\nolimits_\sigma \tau(\mathrm{Spec}(\sigma)^* a) b$$

It follows that $a \mapsto \tau(\mathrm{Spec}(\sigma)^* a)$ is a full set of embeddings of $H^0(X)$ into $\overline{F}$. Applying τ to the formula for $\int_X a$ obtained above we conclude that $\int_X$ is the trace map. By Lemma 45.9.5 we have $\gamma([X]) = 1$. Finally, we have $\int_X \gamma([X]) = \deg(X \to \mathrm{Spec}(k))$ because $\gamma([X]) = 1$ and the trace of 1 is equal to $[k' : k]$ □

0FHR Lemma 45.10.3. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme equidimensional of dimension d over k. The diagram

$$\begin{tikzcd} \mathrm{CH}^d(X) \arrow[r, "\gamma"'] \arrow[d, equal] & H^{2d}(X)(d) \arrow[d, "\int_X"] \\ \mathrm{CH}_0(X) \arrow[r, "\deg"] & F \end{tikzcd}$$

commutes where $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$ is the degree of zero cycles discussed in Chow Homology, Section 42.41.

Proof. Let x be a closed point of X whose residue field is separable over k. View x as a scheme and denote $i : x \to X$ the inclusion morphism. To avoid confusion denote $\gamma' : \mathrm{CH}_0(x) \to H^0(x)$ the cycle class map for x. Then we have

$$\int_X \gamma([x]) = \int_X \gamma(i_*[x]) = \int_X i_* \gamma'([x]) = \int_x \gamma'([x]) = \deg(x \to \mathrm{Spec}(k))$$

The second equality is axiom (C)(b) and the third equality is the definition of i_* on cohomology. The final equality is Lemma 45.10.2. This proves the lemma because $\mathrm{CH}_0(X)$ is generated by the classes of points x as above by Lemma 45.8.1. □

0FHS Lemma 45.10.4. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. We have

$$\sum\nolimits_i (-1)^i \dim_F H^i(X) = \deg(\Delta \cdot \Delta) = \deg(c_d(\mathcal{T}_{X/k}))$$

Proof. Equality on the right. We have $[\Delta] \cdot [\Delta] = \Delta_*(\Delta^![\Delta])$ (Chow Homology, Lemma 42.62.6). Since Δ_* preserves degrees of 0-cycles it suffices to compute the degree of $\Delta^![\Delta]$. The class $\Delta^![\Delta]$ is given by capping $[\Delta]$ with the top Chern class of the normal sheaf of $\Delta \subset X \times X$ (Chow Homology, Lemma 42.54.5). Since the conormal sheaf of Δ is $\Omega_{X/k}$ (Morphisms, Lemma 29.32.7) we see that the normal sheaf is equal to the tangent sheaf $\mathcal{T}_{X/k} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ as desired.

Equality on the left. By Lemma 45.10.3 we have

$$\begin{aligned} \deg([\Delta] \cdot [\Delta]) &= \int_{X \times X} \gamma([\Delta]) \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_* 1 \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_*(\Delta^* \gamma([\Delta])) \\ &= \int_X \Delta^* \gamma([\Delta]) \end{aligned}$$

We have used Lemmas 45.9.6 and 45.9.1. Write $\gamma([\Delta]) = \sum e_{i,j} \otimes e'_{2d-i,j}$ as in Lemma 45.9.7. Recalling that Δ^* is given by cup product (Remark 45.9.3) we obtain

$$\int_X \sum\nolimits_{i,j} e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} \int_X e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} (-1)^i = \sum (-1)^i \beta_i$$

as desired. $\square$

0FHT Lemma 45.10.5. Let F be a field of characteristic 0. Let F' and F_i, $i = 1, \ldots, r$ be finite separable F-algebras. Let A be a finite F-algebra. Let $\sigma, \sigma' : A \to F'$ and $\sigma_i : A \to F_i$ be F-algebra maps. Assume σ and σ' surjective. If there is a relation

$$\mathrm{Tr}_{F'/F} \circ \sigma - \mathrm{Tr}_{F'/F} \circ \sigma' = n(\sum m_i \mathrm{Tr}_{F_i/F} \circ \sigma_i)$$

where $n > 1$ and m_i are integers, then $\sigma = \sigma'$.

Proof. We may write $A = \prod A_j$ as a finite product of local Artinian F-algebras $(A_j, \mathfrak{m}_j, \kappa_j)$, see Algebra, Lemma 10.53.2 and Proposition 10.60.7. Denote $A' = \prod \kappa_j$ where the product is over those j such that κ_j/k is separable. Then each of the maps $\sigma, \sigma', \sigma_i$ factors over the map $A \to A'$. After replacing A by A' we may assume A is a finite separable F-algebra.

Choose an algebraic closure $\overline{F}$. Set $\overline{A} = A \otimes_F \overline{F}$, $\overline{F}' = F' \otimes_F \overline{F}$, and $\overline{F}_i = F_i \otimes_F \overline{F}$. We can base change σ, σ', σ_i to get $\overline{F}$ algebra maps $\overline{A} \to \overline{F}'$ and $\overline{A} \to \overline{F}_i$. Moreover $\mathrm{Tr}_{\overline{F}'/\overline{F}}$ is the base change of $\mathrm{Tr}_{F'/F}$ and similarly for $\mathrm{Tr}_{F_i/F}$. Thus we may replace F by $\overline{F}$ and we reduce to the case discussed in the next paragraph.

Assume F is algebraically closed and A a finite separable F-algebra. Then each of A, F', F_i is a product of copies of F. Let us say an element e of a product $F \times \ldots \times F$ of copies of F is a minimal idempotent if it generates one of the factors, i.e., if $e = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $e \in A$ be a minimal idempotent. Since σ and σ' are surjective, we see that $\sigma(e)$ and $\sigma'(e)$ are minimal idempotents or zero. If $\sigma \neq \sigma'$, then we can choose a minimal idempotent $e \in A$ such that $\sigma(e) = 0$ and $\sigma'(e) \neq 0$ or vice versa. Then $\mathrm{Tr}_{F'/F}(\sigma(e)) = 0$ and $\mathrm{Tr}_{F'/F}(\sigma'(e)) = 1$ or vice versa. On the other hand, $\sigma_i(e)$ is an idempotent and hence $\mathrm{Tr}_{F_i/F}(\sigma_i(e)) = r_i$ is an integer. We conclude that

$$-1 = \sum n m_i r_i = n(\sum m_i r_i) \quad \text{or} \quad 1 = \sum n m_i r_i = n(\sum m_i r_i)$$

which is impossible. $\square$

0FHU Lemma 45.10.6. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let k'/k be a finite separable extension. Let X be a smooth projective scheme over k'. Let $x, x' \in X$ be k'-rational points. If $\gamma(x) \neq \gamma(x')$, then $[x] - [x']$ is not divisible by any integer $n > 1$ in $\mathrm{CH}_0(X)$.

Proof. If x and x' lie on distinct irreducible components of X, then the result is obvious. Thus we may X irreducible of dimension d. Say $[x] - [x']$ is divisible by $n > 1$ in $\mathrm{CH}_0(X)$. We may write $[x] - [x'] = n(\sum m_i[x_i])$ in $\mathrm{CH}_0(X)$ for some $x_i \in X$ closed points whose residue fields are separable over k by Lemma 45.8.1. Then

$$\gamma([x]) - \gamma([x']) = n(\sum m_i \gamma([x_i]))$$

in $H^{2d}(X)(d)$. Denote $i^*, (i')^*, i_i^*$ the pullback maps $H^0(X) \to H^0(x)$, $H^0(X) \to H^0(x')$, $H^0(X) \to H^0(x_i)$. Recall that $H^0(x)$ is a finite separable F-algebra and that $\int_x : H^0(x) \to F$ is the trace map (Lemma 45.10.2) which we will denote Tr_x. Similarly for x' and x_i. Then by Poincaré duality in the form of axiom (A)(b) the equation above is dual to

$$\mathrm{Tr}_x \circ i^* - \mathrm{Tr}_{x'} \circ (i')^* = n(\sum m_i \mathrm{Tr}_{x_i} \circ i_i^*)$$

which takes place in $\mathrm{Hom}_F(H^0(X), F)$. Finally, observe that i^* and $(i')^*$ are surjective as x and x' are k'-rational points and hence the compositions $H^0(\mathrm{Spec}(k')) \to H^0(X) \to H^0(x)$ and $H^0(\mathrm{Spec}(k')) \to H^0(X) \to H^0(x')$ are isomorphisms. By Lemma 45.10.5 we conclude that $i^* = (i')^*$ which contradicts the assumption that $\gamma([x]) \neq \gamma([x'])$. □

0FHV Lemma 45.10.7. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let k'/k be a finite separable extension. Let X be a geometrically irreducible smooth projective scheme over k' of dimension d. Then $\gamma : \mathrm{CH}_0(X) \to H^{2d}(X)(d)$ factors through $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$.

Proof. By Lemma 45.8.1 it suffices to show: given closed points $x, x' \in X$ whose residue fields are separable over k we have $\deg(x')\gamma([x]) = \deg(x)\gamma([x'])$.

We first reduce to the case of k'-rational points. Let k''/k' be a Galois extension such that $\kappa(x)$ and $\kappa(x')$ embed into k'' over k. Set $Y = X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')$ and denote $p : Y \to X$ the projection. By our choice of k''/k' there exists a k''-rational point y, resp. y' on Y mapping to x, resp. x'. Then $p_*[y] = [k'' : \kappa(x)][x]$ and $p_*[y'] = [k'' : \kappa(x')][x']$ in $\mathrm{CH}_0(X)$. By compatibility with pushforwards given in axiom (C)(b) it suffices to prove $\gamma([y]) = \gamma([y'])$ in $\mathrm{CH}^{2d}(Y)(d)$. This reduces us to the discussion in the next paragraph.

Assume x and x' are k'-rational points. By Lemma 45.8.3 there exists a finite separable extension k''/k' of fields such that the pullback $[y] - [y']$ of the difference $[x] - [x']$ becomes divisible by an integer $n > 1$ on $Y = X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')$. (Note that $y, y' \in Y$ are k''-rational points.) By Lemma 45.10.6 we have $\gamma([y]) = \gamma([y'])$ in $H^{2d}(Y)(d)$. By compatibility with pushforward in axiom (C)(b) we conclude the same for x and x'. □

0FHW Lemma 45.10.8. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $f : X \to Y$ be a dominant morphism of irreducible smooth projective schemes over k. Then $H^*(Y) \to H^*(X)$ is injective.

Proof. There exists an integral closed subscheme $Z \subset X$ of the same dimension as Y mapping onto Y. Thus $f_*[Z] = m[Y]$ for some $m > 0$. Then $f_*\gamma([Z]) = m\gamma([Y]) = m$ in $H^*(Y)$ because of Lemma 45.9.5. Hence by the projection formula (Lemma 45.9.1) we have $f_*(f^*a \cup \gamma([Z])) = ma$ and we conclude. □

0FHX Lemma 45.10.9. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $k''/k'/k$ be finite separable algebras and let X be a smooth projective scheme over k'. Then

$$H^*(X) \otimes_{H^0(\mathrm{Spec}(k'))} H^0(\mathrm{Spec}(k'')) = H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k''))$$

Proof. We will use the results of Lemma 45.10.2 without further mention. Write

$$k' \otimes_k k'' = k'' \times l$$

for some finite separable k'-algebra l. Write $F' = H^0(\mathrm{Spec}(k'))$, $F'' = H^0(\mathrm{Spec}(k''))$, and $G = H^0(\mathrm{Spec}(l))$. Since $\mathrm{Spec}(k') \times \mathrm{Spec}(k'') = \mathrm{Spec}(k'') \amalg \mathrm{Spec}(l)$ we deduce from axiom (B)(a) and Lemma 45.9.9 that we have

$$F' \otimes_F F'' = F'' \times G$$

The map from left to right identifies F'' with $F' \otimes_{F'} F''$. By the same token we have

$$H^*(X) \otimes_F F'' = H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')) \times H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(l))$$

as modules over $F' \otimes_F F'' = F'' \times G$. This proves the lemma. □

45.11. Weil cohomology theories, II

0FHY For us a Weil cohomology theory will be the analogue of a classical Weil cohomology theory (Section 45.7) when the ground field k is not algebraically closed. In Section 45.9 we listed axioms which guarantee our cohomology theory comes from a symmetric monoidal functor on the category of motives over k. Missing from our axioms so far are the condition $H^i(X) = 0$ for $i < 0$ and a condition on $H^{2d}(X)(d)$ for X equidimensional of dimension d corresponding to the classical axioms (A)(c) and (A)(d). Let us first convince the reader that it is necessary to impose such conditions.

0FHZ Example 45.11.1. Let $k = \mathbf{C}$ and $F = \mathbf{C}$ both be equal to the field of complex numbers. For X smooth projective over k denote $H^{p,q}(X) = H^q(X, \Omega^p_{X/k})$. Let $(H')^*$ be the functor which sends X to $(H')^*(X) = \bigoplus H^{p,q}(X)$ with the usual cup product. This is a classical Weil cohomology theory (insert future reference here). By Proposition 45.7.11 we obtain a $\mathbf{Q}$-linear symmetric monoidal functor G' from M_k to the category of graded F-vector spaces. Of course, in this case for every M in M_k the value $G'(M)$ is naturally bigraded, i.e., we have

$$(G')(M) = \bigoplus (G')^{p,q}(M), \quad (G')^n = \bigoplus\nolimits_{n=p+q} (G')^{p,q}(M)$$

with $(G')^{p,q}$ sitting in total degree $p+q$ as indicated. Now we are going to construct a $\mathbf{Q}$-linear symmetric monoidal functor G to the category of graded F-vector spaces by setting

$$G^n(M) = \bigoplus\nolimits_{n=3p-q} (G')^{p,q}(M)$$

We omit the verification that this defines a symmetric monoidal functor (a technical point is that because we chose odd numbers 3 and -1 above the functor G

is compatible with the commutativity constraints). Observe that $G(\mathbf{1}(1))$ is still sitting in degree -2! Hence by Lemma 45.7.9 we obtain a functor H^*, cycle classes γ, and trace maps satisfying all classical axioms (A), (B), (C), except for possibly the classical axioms (A)(a) and (A)(d). However, if E is an elliptic curve over k, then we find $\dim H^{-1}(E) = 1$, i.e., axiom (A)(a) is indeed violated.

0FI0 Lemma 45.11.2. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme over k. Set $k' = \Gamma(X, \mathcal{O}_X)$. The following are equivalent

(1) there exist finitely many closed points $x_1, \ldots, x_r \in X$ whose residue fields are separable over k such that $H^0(X) \to H^0(x_1) \oplus \ldots \oplus H^0(x_r)$ is injective,
(2) the map $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism.

If this is true, then $H^0(X)$ is a finite separable algebra over F. If X is equidimensional of dimension d, then (1) and (2) are also equivalent to

(3) the classes of closed points generate $H^{2d}(X)(d)$ as a module over $H^0(X)$.

Proof. We observe that the statement makes sense because k' is a finite separable algebra over k (Varieties, Lemma 33.9.3) and hence $\mathrm{Spec}(k')$ is smooth and projective over k. The compatibility of H^* with direct sums (Lemmas 45.9.9 and 45.10.1) shows that it suffices to prove the lemma when X is connected. Hence we may assume X is irreducible and we have to show the equivalence of (1), (2), and (3). Set $d = \dim(X)$. This implies that k' is a field finite separable over k and that X is geometrically irreducible over k', see Varieties, Lemmas 33.9.3 and 33.9.4.

By Lemma 45.8.1 we see that the closed points in (3) may be assumed to have separable residue fields over k. By axioms (A)(a) and (A)(b) we see that conditions (1) and (3) are equivalent.

If (2) holds, then pick any closed point $x \in X$ whose residue field is finite separable over k'. Then $H^0(\mathrm{Spec}(k')) = H^0(X) \to H^0(x)$ is injective for example by Lemma 45.10.8.

Assume the equivalent conditions (1) and (3) hold. Choose $x_1, \ldots, x_r \in X$ as in (1). Choose a finite separable extension k''/k'. By Lemma 45.10.9 we have

$$H^0(X) \otimes_{H^0(\mathrm{Spec}(k'))} H^0(\mathrm{Spec}(k'')) = H^0(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k''))$$

Thus in order to show that $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism we may replace k' by k''. Thus we may assume $x_1, \ldots, x_r$ are k'-rational points (this replaces each x_i with multiple points, so r is increased in this step). By Lemma 45.10.7 $\gamma(x_1) = \gamma(x_2) = \ldots = \gamma(x_r)$. By axiom (A)(b) all the maps $H^0(X) \to H^0(x_i)$ are the same. This means (2) holds.

Finally, Lemma 45.10.2 implies $H^0(X)$ is a separable F-algebra if (1) holds. □

0FI1 Lemma 45.11.3. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). If there exists a smooth projective scheme Y over k such that $H^i(Y)$ is nonzero for some $i < 0$, then there exists an equidimensional smooth projective scheme X over k such that the equivalent conditions of Lemma 45.11.2 fail for X.

Proof. By Lemma 45.9.9 we may assume Y is irreducible and a fortiori equidimensional. If i is odd, then after replacing Y by $Y \times Y$ we find an example where Y

is equidimensional and $i = -2l$ for some $l > 0$. Set $X = Y \times (\mathbf{P}^1_k)^l$. Using axiom (B)(a) we obtain

$$H^0(X) \supset H^0(Y) \oplus H^i(Y) \otimes_F H^2(\mathbf{P}^1_k)^{\otimes_F l}$$

with both summands nonzero. Thus it is clear that $H^0(X)$ cannot be isomorphic to H^0 of the spectrum of $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$ as this falls into the first summand. $\square$

Thus it makes sense to finally make the following definition.

0FI2 Definition 45.11.4. Let k be a field. Let F be a field of characteristic 0. A Weil cohomology theory over k with coefficients in F is given by data (D0), (D1), (D2), and (D3) satisfying Poincaré duality, the Künneth formula, and compatibility with cycle classes, more precisely, satisfying axioms (A), (B), and (C) of Section 45.9 and in addition such that the equivalent conditions (1) and (2) of Lemma 45.11.2 hold for every smooth projective X over k.

By Lemma 45.11.3 this means also that there are no nonzero negative cohomology groups. In particular, if k is algebraically closed, then a Weil cohomology theory as above together with an isomorphism $F \to F(1)$ is the same thing as a classical Weil cohomology theory.

0FI3 Remark 45.11.5. Let H^* be a Weil cohomology theory (Definition 45.11.4). Let X be a geometrically irreducible smooth projective scheme of dimension d over k' with k'/k a finite separable extension of fields. Suppose that

$$H^0(\mathrm{Spec}(k')) = F_1 \times \ldots \times F_r$$

for some fields F_i. Then we accordingly can write

$$H^*(X) = \prod\nolimits_{i=1,\ldots,r} H^*(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$$

Now, our final assumption in Definition 45.11.4 tells us that $H^0(X)$ is free of rank 1 over $\prod F_i$. In other words, each of the factors $H^0(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$ has dimension 1 over F_i. Poincaré duality then tells us that the same is true for cohomology in degree $2d$. What isn't clear however is that the same holds in other degrees. Namely, we don't know that given $0 < n < \dim(X)$ the integers

$$\dim_{F_i} H^n(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$$

are independent of i! This question is closely related to the following open question: given an algebraically closed base field $\overline{k}$, a field of characteristic zero F, a classical Weil cohomology theory H^* over $\overline{k}$ with coefficient field F, and a smooth projective variety X over $\overline{k}$ is it true that the betti numbers of X

$$\beta_i = \dim_F H^i(X)$$

are independent of F and the Weil cohomology theory H^*?

0GIJ Proposition 45.11.6. Let k be a field. Let F be a field of characteristic 0. A Weil cohomology theory is the same thing as a $\mathbf{Q}$-linear symmetric monoidal functor

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

such that

(1) $G(\mathbf{1}(1))$ is nonzero only in degree -2, and

(2) for every smooth projective scheme X over k with $k' = \Gamma(X, \mathcal{O}_X)$ the homomorphism $G(h(\text{Spec}(k'))) \to G(h(X))$ of graded F-vector spaces is an isomorphism in degree 0.

Proof. Immediate consequence of Proposition 45.9.12 and Definition 45.11.4. Of course we could replace (2) by the condition that $G(h(X)) \to \bigoplus G(h(x_i))$ is injective in degree 0 for some choice of closed points $x_1, \ldots, x_r \in X$ whose residue fields are separable over k. $\square$

45.12. Chern classes

0FI4 In this section we discuss how given a first Chern class and a projective space bundle formula we can get all Chern classes. A reference for this section is [Gro58] although our axioms are slightly different.

Let $\mathcal{C}$ be a category of schemes with the following properties

1. Every $X \in \text{Ob}(\mathcal{C})$ is quasi-compact and quasi-separated.
2. If $X \in \text{Ob}(\mathcal{C})$ and $U \subset X$ is open and closed, then $U \to X$ is a morphism of $\mathcal{C}$. If $X' \to X$ is a morphism of $\mathcal{C}$ factoring through U, then $X' \to U$ is a morphism of $\mathcal{C}$.
3. If $X \in \text{Ob}(\mathcal{C})$ and if $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module, then
   (a) $p : \mathbf{P}(\mathcal{E}) \to X$ is a morphism of $\mathcal{C}$,
   (b) for a morphism $f : X' \to X$ in $\mathcal{C}$ the induced morphism $\mathbf{P}(f^*\mathcal{E}) \to \mathbf{P}(\mathcal{E})$ is a morphism of $\mathcal{C}$,
   (c) if $\mathcal{E} \to \mathcal{F}$ is a surjection onto another finite locally free $\mathcal{O}_X$-module then the closed immersion $\mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E})$ is a morphism of $\mathcal{C}$.

Next, assume given a contravariant functor A from the category $\mathcal{C}$ to the category of graded algebras. Here a graded algebra A is a unital, associative, not necessarily commutative $\mathbf{Z}$-algebra A endowed with a grading $A = \bigoplus_{i \geq 0} A^i$. Given a morphism $f : X' \to X$ of $\mathcal{C}$ we denote $f^* : A(X) \to A(X')$ the induced algebra map. We will denote the product of $a, b \in A(X)$ by $a \cup b$. Finally, we assume given for every object X of $\mathcal{C}$ an additive map

$$c_1^A : \text{Pic}(X) \longrightarrow A^1(X)$$

We assume the following axioms are satisfied

1. Given $X \in \text{Ob}(\mathcal{C})$ and $\mathcal{L} \in \text{Pic}(X)$ the element $c_1^A(\mathcal{L})$ is in the center of the algebra $A(X)$.
2. If $X \in \text{Ob}(\mathcal{C})$ and $X = U \amalg V$ with U and V open and closed, then $A(X) = A(U) \times A(V)$ via the induced maps $A(X) \to A(U)$ and $A(X) \to A(V)$.
3. If $f : X' \to X$ is a morphism of $\mathcal{C}$ and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then $f^* c_1^A(\mathcal{L}) = c_1^A(f^*\mathcal{L})$.
4. Given $X \in \text{Ob}(\mathcal{C})$ and locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank r consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$ of $\mathcal{C}$. Then the map
$$\bigoplus\nolimits_{i=0,\ldots,r-1} A(X) \longrightarrow A(P), \quad (a_0, \ldots, a_{r-1}) \longmapsto \sum c_1^A(\mathcal{O}_P(1))^i \cup p^*(a_i)$$
is bijective.
5. Let $X \in \text{Ob}(\mathcal{C})$ and let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free $\mathcal{O}_X$-modules of ranks $r + 1$ and r. Denote $i : P' = \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E}) = P$ the corresponding incusion morphism. This is a morphism of $\mathcal{C}$ which exhibits

P' as an effective Cartier divisor on P. Then for $a \in A(P)$ with $i^*a = 0$ we have $a \cup c_1^A(\mathcal{O}_P(P')) = 0$.

To formulate our result recall that $\text{Vect}(X)$ denotes the (exact) category of finite locally free $\mathcal{O}_X$-modules. In Derived Categories of Schemes, Section 36.38 we have defined the zeroth K-group $K_0(\text{Vect}(X))$ of this category. Moreover, we have seen that $K_0(\text{Vect}(X))$ is a ring, see Derived Categories of Schemes, Remark 36.38.6.

0FI5 Proposition 45.12.1. In the situation above there is a unique rule which assigns to every $X \in \text{Ob}(\mathcal{C})$ a "total Chern class"

$$c^A : K_0(\text{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

with the following properties

(1) For $X \in \text{Ob}(\mathcal{C})$ we have $c^A(\alpha + \beta) = c^A(\alpha)c^A(\beta)$ and $c^A(0) = 1$.
(2) If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* \circ c^A = c^A \circ f^*$.
(3) Given $X \in \text{Ob}(\mathcal{C})$ and $\mathcal{L} \in \text{Pic}(X)$ we have $c^A([\mathcal{L}]) = 1 + c_1^A(\mathcal{L})$.

Proof. Let $X \in \text{Ob}(\mathcal{C})$ and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We first show how to define an element $c^A(\mathcal{E}) \in A(X)$.

As a first step, let $X = \bigcup X_r$ be the decomposition into open and closed subschemes such that $\mathcal{E}|_{X_r}$ has constant rank r. Since X is quasi-compact, this decomposition is finite. Hence $A(X) = \prod A(X_r)$. Thus it suffices to define $c^A(\mathcal{E})$ when $\mathcal{E}$ has constant rank r. In this case let $p : P \to X$ be the projective bundle of $\mathcal{E}$. We can uniquely define elements $c_i^A(\mathcal{E}) \in A^i(X)$ for $i \geq 0$ such that $c_0^A(\mathcal{E}) = 1$ and the equation

0FI6 (45.12.1.1)
$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cup p^* c_{r-i}^A(\mathcal{E}) = 0$$

is true. As usual we set $c^A(\mathcal{E}) = c_0^A(\mathcal{E}) + c_1^A(\mathcal{E}) + \ldots + c_r^A(\mathcal{E})$ in $A(X)$.

If $\mathcal{E}$ is invertible, then $c^A(\mathcal{E}) = 1 + c_1^A(\mathcal{L})$. This follows immediately from the construction above.

The elements $c_i^A(\mathcal{E})$ are in the center of $A(X)$. Namely, to prove this we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ be the corresponding projective bundle. if $a \in A(X)$ then $p^*a \cup (-1)^r c_1(\mathcal{O}_P(1))^r = (-1)^r c_1(\mathcal{O}_P(1))^r \cup p^*a$ and hence we must have the same for all the other terms in the expression defining $c_i^A(\mathcal{E})$ as well and we conclude.

If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^*c_i^A(\mathcal{E}) = c_i^A(f^*\mathcal{E})$. Namely, to prove this we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ and $p' : P' \to X'$ be the projective bundles corresponding to $\mathcal{E}$ and $f^*\mathcal{E}$. The induced morphism $g : P' \to P$ is a morphism of $\mathcal{C}$. The pullback by g of the equality defining $c_i^A(\mathcal{E})$ is the corresponding equation for $f^*\mathcal{E}$ and we conclude.

Let $X \in \text{Ob}(\mathcal{C})$. Consider a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

of finite locally free $\mathcal{O}_X$-modules with $\mathcal{L}$ invertible. Then

$$c^A(\mathcal{E}) = c^A(\mathcal{L})c^A(\mathcal{F})$$

Namely, by the construction of c_i^A we may assume $\mathcal{E}$ has constant rank $r+1$ and $\mathcal{F}$ has constant rank r. The inclusion

$$i : P' = \mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}(\mathcal{E}) = P$$

is a morphism of $\mathcal{C}$ and it is the zero scheme of a regular section of the invertible module $\mathcal{L}^{\otimes -1} \otimes \mathcal{O}_P(1)$. The element

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1^A(\mathcal{O}_P(1))^i \cup p^* c_i^A(\mathcal{F})$$

pulls back to zero on P' by definition. Hence we see that

$$\left(c_1^A(\mathcal{O}_P(1)) - c_1^A(\mathcal{L})\right) \cup \left(\sum\nolimits_{i=0}^{r} (-1)^i c_1^A(\mathcal{O}_P(1))^i \cup p^* c_i^A(\mathcal{F})\right) = 0$$

in $A^*(P)$ by assumption (5) on our cohomology A. By definition of $c_1^A(\mathcal{E})$ this gives the desired equality.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of finite locally free $\mathcal{O}_X$-modules. Then

$$c^A(\mathcal{F}) = c^A(\mathcal{E}) c^A(\mathcal{G})$$

Namely, by the construction of c_i^A we may assume $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ have constant ranks r, s, and t. We prove it by induction on r. The case $r = 1$ was done above. If $r > 1$, then it suffices to check this after pulling back by the morphism $\mathbf{P}(\mathcal{E}^\vee) \to X$. Thus we may assume we have an invertible submodule $\mathcal{L} \subset \mathcal{E}$ such that both $\mathcal{E}' = \mathcal{E}/\mathcal{L}$ and $\mathcal{F}' = \mathcal{E}/\mathcal{L}$ are finite locally free (of ranks $s-1$ and $t-1$). Then we have

$$c^A(\mathcal{E}) = c^A(\mathcal{L}) c^A(\mathcal{E}') \quad \text{and} \quad c^A(\mathcal{F}) = c^A(\mathcal{L}) c^A(\mathcal{F}')$$

Since we have the short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{F}' \to \mathcal{G} \to 0$$

we see by induction hypothesis that

$$c^A(\mathcal{F}') = c^A(\mathcal{E}') c^A(\mathcal{G})$$

Thus the result follows from a formal calculation.

At this point for $X \in \mathrm{Ob}(\mathcal{C})$ we can define $c^A : K_0(\mathrm{Vect}(X)) \to A(X)$. Namely, we send a generator $[\mathcal{E}]$ to $c^A(\mathcal{E})$ and we extend multiplicatively. Thus for example $c^A(-[\mathcal{E}]) = c^A(\mathcal{E})^{-1}$ is the formal inverse of $a^A([\mathcal{E}])$. The multiplicativity in short exact sequences shown above guarantees that this works.

Uniqueness. Suppose $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module. We want to show that conditions (1), (2), and (3) of the lemma uniquely determine $c^A([\mathcal{E}])$. To prove this we may assume $\mathcal{E}$ has constant rank r; this already uses (2). Then we may use induction on r. If $r = 1$, then uniqueness follows from (3). If $r > 1$ we pullback using (2) to the projective bundle $p : P \to X$ and we see that we may assume we have a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ with $\mathcal{E}'$ and $\mathcal{E}''$ having lower rank. By induction hypothesis $c^A(\mathcal{E}')$ and $c^A(\mathcal{E}'')$ are uniquely determined. Thus uniqueness for $\mathcal{E}$ by the axiom (1). □

0FI7 **Lemma 45.12.2.** In the situation above. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a morphism $p : P \to X$ in $\mathcal{C}$ such that

(1) $p^* : A(X) \to A(P)$ is injective,
(2) each $p^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.

Proof. We may assume $r_i \geq 1$ for all i. We will prove the lemma by induction on $\sum(r_i - 1)$. If this integer is 0, then $\mathcal{E}_i$ is invertible for all i and we conclude by taking $\pi = \text{id}_X$. If not, then we can pick an i such that $r_i > 1$ and consider the projective bundle $p : P \to X$ associated to $\mathcal{E}_i$. We have a short exact sequence

$$0 \to \mathcal{F} \to p^*\mathcal{E}_i \to \mathcal{O}_P(1) \to 0$$

of finite locally free $\mathcal{O}_P$-modules of ranks $r_i - 1$, r_i, and 1. Observe that $p^* : A(X) \to A(P)$ is injective by assumption. By the induction hypothesis applied to the finite locally free $\mathcal{O}_P$-modules $\mathcal{F}$ and $p^*\mathcal{E}_{i'}$ for $i' \neq i$, we find a morphism $p' : P' \to P$ with properties stated as in the lemma. Then the composition $p \circ p' : P' \to X$ does the job. $\square$

0FI8 Lemma 45.12.3. Let $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

$$c_i^A(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r - i + j}{j} c_{i-j}^A(\mathcal{E}) \cup c_1^A(\mathcal{L})^j$$

Proof. By the construction of c_i^A we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ and $p' : P' \to X$ be the projective bundle associated to $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}$. Then there is an isomorphism $g : P \to P'$ such that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes p^*\mathcal{L}$. See Constructions, Lemma 27.20.1. Thus

$$g^*c_1^A(\mathcal{O}_{P'}(1)) = c_1^A(\mathcal{O}_P(1)) + p^*c_1^A(\mathcal{L})$$

The desired equality follows formally from this and the definition of Chern classes using equation (45.12.1.1). $\square$

0FI9 Proposition 45.12.4. In the situation above assume $A(X)$ is a $\mathbf{Q}$-algebra for all $X \in \text{Ob}(\mathcal{C})$. Then there is a unique rule which assigns to every $X \in \text{Ob}(\mathcal{C})$ a "chern character"

$$ch^A : K_0(\text{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

with the following properties

(1) ch^A is a ring map for all $X \in \text{Ob}(\mathcal{C})$.
(2) If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* \circ ch^A = ch^A \circ f^*$.
(3) Given $X \in \text{Ob}(\mathcal{C})$ and $\mathcal{L} \in \text{Pic}(X)$ we have $ch^A([\mathcal{L}]) = \exp(c_1^A(\mathcal{L}))$.

Proof. Let $X \in \text{Ob}(\mathcal{C})$ and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We first show how to define the rank $r^A(\mathcal{E}) \in A^0(X)$. Namely, let $X = \bigcup X_r$ be the decomposition into open and closed subschemes such that $\mathcal{E}|_{X_r}$ has constant rank r. Since X is quasi-compact, this decomposition is finite, say $X = X_0 \amalg X_1 \amalg \ldots \amalg X_n$. Then $A(X) = A(X_0) \times A(X_1) \times \ldots \times A(X_n)$. Thus we can define $r^A(\mathcal{E}) = (0, 1, \ldots, n) \in A^0(X)$.

Let $P_p(c_1, \ldots, c_p)$ be the polynomials constructed in Chow Homology, Example 42.43.6. Then we can define

$$ch^A(\mathcal{E}) = r^A(\mathcal{E}) + \sum\nolimits_{i \geq 1} (1/i!) P_i(c_1^A(\mathcal{E}), \ldots, c_i^A(\mathcal{E})) \in \prod\nolimits_{i \geq 0} A^i(X)$$

where ci^A are the Chern classes of Proposition 45.12.1. It follows immediately that we have property (2) and (3) of the lemma.

We still have to show the following three statements

(1) If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules on $X \in \mathrm{Ob}(\mathcal{C})$, then $ch^A(\mathcal{E}) = ch^A(\mathcal{E}_1) + ch^A(\mathcal{E}_2)$.
(2) If $\mathcal{E}_1$ and $\mathcal{E}_2 \to 0$ are finite locally free $\mathcal{O}_X$-modules on $X \in \mathrm{Ob}(\mathcal{C})$, then $ch^A(\mathcal{E}_1 \otimes \mathcal{E}_2) = ch^A(\mathcal{E}_1)ch^A(\mathcal{E}_2)$.

Namely, the first will prove that ch^A factors through $K_0(\mathrm{Vect}(X))$ and the first and the second will combined show that ch^A is a ring map.

To prove these statements we can reduce to the case where $\mathcal{E}_1$ and $\mathcal{E}_2$ have constant ranks r_1 and r_2. In this case the equalities in $A^0(X)$ are immediate. To prove the equalities in higher degrees, by Lemma 45.12.2 we may assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ have filtrations whose graded pieces are invertible modules $\mathcal{L}_{1,j}$, $j = 1, \ldots, r_1$ and $\mathcal{L}_{2,j}$, $j = 1, \ldots, r_2$. Using the multiplicativity of Chern classes we get

$$c_i^A(\mathcal{E}_1) = s_i(c_1^A(\mathcal{L}_{1,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}))$$

where s_i is the ith elementary symmetric function as in Chow Homology, Example 42.43.6. Similarly for $c_i^A(\mathcal{E}_2)$. In case (1) we get

$$c_i^A(\mathcal{E}) = s_i(c_1^A(\mathcal{L}_{1,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}), c_1^A(\mathcal{L}_{2,1}), \ldots, c_1^A(\mathcal{L}_{2,r_2}))$$

and for case (2) we get

$$c_i^A(\mathcal{E}_1 \otimes \mathcal{E}_2) = s_i(c_1^A(\mathcal{L}_{1,1}) + c_1^A(\mathcal{L}_{2,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}) + c_1^A(\mathcal{L}_{2,r_2}))$$

By the definition of the polynomials P_i we see that this means

$$P_i(c_1^A(\mathcal{E}_1), \ldots, c_i^A(\mathcal{E}_1)) = \sum\nolimits_{j=1,\ldots,r_1} c_1^A(\mathcal{L}_{1,j})^i$$

and similarly for $\mathcal{E}_2$. In case (1) we have also

$$P_i(c_1^A(\mathcal{E}), \ldots, c_i^A(\mathcal{E})) = \sum\nolimits_{j=1,\ldots,r_1} c_1^A(\mathcal{L}_{1,j})^i + \sum\nolimits_{j=1,\ldots,r_2} c_1^A(\mathcal{L}_{2,j})^i$$

In case (2) we get accordingly

$$P_i(c_1^A(\mathcal{E}_1 \otimes \mathcal{E}_2), \ldots, c_i^A(\mathcal{E}_1 \otimes \mathcal{E}_2)) = \sum\nolimits_{j=1,\ldots,r_1} \sum\nolimits_{j'=1,\ldots,r_2} (c_1^A(\mathcal{L}_{1,j}) + c_1^A(\mathcal{L}_{2,j'}))^i$$

Thus the desired equalities are now consequences of elementary identities between symmetric polynomials.

We omit the proof of uniqueness. □

0FIA **Lemma 45.12.5.** In the situation above let $X \in \mathrm{Ob}(\mathcal{C})$. If ψ^2 is as in Chow Homology, Lemma 42.56.1 and c^A and ch^A are as in Propositions 45.12.1 and 45.12.4 then we have $c_i^A(\psi^2(\alpha)) = 2^i c_i^A(\alpha)$ and $ch_i^A(\psi^2(\alpha)) = 2^i ch_i^A(\alpha)$ for all $\alpha \in K_0(\mathrm{Vect}(X))$.

Proof. Observe that the map $\prod_{i \geq 0} A^i(X) \to \prod_{i \geq 0} A^i(X)$ multiplying by 2^i on $A^i(X)$ is a ring map. Hence, since ψ^2 is also a ring map, it suffices to prove the formulas for additive generators of $K_0(\mathrm{Vect}(X))$. Thus we may assume $\alpha = [\mathcal{E}]$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. By construction of the Chern classes of $\mathcal{E}$ we immediately reduce to the case where $\mathcal{E}$ has constant rank r. In this case, we can choose a projective smooth morphism $p : P \to X$ such that restriction $A^*(X) \to A^*(P)$ is injective and such that $p^*\mathcal{E}$ has a finite filtration whose graded parts are invertible $\mathcal{O}_P$-modules $\mathcal{L}_j$, see Lemma 45.12.2. Then $[p^*\mathcal{E}] = \sum[\mathcal{L}_j]$ and hence $\psi^2([p^*\mathcal{E}]) = \sum[\mathcal{L}_j^{\otimes 2}]$ by definition of ψ^2. Setting $x_j = c_1^A(\mathcal{L}_j)$ we have

$$c^A(\alpha) = \prod(1 + x_j) \quad \text{and} \quad c^A(\psi^2(\alpha)) = \prod(1 + 2x_j)$$

in $\prod A^i(P)$ and we have

$$ch^A(\alpha) = \sum \exp(x_j) \quad \text{and} \quad ch^A(\psi^2(\alpha)) = \sum \exp(2x_j)$$

in $\prod A^i(P)$. From these formulas the desired result follows. ☐

45.13. Exterior powers and K-groups

0FIB We do the minimal amount of work to define the lambda operators. Let X be a scheme. Recall that $\text{Vect}(X)$ denotes the category of finite locally free $\mathcal{O}_X$-modules. Moreover, recall that we have constructed a zeroth K-group $K_0(\text{Vect}(X))$ associated to this category in Derived Categories of Schemes, Section 36.38. Finally, $K_0(\text{Vect}(X))$ is a ring, see Derived Categories of Schemes, Remark 36.38.6.

0FIC **Lemma 45.13.1.** Let X be a scheme. There are maps

$$\lambda^r : K_0(\text{Vect}(X)) \longrightarrow K_0(\text{Vect}(X))$$

which sends $[\mathcal{E}]$ to $[\wedge^r(\mathcal{E})]$ when $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and which are compatible with pullbacks.

Proof. Consider the ring $R = K_0(\text{Vect}(X))[[t]]$ where t is a variable. For a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we set

$$c(\mathcal{E}) = \sum\nolimits_{i=0}^{\infty} [\wedge^i(\mathcal{E})]t^i$$

in R. We claim that given a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

of finite locally free $\mathcal{O}_X$-modules we have $c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'')$. The claim implies that c extends to a map

$$c : K_0(\text{Vect}(X)) \longrightarrow R$$

which converts addition in $K_0(\text{Vect}(X))$ to multiplication in R. Writing $c(\alpha) = \sum \lambda^i(\alpha)t^i$ we obtain the desired operators λ^i.

To see the claim, we consider the short exact sequence as a filtration on $\mathcal{E}$ with 2 steps. We obtain an induced filtration on $\wedge^r(\mathcal{E})$ with $r+1$ steps and subquotients

$$\wedge^r(\mathcal{E}'), \wedge^{r-1}(\mathcal{E}') \otimes \mathcal{E}'', \wedge^{r-2}(\mathcal{E}') \otimes \wedge^2(\mathcal{E}''), \ldots \wedge^r(\mathcal{E}'')$$

Thus we see that $[\wedge^r(\mathcal{E})]$ is equal to

$$\sum\nolimits_{i=0}^{r} [\wedge^{r-i}(\mathcal{E}')][\wedge^i(\mathcal{E}'')]$$

and the result follows easily from this and elementary algebra. ☐

45.14. Weil cohomology theories, III

0FID Let k be a field. Let F be a field of characteristic zero. Suppose we are given the following data

(D0) A 1-dimensional F-vector space $F(1)$.
(D1) A contravariant functor $H^*(-)$ from the category of smooth projective schemes over k to the category of graded commutative F-algebras.
(D2') For every smooth projective scheme X over k a homomorphism $c_1^H : \text{Pic}(X) \to H^2(X)(1)$ of abelian groups.

We will use the terminology, notation, and conventions regarding (D0) and (D1) as discussed in Section 45.9. Given a smooth projective scheme X over k and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ the cohomology class $c_1^H(\mathcal{L}) \in H^2(X)(1)$ of (D2') is sometimes called the first Chern class of $\mathcal{L}$ in cohomology.

Here is the list of axioms.

(A1) H^* is compatible with finite coproducts
(A2) c_1^H is compatible with pullbacks
(A3) Let X be a smooth projective scheme over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r \geq 1$. Consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$. Then the map

$$\bigoplus\nolimits_{i=0,\ldots,r-1} H^*(X)(-i) \longrightarrow H^*(P), \quad (a_0, \ldots, a_{r-1}) \longmapsto \sum c_1^H(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

is an isomorphism of F-vector spaces.
(A4) Let $i : Y \to X$ be the inclusion of an effective Cartier divisor over k with both X and Y smooth and projective over k. For $a \in H^*(X)$ with $i^*a = 0$ we have $a \cup c_1^H(\mathcal{O}_X(Y)) = 0$.
(A5) H^* is compatible with finite products
(A6) Let X be a nonempty smooth, projective scheme over k equidimensional of dimension d. Then there exists an F-linear map $\lambda : H^{2d}(X)(d) \to F$ such that $(\text{id} \otimes \lambda)\gamma([\Delta]) = 1$ in $H^*(X)$.
(A7) If $b : X' \to X$ is the blowing up of a smooth center in a smooth projective scheme X over k[2], then $b^* : H^*(X) \to H^*(X')$ is injective.
(A8) If X is a smooth projective scheme over k and $k' = \Gamma(X, \mathcal{O}_X)$, then the map $H^0(\text{Spec}(k')) \to H^0(X)$ is an isomorphism.
(A9) Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Let $i : Y \to X$ be a nonempty effective Cartier divisor smooth over k. For $a \in H^{2d-2}(X)(d-1)$ we have $\lambda_Y(i^*(a)) = \lambda_X(a \cup c_1^H(\mathcal{O}_X(Y))$ where λ_Y and λ_X are as in axiom (A6) for X and Y.

Let us explain more precisely what we mean by each of these axioms. Axioms (A3), (A4), and (A7) are clear as stated.

Ad (A1). This means that $H^*(\emptyset) = 0$ and that $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism where i and j are the coprojections.

Ad (A2). This means that given a morphism $f : X \to Y$ of smooth projective schemes over k and an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ we have $f^*c_1^H(\mathcal{L}) = c_1^H(f^*\mathcal{L})$.

[2] Then X' is smooth and projective over k as well, see More on Morphisms, Lemma 37.17.3.

Ad (A5). This means that $H^*(\mathrm{Spec}(k)) = F$ and that for X and Y smooth projective over k the map $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $a \otimes b \mapsto p^*(a) \cup q^*(b)$ is an isomorphism where p and q are the projections.

Ad (A6). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. By Lemma 45.14.2 if we have axioms (A1) – (A4) we can consider the class of the diagonal

$$\gamma([\Delta]) \in H^{2d}(X \times X)(d) = \bigoplus\nolimits_i H^i(X) \otimes_F H^{2d-i}(X)(d)$$

where the tensor decomposition comes from axiom (A5). Given an F-linear map $\lambda : H^{2d}(X)(d) \to F$ we may also view λ as an F-linear map $\lambda : H^*(X)(d) \to F$ by precomposing with the projection onto $H^{2d}(X)(d)$. Having said this axiom (A6) makes sense.

Ad (A8). Let X be a smooth projective scheme over k. Then $k' = \Gamma(X, \mathcal{O}_X)$ is a finite separable k-algebra (Varieties, Lemma 33.9.3) and hence $\mathrm{Spec}(k')$ is smooth and projective over k. Thus we may apply H^* to $\mathrm{Spec}(k')$ and axiom (A8) makes sense.

Ad (A9). We will see in Remark 45.14.6 that if we have axioms (A1) – (A7) then the map λ of axiom (A6) is unique.

0FIE Lemma 45.14.1. Assume given (D0), (D1), and (D2') satisfying axioms (A1), (A2), (A3), and (A4). There is a unique rule which assigns to every smooth projective X over k a ring homomorphism

$$ch^H : K_0(\mathrm{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}(X)(i)$$

compatible with pullbacks such that $ch^H(\mathcal{L}) = \exp(c_1^H(\mathcal{L}))$ for any invertible $\mathcal{O}_X$-module $\mathcal{L}$.

Proof. Immediate from Proposition 45.12.4 applied to the category of smooth projective schemes over k, the functor $A : X \mapsto \bigoplus_{i \geq 0} H^{2i}(X)(i)$, and the map c_1^H. $\square$

0FIF Lemma 45.14.2. Assume given (D0), (D1), and (D2') satisfying axioms (A1), (A2), (A3), and (A4). There is a unique rule which assigns to every smooth projective X over k a graded ring homomorphism

$$\gamma : \mathrm{CH}^*(X) \longrightarrow \bigoplus\nolimits_{i \geq 0} H^{2i}(X)(i)$$

compatible with pullbacks such that $ch^H(\alpha) = \gamma(ch(\alpha))$ for α in $K_0(\mathrm{Vect}(X))$.

Proof. Recall that we have an isomorphism

$$K_0(\mathrm{Vect}(X)) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}^*(X) \otimes \mathbf{Q}, \quad \alpha \longmapsto ch(\alpha) \cap [X]$$

see Chow Homology, Lemma 42.58.1. It is an isomorphism of rings by Chow Homology, Remark 42.56.5. We define γ by the formula $\gamma(\alpha) = ch^H(\alpha')$ where ch^H is as in Lemma 45.14.1 and $\alpha' \in K_0(\mathrm{Vect}(X))$ is such that $ch(\alpha') \cap [X] = \alpha$ in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$.

The construction $\alpha \mapsto \gamma(\alpha)$ is compatible with pullbacks because both ch^H and taking Chern classes is compatible with pullbacks, see Lemma 45.14.1 and Chow Homology, Remark 42.59.9.

We still have to see that γ is graded. Let $\psi^2 : K_0(\text{Vect}(X)) \to K_0(\text{Vect}(X))$ be the second Adams operator, see Chow Homology, Lemma 42.56.1. If $\alpha \in \text{CH}^i(X)$ and $\alpha' \in K_0(\text{Vect}(X)) \otimes \mathbf{Q}$ is the unique element with $ch(\alpha') \cap [X] = \alpha$, then we have seen in Chow Homology, Section 42.58 that $\psi^2(\alpha') = 2^i\alpha'$. Hence we conclude that $ch^H(\alpha') \in H^{2i}(X)(i)$ by Lemma 45.12.5 as desired. $\square$

0FIH Lemma 45.14.3. Let $b : X' \to X$ be the blowing up of a smooth projective scheme over k in a smooth closed subscheme $Z \subset X$. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Assume there exists an element of $K_0(X)$ whose restriction to Z is equal to the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$. Assume every irreducible component of Z has codimension r in X. Then there exists a cycle $\theta \in \text{CH}^{r-1}(X')$ such that $b^![Z] = [E] \cdot \theta$ in $\text{CH}^r(X')$ and $\pi_* j^!(\theta) = [Z]$ in $\text{CH}^r(Z)$.

Proof. The scheme X is smooth and projective over k and hence we have $K_0(X) = K_0(\text{Vect}(X))$. See Derived Categories of Schemes, Lemmas 36.36.2 and 36.38.5. Let $\alpha \in K_0(\text{Vect}(X))$ be an element whose restriction to Z is $\mathcal{C}_{Z/X}$. By Chow Homology, Lemma 42.56.3 there exists an element $\alpha^\vee$ which restricts to $\mathcal{C}_{Z/X}^\vee$. By the blow up formula (Chow Homology, Lemma 42.59.11) we have

$$b^![Z] = b^! i_*[Z] = j_* res(b^!)([Z]) = j_*(c_{r-1}(\mathcal{F}^\vee) \cap \pi^*[Z]) = j_*(c_{r-1}(\mathcal{F}^\vee) \cap [E])$$

where $\mathcal{F}$ is the kernel of the surjection $\pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'}$. Observe that $b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]$ is an element of $K_0(\text{Vect}(X'))$ which restricts to $[\pi^*\mathcal{C}_{Z/X}^\vee] - [\mathcal{C}_{E/X'}^\vee] = [\mathcal{F}^\vee]$ on E. Since capping with Chern classes commutes with j_* we conclude that the above is equal to

$$c_{r-1}(b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [E]$$

in the chow group of X'. Hence we see that setting

$$\theta = c_{r-1}(b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [X']$$

we get the first relation $\theta \cdot [E] = b^![Z]$ for example by Chow Homology, Lemma 42.62.2. For the second relation observe that

$$j^!\theta = j^!(c_{r-1}(b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [X']) = c_{r-1}(\mathcal{F}^\vee) \cap j^![X'] = c_{r-1}(\mathcal{F}^\vee) \cap [E]$$

in the chow groups of E. To prove that π_* of this is equal to $[Z]$ it suffices to prove that the degree of the codimension $r-1$ cycle $(-1)^{r-1}c_{r-1}(\mathcal{F}) \cap [E]$ on the fibres of π is 1. This is a computation we omit. $\square$

0FII Lemma 45.14.4. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A4) and (A7). Let X be a smooth projective scheme over k. Let $Z \subset X$ be a smooth closed subscheme such that every irreducible component of Z has codimension r in X. Assume the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$ is the restriction of an element of $K_0(X)$. If $a \in H^*(X)$ and $a|_Z = 0$ in $H^*(Z)$, then $\gamma([Z]) \cup a = 0$.

Proof. Let $b : X' \to X$ be the blowing up. By (A7) it suffices to show that

$$b^*(\gamma([Z]) \cup a) = b^*\gamma([Z]) \cup b^*a = 0$$

By Lemma 45.14.3 we have

$$b^*\gamma([Z]) = \gamma(b^*[Z]) = \gamma([E] \cdot \theta) = \gamma([E]) \cup \gamma(\theta)$$

Hence because b^*a restricts to zero on E and since $\gamma([E]) = c_1^H(\mathcal{O}_{X'}(E))$ we get what we want from (A4). $\square$

0FIJ Lemma 45.14.5. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (A) of Section 45.9 holds with $\int_X = \lambda$ as in axiom (A6).

Proof. Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. We will show that the graded F-vector space $H^*(X)(d)[2d]$ is a left dual to $H^*(X)$. This will prove what we want by Homology, Lemma 12.17.5. We are going to use axiom (A5) which in particular says that

$$H^*(X \times X)(d) = \bigoplus H^i(X) \otimes H^j(X)(d) = \bigoplus H^i(X)(d) \otimes H^j(X)$$

Define a map

$$\eta : F \longrightarrow H^*(X \times X)(d)$$

by multiplying by $\gamma([\Delta]) \in H^{2d}(X \times X)(d)$. On the other hand, define a map

$$\epsilon : H^*(X \times X)(d) \longrightarrow H^*(X)(d) \xrightarrow{\lambda} F$$

by first using pullback Δ^* by the diagonal morphism $\Delta : X \to X \times X$ and then using the F-linear map $\lambda : H^{2d}(X)(d) \to F$ of axiom (A6) precomposed by the projection $H^*(X)(d) \to H^{2d}(X)(d)$. In order to show that $H^*(X)(d)$ is a left dual to $H^*(X)$ we have to show that the composition of the maps

$$\eta \otimes 1 : H^*(X) \longrightarrow H^*(X \times X \times X)(d)$$

and

$$1 \otimes \epsilon : H^*(X \times X \times X)(d) \longrightarrow H^*(X)$$

is the identity. If $a \in H^*(X)$ then we see that the composition maps a to

$$(1 \otimes \lambda)(\Delta_{23}^*(q_{12}^*\gamma([\Delta]) \cup q_3^*a)) = (1 \otimes \lambda)(\gamma([\Delta]) \cup p_2^*a)$$

where $q_i : X \times X \times X \to X$ and $q_{ij} : X \times X \times X \to X \times X$ are the projections, $\Delta_{23} : X \times X \to X \times X \times X$ is the diagonal, and $p_i : X \times X \to X$ are the projections. The equality holds because $\Delta_{23}^*(q_{12}^*\gamma([\Delta]) = \Delta_{23}^*\gamma([\Delta \times X]) = \gamma([\Delta])$ and because $\Delta_{23}^*q_3^*a = p_2^*a$. Since $\gamma([\Delta]) \cup p_1^*a = \gamma([\Delta]) \cup p_2^*a$ (see below) the above simplifies to

$$(1 \otimes \lambda)(\gamma([\Delta]) \cup p_1^*a) = a$$

by our choice of λ as desired. The second condition $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}$ of Categories, Definition 4.43.5 is proved in exactly the same manner.

Note that p_1^*a and pr_2^*a restrict to the same cohomology class on $\Delta \subset X \times X$. Moreover we have $\mathcal{C}_{\Delta/X \times X} = \Omega^1_\Delta$ which is the restriction of $p_1^*\Omega^1_X$. Hence Lemma 45.14.4 implies $\gamma([\Delta]) \cup p_1^*a = \gamma([\Delta]) \cup p_2^*a$ and the proof is complete. $\square$

0FIK Remark 45.14.6 (Uniqueness of trace maps). Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Let X be a smooth projective scheme over k which is nonempty and equidimensional of dimension d. Combining what was said in the proofs of Lemma 45.14.5 and Homology, Lemma 12.17.5 we see that

$$\gamma([\Delta]) \in \bigoplus\nolimits_i H^i(X) \otimes H^{2d-i}(X)(d)$$

defines a perfect duality between $H^i(X)$ and $H^{2d-i}(X)(d)$ for all i. In particular, the linear map $\int_X = \lambda : H^{2d}(X)(d) \to F$ of axiom (A6) is unique! We will call the linear map $\int_X$ the trace map of X from now on.

0FIL Lemma 45.14.7. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (B) of Section 45.9 holds.

Proof. Axiom (B)(a) is immediate from axiom (A5). Let X and Y be nonempty smooth projective schemes over k equidimensional of dimensions d and e. To see that axiom (B)(b) holds, observe that the diagonal $\Delta_{X \times Y}$ of $X \times Y$ is the intersection product of the pullbacks of the diagonals Δ_X of X and Δ_Y of Y by the projections $p : X \times Y \times X \times Y \to X \times X$ and $q : X \times Y \times X \times Y \to Y \times Y$. Compatibility of γ with intersection products then gives that

$$\gamma([\Delta_{X \times Y}]) \in H^{2d+2e}(X \times Y \times X \times Y)(d+e)$$

is the cup product of the pullbacks of $\gamma([\Delta_X])$ and $\gamma([\Delta_Y])$ by p and q. Write

$$\gamma([\Delta_{X \times Y}]) = \sum \eta_{X \times Y,i} \text{ with } \eta_{X \times Y,i} \in H^i(X \times Y) \otimes H^{2d+2e-i}(X \times Y)(d+e)$$

and simiarly $\gamma([\Delta_X]) = \sum \eta_{X,i}$ and $\gamma([\Delta_Y]) = \sum \eta_{Y,i}$. The observation above implies we have

$$\eta_{X \times Y,0} = \sum\nolimits_{i \in \mathbf{Z}} p^*\eta_{X,i} \cup q^*\eta_{Y,-i}$$

(If our cohomology theory vanishes in negative degrees, which will be true in almost all cases, then only the term for $i = 0$ contributes and $\eta_{X\times Y,0}$ lies in $H^0(X) \otimes H^0(Y) \otimes H^{2d}(X)(d) \otimes H^{2e}(Y)(e)$ as expected, but we don't need this.) Since $\lambda_X : H^{2d}(X)(d) \to F$ and $\lambda_Y : H^{2e}(Y)(e) \to F$ send $\eta_{X,0}$ and $\eta_{Y,0}$ to 1 in $H^*(X)$ and $H^*(Y)$, we see that $\lambda_X \otimes \lambda_Y$ sends $\eta_{X\times Y,0}$ to 1 in $H^*(X) \otimes H^*(Y) = H^*(X \times Y)$ and the proof is complete. □

0FIM Lemma 45.14.8. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (C)(d) of Section 45.9 holds.

Proof. We have $\gamma([\mathrm{Spec}(k)]) = 1 \in H^*(\mathrm{Spec}(k))$ by construction. Since

$$H^0(\mathrm{Spec}(k)) = F, \quad H^0(\mathrm{Spec}(k) \times \mathrm{Spec}(k)) = H^0(\mathrm{Spec}(k)) \otimes_F H^0(\mathrm{Spec}(k))$$

the map $\int_{\mathrm{Spec}(k)} = \lambda$ of axiom (A6) must send 1 to 1 because we have seen that $\int_{\mathrm{Spec}(k)\times\mathrm{Spec}(k)} = \int_{\mathrm{Spec}(k)} \int_{\mathrm{Spec}(k)}$ in Lemma 45.14.7. □

Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then we obtain data (D0), (D1), (D2), and (D3) of Section 45.9 satisfying axioms (A), (B) and (C)(a), (C)(c), and (C)(d) of Section 45.9, see Lemmas 45.14.5, 45.14.7, and 45.14.8. Moreover, we have the pushforwards $f_* : H^*(X) \to H^*(Y)$ as constructed in Section 45.9. The only axiom of Section 45.9 which isn't clear yet is axiom (C)(b).

0FIN Lemma 45.14.9. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Let $p : P \to X$ be as in axiom (A3) with X nonempty equidimensional. Then γ commutes with pushforward along p.

Proof. It suffices to prove this on generators for $\mathrm{CH}_*(P)$. Thus it suffices to prove this for a cycle class of the form $\xi^i \cdot p^*\alpha$ where $0 \leq i \leq r-1$ and $\alpha \in \mathrm{CH}_*(X)$. Note that $p_*(\xi^i \cdot p^*\alpha) = 0$ if $i < r-1$ and $p_*(\xi^{r-1} \cdot p^*\alpha) = \alpha$. On the other hand, we have $\gamma(\xi^i \cdot p^*\alpha) = c^i \cup p^*\gamma(\alpha)$ and by the projection formula (Lemma 45.9.1) we have

$$p_*\gamma(\xi^i \cdot p^*\alpha) = p_*(c^i) \cup \gamma(\alpha)$$

Thus it suffices to show that $p_*c^i = 0$ for $i < r-1$ and $p_*c^{r-1} = 1$. Equivalently, it suffices to prove that $\lambda_P : H^{2d+2r-2}(P)(d+r-1) \to F$ defined by the rules

$$\lambda_P(c^i \cup p^*(a)) = \begin{cases} 0 & \text{if} \quad i < r-1 \\ \int_X(a) & \text{if} \quad i = r-1 \end{cases}$$

satisfies the condition of axiom (A5). This follows from the computation of the class of the diagonal of P in Lemma 45.6.2. □

0FVR Lemma 45.14.10. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). If k'/k is a Galois extension, then we have $\int_{\mathrm{Spec}(k')} 1 = [k' : k]$.

Proof. We observe that

$$\mathrm{Spec}(k') \times \mathrm{Spec}(k') = \coprod\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} (\mathrm{Spec}(\sigma) \times \mathrm{id})^{-1}\Delta$$

as cycles on $\mathrm{Spec}(k') \times \mathrm{Spec}(k')$. Our construction of γ always sends $[X]$ to 1 in $H^0(X)$. Thus $1 \otimes 1 = 1 = \sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^*\gamma([\Delta])$. Denote $\lambda : H^0(\mathrm{Spec}(k')) \to F$ the map from axiom (A6), in other words $(\mathrm{id} \otimes \lambda)(\gamma(\Delta)) = 1$ in $H^0(\mathrm{Spec}(k'))$. We obtain

$$\begin{aligned} \lambda(1)1 &= (\mathrm{id} \otimes \lambda)(1 \otimes 1) \\ &= (\mathrm{id} \otimes \lambda)(\sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^*\gamma([\Delta])) \\ &= \sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^*((\mathrm{id} \otimes \lambda)(\gamma([\Delta])) \\ &= \sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^*(1) \\ &= [k' : k] \end{aligned}$$

Since λ is another name for $\int_{\mathrm{Spec}(k')}$ (Remark 45.14.6) the proof is complete. □

0FIP Lemma 45.14.11. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). In order to show that γ commutes with pushforward it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k.

Proof. We will use without further mention that we've constructed our cycle class map γ in Lemma 45.14.2 compatible with intersection products and pullbacks and that we've already shown axioms (A), (B), (C)(a), (C)(c), and (C)(d) of Section 45.9, see Lemma 45.14.5, Remark 45.14.6, and Lemmas 45.14.7 and 45.14.8. In particular, we may use (for example) Lemma 45.9.1 to see that pushforward on H^* is compatible with composition and satisfies the projection formula.

Let $f : X \to Y$ be a morphism of nonempty equidimensional smooth projective schemes over k. We are trying to show $f_*\gamma(\alpha) = \gamma(f_*\alpha)$ for any cycle class α on X. We can write α as a $\mathbf{Q}$-linear combination of products of Chern classes of locally free $\mathcal{O}_X$-modules (Chow Homology, Lemma 42.58.1). Thus we may assume α is a product of Chern classes of finite locally free $\mathcal{O}_X$-modules $\mathcal{E}_1, \ldots, \mathcal{E}_r$. Pick $p : P \to X$ as in the splitting principle (Chow Homology, Lemma 42.43.1). By Chow Homology, Remark 42.43.2 we see that p is a composition of projective space bundles and that $\alpha = p_*(\xi_1 \cap \ldots \cap \xi_d \cap \cdot p^*\alpha)$ where ξ_i are first Chern classes of invertible modules. By Lemma 45.14.9 we know that p_* commutes with cycle classes. Thus it suffices to prove the property for the composition $f \circ p$. Since $p^*\mathcal{E}_1, \ldots, p^*\mathcal{E}_r$ have filtrations whose successive quotients are invertible modules, this reduces us to the case where α is of the form $\xi_1 \cap \ldots \cap \xi_t \cap [X]$ for some first Chern classes ξ_i of invertible modules $\mathcal{L}_i$.

Assume $\alpha = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_t) \cap [X]$ for some invertible modules $\mathcal{L}_i$ on X. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For $n \gg 0$ the invertible $\mathcal{O}_X$-modules $\mathcal{L}^{\otimes n}$ and $\mathcal{L}_1 \otimes \mathcal{L}^{\otimes n}$ are both very ample on X over k, see Morphisms, Lemma 29.39.8. Since $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_1 \otimes \mathcal{L}^{\otimes n}) - c_1(\mathcal{L}^{\otimes n})$ this reduces us to the case where $\mathcal{L}_1$ is very ample. Repeating this with $\mathcal{L}_i$ for $i = 2, \ldots, t$ we reduce to the case where $\mathcal{L}_i$ is very ample on X over k for all $i = 1, \ldots, t$.

Assume k is infinite and $\alpha = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_t) \cap [X]$ for some very ample invertible modules $\mathcal{L}_i$ on X over k. By Bertini in the form of Varieties, Lemma 33.47.3 we can successively find regular sections s_i of $\mathcal{L}_i$ such that the schemes $Z(s_1) \cap \ldots \cap Z(s_i)$ are smooth over k and of codimension i in X. By the construction of capping with the first class of an invertible module (going back to Chow Homology, Definition 42.24.1), this reduces us to the case where $\alpha = [Z]$ for some nonempty smooth closed subscheme $Z \subset X$ which is equidimensional.

Assume $\alpha = [Z]$ where $Z \subset X$ is a smooth closed subscheme. Choose a closed embedding $X \to \mathbf{P}^n$. We can factor f as

$$X \to Y \times \mathbf{P}^n \to Y$$

Since we know the result for the second morphism by Lemma 45.14.9 it suffices to prove the result when $\alpha = [Z]$ where $i : Z \to X$ is a closed immersion and f is a closed immersion. Then $j = f \circ i$ is a closed embedding too. Using the hypothesis for i and j we win.

We still have to prove the lemma in case k is finite. We urge the reader to skip the rest of the proof. Everything we said above continues to work, except that we do not know we can choose the sections s_i cutting out our Z over k as k is finite. However, we do know that we can find s_i over the algebraic closure $\overline{k}$ of k (by the same lemma). This means that we can find a finite extension k'/k such that our sections s_i are defined over k'. Denote $\pi : X_{k'} \to X$ the projection. The arguments above shows that we get the desired conclusion (from the assumption in the lemma) for the cycle $\pi^*\alpha$ and the morphism $f \circ \pi : X_{k'} \to Y$. We have $\pi_*\pi^*\alpha = [k' : k]\alpha$, see Chow Homology, Lemma 42.15.2. On the other hand, we have

$$\pi_*\gamma(\pi^*\alpha) = \pi_*\pi^*\gamma(\alpha) = \gamma(\alpha)\pi_*1$$

by the projection formula for our cohomology theory. Observe that π is a projection (!) and hence we have $\pi_*(1) = \int_{\mathrm{Spec}(k')}(1)1$ by Lemma 45.9.2. Thus to finish the proof in the finite field case, it suffices to prove that $\int_{\mathrm{Spec}(k')}(1) = [k' : k]$ which we do in Lemma 45.14.10. $\square$

In the lemmas below we use the Grassmannians defined and constructed in Constructions, Section 27.22.

0FIQ **Lemma 45.14.12.** Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Given integers $0 < l < n$ and a nonempty equidimensional smooth projective scheme X over k consider the projection morphism $p : X \times \mathbf{G}(l, n) \to X$. Then γ commutes with pushforward along p.

Proof. If $l = 1$ or $l = n-1$ then p is a projective bundle and the result follows from Lemma 45.14.9. In general there exists a morphism

$$h : Y \to X \times \mathbf{G}(l, n)$$

such that both h and $p \circ h$ are compositions of projective space bundles. Namely, denote $\mathbf{G}(1, 2, \ldots, l; n)$ the partial flag variety. Then the morphism

$$\mathbf{G}(1, 2, \ldots, l; n) \to \mathbf{G}(l, n)$$

is a compostion of projective space bundles and similarly the structure morphism $\mathbf{G}(1, 2, \ldots, l; n) \to \mathrm{Spec}(k)$ is of this form. Thus we may set $Y = X \times \mathbf{G}(1, 2, \ldots, l; n)$. Since every cycle on $X \times \mathbf{G}(l, n)$ is the pushforward of a cycle on Y, the result for $Y \to X$ and the result for $Y \to X \times \mathbf{G}(l, n)$ imply the result for p. $\square$

0FIR Lemma 45.14.13. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). In order to show that γ commutes with pushforward it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k such that the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$ is the pullback of a class in $K_0(X)$.

Proof. By Lemma 45.14.11 it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k. Say Z has codimension r in X. Let $\mathcal{L}$ be a sufficiently ample invertible module on X. Choose $n > 0$ and a surjection

$$\mathcal{O}_Z^{\oplus n} \to \mathcal{C}_{Z/X} \otimes \mathcal{L}|_Z$$

This gives a morphism $g : Z \to \mathbf{G}(n - r, n)$ to the Grassmannian over k, see Constructions, Section 27.22. Consider the composition

$$Z \to X \times \mathbf{G}(n - r, n) \to X$$

Pushforward along the second morphism is compatible with classes of cycles by Lemma 45.14.12. The conormal sheaf $\mathcal{C}$ of the closed immersion $Z \to X \times \mathbf{G}(n-r, n)$ sits in a short exact sequence

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{C} \to g^*\Omega_{\mathbf{G}(n-r,n)} \to 0$$

See More on Morphisms, Lemma 37.11.13. Since $\mathcal{C}_{Z/X} \otimes \mathcal{L}|_Z$ is the pull back of a finite locally free sheaf on $\mathbf{G}(n - r, n)$ we conclude that the class of $\mathcal{C}$ in $K_0(Z)$ is the pullback of a class in $K_0(X \times \mathbf{G}(n - r, n))$. Hence we have the property for $Z \to X \times \mathbf{G}(n - r, n)$ and we conclude. $\square$

0FVS Lemma 45.14.14. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). If $k''/k'/k$ are finite separable field extensions, then $H^0(\mathrm{Spec}(k')) \to H^0(\mathrm{Spec}(k''))$ is injective.

Proof. We may replace k'' by its normal closure over k which is Galois over k, see Fields, Lemma 9.21.5. Then k'' is Galois over k' as well, see Fields, Lemma 9.21.4. We deduce we have an isomorphism

$$k' \otimes_k k'' \longrightarrow \prod\nolimits_{\sigma \in \mathrm{Gal}(k''/k')} k'', \quad \eta \otimes \zeta \longmapsto (\eta\sigma(\zeta))_\sigma$$

This produces an isomorphism $\coprod_\sigma \mathrm{Spec}(k'') \to \mathrm{Spec}(k') \times \mathrm{Spec}(k'')$ which on cohomology produces the isomorphism

$$H^*(\mathrm{Spec}(k'))\otimes_F H^*(\mathrm{Spec}(k'')) \to \prod\nolimits_\sigma H^*(\mathrm{Spec}(k'')), \quad a'\otimes a'' \longmapsto (\pi^*a' \cup \mathrm{Spec}(\sigma)^*a'')_\sigma$$

where $\pi : \mathrm{Spec}(k'') \to \mathrm{Spec}(k')$ is the morphism corresponding to the inclusion of k' in k''. We conclude the lemma is true by taking $a'' = 1$. □

0FIS Lemma 45.14.15. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Let $b : X' \to X$ be a blowing up of a smooth projective scheme X over k which is nonempty equidimensional of dimension d in a nonwhere dense smooth center Z. Then $b_*(1) = 1$.

Proof. We may replace X by a connected component of X (some details omitted). Thus we may assume X is connected and hence irreducible. Set $k' = \Gamma(X, \mathcal{O}_X) = \Gamma(X', \mathcal{O}_{X'})$; we omit the proof of the equality. Choose a closed point $x' \in X'$ which isn't contained in the exceptional divisor and whose residue field k'' is separable over k; this is possible by Varieties, Lemma 33.25.6. Denote $x \in X$ the image (whose residue field is equal to k'' as well of course). Consider the diagram

$$\begin{tikzcd} x' \times X' \arrow[r] \arrow[d] & X' \times X' \arrow[d] \\ x \times X \arrow[r] & X \times X \end{tikzcd}$$

The class of the diagonal $\Delta = \Delta_X$ pulls back to the class of the "diagonal point" $\delta_x : x \to x \times X$ and similarly for the class of the diagonal Δ'. On the other hand, the diagonal point δ_x pulls back to the diagonal point $\delta_{x'}$ by the left vertical arrow. Write $\gamma([\Delta]) = \sum \eta_i$ with $\eta_i \in H^i(X) \otimes H^{2d-i}(X)(d)$ and $\gamma([\Delta']) = \sum \eta'_i$ with $\eta'_i \in H^i(X') \otimes H^{2d-i}(X')(d)$. The arguments above show that η_0 and η'_0 map to the same class in

$$H^0(x') \otimes_F H^{2d}(X')(d)$$

We have $H^0(\mathrm{Spec}(k')) = H^0(X) = H^0(X')$ by axiom (A8). By Lemma 45.14.14 this common value maps injectively into $H^0(x')$. We conclude that η_0 maps to η'_0 by the map

$$H^0(X) \otimes_F H^{2d}(X)(d) \longrightarrow H^0(X') \otimes_F H^{2d}(X')(d)$$

This means that $\int_X$ is equal to $\int_{X'}$ composed with the pullback map. This proves the lemma. □

0FIT Lemma 45.14.16. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Then the cycle class map γ commutes with pushforward.

Proof. Let $i : Z \to X$ be as in Lemma 45.14.13. Consider the diagram

$$\begin{tikzcd} E \arrow[r, "j"'] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Let $\theta \in \mathrm{CH}^{r-1}(X')$ be as in Lemma 45.14.3. Then $\pi_* j^! \theta = [Z]$ in $\mathrm{CH}_*(Z)$ implies that $\pi_*\gamma(j^!\theta) = 1$ by Lemma 45.14.9 because π is a projective space bundle. Hence we see that

$$i_*(1) = i_*(\pi_*(\gamma(j^!\theta))) = b_*j_*(j^*\gamma(\theta)) = b_*(j_*(1) \cup \gamma(\theta))$$

We have $j_*(1) = \gamma([E])$ by (A9). Thus this is equal to

$$b_*(\gamma([E]) \cup \gamma(\theta)) = b_*(\gamma([E] \cdot \theta)) = b_*(\gamma(b^*[Z])) = b_*b^*\gamma([Z]) = b_*(1) \cup \gamma([Z])$$

Since $b_*(1) = 1$ by Lemma 45.14.15 the proof is complete. □

0FIU Proposition 45.14.17. Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Then we have a Weil cohomology theory.

Proof. We have axioms (A), (B) and (C)(a), (C)(c), and (C)(d) of Section 45.9 by Lemmas 45.14.5, 45.14.7, and 45.14.8. We have axiom (C)(b) by Lemma 45.14.16. Finally, the additional condition of Definition 45.11.4 holds because it is the same as our axiom (A8). □

The following lemma is sometimes useful to show that we get a Weil cohomology theory over a nonclosed field by reducing to a closed one.

0FVT Lemma 45.14.18. Let k'/k be an extension of fields. Let F'/F be an extension of fields of characteristic 0. Assume given

(1) data (D0), (D1), (D2') for k and F denoted $F(1), H^*, c_1^H$,
(2) data (D0), (D1), (D2') for k' and F' denoted $F'(1), (H')^*, c_1^{H'}$, and
(3) an isomorphism $F(1) \otimes_F F' \to F'(1)$, functorial isomorphisms $H^*(X) \otimes_F F' \to (H')^*(X_{k'})$ on the category of smooth projective schemes X over k such that the diagrams

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "c_1^H"'] \arrow[d] & H^2(X)(1) \arrow[d] \\ \mathrm{Pic}(X_{k'}) \arrow[r, "c_1^{H'}"] & (H')^2(X_{k'})(1) \end{tikzcd}$$

commute.

In this case, if $F'(1), (H')^*, c_1^{H'}$ satisfy axioms (A1) – (A9), then the same is true for $F(1), H^*, c_1^H$.

Proof. We go by the axioms one by one.

Axiom (A1). We have to show $H^*(\emptyset) = 0$ and that $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism where i and j are the coprojections. By the functorial nature of the isomorphisms $H^*(X) \otimes_F F' \to (H')^*(X_{k'})$ this follows from linear algebra: if $\varphi : V \to W$ is an F-linear map of F-vector spaces, then φ is an isomorphism if and only if $\varphi_{F'} : V \otimes_F F' \to W \otimes_F F'$ is an isomorphism.

Axiom (A2). This means that given a morphism $f : X \to Y$ of smooth projective schemes over k and an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ we have $f^*c_1^H(\mathcal{L}) = c_1^H(f^*\mathcal{L})$. This is immediately clear from the corresponding property for $c_1^{H'}$, the commutative diagrams in the lemma, and the fact that the canonical map $V \to V \otimes_F F'$ is injective for any F-vector space V.

Axiom (A3). This follows from the principle stated in the proof of axiom (A1) and compatibility of c_1^H and $c_1^{H'}$.

Axiom (A4). Let $i : Y \to X$ be the inclusion of an effective Cartier divisor over k with both X and Y smooth and projective over k. For $a \in H^*(X)$ with $i^*a = 0$ we have to show $a \cup c_1^H(\mathcal{O}_X(Y)) = 0$. Denote $a' \in (H')^*(X_{k'})$ the image of a. The assumption implies that $(i')^*a' = 0$ where $i' : Y_{k'} \to X_{k'}$ is the base change of i. Hence we get $a' \cup c_1^{H'}(\mathcal{O}_{X_{k'}}(Y_{k'})) = 0$ by the axiom for $(H')^*$. Since $a' \cup c_1^{H'}(\mathcal{O}_{X_{k'}}(Y_{k'}))$ is the image of $a \cup c_1^H(\mathcal{O}_X(Y))$ we conclude by the princple stated in the proof of axiom (A2).

Axiom (A5). This means that $H^*(\text{Spec}(k)) = F$ and that for X and Y smooth projective over k the map $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $a \otimes b \mapsto p^*(a) \cup q^*(b)$ is an isomorphism where p and q are the projections. This follows from the principle stated in the proof of axiom (A1).

We interrupt the flow of the arguments to show that for every smooth projective scheme X over k the diagram

$$\begin{tikzcd} \mathrm{CH}^*(X) \arrow[r, "\gamma"'] \arrow[d, "g^*"'] & \bigoplus H^{2i}(X)(i) \arrow[d] \\ \mathrm{CH}^*(X_{k'}) \arrow[r, "\gamma'"] & \bigoplus (H')^{2i}(X_{k'})(i) \end{tikzcd}$$

commutes. Observe that we have γ as we know axioms (A1) – (A4) already; see Lemma 45.14.2. Also, the left vertical arrow is the one discussed in Chow Homology, Section 42.67 for the morphism of base schemes $g : \text{Spec}(k') \to \text{Spec}(k)$. More precisely, it is the map given in Chow Homology, Lemma 42.67.4. Pick $\alpha \in \mathrm{CH}^*(X)$. Write $\alpha = ch(\beta) \cap [X]$ in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$ for some $\beta \in K_0(\text{Vect}(X)) \otimes \mathbf{Q}$ so that $\gamma(\alpha) = ch^H(\beta)$; this is our construction of γ. Since the map of Chow Homology, Lemma 42.67.4 is compatible with capping with Chern classes by Chow Homology, Lemma 42.67.8 we see that $g^*\alpha = ch((X_{k'} \to X)^*\beta) \cap [X_{k'}]$. Hence $\gamma'(g^*\alpha) = ch^{H'}((X_{k'} \to X)^*\beta)$. Thus commutativity of the diagram will hold if for any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r and $0 \leq i \leq r$ the element $c_i^H(\mathcal{E})$ of $H^{2i}(X)(i)$ maps to the element $c_i^{H'}(\mathcal{E}_{k'})$ in $(H')^{2i}(X_{k'})(i)$. Because we have the projective space bundle formula for both X and X' we may replace X by a projective space bundle over X finitely many times to show this. Thus we may assume $\mathcal{E}$ has a filtration whose graded pieces are invertible $\mathcal{O}_X$-modules $\mathcal{L}_1, \ldots, \mathcal{L}_r$. See Chow Homology, Lemma 42.43.1 and Remark 42.43.2. Then $c_i^H(\mathcal{E}$ is the ith elementary symmetric polynomial in $c_1^H(\mathcal{L}_1), \ldots, c_1^H(\mathcal{L}_r)$ and we conclude by our assumption that we have agreement for first Chern classes.

Axiom (A6). Suppose given F-vector spaces V, W, an element $v \in V$, and a tensor $\xi \in V \otimes_F W$. Denote $V' = V \otimes_F F'$, $W' = W \otimes_F F'$ and v', ξ' the images of v, ξ in V', $V' \otimes_{F'} W'$. The linear algebra principle we will use in the proof of axiom (A6) is the following: there exists an F-linear map $\lambda : W \to F$ such that $(1 \otimes \lambda)\xi = v$ if and only if there exists an F'-linear map $\lambda' : W \otimes_F F' \to F'$ such that $(1 \otimes \lambda')\xi' = v'$.

Let X be a nonempty equidimensional smooth projective scheme over k of dimension d. Denote $\gamma = \gamma([\Delta])$ in $H^{2d}(X \times X)(d)$ (unadorned fibre products will be over k). Observe/recall that this makes sense as we know axioms (A1) – (A4) already; see Lemma 45.14.2. We may decompose

$$\gamma = \sum \gamma_i, \quad \gamma_i \in H^i(X) \otimes_F H^{2d-i}(X)(d)$$

in the Künneth decomposition. Similarly, denote $\gamma' = \gamma([\Delta']) = \sum \gamma'_i$ in $(H')^{2d}(X_{k'} \times_{k'} X_{k'})(d)$. By the linear algebra princple mentioned above, it suffices to show that γ_0 maps to γ'_0 in $(H')^0(X) \otimes_{F'} (H')^{2d}(X')(d)$. By the compatibility of Künneth decompositions we see that it suffice to show that γ maps to γ' in

$$(H')^{2d}(X_{k'} \times_{k'} X_{k'})(d) = (H')^{2d}((X \times X)_{k'})(d)$$

Since $\Delta_{k'} = \Delta'$ this follows from the discussion above.

Axiom (A7). This follows from the linear algebra fact: a linear map $V \to W$ of F-vector spaces is injective if and only if $V \otimes_F F' \to W \otimes_F F'$ is injective.

Axiom (A8). Follows from the linear algebra fact used in the proof of axiom (A1).

Axiom (A9). Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Let $i : Y \to X$ be a nonempty effective Cartier divisor smooth over k. Let λ_Y and λ_X be as in axiom (A6) for X and Y. We have to show: for $a \in H^{2d-2}(X)(d-1)$ we have $\lambda_Y(i^*(a)) = \lambda_X(a \cup c_1^H(\mathcal{O}_X(Y))$. By Remark 45.14.6 we know that $\lambda_X : H^{2d}(X)(d) \to F$ and $\lambda_Y : H^{2d-2}(Y)(d-1)$ are uniquely determined by the requirement in axiom (A6). Having said this, it follows from our proof of axiom (A6) for H^* above that $\lambda_X \otimes \text{id}_{F'}$ corresponds to $\lambda_{X_{k'}}$ via the given identification $H^{2d}(X)(d) \otimes_F F' = H^{2d}(X_{k'})(d)$. Thus the fact that we know axiom (A9) for $F'(1), (H')^*, c_1^{H'}$ implies the axiom for $F(1), H^*, c_1^H$ by a diagram chase. This completes the proof of the lemma. $\square$

45.15. Other chapters

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CHAPTER 46

Adequate Modules

06Z1 46.1. Introduction

06Z2 For any scheme X the category $QCoh(\mathcal{O}_X)$ of quasi-coherent modules is abelian and a weak Serre subcategory of the abelian category of all $\mathcal{O}_X$-modules. The same thing works for the category of quasi-coherent modules on an algebraic space X viewed as a subcategory of the category of all $\mathcal{O}_X$-modules on the small étale site of X. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the pushforward f_* and higher direct images preserve quasi-coherence.

Next, let X be a scheme and let $\mathcal{O}$ be the structure sheaf on one of the big sites of X, say, the big fppf site. The category of quasi-coherent $\mathcal{O}$-modules is abelian (in fact it is equivalent to the category of usual quasi-coherent $\mathcal{O}_X$-modules on the scheme X we mentioned above) but its imbedding into $\text{Mod}(\mathcal{O})$ is not exact. An example is the map of quasi-coherent modules

$$\mathcal{O}_{\mathbf{A}^1_k} \longrightarrow \mathcal{O}_{\mathbf{A}^1_k}$$

on $\mathbf{A}^1_k = \text{Spec}(k[x])$ given by multiplication by x. In the abelian category of quasi-coherent sheaves this map is injective, whereas in the abelian category of all $\mathcal{O}$-modules on the big site of $\mathbf{A}^1_k$ this map has a nontrivial kernel as we see by evaluating on sections over $\text{Spec}(k[x]/(x)) = \text{Spec}(k)$. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the functor $f_{big,*}$ does not preserve quasi-coherence.

In this chapter we introduce the category of what we will call adequate modules, closely related to quasi-coherent modules, which "fixes" the two problems mentioned above. Another solution, which we will implement when we talk about quasi-coherent modules on algebraic stacks, is to consider $\mathcal{O}$-modules which are locally quasi-coherent and satisfy the flat base change property. See Cohomology of Stacks, Section 103.8, Cohomology of Stacks, Remark 103.10.7, and Derived Categories of Stacks, Section 104.5.

46.2. Conventions

06Z3 In this chapter we fix $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ and we fix a big τ-site Sch_τ as in Topologies, Section 34.2. All schemes will be objects of Sch_τ. In particular, given a scheme S we obtain sites $(\text{Aff}/S)_\tau \subset (Sch/S)_\tau$. The structure sheaf $\mathcal{O}$ on these sites is defined by the rule $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$.

All rings A will be such that $\text{Spec}(A)$ is (isomorphic to) an object of Sch_τ. Given a ring A we denote Alg_A the category of A-algebras whose objects are the A-algebras B of the form $B = \Gamma(U, \mathcal{O}_U)$ where S is an affine object of Sch_τ. Thus given an affine scheme $S = \mathrm{Spec}(A)$ the functor

$$(\mathrm{Aff}/S)_\tau \longrightarrow \mathrm{Alg}_A, \quad U \longmapsto \mathcal{O}(U)$$

is an equivalence.

46.3. Adequate functors

06US In this section we discuss a topic closely related to direct images of quasi-coherent sheaves. Most of this material was taken from the paper [Jaf97].

06Z4 Definition 46.3.1. Let A be a ring. A module-valued functor is a functor $F : \mathrm{Alg}_A \to \mathrm{Ab}$ such that

(1) for every object B of Alg_A the group $F(B)$ is endowed with the structure of a B-module, and
(2) for any morphism $B \to B'$ of Alg_A the map $F(B) \to F(B')$ is B-linear.

A morphism of module-valued functors is a transformation of functors $\varphi : F \to G$ such that $F(B) \to G(B)$ is B-linear for all $B \in \mathrm{Ob}(\mathrm{Alg}_A)$.

Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of module-valued functors on Alg_A is equivalent to the category $\mathrm{PMod}((\mathrm{Aff}/S)_\tau, \mathcal{O})$ of presheaves of $\mathcal{O}$-modules. The equivalence is given by the rule which assigns to the module-valued functor F the presheaf $\mathcal{F}$ defined by the rule $\mathcal{F}(U) = F(\mathcal{O}(U))$. This is clear from the equivalence $(\mathrm{Aff}/S)_\tau \to \mathrm{Alg}_A$, $U \mapsto \mathcal{O}(U)$ given in Section 46.2. The quasi-inverse sets $F(B) = \mathcal{F}(\mathrm{Spec}(B))$.

An important special case of a module-valued functor comes about as follows. Let M be an A-module. Then we will denote $\underline{M}$ the module-valued functor $B \mapsto M \otimes_A B$ (with obvious B-module structure). Note that if $M \to N$ is a map of A-modules then there is an associated morphism $\underline{M} \to \underline{N}$ of module-valued functors. Conversely, any morphism of module-valued functors $\underline{M} \to \underline{N}$ comes from an A-module map $M \to N$ as the reader can see by evaluating on $B = A$. In other words Mod_A is a full subcategory of the category of module-valued functors on Alg_A.

Given and A-module map $\varphi : M \to N$ then $\mathrm{Coker}(\underline{M} \to \underline{N}) = \underline{Q}$ where $Q = \mathrm{Coker}(M \to N)$ because $\otimes$ is right exact. But this isn't the case for the kernel in general: for example an injective map of A-modules need not be injective after base change. Thus the following definition makes sense.

06UT Definition 46.3.2. Let A be a ring. A module-valued functor F on Alg_A is called

(1) adequate if there exists a map of A-modules $M \to N$ such that F is isomorphic to $\mathrm{Ker}(\underline{M} \to \underline{N})$.
(2) linearly adequate if F is isomorphic to the kernel of a map $\underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Note that F is adequate if and only if there exists an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and F is linearly adequate if and only if there exists an exact sequence $0 \to F \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Let A be a ring. In this section we will show the category of adequate functors on Alg_A is abelian (Lemmas 46.3.10 and 46.3.11) and has a set of generators (Lemma 46.3.6). We will also see that it is a weak Serre subcategory of the category of all module-valued functors on Alg_A (Lemma 46.3.16) and that it has arbitrary colimits (Lemma 46.3.12).

06UU **Lemma 46.3.3.** Let A be a ring. Let F be an adequate functor on $\textit{Alg}_A$. If $B = \operatorname{colim} B_i$ is a filtered colimit of A-algebras, then $F(B) = \operatorname{colim} F(B_i)$.

Proof. This holds because for any A-module M we have $M \otimes_A B = \operatorname{colim} M \otimes_A B_i$ (see Algebra, Lemma 10.12.9) and because filtered colimits commute with exact sequences, see Algebra, Lemma 10.8.8. $\square$

06UV **Remark 46.3.4.** Consider the category $\textit{Alg}_{fp,A}$ whose objects are A-algebras B of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and whose morphisms are A-algebra maps. Every A-algebra B is a filtered colimit of finitely presented A-algebra, i.e., a filtered colimit of objects of $\textit{Alg}_{fp,A}$. By Lemma 46.3.3 we conclude every adequate functor F is determined by its restriction to $\textit{Alg}_{fp,A}$. For some questions we can therefore restrict to functors on $\textit{Alg}_{fp,A}$. For example, the category of adequate functors does not depend on the choice of the big τ-site chosen in Section 46.2.

06UW **Lemma 46.3.5.** Let A be a ring. Let F be an adequate functor on $\textit{Alg}_A$. If $B \to B'$ is flat, then $F(B) \otimes_B B' \to F(B')$ is an isomorphism.

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. This gives the diagram

$$\begin{tikzcd} 0 \arrow[r] & F(B) \otimes_B B' \arrow[r] \arrow[d] & (M \otimes_A B) \otimes_B B' \arrow[r] \arrow[d] & (N \otimes_A B) \otimes_B B' \arrow[d] \\ 0 \arrow[r] & F(B') \arrow[r] & M \otimes_A B' \arrow[r] & N \otimes_A B' \end{tikzcd}$$

where the rows are exact (the top one because $B \to B'$ is flat). Since the right two vertical arrows are isomorphisms, so is the left one. $\square$

06UX **Lemma 46.3.6.** Let A be a ring. Let F be an adequate functor on $\textit{Alg}_A$. Then there exists a surjection $L \to F$ with L a direct sum of linearly adequate functors.

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ where $\underline{M} \to \underline{N}$ is given by $\varphi : M \to N$. By Lemma 46.3.3 it suffices to construct $L \to F$ such that $L(B) \to F(B)$ is surjective for every finitely presented A-algebra B. Hence it suffices to construct, given a finitely presented A-algebra B and an element $\xi \in F(B)$ a map $L \to F$ with L linearly adequate such that ξ is in the image of $L(B) \to F(B)$. (Because there is a set worth of such pairs (B, ξ) up to isomorphism.)

To do this write $\sum_{i=1,\ldots,n} m_i \otimes b_i$ the image of ξ in $\underline{M}(B) = M \otimes_A B$. We know that $\sum \varphi(m_i) \otimes b_i = 0$ in $N \otimes_A B$. As N is a filtered colimit of finitely presented A-modules, we can find a finitely presented A-module N', a commutative diagram of A-modules

$$\begin{tikzcd} A^{\oplus n} \arrow[r] \arrow[d, "m_1,\ldots,m_n"'] & N' \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

such that $(b_1, \ldots, b_n)$ maps to zero in $N' \otimes_A B$. Choose a presentation $A^{\oplus l} \to A^{\oplus k} \to N' \to 0$. Choose a lift $A^{\oplus n} \to A^{\oplus k}$ of the map $A^{\oplus n} \to N'$ of the diagram. Then we see that there exist $(c_1, \ldots, c_l) \in B^{\oplus l}$ such that $(b_1, \ldots, b_n, c_1, \ldots, c_l)$ maps to zero in $B^{\oplus k}$ under the map $B^{\oplus n} \oplus B^{\oplus l} \to B^{\oplus k}$. Consider the commutative

diagram

$$\begin{tikzcd} A^{\oplus n} \oplus A^{\oplus l} \arrow[r] \arrow[d] & A^{\oplus k} \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

where the left vertical arrow is zero on the summand $A^{\oplus l}$. Then we see that L equal to the kernel of $\underline{A^{\oplus n+l}} \to \underline{A^{\oplus k}}$ works because the element $(b_1, \ldots, b_n, c_1, \ldots, c_l) \in L(B)$ maps to ξ. □

Consider a graded A-algebra $B = \bigoplus_{d \geq 0} B_d$. Then there are two A-algebra maps $p, a : B \to B[t, t^{-1}]$, namely $p : b \mapsto b$ and $a : b \mapsto t^{\deg(b)} b$ where b is homogeneous. If F is a module-valued functor on Alg_A, then we define

06UY (46.3.6.1) $$F(B)^{(k)} = \{\xi \in F(B) \mid t^k F(p)(\xi) = F(a)(\xi)\}.$$

For functors which behave well with respect to flat ring extensions this gives a direct sum decomposition. This amounts to the fact that representations of $\mathbf{G}_m$ are completely reducible.

06UZ **Lemma 46.3.7.** Let A be a ring. Let F be a module-valued functor on Alg_A. Assume that for $B \to B'$ flat the map $F(B) \otimes_B B' \to F(B')$ is an isomorphism. Let B be a graded A-algebra. Then

(1) $F(B) = \bigoplus_{k \in \mathbf{Z}} F(B)^{(k)}$, and
(2) the map $B \to B_0 \to B$ induces map $F(B) \to F(B)$ whose image is contained in $F(B)^{(0)}$.

Proof. Let $x \in F(B)$. The map $p : B \to B[t, t^{-1}]$ is free hence we know that

$$F(B[t, t^{-1}]) = \bigoplus\nolimits_{k \in \mathbf{Z}} F(p)(F(B)) \cdot t^k = \bigoplus\nolimits_{k \in \mathbf{Z}} F(B) \cdot t^k$$

as indicated we drop the $F(p)$ in the rest of the proof. Write $F(a)(x) = \sum t^k x_k$ for some $x_k \in F(B)$. Denote $\epsilon : B[t, t^{-1}] \to B$ the B-algebra map $t \mapsto 1$. Note that the compositions $\epsilon \circ p, \epsilon \circ a : B \to B[t, t^{-1}] \to B$ are the identity. Hence we see that

$$x = F(\epsilon)(F(a)(x)) = F(\epsilon)(\sum t^k x_k) = \sum x_k.$$

On the other hand, we claim that $x_k \in F(B)^{(k)}$. Namely, consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "a"'] \arrow[d, "a'"'] & B[t, t^{-1}] \arrow[d, "f"] \\ B[s, s^{-1}] \arrow[r, "g"] & B[t, s, t^{-1}, s^{-1}] \end{tikzcd}$$

where $a'(b) = s^{\deg(b)} b$, $f(b) = b$, $f(t) = st$ and $g(b) = t^{\deg(b)} b$ and $g(s) = s$. Then

$$F(g)(F(a'))(x) = F(g)(\sum s^k x_k) = \sum s^k F(a)(x_k)$$

and going the other way we see

$$F(f)(F(a))(x) = F(f)(\sum t^k x_k) = \sum (st)^k x_k.$$

Since $B \to B[s, t, s^{-1}, t^{-1}]$ is free we see that $F(B[t, s, t^{-1}, s^{-1}]) = \bigoplus_{k, l \in \mathbf{Z}} F(B) \cdot t^k s^l$ and comparing coefficients in the expressions above we find $F(a)(x_k) = t^k x_k$ as desired.

Finally, the image of $F(B_0) \to F(B)$ is contained in $F(B)^{(0)}$ because $B_0 \to B \xrightarrow{a} B[t, t^{-1}]$ is equal to $B_0 \to B \xrightarrow{p} B[t, t^{-1}]$. $\square$

As a particular case of Lemma 46.3.7 note that

$$\underline{M}(B)^{(k)} = M \otimes_A B_k$$

where B_k is the degree k part of the graded A-algebra B.

06V0 Lemma 46.3.8. Let A be a ring. Given a solid diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\varphi"'] & \underline{A^{\oplus n}} \arrow[r] \arrow[dl, dotted] & \underline{A^{\oplus m}} \\ & \underline{M} & & \end{tikzcd}$$

of module-valued functors on $\textit{Alg}_A$ with exact row there exists a dotted arrow making the diagram commute.

Proof. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the ring $B = A[x_1, \ldots, x_n]/(\sum a_{ij} x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Note that ξ has the following universal property: for any A-algebra C and any $\xi' \in L(C)$ there exists an A-algebra map $B \to C$ such that ξ maps to ξ' via the map $L(B) \to L(C)$.

Note that B is a graded A-algebra, hence we can use Lemmas 46.3.7 and 46.3.5 to decompose the values of our functors on B into graded pieces. Note that $\xi \in L(B)^{(1)}$ as $(x_1, \ldots, x_n)$ is an element of degree one in $\underline{A^{\oplus n}}(B)$. Hence we see that $\varphi(\xi) \in \underline{M}(B)^{(1)} = M \otimes_A B_1$. Since B_1 is generated by $x_1, \ldots, x_n$ as an A-module we can write $\varphi(\xi) = \sum m_i \otimes x_i$. Consider the map $A^{\oplus n} \to M$ which maps the ith basis vector to m_i. By construction the associated map $\underline{A^{\oplus n}} \to \underline{M}$ maps the element ξ to $\varphi(\xi)$. It follows from the universal property mentioned above that the diagram commutes. $\square$

06V1 Lemma 46.3.9. Let A be a ring. Let $\varphi : F \to \underline{M}$ be a map of module-valued functors on $\textit{Alg}_A$ with F adequate. Then $\text{Coker}(\varphi)$ is adequate.

Proof. By Lemma 46.3.6 we may assume that $F = \bigoplus L_i$ is a direct sum of linearly adequate functors. Choose exact sequences $0 \to L_i \to \underline{A^{\oplus n_i}} \to \underline{A^{\oplus m_i}}$. For each i choose a map $A^{\oplus n_i} \to M$ as in Lemma 46.3.8. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus L_i \arrow[r] \arrow[d] & \bigoplus \underline{A^{\oplus n_i}} \arrow[r] \arrow[dl] & \bigoplus \underline{A^{\oplus m_i}} \\ & \underline{M} & & \end{tikzcd}$$

Consider the A-modules

$$Q = \text{Coker}(\bigoplus A^{\oplus n_i} \to M \oplus \bigoplus A^{\oplus m_i}) \quad \text{and} \quad P = \text{Coker}(\bigoplus A^{\oplus n_i} \to \bigoplus A^{\oplus m_i}).$$

Then we see that $\text{Coker}(\varphi)$ is isomorphic to the kernel of $\underline{Q} \to \underline{P}$. $\square$

06V2 Lemma 46.3.10. Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on $\textit{Alg}_A$. Then $\text{Coker}(\varphi)$ is adequate.

Proof. Choose an injection $G \to \underline{M}$. Then we have an injection $G/F \to \underline{M}/F$. By Lemma 46.3.9 we see that $\underline{M}/F$ is adequate, hence we can find an injection $\underline{M}/F \to \underline{N}$. Composing we obtain an injection $G/F \to \underline{N}$. By Lemma 46.3.9 the cokernel of the induced map $G \to \underline{N}$ is adequate hence we can find an injection $\underline{N}/G \to \underline{K}$. Then $0 \to G/F \to \underline{N} \to \underline{K}$ is exact and we win. $\square$

06V3 **Lemma 46.3.11.** *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then $\mathrm{Ker}(\varphi)$ is adequate.*

Proof. Choose an injection $F \to \underline{M}$ and an injection $G \to \underline{N}$. Denote $F \to \underline{M \oplus N}$ the diagonal map so that

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d] \\ \underline{M \oplus N} \arrow[r] & \underline{N} \end{tikzcd}$$

commutes. By Lemma 46.3.10 we can find a module map $M \oplus N \to K$ such that F is the kernel of $\underline{M \oplus N} \to \underline{K}$. Then $\mathrm{Ker}(\varphi)$ is the kernel of $\underline{M \oplus N} \to \underline{K \oplus N}$. $\square$

06V4 **Lemma 46.3.12.** *Let A be a ring. An arbitrary direct sum of adequate functors on Alg_A is adequate. A colimit of adequate functors is adequate.*

Proof. The statement on direct sums is immediate. A general colimit can be written as a kernel of a map between direct sums, see Categories, Lemma 4.14.12. Hence this follows from Lemma 46.3.11. $\square$

06V5 **Lemma 46.3.13.** *Let A be a ring. Let F, G be module-valued functors on Alg_A. Let $\varphi : F \to G$ be a transformation of functors. Assume*

1. *φ is additive,*
2. *for every A-algebra B and $\xi \in F(B)$ and unit $u \in B^*$ we have $\varphi(u\xi) = u\varphi(\xi)$ in $G(B)$, and*
3. *for any flat ring map $B \to B'$ we have $G(B) \otimes_B B' = G(B')$.*

Then φ is a morphism of module-valued functors.

Proof. Let B be an A-algebra, $\xi \in F(B)$, and $b \in B$. We have to show that $\varphi(b\xi) = b\varphi(\xi)$. Consider the ring map

$$B \to B' = B[x, y, x^{-1}, y^{-1}]/(x + y - b).$$

This ring map is faithfully flat, hence $G(B) \subset G(B')$. On the other hand

$$\varphi(b\xi) = \varphi((x + y)\xi) = \varphi(x\xi) + \varphi(y\xi) = x\varphi(\xi) + y\varphi(\xi) = (x + y)\varphi(\xi) = b\varphi(\xi)$$

because x, y are units in B'. Hence we win. $\square$

06V6 **Lemma 46.3.14.** *Let A be a ring. Let $0 \to \underline{M} \to G \to L \to 0$ be a short exact sequence of module-valued functors on Alg_A with L linearly adequate. Then G is adequate.*

Proof. We first point out that for any flat A-algebra map $B \to B'$ the map $G(B) \otimes_B B' \to G(B')$ is an isomorphism. Namely, this holds for $\underline{M}$ and L, see Lemma 46.3.5 and hence follows for G by the five lemma. In particular, by Lemma 46.3.7 we see that $G(B) = \bigoplus_{k \in \mathbf{Z}} G(B)^{(k)}$ for any graded A-algebra B.

Choose an exact sequence $0 \to L \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the graded A-algebra

$B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Observe that $\xi \in L(B)^{(1)}$. The map

$$\mathrm{Hom}_A(B, C) \longrightarrow L(C), \quad f \longmapsto L(f)(\xi)$$

defines an isomorphism of functors. The reason is that f is determined by the images $c_i = f(x_i) \in C$ which have to satisfy the relations $\sum a_{ij}c_j = 0$. And $L(C)$ is the set of n-tuples $(c_1, \ldots, c_n)$ satisfying the relations $\sum a_{ij}c_j = 0$.

Since the value of each of the functors $\underline{M}$, G, L on B is a direct sum of its weight spaces (by the lemma mentioned above) exactness of $0 \to \underline{M} \to G \to L \to 0$ implies the sequence $0 \to \underline{M}(B)^{(1)} \to G(B)^{(1)} \to L(B)^{(1)} \to 0$ is exact. Thus we may choose an element $\theta \in G(B)^{(1)}$ mapping to ξ.

Consider the graded A-algebra

$$C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$$

There are three graded A-algebra homomorphisms $p_1, p_2, m : B \to C$ defined by the rules

$$p_1(x_i) = x_i, \quad p_1(x_i) = y_i, \quad m(x_i) = x_i + y_i.$$

We will show that the element

$$\tau = G(m)(\theta) - G(p_1)(\theta) - G(p_2)(\theta) \in G(C)$$

is zero. First, τ maps to zero in $L(C)$ by a direct calculation. Hence τ is an element of $\underline{M}(C)$. Moreover, since m, p_1, p_2 are graded algebra maps we see that $\tau \in G(C)^{(1)}$ and since $\underline{M} \subset G$ we conclude

$$\tau \in \underline{M}(C)^{(1)} = M \otimes_A C_1.$$

We may write uniquely $\tau = \underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)$ with $\tau_i \in M \otimes_A B_1 = \underline{M}(B)^{(1)}$ because $C_1 = p_1(B_1) \oplus p_2(B_1)$. Consider the ring map $q_1 : C \to B$ defined by $x_i \mapsto x_i$ and $y_i \mapsto 0$. Then $\underline{M}(q_1)(\tau) = \underline{M}(q_1)(\underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)) = \tau_1$. On the other hand, because $q_1 \circ m = q_1 \circ p_1$ we see that $G(q_1)(\tau) = -G(q_1 \circ p_2)(\tau)$. Since $q_1 \circ p_2$ factors as $B \to A \to B$ we see that $G(q_1 \circ p_2)(\tau)$ is in $G(B)^{(0)}$, see Lemma 46.3.7. Hence $\tau_1 = 0$ because it is in $G(B)^{(0)} \cap \underline{M}(B)^{(1)} \subset G(B)^{(0)} \cap G(B)^{(1)} = 0$. Similarly $\tau_2 = 0$, whence $\tau = 0$.

Since $\theta \in G(B)$ we obtain a transformation of functors

$$\psi : L(-) = \mathrm{Hom}_A(B, -) \longrightarrow G(-)$$

by mapping $f : B \to C$ to $G(f)(\theta)$. Since θ is a lift of ξ the map ψ is a right inverse of $G \to L$. In terms of ψ the statements proved above have the following meaning: $\tau = 0$ means that ψ is additive and $\theta \in G(B)^{(1)}$ implies that for any A-algebra D we have $\psi(ul) = u\psi(l)$ in $G(D)$ for $l \in L(D)$ and $u \in D^*$ a unit. This implies that ψ is a morphism of module-valued functors, see Lemma 46.3.13. Clearly this implies that $G \cong \underline{M} \oplus L$ and we win. $\square$

06V7 Remark 46.3.15. Let A be a ring. The proof of Lemma 46.3.14 shows that any extension $0 \to \underline{M} \to E \to L \to 0$ of module-valued functors on Alg_A with L linearly adequate splits. It uses only the following properties of the module-valued functor $F = \underline{M}$:

(1) $F(B) \otimes_B B' \to F(B')$ is an isomorphism for a flat ring map $B \to B'$, and

(2) $F(C)^{(1)} = F(p_1)(F(B)^{(1)}) \oplus F(p_2)(F(B)^{(1)})$ where $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$ and $C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$.

These two properties hold for any adequate functor F; details omitted. Hence we see that L is a projective object of the abelian category of adequate functors.

06V8 Lemma 46.3.16. Let A be a ring. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A. If F and H are adequate, so is G.

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. If we can show that $(\underline{M} \oplus G)/F$ is adequate, then G is the kernel of the map of adequate functors $(\underline{M} \oplus G)/F \to \underline{N}$, hence adequate by Lemma 46.3.11. Thus we may assume $F = \underline{M}$.

We can choose a surjection $L \to H$ where L is a direct sum of linearly adequate functors, see Lemma 46.3.6. If we can show that the pullback $G \times_H L$ is adequate, then G is the cokernel of the map $\text{Ker}(L \to H) \to G \times_H L$ hence adequate by Lemma 46.3.10. Thus we may assume that $H = \bigoplus L_i$ is a direct sum of linearly adequate functors. By Lemma 46.3.14 each of the pullbacks $G \times_H L_i$ is adequate. By Lemma 46.3.12 we see that $\bigoplus G \times_H L_i$ is adequate. Then G is the cokernel of

$$\bigoplus\nolimits_{i \neq i'} F \longrightarrow \bigoplus G \times_H L_i$$

where ξ in the summand (i, i') maps to $(0, \ldots, 0, \xi, 0, \ldots, 0, -\xi, 0, \ldots, 0)$ with nonzero entries in the summands i and i'. Thus G is adequate by Lemma 46.3.10. $\square$

06V9 Lemma 46.3.17. Let $A \to A'$ be a ring map. If F is an adequate functor on Alg_A, then its restriction F' to $\text{Alg}_{A'}$ is adequate too.

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. Then $F'(B') = F(B') = \text{Ker}(M \otimes_A B' \to N \otimes_A B')$. Since $M \otimes_A B' = M \otimes_A A' \otimes_{A'} B'$ and similarly for N we see that F' is the kernel of $\underline{M \otimes_A A'} \to \underline{N \otimes_A A'}$. $\square$

06VA Lemma 46.3.18. Let $A \to A'$ be a ring map. If F' is an adequate functor on $\text{Alg}_{A'}$, then the module-valued functor $F : B \mapsto F'(A' \otimes_A B)$ on Alg_A is adequate too.

Proof. Choose an exact sequence $0 \to F' \to \underline{M'} \to \underline{N'}$. Then

$$\begin{aligned} F(B) &= F'(A' \otimes_A B) \\ &= \text{Ker}(M' \otimes_{A'} (A' \otimes_A B) \to N' \otimes_{A'} (A' \otimes_A B)) \\ &= \text{Ker}(M' \otimes_A B \to N' \otimes_A B) \end{aligned}$$

Thus F is the kernel of $\underline{M} \to \underline{N}$ where $M = M'$ and $N = N'$ viewed as A-modules. $\square$

06VB Lemma 46.3.19. Let $A = A_1 \times \ldots \times A_n$ be a product of rings. An adequate functor over A is the same thing as a sequence $F_1, \ldots, F_n$ of adequate functors F_i over A_i.

Proof. This is true because an A-algebra B is canonically a product $B_1 \times \ldots \times B_n$ and the same thing holds for A-modules. Setting $F(B) = \coprod F_i(B_i)$ gives the correspondence. Details omitted. $\square$

06VH Lemma 46.3.20. Let $A \to A'$ be a ring map and let F be a module-valued functor on Alg_A such that

(1) the restriction F' of F to the category of A'-algebras is adequate, and

(2) for any A-algebra B the sequence

$$0 \to F(B) \to F(B \otimes_A A') \to F(B \otimes_A A' \otimes_A A')$$

is exact.

Then F is adequate.

Proof. The functors $B \to F(B \otimes_A A')$ and $B \mapsto F(B \otimes_A A' \otimes_A A')$ are adequate, see Lemmas 46.3.18 and 46.3.17. Hence F as a kernel of a map of adequate functors is adequate, see Lemma 46.3.11. □

46.4. Higher exts of adequate functors

06Z5 Let A be a ring. In Lemma 46.3.16 we have seen that any extension of adequate functors in the category of module-valued functors on Alg_A is adequate. In this section we show that the same remains true for higher ext groups.

06Z6 **Lemma 46.4.1.** Let A be a ring. For every module-valued functor F on Alg_A there exists a morphism $Q(F) \to F$ of module-valued functors on Alg_A such that (1) $Q(F)$ is adequate and (2) for every adequate functor G the map $\text{Hom}(G, Q(F)) \to \text{Hom}(G, F)$ is a bijection.

Proof. Choose a set $\{L_i\}_{i \in I}$ of linearly adequate functors such that every linearly adequate functor is isomorphic to one of the L_i. This is possible. Suppose that we can find $Q(F) \to F$ with (1) and (2)' or every $i \in I$ the map $\text{Hom}(L_i, Q(F)) \to \text{Hom}(L_i, F)$ is a bijection. Then (2) holds. Namely, combining Lemmas 46.3.6 and 46.3.11 we see that every adequate functor G sits in an exact sequence

$$K \to L \to G \to 0$$

with K and L direct sums of linearly adequate functors. Hence (2)' implies that $\text{Hom}(L, Q(F)) \to \text{Hom}(L, F)$ and $\text{Hom}(K, Q(F)) \to \text{Hom}(K, F)$ are bijections, whence the same thing for G.

Consider the category $\mathcal{I}$ whose objects are pairs (i, φ) where $i \in I$ and $\varphi : L_i \to F$ is a morphism. A morphism $(i, \varphi) \to (i', \varphi')$ is a map $\psi : L_i \to L_{i'}$ such that $\varphi' \circ \psi = \varphi$. Set

$$Q(F) = \text{colim}_{(i,\varphi) \in \text{Ob}(\mathcal{I})} L_i$$

There is a natural map $Q(F) \to F$, by Lemma 46.3.12 it is adequate, and by construction it has property (2)'. □

06Z7 **Lemma 46.4.2.** Let A be a ring. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 46.4.1. Then

(1) i is fully faithful, exact, and its image is a weak Serre subcategory,
(2) $\mathcal{P}$ has enough injectives,
(3) the functor Q is a right adjoint to i hence left exact,
(4) Q transforms injectives into injectives,
(5) $\mathcal{A}$ has enough injectives.

Proof. This lemma just collects some facts we have already seen so far. Part (1) is clear from the definitions, the characterization of weak Serre subcategories (see Homology, Lemma 12.10.3), and Lemmas 46.3.10, 46.3.11, and 46.3.16. Recall that $\mathcal{P}$ is equivalent to the category $\text{PMod}((\text{Aff}/\text{Spec}(A))_\tau, \mathcal{O})$. Hence (2) by Injectives, Proposition 19.8.5. Part (3) follows from Lemma 46.4.1 and Categories, Lemma 4.24.5. Parts (4) and (5) follow from Homology, Lemmas 12.29.1 and 12.29.3. $\square$

Let A be a ring. As in Formal Deformation Theory, Section 90.11 given an A-algebra B and an B-module N we set $B[N]$ equal to the R-algebra with underlying B-module $B \oplus N$ with multiplication given by $(b, m)(b', m') = (bb', bm' + b'm)$. Note that this construction is functorial in the pair (B, N) where morphism $(B, N) \to (B', N')$ is given by an A-algebra map $B \to B'$ and an B-module map $N \to N'$. In some sense the functor TF of pairs defined in the following lemma is the tangent space of F. Below we will only consider pairs (B, N) such that $B[N]$ is an object of Alg_A.

06Z8 Lemma 46.4.3. Let A be a ring. Let F be a module valued functor. For every $B \in \text{Ob}(\text{Alg}_A)$ and B-module N there is a canonical decomposition

$$F(B[N]) = F(B) \oplus TF(B, N)$$

characterized by the following properties

(1) $TF(B, N) = \text{Ker}(F(B[N]) \to F(B))$,
(2) there is a B-module structure $TF(B, N)$ compatible with $B[N]$-module structure on $F(B[N])$,
(3) TF is a functor from the category of pairs (B, N),
06Z9 (4) there are canonical maps $N \otimes_B F(B) \to TF(B, N)$ inducing a transformation between functors defined on the category of pairs (B, N),
(5) $TF(B, 0) = 0$ and the map $TF(B, N) \to TF(B, N')$ is zero when $N \to N'$ is the zero map.

Proof. Since $B \to B[N] \to B$ is the identity we see that $F(B) \to F(B[N])$ is a direct summand whose complement is $TF(N, B)$ as defined in (1). This construction is functorial in the pair (B, N) simply because given a morphism of pairs $(B, N) \to (B', N')$ we obtain a commutative diagram

$$\begin{tikzcd} B' \arrow[r] & B'[N'] \arrow[r] & B' \\ B \arrow[r] \arrow[u] & B[N] \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

in Alg_A. The B-module structure comes from the $B[N]$-module structure and the ring map $B \to B[N]$. The map in (4) is the composition

$$N \otimes_B F(B) \longrightarrow B[N] \otimes_{B[N]} F(B[N]) \longrightarrow F(B[N])$$

whose image is contained in $TF(B, N)$. (The first arrow uses the inclusions $N \to B[N]$ and $F(B) \to F(B[N])$ and the second arrow is the multiplication map.) If $N = 0$, then $B = B[N]$ hence $TF(B, 0) = 0$. If $N \to N'$ is zero then it factors as $N \to 0 \to N'$ hence the induced map is zero since $TF(B, 0) = 0$. $\square$

Let A be a ring. Let M be an A-module. Then the module-valued functor $\underline{M}$ has tangent space $T\underline{M}$ given by the rule $T\underline{M}(B, N) = N \otimes_A M$. In particular, for B given, the functor $N \mapsto T\underline{M}(B, N)$ is additive and right exact. It turns out this also holds for injective module-valued functors.

06ZA Lemma 46.4.4. Let A be a ring. Let I be an injective object of the category of module-valued functors. Then for any $B \in \mathrm{Ob}(\mathrm{Alg}_A)$ and short exact sequence $0 \to N_1 \to N \to N_2 \to 0$ of B-modules the sequence

$$TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$$

is exact.

Proof. We will use the results of Lemma 46.4.3 without further mention. Denote $h : \mathrm{Alg}_A \to \mathrm{Sets}$ the functor given by $h(C) = \mathrm{Mor}_A(B[N], C)$. Similarly for h_1 and h_2. The map $B[N] \to B[N_2]$ corresponding to the surjection $N \to N_2$ is surjective. It corresponds to a map $h_2 \to h$ such that $h_2(C) \to h(C)$ is injective for all A-algebras C. On the other hand, there are two maps $p, q : h \to h_1$, corresponding to the zero map $N_1 \to N$ and the injection $N_1 \to N$. Note that

$$\begin{tikzcd} h_2 \arrow[r] & h \arrow[r, shift left] \arrow[r, shift right] & h_1 \end{tikzcd}$$

is an equalizer diagram. Denote $\mathcal{O}_h$ the module-valued functor $C \mapsto \bigoplus_{h(C)} C$. Similarly for $\mathcal{O}_{h_1}$ and $\mathcal{O}_{h_2}$. Note that

$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{O}_h, F) = F(B[N])$$

where $\mathcal{P}$ is the category of module-valued functors on Alg_A. We claim there is an equalizer diagram

$$\begin{tikzcd} \mathcal{O}_{h_2} \arrow[r] & \mathcal{O}_h \arrow[r, shift left] \arrow[r, shift right] & \mathcal{O}_{h_1} \end{tikzcd}$$

in $\mathcal{P}$. Namely, suppose that $C \in \mathrm{Ob}(\mathrm{Alg}_A)$ and $\xi = \sum_{i=1,\ldots,n} c_i \cdot f_i$ where $c_i \in C$ and $f_i : B[N] \to C$ is an element of $\mathcal{O}_h(C)$. If $p(\xi) = q(\xi)$, then we see that

$$\sum c_i \cdot f_i \circ z = \sum c_i \cdot f_i \circ y$$

where $z, y : B[N_1] \to B[N]$ are the maps $z : (b, m_1) \mapsto (b, 0)$ and $y : (b, m_1) \mapsto (b, m_1)$. This means that for every i there exists a j such that $f_j \circ z = f_i \circ y$. Clearly, this implies that $f_i(N_1) = 0$, i.e., f_i factors through a unique map $\overline{f}_i : B[N_2] \to C$. Hence ξ is the image of $\overline{\xi} = \sum c_i \cdot \overline{f}_i$. Since I is injective, it transforms this equalizer diagram into a coequalizer diagram

$$\begin{tikzcd} I(B[N_1]) \arrow[r, shift left] \arrow[r, shift right] & I(B[N]) \arrow[r] & I(B[N_2]) \end{tikzcd}$$

This diagram is compatible with the direct sum decompositions $I(B[N]) = I(B) \oplus TI(B, N)$ and $I(B[N_i]) = I(B) \oplus TI(B, N_i)$. The zero map $N \to N_1$ induces the zero map $TI(B, N) \to TI(B, N_1)$. Thus we see that the coequalizer property above means we have an exact sequence $TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$ as desired. □

06ZB Lemma 46.4.5. Let A be a ring. Let F be a module-valued functor such that for any $B \in \mathrm{Ob}(\mathrm{Alg}_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Then

(1) $TF(B, N_1 \oplus N_2) = TF(B, N_1) \oplus TF(B, N_2)$,
(2) there is a second functorial B-module structure on $TF(B, N)$ defined by setting $x \cdot b = TF(B, b \cdot 1_N)(x)$ for $x \in TF(B, N)$ and $b \in B$,
06ZC (3) the canonical map $N \otimes_B F(B) \to TF(B, N)$ of Lemma 46.4.3 is B-linear also with respect to the second B-module structure,

06ZD (4) given a finitely presented B-module N there is a canonical isomorphism $TF(B,B) \otimes_B N \to TF(B,N)$ where the tensor product uses the second B-module structure on $TF(B,B)$.

Proof. We will use the results of Lemma 46.4.3 without further mention. The maps $N_1 \to N_1 \oplus N_2$ and $N_2 \to N_1 \oplus N_2$ give a map $TF(B,N_1) \oplus TF(B,N_2) \to TF(B,N_1 \oplus N_2)$ which is injective since the maps $N_1 \oplus N_2 \to N_1$ and $N_1 \oplus N_2 \to N_2$ induce an inverse. Since TF is right exact we see that $TF(B,N_1) \to TF(B,N_1 \oplus N_2) \to TF(B,N_2) \to 0$ is exact. Hence $TF(B,N_1) \oplus TF(B,N_2) \to TF(B,N_1 \oplus N_2)$ is an isomorphism. This proves (1).

To see (2) the only thing we need to show is that $x \cdot (b_1 + b_2) = x \cdot b_1 + x \cdot b_2$. (Associativity and additivity are clear.) To see this consider

$$N \xrightarrow{(b_1,b_2)} N \oplus N \xrightarrow{+} N$$

and apply $TF(B,-)$.

Part (3) follows immediately from the fact that $N \otimes_B F(B) \to TF(B,N)$ is functorial in the pair (B,N).

Suppose N is a finitely presented B-module. Choose a presentation $B^{\oplus m} \to B^{\oplus n} \to N \to 0$. This gives an exact sequence

$$TF(B,B^{\oplus m}) \to TF(B,B^{\oplus n}) \to TF(B,N) \to 0$$

by right exactness of $TF(B,-)$. By part (1) we can write $TF(B,B^{\oplus m}) = TF(B,B)^{\oplus m}$ and $TF(B,B^{\oplus n}) = TF(B,B)^{\oplus n}$. Next, suppose that $B^{\oplus m} \to B^{\oplus n}$ is given by the matrix $T = (b_{ij})$. Then the induced map $TF(B,B)^{\oplus m} \to TF(B,B)^{\oplus n}$ is given by the matrix with entries $TF(B, b_{ij} \cdot 1_B)$. This combined with right exactness of $\otimes$ proves (4). $\square$

06ZE Example 46.4.6. Let F be a module-valued functor as in Lemma 46.4.5. It is not always the case that the two module structures on $TF(B,N)$ agree. Here is an example. Suppose $A = \mathbf{F}_p$ where p is a prime. Set $F(B) = B$ but with B-module structure given by $b \cdot x = b^p x$. Then $TF(B,N) = N$ with B-module structure given by $b \cdot x = b^p x$ for $x \in N$. However, the second B-module structure is given by $x \cdot b = bx$. Note that in this case the canonical map $N \otimes_B F(B) \to TF(B,N)$ is zero as raising an element $n \in B[N]$ to the pth power is zero.

In the following lemma we will frequently use the observation that if $0 \to F \to G \to H \to 0$ is an exact sequence of module-valued functors on Alg_A, then for any pair (B,N) the sequence $0 \to TF(B,N) \to TG(B,N) \to TH(B,N) \to 0$ is exact. This follows from the fact that $0 \to F(B[N]) \to G(B[N]) \to H(B[N]) \to 0$ is exact.

06ZF Lemma 46.4.7. Let A be a ring. For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \text{Ob}(\text{Alg}_A)$ the functor $TF(B,-)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A.

(1) If $(*)$ holds for F, G then $(*)$ holds for H.
(2) If $(*)$ holds for F, H then $(*)$ holds for G.
(3) If $H' \to H$ is morphism of module-valued functors on Alg_A and $(*)$ holds for F, G, H, and H', then $(*)$ holds for $G \times_H H'$.

Proof. Let B be given. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of B-modules. Part (1) follows from a diagram chase in the diagram

$$\begin{tikzcd}
0 \arrow[r] & TF(B, N_1) \arrow[r] \arrow[d] & TG(B, N_1) \arrow[r] \arrow[d] & TH(B, N_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_2) \arrow[r] \arrow[d] & TG(B, N_2) \arrow[r] \arrow[d] & TH(B, N_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_3) \arrow[r] \arrow[d] & TG(B, N_3) \arrow[r] \arrow[d] & TH(B, N_3) \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

with exact horizontal rows and exact columns involving TF and TG. To prove part (2) we do a diagram chase in the diagram

$$\begin{tikzcd}
0 \arrow[r] & TF(B, N_1) \arrow[r] \arrow[d] & TG(B, N_1) \arrow[r] \arrow[d] & TH(B, N_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_2) \arrow[r] \arrow[d] & TG(B, N_2) \arrow[r] \arrow[d] & TH(B, N_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_3) \arrow[r] \arrow[d] & TG(B, N_3) \arrow[r] & TH(B, N_3) \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 &
\end{tikzcd}$$

with exact horizontal rows and exact columns involving TF and TH. Part (3) follows from part (2) as $G \times_H H'$ sits in the exact sequence $0 \to F \to G \times_H H' \to H' \to 0$. $\square$

Most of the work in this section was done in order to prove the following key vanishing result.

06ZG Lemma 46.4.8. Let A be a ring. Let M, P be A-modules with P of finite presentation. Then $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.

Proof. Choose an injective resolution $\underline{M} \to I^\bullet$ in $\mathcal{P}$, see Lemma 46.4.2. By Derived Categories, Lemma 13.27.2 any element of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ comes from a morphism $\varphi : \underline{P} \to I^i$ with $d^i \circ \varphi = 0$. We will prove that the Yoneda extension

$$E : 0 \to \underline{M} \to I^0 \to \ldots \to I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P} \to \underline{P} \to 0$$

of $\underline{P}$ by $\underline{M}$ associated to φ is trivial, which will prove the lemma by Derived Categories, Lemma 13.27.5.

For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(\mathrm{Alg}_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Recall that the module-valued functors $\underline{M}, I^n, \underline{P}$ each have property $(*)$, see Lemma 46.4.4 and the remarks preceding it. By splitting $0 \to$

$\underline{M} \to I^\bullet$ into short exact sequences we find that each of the functors $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n) \subset I^n$ has property $(*)$ by Lemma 46.4.7 and also that $I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P}$ has property $(*)$.

Thus we may assume the Yoneda extension is given as

$$E : 0 \to \underline{M} \to F_{i-1} \to \ldots \to F_0 \to \underline{P} \to 0$$

where each of the module-valued functors F_j has property $(*)$. Set $G_j(B) = TF_j(B, B)$ viewed as a B-module via the second B-module structure defined in Lemma 46.4.5. Since TF_j is a functor on pairs we see that G_j is a module-valued functor on Alg_A. Moreover, since E is an exact sequence the sequence $G_{j+1} \to G_j \to G_{j-1}$ is exact (see remark preceding Lemma 46.4.7). Observe that $T\underline{M}(B, B) = M \otimes_A B = \underline{M}(B)$ and that the two B-module structures agree on this. Thus we obtain a Yoneda extension

$$E' : 0 \to \underline{M} \to G_{i-1} \to \ldots \to G_0 \to \underline{P} \to 0$$

Moreover, the canonical maps

$$F_j(B) = B \otimes_B F_j(B) \longrightarrow TF_j(B, B) = G_j(B)$$

of Lemma 46.4.3 (4) are B-linear by Lemma 46.4.5 (3) and functorial in B. Hence a map

$$\begin{tikzcd} 0 \arrow[r] & \underline{M} \arrow[r] \arrow[d, "1"] & F_{i-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & F_0 \arrow[r] \arrow[d] & \underline{P} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \underline{M} \arrow[r] & G_{i-1} \arrow[r] & \ldots \arrow[r] & G_0 \arrow[r] & \underline{P} \arrow[r] & 0 \end{tikzcd}$$

of Yoneda extensions. In particular we see that E and E' have the same class in $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ by the lemma on Yoneda Exts mentioned above. Finally, let N be a A-module of finite presentation. Then we see that

$$0 \to T\underline{M}(A, N) \to TF_{i-1}(A, N) \to \ldots \to TF_0(A, N) \to T\underline{P}(A, N) \to 0$$

is exact. By Lemma 46.4.5 (4) with $B = A$ this translates into the exactness of the sequence of A-modules

$$0 \to M \otimes_A N \to G_{i-1}(A) \otimes_A N \to \ldots \to G_0(A) \otimes_A N \to P \otimes_A N \to 0$$

Hence the sequence of A-modules $0 \to M \to G_{i-1}(A) \to \ldots \to G_0(A) \to P \to 0$ is universally exact, in the sense that it remains exact on tensoring with any finitely presented A-module N. Let $K = \mathrm{Ker}(G_0(A) \to P)$ so that we have exact sequences

$$0 \to K \to G_0(A) \to P \to 0 \quad \text{and} \quad G_2(A) \to G_1(A) \to K \to 0$$

Tensoring the second sequence with N we obtain that $K \otimes_A N = \mathrm{Coker}(G_2(A) \otimes_A N \to G_1(A) \otimes_A N)$. Exactness of $G_2(A) \otimes_A N \to G_1(A) \otimes_A N \to G_0(A) \otimes_A N$ then implies that $K \otimes_A N \to G_0(A) \otimes_A N$ is injective. By Algebra, Theorem 10.82.3 this means that the A-module extension $0 \to K \to G_0(A) \to P \to 0$ is exact, and because P is assumed of finite presentation this means the sequence is split, see Algebra, Lemma 10.82.4. Any splitting $P \to G_0(A)$ defines a map $\underline{P} \to G_0$ which splits the surjection $G_0 \to \underline{P}$. Thus the Yoneda extension E' is equivalent to the trivial Yoneda extension and we win. □

06ZH Lemma 46.4.9. Let A be a ring. Let M be an A-module. Let L be a linearly adequate functor on Alg_A. Then $\text{Ext}^i_{\mathcal{P}}(L, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.

Proof. Since L is linearly adequate there exists an exact sequence

$$0 \to L \to \underline{A^{\oplus m}} \to \underline{A^{\oplus n}} \to \underline{P} \to 0$$

Here $P = \text{Coker}(A^{\oplus m} \to A^{\oplus n})$ is the cokernel of the map of finite free A-modules which is given by the definition of linearly adequate functors. By Lemma 46.4.8 we have the vanishing of $\text{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ and $\text{Ext}^i_{\mathcal{P}}(\underline{A}, \underline{M})$ for $i > 0$. Let $K = \text{Ker}(\underline{A^{\oplus n}} \to \underline{P})$. By the long exact sequence of Ext groups associated to the exact sequence $0 \to K \to \underline{A^{\oplus n}} \to \underline{P} \to 0$ we conclude that $\text{Ext}^i_{\mathcal{P}}(K, \underline{M}) = 0$ for $i > 0$. Repeating with the sequence $0 \to L \to \underline{A^{\oplus m}} \to K \to 0$ we win. □

06ZI Lemma 46.4.10. With notation as in Lemma 46.4.2 we have $R^pQ(F) = 0$ for all $p > 0$ and any adequate functor F.

Proof. Choose an exact sequence $0 \to F \to \underline{M^0} \to \underline{M^1}$. Set $M^2 = \text{Coker}(M^0 \to M^1)$ so that $0 \to F \to \underline{M^0} \to \underline{M^1} \to \underline{M^2} \to 0$ is a resolution. By Derived Categories, Lemma 13.21.3 we obtain a spectral sequence

$$R^pQ(\underline{M^q}) \Rightarrow R^{p+q}Q(F)$$

Since $Q(\underline{M^q}) = \underline{M^q}$ it suffices to prove $R^pQ(\underline{M}) = 0$, $p > 0$ for any A-module M.

Choose an injective resolution $\underline{M} \to I^\bullet$ in the category $\mathcal{P}$. Suppose that $R^iQ(\underline{M})$ is nonzero. Then $\text{Ker}(Q(I^i) \to Q(I^{i+1}))$ is strictly bigger than the image of $Q(I^{i-1}) \to Q(I^i)$. Hence by Lemma 46.3.6 there exists a linearly adequate functor L and a map $\varphi : L \to Q(I^i)$ mapping into the kernel of $Q(I^i) \to Q(I^{i+1})$ which does not factor through the image of $Q(I^{i-1}) \to Q(I^i)$. Because Q is a left adjoint to the inclusion functor the map φ corresponds to a map $\varphi' : L \to I^i$ with the same properties. Thus φ' gives a nonzero element of $\text{Ext}^i_{\mathcal{P}}(L, \underline{M})$ contradicting Lemma 46.4.9. □

46.5. Adequate modules

06VF In Descent, Section 35.8 we have seen that quasi-coherent modules on a scheme S are the same as quasi-coherent modules on any of the big sites $(Sch/S)_\tau$ associated to S. We have seen that there are two issues with this identification:

(1) $QCoh(\mathcal{O}_S) \to \text{Mod}((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is not exact in general (Descent, Lemma 35.10.2), and
(2) given a quasi-compact and quasi-separated morphism $f : X \to S$ the functor f_* does not preserve quasi-coherent sheaves on the big sites in general (Descent, Proposition 35.9.4).

Part (1) means that we cannot define a triangulated subcategory of $D(\mathcal{O})$ consisting of complexes whose cohomology sheaves are quasi-coherent. Part (2) means that $Rf_*\mathcal{F}$ isn't a complex with quasi-coherent cohomology sheaves even when $\mathcal{F}$ is quasi-coherent and f is quasi-compact and quasi-separated. Moreover, the examples given in the proofs of Descent, Lemma 35.10.2 and Descent, Proposition 35.9.4 are not of a pathological nature.

In this section we discuss a slightly larger category of $\mathcal{O}$-modules on $(Sch/S)_\tau$ with contains the quasi-coherent modules, is abelian, and is preserved under f_* when f

is quasi-compact and quasi-separated. To do this, suppose that S is a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. For any affine object $U = \mathrm{Spec}(A)$ of $(Sch/S)_\tau$ we can restrict $\mathcal{F}$ to $(\mathrm{Aff}/U)_\tau$ to get a presheaf of $\mathcal{O}$-modules on this site. The corresponding module-valued functor, see Section 46.3, will be denoted

$$F = F_{\mathcal{F},A} : \mathrm{Alg}_A \longrightarrow \mathrm{Ab}, \quad B \longmapsto \mathcal{F}(\mathrm{Spec}(B))$$

The assignment $\mathcal{F} \mapsto F_{\mathcal{F},A}$ is an exact functor of abelian categories.

06VG Definition 46.5.1. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $(Sch/S)_\tau$ is adequate if there exists a τ-covering $\{\mathrm{Spec}(A_i) \to S\}_{i \in I}$ such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$.

We will see below that the category of adequate $\mathcal{O}$-modules is independent of the chosen topology τ.

06VI Lemma 46.5.2. Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate.

Proof. Let $\{\mathrm{Spec}(A_i) \to S\}_{i \in I}$ be a τ-covering such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$. We can find a standard affine τ-covering $\{\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A)\}_{j=1,\ldots,m}$ such that $\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A) \to S$ factors through $\mathrm{Spec}(A_{i(j)})$ for some $i(j) \in I$. Then we see that $F_{\mathcal{F},A'_j}$ is the restriction of $F_{\mathcal{F},A_{i(j)}}$ to the category of A'_j-algebras. Hence $F_{\mathcal{F},A'_j}$ is adequate by Lemma 46.3.17. By Lemma 46.3.19 the sequence $F_{\mathcal{F},A'_j}$ corresponds to an adequate "product" functor F' over $A' = A'_1 \times \ldots \times A'_m$. As $\mathcal{F}$ is a sheaf (for the Zariski topology) this product functor F' is equal to $F_{\mathcal{F},A'}$, i.e., is the restriction of F to A'-algebras. Finally, $\{\mathrm{Spec}(A') \to \mathrm{Spec}(A)\}$ is a τ-covering. It follows from Lemma 46.3.20 that $F_{\mathcal{F},A}$ is adequate. □

06ZJ Lemma 46.5.3. Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is equivalent to the category of adequate module-valued functors on Alg_A.

Proof. Given an adequate module $\mathcal{F}$ the functor $F_{\mathcal{F},A}$ is adequate by Lemma 46.5.2. Given an adequate functor F we choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and we consider the $\mathcal{O}$-module $\mathcal{F} = \mathrm{Ker}(M^a \to N^a)$ where M^a denotes the quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on S. Note that $F = F_{\mathcal{F},A}$, in particular the module $\mathcal{F}$ is adequate by definition. We omit the proof that the constructions define mutually inverse equivalences of categories. □

06VJ Lemma 46.5.4. Let $f : T \to S$ be a morphism of schemes. The pullback $f^*\mathcal{F}$ of an adequate $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_\tau$ is an adequate $\mathcal{O}$-module on $(Sch/T)_\tau$.

Proof. The pullback map $f^* : \mathrm{Mod}((Sch/S)_\tau, \mathcal{O}) \to \mathrm{Mod}((Sch/T)_\tau, \mathcal{O})$ is given by restriction, i.e., $f^*\mathcal{F}(V) = \mathcal{F}(V)$ for any scheme V over T. Hence this lemma follows immediately from Lemma 46.5.2 and the definition. □

Here is a characterization of the category of adequate $\mathcal{O}$-modules. To understand the significance, consider a map $\mathcal{G} \to \mathcal{H}$ of quasi-coherent $\mathcal{O}_S$-modules on a scheme S. The cokernel of the associated map $\mathcal{G}^a \to \mathcal{H}^a$ of $\mathcal{O}$-modules is quasi-coherent because it is equal to $(\mathcal{H}/\mathcal{G})^a$. But the kernel of $\mathcal{G}^a \to \mathcal{H}^a$ in general isn't quasi-coherent. However, it is adequate.

06VK Lemma 46.5.5. Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent

(1) $\mathcal{F}$ is adequate,
(2) there exists an affine open covering $S = \bigcup S_i$ and maps of quasi-coherent $\mathcal{O}_{S_i}$-modules $\mathcal{G}_i \to \mathcal{H}_i$ such that $\mathcal{F}|_{(Sch/S_i)_\tau}$ is the kernel of $\mathcal{G}_i^a \to \mathcal{H}_i^a$
(3) there exists a τ-covering $\{S_i \to S\}_{i \in I}$ and maps of $\mathcal{O}_{S_i}$-quasi-coherent modules $\mathcal{G}_i \to \mathcal{H}_i$ such that $\mathcal{F}|_{(Sch/S_i)_\tau}$ is the kernel of $\mathcal{G}_i^a \to \mathcal{H}_i^a$,
(4) there exists a τ-covering $\{f_i : S_i \to S\}_{i \in I}$ such that each $f_i^*\mathcal{F}$ is adequate,
(5) for any affine scheme U over S the restriction $\mathcal{F}|_{(Sch/U)_\tau}$ is the kernel of a map $\mathcal{G}^a \to \mathcal{H}^a$ of quasi-coherent $\mathcal{O}_U$-modules.

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. By definition, the functor F is adequate if and only if there exists a map of A-modules $M \to N$ such that $F = \mathrm{Ker}(\underline{M} \to \underline{N})$. Combining with Lemmas 46.5.2 and 46.5.3 we see that (1) and (5) are equivalent.

It is clear that (5) implies (2) and (2) implies (3). If (3) holds then we can refine the covering $\{S_i \to S\}$ such that each $S_i = \mathrm{Spec}(A_i)$ is affine. Then we see, by the preliminary remarks of the proof, that $F_{\mathcal{F},A_i}$ is adequate. Thus $\mathcal{F}$ is adequate by definition. Hence (3) implies (1).

Finally, (4) is equivalent to (1) using Lemma 46.5.4 for one direction and that a composition of τ-coverings is a τ-covering for the other. $\square$

Just like is true for quasi-coherent sheaves the category of adequate modules is independent of the topology.

06VL Lemma 46.5.6. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any surjective flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affines over S the extended Čech complex

$$0 \to \mathcal{F}(\mathrm{Spec}(A)) \to \mathcal{F}(\mathrm{Spec}(B)) \to \mathcal{F}(\mathrm{Spec}(B \otimes_A B)) \to \ldots$$

is exact. In particular $\mathcal{F}$ satisfies the sheaf condition for fpqc coverings, and is a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$.

Proof. With $A \to B$ as in the lemma let $F = F_{\mathcal{F},A}$. This functor is adequate by Lemma 46.5.2. By Lemma 46.3.5 since $A \to B$, $A \to B \otimes_A B$, etc are flat we see that $F(B) = F(A) \otimes_A B$, $F(B \otimes_A B) = F(A) \otimes_A B \otimes_A B$, etc. Exactness follows from Descent, Lemma 35.3.6.

Thus $\mathcal{F}$ satisfies the sheaf condition for τ-coverings (in particular Zariski coverings) and any faithfully flat covering of an affine by an affine. Arguing as in the proofs of Descent, Lemma 35.5.1 and Descent, Proposition 35.5.2 we conclude that $\mathcal{F}$ satisfies the sheaf condition for all fpqc coverings (made out of objects of $(Sch/S)_\tau$). Details omitted. $\square$

Lemma 46.5.6 shows in particular that for any pair of topologies τ, τ' the collection of adequate modules for the τ-topology and the τ'-topology are identical (as presheaves of modules on the underlying category Sch/S).

07AH Definition 46.5.7. Let S be a scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is denoted $\mathrm{Adeq}(\mathcal{O})$ or $\mathrm{Adeq}((Sch/S)_\tau, \mathcal{O})$. If we want to think just about the abelian category of adequate modules without choosing a topology we simply write $\mathrm{Adeq}(S)$.

06VM Lemma 46.5.8. Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$.

(1) The restriction $\mathcal{F}|_{S_{Zar}}$ is a quasi-coherent $\mathcal{O}_S$-module on the scheme S.

(2) The restriction $\mathcal{F}|_{S_{\acute{e}tale}}$ is the quasi-coherent module associated to $\mathcal{F}|_{S_{Zar}}$.
(3) For any affine scheme U over S we have $H^q(U, \mathcal{F}) = 0$ for all $q > 0$.
(4) There is a canonical isomorphism

$$H^q(S, \mathcal{F}|_{S_{Zar}}) = H^q((Sch/S)_\tau, \mathcal{F}).$$

Proof. By Lemma 46.3.5 and Lemma 46.5.2 we see that for any flat morphism of affines $U \to V$ over S we have $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. This works in particular if $U \subset V \subset S$ are affine opens of S, hence $\mathcal{F}|_{S_{Zar}}$ is quasi-coherent. Thus (1) holds.

Let $S' \to S$ be an étale morphism of schemes. Then for $U \subset S'$ affine open mapping into an affine open $V \subset S$ we see that $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$ because $U \to V$ is étale, hence flat. Therefore $\mathcal{F}|_{S'_{Zar}}$ is the pullback of $\mathcal{F}|_{S_{Zar}}$. This proves (2).

We are going to apply Cohomology on Sites, Lemma 21.10.9 to the site $(Sch/S)_\tau$ with $\mathcal{B}$ the set of affine schemes over S and Cov the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Descent, Lemma 35.9.1 and Lemma 46.5.6 for the case of a covering by a single affine. Hence we conclude that $H^p(U, \mathcal{F}) = 0$ for every affine scheme U over S. This proves (3). In exactly the same way as in the proof of Descent, Proposition 35.9.3 this implies the equality of cohomologies (4). ☐

06VN Remark 46.5.9. Let S be a scheme. We have functors $u : QCoh(\mathcal{O}_S) \to \text{Adeq}(\mathcal{O})$ and $v : \text{Adeq}(\mathcal{O}) \to QCoh(\mathcal{O}_S)$. Namely, the functor $u : \mathcal{F} \mapsto \mathcal{F}^a$ comes from taking the associated $\mathcal{O}$-module which is adequate by Lemma 46.5.5. Conversely, the functor v comes from restriction $v : \mathcal{G} \mapsto \mathcal{G}|_{S_{Zar}}$, see Lemma 46.5.8. Since $\mathcal{F}^a$ can be described as the pullback of $\mathcal{F}$ under a morphism of ringed topoi $((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Descent, Remark 35.8.6 and since restriction is the pushforward we see that u and v are adjoint as follows

$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(u\mathcal{F}, \mathcal{G})$$

where $\mathcal{O}$ denotes the structure sheaf on the big site. It is immediate from the description that the adjunction mapping $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for all quasi-coherent sheaves.

06VP Lemma 46.5.10. Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. If for every affine scheme $\text{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate, then the sheafification of $\mathcal{F}$ is an adequate $\mathcal{O}$-module.

Proof. Let $U = \text{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. The sheafification $\mathcal{F}^\# = (\mathcal{F}^+)^+$, see Sites, Section 7.10. By construction

$$(\mathcal{F})^+(U) = \text{colim}_\mathcal{U} \check{H}^0(\mathcal{U}, \mathcal{F})$$

where the colimit is over coverings in the site $(Sch/S)_\tau$. Since U is affine it suffices to take the limit over standard affine τ-coverings $\mathcal{U} = \{U_i \to U\}_{i \in I} = \{\text{Spec}(A_i) \to \text{Spec}(A)\}_{i \in I}$ of U. Since each $A \to A_i$ and $A \to A_i \otimes_A A_j$ is flat we see that

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \text{Ker}(\prod F(A) \otimes_A A_i \to \prod F(A) \otimes_A A_i \otimes_A A_j)$$

by Lemma 46.3.5. Since $A \to \prod A_i$ is faithfully flat we see that this always is canonically isomorphic to $F(A)$ by Descent, Lemma 35.3.6. Thus the presheaf $(\mathcal{F})^+$ has the same value as $\mathcal{F}$ on all affine schemes over S. Repeating the argument once more we deduce the same thing for $\mathcal{F}^\# = ((\mathcal{F})^+)^+$. Thus $F_{\mathcal{F},A} = F_{\mathcal{F}^\#,A}$ and we conclude that $\mathcal{F}^\#$ is adequate. ☐

06VQ Lemma 46.5.11. Let S be a scheme.

(1) The category $\text{Adeq}(\mathcal{O})$ is abelian.
(2) The functor $\text{Adeq}(\mathcal{O}) \to \text{Mod}((Sch/S)_\tau, \mathcal{O})$ is exact.
(3) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate, then $\mathcal{F}_2$ is adequate.
(4) The category $\text{Adeq}(\mathcal{O})$ has colimits and $\text{Adeq}(\mathcal{O}) \to \text{Mod}((Sch/S)_\tau, \mathcal{O})$ commutes with them.

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of adequate $\mathcal{O}$-modules. To prove (1) and (2) it suffices to show that $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ computed in $\text{Mod}((Sch/S)_\tau, \mathcal{O})$ are adequate. Let $U = \text{Spec}(A)$ be an affine scheme over S. Let $F = F_{\mathcal{F},A}$ and $G = F_{\mathcal{G},A}$. By Lemmas 46.3.11 and 46.3.10 the kernel K and cokernel Q of the induced map $F \to G$ are adequate functors. Because the kernel is computed on the level of presheaves, we see that $K = F_{\mathcal{K},A}$ and we conclude $\mathcal{K}$ is adequate. To prove the result for the cokernel, denote $\mathcal{Q}'$ the presheaf cokernel of φ. Then $Q = F_{\mathcal{Q}',A}$ and $\mathcal{Q} = (\mathcal{Q}')^\#$. Hence $\mathcal{Q}$ is adequate by Lemma 46.5.10.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate. Let $U = \text{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. The sequence of functors

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact, because for $V = \text{Spec}(B)$ affine over U we have $H^1(V, \mathcal{F}_1) = 0$ by Lemma 46.5.8. Since F_1 and F_3 are adequate functors by Lemma 46.5.2 we see that F_2 is adequate by Lemma 46.3.16. Thus $\mathcal{F}_2$ is adequate.

Let $\mathcal{I} \to \text{Adeq}(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Denote $\mathcal{F} = \text{colim}_i \mathcal{F}_i$ the colimit computed in $\text{Mod}((Sch/S)_\tau, \mathcal{O})$. To prove (4) it suffices to show that $\mathcal{F}$ is adequate. Let $\mathcal{F}' = \text{colim}_i \mathcal{F}_i$ be the colimit computed in presheaves of $\mathcal{O}$-modules. Then $\mathcal{F} = (\mathcal{F}')^\#$. Let $U = \text{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. By Lemma 46.3.12 the functor $\text{colim}_i F_i = F_{\mathcal{F}',A}$ is adequate. Lemma 46.5.10 shows that $\mathcal{F}$ is adequate. $\square$

The following lemma tells us that the total direct image $Rf_*\mathcal{F}$ of an adequate module under a quasi-compact and quasi-separated morphism is a complex whose cohomology sheaves are adequate.

06VR Lemma 46.5.12. Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^if_*\mathcal{F}$ are adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.

Proof. First we explain how to compute the higher direct images. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then $R^if_*\mathcal{F}$ is the ith cohomology sheaf of the complex $f_*\mathcal{I}^\bullet$. Hence $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to an object U/S of $(Sch/S)_\tau$ the module

$$\begin{aligned}
\frac{\text{Ker}(f_*\mathcal{I}^i(U) \to f_*\mathcal{I}^{i+1}(U))}{\text{Im}(f_*\mathcal{I}^{i-1}(U) \to f_*\mathcal{I}^i(U))} &= \frac{\text{Ker}(\mathcal{I}^i(U \times_S T) \to \mathcal{I}^{i+1}(U \times_S T))}{\text{Im}(\mathcal{I}^{i-1}(U \times_S T) \to \mathcal{I}^i(U \times_S T))} \\
&= H^i(U \times_S T, \mathcal{F}) \\
&= H^i((Sch/U \times_S T)_\tau, \mathcal{F}|_{(Sch/U \times_S T)_\tau}) \\
&= H^i(U \times_S T, \mathcal{F}|_{(U \times_S T)_{Zar}})
\end{aligned}$$

The first equality by Topologies, Lemma 34.7.12 (and its analogues for other topologies), the second equality by definition of cohomology of $\mathcal{F}$ over an object of $(Sch/T)_\tau$, the third equality by Cohomology on Sites, Lemma 21.7.1, and the last equality by Lemma 46.5.8. Thus by Lemma 46.5.10 it suffices to prove the claim stated in the following paragraph.

Let A be a ring. Let T be a scheme quasi-compact and quasi-separated over A. Let $\mathcal{F}$ be an adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$. For an A-algebra B set $T_B = T \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and denote $\mathcal{F}_B = \mathcal{F}|_{(T_B)_{Zar}}$ the restriction of $\mathcal{F}$ to the small Zariski site of T_B. (Recall that this is a "usual" quasi-coherent sheaf on the scheme T_B, see Lemma 46.5.8.) Claim: The functor

$$B \longmapsto H^q(T_B, \mathcal{F}_B)$$

is adequate. We will prove the lemma by the usual procedure of cutting T into pieces.

Case I: T is affine. In this case the schemes T_B are all affine and $H^q(T_B, \mathcal{F}_B) = 0$ for all $q \geq 1$. The functor $B \mapsto H^0(T_B, \mathcal{F}_B)$ is adequate by Lemma 46.3.18.

Case II: T is separated. Let n be the minimal number of affines needed to cover T. We argue by induction on n. The base case is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Observe that

$$U \cap V = (V_1 \cap V_n) \cup \ldots \cup (V_{n-1} \cap V_n)$$

is also a union of $n-1$ affine opens as T is separated, see Schemes, Lemma 26.21.7. Note that for each B the base changes U_B, V_B and $(U \cap V)_B = U_B \cap V_B$ behave in the same way. Hence we see that for each B we have a long exact sequence

$$0 \to H^0(T_B, \mathcal{F}_B) \to H^0(U_B, \mathcal{F}_B) \oplus H^0(V_B, \mathcal{F}_B) \to H^0((U \cap V)_B, \mathcal{F}_B) \to H^1(T_B, \mathcal{F}_B) \to \ldots$$

functorial in B, see Cohomology, Lemma 20.8.2. By induction hypothesis the functors $B \mapsto H^q(U_B, \mathcal{F}_B)$, $B \mapsto H^q(V_B, \mathcal{F}_B)$, and $B \mapsto H^q((U \cap V)_B, \mathcal{F}_B)$ are adequate. Using Lemmas 46.3.11 and 46.3.10 we see that our functor $B \mapsto H^q(T_B, \mathcal{F}_B)$ sits in the middle of a short exact sequence whose outer terms are adequate. Thus the claim follows from Lemma 46.3.16.

Case III: General quasi-compact and quasi-separated case. The proof is again by induction on the number n of affines needed to cover T. The base case $n = 1$ is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Note that since T is quasi-separated $U \cap V$ is a quasi-compact open of an affine scheme, hence Case II applies to it. The rest of the argument proceeds in exactly the same manner as in the paragraph above and is omitted. $\square$

46.6. Parasitic adequate modules

06ZK In this section we start comparing adequate modules and quasi-coherent modules on a scheme S. Recall that there are functors $u : QCoh(\mathcal{O}_S) \to \mathrm{Adeq}(\mathcal{O})$ and $v : \mathrm{Adeq}(\mathcal{O}) \to QCoh(\mathcal{O}_S)$ satisfying the adjunction

$$\mathcal{H}om_{QCoh(\mathcal{O}_S)}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{\mathrm{Adeq}(\mathcal{O})}(u\mathcal{F}, \mathcal{G})$$

and such that $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for every quasi-coherent sheaf $\mathcal{F}$, see Remark 46.5.9. Hence u is a fully faithful embedding and we can identify $QCoh(\mathcal{O}_S)$ with a full subcategory of $\mathrm{Adeq}(\mathcal{O})$. The functor v is exact but u is not left exact in general. The kernel of v is the subcategory of parasitic adequate modules.

In Descent, Definition 35.12.1 we give the definition of a parasitic module. For adequate modules the notion does not depend on the chosen topology.

06ZM Lemma 46.6.1. Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent:

(1) $v\mathcal{F} = 0$,
(2) $\mathcal{F}$ is parasitic,
(3) $\mathcal{F}$ is parasitic for the τ-topology,
(4) $\mathcal{F}(U) = 0$ for all $U \subset S$ open, and
(5) there exists an affine open covering $S = \bigcup U_i$ such that $\mathcal{F}(U_i) = 0$ for all i.

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate from the definitions. Assume (5). Suppose that $S = \bigcup U_i$ is an affine open covering such that $\mathcal{F}(U_i) = 0$ for all i. Let $V \to S$ be a flat morphism. There exists an affine open covering $V = \bigcup V_j$ such that each V_j maps into some U_i. As the morphism $V_j \to S$ is flat, also $V_j \to U_i$ is flat. Hence the corresponding ring map $A_i = \mathcal{O}(U_i) \to \mathcal{O}(V_j) = B_j$ is flat. Thus by Lemma 46.5.2 and Lemma 46.3.5 we see that $\mathcal{F}(U_i) \otimes_{A_i} B_j \to \mathcal{F}(V_j)$ is an isomorphism. Hence $\mathcal{F}(V_j) = 0$. Since $\mathcal{F}$ is a sheaf for the Zariski topology we conclude that $\mathcal{F}(V) = 0$. In this way we see that (5) implies (2).

This proves the equivalence of (2), (3), (4), and (5). As (1) is equivalent to (3) (see Remark 46.5.9) we conclude that all five conditions are equivalent. $\square$

Let S be a scheme. The subcategory of parasitic adequate modules is a Serre subcategory of $\text{Adeq}(\mathcal{O})$. The quotient is the category of quasi-coherent modules.

06ZN Lemma 46.6.2. Let S be a scheme. The subcategory $\mathcal{C} \subset \text{Adeq}(\mathcal{O})$ of parasitic adequate modules is a Serre subcategory. Moreover, the functor v induces an equivalence of categories

$$\text{Adeq}(\mathcal{O})/\mathcal{C} = QCoh(\mathcal{O}_S).$$

Proof. The category $\mathcal{C}$ is the kernel of the exact functor $v : \text{Adeq}(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, see Lemma 46.6.1. Hence it is a Serre subcategory by Homology, Lemma 12.10.4. By Homology, Lemma 12.10.6 we obtain an induced exact functor $\overline{v} : \text{Adeq}(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$. Because u is a right inverse to v we see right away that $\overline{v}$ is essentially surjective. We see that $\overline{v}$ is faithful by Homology, Lemma 12.10.7. Because u is a right inverse to v we finally conclude that $\overline{v}$ is fully faithful. $\square$

06ZP Lemma 46.6.3. Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any parasitic adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^if_*\mathcal{F}$ are parasitic adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.

Proof. We have already seen in Lemma 46.5.12 that these higher direct images are adequate. Hence it suffices to show that $(R^if_*\mathcal{F})(U_i) = 0$ for any τ-covering $\{U_i \to S\}$ open. And $R^if_*\mathcal{F}$ is parasitic by Descent, Lemma 35.12.3. $\square$

46.7. Derived categories of adequate modules, I

06VS Let S be a scheme. We continue the discussion started in Section 46.6. The exact functor v induces a functor

$$D(\text{Adeq}(\mathcal{O})) \longrightarrow D(QCoh(\mathcal{O}_S))$$

and similarly for bounded versions.

06ZQ **Lemma 46.7.1.** Let S be a scheme. Let $\mathcal{C} \subset \text{Adeq}(\mathcal{O})$ denote the full subcategory consisting of parasitic adequate modules. Then

$$D(\text{Adeq}(\mathcal{O}))/D_{\mathcal{C}}(\text{Adeq}(\mathcal{O})) = D(QCoh(\mathcal{O}_S))$$

and similarly for the bounded versions.

Proof. Follows immediately from Derived Categories, Lemma 13.17.3. ☐

Next, we look for a description the other way around by looking at the functors

$$K^+(QCoh(\mathcal{O}_S)) \longrightarrow K^+(\text{Adeq}(\mathcal{O})) \longrightarrow D^+(\text{Adeq}(\mathcal{O})) \longrightarrow D^+(QCoh(\mathcal{O}_S)).$$

In some cases the derived category of adequate modules is a localization of the homotopy category of complexes of quasi-coherent modules at universal quasi-isomorphisms. Let S be a scheme. A map of complexes $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules is said to be a *universal quasi-isomorphism* if for every morphism of schemes $f : T \to S$ the pullback $f^*\varphi$ is a quasi-isomorphism.

06ZR **Lemma 46.7.2.** Let $U = \text{Spec}(A)$ be an affine scheme. The bounded below derived category $D^+(\text{Adeq}(\mathcal{O}))$ is the localization of $K^+(QCoh(\mathcal{O}_U))$ at the multiplicative subset of universal quasi-isomorphisms.

Proof. If $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes of quasi-coherent $\mathcal{O}_U$-modules, then $u\varphi : u\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ is a quasi-isomorphism if and only if φ is a universal quasi-isomorphism. Hence the collection S of universal quasi-isomorphisms is a saturated multiplicative system compatible with the triangulated structure by Derived Categories, Lemma 13.5.4. Hence $S^{-1}K^+(QCoh(\mathcal{O}_U))$ exists and is a triangulated category, see Derived Categories, Proposition 13.5.6. We obtain a canonical functor $can : S^{-1}K^+(QCoh(\mathcal{O}_U)) \to D^+(\text{Adeq}(\mathcal{O}))$ by Derived Categories, Lemma 13.5.7.

Note that, almost by definition, every adequate module on U has an embedding into a quasi-coherent sheaf, see Lemma 46.5.5. Hence by Derived Categories, Lemma 13.15.5 given $\mathcal{F}^\bullet \in \text{Ob}(K^+(\text{Adeq}(\mathcal{O})))$ there exists a quasi-isomorphism $\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ where $\mathcal{G}^\bullet \in \text{Ob}(K^+(QCoh(\mathcal{O}_U)))$. This proves that can is essentially surjective.

Similarly, suppose that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are bounded below complexes of quasi-coherent $\mathcal{O}_U$-modules. A morphism in $D^+(\text{Adeq}(\mathcal{O}))$ between these consists of a pair $f : u\mathcal{F}^\bullet \to \mathcal{H}^\bullet$ and $s : u\mathcal{G}^\bullet \to \mathcal{H}^\bullet$ where s is a quasi-isomorphism. Pick a quasi-isomorphism $s' : \mathcal{H}^\bullet \to u\mathcal{E}^\bullet$. Then we see that $s' \circ f : \mathcal{F} \to \mathcal{E}^\bullet$ and the universal quasi-isomorphism $s' \circ s : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ give a morphism in $S^{-1}K^+(QCoh(\mathcal{O}_U))$ mapping to the given morphism. This proves the "fully" part of full faithfulness. Faithfulness is proved similarly. ☐

06ZS **Lemma 46.7.3.** Let $U = \text{Spec}(A)$ be an affine scheme. The inclusion functor

$$\text{Adeq}(\mathcal{O}) \to \text{Mod}((Sch/U)_\tau, \mathcal{O})$$

has a right adjoint A[1]. Moreover, the adjunction mapping $A(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every adequate module $\mathcal{F}$.

[1]This is the “adequator”.

Proof. By Topologies, Lemma 34.7.11 (and similarly for the other topologies) we may work with $\mathcal{O}$-modules on $(\textit{Aff}/U)_\tau$. Denote $\mathcal{P}$ the category of module-valued functors on $\textit{Alg}_A$ and $\mathcal{A}$ the category of adequate functors on $\textit{Alg}_A$. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 46.4.1. We have the commutative diagram

06ZT (46.7.3.1)
$$\begin{tikzcd} \textit{Adeq}(\mathcal{O}) \arrow[r, "k"'] \arrow[d, equal] & \textit{Mod}((\textit{Aff}/U)_\tau, \mathcal{O}) \arrow[r, "j"'] & \textit{PMod}((\textit{Aff}/U)_\tau, \mathcal{O}) \arrow[d, equal] \\ \mathcal{A} \arrow[rr, "i"] & & \mathcal{P} \end{tikzcd}$$

The left vertical equality is Lemma 46.5.3 and the right vertical equality was explained in Section 46.3. Define $A(\mathcal{F}) = Q(j(\mathcal{F}))$. Since j is fully faithful it follows immediately that A is a right adjoint of the inclusion functor k. Also, since k is fully faithful too, the final assertion follows formally. □

The functor A is a right adjoint hence left exact. Since the inclusion functor is exact, see Lemma 46.5.11 we conclude that A transforms injectives into injectives, and that the category $\textit{Adeq}(\mathcal{O})$ has enough injectives, see Homology, Lemma 12.29.3 and Injectives, Theorem 19.8.4. This also follows from the equivalence in (46.7.3.1) and Lemma 46.4.2.

06ZU Lemma 46.7.4. Let $U = \mathrm{Spec}(A)$ be an affine scheme. For any object $\mathcal{F}$ of $\textit{Adeq}(\mathcal{O})$ we have $R^pA(\mathcal{F}) = 0$ for all $p > 0$ where A is as in Lemma 46.7.3.

Proof. With notation as in the proof of Lemma 46.7.3 choose an injective resolution $k(\mathcal{F}) \to \mathcal{I}^\bullet$ in the category of $\mathcal{O}$-modules on $(\textit{Aff}/U)_\tau$. By Cohomology on Sites, Lemmas 21.12.2 and Lemma 46.5.8 the complex $j(\mathcal{I}^\bullet)$ is exact. On the other hand, each $j(\mathcal{I}^n)$ is an injective object of the category of presheaves of modules by Cohomology on Sites, Lemma 21.12.1. It follows that $R^pA(\mathcal{F}) = R^pQ(j(k(\mathcal{F})))$. Hence the result now follows from Lemma 46.4.10. □

Let S be a scheme. By the discussion in Section 46.5 the embedding $\textit{Adeq}(\mathcal{O}) \subset \textit{Mod}((\textit{Sch}/S)_\tau, \mathcal{O})$ exhibits $\textit{Adeq}(\mathcal{O})$ as a weak Serre subcategory of the category of all $\mathcal{O}$-modules. Denote

$$D_{\textit{Adeq}}(\mathcal{O}) \subset D(\mathcal{O}) = D(\textit{Mod}((\textit{Sch}/S)_\tau, \mathcal{O}))$$

the triangulated subcategory of complexes whose cohomology sheaves are adequate, see Derived Categories, Section 13.17. We obtain a canonical functor

$$D(\textit{Adeq}(\mathcal{O})) \longrightarrow D_{\textit{Adeq}}(\mathcal{O})$$

see Derived Categories, Equation (13.17.1.1).

06ZV Lemma 46.7.5. If $U = \mathrm{Spec}(A)$ is an affine scheme, then the bounded below version

06VV (46.7.5.1)
$$D^+(\textit{Adeq}(\mathcal{O})) \longrightarrow D^+_{\textit{Adeq}}(\mathcal{O})$$

of the functor above is an equivalence.

Proof. Let $A : \textit{Mod}(\mathcal{O}) \to \textit{Adeq}(\mathcal{O})$ be the right adjoint to the inclusion functor constructed in Lemma 46.7.3. Since A is left exact and since $\textit{Mod}(\mathcal{O})$ has enough injectives, A has a right derived functor $RA : D^+_{\textit{Adeq}}(\mathcal{O}) \to D^+(\textit{Adeq}(\mathcal{O}))$. We claim that RA is a quasi-inverse to (46.7.5.1). To see this the key fact is that if $\mathcal{F}$ is an adequate module, then the adjunction map $\mathcal{F} \to RA(\mathcal{F})$ is a quasi-isomorphism by Lemma 46.7.4.

Namely, to prove the lemma in full it suffices to show:

(1) Given $\mathcal{F}^\bullet \in K^+(\text{Adeq}(\mathcal{O}))$ the canonical map $\mathcal{F}^\bullet \to RA(\mathcal{F}^\bullet)$ is a quasi-isomorphism, and
(2) given $\mathcal{G}^\bullet \in K^+(\text{Mod}(\mathcal{O}))$ the canonical map $RA(\mathcal{G}^\bullet) \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Both (1) and (2) follow from the key fact via a spectral sequence argument using one of the spectral sequences of Derived Categories, Lemma 13.21.3. Some details omitted. □

06ZW Lemma 46.7.6. Let $U = \text{Spec}(A)$ be an affine scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be adequate $\mathcal{O}$-modules. For any $i \geq 0$ the natural map

$$\text{Ext}^i_{\text{Adeq}(\mathcal{O})}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Ext}^i_{\text{Mod}(\mathcal{O})}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. By definition these ext groups are computed as hom sets in the derived category. Hence this follows immediately from Lemma 46.7.5. □

46.8. Pure extensions

06ZX We want to characterize extensions of quasi-coherent sheaves on the big site of an affine schemes in terms of algebra. To do this we introduce the following notion.

06ZY Definition 46.8.1. Let A be a ring.

(1) An A-module P is said to be pure projective if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \text{Hom}_A(P, K) \to \text{Hom}_A(P, M) \to \text{Hom}_A(P, N) \to 0$ is exact.
(2) An A-module I is said to be pure injective if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \text{Hom}_A(N, I) \to \text{Hom}_A(M, I) \to \text{Hom}_A(K, I) \to 0$ is exact.

Let's characterize pure projectives.

06ZZ Lemma 46.8.2. Let A be a ring.

(1) A module is pure projective if and only if it is a direct summand of a direct sum of finitely presented A-modules.
(2) For any module M there exists a universally exact sequence $0 \to N \to P \to M \to 0$ with P pure projective.

Proof. First note that a finitely presented A-module is pure projective by Algebra, Theorem 10.82.3. Hence a direct summand of a direct sum of finitely presented A-modules is indeed pure projective. Let M be any A-module. Write $M = \text{colim}_{i \in I} P_i$ as a filtered colimit of finitely presented A-modules. Consider the sequence

$$0 \to N \to \bigoplus P_i \to M \to 0.$$

For any finitely presented A-module P the map $\text{Hom}_A(P, \bigoplus P_i) \to \text{Hom}_A(P, M)$ is surjective, as any map $P \to M$ factors through some P_i. Hence by Algebra, Theorem 10.82.3 this sequence is universally exact. This proves (2). If now M is pure projective, then the sequence is split and we see that M is a direct summand of $\bigoplus P_i$. □

Let's characterize pure injectives.

0700 **Lemma 46.8.3.** Let A be a ring. For any A-module M set $M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$.

1. For any A-module M the A-module $M^\vee$ is pure injective.
2. An A-module I is pure injective if and only if the map $I \to (I^\vee)^\vee$ splits.
3. For any module M there exists a universally exact sequence $0 \to M \to I \to N \to 0$ with I pure injective.

Proof. We will use the properties of the functor $M \mapsto M^\vee$ found in More on Algebra, Section 15.55 without further mention. Part (1) holds because $\mathrm{Hom}_A(N, M^\vee) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_A M, \mathbf{Q}/\mathbf{Z})$ and because $\mathbf{Q}/\mathbf{Z}$ is injective in the category of abelian groups. Hence if $I \to (I^\vee)^\vee$ is split, then I is pure injective. We claim that for any A-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is universally injective. To see this note that $ev^\vee : ((M^\vee)^\vee)^\vee \to M^\vee$ has a right inverse, namely $ev' : M^\vee \to ((M^\vee)^\vee)^\vee$. Then for any A-module N applying the exact faithful functor ${}^\vee$ to the map $N \otimes_A M \to N \otimes_A (M^\vee)^\vee$ gives

$$\mathrm{Hom}_A(N, ((M^\vee)^\vee)^\vee) = \Big(N \otimes_A (M^\vee)^\vee\Big)^\vee \to \Big(N \otimes_A M\Big)^\vee = \mathrm{Hom}_A(N, M^\vee)$$

which is surjective by the existence of the right inverse. The claim follows. The claim implies (3) and the necessity of the condition in (2). $\square$

Before we continue we make the following observation which we will use frequently in the rest of this section.

0701 **Lemma 46.8.4.** Let A be a ring.

1. Let $L \to M \to N$ be a universally exact sequence of A-modules. Let $K = \mathrm{Im}(M \to N)$. Then $K \to N$ is universally injective.
2. Any universally exact complex can be split into universally exact short exact sequences.

Proof. Proof of (1). For any A-module T the sequence $L \otimes_A T \to M \otimes_A T \to K \otimes_A T \to 0$ is exact by right exactness of $\otimes$. By assumption the sequence $L \otimes_A T \to M \otimes_A T \to N \otimes_A T$ is exact. Combined this shows that $K \otimes_A T \to N \otimes_A T$ is injective.

Part (2) means the following: Suppose that $M^\bullet$ is a universally exact complex of A-modules. Set $K^i = \mathrm{Ker}(d^i) \subset M^i$. Then the short exact sequences $0 \to K^i \to M^i \to K^{i+1} \to 0$ are universally exact. This follows immediately from part (1). $\square$

0702 **Definition 46.8.5.** Let A be a ring. Let M be an A-module.

1. A *pure projective resolution* $P_\bullet \to M$ is a universally exact sequence
$$\ldots \to P_1 \to P_0 \to M \to 0$$
with each P_i pure projective.
2. A *pure injective resolution* $M \to I^\bullet$ is a universally exact sequence
$$0 \to M \to I^0 \to I^1 \to \ldots$$
with each I^i pure injective.

These resolutions satisfy the usual uniqueness properties among the class of all universally exact left or right resolutions.

0703 **Lemma 46.8.6.** Let A be a ring.

(1) Any A-module has a pure projective resolution.

Let $M \to N$ be a map of A-modules. Let $P_\bullet \to M$ be a pure projective resolution and let $N_\bullet \to N$ be a universally exact resolution.

(2) There exists a map of complexes $P_\bullet \to N_\bullet$ inducing the given map

$$M = \mathrm{Coker}(P_1 \to P_0) \to \mathrm{Coker}(N_1 \to N_0) = N$$

(3) two maps $\alpha, \beta : P_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.

Proof. Part (1) follows immediately from Lemma 46.8.2. Before we prove (2) and (3) note that by Lemma 46.8.4 we can split the universally exact complex $N_\bullet \to N \to 0$ into universally exact short exact sequences $0 \to K_0 \to N_0 \to N \to 0$ and $0 \to K_i \to N_i \to K_{i-1} \to 0$.

Proof of (2). Because P_0 is pure projective we can find a map $P_0 \to N_0$ lifting the map $P_0 \to M \to N$. We obtain an induced map $P_1 \to F_0 \to N_0$ wich ends up in K_0. Since P_1 is pure projective we may lift this to a map $P_1 \to N_1$. This in turn induces a map $P_2 \to P_1 \to N_1$ which maps to zero into N_0, i.e., into K_1. Hence we may lift to get a map $P_2 \to N_2$. Repeat.

Proof of (3). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : P_0 \to N_0$ is contained in K_0. Hence we may lift γ_0 to a map $h_0 : P_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{P,1} : P_1 \to N_1$. By our choice of h_0 we see that the image of γ_1' is contained in K_1. Since P_1 is pure projective may lift γ_1' to a map $h_1 : P_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

0704 Lemma 46.8.7. Let A be a ring.

(1) Any A-module has a pure injective resolution.

Let $M \to N$ be a map of A-modules. Let $M \to M^\bullet$ be a universally exact resolution and let $N \to I^\bullet$ be a pure injective resolution.

(2) There exists a map of complexes $M^\bullet \to I^\bullet$ inducing the given map

$$M = \mathrm{Ker}(M^0 \to M^1) \to \mathrm{Ker}(I^0 \to I^1) = N$$

(3) two maps $\alpha, \beta : M^\bullet \to I^\bullet$ inducing the same map $M \to N$ are homotopic.

Proof. This lemma is dual to Lemma 46.8.6. The proof is identical, except one has to reverse all the arrows. □

Using the material above we can define pure extension groups as follows. Let A be a ring and let M, N be A-modules. Choose a pure injective resolution $N \to I^\bullet$. By Lemma 46.8.7 the complex

$$\mathrm{Hom}_A(M, I^\bullet)$$

is well defined up to homotopy. Hence its ith cohomology module is a well defined invariant of M and N.

0705 Definition 46.8.8. Let A be a ring and let M, N be A-modules. The ith pure extension module $\mathrm{Pext}^i_A(M, N)$ is the ith cohomology module of the complex $\mathrm{Hom}_A(M, I^\bullet)$ where $I^\bullet$ is a pure injective resolution of N.

Warning: It is not true that an exact sequence of A-modules gives rise to a long exact sequence of pure extensions groups. (You need a universally exact sequence for this.) We collect some facts which are obvious from the material above.

0706 Lemma 46.8.9. Let A be a ring.

(1) $\text{Pext}^i_A(M, N) = 0$ for $i > 0$ whenever N is pure injective,
(2) $\text{Pext}^i_A(M, N) = 0$ for $i > 0$ whenever M is pure projective, in particular if M is an A-module of finite presentation,
(3) $\text{Pext}^i_A(M, N)$ is also the ith cohomology module of the complex $\text{Hom}_A(P_\bullet, N)$ where $P_\bullet$ is a pure projective resolution of M.

Proof. To see (3) consider the double complex

$$A^{\bullet,\bullet} = \text{Hom}_A(P_\bullet, I^\bullet)$$

Each of its rows is exact except in degree 0 where its cohomology is $\text{Hom}_A(M, I^q)$. Each of its columns is exact except in degree 0 where its cohomology is $\text{Hom}_A(P_p, N)$. Hence the two spectral sequences associated to this complex in Homology, Section 12.25 degenerate, giving the equality. □

46.9. Higher exts of quasi-coherent sheaves on the big site

0707 It turns out that the module-valued functor $\underline{I}$ associated to a pure injective module I gives rise to an injective object in the category of adequate functors on Alg_A. Warning: It is not true that a pure projective module gives rise to a projective object in the category of adequate functors. We do have plenty of projective objects, namely, the linearly adequate functors.

0708 Lemma 46.9.1. Let A be a ring. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. The injective objects of $\mathcal{A}$ are exactly the functors $\underline{I}$ where I is a pure injective A-module.

Proof. Let I be an injective object of $\mathcal{A}$. Choose an embedding $I \to \underline{M}$ for some A-module M. As I is injective we see that $\underline{M} = I \oplus F$ for some module-valued functor F. Then $M = I(A) \oplus F(A)$ and it follows that $I = \underline{I(A)}$. Thus we see that any injective object is of the form $\underline{I}$ for some A-module I. It is clear that the module I has to be pure injective since any universally exact sequence $0 \to M \to N \to L \to 0$ gives rise to an exact sequence $0 \to \underline{M} \to \underline{N} \to \underline{L} \to 0$ of $\mathcal{A}$.

Finally, suppose that I is a pure injective A-module. Choose an embedding $\underline{I} \to J$ into an injective object of $\mathcal{A}$ (see Lemma 46.4.2). We have seen above that $J = \underline{I'}$ for some A-module I' which is pure injective. As $\underline{I} \to \underline{I'}$ is injective the map $I \to I'$ is universally injective. By assumption on I it splits. Hence $\underline{I}$ is a summand of $J = \underline{I'}$ whence an injective object of the category $\mathcal{A}$. □

Let $U = \text{Spec}(A)$ be an affine scheme. Let M be an A-module. We will use the notation M^a to denote the quasi-coherent sheaf of $\mathcal{O}$-modules on $(Sch/U)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on U. Now we have all the notation in place to formulate the following lemma.

0709 Lemma 46.9.2. Let $U = \text{Spec}(A)$ be an affine scheme. Let M, N be A-modules. For all i we have a canonical isomorphism

$$\text{Ext}^i_{\textit{Mod}(\mathcal{O})}(M^a, N^a) = \text{Pext}^i_A(M, N)$$

functorial in M and N.

Proof. Let us construct a canonical arrow from right to left. Namely, if $N \to I^\bullet$ is a pure injective resolution, then $M^a \to (I^\bullet)^a$ is an exact complex of (adequate) $\mathcal{O}$-modules. Hence any element of $\text{Pext}^i_A(M, N)$ gives rise to a map $N^a \to M^a[i]$ in $D(\mathcal{O})$, i.e., an element of the group on the left.

To prove this map is an isomorphism, note that we may replace $\text{Ext}^i_{\text{Mod}(\mathcal{O})}(M^a, N^a)$ by $\text{Ext}^i_{\text{Adeq}(\mathcal{O})}(M^a, N^a)$, see Lemma 46.7.6. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. We have seen that $\mathcal{A}$ is equivalent to $\text{Adeq}(\mathcal{O})$, see Lemma 46.5.3; see also the proof of Lemma 46.7.3. Hence now it suffices to prove that

$$\text{Ext}^i_{\mathcal{A}}(\underline{M}, \underline{N}) = \text{Pext}^i_A(M, N)$$

However, this is clear from Lemma 46.9.1 as a pure injective resolution $N \to I^\bullet$ exactly corresponds to an injective resolution of $\underline{N}$ in $\mathcal{A}$. □

46.10. Derived categories of adequate modules, II

070T Let S be a scheme. Denote $\mathcal{O}_S$ the structure sheaf of S and $\mathcal{O}$ the structure sheaf of the big site $(Sch/S)_\tau$. In Descent, Remark 35.8.4 we constructed a morphism of ringed sites

070U (46.10.0.1)
$$f : ((Sch/S)_\tau, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_S).$$

In the previous sections have seen that the functor $f_* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_S)$ transforms adequate sheaves into quasi-coherent sheaves, and induces an exact functor $v : \text{Adeq}(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, and in fact that $f_* = v$ induces an equivalence $\text{Adeq}(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$ where $\mathcal{C}$ is the subcategory of parasitic adequate modules. Moreover, the functor f^* transforms quasi-coherent modules into adequate modules, and induces a functor $u : QCoh(\mathcal{O}_S) \to \text{Adeq}(\mathcal{O})$ which is a left adjoint to v.

There is a very similar relationship between $D_{\text{Adeq}}(\mathcal{O})$ and $D_{QCoh}(S)$. First we explain why the category $D_{\text{Adeq}}(\mathcal{O})$ is independent of the chosen topology.

070V Remark 46.10.1. Let S be a scheme. Let $\tau, \tau' \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Denote $\mathcal{O}_\tau$, resp. $\mathcal{O}_{\tau'}$ the structure sheaf $\mathcal{O}$ viewed as a sheaf on $(Sch/S)_\tau$, resp. $(Sch/S)_{\tau'}$. Then $D_{\text{Adeq}}(\mathcal{O}_\tau)$ and $D_{\text{Adeq}}(\mathcal{O}_{\tau'})$ are canonically isomorphic. This follows from Cohomology on Sites, Lemma 21.29.1. Namely, assume τ is stronger than the topology τ', let $\mathcal{C} = (Sch/S)_{fppf}$, and let $\mathcal{B}$ the collection of affine schemes over S. Assumptions (1) and (2) we've seen above. Assumption (3) is clear and assumption (4) follows from Lemma 46.5.8.

070W Remark 46.10.2. Let S be a scheme. The morphism f see (46.10.0.1) induces adjoint functors $Rf_* : D_{\text{Adeq}}(\mathcal{O}) \to D_{QCoh}(S)$ and $Lf^* : D_{QCoh}(S) \to D_{\text{Adeq}}(\mathcal{O})$. Moreover $Rf_* Lf^* \cong \text{id}_{D_{QCoh}(S)}$.

We sketch the proof. By Remark 46.10.1 we may assume the topology τ is the Zariski topology. We will use the existence of the unbounded total derived functors Lf^* and Rf_* on $\mathcal{O}$-modules and their adjointness, see Cohomology on Sites, Lemma 21.19.1. In this case f_* is just the restriction to the subcategory S_{Zar} of $(Sch/S)_{Zar}$. Hence it is clear that $Rf_* = f_*$ induces $Rf_* : D_{\text{Adeq}}(\mathcal{O}) \to D_{QCoh}(S)$. Suppose that $\mathcal{G}^\bullet$ is an object of $D_{QCoh}(S)$. We may choose a system $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ of bounded above complexes of flat $\mathcal{O}_S$-modules whose transition maps are termwise split injectives and a diagram

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the properties (1), (2), (3) listed in Derived Categories, Lemma 13.29.1 where $\mathcal{P}$ is the collection of flat $\mathcal{O}_S$-modules. Then $Lf^*\mathcal{G}^\bullet$ is computed by $\operatorname{colim} f^*\mathcal{K}_n^\bullet$, see Cohomology on Sites, Lemmas 21.18.1 and 21.18.2 (note that our sites have enough points by Étale Cohomology, Lemma 59.30.1). We have to see that $H^i(Lf^*\mathcal{G}^\bullet) = \operatorname{colim} H^i(f^*\mathcal{K}_n^\bullet)$ is adequate for each i. By Lemma 46.5.11 we conclude that it suffices to show that each $H^i(f^*\mathcal{K}_n^\bullet)$ is adequate.

The adequacy of $H^i(f^*\mathcal{K}_n^\bullet)$ is local on S, hence we may assume that $S = \operatorname{Spec}(A)$ is affine. Because S is affine $D_{QCoh}(S) = D(QCoh(\mathcal{O}_S))$, see the discussion in Derived Categories of Schemes, Section 36.3. Hence there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{K}_n^\bullet$ where $\mathcal{F}^\bullet$ is a bounded above complex of flat quasi-coherent modules. Then $f^*\mathcal{F}^\bullet \to f^*\mathcal{K}_n^\bullet$ is a quasi-isomorphism, and the cohomology sheaves of $f^*\mathcal{F}^\bullet$ are adequate.

The final assertion $Rf_*Lf^* \cong \text{id}_{D_{QCoh}(S)}$ follows from the explicit description of the functors above. (In plain English: if $\mathcal{F}$ is quasi-coherent and $p > 0$, then $L_pf^*\mathcal{F}$ is a parasitic adequate module.)

070X Remark 46.10.3. Remark 46.10.2 above implies we have an equivalence of derived categories

$$D_{\text{Adeq}}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O}) \longrightarrow D_{QCoh}(S)$$

where $\mathcal{C}$ is the category of parasitic adequate modules. Namely, it is clear that $D_{\mathcal{C}}(\mathcal{O})$ is the kernel of Rf_*, hence a functor as indicated. For any object X of $D_{\text{Adeq}}(\mathcal{O})$ the map $Lf^*Rf_*X \to X$ maps to a quasi-isomorphism in $D_{QCoh}(S)$, hence $Lf^*Rf_*X \to X$ is an isomorphism in $D_{\text{Adeq}}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})$. Finally, for X, Y objects of $D_{\text{Adeq}}(\mathcal{O})$ the map

$$Rf_* : \operatorname{Hom}_{D_{\text{Adeq}}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})}(X, Y) \to \operatorname{Hom}_{D_{QCoh}(S)}(Rf_*X, Rf_*Y)$$

is bijective as Lf^* gives an inverse (by the remarks above).

46.11. Other chapters

Preliminaries

CHAPTER 47

Dualizing Complexes

08XG

47.1. Introduction

08XH In this chapter we discuss dualizing complexes in commutative algebra. A reference is [Har66].

We begin with a discussion of essential surjections and essential injections, projective covers, injective hulls, duality for Artinian rings, and study injective hulls of residue fields, leading quickly to a proof of Matlis duality. See Sections 47.2, 47.3, 47.4, 47.5, 47.6, and 47.7 and Proposition 47.7.8.

This is followed by three sections discussing local cohomology in great generality, see Sections 47.8, 47.9, and 47.10. We apply some of this to a discussion of depth in Section 47.11. In another application we show how, given a finitely generated ideal I of a ring A, the "I-complete" and "I-torsion" objects of the derived category of A are equivalent, see Section 47.12. To learn more about local cohomology, for example the finiteness theorem (which relies on local duality – see below) please visit Local Cohomology, Section 51.1.

The bulk of this chapter is devoted to duality for a ring map and dualizing complexes. See Sections 47.13, 47.14, 47.15, 47.16, 47.17, 47.18, 47.19, 47.20, 47.21, 47.22, and 47.23. The key definition is that of a dualizing complex $\omega_A^\bullet$ over a Noetherian ring A as an object $\omega_A^\bullet \in D^+(A)$ whose cohomology modules $H^i(\omega_A^\bullet)$ are finite A-modules, which has finite injective dimension, and is such that the map

$$A \longrightarrow R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$$

is a quasi-isomorphism. After establishing some elementary properties of dualizing complexes, we show a dualizing complex gives rise to a dimension function. Next, we prove Grothendieck's local duality theorem. After briefly discussing dualizing modules and Cohen-Macaulay rings, we introduce Gorenstein rings and we show many familiar Noetherian rings have dualizing complexes. In a last section we apply the material to show there is a good theory of Noetherian local rings whose formal fibres are Gorenstein or local complete intersections.

In the last few sections, we describe an algebraic construction of the "upper shriek functors" used in algebraic geometry, for example in the book [Har66]. This topic is continued in the chapter on duality for schemes. See Duality for Schemes, Section 48.1.

47.2. Essential surjections and injections

08XI We will mostly work in categories of modules, but we may as well make the definition in general.

08XJ Definition 47.2.1. Let $\mathcal{A}$ be an abelian category.

(1) An injection $A \subset B$ of $\mathcal{A}$ is essential, or we say that B is an essential extension of A, if every nonzero subobject $B' \subset B$ has nonzero intersection with A.
(2) A surjection $f : A \to B$ of $\mathcal{A}$ is essential if for every proper subobject $A' \subset A$ we have $f(A') \neq B$.

Some lemmas about this notion.

08XK Lemma 47.2.2. Let $\mathcal{A}$ be an abelian category.
(1) If $A \subset B$ and $B \subset C$ are essential extensions, then $A \subset C$ is an essential extension.
(2) If $A \subset B$ is an essential extension and $C \subset B$ is a subobject, then $A \cap C \subset C$ is an essential extension.
(3) If $A \to B$ and $B \to C$ are essential surjections, then $A \to C$ is an essential surjection.
(4) Given an essential surjection $f : A \to B$ and a surjection $A \to C$ with kernel K, the morphism $C \to B/f(K)$ is an essential surjection.

Proof. Omitted. $\square$

08XL Lemma 47.2.3. Let R be a ring. Let M be an R-module. Let $E = \operatorname{colim} E_i$ be a filtered colimit of R-modules. Suppose given a compatible system of essential injections $M \to E_i$ of R-modules. Then $M \to E$ is an essential injection.

Proof. Immediate from the definitions and the fact that filtered colimits are exact (Algebra, Lemma 10.8.8). $\square$

08XM Lemma 47.2.4. Let R be a ring. Let $M \subset N$ be R-modules. The following are equivalent
(1) $M \subset N$ is an essential extension,
(2) for all $x \in N$ nonzero there exists an $f \in R$ such that $fx \in M$ and $fx \neq 0$.

Proof. Assume (1) and let $x \in N$ be a nonzero element. By (1) we have $Rx \cap M \neq 0$. This implies (2).

Assume (2). Let $N' \subset N$ be a nonzero submodule. Pick $x \in N'$ nonzero. By (2) we can find $f \in R$ with $fx \in M$ and $fx \neq 0$. Thus $N' \cap M \neq 0$. $\square$

47.3. Injective modules

08XN Some results about injective modules over rings.

08XP Lemma 47.3.1. Let R be a ring. Any product of injective R-modules is injective.

Proof. Special case of Homology, Lemma 12.27.3. $\square$

08XQ Lemma 47.3.2. Let $R \to S$ be a flat ring map. If E is an injective S-module, then E is injective as an R-module.

Proof. This is true because $\operatorname{Hom}_R(M, E) = \operatorname{Hom}_S(M \otimes_R S, E)$ by Algebra, Lemma 10.14.3 and the fact that tensoring with S is exact. $\square$

08YV Lemma 47.3.3. Let $R \to S$ be an epimorphism of rings. Let E be an S-module. If E is injective as an R-module, then E is an injective S-module.

Proof. This is true because $\operatorname{Hom}_R(N, E) = \operatorname{Hom}_S(N, E)$ for any S-module N, see Algebra, Lemma 10.107.14. $\square$

08XR Lemma 47.3.4. Let $R \to S$ be a ring map. If E is an injective R-module, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.

Proof. This is true because $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, E)) = \mathrm{Hom}_R(N, E)$ by Algebra, Lemma 10.14.4. □

08XS Lemma 47.3.5. Let R be a ring. Let I be an injective R-module. Let $E \subset I$ be a submodule. The following are equivalent

(1) E is injective, and
(2) for all $E \subset E' \subset I$ with $E \subset E'$ essential we have $E = E'$.

In particular, an R-module is injective if and only if every essential extension is trivial.

Proof. The final assertion follows from the first and the fact that the category of R-modules has enough injectives (More on Algebra, Section 15.55).

Assume (1). Let $E \subset E' \subset I$ as in (2). Then the map $\mathrm{id}_E : E \to E$ can be extended to a map $\alpha : E' \to E$. The kernel of α has to be zero because it intersects E trivially and E' is an essential extension. Hence $E = E'$.

Assume (2). Let $M \subset N$ be R-modules and let $\varphi : M \to E$ be an R-module map. In order to prove (1) we have to show that φ extends to a morphism $N \to E$. Consider the set $\mathcal{S}$ of pairs (M', φ') where $M \subset M' \subset N$ and $\varphi' : M' \to E$ is an R-module map agreeing with φ on M. We define an ordering on $\mathcal{S}$ by the rule $(M', \varphi') \leq (M'', \varphi'')$ if and only if $M' \subset M''$ and $\varphi''|_{M'} = \varphi'$. It is clear that we can take the maximum of a totally ordered subset of $\mathcal{S}$. Hence by Zorn's lemma we may assume (M, φ) is a maximal element.

Choose an extension $\psi : N \to I$ of φ composed with the inclusion $E \to I$. This is possible as I is injective. If $\psi(N) \subset E$, then ψ is the desired extension. If $\psi(N)$ is not contained in E, then by (2) the inclusion $E \subset E + \psi(N)$ is not essential. hence we can find a nonzero submodule $K \subset E + \psi(N)$ meeting E in 0. This means that $M' = \psi^{-1}(E + K)$ strictly contains M. Thus we can extend φ to M' using

$$M' \xrightarrow{\psi|_{M'}} E + K \to (E + K)/K = E$$

This contradicts the maximality of (M, φ). □

08XT Example 47.3.6. Let R be a reduced ring. Let $\mathfrak{p} \subset R$ be a minimal prime so that $K = R_\mathfrak{p}$ is a field (Algebra, Lemma 10.25.1). Then K is an injective R-module. Namely, we have $\mathrm{Hom}_R(M, K) = \mathrm{Hom}_K(M_\mathfrak{p}, K)$ for any R-module M. Since localization is an exact functor and taking duals is an exact functor on K-vector spaces we conclude $\mathrm{Hom}_R(-, K)$ is an exact functor, i.e., K is an injective R-module.

08XV Lemma 47.3.7. Let R be a Noetherian ring. A direct sum of injective modules is injective.

Proof. Let E_i be a family of injective modules parametrized by a set I. Set $E = \bigoplus E_i$. To show that E is injective we use Injectives, Lemma 19.2.6. Thus let $\varphi : I \to E$ be a module map from an ideal of R into E. As I is a finite R-module (because R is Noetherian) we can find finitely many elements $i_1, \ldots, i_r \in I$ such that φ maps into $\bigoplus_{j=1,\ldots,r} E_{i_j}$. Then we can extend φ into $\bigoplus_{j=1,\ldots,r} E_{i_j}$ using the injectivity of the modules E_{i_j}. □

0A6I **Lemma 47.3.8.** Let R be a Noetherian ring. Let $S \subset R$ be a multiplicative subset. If E is an injective R-module, then $S^{-1}E$ is an injective $S^{-1}R$-module.

Proof. Since $R \to S^{-1}R$ is an epimorphism of rings, it suffices to show that $S^{-1}E$ is injective as an R-module, see Lemma 47.3.3. To show this we use Injectives, Lemma 19.2.6. Thus let $I \subset R$ be an ideal and let $\varphi : I \to S^{-1}E$ be an R-module map. As I is a finitely presented R-module (because R is Noetherian) we can find an $f \in S$ and an R-module map $I \to E$ such that $f\varphi$ is the composition $I \to E \to S^{-1}E$ (Algebra, Lemma 10.10.2). Then we can extend $I \to E$ to a homomorphism $R \to E$. Then the composition

$$R \to E \to S^{-1}E \xrightarrow{f^{-1}} S^{-1}E$$

is the desired extension of φ to R. $\square$

08XW **Lemma 47.3.9.** Let R be a Noetherian ring. Let I be an injective R-module.

(1) Let $f \in R$. Then $E = \bigcup I[f^n] = I[f^\infty]$ is an injective submodule of I.
(2) Let $J \subset R$ be an ideal. Then the J-power torsion submodule $I[J^\infty]$ is an injective submodule of I.

Proof. We will use Lemma 47.3.5 to prove (1). Suppose that $E \subset E' \subset I$ and that E' is an essential extension of E. We will show that $E' = E$. If not, then we can find $x \in E'$ and $x \notin E$. Let $J = \{a \in R \mid ax \in E\}$. Since R is Noetherian, we may write $J = (g_1, \ldots, g_t)$ for some $g_i \in R$. By definition E is the set of elements of I annihilated by powers of f, so we may choose integers n_i so that $f^{n_i}g_ix = 0$. Set $n = \max\{n_i\}$. Then $x' = f^n x$ is an element of E' not in E and is annihilated by J. Set $J' = \{a \in R \mid ax' \in E\}$ so $J \subset J'$. Conversely, we have $a \in J'$ if and only if $ax' \in E$ if and only if $f^m ax' = 0$ for some $m \geq 0$. But then $f^m ax' = f^{m+n}ax$ implies $ax \in E$, i.e., $a \in J$. Hence $J = J'$. Thus $J = J' = \text{Ann}(x')$, so $Rx' \cap E = 0$. Hence E' is not an essential extension of E, a contradiction.

To prove (2) write $J = (f_1, \ldots, f_t)$. Then $I[J^\infty]$ is equal to

$$(\ldots((I[f_1^\infty])[f_2^\infty])\ldots)[f_t^\infty]$$

and the result follows from (1) and induction. $\square$

0A6J **Lemma 47.3.10.** Let A be a Noetherian ring. Let E be an injective A-module. Then $E \otimes_A A[x]$ has injective-amplitude $[0, 1]$ as an object of $D(A[x])$. In particular, $E \otimes_A A[x]$ has finite injective dimension as an $A[x]$-module.

Proof. Let us write $E[x] = E \otimes_A A[x]$. Consider the short exact sequence of $A[x]$-modules

$$0 \to E[x] \to \text{Hom}_A(A[x], E[x]) \to \text{Hom}_A(A[x], E[x]) \to 0$$

where the first map sends $p \in E[x]$ to $f \mapsto fp$ and the second map sends φ to $f \mapsto \varphi(xf) - x\varphi(f)$. The second map is surjective because $\text{Hom}_A(A[x], E[x]) = \prod_{n \geq 0} E[x]$ as an abelian group and the map sends (e_n) to $(e_{n+1} - xe_n)$ which is surjective. As an A-module we have $E[x] \cong \bigoplus_{n \geq 0} E$ which is injective by Lemma 47.3.7. Hence the $A[x]$-module $\text{Hom}_A(A[x], E[x])$ is injective by Lemma 47.3.4 and the proof is complete. $\square$

47.4. Projective covers

08XX In this section we briefly discuss projective covers.

08XY Definition 47.4.1. Let R be a ring. A surjection $P \to M$ of R-modules is said to be a projective cover, or sometimes a projective envelope, if P is a projective R-module and $P \to M$ is an essential surjection.

Projective covers do not always exist. For example, if k is a field and $R = k[x]$ is the polynomial ring over k, then the module $M = R/(x)$ does not have a projective cover. Namely, for any surjection $f : P \to M$ with P projective over R, the proper submodule $(x-1)P$ surjects onto M. Hence f is not essential.

08XZ Lemma 47.4.2. Let R be a ring and let M be an R-module. If a projective cover of M exists, then it is unique up to isomorphism.

Proof. Let $P \to M$ and $P' \to M$ be projective covers. Because P is a projective R-module and $P' \to M$ is surjective, we can find an R-module map $\alpha : P \to P'$ compatible with the maps to M. Since $P' \to M$ is essential, we see that α is surjective. As P' is a projective R-module we can choose a direct sum decomposition $P = \text{Ker}(\alpha) \oplus P'$. Since $P' \to M$ is surjective and since $P \to M$ is essential we conclude that $\text{Ker}(\alpha)$ is zero as desired. $\square$

Here is an example where projective covers exist.

08Y0 Lemma 47.4.3. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Any finite R-module has a projective cover.

Proof. Let M be a finite R-module. Let $r = \dim_\kappa(M/\mathfrak{m}M)$. Choose $x_1, \ldots, x_r \in M$ mapping to a basis of $M/\mathfrak{m}M$. Consider the map $f : R^{\oplus r} \to M$. By Nakayama's lemma this is a surjection (Algebra, Lemma 10.20.1). If $N \subset R^{\oplus r}$ is a proper submodule, then $N/\mathfrak{m}N \to \kappa^{\oplus r}$ is not surjective (by Nakayama's lemma again) hence $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is not surjective. Thus f is an essential surjection. $\square$

47.5. Injective hulls

08Y1 In this section we briefly discuss injective hulls.

08Y2 Definition 47.5.1. Let R be a ring. A injection $M \to I$ of R-modules is said to be an injective hull if I is a injective R-module and $M \to I$ is an essential injection.

Injective hulls always exist.

08Y3 Lemma 47.5.2. Let R be a ring. Any R-module has an injective hull.

Proof. Let M be an R-module. By More on Algebra, Section 15.55 the category of R-modules has enough injectives. Choose an injection $M \to I$ with I an injective R-module. Consider the set $\mathcal{S}$ of submodules $M \subset E \subset I$ such that E is an essential extension of M. We order $\mathcal{S}$ by inclusion. If $\{E_\alpha\}$ is a totally ordered subset of $\mathcal{S}$, then $\bigcup E_\alpha$ is an essential extension of M too (Lemma 47.2.3). Thus we can apply Zorn's lemma and find a maximal element $E \in \mathcal{S}$. We claim $M \subset E$ is an injective hull, i.e., E is an injective R-module. This follows from Lemma 47.3.5. $\square$

08Y4 Lemma 47.5.3. Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then

(1) for any R-module map $\varphi : M \to N$ there exists an R-module map $\psi : E \to E'$ such that

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\varphi"'] & E \arrow[d, "\psi"] \\ N \arrow[r] & E' \end{tikzcd}$$

commutes,
(2) if φ is injective, then ψ is injective,
(3) if φ is an essential injection, then ψ is an isomorphism,
(4) if φ is an isomorphism, then ψ is an isomorphism,
(5) if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \oplus I'$ compatible with the embeddings of M,

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part (1) follows from the fact that E' is an injective R-module. Part (2) follows as $\mathrm{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential injection. Then $E \cong \psi(E) \subset E'$ by (2) which implies $E' = \psi(E) \oplus E''$ because E is injective. Since E' is an essential extension of M (Lemma 47.2.2) we get $E'' = 0$. Part (4) is a special case of (3). Assume $M \to I$ as in (5). Choose a map $\alpha : E \to I$ extending the map $M \to I$. Arguing as before we see that α is injective. Thus as before $\alpha(E)$ splits off from I. This proves (5). □

08Y5 Example 47.5.4. Let R be a domain with fraction field K. Then $R \subset K$ is an injective hull of R. Namely, by Example 47.3.6 we see that K is an injective R-module and by Lemma 47.2.4 we see that $R \subset K$ is an essential extension.

08Y6 Definition 47.5.5. An object X of an additive category is called indecomposable if it is nonzero and if $X = Y \oplus Z$, then either $Y = 0$ or $Z = 0$.

08Y7 Lemma 47.5.6. Let R be a ring. Let E be an indecomposable injective R-module. Then

(1) E is the injective hull of any nonzero submodule of E,
(2) the intersection of any two nonzero submodules of E is nonzero,
(3) $\mathrm{End}_R(E, E)$ is a noncommutative local ring with maximal ideal those $\varphi : E \to E$ whose kernel is nonzero, and
(4) the set of zerodivisors on E is a prime ideal $\mathfrak{p}$ of R and E is an injective $R_\mathfrak{p}$-module.

Proof. Part (1) follows from Lemma 47.5.3. Part (2) follows from part (1) and the definition of injective hulls.

Proof of (3). Set $A = \mathrm{End}_R(E, E)$ and $I = \{\varphi \in A \mid \mathrm{Ker}(\varphi) \neq 0\}$. The statement means that I is a two sided ideal and that any $\varphi \in A$, $\varphi \notin I$ is invertible. Suppose φ and ψ are not injective. Then $\mathrm{Ker}(\varphi) \cap \mathrm{Ker}(\psi)$ is nonzero by (2). Hence $\varphi + \psi \in I$. It follows that I is a two sided ideal. If $\varphi \in A$, $\varphi \notin I$, then $E \cong \varphi(E) \subset E$ is an injective submodule, hence $E = \varphi(E)$ because E is indecomposable.

Proof of (4). Consider the ring map $R \to A$ and let $\mathfrak{p} \subset R$ be the inverse image of the maximal ideal I. Then it is clear that $\mathfrak{p}$ is a prime ideal and that $R \to A$ extends to $R_\mathfrak{p} \to A$. Thus E is an $R_\mathfrak{p}$-module. It follows from Lemma 47.3.3 that E is injective as an $R_\mathfrak{p}$-module. □

08Y8 Lemma 47.5.7. Let $\mathfrak{p} \subset R$ be a prime of a ring R. Let E be the injective hull of $R/\mathfrak{p}$. Then

(1) E is indecomposable,
(2) E is the injective hull of $\kappa(\mathfrak{p})$,
(3) E is the injective hull of $\kappa(\mathfrak{p})$ over the ring $R_\mathfrak{p}$.

Proof. By Lemma 47.2.4 the inclusion $R/\mathfrak{p} \subset \kappa(\mathfrak{p})$ is an essential extension. Then Lemma 47.5.3 shows (2) holds. For $f \in R$, $f \notin \mathfrak{p}$ the map $f : \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})$ is an isomorphism hence the map $f : E \to E$ is an isomorphism, see Lemma 47.5.3. Thus E is an $R_\mathfrak{p}$-module. It is injective as an $R_\mathfrak{p}$-module by Lemma 47.3.3. Finally, let $E' \subset E$ be a nonzero injective R-submodule. Then $J = (R/\mathfrak{p}) \cap E'$ is nonzero. After shrinking E' we may assume that E' is the injective hull of J (see Lemma 47.5.3 for example). Observe that $R/\mathfrak{p}$ is an essential extension of J for example by Lemma 47.2.4. Hence $E' \to E$ is an isomorphism by Lemma 47.5.3 part (3). Hence E is indecomposable. □

08Y9 Lemma 47.5.8. Let R be a Noetherian ring. Let E be an indecomposable injective R-module. Then there exists a prime ideal $\mathfrak{p}$ of R such that E is the injective hull of $\kappa(\mathfrak{p})$.

Proof. Let $\mathfrak{p}$ be the prime ideal found in Lemma 47.5.6. Say $\mathfrak{p} = (f_1, \ldots, f_r)$. Pick a nonzero element $x \in \bigcap \mathrm{Ker}(f_i : E \to E)$, see Lemma 47.5.6. Then $(R_\mathfrak{p})x$ is a module isomorphic to $\kappa(\mathfrak{p})$ inside E. We conclude by Lemma 47.5.6. □

08YA Proposition 47.5.9 (Structure of injective modules over Noetherian rings). Let R be a Noetherian ring. Every injective module is a direct sum of indecomposable injective modules. Every indecomposable injective module is the injective hull of the residue field at a prime.

Proof. The second statement is Lemma 47.5.8. For the first statement, let I be an injective R-module. We will use transfinite recursion to construct $I_\alpha \subset I$ for ordinals α which are direct sums of indecomposable injective R-modules $E_{\beta+1}$ for $\beta < \alpha$. For $\alpha = 0$ we let $I_0 = 0$. Suppose given an ordinal α such that I_α has been constructed. Then I_α is an injective R-module by Lemma 47.3.7. Hence $I \cong I_\alpha \oplus I'$. If $I' = 0$ we are done. If not, then I' has an associated prime by Algebra, Lemma 10.63.7. Thus I' contains a copy of $R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Hence I' contains an indecomposable submodule E by Lemmas 47.5.3 and 47.5.7. Set $I_{\alpha+1} = I_\alpha \oplus E_\alpha$. If α is a limit ordinal and I_β has been constructed for $\beta < \alpha$, then we set $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Observe that $I_\alpha = \bigoplus_{\beta<\alpha} E_{\beta+1}$. This concludes the proof. □

47.6. Duality over Artinian local rings

08YW Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Recall that this implies R is Noetherian and that R has finite length as an R-module. Moreover an R-module is finite if and only if it has finite length. We will use these facts without further mention in this section. Please see Algebra, Sections 10.52 and 10.53 and Algebra, Proposition 10.60.7 for more details.

08YX Lemma 47.6.1. Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For every finite R-module M we have

$$\mathrm{length}_R(M) = \mathrm{length}_R(\mathrm{Hom}_R(M, E))$$

In particular, the injective hull E of κ is a finite R-module.

Proof. Because E is an essential extension of κ we have $\kappa = E[\mathfrak{m}]$ where $E[\mathfrak{m}]$ is the $\mathfrak{m}$-torsion in E (notation as in More on Algebra, Section 15.88). Hence $\mathrm{Hom}_R(\kappa, E) \cong \kappa$ and the equality of lengths holds for $M = \kappa$. We prove the displayed equality of the lemma by induction on the length of M. If M is nonzero there exists a surjection $M \to \kappa$ with kernel M'. Since the functor $M \mapsto \mathrm{Hom}_R(M, E)$ is exact we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(\kappa, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E) \to 0.$$

Additivity of length for this sequence and the sequence $0 \to M' \to M \to \kappa \to 0$ and the equality for M' (induction hypothesis) and κ implies the equality for M. The final statement of the lemma follows as $E = \mathrm{Hom}_R(R, E)$. $\square$

08YY Lemma 47.6.2. Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For any finite R-module M the evaluation map

$$M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$$

is an isomorphism. In particular $R = \mathrm{Hom}_R(E, E)$.

Proof. Observe that the displayed arrow is injective. Namely, if $x \in M$ is a nonzero element, then there is a nonzero map $Rx \to \kappa$ which we can extend to a map $\varphi : M \to E$ that doesn't vanish on x. Since the source and target of the arrow have the same length by Lemma 47.6.1 we conclude it is an isomorphism. The final statement follows on taking $M = R$. $\square$

To state the next lemma, denote Mod_R^{fg} the category of finite R-modules over a ring R.

08YZ Lemma 47.6.3. Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an exact anti-equivalence $\mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R^{fg}$ and $D \circ D \cong \mathrm{id}$.

Proof. We have seen that $D \circ D = \mathrm{id}$ on Mod_R^{fg} in Lemma 47.6.2. It follows immediately that D is an anti-equivalence. $\square$

08Z0 Lemma 47.6.4. Assumptions and notation as in Lemma 47.6.3. Let $I \subset R$ be an ideal and M a finite R-module. Then

$$D(M[I]) = D(M)/ID(M) \quad \text{and} \quad D(M/IM) = D(M)[I]$$

Proof. Say $I = (f_1, \ldots, f_t)$. Consider the map

$$M^{\oplus t} \xrightarrow{f_1, \ldots, f_t} M$$

with cokernel M/IM. Applying the exact functor D we conclude that $D(M/IM)$ is $D(M)[I]$. The other case is proved in the same way. $\square$

47.7. Injective hull of the residue field

08Z1 Most of our results will be for Noetherian local rings in this section.

08Z2 Lemma 47.7.1. Let $R \to S$ be a surjective map of local rings with kernel I. Let E be the injective hull of the residue field of R over R. Then $E[I]$ is the injective hull of the residue field of S over S.

Proof. Observe that $E[I] = \mathrm{Hom}_R(S, E)$ as $S = R/I$. Hence $E[I]$ is an injective S-module by Lemma 47.3.4. Since E is an essential extension of $\kappa = R/\mathfrak{m}_R$ it follows that $E[I]$ is an essential extension of κ as well. The result follows. □

08Z3 Lemma 47.7.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let E be the injective hull of κ. Let M be a $\mathfrak{m}$-power torsion R-module with $n = \dim_\kappa(M[\mathfrak{m}]) < \infty$. Then M is isomorphic to a submodule of $E^{\oplus n}$.

Proof. Observe that $E^{\oplus n}$ is the injective hull of $\kappa^{\oplus n} = M[\mathfrak{m}]$. Thus there is an R-module map $M \to E^{\oplus n}$ which is injective on $M[\mathfrak{m}]$. Since M is $\mathfrak{m}$-power torsion the inclusion $M[\mathfrak{m}] \subset M$ is an essential extension (for example by Lemma 47.2.4) we conclude that the kernel of $M \to E^{\oplus n}$ is zero. □

08Z4 Lemma 47.7.3. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Let E_n be an injective hull of κ over $R/\mathfrak{m}^n$. Then $E = \bigcup E_n$ and $E_n = E[\mathfrak{m}^n]$.

Proof. We have $E_n = E[\mathfrak{m}^n]$ by Lemma 47.7.1. We have $E = \bigcup E_n$ because $\bigcup E_n = E[\mathfrak{m}^\infty]$ is an injective R-submodule which contains κ, see Lemma 47.3.9. □

The following lemma tells us the injective hull of the residue field of a Noetherian local ring only depends on the completion.

08Z5 Lemma 47.7.4. Let $R \to S$ be a flat local homomorphism of local Noetherian rings such that $R/\mathfrak{m}_R \cong S/\mathfrak{m}_R S$. Then the injective hull of the residue field of R is the injective hull of the residue field of S.

Proof. Note that $\mathfrak{m}_R S = \mathfrak{m}_S$ as the quotient by the former is a field. Set $\kappa = R/\mathfrak{m}_R = S/\mathfrak{m}_S$. Let E_R be the injective hull of κ over R. Let E_S be the injective hull of κ over S. Observe that E_S is an injective R-module by Lemma 47.3.2. Choose an extension $E_R \to E_S$ of the identification of residue fields. This map is an isomorphism by Lemma 47.7.3 because $R \to S$ induces an isomorphism $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ for all n. □

08Z6 Lemma 47.7.5. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then $\mathrm{Hom}_R(E, E)$ is canonically isomorphic to the completion of R.

Proof. Write $E = \bigcup E_n$ with $E_n = E[\mathfrak{m}^n]$ as in Lemma 47.7.3. Any endomorphism of E preserves this filtration. Hence

$$\mathrm{Hom}_R(E, E) = \lim \mathrm{Hom}_R(E_n, E_n)$$

The lemma follows as $\mathrm{Hom}_R(E_n, E_n) = \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = R/\mathfrak{m}^n$ by Lemma 47.6.2. □

08Z7 Lemma 47.7.6. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then E satisfies the descending chain condition.

Proof. If $E \supset M_1 \supset M_2 \supset \ldots$ is a sequence of submodules, then

$$\mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(M_1, E) \to \mathrm{Hom}_R(M_2, E) \to \ldots$$

is a sequence of surjections. By Lemma 47.7.5 each of these is a module over the completion $R^\wedge = \mathrm{Hom}_R(E, E)$. Since $R^\wedge$ is Noetherian (Algebra, Lemma 10.97.6) the sequence stabilizes: $\mathrm{Hom}_R(M_n, E) = \mathrm{Hom}_R(M_{n+1}, E) = \ldots$. Since E is injective, this can only happen if $\mathrm{Hom}_R(M_n/M_{n+1}, E)$ is zero. However, if M_n/M_{n+1} is nonzero, then it contains a nonzero element annihilated by $\mathfrak{m}$, because

E is $\mathfrak{m}$-power torsion by Lemma 47.7.3. In this case M_n/M_{n+1} has a nonzero map into E, contradicting the assumed vanishing. This finishes the proof. □

08Z8 Lemma 47.7.7. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ.

1. For an R-module M the following are equivalent:
   (a) M satisfies the ascending chain condition,
   (b) M is a finite R-module, and
   (c) there exist n, m and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$.
2. For an R-module M the following are equivalent:
   (a) M satisfies the descending chain condition,
   (b) M is $\mathfrak{m}$-power torsion and $\dim_\kappa(M[\mathfrak{m}]) < \infty$, and
   (c) there exist n, m and an exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Proof. We omit the proof of (1).

Let M be an R-module with the descending chain condition. Let $x \in M$. Then $\mathfrak{m}^n x$ is a descending chain of submodules, hence stabilizes. Thus $\mathfrak{m}^n x = \mathfrak{m}^{n+1} x$ for some n. By Nakayama's lemma (Algebra, Lemma 10.20.1) this implies $\mathfrak{m}^n x = 0$, i.e., x is $\mathfrak{m}$-power torsion. Since $M[\mathfrak{m}]$ is a vector space over κ it has to be finite dimensional in order to have the descending chain condition.

Assume that M is $\mathfrak{m}$-power torsion and has a finite dimensional $\mathfrak{m}$-torsion submodule $M[\mathfrak{m}]$. By Lemma 47.7.2 we see that M is a submodule of $E^{\oplus n}$ for some n. Consider the quotient $N = E^{\oplus n}/M$. By Lemma 47.7.6 the module E has the descending chain condition hence so do $E^{\oplus n}$ and N. Therefore N satisfies (2)(a) which implies N satisfies (2)(b) by the second paragraph of the proof. Thus by Lemma 47.7.2 again we see that N is a submodule of $E^{\oplus m}$ for some m. Thus we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Assume we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$. Since E satisfies the descending chain condition by Lemma 47.7.6 so does M. □

08Z9 Proposition 47.7.8 (Matlis duality). Let $(R, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Let E be an injective hull of κ over R. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an anti-equivalence

$$\left\{ \begin{matrix} R\text{-modules with the} \\ \text{descending chain condition} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} R\text{-modules with the} \\ \text{ascending chain condition} \end{matrix} \right\}$$

and we have $D \circ D = \mathrm{id}$ on either side of the equivalence.

Proof. By Lemma 47.7.5 we have $R = \mathrm{Hom}_R(E, E) = D(E)$. Of course we have $E = \mathrm{Hom}_R(R, E) = D(R)$. Since E is injective the functor D is exact. The result now follows immediately from the description of the categories in Lemma 47.7.7. □

0EGL Remark 47.7.9. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Here is an addendum to Matlis duality: If N is an $\mathfrak{m}$-power torsion module and $M = \mathrm{Hom}_R(N, E)$ is a finite module over the completion of R, then N satisfies the descending chain condition. Namely, for any submodules $N'' \subset N' \subset N$ with $N'' \neq N'$, we can find an embedding $\kappa \subset N''/N'$ and hence a nonzero map $N' \to E$ annihilating N'' which we can extend to a map $N \to E$ annihilating N''. Thus $N \supset N' \mapsto M' = \mathrm{Hom}_R(N/N', E) \subset M$ is an inclusion preserving map from submodules of N to submodules of M, whence the conclusion.

47.8. Deriving torsion

0BJA Let A be a ring and let $I \subset A$ be a finitely generated ideal (if I is not finitely generated perhaps a different definition should be used). Let $Z = V(I) \subset \text{Spec}(A)$. Recall that the category I^∞-torsion of I-power torsion modules only depends on the closed subset Z and not on the choice of the finitely generated ideal I such that $Z = V(I)$, see More on Algebra, Lemma 15.88.6. In this section we will consider the functor

$$H^0_I : \text{Mod}_A \longrightarrow I^\infty\text{-torsion}, \quad M \longmapsto M[I^\infty] = \bigcup M[I^n]$$

which sends M to the submodule of I-power torsion.

Let A be a ring and let I be a finitely generated ideal. Note that I^∞-torsion is a Grothendieck abelian category (direct sums exist, filtered colimits are exact, and $\bigoplus A/I^n$ is a generator by More on Algebra, Lemma 15.88.2). Hence the derived category $D(I^\infty\text{-torsion})$ exists, see Injectives, Remark 19.13.3. Our functor H^0_I is left exact and has a derived extension which we will denote

$$R\Gamma_I : D(A) \longrightarrow D(I^\infty\text{-torsion}).$$

Warning: this functor does not deserve the name local cohomology unless the ring A is Noetherian. The functors H^0_I, $R\Gamma_I$, and the satellites H^p_I only depend on the closed subset $Z \subset \text{Spec}(A)$ and not on the choice of the finitely generated ideal I such that $V(I) = Z$. However, we insist on using the subscript I for the functors above as the notation $R\Gamma_Z$ is going to be used for a different functor, see (47.9.0.1), which agrees with the functor $R\Gamma_I$ only (as far as we know) in case A is Noetherian (see Lemma 47.10.1).

0A6L **Lemma 47.8.1.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor $R\Gamma_I$ is right adjoint to the functor $D(I^\infty\text{-torsion}) \to D(A)$.

Proof. This follows from the fact that taking I-power torsion submodules is the right adjoint to the inclusion functor $I^\infty\text{-torsion} \to \text{Mod}_A$. See Derived Categories, Lemma 13.30.3. □

0954 **Lemma 47.8.2.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. For any object K of $D(A)$ we have

$$R\Gamma_I(K) = \text{hocolim } R\,\text{Hom}_A(A/I^n, K)$$

in $D(A)$ and

$$R^q\Gamma_I(K) = \text{colim}_n \text{Ext}^q_A(A/I^n, K)$$

as modules for all $q \in \mathbf{Z}$.

Proof. Let $J^\bullet$ be a K-injective complex representing K. Then

$$R\Gamma_I(K) = J^\bullet[I^\infty] = \text{colim } J^\bullet[I^n] = \text{colim}\,\text{Hom}_A(A/I^n, J^\bullet)$$

where the first equality is the definition of $R\Gamma_I(K)$. By Derived Categories, Lemma 13.33.7 we obtain the first displayed equality in the statement of the lemma. The second displayed equality in the statement of the lemma then follows because $H^q(\text{Hom}_A(A/I^n, J^\bullet)) = \text{Ext}^q_A(A/I^n, K)$ and because filtered colimits are exact in the category of abelian groups. □

0A6M **Lemma 47.8.3.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_I(K^\bullet) = 0$.

Proof. Namely, in this case the cohomology modules of $R\Gamma_I(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. □

Let A be a ring and $I \subset A$ a finitely generated ideal. By More on Algebra, Lemma 15.88.5 the category of I-power torsion modules is a Serre subcategory of the category of all A-modules, hence there is a functor

0A6N (47.8.3.1)
$$D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$$

see Derived Categories, Section 13.17.

0A6P **Lemma 47.8.4.** Let A be a ring and let I be a finitely generated ideal. Let M and N be I-power torsion modules.

(1) $\mathrm{Hom}_{D(A)}(M, N) = \mathrm{Hom}_{D(I^\infty\text{-torsion})}(M, N)$,
(2) $\mathrm{Ext}^1_{D(A)}(M, N) = \mathrm{Ext}^1_{D(I^\infty\text{-torsion})}(M, N)$,
(3) $\mathrm{Ext}^2_{D(I^\infty\text{-torsion})}(M, N) \to \mathrm{Ext}^2_{D(A)}(M, N)$ is not surjective in general,
(4) (47.8.3.1) is not an equivalence in general.

Proof. Parts (1) and (2) follow immediately from the fact that I-power torsion forms a Serre subcategory of Mod_A. Part (4) follows from part (3).

For part (3) let A be a ring with an element $f \in A$ such that $A[f]$ contains a nonzero element x annihilated by f and A contains elements x_n with $f^n x_n = x$. Such a ring A exists because we can take

$$A = \mathbf{Z}[f, x, x_n]/(fx, f^n x_n - x)$$

Given A set $I = (f)$. Then the exact sequence

$$0 \to A[f] \to A \xrightarrow{f} A \to A/fA \to 0$$

defines an element in $\mathrm{Ext}^2_A(A/fA, A[f])$. We claim this element does not come from an element of $\mathrm{Ext}^2_{D(f^\infty\text{-torsion})}(A/fA, A[f])$. Namely, if it did, then there would be an exact sequence

$$0 \to A[f] \to M \to N \to A/fA \to 0$$

where M and N are f-power torsion modules defining the same 2 extension class. Since $A \to A$ is a complex of free modules and since the 2 extension classes are the same we would be able to find a map

$$\begin{tikzcd}
0 \arrow[r] & A[f] \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varphi"'] & A \arrow[r] \arrow[d, "\psi"'] & A/fA \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A[f] \arrow[r] & M \arrow[r] & N \arrow[r] & A/fA \arrow[r] & 0
\end{tikzcd}$$

(some details omitted). Then we could replace M by the image of φ and N by the image of ψ. Then M would be a cyclic module, hence $f^n M = 0$ for some n. Considering $\varphi(x_{n+1})$ we get a contradiction with the fact that $f^{n+1} x_n = x$ is nonzero in $A[f]$. □

47.9. Local cohomology

0952 Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \text{Spec}(A)$. We will construct a functor

0A6Q (47.9.0.1)
$$R\Gamma_Z : D(A) \longrightarrow D_{I^\infty\text{-torsion}}(A).$$

which is right adjoint to the inclusion functor. For notation see Section 47.8. The cohomology modules of $R\Gamma_Z(K)$ are the local cohomology groups of K with respect to Z. By Lemma 47.8.4 this functor will in general not be equal to $R\Gamma_I(-)$ even viewed as functors into $D(A)$. In Section 47.10 we will show that if A is Noetherian, then the two agree.

We will continue the discussion of local cohomology in the chapter on local cohomology, see Local Cohomology, Section 51.1. For example, there we will show that $R\Gamma_Z$ computes cohomology with support in Z for the associated complex of quasi-coherent sheaves on $\text{Spec}(A)$. See Local Cohomology, Lemma 51.2.1.

0A6R **Lemma 47.9.1.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. There exists a right adjoint $R\Gamma_Z$ (47.9.0.1) to the inclusion functor $D_{I^\infty\text{-torsion}}(A) \to D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have

$$R\Gamma_Z(K) = (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \otimes_A^{\mathbf{L}} K$$

functorially in $K \in D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$ is an ideal. Let $K^\bullet$ be a complex of A-modules. There is a canonical map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \longrightarrow A.$$

from the extended Čech complex to A. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\text{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right) \longrightarrow K^\bullet$$

in $D(A)$. We claim the cohomology modules of the complex on the left are I-power torsion, i.e., the LHS is an object of $D_{I^\infty\text{-torsion}}(A)$. Namely, we have

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) = \text{colim}\, K(A, f_1^n, \ldots, f_r^n)$$

by More on Algebra, Lemma 15.29.6. Moreover, multiplication by f_i^n on the complex $K(A, f_1^n, \ldots, f_r^n)$ is homotopic to zero by More on Algebra, Lemma 15.28.6. Since

$$H^q(LHS) = \text{colim}\, H^q(\text{Tot}(K^\bullet \otimes_A K(A, f_1^n, \ldots, f_r^n)))$$

we obtain our claim. On the other hand, if $K^\bullet$ is an object of $D_{I^\infty\text{-torsion}}(A)$, then the complexes $K^\bullet \otimes_A A_{f_{i_0}\ldots f_{i_p}}$ have vanishing cohomology. Hence in this case the map $LHS \to K^\bullet$ is an isomorphism in $D(A)$. The construction

$$R\Gamma_Z(K^\bullet) = \text{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right)$$

is functorial in $K^\bullet$ and defines an exact functor $D(A) \to D_{I^\infty\text{-torsion}}(A)$ between triangulated categories. It follows formally from the existence of the natural transformation $R\Gamma_Z \to \text{id}$ given above and the fact that this evaluates to an isomorphism on $K^\bullet$ in the subcategory, that $R\Gamma_Z$ is the desired right adjoint. $\square$

0BJB **Lemma 47.9.2.** Let $A \to B$ be a ring homomorphism and let $I \subset A$ be a finitely generated ideal. Set $J = IB$. Set $Z = V(I)$ and $Y = V(J)$. Then

$$R\Gamma_Z(M_A) = R\Gamma_Y(M)_A$$

functorially in $M \in D(B)$. Here $(-)_A$ denotes the restriction functors $D(B) \to D(A)$ and $D_{J^\infty\text{-torsion}}(B) \to D_{I^\infty\text{-torsion}}(A)$.

Proof. This follows from uniqueness of adjoint functors as both $R\Gamma_Z((-)_A)$ and $R\Gamma_Y(-)_A$ are right adjoint to the functor $D_{I^\infty\text{-torsion}}(A) \to D(B)$, $K \mapsto K \otimes_A^{\mathbf{L}} B$. Alternatively, one can use the description of $R\Gamma_Z$ and $R\Gamma_Y$ in terms of alternating Čech complexes (Lemma 47.9.1). Namely, if $I = (f_1, \ldots, f_r)$ then J is generated by the images $g_1, \ldots, g_r \in B$ of $f_1, \ldots, f_r$. Then the statement of the lemma follows from the existence of a canonical isomorphism

$$\begin{aligned} M_A \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \\ = M \otimes_B (B \to \prod\nolimits_{i_0} B_{g_{i_0}} \to \prod\nolimits_{i_0 < i_1} B_{g_{i_0} g_{i_1}} \to \ldots \to B_{g_1 \ldots g_r}) \end{aligned}$$

for any B-module M. $\square$

0ALZ **Lemma 47.9.3.** Let $A \to B$ be a ring homomorphism and let $I \subset A$ be a finitely generated ideal. Set $J = IB$. Let $Z = V(I)$ and $Y = V(J)$. Then

$$R\Gamma_Z(K) \otimes_A^{\mathbf{L}} B = R\Gamma_Y(K \otimes_A^{\mathbf{L}} B)$$

functorially in $K \in D(A)$.

Proof. Write $I = (f_1, \ldots, f_r)$. Then J is generated by the images $g_1, \ldots, g_r \in B$ of $f_1, \ldots, f_r$. Then we have

$$(A \to \prod A_{f_{i_0}} \to \ldots \to A_{f_1 \ldots f_r}) \otimes_A B = (B \to \prod B_{g_{i_0}} \to \ldots \to B_{g_1 \ldots g_r})$$

as complexes of B-modules. Represent K by a K-flat complex $K^\bullet$ of A-modules. Since the total complexes associated to

$$K^\bullet \otimes_A (A \to \prod A_{f_{i_0}} \to \ldots \to A_{f_1 \ldots f_r}) \otimes_A B$$

and

$$K^\bullet \otimes_A B \otimes_B (B \to \prod B_{g_{i_0}} \to \ldots \to B_{g_1 \ldots g_r})$$

represent the left and right hand side of the displayed formula of the lemma (see Lemma 47.9.1) we conclude. $\square$

0A6S **Lemma 47.9.4.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_Z(K^\bullet) = 0$.

Proof. Namely, in this case the cohomology modules of $R\Gamma_Z(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. $\square$

0ALY **Lemma 47.9.5.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. For $K, L \in D(A)$ we have

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L) = R\Gamma_Z(K) \otimes_A^{\mathbf{L}} L = R\Gamma_Z(K) \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

If K or L is in $D_{I^\infty\text{-torsion}}(A)$ then so is $K \otimes_A^{\mathbf{L}} L$.

Proof. By Lemma 47.9.1 we know that $R\Gamma_Z$ is given by $C \otimes^{\mathbf{L}} -$ for some $C \in D(A)$. Hence, for $K, L \in D(A)$ general we have

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes^{\mathbf{L}} L \otimes_A^{\mathbf{L}} C = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

The other equalities follow formally from this one. This also implies the last statement of the lemma. $\square$

0BJC **Lemma 47.9.6.** Let A be a ring and let $I, J \subset A$ be finitely generated ideals. Set $Z = V(I)$ and $Y = V(J)$. Then $Z \cap Y = V(I + J)$ and $R\Gamma_Y \circ R\Gamma_Z = R\Gamma_{Y\cap Z}$ as functors $D(A) \to D_{(I+J)^\infty\text{-torsion}}(A)$. For $K \in D^+(A)$ there is a spectral sequence

$$E_2^{p,q} = H^p_Y(H^q_Z(K)) \Rightarrow H^{p+q}_{Y\cap Z}(K)$$

as in Derived Categories, Lemma 13.22.2.

Proof. There is a bit of abuse of notation in the lemma as strictly speaking we cannot compose $R\Gamma_Y$ and $R\Gamma_Z$. The meaning of the statement is simply that we are composing $R\Gamma_Z$ with the inclusion $D_{I^\infty\text{-torsion}}(A) \to D(A)$ and then with $R\Gamma_Y$. Then the equality $R\Gamma_Y \circ R\Gamma_Z = R\Gamma_{Y\cap Z}$ follows from the fact that

$$D_{I^\infty\text{-torsion}}(A) \to D(A) \xrightarrow{R\Gamma_Y} D_{(I+J)^\infty\text{-torsion}}(A)$$

is right adjoint to the inclusion $D_{(I+J)^\infty\text{-torsion}}(A) \to D_{I^\infty\text{-torsion}}(A)$. Alternatively one can prove the formula using Lemma 47.9.1 and the fact that the tensor product of extended Čech complexes on $f_1, \ldots, f_r$ and $g_1, \ldots, g_m$ is the extended Čech complex on $f_1, \ldots, f_n.g_1, \ldots, g_m$. The final assertion follows from this and the cited lemma. $\square$

The following lemma is the analogue of More on Algebra, Lemma 15.91.24 for complexes with torsion cohomologies.

0AM0 **Lemma 47.9.7.** Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I = B/IB$. Then base change and restriction induce quasi-inverse equivalences $D_{I^\infty\text{-torsion}}(A) = D_{(IB)^\infty\text{-torsion}}(B)$.

Proof. More precisely the functors are $K \mapsto K \otimes_A^{\mathbf{L}} B$ for K in $D_{I^\infty\text{-torsion}}(A)$ and $M \mapsto M_A$ for M in $D_{(IB)^\infty\text{-torsion}}(B)$. The reason this works is that $H^i(K \otimes_A^{\mathbf{L}} B) = H^i(K) \otimes_A B = H^i(K)$. The first equality holds as $A \to B$ is flat and the second by More on Algebra, Lemma 15.89.2. $\square$

The following lemma was shown for Hom and Ext^1 of modules in More on Algebra, Lemmas 15.89.3 and 15.89.8.

05EH **Lemma 47.9.8.** Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I \to B/IB$ is an isomorphism. For $K \in D_{I^\infty\text{-torsion}}(A)$ and $L \in D(A)$ the map

$$R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_B(K \otimes_A B, L \otimes_A B)$$

is a quasi-isomorphism. In particular, if M, N are A-modules and M is I-power torsion, then the canonical map

$$\text{Ext}^i_A(M, N) \longrightarrow \text{Ext}^i_B(M \otimes_A B, N \otimes_A B)$$

is an isomorphism for all i.

Proof. Let $Z = V(I) \subset \mathrm{Spec}(A)$ and $Y = V(IB) \subset \mathrm{Spec}(B)$. Since the cohomology modules of K are I power torsion, the canonical map $R\Gamma_Z(L) \to L$ induces an isomorphism

$$R\operatorname{Hom}_A(K, R\Gamma_Z(L)) \to R\operatorname{Hom}_A(K, L)$$

in $D(A)$. Similarly, the cohomology modules of $K \otimes_A B$ are IB power torsion and we have an isomorphism

$$R\operatorname{Hom}_B(K \otimes_A B, R\Gamma_Y(L \otimes_A B)) \to R\operatorname{Hom}_B(K \otimes_A B, L \otimes_A B)$$

in $D(B)$. By Lemma 47.9.3 we have $R\Gamma_Z(L) \otimes_A B = R\Gamma_Y(L \otimes_A B)$. Hence it suffices to show that the map

$$R\operatorname{Hom}_A(K, R\Gamma_Z(L)) \to R\operatorname{Hom}_B(K \otimes_A B, R\Gamma_Z(L) \otimes_A B)$$

is a quasi-isomorphism. This follows from Lemma 47.9.7. □

47.10. Local cohomology for Noetherian rings

0BJD Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \mathrm{Spec}(A)$. Recall that (47.8.3.1) is the functor

$$D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$$

In fact, there is a natural transformation of functors

0A6U (47.10.0.1) $$(47.8.3.1) \circ R\Gamma_I(-) \longrightarrow R\Gamma_Z(-)$$

Namely, given a complex of A-modules $K^\bullet$ the canonical map $R\Gamma_I(K^\bullet) \to K^\bullet$ in $D(A)$ factors (uniquely) through $R\Gamma_Z(K^\bullet)$ as $R\Gamma_I(K^\bullet)$ has I-power torsion cohomology modules (see Lemma 47.8.1). In general this map is not an isomorphism (we've seen this in Lemma 47.8.4).

0955 Lemma 47.10.1. Let A be a Noetherian ring and let $I \subset A$ be an ideal.

(1) the adjunction $R\Gamma_I(K) \to K$ is an isomorphism for $K \in D_{I^\infty\text{-torsion}}(A)$,
(2) the functor (47.8.3.1) $D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$ is an equivalence,
(3) the transformation of functors (47.10.0.1) is an isomorphism, in other words $R\Gamma_I(K) = R\Gamma_Z(K)$ for $K \in D(A)$.

Proof. A formal argument, which we omit, shows that it suffices to prove (1).

Let M be an I-power torsion A-module. Choose an embedding $M \to J$ into an injective A-module. Then $J[I^\infty]$ is an injective A-module, see Lemma 47.3.9, and we obtain an embedding $M \to J[I^\infty]$. Thus every I-power torsion module has an injective resolution $M \to J^\bullet$ with J^n also I-power torsion. It follows that $R\Gamma_I(M) = M$ (this is not a triviality and this is not true in general if A is not Noetherian). Next, suppose that $K \in D^+_{I^\infty\text{-torsion}}(A)$. Then the spectral sequence

$$R^q\Gamma_I(H^p(K)) \Rightarrow R^{p+q}\Gamma_I(K)$$

(Derived Categories, Lemma 13.21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $R\Gamma_I(K) \to K$ is an isomorphism.

Suppose K is an arbitrary object of $D_{I^\infty\text{-torsion}}(A)$. We have

$$R^q\Gamma_I(K) = \operatorname{colim} \operatorname{Ext}^q_A(A/I^n, K)$$

by Lemma 47.8.2. Choose $f_1, \ldots, f_r \in A$ generating I. Let $K_n^\bullet = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex with terms in degrees $-r, \ldots, 0$. Since the pro-objects

$\{A/I^n\}$ and $\{K_n^\bullet\}$ in $D(A)$ are the same by More on Algebra, Lemma 15.94.1, we see that

$$R^q\Gamma_I(K) = \mathrm{colim}\,\mathrm{Ext}^q_A(K_n^\bullet, K)$$

Pick any complex $K^\bullet$ of A-modules representing K. Since $K_n^\bullet$ is a finite complex of finite free modules we see that

$$\mathrm{Ext}^q_A(K_n, K) = H^q(\mathrm{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet))$$

where $(K_n^\bullet)^\vee$ is the dual of the complex $K_n^\bullet$. See More on Algebra, Lemma 15.73.2. As $(K_n^\bullet)^\vee$ is a complex of finite free A-modules sitting in degrees $0, \ldots, r$ we see that the terms of the complex $\mathrm{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet)$ are the same as the terms of the complex $\mathrm{Tot}((K_n^\bullet)^\vee \otimes_A \tau_{\geq q-r-2}K^\bullet)$ in degrees $q-1$ and higher. Hence we see that

$$\mathrm{Ext}^q_A(K_n, K) = \mathrm{Ext}^q_A(K_n, \tau_{\geq q-r-2}K)$$

for all n. It follows that

$$R^q\Gamma_I(K) = R^q\Gamma_I(\tau_{\geq q-r-2}K) = H^q(\tau_{\geq q-r-2}K) = H^q(K)$$

Thus we see that the map $R\Gamma_I(K) \to K$ is an isomorphism. □

0956 Lemma 47.10.2. Let A be a Noetherian ring and let $I = (f_1, \ldots, f_r)$ be an ideal of A. Set $Z = V(I) \subset \mathrm{Spec}(A)$. There are canonical isomorphisms

$$R\Gamma_I(A) \to (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \to R\Gamma_Z(A)$$

in $D(A)$. If M is an A-module, then we have similarly

$$R\Gamma_I(M) \cong (M \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1\ldots f_r}) \cong R\Gamma_Z(M)$$

in $D(A)$.

Proof. This follows from Lemma 47.10.1 and the computation of the functor $R\Gamma_Z$ in Lemma 47.9.1. □

0957 Lemma 47.10.3. If $A \to B$ is a homomorphism of Noetherian rings and $I \subset A$ is an ideal, then in $D(B)$ we have

$$R\Gamma_I(A) \otimes_A^{\mathbf{L}} B = R\Gamma_Z(A) \otimes_A^{\mathbf{L}} B = R\Gamma_Y(B) = R\Gamma_{IB}(B)$$

where $Y = V(IB) \subset \mathrm{Spec}(B)$.

Proof. Combine Lemmas 47.10.2 and 47.9.3. □

47.11. Depth

0AVY In this section we revisit the notion of depth introduced in Algebra, Section 10.72.

0AVZ Lemma 47.11.1. Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let M be a finite A-module such that $IM \neq M$. Then the following integers are equal:

(1) $\mathrm{depth}_I(M)$,
(2) the smallest integer i such that $\mathrm{Ext}^i_A(A/I, M)$ is nonzero, and
(3) the smallest integer i such that $H^i_I(M)$ is nonzero.

Moreover, we have $\mathrm{Ext}^i_A(N, M) = 0$ for $i < \mathrm{depth}_I(M)$ for any finite A-module N annihilated by a power of I.

Proof. We prove the equality of (1) and (2) by induction on $\text{depth}_I(M)$ which is allowed by Algebra, Lemma 10.72.4.

Base case. If $\text{depth}_I(M) = 0$, then I is contained in the union of the associated primes of M (Algebra, Lemma 10.63.9). By prime avoidance (Algebra, Lemma 10.15.2) we see that $I \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Hence $\text{Hom}_A(A/I, M)$ is nonzero. Thus equality holds in this case.

Assume that $\text{depth}_I(M) > 0$. Let $f \in I$ be a nonzerodivisor on M such that $\text{depth}_I(M/fM) = \text{depth}_I(M) - 1$. Consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

and the associated long exact sequence for $\text{Ext}^*_A(A/I, -)$. Note that $\text{Ext}^i_A(A/I, M)$ is a finite A/I-module (Algebra, Lemmas 10.71.9 and 10.71.8). Hence we obtain

$$\text{Hom}_A(A/I, M/fM) = \text{Ext}^1_A(A/I, M)$$

and short exact sequences

$$0 \to \text{Ext}^i_A(A/I, M) \to \text{Ext}^i_A(A/I, M/fM) \to \text{Ext}^{i+1}_A(A/I, M) \to 0$$

Thus the equality of (1) and (2) by induction.

Observe that $\text{depth}_I(M) = \text{depth}_{I^n}(M)$ for all $n \geq 1$ for example by Algebra, Lemma 10.68.9. Hence by the equality of (1) and (2) we see that $\text{Ext}^i_A(A/I^n, M) = 0$ for all n and $i < \text{depth}_I(M)$. Let N be a finite A-module annihilated by a power of I. Then we can choose a short exact sequence

$$0 \to N' \to (A/I^n)^{\oplus m} \to N \to 0$$

for some $n, m \geq 0$. Then $\text{Hom}_A(N, M) \subset \text{Hom}_A((A/I^n)^{\oplus m}, M)$ and $\text{Ext}^i_A(N, M) \subset \text{Ext}^{i-1}_A(N', M)$ for $i < \text{depth}_I(M)$. Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have $H^p_I(M) = \text{colim}\, \text{Ext}^p_A(A/I^n, M)$ by Lemma 47.8.2. Thus we see that $H^i_I(M) = 0$ for $i < \text{depth}_I(M)$. For $i = \text{depth}_I(M)$, using the vanishing of $\text{Ext}^{i-1}_A(I/I^n, M)$ we see that the map $\text{Ext}^i_A(A/I, M) \to H^i_I(M)$ is injective which proves nonvanishing in the correct degree. □

0BUV Lemma 47.11.2. Let A be a Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of finite A-modules. Let $I \subset A$ be an ideal.

(1) $\text{depth}_I(N) \geq \min\{\text{depth}_I(N'), \text{depth}_I(N'')\}$
(2) $\text{depth}_I(N'') \geq \min\{\text{depth}_I(N), \text{depth}_I(N') - 1\}$
(3) $\text{depth}_I(N') \geq \min\{\text{depth}_I(N), \text{depth}_I(N'') + 1\}$

Proof. Assume $IN \neq N$, $IN' \neq N'$, and $IN'' \neq N''$. Then we can use the characterization of depth using the Ext groups $\text{Ext}^i(A/I, N)$, see Lemma 47.11.1, and use the long exact cohomology sequence

$$\begin{aligned} 0 &\to \text{Hom}_A(A/I, N') \to \text{Hom}_A(A/I, N) \to \text{Hom}_A(A/I, N'') \\ &\to \text{Ext}^1_A(A/I, N') \to \text{Ext}^1_A(A/I, N) \to \text{Ext}^1_A(A/I, N'') \to \ldots \end{aligned}$$

from Algebra, Lemma 10.71.6. This argument also works if $IN = N$ because in this case $\text{Ext}^i_A(A/I, N) = 0$ for all i. Similarly in case $IN' \neq N'$ and/or $IN'' \neq N''$. □

0BUW Lemma 47.11.3. Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let M a finite A-module with $IM \neq M$.

(1) If $x \in I$ is a nonzerodivisor on M, then $\text{depth}_I(M/xM) = \text{depth}_I(M) - 1$.
(2) Any M-regular sequence $x_1, \ldots, x_r$ in I can be extended to an M-regular sequence in I of length $\text{depth}_I(M)$.

Proof. Part (2) is a formal consequence of part (1). Let $x \in I$ be as in (1). By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 47.11.2 we see that $\text{depth}_I(M/xM) \geq \text{depth}_I(M) - 1$. On the other hand, if $x_1, \ldots, x_r \in I$ is a regular sequence for M/xM, then $x, x_1, \ldots, x_r$ is a regular sequence for M. Hence (1) holds. $\square$

0BUX Lemma 47.11.4. Let R be a Noetherian local ring. If M is a finite Cohen-Macaulay R-module and $I \subset R$ a nontrivial ideal. Then

$$\text{depth}_I(M) = \dim(\text{Supp}(M)) - \dim(\text{Supp}(M/IM)).$$

Proof. We will prove this by induction on $\text{depth}_I(M)$.

If $\text{depth}_I(M) = 0$, then I is contained in one of the associated primes $\mathfrak{p}$ of M (Algebra, Lemma 10.63.18). Then $\mathfrak{p} \in \text{Supp}(M/IM)$, hence $\dim(\text{Supp}(M/IM)) \geq \dim(R/\mathfrak{p}) = \dim(\text{Supp}(M))$ where equality holds by Algebra, Lemma 10.103.7. Thus the lemma holds in this case.

If $\text{depth}_I(M) > 0$, we pick $x \in I$ which is a nonzerodivisor on M. Note that $(M/xM)/I(M/xM) = M/IM$. On the other hand we have $\text{depth}_I(M/xM) = \text{depth}_I(M) - 1$ by Lemma 47.11.3 and $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$ by Algebra, Lemma 10.63.10. Thus the result by induction hypothesis. $\square$

0BUY Lemma 47.11.5. Let $R \to S$ be a flat local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m} \subset R$ the maximal ideal. Let $I \subset S$ be an ideal. If $S/\mathfrak{m}S$ is Cohen-Macaulay, then

$$\text{depth}_I(S) \geq \dim(S/\mathfrak{m}S) - \dim(S/\mathfrak{m}S + I)$$

Proof. By Algebra, Lemma 10.99.3 any sequence in S which maps to a regular sequence in $S/\mathfrak{m}S$ is a regular sequence in S. Thus it suffices to prove the lemma in case R is a field. This is a special case of Lemma 47.11.4. $\square$

0AW0 Lemma 47.11.6. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Let $Z = V(I)$. Then $H^0_I(M) = H^0_Z(M)$. Let N be the common value and set $M' = M/N$. Then

(1) $H^0_I(M') = 0$ and $H^p_I(M) = H^p_I(M')$ and $H^p_I(N) = 0$ for all $p > 0$,
(2) $H^0_Z(M') = 0$ and $H^p_Z(M) = H^p_Z(M')$ and $H^p_Z(N) = 0$ for all $p > 0$.

Proof. By definition $H^0_I(M) = M[I^\infty]$ is I-power torsion. By Lemma 47.9.1 we see that

$$H^0_Z(M) = \text{Ker}(M \longrightarrow M_{f_1} \times \ldots \times M_{f_r})$$

if $I = (f_1, \ldots, f_r)$. Thus $H^0_I(M) \subset H^0_Z(M)$ and conversely, if $x \in H^0_Z(M)$, then it is annihilated by a $f_i^{e_i}$ for some $e_i \geq 1$ hence annihilated by some power of I. This proves the first equality and moreover N is I-power torsion. By Lemma 47.8.1 we see that $R\Gamma_I(N) = N$. By Lemma 47.9.1 we see that $R\Gamma_Z(N) = N$. This proves the higher vanishing of $H^p_I(N)$ and $H^p_Z(N)$ in (1) and (2). The vanishing of $H^0_I(M')$ and $H^0_Z(M')$ follow from the preceding remarks and the fact that M' is I-power torsion free by More on Algebra, Lemma 15.88.4. The equality of higher cohomologies for M and M' follow immediately from the long exact cohomology sequence. $\square$

47.12. Torsion versus complete modules

0A6V Let A be a ring and let I be a finitely generated ideal. In this case we can consider the derived category $D_{I^\infty\text{-torsion}}(A)$ of complexes with I-power torsion cohomology modules (Section 47.9) and the derived category $D_{comp}(A, I)$ of derived complete complexes (More on Algebra, Section 15.91). In this section we show these categories are equivalent. A more general statement can be found in [DG02].

0A6W **Lemma 47.12.1.** Let A be a ring and let I be a finitely generated ideal. Let $R\Gamma_Z$ be as in Lemma 47.9.1. Let $^\wedge$ denote derived completion as in More on Algebra, Lemma 15.91.10. For an object K in $D(A)$ we have

$$R\Gamma_Z(K^\wedge) = R\Gamma_Z(K) \quad \text{and} \quad (R\Gamma_Z(K))^\wedge = K^\wedge$$

in $D(A)$.

Proof. Choose $f_1, \ldots, f_r \in A$ generating I. Recall that

$$K^\wedge = R\operatorname{Hom}_A\left((A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1\ldots f_r}), K\right)$$

by More on Algebra, Lemma 15.91.10. Hence the cone $C = \text{Cone}(K \to K^\wedge)$ is given by

$$R\operatorname{Hom}_A\left((\prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1\ldots f_r}), K\right)$$

which can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

$$R\operatorname{Hom}_A(A_{f_{i_0}\ldots f_{i_p}}, K), \quad p > 0$$

These complexes vanish on applying $R\Gamma_Z$, see Lemma 47.9.4. Applying $R\Gamma_Z$ to the distinguished triangle $K \to K^\wedge \to C \to K[1]$ we see that the first formula of the lemma is correct.

Recall that

$$R\Gamma_Z(K) = K \otimes^{\mathbf{L}} (A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1\ldots f_r})$$

by Lemma 47.9.1. Hence the cone $C = \text{Cone}(R\Gamma_Z(K) \to K)$ can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

$$K \otimes_A A_{f_{i_0}\ldots f_{i_p}}, \quad p > 0$$

These complexes vanish on applying $^\wedge$, see More on Algebra, Lemma 15.91.12. Applying derived completion to the distinguished triangle $R\Gamma_Z(K) \to K \to C \to R\Gamma_Z(K)[1]$ we see that the second formula of the lemma is correct. □

The following result is a special case of a very general phenomenon concerning admissible subcategories of a triangulated category.

0A6X **Proposition 47.12.2.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functors $R\Gamma_Z$ and $^\wedge$ define quasi-inverse equivalences of categories

This is a special case of [PSY14b, Theorem 1.1].

$$D_{I^\infty\text{-torsion}}(A) \leftrightarrow D_{comp}(A, I)$$

Proof. Follows immediately from Lemma 47.12.1. □

The following addendum of the proposition above makes the correspondence on morphisms more precise.

0A6Y Lemma 47.12.3. With notation as in Lemma 47.12.1. For objects K, L in $D(A)$ there is a canonical isomorphism

$$R\operatorname{Hom}_A(K^\wedge, L^\wedge) \longrightarrow R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L))$$

in $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Denote $C = (A \to \prod A_{f_i} \to \ldots \to A_{f_1\ldots f_r})$ the alternating Čech complex. Then derived completion is given by $R\operatorname{Hom}_A(C, -)$ (More on Algebra, Lemma 15.91.10) and local cohomology by $C \otimes^{\mathbf{L}} -$ (Lemma 47.9.1). Combining the isomorphism

$$R\operatorname{Hom}_A(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C) = R\operatorname{Hom}_A(K, R\operatorname{Hom}_A(C, L \otimes^{\mathbf{L}} C))$$

(More on Algebra, Lemma 15.73.1) and the map

$$L \to R\operatorname{Hom}_A(C, L \otimes^{\mathbf{L}} C)$$

(More on Algebra, Lemma 15.73.6) we obtain a map

$$\gamma : R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_A(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C)$$

On the other hand, the right hand side is derived complete as it is equal to

$$R\operatorname{Hom}_A(C, R\operatorname{Hom}_A(K, L \otimes^{\mathbf{L}} C)).$$

Thus γ factors through the derived completion of $R\operatorname{Hom}_A(K, L)$ by the universal property of derived completion. However, the derived completion goes inside the $R\operatorname{Hom}_A$ by More on Algebra, Lemma 15.91.13 and we obtain the desired map.

To show that the map of the lemma is an isomorphism we may assume that K and L are derived complete, i.e., $K = K^\wedge$ and $L = L^\wedge$. In this case we are looking at the map

$$\gamma : R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L))$$

By Proposition 47.12.2 we know that the cohomology groups of the left and the right hand side coincide. In other words, we have to check that the map γ sends a morphism $\alpha : K \to L$ in $D(A)$ to the morphism $R\Gamma_Z(\alpha) : R\Gamma_Z(K) \to R\Gamma_Z(L)$. We omit the verification (hint: note that $R\Gamma_Z(\alpha)$ is just the map $\alpha \otimes \mathrm{id}_C : K \otimes^{\mathbf{L}} C \to L \otimes^{\mathbf{L}} C$ which is almost the same as the construction of the map in More on Algebra, Lemma 15.73.6). □

0EEW Lemma 47.12.4. Let I and J be ideals in a Noetherian ring A. Let M be a finite A-module. Set $Z = V(J)$. Consider the derived I-adic completion $R\Gamma_Z(M)^\wedge$ of local cohomology. Then

(1) we have $R\Gamma_Z(M)^\wedge = R\lim R\Gamma_Z(M/I^nM)$, and
(2) there are short exact sequences

$$0 \to R^1\lim H^{i-1}_Z(M/I^nM) \to H^i(R\Gamma_Z(M)^\wedge) \to \lim H^i_Z(M/I^nM) \to 0$$

In particular $R\Gamma_Z(M)^\wedge$ has vanishing cohomology in negative degrees.

Proof. Suppose that $J = (g_1, \ldots, g_m)$. Then $R\Gamma_Z(M)$ is computed by the complex

$$M \to \prod M_{g_{j_0}} \to \prod M_{g_{j_0}g_{j_1}} \to \ldots \to M_{g_1g_2\ldots g_m}$$

by Lemma 47.9.1. By More on Algebra, Lemma 15.94.6 the derived I-adic completion of this complex is given by the complex

$$\lim M/I^nM \to \prod \lim (M/I^nM)_{g_{j_0}} \to \ldots \to \lim (M/I^nM)_{g_1g_2\ldots g_m}$$

of usual completions. Since $R\Gamma_Z(M/I^nM)$ is computed by the complex $M/I^nM \to \prod (M/I^nM)_{g_{j_0}} \to \ldots \to (M/I^nM)_{g_1g_2\ldots g_m}$ and since the transition maps between these complexes are surjective, we conclude that (1) holds by More on Algebra, Lemma 15.87.1. Part (2) then follows from More on Algebra, Lemma 15.87.4. □

0EEX Lemma 47.12.5. With notation and hypotheses as in Lemma 47.12.4 assume A is I-adically complete. Then

$$H^0(R\Gamma_Z(M)^\wedge) = \mathrm{colim}\, H^0_{V(J')}(M)$$

where the filtered colimit is over $J' \subset J$ such that $V(J') \cap V(I) = V(J) \cap V(I)$.

Proof. Since M is a finite A-module, we have that M is I-adically complete. The proof of Lemma 47.12.4 shows that

$$H^0(R\Gamma_Z(M)^\wedge) = \mathrm{Ker}(M^\wedge \to \prod M_{g_j}^\wedge) = \mathrm{Ker}(M \to \prod M_{g_j}^\wedge)$$

where on the right hand side we have usual I-adic completion. The kernel K_j of $M_{g_j} \to M_{g_j}^\wedge$ is $\bigcap I^n M_{g_j}$. By Algebra, Lemma 10.51.5 for every $\mathfrak p \in V(IA_{g_j})$ we find an $f \in A_{g_j}$, $f \notin \mathfrak p$ such that $(K_j)_f = 0$.

Let $s \in H^0(R\Gamma_Z(M)^\wedge)$. By the above we may think of s as an element of M. The support Z' of s intersected with $D(g_j)$ is disjoint from $D(g_j) \cap V(I)$ by the arguments above. Thus Z' is a closed subset of $\mathrm{Spec}(A)$ with $Z' \cap V(I) \subset V(J)$. Then $Z' \cup V(J) = V(J')$ for some ideal $J' \subset J$ with $V(J') \cap V(I) \subset V(J)$ and we have $s \in H^0_{V(J')}(M)$. Conversely, any $s \in H^0_{V(J')}(M)$ with $J' \subset J$ and $V(J') \cap V(I) \subset V(J)$ maps to zero in $M_{g_j}^\wedge$ for all j. This proves the lemma. □

47.13. Trivial duality for a ring map

0A6Z Let $A \to B$ be a ring homomorphism. Consider the functor

$$\mathrm{Hom}_A(B, -) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto \mathrm{Hom}_A(B, M)$$

This functor is left exact and has a derived extension $R\,\mathrm{Hom}(B, -) : D(A) \to D(B)$.

0A70 Lemma 47.13.1. Let $A \to B$ be a ring homomorphism. The functor $R\,\mathrm{Hom}(B, -)$ constructed above is right adjoint to the restriction functor $D(B) \to D(A)$.

Proof. This is a consequence of the fact that restriction and $\mathrm{Hom}_A(B, -)$ are adjoint functors by Algebra, Lemma 10.14.4. See Derived Categories, Lemma 13.30.3. □

0C0F Lemma 47.13.2. Let $A \to B \to C$ be ring maps. Then $R\,\mathrm{Hom}(C, -) \circ R\,\mathrm{Hom}(B, -) : D(A) \to D(C)$ is the functor $R\,\mathrm{Hom}(C, -) : D(A) \to D(C)$.

Proof. Follows from uniqueness of right adjoints and Lemma 47.13.1. □

0A71 Lemma 47.13.3. Let $\varphi : A \to B$ be a ring homomorphism. For K in $D(A)$ we have

$$\varphi_* R\,\mathrm{Hom}(B, K) = R\,\mathrm{Hom}_A(B, K)$$

where $\varphi_* : D(B) \to D(A)$ is restriction. In particular $R^q\,\mathrm{Hom}(B, K) = \mathrm{Ext}^q_A(B, K)$.

Proof. Choose a K-injective complex $I^\bullet$ representing K. Then $R\,\mathrm{Hom}(B, K)$ is represented by the complex $\mathrm{Hom}_A(B, I^\bullet)$ of B-modules. Since this complex, as a complex of A-modules, represents $R\,\mathrm{Hom}_A(B, K)$ we see that the lemma is true. □

Let A be a Noetherian ring. We will denote

$$D_{\text{Coh}}(A) \subset D(A)$$

the full subcategory consisting of those objects K of $D(A)$ whose cohomology modules are all finite A-modules. This makes sense by Derived Categories, Section 13.17 because as A is Noetherian, the subcategory of finite A-modules is a Serre subcategory of Mod_A.

0A72 Lemma 47.13.4. With notation as above, assume $A \to B$ is a finite ring map of Noetherian rings. Then $R\operatorname{Hom}(B, -)$ maps $D^+_{\text{Coh}}(A)$ into $D^+_{\text{Coh}}(B)$.

Proof. We have to show: if $K \in D^+(A)$ has finite cohomology modules, then the complex $R\operatorname{Hom}(B, K)$ has finite cohomology modules too. This follows for example from Lemma 47.13.3 if we can show the ext modules $\operatorname{Ext}^i_A(B, K)$ are finite A-modules. Since K is bounded below there is a convergent spectral sequence

$$\operatorname{Ext}^p_A(B, H^q(K)) \Rightarrow \operatorname{Ext}^{p+q}_A(B, K)$$

This finishes the proof as the modules $\operatorname{Ext}^p_A(B, H^q(K))$ are finite by Algebra, Lemma 10.71.9. $\square$

0A73 Remark 47.13.5. Let A be a ring and let $I \subset A$ be an ideal. Set $B = A/I$. In this case the functor $\operatorname{Hom}_A(B, -)$ is equal to the functor

$$\text{Mod}_A \longrightarrow \text{Mod}_B, \quad M \longmapsto M[I]$$

which sends M to the submodule of I-torsion.

0BZB Situation 47.13.6. Let $R \to A$ be a ring map. We will give an alternative construction of $R\operatorname{Hom}(A, -)$ which will stand us in good stead later in this chapter. Namely, suppose we have a differential graded algebra (E, d) over R and a quasi-isomorphism $E \to A$ where we view A as a differential graded algebra over R with zero differential. Then we have commutative diagrams

$$\begin{tikzcd} D(E, \text{d}) \arrow[dr] & & D(A) \arrow[ll] \arrow[dl] \\ & D(R) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D(E, \text{d}) \arrow[rr, "- \otimes^{\mathbf{L}}_E A"] & & D(A) \\ & D(R) \arrow[ul, "- \otimes^{\mathbf{L}}_R E"] \arrow[ur, "- \otimes^{\mathbf{L}}_R A"'] & \end{tikzcd}$$

where the horizontal arrows are equivalences of categories (Differential Graded Algebra, Lemma 22.37.1). It is clear that the first diagram commutes. The second diagram commutes because the first one does and our functors are their left adjoints (Differential Graded Algebra, Example 22.33.6) or because we have $E \otimes^{\mathbf{L}}_E A = E \otimes_E A$ and we can use Differential Graded Algebra, Lemma 22.34.1.

0BZC Lemma 47.13.7. In Situation 47.13.6 the functor $R\operatorname{Hom}(A, -)$ is equal to the composition of $R\operatorname{Hom}(E, -) : D(R) \to D(E, \text{d})$ and the equivalence $- \otimes^{\mathbf{L}}_E A : D(E, \text{d}) \to D(A)$.

Proof. This is true because $R\operatorname{Hom}(E, -)$ is the right adjoint to $- \otimes^{\mathbf{L}}_R E$, see Differential Graded Algebra, Lemma 22.33.5. Hence this functor plays the same role as the functor $R\operatorname{Hom}(A, -)$ for the map $R \to A$ (Lemma 47.13.1), whence these functors must correspond via the equivalence $- \otimes^{\mathbf{L}}_E A : D(E, \text{d}) \to D(A)$. $\square$

0BZD Lemma 47.13.8. In Situation 47.13.6 assume that

(1) E viewed as an object of $D(R)$ is compact, and
(2) $N = \mathrm{Hom}^\bullet_R(E^\bullet, R)$ computes $R\mathrm{Hom}(E, R)$.

Then $R\mathrm{Hom}(E, -) : D(R) \to D(E)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N$.

Proof. Special case of Differential Graded Algebra, Lemma 22.33.9. □

0BZE Lemma 47.13.9. In Situation 47.13.6 assume A is a perfect R-module. Then

$$R\mathrm{Hom}(A, -) : D(R) \to D(A)$$

is given by $K \mapsto K \otimes_R^{\mathbf{L}} M$ where $M = R\mathrm{Hom}(A, R) \in D(A)$.

Proof. We apply Divided Power Algebra, Lemma 23.6.10 to choose a Tate resolution (E, d) of A over R. Note that $E^i = 0$ for $i > 0$, $E^0 = R[x_1, \ldots, x_n]$ is a polynomial algebra, and E^i is a finite free E^0-module for $i < 0$. It follows that E viewed as a complex of R-modules is a bounded above complex of free R-modules. We check the assumptions of Lemma 47.13.8. The first holds because A is perfect (hence compact by More on Algebra, Proposition 15.78.3) and the second by More on Algebra, Lemma 15.73.2. From the lemma conclude that $K \mapsto R\mathrm{Hom}(E, K)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N$ for some differential graded E-module N. Observe that

$$(R \otimes_R E) \otimes_E^{\mathbf{L}} A = R \otimes_E E \otimes_E A$$

in $D(A)$. Hence by Differential Graded Algebra, Lemma 22.34.2 we conclude that the composition of $- \otimes_R^{\mathbf{L}} N$ and $- \otimes_R^{\mathbf{L}} A$ is of the form $- \otimes_R M$ for some $M \in D(A)$. To finish the proof we apply Lemma 47.13.7. □

0BZH Lemma 47.13.10. Let $R \to A$ be a surjective ring map whose kernel I is an invertible R-module. The functor $R\mathrm{Hom}(A, -) : D(R) \to D(A)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N[-1]$ where N is inverse of the invertible A-module $I \otimes_R A$.

Proof. Since A has the finite projective resolution

$$0 \to I \to R \to A \to 0$$

we see that A is a perfect R-module. By Lemma 47.13.9 it suffices to prove that $R\mathrm{Hom}(A, R)$ is represented by $N[-1]$ in $D(A)$. This means $R\mathrm{Hom}(A, R)$ has a unique nonzero cohomology module, namely N in degree 1. As $\mathrm{Mod}_A \to \mathrm{Mod}_R$ is fully faithful it suffice to prove this after applying the restriction functor $i_* : D(A) \to D(R)$. By Lemma 47.13.3 we have

$$i_* R\mathrm{Hom}(A, R) = R\mathrm{Hom}_R(A, R)$$

Using the finite projective resolution above we find that the latter is represented by the complex $R \to I^{\otimes -1}$ with R in degree 0. The map $R \to I^{\otimes -1}$ is injective and the cokernel is N. □

47.14. Base change for trivial duality

0E28 In this section we consider a cocartesian square of rings

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A' \\ R \arrow[u, "\varphi"] \arrow[r, "\rho"] & R' \arrow[u, "\varphi'"'] \end{tikzcd}$$

In other words, we have $A' = A \otimes_R R'$. If A and R' are tor independent over R then there is a canonical base change map

0E29 (47.14.0.1)
$$R\operatorname{Hom}(A, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}(A', K \otimes_R^{\mathbf{L}} R')$$

in $D(A')$ functorial for K in $D(R)$. Namely, by the adjointness of Lemma 47.13.1 such an arrow is the same thing as a map

$$\varphi'_*\left(R\operatorname{Hom}(A, K) \otimes_A^{\mathbf{L}} A'\right) \longrightarrow K \otimes_R^{\mathbf{L}} R'$$

in $D(R')$ where $\varphi'_* : D(A') \to D(R')$ is the restriction functor. We may apply More on Algebra, Lemma 15.61.2 to the left hand side to get that this is the same thing as a map

$$\varphi_*(R\operatorname{Hom}(A, K)) \otimes_R^{\mathbf{L}} R' \longrightarrow K \otimes_R^{\mathbf{L}} R'$$

in $D(R')$ where $\varphi_* : D(A) \to D(R)$ is the restriction functor. For this we can choose $can \otimes^{\mathbf{L}} \mathrm{id}_{R'}$ where $can : \varphi_*(R\operatorname{Hom}(A, K)) \to K$ is the counit of the adjunction between $R\operatorname{Hom}(A, -)$ and φ_*.

0E2A **Lemma 47.14.1.** *In the situation above, the map (47.14.0.1) is an isomorphism if and only if the map*

$$R\operatorname{Hom}_R(A, K) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_R(A, K \otimes_R^{\mathbf{L}} R')$$

of More on Algebra, Lemma 15.73.5 is an isomorphism.

Proof. To see that the map is an isomorphism, it suffices to prove it is an isomorphism after applying φ'_*. Applying the functor φ'_* to (47.14.0.1) and using that $A' = A \otimes_R^{\mathbf{L}} R'$ we obtain the base change map $R\operatorname{Hom}_R(A, K) \otimes_R^{\mathbf{L}} R' \to R\operatorname{Hom}_{R'}(A \otimes_R^{\mathbf{L}} R', K \otimes_R^{\mathbf{L}} R')$ for derived hom of More on Algebra, Equation (15.99.1.1). Unwinding the left and right hand side exactly as in the proof of More on Algebra, Lemma 15.99.2 and in particular using More on Algebra, Lemma 15.99.1 gives the desired result. $\square$

0BZM **Lemma 47.14.2.** *Let $R \to A$ and $R \to R'$ be ring maps and $A' = A \otimes_R R'$. Assume*

1. *A is pseudo-coherent as an R-module,*
2. *R' has finite tor dimension as an R-module (for example $R \to R'$ is flat),*
3. *A and R' are tor independent over R.*

Then (47.14.0.1) is an isomorphism for $K \in D^+(R)$.

Proof. Follows from Lemma 47.14.1 and More on Algebra, Lemma 15.98.3 part (4). $\square$

0BZP **Lemma 47.14.3.** *Let $R \to A$ and $R \to R'$ be ring maps and $A' = A \otimes_R R'$. Assume*

1. *A is perfect as an R-module,*
2. *A and R' are tor independent over R.*

Then (47.14.0.1) is an isomorphism for all $K \in D(R)$.

Proof. Follows from Lemma 47.14.1 and More on Algebra, Lemma 15.98.3 part (1). $\square$

47.15. Dualizing complexes

0A7A In this section we define dualizing complexes for Noetherian rings.

0A7B Definition 47.15.1. Let A be a Noetherian ring. A dualizing complex is a complex of A-modules $\omega_A^\bullet$ such that

(1) $\omega_A^\bullet$ has finite injective dimension,
(2) $H^i(\omega_A^\bullet)$ is a finite A-module for all i, and
(3) $A \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism.

This definition takes some time getting used to. It is perhaps a good idea to prove some of the following lemmas yourself without reading the proofs.

0G4H Lemma 47.15.2. Let A be a Noetherian ring. Let $K, L \in D_{Coh}(A)$ and assume L has finite injective dimension. Then $R\operatorname{Hom}_A(K, L)$ is in $D_{Coh}(A)$.

Proof. Pick an integer n and consider the distinguished triangle

$$\tau_{\leq n}K \to K \to \tau_{\geq n+1}K \to \tau_{\leq n}K[1]$$

see Derived Categories, Remark 13.12.4. Since L has finite injective dimension we see that $R\operatorname{Hom}_A(\tau_{\geq n+1}K, L)$ has vanishing cohomology in degrees $\geq c - n$ for some constant c. Hence, given i, we see that $\operatorname{Ext}^i_A(K, L) \to \operatorname{Ext}^i_A(\tau_{\leq n}K, L)$ is an isomorphism for some $n \gg -i$. By Derived Categories of Schemes, Lemma 36.11.5 applied to $\tau_{\leq n}K$ and L we see conclude that $\operatorname{Ext}^i_A(K, L)$ is a finite A-module for all i. Hence $R\operatorname{Hom}_A(K, L)$ is indeed an object of $D_{Coh}(A)$. $\square$

0A7C Lemma 47.15.3. Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then the functor

$$D : K \longmapsto R\operatorname{Hom}_A(K, \omega_A^\bullet)$$

is an anti-equivalence $D_{Coh}(A) \to D_{Coh}(A)$ which exchanges $D^+_{Coh}(A)$ and $D^-_{Coh}(A)$ and induces an anti-equivalence $D^b_{Coh}(A) \to D^b_{Coh}(A)$. Moreover $D \circ D$ is isomorphic to the identity functor.

Proof. Let K be an object of $D_{Coh}(A)$. From Lemma 47.15.2 we see $R\operatorname{Hom}_A(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. By More on Algebra, Lemma 15.98.2 and the assumptions on the dualizing complex we obtain a canonical isomorphism

$$K = R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_A(R\operatorname{Hom}_A(K, \omega_A^\bullet), \omega_A^\bullet)$$

Thus our functor has a quasi-inverse and the proof is complete. $\square$

Let R be a ring. Recall that an object L of $D(R)$ is invertible if it is an invertible object for the symmetric monoidal structure on $D(R)$ given by derived tensor product. In More on Algebra, Lemma 15.126.4 we have seen this means L is perfect, $L = \bigoplus H^n(L)[-n]$, this is a finite sum, each $H^n(L)$ is finite projective, and there is an open covering $\operatorname{Spec}(R) = \bigcup D(f_i)$ such that $L \otimes_R R_{f_i} \cong R_{f_i}[-n_i]$ for some integers n_i.

0A7E Lemma 47.15.4. Let A be a Noetherian ring. Let $F : D^b_{Coh}(A) \to D^b_{Coh}(A)$ be an A-linear equivalence of categories. Then $F(A)$ is an invertible object of $D(A)$.

Proof. Let $\mathfrak{m} \subset A$ be a maximal ideal with residue field κ. Consider the object $F(\kappa)$. Since $\kappa = \operatorname{Hom}_{D(A)}(\kappa, \kappa)$ we find that all cohomology groups of $F(\kappa)$ are annihilated by $\mathfrak{m}$. We also see that

$$\operatorname{Ext}^i_A(\kappa, \kappa) = \operatorname{Ext}^i_A(F(\kappa), F(\kappa)) = \operatorname{Hom}_{D(A)}(F(\kappa), F(\kappa)[i])$$

is zero for $i < 0$. Say $H^a(F(\kappa)) \neq 0$ and $H^b(F(\kappa)) \neq 0$ with a minimal and b maximal (so in particular $a \leq b$). Then there is a nonzero map

$$F(\kappa) \to H^b(F(\kappa))[-b] \to H^a(F(\kappa))[-b] \to F(\kappa)[a-b]$$

in $D(A)$ (nonzero because it induces a nonzero map on cohomology). This proves that $b = a$. We conclude that $F(\kappa) = \kappa[-a]$.

Let G be a quasi-inverse to our functor F. Arguing as above we find an integer b such that $G(\kappa) = \kappa[-b]$. On composing we find $a + b = 0$. Let E be a finite A-module wich is annihilated by a power of $\mathfrak{m}$. Arguing by induction on the length of E we find that $G(E) = E'[-b]$ for some finite A-module E' annihilated by a power of $\mathfrak{m}$. Then $E[-a] = F(E')$. Next, we consider the groups

$$\mathrm{Ext}^i_A(A, E') = \mathrm{Ext}^i_A(F(A), F(E')) = \mathrm{Hom}_{D(A)}(F(A), E[-a+i])$$

The left hand side is nonzero if and only if $i = 0$ and then we get E'. Applying this with $E = E' = \kappa$ and using Nakayama's lemma this implies that $H^j(F(A))_\mathfrak{m}$ is zero for $j > a$ and generated by 1 element for $j = a$. On the other hand, if $H^j(F(A))_\mathfrak{m}$ is not zero for some $j < a$, then there is a map $F(A) \to E[-a+i]$ for some $i < 0$ and some E (More on Algebra, Lemma 15.65.7) which is a contradiction. Thus we see that $F(A)_\mathfrak{m} = M[-a]$ for some $A_\mathfrak{m}$-module M generated by 1 element. However, since

$$A_\mathfrak{m} = \mathrm{Hom}_{D(A)}(A, A)_\mathfrak{m} = \mathrm{Hom}_{D(A)}(F(A), F(A))_\mathfrak{m} = \mathrm{Hom}_{A_\mathfrak{m}}(M, M)$$

we see that $M \cong A_\mathfrak{m}$. We conclude that there exists an element $f \in A$, $f \notin \mathfrak{m}$ such that $F(A)_f$ is isomorphic to $A_f[-a]$. This finishes the proof. $\square$

0A7F Lemma 47.15.5. Let A be a Noetherian ring. If $\omega_A^\bullet$ and $(\omega'_A)^\bullet$ are dualizing complexes, then $(\omega'_A)^\bullet$ is quasi-isomorphic to $\omega_A^\bullet \otimes_A^{\mathbf{L}} L$ for some invertible object L of $D(A)$.

Proof. By Lemmas 47.15.3 and 47.15.4 the functor $K \mapsto R\,\mathrm{Hom}_A(R\,\mathrm{Hom}_A(K, \omega_A^\bullet), (\omega'_A)^\bullet)$ maps A to an invertible object L. In other words, there is an isomorphism

$$L \longrightarrow R\,\mathrm{Hom}_A(\omega_A^\bullet, (\omega'_A)^\bullet)$$

Since L has finite tor dimension, this means that we can apply More on Algebra, Lemma 15.98.2 to see that

$$R\,\mathrm{Hom}_A(\omega_A^\bullet, (\omega'_A)^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow R\,\mathrm{Hom}_A(R\,\mathrm{Hom}_A(K, \omega_A^\bullet), (\omega'_A)^\bullet)$$

is an isomorphism for K in $D^b_{Coh}(A)$. In particular, setting $K = \omega_A^\bullet$ finishes the proof. $\square$

0A7G Lemma 47.15.6. Let A be a Noetherian ring. Let $B = S^{-1}A$ be a localization. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $S^{-1}I^\bullet$ is a bounded complex of injective $B = S^{-1}A$-modules (Lemma 47.3.8) representing $\omega_A^\bullet \otimes_A B$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension. Since $H^i(\omega_A^\bullet \otimes_A B) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\,\mathrm{Hom}_A(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 15.99.2. $\square$

0A7H **Lemma 47.15.7.** *Let A be a Noetherian ring. Let $f_1, \ldots, f_n \in A$ generate the unit ideal. If $\omega_A^\bullet$ is a complex of A-modules such that $(\omega_A^\bullet)_{f_i}$ is a dualizing complex for A_{f_i} for all i, then $\omega_A^\bullet$ is a dualizing complex for A.*

Proof. Consider the double complex

$$\prod\nolimits_{i_0} (\omega_A^\bullet)_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} (\omega_A^\bullet)_{f_{i_0} f_{i_1}} \to \ldots$$

The associated total complex is quasi-isomorphic to $\omega_A^\bullet$ for example by Descent, Remark 35.3.10 or by Derived Categories of Schemes, Lemma 36.9.4. By assumption the complexes $(\omega_A^\bullet)_{f_i}$ have finite injective dimension as complexes of A_{f_i}-modules. This implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over $A_{f_{i_0} \ldots f_{i_p}}$, see Lemma 47.3.8. This in turn implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over A, see Lemma 47.3.2. Hence $\omega_A^\bullet$ has finite injective dimension as a complex of A-modules (as it can be represented by a complex endowed with a finite filtration whose graded parts have finite injective dimension). Since $H^n(\omega_A^\bullet)_{f_i}$ is a finite A_{f_i} module for each i we see that $H^i(\omega_A^\bullet)$ is a finite A-module, see Algebra, Lemma 10.23.2. Finally, the (derived) base change of the map $A \to R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ to A_{f_i} is the map $A_{f_i} \to R\mathrm{Hom}_A((\omega_A^\bullet)_{f_i}, (\omega_A^\bullet)_{f_i})$ by More on Algebra, Lemma 15.99.2. Hence we deduce that $A \to R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is an isomorphism and the proof is complete. □

0AX0 **Lemma 47.15.8.** *Let $A \to B$ be a finite ring map of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\mathrm{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $\mathrm{Hom}_A(B, I^\bullet)$ is a bounded complex of injective B-modules (Lemma 47.3.4) representing $R\mathrm{Hom}(B, \omega_A^\bullet)$. Thus $R\mathrm{Hom}(B, \omega_A^\bullet)$ has finite injective dimension. By Lemma 47.13.4 it is an object of $D_{\mathrm{Coh}}(B)$. Finally, we compute

$$\mathrm{Hom}_{D(B)}(R\mathrm{Hom}(B, \omega_A^\bullet), R\mathrm{Hom}(B, \omega_A^\bullet)) = \mathrm{Hom}_{D(A)}(R\mathrm{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B$$

and for $n \neq 0$ we compute

$$\mathrm{Hom}_{D(B)}(R\mathrm{Hom}(B, \omega_A^\bullet), R\mathrm{Hom}(B, \omega_A^\bullet)[n]) = \mathrm{Hom}_{D(A)}(R\mathrm{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0$$

which proves the last property of a dualizing complex. In the displayed equations, the first equality holds by Lemma 47.13.1 and the second equality holds by Lemma 47.15.3. □

0A7I **Lemma 47.15.9.** *Let $A \to B$ be a surjective homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\mathrm{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Special case of Lemma 47.15.8. □

0A7J **Lemma 47.15.10.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A A[x]$ is a dualizing complex for $A[x]$.*

Proof. Set $B = A[x]$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$. It follows from Lemma 47.3.10 and More on Algebra, Lemma 15.69.5 that $\omega_B^\bullet$ has finite injective dimension. Since $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\mathrm{Hom}_B(\omega_B^\bullet, \omega_B^\bullet)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 15.99.2. □

0A7K Proposition 47.15.11. Let A be a Noetherian ring which has a dualizing complex. Then any A-algebra essentially of finite type over A has a dualizing complex.

Proof. This follows from a combination of Lemmas 47.15.6, 47.15.9, and 47.15.10. $\square$

0A7L Lemma 47.15.12. Let A be a Noetherian ring. Let $\omega_A^\bullet$ be a dualizing complex. Let $\mathfrak m \subset A$ be a maximal ideal and set $\kappa = A/\mathfrak m$. Then $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$ for some $n \in \mathbf{Z}$.

Proof. This is true because $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet)$ is a dualizing complex over κ (Lemma 47.15.9), because dualizing complexes over κ are unique up to shifts (Lemma 47.15.5), and because κ is a dualizing complex over κ. $\square$

47.16. Dualizing complexes over local rings

0A7M In this section $(A, \mathfrak m, \kappa)$ will be a Noetherian local ring endowed with a dualizing complex $\omega_A^\bullet$ such that the integer n of Lemma 47.15.12 is zero. More precisely, we assume that $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) = \kappa[0]$. In this case we will say that the dualizing complex is normalized. Observe that a normalized dualizing complex is unique up to isomorphism and that any other dualizing complex for A is isomorphic to a shift of a normalized one (Lemma 47.15.5).

0AX1 Lemma 47.16.1. Let $(A, \mathfrak m, \kappa) \to (B, \mathfrak m', \kappa')$ be a finite local map of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\omega_B^\bullet = R\operatorname{Hom}(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.

Proof. By Lemma 47.15.8 the complex $\omega_B^\bullet$ is dualizing for B. We have

$$R\operatorname{Hom}_B(\kappa', \omega_B^\bullet) = R\operatorname{Hom}_B(\kappa', R\operatorname{Hom}(B, \omega_A^\bullet)) = R\operatorname{Hom}_A(\kappa', \omega_A^\bullet)$$

by Lemma 47.13.1. Since κ' is isomorphic to a finite direct sum of copies of κ as an A-module and since $\omega_A^\bullet$ is normalized, we see that this complex only has cohomology placed in degree 0. Thus $\omega_B^\bullet$ is a normalized dualizing complex as well. $\square$

0A7N Lemma 47.16.2. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $A \to B$ be surjective. Then $\omega_B^\bullet = R\operatorname{Hom}_A(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.

Proof. Special case of Lemma 47.16.1. $\square$

0A7P Lemma 47.16.3. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local ring. Let F be an A-linear self-equivalence of the category of finite length A-modules. Then F is isomorphic to the identity functor.

Proof. Since κ is the unique simple object of the category we have $F(\kappa) \cong \kappa$. Since our category is abelian, we find that F is exact. Hence $F(E)$ has the same length as E for all finite length modules E. Since $\operatorname{Hom}(E, \kappa) = \operatorname{Hom}(F(E), F(\kappa)) \cong \operatorname{Hom}(F(E), \kappa)$ we conclude from Nakayama's lemma that E and $F(E)$ have the same number of generators. Hence $F(A/\mathfrak m^n)$ is a cyclic A-module. Pick a generator $e \in F(A/\mathfrak m^n)$. Since F is A-linear we conclude that $\mathfrak m^n e = 0$. The map $A/\mathfrak m^n \to F(A/\mathfrak m^n)$ has to be an isomorphism as the lengths are equal. Pick an element

$$e \in \lim F(A/\mathfrak m^n)$$

which maps to a generator for all n (small argument omitted). Then we obtain a system of isomorphisms $A/\mathfrak m^n \to F(A/\mathfrak m^n)$ compatible with all A-module maps

$A/\mathfrak{m}^n \to A/\mathfrak{m}^{n'}$ (by A-linearity of F again). Since any finite length module is a cokernel of a map between direct sums of cyclic modules, we obtain the isomorphism of the lemma. $\square$

0A7Q Lemma 47.16.4. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let E be an injective hull of κ. Then there exists a functorial isomorphism

$$R\operatorname{Hom}_A(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)[0]$$

for N running through the finite length A-modules.

Proof. By induction on the length of N we see that $R\operatorname{Hom}_A(N, \omega_A^\bullet)$ is a module of finite length sitting in degree 0. Thus $R\operatorname{Hom}_A(-, \omega_A^\bullet)$ induces an anti-equivalence on the category of finite length modules. Since the same is true for $\operatorname{Hom}_A(-, E)$ by Proposition 47.7.8 we see that

$$N \longmapsto \operatorname{Hom}_A(R\operatorname{Hom}_A(N, \omega_A^\bullet), E)$$

is an equivalence as in Lemma 47.16.3. Hence it is isomorphic to the identity functor. Since $\operatorname{Hom}_A(-, E)$ applied twice is the identity (Proposition 47.7.8) we obtain the statement of the lemma. $\square$

0A7U Lemma 47.16.5. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module and let $d = \dim(\operatorname{Supp}(M))$. Then

1. if $\operatorname{Ext}^i_A(M, \omega_A^\bullet)$ is nonzero, then $i \in \{-d, \ldots, 0\}$,
2. the dimension of the support of $\operatorname{Ext}^i_A(M, \omega_A^\bullet)$ is at most $-i$,
3. $\operatorname{depth}(M)$ is the smallest integer $\delta \geq 0$ such that $\operatorname{Ext}^{-\delta}_A(M, \omega_A^\bullet) \neq 0$.

Proof. We prove this by induction on d. If $d = 0$, this follows from Lemma 47.16.4 and Matlis duality (Proposition 47.7.8) which guarantees that $\operatorname{Hom}_A(M, E)$ is nonzero if M is nonzero.

Assume the result holds for modules with support of dimension $< d$ and that M has depth > 0. Choose an $f \in \mathfrak{m}$ which is a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

Since $\dim(\operatorname{Supp}(M/fM)) = d - 1$ (Algebra, Lemma 10.63.10) we may apply the induction hypothesis. Writing $E^i = \operatorname{Ext}^i_A(M, \omega_A^\bullet)$ and $F^i = \operatorname{Ext}^i_A(M/fM, \omega_A^\bullet)$ we obtain a long exact sequence

$$\ldots \to F^i \to E^i \xrightarrow{f} E^i \to F^{i+1} \to \ldots$$

By induction $E^i/fE^i = 0$ for $i+1 \notin \{-\dim(\operatorname{Supp}(M/fM)), \ldots, -\operatorname{depth}(M/fM)\}$. By Nakayama's lemma (Algebra, Lemma 10.20.1) and Algebra, Lemma 10.72.7 we conclude $E^i = 0$ for $i \notin \{-\dim(\operatorname{Supp}(M)), \ldots, -\operatorname{depth}(M)\}$. Moreover, in the boundary case $i = -\operatorname{depth}(M)$ we deduce that E^i is nonzero as F^{i+1} is nonzero by induction. Since $E^i/fE^i \subset F^{i+1}$ we get

$$\dim(\operatorname{Supp}(F^{i+1})) \geq \dim(\operatorname{Supp}(E^i/fE^i)) \geq \dim(\operatorname{Supp}(E^i)) - 1$$

(see lemma used above) we also obtain the dimension estimate (2).

If M has depth 0 and $d > 0$ we let $N = M[\mathfrak{m}^\infty]$ and set $M' = M/N$ (compare with Lemma 47.11.6). Then M' has depth > 0 and $\dim(\operatorname{Supp}(M')) = d$. Thus we know the result for M' and since $R\operatorname{Hom}_A(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)$ (Lemma 47.16.4) the long exact cohomology sequence of Ext's implies the result for M. $\square$

0BUJ Remark 47.16.6. Let $(A, \mathfrak{m})$ and $\omega_A^\bullet$ be as in Lemma 47.16.5. By More on Algebra, Lemma 15.69.2 we see that $\omega_A^\bullet$ has injective-amplitude in $[-d, 0]$ because part (3) of that lemma applies. In particular, for any A-module M (not necessarily finite) we have $\text{Ext}^i_A(M, \omega_A^\bullet) = 0$ for $i \not\in \{-d, \ldots, 0\}$.

0B5A Lemma 47.16.7. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. The following are equivalent

(1) M is Cohen-Macaulay,
(2) $\text{Ext}^i_A(M, \omega_A^\bullet)$ is nonzero for a single i,
(3) $\text{Ext}^{-i}_A(M, \omega_A^\bullet)$ is zero for $i \neq \dim(\text{Supp}(M))$.

Denote CM_d the category of finite Cohen-Macaulay A-modules of depth d. Then $M \mapsto \text{Ext}^{-d}_A(M, \omega_A^\bullet)$ defines an anti-auto-equivalence of CM_d.

Proof. We will use the results of Lemma 47.16.5 without further mention. Fix a finite module M. If M is Cohen-Macaulay, then only $\text{Ext}^{-d}_A(M, \omega_A^\bullet)$ can be nonzero, hence (1) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) is immediate. Assume (2) and let $N = \text{Ext}^{-\delta}_A(M, \omega_A^\bullet)$ be the nonzero Ext where $\delta = \text{depth}(M)$. Then, since

$$M[0] = R\operatorname{Hom}_A(R\operatorname{Hom}_A(M, \omega_A^\bullet), \omega_A^\bullet) = R\operatorname{Hom}_A(N[\delta], \omega_A^\bullet)$$

(Lemma 47.15.3) we conclude that $M = \text{Ext}^{-\delta}_A(N, \omega_A^\bullet)$. Thus $\delta \geq \dim(\text{Supp}(M))$. However, since we also know that $\delta \leq \dim(\text{Supp}(M))$ (Algebra, Lemma 10.72.3) we conclude that M is Cohen-Macaulay.

To prove the final statement, it suffices to show that $N = \text{Ext}^{-d}_A(M, \omega_A^\bullet)$ is in CM_d for M in CM_d. Above we have seen that $M[0] = R\operatorname{Hom}_A(N[d], \omega_A^\bullet)$ and this proves the desired result by the equivalence of (1) and (3). $\square$

0A7R Lemma 47.16.8. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. If $\dim(A) = 0$, then $\omega_A^\bullet \cong E[0]$ where E is an injective hull of the residue field.

Proof. Immediate from Lemma 47.16.4. $\square$

0A7S Lemma 47.16.9. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex. Let $I \subset \mathfrak{m}$ be an ideal of finite length. Set $B = A/I$. Then there is a distinguished triangle

$$\omega_B^\bullet \to \omega_A^\bullet \to \operatorname{Hom}_A(I, E)[0] \to \omega_B^\bullet[1]$$

in $D(A)$ where E is an injective hull of κ and $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to I \to A \to B \to 0$ and Lemmas 47.16.4 and 47.16.2. $\square$

0A7T Lemma 47.16.10. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then there is a distinguished triangle

$$\omega_B^\bullet \to \omega_A^\bullet \to \omega_A^\bullet \to \omega_B^\bullet[1]$$

in $D(A)$ where $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to A \to A \to B \to 0$ and Lemma 47.16.2. $\square$

0A7V **Lemma 47.16.11.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $\mathfrak{p}$ be a minimal prime of A with $\dim(A/\mathfrak{p}) = e$. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero if and only if $i = -e$.

Proof. Since $A_\mathfrak{p}$ has dimension zero, there exists an integer $n > 0$ such that $\mathfrak{p}^n A_\mathfrak{p}$ is zero. Set $B = A/\mathfrak{p}^n$ and $\omega_B^\bullet = R\operatorname{Hom}_A(B, \omega_A^\bullet)$. Since $B_\mathfrak{p} = A_\mathfrak{p}$ we see that

$$(\omega_B^\bullet)_\mathfrak{p} = R\operatorname{Hom}_A(B, \omega_A^\bullet) \otimes_A^{\mathbf{L}} A_\mathfrak{p} = R\operatorname{Hom}_{A_\mathfrak{p}}(B_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}) = (\omega_A^\bullet)_\mathfrak{p}$$

The second equality holds by More on Algebra, Lemma 15.99.2. By Lemma 47.16.2 we may replace A by B. After doing so, we see that $\dim(A) = e$. Then we see that $H^i(\omega_A^\bullet)_\mathfrak{p}$ can only be nonzero if $i = -e$ by Lemma 47.16.5 parts (1) and (2). On the other hand, since $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex for the nonzero ring $A_\mathfrak{p}$ (Lemma 47.15.6) we see that the remaining module has to be nonzero. $\square$

47.17. Dualizing complexes and dimension functions

0A7W Our results in the local setting have the following consequence: a Noetherian ring which has a dualizing complex is a universally catenary ring of finite dimension.

0A7X **Lemma 47.17.1.** Let A be a Noetherian ring. Let $\mathfrak{p}$ be a minimal prime of A. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero for exactly one i.

Proof. The complex $\omega_A^\bullet \otimes_A A_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (Lemma 47.15.6). The dimension of $A_\mathfrak{p}$ is zero as $\mathfrak{p}$ is minimal. Hence the result follows from Lemma 47.16.8. $\square$

Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Lemma 47.15.12 allows us to define a function

$$\delta = \delta_{\omega_A^\bullet} : \operatorname{Spec}(A) \longrightarrow \mathbf{Z}$$

by mapping $\mathfrak{p}$ to the integer of Lemma 47.15.12 for the dualizing complex $(\omega_A^\bullet)_\mathfrak{p}$ over $A_\mathfrak{p}$ (Lemma 47.15.6) and the residue field $\kappa(\mathfrak{p})$. To be precise, we define $\delta(\mathfrak{p})$ to be the unique integer such that

$$(\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})]$$

is a normalized dualizing complex over the Noetherian local ring $A_\mathfrak{p}$.

0A7Y **Lemma 47.17.2.** Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Let $A \to B$ be a surjective ring map and let $\omega_B^\bullet = R\operatorname{Hom}(B, \omega_A^\bullet)$ be the dualizing complex for B of Lemma 47.15.9. Then we have

$$\delta_{\omega_B^\bullet} = \delta_{\omega_A^\bullet}|_{\operatorname{Spec}(B)}$$

Proof. This follows from the definition of the functions and Lemma 47.16.2. $\square$

0A7Z **Lemma 47.17.3.** Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. The function $\delta = \delta_{\omega_A^\bullet}$ defined above is a dimension function (Topology, Definition 5.20.1).

Proof. Let $\mathfrak{p} \subset \mathfrak{q}$ be an immediate specialization. We have to show that $\delta(\mathfrak{p}) = \delta(\mathfrak{q}) + 1$. We may replace A by $A/\mathfrak{p}$, the complex $\omega_A^\bullet$ by $\omega_{A/\mathfrak{p}}^\bullet = R\operatorname{Hom}(A/\mathfrak{p}, \omega_A^\bullet)$, the prime $\mathfrak{p}$ by (0), and the prime $\mathfrak{q}$ by $\mathfrak{q}/\mathfrak{p}$, see Lemma 47.17.2. Thus we may assume that A is a domain, $\mathfrak{p} = (0)$, and $\mathfrak{q}$ is a prime ideal of height 1.

Then $H^i(\omega_A^\bullet)_{(0)}$ is nonzero for exactly one i, say i_0, by Lemma 47.17.1. In fact $i_0 = -\delta((0))$ because $(\omega_A^\bullet)_{(0)}[-\delta((0))]$ is a normalized dualizing complex over the field $A_{(0)}$.

On the other hand $(\omega_A^\bullet)_\mathfrak q[-\delta(\mathfrak q)]$ is a normalized dualizing complex for $A_\mathfrak q$. By Lemma 47.16.11 we see that

$$H^e((\omega_A^\bullet)_\mathfrak q[-\delta(\mathfrak q)])_{(0)} = H^{e-\delta(\mathfrak q)}(\omega_A^\bullet)_{(0)}$$

is nonzero only for $e = -\dim(A_\mathfrak q) = -1$. We conclude

$$-\delta((0)) = -1 - \delta(\mathfrak q)$$

as desired. □

0A80 Lemma 47.17.4. Let A be a Noetherian ring which has a dualizing complex. Then A is universally catenary of finite dimension.

Proof. Because $\text{Spec}(A)$ has a dimension function by Lemma 47.17.3 it is catenary, see Topology, Lemma 5.20.2. Hence A is catenary, see Algebra, Lemma 10.105.2. It follows from Proposition 47.15.11 that A is universally catenary.

Because any dualizing complex $\omega_A^\bullet$ is in $D^b_{Coh}(A)$ the values of the function $\delta_{\omega_A^\bullet}$ in minimal primes are bounded by Lemma 47.17.1. On the other hand, for a maximal ideal $\mathfrak m$ with residue field κ the integer $i = -\delta(\mathfrak m)$ is the unique integer such that $\text{Ext}^i_A(\kappa, \omega_A^\bullet)$ is nonzero (Lemma 47.15.12). Since $\omega_A^\bullet$ has finite injective dimension these values are bounded too. Since the dimension of A is the maximal value of $\delta(\mathfrak p) - \delta(\mathfrak m)$ where $\mathfrak p \subset \mathfrak m$ are a pair consisting of a minimal prime and a maximal prime we find that the dimension of $\text{Spec}(A)$ is bounded. □

0AWE Lemma 47.17.5. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $d = \dim(A)$ and $\omega_A = H^{-d}(\omega_A^\bullet)$. Then

(1) the support of ω_A is the union of the irreducible components of $\text{Spec}(A)$ of dimension d,
(2) ω_A satisfies (S_2), see Algebra, Definition 10.157.1.

Proof. We will use Lemma 47.16.5 without further mention. By Lemma 47.16.11 the support of ω_A contains the irreducible components of dimension d. Let $\mathfrak p \subset A$ be a prime. By Lemma 47.17.3 the complex $(\omega_A^\bullet)_\mathfrak p[-\dim(A/\mathfrak p)]$ is a normalized dualizing complex for $A_\mathfrak p$. Hence if $\dim(A/\mathfrak p) + \dim(A_\mathfrak p) < d$, then $(\omega_A)_\mathfrak p = 0$. This proves the support of ω_A is the union of the irreducible components of dimension d, because the complement of this union is exactly the primes $\mathfrak p$ of A for which $\dim(A/\mathfrak p) + \dim(A_\mathfrak p) < d$ as A is catenary (Lemma 47.17.4). On the other hand, if $\dim(A/\mathfrak p) + \dim(A_\mathfrak p) = d$, then

$$(\omega_A)_\mathfrak p = H^{-\dim(A_\mathfrak p)}((\omega_A^\bullet)_\mathfrak p[-\dim(A/\mathfrak p)])$$

Hence in order to prove ω_A has (S_2) it suffices to show that the depth of ω_A is at least $\min(\dim(A), 2)$. We prove this by induction on $\dim(A)$. The case $\dim(A) = 0$ is trivial.

Assume $\text{depth}(A) > 0$. Choose a nonzerodivisor $f \in \mathfrak m$ and set $B = A/fA$. Then $\dim(B) = \dim(A) - 1$ and we may apply the induction hypothesis to B. By Lemma 47.16.10 we see that multiplication by f is injective on ω_A and we get $\omega_A/f\omega_A \subset \omega_B$. This proves the depth of ω_A is at least 1. If $\dim(A) > 1$, then $\dim(B) > 0$ and ω_B has depth > 0. Hence ω_A has depth > 1 and we conclude in this case.

Assume $\dim(A) > 0$ and $\text{depth}(A) = 0$. Let $I = A[\mathfrak{m}^\infty]$ and set $B = A/I$. Then B has depth ≥ 1 and $\omega_A = \omega_B$ by Lemma 47.16.9. Since we proved the result for ω_B above the proof is done. □

47.18. The local duality theorem

0A81 The main result in this section is due to Grothendieck.

0A82 **Lemma 47.18.1.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let $Z = V(\mathfrak{m}) \subset \text{Spec}(A)$. Then $E = R^0\Gamma_Z(\omega_A^\bullet)$ is an injective hull of κ and $R\Gamma_Z(\omega_A^\bullet) = E[0]$.

Proof. By Lemma 47.10.1 we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$. Thus

$$R\Gamma_Z(\omega_A^\bullet) = R\Gamma_\mathfrak{m}(\omega_A^\bullet) = \text{hocolim } R\operatorname{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet)$$

by Lemma 47.8.2. Let E' be an injective hull of the residue field. By Lemma 47.16.4 we can find isomorphisms

$$R\operatorname{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet) \cong \operatorname{Hom}_A(A/\mathfrak{m}^n, E')[0]$$

compatible with transition maps. Since $E' = \bigcup E'[\mathfrak{m}^n] = \text{colim} \operatorname{Hom}_A(A/\mathfrak{m}^n, E')$ by Lemma 47.7.3 we conclude that $E \cong E'$ and that all other cohomology groups of the complex $R\Gamma_Z(\omega_A^\bullet)$ are zero. □

0A83 **Remark 47.18.2.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with a normalized dualizing complex $\omega_A^\bullet$. By Lemma 47.18.1 above we see that $R\Gamma_Z(\omega_A^\bullet)$ is an injective hull of the residue field placed in degree 0. In fact, this gives a "construction" or "realization" of the injective hull which is slightly more canonical than just picking any old injective hull. Namely, a normalized dualizing complex is unique up to isomorphism, with group of automorphisms the group of units of A, whereas an injective hull of κ is unique up to isomorphism, with group of automorphisms the group of units of the completion $A^\wedge$ of A with respect to $\mathfrak{m}$.

Here is the main result of this section.

0A84 **Theorem 47.18.3.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $Z = V(\mathfrak{m}) \subset \text{Spec}(A)$. Denote $^\wedge$ derived completion with respect to $\mathfrak{m}$. Then

$$R\operatorname{Hom}_A(K, \omega_A^\bullet)^\wedge \cong R\operatorname{Hom}_A(R\Gamma_Z(K), E[0])$$

for K in $D(A)$.

Proof. Observe that $E[0] \cong R\Gamma_Z(\omega_A^\bullet)$ by Lemma 47.18.1. By More on Algebra, Lemma 15.91.13 completion on the left hand side goes inside. Thus we have to prove

$$R\operatorname{Hom}_A(K^\wedge, (\omega_A^\bullet)^\wedge) = R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(\omega_A^\bullet))$$

This follows from the equivalence between $D_{comp}(A, \mathfrak{m})$ and $D_{\mathfrak{m}^\infty\text{-torsion}}(A)$ given in Proposition 47.12.2. More precisely, it is a special case of Lemma 47.12.3. □

Here is a special case of the theorem above.

0AAK **Lemma 47.18.4.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $K \in D_{\text{Coh}}(A)$. Then

$$\operatorname{Ext}_A^{-i}(K, \omega_A^\bullet)^\wedge = \operatorname{Hom}_A(H^i_\mathfrak{m}(K), E)$$

where $^\wedge$ denotes $\mathfrak{m}$-adic completion.

Proof. By Lemma 47.15.3 we see that $R\operatorname{Hom}_A(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. It follows that the cohomology modules of the derived completion of $R\operatorname{Hom}_A(K, \omega_A^\bullet)$ are equal to the usual completions $\operatorname{Ext}^i_A(K, \omega_A^\bullet)^\wedge$ by More on Algebra, Lemma 15.94.4. On the other hand, we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$ for $Z = V(\mathfrak{m})$ by Lemma 47.10.1. Moreover, the functor $\operatorname{Hom}_A(-, E)$ is exact hence factors through cohomology. Hence the lemma is consequence of Theorem 47.18.3. □

47.19. Dualizing modules

0DW3 If $(A, \mathfrak{m}, \kappa)$ is a Noetherian local ring and $\omega_A^\bullet$ is a normalized dualizing complex, then we say the module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$, described in Lemma 47.17.5, is a *dualizing module* for A. This module is a canonical module of A. It seems generally agreed upon to define a *canonical module* for a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ to be a finite A-module K such that

$$\operatorname{Hom}_A(K, E) \cong H^{\dim(A)}_\mathfrak{m}(A)$$

where E is an injective hull of the residue field. A dualizing module is canonical because

$$\operatorname{Hom}_A(H^{\dim(A)}_\mathfrak{m}(A), E) = (\omega_A)^\wedge$$

by Lemma 47.18.4 and hence applying $\operatorname{Hom}_A(-, E)$ we get

$$\begin{aligned}
\operatorname{Hom}_A(\omega_A, E) &= \operatorname{Hom}_A((\omega_A)^\wedge, E) \\
&= \operatorname{Hom}_A(\operatorname{Hom}_A(H^{\dim(A)}_\mathfrak{m}(A), E), E) \\
&= H^{\dim(A)}_\mathfrak{m}(A)
\end{aligned}$$

the first equality because E is $\mathfrak{m}$-power torsion, the second by the above, and the third by Matlis duality (Proposition 47.7.8). The utility of the definition of a canonical module given above lies in the fact that it makes sense even if A does not have a dualizing complex.

47.20. Cohen-Macaulay rings

0DW4 Cohen-Macaulay modules and rings were studied in Algebra, Sections 10.103 and 10.104.

0AWR **Lemma 47.20.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Then $\operatorname{depth}(A)$ is equal to the smallest integer $\delta \geq 0$ such that $H^{-\delta}(\omega_A^\bullet) \neq 0$.*

Proof. This follows immediately from Lemma 47.16.5. Here are two other ways to see that it is true.

First alternative. By Nakayama's lemma we see that δ is the smallest integer such that $\operatorname{Hom}_A(H^{-\delta}(\omega_A^\bullet), \kappa) \neq 0$. In other words, it is the smallest integer such that $\operatorname{Ext}^{-\delta}_A(\omega_A^\bullet, \kappa)$ is nonzero. Using Lemma 47.15.3 and the fact that $\omega_A^\bullet$ is normalized this is equal to the smallest integer such that $\operatorname{Ext}^\delta_A(\kappa, A)$ is nonzero. This is equal to the depth of A by Algebra, Lemma 10.72.5.

Second alternative. By the local duality theorem (in the form of Lemma 47.18.4) δ is the smallest integer such that $H^\delta_\mathfrak{m}(A)$ is nonzero. This is equal to the depth of A by Lemma 47.11.1. □

0AWS **Lemma 47.20.2.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$ and dualizing module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$. The following are equivalent

(1) A is Cohen-Macaulay,
(2) $\omega_A^\bullet$ is concentrated in a single degree, and
(3) $\omega_A^\bullet = \omega_A[\dim(A)]$.

In this case ω_A is a maximal Cohen-Macaulay module.

Proof. Follows immediately from Lemma 47.16.7. □

0DW5 **Lemma 47.20.3.** Let A be a Noetherian ring. If there exists a finite A-module ω_A such that $\omega_A[0]$ is a dualizing complex, then A is Cohen-Macaulay.

Proof. We may replace A by the localization at a prime (Lemma 47.15.6 and Algebra, Definition 10.104.6). In this case the result follows immediately from Lemma 47.20.2. □

0EHS **Lemma 47.20.4.** Let A be a Noetherian ring with dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Then

$$U = \{\mathfrak{p} \in \text{Spec}(A) \mid M_\mathfrak{p} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\text{Spec}(A)$ whose intersection with $\text{Supp}(M)$ is dense.

Proof. If $\mathfrak{p}$ is a generic point of $\text{Supp}(M)$, then $\text{depth}(M_\mathfrak{p}) = \dim(M_\mathfrak{p}) = 0$ and hence $\mathfrak{p} \in U$. This proves denseness. If $\mathfrak{p} \in U$, then we see that

$$R\,\text{Hom}_A(M, \omega_A^\bullet)_\mathfrak{p} = R\,\text{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p})$$

has a unique nonzero cohomology module, say in degree i_0, by Lemma 47.16.7. Since $R\,\text{Hom}_A(M, \omega_A^\bullet)$ has only a finite number of nonzero cohomology modules H^i and since each of these is a finite A-module, we can find an $f \in A$, $f \notin \mathfrak{p}$ such that $(H^i)_f = 0$ for $i \neq i_0$. Then $R\,\text{Hom}_A(M, \omega_A^\bullet)_f$ has a unique nonzero cohomology module and reversing the arguments just given we find that $D(f) \subset U$. □

0EHT **Lemma 47.20.5.** Let A be a Noetherian ring. If A has a dualizing complex $\omega_A^\bullet$, then $\{\mathfrak{p} \in \text{Spec}(A) \mid A_\mathfrak{p} \text{ is Cohen-Macaulay}\}$ is a dense open subset of $\text{Spec}(A)$.

Proof. Immediate consequence of Lemma 47.20.4 and the definitions. □

47.21. Gorenstein rings

0DW6 So far, the only explicit dualizing complex we've seen is κ on κ for a field κ, see proof of Lemma 47.15.12. By Proposition 47.15.11 this means that any finite type algebra over a field has a dualizing complex. However, it turns out that there are Noetherian (local) rings which do not have a dualizing complex. Namely, we have seen that a ring which has a dualizing complex is universally catenary (Lemma 47.17.4) but there are examples of Noetherian local rings which are not catenary, see Examples, Section 110.18.

Nonetheless many rings in algebraic geometry have dualizing complexes simply because they are quotients of Gorenstein rings. This condition is in fact both necessary and sufficient. That is: a Noetherian ring has a dualizing complex if and only if it is a quotient of a finite dimensional Gorenstein ring. This is Sharp's conjecture ([Sha79]) which can be found as [Kaw02, Corollary 1.4] in the literature. Returning to our current topic, here is the definition of Gorenstein rings.

0DW7 **Definition 47.21.1.** Gorenstein rings.

(1) Let A be a Noetherian local ring. We say A is Gorenstein if $A[0]$ is a dualizing complex for A.
(2) Let A be a Noetherian ring. We say A is Gorenstein if $A_{\mathfrak p}$ is Gorenstein for every prime $\mathfrak p$ of A.

This definition makes sense, because if $A[0]$ is a dualizing complex for A, then $S^{-1}A[0]$ is a dualizing complex for $S^{-1}A$ by Lemma 47.15.6. We will see later that a finite dimensional Noetherian ring is Gorenstein if it has finite injective dimension as a module over itself.

0DW8 **Lemma 47.21.2.** A Gorenstein ring is Cohen-Macaulay.

Proof. Follows from Lemma 47.20.2. □

An example of a Gorenstein ring is a regular ring.

0AWX **Lemma 47.21.3.** A regular local ring is Gorenstein. A regular ring is Gorenstein.

Proof. Let A be a regular ring of finite dimension d. Then A has finite global dimension d, see Algebra, Lemma 10.110.8. Hence $\mathrm{Ext}^{d+1}_A(M, A) = 0$ for all A-modules M, see Algebra, Lemma 10.109.8. Thus A has finite injective dimension as an A-module by More on Algebra, Lemma 15.69.2. It follows that $A[0]$ is a dualizing complex, hence A is Gorenstein by the remark following the definition. □

0DW9 **Lemma 47.21.4.** Let A be a Noetherian ring.

(1) If A has a dualizing complex $\omega_A^\bullet$, then
    (a) A is Gorenstein $\Leftrightarrow$ $\omega_A^\bullet$ is an invertible object of $D(A)$,
    (b) $A_{\mathfrak p}$ is Gorenstein $\Leftrightarrow$ $(\omega_A^\bullet)_{\mathfrak p}$ is an invertible object of $D(A_{\mathfrak p})$,
    (c) $\{\mathfrak p \in \mathrm{Spec}(A) \mid A_{\mathfrak p} \text{ is Gorenstein}\}$ is an open subset.
(2) If A is Gorenstein, then A has a dualizing complex if and only if $A[0]$ is a dualizing complex.

Proof. For invertible objects of $D(A)$, see More on Algebra, Lemma 15.126.4 and the discussion in Section 47.15.

By Lemma 47.15.6 for every $\mathfrak p$ the complex $(\omega_A^\bullet)_{\mathfrak p}$ is a dualizing complex over $A_{\mathfrak p}$. By definition and uniqueness of dualizing complexes (Lemma 47.15.5) we see that (1)(b) holds.

To see (1)(c) assume that $A_{\mathfrak p}$ is Gorenstein. Let n_x be the unique integer such that $H^{n_x}((\omega_A^\bullet)_{\mathfrak p})$ is nonzero and isomorphic to $A_{\mathfrak p}$. Since $\omega_A^\bullet$ is in $D^b_{\mathrm{Coh}}(A)$ there are finitely many nonzero finite A-modules $H^i(\omega_A^\bullet)$. Thus there exists some $f \in A$, $f \notin \mathfrak p$ such that only $H^{n_x}((\omega_A^\bullet)_f)$ is nonzero and generated by 1 element over A_f. Since dualizing complexes are faithful (by definition) we conclude that $A_f \cong H^{n_x}((\omega_A^\bullet)_f)$. In this way we see that $A_{\mathfrak q}$ is Gorenstein for every $\mathfrak q \in D(f)$. This proves that the set in (1)(c) is open.

Proof of (1)(a). The implication $\Leftarrow$ follows from (1)(b). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $A_{\mathfrak p}$ is Gorenstein, then for some $f \in A$, $f \notin \mathfrak p$ the complex $(\omega_A^\bullet)_f$ has only one nonzero cohomology module which is invertible.

If $A[0]$ is a dualizing complex then A is Gorenstein by part (1). Conversely, we see that part (1) shows that $\omega_A^\bullet$ is locally isomorphic to a shift of A. Since being a dualizing complex is local (Lemma 47.15.7) the result is clear. □

0BJI Lemma 47.21.5. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then A is Gorenstein if and only if $\text{Ext}^i_A(\kappa, A)$ is zero for $i \gg 0$.

Proof. Observe that $A[0]$ is a dualizing complex for A if and only if A has finite injective dimension as an A-module (follows immediately from Definition 47.15.1). Thus the lemma follows from More on Algebra, Lemma 15.69.7. □

0BJJ Lemma 47.21.6. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then A is Gorenstein if and only if B is Gorenstein.

Proof. If A is Gorenstein, then B is Gorenstein by Lemma 47.16.10. Conversely, suppose that B is Gorenstein. Then $\text{Ext}^i_B(\kappa, B)$ is zero for $i \gg 0$ (Lemma 47.21.5). Recall that $R\text{Hom}(B, -) : D(A) \to D(B)$ is a right adjoint to restriction (Lemma 47.13.1). Hence

$$R\text{Hom}_A(\kappa, A) = R\text{Hom}_B(\kappa, R\text{Hom}(B, A)) = R\text{Hom}_B(\kappa, B[1])$$

The final equality by direct computation or by Lemma 47.13.10. Thus we see that $\text{Ext}^i_A(\kappa, A)$ is zero for $i \gg 0$ and A is Gorenstein (Lemma 47.21.5). □

0DWA Lemma 47.21.7. If $A \to B$ is a local complete intersection homomorphism of rings and A is a Noetherian Gorenstein ring, then B is a Gorenstein ring.

Proof. By More on Algebra, Definition 15.33.2 we can write $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul-regular ideal. Observe that a polynomial ring over a Gorenstein ring A is Gorenstein: reduce to A local and then use Lemmas 47.15.10 and 47.21.4. A Koszul-regular ideal is by definition locally generated by a Koszul-regular sequence, see More on Algebra, Section 15.32. Looking at local rings of $A[x_1, \ldots, x_n]$ we see it suffices to show: if R is a Noetherian local Gorenstein ring and $f_1, \ldots, f_c \in \mathfrak{m}_R$ is a Koszul regular sequence, then $R/(f_1, \ldots, f_c)$ is Gorenstein. This follows from Lemma 47.21.6 and the fact that a Koszul regular sequence in R is just a regular sequence (More on Algebra, Lemma 15.30.7). □

0BJL Lemma 47.21.8. Let $A \to B$ be a flat local homomorphism of Noetherian local rings. The following are equivalent

(1) B is Gorenstein, and
(2) A and $B/\mathfrak{m}_A B$ are Gorenstein.

Proof. Below we will use without further mention that a local Gorenstein ring has finite injective dimension as well as Lemma 47.21.5. By More on Algebra, Lemma 15.65.4 we have

$$\text{Ext}^i_A(\kappa_A, A) \otimes_A B = \text{Ext}^i_B(B/\mathfrak{m}_A B, B)$$

for all i.

Assume (2). Using that $R\text{Hom}(B/\mathfrak{m}_A B, -) : D(B) \to D(B/\mathfrak{m}_A B)$ is a right adjoint to restriction (Lemma 47.13.1) we obtain

$$R\text{Hom}_B(\kappa_B, B) = R\text{Hom}_{B/\mathfrak{m}_A B}(\kappa_B, R\text{Hom}(B/\mathfrak{m}_A B, B))$$

The cohomology modules of $R\text{Hom}(B/\mathfrak{m}_A B, B)$ are the modules $\text{Ext}^i_B(B/\mathfrak{m}_A B, B) = \text{Ext}^i_A(\kappa_A, A) \otimes_A B$. Since A is Gorenstein, we conclude only a finite number of these are nonzero and each is isomorphic to a direct sum of copies of $B/\mathfrak{m}_A B$. Hence since $B/\mathfrak{m}_A B$ is Gorenstein we conclude that $R\text{Hom}_B(B/\mathfrak{m}_B, B)$ has only a finite number of nonzero cohomology modules. Hence B is Gorenstein.

Assume (1). Since B has finite injective dimension, $\text{Ext}^i_B(B/\mathfrak{m}_A B, B)$ is 0 for $i \gg 0$. Since $A \to B$ is faithfully flat we conclude that $\text{Ext}^i_A(\kappa_A, A)$ is 0 for $i \gg 0$. We conclude that A is Gorenstein. This implies that $\text{Ext}^i_A(\kappa_A, A)$ is nonzero for exactly one i, namely for $i = \dim(A)$, and $\text{Ext}^{\dim(A)}_A(\kappa_A, A) \cong \kappa_A$ (see Lemmas 47.16.1, 47.20.2, and 47.21.2). Thus we see that $\text{Ext}^i_B(B/\mathfrak{m}_A B, B)$ is zero except for one i, namely $i = \dim(A)$ and $\text{Ext}^{\dim(A)}_B(B/\mathfrak{m}_A B, B) \cong B/\mathfrak{m}_A B$. Thus $B/\mathfrak{m}_A B$ is Gorenstein by Lemma 47.16.1. □

0EBT Lemma 47.21.9. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local Gorenstein ring of dimension d. Let E be the injective hull of κ. Then $\text{Tor}^A_i(E, \kappa)$ is zero for $i \neq d$ and $\text{Tor}^A_d(E, \kappa) = \kappa$.

Proof. Since A is Gorenstein $\omega_A^\bullet = A[d]$ is a normalized dualizing complex for A. Also E is the only nonzero cohomology module of $R\Gamma_\mathfrak{m}(\omega_A^\bullet)$ sitting in degree 0, see Lemma 47.18.1. By Lemma 47.9.5 we have

$$E \otimes_A^{\mathbf{L}} \kappa = R\Gamma_\mathfrak{m}(\omega_A^\bullet) \otimes_A^{\mathbf{L}} \kappa = R\Gamma_\mathfrak{m}(\omega_A^\bullet \otimes_A^{\mathbf{L}} \kappa) = R\Gamma_\mathfrak{m}(\kappa[d]) = \kappa[d]$$

and the lemma follows. □

47.22. The ubiquity of dualizing complexes

0DWB Many Noetherian rings have dualizing complexes.

0AWD Lemma 47.22.1. Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. If $A \to B$ is flat and $\mathfrak{m}_A B = \mathfrak{m}_B$, then $\omega_A^\bullet \otimes_A B$ is a normalized dualizing complex for B.

Proof. It is clear that $\omega_A^\bullet \otimes_A B$ is in $D^b_{Coh}(B)$. Let κ_A and κ_B be the residue fields of A and B. By More on Algebra, Lemma 15.99.2 we see that

$$R\,\text{Hom}_B(\kappa_B, \omega_A^\bullet \otimes_A B) = R\,\text{Hom}_A(\kappa_A, \omega_A^\bullet) \otimes_A B = \kappa_A[0] \otimes_A B = \kappa_B[0]$$

Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 15.69.7. Finally, we can use the same arguments to see that

$$R\,\text{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = R\,\text{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. □

0DWC Lemma 47.22.2. Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal such that $A/I = B/IB$ and such that IB is contained in the Jacobson radical of B. Let $\omega_A^\bullet$ be a dualizing complex. Then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.

Proof. It is clear that $\omega_A^\bullet \otimes_A B$ is in $D^b_{Coh}(B)$. By More on Algebra, Lemma 15.99.2 we see that

$$R\,\text{Hom}_B(K \otimes_A B, \omega_A^\bullet \otimes_A B) = R\,\text{Hom}_A(K, \omega_A^\bullet) \otimes_A B$$

for any $K \in D^b_{Coh}(A)$. For any ideal $IB \subset J \subset B$ there is a unique ideal $I \subset J' \subset A$ such that $A/J' \otimes_A B = B/J$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 15.69.6. Finally, we also have

$$R\,\text{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = R\,\text{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. □

0DWD Lemma 47.22.3. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $\omega_A^\bullet$ be a dualizing complex.

(1) $\omega_A^\bullet \otimes_A A^h$ is a dualizing complex on the henselization (A^h, I^h) of the pair (A, I),
(2) $\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex on the I-adic completion $A^\wedge$, and
(3) if A is local, then $\omega_A^\bullet \otimes_A A^h$, resp. $\omega_A^\bullet \otimes_A A^{sh}$ is a dualzing complex on the henselization, resp. strict henselization of A.

Proof. Immediate from Lemmas 47.22.1 and 47.22.2. See More on Algebra, Sections 15.11, 15.43, and 15.45 and Algebra, Sections 10.96 and 10.97 for information on completions and henselizations. □

0BFR Lemma 47.22.4. The following types of rings have a dualizing complex:
(1) fields,
(2) Noetherian complete local rings,
(3) $\mathbf{Z}$,
(4) Dedekind domains,
(5) any ring which is obtained from one of the rings above by taking an algebra essentially of finite type, or by taking an ideal-adic completion, or by taking a henselization, or by taking a strict henselization.

Proof. Part (5) follows from Proposition 47.15.11 and Lemma 47.22.3. By Lemma 47.21.3 a regular local ring has a dualizing complex. A complete Noetherian local ring is the quotient of a regular local ring by the Cohen structure theorem (Algebra, Theorem 10.160.8). Let A be a Dedekind domain. Then every ideal I is a finite projective A-module (follows from Algebra, Lemma 10.78.2 and the fact that the local rings of A are discrete valuation ring and hence PIDs). Thus every A-module has finite injective dimension at most 1 by More on Algebra, Lemma 15.69.2. It follows easily that $A[0]$ is a dualizing complex. □

47.23. Formal fibres

0BJM This section is a continuation of More on Algebra, Section 15.51. There we saw there is a (fairly) good theory of Noetherian rings A whose local rings have Cohen-Macaulay formal fibres. Namely, we proved (1) it suffices to check the formal fibres of localizations at maximal ideals are Cohen-Macaulay, (2) the property is inherited by rings of finite type over A, (3) the fibres of $A \to A^\wedge$ are Cohen-Macaulay for any completion $A^\wedge$ of A, and (4) the property is inherited by henselizations of A. See More on Algebra, Lemma 15.51.4, Proposition 15.51.5, Lemma 15.51.6, and Lemma 15.51.7. Similarly, for Noetherian rings whose local rings have formal fibres which are geometrically reduced, geometrically normal, (S_n), and geometrically (R_n). In this section we will see that the same is true for Noetherian rings whose local rings have formal fibres which are Gorenstein or local complete intersections. This is relevant to this chapter because a Noetherian ring which has a dualizing complex is an example.

0BJN Lemma 47.23.1. Properties (A), (B), (C), (D), and (E) of More on Algebra, Section 15.51 hold for $P(k \to R) =$"R is a Gorenstein ring".

Proof. Since we already know the result holds for Cohen-Macaulay instead of Gorenstein, we may in each step assume the ring we have is Cohen-Macaulay. This is not particularly helpful for the proof, but psychologically may be useful.

Part (A). Let K/k be a finitely generated field extension. Let R be a Gorenstein k-algebra. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$

over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.158.11. Then $R \to R \otimes_k A$ is a relative global complete intersection. Hence $R \otimes_k A$ is Gorenstein by Lemma 47.21.7. Thus $R \otimes_k K$ is too as a localization.

Proof of (B). This is clear because a ring is Gorenstein if and only if all of its local rings are Gorenstein.

Part (C). Let $A \to B \to C$ be flat maps of Noetherian rings. Assume the fibres of $A \to B$ are Gorenstein and $B \to C$ is regular. We have to show the fibres of $A \to C$ are Gorenstein. Clearly, we may assume $A = k$ is a field. Then we may assume that $B \to C$ is a regular local homomorphism of Noetherian local rings. Then B is Gorenstein and $C/\mathfrak{m}_B C$ is regular, in particular Gorenstein (Lemma 47.21.3). Then C is Gorenstein by Lemma 47.21.8.

Part (D). This follows from Lemma 47.21.8. Part (E) is immediate as the condition does not refer to the ground field. $\square$

0AWY Lemma 47.23.2. Let A be a Noetherian local ring. If A has a dualizing complex, then the formal fibres of A are Gorenstein.

Proof. Let $\mathfrak{p}$ be a prime of A. The formal fibre of A at $\mathfrak{p}$ is isomorphic to the formal fibre of $A/\mathfrak{p}$ at (0). The quotient $A/\mathfrak{p}$ has a dualizing complex (Lemma 47.15.9). Thus it suffices to check the statement when A is a local domain and $\mathfrak{p} = (0)$. Let $\omega_A^\bullet$ be a dualizing complex for A. Then $\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex for the completion $A^\wedge$ (Lemma 47.22.1). Then $\omega_A^\bullet \otimes_A K$ is a dualizing complex for the fraction field K of A (Lemma 47.15.6). Hence $\omega_A^\bullet \otimes_A K$ is isomorphic ot $K[n]$ for some $n \in \mathbf{Z}$. Similarly, we conclude a dualizing complex for the formal fibre $A^\wedge \otimes_A K$ is

$$\omega_A^\bullet \otimes_A A^\wedge \otimes_{A^\wedge} (A^\wedge \otimes_A K) = (\omega_A^\bullet \otimes_A K) \otimes_K (A^\wedge \otimes_A K) \cong (A^\wedge \otimes_A K)[n]$$

as desired. $\square$

Here is the verification promised in Divided Power Algebra, Remark 23.9.3.

0BJP Lemma 47.23.3. Properties (A), (B), (C), (D), and (E) of More on Algebra, Section 15.51 hold for $P(k \to R) =$"R is a local complete intersection". See Divided Power Algebra, Definition 23.8.5.

Proof. Part (A). Let K/k be a finitely generated field extension. Let R be a k-algebra which is a local complete intersection. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.158.11. Then $R \to R \otimes_k A$ is a relative global complete intersection. It follows that $R \otimes_k A$ is a local complete intersection by Divided Power Algebra, Lemma 23.8.9.

Proof of (B). This is clear because a ring is a local complete intersection if and only if all of its local rings are complete intersections.

Part (C). Let $A \to B \to C$ be flat maps of Noetherian rings. Assume the fibres of $A \to B$ are local complete intersections and $B \to C$ is regular. We have to show the fibres of $A \to C$ are local complete intersections. Clearly, we may assume $A = k$ is a field. Then we may assume that $B \to C$ is a regular local homomorphism of Noetherian local rings. Then B is a complete intersection and $C/\mathfrak{m}_B C$ is regular, in particular a complete intersection (by definition). Then C is a complete intersection by Divided Power Algebra, Lemma 23.8.9.

Part (D). This follows by the same arguments as in (C) from the other implication in Divided Power Algebra, Lemma 23.8.9. Part (E) is immediate as the condition does not refer to the ground field. □

47.24. Upper shriek algebraically

0BZI For a finite type homomorphism $R \to A$ of Noetherian rings we will construct a functor $\varphi^! : D(R) \to D(A)$ well defined up to nonunique isomorphism which as we will see in Duality for Schemes, Remark 48.17.5 agrees up to isomorphism with the upper shriek functors one encounters in the duality theory for schemes. To motivate the construction we mention two additional properties:

(1) $\varphi^!$ sends a dualizing complex for R (if it exists) to a dualizing complex for A, and
(2) $\omega_{A/R}^\bullet = \varphi^!(R)$ is a kind of relative dualizing complex: it lies in $D^b_{Coh}(A)$ and restricts to a dualizing complex on the fibres provided $R \to A$ is flat.

These statemens are Lemmas 47.24.3 and 47.25.2.

Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. We will define a functor $\varphi^! : D(R) \to D(A)$ in the following way

(1) If $\varphi : R \to A$ is surjective we set $\varphi^!(K) = R\operatorname{Hom}(A, K)$. Here we use the functor $R\operatorname{Hom}(A, -) : D(R) \to D(A)$ of Section 47.13, and
(2) in general we choose a surjection $\psi : P \to A$ with $P = R[x_1, \ldots, x_n]$ and we set $\varphi^!(K) = \psi^!(K \otimes_R^{\mathbf{L}} P)[n]$. Here we use the functor $- \otimes_R^{\mathbf{L}} P : D(R) \to D(P)$ of More on Algebra, Section 15.60.

Note the shift $[n]$ by the number of variables in the polynomial ring. This construction is not canonical and the functor $\varphi^!$ will only be well defined up to a (nonunique) isomorphism of functors[1].

0BZJ **Lemma 47.24.1.** Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. The functor $\varphi^!$ is well defined up to isomorphism.

Proof. Suppose that $\psi_1 : P_1 = R[x_1, \ldots, x_n] \to A$ and $\psi_2 : P_2 = R[y_1, \ldots, y_m] \to A$ are two surjections from polynomial rings onto A. Then we get a commutative diagram

$$\begin{tikzcd}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\
R[y_1, \ldots, y_m] \arrow[r] & A
\end{tikzcd}$$

where f_j and g_i are chosen such that $\psi_1(f_j) = \psi_2(y_j)$ and $\psi_2(g_i) = \psi_1(x_i)$. By symmetry it suffices to prove the functors defined using $P \to A$ and $P[y_1, \ldots, y_m] \to A$ are isomorphic. By induction we may assume $m = 1$. This reduces us to the case discussed in the next paragraph.

Here $\psi : P \to A$ is given and $\chi : P[y] \to A$ induces ψ on P. Write $Q = P[y]$. Choose $g \in P$ with $\psi(g) = \chi(y)$. Denote $\pi : Q \to P$ the P-algebra map with $\pi(y) = g$.

[1] It is possible to make the construction canonical: use $\Omega^n_{P/R}[n]$ instead of $P[n]$ in the construction and use this in Lemma 47.24.1. The material in this section becomes a lot more involved if one wants to do this.

Then $\chi = \psi \circ \pi$ and hence $\chi^! = \psi^! \circ \pi^!$ as both are adjoint to the restriction functor $D(A) \to D(Q)$ by the material in Section 47.13. Thus

$$\chi^! \left(K \otimes_R^{\mathbf{L}} Q\right)[n+1] = \psi^! \left(\pi^! \left(K \otimes_R^{\mathbf{L}} Q\right)[1]\right)[n]$$

Hence it suffices to show that $\pi^!(K \otimes_R^{\mathbf{L}} Q[1]) = K \otimes_R^{\mathbf{L}} P$ Thus it suffices to show that the functor $\pi^!(-) : D(Q) \to D(P)$ is isomorphic to $K \mapsto K \otimes_Q^{\mathbf{L}} P[-1]$. This follows from Lemma 47.13.10. $\square$

0BZK Lemma 47.24.2. Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings.

(1) $\varphi^!$ maps $D^+(R)$ into $D^+(A)$ and $D^+_{\text{Coh}}(R)$ into $D^+_{\text{Coh}}(A)$.
(2) if φ is perfect, then $\varphi^!$ maps $D^-(R)$ into $D^-(A)$, $D^-_{\text{Coh}}(R)$ into $D^-_{\text{Coh}}(A)$, and $D^b_{\text{Coh}}(R)$ into $D^b_{\text{Coh}}(A)$.

Proof. Choose a factorization $R \to P \to A$ as in the definition of $\varphi^!$. The functor $-\otimes_R^{\mathbf{L}} : D(R) \to D(P)$ preserves the subcategories $D^+, D^+_{\text{Coh}}, D^-, D^-_{\text{Coh}}, D^b_{\text{Coh}}$. The functor $R\operatorname{Hom}(A, -) : D(P) \to D(A)$ preserves D^+ and D^+_{Coh} by Lemma 47.13.4. If $R \to A$ is perfect, then A is perfect as a P-module, see More on Algebra, Lemma 15.82.2. Recall that the restriction of $R\operatorname{Hom}(A, K)$ to $D(P)$ is $R\operatorname{Hom}_P(A, K)$. By More on Algebra, Lemma 15.74.15 we have $R\operatorname{Hom}_P(A, K) = E \otimes_P^{\mathbf{L}} K$ for some perfect $E \in D(P)$. Since we can represent E by a finite complex of finite projective P-modules it is clear that $R\operatorname{Hom}_P(A, K)$ is in $D^-(P), D^-_{\text{Coh}}(P), D^b_{\text{Coh}}(P)$ as soon as K is. Since the restriction functor $D(A) \to D(P)$ reflects these subcategories, the proof is complete. $\square$

0BZL Lemma 47.24.3. Let φ be a finite type homomorphism of Noetherian rings. If $\omega_R^\bullet$ is a dualizing complex for R, then $\varphi^!(\omega_R^\bullet)$ is a dualizing complex for A.

Proof. Follows from Lemmas 47.15.10 and 47.15.9, $\square$

0BZN Lemma 47.24.4. Let $R \to R'$ be a flat homomorphism of Noetherian rings. Let $\varphi : R \to A$ be a finite type ring map. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial maps

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' \longrightarrow (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$ which are isomorphisms for $K \in D^+(R)$.

Proof. Choose a factorization $R \to P \to A$ where P is a polynomial ring over R. This gives a corresponding factorization $R' \to P' \to A'$ by base change. Since we have $(K \otimes_R^{\mathbf{L}} P) \otimes_P^{\mathbf{L}} P' = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} P'$ by More on Algebra, Lemma 15.60.5 it suffices to construct maps

$$R\operatorname{Hom}(A, K \otimes_R^{\mathbf{L}} P[n]) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}(A', (K \otimes_R^{\mathbf{L}} P[n]) \otimes_P^{\mathbf{L}} P')$$

functorial in K. For this we use the map (47.14.0.1) constructed in Section 47.14 for P, A, P', A'. The map is an isomorphism for $K \in D^+(R)$ by Lemma 47.14.2. $\square$

0BZR Lemma 47.24.5. Let $R \to R'$ be a homomorphism of Noetherian rings. Let $\varphi : R \to A$ be a perfect ring map (More on Algebra, Definition 15.82.1) such that R' and A are tor independent over R. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial isomorphism

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' = (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$.

Proof. We may choose a factorization $R \to P \to A$ where P is a polynomial ring over R such that A is a perfect P-module, see More on Algebra, Lemma 15.82.2. This gives a corresponding factorization $R' \to P' \to A'$ by base change. Since we have $(K \otimes_R^{\mathbf{L}} P) \otimes_P^{\mathbf{L}} P' = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} P'$ by More on Algebra, Lemma 15.60.5 it suffices to construct maps

$$R\operatorname{Hom}(A, K \otimes_R^{\mathbf{L}} P[n]) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}(A', (K \otimes_R^{\mathbf{L}} P[n]) \otimes_P^{\mathbf{L}} P')$$

functorial in K. We have

$$A \otimes_P^{\mathbf{L}} P' = A \otimes_R^{\mathbf{L}} R' = A'$$

The first equality by More on Algebra, Lemma 15.61.2 applied to R, R', P, P'. The second equality because A and R' are tor independent over R. Hence A and P' are tor independent over P and we can use the map (47.14.0.1) constructed in Section 47.14 for P, A, P', A' get the desired arrow. By Lemma 47.14.3 to finish the proof it suffices to prove that A is a perfect P-module which we saw above. □

0BZS Lemma 47.24.6. Let $R \to R'$ be a homomorphism of Noetherian rings. Let $\varphi : R \to A$ be flat of finite type. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial isomorphism

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' = (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$.

Proof. Special case of Lemma 47.24.5 by More on Algebra, Lemma 15.82.4. □

0BZT Lemma 47.24.7. Let $A \xrightarrow{a} B \xrightarrow{b} C$ be finite type homomorphisms of Noetherian rings. Then there is a transformation of functors $b^! \circ a^! \to (b \circ a)^!$ which is an isomorphism on $D^+(A)$.

Proof. Choose a polynomial ring $P = A[x_1, \ldots, x_n]$ over A and a surjection $P \to B$. Choose elements $c_1, \ldots, c_m \in C$ generating C over B. Set $Q = P[y_1, \ldots, y_m]$ and denote $Q' = Q \otimes_P B = B[y_1, \ldots, y_m]$. Let $\chi : Q' \to C$ be the surjection sending y_j to c_j. Picture

$$\begin{tikzcd} & Q \arrow[r, "\psi'"'] & Q' \arrow[r, "\chi"'] & C \\ A \arrow[r] & P \arrow[u] \arrow[r, "\psi"] & B \arrow[u] \end{tikzcd}$$

By Lemma 47.14.2 for $M \in D(P)$ we have an arrow $\psi^!(M) \otimes_B^{\mathbf{L}} Q' \to (\psi')^!(M \otimes_P^{\mathbf{L}} Q)$ which is an isomorphism whenever M is bounded below. Also we have $\chi^! \circ (\psi')^! = (\chi \circ \psi')^!$ as both functors are adjoint to the restriction functor $D(C) \to D(Q)$ by Section 47.13. Then we see

$$\begin{aligned} b^!(a^!(K)) &= \chi^!(\psi^!(K \otimes_A^{\mathbf{L}} P)[n] \otimes_B^{\mathbf{L}} Q)[m] \\ &\to \chi^!((\psi')^!(K \otimes_A^{\mathbf{L}} P \otimes_P^{\mathbf{L}} Q))[n+m] \\ &= (\chi \circ \psi')^!(K \otimes_A^{\mathbf{L}} Q)[n+m] \\ &= (b \circ a)^!(K) \end{aligned}$$

where we have used in addition to the above More on Algebra, Lemma 15.60.5. □

0C0G Lemma 47.24.8. Let $\varphi : R \to A$ be a finite map of Noetherian rings. Then $\varphi^!$ is isomorphic to the functor $R\operatorname{Hom}(A, -) : D(R) \to D(A)$ from Section 47.13.

Proof. Suppose that A is generated by $n > 1$ elements over R. Then can factor $R \to A$ as a composition of two finite ring maps where in both steps the number of generators is $< n$. Since we have Lemma 47.24.7 and Lemma 47.13.2 we conclude that it suffices to prove the lemma when A is generated by one element over R. Since A is finite over R, it follows that A is a quotient of $B = R[x]/(f)$ where f is a monic polynomial in x (Algebra, Lemma 10.36.3). Again using the lemmas on composition and the fact that we have agreement for surjections by definition, we conclude that it suffices to prove the lemma for $R \to B = R[x]/(f)$. In this case, the functor $\varphi^!$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} B$; you prove this by using Lemma 47.13.10 for the map $R[x] \to B$ (note that the shift in the definition of $\varphi^!$ and in the lemma add up to zero). For the functor $R\operatorname{Hom}(B, -) : D(R) \to D(B)$ we can use Lemma 47.13.9 to see that it suffices to show $\operatorname{Hom}_R(B, R) \cong B$ as B-modules. Suppose that f has degree d. Then an R-basis for B is given by $1, x, \ldots, x^{d-1}$. Let $\delta_i : B \to R$, $i = 0, \ldots, d-1$ be the R-linear map which picks off the coefficient of x^i with respect to the given basis. Then $\delta_0, \ldots, \delta_{d-1}$ is a basis for $\operatorname{Hom}_R(B, R)$. Finally, for $0 \leq i \leq d-1$ a computation shows that

$$x^i \delta_{d-1} = \delta_{d-1-i} + b_1 \delta_{d-i} + \ldots + b_i \delta_{d-1}$$

for some $c_1, \ldots, c_d \in R$[2]. Hence $\operatorname{Hom}_R(B, R)$ is a principal B-module with generator δ_{d-1}. By looking at ranks we conclude that it is a rank 1 free B-module. □

0C0H Lemma 47.24.9. Let R be a Noetherian ring and let $f \in R$. If φ denotes the map $R \to R_f$, then $\varphi^!$ is isomorphic to $- \otimes_R^{\mathbf{L}} R_f$. More generally, if $\varphi : R \to R'$ is a map such that $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is an open immersion, then $\varphi^!$ is isomorphic to $- \otimes_R^{\mathbf{L}} R'$.

Proof. Choose the presentation $R \to R[x] \to R[x]/(fx - 1) = R_f$ and observe that $fx - 1$ is a nonzerodivisor in $R[x]$. Thus we can apply using Lemma 47.13.10 to compute the functor $\varphi^!$. Details omitted; note that the shift in the definition of $\varphi^!$ and in the lemma add up to zero.

In the general case note that $R' \otimes_R R' = R'$. Hence the result follows from the base change results above. Either Lemma 47.24.4 or Lemma 47.24.5 will do. □

0BZU Lemma 47.24.10. Let $\varphi : R \to A$ be a perfect homomorphism of Noetherian rings (for example φ is flat of finite type). Then $\varphi^!(K) = K \otimes_R^{\mathbf{L}} \varphi^!(R)$ for $K \in D(R)$.

Proof. (The parenthetical statement follows from More on Algebra, Lemma 15.82.4.) We can choose a factorization $R \to P \to A$ where P is a polynomial ring in n variables over R and then A is a perfect P-module, see More on Algebra, Lemma 15.82.2. Recall that $\varphi^!(K) = R\operatorname{Hom}(A, K \otimes_R^{\mathbf{L}} P[n])$. Thus the result follows from Lemma 47.13.9 and More on Algebra, Lemma 15.60.5. □

0E9L Lemma 47.24.11. Let $\varphi : A \to B$ be a finite type homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex for A. Set $\omega_B^\bullet = \varphi^!(\omega_A^\bullet)$. Denote $D_A(K) = R\operatorname{Hom}_A(K, \omega_A^\bullet)$ for $K \in D_{Coh}(A)$ and $D_B(L) = R\operatorname{Hom}_B(L, \omega_B^\bullet)$ for $L \in D_{Coh}(B)$. Then there is a functorial isomorphism

$$\varphi^!(K) = D_B(D_A(K) \otimes_A^{\mathbf{L}} B)$$

for $K \in D_{Coh}(A)$.

[2]If $f = x^d + a_1 x^{d-1} + \ldots + a_d$, then $c_1 = -a_1$, $c_2 = a_1^2 - a_2$, $c_3 = -a_1^3 + 2a_1 a_2 - a_3$, etc.

Proof. Observe that $\omega_B^\bullet$ is a dualizing complex for B by Lemma 47.24.3. Let $A \to B \to C$ be finite type homomorphisms of Noetherian rings. If the lemma holds for $A \to B$ and $B \to C$, then the lemma holds for $A \to C$. This follows from Lemma 47.24.7 and the fact that $D_B \circ D_B \cong \text{id}$ by Lemma 47.15.3. Thus it suffices to prove the lemma in case $A \to B$ is a surjection and in the case where B is a polynomial ring over A.

Assume $B = A[x_1, \ldots, x_n]$. Since $D_A \circ D_A \cong \text{id}$, it suffices to prove $D_B(K \otimes_A B) \cong D_A(K) \otimes_A B[n]$ for K in $D_{\text{Coh}}(A)$. Choose a bounded complex $I^\bullet$ of injectives representing $\omega_A^\bullet$. Choose a quasi-isomorphism $I^\bullet \otimes_A B \to J^\bullet$ where $J^\bullet$ is a bounded complex of B-modules. Given a complex $K^\bullet$ of A-modules, consider the obvious map of complexes

$$\text{Hom}^\bullet(K^\bullet, I^\bullet) \otimes_A B[n] \longrightarrow \text{Hom}^\bullet(K^\bullet \otimes_A B, J^\bullet[n])$$

The left hand side represents $D_A(K) \otimes_A B[n]$ and the right hand side represents $D_B(K \otimes_A B)$. Thus it suffices to prove this map is a quasi-isomorphism if the cohomology modules of $K^\bullet$ are finite A-modules. Observe that the cohomology of the complex in degree r (on either side) only depends on finitely many of the K^i. Thus we may replace $K^\bullet$ by a truncation, i.e., we may assume $K^\bullet$ represents an object of $D^-_{\text{Coh}}(A)$. Then $K^\bullet$ is quasi-isomorphic to a bounded above complex of finite free A-modules. Therefore we may assume $K^\bullet$ is a bounded above complex of finite free A-modules. In this case it is easy to that the displayed map is an isomorphism of complexes which finishes the proof in this case.

Assume that $A \to B$ is surjective. Denote $i_* : D(B) \to D(A)$ the restriction functor and recall that $\varphi^!(-) = R\,\text{Hom}(A, -)$ is a right adjoint to i_* (Lemma 47.13.1). For $F \in D(B)$ we have

$$\begin{aligned} \text{Hom}_B(F, D_B(D_A(K) \otimes_A^{\mathbf{L}} B)) &= \text{Hom}_B((D_A(K) \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} F, \omega_B^\bullet) \\ &= \text{Hom}_A(D_A(K) \otimes_A^{\mathbf{L}} i_*F, \omega_A^\bullet) \\ &= \text{Hom}_A(i_*F, D_A(D_A(K))) \\ &= \text{Hom}_A(i_*F, K) \\ &= \text{Hom}_B(F, \varphi^!(K)) \end{aligned}$$

The first equality follows from More on Algebra, Lemma 15.73.1 and the definition of D_B. The second equality by the adjointness mentioned above and the equality $i_*((D_A(K) \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} F) = D_A(K) \otimes_A^{\mathbf{L}} i_*F$ (More on Algebra, Lemma 15.60.1). The third equality follows from More on Algebra, Lemma 15.73.1. The fourth because $D_A \circ D_A = \text{id}$. The final equality by adjointness again. Thus the result holds by the Yoneda lemma. □

47.25. Relative dualizing complexes in the Noetherian case

0E9M Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. Then we define the relative dualizing complex of A over R as the object

$$\omega_{A/R}^\bullet = \varphi^!(R)$$

of $D(A)$. Here $\varphi^!$ is as in Section 47.24. From the material in that section we see that $\omega_{A/R}^\bullet$ is well defined up to (non-unique) isomorphism.

0BZV Lemma 47.25.1. Let $R \to R'$ be a homomorphism of Noetherian rings. Let $R \to A$ be of finite type. Set $A' = A \otimes_R R'$. If

(1) $R \to R'$ is flat, or
(2) $R \to A$ is flat, or
(3) $R \to A$ is perfect and R' and A are tor independent over R,

then there is an isomorphism $\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} A' \to \omega_{A'/R'}^\bullet$ in $D(A')$.

Proof. Follows from Lemmas 47.24.4, 47.24.6, and 47.24.5 and the definitions. $\square$

0BZW Lemma 47.25.2. Let $\varphi : R \to A$ be a flat finite type map of Noetherian rings. Then

(1) $\omega_{A/R}^\bullet$ is in $D^b_{\text{Coh}}(A)$ and R-perfect (More on Algebra, Definition 15.83.1),
(2) $A \to R\operatorname{Hom}_A(\omega_{A/R}^\bullet, \omega_{A/R}^\bullet)$ is an isomorphism, and
(3) for every map $R \to k$ to a field the base change $\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} (A \otimes_R k)$ is a dualizing complex for $A \otimes_R k$.

Proof. Choose $R \to P \to A$ as in the definition of $\varphi^!$. Recall that $R \to A$ is a perfect ring map (More on Algebra, Lemma 15.82.4) and hence A is perfect as a P-modue (More on Algebra, Lemma 15.82.2). This shows that $\omega_{A/R}^\bullet$ is in $D^b_{\text{Coh}}(A)$ by Lemma 47.24.2. To show $\omega_{A/R}^\bullet$ is R-perfect it suffices to show it has finite tor dimension as a complex of R-modules. This is true because $\omega_{A/R}^\bullet = \varphi^!(R) = R\operatorname{Hom}(A, P)[n]$ maps to $R\operatorname{Hom}_P(A, P)[n]$ in $D(P)$, which is perfect in $D(P)$ (More on Algebra, Lemma 15.74.15), hence has finite tor dimension in $D(R)$ as $R \to P$ is flat. This proves (1).

Proof of (2). The object $R\operatorname{Hom}_A(\omega_{A/R}^\bullet, \omega_{A/R}^\bullet)$ of $D(A)$ maps in $D(P)$ to

$$
\begin{aligned}
R\operatorname{Hom}_P(\omega_{A/R}^\bullet, R\operatorname{Hom}(A, P)[n]) &= R\operatorname{Hom}_P(R\operatorname{Hom}_P(A, P)[n], P)[n] \\
&= R\operatorname{Hom}_P(R\operatorname{Hom}_P(A, P), P)
\end{aligned}
$$

This is equal to A by the already used More on Algebra, Lemma 15.74.15.

Proof of (3). By Lemma 47.25.1 there is an isomorphism

$$\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} (A \otimes_R k) \cong \omega_{A \otimes_R k/k}^\bullet$$

and the right hand side is a dualizing complex by Lemma 47.24.3. $\square$

0E0P Lemma 47.25.3. Let K/k be an extension of fields. Let A be a finite type k-algebra. Let $A_K = A \otimes_k K$. If $\omega_A^\bullet$ is a dualizing complex for A, then $\omega_A^\bullet \otimes_A A_K$ is a dualizing complex for A_K.

Proof. By the uniqueness of dualizing complexes, it doesn't matter which dualizing complex we pick for A; we omit the detailed proof. Denote $\varphi : k \to A$ the algebra structure. We may take $\omega_A^\bullet = \varphi^!(k[0])$ by Lemma 47.24.3. We conclude by Lemma 47.25.2. $\square$

0E4B Lemma 47.25.4. Let $\varphi : R \to A$ be a local complete intersection homomorphism of Noetherian rings. Then $\omega_{A/R}^\bullet$ is an invertible object of $D(A)$ and $\varphi^!(K) = K \otimes_R^{\mathbf{L}} \omega_{A/R}^\bullet$ for all $K \in D(R)$.

Proof. Recall that a local complete intersection homomorphism is a perfect ring map by More on Algebra, Lemma 15.82.6. Hence the final statement holds by Lemma 47.24.10. By More on Algebra, Definition 15.33.2 we can write $A = R[x_1, \ldots, x_n]/I$ where I is a Koszul-regular ideal. The construction of $\varphi^!$ in Section

47.24 shows that it suffices to show the lemma in case $A = R/I$ where $I \subset R$ is a Koszul-regular ideal. Checking $\omega_{A/R}^\bullet$ is invertible in $D(A)$ is local on $\text{Spec}(A)$ by More on Algebra, Lemma 15.126.4. Moreover, formation of $\omega_{A/R}^\bullet$ commutes with localization on R by Lemma 47.24.4. Combining More on Algebra, Definition 15.32.1 and Lemma 15.30.7 and Algebra, Lemma 10.68.6 we can find $g_1, \ldots, g_r \in R$ generating the unit ideal in A such that $I_{g_j} \subset R_{g_j}$ is generated by a regular sequence. Thus we may assume $A = R/(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in R. Then we consider the ring maps

$$R \to R/(f_1) \to R/(f_1, f_2) \to \ldots \to R/(f_1, \ldots, f_c) = A$$

and we use Lemma 47.24.7 (and the final statement already proven) to see that it suffices to prove the lemma for each step. Finally, in case $A = R/(f)$ for some nonzerodivisor f we see that the lemma is true since $\varphi^!(R) = R\operatorname{Hom}(A, R)$ is invertible by Lemma 47.13.10. $\square$

0E4C Lemma 47.25.5. Let $\varphi : R \to A$ be a flat finite type homomorphism of Noetherian rings. The following are equivalent

(1) the fibres $A \otimes_R \kappa(\mathfrak{p})$ are Gorenstein for all primes $\mathfrak{p} \subset R$, and
(2) $\omega_{A/R}^\bullet$ is an invertible object of $D(A)$, see More on Algebra, Lemma 15.126.4.

Proof. If (2) holds, then the fibre rings $A \otimes_R \kappa(\mathfrak{p})$ have invertible dualizing complexes, and hence are Gorenstein. See Lemmas 47.25.2 and 47.21.4.

For the converse, assume (1). Observe that $\omega_{A/R}^\bullet$ is in $D^b_{\text{Coh}}(A)$ by Lemma 47.24.2 (since flat finite type homomorphisms of Noetherian rings are perfect, see More on Algebra, Lemma 15.82.4). Take a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Then

$$\omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} \kappa(\mathfrak{q}) = \omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} (A \otimes_R \kappa(\mathfrak{p})) \otimes_{(A \otimes_R \kappa(\mathfrak{p}))}^{\mathbf{L}} \kappa(\mathfrak{q})$$

Applying Lemmas 47.25.2 and 47.21.4 and assumption (1) we find that this complex has 1 nonzero cohomology group which is a 1-dimensional $\kappa(\mathfrak{q})$-vector space. By More on Algebra, Lemma 15.77.1 we conclude that $(\omega_{A/R}^\bullet)_f$ is an invertible object of $D(A_f)$ for some $f \in A$, $f \notin \mathfrak{q}$. This proves (2) holds. $\square$

The following lemma is useful to see how dimension functions change when passing to a finite type algebra over a Noetherian ring.

0E9N Lemma 47.25.6. Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. Assume R local and let $\mathfrak{m} \subset A$ be a maximal ideal lying over the maximal ideal of R. If $\omega_R^\bullet$ is a normalized dualizing complex for R, then $\varphi^!(\omega_R^\bullet)_\mathfrak{m}$ is a normalized dualizing complex for $A_\mathfrak{m}$.

Proof. We already know that $\varphi^!(\omega_R^\bullet)$ is a dualizing complex for A, see Lemma 47.24.3. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in the construction of $\varphi^!$. If we can prove the lemma for $R \to P$ and the maximal ideal $\mathfrak{m}'$ of P corresponding to $\mathfrak{m}$, then we obtain the result for $R \to A$ by applying Lemma 47.16.2 to $P_{\mathfrak{m}'} \to A_\mathfrak{m}$ or by applying Lemma 47.17.2 to $P \to A$. In the case $A = R[x_1, \ldots, x_n]$ we see that $\dim(A_\mathfrak{m}) = \dim(R) + n$ for example by Algebra, Lemma 10.112.7 (combined with Algebra, Lemma 10.114.1 to compute the dimension of the fibre). The fact that $\omega_R^\bullet$ is normalized means that $i = -\dim(R)$ is the smallest index such that $H^i(\omega_R^\bullet)$ is nonzero (follows from Lemmas 47.16.5 and 47.16.11). Then $\varphi^!(\omega_R^\bullet)_\mathfrak{m} = \omega_R^\bullet \otimes_R A_\mathfrak{m}[n]$ has its first nonzero cohomology module in degree $-\dim(R) - n$ and therefore is the normalized dualizing complex for $A_\mathfrak{m}$. $\square$

0E9P Lemma 47.25.7. Let $R \to A$ be a finite type homomorphism of Noetherian rings. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Then

$$H^i(\omega_{A/R}^\bullet)_\mathfrak{q} \neq 0 \Rightarrow -d \leq i$$

where d is the dimension of the fibre of $\text{Spec}(A) \to \text{Spec}(R)$ over $\mathfrak{p}$ at the point $\mathfrak{q}$.

Proof. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in Section 47.24 so that $\omega_{A/R}^\bullet = R\,\text{Hom}(A, P)[n]$. We have to show that $R\,\text{Hom}(A, P)_\mathfrak{q}$ has vanishing cohomology in degrees $< n-d$. By Lemma 47.13.3 this means we have to show that $\text{Ext}^i_P(P/I, P)_\mathfrak{r} = 0$ for $i < n-d$ where $\mathfrak{r} \subset P$ is the prime corresponding to $\mathfrak{q}$ and I is the kernel of $P \to A$. We may rewrite this as $\text{Ext}^i_{P_\mathfrak{r}}(P_\mathfrak{r}/IP_\mathfrak{r}, P_\mathfrak{r})$ by More on Algebra, Lemma 15.65.4. Thus we have to show

$$\text{depth}_{IP_\mathfrak{r}}(P_\mathfrak{r}) \geq n - d$$

by Lemma 47.11.1. By Lemma 47.11.5 we have

$$\text{depth}_{IP_\mathfrak{r}}(P_\mathfrak{r}) \geq \dim((P \otimes_R \kappa(\mathfrak{p}))_\mathfrak{r}) - \dim((P/I \otimes_R \kappa(\mathfrak{p}))_\mathfrak{r})$$

The two expressions on the right hand side agree by Algebra, Lemma 10.116.4. □

0E9Q Lemma 47.25.8. Let $R \to A$ be a flat finite type homomorphism of Noetherian rings. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Then

$$H^i(\omega_{A/R}^\bullet)_\mathfrak{q} \neq 0 \Rightarrow -d \leq i \leq 0$$

where d is the dimension of the fibre of $\text{Spec}(A) \to \text{Spec}(R)$ over $\mathfrak{p}$ at the point $\mathfrak{q}$. If all fibres of $\text{Spec}(A) \to \text{Spec}(R)$ have dimension $\leq d$, then $\omega_{A/R}^\bullet$ has tor amplitude in $[-d, 0]$ as a complex of R-modules.

Proof. The lower bound has been shown in Lemma 47.25.7. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in Section 47.24 so that $\omega_{A/R}^\bullet = R\,\text{Hom}(A, P)[n]$. The upper bound means that $\text{Ext}^i_P(A, P)$ is zero for $i > n$. This follows from More on Algebra, Lemma 15.77.5 which shows that A is a perfect P-module with tor amplitude in $[-n, 0]$.

Proof of the final statement. Let $R \to R'$ be a ring homomorphism of Noetherian rings. Set $A' = A \otimes_R R'$. Then

$$\omega_{A'/R'}^\bullet = \omega_{A/R}^\bullet \otimes_A^{\mathbf{L}} A' = \omega_{A/R}^\bullet \otimes_R^{\mathbf{L}} R'$$

The first isomorphism by Lemma 47.25.1 and the second, which takes place in $D(R')$, by More on Algebra, Lemma 15.61.2. By the first part of the proof (note that the fibres of $\text{Spec}(A') \to \text{Spec}(R')$ have dimension $\leq d$) we conclude that $\omega_{A/R}^\bullet \otimes_R^{\mathbf{L}} R'$ has cohomology only in degrees $[-d, 0]$. Taking $R' = R \oplus M$ to be the square zero thickening of R by a finite R-module M, we see that $R\,\text{Hom}(A, P) \otimes_R^{\mathbf{L}} M$ has cohomology only in the interval $[-d, 0]$ for any finite R-module M. Since any R-module is a filtered colimit of finite R-modules and since tensor products commute with colimits we conclude. □

0E9R Lemma 47.25.9. Let $R \to A$ be a finite type homomorphism of Noetherian rings. Let $\mathfrak{p} \subset R$ be a prime ideal. Assume

(1) $R_\mathfrak{p}$ is Cohen-Macaulay, and
(2) for any minimal prime $\mathfrak{q} \subset A$ we have $\text{trdeg}_{\kappa(R \cap \mathfrak{q})} \kappa(\mathfrak{q}) \leq r$.

Then

$$H^i(\omega_{A/R}^\bullet)_\mathfrak{p} \neq 0 \Rightarrow -r \leq i$$

and $H^{-r}(\omega_{A/R}^\bullet)_\mathfrak{p}$ is (S_2) as an $A_\mathfrak{p}$-module.

Proof. We may replace R by $R_\mathfrak{p}$ by Lemma 47.25.1. Thus we may assume R is a Cohen-Macaulay local ring and we have to show the assertions of the lemma for the A-modules $H^i(\omega_{A/R}^\bullet)$.

Let $R^\wedge$ be the completion of R. The map $R \to R^\wedge$ is flat and $R^\wedge$ is Cohen-Macaulay (More on Algebra, Lemma 15.43.3). Observe that the minimal primes of $A \otimes_R R^\wedge$ lie over minimal primes of A by the flatness of $A \to A \otimes_R R^\wedge$ (and going down for flatness, see Algebra, Lemma 10.39.19). Thus condition (2) holds for the finite type ring map $R^\wedge \to A \otimes_R R^\wedge$ by Morphisms, Lemma 29.28.3. Appealing to Lemma 47.25.1 once again it suffices to prove the lemma for $R^\wedge \to A \otimes_R R^\wedge$. In this way, using Lemma 47.22.4, we may assume R is a Noetherian local Cohen-Macaulay ring which has a dualizing complex $\omega_R^\bullet$.

Let $\mathfrak{m} \subset A$ be a maximal ideal. It suffices to show that the assertions of the lemma hold for $H^i(\omega_{A/R}^\bullet)_\mathfrak{m}$. If $\mathfrak{m}$ does not lie over the maximal ideal of R, then we replace R by a localization to reduce to this case (small detail omitted).

We may assume $\omega_R^\bullet$ is normalized. Setting $d = \dim(R)$ we see that $\omega_R^\bullet = \omega_R[d]$ for some R-module ω_R, see Lemma 47.20.2. Set $\omega_A^\bullet = \varphi^!(\omega_R^\bullet)$. By Lemma 47.24.11 we have

$$\omega_{A/R}^\bullet = R\operatorname{Hom}_A(\omega_R[d] \otimes_R^{\mathbf{L}} A, \omega_A^\bullet)$$

By the dimension formula we have $\dim(A_\mathfrak{m}) \leq d+r$, see Morphisms, Lemma 29.52.2 and use that $\kappa(\mathfrak{m})$ is finite over the residue field of R by the Hilbert Nullstellensatz. By Lemma 47.25.6 we see that $(\omega_A^\bullet)_\mathfrak{m}$ is a normalized dualizing complex for $A_\mathfrak{m}$. Hence $H^i((\omega_A^\bullet)_\mathfrak{m})$ is nonzero only for $-d - r \leq i \leq 0$, see Lemma 47.16.5. Since $\omega_R[d] \otimes_R^{\mathbf{L}} A$ lives in degrees $\leq -d$ we conclude the vanishing holds. Finally, we also see that

$$H^{-r}(\omega_{A/R}^\bullet)_\mathfrak{m} = \operatorname{Hom}_A(\omega_R \otimes_R A, H^{-d-r}(\omega_A^\bullet))_\mathfrak{m}$$

Since $H^{-d-r}(\omega_A^\bullet)_\mathfrak{m}$ is (S_2) by Lemma 47.17.5 we find that the final statement is true by More on Algebra, Lemma 15.23.11. $\square$

47.26. More on dualizing complexes

0E49 Some lemmas which don't fit anywhere else very well.

0E4A **Lemma 47.26.1.** Let $A \to B$ be a faithfully flat map of Noetherian rings. If $K \in D(A)$ and $K \otimes_A^{\mathbf{L}} B$ is a dualizing complex for B, then K is a dualizing complex for A.

Proof. Since $A \to B$ is flat we have $H^i(K) \otimes_A B = H^i(K \otimes_A^{\mathbf{L}} B)$. Since $K \otimes_A^{\mathbf{L}} B$ is in $D^b_{\textit{Coh}}(B)$ we first find that K is in $D^b(A)$ and then we see that $H^i(K)$ is a finite A-module by Algebra, Lemma 10.83.2. Let M be a finite A-module. Then

$$R\operatorname{Hom}_A(M, K) \otimes_A B = R\operatorname{Hom}_B(M \otimes_A B, K \otimes_A^{\mathbf{L}} B)$$

by More on Algebra, Lemma 15.99.2. Since $K \otimes_A^{\mathbf{L}} B$ has finite injective dimension, say injective-amplitude in $[a, b]$, we see that the right hand side has vanishing cohomology in degrees $> b$. Since $A \to B$ is faithfully flat, we find that $R\operatorname{Hom}_A(M, K)$ has vanishing cohomology in degrees $> b$. Thus K has finite injective dimension by More on Algebra, Lemma 15.69.2. To finish the proof we have to show that the map $A \to R\operatorname{Hom}_A(K, K)$ is an isomorphism. For this we again use More on Algebra, Lemma 15.99.2 and the fact that $B \to R\operatorname{Hom}_B(K \otimes_A^{\mathbf{L}} B, K \otimes_A^{\mathbf{L}} B)$ is an isomorphism. □

0E4D Lemma 47.26.2. Let $\varphi : A \to B$ be a homomorphism of Noetherian rings. Assume

(1) $A \to B$ is syntomic and induces a surjective map on spectra, or
(2) $A \to B$ is a faithfully flat local complete intersection, or
(3) $A \to B$ is faithfully flat of finite type with Gorenstein fibres.

Then $K \in D(A)$ is a dualizing complex for A if and only if $K \otimes_A^{\mathbf{L}} B$ is a dualizing complex for B.

Proof. Observe that $A \to B$ satisfies (1) if and only if $A \to B$ satisfies (2) by More on Algebra, Lemma 15.33.5. Observe that in both (2) and (3) the relative dualzing complex $\varphi^!(A) = \omega_{B/A}^\bullet$ is an invertible object of $D(B)$, see Lemmas 47.25.4 and 47.25.5. Moreover we have $\varphi^!(K) = K \otimes_A^{\mathbf{L}} \omega_{B/A}^\bullet$ in both cases, see Lemma 47.24.10 for case (3). Thus $\varphi^!(K)$ is the same as $K \otimes_A^{\mathbf{L}} B$ up to tensoring with an invertible object of $D(B)$. Hence $\varphi^!(K)$ is a dualizing complex for B if and only if $K \otimes_A^{\mathbf{L}} B$ is (as being a dualizing complex is local and invariant under shifts). Thus we see that if K is dualizing for A, then $K \otimes_A^{\mathbf{L}} B$ is dualizing for B by Lemma 47.24.3. To descend the property, see Lemma 47.26.1. □

0E4E Lemma 47.26.3. Let $(A, \mathfrak{m}, \kappa) \to (B, \mathfrak{n}, l)$ be a flat local homorphism of Noetherian rings such that $\mathfrak{n} = \mathfrak{m}B$. If E is the injective hull of κ, then $E \otimes_A B$ is the injective hull of l.

Proof. Write $E = \bigcup E_n$ as in Lemma 47.7.3. It suffices to show that $E_n \otimes_{A/\mathfrak{m}^n} B/\mathfrak{n}^n$ is the injective hull of l over $B/\mathfrak{n}$. This reduces us to the case where A and B are Artinian local. Observe that $\text{length}_A(A) = \text{length}_B(B)$ and $\text{length}_A(E) = \text{length}_B(E \otimes_A B)$ by Algebra, Lemma 10.52.13. By Lemma 47.6.1 we have $\text{length}_A(E) = \text{length}_A(A)$ and $\text{length}_B(E') = \text{length}_B(B)$ where E' is the injective hull of l over B. We conclude $\text{length}_B(E') = \text{length}_B(E \otimes_A B)$. Observe that

$$\dim_l((E \otimes_A B)[\mathfrak{n}]) = \dim_l(E[\mathfrak{m}] \otimes_A B) = \dim_\kappa(E[\mathfrak{m}]) = 1$$

where we have used flatness of $A \to B$ and $\mathfrak{n} = \mathfrak{m}B$. Thus there is an injective B-module map $E \otimes_A B \to E'$ by Lemma 47.7.2. By equality of lengths shown above this is an isomorphism. □

0E4F Lemma 47.26.4. Let $\varphi : A \to B$ be a flat homorphism of Noetherian rings such that for all primes $\mathfrak{q} \subset B$ we have $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, for example if φ is étale. If I is an injective A-module, then $I \otimes_A B$ is an injective B-module.

Proof. Étale maps satisfy the assumption by Algebra, Lemma 10.143.5. By Lemma 47.3.7 and Proposition 47.5.9 we may assume I is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$. Then I is a module over $A_\mathfrak{p}$. It suffices to prove $I \otimes_A B = I \otimes_{A_\mathfrak{p}} B_\mathfrak{p}$ is injective as a $B_\mathfrak{p}$-module, see Lemma 47.3.2. Thus we may assume $(A, \mathfrak{m}, \kappa)$ is local Noetherian and $I = E$ is the injective hull of the residue field κ. Our assumption implies that the Noetherian ring $B/\mathfrak{m}B$ is a product of fields (details omitted). Thus there are finitely many prime ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ in B lying over $\mathfrak{m}$ and they are all maximal ideals. Write $E = \bigcup E_n$ as in Lemma 47.7.3. Then $E \otimes_A B = \bigcup E_n \otimes_A B$ and $E_n \otimes_A B$ is a finite B-module with support $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ hence decomposes as a product over the localizations at $\mathfrak{m}_i$. Thus $E \otimes_A B = \prod (E \otimes_A B)_{\mathfrak{m}_i}$. Since $(E \otimes_A B)_{\mathfrak{m}_i} = E \otimes_A B_{\mathfrak{m}_i}$ is the injective hull of the residue field of $\mathfrak{m}_i$ by Lemma 47.26.3 we conclude. □

47.27. Relative dualizing complexes

0E2B For a finite type ring map $\varphi : R \to A$ of Noetherian rings we have the relative dualizing complex $\omega_{A/R}^\bullet = \varphi^!(R)$ considered in Section 47.25. If R is not Noetherian, a similarly constructed complex will in general not have good properties. In this section, we give a definition of a relative dualizing complex for a flat and finitely presented ring maps $R \to A$ of non-Noetherian rings. The definition is chosen to globalize to flat and finitely presented morphisms of schemes, see Duality for Schemes, Section 48.28. We will show that relative dualizing complexes exist (when the definition applies), are unique up to (noncanonical) isomorphism, and that in the Noetherian case we recover the complex of Section 47.25.

The Noetherian reader may safely skip this section!

0E2C **Definition 47.27.1.** Let $R \to A$ be a flat ring map of finite presentation. A relative dualizing complex is an object $K \in D(A)$ such that

1. K is R-perfect (More on Algebra, Definition 15.83.1), and
2. $R\operatorname{Hom}_{A \otimes_R A}(A, K \otimes_A^{\mathbf{L}} (A \otimes_R A))$ is isomorphic to A.

To understand this definition you may have to read and understand some of the following lemmas. Lemmas 47.27.3 and 47.27.2 show this definition does not clash with the definition in Section 47.25.

0E2D **Lemma 47.27.2.** Let $R \to A$ be a flat ring map of finite presentation. Any two relative dualizing complexes for $R \to A$ are isomorphic.

Proof. Let K and L be two relative dualizing complexes for $R \to A$. Denote $K_1 = K \otimes_A^{\mathbf{L}} (A \otimes_R A)$ and $L_2 = (A \otimes_R A) \otimes_A^{\mathbf{L}} L$ the derived base changes via the first and second coprojections $A \to A \otimes_R A$. By symmetry the assumption on L_2 implies that $R\operatorname{Hom}_{A \otimes_R A}(A, L_2)$ is isomorphic to A. By More on Algebra, Lemma 15.98.3 part (3) applied twice we have

$$A \otimes_{A \otimes_R A}^{\mathbf{L}} L_2 \cong R\operatorname{Hom}_{A \otimes_R A}(A, K_1 \otimes_{A \otimes_R A}^{\mathbf{L}} L_2) \cong A \otimes_{A \otimes_R A}^{\mathbf{L}} K_1$$

Applying the restriction functor $D(A \otimes_R A) \to D(A)$ for either coprojection we obtain the desired result. □

0E2E **Lemma 47.27.3.** Let $\varphi : R \to A$ be a flat finite type ring map of Noetherian rings. Then the relative dualizing complex $\omega_{A/R}^\bullet = \varphi^!(R)$ of Section 47.25 is a relative dualizing complex in the sense of Definition 47.27.1.

Proof. From Lemma 47.25.2 we see that $\varphi^!(R)$ is R-perfect. Denote $\delta : A \otimes_R A \to A$ the multiplication map and $p_1, p_2 : A \to A \otimes_R A$ the coprojections. Then

$$\varphi^!(R) \otimes_A^{\mathbf{L}} (A \otimes_R A) = \varphi^!(R) \otimes_{A,p_1}^{\mathbf{L}} (A \otimes_R A) = p_2^!(A)$$

by Lemma 47.24.4. Recall that $R\operatorname{Hom}_{A \otimes_R A}(A, \varphi^!(R) \otimes_A^{\mathbf{L}} (A \otimes_R A))$ is the image of $\delta^!(\varphi^!(R) \otimes_A^{\mathbf{L}} (A \otimes_R A))$ under the restriction map $\delta_* : D(A) \to D(A \otimes_R A)$. Use the definition of $\delta^!$ from Section 47.24 and Lemma 47.13.3. Since $\delta^!(p_2^!(A)) \cong A$ by Lemma 47.24.7 we conclude. □

0E2F **Lemma 47.27.4.** Let $R \to A$ be a flat ring map of finite presentation. Then

(1) there exists a relative dualizing complex K in $D(A)$, and
(2) for any ring map $R \to R'$ setting $A' = A \otimes_R R'$ and $K' = K \otimes_A^{\mathbf{L}} A'$, then K' is a relative dualizing complex for $R' \to A'$.

Moreover, if

$$\xi : A \longrightarrow K \otimes_A^{\mathbf{L}} (A \otimes_R A)$$

is a generator for the cyclic module $\mathrm{Hom}_{D(A \otimes_R A)}(A, K \otimes_A^{\mathbf{L}} (A \otimes_R A))$ then in (2) the derived base change of ξ by $A \otimes_R A \to A' \otimes_{R'} A'$ is a generator for the cyclic module $\mathrm{Hom}_{D(A' \otimes_{R'} A')}(A', K' \otimes_{A'}^{\mathbf{L}} (A' \otimes_{R'} A'))$

Proof. We first reduce to the Noetherian case. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. By Lemma 47.27.3 there exists a relative dualizing complex $K_0 \in D(A_0)$. Thus if we show (2) for K_0, then we find that $K_0 \otimes_{A_0}^{\mathbf{L}} A$ is a dualizing complex for $R \to A$ and that it also satisfies (2) by transitivity of derived base change. The uniqueness of relative dualizing complexes (Lemma 47.27.2) then shows that this holds for any relative dualizing complex.

Assume R Noetherian and let K be a relative dualizing complex for $R \to A$. Given a ring map $R \to R'$ set $A' = A \otimes_R R'$ and $K' = K \otimes_A^{\mathbf{L}} A'$. To finish the proof we have to show that K' is a relative dualizing complex for $R' \to A'$. By More on Algebra, Lemma 15.83.5 we see that K' is R'-perfect in all cases. By Lemmas 47.25.1 and 47.27.3 if R' is Noetherian, then K' is a relative dualizing complex for $R' \to A'$ (in either sense). Transitivity of derived tensor product shows that $K \otimes_A^{\mathbf{L}} (A \otimes_R A) \otimes_{A \otimes_R A}^{\mathbf{L}} (A' \otimes_{R'} A') = K' \otimes_{A'}^{\mathbf{L}} (A' \otimes_{R'} A')$. Flatness of $R \to A$ guarantees that $A \otimes_{A \otimes_R A}^{\mathbf{L}} (A' \otimes_{R'} A') = A'$; namely $A \otimes_R A$ and R' are tor independent over R so we can apply More on Algebra, Lemma 15.61.2. Finally, A is pseudo-coherent as an $A \otimes_R A$-module by More on Algebra, Lemma 15.82.8. Thus we have checked all the assumptions of More on Algebra, Lemma 15.83.6. We find there exists a bounded below complex $E^\bullet$ of R-flat finitely presented $A \otimes_R A$-modules such that $E^\bullet \otimes_R R'$ represents $R\mathrm{Hom}_{A' \otimes_{R'} A'}(A', K' \otimes_{A'}^{\mathbf{L}} (A' \otimes_{R'} A'))$ and these identifications are compatible with derived base change. Let $n \in \mathbf{Z}$, $n \neq 0$. Define Q^n by the sequence

$$E^{n-1} \to E^n \to Q^n \to 0$$

Since $\kappa(\mathfrak p)$ is a Noetherian ring, we know that $H^n(E^\bullet \otimes_R \kappa(\mathfrak p)) = 0$, see remarks above. Chasing diagrams this means that

$$Q^n \otimes_R \kappa(\mathfrak p) \to E^{n+1} \otimes_R \kappa(\mathfrak p)$$

is injective. Hence for a prime $\mathfrak q$ of $A \otimes_R A$ lying over $\mathfrak p$ we have $Q^n_{\mathfrak q}$ is $R_{\mathfrak p}$-flat and $Q^n_{\mathfrak p} \to E^{n+1}_{\mathfrak q}$ is $R_{\mathfrak p}$-universally injective, see Algebra, Lemma 10.99.1. Since this holds for all primes, we conclude that Q^n is R-flat and $Q^n \to E^{n+1}$ is R-universally injective. In particular $H^n(E^\bullet \otimes_R R') = 0$ for any ring map $R \to R'$. Let $Z^0 = \mathrm{Ker}(E^0 \to E^1)$. Since there is an exact sequence $0 \to Z^0 \to E^0 \to E^1 \to Q^1 \to 0$ we see that Z^0 is R-flat and that $Z^0 \otimes_R R' = \mathrm{Ker}(E^0 \otimes_R R' \to E^1 \otimes_R R')$ for all $R \to R'$. Then the short exact sequence $0 \to Q^{-1} \to Z^0 \to H^0(E^\bullet) \to 0$ shows that

$$H^0(E^\bullet \otimes_R R') = H^0(E^\bullet) \otimes_R R' = A \otimes_R R' = A'$$

as desired. This equality furthermore gives the final assertion of the lemma. $\square$

0E2G **Lemma 47.27.5.** Let $R \to A$ be a flat ring map of finite presentation. Let K be a relative dualizing complex. Then $A \to R\operatorname{Hom}_A(K, K)$ is an isomorphism.

Proof. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. By Lemmas 47.27.2, 47.27.3, and 47.27.4 there exists a relative dualizing complex $K_0 \in D(A_0)$ and its derived base change is K. This reduces us to the situation discussed in the next paragraph.

Assume R Noetherian and let K be a relative dualizing complex for $R \to A$. Given a ring map $R \to R'$ set $A' = A \otimes_R R'$ and $K' = K \otimes_A^{\mathbf{L}} A'$. To finish the proof we show $R\operatorname{Hom}_{A'}(K', K') = A'$. By Lemma 47.25.2 we know this is true whenever R' is Noetherian. Since a general R' is a filtered colimit of Noetherian R-algebras, we find the result holds by More on Algebra, Lemma 15.83.7. $\square$

0E2H **Lemma 47.27.6.** Let $R \to A \to B$ be a ring maps which are flat and of finite presentation. Let $K_{A/R}$ and $K_{B/A}$ be relative dualizing complexes for $R \to A$ and $A \to B$. Then $K = K_{A/R} \otimes_A^{\mathbf{L}} K_{B/A}$ is a relative dualizing complex for $R \to B$.

Proof. We will use reduction to the Noetherian case. Namely, by Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. After increasing R_0 and correspondingly replacing A_0 we may assume there is a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{R_0} R$ (use the same lemma). If we prove the lemma for $R_0 \to A_0 \to B_0$, then the lemma follows by Lemmas 47.27.2, 47.27.3, and 47.27.4. This reduces us to the situation discussed in the next paragraph.

Assume R is Noetherian and denote $\varphi : R \to A$ and $\psi : A \to B$ the given ring maps. Then $K_{A/R} \cong \varphi^!(R)$ and $K_{B/A} \cong \psi^!(A)$, see references given above. Then

$$K = K_{A/R} \otimes_A^{\mathbf{L}} K_{B/A} \cong \varphi^!(R) \otimes_A^{\mathbf{L}} \psi^!(A) \cong \psi^!(\varphi^!(R)) \cong (\psi \circ \varphi)^!(R)$$

by Lemmas 47.24.10 and 47.24.7. Thus K is a relative dualizing complex for $R \to B$. $\square$

47.28. Other chapters

Preliminaries

1. (1) Introduction
2. (2) Conventions
3. (3) Set Theory
4. (4) Categories
5. (5) Topology
6. (6) Sheaves on Spaces
7. (7) Sites and Sheaves
8. (8) Stacks
9. (9) Fields
10. (10) Commutative Algebra
11. (11) Brauer Groups
12. (12) Homological Algebra
13. (13) Derived Categories
14. (14) Simplicial Methods
15. (15) More on Algebra
16. (16) Smoothing Ring Maps
17. (17) Sheaves of Modules
18. (18) Modules on Sites
19. (19) Injectives
20. (20) Cohomology of Sheaves
21. (21) Cohomology on Sites
22. (22) Differential Graded Algebra
23. (23) Divided Power Algebra
24. (24) Differential Graded Sheaves
25. (25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes

CHAPTER 48

Duality for Schemes

0DWE 48.1. Introduction

0DWF This chapter studies relative duality for morphisms of schemes and the dualizing complex on a scheme. A reference is [Har66].

Dualizing complexes for Noetherian rings were defined and studied in Dualizing Complexes, Section 47.15 ff. In this chapter we continue this by studying dualizing complexes on schemes, see Section 48.2.

The bulk of this chapter is devoted to studying the right adjoint of pushforward in the setting of derived categories of sheaves of modules with quasi-coherent cohomology sheaves. See Sections 48.3, 48.4, 48.5, 48.6, 48.7, 48.8, 48.9, 48.11, 48.13, 48.14, and 48.15. Here we follow the papers [Nee96], [LN07], [Lip09], and [Nee14].

We discuss the important and useful upper shriek functors $f^!$ for separated morphisms of finite type between Noetherian schemes in Sections 48.16, 48.17, and 48.18 culminating in the overview Section 48.19.

In Section 48.20 we explain alternative theory of duality and dualizing complexes when working over a fixed locally Noetherian base endowed with a dualizing complex (this section corresponds to a remark in Hartshorne's book).

In the remaining sections we give a few applications.

This chapter is continued by the chapter on duality on algebraic spaces, see Duality for Spaces, Section 86.1.

48.2. Dualizing complexes on schemes

0A85 We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

0A86 Lemma 48.2.1. Let X be a locally Noetherian scheme. Let K be an object of $D(\mathcal{O}_X)$. The following are equivalent

(1) For every affine open $U = \mathrm{Spec}(A) \subset X$ there exists a dualizing complex $\omega_A^\bullet$ for A such that $K|_U$ is isomorphic to the image of $\omega_A^\bullet$ by the functor $\widetilde{\ }: D(A) \to D(\mathcal{O}_U)$.
(2) There is an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that for each i there exists a dualizing complex $\omega_i^\bullet$ for A_i such that $K|_{U_i}$ is isomorphic to the image of $\omega_i^\bullet$ by the functor $\widetilde{\ }: D(A_i) \to D(\mathcal{O}_{U_i})$.

Proof. Assume (2) and let $U = \mathrm{Spec}(A)$ be an affine open of X. Since condition (2) implies that K is in $D_{QCoh}(\mathcal{O}_X)$ we find an object $\omega_A^\bullet$ in $D(A)$ whose associated complex of quasi-coherent sheaves is isomorphic to $K|_U$, see Derived Categories of Schemes, Lemma 36.3.5. We will show that $\omega_A^\bullet$ is a dualizing complex for A which will finish the proof.

Since $X = \bigcup U_i$ is an open covering, we can find a standard open covering $U = D(f_1) \cup \ldots \cup D(f_m)$ such that each $D(f_j)$ is a standard open in one of the affine opens U_i, see Schemes, Lemma 26.11.5. Say $D(f_j) = D(g_j)$ for $g_j \in A_{i_j}$. Then $A_{f_j} \cong (A_{i_j})_{g_j}$ and we have

$$(\omega_A^\bullet)_{f_j} \cong (\omega_i^\bullet)_{g_j}$$

in the derived category by Derived Categories of Schemes, Lemma 36.3.5. By Dualizing Complexes, Lemma 47.15.6 we find that the complex $(\omega_A^\bullet)_{f_j}$ is a dualizing complex over A_{f_j} for $j = 1, \ldots, m$. This implies that $\omega_A^\bullet$ is dualizing by Dualizing Complexes, Lemma 47.15.7. $\square$

0A87 Definition 48.2.2. Let X be a locally Noetherian scheme. An object K of $D(\mathcal{O}_X)$ is called a dualizing complex if K satisfies the equivalent conditions of Lemma 48.2.1.

Please see remarks made at the beginning of this section.

0A88 Lemma 48.2.3. Let A be a Noetherian ring and let $X = \mathrm{Spec}(A)$. Let K, L be objects of $D(A)$. If $K \in D_{\mathrm{Coh}}(A)$ and L has finite injective dimension, then

$$R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) = R\widetilde{\mathrm{Hom}_A(K, L)}$$

in $D(\mathcal{O}_X)$.

Proof. We may assume that L is given by a finite complex $I^\bullet$ of injective A-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A-module. In this case, Derived Categories of Schemes, Lemma 36.10.8, tells us that the nth cohomology sheaf of $R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f)$$

Since A is Noetherian, the A_f-module $I \otimes_A A_f$ is injective (Dualizing Complexes, Lemma 47.3.8). Hence we see that

$$\begin{aligned} \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f) &= \mathrm{Hom}_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f) \\ &= \mathrm{Hom}_{A_f}(H^{-n}(K) \otimes_A A_f, I \otimes_A A_f) \\ &= \mathrm{Hom}_A(H^{-n}(K), I) \otimes_A A_f \end{aligned}$$

The last equality because $H^{-n}(K)$ is a finite A-module, see Algebra, Lemma 10.10.2. This proves that the canonical map

$$R\widetilde{\mathrm{Hom}_A(K, L)} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

is a quasi-isomorphism in this case and the proof is done. $\square$

0G4I Lemma 48.2.4. Let X be a Noetherian scheme. Let $K, L, M \in D_{QCoh}(\mathcal{O}_X)$. Then the map

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

of Cohomology, Lemma 20.42.9 is an isomorphism in the following two cases

(1) $K \in D^-_{\mathrm{Coh}}(\mathcal{O}_X)$, $L \in D^+_{\mathrm{Coh}}(\mathcal{O}_X)$, and M affine locally has finite injective dimension (see proof), or

(2) K and L are in $D_{\text{Coh}}(\mathcal{O}_X)$, the object $R\mathcal{H}om(L, M)$ has finite tor dimension, and L and M affine locally have finite injective dimension (in particular L and M are bounded).

Proof. Proof of (1). We say M has affine locally finite injective dimension if X has an open covering by affines $U = \text{Spec}(A)$ such that the object of $D(A)$ corresponding to $M|_U$ (Derived Categories of Schemes, Lemma 36.3.5) has finite injective dimension[1]. To prove the lemma we may replace X by U, i.e., we may assume $X = \text{Spec}(A)$ for some Noetherian ring A. Observe that $R\mathcal{H}om(K, L)$ is in $D^+_{\text{Coh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.5. Moreover, the formation of the left and right hand side of the arrow commutes with the functor $D(A) \to D_{QCoh}(\mathcal{O}_X)$ by Lemma 48.2.3 and Derived Categories of Schemes, Lemma 36.10.8 (to be sure this uses the assumptions on K, L, M and what we just proved about $R\mathcal{H}om(K, L)$). Then finally the arrow is an isomorphism by More on Algebra, Lemmas 15.98.1 part (2).

Proof of (2). We argue as above. A small change is that here we get $R\mathcal{H}om(K, L)$ in $D_{\text{Coh}}(\mathcal{O}_X)$ because affine locally (which is allowable by Lemma 48.2.3) we may appeal to Dualizing Complexes, Lemma 47.15.2. Then we finally conclude by More on Algebra, Lemma 15.98.2. □

0A89 Lemma 48.2.5. Let K be a dualizing complex on a locally Noetherian scheme X. Then K is an object of $D_{\text{Coh}}(\mathcal{O}_X)$ and $D = R\mathcal{H}om_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence

$$D : D_{\text{Coh}}(\mathcal{O}_X) \longrightarrow D_{\text{Coh}}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $\text{id} \to D \circ D$. If X is quasi-compact, then D exchanges $D^+_{\text{Coh}}(\mathcal{O}_X)$ and $D^-_{\text{Coh}}(\mathcal{O}_X)$ and induces an equivalence $D^b_{\text{Coh}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_X)$.

Proof. Let $U \subset X$ be an affine open. Say $U = \text{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 48.2.1. By Lemma 48.2.3 the diagram

$$\begin{tikzcd} D_{\text{Coh}}(A) \arrow[r] \arrow[d, "R\text{Hom}_A(-, \omega_A^\bullet)"'] & D_{\text{Coh}}(\mathcal{O}_U) \arrow[d, "R\mathcal{H}om_{\mathcal{O}_X}(-, K|_U)"] \\ D_{\text{Coh}}(A) \arrow[r] & D(\mathcal{O}_U) \end{tikzcd}$$

commutes. We conclude that D sends $D_{\text{Coh}}(\mathcal{O}_X)$ into $D_{\text{Coh}}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(K, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(using Cohomology, Lemma 20.42.9 for the second arrow) is an isomorphism for all L because this is true on affines by Dualizing Complexes, Lemma 47.15.3[2] and we have already seen on affines that we recover what happens in algebra. The statement on boundedness properties of the functor D in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 47.15.3. □

[1]This condition is independent of the choice of the affine open cover of the Noetherian scheme X. Details omitted.

[2]An alternative is to first show that $R\mathcal{H}om_{\mathcal{O}_X}(K, K) = \mathcal{O}_X$ by working affine locally and then use Lemma 48.2.4 part (2) to see the map is an isomorphism.

Let X be a locally ringed space. Recall that an object L of $D(\mathcal{O}_X)$ is invertible if it is an invertible object for the symmetric monoidal structure on $D(\mathcal{O}_X)$ given by derived tensor product. In Cohomology, Lemma 20.52.2 we have seen this means L is perfect and there is an open covering $X = \bigcup U_i$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers n_i. In this case, the function

$$x \mapsto n_x, \quad \text{where } n_x \text{ is the unique integer such that } H^{n_x}(L_x) \neq 0$$

is locally constant on X. In particular, we have $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$-modules (with zero differentials) representing L.

0ATP Lemma 48.2.6. Let X be a locally Noetherian scheme. If K and K' are dualizing complexes on X, then K' is isomorphic to $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for some invertible object L of $D(\mathcal{O}_X)$.

Proof. Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of $D(\mathcal{O}_X)$, because affine locally this is true, see Dualizing Complexes, Lemma 47.15.5 and its proof. The evaluation map $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K'$ is an isomorphism for the same reason. $\square$

0AWF Lemma 48.2.7. Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X. Then X is universally catenary and the function $X \to \mathbf{Z}$ defined by

$$x \longmapsto \delta(x) \text{ such that } \omega_{X,x}^\bullet[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,x}$$

is a dimension function.

Proof. Immediate from the affine case Dualizing Complexes, Lemma 47.17.3 and the definitions. $\square$

0ECM Lemma 48.2.8. Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X with associated dimension function δ. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $\mathcal{E}^i = \mathcal{E}xt^{-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)$. Then $\mathcal{E}^i$ is a coherent $\mathcal{O}_X$-module and for $x \in X$ we have

(1) $\mathcal{E}^i_x$ is nonzero only for $\delta(x) \leq i \leq \delta(x) + \dim(\text{Supp}(\mathcal{F}_x))$,
(2) $\dim(\text{Supp}(\mathcal{E}^{i+\delta(x)}_x)) \leq i$,
(3) $\text{depth}(\mathcal{F}_x)$ is the smallest integer $i \geq 0$ such that $\mathcal{E}^{i+\delta(x)}_x \neq 0$, and
(4) we have $x \in \text{Supp}(\bigoplus_{j \leq i} \mathcal{E}^j) \Leftrightarrow \text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \delta(x) \leq i$.

Proof. Lemma 48.2.5 tells us that $\mathcal{E}^i$ is coherent. Choosing an affine neighbourhood of x and using Derived Categories of Schemes, Lemma 36.10.8 and More on Algebra, Lemma 15.99.2 part (3) we have

$$\mathcal{E}^i_x = \mathcal{E}xt^{-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)_x = \text{Ext}^{-i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega_{X,x}^\bullet) = \text{Ext}^{\delta(x)-i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega_{X,x}^\bullet[-\delta(x)])$$

By construction of δ in Lemma 48.2.7 this reduces parts (1), (2), and (3) to Dualizing Complexes, Lemma 47.16.5. Part (4) is a formal consequence of (3) and (1). $\square$

48.3. Right adjoint of pushforward

0A9D References for this section and the following are [Nee96], [LN07], [Lip09], and [Nee14].

Let $f : X \to Y$ be a morphism of schemes. In this section we consider the right adjoint to the functor $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. In the literature, if this functor exists, then it is sometimes denoted $f^\times$. This notation is not universally accepted and we refrain from using it. We will not use the notation $f^!$ for such a functor, as this would clash (for general morphisms f) with the notation in [Har66].

0A9E Lemma 48.3.1. Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact schemes. The functor $Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y)$ has a right adjoint.

This is almost the same as [Nee96, Example 4.2].

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 13.38.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Schemes, Lemma 36.3.1. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Schemes, Theorem 36.15.3. Since X and Y are quasi-compact and quasi-separated, so is f, see Schemes, Lemmas 26.21.13 and 26.21.14. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Schemes, Lemma 36.4.5. This finishes the proof. $\square$

0A9F Example 48.3.2. Let $A \to B$ be a ring map. Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ corresponds to restriction $D(B) \to D(A)$ via the equivalences $D(B) \to D_{QCoh}(\mathcal{O}_X)$ and $D(A) \to D_{QCoh}(\mathcal{O}_Y)$. Hence the right adjoint corresponds to the functor $K \longmapsto R\mathrm{Hom}(B, K)$ of Dualizing Complexes, Section 47.13.

0A9G Example 48.3.3. If $f : X \to Y$ is a separated finite type morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{\mathrm{Coh}}(\mathcal{O}_Y)$ into $D_{\mathrm{Coh}}(\mathcal{O}_X)$. Namely, let k be a field and consider the morphism $f : \mathbf{A}^1_k \to \mathrm{Spec}(k)$. By Example 48.3.2 this corresponds to the question of whether $R\mathrm{Hom}(B, -)$ maps $D_{\mathrm{Coh}}(A)$ into $D_{\mathrm{Coh}}(B)$ where $A = k$ and $B = k[x]$. This is not true because

$$R\mathrm{Hom}(k[x], k) = \left(\prod\nolimits_{n \geq 0} k\right)[0]$$

which is not a finite $k[x]$-module. Hence $a(\mathcal{O}_Y)$ does not have coherent cohomology sheaves.

0A9H Example 48.3.4. If $f : X \to Y$ is a proper or even finite morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D^-_{QCoh}(\mathcal{O}_Y)$ into $D^-_{QCoh}(\mathcal{O}_X)$. Namely, let k be a field, let $k[\epsilon]$ be the dual numbers over k, let $X = \mathrm{Spec}(k)$, and let $Y = \mathrm{Spec}(k[\epsilon])$. Then $\mathrm{Ext}^i_{k[\epsilon]}(k, k)$ is nonzero for all $i \geq 0$. Hence $a(\mathcal{O}_Y)$ is not bounded above by Example 48.3.2.

0A9I Lemma 48.3.5. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_* of Lemma 48.3.1. Then a maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(\mathcal{O}_X)$. In fact, there exists an integer N such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.

Proof. By Derived Categories of Schemes, Lemma 36.4.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that

$H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D^+_{QCoh}(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. $\square$

Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. Let a denote the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. For every $K \in D_{QCoh}(\mathcal{O}_Y)$ and $L \in D_{QCoh}(\mathcal{O}_X)$ we obtain a canonical map

0B6H (48.3.5.1)
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology, Remark 20.42.11 and the second arrow is the counit $Rf_*a(K) \to K$ of the adjunction.

0A9Q **Lemma 48.3.6.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$. Then the map (48.3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

becomes an isomorphism after applying the functor $DQ_Y : D(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Y)$ discussed in Derived Categories of Schemes, Section 36.21.

Proof. The statement makes sense as DQ_Y exists by Derived Categories of Schemes, Lemma 36.21.1. Since DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$ to prove the lemma we have to show that for any $M \in D_{QCoh}(\mathcal{O}_Y)$ the map (48.3.5.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

To see this we use the following string of equalities

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\ &= \mathrm{Hom}_X(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, a(K)) \\ &= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L), K) \\ &= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L, K) \\ &= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)) \end{aligned}$$

The first equality holds by Cohomology, Lemma 20.28.1. The second equality by Cohomology, Lemma 20.42.2. The third equality by construction of a. The fourth equality by Derived Categories of Schemes, Lemma 36.22.1 (this is the important step). The fifth by Cohomology, Lemma 20.42.2. $\square$

0GEU **Example 48.3.7.** The statement of Lemma 48.3.6 is not true without applying the "coherator" DQ_Y. Indeed, suppose $Y = \mathrm{Spec}(R)$ and $X = \mathbf{A}^1_R$. Take $L = \mathcal{O}_X$ and $K = \mathcal{O}_Y$. The left hand side of the arrow is in $D_{QCoh}(\mathcal{O}_Y)$ but the right hand side of the arrow is isomorphic to $\prod_{n \geq 0} \mathcal{O}_Y$ which is not quasi-coherent.

0GEV Remark 48.3.8. In the situation of Lemma 48.3.6 we have

$$DQ_Y(Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = Rf_*DQ_X(R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

by Derived Categories of Schemes, Lemma 36.21.2. Thus if $R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \in D_{QCoh}(\mathcal{O}_X)$, then we can "erase" the DQ_Y on the left hand side of the arrow. On the other hand, if we know that $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \in D_{QCoh}(\mathcal{O}_Y)$, then we can "erase" the DQ_Y from the right hand side of the arrow. If both are true then we see that (48.3.5.1) is an isomorphism. Combining this with Derived Categories of Schemes, Lemma 36.10.8 we see that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism if

(1) L and Rf_*L are perfect, or
(2) K is bounded below and L and Rf_*L are pseudo-coherent.

For (2) we use that $a(K)$ is bounded below if K is bounded below, see Lemma 48.3.5.

0GEW Example 48.3.9. Let $f : X \to Y$ be a proper morphism of Noetherian schemes, $L \in D^-_{\text{Coh}}(X)$ and $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Then the map $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism. Namely, the complexes L and Rf_*L are pseudo-coherent by Derived Categories of Schemes, Lemmas 36.10.3 and 36.11.3 and the discussion in Remark 48.3.8 applies.

0B6I Lemma 48.3.10. Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. For all $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ (48.3.5.1) induces an isomorphism $R\operatorname{Hom}_X(L, a(K)) \to R\operatorname{Hom}_Y(Rf_*L, K)$ of global derived homs.

Proof. By the construction in Cohomology, Section 20.44 we have

$$R\operatorname{Hom}_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\operatorname{Hom}_Y(Rf_*L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

Thus the lemma is a consequence of Lemma 48.3.6. Namely, a map $E \to E'$ in $D(\mathcal{O}_Y)$ which induces an isomorphism $DQ_Y(E) \to DQ_Y(E')$ induces a quasi-isomorphism $R\Gamma(Y, E) \to R\Gamma(Y, E')$. Indeed we have $H^i(Y, E) = \operatorname{Ext}^i_Y(\mathcal{O}_Y, E) = \operatorname{Hom}(\mathcal{O}_Y[-i], E) = \operatorname{Hom}(\mathcal{O}_Y[-i], DQ_Y(E))$ because $\mathcal{O}_Y[-i]$ is in $D_{QCoh}(\mathcal{O}_Y)$ and DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$. $\square$

48.4. Right adjoint of pushforward and restriction to opens

0E4G In this section we study the question to what extend the right adjoint of pushforward commutes with restriction to open subschemes. This is a base change question, so let's first discuss this more generally.

We often want to know whether the right adjoints to pushforward commutes with base change. Thus we consider a cartesian square

0A9J (48.4.0.1)

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes. Denote

$$a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{QCoh}(\mathcal{O}_{Y'}) \to D_{QCoh}(\mathcal{O}_{X'})$$

the right adjoints to Rf_* and Rf'_* (Lemma 48.3.1). Consider the base change map of Cohomology, Remark 20.28.3. It induces a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0A9K Lemma 48.4.1. In diagram (48.4.0.1) assume that g is flat or more generally that f and g are Tor independent. Then $a \circ Rg_* \leftarrow Rg'_* \circ a'$ is an isomorphism.

Proof. In this case the base change map $Lg^* \circ Rf_*K \longrightarrow Rf'_* \circ L(g')^*K$ is an isomorphism for every K in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.22.5. Thus the corresponding transformation between adjoint functors is an isomorphism as well. $\square$

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open subscheme and set $U = f^{-1}(V)$. This gives a cartesian square

$$\begin{tikzcd} U \arrow[r, "j'"'] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r, "j"] & Y \end{tikzcd}$$

as in (48.4.0.1). By Lemma 48.4.1 the map $\xi : a \circ Rj_* \leftarrow Rj'_* \circ a'$ is an isomorphism where a and a' are the right adjoints to Rf_* and $R(f|_U)_*$. We obtain a transformation of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_U)$

0A9L (48.4.1.1)
$$(j')^* \circ a \to (j')^* \circ a \circ Rj_* \circ j^* \xrightarrow{\xi^{-1}} (j')^* \circ Rj'_* \circ a' \circ j^* \to a' \circ j^*$$

where the first arrow comes from $\text{id} \to Rj_* \circ j^*$ and the final arrow from the isomorphism $(j')^* \circ Rj'_* \to \text{id}$. In particular, we see that (48.4.1.1) is an isomorphism when evaluated on K if and only if $a(K)|_U \to a(Rj_*(K|_V))|_U$ is an isomorphism.

0A9M Example 48.4.2. There is a finite morphism $f : X \to Y$ of Noetherian schemes such that (48.4.1.1) is not an isomorphism when evaluated on some $K \in D_{\text{Coh}}(\mathcal{O}_Y)$. Namely, let $X = \text{Spec}(B) \to Y = \text{Spec}(A)$ with $A = k[x, \epsilon]$ where k is a field and $\epsilon^2 = 0$ and $B = k[x] = A/(\epsilon)$. For $n \in \mathbf{N}$ set $M_n = A/(\epsilon, x^n)$. Observe that

$$\text{Ext}^i_A(B, M_n) = M_n, \quad i \geq 0$$

because B has the free periodic resolution $\ldots \to A \to A \to A$ with maps given by multiplication by ϵ. Consider the object $K = \bigoplus M_n[n] = \prod M_n[n]$ of $D_{\text{Coh}}(A)$ (equality in $D(A)$ by Derived Categories, Lemmas 13.33.5 and 13.34.2). Then we see that $a(K)$ corresponds to $R\,\text{Hom}(B, K)$ by Example 48.3.2 and

$$H^0(R\,\text{Hom}(B, K)) = \text{Ext}^0_A(B, K) = \prod\nolimits_{n \geq 1} \text{Ext}^n_A(B.M_n) = \prod\nolimits_{n \geq 1} M_n$$

by the above. But this module has elements which are not annihilated by any power of x, whereas the complex K does have every element of its cohomology annihilated by a power of x. In other words, for the map (48.4.1.1) with $V = D(x)$ and $U = D(x)$ and the complex K cannot be an isomorphism because $(j')^*(a(K))$ is nonzero and $a'(j^*K)$ is zero.

0A9N Lemma 48.4.3. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $V \subset Y$ be quasi-compact open with inverse image $U \subset X$.

(1) For every $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (48.4.1.1) is an isomorphism on all K in $D^+_{QCoh}(\mathcal{O}_Y)$.
(2) For every $Q \in D_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (48.4.1.1) is an isomorphism on all K in $D_{QCoh}(\mathcal{O}_Y)$.
(3) If a commutes with direct sums, then the equivalent conditions of (1) imply the equivalent conditions of (2).

Proof. Proof of (1). Let $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Choose a distinguished triangle

$$K \to Rj_*K|_V \to Q \to K[1]$$

Observe that Q is in $D^+_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 36.4.1) and is supported on $Y \setminus V$ (Derived Categories of Schemes, Definition 36.6.1). Applying a we obtain a distinguished triangle

$$a(K) \to a(Rj_*K|_V) \to a(Q) \to a(K)[1]$$

on X. If $a(Q)$ is supported on $X \setminus U$, then restricting to U the map $a(K)|_U \to a(Rj_*K|_V)|_U$ is an isomorphism, i.e., (48.4.1.1) is an isomorphism on K. The converse is immediate.

The proof of (2) is exactly the same as the proof of (1).

Proof of (3). Assume the equivalent conditions of (1) hold. Set $T = Y \setminus V$. We will use the notation $D_{QCoh,T}(\mathcal{O}_Y)$ and $D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)$ to denote complexes whose cohomology sheaves are supported on T and $f^{-1}(T)$. Since a commutes with direct sums, the strictly full, saturated, triangulated subcategory $\mathcal{D}$ with objects

$$\{Q \in D_{QCoh,T}(\mathcal{O}_Y) \mid a(Q) \in D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)\}$$

is preserved by direct sums and hence derived colimits. On the other hand, the category $D_{QCoh,T}(\mathcal{O}_Y)$ is generated by a perfect object E (see Derived Categories of Schemes, Lemma 36.15.4). By assumption we see that $E \in \mathcal{D}$. By Derived Categories, Lemma 13.37.3 every object Q of $D_{QCoh,T}(\mathcal{O}_Y)$ is a derived colimit of a system $Q_1 \to Q_2 \to Q_3 \to \ldots$ such that the cones of the transition maps are direct sums of shifts of E. Arguing by induction we see that $Q_n \in \mathcal{D}$ for all n and finally that Q is in $\mathcal{D}$. Thus the equivalent conditions of (2) hold. □

0A9P Lemma 48.4.4. Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper morphism. If[3]

(1) f is flat and of finite presentation, or
(2) Y is Noetherian

then the equivalent conditions of Lemma 48.4.3 part (1) hold for all quasi-compact opens V of Y.

[3]This proof works for those morphisms of quasi-compact and quasi-separated schemes such that Rf_*P is pseudo-coherent for all P perfect on X. It follows easily from a theorem of Kiehl [Kie72] that this holds if f is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this chapter.

Proof. Let $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex P_U on U and a nonzero map $P_U \to a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 36.15.3). Then using Derived Categories of Schemes, Lemma 36.13.10 we may assume there is a perfect complex P on X and a map $P \to a(Q)$ whose restriction to U is nonzero. By definition of a this map is adjoint to a map $Rf_*P \to Q$.

The complex Rf_*P is pseudo-coherent. In case (1) this follows from Derived Categories of Schemes, Lemma 36.30.5. In case (2) this follows from Derived Categories of Schemes, Lemmas 36.11.3 and 36.10.3. Thus we may apply Derived Categories of Schemes, Lemma 36.17.5 and get a map $I \to \mathcal{O}_Y$ of perfect complexes whose restriction to V is an isomorphism such that the composition $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P \to Rf_*P \to Q$ is zero. By Derived Categories of Schemes, Lemma 36.22.1 we have $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P = Rf_*(Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)$. We conclude that the composition

$$Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P \to P \to a(Q)$$

is zero. However, the restriction to U is the map $P|_U \to a(Q)|_U$ which we assumed to be nonzero. This contradiction finishes the proof. □

48.5. Right adjoint of pushforward and base change, I

0AA5 The map (48.4.1.1) is a special case of a base change map. Namely, suppose that we have a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes, i.e., a diagram as in (48.4.0.1). Assume f and g are Tor independent. Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

0AA6 (48.5.0.1) $$L(g')^* \circ a \to L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^*$$

The first arrow comes from the adjunction map $\text{id} \to Rg_*Lg^*$ and the last arrow from the adjunction map $L(g')^*Rg'_* \to \text{id}$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 48.4.1. Alternatively, we can think of (48.5.0.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation

$$a \to a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$\begin{aligned} \text{Hom}_X(M, a(K)) &= \text{Hom}_Y(Rf_*M, K) \\ &\to \text{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\ &= \text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K) \\ &\leftarrow \text{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K) \\ &= \text{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) \\ &= \text{Hom}_X(M, Rg'_*a'(Lg^*K)) \end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Schemes, Lemma 36.22.5) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology, Remarks 20.28.4 and 20.28.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0ATQ **Lemma 48.5.1.** Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "k"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "m"] & Z \end{tikzcd}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and l as well as g and m are Tor independent. Then the maps (48.5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that $g \circ f$ and m are Tor independent (details omitted), hence the statement makes sense. In this proof we write k^* in place of Lk^* and f_* instead of Rf_*. Let a, b, and c be the right adjoints of Lemma 48.3.1 for f, g, and $g \circ f$ and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \to k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \to a' \circ l^*$$

where $\xi_{top} : k_* \circ a' \to a \circ l_*$ is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map $l^* \circ f_* \to f'_* \circ k^*$. The outer arrows come from the canonical maps $1 \to l_* \circ l^*$ and $k^* \circ k_* \to 1$. Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \to l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \to b' \circ m^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \to k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \to c' \circ m^*$$

We have $(g \circ f)_* = g_* \circ f_*$ and hence $c = a \circ b$ and similarly $c' = a' \circ b'$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

$$\begin{tikzcd}
 & & k^* \circ a \circ b \arrow[d] \arrow[ddll] \\
 & & k^* \circ a \circ l_* \circ l^* \circ b \arrow[dl] \\
k^* \circ a \circ b \circ m_* \circ m^* \arrow[r] & k^* \circ a \circ l_* \circ l^* \circ b \circ m_* \circ m^* & k^* \circ k_* \circ a' \circ l^* \circ b \arrow[u, "\xi_{top}"'] \arrow[dl] \arrow[d] \\
 & k^* \circ k_* \circ a' \circ l^* \circ b \circ m_* \circ m^* \arrow[u, "\xi_{top}"'] \arrow[dr] & a' \circ l^* \circ b \arrow[d] \\
k^* \circ k_* \circ a' \circ b' \circ m^* \arrow[uu, "\xi_{rect}"'] \arrow[ddrr] & k^* \circ k_* \circ a' \circ l^* \circ l_* \circ b' \circ m^* \arrow[l] \arrow[u, "\xi_{bot}"'] \arrow[dr] & a' \circ l^* \circ b \circ m_* \circ m^* \\
 & & a' \circ l^* \circ l_* \circ b' \circ m^* \arrow[u, "\xi_{bot}"'] \arrow[d] \\
 & & a' \circ b' \circ m^*
\end{tikzcd}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show the diagram

$$\begin{tikzcd}
a \circ l_* \circ l^* \circ b \circ m_* & a \circ b \circ m_* \arrow[l] \\
k_* \circ a' \circ l^* \circ b \circ m_* \arrow[u, "\xi_{top}"'] & \\
k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[u, "\xi_{bot}"'] \arrow[r] & k_* \circ a' \circ b' \arrow[uu, "\xi_{rect}"']
\end{tikzcd}$$

becomes commutative if we invert the arrows ξ_{top}, ξ_{bot}, and ξ_{rect} (note that this is different from asking the diagram to be commutative). However, the diagram

$$\begin{tikzcd}
 & a \circ l_* \circ l^* \circ b \circ m_* & \\
a \circ l_* \circ l^* \circ l_* \circ b' \arrow[ur, "\xi_{bot}"] & & k_* \circ a' \circ l^* \circ b \circ m_* \arrow[ul, "\xi_{top}"'] \\
 & k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[ul, "\xi_{top}"] \arrow[ur, "\xi_{bot}"'] &
\end{tikzcd}$$

commutes by Categories, Lemma 4.28.2. Since the diagrams

$$\begin{tikzcd} a \circ l_* \circ l^* \circ b \circ m_* & a \circ b \circ m \arrow[l] \\ a \circ l_* \circ l^* \circ l_* \circ b' \arrow[u] & a \circ l_* \circ b' \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \circ l_* \circ l^* \circ l_* \circ b' \arrow[r] & a \circ l_* \circ b' \\ k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[r] \arrow[u] & k_* \circ a' \circ b' \arrow[u] \end{tikzcd}$$

commute (see references cited) and since the composition of $l_* \to l_* \circ l^* \circ l_* \to l_*$ is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to ξ_{rect} (via the identifications $a \circ b = c$ and $a' \circ b' = c'$). This is the statement dual to Cohomology, Remark 20.28.4 and the proof is complete. □

0ATR Lemma 48.5.2. Consider a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "g'"'] \arrow[d, "f''"'] & X' \arrow[r, "g"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and h as well as f' and h' are Tor independent. Then the maps (48.5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that f and $h \circ h'$ are Tor independent (details omitted), hence the statement makes sense. In this proof we write g^* in place of Lg^* and f_* instead of Rf_*. Let a, a', and a'' be the right adjoints of Lemma 48.3.1 for f, f', and f''. The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \to g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \to a' \circ h^*$$

where $\xi_{right} : g_* \circ a' \to a \circ h_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $h^* \circ f_* \to f'_* \circ g^*$. The outer arrows come from the canonical maps $1 \to h_* \circ h^*$ and $g^* \circ g_* \to 1$. Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \to a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \to k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \to a'' \circ m^*$$

where $k = g \circ g'$ and $m = h \circ h'$. We have $k^* = (g')^* \circ g^*$ and $m^* = (h')^* \circ h^*$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

$$
\begin{tikzcd}
 & (g')^* \circ g^* \circ a \arrow[d] \arrow[ddl] \\
 & (g')^* \circ g^* \circ a \circ h_* \circ h^* \arrow[dl] \\
(g')^* \circ g^* \circ a \circ h_* \circ (h')_* \circ (h')^* \circ h^* & (g')^* \circ g^* \circ g_* \circ a' \circ h^* \arrow[u, "\xi_{right}"'] \arrow[d] \arrow[dl] \\
(g')^* \circ g^* \circ g_* \circ a' \circ (h')_* \circ (h')^* \circ h^* \arrow[u, "\xi_{right}"'] \arrow[dr] & (g')^* \circ a' \circ h^* \arrow[d] \\
(g')^* \circ g^* \circ g_* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \arrow[u, "\xi_{left}"'] \arrow[dr] \arrow[ddr] & (g')^* \circ a' \circ (h')_* \circ (h')^* \circ h^* \\
 & (g')^* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \arrow[u, "\xi_{left}"'] \arrow[d] \\
 & a'' \circ (h')^* \circ h^*
\end{tikzcd}
$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to ξ_{rect}. This is the statement dual to Cohomology, Remark 20.28.5 and the proof is complete. $\square$

0ATS Remark 48.5.3. Consider a commutative diagram

$$
\begin{tikzcd}
X'' \arrow[r, "k'"'] \arrow[d, "f''"'] & X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y'' \arrow[r, "l'"] \arrow[d, "g''"'] & Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z'' \arrow[r, "m'"] & Z' \arrow[r, "m"] & Z
\end{tikzcd}
$$

of quasi-compact and quasi-separated schemes where all squares are cartesian and where (f, l), (g, m), (f', l'), (g', m') are Tor independent pairs of maps. Let a, a', a'', b, b', b'' be the right adjoints of Lemma 48.3.1 for f, f', f'', g, g', g''. Let us label the squares of the diagram A, B, C, D as follows

$$
\begin{matrix}
A & B \\
C & D
\end{matrix}
$$

Then the maps (48.5.0.1) for the squares are (where we use $k^* = Lk^*$, etc)

$$\gamma_A : (k')^* \circ a' \to a'' \circ (l')^* \quad \gamma_B : k^* \circ a \to a' \circ l^*$$
$$\gamma_C : (l')^* \circ b' \to b'' \circ (m')^* \quad \gamma_D : l^* \circ b \to b' \circ m^*$$

For the 2×1 and 1×2 rectangles we have four further base change maps

$$\gamma_{A+B} : (k \circ k')^* \circ a \to a'' \circ (l \circ l')^*$$
$$\gamma_{C+D} : (l \circ l')^* \circ b \to b'' \circ (m \circ m')^*$$
$$\gamma_{A+C} : (k')^* \circ (a' \circ b') \to (a'' \circ b'') \circ (m')^*$$
$$\gamma_{B+D} : k^* \circ (a \circ b) \to (a' \circ b') \circ m^*$$

By Lemma 48.5.2 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 48.5.1 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \mathrm{id}_{l^*}) \circ (\mathrm{id}_{(k')^*} \star \gamma_B)$ with notation as in Categories, Section 4.28 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 48.5.1 and 48.5.2 of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{tikzcd}
(k')^* \circ a' \circ l^* \circ b \arrow[r, "\gamma_D"'] \arrow[d, "\gamma_A"'] & (k')^* \circ a' \circ b' \circ m^* \arrow[d, "\gamma_A"] \\
a'' \circ (l')^* \circ l^* \circ b \arrow[r, "\gamma_D"] & a'' \circ (l')^* \circ b' \circ m^*
\end{tikzcd}$$

commutes. This is true by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1.

48.6. Right adjoint of pushforward and base change, II

0BZF In this section we prove that the base change map of Section 48.5 is an isomorphism in some cases. We first observe that it suffices to check over affine opens, provided formation of the right adjoint of pushforward commutes with restriction to opens.

0E9S **Remark 48.6.1.** Consider a cartesian diagram

$$\begin{tikzcd}
X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

of quasi-compact and quasi-separated schemes with (g, f) Tor independent. Let $V \subset Y$ and $V' \subset Y'$ be affine opens with $g(V') \subset V$. Form the cartesian diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d] \\ V' \arrow[r] & Y' \end{tikzcd}$$

Assume (48.4.1.1) with respect to K and the first diagram and (48.4.1.1) with respect to Lg^*K and the second diagram are isomorphisms. Then the restriction of the base change map (48.5.0.1)

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

to U' is isomorphic to the base change map (48.5.0.1) for $K|_V$ and the cartesian diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

This follows from the fact that (48.4.1.1) is a special case of the base change map (48.5.0.1) and that the base change maps compose correctly if we stack squares horizontally, see Lemma 48.5.2. Thus in order to check the base change map restricted to U' is an isomorphism it suffices to work with the last diagram.

0AA8 **Lemma 48.6.2.** In diagram (48.4.0.1) assume

(1) $g : Y' \to Y$ is a morphism of affine schemes,
(2) $f : X \to Y$ is proper, and
(3) f and g are Tor independent.

Then the base change map (48.5.0.1) induces an isomorphism

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

in the following cases

(1) for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is flat of finite presentation,
(2) for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is perfect and Y Noetherian,
(3) for $K \in D^+_{QCoh}(\mathcal{O}_X)$ if g has finite Tor dimension and Y Noetherian.

Proof. Write $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Let M be a perfect generator for $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 36.15.3. Then $L(g')^*M$ is a generator for $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Remark 36.16.4. Hence it suffices to show that (48.5.0.1) induces an isomorphism

0E45 (48.6.2.1) $$R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology, Section 20.44, as this will imply the cone of $L(g')^*a(K) \to a'(Lg^*K)$ is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (48.6.2.1).

The left hand side. Because M is perfect, the canonical map

$$R\mathrm{Hom}_X(M, a(K)) \otimes^{\mathbf{L}}_A A' \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Schemes, Lemma 36.22.6. We can combine this with the isomorphism $R\operatorname{Hom}_Y(Rf_*M, K) = R\operatorname{Hom}_X(M, a(K))$ of Lemma 48.3.10 to get that the left hand side equals $R\operatorname{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A'$.

The right hand side. Here we first use the isomorphism

$$R\operatorname{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\operatorname{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 48.3.10. Then we use the base change map $Lg^*Rf_*M \to Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Schemes, Lemma 36.22.5. Hence we may rewrite this as $R\operatorname{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since Y, Y' are affine and K, Rf_*M are in $D_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 36.4.1) we have a canonical map

$$\beta : R\operatorname{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (15.99.1.1) where we have used Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.8 to translate back and forth into algebra.

(1) If f is flat and of finite presentation, the complex Rf_*M is perfect on Y by Derived Categories of Schemes, Lemma 36.30.4 and β is an isomorphism by More on Algebra, Lemma 15.99.2 part (1).
(2) If f is perfect and Y Noetherian, the complex Rf_*M is perfect on Y by More on Morphisms, Lemma 37.61.13 and β is an isomorphism as before.
(3) If g has finite tor dimension and Y is Noetherian, the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Schemes, Lemmas 36.11.3 and 36.10.3) and β is an isomorphism by More on Algebra, Lemma 15.99.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (48.6.2.1) given in the second and third paragraph are in fact given by (48.6.2.1). To make our formulas manageable we will use $(-,-)_X = R\operatorname{Hom}_X(-,-)$, use $-\otimes A'$ in stead of $-\otimes_A^{\mathbf{L}} A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

$$\begin{tikzcd}
((g')^*M, (g')^*a(K))_{X'} \arrow[d] & (M, a(K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[r, equal] \arrow[d] & (f_*M, K)_Y \otimes A' \arrow[d] \\
((g')^*M, (g')^*a(g_*g^*K))_{X'} & (M, a(g_*g^*K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[r, equal] & (f_*M, g_*g^*K)_Y \otimes A' \arrow[dd, bend right=60, "\mu'"'] \\
((g')^*M, (g')^*g'_*a'(g^*K))_{X'} \arrow[u] \arrow[d] & (M, g'_*a'(g^*K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[u] \arrow[dl, "\mu"] & (f_*M, K) \otimes A' \arrow[d, "\beta"] \\
((g')^*M, a'(g^*K))_{X'} \arrow[r, equal] & (f'_*(g')^*M, g^*K)_{Y'} \arrow[r] & (g^*f_*M, g^*K)_{Y'}
\end{tikzcd}$$

The arrows labeled α are the maps from Derived Categories of Schemes, Lemma 36.22.6 for the diagram with corners X', X, Y', Y. The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ a' \to a \circ g_*$ of Lemma 48.4.1 and therefore the middle square is commutative. Going down the left hand side is (48.6.2.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including β provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \to E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \to E'$ the map induced by c and the adjointness of restriction and base change; if c is clear we write $\mu = \mu_c$, i.e., we drop c from the notation. The map μ in the diagram is of this form with c given by the identification $(M, g'_*a(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$; the triangle involving μ is commutative by Derived Categories of Schemes, Remark 36.22.7.

Observe that

$$\begin{tikzcd}
(M, a(g_*g^*K))_X \arrow[r, equal] & (f_*M, g_*g^*K)_Y \arrow[r, equal] & (g^*f_*M, g^*K)_{Y'} \\
(M, g'_*a'(g^*K))_X \arrow[u] \arrow[r, equal] & ((g')^*M, a'(g^*K))_{X'} \arrow[r, equal] & (f'_*(g')^*M, g^*K)_{Y'} \arrow[u]
\end{tikzcd}$$

is commutative by the very definition of the transformation $g'_* \circ a' \to a \circ g_*$. Letting μ' be as above corresponding to the identification $(f_*M, g_*g^*K)_X = (g^*f_*M, g^*K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that β is equal to the composition of $(f_*M, K)_Y \otimes A' \to (f_*M, g_*g^*K)_X \otimes A'$ and μ'. To do this, it suffices to prove the two induced maps $(f_*M, K)_Y \to (g^*f_*M, g^*K)_{Y'}$ are the same. In other words, it suffices to show the diagram

$$\begin{tikzcd}
R\operatorname{Hom}_A(E, K) \arrow[rr, "\text{induced by } \beta"'] \arrow[dr] & & R\operatorname{Hom}_{A'}(E \otimes_A^{\mathbf{L}} A', K \otimes_A^{\mathbf{L}} A') \\
& R\operatorname{Hom}_A(E, K \otimes_A^{\mathbf{L}} A') \arrow[ur] &
\end{tikzcd}$$

commutes for all $E, K \in D(A)$. Since this is how β is constructed in More on Algebra, Section 15.99 the proof is complete. □

48.7. Right adjoint of pushforward and trace maps

0AWG Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 48.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\operatorname{Tr}_f : Rf_* \circ a \longrightarrow \operatorname{id}$$

The corresponding map $\operatorname{Tr}_{f,K} : Rf_*a(K) \longrightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the trace map. This is the map which has the property that the bijection

$$\operatorname{Hom}_X(L, a(K)) \longrightarrow \operatorname{Hom}_Y(Rf_*L, K)$$

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

$$\varphi \longmapsto \operatorname{Tr}_{f,K} \circ Rf_*\varphi$$

The map (48.3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

comes about by composition with $\operatorname{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0B6J Lemma 48.7.1 (Trace map and base change). Suppose we have a diagram (48.4.0.1) where f and g are tor independent. Then the maps $1 \star \text{Tr}_f : Lg^* \circ Rf_* \circ a \to Lg^*$ and $\text{Tr}_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \to Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 20.28.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (48.5.0.1). More precisely, the diagram

$$\begin{tikzcd}
Lg^* \circ Rf_* \circ a \arrow[r, "1\star\text{Tr}_f"'] \arrow[d, "\beta\star 1"'] & Lg^* \\
Rf'_* \circ L(g')^* \circ a \arrow[r, "1\star\alpha"] & Rf'_* \circ a' \circ Lg^* \arrow[u, "\text{Tr}_{f'}\star 1"']
\end{tikzcd}$$

of transformations of functors commutes.

Proof. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 48.4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \to g^* \circ f_* \circ a \circ g_* \circ g^* \to g^* \circ g_* \circ g^* \to g^*$$

where the first arrow is the unit for (g^*, g_*), the second arrow is Tr_f, and the third arrow is the counit for (g^*, g_*). This is a simple consequence of the fact that the composition $g^* \to g^* \circ g_* \circ g^* \to g^*$ of unit and counit is the identity. Consider the diagram

$$\begin{tikzcd}
 & g^* \circ f_* \circ a \arrow[r, "\text{Tr}_f"'] \arrow[ld, "\beta"'] \arrow[d] & g^* & \\
f'_* \circ (g')^* \circ a \arrow[rd] & g^* \circ f_* \circ a \circ g_* \circ g^* \arrow[ru] \arrow[d, "\beta"] & g^* \circ f_* \circ g'_* \circ a' \circ g^* \arrow[l, "\beta^\vee"'] \arrow[d, "\beta"] & f'_* \circ a' \circ g^* \arrow[lu, "\text{Tr}_{f'}"'] \\
 & f'_* \circ (g')^* \circ a \circ g_* \circ g^* & f'_* \circ (g')^* \circ g'_* \circ a' \circ g^* \arrow[l, "\beta^\vee"'] \arrow[ru] &
\end{tikzcd}$$

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By Categories, Lemma 4.24.8 the square

$$\begin{tikzcd}
g^* \circ f_* \circ g'_* \circ a' \arrow[r, "\beta"'] \arrow[d, "\beta^\vee"'] & f'_* \circ (g')^* \circ g'_* \circ a' \arrow[d] \\
g^* \circ f_* \circ a \circ g_* \arrow[r] & \text{id}
\end{tikzcd}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\text{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma. $\square$

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* as in Lemma 48.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\eta_f : \text{id} \to a \circ Rf_*$$

which is called the unit of the adjunction.

0B6K **Lemma 48.7.2.** Suppose we have a diagram (48.4.0.1) where f and g are tor independent. Then the maps $1 \star \eta_f : L(g')^* \to L(g')^* \circ a \circ Rf_*$ and $\eta_{f'} \star 1 : L(g')^* \to a' \circ Rf'_* \circ L(g')^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 20.28.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (48.5.0.1). More precisely, the diagram

$$
\begin{tikzcd}
L(g')^* \arrow[r, "1\star\eta_f"'] \arrow[d, "\eta_{f'}\star 1"'] & L(g')^* \circ a \circ Rf_* \arrow[d, "\alpha"] \\
a' \circ Rf'_* \circ L(g')^* & a' \circ Lg^* \circ Rf_* \arrow[l, "\beta"']
\end{tikzcd}
$$

of transformations of functors commutes.

Proof. This proof is dual to the proof of Lemma 48.7.1. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 48.4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$
(g')^* \to (g')^* \circ g'_* \circ (g')^* \to (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \to a' \circ f'_* \circ (g')^*
$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_{f'}$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \to (g')^* \circ (g')_* \circ (g')^* \to (g')^*$ of unit and counit is the identity. Consider the diagram

$$
\begin{tikzcd}
 & (g')^* \circ a \circ f_* \arrow[r] & (g')^* \circ a \circ g_* \circ g^* \circ f_* \arrow[dl, "\beta"'] \\
(g')^* \arrow[ur, "\eta_f"] \arrow[dr] \arrow[dd, "\eta_{f'}"'] & (g')^* \circ a \circ g_* \circ f'_* \circ (g')^* & (g')^* \circ g'_* \circ a' \circ g^* \circ f_* \arrow[u, "\beta^\vee"'] \arrow[dl, "\beta"'] \arrow[d] \\
 & (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \arrow[u, "\beta^\vee"'] \arrow[dl] & a' \circ g^* \circ f_* \arrow[dll, "\beta"] \\
a' \circ f'_* \circ (g')^* & &
\end{tikzcd}
$$

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 4.24.8 the square

$$
\begin{tikzcd}
\mathrm{id} \arrow[r] \arrow[d] & g'_* \circ a' \circ g^* \circ f_* \arrow[d, "\beta"] \\
g'_* \circ a' \circ g^* \circ f_* \arrow[r, "\beta^\vee"] & a \circ g_* \circ f'_* \circ (g')^*
\end{tikzcd}
$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\beta \circ \alpha \circ \eta_f$ by definition this proves the lemma. $\square$

0B6L Example 48.7.3. Let $A \to B$ be a ring map. Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. As seen in Example 48.3.2 the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ sends an object K of $D(A) = D_{QCoh}(\mathcal{O}_Y)$ to $R\mathrm{Hom}(B, K)$ in $D(B) = D_{QCoh}(\mathcal{O}_X)$. The trace map is the map

$$\mathrm{Tr}_{f,K} : R\mathrm{Hom}(B, K) \longrightarrow R\mathrm{Hom}(A, K) = K$$

induced by the A-module map $A \to B$.

48.8. Right adjoint of pushforward and pullback

0B6N Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint of pushforward as in Lemma 48.3.1. For $K, L \in D_{QCoh}(\mathcal{O}_Y)$ there is a canonical map

$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \longrightarrow a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$$

Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(a(L)) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L$$

(equality by Derived Categories of Schemes, Lemma 36.22.1) for which we use the trace map $Rf_*a(L) \to L$. When $L = \mathcal{O}_Y$ we obtain a map

0A9S (48.8.0.1) $$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in K and compatible with distinguished triangles.

0A9T Lemma 48.8.1. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. The map $Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \to a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$ defined above for $K, L \in D_{QCoh}(\mathcal{O}_Y)$ is an isomorphism if K is perfect. In particular, (48.8.0.1) is an isomorphism if K is perfect.

Proof. Let $K^\vee$ be the "dual" to K, see Cohomology, Lemma 20.50.5. For $M \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K^\vee, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K^\vee, a(L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) \end{aligned}$$

Second equality by the definition of a and the projection formula (Cohomology, Lemma 20.54.3) or the more general Derived Categories of Schemes, Lemma 36.22.1. Hence the result by the Yoneda lemma. ☐

0B6P Lemma 48.8.2. Suppose we have a diagram (48.4.0.1) where f and g are tor independent. Let $K \in D_{QCoh}(\mathcal{O}_Y)$. The diagram

$$\begin{tikzcd} L(g')^*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)) \arrow[r] \arrow[d] & L(g')^*a(K) \arrow[d] \\ L(f')^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} a'(\mathcal{O}_{Y'}) \arrow[r] & a'(Lg^*K) \end{tikzcd}$$

commutes where the horizontal arrows are the maps (48.8.0.1) for K and Lg^*K and the vertical maps are constructed using Cohomology, Remark 20.28.3 and (48.5.0.1).

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. Let us write (48.8.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) & \to a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ & \leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ & \to a(K \otimes \mathcal{O}_Y) \\ & \to a(K) \end{aligned}$$

Here the first arrow is the unit η_f, the second arrow is a applied to Cohomology, Equation (20.54.2.1) which is an isomorphism by Derived Categories of Schemes, Lemma 36.22.1, the third arrow is a applied to $\text{id}_K \otimes \text{Tr}_f$, and the fourth arrow is a applied to the isomorphism $K \otimes \mathcal{O}_Y = K$. The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For η_f and Tr_f this is Lemmas 48.7.2 and 48.7.1. For the arrow using Cohomology, Equation (20.54.2.1) this is Cohomology, Remark 20.54.5. For the multiplication map it is clear. This finishes the proof. □

0B6Q Lemma 48.8.3. Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open such that $f^{-1}(V) \to V$ is an isomorphism. Then for $K \in D^+_{QCoh}(\mathcal{O}_Y)$ the map (48.8.0.1) restricts to an isomorphism over $f^{-1}(V)$.

Proof. By Lemma 48.4.4 the map (48.4.1.1) is an isomorphism for objects of $D^+_{QCoh}(\mathcal{O}_Y)$. Hence Lemma 48.8.2 tells us the restriction of (48.8.0.1) for K to $f^{-1}(V)$ is the map (48.8.0.1) for $K|_V$ and $f^{-1}(V) \to V$. Thus it suffices to show that the map is an isomorphism when f is the identity morphism. This is clear. □

0B6R Lemma 48.8.4. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of quasi-compact and quasi-separated schemes and set $h = g \circ f$. Let a, b, c be the adjoints of Lemma 48.3.1 for f, g, h. For any $K \in D_{QCoh}(\mathcal{O}_Z)$ the diagram

$$\begin{tikzcd} Lf^*(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} b(\mathcal{O}_Z)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \arrow[r] \arrow[d, equal] & a(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} b(\mathcal{O}_Z)) \arrow[r] & a(b(K)) \arrow[d, equal] \\ Lh^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*b(\mathcal{O}_Z) \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \arrow[r] & Lh^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} c(\mathcal{O}_Z) \arrow[r] & c(K) \end{tikzcd}$$

is commutative where the arrows are (48.8.0.1) and we have used $Lh^* = Lf^* \circ Lg^*$ and $c = a \circ b$.

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. The composition of the top arrows is adjoint to a map

$$g_*f_*(f^*(g^*K \otimes b(\mathcal{O}_Z)) \otimes a(\mathcal{O}_Y)) \to K$$

The left hand side is equal to $K \otimes g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y))$ by Derived Categories of Schemes, Lemma 36.22.1 and inspection of the definitions shows the map comes from the map

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{g_*\epsilon} g_*(b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y)) \xrightarrow{g_*\alpha} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

tensored with id_K. Here ϵ is the isomorphism from Derived Categories of Schemes, Lemma 36.22.1 and β comes from the counit map $g_*b \to \text{id}$. Similarly, the composition of the lower horizontal arrows is adjoint to id_K tensored with the composition

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xrightarrow{g_*f_*\delta} g_*f_*(ab(\mathcal{O}_Z)) \xrightarrow{g_*\gamma} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

where γ comes from the counit map $f_*a \to \mathrm{id}$ and δ is the map whose adjoint is the composition

$$f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{\epsilon} b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y) \xrightarrow{\alpha} b(\mathcal{O}_Z)$$

By general properties of adjoint functors, adjoint maps, and counits (see Categories, Section 4.24) we have $\gamma \circ f_*\delta = \alpha \circ \epsilon^{-1}$ as desired. □

48.9. Right adjoint of pushforward for closed immersions

0A74 Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces such that i is a homeomorphism onto a closed subset and such that $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. (For example a closed immersion of schemes.) Let $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

a sheaf of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. Hence by Modules, Lemma 17.13.4 there is a sheaf of $\mathcal{O}_Z$-modules, which we will denote $\mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$, such that

$$i_*\,\mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

as $\mathcal{O}_X$-modules. We spell out what this means.

0A75 Lemma 48.9.1. With notation as above. The functor $\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to the functor $i_* : \mathrm{Mod}(\mathcal{O}_Z) \to \mathrm{Mod}(\mathcal{O}_X)$. For $V \subset Z$ open we have

$$\Gamma(V, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})) = \{s \in \Gamma(U, \mathcal{F}) \mid \mathcal{I}s = 0\}$$

where $U \subset X$ is an open whose intersection with Z is V.

Proof. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Z$-modules. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{i_*\mathcal{O}_Z}(i_*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}))$$

The first equality by Modules, Lemma 17.22.3 and the second by the fully faithfulness of i_*, see Modules, Lemma 17.13.4. The description of sections is left to the reader. □

The functor

$$\mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{O}_Z), \quad \mathcal{F} \longmapsto \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$$

is left exact and has a derived extension

$$R\,\mathcal{H}om(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_Z).$$

0A76 Lemma 48.9.2. With notation as above. The functor $R\,\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint of the functor $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.

Proof. This is a consequence of the fact that i_* and $\mathcal{H}om(\mathcal{O}_Z, -)$ are adjoint functors by Lemma 48.9.1. See Derived Categories, Lemma 13.30.3. □

0A77 Lemma 48.9.3. With notation as above. We have

$$Ri_*R\,\mathcal{H}om(\mathcal{O}_Z, K) = R\,\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$$

in $D(\mathcal{O}_X)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\,\mathcal{H}om(\mathcal{O}_Z, -)$. □

0E2I Lemma 48.9.4. With notation as above. For $M \in D(\mathcal{O}_Z)$ we have

$$R\,\mathcal{H}om_{\mathcal{O}_X}(Ri_*M, K) = Ri_*R\,\mathcal{H}om_{\mathcal{O}_Z}(M, R\,\mathcal{H}om(\mathcal{O}_Z, K))$$

in $D(\mathcal{O}_Z)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ and the fact that if $\mathcal{K}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}om(\mathcal{O}_Z, \mathcal{K}^\bullet)$ is a K-injective complex of $\mathcal{O}_Z$-modules, see Derived Categories, Lemma 13.31.9. $\square$

0A78 **Lemma 48.9.5.** Let $i : Z \to X$ be a pseudo-coherent closed immersion of schemes (any closed immersion if X is locally Noetherian). Then

(1) $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$, and
(2) if $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, then the diagram

$$\begin{tikzcd} D^+(B) \arrow[r] & D^+_{QCoh}(\mathcal{O}_Z) \\ D^+(A) \arrow[r] \arrow[u, "R\,\mathrm{Hom}(B,-)"] & D^+_{QCoh}(\mathcal{O}_X) \arrow[u, "R\mathcal{H}om(\mathcal{O}_Z,-)"'] \end{tikzcd}$$

is commutative.

Proof. To explain the parenthetical remark, if X is locally Noetherian, then i is pseudo-coherent by More on Morphisms, Lemma 37.60.9.

Let K be an object of $D^+_{QCoh}(\mathcal{O}_X)$. To prove (1), by Morphisms, Lemma 29.4.1 it suffices to show that i_* applied to $H^n(R\mathcal{H}om(\mathcal{O}_Z, K))$ produces a quasi-coherent module on X. By Lemma 48.9.3 this means we have to show that $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$ is in $D_{QCoh}(\mathcal{O}_X)$. Since i is pseudo-coherent the sheaf $\mathcal{O}_Z$ is a pseudo-coherent $\mathcal{O}_X$-module. Hence the result follows from Derived Categories of Schemes, Lemma 36.10.8.

Assume $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective A-modules representing an object K of $D^+(A)$. Then we know that $R\,\mathrm{Hom}(B, K) = \mathrm{Hom}_A(B, I^\bullet)$ viewed as a complex of B-modules. Choose a quasi-isomorphism

$$\widetilde{I^\bullet} \longrightarrow \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex of injective $\mathcal{O}_X$-modules. It follows from the description of the functor $\mathcal{H}om(\mathcal{O}_Z, -)$ in Lemma 48.9.1 that there is a map

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \Gamma(Z, \mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet))$$

Observe that $\mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet)$ represents $R\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$. Applying the universal property of the $\widetilde{\ }$ functor we obtain a map

$$\widetilde{\mathrm{Hom}_A(B, I^\bullet)} \longrightarrow R\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$$

in $D(\mathcal{O}_Z)$. We may check that this map is an isomorphism in $D(\mathcal{O}_Z)$ after applying i_*. However, once we apply i_* we obtain the isomorphism of Derived Categories of Schemes, Lemma 36.10.8 via the identification of Lemma 48.9.3. $\square$

0A79 **Lemma 48.9.6.** Let $i : Z \to X$ be a closed immersion of schemes. Assume X is a locally Noetherian. Then $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{\mathrm{Coh}}(\mathcal{O}_X)$ into $D^+_{\mathrm{Coh}}(\mathcal{O}_Z)$.

Proof. The question is local on X, hence we may assume that X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ with A Noetherian and $A \to B$ surjective. In this case, we can apply Lemma 48.9.5 to translate the question into algebra. The corresponding algebra result is a consequence of Dualizing Complexes, Lemma 47.13.4. $\square$

0A9X Lemma 48.9.7. Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a pseudo-coherent closed immersion (if X is Noetherian, then any closed immersion is pseudo-coherent). Let $a : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Z)$ be the right adjoint to Ri_*. Then there is a functorial isomorphism

$$a(K) = R\mathcal{H}om(\mathcal{O}_Z, K)$$

for $K \in D^+_{QCoh}(\mathcal{O}_X)$.

Proof. (The parenthetical statement follows from More on Morphisms, Lemma 37.60.9.) By Lemma 48.9.2 the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. Moreover, by Lemma 48.9.5 and Lemma 48.3.5 both $R\mathcal{H}om(\mathcal{O}_Z, -)$ and a map $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. $\square$

0B6M Example 48.9.8. If $i : Z \to X$ is closed immersion of Noetherian schemes, then the diagram

$$\begin{tikzcd} i_*a(K) \arrow[rr, "\mathrm{Tr}_{i,K}"'] \arrow[d, equal] & & K \arrow[d, equal] \\ i_*R\mathcal{H}om(\mathcal{O}_Z, K) \arrow[r, equal] & R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K) \arrow[r] & K \end{tikzcd}$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. Here the horizontal equality sign is Lemma 48.9.3 and the lower horizontal arrow is induced by the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$. The commutativity of the diagram is a consequence of Lemma 48.9.7.

48.10. Right adjoint of pushforward for closed immersions and base change

0E2J Consider a cartesian diagram of schemes

$$\begin{tikzcd} Z' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed immersion. If Z and X' are tor independent over X, then there is a canonical base change map

0E2K (48.10.0.1) $$Lg^*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^*K)$$

in $D(\mathcal{O}_{Z'})$ functorial for K in $D(\mathcal{O}_X)$. Namely, by adjointness of Lemma 48.9.2 such an arrow is the same thing as a map

$$Ri'_*Lg^*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^*K$$

in $D(\mathcal{O}_{X'})$. By tor independence we have $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ (see Derived Categories of Schemes, Lemma 36.22.9). Thus this is the same thing as a map

$$Lf^*Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^*K$$

For this we can use $Lf^*(can)$ where $can : Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \to K$ is the counit of the adjunction.

0E2L Lemma 48.10.1. In the situation above, the map (48.10.0.1) is an isomorphism if and only if the base change map

$$Lf^*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_{Z'}, Lf^*K)$$

of Cohomology, Remark 20.42.13 is an isomorphism.

Proof. The statement makes sense because $\mathcal{O}_{Z'} = Lf^*\mathcal{O}_Z$ by the assumed tor independence. Since i'_* is exact and faithful we see that it suffices to show the map (48.10.0.1) is an isomorphism after applying Ri'_*. Since $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ by the assumed tor indepence and Derived Categories of Schemes, Lemma 36.22.9 we obtain a map

$$Lf^*Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Ri'_*R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^*K)$$

whose source and target are as in the statement of the lemma by Lemma 48.9.3. We omit the verification that this is the same map as the one constructed in Cohomology, Remark 20.42.13. □

0E2M Lemma 48.10.2. In the situation above, assume f is flat and i pseudo-coherent. Then (48.10.0.1) is an isomorphism for K in $D^+_{QCoh}(\mathcal{O}_X)$.

Proof. First proof. To prove this map is an isomorphism, we may work locally. Hence we may assume X, X', Z, Z' are affine, say corresponding to the rings A, A', B, B'. Then B and A' are tor independent over A. By Lemma 48.10.1 it suffices to check that

$$R\operatorname{Hom}_A(B, K) \otimes_A^{\mathbf{L}} A' = R\operatorname{Hom}_{A'}(B', K \otimes_A^{\mathbf{L}} A')$$

in $D(A')$ for all $K \in D^+(A)$. Here we use Derived Categories of Schemes, Lemma 36.10.8 and the fact that B, resp. B' is pseudo-coherent as an A-module, resp. A'-module to compare derived hom on the level of rings and schemes. The displayed equality follows from More on Algebra, Lemma 15.98.3 part (3). See also the discussion in Dualizing Complexes, Section 47.14.

Second proof[4]. Let $z' \in Z'$ with image $z \in Z$. First show that (48.10.0.1) on stalks at z' induces the map

$$R\operatorname{Hom}(\mathcal{O}_{Z,z}, K_z) \otimes_{\mathcal{O}_{Z,x}}^{\mathbf{L}} \mathcal{O}_{Z',z'} \longrightarrow R\operatorname{Hom}(\mathcal{O}_{Z',z'}, K_z \otimes_{\mathcal{O}_{X,z}}^{\mathbf{L}} \mathcal{O}_{X',z'})$$

from Dualizing Complexes, Equation (47.14.0.1). Namely, the constructions of these maps are identical. Then apply Dualizing Complexes, Lemma 47.14.2. □

0E2N Lemma 48.10.3. Let $i : Z \to X$ be a pseudo-coherent closed immersion of schemes. Let $M \in D_{QCoh}(\mathcal{O}_X)$ locally have tor-amplitude in $[a, \infty)$. Let $K \in D^+_{QCoh}(\mathcal{O}_X)$. Then there is a canonical isomorphism

$$R\mathcal{H}om(\mathcal{O}_Z, K) \otimes_{\mathcal{O}_Z}^{\mathbf{L}} Li^*M = R\mathcal{H}om(\mathcal{O}_Z, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

in $D(\mathcal{O}_Z)$.

Proof. A map from LHS to RHS is the same thing as a map

$$Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} M \longrightarrow K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$$

by Lemmas 48.9.2 and 48.9.3. For this map we take the counit $Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \to K$ tensored with id_M. To see this map is an isomorphism under the hypotheses given, translate into algebra using Lemma 48.9.5 and then for example use More on Algebra, Lemma 15.98.3 part (3). Instead of using Lemma 48.9.5 you can look at stalks as in the second proof of Lemma 48.10.2. □

[4]This proof shows it suffices to assume K is in $D^+(\mathcal{O}_X)$.

48.11. Right adjoint of pushforward for finite morphisms

0AWZ If $i : Z \to X$ is a closed immersion of schemes, then there is a right adjoint $\mathcal{H}om(\mathcal{O}_Z, -)$ to the functor $i_* : \text{Mod}(\mathcal{O}_Z) \to \text{Mod}(\mathcal{O}_X)$ whose derived extension $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. See Section 48.9. In the case of a finite morphism $f : Y \to X$ this strategy cannot work, as the functor $f_* : \text{Mod}(\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X)$ is not exact in general and hence does not have a right adjoint. A replacement is to consider the exact functor $\text{Mod}(f_*\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X)$ and consider the corresponding right adjoint and its derived extension.

Let $f : Y \to X$ be an affine morphism of schemes. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F})$$

is a sheaf of $f_*\mathcal{O}_Y$-modules. We obtain a functor $\text{Mod}(\mathcal{O}_X) \to \text{Mod}(f_*\mathcal{O}_Y)$ which we will denote $\mathcal{H}om(f_*\mathcal{O}_Y, -)$.

0BUZ **Lemma 48.11.1.** With notation as above. The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is a right adjoint to the restriction functor $\text{Mod}(f_*\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X)$. For an affine open $U \subset X$ we have

$$\Gamma(U, \mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{F})) = \text{Hom}_A(B, \mathcal{F}(U))$$

where $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(f^{-1}(U))$.

Proof. Adjointness follows from Modules, Lemma 17.22.3. As f is affine we see that $f_*\mathcal{O}_Y$ is the quasi-coherent sheaf corresponding to B viewed as an A-module. Hence the description of sections over U follows from Schemes, Lemma 26.7.1. $\square$

The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is left exact. Let

$$R\mathcal{H}om(f_*\mathcal{O}_Y, -) : D(\mathcal{O}_X) \longrightarrow D(f_*\mathcal{O}_Y)$$

be its derived extension.

0BV0 **Lemma 48.11.2.** With notation as above. The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is the right adjoint of the functor $D(f_*\mathcal{O}_Y) \to D(\mathcal{O}_X)$.

Proof. Follows from Lemma 48.11.1 and Derived Categories, Lemma 13.30.3. $\square$

0BV1 **Lemma 48.11.3.** With notation as above. The composition

$$D(\mathcal{O}_X) \xrightarrow{R\mathcal{H}om(f_*\mathcal{O}_Y, -)} D(f_*\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

is the functor $K \mapsto R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K)$.

Proof. This is immediate from the construction. $\square$

0AX2 **Lemma 48.11.4.** Let $f : Y \to X$ be a finite pseudo-coherent morphism of schemes (a finite morphism of Noetherian schemes is pseudo-coherent). The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(f_*\mathcal{O}_Y)$. If X is quasi-compact and quasi-separated, then the diagram

$$\begin{tikzcd} D^+_{QCoh}(\mathcal{O}_X) \arrow[rr, "a"'] \arrow[dr, "R\mathcal{H}om(f_*\mathcal{O}_Y, -)"'] & & D^+_{QCoh}(\mathcal{O}_Y) \arrow[dl, "\Phi"] \\ & D^+_{QCoh}(f_*\mathcal{O}_Y) & \end{tikzcd}$$

is commutative, where a is the right adjoint of Lemma 48.3.1 for f and Φ is the equivalence of Derived Categories of Schemes, Lemma 36.5.4.

Proof. (The parenthetical remark follows from More on Morphisms, Lemma 37.60.9.) Since f is pseudo-coherent, the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ is pseudo-coherent, see More on Morphisms, Lemma 37.60.8. Thus $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(f_*\mathcal{O}_Y)$, see Derived Categories of Schemes, Lemma 36.10.8. Then $\Phi \circ a$ and $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ agree on $D^+_{QCoh}(\mathcal{O}_X)$ because these functors are both right adjoint to the restriction functor $D^+_{QCoh}(f_*\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$. To see this use Lemmas 48.3.5 and 48.11.2. $\square$

0AX3 Remark 48.11.5. If $f : Y \to X$ is a finite morphism of Noetherian schemes, then the diagram

$$\begin{tikzcd} Rf_*a(K) \arrow[r, "\mathrm{Tr}_{f,K}"'] \arrow[d, equal] & K \arrow[d, equal] \\ R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K) \arrow[r] & K \end{tikzcd}$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. This follows from Lemma 48.11.4. The lower horizontal arrow is induced by the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ and the upper horizontal arrow is the trace map discussed in Section 48.7.

48.12. Right adjoint of pushforward for proper flat morphisms

0E4H For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated schemes the right adjoint of pushforward enjoys some remarkable properties.

0E4I Lemma 48.12.1. Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then a commutes with direct sums.

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Lemma 36.30.4 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 36.17.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 36.15.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. $\square$

0E4J Lemma 48.12.2. Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then

(1) for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,
(2) for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (48.4.1.1) is an isomorphism, and

Proof. This follows from Lemmas 48.4.3, 48.4.4, and 48.12.1. $\square$

0E4K Lemma 48.12.3. Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. The map (48.8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 48.12.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (48.8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 36.18.3) generated by a single perfect object (Derived Categories of Schemes, Theorem 36.15.3) we can argue as in More on Algebra, Remark 15.59.11 to see that it suffices to prove (48.8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 48.8.1. $\square$

The following lemma shows that the base change map (48.5.0.1) is an isomorphism for proper, flat morphisms of finite presentation. We will see in Example 48.15.2 that this does not remain true for perfect proper morphisms; in that case one has to make a tor independence condition.

0AAB Lemma 48.12.4. Let $g : Y' \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Then the base change map (48.5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 48.12.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 48.6.1. In this case the statement follows from Lemma 48.6.2. $\square$

0B6S Remark 48.12.5. Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Let a be the adjoint of Lemma 48.3.1 for f. In this situation, $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$ is sometimes called the relative dualizing complex. By Lemma 48.12.3 there is a functorial isomorphism $a(K) = Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet$ for $K \in D_{QCoh}(\mathcal{O}_Y)$. Moreover, the trace map

$$\mathrm{Tr}_{f,\mathcal{O}_Y} : Rf_*\omega_{X/Y}^\bullet \to \mathcal{O}_Y$$

of Section 48.7 induces the trace map for all K in $D_{QCoh}(\mathcal{O}_Y)$. More precisely the diagram

$$\begin{tikzcd}
Rf_*a(K) \arrow[rr, "\mathrm{Tr}_{f,K}"'] \arrow[d, equal] & & K \arrow[d, equal] \\
Rf_*(Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet) \arrow[r, equal] & K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*\omega_{X/Y}^\bullet \arrow[r, "\mathrm{id}_K \otimes \mathrm{Tr}_{f,\mathcal{O}_Y}"] & K
\end{tikzcd}$$

where the equality on the lower right is Derived Categories of Schemes, Lemma 36.22.1. If $g : Y' \to Y$ is a morphism of quasi-compact and quasi-separated schemes and $X' = Y' \times_Y X$, then by Lemma 48.12.4 we have $\omega^\bullet_{X'/Y'} = L(g')^*\omega^\bullet_{X/Y}$ where $g' : X' \to X$ is the projection and by Lemma 48.7.1 the trace map

$$\mathrm{Tr}_{f',\mathcal{O}_{Y'}} : Rf'_*\omega^\bullet_{X'/Y'} \to \mathcal{O}_{Y'}$$

for $f' : X' \to Y'$ is the base change of $\mathrm{Tr}_{f,\mathcal{O}_Y}$ via the base change isomorphism.

0G81 Remark 48.12.6. Let $f : X \to Y$, $\omega^\bullet_{X/Y}$, and $\mathrm{Tr}_{f,\mathcal{O}_Y}$ be as in Remark 48.12.5. Let K and M be in $D_{QCoh}(\mathcal{O}_X)$ with M pseudo-coherent (for example perfect). Suppose given a map $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to \omega^\bullet_{X/Y}$ which corresponds to an isomorphism $K \to R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/Y})$ via Cohomology, Equation (20.42.0.1). Then the relative cup product (Cohomology, Remark 20.28.7)

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M \to Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \to Rf_*\omega^\bullet_{X/Y} \xrightarrow{\mathrm{Tr}_{f,\mathcal{O}_Y}} \mathcal{O}_Y$$

determines an isomorphism $Rf_*K \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*M, \mathcal{O}_Y)$. Namely, since $\omega^\bullet_{X/Y} = a(\mathcal{O}_Y)$ the canonical map (48.3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/Y}) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*M, \mathcal{O}_Y)$$

is an isomorphism by Lemma 48.3.6 and Remark 48.3.8 and the fact that M and Rf_*M are pseudo-coherent, see Derived Categories of Schemes, Lemma 36.30.5. To see that the relative cup product induces this isomorphism use the commutativity of the diagram in Cohomology, Remark 20.42.12.

0E4L Lemma 48.12.7. Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation with relative dualizing complex $\omega^\bullet_{X/Y}$ (Remark 48.12.5). Then

(1) $\omega^\bullet_{X/Y}$ is a Y-perfect object of $D(\mathcal{O}_X)$,
(2) $Rf_*\omega^\bullet_{X/Y}$ has vanishing cohomology sheaves in positive degrees,
(3) $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y})$ is an isomorphism.

Proof. In view of the fact that formation of $\omega^\bullet_{X/Y}$ commutes with base change (see Remark 48.12.5), we may and do assume that Y is affine. For a perfect object E of $D(\mathcal{O}_X)$ we have

$$\begin{aligned} Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) &= Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) \\ &= R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y) \\ &= (Rf_*E^\vee)^\vee \end{aligned}$$

For the first equality, see Cohomology, Lemma 20.50.5. For the second equality, see Lemma 48.3.6, Remark 48.3.8, and Derived Categories of Schemes, Lemma 36.30.4. The third equality is the definition of the dual. In particular these references also show that the outcome is a perfect object of $D(\mathcal{O}_Y)$. We conclude that $\omega^\bullet_{X/Y}$ is Y-perfect by More on Morphisms, Lemma 37.69.6. This proves (1).

Let M be an object of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*\omega^\bullet_{X/Y}) &= \mathrm{Hom}_X(Lf^*M, \omega^\bullet_{X/Y}) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\mathcal{O}_X, \mathcal{O}_Y) \end{aligned}$$

The first equality holds by Cohomology, Lemma 20.28.1. The second equality by construction of a. The third equality by Derived Categories of Schemes, Lemma

36.22.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some N, see Derived Categories of Schemes, Lemma 36.30.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 15.74.2 and the fact that Y is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since Y is affine we conclude that $H^i(Rf_*\omega_{X/Y}^\bullet) = 0$ for $i > 0$. This proves (2).

Let E be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have

$$\begin{aligned}
\mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet) &= \mathrm{Hom}_X(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet) \\
&= \mathrm{Hom}_Y(Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}^\bullet), \mathcal{O}_Y) \\
&= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet)), \mathcal{O}_Y) \\
&= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\
&= R\Gamma(Y, Rf_*E^\vee) \\
&= \mathrm{Hom}_X(E, \mathcal{O}_X)
\end{aligned}$$

The first equality holds by Cohomology, Lemma 20.42.2. The second equality is the definition of $\omega_{X/Y}^\bullet$. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology, Lemma 20.50.5. The fourth equality follows from the equality $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$ shown in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology, Lemma 20.50.5) and the fact that Rf_*E is perfect by Derived Categories of Schemes, Lemma 36.30.4. The last equality is Leray for f. This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\mathcal{H}om(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.10.8. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ corresponding to $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in E we see that the cone on α is an object C of $D_{QCoh}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E, C) = 0$ for all perfect E. Since the perfect objects generate (Derived Categories of Schemes, Theorem 36.15.3) we conclude that α is an isomorphism. $\square$

0E2P Lemma 48.12.8 (Rigidity). Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation with relative dualizing complex $\omega_{X/Y}^\bullet$ (Remark 48.12.5). There is a canonical isomorphism

0E2Q (48.12.8.1)
$$\mathcal{O}_X = c(L\mathrm{pr}_1^*\omega_{X/Y}^\bullet) = c(L\mathrm{pr}_2^*\omega_{X/Y}^\bullet)$$

and a canonical isomorphism

0E2R (48.12.8.2)
$$\omega_{X/Y}^\bullet = c\left(L\mathrm{pr}_1^*\omega_{X/Y}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} L\mathrm{pr}_2^*\omega_{X/Y}^\bullet\right)$$

where c is the right adjoint of Lemma 48.3.1 for the diagonal $\Delta : X \to X \times_Y X$.

Proof. Let a be the right adjoint to Rf_* as in Lemma 48.3.1. Consider the cartesian square

$$\begin{array}{ccc}
X \times_Y X & \xrightarrow{q} & X \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{f} & Y
\end{array}$$

Let b be the right adjoint for p as in Lemma 48.3.1. Then

$$\begin{aligned}\omega^\bullet_{X/Y} &= c(b(\omega^\bullet_{X/Y})) \\ &= c(Lp^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} b(\mathcal{O}_X)) \\ &= c(Lp^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} Lq^*a(\mathcal{O}_Y)) \\ &= c(Lp^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} Lq^*\omega^\bullet_{X/Y})\end{aligned}$$

as in (48.12.8.2). Explanation as follows:

(1) The first equality holds as $\mathrm{id} = c \circ b$ because $\mathrm{id}_X = p \circ \Delta$.
(2) The second equality holds by Lemma 48.12.3.
(3) The third holds by Lemma 48.12.4 and the fact that $\mathcal{O}_X = Lf^*\mathcal{O}_Y$.
(4) The fourth holds because $\omega^\bullet_{X/Y} = a(\mathcal{O}_Y)$.

Equation (48.12.8.1) is proved in exactly the same way. □

0BRU Remark 48.12.9. Lemma 48.12.8 means our relative dualizing complex is rigid in a sense analogous to the notion introduced in [vdB97]. Namely, since the functor on the right of (48.12.8.2) is "quadratic" in $\omega^\bullet_{X/Y}$ and the functor on the left of (48.12.8.2) is "linear" this "pins down" the complex $\omega^\bullet_{X/Y}$ to some extent. There is an approach to duality theory using "rigid" (relative) dualizing complexes, see for example [Nee11], [Yek10], and [YZ09]. We will return to this in Section 48.28.

48.13. Right adjoint of pushforward for perfect proper morphisms

0AA9 The correct generality for this section would be to consider perfect proper morphisms of quasi-compact and quasi-separated schemes, see [LN07].

0A9R Lemma 48.13.1. Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then a commutes with direct sums.

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By More on Morphisms, Lemma 37.61.13 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i))\end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 36.17.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 36.15.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. □

0AAA Lemma 48.13.2. Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then

(1) for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,
(2) for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (48.4.1.1) is an isomorphism, and

Proof. This follows from Lemmas 48.4.3, 48.4.4, and 48.13.1. $\square$

0A9U Lemma 48.13.3. Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. The map (48.8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 48.13.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (48.8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 36.18.3) generated by a single perfect object (Derived Categories of Schemes, Theorem 36.15.3) we can argue as in More on Algebra, Remark 15.59.11 to see that it suffices to prove (48.8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 48.8.1. $\square$

0BZG Lemma 48.13.4. Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let $g : Y' \to Y$ be a morphism with Y' Noetherian. If X and Y' are tor independent over Y, then the base change map (48.5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 48.13.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 48.6.1. In this case the statement follows from Lemma 48.6.2. $\square$

48.14. Right adjoint of pushforward for effective Cartier divisors

0B4A Let X be a scheme and let $i : D \to X$ be the inclusion of an effective Cartier divisor. Denote $\mathcal{N} = i^*\mathcal{O}_X(D)$ the normal sheaf of i, see Morphisms, Section 29.31 and Divisors, Section 31.13. Recall that $R\mathcal{H}om(\mathcal{O}_D, -)$ denotes the right adjoint to $i_* : D(\mathcal{O}_D) \to D(\mathcal{O}_X)$ and has the property $i_*R\mathcal{H}om(\mathcal{O}_D, -) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, -)$, see Section 48.9.

0B4B Lemma 48.14.1. As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. There is a canonical isomorphism $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = \mathcal{N}[-1]$ in $D(\mathcal{O}_D)$.

Proof. Equivalently, we are saying that $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ has a unique nonzero cohomology sheaf in degree 1 and that this sheaf is isomorphic to $\mathcal{N}$. Since i_* is exact and fully faithful, it suffices to prove that $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ is isomorphic to $i_*\mathcal{N}[-1]$. We have $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X)$ by Lemma 48.9.3. We have a resolution

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where $\mathcal{I}$ is the ideal sheaf of D which we can use to compute. Since $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ and $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) = \mathcal{O}_X(D)$ by a local computation, we see that

$$R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

where on the right hand side we have $\mathcal{O}_X$ in degree 0 and $\mathcal{O}_X(D)$ in degree 1. The result follows from the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$$

coming from the fact that D is the zero scheme of the canonical section of $\mathcal{O}_X(D)$ and from the fact that $\mathcal{N} = i^*\mathcal{O}_X(D)$. $\square$

For every object K of $D(\mathcal{O}_X)$ there is a canonical map

0B4C (48.14.1.1) $$Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

in $D(\mathcal{O}_D)$ functorial in K and compatible with distinguished triangles. Namely, this map is adjoint to a map

$$i_*(Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \longrightarrow K$$

where the equality is Cohomology, Lemma 20.54.4 and the arrow comes from the canonical map $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \to \mathcal{O}_X$ induced by $\mathcal{O}_X \to i_*\mathcal{O}_D$.

If $K \in D_{QCoh}(\mathcal{O}_X)$, then (48.14.1.1) is equal to (48.8.0.1) via the identification $a(K) = R\mathcal{H}om(\mathcal{O}_D, K)$ of Lemma 48.9.7. If $K \in D_{QCoh}(\mathcal{O}_X)$ and X is Noetherian, then the following lemma is a special case of Lemma 48.13.3.

0AA4 Lemma 48.14.2. As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Then (48.14.1.1) combined with Lemma 48.14.1 defines an isomorphism

$$Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} \mathcal{N}[-1] \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

functorial in K in $D(\mathcal{O}_X)$.

Proof. Since i_* is exact and fully faithful on modules, to prove the map is an isomorphism, it suffices to show that it is an isomorphism after applying i_*. We will use the short exact sequences $0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$ and $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$ used in the proof of Lemma 48.14.1 without further mention. By Cohomology, Lemma 20.54.4 which was used to define the map (48.14.1.1) the left hand side becomes

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} i_*\mathcal{N}[-1] = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{O}_X \to \mathcal{O}_X(D))$$

The right hand side becomes

$$\begin{aligned} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, K) &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), K) \\ &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \end{aligned}$$

the final equality by Cohomology, Lemma 20.50.5. Since the map comes from the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

the lemma is clear. □

48.15. Right adjoint of pushforward in examples

0BQV In this section we compute the right adjoint to pushforward in some examples. The isomorphisms are canonical but only in the weakest possible sense, i.e., we do not prove or claim that these isomorphisms are compatible with various operations such as base change and compositions of morphisms. There is a huge literature on these types of issues; the reader can start with the material in [Har66], [Con00] (these citations use a different starting point for duality but address the issue of constructing canonical representatives for relative dualizing complexes) and then continue looking at works by Joseph Lipman and collaborators.

0A9W Lemma 48.15.1. Let Y be a Noetherian scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$-module of rank $n+1$ with determinant $\mathcal{L} = \wedge^{n+1}(\mathcal{E})$. Let $f : X = \mathbf{P}(\mathcal{E}) \to Y$

be the projection. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then there is an isomorphism

$$c : f^*\mathcal{L}(-n-1)[n] \longrightarrow a(\mathcal{O}_Y)$$

In particular, if $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$, then $X = \mathbf{P}^n_Y$ and we obtain $a(\mathcal{O}_Y) = \mathcal{O}_X(-n-1)[n]$.

Proof. In (the proof of) Cohomology of Schemes, Lemma 30.8.4 we constructed a canonical isomorphism

$$R^n f_*(f^*\mathcal{L}(-n-1)) \longrightarrow \mathcal{O}_Y$$

Moreover, $Rf_*(f^*\mathcal{L}(-n-1))[n] = R^n f_*(f^*\mathcal{L}(-n-1))$, i.e., the other higher direct images are zero. Thus we find an isomorphism

$$Rf_*(f^*\mathcal{L}(-n-1)[n]) \longrightarrow \mathcal{O}_Y$$

This isomorphism determines c as in the statement of the lemma because a is the right adjoint of Rf_*. By Lemma 48.4.4 construction of the a is local on the base. In particular, to check that c is an isomorphism, we may work locally on Y. In other words, we may assume Y is affine and $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$. In this case the sheaves $\mathcal{O}_X, \mathcal{O}_X(-1), \ldots, \mathcal{O}_X(-n)$ generate $D_{QCoh}(X)$, see Derived Categories of Schemes, Lemma 36.16.3. Hence it suffices to show that $c : \mathcal{O}_X(-n-1)[n] \to a(\mathcal{O}_Y)$ is transformed into an isomorphism under the functors

$$F_{i,p}(-) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), (-)[p])$$

for $i \in \{-n, \ldots, 0\}$ and $p \in \mathbf{Z}$. For $F_{0,p}$ this holds by construction of the arrow c! For $i \in \{-n, \ldots, -1\}$ we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), \mathcal{O}_X(-n-1)[n+p]) = H^p(X, \mathcal{O}_X(-n-1-i)) = 0$$

by the computation of cohomology of projective space (Cohomology of Schemes, Lemma 30.8.1) and we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), a(\mathcal{O}_Y)[p]) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\mathcal{O}_X(i), \mathcal{O}_Y[p]) = 0$$

because $Rf_*\mathcal{O}_X(i) = 0$ by the same lemma. Hence the source and the target of $F_{i,p}(c)$ vanish and $F_{i,p}(c)$ is necessarily an isomorphism. This finishes the proof. $\square$

0AAC Example 48.15.2. The base change map (48.5.0.1) is not an isomorphism if f is perfect proper and g is perfect. Let k be a field. Let $Y = \mathbf{A}^2_k$ and let $f : X \to Y$ be the blowup of Y in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor f as

$$X \xrightarrow{i} \mathbf{P}^1_Y \xrightarrow{p} Y$$

This gives a factorization $a = c \circ b$ where a, b, and c are the right adjoints of Lemma 48.3.1 of Rf_*, Rp_*, and Ri_*. Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbf{P}^1_Y$ and denote $\mathcal{O}_X(n)$ its restriction to X. Note that $X \subset \mathbf{P}^1_Y$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 48.15.1 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 48.9.7 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$

Last equality by Lemma 48.14.2. Let $Y' = \mathrm{Spec}(k)$ be the origin in Y. The restriction of $a(\mathcal{O}_Y)$ to $X' = E = \mathbf{P}^1_k$ is an invertible sheaf of degree -1 placed in cohomological degree 0. But on the other hand, $a'(\mathcal{O}_{\mathrm{Spec}(k)}) = \mathcal{O}_E(-2)[1]$ which is an invertible sheaf of degree -2 placed in cohomological degree -1, so different. In this example the hypothesis of Tor indepence in Lemma 48.6.2 is violated.

0BQW Lemma 48.15.3. Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. There is a canonical isomorphism $c : \mathcal{N} \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Consider the canonical short exact sequence

0BQX (48.15.3.1)
$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_Y/\mathcal{I}^2 \to \mathcal{O}_X \to 0$$

Let $U \subset X$ be open and let $s \in \mathcal{N}(U)$. Then we can pushout (48.15.3.1) via s to get an extension E_s of $\mathcal{O}_X|_U$ by $\mathcal{O}_X|_U$. This in turn defines a section $c(s)$ of $\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ over U. See Cohomology, Lemma 20.42.1 and Derived Categories, Lemma 13.27.6. Conversely, given an extension

$$0 \to \mathcal{O}_X|_U \to \mathcal{E} \to \mathcal{O}_X|_U \to 0$$

of $\mathcal{O}_U$-modules, we can find an open covering $U = \bigcup U_i$ and sections $e_i \in \mathcal{E}(U_i)$ mapping to $1 \in \mathcal{O}_X(U_i)$. Then e_i defines a map $\mathcal{O}_Y|_{U_i} \to \mathcal{E}|_{U_i}$ whose kernel contains $\mathcal{I}^2$. In this way we see that $\mathcal{E}|_{U_i}$ comes from a pushout as above. This shows that c is surjective. We omit the proof of injectivity. $\square$

0BQY Lemma 48.15.4. Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 31.20.2) then composition on $R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ defines isomorphisms

$$\wedge^i(\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)) \longrightarrow \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

for all i.

Proof. By composition we mean the map

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

of Cohomology, Lemma 20.42.5. This induces multiplication maps

$$\mathcal{E}xt^a_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{E}xt^b_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^{a+b}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

Please compare with More on Algebra, Equation (15.63.0.1). The statement of the lemma means that the induced map

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes \ldots \otimes \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

factors through the wedge product and then induces an isomorphism. To see this is true we may work locally on Y. Hence we may assume that we have global sections $f_1, \ldots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Denote

$$\mathcal{A} = \mathcal{O}_Y\langle \xi_1, \ldots, \xi_r \rangle$$

the sheaf of strictly commutative differential graded $\mathcal{O}_Y$-algebras which is a (divided power) polynomial algebra on $\xi_1, \ldots, \xi_r$ in degree -1 over $\mathcal{O}_Y$ with differential d given by the rule $\text{d}\xi_i = f_i$. Let us denote $\mathcal{A}^\bullet$ the underlying complex of $\mathcal{O}_Y$-modules which is the Koszul complex mentioned above. Thus the canonical map $\mathcal{A}^\bullet \to \mathcal{O}_X$ is a quasi-isomorphism. We obtain quasi-isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{A}^\bullet) \to \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

by Cohomology, Lemma 20.46.9. The differentials of the latter complex are zero, and hence

$$\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$$

For $j \in \{1, \ldots, r\}$ let $\delta_j : \mathcal{A} \to \mathcal{A}$ be the derivation of degree 1 with $\delta_j(\xi_i) = \delta_{ij}$ (Kronecker delta). A computation shows that $\delta_j \circ \mathrm{d} = -\mathrm{d} \circ \delta_j$ which shows that we get a morphism of complexes.

$$\delta_j : \mathcal{A}^\bullet \to \mathcal{A}^\bullet[1].$$

Whence δ_j defines a section of the corresponding $\mathcal{E}xt$-sheaf. Another computation shows that $\delta_1, \ldots, \delta_r$ map to a basis for $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-1}, \mathcal{O}_X)$ over $\mathcal{O}_X$. Since it is clear that $\delta_j \circ \delta_j = 0$ and $\delta_j \circ \delta_{j'} = -\delta_{j'} \circ \delta_j$ as endomorphisms of $\mathcal{A}$ and hence in the $\mathcal{E}xt$-sheaves we obtain the statement that our map above factors through the exterior power. To see we get the desired isomorphism the reader checks that the elements

$$\delta_{j_1} \circ \ldots \circ \delta_{j_i}$$

for $j_1 < \ldots < j_i$ map to a basis of the sheaf $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$ over $\mathcal{O}_X$. $\square$

0BQZ Lemma 48.15.5. Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 31.20.2) then

$$R\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r \mathcal{N}[r]$$

where $r : Y \to \{1, 2, 3, \ldots\}$ sends y to the minimal number of generators of $\mathcal{I}$ needed in a neighbourhood of y.

Proof. We can use Lemmas 48.15.3 and 48.15.4 to see that we have isomorphisms $\wedge^i \mathcal{N} \to \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ for $i \geq 0$. Thus it suffices to show that the map $\mathcal{O}_Y \to \mathcal{O}_X$ induces an isomorphism

$$\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

and that $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$ is zero for $i \neq r$. These statements are local on Y. Thus we may assume that we have global sections $f_1, \ldots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Let $\mathcal{A}^\bullet$ be the Koszul complex on $f_1, \ldots, f_r$ as introduced in the proof of Lemma 48.15.4. Then

$$R\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by Cohomology, Lemma 20.46.9. Denote $1 \in H^0(\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y))$ the identity map of $\mathcal{A}^0 = \mathcal{O}_Y \to \mathcal{O}_Y$. With δ_j as in the proof of Lemma 48.15.4 we get an isomorphism of graded $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y\langle \delta_1, \ldots, \delta_r \rangle \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by mapping $\delta_{j_1} \ldots \delta_{j_i}$ to $1 \circ \delta_{j_1} \circ \ldots \circ \delta_{j_i}$ in degree i. Via this isomorphism the differential on the right hand side induces a differential d on the left hand side. By our sign rules we have $\mathrm{d}(1) = -\sum f_j \delta_j$. Since $\delta_j : \mathcal{A}^\bullet \to \mathcal{A}^\bullet[1]$ is a morphism of complexes, it follows that

$$\mathrm{d}(\delta_{j_1} \ldots \delta_{j_i}) = (-\sum f_j \delta_j)\delta_{j_1} \ldots \delta_{j_i}$$

Observe that we have $\mathrm{d} = \sum f_j \delta_j$ on the differential graded algebra $\mathcal{A}$. Therefore the map defined by the rule

$$1 \circ \delta_{j_1} \ldots \delta_{j_i} \longmapsto (\delta_{j_1} \circ \ldots \circ \delta_{j_i})(\xi_1 \ldots \xi_r)$$

will define an isomorphism of complexes

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{A}^\bullet[-r]$$

if r is odd and commuting with differentials up to sign if r is even. In any case these complexes have isomorphic cohomology, which shows the desired vanishing. The isomorphism on cohomology in degree r under the map

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

also follows in a straightforward manner from this. (We observe that our choice of conventions regarding Koszul complexes does intervene in the definition of the isomorphism $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r \mathcal{N}[r]$.) $\square$

0BR0 Lemma 48.15.6. Let Y be a quasi-compact and quasi-separated scheme. Let $i : X \to Y$ be a Koszul-regular closed immersion. Let a be the right adjoint of $Ri_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then there is an isomorphism

$$\wedge^r \mathcal{N}[-r] \longrightarrow a(\mathcal{O}_Y)$$

where $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$ is the normal sheaf of i (Morphisms, Section 29.31) and r is its rank viewed as a locally constant function on X.

Proof. Recall, from Lemmas 48.9.7 and 48.9.3, that $a(\mathcal{O}_Y)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ whose pushforward to Y is $R\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \mathcal{O}_Y)$. Thus the result follows from Lemma 48.15.5. $\square$

0BRT Lemma 48.15.7. Let S be a Noetherian scheme. Let $f : X \to S$ be a smooth proper morphism of relative dimension d. Let a be the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ as in Lemma 48.3.1. Then there is an isomorphism

$$\wedge^d \Omega_{X/S}[d] \longrightarrow a(\mathcal{O}_S)$$

in $D(\mathcal{O}_X)$.

Proof. Set $\omega_{X/S}^\bullet = a(\mathcal{O}_S)$ as in Remark 48.12.5. Let c be the right adjoint of Lemma 48.3.1 for $\Delta : X \to X \times_S X$. Because Δ is the diagonal of a smooth morphism it is a Koszul-regular immersion, see Divisors, Lemma 31.22.11. In particular, Δ is a perfect proper morphism (More on Morphisms, Lemma 37.61.7) and we obtain

$$\begin{aligned}
\mathcal{O}_X &= c(L\text{pr}_1^*\omega_{X/S}^\bullet) \\
&= L\Delta^*(L\text{pr}_1^*\omega_{X/S}^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_{X \times_S X}) \\
&= \omega_{X/S}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_{X \times_S X}) \\
&= \omega_{X/S}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \wedge^d(\mathcal{N}_\Delta)[-d]
\end{aligned}$$

The first equality is (48.12.8.1) because $\omega_{X/S}^\bullet = a(\mathcal{O}_S)$. The second equality by Lemma 48.13.3. The third equality because $\text{pr}_1 \circ \Delta = \text{id}_X$. The fourth equality by Lemma 48.15.6. Observe that $\wedge^d(\mathcal{N}_\Delta)$ is an invertible $\mathcal{O}_X$-module. Hence $\wedge^d(\mathcal{N}_\Delta)[-d]$ is an invertible object of $D(\mathcal{O}_X)$ and we conclude that $a(\mathcal{O}_S) = \omega_{X/S}^\bullet = \wedge^d(\mathcal{C}_\Delta)[d]$. Since the conormal sheaf $\mathcal{C}_\Delta$ of Δ is $\Omega_{X/S}$ by Morphisms, Lemma 29.32.7 the proof is complete. $\square$

48.16. Upper shriek functors

0A9Y In this section, we construct the functors $f^!$ for morphisms between schemes which are of finite type and separated over a fixed Noetherian base using compactifications. As is customary in coherent duality, there are a number of diagrams that have to be shown to be commutative. We suggest the reader, after reading the construction, skips the verification of the lemmas and continues to the next section where we discuss properties of the upper shriek functors.

0F42 **Situation 48.16.1.** Here S is a Noetherian scheme and $\textit{FTS}_S$ is the category whose

(1) objects are schemes X over S such that the structure morphism $X \to S$ is both separated and of finite type, and
(2) morphisms $f : X \to Y$ between objects are morphisms of schemes over S.

In Situation 48.16.1 given a morphism $f : X \to Y$ in $\textit{FTS}_S$, we will define an exact functor

$$f^! : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$$

of triangulated categories. Namely, we choose a compactification $X \to \overline{X}$ over Y which is possible by More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $\overline{a} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{\overline{X}})$ be the right adjoint of $R\overline{f}_*$ constructed in Lemma 48.3.1. Then we set

$$f^!K = \overline{a}(K)|_X$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$. The result is an object of $D^+_{QCoh}(\mathcal{O}_X)$ by Lemma 48.3.5.

0AA0 **Lemma 48.16.2.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. The functor $f^!$ is, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. The category of compactifications of X over Y is defined in More on Flatness, Section 38.32. By More on Flatness, Theorem 38.33.8 and Lemma 38.32.2 it is nonempty. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $j^* \circ \overline{a} : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$ where $\overline{a}$ is the right adjoint of $R\overline{f}_*$ constructed in Lemma 48.3.1.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y such that $g^{-1}(j_2(X)) = j_1(X)$[5]. Let $\overline{c}$ be the right adjoint of Lemma 48.3.1 for g. Then $\overline{c} \circ \overline{a}_2 = \overline{a}_1$ because these functors are adjoint to $R\overline{f}_{2,*} \circ Rg_* = R(\overline{f}_2 \circ g)_*$. By (48.4.1.1) we have a canonical transformation

$$j_1^* \circ \overline{c} \longrightarrow j_2^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_2}) \to D^+_{QCoh}(\mathcal{O}_X)$ which is an isomorphism by Lemma 48.4.4. The composition

$$j_1^* \circ \overline{a}_1 \longrightarrow j_1^* \circ \overline{c} \circ \overline{a}_2 \longrightarrow j_2^* \circ \overline{a}_2$$

is an isomorphism of functors which we will denote by α_g.

Consider two compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$ of X over Y. By More on Flatness, Lemma 38.32.1 part (b) we can find a compactification $j : X \to \overline{X}$ with dense image and morphisms $g_i : \overline{X} \to \overline{X}_i$ of compactifications. By More on Flatness, Lemma 38.32.1 part (c) we have $g_i^{-1}(j_i(X)) = j(X)$. Hence we get isomorpisms

$$\alpha_{g_i} : j^* \circ \overline{a} \longrightarrow j_i^* \circ \overline{a}_i$$

[5] This may fail with our definition of compactification. See More on Flatness, Section 38.32.

by the previous paragraph. We obtain an isomorphism

$$\alpha_{g_2} \circ \alpha_{g_1}^{-1} : j_1^* \circ \overline{a}_1 \to j_2^* \circ \overline{a}_2$$

To finish the proof we have to show that these isomorphisms are well defined. We claim it suffices to show the composition of isomorphisms constructed in the previous paragraph is another (for a precise statement see the next paragraph). We suggest the reader check this is true on a napkin, but we will also completely spell it out in the rest of this paragraph. Namely, consider a second choice of a compactification $j' : X \to \overline{X}'$ with dense image and morphisms of compactifications $g'_i : \overline{X}' \to \overline{X}_i$. By More on Flatness, Lemma 38.32.1 we can find a compactification $j'' : X \to \overline{X}''$ with dense image and morphisms of compactifications $h : \overline{X}'' \to \overline{X}$ and $h' : \overline{X}'' \to \overline{X}'$. We may even assume $g_1 \circ h = g'_1 \circ h'$ and $g_2 \circ h = g'_2 \circ h'$. The result of the next paragraph gives

$$\alpha_{g_i} \circ \alpha_h = \alpha_{g_i \circ h} = \alpha_{g'_i \circ h'} = \alpha_{g'_i} \circ \alpha_{h'}$$

for $i = 1, 2$. Since these are all isomorphisms of functors we conclude that $\alpha_{g_2} \circ \alpha_{g_1}^{-1} = \alpha_{g'_2} \circ \alpha_{g'_1}^{-1}$ as desired.

Suppose given compactifications $j_i : X \to \overline{X}_i$ for $i = 1, 2, 3$. Suppose given morphisms $g : \overline{X}_1 \to \overline{X}_2$ and $h : \overline{X}_2 \to \overline{X}_3$ of compactifications such that $g^{-1}(j_2(X)) = j_1(X)$ and $h^{-1}(j_2(X)) = j_3(X)$. Let $\overline{a}_i$ be as above. The claim above means that

$$\alpha_g \circ \alpha_h = \alpha_{g \circ h} : j_1^* \circ \overline{a}_1 \to j_3^* \circ \overline{a}_3$$

Let $\overline{c}$, resp. $\overline{d}$ be the right adjoint of Lemma 48.3.1 for g, resp. h. Then $\overline{c} \circ \overline{a}_2 = \overline{a}_1$ and $\overline{d} \circ \overline{a}_3 = \overline{a}_2$ and there are canonical transformations

$$j_1^* \circ \overline{c} \longrightarrow j_2^* \quad \text{and} \quad j_2^* \circ \overline{d} \longrightarrow j_3^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_2}) \to D^+_{QCoh}(\mathcal{O}_X)$ and $D^+_{QCoh}(\mathcal{O}_{\overline{X}_3}) \to D^+_{QCoh}(\mathcal{O}_X)$ for the same reasons as above. Denote $\overline{e}$ the right adjoint of Lemma 48.3.1 for $h \circ g$. There is a canonical transformation

$$j_1^* \circ \overline{e} \longrightarrow j_3^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_3}) \to D^+_{QCoh}(\mathcal{O}_X)$ given by (48.4.1.1). Spelling things out we have to show that the composition

$$\alpha_h \circ \alpha_g : j_1^* \circ \overline{a}_1 \to j_1^* \circ \overline{c} \circ \overline{a}_2 \to j_2^* \circ \overline{a}_2 \to j_2^* \circ \overline{d} \circ \overline{a}_3 \to j_3^* \circ \overline{a}_3$$

is the same as the composition

$$\alpha_{h \circ g} : j_1^* \circ \overline{a}_1 \to j_1^* \circ \overline{e} \circ \overline{a}_3 \to j_3^* \circ \overline{a}_3$$

We split this into two parts. The first is to show that the diagram

$$\begin{tikzcd} \overline{a}_1 \arrow[r] \arrow[d] & \overline{c} \circ \overline{a}_2 \arrow[d] \\ \overline{e} \circ \overline{a}_3 \arrow[r] & \overline{c} \circ \overline{d} \circ \overline{a}_3 \end{tikzcd}$$

commutes where the lower horizontal arrow comes from the identification $\overline{e} = \overline{c} \circ \overline{d}$. This is true because the corresponding diagram of total direct image functors

$$\begin{tikzcd} R\overline{f}_{1,*} \arrow[r] \arrow[d] & Rg_* \circ R\overline{f}_{2,*} \arrow[d] \\ R(h \circ g)_* \circ R\overline{f}_{3,*} \arrow[r] & Rg_* \circ Rh_* \circ R\overline{f}_{3,*} \end{tikzcd}$$

is commutative (insert future reference here). The second part is to show that the composition

$$j_1^* \circ \overline{c} \circ \overline{d} \to j_2^* \circ \overline{d} \to j_3^*$$

is equal to the map

$$j_1^* \circ \overline{e} \to j_3^*$$

via the identification $\overline{e} = \overline{c} \circ \overline{d}$. This was proven in Lemma 48.5.1 (note that in the current case the morphisms f', g' of that lemma are equal to id_X). $\square$

0ATX Lemma 48.16.3. In Situation 48.16.1 let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of FTS_S. Then there is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$.

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and Lemma 38.32.2 twice. Let $\overline{a}$ be the right adjoint of Lemma 48.3.1 for $\overline{X} \to \overline{Y}$ and let $\overline{b}$ be the right adjoint of Lemma 48.3.1 for $\overline{Y} \to Z$. Then $\overline{a} \circ \overline{b}$ is the right adjoint of Lemma 48.3.1 for the composition $\overline{X} \to Z$. Hence $g^! = i^* \circ \overline{b}$ and $(g \circ f)^! = (X \to \overline{X})^* \circ \overline{a} \circ \overline{b}$. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"'] \arrow[d] & U \arrow[r, "j'"'] \arrow[dl] & \overline{X} \arrow[dl] \\ Y \arrow[r, "i"'] \arrow[d] & \overline{Y} \arrow[dl] & \\ Z & & \end{tikzcd}$$

Let $\overline{a}'$ be the right adjoint of Lemma 48.3.1 for $U \to Y$. Then $f^! = j^* \circ \overline{a}'$. We obtain

$$\gamma : (j')^* \circ \overline{a} \to \overline{a}' \circ i^*$$

by (48.4.1.1) and we can use it to define

$$(g \circ f)^! = (j' \circ j)^* \circ \overline{a} \circ \overline{b} = j^* \circ (j')^* \circ \overline{a} \circ \overline{b} \to j^* \circ \overline{a}' \circ i^* \circ \overline{b} = f^! \circ g^!$$

which is an isomorphism on objects of $D^+_{QCoh}(\mathcal{O}_Z)$ by Lemma 48.4.4. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d] & U_1 \arrow[r, "j_1'"'] \arrow[dl] & \overline{X}_1 \arrow[dl] \\
Y \arrow[r, "i_1"'] \arrow[d] & \overline{Y}_1 \arrow[dl] & \\
Z & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "j_2"'] \arrow[d] & U_2 \arrow[r, "j_2'"'] \arrow[dl] & \overline{X}_2 \arrow[dl] \\
Y \arrow[r, "i_2"'] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\
Z & &
\end{tikzcd}
$$

We can first choose a compactification $i : Y \to \overline{Y}$ with dense image of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 38.32.1. By More on Flatness, Lemma 38.32.3 and Categories, Lemmas 4.27.13 and 4.27.14 we can choose a compactification $X \to \overline{X}$ with dense image of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when we have a commutative diagram as follows

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, equal] & U_1 \arrow[r, "j_1'"'] \arrow[d] \arrow[ddl] & \overline{X}_1 \arrow[d] \arrow[ddl] \\
X \arrow[r, "j_2"] \arrow[d] & U_2 \arrow[r, "j_2'"] \arrow[ddl] & \overline{X}_2 \arrow[ddl] \\
Y \arrow[r, "i_1"] \arrow[d, equal] & \overline{Y}_1 \arrow[d] & \\
Y \arrow[r, "i_2"] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\
Z & &
\end{tikzcd}
$$

and moreover the compactifications $X \to \overline{X}_1$ and $Y \to \overline{Y}_2$ have dense image. We use $\overline{a}_i$, $\overline{a}'_i$, $\overline{c}$, and $\overline{c}'$ for the right adjoint of Lemma 48.3.1 for $\overline{X}_i \to \overline{Y}_i$, $U_i \to Y$, $\overline{X}_1 \to \overline{X}_2$, and $U_1 \to U_2$. Each of the squares

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & U_1 \arrow[d] \\
X \arrow[r] & U_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X}_2 \arrow[d] \\
Y \arrow[r] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{X}_1 \arrow[d] \\
Y \arrow[r] & \overline{Y}_1
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "D"] & \overline{Y}_1 \arrow[d] \\
Y \arrow[r] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "E"] & \overline{X}_1 \arrow[d] \\
X \arrow[r] & \overline{X}_2
\end{tikzcd}
$$

is cartesian (see More on Flatness, Lemma 38.32.1 part (c) for A, D, E and recall that U_i is the inverse image of Y by $\overline{X}_i \to \overline{Y}_i$ for B, C) and hence gives rise to a base change map (48.4.1.1) as follows

$$
\begin{matrix}
\gamma_A : j_1^* \circ \overline{c}' \to j_2^* & \gamma_B : (j_2')^* \circ \overline{a}_2 \to \overline{a}'_2 \circ i_2^* & \gamma_C : (j_1')^* \circ \overline{a}_1 \to \overline{a}'_1 \circ i_1^* \\
\gamma_D : i_1^* \circ \overline{d} \to i_2^* & \gamma_E : (j_1' \circ j_1)^* \circ \overline{c} \to (j_2' \circ j_2)^* &
\end{matrix}
$$

Denote $f_1^! = j_1^* \circ \overline{a}'_1$, $f_2^! = j_2^* \circ \overline{a}'_2$, $g_1^! = i_1^* \circ \overline{b}_1$, $g_2^! = i_2^* \circ \overline{b}_2$, $(g \circ f)_1^! = (j'_1 \circ j_1)^* \circ \overline{a}_1 \circ \overline{b}_1$, and $(g \circ f)_2^! = (j'_2 \circ j_2)^* \circ \overline{a}_2 \circ \overline{b}_2$. The construction given in the first paragraph of the proof and in Lemma 48.16.2 uses

(1) γ_C for the map $(g \circ f)_1^! \to f_1^! \circ g_1^!$,
(2) γ_B for the map $(g \circ f)_2^! \to f_2^! \circ g_2^!$,
(3) γ_A for the map $f_1^! \to f_2^!$,
(4) γ_D for the map $g_1^! \to g_2^!$, and
(5) γ_E for the map $(g \circ f)_1^! \to (g \circ f)_2^!$.

We have to show that the diagram

$$\begin{tikzcd} (g \circ f)_1^! \arrow[r, "\gamma_E"'] \arrow[d, "\gamma_C"'] & (g \circ f)_2^! \arrow[d, "\gamma_B"'] \\ f_1^! \circ g_1^! \arrow[r, "\gamma_A \circ \gamma_D"] & f_2^! \circ g_2^! \end{tikzcd}$$

is commutative. We will use Lemmas 48.5.1 and 48.5.2 and with (abuse of) notation as in Remark 48.5.3 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_A \circ \gamma_F$ where

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "F"] & \overline{X}_1 \arrow[d] \\ U_2 \arrow[r] & \overline{X}_2 \end{tikzcd}$$

Thus we see that

$$\gamma_B \circ \gamma_E = \gamma_B \circ \gamma_A \circ \gamma_F = \gamma_A \circ \gamma_B \circ \gamma_F$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 48.5.3). Thus it suffices to prove that $\gamma_D \circ \gamma_C = \gamma_B \circ \gamma_F$. Since both of these are equal to the map (48.4.1.1) for the square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & \overline{X}_1 \arrow[d] \\ Y \arrow[r] & \overline{Y}_2 \end{tikzcd}$$

we conclude. □

0ATY **Lemma 48.16.4.** In Situation 48.16.1 the constructions of Lemmas 48.16.2 and 48.16.3 define a pseudo functor from the category FTS_S into the 2-category of categories (see Categories, Definition 4.29.5).

Proof. To show this we have to prove given morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ that

$$\begin{tikzcd} (h \circ g \circ f)^! \arrow[r, "\gamma_{A+B}"'] \arrow[d, "\gamma_{B+C}"'] & f^! \circ (h \circ g)^! \arrow[d, "\gamma_C"] \\ (g \circ f)^! \circ h^! \arrow[r, "\gamma_A"] & f^! \circ g^! \circ h^! \end{tikzcd}$$

is commutative (for the meaning of the γ's, see below). To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under $\overline{X} \to \overline{Y}$. This produces the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X} \arrow[d] \\
Y \arrow[r] \arrow[d, "g"] & Y \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{Y} \arrow[d] \\
Z \arrow[r] \arrow[d, "h"] & Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\
T \arrow[r] & T \arrow[r] & T \arrow[r] & T
\end{tikzcd}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 48.16.3 we see that the maps in the first displayed diagram use the maps (48.4.1.1) for the rectangles $A+B$, $B+C$, A, and C as indicated. Since by Lemmas 48.5.1 and 48.5.2 we have $\gamma_{A+B} = \gamma_A \circ \gamma_B$ and $\gamma_{B+C} = \gamma_C \circ \gamma_B$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 48.5.3). $\square$

0B6T Lemma 48.16.5. In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. There are canonical maps

$$\mu_{f,K} : Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \longrightarrow f^!K$$

functorial in K in $D^+_{QCoh}(\mathcal{O}_Y)$. If $g : Y \to Z$ is another morphism of $\textit{FTS}_S$, then the diagram

$$\begin{tikzcd}
Lf^*(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g^!\mathcal{O}_Z) \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \arrow[r, "\mu_f"'] \arrow[d, equal] & f^!(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g^!\mathcal{O}_Z) \arrow[r, "f^!\mu_g"'] & f^!g^!K \arrow[d, equal] \\
Lf^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*g^!\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \arrow[r, "\mu_f"] & Lf^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!g^!\mathcal{O}_Z \arrow[r, "\mu_{g\circ f}"] & f^!g^!K
\end{tikzcd}$$

commutes for all $K \in D^+_{QCoh}(\mathcal{O}_Z)$.

Proof. If f is proper, then $f^! = a$ and we can use (48.8.0.1) and if g is also proper, then Lemma 48.8.4 proves the commutativity of the diagram (in greater generality).

Let us define the map $\mu_{f,K}$. Choose a compactification $j : X \to \overline{X}$ of X over Y. Since $f^!$ is defined as $j^* \circ \overline{a}$ we obtain $\mu_{f,K}$ as the restriction of the map (48.8.0.1)

$$L\overline{f}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}(\mathcal{O}_Y) \longrightarrow \overline{a}(K)$$

to X. To see this is independent of the choice of the compactification we argue as in the proof of Lemma 48.16.2. We urge the reader to read the proof of that lemma first.

Assume given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y such that $g^{-1}(j_2(X)) = j_1(X)$. Denote $\overline{c}$ the right adjoint for pushforward of Lemma 48.3.1 for the morphism g. The maps

$$L\overline{f}_1^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}_1(\mathcal{O}_Y) \longrightarrow \overline{a}_1(K) \quad \text{and} \quad L\overline{f}_2^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}_2(\mathcal{O}_Y) \longrightarrow \overline{a}_2(K)$$

fit into the commutative diagram

$$\begin{tikzcd} Lg^*(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)) \otimes^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) \arrow[r, "\sigma"'] \arrow[d, equal] & \overline{c}(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)) \arrow[r] & \overline{c}(\overline{a}_2(K)) \arrow[d, equal] \\ L\overline{f}_1^*K \otimes^{\mathbf{L}} Lg^*\overline{a}_2(\mathcal{O}_Y) \otimes^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) \arrow[r, "1\otimes\tau"] & L\overline{f}_1^*K \otimes^{\mathbf{L}} \overline{a}_1(\mathcal{O}_Y) \arrow[r] & \overline{a}_1(K) \end{tikzcd}$$

by Lemma 48.8.4. By Lemma 48.8.3 the maps σ and τ restrict to an isomorphism over X. In fact, we can say more. Recall that in the proof of Lemma 48.16.2 we used the map (48.4.1.1) $\gamma : j_1^* \circ \overline{c} \to j_2^*$ to construct our isomorphism $\alpha_g : j_1^* \circ \overline{a}_1 \to j_2^* \circ \overline{a}_2$. Pulling back to map σ by j_1 we obtain the identity map on $j_2^*\left(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)\right)$ if we identify $j_1^*\overline{c}(\mathcal{O}_{\overline{X}_2})$ with $\mathcal{O}_X$ via $j_1^* \circ \overline{c} \to j_2^*$, see Lemma 48.8.2. Similarly, the map $\tau : Lg^*\overline{a}_2(\mathcal{O}_Y) \otimes^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) \to \overline{a}_1(\mathcal{O}_Y) = \overline{c}(\overline{a}_2(\mathcal{O}_Y))$ pulls back to the identity map on $j_2^*\overline{a}_2(\mathcal{O}_Y)$. We conclude that pulling back by j_1 and applying γ wherever we can we obtain a commutative diagram

$$\begin{tikzcd} j_2^*\left(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)\right) \arrow[rr] \arrow[d] & & j_2^*\overline{a}_2(K) \\ j_1^*L\overline{f}_1^*K \otimes^{\mathbf{L}} j_2^*\overline{a}_2(\mathcal{O}_Y) & j_1^*(L\overline{f}_1^*K \otimes^{\mathbf{L}} \overline{a}_1(\mathcal{O}_Y)) \arrow[l, "1\otimes\alpha_g"'] \arrow[r] & j_1^*\overline{a}_1(K) \arrow[u, "\alpha_g"'] \end{tikzcd}$$

The commutativity of this diagram exactly tells us that the map $\mu_{f,K}$ constructed using the compactification $\overline{X}_1$ is the same as the map $\mu_{f,K}$ constructed using the compactification $\overline{X}_2$ via the identification α_g used in the proof of Lemma 48.16.2. Some categorical arguments exactly as in the proof of Lemma 48.16.2 now show that $\mu_{f,K}$ is well defined (small detail omitted).

Having said this, the commutativity of the diagram in the statement of our lemma follows from the construction of the isomorphism $(g \circ f)^! \to f^! \circ g^!$ (first part of the proof of Lemma 48.16.3 using $\overline{X} \to \overline{Y} \to Z$) and the result of Lemma 48.8.4 for $\overline{X} \to \overline{Y} \to Z$. □

48.17. Properties of upper shriek functors

0ATZ Here are some properties of the upper shriek functors.

0AU0 Lemma 48.17.1. In Situation 48.16.1 let Y be an object of FTS_S and let $j : X \to Y$ be an open immersion. Then there is a canonical isomorphism $j^! = j^*$ of functors.

For an étale morphism $f : X \to Y$ of FTS_S we also have $f^* \cong f^!$, see Lemma 48.18.2.

Proof. In this case we may choose $\overline{X} = Y$ as our compactification. Then the right adjoint of Lemma 48.3.1 for $\text{id} : Y \to Y$ is the identity functor and hence $j^! = j^*$ by definition. □

0G4J Lemma 48.17.2. In Situation 48.16.1 let

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ V \arrow[r, "j'"] & Y \end{tikzcd}$$

be a commutative diagram of FTS_S where j and j' are open immersions. Then $j^* \circ f^! = g^! \circ (j')^*$ as functors $D^+_{QCoh}(\mathcal{O}_Y) \to D^+(\mathcal{O}_U)$.

Proof. Let $h = f \circ j = j' \circ g$. By Lemma 48.16.3 we have $h^! = j^! \circ f^! = g^! \circ (j')^!$. By Lemma 48.17.1 we have $j^! = j^*$ and $(j')^! = (j')^*$. $\square$

0AA1 Lemma 48.17.3. In Situation 48.16.1 let Y be an object of FTS_S and let $f : X = \mathbf{A}^1_Y \to Y$ be the projection. Then there is a (noncanonical) isomorphism $f^!(-) \cong Lf^*(-)[1]$ of functors.

Proof. Since $X = \mathbf{A}^1_Y \subset \mathbf{P}^1_Y$ and since $\mathcal{O}_{\mathbf{P}^1_Y}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 48.15.1 and 48.13.3. $\square$

0AA2 Lemma 48.17.4. In Situation 48.16.1 let Y be an object of FTS_S and let $i : X \to Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$ of functors.

Proof. This is a restatement of Lemma 48.9.7. $\square$

0BV2 Remark 48.17.5 (Local description upper shriek). In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Using the lemmas above we can compute $f^!$ locally as follows. Suppose that we are given affine opens

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ V \arrow[r, "i"] & Y \end{tikzcd}$$

Since $j^! \circ f^! = g^! \circ i^!$ (Lemma 48.16.3) and since $j^!$ and $i^!$ are given by restriction (Lemma 48.17.1) we see that

$$(f^! E)|_U = g^!(E|_V)$$

for any $E \in D^+_{QCoh}(\mathcal{O}_X)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ and let $\varphi : R \to A$ be the finite type ring map corresponding to g. Choose a presentation $A = P/I$ where $P = R[x_1, \ldots, x_n]$ is a polynomial algebra in n variables over R. Choose an object $K \in D^+(R)$ corresponding to $E|_V$ (Derived Categories of Schemes, Lemma 36.3.5). Then we claim that $f^! E|_U$ corresponds to

$$\varphi^!(K) = R\,\mathrm{Hom}(A, K \otimes_R^{\mathbf{L}} P)[n]$$

where $R\,\mathrm{Hom}(A, -) : D(P) \to D(A)$ is the functor of Dualizing Complexes, Section 47.13 and where $\varphi^! : D(R) \to D(A)$ is the functor of Dualizing Complexes, Section 47.24. Namely, the choice of presentation gives a factorization

$$U \to \mathbf{A}^n_V \to \mathbf{A}^{n-1}_V \to \ldots \to \mathbf{A}^1_V \to V$$

Applying Lemma 48.17.3 exactly n times we see that $(\mathbf{A}^n_V \to V)^!(E|_V)$ corresponds to $K \otimes_R^{\mathbf{L}} P[n]$. By Lemmas 48.9.5 and 48.17.4 the last step corresponds to applying $R\,\mathrm{Hom}(A, -)$.

0AU1 **Lemma 48.17.6.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Then $f^!$ maps $D^+_{\text{Coh}}(\mathcal{O}_Y)$ into $D^+_{\text{Coh}}(\mathcal{O}_X)$.

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. The lemma follows from By Lemmas 48.17.3 and 48.9.6 and Dualizing Complexes, Lemma 47.15.10 and induction. $\square$

0AA3 **Lemma 48.17.7.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If K is a dualizing complex for Y, then $f^!K$ is a dualizing complex for X.

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. By Lemma 48.17.3 and Dualizing Complexes, Lemma 47.15.10 and induction we see that the $p^!K$ is a dualizing complex on $\mathbf{A}^n_Y$ where $p : \mathbf{A}^n_Y \to Y$ is the projection. Similarly, by Dualizing Complexes, Lemma 47.15.9 and Lemmas 48.9.5 and 48.17.4 we see that $i^!$ transforms dualizing complexes into dualizing complexes. $\square$

0AU2 **Lemma 48.17.8.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let K be a dualizing complex on Y. Set $D_Y(M) = R\mathcal{H}om_{\mathcal{O}_Y}(M, K)$ for $M \in D_{\text{Coh}}(\mathcal{O}_Y)$ and $D_X(E) = R\mathcal{H}om_{\mathcal{O}_X}(E, f^!K)$ for $E \in D_{\text{Coh}}(\mathcal{O}_X)$. Then there is a canonical isomorphism

$$f^!M \longrightarrow D_X(Lf^*D_Y(M))$$

for $M \in D^+_{\text{Coh}}(\mathcal{O}_Y)$.

Proof. Choose compactification $j : X \subset \overline{X}$ of X over Y (More on Flatness, Theorem 38.33.8 and Lemma 38.32.2). Let a be the right adjoint of Lemma 48.3.1 for $\overline{X} \to Y$. Set $D_{\overline{X}}(E) = R\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(E, a(K))$ for $E \in D_{\text{Coh}}(\mathcal{O}_{\overline{X}})$. Since formation of $R\mathcal{H}om$ commutes with restriction to opens and since $f^! = j^* \circ a$ we see that it suffices to prove that there is a canonical isomorphism

$$a(M) \longrightarrow D_{\overline{X}}(L\overline{f}^*D_Y(M))$$

for $M \in D_{\text{Coh}}(\mathcal{O}_Y)$. For $F \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned}
\text{Hom}_{\overline{X}}(F, D_{\overline{X}}(L\overline{f}^*D_Y(M))) &= \text{Hom}_{\overline{X}}(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} L\overline{f}^*D_Y(M), a(K)) \\
&= \text{Hom}_Y(R\overline{f}_*(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} L\overline{f}^*D_Y(M)), K) \\
&= \text{Hom}_Y(R\overline{f}_*(F) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} D_Y(M), K) \\
&= \text{Hom}_Y(R\overline{f}_*(F), D_Y(D_Y(M))) \\
&= \text{Hom}_Y(R\overline{f}_*(F), M) \\
&= \text{Hom}_{\overline{X}}(F, a(M))
\end{aligned}$$

The first equality by Cohomology, Lemma 20.42.2. The second by definition of a. The third by Derived Categories of Schemes, Lemma 36.22.1. The fourth equality by Cohomology, Lemma 20.42.2 and the definition of D_Y. The fifth equality by

Lemma 48.2.5. The final equality by definition of a. Hence we see that $a(M) = D_{\overline{X}}(L\overline{f}^* D_Y(M))$ by Yoneda's lemma. $\square$

0B6U Lemma 48.17.9. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Assume f is perfect (e.g., flat). Then

(a) $f^!$ maps $D^b_{\text{Coh}}(\mathcal{O}_Y)$ into $D^b_{\text{Coh}}(\mathcal{O}_X)$,
(b) the map $\mu_{f,K} : Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \to f^!K$ of Lemma 48.16.5 is an isomorphism for all $K \in D^+_{QCoh}(\mathcal{O}_Y)$.

Proof. (A flat morphism of finite presentation is perfect, see More on Morphisms, Lemma 37.61.5.) We begin with a series of preliminary remarks.

(1) We already know that $f^!$ sends $D^+_{\text{Coh}}(\mathcal{O}_Y)$ into $D^+_{\text{Coh}}(\mathcal{O}_X)$, see Lemma 48.17.6.
(2) If f is an open immersion, then (a) and (b) are true because we can take $\overline{X} = Y$ in the construction of $f^!$ and μ_f. See also Lemma 48.17.1.
(3) If f is a perfect proper morphism, then (b) is true by Lemma 48.13.3.
(4) If there exists an open covering $X = \bigcup U_i$ and (a) is true for $U_i \to Y$, then (a) is true for $X \to Y$. Same for (b). This holds because the construction of $f^!$ and μ_f commutes with passing to open subschemes.
(5) If $g : Y \to Z$ is a second perfect morphism in FTS_S and (b) holds for f and g, then $f^!g^!\mathcal{O}_Z = Lf^*g^!\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y$ and (b) holds for $g \circ f$ by the commutative diagram of Lemma 48.16.5.
(6) If (a) and (b) hold for both f and g, then (a) and (b) hold for $g \circ f$. Namely, then $f^!g^!\mathcal{O}_Z$ is bounded above (by the previous point) and $L(g \circ f)^*$ has finite cohomological dimension and (a) follows from (b) which we saw above.

From these points we see it suffices to prove the result in case X is affine. Choose an immersion $X \to \mathbf{A}^n_Y$ (Morphisms, Lemma 29.39.2) which we factor as $X \to U \to \mathbf{A}^n_Y \to Y$ where $X \to U$ is a closed immersion and $U \subset \mathbf{A}^n_Y$ is open. Note that $X \to U$ is a perfect closed immersion by More on Morphisms, Lemma 37.61.8. Thus it suffices to prove the lemma for a perfect closed immersion and for the projection $\mathbf{A}^n_Y \to Y$.

Let $f : X \to Y$ be a perfect closed immersion. We already know (b) holds. Let $K \in D^b_{\text{Coh}}(\mathcal{O}_Y)$. Then $f^!K = R\mathcal{H}om(\mathcal{O}_X, K)$ (Lemma 48.17.4) and $f_*f^!K = R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$. Since f is perfect, the complex $f_*\mathcal{O}_X$ is perfect and hence $R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$ is bounded above. This proves that (a) holds. Some details omitted.

Let $f : \mathbf{A}^n_Y \to Y$ be the projection. Then (a) holds by repeated application of Lemma 48.17.3. Finally, (b) is true because it holds for $\mathbf{P}^n_Y \to Y$ (flat and proper) and because $\mathbf{A}^n_Y \subset \mathbf{P}^n_Y$ is an open. $\square$

0E9T Lemma 48.17.10. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If f is flat, then $f^!\mathcal{O}_Y$ is a Y-perfect object of $D(\mathcal{O}_X)$ and $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$ is an isomorphism.

Proof. Both assertions are local on X. Thus we may assume X and Y are affine. Then Remark 48.17.5 turns the lemma into an algebra lemma, namely Dualizing Complexes, Lemma 47.25.2. (Use Derived Categories of Schemes, Lemma 36.35.3 to match the languages.) $\square$

0B6V **Lemma 48.17.11.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. Assume $f : X \to Y$ is a local complete intersection morphism. Then

1. $f^!\mathcal{O}_Y$ is an invertible object of $D(\mathcal{O}_X)$, and
2. $f^!$ maps perfect complexes to perfect complexes.

Proof. Recall that a local complete intersection morphism is perfect, see More on Morphisms, Lemma 37.62.4. By Lemma 48.17.9 it suffices to show that $f^!\mathcal{O}_Y$ is an invertible object in $D(\mathcal{O}_X)$. This question is local on X and Y. Hence we may assume that $X \to Y$ factors as $X \to \mathbf{A}^n_Y \to Y$ where the first arrow is a Koszul regular immersion. See More on Morphisms, Section 37.62. The result holds for $\mathbf{A}^n_Y \to Y$ by Lemma 48.17.3. Thus it suffices to prove the lemma when f is a Koszul regular immersion. Working locally once again we reduce to the case $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, where $A = B/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r \in B$ (use that for Noetherian local rings the notion of Koszul regular and regular are the same, see More on Algebra, Lemma 15.30.7). Thus $X \to Y$ is a composition

$$X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = Y$$

where each arrow is the inclusion of an effective Cartier divisor. In this way we reduce to the case of an inclusion of an effective Cartier divisor $i : D \to X$. In this case $i^!\mathcal{O}_X = \mathcal{N}[1]$ by Lemma 48.14.1 and the proof is complete. $\square$

48.18. Base change for upper shriek

0BZX In Situation 48.16.1 let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram in $\textit{FTS}_S$ such that X and Y' are Tor independent over Y. Our setup is currently not sufficient to construct a base change map $L(g')^* \circ f^! \to (f')^! \circ Lg^*$ in this generality. The reason is that in general it will not be possible to choose a compactification $j : X \to \overline{X}$ over Y such that $\overline{X}$ and Y' are tor independent over Y and hence our construction of the base change map in Section 48.5 does not apply[6].

A partial remedy will be found in Section 48.28. Namely, if the morphism f is flat, then there is a good notion of a relative dualizing complex and using Lemmas 48.28.9 48.28.6, and 48.17.9 we may construct a canonical base change isomorphism. If we ever need to use this, we will add precise statements and proofs later in this chapter.

[6]The reader who is well versed with derived algebraic geometry will realize this is not a "real" problem. Namely, taking $\overline{X}'$ to be the derived fibre product of $\overline{X}$ and Y' over Y, one can argue exactly as in the proof of Lemma 48.18.1 to define this map. After all, the Tor independence of X and Y' guarantees that X' will be an open subscheme of the derived scheme $\overline{X}'$.

0E9U **Lemma 48.18.1.** In Situation 48.16.1 let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of FTS_S with g flat. Then there is an isomorphism $L(g')^* \circ f^! \to (f')^! \circ Lg^*$ on $D^+_{QCoh}(\mathcal{O}_Y)$.

Proof. Namely, because g is flat, for every choice of compactification $j : X \to \overline{X}$ of X over Y the scheme $\overline{X}$ is Tor independent of Y'. Denote $j' : X' \to \overline{X}'$ the base change of j and $\overline{g}' : \overline{X}' \to \overline{X}$ the projection. We define the base change map as the composition

$$L(g')^* \circ f^! = L(g')^* \circ j^* \circ a = (j')^* \circ L(\overline{g}')^* \circ a \longrightarrow (j')^* \circ a' \circ Lg^* = (f')^! \circ Lg^*$$

where the middle arrow is the base change map (48.5.0.1) and a and a' are the right adjoints to pushforward of Lemma 48.3.1 for $\overline{X} \to Y$ and $\overline{X}' \to Y'$. This construction is independent of the choice of compactification (we will formulate a precise lemma and prove it, if we ever need this result).

To finish the proof it suffices to show that the base change map $L(g')^* \circ a \to a' \circ Lg^*$ is an isomorphism on $D^+_{QCoh}(\mathcal{O}_Y)$. By Lemma 48.4.4 formation of a and a' commutes with restriction to affine opens of Y and Y'. Thus by Remark 48.6.1 we may assume that Y and Y' are affine. Thus the result by Lemma 48.6.2. $\square$

0FWI **Lemma 48.18.2.** In Situation 48.16.1 let $f : X \to Y$ be an étale morphism of FTS_S. Then $f^! \cong f^*$ as functors on $D^+_{QCoh}(\mathcal{O}_Y)$.

Proof. We are going to use that an étale morphism is flat, syntomic, and a local complete intersection morphism (Morphisms, Lemma 29.36.10 and 29.36.12 and More on Morphisms, Lemma 37.62.8). By Lemma 48.17.9 it suffices to show $f^!\mathcal{O}_Y = \mathcal{O}_X$. By Lemma 48.17.11 we know that $f^!\mathcal{O}_Y$ is an invertible module. Consider the commutative diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and the diagonal $\Delta : X \to X \times_Y X$. Since Δ is an open immersion (by Morphisms, Lemmas 29.35.13 and 29.36.5), by Lemma 48.17.1 we have $\Delta^! = \Delta^*$. By Lemma 48.16.3 we have $\Delta^! \circ p_1^! \circ f^! = f^!$. By Lemma 48.18.1 applied to the diagram we have $p_1^!\mathcal{O}_X = p_2^* f^!\mathcal{O}_Y$. Hence we conclude

$$f^!\mathcal{O}_Y = \Delta^! p_1^! f^!\mathcal{O}_Y = \Delta^*(p_1^* f^!\mathcal{O}_Y \otimes p_1^!\mathcal{O}_X) = \Delta^*(p_2^* f^!\mathcal{O}_Y \otimes p_1^* f^!\mathcal{O}_Y) = (f^!\mathcal{O}_Y)^{\otimes 2}$$

where in the second step we have used Lemma 48.17.9 once more. Thus $f^!\mathcal{O}_Y = \mathcal{O}_X$ as desired. $\square$

In the rest of this section, we formulate some easy to prove results which would be consequences of a good theory of the base change map.

0BZY **Lemma 48.18.3** (Makeshift base change). In Situation 48.16.1 let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of FTS_S. Let $E \in D^+_{QCoh}(\mathcal{O}_Y)$ be an object such that Lg^*E is in $D^+(\mathcal{O}_{Y'})$. If f is flat, then $L(g')^*f^!E$ and $(f')^!Lg^*E$ restrict to isomorphic objects of $D(\mathcal{O}_{U'})$ for $U' \subset X'$ affine open mapping into affine opens of Y, Y', and X.

Proof. By our assumptions we immediately reduce to the case where X, Y, Y', and X' are affine. Say $Y = \mathrm{Spec}(R)$, $Y' = \mathrm{Spec}(R')$, $X = \mathrm{Spec}(A)$, and $X' = \mathrm{Spec}(A')$. Then $A' = A \otimes_R R'$. Let E correspond to $K \in D^+(R)$. Denoting $\varphi : R \to A$ and $\varphi' : R' \to A'$ the given maps we see from Remark 48.17.5 that $L(g')^*f^!E$ and $(f')^!Lg^*E$ correspond to $\varphi^!(K) \otimes_A^{\mathbf{L}} A'$ and $(\varphi')^!(K \otimes_R^{\mathbf{L}} R')$ where $\varphi^!$ and $(\varphi')^!$ are the functors from Dualizing Complexes, Section 47.24. The result follows from Dualizing Complexes, Lemma 47.24.6. $\square$

0BZZ **Lemma 48.18.4.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Assume f is flat. Set $\omega^\bullet_{X/Y} = f^!\mathcal{O}_Y$ in $D^b_{\mathrm{Coh}}(X)$. Let $y \in Y$ and $h : X_y \to X$ the projection. Then $Lh^*\omega^\bullet_{X/Y}$ is a dualizing complex on X_y.

Proof. The complex $\omega^\bullet_{X/Y}$ is in D^b_{Coh} by Lemma 48.17.9. Being a dualizing complex is a local property. Hence by Lemma 48.18.3 it suffices to show that $(X_y \to y)^!\mathcal{O}_y$ is a dualizing complex on X_y. This follows from Lemma 48.17.7. $\square$

48.19. A duality theory

0AU3 In this section we spell out what kind of a duality theory our very general results above give for finite type separated schemes over a fixed Noetherian base scheme.

Recall that a dualizing complex on a Noetherian scheme X, is an object of $D(\mathcal{O}_X)$ which affine locally gives a dualizing complex for the corresponding rings, see Definition 48.2.2.

Given a Noetherian scheme S denote FTS_S the category of schemes which are of finite type and separated over S. Then:

(1) the functors $f^!$ turn D^+_{QCoh} into a pseudo functor on FTS_S,
(2) if $f : X \to Y$ is a proper morphism in FTS_S, then $f^!$ is the restriction of the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ to $D^+_{QCoh}(\mathcal{O}_Y)$ and there is a canonical isomorphism
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, f^!M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, M)$$
for all $K \in D^-_{\mathrm{Coh}}(\mathcal{O}_X)$ and $M \in D^+_{QCoh}(\mathcal{O}_Y)$,
(3) if an object X of FTS_S has a dualizing complex $\omega^\bullet_X$, then the functor $D_X = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega^\bullet_X)$ defines an involution of $D_{\mathrm{Coh}}(\mathcal{O}_X)$ switching $D^+_{\mathrm{Coh}}(\mathcal{O}_X)$ and $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$ and fixing $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$,
(4) if $f : X \to Y$ is a morphism of FTS_S and $\omega^\bullet_Y$ is a dualizing complex on Y, then
    (a) $\omega^\bullet_X = f^!\omega^\bullet_Y$ is a dualizing complex for X,

   (b) $f^!M = D_X(Lf^*D_Y(M))$ canonically for $M \in D^+_{Coh}(\mathcal{O}_Y)$, and
   (c) if in addition f is proper then
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, \omega_Y^\bullet)$$
   for K in $D^-_{Coh}(\mathcal{O}_X)$,

(5) if $f : X \to Y$ is a closed immersion in FTS_S, then $f^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$,
(6) if $f : Y \to X$ is a finite morphism in FTS_S, then $f_*f^!(-) = R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -)$,
(7) if $f : X \to Y$ is the inclusion of an effective Cartier divisor into an object of FTS_S, then $f^!(-) = Lf^*(-) \otimes_{\mathcal{O}_X} \mathcal{O}_Y(-X)[-1]$,
(8) if $f : X \to Y$ is a Koszul regular immersion of codimension c into an object of FTS_S, then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \wedge^c\mathcal{N}[-c]$, and
(9) if $f : X \to Y$ is a smooth proper morphism of relative dimension d in FTS_S, then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \Omega^d_{X/Y}[d]$.

This follows from Lemmas 48.2.5, 48.3.6, 48.9.7, 48.11.4, 48.14.2, 48.15.6, 48.15.7, 48.16.3, 48.16.4, 48.17.4, 48.17.7, 48.17.8, and 48.17.9 and Example 48.3.9. We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 13.38.2). This means we have, in general, no explicit description of the functors $f^!$. This can sometimes be a problem. But in fact, it is often enough to know the existence of a dualizing complex and the duality isomorphism to pin down $f^!$.

48.20. Glueing dualizing complexes

0AU5 We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over S given a pair $(S, \omega_S^\bullet)$ as in Situation 48.20.1. This is similar to [Har66, Remark on page 310].

0AU4 Situation 48.20.1. Here S is a Noetherian scheme and $\omega_S^\bullet$ is a dualizing complex.

In Situation 48.20.1 let X be a scheme of finite type over S. Let $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$ be a finite open covering of X by objects of FTS_S, see Situation 48.16.1. All this means is that the morphisms $U_i \to S$ are separated (as they are already of finite type). Every affine scheme of finite type over S is an object of FTS_S by Schemes, Lemma 26.21.13 hence such open coverings certainly exist. Then for each $i, j, k \in \{1, \ldots, n\}$ the morphisms $p_i : U_i \to S$, $p_{ij} : U_i \cap U_j \to S$, and $p_{ijk} : U_i \cap U_j \cap U_k \to S$ are separated and each of these schemes is an object of FTS_S. From such an open covering we obtain

(1) $\omega_i^\bullet = p_i^!\omega_S^\bullet$ a dualizing complex on U_i, see Section 48.19,
(2) for each i, j a canonical isomorphism $\varphi_{ij} : \omega_i^\bullet|_{U_i \cap U_j} \to \omega_j^\bullet|_{U_i \cap U_j}$, and

0AU6 (3) for each i, j, k we have
$$\varphi_{ik}|_{U_i \cap U_j \cap U_k} = \varphi_{jk}|_{U_i \cap U_j \cap U_k} \circ \varphi_{ij}|_{U_i \cap U_j \cap U_k}$$
   in $D(\mathcal{O}_{U_i \cap U_j \cap U_k})$.

Here, in (2) we use that $(U_i \cap U_j \to U_i)^!$ is given by restriction (Lemma 48.17.1) and that we have canonical isomorphisms
$$(U_i \cap U_j \to U_i)^! \circ p_i^! = p_{ij}^! = (U_i \cap U_j \to U_j)^! \circ p_j^!$$
by Lemma 48.16.3 and to get (3) we use that the upper shriek functors form a pseudo functor by Lemma 48.16.4.

In the situation just described a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ is a pair (K, α_i) where $K \in D(\mathcal{O}_X)$ and $\alpha_i : K|_{U_i} \to \omega_i^\bullet$ are isomorphisms such that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. Since being a dualizing complex on a scheme is a local property we see that dualizing complexes normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ are indeed dualizing complexes.

0AU7 Lemma 48.20.2. In Situation 48.20.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by schemes separated over S. If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then it is unique up to unique isomorphism.

Proof. If (K, α_i) and (K', α_i') are two, then we consider $L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$. By Lemma 48.2.6 and its proof, this is an invertible object of $D(\mathcal{O}_X)$. Using α_i and α_i' we obtain an isomorphism

$$\alpha_i^t \otimes \alpha_i' : L|_{U_i} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_i^\bullet, \omega_i^\bullet) = \mathcal{O}_{U_i}[0]$$

This already implies that $L = H^0(L)[0]$ in $D(\mathcal{O}_X)$. Moreover, $H^0(L)$ is an invertible sheaf with given trivializations on the opens U_i of X. Finally, the condition that $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$ and $\alpha_j'|_{U_i \cap U_j} \circ (\alpha_i')^{-1}|_{U_i \cap U_j}$ both give φ_{ij} implies that the transition maps are 1 and we get an isomorphism $H^0(L) = \mathcal{O}_X$. □

0AU8 Lemma 48.20.3. In Situation 48.20.1 let X be a scheme of finite type over S and let $\mathcal{U}$, $\mathcal{V}$ be two finite open coverings of X by schemes separated over S. If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ and these complexes are canonically isomorphic.

Proof. It suffices to prove this when $\mathcal{U}$ is given by the opens $U_1, \ldots, U_n$ and $\mathcal{V}$ by the opens $U_1, \ldots, U_{n+m}$. In fact, we may and do even assume $m = 1$. To go from a dualizing complex (K, α_i) normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ to a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ is achieved by forgetting about α_i for $i = n + 1$. Conversely, let (K, α_i) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. To finish the proof we need to construct a map $\alpha_{n+1} : K|_{U_{n+1}} \to \omega_{n+1}^\bullet$ satisfying the desired conditions. To do this we observe that $U_{n+1} = \bigcup U_i \cap U_{n+1}$ is an open covering. It is clear that $(K|_{U_{n+1}}, \alpha_i|_{U_i \cap U_{n+1}})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. On the other hand, by condition (3) the pair $(\omega_{n+1}^\bullet|_{U_{n+1}}, \varphi_{n+1i})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. By Lemma 48.20.2 we obtain a unique isomorphism

$$\alpha_{n+1} : K|_{U_{n+1}} \longrightarrow \omega_{n+1}^\bullet$$

compatible with the given local isomorphisms. It is a pleasant exercise to show that this means it satisfies the required property. □

0AU9 Lemma 48.20.4. In Situation 48.20.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by schemes separated over S. Then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$.

Proof. Say $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$. We prove the lemma by induction on n. The base case $n = 1$ is immediate. Assume $n > 1$. Set $X' = U_1 \cup \ldots \cup U_{n-1}$ and let $(K', \{\alpha_i'\}_{i=1,\ldots,n-1})$ be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}' : X' = \bigcup_{i=1,\ldots,n-1} U_i$. It is clear that $(K'|_{X' \cap U_n}, \alpha_i'|_{U_i \cap U_n})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. On the other hand, by condition (3) the pair $(\omega_n^\bullet|_{X' \cap U_n}, \varphi_{ni})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. By Lemma 48.20.2 we obtain a unique isomorphism

$$\epsilon : K'|_{X' \cap U_n} \longrightarrow \omega_i^\bullet|_{X' \cap U_n}$$

compatible with the given local isomorphisms. By Cohomology, Lemma 20.45.1 we obtain $K \in D(\mathcal{O}_X)$ together with isomorphisms $\beta : K|_{X'} \to K'$ and $\gamma : K|_{U_n} \to \omega_n^\bullet$ such that $\epsilon = \gamma|_{X' \cap U_n} \circ \beta|_{X' \cap U_n}^{-1}$. Then we define

$$\alpha_i = \alpha_i' \circ \beta|_{U_i}, i = 1, \ldots, n-1, \text{ and } \alpha_n = \gamma$$

We still need to verify that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. For $i, j \leq n-1$ this follows from the corresponding condition for α_i'. For $i = j = n$ it is clear as well. If $i < j = n$, then we get

$$\alpha_n|_{U_i \cap U_n} \circ \alpha_i^{-1}|_{U_i \cap U_n} = \gamma|_{U_i \cap U_n} \circ \beta^{-1}|_{U_i \cap U_n} \circ (\alpha_i')^{-1}|_{U_i \cap U_n} = \epsilon|_{U_i \cap U_n} \circ (\alpha_i')^{-1}|_{U_i \cap U_n}$$

This is equal to α_{in} exactly because ϵ is the unique map compatible with the maps α_i' and α_{ni}. $\square$

Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. The upshot of the lemmas above is that given any scheme X of finite type over S, there is a pair (K, α_U) given up to unique isomorphism, consisting of an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\alpha_U : K|_U \to \omega_U^\bullet$ for every open subscheme $U \subset X$ which is separated over S. Here $\omega_U^\bullet = (U \to S)^! \omega_S^\bullet$ is a dualizing complex on U, see Section 48.19. Moreover, if $\mathcal{U} : X = \bigcup U_i$ is a finite open covering by opens which are separated over S, then (K, α_{U_i}) is a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. Namely, uniqueness up to unique isomorphism by Lemma 48.20.2, existence for one open covering by Lemma 48.20.4, and the fact that K then works for all open coverings is Lemma 48.20.3.

0AUA Definition 48.20.5. Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex on S. Let X be a scheme of finite type over S. The complex K constructed above is called the dualizing complex normalized relative to $\omega_S^\bullet$ and is denoted $\omega_X^\bullet$.

As the terminology suggest, a dualizing complex normalized relative to $\omega_S^\bullet$ is not just an object of the derived category of X but comes equipped with the local isomorphisms described above. This does not conflict with setting $\omega_X^\bullet = p^! \omega_S^\bullet$ where $p : X \to S$ is the structure morphism if X is separated over S. More generally we have the following sanity check.

0AUB Lemma 48.20.6. Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. Let $f : X \to Y$ be a morphism of finite type schemes over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then $\omega_X^\bullet$ is a dualizing complex normalized relative to $\omega_Y^\bullet$.

Proof. This is just a matter of bookkeeping. Choose a finite affine open covering $\mathcal{V} : Y = \bigcup V_j$. For each j choose a finite affine open covering $f^{-1}(V_j) = U_{ji}$. Set $\mathcal{U} : X = \bigcup U_{ji}$. The schemes V_j and U_{ji} are separated over S, hence we have the upper shriek functors for $q_j : V_j \to S$, $p_{ji} : U_{ji} \to S$ and $f_{ji} : U_{ji} \to V_j$ and $f'_{ji} : U_{ji} \to Y$. Let (L, β_j) be a dualizing complex normalized relative to $\omega_S^\bullet$ and

$\mathcal{V}$. Let (K, γ_{ji}) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. (In other words, $L = \omega_Y^\bullet$ and $K = \omega_X^\bullet$.) We can define

$$\alpha_{ji} : K|_{U_{ji}} \xrightarrow{\gamma_{ji}} p_{ji}^!\omega_S^\bullet = f_{ji}^! q_j^! \omega_S^\bullet \xrightarrow{f_{ji}^!\beta_j^{-1}} f_{ji}^!(L|_{V_j}) = (f'_{ji})^!(L)$$

To finish the proof we have to show that $\alpha_{ji}|_{U_{ji}\cap U_{j'i'}} \circ \alpha_{j'i'}^{-1}|_{U_{ji}\cap U_{j'i'}}$ is the canonical isomorphism $(f'_{ji})^!(L)|_{U_{ji}\cap U_{j'i'}} \to (f'_{j'i'})^!(L)|_{U_{ji}\cap U_{j'i'}}$. This is formal and we omit the details. □

0AUC Lemma 48.20.7. Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. Let $j : X \to Y$ be an open immersion of schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then there is a canonical isomorphism $\omega_X^\bullet = \omega_Y^\bullet|_X$.

Proof. Immediate from the construction of normalized dualizing complexes given just above Definition 48.20.5. □

0AUD Lemma 48.20.8. Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. Let $f : X \to Y$ be a proper morphism of schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Let a be the right adjoint of Lemma 48.3.1 for f. Then there is a canonical isomorphism $a(\omega_Y^\bullet) = \omega_X^\bullet$.

Proof. Let $p : X \to S$ and $q : Y \to S$ be the structure morphisms. If X and Y are separated over S, then this follows from the fact that $\omega_X^\bullet = p^!\omega_S^\bullet$, $\omega_Y^\bullet = q^!\omega_S^\bullet$, $f^! = a$, and $f^! \circ q^! = p^!$ (Lemma 48.16.3). In the general case we first use Lemma 48.20.6 to reduce to the case $Y = S$. In this case X and Y are separated over S and we've just seen the result. □

Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. For a scheme X of finite type over S denote $\omega_X^\bullet$ the dualizing complex for X normalized relative to $\omega_S^\bullet$. Define $D_X(-) = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$ as in Lemma 48.2.5. Let $f : X \to Y$ be a morphism of finite type schemes over S. Define

$$f_{new}^! = D_X \circ Lf^* \circ D_Y : D_{\text{Coh}}^+(\mathcal{O}_Y) \to D_{\text{Coh}}^+(\mathcal{O}_X)$$

If $f : X \to Y$ and $g : Y \to Z$ are composable morphisms between schemes of finite type over S, define

$$\begin{aligned} (g \circ f)_{new}^! &= D_X \circ L(g \circ f)^* \circ D_Z \\ &= D_X \circ Lf^* \circ Lg^* \circ D_Z \\ &\to D_X \circ Lf^* \circ D_Y \circ D_Y \circ Lg^* \circ D_Z \\ &= f_{new}^! \circ g_{new}^! \end{aligned}$$

where the arrow is defined in Lemma 48.2.5. We collect the results together in the following lemma.

0AUE Lemma 48.20.9. Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. With $f_{new}^!$ and $\omega_X^\bullet$ defined for all (morphisms of) schemes of finite type over S as above:

(1) the functors $f_{new}^!$ and the arrows $(g \circ f)_{new}^! \to f_{new}^! \circ g_{new}^!$ turn D_{Coh}^+ into a pseudo functor from the category of schemes of finite type over S into the 2-category of categories,
(2) $\omega_X^\bullet = (X \to S)_{new}^! \omega_S^\bullet$,
(3) the functor D_X defines an involution of $D_{\text{Coh}}(\mathcal{O}_X)$ switching $D_{\text{Coh}}^+(\mathcal{O}_X)$ and $D_{\text{Coh}}^-(\mathcal{O}_X)$ and fixing $D_{\text{Coh}}^b(\mathcal{O}_X)$,

(4) $\omega_X^\bullet = f_{new}^! \omega_Y^\bullet$ for $f : X \to Y$ a morphism of finite type schemes over S,
(5) $f_{new}^! M = D_X(Lf^* D_Y(M))$ for $M \in D^+_{\text{Coh}}(\mathcal{O}_Y)$, and
(6) if in addition f is proper, then $f_{new}^!$ is isomorphic to the restriction of the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ to $D^+_{\text{Coh}}(\mathcal{O}_Y)$ and there is a canonical isomorphism

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, f_{new}^! M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, M)$$

for $K \in D^-_{\text{Coh}}(\mathcal{O}_X)$ and $M \in D^+_{\text{Coh}}(\mathcal{O}_Y)$, and

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, \omega_Y^\bullet)$$

for $K \in D^-_{\text{Coh}}(\mathcal{O}_X)$ and

If X is separated over S, then $\omega_X^\bullet$ is canonically isomorphic to $(X \to S)^! \omega_S^\bullet$ and if f is a morphism between schemes separated over S, then there is a canonical isomorphism[7] $f_{new}^! K = f^! K$ for K in D^+_{Coh}.

Proof. Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be morphisms of schemes of finite type over S. We have to show that

$$\begin{tikzcd}
(h \circ g \circ f)_{new}^! \arrow[r] \arrow[d] & f_{new}^! \circ (h \circ g)_{new}^! \arrow[d] \\
(g \circ f)_{new}^! \circ h_{new}^! \arrow[r] & f_{new}^! \circ g_{new}^! \circ h_{new}^!
\end{tikzcd}$$

is commutative. Let $\eta_Y : \text{id} \to D_Y^2$ and $\eta_Z : \text{id} \to D_Z^2$ be the canonical isomorphisms of Lemma 48.2.5. Then, using Categories, Lemma 4.28.2, a computation (omitted) shows that both arrows $(h \circ g \circ f)_{new}^! \to f_{new}^! \circ g_{new}^! \circ h_{new}^!$ are given by

$$1 \star \eta_Y \star 1 \star \eta_Z \star 1 : D_X \circ Lf^* \circ Lg^* \circ Lh^* \circ D_T \longrightarrow D_X \circ Lf^* \circ D_Y^2 \circ Lg^* \circ D_Z^2 \circ Lh^* \circ D_T$$

This proves (1). Part (2) is immediate from the definition of $(X \to S)_{new}^!$ and the fact that $D_S(\omega_S^\bullet) = \mathcal{O}_S$. Part (3) is Lemma 48.2.5. Part (4) follows by the same argument as part (2). Part (5) is the definition of $f_{new}^!$.

Proof of (6). Let a be the right adjoint of Lemma 48.3.1 for the proper morphism $f : X \to Y$ of schemes of finite type over S. The issue is that we do not know X or Y is separated over S (and in general this won't be true) hence we cannot immediately apply Lemma 48.17.8 to f over S. To get around this we use the canonical identification $\omega_X^\bullet = a(\omega_Y^\bullet)$ of Lemma 48.20.8. Hence $f_{new}^!$ is the restriction of a to $D^+_{\text{Coh}}(\mathcal{O}_Y)$ by Lemma 48.17.8 applied to $f : X \to Y$ over the base scheme Y! The displayed equalities hold by Example 48.3.9.

The final assertions follow from the construction of normalized dualizing complexes and the already used Lemma 48.17.8. $\square$

0BV3 Remark 48.20.10. Let S be a Noetherian scheme which has a dualizing complex. Let $f : X \to Y$ be a morphism of schemes of finite type over S. Then the functor

$$f_{new}^! : D^+_{Coh}(\mathcal{O}_Y) \to D^+_{Coh}(\mathcal{O}_X)$$

is independent of the choice of the dualizing complex $\omega_S^\bullet$ up to canonical isomorphism. We sketch the proof. Any second dualizing complex is of the form $\omega_S^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{L}$ where $\mathcal{L}$ is an invertible object of $D(\mathcal{O}_S)$, see Lemma 48.2.6. For any separated

[7]We haven't checked that these are compatible with the isomorphisms $(g \circ f)^! \to f^! \circ g^!$ and $(g \circ f)_{new}^! \to f_{new}^! \circ g_{new}^!$. We will do this here if we need this later.

morphism $p : U \to S$ of finite type we have $p^!(\omega_S^\bullet \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{L}) = p^!(\omega_S^\bullet) \otimes_{\mathcal{O}_U}^{\mathbf{L}} Lp^*\mathcal{L}$ by Lemma 48.8.1. Hence, if $\omega_X^\bullet$ and $\omega_Y^\bullet$ are the dualizing complexes normalized relative to $\omega_S^\bullet$ we see that $\omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}$ and $\omega_Y^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lb^*\mathcal{L}$ are the dualizing complexes normalized relative to $\omega_S^\bullet \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{L}$ (where $a : X \to S$ and $b : Y \to S$ are the structure morphisms). Then the result follows as

$$\begin{aligned}
& R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lb^*\mathcal{L}), \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R(\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lb^*\mathcal{L}), \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet), \omega_X^\bullet)
\end{aligned}$$

for $K \in D_{Coh}^+(\mathcal{O}_Y)$. The last equality because $La^*\mathcal{L}$ is invertible in $D(\mathcal{O}_X)$.

0B6X Example 48.20.11. Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to Y$ be a proper morphism of finite type schemes over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. In this situation we have $a(\omega_Y^\bullet) = \omega_X^\bullet$ (Lemma 48.20.8) and hence the trace map (Section 48.7) is a canonical arrow

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet \longrightarrow \omega_Y^\bullet$$

which produces the isomorphisms (Lemma 48.20.9)

$$\mathrm{Hom}_X(L, \omega_X^\bullet) = \mathrm{Hom}_Y(Rf_*L, \omega_Y^\bullet)$$

and

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, \omega_Y^\bullet)$$

for L in $D_{QCoh}(\mathcal{O}_X)$.

0AX4 Remark 48.20.12. Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to Y$ be a finite morphism between schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then we have

$$f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet)$$

in $D_{QCoh}^+(f_*\mathcal{O}_X)$ by Lemmas 48.11.4 and 48.20.8 and the trace map of Example 48.20.11 is the map

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet = f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet) \longrightarrow \omega_Y^\bullet$$

which often goes under the name “evaluation at 1”.

0B6W Remark 48.20.13. Let $f : X \to Y$ be a flat proper morphism of finite type schemes over a pair $(S, \omega_S^\bullet)$ as in Situation 48.20.1. The relative dualizing complex (Remark 48.12.5) is $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$. By Lemma 48.20.8 we have the first canonical isomorphism in

$$\omega_X^\bullet = a(\omega_Y^\bullet) = Lf^*\omega_Y^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet$$

in $D(\mathcal{O}_X)$. The second canonical isomorphism follows from the discussion in Remark 48.12.5.

48.21. Dimension functions

0BV4 We need a bit more information about how the dimension functions change when passing to a scheme of finite type over another.

0AWL **Lemma 48.21.1.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let X be a scheme of finite type over S and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. If $x \in X$ is a closed point lying over a closed point s of S, then $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$ provided that $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$.*

Proof. We may replace X by an affine neighbourhood of x, hence we may and do assume that $f : X \to S$ is separated. Then $\omega_X^\bullet = f^!\omega_S^\bullet$. We have to show that $R\operatorname{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ is sitting in degree 0. Let $i_x : x \to X$ denote the inclusion morphism which is a closed immersion as x is a closed point. Hence $R\operatorname{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ represents $i_x^!\omega_X^\bullet$ by Lemma 48.17.4. Consider the commutative diagram

$$\begin{tikzcd} x \arrow[r, "i_x"] \arrow[d, "\pi"] & X \arrow[d, "f"] \\ s \arrow[r, "i_s"] & S \end{tikzcd}$$

By Morphisms, Lemma 29.20.3 the extension $\kappa(x)/\kappa(s)$ is finite and hence π is a finite morphism. We conclude that

$$i_x^!\omega_X^\bullet = i_x^! f^!\omega_S^\bullet = \pi^! i_s^!\omega_S^\bullet$$

Thus if $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$, then $i_s^!\omega_S^\bullet = \kappa(s)[0]$ by the same reasoning as above. We have

$$R\pi_*(\pi^!(\kappa(s)[0])) = R\mathcal{H}om_{\mathcal{O}_s}(R\pi_*(\kappa(x)[0]), \kappa(s)[0]) = \widetilde{\operatorname{Hom}_{\kappa(s)}(\kappa(x), \kappa(s))}$$

The first equality by Example 48.3.9 applied with $L = \kappa(x)[0]$. The second equality holds because π_* is exact. Thus $\pi^!(\kappa(s)[0])$ is supported in degree 0 and we win. $\square$

0AWM **Lemma 48.21.2.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to S$ be of finite type and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. For all $x \in X$ we have*

$$\delta_X(x) - \delta_S(f(x)) = \operatorname{trdeg}_{\kappa(f(x))}(\kappa(x))$$

where δ_S, resp. δ_X is the dimension function of $\omega_S^\bullet$, resp. $\omega_X^\bullet$, see Lemma 48.2.7.

Proof. We may replace X by an affine neighbourhood of x. Hence we may and do assume there is a compactification $X \subset \overline{X}$ over S. Then we may replace X by $\overline{X}$ and assume that X is proper over S. We may also assume X is connected by replacing X by the connected component of X containing x. Next, recall that both δ_X and the function $x \mapsto \delta_S(f(x)) + \operatorname{trdeg}_{\kappa(f(x))}(\kappa(x))$ are dimension functions on X, see Morphisms, Lemma 29.52.3 (and the fact that S is universally catenary by Lemma 48.2.7). By Topology, Lemma 5.20.3 we see that the difference is locally constant, hence constant as X is connected. Thus it suffices to prove equality in any point of X. By Properties, Lemma 28.5.9 the scheme X has a closed point x. Since $X \to S$ is proper the image s of x is closed in S. Thus we may apply Lemma 48.21.1 to conclude. $\square$

0BV5 **Lemma 48.21.3.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. Let $x \in X$ with image $y \in Y$. Then

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i.$$

Proof. Since the statement is local on X we may assume X and Y are affine schemes. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$. Then $f^!\mathcal{O}_Y$ corresponds to the relative dualizing complex $\omega^\bullet_{A/R}$ of Dualizing Complexes, Section 47.25 by Remark 48.17.5. Thus the lemma follows from Dualizing Complexes, Lemma 47.25.7. $\square$

0BV6 **Lemma 48.21.4.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. Let $x \in X$ with image $y \in Y$. If f is flat, then

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i \leq 0.$$

In fact, if all fibres of f have dimension $\leq d$, then $f^!\mathcal{O}_Y$ has tor-amplitude in $[-d, 0]$ as an object of $D(X, f^{-1}\mathcal{O}_Y)$.

Proof. Arguing exactly as in the proof of Lemma 48.21.3 this follows from Dualizing Complexes, Lemma 47.25.8. $\square$

0E9V **Lemma 48.21.5.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. Let $x \in X$ with image $y \in Y$. Assume

(1) $\mathcal{O}_{Y,y}$ is Cohen-Macaulay, and
(2) $\mathrm{trdeg}_{\kappa(f(\xi))}(\kappa(\xi)) \leq r$ for any generic point ξ of an irreducible component of X containing x.

Then

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -r \leq i$$

and the stalk $H^{-r}(f^!\mathcal{O}_Y)_x$ is (S_2) as an $\mathcal{O}_{X,x}$-module.

Proof. After replacing X by an open neighbourhood of x, we may assume every irreducible component of X passes through x. Then arguing exactly as in the proof of Lemma 48.21.3 this follows from Dualizing Complexes, Lemma 47.25.9. $\square$

0BV7 **Lemma 48.21.6.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. If f is flat and quasi-finite, then

$$f^!\mathcal{O}_Y = \omega_{X/Y}[0]$$

for some coherent $\mathcal{O}_X$-module $\omega_{X/Y}$ flat over Y.

Proof. Consequence of Lemma 48.21.4 and the fact that the cohomology sheaves of $f^!\mathcal{O}_Y$ are coherent by Lemma 48.17.6. $\square$

0BV8 **Lemma 48.21.7.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. If f is Cohen-Macaulay (More on Morphisms, Definition 37.22.1), then

$$f^!\mathcal{O}_Y = \omega_{X/Y}[d]$$

for some coherent $\mathcal{O}_X$-module $\omega_{X/Y}$ flat over Y where d is the locally constant function on X which gives the relative dimension of X over Y.

Proof. The relative dimension d is well defined and locally constant by Morphisms, Lemma 29.29.4. The cohomology sheaves of $f^!\mathcal{O}_Y$ are coherent by Lemma 48.17.6. We will get flatness of $\omega_{X/Y}$ from Lemma 48.21.4 if we can show the other cohomology sheaves of $f^!\mathcal{O}_Y$ are zero.

The question is local on X, hence we may assume X and Y are affine and the morphism has relative dimension d. If $d = 0$, then the result follows directly from Lemma 48.21.6. If $d > 0$, then we may assume there is a factorization

$$X \xrightarrow{g} \mathbf{A}^d_Y \xrightarrow{p} Y$$

with g quasi-finite and flat, see More on Morphisms, Lemma 37.22.8. Then $f^! = g^! \circ p^!$. By Lemma 48.17.3 we see that $p^!\mathcal{O}_Y \cong \mathcal{O}_{\mathbf{A}^d_Y}[-d]$. We conclude by the case $d = 0$. □

0BV9 Remark 48.21.8. Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. In this case Lemmas 48.21.3, 48.21.4, 48.21.6, and 48.21.7 are true for any morphism $f : X \to Y$ of finite type schemes over S but with $f^!$ replaced by $f^!_{new}$. This is clear because in each case the proof reduces immediately to the affine case and then $f^! = f^!_{new}$ by Lemma 48.20.9.

48.22. Dualizing modules

0AWH This section is a continuation of Dualizing Complexes, Section 47.19.

Let X be a Noetherian scheme and let $\omega_X^\bullet$ be a dualizing complex. Let $n \in \mathbf{Z}$ be the smallest integer such that $H^n(\omega_X^\bullet)$ is nonzero. In other words, $-n$ is the maximal value of the dimension function associated to $\omega_X^\bullet$ (Lemma 48.2.7). Sometimes $H^n(\omega_X^\bullet)$ is called a dualizing module or dualizing sheaf for X and then it is often denoted by ω_X. We will say "let ω_X be a dualizing module" to indicate the above.

Care has to be taken when using dualizing modules ω_X on Noetherian schemes X:

(1) the integer n may change when passing from X to an open U of X and then it won't be true that $\omega_X|_U = \omega_U$,
(2) the dualizing complex isn't unique; the dualizing module is only unique up to tensoring by an invertible module.

The second problem will often be irrelevant because we will work with X of finite type over a base change S which is endowed with a fixed dualizing complex $\omega_S^\bullet$ and $\omega_X^\bullet$ will be the dualizing complex normalized relative to $\omega_S^\bullet$. The first problem will not occur if X is equidimensional, more precisely, if the dimension function associated to $\omega_X^\bullet$ (Lemma 48.2.7) maps every generic point of X to the same integer.

0AWI Example 48.22.1. Say $S = \mathrm{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring, and $\omega_S^\bullet$ corresponds to a normalized dualizing complex $\omega_A^\bullet$. Then if $f : X \to S$ is proper over S and $\omega_X^\bullet = f^!\omega_S^\bullet$ the coherent sheaf

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

is a dualizing module and is often called the dualizing module of X (with S and $\omega_S^\bullet$ being understood). We will see that this has good properties.

0AWJ Example 48.22.2. Say X is an equidimensional scheme of finite type over a field k. Then it is customary to take $\omega_X^\bullet$ the dualizing complex normalized relative to $k[0]$ and to refer to

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

as the dualizing module of X. If X is separated over k, then $\omega_X^\bullet = f^!\mathcal{O}_{\mathrm{Spec}(k)}$ where $f : X \to \mathrm{Spec}(k)$ is the structure morphism by Lemma 48.20.9. If X is proper over k, then this is a special case of Example 48.22.1.

0AWK Lemma 48.22.3. Let X be a connected Noetherian scheme and let ω_X be a dualizing module on X. The support of ω_X is the union of the irreducible components of maximal dimension with respect to any dimension function and ω_X is a coherent $\mathcal{O}_X$-module having property (S_2).

Proof. By our conventions discussed above there exists a dualizing complex $\omega_X^\bullet$ such that ω_X is the leftmost nonvanishing cohomology sheaf. Since X is connected, any two dimension functions differ by a constant (Topology, Lemma 5.20.3). Hence we may use the dimension function associated to $\omega_X^\bullet$ (Lemma 48.2.7). With these remarks in place, the lemma now follows from Dualizing Complexes, Lemma 47.17.5 and the definitions (in particular Cohomology of Schemes, Definition 30.11.1). $\square$

0AWN Lemma 48.22.4. Let X/A with $\omega_X^\bullet$ and ω_X be as in Example 48.22.1. Then

(1) $H^i(\omega_X^\bullet) \neq 0 \Rightarrow i \in \{-\dim(X), \ldots, 0\}$,
(2) the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,
(3) $\text{Supp}(\omega_X)$ is the union of the components of dimension $\dim(X)$, and
(4) ω_X has property (S_2).

Proof. Let δ_X and δ_S be the dimension functions associated to $\omega_X^\bullet$ and $\omega_S^\bullet$ as in Lemma 48.21.2. As X is proper over A, every closed subscheme of X contains a closed point x which maps to the closed point $s \in S$ and $\delta_X(x) = \delta_S(s) = 0$. Hence $\delta_X(\xi) = \dim(\overline{\{\xi\}})$ for any point $\xi \in X$. Hence we can check each of the statements of the lemma by looking at what happens over $\text{Spec}(\mathcal{O}_{X,x})$ in which case the result follows from Dualizing Complexes, Lemmas 47.16.5 and 47.17.5. Some details omitted. The last two statements can also be deduced from Lemma 48.22.3. $\square$

0AWP Lemma 48.22.5. Let X/A with dualizing module ω_X be as in Example 48.22.1. Let $d = \dim(X_s)$ be the dimension of the closed fibre. If $\dim(X) = d + \dim(A)$, then the dualizing module ω_X represents the functor

$$\mathcal{F} \longmapsto \text{Hom}_A(H^d(X, \mathcal{F}), \omega_A)$$

on the category of coherent $\mathcal{O}_X$-modules.

Proof. We have

$$\begin{aligned}
\text{Hom}_X(\mathcal{F}, \omega_X) &= \text{Ext}_X^{-\dim(X)}(\mathcal{F}, \omega_X^\bullet) \\
&= \text{Hom}_X(\mathcal{F}[\dim(X)], \omega_X^\bullet) \\
&= \text{Hom}_X(\mathcal{F}[\dim(X)], f^!(\omega_A^\bullet)) \\
&= \text{Hom}_S(Rf_*\mathcal{F}[\dim(X)], \omega_A^\bullet) \\
&= \text{Hom}_A(H^d(X, \mathcal{F}), \omega_A)
\end{aligned}$$

The first equality because $H^i(\omega_X^\bullet) = 0$ for $i < -\dim(X)$, see Lemma 48.22.4 and Derived Categories, Lemma 13.27.3. The second equality is follows from the definition of Ext groups. The third equality is our choice of $\omega_X^\bullet$. The fourth equality holds because $f^!$ is the right adjoint of Lemma 48.3.1 for f, see Section 48.19. The final equality holds because $R^if_*\mathcal{F}$ is zero for $i > d$ (Cohomology of Schemes, Lemma 30.20.9) and $H^j(\omega_A^\bullet)$ is zero for $j < -\dim(A)$. $\square$

48.23. Cohen-Macaulay schemes

0AWQ This section is the continuation of Dualizing Complexes, Section 47.20. Duality takes a particularly simple form for Cohen-Macaulay schemes.

0AWT **Lemma 48.23.1.** Let X be a locally Noetherian scheme with dualizing complex $\omega_X^\bullet$.

(1) X is Cohen-Macaulay $\Leftrightarrow$ $\omega_X^\bullet$ locally has a unique nonzero cohomology sheaf,
(2) $\mathcal{O}_{X,x}$ is Cohen-Macaulay $\Leftrightarrow$ $\omega_{X,x}^\bullet$ has a unique nonzero cohomology,
(3) $U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is Cohen-Macaulay}\}$ is open and Cohen-Macaulay.

If X is connected and Cohen-Macaulay, then there is an integer n and a coherent Cohen-Macaulay $\mathcal{O}_X$-module ω_X such that $\omega_X^\bullet = \omega_X[-n]$.

Proof. By definition and Dualizing Complexes, Lemma 47.15.6 for every $x \in X$ the complex $\omega_{X,x}^\bullet$ is a dualizing complex over $\mathcal{O}_{X,x}$. By Dualizing Complexes, Lemma 47.20.2 we see that (2) holds.

To see (3) assume that $\mathcal{O}_{X,x}$ is Cohen-Macaulay. Let n_x be the unique integer such that $H^{n_x}(\omega_{X,x}^\bullet)$ is nonzero. For an affine neighbourhood $V \subset X$ of x we have $\omega_X^\bullet|_V$ is in $D^b_{\text{Coh}}(\mathcal{O}_V)$ hence there are finitely many nonzero coherent modules $H^i(\omega_X^\bullet)|_V$. Thus after shrinking V we may assume only H^{n_x} is nonzero, see Modules, Lemma 17.9.5. In this way we see that $\mathcal{O}_{X,v}$ is Cohen-Macaulay for every $v \in V$. This proves that U is open as well as a Cohen-Macaulay scheme.

Proof of (1). The implication $\Leftarrow$ follows from (2). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then in a neighbourhood of x the complex $\omega_X^\bullet$ has only one nonzero cohomology sheaf.

Assume X is connected and Cohen-Macaulay. The above shows that the map $x \mapsto n_x$ is locally constant. Since X is connected it is constant, say equal to n. Setting $\omega_X = H^n(\omega_X^\bullet)$ we see that the lemma holds because ω_X is Cohen-Macaulay by Dualizing Complexes, Lemma 47.20.2 (and Cohomology of Schemes, Definition 30.11.4). $\square$

0AWU **Lemma 48.23.2.** Let X be a locally Noetherian scheme. If there exists a coherent sheaf ω_X such that $\omega_X[0]$ is a dualizing complex on X, then X is a Cohen-Macaulay scheme.

Proof. This follows immediately from Dualizing Complexes, Lemma 47.20.3 and our definitions. $\square$

0C0Z **Lemma 48.23.3.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. Let $x \in X$. If f is flat, then the following are equivalent

(1) f is Cohen-Macaulay at x,
(2) $f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf in a neighbourhood of x.

Proof. One direction of the lemma follows from Lemma 48.21.7. To prove the converse, we may assume $f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf. Let $y = f(x)$. Let $\xi_1, \ldots, \xi_n \in X_y$ be the generic points of the fibre X_y specializing to x. Let $d_1, \ldots, d_n$ be the dimensions of the corresponding irreducible components of X_y. The morphism $f : X \to Y$ is Cohen-Macaulay at η_i by More on Morphisms, Lemma 37.22.7. Hence by Lemma 48.21.7 we see that $d_1 = \ldots = d_n$. If d denotes the common value, then $d = \dim_x(X_y)$. After shrinking X we may assume all fibres have dimension at most d (Morphisms, Lemma 29.28.4). Then the only nonzero cohomology sheaf $\omega = H^{-d}(f^!\mathcal{O}_Y)$ is flat over Y by Lemma 48.21.4. Hence, if $h : X_y \to X$ denotes the canonical morphism, then $Lh^*(f^!\mathcal{O}_Y) = Lh^*(\omega[d]) = (h^*\omega)[d]$ by Derived Categories of Schemes, Lemma 36.22.8. Thus $h^*\omega[d]$ is the dualizing complex of X_y by Lemma 48.18.4. Hence X_y is Cohen-Macaulay by Lemma 48.23.1. This proves f is Cohen-Macaulay at x as desired. □

0C10 Remark 48.23.4. In Situation 48.16.1 let $f : X \to Y$ be a morphism of $\textit{FTS}_S$. Assume f is a Cohen-Macaulay morphism of relative dimension d. Let $\omega_{X/Y} = H^{-d}(f^!\mathcal{O}_Y)$ be the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 48.21.7. Then there is a canonical isomorphism

$$f^!K = Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}[d]$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$, see Lemma 48.17.9. In particular, if S has a dualizing complex $\omega_S^\bullet$, $\omega_Y^\bullet = (Y \to S)^!\omega_S^\bullet$, and $\omega_X^\bullet = (X \to S)^!\omega_S^\bullet$ then we have

$$\omega_X^\bullet = Lf^*\omega_Y^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}[d]$$

Thus if further X and Y are connected and Cohen-Macaulay and if ω_Y and ω_X denote the unique nonzero cohomology sheaves of $\omega_Y^\bullet$ and $\omega_X^\bullet$, then we have

$$\omega_X = f^*\omega_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}.$$

Similar results hold for X and Y arbitrary finite type schemes over S (i.e., not necessarily separated over S) with dualizing complexes normalized with respect to $\omega_S^\bullet$ as in Section 48.20.

48.24. Gorenstein schemes

0AWV This section is the continuation of Dualizing Complexes, Section 47.21.

0AWW Definition 48.24.1. Let X be a scheme. We say X is *Gorenstein* if X is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$.

This definition makes sense because a Noetherian ring is said to be Gorenstein if and only if all of its local rings are Gorenstein, see Dualizing Complexes, Definition 47.21.1.

0C00 Lemma 48.24.2. A Gorenstein scheme is Cohen-Macaulay.

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 47.21.2. □

0DWG Lemma 48.24.3. A regular scheme is Gorenstein.

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 47.21.3. □

0BFQ Lemma 48.24.4. Let X be a locally Noetherian scheme.

1. If X has a dualizing complex $\omega_X^\bullet$, then
   (a) X is Gorenstein $\Leftrightarrow$ $\omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$,
   (b) $\mathcal{O}_{X,x}$ is Gorenstein $\Leftrightarrow$ $\omega_{X,x}^\bullet$ is an invertible object of $D(\mathcal{O}_{X,x})$,
   (c) $U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is Gorenstein}\}$ is an open Gorenstein subscheme.
2. If X is Gorenstein, then X has a dualizing complex if and only if $\mathcal{O}_X[0]$ is a dualizing complex.

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 47.21.4. ☐

0BVA Lemma 48.24.5. If $f : Y \to X$ is a local complete intersection morphism with X a Gorenstein scheme, then Y is Gorenstein.

Proof. By More on Morphisms, Lemma 37.62.5 it suffices to prove the corresponding statement about ring maps. This is Dualizing Complexes, Lemma 47.21.7. ☐

0C01 Lemma 48.24.6. The property $\mathcal{P}(S) =$"S is Gorenstein" is local in the syntomic topology.

Proof. Let $\{S_i \to S\}$ be a syntomic covering. The scheme S is locally Noetherian if and only if each S_i is Noetherian, see Descent, Lemma 35.16.1. Thus we may now assume S and S_i are locally Noetherian. If S is Gorenstein, then each S_i is Gorenstein by Lemma 48.24.5. Conversely, if each S_i is Gorenstein, then for each point $s \in S$ we can pick i and $t \in S_i$ mapping to s. Then $\mathcal{O}_{S,s} \to \mathcal{O}_{S_i,t}$ is a flat local ring homomorphism with $\mathcal{O}_{S_i,t}$ Gorenstein. Hence $\mathcal{O}_{S,s}$ is Gorenstein by Dualizing Complexes, Lemma 47.21.8. ☐

48.25. Gorenstein morphisms

0C02 This section is one in a series. The corresponding sections for normal morphisms, regular morphisms, and Cohen-Macaulay morphisms can be found in More on Morphisms, Sections 37.20, 37.21, and 37.22.

The following lemma says that it does not make sense to define geometrically Gorenstein schemes, since these would be the same as Gorenstein schemes.

0C03 Lemma 48.25.1. Let X be a locally Noetherian scheme over the field k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have

$$\mathcal{O}_{X,x} \text{ is Gorenstein} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Gorenstein}$$

If X is locally of finite type over k, the same holds for any field extension k'/k.

Proof. In both cases the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{k'},x'}$ is a faithfully flat local homomorphism of Noetherian local rings. Thus if $\mathcal{O}_{X_{k'},x'}$ is Gorenstein, then so is $\mathcal{O}_{X,x}$ by Dualizing Complexes, Lemma 47.21.8. To go up, we use Dualizing Complexes, Lemma 47.21.8 as well. Thus we have to show that

$$\mathcal{O}_{X_{k'},x'}/\mathfrak{m}_x\mathcal{O}_{X_{k'},x'} = \kappa(x) \otimes_k k'$$

is Gorenstein. Note that in the first case $k \to k'$ is finitely generated and in the second case $k \to \kappa(x)$ is finitely generated. Hence this follows as property (A) holds for Gorenstein, see Dualizing Complexes, Lemma 47.23.1. ☐

The lemma above guarantees that the following is the correct definition of Gorenstein morphisms.

0C04 Definition 48.25.2. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is Gorenstein at x if f is flat at x, and the local ring of the scheme X_y at x is Gorenstein.
(2) We say f is a Gorenstein morphism if f is Gorenstein at every point of X.

Here is a translation.

0C05 Lemma 48.25.3. Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent

(1) f is Gorenstein, and
(2) f is flat and its fibres are Gorenstein schemes.

Proof. This follows directly from the definitions. $\square$

0C06 Lemma 48.25.4. A Gorenstein morphism is Cohen-Macaulay.

Proof. Follows from Lemma 48.24.2 and the definitions. $\square$

0C15 Lemma 48.25.5. A syntomic morphism is Gorenstein. Equivalently a flat local complete intersection morphism is Gorenstein.

Proof. Recall that a syntomic morphism is flat and its fibres are local complete intersections over fields, see Morphisms, Lemma 29.30.11. Since a local complete intersection over a field is a Gorenstein scheme by Lemma 48.24.5 we conclude. The properties "syntomic" and "flat and local complete intersection morphism" are equivalent by More on Morphisms, Lemma 37.62.8. $\square$

0C11 Lemma 48.25.6. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Assume that the fibres X_y, Y_z and X_z of f, g, and $g \circ f$ are locally Noetherian.

(1) If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x.
(2) If f and g are Gorenstein, then $g \circ f$ is Gorenstein.
(3) If $g \circ f$ is Gorenstein at x and f is flat at x, then f is Gorenstein at x and g is Gorenstein at $f(x)$.
(4) If $g \circ f$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f.

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 47.21.8. $\square$

0C12 Lemma 48.25.7. Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(x)}$ is Gorenstein for all $x \in X$.

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 47.21.8. $\square$

0C07 Lemma 48.25.8. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.

(1) If f is Gorenstein at x, then $f' : X' \to Y'$ is Gorenstein at x'.
(2) If f is flat at x and f' is Gorenstein at x', then f is Gorenstein at x.
(3) If $Y' \to Y$ is flat at $f'(x')$ and f' is Gorenstein at x', then f is Gorenstein at x.

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y')/\kappa(y)$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 33.11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{Y',y'} \arrow[u] & \mathcal{O}_{Y,y} \arrow[l] \arrow[u] \end{tikzcd}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Gorenstein at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein, see Lemma 48.25.1. Hence we see that f' is Gorenstein at x'.

Assume f is flat at x and f' is Gorenstein at x'. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 48.25.1. Hence we see that f is Gorenstein at x.

Assume $Y' \to Y$ is flat at y' and f' is Gorenstein at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 48.25.1. Hence we see that f is Gorenstein at x. ☐

0E0Q Lemma 48.25.9. Let $f : X \to Y$ be a morphism of schemes which is flat and locally of finite type. Then formation of the set $\{x \in X \mid f \text{ is Gorenstein at } x\}$ commutes with arbitrary base change.

Proof. The assumption implies any fibre of f is locally of finite type over a field and hence locally Noetherian and the same is true for any base change. Thus the statement makes sense. Looking at fibres we reduce to the following problem: let X be a scheme locally of finite type over a field k, let K/k be a field extension, and let $x_K \in X_K$ be a point with image $x \in X$. Problem: show that $\mathcal{O}_{X_K,x_K}$ is Gorenstein if and only if $\mathcal{O}_{X,x}$ is Gorenstein.

The problem can be solved using a bit of algebra as follows. Choose an affine open $\text{Spec}(A) \subset X$ containing x. Say x corresponds to $\mathfrak{p} \subset A$. With $A_K = A \otimes_k K$ we see that $\text{Spec}(A_K) \subset X_K$ contains x_K. Say x_K corresponds to $\mathfrak{p}_K \subset A_K$. Let $\omega_A^\bullet$ be a dualizing complex for A. By Dualizing Complexes, Lemma 47.25.3 $\omega_A^\bullet \otimes_A A_K$ is a dualizing complex for A_K. Now we are done because $A_\mathfrak{p} \to (A_K)_{\mathfrak{p}_K}$ is a flat local homomorphism of Noetherian rings and hence $(\omega_A^\bullet)_\mathfrak{p}$ is an invertible object of $D(A_\mathfrak{p})$ if and only if $(\omega_A^\bullet)_\mathfrak{p} \otimes_{A_\mathfrak{p}} (A_K)_{\mathfrak{p}_K}$ is an invertible object of $D((A_K)_{\mathfrak{p}_K})$. Some details omitted; hint: look at cohomology modules. ☐

0C08 Lemma 48.25.10. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let $x \in X$. If f is flat, then the following are equivalent

(1) f is Gorenstein at x,
(2) $f^!\mathcal{O}_Y$ is isomorphic to an invertible object in a neighbourhood of x.

In particular, the set of points where f is Gorenstein is open in X.

Proof. Set $\omega^\bullet = f^!\mathcal{O}_Y$. By Lemma 48.18.4 this is a bounded complex with coherent cohomology sheaves whose derived restriction $Lh^*\omega^\bullet$ to the fibre X_y is a dualizing complex on X_y. Denote $i : x \to X_y$ the inclusion of a point. Then the following are equivalent

(1) f is Gorenstein at x,
(2) $\mathcal{O}_{X_y,x}$ is Gorenstein,
(3) $Lh^*\omega^\bullet$ is invertible in a neighbourhood of x,
(4) $Li^*Lh^*\omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
(5) $L(h\circ i)^*\omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
(6) $\omega^\bullet$ is invertible in a neighbourhood of x.

The equivalence of (1) and (2) is by definition (as f is flat). The equivalence of (2) and (3) follows from Lemma 48.24.4. The equivalence of (3) and (4) follows from More on Algebra, Lemma 15.77.1. The equivalence of (4) and (5) holds because $Li^*Lh^* = L(h \circ i)^*$. The equivalence of (5) and (6) holds by More on Algebra, Lemma 15.77.1. Thus the lemma is clear. ☐

0C09 Lemma 48.25.11. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent

(1) f is Gorenstein at x,
(2) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \to \mathbf{A}^d_S$ over S which is Gorenstein at x,
(3) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite Gorenstein morphism $U \to \mathbf{A}^d_S$ over S,
(4) for any S-morphism $g : U \to \mathbf{A}^d_S$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is Gorenstein at x.

In particular, the set of points where f is Gorenstein is open in X.

Proof. Choose affine open $U = \mathrm{Spec}(A) \subset X$ with $x \in U$ and $V = \mathrm{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma 10.125.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally[8] quasi-finite morphism $g : U \to \mathbf{A}^d_S$.

Having said this, the lemma translates into the following algebra problem (translation omitted). Given $R \to A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \ldots, x_d] \to A$ quasi-finite at $\mathfrak{p}$ the following are equivalent

(a) $\mathrm{Spec}(\varphi)$ is Gorenstein at $\mathfrak{p}$, and
(b) $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Gorenstein at $\mathfrak{p}$.
(c) $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Gorenstein in an open neighbourhood of $\mathfrak{p}$.

In each case $R[x_1, \ldots, x_n] \to A$ is flat at $\mathfrak{p}$ hence by openness of flatness (Algebra, Theorem 10.129.4), we may assume $R[x_1, \ldots, x_n] \to A$ is flat (replace A by a suitable principal localization). By Algebra, Lemma 10.168.1 there exists $R_0 \subset R$ and $R_0[x_1, \ldots, x_n] \to A_0$ such that R_0 is of finite type over $\mathbf{Z}$ and $R_0 \to A_0$ is of finite type and $R_0[x_1, \ldots, x_n] \to A_0$ is flat. Note that the set of points where a flat finite type morphism is Gorenstein commutes with base change by Lemma 48.25.8. In this way we reduce to the case where R is Noetherian.

[8]If S is quasi-separated, then g will be quasi-finite.

Thus we may assume X and S affine and that we have a factorization of f of the form

$$X \xrightarrow{g} \mathbf{A}^n_S \xrightarrow{p} S$$

with g flat and quasi-finite and S Noetherian. Then X and $\mathbf{A}^n_S$ are separated over S and we have

$$f^!\mathcal{O}_S = g^!p^!\mathcal{O}_S = g^!\mathcal{O}_{\mathbf{A}^n_S}[n]$$

by know properties of upper shriek functors (Lemmas 48.16.3 and 48.17.3). Hence the equivalence of (a), (b), and (c) by Lemma 48.25.10. □

0C0A Lemma 48.25.12. The property $\mathcal{P}(f)$ =“the fibres of f are locally Noetherian and f is Gorenstein” is local in the fppf topology on the target and local in the syntomic topology on the source.

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f)$ =“f is flat”, and $\mathcal{P}_2(f)$ =“the fibres of f are locally Noetherian and Gorenstein”. We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 35.23.15 and 35.27.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y_i)/\kappa(y)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 33.11.1. And if in addition X_y is Gorenstein, then X_{i,y_i} is Gorenstein, see Lemma 48.25.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Lemma 48.24.6. □

48.26. More on dualizing complexes

0E4M Some lemmas which don't fit anywhere else very well.

0E4N Lemma 48.26.1. Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Assume

(1) f is syntomic and surjective, or
(2) f is a surjective flat local complete intersection morphism, or
(3) f is a surjective Gorenstein morphism of finite type.

Then $K \in D_{QCoh}(\mathcal{O}_Y)$ is a dualizing complex on Y if and only if Lf^*K is a dualizing complex on X.

Proof. Taking affine opens and using Derived Categories of Schemes, Lemma 36.3.5 this translates into Dualizing Complexes, Lemma 47.26.2. □

48.27. Duality for proper schemes over fields

0FVU In this section we work out the consequences of the very general material above on dualizing complexes and duality for proper schemes over fields.

0FVV **Lemma 48.27.1.** Let X be a proper scheme over a field k. There exists a dualizing complex $\omega_X^\bullet$ with the following properties

1. $H^i(\omega_X^\bullet)$ is nonzero only for $i \in [-\dim(X), 0]$,
2. $\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$ is a coherent (S_2)-module whose support is the irreducible components of dimension $\dim(X)$,
3. the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,
4. for $x \in X$ closed the module $H^i(\omega_{X,x}^\bullet) \oplus \ldots \oplus H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if $\text{depth}(\mathcal{O}_{X,x}) \leq -i$,
5. for $K \in D_{QCoh}(\mathcal{O}_X)$ there are functorial isomorphisms[9]
$$\text{Ext}^i_X(K, \omega_X^\bullet) = \text{Hom}_k(H^{-i}(X, K), k)$$
compatible with shifts and distinguished triangles,
6. there are functorial isomorphisms $\text{Hom}(\mathcal{F}, \omega_X) = \text{Hom}_k(H^{\dim(X)}(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X, and
7. if $X \to \text{Spec}(k)$ is smooth of relative dimension d, then $\omega_X^\bullet \cong \wedge^d \Omega_{X/k}[d]$ and $\omega_X \cong \wedge^d \Omega_{X/k}$.

Proof. Denote $f : X \to \text{Spec}(k)$ the structure morphism. Let a be the right adjoint of pushforward of this morphism, see Lemma 48.3.1. Consider the relative dualizing complex
$$\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$$
Compare with Remark 48.12.5. Since f is proper we have $f^!(\mathcal{O}_{\text{Spec}(k)}) = a(\mathcal{O}_{\text{Spec}(k)})$ by definition, see Section 48.16. Applying Lemma 48.17.7 we find that $\omega_X^\bullet$ is a dualizing complex. Moreover, we see that $\omega_X^\bullet$ and ω_X are as in Example 48.22.1 and as in Example 48.22.2.

Parts (1), (2), and (3) follow from Lemma 48.22.4.

For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Lemma 48.21.1. Part (4) then follows from Dualizing Complexes, Lemma 47.20.1.

Part (5) holds by construction as a is the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_{\text{Spec}(k)}) = D(k)$ which we can identify with $K \mapsto R\Gamma(X, K)$. We also use that the derived category $D(k)$ of k-modules is the same as the category of graded k-vector spaces.

Part (6) follows from Lemma 48.22.5 for coherent $\mathcal{F}$ and in general by unwinding (5) for $K = \mathcal{F}[0]$ and $i = -\dim(X)$.

Part (7) follows from Lemma 48.15.7. □

0FVW **Remark 48.27.2.** Let k, X, and $\omega_X^\bullet$ be as in Lemma 48.27.1. The identity on the complex $\omega_X^\bullet$ corresponds, via the functorial isomorphism in part (5), to a map
$$t : H^0(X, \omega_X^\bullet) \longrightarrow k$$

[9] This property characterizes $\omega_X^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$ up to unique isomorphism by the Yoneda lemma. Since $\omega_X^\bullet$ is in $D^b_{\text{Coh}}(\mathcal{O}_X)$ in fact it suffices to consider $K \in D^b_{\text{Coh}}(\mathcal{O}_X)$.

For an arbitrary K in $D_{QCoh}(\mathcal{O}_X)$ the identification $\mathrm{Hom}(K, \omega_X^\bullet)$ with $H^0(X, K)^\vee$ in part (5) corresponds to the pairing

$$\mathrm{Hom}_X(K, \omega_X^\bullet) \times H^0(X, K) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

This follows from the functoriality of the isomorphisms in (5). Similarly for any $i \in \mathbf{Z}$ we get the pairing

$$\mathrm{Ext}^i_X(K, \omega_X^\bullet) \times H^{-i}(X, K) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

Here we think of α as a morphism $K[-i] \to \omega_X^\bullet$ and β as an element of $H^0(X, K[-i])$ in order to define $\alpha(\beta)$. Observe that if K is general, then we only know that this pairing is nondegenerate on one side: the pairing induces an isomorphism of $\mathrm{Hom}_X(K, \omega_X^\bullet)$, resp. $\mathrm{Ext}^i_X(K, \omega_X^\bullet)$ with the k-linear dual of $H^0(X, K)$, resp. $H^{-i}(X, K)$ but in general not vice versa. If K is in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$, then $\mathrm{Hom}_X(K, \omega_X^\bullet)$, $\mathrm{Ext}_X(K, \omega_X^\bullet)$, $H^0(X, K)$, and $H^i(X, K)$ are finite dimensional k-vector spaces (by Derived Categories of Schemes, Lemmas 36.11.5 and 36.11.4) and the pairings are perfect in the usual sense.

0FVX Remark 48.27.3. We continue the discussion in Remark 48.27.2 and we use the same notation k, X, $\omega_X^\bullet$, and t. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module we obtain perfect pairings

$$\langle -, - \rangle : \mathrm{Ext}^i_X(\mathcal{F}, \omega_X^\bullet) \times H^{-i}(X, \mathcal{F}) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

of finite dimensional k-vector spaces. These pairings satisfy the following (obvious) functoriality: if $\varphi : \mathcal{F} \to \mathcal{G}$ is a homomorphism of coherent $\mathcal{O}_X$-modules, then we have

$$\langle \alpha \circ \varphi, \beta \rangle = \langle \alpha, \varphi(\beta) \rangle$$

for $\alpha \in \mathrm{Ext}^i_X(\mathcal{G}, \omega_X^\bullet)$ and $\beta \in H^{-i}(X, \mathcal{F})$. In other words, the k-linear map $\mathrm{Ext}^i_X(\mathcal{G}, \omega_X^\bullet) \to \mathrm{Ext}^i_X(\mathcal{F}, \omega_X^\bullet)$ induced by φ is, via the pairings, the k-linear dual of the k-linear map $H^{-i}(X, \mathcal{F}) \to H^{-i}(X, \mathcal{G})$ induced by φ. Formulated in this manner, this still works if φ is a homomorphism of quasi-coherent $\mathcal{O}_X$-modules.

0FVY Lemma 48.27.4. Let k, X, and $\omega_X^\bullet$ be as in Lemma 48.27.1. Let $t : H^0(X, \omega_X^\bullet) \to k$ be as in Remark 48.27.2. Let $E \in D(\mathcal{O}_X)$ be perfect. Then the pairings

$$H^i(X, \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee) \times H^{-i}(X, E) \longrightarrow k, \quad (\xi, \eta) \longmapsto t((1_{\omega_X^\bullet} \otimes \epsilon)(\xi \cup \eta))$$

are perfect for all i. Here $\cup$ denotes the cupproduct of Cohomology, Section 20.31 and $\epsilon : E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to \mathcal{O}_X$ is as in Cohomology, Example 20.50.7.

Proof. By replacing E with $E[-i]$ this reduces to the case $i = 0$. By Cohomology, Lemma 20.51.2 we see that the pairing is the same as the one discussed in Remark 48.27.2 whence the result by the discussion in that remark. □

0FVZ Lemma 48.27.5. Let X be a proper scheme over a field k which is Cohen-Macaulay and equidimensional of dimension d. The module ω_X of Lemma 48.27.1 has the following properties

(1) ω_X is a dualizing module on X (Section 48.22),
(2) ω_X is a coherent Cohen-Macaulay module whose support is X,
(3) there are functorial isomorphisms $\mathrm{Ext}^i_X(K, \omega_X[d]) = \mathrm{Hom}_k(H^{-i}(X, K), k)$ compatible with shifts and distinguished triangles for $K \in D_{QCoh}(X)$,
(4) there are functorial isomorphisms $\mathrm{Ext}^{d-i}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^i(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X.

Proof. It is clear from Lemma 48.27.1 that ω_X is a dualizing module (as it is the left most nonvanishing cohomology sheaf of a dualizing complex). We have $\omega_X^\bullet = \omega_X[d]$ and ω_X is Cohen-Macaulay as X is Cohen-Macualay, see Lemma 48.23.1. The other statements follow from this combined with the corresponding statements of Lemma 48.27.1. □

0FW0 Remark 48.27.6. Let X be a proper Cohen-Macaulay scheme over a field k which is equidimensional of dimension d. Let $\omega_X^\bullet$ and ω_X be as in Lemma 48.27.1. By Lemma 48.27.5 we have $\omega_X^\bullet = \omega_X[d]$. Let $t : H^d(X, \omega_X) \to k$ be the map of Remark 48.27.2. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then we have perfect pairings

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{E}^\vee) \times H^{d-i}(X, \mathcal{E}) \longrightarrow k, \quad (\xi, \eta) \longmapsto t(1 \otimes \epsilon)(\xi \cup \eta))$$

where $\cup$ is the cup-product and $\epsilon : \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_X$ is the evaluation map. This is a special case of Lemma 48.27.4.

Here is a sanity check for the dualizing complex.

0FW1 Lemma 48.27.7. Let X be a proper scheme over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 48.27.1.

(1) If $X \to \mathrm{Spec}(k)$ factors as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for some field k', then $\omega_X^\bullet$ and ω_X are as in Lemma 48.27.1 for the morphism $X \to \mathrm{Spec}(k')$.
(2) If K/k is a field extension, then the pullback of $\omega_X^\bullet$ and ω_X to the base change X_K are as in Lemma 48.27.1 for the morphism $X_K \to \mathrm{Spec}(K)$.

Proof. Denote $f : X \to \mathrm{Spec}(k)$ the structure morphism and denote $f' : X \to \mathrm{Spec}(k')$ the given factorization. In the proof of Lemma 48.27.1 we took $\omega_X^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$ where a be is the right adjoint of Lemma 48.3.1 for f. Thus we have to show $a(\mathcal{O}_{\mathrm{Spec}(k)}) \cong a'(\mathcal{O}_{\mathrm{Spec}(k)})$ where a' be is the right adjoint of Lemma 48.3.1 for f'. Since $k' \subset H^0(X, \mathcal{O}_X)$ we see that k'/k is a finite extension (Cohomology of Schemes, Lemma 30.19.2). By uniqueness of adjoints we have $a = a' \circ b$ where b is the right adjoint of Lemma 48.3.1 for $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Another way to say this: we have $f^! = (f')^! \circ g^!$. Thus it suffices to show that $\mathrm{Hom}_k(k', k) \cong k'$ as k'-modules, see Example 48.3.2. This holds because these are k'-vector spaces of the same dimension (namely dimension 1).

Proof of (2). This holds because we have base change for a by Lemma 48.6.2. See discussion in Remark 48.12.5. □

48.28. Relative dualizing complexes

0E2S For a proper, flat morphism of finite presentation we have a rigid relative dualizing complex, see Remark 48.12.5 and Lemma 48.12.8. For a separated and finite type morphism $f : X \to Y$ of Noetherian schemes, we can consider $f^!\mathcal{O}_Y$. In this section we define relative dualizing complexes for morphisms which are flat and locally of finite presentation (but not necessarily quasi-separated or quasi-compact) between schemes (not necessarily locally Noetherian). We show such complexes exist, are unique up to unique isomorphism, and agree with the cases mentioned above. Before reading this section, please read Dualizing Complexes, Section 47.27.

0E2T Definition 48.28.1. Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $W \subset X \times_S X$ be any open such that the diagonal

$\Delta_{X/S} : X \to X \times_S X$ factors through a closed immersion $\Delta : X \to W$. A relative dualizing complex is a pair (K, ξ) consisting of an object $K \in D(\mathcal{O}_X)$ and a map

$$\xi : \Delta_*\mathcal{O}_X \longrightarrow L\text{pr}_1^*K|_W$$

in $D(\mathcal{O}_W)$ such that

(1) K is S-perfect (Derived Categories of Schemes, Definition 36.35.1), and
(2) ξ defines an isomorphism of $\Delta_*\mathcal{O}_X$ with $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)$.

By Lemma 48.9.3 condition (2) is equivalent to the existence of an isomorphism

$$\mathcal{O}_X \longrightarrow R\mathcal{H}om(\mathcal{O}_X, L\text{pr}_1^*K|_W)$$

in $D(\mathcal{O}_X)$ whose pushforward via Δ is equal to ξ. Since $R\mathcal{H}om(\mathcal{O}_X, L\text{pr}_1^*K|_W)$ is independent of the choice of the open W, so is the category of pairs (K, ξ). If $X \to S$ is separated, then we can choose $W = X \times_S X$. We will reduce many of the arguments to the case of rings using the following lemma.

0E2U Lemma 48.28.2. Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then for any commutative diagram

$$\begin{tikzcd} \text{Spec}(A) \arrow[r] \arrow[d] & X \arrow[d] \\ \text{Spec}(R) \arrow[r] & S \end{tikzcd}$$

whose horizontal arrows are open immersions, the restriction of K to $\text{Spec}(A)$ corresponds via Derived Categories of Schemes, Lemma 36.3.5 to a relative dualizing complex for $R \to A$ in the sense of Dualizing Complexes, Definition 47.27.1.

Proof. Since formation of $R\mathcal{H}om$ commutes with restrictions to opens we may as well assume $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$. Observe that relatively perfect objects of $D(\mathcal{O}_X)$ are pseudo-coherent and hence are in $D_{QCoh}(\mathcal{O}_X)$ (Derived Categories of Schemes, Lemma 36.10.1). Thus the statement makes sense. Observe that taking Δ_*, $L\text{pr}_1^*$, and $R\mathcal{H}om$ is compatible with what happens on the algebraic side by Derived Categories of Schemes, Lemmas 36.3.7, 36.3.8, 36.10.8. For the last one we observe that $L\text{pr}_1^*K$ is S-perfect (hence bounded below) and that $\Delta_*\mathcal{O}_X$ is a pseudo-coherent object of $D(\mathcal{O}_W)$; translated into algebra this means that A is pseudo-coherent as an $A \otimes_R A$-module which follows from More on Algebra, Lemma 15.82.8 applied to $R \to A \otimes_R A \to A$. Thus we recover exactly the conditions in Dualizing Complexes, Definition 47.27.1. $\square$

0E2V Lemma 48.28.3. Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(K, K)$ is an isomorphism.

Proof. Looking affine locally this reduces using Lemma 48.28.2 to the algebraic case which is Dualizing Complexes, Lemma 47.27.5. $\square$

0E2W Lemma 48.28.4. Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. If (K, ξ) and (L, η) are two relative dualizing complexes on X/S, then there is a unique isomorphism $K \to L$ sending ξ to η.

Proof. Let $U \subset X$ be an affine open mapping into an affine open of S. Then there is an isomorphism $K|_U \to L|_U$ by Lemma 48.28.2 and Dualizing Complexes, Lemma 47.27.2. The reader can reuse the argument of that lemma in the schemes case to obtain a proof in this case. We will instead use a glueing argument.

Suppose we have an isomorphism $\alpha : K \to L$. Then $\alpha(\xi) = u\eta$ for some invertible section $u \in H^0(W, \Delta_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. (Because both η and $\alpha(\xi)$ are generators of an invertible $\Delta_*\mathcal{O}_X$-module by assumption.) Hence after replacing α by $u^{-1}\alpha$ we see that $\alpha(\xi) = \eta$. Since the automorphism group of K is $H^0(X, \mathcal{O}_X^*)$ by Lemma 48.28.3 there is at most one such α.

Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S. For each $U \in \mathcal{B}$ we have a unique isomorphism $\alpha_U : K|_U \to L|_U$ mapping ξ to η by the discussion in the previous two paragraphs. Observe that $\mathrm{Ext}^i(K|_U, K|_U) = 0$ for $i < 0$ and any open U of X by Lemma 48.28.3. By Cohomology, Lemma 20.45.2 applied to $\mathrm{id} : X \to X$ we get a unique morphism $\alpha : K \to L$ agreeing with α_U for all $U \in \mathcal{B}$. Then α sends ξ to η as this is true locally. $\square$

0E2X Lemma 48.28.5. Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. There exists a relative dualizing complex (K, ξ).

Proof. Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S. For each U we have a relative dualizing complex (K_U, ξ_U) for U over S. Namely, choose an affine open $V \subset S$ such that $U \to X \to S$ factors through V. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$. By Dualizing Complexes, Lemma 47.27.4 there exists a relative dualizing complex $K_A \in D(A)$ for $R \to A$. Arguing backwards through the proof of Lemma 48.28.2 this determines an V-perfect object $K_U \in D(\mathcal{O}_U)$ and a map

$$\xi : \Delta_*\mathcal{O}_U \to L\mathrm{pr}_1^*K_U$$

in $D(\mathcal{O}_{U \times_V U})$. Since being V-perfect is the same as being S-perfect and since $U \times_V U = U \times_S U$ we find that (K_U, ξ_U) is as desired.

If $U' \subset U \subset X$ with $U', U \in \mathcal{B}$, then we have a unique isomorphism $\rho^U_{U'} : K_U|_{U'} \to K_{U'}$ in $D(\mathcal{O}_{U'})$ sending $\xi_U|_{U' \times_S U'}$ to $\xi_{U'}$ by Lemma 48.28.4 (note that trivially the restriction of a relative dualizing complex to an open is a relative dualizing complex). The uniqueness guarantees that $\rho^U_{U''} = \rho^V_{U''} \circ \rho^U_{U'}|_{U''}$ for $U'' \subset U' \subset U$ in $\mathcal{B}$. Observe that $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$ for $U \in \mathcal{B}$ by Lemma 48.28.3 applied to U/S and K_U. Thus the BBD glueing lemma (Cohomology, Theorem 20.45.8) tells us there is a unique solution, namely, an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\rho_U : K|_U \to K_U$ such that we have $\rho^U_{U'} \circ \rho_U|_{U'} = \rho_{U'}$ for all $U' \subset U$, $U, U' \in \mathcal{B}$.

To finish the proof we have to construct the map

$$\xi : \Delta_*\mathcal{O}_X \longrightarrow L\mathrm{pr}_1^*K|_W$$

in $D(\mathcal{O}_W)$ inducing an isomorphism from $\Delta_*\mathcal{O}_X$ to $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*K|_W)$. Since we may change W, we choose $W = \bigcup_{U \in \mathcal{B}} U \times_S U$. We can use ρ_U to get isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*K|_W)|_{U \times_S U} \xrightarrow{\rho_U} R\mathcal{H}om_{\mathcal{O}_{U \times_S U}}(\Delta_*\mathcal{O}_U, L\mathrm{pr}_1^*K_U)$$

As W is covered by the opens $U \times_S U$ we conclude that the cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*K|_W)$ are zero except in degree 0. Moreover, we obtain isomorphisms

$$H^0\left(U \times_S U, R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*K|_W)\right) \xrightarrow{\rho_U} H^0\left((R\mathcal{H}om_{\mathcal{O}_{U\times_S U}}(\Delta_*\mathcal{O}_U, L\mathrm{pr}_1^*K_U)\right)$$

Let τ_U in the LHS be an element mapping to ξ_U under this map. The compatibilities between $\rho^U_{U'}$, ξ_U, $\xi_{U'}$, ρ_U, and $\rho_{U'}$ for $U' \subset U \subset X$ open $U', U \in \mathcal{B}$ imply that $\tau_U|_{U'\times_S U'} = \tau_{U'}$. Thus we get a global section τ of the 0th cohomology sheaf $H^0(R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*K|_W))$. Since the other cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*K|_W)$ are zero, this global section τ determines a morphism ξ as desired. Since the restriction of ξ to $U \times_S U$ gives ξ_U, we see that it satisfies the final condition of Definition 48.28.1. $\square$

0E2Y Lemma 48.28.6. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes. Assume $X \to S$ is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex for f. Set $K' = L(g')^*K$. Let ξ' be the derived base change of ξ (see proof). Then (K', ξ') is a relative dualizing complex for f'.

Proof. Consider the cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\Delta_{X'/S'}"'] & X \arrow[d, "\Delta_{X/S}"] \\ X' \times_{S'} X' \arrow[r, "g' \times g'"] & X \times_S X \end{tikzcd}$$

Choose $W \subset X \times_S X$ open such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \to W$. Choose $W' \subset X' \times_{S'} X'$ open such that $\Delta_{X'/S'}$ factors through a closed immersion $\Delta' : X \to W'$ and such that $(g' \times g')(W') \subset W$. Let us still denote $g' \times g' : W' \to W$ the induced morphism. We have

$$L(g' \times g')^*\Delta_*\mathcal{O}_X = \Delta'_*\mathcal{O}_{X'} \quad \text{and} \quad L(g' \times g')^*L\mathrm{pr}_1^*K|_W = L\mathrm{pr}_1^*K'|_{W'}$$

The first equality holds because X and $X' \times_{S'} X'$ are tor independent over $X \times_S X$ (see for example More on Morphisms, Lemma 37.69.1). The second holds by transitivity of derived pullback (Cohomology, Lemma 20.27.2). Thus $\xi' = L(g' \times g')^*\xi$ can be viewed as a map

$$\xi' : \Delta'_*\mathcal{O}_{X'} \longrightarrow L\mathrm{pr}_1^*K'|_{W'}$$

Having said this the proof of the lemma is straightforward. First, K' is S'-perfect by Derived Categories of Schemes, Lemma 36.35.6. To check that ξ' induces an isomorphism of $\Delta'_*\mathcal{O}_{X'}$ to $R\mathcal{H}om_{\mathcal{O}_{W'}}(\Delta'_*\mathcal{O}_{X'}, L\mathrm{pr}_1^*K'|_{W'})$ we may work affine locally. By Lemma 48.28.2 we reduce to the corresponding statement in algebra which is proven in Dualizing Complexes, Lemma 47.27.4. $\square$

0E2Z Lemma 48.28.7. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a proper, flat morphism of finite presentation. The relative dualizing complex $\omega^\bullet_{X/S}$ of Remark 48.12.5 together with (48.12.8.1) is a relative dualizing complex in the sense of Definition 48.28.1.

Proof. In Lemma 48.12.7 we proved that $\omega^\bullet_{X/S}$ is S-perfect. Let c be the right adjoint of Lemma 48.3.1 for the diagonal $\Delta : X \to X \times_S X$. Then we can apply Δ_* to (48.12.8.1) to get an isomorphism

$$\Delta_*\mathcal{O}_X \to \Delta_*(c(L\text{pr}_1^*\omega^\bullet_{X/S})) = R\mathcal{H}om_{\mathcal{O}_{X\times_S X}}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*\omega^\bullet_{X/S})$$

The equality holds by Lemmas 48.9.7 and 48.9.3. This finishes the proof. □

0E4P Remark 48.28.8. Let $X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. By Lemma 48.28.5 there exists a relative dualizing complex $(\omega^\bullet_{X/S}, \xi)$ in the sense of Definition 48.28.1. Consider any morphism $g : S' \to S$ where S' is quasi-compact and quasi-separated (for example an affine open of S). By Lemma 48.28.6 we see that $(L(g')^*\omega^\bullet_{X/S}, L(g')^*\xi)$ is a relative dualizing complex for the base change $f' : X' \to S'$ in the sense of Definition 48.28.1. Let $\omega^\bullet_{X'/S'}$ be the relative dualizing complex for $X' \to S'$ in the sense of Remark 48.12.5. Combining Lemmas 48.28.7 and 48.28.4 we see that there is a unique isomorphism

$$\omega^\bullet_{X'/S'} \longrightarrow L(g')^*\omega^\bullet_{X/S}$$

compatible with (48.12.8.1) and $L(g')^*\xi$. These isomorphisms are compatible with morphisms between quasi-compact and quasi-separated schemes over S and the base change isomorphisms of Lemma 48.12.4 (if we ever need this compatibility we will carefully state and prove it here).

0E9W Lemma 48.28.9. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If f is flat, then $f^!\mathcal{O}_Y$ is (the first component of) a relative dualizing complex for X over Y in the sense of Definition 48.28.1.

Proof. By Lemma 48.17.10 we have that $f^!\mathcal{O}_Y$ is Y-perfect. As f is separated the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion and $\Delta_*\Delta^!(-) = R\mathcal{H}om_{\mathcal{O}_{X\times_Y X}}(\mathcal{O}_X, -)$, see Lemmas 48.9.7 and 48.9.3. Hence to finish the proof it suffices to show $\Delta^!(L\text{pr}_1^*f^!(\mathcal{O}_Y)) \cong \mathcal{O}_X$ where $\text{pr}_1 : X \times_Y X \to X$ is the first projection. We have

$$\mathcal{O}_X = \Delta^!\text{pr}_1^!\mathcal{O}_X = \Delta^!\text{pr}_1^!L\text{pr}_2^*\mathcal{O}_Y = \Delta^!(L\text{pr}_1^*f^!\mathcal{O}_Y)$$

where $\text{pr}_2 : X \times_Y X \to X$ is the second projection and where we have used the base change isomorphism $\text{pr}_1^! \circ L\text{pr}_2^* = L\text{pr}_1^* \circ f^!$ of Lemma 48.18.1. □

0E30 Lemma 48.28.10. Let $f : Y \to X$ and $X \to S$ be morphisms of schemes which are flat and of finite presentation. Let (K, ξ) and (M, η) be a relative dualizing complex for $X \to S$ and $Y \to X$. Set $E = M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*K$. Then (E, ζ) is a relative dualizing complex for $Y \to S$ for a suitable ζ.

Proof. Using Lemma 48.28.2 and the algebraic version of this lemma (Dualizing Complexes, Lemma 47.27.6) we see that E is affine locally the first component of a relative dualizing complex. In particular we see that E is S-perfect since this may be checked affine locally, see Derived Categories of Schemes, Lemma 36.35.3.

Let us first prove the existence of ζ in case the morphisms $X \to S$ and $Y \to X$ are separated so that $\Delta_{X/S}$, $\Delta_{Y/X}$, and $\Delta_{Y/S}$ are closed immersions. Consider the following diagram

$$\begin{tikzcd}
 & & Y \arrow[r, equal] & Y \arrow[d, "f"] \\
Y \arrow[r, "\Delta_{Y/X}"'] & Y \times_X Y \arrow[r, "\delta"'] \arrow[ur, "q"] \arrow[d, "m"'] & Y \times_S Y \arrow[ur, "p"] \arrow[d, "f \times f"] & X \\
 & X \arrow[r, "\Delta_{X/S}"] & X \times_S X \arrow[ur, "r"'] &
\end{tikzcd}$$

where p, q, r are the first projections. By Lemma 48.9.4 we have

$$R\mathcal{H}om_{\mathcal{O}_{Y \times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_* \left(R\mathcal{H}om_{\mathcal{O}_{Y \times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, Lp^*E))\right)$$

By Lemma 48.10.3 we have

$$R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, Lp^*E) = R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, L(f \times f)^*Lr^*K) \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times_S Y}} Lq^*M$$

By Lemma 48.10.2 we have

$$R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, L(f \times f)^*Lr^*K) = Lm^*R\mathcal{H}om(\mathcal{O}_X, Lr^*K)$$

The last expression is isomorphic (via ξ) to $Lm^*\mathcal{O}_X = \mathcal{O}_{Y \times_X Y}$. Hence the expression preceding is isomorphic to Lq^*M. Hence

$$R\mathcal{H}om_{\mathcal{O}_{Y \times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_* \left(R\mathcal{H}om_{\mathcal{O}_{Y \times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, Lq^*M)\right)$$

The material inside the parentheses is isomorphic to $\Delta_{Y/X,*} * \mathcal{O}_X$ via η. This finishes the proof in the separated case.

In the general case we choose an open $W \subset X \times_S X$ such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \to W$ and we choose an open $V \subset Y \times_X Y$ such that $\Delta_{Y/X}$ factors through a closed immersion $\Delta' : Y \to V$. Finally, choose an open $W' \subset Y \times_S Y$ whose intersection with $Y \times_X Y$ gives V and which maps into W. Then we consider the diagram

$$\begin{tikzcd}
 & & Y \arrow[r, equal] & Y \arrow[d, "f"] \\
Y \arrow[r, "\Delta'"'] & V \arrow[r, "\delta"'] \arrow[ur, "q"] \arrow[d, "m"'] & W' \arrow[ur, "p"] \arrow[d, "f \times f"] & X \\
 & X \arrow[r, "\Delta"] & W \arrow[ur, "r"'] &
\end{tikzcd}$$

and we use exactly the same argument as before. $\square$

48.29. The fundamental class of an lci morphism

0E9X In this section we will use the computations made in Section 48.15. Thus our result will suffer from the same kind of non-uniqueness as we have in that section.

0E9Y **Lemma 48.29.1.** Let X be a locally ringed space. Let

$$\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_0 \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{E}_1$ and $\mathcal{E}_0$ are locally free of ranks r_1, r_0. Then there is a canonical map

$$\wedge^{r_0 - r_1}\mathcal{F} \longrightarrow \wedge^{r_1}(\mathcal{E}_1^\vee) \otimes \wedge^{r_0}\mathcal{E}_0$$

which is an isomorphism on the stalk at $x \in X$ if and only if $\mathcal{F}$ is locally free of rank $r_0 - r_1$ in an open neighbourhood of x.

Proof. If $r_1 > r_0$ then $\wedge^{r_0-r_1}\mathcal{F} = 0$ by convention and the unique map cannot be an isomorphism. Thus we may assume $r = r_0 - r_1 \geq 0$. Define the map by the formula

$$s_1 \wedge \ldots \wedge s_r \mapsto t_1^\vee \wedge \ldots \wedge t_{r_1}^\vee \otimes \alpha(t_1) \wedge \ldots \wedge \alpha(t_{r_1}) \wedge \tilde{s}_1 \wedge \ldots \wedge \tilde{s}_r$$

where $t_1, \ldots, t_{r_1}$ is a local basis for $\mathcal{E}_1$, correspondingly $t_1^\vee, \ldots, t_{r_1}^\vee$ is the dual basis for $\mathcal{E}_1^\vee$, and s_i' is a local lift of s_i to a section of $\mathcal{E}_0$. We omit the proof that this is well defined.

If $\mathcal{F}$ is locally free of rank r, then it is straightforward to verify that the map is an isomorphism. Conversely, assume the map is an isomorphism on stalks at x. Then $\wedge^r\mathcal{F}_x$ is invertible. This implies that $\mathcal{F}_x$ is generated by at most r elements. This can only happen if α has rank r modulo $\mathfrak{m}_x$, i.e., α has maximal rank modulo $\mathfrak{m}_x$. This implies that α has maximal rank in a neighbourhood of x and hence $\mathcal{F}$ is locally free of rank r in a neighbourhood as desired. □

0E9Z Lemma 48.29.2. Let Y be a Noetherian scheme. Let $f : X \to Y$ be a local complete intersection morphism which factors as an immersion $X \to P$ followed by a proper smooth morphism $P \to Y$. Let r be the locally constant function on X such that $\omega_{X/Y} = H^{-r}(f^!\mathcal{O}_Y)$ is the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 48.17.11. Then there is a map

$$\wedge^r\Omega_{X/Y} \longrightarrow \omega_{X/Y}$$

which is an isomorphism on the stalk at a point x if and only if f is smooth at x.

Proof. The assumption implies that X is compactifiable over Y hence $f^!$ is defined, see Section 48.16. Let $j : W \to P$ be an open subscheme such that $X \to P$ factors through a closed immersion $i : X \to W$. Moreover, we have $f^! = i^! \circ j^! \circ g^!$ where $g : P \to Y$ is the given morphism. We have $g^!\mathcal{O}_Y = \wedge^d\Omega_{P/Y}[d]$ by Lemma 48.15.7 where d is the locally constant function giving the relative dimension of P over Y. We have $j^! = j^*$. We have $i^!\mathcal{O}_W = \wedge^c\mathcal{N}[-c]$ where c is the codimension of X in W (a locally constant function on X) and where $\mathcal{N}$ is the normal sheaf of the Koszul-regular immersion i, see Lemma 48.15.6. Combining the above we find

$$f^!\mathcal{O}_Y = \left(\wedge^c\mathcal{N} \otimes_{\mathcal{O}_X} \wedge^d\Omega_{P/Y}|_X\right)[d-c]$$

where we have also used Lemma 48.17.9. Thus $r = d|_X - c$ as locally constant functions on X. The conormal sheaf of $X \to P$ is the module $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of i, see Morphisms, Section 29.31. Consider the canonical exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{P/Y}|_X \to \Omega_{X/Y} \to 0$$

of Morphisms, Lemma 29.32.15. We obtain our map by an application of Lemma 48.29.1.

If f is smooth at x, then the map is an isomorphism by an application of Lemma 48.29.1 and the fact that $\Omega_{X/Y}$ is locally free at x of rank r. Conversely, assume that our map is an isomorphism on stalks at x. Then the lemma shows that $\Omega_{X/Y}$ is free of rank r after replacing X by an open neighbourhood of x. On the other hand, we may also assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$ where $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and where $f_1, \ldots, f_m$ is a Koszul regular sequence (this follows from the definition of local complete intersection morphisms). Clearly this implies $r = n - m$. We conclude that the rank of the matrix of partials $\partial f_j/\partial x_i$ in the residue field at x is m. Thus after reordering the variables we may assume the determinant of $(\partial f_j/\partial x_i)_{1 \leq i,j \leq m}$ is invertible in an open neighbourhood of x. It follows that $R \to A$ is smooth at this point, see for example Algebra, Example 10.137.8. □

0EA0 Lemma 48.29.3. Let $f : X \to Y$ be a morphism of schemes. Let $r \geq 0$. Assume

(1) Y is Cohen-Macaulay (Properties, Definition 28.8.1),
(2) f factors as $X \to P \to Y$ where the first morphism is an immersion and the second is smooth and proper,
(3) if $x \in X$ and $\dim(\mathcal{O}_{X,x}) \leq 1$, then f is Koszul at x (More on Morphisms, Definition 37.62.2), and
(4) if ξ is a generic point of an irreducible component of X, then we have $\mathrm{trdeg}_{\kappa(f(\xi))}\kappa(\xi) = r$.

Then with $\omega_{X/Y} = H^{-r}(f^!\mathcal{O}_Y)$ there is a map

$$\wedge^r \Omega_{X/Y} \longrightarrow \omega_{X/Y}$$

which is an isomorphism on the locus where f is smooth.

Proof. Let $U \subset X$ be the open subscheme over which f is a local complete intersection morphism. Since f has relative dimension r at all generic points by assumption (4) we see that the locally constant function of Lemma 48.29.2 is constant with value r and we obtain a map

$$\wedge^r \Omega_{X/Y}|_U = \wedge^r \Omega_{U/Y} \longrightarrow \omega_{U/Y} = \omega_{X/Y}|_U$$

which is an isomorphism in the smooth points of f (this locus is contained in U because a smooth morphism is a local complete intersection morphism). By Lemma 48.21.5 and the assumption that Y is Cohen-Macaulay the module $\omega_{X/Y}$ is (S_2). Since U contains all the points of codimension 1 by condition (3) and using Divisors, Lemma 31.5.11 we see that $j_*\omega_{U/Y} = \omega_{X/Y}$. Hence the map over U extends to X and the proof is complete. □

48.30. Extension by zero for coherent modules

0G2G The material in this section and the next few can be found in the appendix by Deligne of [Har66].

In this section $j : U \to X$ will be an open immersion of Noetherian schemes. We are going to consider inverse systems (K_n) in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ constructed as follows. Let $\mathcal{F}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals with $V(\mathcal{I}) = X \setminus U$. Then we can set

$$K_n = \mathcal{I}^n \mathcal{F}^\bullet$$

More precisely, K_n is the object of $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ represented by the complex whose term in degree q is the coherent submodule $\mathcal{I}^n\mathcal{F}^q$ of $\mathcal{F}^q$. Observe that the maps

$\ldots \to K_3 \to K_2 \to K_1$ induce isomorphisms on restriction to U. Let us call such a system a Deligne system.

0G2H Lemma 48.30.1. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let (K_n) be a Deligne system and denote $K \in D^b_{Coh}(\mathcal{O}_U)$ the value of the constant system $(K_n|_U)$. Let L be an object of $D^b_{Coh}(\mathcal{O}_X)$. Then $\mathop{\mathrm{colim}} \mathop{\mathrm{Hom}}_X(K_n, L) = \mathop{\mathrm{Hom}}_U(K, L|_U)$.

Proof. Let $L \to M \to N \to L[1]$ be a distinguished triangle in $D^b_{Coh}(\mathcal{O}_X)$. Then we obtain a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & \mathop{\mathrm{colim}} \mathop{\mathrm{Hom}}_X(K_n, L) \arrow[r] \arrow[d] & \mathop{\mathrm{colim}} \mathop{\mathrm{Hom}}_X(K_n, M) \arrow[r] \arrow[d] & \mathop{\mathrm{colim}} \mathop{\mathrm{Hom}}_X(K_n, N) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \mathop{\mathrm{Hom}}_U(K, L|_U) \arrow[r] & \mathop{\mathrm{Hom}}_U(K, M|_U) \arrow[r] & \mathop{\mathrm{Hom}}_U(K, N|_U) \arrow[r] & \ldots \end{tikzcd}$$

whose rows are exact by Derived Categories, Lemma 13.4.2 and Algebra, Lemma 10.8.8. Hence if the statement of the lemma holds for $N[-1]$, L, N, and $L[1]$ then it holds for M by the 5-lemma. Thus, using the distinguished triangles for the canonical truncations of L (see Derived Categories, Remark 13.12.4) we reduce to the case that L has only one nonzero cohomology sheaf.

Choose a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$-modules and a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ cutting out $X \setminus U$ such that K_n is represented by $\mathcal{I}^n\mathcal{F}^\bullet$. Using "stupid" truncations we obtain compatible termwise split short exact sequences of complexes

$$0 \to \sigma_{\geq a+1}\mathcal{I}^n\mathcal{F}^\bullet \to \mathcal{I}^n\mathcal{F}^\bullet \to \sigma_{\leq a}\mathcal{I}^n\mathcal{F}^\bullet \to 0$$

which in turn correspond to compatible systems of distinguished triangles in $D^b_{Coh}(\mathcal{O}_X)$. Arguing as above we reduce to the case where $\mathcal{F}^\bullet$ has only one nonzero term. This reduces us to the case discussed in the next paragraph.

Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a coherent $\mathcal{O}_X$-module $\mathcal{G}$ we have to show that the canonical map

$$\mathop{\mathrm{colim}} \mathop{\mathrm{Ext}}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G}) \longrightarrow \mathop{\mathrm{Ext}}^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$$

is an isomorphism for all $i \geq 0$. For $i = 0$ this is Cohomology of Schemes, Lemma 30.10.5. Assume $i > 0$.

Injectivity. Let $\xi \in \mathop{\mathrm{Ext}}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G})$ be an element whose restriction to U is zero. We have to show there exists an $m \geq n$ such that the restriction of ξ to $\mathcal{I}^m\mathcal{F} = \mathcal{I}^{m-n}\mathcal{I}^n\mathcal{F}$ is zero. After replacing $\mathcal{F}$ by $\mathcal{I}^n\mathcal{F}$ we may assume $n = 0$, i.e., we have $\xi \in \mathop{\mathrm{Ext}}^i_X(\mathcal{F}, \mathcal{G})$ whose restriction to U is zero. By Derived Categories of Schemes, Proposition 36.11.2 we have $D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X))$. Hence we can compute the Ext group in the abelian category of coherent $\mathcal{O}_X$-modules. This implies there exists an surjection $\alpha : \mathcal{F}'' \to \mathcal{F}$ such that $\xi \circ \alpha = 0$ (this is where we use that $i > 0$). Set $\mathcal{F}' = \mathop{\mathrm{Ker}}(\alpha)$ so that we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F}'' \to \mathcal{F} \to 0$$

It follows that ξ is the image of an element $\xi' \in \mathop{\mathrm{Ext}}^{i-1}_X(\mathcal{F}', \mathcal{G})$ whose restriction to U is in the image of $\mathop{\mathrm{Ext}}^{i-1}_U(\mathcal{F}''|_U, \mathcal{G}|_U) \to \mathop{\mathrm{Ext}}^{i-1}_U(\mathcal{F}'|_U, \mathcal{G}|_U)$. By Artin-Rees the inverse systems $(\mathcal{I}^n\mathcal{F}')$ and $(\mathcal{I}^n\mathcal{F}'' \cap \mathcal{F}')$ are pro-isomorphic, see Cohomology of Schemes, Lemma 30.10.3. Since we have the compatible system of short exact sequences

$$0 \to \mathcal{F}' \cap \mathcal{I}^n\mathcal{F}'' \to \mathcal{I}^n\mathcal{F}'' \to \mathcal{I}^n\mathcal{F} \to 0$$

we obtain a commutativew diagram

$$\begin{tikzcd} \mathrm{colim}\,\mathrm{Ext}^{i-1}_X(\mathcal{I}^n\mathcal{F}'', \mathcal{G}) \arrow[r] \arrow[d] & \mathrm{colim}\,\mathrm{Ext}^{i-1}_X(\mathcal{F}' \cap \mathcal{I}^n\mathcal{F}'', \mathcal{G}) \arrow[r] \arrow[d] & \mathrm{colim}\,\mathrm{Ext}^{i}_X(\mathcal{I}^n\mathcal{F}, \mathcal{G}) \arrow[d] \\ \mathrm{Ext}^{i-1}_U(\mathcal{F}''|_U, \mathcal{G}|_U) \arrow[r] & \mathrm{Ext}^{i-1}_U(\mathcal{F}'|_U, \mathcal{G}|_U) \arrow[r] & \mathrm{Ext}^{i-1}_U(\mathcal{F}|_U, \mathcal{G}|_U) \end{tikzcd}$$

with exact rows. By induction on i and the comment on inverse systems above we find that the left two vertical arrows are isomorphisms. Now ξ gives an element in the top right group which is the image of ξ' in the middle top group, which in turn maps to an element of the bottom middle group coming from some element in the left bottom group. We conclude that ξ maps to zero in $\mathrm{Ext}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G})$ for some n as desired.

Surjectivity. Let $\xi \in \mathrm{Ext}^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$. Arguing as above using that $i > 0$ we can find an surjection $\mathcal{H} \to \mathcal{F}|_U$ of coherent $\mathcal{O}_U$-modules such that ξ maps to zero in $\mathrm{Ext}^i_U(\mathcal{H}, \mathcal{G}|_U)$. Then we can find a map $\varphi : \mathcal{F}'' \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to U is $\mathcal{H} \to \mathcal{F}|_U$, see Properties, Lemma 28.22.4. Observe that the lemma doesn't guarantee φ is surjective but this won't matter (it is possible to pick a surjective φ with a little bit of additional work). Denote $\mathcal{F}' = \mathrm{Ker}(\varphi)$. The short exact sequence

$$0 \to \mathcal{F}'|_U \to \mathcal{F}''|_U \to \mathcal{F}|_U \to 0$$

shows that ξ is the image of ξ' in $\mathrm{Ext}^{i-1}_U(\mathcal{F}'|_U, \mathcal{G}|_U)$. By induction on i we can find an n such that ξ' is the image of some ξ'_n in $\mathrm{Ext}^{i-1}_X(\mathcal{I}^n\mathcal{F}', \mathcal{G})$. By Artin-Rees we can find an $m \geq n$ such that $\mathcal{F}' \cap \mathcal{I}^m\mathcal{F}'' \subset \mathcal{I}^n\mathcal{F}'$. Using the short exact sequence

$$0 \to \mathcal{F}' \cap \mathcal{I}^m\mathcal{F}'' \to \mathcal{I}^m\mathcal{F}'' \to \mathcal{I}^m\,\mathrm{Im}(\varphi) \to 0$$

the image of ξ'_n in $\mathrm{Ext}^{i-1}_X(\mathcal{F}' \cap \mathcal{I}^m\mathcal{F}'', \mathcal{G})$ maps by the boundary map to an element ξ_m of $\mathrm{Ext}^i_X(\mathcal{I}^m\,\mathrm{Im}(\varphi), \mathcal{G})$ which maps to ξ. Since $\mathrm{Im}(\varphi)$ and $\mathcal{F}$ agree over U we see that $\mathcal{F}/\mathcal{I}^m\,\mathrm{Im}(\varphi)$ is supported on $X \setminus U$. Hence there exists an $l \geq m$ such that $\mathcal{I}^l\mathcal{F} \subset \mathcal{I}^m\,\mathrm{Im}(\varphi)$, see Cohomology of Schemes, Lemma 30.10.2. Taking the image of ξ_m in $\mathrm{Ext}^i_X(\mathcal{I}^l\mathcal{F}, \mathcal{G})$ we win. □

0G4K Lemma 48.30.2. The result of Lemma 48.30.1 holds even for $L \in D^+_{\mathrm{Coh}}(\mathcal{O}_X)$.

Proof. Namely, if (K_n) is a Deligne system then there exists a $b \in \mathbf{Z}$ such that $H^i(K_n) = 0$ for $i > b$. Then $\mathrm{Hom}(K_n, L) = \mathrm{Hom}(K_n, \tau_{\leq b}L)$ and $\mathrm{Hom}(K, L) = \mathrm{Hom}(K, \tau_{\leq b}L)$. Hence using the result of the lemma for $\tau_{\leq b}L$ we win. □

0G4L Lemma 48.30.3. Let $j : U \to X$ be an open immersion of Noetherian schemes.

(1) Let (K_n) and (L_n) be Deligne systems. Let K and L be the values of the constant systems $(K_n|_U)$ and $(L_n|_U)$. Given a morphism $\alpha : K \to L$ of $D(\mathcal{O}_U)$ there is a unique morphism of pro-systems $(K_n) \to (L_n)$ of $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ whose restriction to U is α.
(2) Given $K \in D^b_{\mathrm{Coh}}(\mathcal{O}_U)$ there exists a Deligne system (K_n) such that $(K_n|_U)$ is constant with value K.

(3) The pro-object (K_n) of $D^b_{\text{Coh}}(\mathcal{O}_X)$ of (2) is unique up to unique isomorphism (as a pro-object).

Proof. Part (1) is an immediate consequence of Lemma 48.30.1 and the fact that morphisms between pro-systems are the same as morphisms between the functors they corepresent, see Categories, Remark 4.22.7.

Let K be as in (2). We can choose $K' \in D^b_{\text{Coh}}(\mathcal{O}_X)$ whose restriction to U is isomorphic to K, see Derived Categories of Schemes, Lemma 36.13.2. By Derived Categories of Schemes, Proposition 36.11.2 we can represent K' by a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$-modules. Choose a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing locus is $X \setminus U$ (for example choose $\mathcal{I}$ to correspond to the reduced induced subscheme structure on $X \setminus U$). Then we can set K_n equal to the object represented by the complex $\mathcal{I}^n\mathcal{F}^\bullet$ as in the introduction to this section.

Part (3) is immediate from parts (1) and (2). $\square$

0G4M Lemma 48.30.4. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let

$$K \to L \to M \to K[1]$$

be a distinguished triangle of $D^b_{\text{Coh}}(\mathcal{O}_U)$. Then there exists an inverse system of distinguished triangles

$$K_n \to L_n \to M_n \to K_n[1]$$

in $D^b_{\text{Coh}}(\mathcal{O}_X)$ such that (K_n), (L_n), (M_n) are Deligne systems and such that the restriction of these distinguished triangles to U is isomorphic to the distinguished triangle we started out with.

Proof. Let (K_n) be as in Lemma 48.30.3 part (2). Choose an object L' of $D^b_{\text{Coh}}(\mathcal{O}_X)$ whose restriction to U is L (we can do this as the lemma shows). By Lemma 48.30.1 we can find an n and a morphism $K_n \to L'$ on X whose restriction to U is the given arrow $K \to L$. We conclude there is a morphism $K' \to L'$ of $D^b_{\text{Coh}}(\mathcal{O}_X)$ whose restriction to U is the given arrow $K \to L$.

By Derived Categories of Schemes, Proposition 36.11.2 we can find a morphism $\alpha^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of bounded complexes of coherent $\mathcal{O}_X$-modules representing $K' \to L'$. Choose a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing locus is $X \setminus U$. Then we let $K_n = \mathcal{I}^n\mathcal{F}^\bullet$ and $L_n = \mathcal{I}^n\mathcal{G}^\bullet$. Observe that $\alpha^\bullet$ induces a morphism of complexes $\alpha_n^\bullet : \mathcal{I}^n\mathcal{F}^\bullet \to \mathcal{I}^n\mathcal{G}^\bullet$. From the construction of cones in Derived Categories, Section 13.9 it is clear that

$$C(\alpha_n)^\bullet = \mathcal{I}^n C(\alpha^\bullet)$$

and hence we can set $M_n = C(\alpha_n)^\bullet$. Namely, we have a compatible system of distinguished triangles (see discussion in Derived Categories, Section 13.12)

$$K_n \to L_n \to M_n \to K_n[1]$$

whose restriction to U is isomorphic to the distinguished triangle we started out with by axiom TR3 and Derived Categories, Lemma 13.4.3. $\square$

0G4N Remark 48.30.5. Let $j : U \to X$ be an open immersion of Noetherian schemes. Sending $K \in D^b_{\text{Coh}}(\mathcal{O}_U)$ to a Deligne system whose restriction to U is K determines a functor

$$Rj_! : D^b_{\text{Coh}}(\mathcal{O}_U) \longrightarrow \text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_X)$$

which is "exact" by Lemma 48.30.4 and which is "left adjoint" to the functor $j^* : D^b_{\text{Coh}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_U)$ by Lemma 48.30.1.

0G4P Remark 48.30.6. Let (A_n) and (B_n) be inverse systems of a category $\mathcal{C}$. Let us say a linear-pro-morphism from (A_n) to (B_n) is given by a compatible family of morphisms $\varphi_n : A_{cn+d} \to B_n$ for all $n \geq 1$ for some fixed integers $c, d \geq 1$. We'll say $(\varphi_n : A_{cn+d} \to B_n)$ and $(\psi_n : A_{c'n+d'} \to B_n)$ determine the same morphism if there exist $c'' \geq \max(c, c')$ and $d'' \geq \max(d, d')$ such that the two induced morphisms $A_{c''n+d''} \to B_n$ are the same for all n. It seems likely that Deligne systems (K_n) with given value on U are well defined up to linear-pro-isomorphisms. If we ever need this we will carefully formulate and prove this here.

0G4Q Lemma 48.30.7. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let

$$K_n \to L_n \to M_n \to K_n[1]$$

be an inverse system of distinguished triangles in $D^b_{\text{Coh}}(\mathcal{O}_X)$. If (K_n) and (M_n) are pro-isomorphic to Deligne systems, then so is (L_n).

Proof. Observe that the systems $(K_n|_U)$ and $(M_n|_U)$ are essentially constant as they are pro-isomorphic to constant systems. Denote K and M their values. By Derived Categories, Lemma 13.42.2 we see that the inverse system $L_n|_U$ is essentially constant as well. Denote L its value. Let $N \in D^b_{\text{Coh}}(\mathcal{O}_X)$. Consider the commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & \operatorname{colim} \operatorname{Hom}_X(M_n, N) \arrow[r] \arrow[d] & \operatorname{colim} \operatorname{Hom}_X(L_n, N) \arrow[r] \arrow[d] & \operatorname{colim} \operatorname{Hom}_X(K_n, N) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \operatorname{Hom}_U(M, N|_U) \arrow[r] & \operatorname{Hom}_U(L, N|_U) \arrow[r] & \operatorname{Hom}_U(K, N|_U) \arrow[r] & \ldots \end{tikzcd}$$

By Lemma 48.30.1 and the fact that isomorphic ind-systems have the same colimit, we see that the vertical arrows two to the right and two to the left of the middle one are isomorphisms. By the 5-lemma we conclude that the middle vertical arrow is an isomorphism. Now, if (L'_n) is a Deligne system whose restriction to U has constant value L (which exists by Lemma 48.30.3), then we have $\operatorname{colim} \operatorname{Hom}_X(L'_n, N) = \operatorname{Hom}_U(L, N|_U)$ as well. Hence the pro-systems (L_n) and (L'_n) are pro-isomorphic by Categories, Remark 4.22.7. $\square$

0G4R Lemma 48.30.8. Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}^\bullet$ be a complex of coherent $\mathcal{O}_X$-modules. Let $p \in \mathbf{Z}$. Set $\mathcal{H} = H^p(\mathcal{F}^\bullet)$ and $\mathcal{H}_n = H^p(\mathcal{I}^n\mathcal{F}^\bullet)$. Then there are canonical $\mathcal{O}_X$-module maps

$$\ldots \to \mathcal{H}_3 \to \mathcal{H}_2 \to \mathcal{H}_1 \to \mathcal{H}$$

There exists a $c > 0$ such that for $n \geq c$ the image of $\mathcal{H}_n \to \mathcal{H}$ is contained in $\mathcal{I}^{n-c}\mathcal{H}$ and there is a canonical $\mathcal{O}_X$-module map $\mathcal{I}^n\mathcal{H} \to \mathcal{H}_{n-c}$ such that the compositions

$$\mathcal{I}^n\mathcal{H} \to \mathcal{H}_{n-c} \to \mathcal{I}^{n-2c}\mathcal{H} \quad \text{and} \quad \mathcal{H}_n \to \mathcal{I}^{n-c}\mathcal{H} \to \mathcal{H}_{n-2c}$$

are the canonical ones. In particular, the inverse systems $(\mathcal{H}_n)$ and $(\mathcal{I}^n\mathcal{H})$ are isomorphic as pro-objects of $\text{Mod}(\mathcal{O}_X)$.

Proof. If X is affine, translated into algebra this is More on Algebra, Lemma 15.101.1. In the general case, argue exactly as in the proof of that lemma replacing the reference to Artin-Rees in algebra with a reference to Cohomology of Schemes, Lemma 30.10.3. Details omitted. $\square$

0G4S Lemma 48.30.9. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $a \leq b$ be integers. Let (K_n) be an inverse system of $D^b_{\text{Coh}}(\mathcal{O}_X)$ such that $H^i(K_n) = 0$ for $i \not\in [a, b]$. The following are equivalent

(1) (K_n) is pro-isomorphic to a Deligne system,
(2) for every $p \in \mathbf{Z}$ there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that the pro-systems $(H^p(K_n))$ and $(\mathcal{I}^n\mathcal{F})$ are pro-isomorphic.

Proof. Assume (1). To prove (2) holds we may assume (K_n) is a Deligne system. By definition we may choose a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$-modules and a quasi-coherent sheaf of ideals cutting out $X \setminus U$ such that K_n is represented by $\mathcal{I}^n\mathcal{F}^\bullet$. Thus the result follows from Lemma 48.30.8.

Assume (2). We will prove that (K_n) is as in (1) by induction on $b - a$. If $a = b$ then (1) holds essentially by assumption. If $a < b$ then we consider the compatible system of distinguished triangles

$$\tau_{\leq a}K_n \to K_n \to \tau_{\geq a+1}K_n \to (\tau_{\leq a}K_n)[1]$$

See Derived Categories, Remark 13.12.4. By induction on $b-a$ we know that $\tau_{\leq a}K_n$ and $\tau_{\geq a+1}K_n$ are pro-isomorphic to Deligne systems. We conclude by Lemma 48.30.7. □

0G4T Lemma 48.30.10. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let (K_n) be an inverse system in $D^b_{\text{Coh}}(\mathcal{O}_X)$. Let $X = W_1 \cup \ldots \cup W_r$ be an open covering. The following are equivalent

(1) (K_n) is pro-isomorphic to a Deligne system,
(2) for each i the restriction $(K_n|_{W_i})$ is pro-isomorphic to a Deligne system with respect to the open immersion $U \cap W_i \to W_i$.

Proof. By induction on r. If $r = 1$ then the result is clear. Assume $r > 1$. Set $V = W_1 \cup \ldots \cup W_{r-1}$. By induction we see that $(K_n|_V)$ is a Deligne system. This reduces us to the discussion in the next paragraph.

Assume $X = V \cup W$ is an open covering and $(K_n|_W)$ and $(K_n|_V)$ are pro-isomorphic to Deligne systems. We have to show that (K_n) is pro-isomorphic to a Deligne system. Observe that $(K_n|_{V\cap W})$ is pro-isomorphic to a Deligne system (it follows immediately from the construction of Deligne systems that restrictions to opens preserves them). In particular the pro-systems $(K_n|_{U\cap V})$, $(K_n|_{U\cap W})$, and $(K_n|_{U\cap V\cap W})$ are essentially constant. It follows from the distinguished triangles in Cohomology, Lemma 20.33.2 and Derived Categories, Lemma 13.42.2 that $(K_n|_U)$ is essentially constant. Denote $K \in D^b_{\text{Coh}}(\mathcal{O}_U)$ the value of this system. Let L be an object of $D^b_{\textit{Coh}}(\mathcal{O}_X)$. Consider the diagram

$$\begin{tikzcd}
\mathrm{colim}\,\mathrm{Ext}^{-1}(K_n|_V, L|_V) \oplus \mathrm{colim}\,\mathrm{Ext}^{-1}(K_n|_W, L|_W) \arrow[r] \arrow[d] & \mathrm{Ext}^{-1}(K|_{U \cap V}, L|_{U \cap V}) \oplus \mathrm{Ext}^{-1}(K|_{U \cap W}, L|_{U \cap W}) \arrow[d] \\
\mathrm{colim}\,\mathrm{Ext}^{-1}(K_n|_{V \cap W}, L|_{V \cap W}) \arrow[r] \arrow[d] & \mathrm{Ext}^{-1}(K|_{U \cap V \cap W}, L|_{U \cap V \cap W}) \arrow[d] \\
\mathrm{colim}\,\mathrm{Hom}(K_n, L) \arrow[r] \arrow[d] & \mathrm{Hom}(K|_U, L|_U) \arrow[d] \\
\mathrm{colim}\,\mathrm{Hom}(K_n|_V, L|_V) \oplus \mathrm{colim}\,\mathrm{Hom}(K_n|_W, L|_W) \arrow[r] \arrow[d] & \mathrm{Hom}(K|_{U \cap V}, L|_{U \cap V}) \oplus \mathrm{Hom}(K|_{U \cap W}, L|_{U \cap W}) \arrow[d] \\
\mathrm{colim}\,\mathrm{Hom}(K_n|_{V \cap W}, L|_{V \cap W}) \arrow[r] & \mathrm{Hom}(K|_{U \cap V \cap W}, L|_{U \cap V \cap W})
\end{tikzcd}$$

The vertical sequences are exact by Cohomology, Lemma 20.33.3 and the fact that filtered colimits are exact. All horizontal arrows except for the middle one are isomorphisms by Lemma 48.30.1 and the fact that pro-isomorphic systems have the same colimits. Hence the middle one is an isomorphism too by the 5-lemma. It follows that (K_n) is pro-isomorphic to a Deligne system for K. Namey, if (K'_n) is a Deligne system whose restriction to U has constant value K (which exists by Lemma 48.30.3), then we have $\mathrm{colim}\,\mathrm{Hom}_X(K'_n, L) = \mathrm{Hom}_U(K, L|_U)$ as well. Hence the pro-systems (K_n) and (K'_n) are pro-isomorphic by Categories, Remark 4.22.7. □

0G4U Lemma 48.30.11. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals with $V(\mathcal{I}) = X \setminus U$. Let K be in $D^b_{\textit{Coh}}(\mathcal{O}_X)$. Then

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}^n$$

is pro-isomorphic to a Deligne system with constant value $K|_U$ over U.

Proof. By Lemma 48.30.10 the question is local on X. Thus we may assume X is the spectrum of a Noetherian ring. In this case the statement follows from the algebra version which is More on Algebra, Lemma 15.101.6. □

48.31. Preliminaries to compactly supported cohomology

0G4V In Situation 48.16.1 let $f : X \to Y$ be a morphism in the category $\textit{FTS}_S$. Using the constructions in the previous section, we will construct a functor

$$Rf_! : D^b_{\textit{Coh}}(\mathcal{O}_X) \longrightarrow \text{Pro-}D^b_{\textit{Coh}}(\mathcal{O}_Y)$$

which reduces to the functor of Remark 48.30.5 if f is an open immersion and in general is constructed using a compactification of f. Before we do this, we need the following lemmas to prove our construction is well defined.

0G4W Lemma 48.31.1. Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open subscheme and set $U = f^{-1}(V)$. Picture

$$\begin{tikzcd} U \arrow[r, "j"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "j'"] & Y \end{tikzcd}$$

Then we have a canonical isomorphism $Rj'_! \circ Rg_* \to Rf_* \circ Rj_!$ of functors $D^b_{\text{Coh}}(\mathcal{O}_U) \to \text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)$ where $Rj_!$ and $Rj'_!$ are as in Remark 48.30.5.

First proof. Let K be an object of $D^b_{\text{Coh}}(\mathcal{O}_U)$. Let (K_n) be a Deligne system for $U \to X$ whose restriction to U is constant with value K. Of course this means that (K_n) represents $Rj_!K$ in $\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_X)$. Observe that both $Rj'_!Rg_*K$ and $Rf_*Rj_!K$ restrict to the constant pro-object with value Rg_*K on V. This is immediate for the first one and for the second one it follows from the fact that $(Rf_*K_n)|_V = Rg_*(K_n|_U) = Rg_*K$. By the uniqueness of Deligne systems in Lemma 48.30.3 it suffices to show that (Rf_*K_n) is pro-isomorphic to a Deligne system. The lemma referenced will also show that the isomorphism we obtain is functorial.

Proof that (Rf_*K_n) is pro-isomorphic to a Deligne system. First, we observe that the question is independent of the choice of the Deligne system (K_n) corresponding to K (by the aforementioned uniqueness). By Lemmas 48.30.4 and 48.30.7 if we have a distinguished triangle

$$K \to L \to M \to K[1]$$

in $D^b_{\text{Coh}}(\mathcal{O}_U)$ and the result holds for K and M, then the result holds for L. Using the distinguished triangles of canonical truncations (Derived Categories, Remark 13.12.4) we reduce to the problem studied in the next paragraph.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals cutting out $Y \setminus V$. Denote $\mathcal{J}^n\mathcal{F}$ the image of $f^*\mathcal{J}^n \otimes \mathcal{F} \to \mathcal{F}$. We have to show that $(Rf_*(\mathcal{J}^n\mathcal{F}))$ is a Deligne system. By Lemma 48.30.10 the question is local on Y. Thus we may assume $Y = \text{Spec}(A)$ is affine and $\mathcal{J}$ corresponds to an ideal $I \subset A$. By Lemma 48.30.9 it suffices to show that the inverse system of cohomology modules $(H^p(X, I^n\mathcal{F}))$ is pro-isomorphic to the inverse system (I^nM) for some finite A-module M. This is shown in Cohomology of Schemes, Lemma 30.20.3. $\square$

Second proof. Let K be an object of $D^b_{\text{Coh}}(\mathcal{O}_U)$. Let L be an object of $D^b_{\text{Coh}}(\mathcal{O}_Y)$. We will construct a bijection

$$\text{Hom}_{\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)}(Rj'_!Rg_*K, L) \longrightarrow \text{Hom}_{\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)}(Rf_*Rj_!K, L)$$

functorial in K and L. Fixing K this will determine an isomorphism of pro-objects $Rf_*Rj_!K \to Rj'_!Rg_*K$ by Categories, Remark 4.22.7 and varying K we obtain that this determines an isomorphism of functors. To actually produce the isomorphism we use the sequence of functorial equalities

$$\begin{aligned}\mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_Y)}(Rj'_!Rg_*K, L) &= \mathrm{Hom}_V(Rg_*K, L|_V) \\ &= \mathrm{Hom}_U(K, g^!(L|_V)) \\ &= \mathrm{Hom}_U(K, f^!L|_U)) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_X)}(Rj_!K, f^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_Y)}(Rf_*Rj_!K, L)\end{aligned}$$

The first equality is true by Lemma 48.30.1. The second equality is true because g is proper (as the base change of f to V) and hence $g^!$ is the right adjoint of pushforward by construction, see Section 48.16. The third equality holds as $g^!(L|_V) = f^!L|_U$ by Lemma 48.17.2. Since $f^!L$ is in $D^+_{\mathrm{Coh}}(\mathcal{O}_X)$ by Lemma 48.17.6 the fourth equality follows from Lemma 48.30.2. The fifth equality holds again because $f^!$ is the right adjoint to Rf_* as f is proper. □

0G4X Lemma 48.31.2. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $j' : U \to X'$ be a compactification of U over X (see proof) and denote $f : X' \to X$ the structure morphism. Then we have a canonical isomorphism $Rj_! \to Rf_* \circ R(j')_!$ of functors $D^b_{\mathrm{Coh}}(\mathcal{O}_U) \to \text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ where $Rj_!$ and $Rj'_!$ are as in Remark 48.30.5.

Proof. The fact that X' is a compactification of U over X means precisely that $f : X' \to X$ is proper, that j' is an open immersion, and $j = f \circ j'$. See More on Flatness, Section 38.32. If $j'(U) = f^{-1}(j(U))$, then the lemma follows immediately from Lemma 48.31.1. If $j'(U) \neq f^{-1}(j(U))$, then denote $X'' \subset X'$ the scheme theoretic closure of $j' : U \to X'$ and denote $j'' : U \to X''$ the corresponding open immersion. Picture

$$\begin{tikzcd} & X'' \arrow[d, "f'"] \\ & X' \arrow[d, "f"] \\ U \arrow[r, "j"] \arrow[ur, "j'"] \arrow[uur, "j''"] & X \end{tikzcd}$$

By More on Flatness, Lemma 38.32.1 part (c) and the discussion above we have isomorphisms $Rf'_* \circ Rj''_! = Rj'_!$ and $R(f \circ f')_* \circ Rj''_! = Rj_!$. Since $R(f \circ f')_* = Rf_* \circ Rf'_*$ we conclude. □

0G4Y Remark 48.31.3. Let $X \supset U \supset U'$ be open subschemes of a Noetherian scheme X. Denote $j : U \to X$ and $j' : U' \to X$ the inclusion morphisms. We claim there is a canonical map

$$Rj'_!(K|_{U'}) \longrightarrow Rj_!K$$

functorial for K in $D^b_{\mathrm{Coh}}(\mathcal{O}_U)$. Namely, by Lemma 48.30.1 we have for any L in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ the map

$$\begin{aligned}\mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_X)}(Rj_!K, L) &= \mathrm{Hom}_U(K, L|_U) \\ &\to \mathrm{Hom}_{U'}(K|_{U'}, L|_{U'}) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_X)}(Rj'_!(K|_{U'}), L)\end{aligned}$$

functorial in L and K'. The functoriality in L shows by Categories, Remark 4.22.7 that we obtain a canonical map $Rj'_!(K|_{U'}) \to Rj_!K$ which is functorial in K by the functoriality of the arrow above in K.

Here is an explicit construction of this arrow. Namely, suppose that $\mathcal{F}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules whose restriction to U represents K in the derived category. We have seen in the proof of Lemma 48.30.3 that such a complex always exists. Let $\mathcal{I}$, resp. $\mathcal{I}'$ be a quasi-coherent sheaf of ideals on X with $V(\mathcal{I}) = X \setminus U$, resp. $V(\mathcal{I}') = X \setminus U'$. After replacing $\mathcal{I}$ by $\mathcal{I} + \mathcal{I}'$ we may assume $\mathcal{I}' \subset \mathcal{I}$. By construction $Rj_!K$, resp. $Rj'_!(K|_{U'})$ is represented by the inverse system (K_n), resp. (K'_n) of $D^b_{\text{Coh}}(\mathcal{O}_X)$ with

$$K_n = \mathcal{I}^n\mathcal{F}^\bullet \quad \text{resp.} \quad K'_n = (\mathcal{I}')^n\mathcal{F}^\bullet$$

Clearly the map constructed above is given by the maps $K'_n \to K_n$ coming from the inclusions $(\mathcal{I}')^n \subset \mathcal{I}^n$.

48.32. Compactly supported cohomology for coherent modules

0G4Z In Situation 48.16.1 given a morphism $f : X \to Y$ in FTS_S, we will define a functor

$$Rf_! : D^b_{\text{Coh}}(\mathcal{O}_X) \longrightarrow \text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)$$

Namely, we choose a compactification $j : X \to \overline{X}$ over Y which is possible by More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Then we set

$$Rf_!K = R\overline{f}_*Rj_!K$$

for $K \in D^b_{\text{Coh}}(\mathcal{O}_X)$ where $Rj_!$ is as in Remark 48.30.5.

0G50 Lemma 48.32.1. The functor $Rf_!$ is, up to isomorphism, independent of the choice of the compactification.

In fact, the functor $Rf_!$ will be characterized as a "left adjoint" to $f^!$ which will determine it up to unique isomorphism.

Proof. Consider the category of compactifications of X over Y, which is cofiltered according to More on Flatness, Theorem 38.33.8 and Lemmas 38.32.1 and 38.32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $R\overline{f}_* \circ Rj_!$. Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Then we get an isomorphism

$$R\overline{f}_{2,*} \circ Rj_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ Rj_{1,!}$$

using Lemma 48.31.2 in the first equality. In this way we see our functor is independent of the choice of compactification up to isomorphism. □

0G51 Proposition 48.32.2. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Then the functors $Rf_!$ and $f^!$ are adjoint in the following sense: for all $K \in D^b_{\text{Coh}}(\mathcal{O}_X)$ and $L \in D^+_{\text{Coh}}(\mathcal{O}_Y)$ we have

$$\text{Hom}_X(K, f^!L) = \text{Hom}_{\text{Pro-}D^+_{\text{Coh}}(\mathcal{O}_Y)}(Rf_!K, L)$$

bifunctorially in K and L.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Then we have

$$\begin{aligned} \mathrm{Hom}_X(K, f^!L) &= \mathrm{Hom}_X(K, j^*\overline{f}^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^+_{Coh}(\mathcal{O}_{\overline{X}})}(Rj_!K, \overline{f}^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^+_{Coh}(\mathcal{O}_Y)}(Rf_*Rj_!K, L) \\ &= \mathrm{Hom}_{\text{Pro-}D^+_{Coh}(\mathcal{O}_Y)}(Rf_!K, L) \end{aligned}$$

The first equality follows immediately from the construction of $f^!$ in Section 48.16. By Lemma 48.17.6 we have $\overline{f}^!L$ in $D^+_{Coh}(\mathcal{O}_{\overline{X}})$ hence the second equality follows from Lemma 48.30.2. Since $\overline{f}$ is proper the functor $\overline{f}^!$ is the right adjoint of pushforward by construction. This is why we have the third equality. The fourth equality holds because $Rf_! = Rf_*Rj_!$. □

0G52 Lemma 48.32.3. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let

$$K \to L \to M \to K[1]$$

be a distinguished triangle of $D^b_{Coh}(\mathcal{O}_X)$. Then there exists an inverse system of distinguished triangles

$$K_n \to L_n \to M_n \to K_n[1]$$

in $D^b_{Coh}(\mathcal{O}_Y)$ such that the pro-systems (K_n), (L_n), and (M_n) give $Rf_!K$, $Rf_!L$, and $Rf_!M$.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Choose an inverse system of distinguished triangles

$$\overline{K}_n \to \overline{L}_n \to \overline{M}_n \to \overline{K}_n[1]$$

in $D^b_{Coh}(\mathcal{O}_{\overline{X}})$ as in Lemma 48.30.4 corresponding to the open immersion j and the given distinguished triangle. Take $K_n = R\overline{f}_*\overline{K}_n$ and similarly for L_n and M_n. This works by the very definition of $Rf_!$. □

0G53 Remark 48.32.4. Let $\mathcal{C}$ be a category. Suppose given an inverse system

$$\ldots \xrightarrow{\alpha_4} (M_{3,n}) \xrightarrow{\alpha_3} (M_{2,n}) \xrightarrow{\alpha_2} (M_{1,n})$$

of inverse systems in the category of pro-objects of $\mathcal{C}$. In other words, the arrows α_i are morphisms of pro-objects. By Categories, Example 4.22.6 we can represent each α_i by a pair (m_i, a_i) where $m_i : \mathbf{N} \to \mathbf{N}$ is an increasing function and $a_{i,n} : M_{i,m_i(n)} \to M_{i-1,n}$ is a morphism of $\mathcal{C}$ making the diagrams

$$\begin{tikzcd} \ldots \arrow[r] & M_{i,m_i(3)} \arrow[r] \arrow[d, "a_{i,3}"] & M_{i,m_i(2)} \arrow[r] \arrow[d, "a_{i,2}"] & M_{i,m_i(1)} \arrow[d, "a_{i,1}"] \\ \ldots \arrow[r] & M_{i-1,3} \arrow[r] & M_{i-1,2} \arrow[r] & M_{i-1,1} \end{tikzcd}$$

commute. By replacing $m_i(n)$ by $\max(n, m_i(n))$ and adjusting the morphisms $a_i(n)$ accordingly (as in the example referenced) we may assume that $m_i(n) \geq n$. In this situation consider the inverse system

$$\ldots \to M_{4,m_4(m_3(m_2(4)))} \to M_{3,m_3(m_2(3))} \to M_{2,m_2(2)} \to M_{1,1}$$

with general term

$$M_k = M_{k,m_k(m_{k-1}(\ldots(m_2(k))\ldots))}$$

For any object N of $\mathcal{C}$ we have

$$\mathrm{colim}_i \, \mathrm{colim}_n \, \mathrm{Mor}_\mathcal{C}(M_{i,n}, N) = \mathrm{colim}_k \, \mathrm{Mor}_\mathcal{C}(M_k, N)$$

We omit the details. In other words, we see that the inverse system (M_k) has the property

$$\mathrm{colim}_i \, \mathrm{Mor}_{\text{Pro-}\mathcal{C}}((M_{i,n}), N) = \mathrm{Mor}_{\text{Pro-}\mathcal{C}}((M_k), N)$$

This property determines the inverse system (M_k) up to pro-isomorphism by the discussion in Categories, Remark 4.22.7. In this way we can turn certain inverse systems in Pro-$\mathcal{C}$ into pro-objects with countable index categories.

0G54 Remark 48.32.5. In Situation 48.16.1 let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of FTS_S. Let us define the composition

$$Rg_! \circ Rf_! : D^b_{\mathrm{Coh}}(\mathcal{O}_X) \longrightarrow \text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_Z)$$

Namely, by the very construction of $Rf_!$ for K in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ the output $Rf_!K$ is the pro-isomorphism class of an inverse system (M_n) in $D^b_{\mathrm{Coh}}(\mathcal{O}_Y)$. Then, since $Rg_!$ is constructed similarly, we see that

$$\ldots \to Rg_!M_3 \to Rg_!M_2 \to Rg_!M_1$$

is an inverse system of Pro-$D^b_{\mathrm{Coh}}(\mathcal{O}_Y)$. By the discussion in Remark 48.32.4 there is a unique pro-isomorphism class, which we will denote $Rg_!Rf_!K$, of inverse systems in $D^b_{\mathrm{Coh}}(\mathcal{O}_Z)$ such that

$$\mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_Z)}(Rg_!Rf_!K, L) = \mathrm{colim}_n \, \mathrm{Hom}_{\text{Pro-}D^b_{\mathrm{Coh}}(\mathcal{O}_Z)}(Rg_!M_n, L)$$

We omit the discussion necessary to see that this construction is functorial in K as it will immediately follow from the next lemma.

0G55 Lemma 48.32.6. In Situation 48.16.1 let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of FTS_S. With notation as in Remark 48.32.5 we have $Rg_! \circ Rf_! = R(g \circ f)_!$.

Proof. By the discussion in Categories, Remark 4.22.7 it suffices to show that we obtain the same answer if we compute Hom into L in $D^b_{\mathrm{Coh}}(\mathcal{O}_Z)$. To do this we compute, using the notation in Remark 48.32.5, as follows

$$\begin{aligned}
\mathrm{Hom}_Z(Rg_!Rf_!K, L) &= \mathrm{colim}_n \, \mathrm{Hom}_Z(Rg_!M_n, L) \\
&= \mathrm{colim}_n \, \mathrm{Hom}_Y(M_n, g^!L) \\
&= \mathrm{Hom}_Y(Rf_!K, g^!L) \\
&= \mathrm{Hom}_X(K, f^!g^!L) \\
&= \mathrm{Hom}_X(K, (g \circ f)^!L) \\
&= \mathrm{Hom}_Z(R(g \circ f)_!K, L)
\end{aligned}$$

The first equality is the definition of $Rg_!Rf_!K$. The second equality is Proposition 48.32.2 for g. The third equality is the fact that $Rf_!K$ is given by (M_n). The fourth equality is Proposition 48.32.2 for f. The fifth equality is Lemma 48.16.3. The sixth is Proposition 48.32.2 for $g \circ f$. $\square$

0G56 Remark 48.32.7. In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S and let $U \subset X$ be an open. Set $g = f|_U : U \to Y$. Then there is a canonical morphism

$$Rg_!(K|_U) \longrightarrow Rf_!K$$

functorial in K in $D^b_{\text{Coh}}(\mathcal{O}_X)$ which can be defined in at least 3 ways.

(1) Denote $i : U \to X$ the inclusion morphism. We have $Rg_! = Rf_! \circ Ri_!$ by Lemma 48.32.6 and we can use $Rf_!$ applied to the map $Ri_!(K|_U) \to K$ which is a special case of Remark 48.31.3.
(2) Choose a compactification $j : X \to \overline{X}$ of X over Y with structure morphism $\overline{f} : \overline{X} \to Y$. Set $j' = j \circ i : U \to \overline{X}$. We can use that $Rf_! = R\overline{f}_* \circ Rj_!$ and $Rg_! = R\overline{f}_* \circ Rj'_!$ and we can use $R\overline{f}_*$ applied to the map $Rj'_!(K|_U) \to Rj_!K$ of Remark 48.31.3.
(3) We can use
$$\begin{aligned} \mathrm{Hom}_{\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)}(Rf_!K, L) &= \mathrm{Hom}_X(K, f^!L) \\ &\to \mathrm{Hom}_U(K|_U, f^!L|_U) \\ &= \mathrm{Hom}_U(K|_U, g^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)}(Rg_!(K|_U), L) \end{aligned}$$
functorial in L and K. Here we have used Proposition 48.32.2 twice and the construction of upper shriek functors which shows that $g^! = i^* \circ f^!$. The functoriality in L shows by Categories, Remark 4.22.7 that we obtain a canonical map $Rg_!(K|_U) \to Rf_!K$ in Pro-$D^b_{\text{Coh}}(\mathcal{O}_Y)$ which is functorial in K by the functoriality of the arrow above in K.

Each of these three constructions gives the same arrow; we omit the details.

0G57 Remark 48.32.8. Let us generalize the covariance of compactly supported cohomology given in Remark 48.32.7 to étale morphisms. Namely, in Situation 48.16.1 suppose given a commutative diagram
$$\begin{tikzcd} U \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$
of FTS_S with h étale. Then there is a canonical morphism
$$Rg_!(h^*K) \longrightarrow Rf_!K$$
functorial in K in $D^b_{\text{Coh}}(\mathcal{O}_X)$. We define this transformation using the sequence of maps
$$\begin{aligned} \mathrm{Hom}_{\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)}(Rf_!K, L) &= \mathrm{Hom}_X(K, f^!L) \\ &\to \mathrm{Hom}_U(h^*K, h^*(f^!L)) \\ &= \mathrm{Hom}_U(h^*K, h^!f^!L) \\ &= \mathrm{Hom}_U(h^*K, g^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{\text{Coh}}(\mathcal{O}_Y)}(Rg_!(h^*K), L) \end{aligned}$$
functorial in L and K. Here we have used Proposition 48.32.2 twice, we have used the equality $h^* = h^!$ of Lemma 48.18.2, and we have used the equality $h^! \circ f^! = g^!$ of Lemma 48.16.3. The functoriality in L shows by Categories, Remark 4.22.7 that we obtain a canonical map $Rg_!(h^*K) \to Rf_!K$ in Pro-$D^b_{\text{Coh}}(\mathcal{O}_Y)$ which is functorial in K by the functoriality of the arrow above in K.

0G58 Remark 48.32.9. In Remarks 48.32.7 and 48.32.8 we have seen that the construction of compactly supported cohomology is covariant with respect to open immersions and étale morphisms. In fact, the correct generality is that given a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "h"] \arrow[rd, "g"'] & & X \arrow[ld, "f"] \\ & Y & \end{tikzcd}$$

of FTS_S with h flat and quasi-finite there exists a canonical transformation

$$Rg_! \circ h^* \longrightarrow Rf_!$$

As in Remark 48.32.8 this map can be constructed using a transformation of functors $h^* \to h^!$ on $D^+_{\textit{Coh}}(\mathcal{O}_X)$. Recall that $h^!K = h^*K \otimes \omega_{U/X}$ where $\omega_{U/X} = h^!\mathcal{O}_X$ is the relative dualizing sheaf of the flat quasi-finite morphism h (see Lemmas 48.17.9 and 48.21.6). Recall that $\omega_{U/X}$ is the same as the relative dualizing module which will be constructed in Discriminants, Remark 49.2.11 by Discriminants, Lemma 49.15.1. Thus we can use the trace element $\tau_{U/X} : \mathcal{O}_U \to \omega_{U/X}$ which will be constructed in Discriminants, Remark 49.4.7 to define our transformation. If we ever need this, we will precisely formulate and prove the result here.

48.33. Duality for compactly supported cohomology

0G59 Let k be a field. Let U be a separated scheme of finite type over k. Let K be an object of $D^b_{\textit{Coh}}(\mathcal{O}_U)$. Let us define the compactly supported cohomology $H^i_c(U, K)$ of K as follows. Choose an open immersion $j : U \to X$ into a scheme proper over k and a Deligne system (K_n) for $j : U \to X$ whose restriction to U is constant with value K. Then we set

$$H^i_c(U, K) = \lim H^i(X, K_n)$$

We view this as a topological k-vector space using the limit topology (see More on Algebra, Section 15.36). There are several points to make here.

First, this definition is independent of the choice of X and (K_n). Namely, if $p : U \to \mathrm{Spec}(k)$ denotes the structure morphism, then we already know that $Rp_!K = (R\Gamma(X, K_n))$ is well defined up to pro-isomorphism in $D(k)$ hence so is the limit defining $H^i_c(U, K)$.

Second, it may seem more natural to use the expression

$$H^i(R\lim R\Gamma(X, K_n)) = R\Gamma(X, R\lim K_n)$$

but this would give the same answer: since the k-vector spaces $H^j(X, K_n)$ are finite dimensional, these inverse systems satisfy Mittag-Leffler and hence $R^1\lim$ terms of Cohomology, Lemma 20.37.1 vanish.

If $U' \subset U$ is an open subscheme, then there is a canonical map

$$H^i_c(U', K|_{U'}) \longrightarrow H^i_c(U, K)$$

functorial for K in $D^b_{\textit{Coh}}(\mathcal{O}_U)$. See for example Remark 48.32.7. In fact, using Remark 48.32.8 we see that more generally such a map exists for an étale morphism $U' \to U$ of separated schemes of finite type over k.

If V is a k-vector space then we put a topology on $\mathrm{Hom}_k(V, k)$ as follows: write $V = \bigcup V_i$ as the filtered union of its finite dimensional k-subvector spaces and use the limit topology on $\mathrm{Hom}_k(V, k) = \lim \mathrm{Hom}_k(V_i, k)$. If $\dim_k V < \infty$ then the topology on $\mathrm{Hom}_k(V, k)$ is discrete. More generally, if $V = \mathrm{colim}_n V_n$ is written as a directed colimit of finite dimensional vector spaces, then $\mathrm{Hom}_k(V, k) = \lim \mathrm{Hom}_k(V_n, k)$ as topological vector spaces.

0G5A Lemma 48.33.1. Let $p : U \to \mathrm{Spec}(k)$ be separated of finite type where k is a field. Let $\omega^\bullet_{U/k} = p^!\mathcal{O}_{\mathrm{Spec}(k)}$. There are canonical isomorphisms

$$\mathrm{Hom}_k(H^i(U, K), k) = H^{-i}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(K, \omega^\bullet_{U/k}))$$

of topological k-vector spaces functorial for K in $D^b_{\mathrm{Coh}}(\mathcal{O}_U)$.

Proof. Choose a compactification $j : U \to X$ over k. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent ideal sheaf with $V(\mathcal{I}) = X \setminus U$. By Derived Categories of Schemes, Proposition 36.11.2 we may choose $M \in D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ with $K = M|_U$. We have

$$H^i(U, K) = \mathrm{Ext}^i_U(\mathcal{O}_U, M|_U) = \mathrm{colim}\, \mathrm{Ext}^i_X(\mathcal{I}^n, M) = \mathrm{colim}\, H^i(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M))$$

by Lemma 48.30.1. Since $\mathcal{I}^n$ is a coherent $\mathcal{O}_X$-module, we have $\mathcal{I}^n$ in $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$, hence $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M)$ is in $D^+_{\mathrm{Coh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.5.

Let $\omega^\bullet_{X/k} = q^!\mathcal{O}_{\mathrm{Spec}(k)}$ where $q : X \to \mathrm{Spec}(k)$ is the structure morphism, see Section 48.27. We find that

$$\begin{aligned}
&\mathrm{Hom}_k(H^i(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M)), k) \\
&\quad = \mathrm{Ext}^{-i}_X(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega^\bullet_{X/k}) \\
&\quad = H^{-i}(X, R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega^\bullet_{X/k}))
\end{aligned}$$

by Lemma 48.27.1. By Lemma 48.2.4 part (1) the canonical map

$$R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}^n \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega^\bullet_{X/k})$$

is an isomorphism. Observe that $\omega^\bullet_{U/k} = \omega^\bullet_{X/k}|_U$ because $p^!$ is constructed as $q^!$ composed with restriction to U. Hence $R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/k})$ is an object of $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ which restricts to $R\mathcal{H}om_{\mathcal{O}_U}(K, \omega^\bullet_{U/k})$ on U. Hence by Lemma 48.30.11 we conclude that

$$\lim H^{-i}(X, R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}^n)$$

is an avatar for the right hand side of the equality of the lemma. Combining all the isomorphisms obtained in this manner we get the isomorphism of the lemma. ☐

0G5B Lemma 48.33.2. With notation as in Lemma 48.33.1 suppose $U' \subset U$ is an open subscheme. Then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_k(H^i(U, K), k) \arrow[r] & H^{-i}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(K, \omega^\bullet_{U/k})) \\
\mathrm{Hom}_k(H^i(U', K|_{U'}), k) \arrow[r] \arrow[u] & H^{-i}_c(U', R\mathcal{H}om_{\mathcal{O}_{U'}}(K, \omega^\bullet_{U'/k})) \arrow[u]
\end{tikzcd}$$

is commutative. Here the horizontal arrows are the isomorphisms of Lemma 48.33.1, the vertical arrow on the left is the contragredient to the restriction map $H^i(U, K) \to H^i(U', K|_{U'})$, and the right vertical arrow is Remark 48.32.7 (see discussion before the lemma).

Proof. We strongly urge the reader to skip this proof. Choose X and M as in the proof of Lemma 48.33.1. We are going to drop the subscript $\mathcal{O}_X$ from $R\mathcal{H}om$ and $\otimes^{\mathbf{L}}$. We write

$$H^i(U, K) = \mathrm{colim}\, H^i(X, R\mathcal{H}om(\mathcal{I}^n, M))$$

and

$$H^i(U', K|_{U'}) = \mathrm{colim}\, H^i(X, R\mathcal{H}om((\mathcal{I}')^n, M))$$

as in the proof of Lemma 48.33.1 where we choose $\mathcal{I}' \subset \mathcal{I}$ as in the discussion in Remark 48.31.3 so that the map $H^i(U, K) \to H^i(U', K|_{U'})$ is induced by the maps $(\mathcal{I}')^n \to \mathcal{I}^n$. We similarly write

$$H^i_c(U, R\mathcal{H}om(K, \omega^\bullet_{U/k})) = \lim H^i(X, R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} \mathcal{I}^n)$$

and

$$H^i_c(U', R\mathcal{H}om(K|_{U'}, \omega^\bullet_{U'/k})) = \lim H^i(X, R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} (\mathcal{I}')^n)$$

so that the arrow $H^i_c(U', R\mathcal{H}om(K|_{U'}, \omega^\bullet_{U'/k})) \to H^i_c(U, R\mathcal{H}om(K, \omega^\bullet_{U/k}))$ is similarly deduced from the maps $(\mathcal{I}')^n \to \mathcal{I}^n$. The diagrams

$$\begin{tikzcd}
R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} \mathcal{I}^n \arrow[r] & R\mathcal{H}om(R\mathcal{H}om(\mathcal{I}^n, M), \omega^\bullet_{X/k}) \\
R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} (\mathcal{I}')^n \arrow[r] \arrow[u] & R\mathcal{H}om(R\mathcal{H}om((\mathcal{I}')^n, M), \omega^\bullet_{X/k}) \arrow[u]
\end{tikzcd}$$

commute because the construction of the horizontal arrows in Cohomology, Lemma 20.42.9 is functorial in all three entries. Hence we finally come down to the assertion that the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_k(H^i(X, R\mathcal{H}om(\mathcal{I}^n, M)), k) \arrow[r] & H^{-i}(X, R\mathcal{H}om(R\mathcal{H}om(\mathcal{I}^n, M), \omega^\bullet_{X/k})) \\
\mathrm{Hom}_k(H^i(X, R\mathcal{H}om((\mathcal{I}')^n, M)), k) \arrow[r] \arrow[u] & H^{-i}(X, R\mathcal{H}om(R\mathcal{H}om((\mathcal{I}')^n, M), \omega^\bullet_{X/k})) \arrow[u]
\end{tikzcd}$$

commute. This is true because the duality isomorphism

$$\mathrm{Hom}_k(H^i(X, L), k) = \mathrm{Ext}^{-i}_X(L, \omega^\bullet_{X/k}) = H^{-i}(X, R\mathcal{H}om(L, \omega^\bullet_{X/k}))$$

is functorial for L in $D_{QCoh}(\mathcal{O}_X)$. $\square$

0G5C Lemma 48.33.3. Let X be a proper scheme over a field k. Let $K \in D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ with $H^i(K) = 0$ for $i < 0$. Set $\mathcal{F} = H^0(K)$. Let $Z \subset X$ be closed with complement $U = X \setminus U$. Then

$$H^0_c(U, K|_U) \subset H^0(X, \mathcal{F})$$

is given by those global sections of $\mathcal{F}$ which vanish in an open neighbourhood of Z.

Proof. Consider the map $H^0_c(U, K|_U) \to H^0_X(X, K) = H^0(X, K) = H^0(X, \mathcal{F})$ of Remark 48.32.7. To study this we represent K by a bounded complex $\mathcal{F}^\bullet$ with $\mathcal{F}^i = 0$ for $i < 0$. Then we have by definition

$$H^0_c(U, K|_U) = \lim H^0(X, \mathcal{I}^n\mathcal{F}^\bullet) = \lim \mathrm{Ker}(H^0(X, \mathcal{I}^n\mathcal{F}^0) \to H^0(X, \mathcal{I}^n\mathcal{F}^1))$$

By Artin-Rees (Cohomology of Schemes, Lemma 30.10.3) this is the same as $\lim H^0(X, \mathcal{I}^n\mathcal{F})$. Thus the arrow $H^0_c(U, K|_U) \to H^0(X, \mathcal{F})$ is injective and the image consists of those global sections of $\mathcal{F}$ which are contained in the subsheaf $\mathcal{I}^n\mathcal{F}$ for any n. The characterization of these as the sections which vanish in a neighbourhood of Z comes from Krull's intersection theorem (Algebra, Lemma 10.51.4) by looking at stalks of $\mathcal{F}$. See discussion in Algebra, Remark 10.51.6 for the case of functions. □

48.34. Lichtenbaum's theorem

0G5D The theorem below was conjectured by Lichtenbaum and proved by Grothendieck (see [Har67]). There is a very nice proof of the theorem by Kleiman in [Kle67]. A generalization of the theorem to the case of cohomology with supports can be found in [Lyu91]. The most interesting part of the argument is contained in the proof of the following lemma.

0G5E **Lemma 48.34.1.** *Let U be a variety. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If $H^d(U, \mathcal{F})$ is nonzero, then $\dim(U) \geq d$ and if equality holds, then U is proper.*

Proof. By the Grothendieck's vanishing result in Cohomology, Proposition 20.20.7 we conclude that $\dim(U) \geq d$. Assume $\dim(U) = d$. Choose a compactification $U \to X$ such that U is dense in X. (This is possible by More on Flatness, Theorem 38.33.8 and Lemma 38.32.2.) After replacing X by its reduction we find that X is a proper variety of dimension d and we see that U is proper if and only if $U = X$. Set $Z = X \setminus U$. We will show that $H^d(U, \mathcal{F})$ is zero if Z is nonempty.

Choose a coherent $\mathcal{O}_X$-module $\mathcal{G}$ whose restriction to U is $\mathcal{F}$, see Properties, Lemma 28.22.5. Let $\omega_X^\bullet$ denote the dualizing complex of X as in Section 48.27. Set $\omega_U^\bullet = \omega_X^\bullet|_U$. Then $H^d(U, \mathcal{F})$ is dual to

$$H^{-d}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}, \omega_U^\bullet))$$

by Lemma 48.33.1. By Lemma 48.27.1 we see that the cohomology sheaves of $\omega_X^\bullet$ vanish in degrees $< -d$ and $H^{-d}(\omega_X^\bullet) = \omega_X$ is a coherent $\mathcal{O}_X$-module which is (S_2) and whose support is X. In particular, ω_X is torsion free, see Divisors, Lemma 31.11.10. Thus we see that the cohomology sheaf

$$H^{-d}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \omega_X^\bullet)) = \mathcal{H}om(\mathcal{G}, \omega_X)$$

is torsion free, see Divisors, Lemma 31.11.12. Consequently this sheaf has no nonzero sections vanishing on any nonempty open of X (those would be torsion sections). Thus it follows from Lemma 48.33.3 that $H^{-d}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}, \omega_U^\bullet))$ is zero, and hence $H^d(U, \mathcal{F})$ is zero as desired. □

0G5F **Theorem 48.34.2.** *Let X be a nonempty separated scheme of finite type over a field k. Let $d = \dim(X)$. The following are equivalent*

1. *$H^d(X, \mathcal{F}) = 0$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ on X,*
2. *$H^d(X, \mathcal{F}) = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ on X, and*
3. *no irreducible component $X' \subset X$ of dimension d is proper over k.*

Proof. Assume there exists an irreducible component $X' \subset X$ (which we view as an integral closed subscheme) which is proper and has dimension d. Let $\omega_{X'}$ be a dualizing module of X' over k, see Lemma 48.27.1. Then $H^d(X', \omega_{X'})$ is nonzero as it is dual to $H^0(X', \mathcal{O}_{X'})$ by the lemma. Hence we see that $H^d(X, \omega_{X'}) = H^d(X', \omega_{X'})$ is nonzero and we conclude that (1) does not hold. In this way we see that (1) implies (3).

Let us prove that (3) implies (1). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that $H^d(X, \mathcal{F})$ is nonzero. Choose a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

as in Cohomology of Schemes, Lemma 30.12.3. We obtain exact sequences

$$H^d(X, \mathcal{F}_i) \to H^d(X, \mathcal{F}_{i+1}) \to H^d(X, \mathcal{F}_{i+1}/\mathcal{F}_i)$$

Thus for some $i \in \{1, \ldots, m\}$ we find that $H^d(X, \mathcal{F}_{i+1}/\mathcal{F}_i)$ is nonzero. By our choice of the filtration this means that there exists an integral closed subscheme $Z \subset X$ and a nonzero coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ such that $H^d(Z, \mathcal{I})$ is nonzero. By Lemma 48.34.1 we conclude $\dim(Z) = d$ and Z is proper over k contradicting (3). Hence (3) implies (1).

Finally, let us show that (1) and (2) are equivalent for any Noetherian scheme X. Namely, (2) trivially implies (1). On the other hand, assume (1) and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as the filtered colimit of its coherent submodules, see Properties, Lemma 28.22.3. Then we have $H^d(X, \mathcal{F}) = \operatorname{colim} H^d(X, \mathcal{F}_i) = 0$ by Cohomology, Lemma 20.19.1. Thus (2) is true. $\square$

48.35. Other chapters

Preliminaries

Schemes

Topics in Scheme Theory

CHAPTER 49

Discriminants and Differents

0DWH 49.1. Introduction

0DWI In this chapter we study the different and discriminant of locally quasi-finite morphisms of schemes. A good reference for some of this material is [Kun86].

Given a quasi-finite morphism $f : Y \to X$ of Noetherian schemes there is a relative dualizing module $\omega_{Y/X}$. In Section 49.2 we construct this module from scratch, using Zariski's main theorem and étale localization methods. The key property is that given a diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"'] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with $g : X' \to X$ flat, $Y' \subset X' \times_X Y$ open, and $f' : Y' \to X'$ finite, then there is a canonical isomorphism

$$f'_*(g')^*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_{X'}}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_{X'})$$

as sheaves of $f'_*\mathcal{O}_{Y'}$-modules. In Section 49.4 we prove that if f is flat, then there is a canonical global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which for every commutative diagram as above maps $(g')^*\tau_{Y/X}$ to the trace map of Section 49.3 for the finite locally free morphism f'. In Section 49.9 we define the different for a flat quasi-finite morphism of Noetherian schemes as the annihilator of the cokernel of $\tau_{Y/X} : \mathcal{O}_X \to \omega_{Y/X}$.

The main goal of this chapter is to prove that for quasi-finite syntomic[1] f the different agrees with the Kähler different. The Kähler different is the zeroth fitting ideal of $\Omega_{Y/X}$, see Section 49.7. This agreement is not obvious; we use a slick argument due to Tate, see Section 49.12. On the way we also discuss the Noether different and the Dedekind different.

Only in the end of this chapter, see Sections 49.15 and 49.16, do we make the link with the more advanced material on duality for schemes.

49.2. Dualizing modules for quasi-finite ring maps

0BUK Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. By Zariski's main theorem (Algebra, Lemma 10.123.14) there exists a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. We set

0BSZ (49.2.0.1)
$$\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$$

in this situation. The reader can think of this as a kind of relative dualizing module, see Lemmas 49.15.1 and 49.2.12. In this section we will show by elementary

[1] AKA flat and lci.

commutative algebra methods that $\omega_{B/A}$ is independent of the choice of the factorization and that formation of $\omega_{B/A}$ commutes with flat base change. To help prove the independence of factorizations we compare two given factorizations.

0BT0 Lemma 49.2.1. Let $A \to B$ be a quasi-finite ring map. Given two factorizations $A \to B' \to B$ and $A \to B'' \to B$ with $A \to B'$ and $A \to B''$ finite and $\text{Spec}(B) \to \text{Spec}(B')$ and $\text{Spec}(B) \to \text{Spec}(B'')$ open immersions, there exists an A-subalgebra $B''' \subset B$ finite over A such that $\text{Spec}(B) \to \text{Spec}(B''')$ an open immersion and $B' \to B$ and $B'' \to B$ factor through B'''.

Proof. Let $B''' \subset B$ be the A-subalgebra generated by the images of $B' \to B$ and $B'' \to B$. As B' and B'' are each generated by finitely many elements integral over A, we see that B''' is generated by finitely many elements integral over A and we conclude that B''' is finite over A (Algebra, Lemma 10.36.5). Consider the maps

$$B = B' \otimes_{B'} B \to B''' \otimes_{B'} B \to B \otimes_{B'} B = B$$

The final equality holds because $\text{Spec}(B) \to \text{Spec}(B')$ is an open immersion (and hence a monomorphism). The second arrow is injective as $B' \to B$ is flat. Hence both arrows are isomorphisms. This means that

$$\begin{tikzcd} \text{Spec}(B''') \arrow[d] & \text{Spec}(B) \arrow[l] \arrow[d] \\ \text{Spec}(B') & \text{Spec}(B) \arrow[l] \end{tikzcd}$$

is cartesian. Since the base change of an open immersion is an open immersion we conclude. $\square$

0BT1 Lemma 49.2.2. The module (49.2.0.1) is well defined, i.e., independent of the choice of the factorization.

Proof. Let B', B'', B''' be as in Lemma 49.2.1. We obtain a canonical map

$$\omega''' = \text{Hom}_A(B''', A) \otimes_{B'''} B \longrightarrow \text{Hom}_A(B', A) \otimes_{B'} B = \omega'$$

and a similar one involving B''. If we show these maps are isomorphisms then the lemma is proved. Let $g \in B'$ be an element such that $B'_g \to B_g$ is an isomorphism and hence $B'_g \to (B''')_g \to B_g$ are isomorphisms. It suffices to show that $(\omega''')_g \to \omega'_g$ is an isomorphism. The kernel and cokernel of the ring map $B' \to B'''$ are finite A-modules and g-power torsion. Hence they are annihilated by a power of g. This easily implies the result. $\square$

0BT2 Lemma 49.2.3. Let $A \to B$ be a quasi-finite map of Noetherian rings.

(1) If $A \to B$ factors as $A \to A_f \to B$ for some $f \in A$, then $\omega_{B/A} = \omega_{B/A_f}$.
(2) If $g \in B$, then $(\omega_{B/A})_g = \omega_{B_g/A}$.
(3) If $f \in A$, then $\omega_{B_f/A_f} = (\omega_{B/A})_f$.

Proof. Say $A \to B' \to B$ is a factorization with $A \to B'$ finite and $\text{Spec}(B) \to \text{Spec}(B')$ an open immersion. In case (1) we may use the factorization $A_f \to B'_f \to B$ to compute ω_{B/A_f} and use Algebra, Lemma 10.10.2. In case (2) use the factorization $A \to B' \to B_g$ to see the result. Part (3) follows from a combination of (1) and (2). $\square$

Let $A \to B$ be a quasi-finite ring map of Noetherian rings, let $A \to A_1$ be an arbitrary ring map of Noetherian rings, and set $B_1 = B \otimes_A A_1$. We obtain a cocartesian diagram

$$\begin{tikzcd} B \arrow[r] & B_1 \\ A \arrow[u] \arrow[r] & A_1 \arrow[u] \end{tikzcd}$$

Observe that $A_1 \to B_1$ is quasi-finite as well (Algebra, Lemma 10.122.8). In this situation we will define a canonical B-linear base change map

0BVB (49.2.3.1) $$\omega_{B/A} \longrightarrow \omega_{B_1/A_1}$$

Namely, we choose a factorization $A \to B' \to B$ as in the construction of $\omega_{B/A}$. Then $B'_1 = B' \otimes_A A_1$ is finite over A_1 and we can use the factorization $A_1 \to B'_1 \to B_1$ in the construction of ω_{B_1/A_1}. Thus we have to construct a map

$$\mathrm{Hom}_A(B', A) \otimes_{B'} B \longrightarrow \mathrm{Hom}_{A_1}(B' \otimes_A A_1, A_1) \otimes_{B'_1} B_1$$

Thus it suffices to construct a B'-linear map $\mathrm{Hom}_A(B', A) \to \mathrm{Hom}_{A_1}(B' \otimes_A A_1, A_1)$ which we will denote $\varphi \mapsto \varphi_1$. Namely, given an A-linear map $\varphi : B' \to A$ we let φ_1 be the map such that $\varphi_1(b' \otimes a_1) = \varphi(b')a_1$. This is clearly A_1-linear and the construction is complete.

0BVC **Lemma 49.2.4.** *The base change map (49.2.3.1) is independent of the choice of the factorization $A \to B' \to B$. Given ring maps $A \to A_1 \to A_2$ the composition of the base change maps for $A \to A_1$ and $A_1 \to A_2$ is the base change map for $A \to A_2$.*

Proof. Omitted. Hint: argue in exactly the same way as in Lemma 49.2.2 using Lemma 49.2.1. $\square$

0BT3 **Lemma 49.2.5.** *If $A \to A_1$ is flat, then the base change map (49.2.3.1) induces an isomorphism $\omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1}$.*

Proof. Assume that $A \to A_1$ is flat. By construction of $\omega_{B/A}$ we may assume that $A \to B$ is finite. Then $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ and $\omega_{B_1/A_1} = \mathrm{Hom}_{A_1}(B_1, A_1)$. Since $B_1 = B \otimes_A A_1$ the result follows from More on Algebra, Lemma 15.65.4. $\square$

0BT4 **Lemma 49.2.6.** *Let $A \to B \to C$ be quasi-finite homomorphisms of Noetherian rings. There is a canonical map $\omega_{B/A} \otimes_B \omega_{C/B} \to \omega_{C/A}$.*

Proof. Choose $A \to B' \to B$ with $A \to B'$ finite such that $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion. Then $B' \to C$ is quasi-finite too. Choose $B' \to C' \to C$ with $B' \to C'$ finite and $\mathrm{Spec}(C) \to \mathrm{Spec}(C')$ an open immersion. Then the source of the arrow is

$$\mathrm{Hom}_A(B', A) \otimes_{B'} B \otimes_B \mathrm{Hom}_B(B \otimes_{B'} C', B) \otimes_{B \otimes_{B'} C'} C$$

which is equal to

$$\mathrm{Hom}_A(B', A) \otimes_{B'} \mathrm{Hom}_{B'}(C', B) \otimes_{C'} C$$

This indeed comes with a canonical map to $\mathrm{Hom}_A(C', A) \otimes_{C'} C = \omega_{C/A}$ coming from composition $\mathrm{Hom}_A(B', A) \times \mathrm{Hom}_{B'}(C', B) \to \mathrm{Hom}_A(C', A)$. $\square$

0BT5 **Lemma 49.2.7.** *Let $A \to B$ and $A \to C$ be quasi-finite maps of Noetherian rings. Then $\omega_{B \times C/A} = \omega_{B/A} \times \omega_{C/A}$ as modules over $B \times C$.*

Proof. Choose factorizations $A \to B' \to B$ and $A \to C' \to C$ such that $A \to B'$ and $A \to C'$ are finite and such that $\text{Spec}(B) \to \text{Spec}(B')$ and $\text{Spec}(C) \to \text{Spec}(C')$ are open immersions. Then $A \to B' \times C' \to B \times C$ is a similar factorization. Using this factorization to compute $\omega_{B \times C/A}$ gives the lemma. $\square$

0BVD Lemma 49.2.8. Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Then $\text{Ass}_B(\omega_{B/A})$ is the set of primes of B lying over associated primes of A.

Proof. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion on spectra. As $\omega_{B/A} = \omega_{B'/A} \otimes_{B'} B$ it suffices to prove the statement for $\omega_{B'/A}$. Thus we may assume $A \to B$ is finite.

Assume $\mathfrak p \in \text{Ass}(A)$ and $\mathfrak q$ is a prime of B lying over $\mathfrak p$. Let $x \in A$ be an element whose annihilator is $\mathfrak p$. Choose a nonzero $\kappa(\mathfrak p)$ linear map $\lambda : \kappa(\mathfrak q) \to \kappa(\mathfrak p)$. Since $A/\mathfrak p \subset B/\mathfrak q$ is a finite extension of rings, there is an $f \in A$, $f \not\in \mathfrak p$ such that $f\lambda$ maps $B/\mathfrak q$ into $A/\mathfrak p$. Hence we obtain a nonzero A-linear map

$$B \to B/\mathfrak q \to A/\mathfrak p \to A, \quad b \mapsto f\lambda(b)x$$

An easy computation shows that this element of $\omega_{B/A}$ has annihilator $\mathfrak q$, whence $\mathfrak q \in \text{Ass}(\omega_{B/A})$.

Conversely, suppose that $\mathfrak q \subset B$ is a prime ideal lying over a prime $\mathfrak p \subset A$ which is not an associated prime of A. We have to show that $\mathfrak q \not\in \text{Ass}_B(\omega_{B/A})$. After replacing A by $A_{\mathfrak p}$ and B by $B_{\mathfrak p}$ we may assume that $\mathfrak p$ is a maximal ideal of A. This is allowed by Lemma 49.2.5 and Algebra, Lemma 10.63.16. Then there exists an $f \in \mathfrak m$ which is a nonzerodivisor on A. Then f is a nonzerodivisor on $\omega_{B/A}$ and hence $\mathfrak q$ is not an associated prime of this module. $\square$

0BVE Lemma 49.2.9. Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. Then $\omega_{B/A}$ is a flat A-module.

Proof. Let $\mathfrak q \subset B$ be a prime lying over $\mathfrak p \subset A$. We will show that the localization $\omega_{B/A,\mathfrak q}$ is flat over $A_{\mathfrak p}$. This suffices by Algebra, Lemma 10.39.18. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak p' \subset A'$ lying over $\mathfrak p$ such that $\kappa(\mathfrak p') = \kappa(\mathfrak p)$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak q'$ of $B \otimes_A A'$ lying over $\mathfrak q$ and $\mathfrak p'$ corresponds to a prime of C. By Lemma 49.2.5 and Algebra, Lemma 10.100.1 it suffices to show $\omega_{B'/A',\mathfrak q'}$ is flat over $A'_{\mathfrak p'}$. Since $\omega_{B'/A'} = \omega_{C/A'} \times \omega_{D/A'}$ by Lemma 49.2.7 this reduces us to the case where B is finite flat over A. In this case B is finite locally free as an A-module and $\omega_{B/A} = \text{Hom}_A(B, A)$ is the dual finite locally free A-module. $\square$

0BVF Lemma 49.2.10. If $A \to B$ is flat, then the base change map (49.2.3.1) induces an isomorphism $\omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1}$.

Proof. If $A \to B$ is finite flat, then B is finite locally free as an A-module. In this case $\omega_{B/A} = \text{Hom}_A(B, A)$ is the dual finite locally free A-module and formation of this module commutes with arbitrary base change which proves the lemma in this case. In the next paragraph we reduce the general (quasi-finite flat) case to the finite flat case just discussed.

Let $\mathfrak q_1 \subset B_1$ be a prime. We will show that the localization of the map at the prime $\mathfrak q_1$ is an isomorphism, which suffices by Algebra, Lemma 10.23.1. Let $\mathfrak q \subset B$ and $\mathfrak p \subset A$ be the prime ideals lying under $\mathfrak q_1$. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak p' \subset A'$ lying over $\mathfrak p$ such that $\kappa(\mathfrak p') = \kappa(\mathfrak p)$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak q'$ of $B \otimes_A A'$ lying over $\mathfrak q$ and $\mathfrak p'$ corresponds to a prime of C. Set $A'_1 = A' \otimes_A A_1$ and consider the base change maps (49.2.3.1) for the ring maps $A \to A' \to A'_1$ and $A \to A_1 \to A'_1$ as in the diagram

$$\begin{tikzcd}
\omega_{B'/A'} \otimes_{B'} B'_1 \arrow[r] & \omega_{B'_1/A'_1} \\
\omega_{B/A} \otimes_B B'_1 \arrow[r] \arrow[u] & \omega_{B_1/A_1} \otimes_{B_1} B'_1 \arrow[u]
\end{tikzcd}$$

where $B' = B \otimes_A A'$, $B_1 = B \otimes_A A_1$, and $B'_1 = B \otimes_A (A' \otimes_A A_1)$. By Lemma 49.2.4 the diagram commutes. By Lemma 49.2.5 the vertical arrows are isomorphisms. As $B_1 \to B'_1$ is étale and hence flat it suffices to prove the top horizontal arrow is an isomorphism after localizing at a prime $\mathfrak q'_1$ of B'_1 lying over $\mathfrak q$ (there is such a prime and use Algebra, Lemma 10.39.17). Thus we may assume that $B = C \times D$ with $A \to C$ finite and $\mathfrak q$ corresponding to a prime of C. In this case the dualizing module $\omega_{B/A}$ decomposes in a similar fashion (Lemma 49.2.7) which reduces the question to the finite flat case $A \to C$ handled above. $\square$

0BVG Remark 49.2.11. Let $f : Y \to X$ be a locally quasi-finite morphism of locally Noetherian schemes. It is clear from Lemma 49.2.3 that there is a unique coherent $\mathcal{O}_Y$-module $\omega_{Y/X}$ on Y such that for every pair of affine opens $\text{Spec}(B) = V \subset Y$, $\text{Spec}(A) = U \subset X$ with $f(V) \subset U$ there is a canonical isomorphism

$$H^0(V, \omega_{Y/X}) = \omega_{B/A}$$

and where these isomorphisms are compatible with restriction maps.

0C0I Lemma 49.2.12. Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Let $\omega_{B/A}^\bullet \in D(B)$ be the algebraic relative dualizing complex discussed in Dualizing Complexes, Section 47.25. Then there is a (nonunique) isomorphism $\omega_{B/A} = H^0(\omega_{B/A}^\bullet)$.

Proof. Choose a factorization $A \to B' \to B$ where $A \to B'$ is finite and $\text{Spec}(B') \to \text{Spec}(B)$ is an open immersion. Then $\omega_{B/A}^\bullet = \omega_{B'/A}^\bullet \otimes_B^{\mathbf{L}} B'$ by Dualizing Complexes, Lemmas 47.24.7 and 47.24.9 and the definition of $\omega_{B/A}^\bullet$. Hence it suffices to show there is an isomorphism when $A \to B$ is finite. In this case we can use Dualizing Complexes, Lemma 47.24.8 to see that $\omega_{B/A}^\bullet = R\,\text{Hom}(B, A)$ and hence $H^0(\omega_{B/A}^\bullet) = \text{Hom}_A(B, A)$ as desired. $\square$

49.3. Discriminant of a finite locally free morphism

0BVH Let X be a scheme and let $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-module. Then there is a canonical trace map

$$\text{Trace} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{O}_X$$

See Exercises, Exercise 111.22.6. This map has the property that Trace(id) is the locally constant function on $\mathcal{O}_X$ corresponding to the rank of $\mathcal{F}$.

Let $\pi : X \to Y$ be a morphism of schemes which is finite locally free. Then there exists a canonical trace for π which is an $\mathcal{O}_Y$-linear map

$$\mathrm{Trace}_\pi : \pi_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

sending a local section f of $\pi_*\mathcal{O}_X$ to the trace of multiplication by f on $\pi_*\mathcal{O}_X$. Over affine opens this recovers the construction in Exercises, Exercise 111.22.7. The composition

$$\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_*\mathcal{O}_X \xrightarrow{\mathrm{Trace}_\pi} \mathcal{O}_Y$$

equals multiplication by the degree of π (which is a locally constant function on Y). In analogy with Fields, Section 9.20 we can define the trace pairing

$$Q_\pi : \pi_*\mathcal{O}_X \times \pi_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

by the rule $(f, g) \mapsto \mathrm{Trace}_\pi(fg)$. We can think of Q_π as a linear map $\pi_*\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X, \mathcal{O}_Y)$ between locally free modules of the same rank, and hence obtain a determinant

$$\det(Q_\pi) : \wedge^{top}(\pi_*\mathcal{O}_X) \longrightarrow \wedge^{top}(\pi_*\mathcal{O}_X)^{\otimes -1}$$

or in other words a global section

$$\det(Q_\pi) \in \Gamma(Y, \wedge^{top}(\pi_*\mathcal{O}_X)^{\otimes -2})$$

The discriminant of π is by definition the closed subscheme $D_\pi \subset Y$ cut out by this global section. Clearly, D_π is a locally principal closed subscheme of Y.

0BJF Lemma 49.3.1. Let $\pi : X \to Y$ be a morphism of schemes which is finite locally free. Then π is étale if and only if its discriminant is empty.

Proof. By Morphisms, Lemma 29.36.8 it suffices to check that the fibres of π are étale. Since the construction of the trace pairing commutes with base change we reduce to the following question: Let k be a field and let A be a finite dimensional k-algebra. Show that A is étale over k if and only if the trace pairing $Q_{A/k} : A \times A \to k$, $(a, b) \mapsto \mathrm{Trace}_{A/k}(ab)$ is nondegenerate.

Assume $Q_{A/k}$ is nondegenerate. If $a \in A$ is a nilpotent element, then ab is nilpotent for all $b \in A$ and we conclude that $Q_{A/k}(a, -)$ is identically zero. Hence A is reduced. Then we can write $A = K_1 \times \ldots \times K_n$ as a product where each K_i is a field (see Algebra, Lemmas 10.53.2, 10.53.6, and 10.25.1). In this case the quadratic space $(A, Q_{A/k})$ is the orthogonal direct sum of the spaces $(K_i, Q_{K_i/k})$. It follows from Fields, Lemma 9.20.7 that each K_i is separable over k. This means that A is étale over k by Algebra, Lemma 10.143.4. The converse is proved by reading the argument backwards. $\square$

49.4. Traces for flat quasi-finite ring maps

0BSY The trace referred to in the title of this section is of a completely different nature than the trace discussed in Duality for Schemes, Section 48.7. Namely, it is the trace as discussed in Fields, Section 9.20 and generalized in Exercises, Exercises 111.22.6 and 111.22.7.

Let $A \to B$ be a finite flat map of Noetherian rings. Then B is finite flat as an A-module and hence finite locally free (Algebra, Lemma 10.78.2). Given $b \in B$ we can consider the trace $\text{Trace}_{B/A}(b)$ of the A-linear map $B \to B$ given by multiplication by b on B. By the references above this defines an A-linear map $\text{Trace}_{B/A} : B \to A$. Since $\omega_{B/A} = \text{Hom}_A(B, A)$ as $A \to B$ is finite, we see that $\text{Trace}_{B/A} \in \omega_{B/A}$.

For a general flat quasi-finite ring map we define the notion of a trace as follows.

0BT6 Definition 49.4.1. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. The *trace element* is the unique[2] element $\tau_{B/A} \in \omega_{B/A}$ with the following property: for any Noetherian A-algebra A_1 such that $B_1 = B \otimes_A A_1$ comes with a product decomposition $B_1 = C \times D$ with $A_1 \to C$ finite the image of $\tau_{B/A}$ in ω_{C/A_1} is Trace_{C/A_1}. Here we use the base change map (49.2.3.1) and Lemma 49.2.7 to get $\omega_{B/A} \to \omega_{B_1/A_1} \to \omega_{C/A_1}$.

We first prove that trace elements are unique and then we prove that they exist.

0BT7 Lemma 49.4.2. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Then there is at most one trace element in $\omega_{B/A}$.

Proof. Let $\mathfrak{q} \subset B$ be a prime ideal lying over the prime $\mathfrak{p} \subset A$. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A_1$ and a prime ideal $\mathfrak{p}_1 \subset A_1$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}_1) = \kappa(\mathfrak{p})$ and such that

$$B_1 = B \otimes_A A_1 = C \times D$$

with $A_1 \to C$ finite and such that the unique prime $\mathfrak{q}_1$ of $B \otimes_A A_1$ lying over $\mathfrak{q}$ and $\mathfrak{p}_1$ corresponds to a prime of C. Observe that $\omega_{C/A_1} = \omega_{B/A} \otimes_B C$ (combine Lemmas 49.2.5 and 49.2.7). Since the collection of ring maps $B \to C$ obtained in this manner is a jointly injective family of flat maps and since the image of $\tau_{B/A}$ in ω_{C/A_1} is prescribed the uniqueness follows. □

Here is a sanity check.

0BT8 Lemma 49.4.3. Let $A \to B$ be a finite flat map of Noetherian rings. Then $\text{Trace}_{B/A} \in \omega_{B/A}$ is the trace element.

Proof. Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Of course in this case $A_1 \to D$ is also finite. Set $B_1 = B \otimes_A A_1$. Since the construction of traces commutes with base change we see that $\text{Trace}_{B/A}$ maps to Trace_{B_1/A_1}. Thus the proof is finished by noticing that $\text{Trace}_{B_1/A_1} = (\text{Trace}_{C/A_1}, \text{Trace}_{D/A_1})$ under the isomorphism $\omega_{B_1/A_1} = \omega_{C/A_1} \times \omega_{D/A_1}$ of Lemma 49.2.7. □

0BT9 Lemma 49.4.4. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $\tau \in \omega_{B/A}$ be a trace element.

1. If $A \to A_1$ is a map with A_1 Noetherian, then with $B_1 = A_1 \otimes_A B$ the image of τ in ω_{B_1/A_1} is a trace element.
2. If $A = R_f$, then τ is a trace element in $\omega_{B/R}$.
3. If $g \in B$, then the image of τ in $\omega_{B_g/A}$ is a trace element.
4. If $B = B_1 \times B_2$, then τ maps to a trace element in both $\omega_{B_1/A}$ and $\omega_{B_2/A}$.

Proof. Part (1) is a formal consequence of the definition.

Statement (2) makes sense because $\omega_{B/R} = \omega_{B/A}$ by Lemma 49.2.3. Denote τ' the element τ but viewed as an element of $\omega_{B/R}$. To see that (2) is true suppose that we

[2] Uniqueness and existence will be justified in Lemmas 49.4.2 and 49.4.6.

have $R \to R_1$ with R_1 Noetherian and a product decomposition $B \otimes_R R_1 = C \times D$ with $R_1 \to C$ finite. Then with $A_1 = (R_1)_f$ we see that $B \otimes_A A_1 = C \times D$. Since $R_1 \to C$ is finite, a fortiori $A_1 \to C$ is finite. Hence we can use the defining property of τ to get the corresponding property of τ'.

Statement (3) makes sense because $\omega_{B_g/A} = (\omega_{B/A})_g$ by Lemma 49.2.3. The proof is similar to the proof of (2). Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B_1 = B \otimes_A A_1$. Then $\mathrm{Spec}(C) \to \mathrm{Spec}(B_1)$ is an open immersion as $B_g \otimes_A A_1 = (B_1)_g$ and the image is closed because $B_1 \to C$ is finite (as $A_1 \to C$ is finite). Thus we see that $B_1 = C \times D_1$ and $D = (D_1)_g$. Then we can use the defining property of τ to get the corresponding property for the image of τ in $\omega_{B_g/A}$.

Statement (4) makes sense because $\omega_{B/A} = \omega_{B_1/A} \times \omega_{B_2/A}$ by Lemma 49.2.7. Suppose we have $A \to A'$ with A' Noetherian and a product decomposition $B \otimes_A A' = C \times D$ with $A' \to C$ finite. Then it is clear that we can refine this product decomposition into $B \otimes_A A' = C_1 \times C_2 \times D_1 \times D_2$ with $A' \to C_i$ finite such that $B_i \otimes_A A' = C_i \times D_i$. Then we can use the defining property of τ to get the corresponding property for the image of τ in $\omega_{B_i/A}$. This uses the obvious fact that $\mathrm{Trace}_{C/A'} = (\mathrm{Trace}_{C_1/A'}, \mathrm{Trace}_{C_2/A'})$ under the decomposition $\omega_{C/A'} = \omega_{C_1/A'} \times \omega_{C_2/A'}$. $\square$

0BTA Lemma 49.4.5. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $g_1, \ldots, g_m \in B$ be elements generating the unit ideal. Let $\tau \in \omega_{B/A}$ be an element whose image in $\omega_{B_{g_i}/A}$ is a trace element for $A \to B_{g_i}$. Then τ is a trace element.

Proof. Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. We have to show that the image of τ in ω_{C/A_1} is Trace_{C/A_1}. Observe that $g_1, \ldots, g_m$ generate the unit ideal in $B_1 = B \otimes_A A_1$ and that τ maps to a trace element in $\omega_{(B_1)_{g_i}/A_1}$ by Lemma 49.4.4. Hence we may replace A by A_1 and B by B_1 to get to the situation as described in the next paragraph.

Here we assume that $B = C \times D$ with $A \to C$ is finite. Let τ_C be the image of τ in $\omega_{C/A}$. We have to prove that $\tau_C = \mathrm{Trace}_{C/A}$ in $\omega_{C/A}$. By the compatibility of trace elements with products (Lemma 49.4.4) we see that τ_C maps to a trace element in $\omega_{C_{g_i}/A}$. Hence, after replacing B by C we may assume that $A \to B$ is finite flat.

Assume $A \to B$ is finite flat. In this case $\mathrm{Trace}_{B/A}$ is a trace element by Lemma 49.4.3. Hence $\mathrm{Trace}_{B/A}$ maps to a trace element in $\omega_{B_{g_i}/A}$ by Lemma 49.4.4. Since trace elements are unique (Lemma 49.4.2) we find that $\mathrm{Trace}_{B/A}$ and τ map to the same elements in $\omega_{B_{g_i}/A} = (\omega_{B/A})_{g_i}$. As $g_1, \ldots, g_m$ generate the unit ideal of B the map $\omega_{B/A} \to \prod \omega_{B_{g_i}/A}$ is injective and we conclude that $\tau_C = \mathrm{Trace}_{B/A}$ as desired. $\square$

0BTB Lemma 49.4.6. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. There exists a trace element $\tau \in \omega_{B/A}$.

Proof. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ an open immersion. Let $g_1, \ldots, g_n \in B'$ be elements such that $\mathrm{Spec}(B) = \bigcup D(g_i)$ as opens of $\mathrm{Spec}(B')$. Suppose that we can prove the existence of trace elements τ_i for the quasi-finite flat ring maps $A \to B_{g_i}$. Then for all i, j the elements

τ_i and τ_j map to trace elements of $\omega_{B_{g_ig_j}/A}$ by Lemma 49.4.4. By uniqueness of trace elements (Lemma 49.4.2) they map to the same element. Hence the sheaf condition for the quasi-coherent module associated to $\omega_{B/A}$ (see Algebra, Lemma 10.24.1) produces an element $\tau \in \omega_{B/A}$. Then τ is a trace element by Lemma 49.4.5. In this way we reduce to the case treated in the next paragraph.

Assume we have $A \to B'$ finite and $g \in B'$ with $B = B'_g$ flat over A. It is our task to construct a trace element in $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$. Choose a resolution $F_1 \to F_0 \to B' \to 0$ of B' by finite free A-modules F_0 and F_1. Then we have an exact sequence

$$0 \to \mathrm{Hom}_A(B', A) \to F_0^\vee \to F_1^\vee$$

where $F_i^\vee = \mathrm{Hom}_A(F_i, A)$ is the dual finite free module. Similarly we have the exact sequence

$$0 \to \mathrm{Hom}_A(B', B') \to F_0^\vee \otimes_A B' \to F_1^\vee \otimes_A B'$$

The idea of the construction of τ is to use the diagram

$$B' \xrightarrow{\mu} \mathrm{Hom}_A(B', B') \leftarrow \mathrm{Hom}_A(B', A) \otimes_A B' \xrightarrow{ev} A$$

where the first arrow sends $b' \in B'$ to the A-linear operator given by multiplication by b' and the last arrow is the evaluation map. The problem is that the middle arrow, which sends $\lambda' \otimes b'$ to the map $b'' \mapsto \lambda'(b'')b'$, is not an isomorphism. If B' is flat over A, the exact sequences above show that it is an isomorphism and the composition from left to right is the usual trace $\mathrm{Trace}_{B'/A}$. In the general case, we consider the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_A(B', A) \otimes_A B' \arrow[r] \arrow[d] & \mathrm{Hom}_A(B', A) \otimes_A B'_g \arrow[d] \\
B' \arrow[r, "\mu"'] \arrow[ur, dotted, "\psi"] \arrow[urr, dotted] & \mathrm{Hom}_A(B', B') \arrow[r] & \mathrm{Ker}(F_0^\vee \otimes_A B'_g \to F_1^\vee \otimes_A B'_g)
\end{tikzcd}$$

By flatness of $A \to B'_g$ we see that the right vertical arrow is an isomorphism. Hence we obtain the unadorned dotted arrow. Since $B'_g = \mathrm{colim} \frac{1}{g^n} B'$, since colimits commute with tensor products, and since B' is a finitely presented A-module we can find an $n \geq 0$ and a B'-linear (for right B'-module structure) map $\psi : B' \to \mathrm{Hom}_A(B', A) \otimes_A B'$ whose composition with the left vertical arrow is $g^n \mu$. Composing with ev we obtain an element $ev \circ \psi \in \mathrm{Hom}_A(B', A)$. Then we set

$$\tau = (ev \circ \psi) \otimes g^{-n} \in \mathrm{Hom}_A(B', A) \otimes_{B'} B'_g = \omega_{B'_g/A} = \omega_{B/A}$$

We omit the easy verification that this element does not depend on the choice of n and ψ above.

Let us prove that τ as constructed in the previous paragraph has the desired property in a special case. Namely, say $B' = C' \times D'$ and $g = (f, h)$ where $A \to C'$ flat, D'_h is flat, and f is a unit in C'. To show: τ maps to $\mathrm{Trace}_{C'/A}$ in $\omega_{C'/A}$. In this case we first choose n_D and $\psi_D : D' \to \mathrm{Hom}_A(D', A) \otimes_A D'$ as above for the pair (D', h) and we can let $\psi_C : C' \to \mathrm{Hom}_A(C', A) \otimes_A C' = \mathrm{Hom}_A(C', C')$ be the map seconding $c' \in C'$ to multiplication by c'. Then we take $n = n_D$ and $\psi = (f^{n_D}\psi_C, \psi_D)$ and the desired compatibility is clear because $\mathrm{Trace}_{C'/A} = ev \circ \psi_C$ as remarked above.

To prove the desired property in general, suppose given $A \to A_1$ with A_1 Noetherian and a product decomposition $B'_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B'_1 = B' \otimes_A A_1$. Then $\mathrm{Spec}(C) \to \mathrm{Spec}(B'_1)$ is an open immersion as $B'_g \otimes_A A_1 = (B'_1)_g$ and the image is closed as $B'_1 \to C$ is finite (since $A_1 \to C$ is finite). Thus $B'_1 = C \times D'$ and $D'_g = D$. We conclude that $B'_1 = C \times D'$ and g over A_1 are as in the previous paragraph. Since formation of the displayed diagram above commutes with base change, the formation of τ commutes with the base change $A \to A_1$ (details omitted; use the resolution $F_1 \otimes_A A_1 \to F_0 \otimes_A A_1 \to B'_1 \to 0$ to see this). Thus the desired compatibility follows from the result of the previous paragraph. □

0BVJ Remark 49.4.7. Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 49.2.11. It is clear from the uniqueness, existence, and compatibility with localization of trace elements (Lemmas 49.4.2, 49.4.6, and 49.4.4) that there exists a global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

such that for every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ that element $\tau_{Y/X}$ maps to $\tau_{B/A}$ under the canonical isomorphism $H^0(V, \omega_{Y/X}) = \omega_{B/A}$.

0C13 Lemma 49.4.8. Let k be a field and let A be a finite k-algebra. Assume A is local with residue field k'. The following are equivalent

(1) $\mathrm{Trace}_{A/k}$ is nonzero,
(2) $\tau_{A/k} \in \omega_{A/k}$ is nonzero, and
(3) k'/k is separable and $\mathrm{length}_A(A)$ is prime to the characteristic of k.

Proof. Conditions (1) and (2) are equivalent by Lemma 49.4.3. Let $\mathfrak{m} \subset A$. Since $\dim_k(A) < \infty$ it is clear that A has finite length over A. Choose a filtration

$$A = I_0 \supset \mathfrak{m} = I_1 \supset I_2 \supset \ldots I_n = 0$$

by ideals such that $I_i/I_{i+1} \cong k'$ as A-modules. See Algebra, Lemma 10.52.11 which also shows that $n = \mathrm{length}_A(A)$. If $a \in \mathfrak{m}$ then $aI_i \subset I_{i+1}$ and it is immediate that $\mathrm{Trace}_{A/k}(a) = 0$. If $a \not\in \mathfrak{m}$ with image $\lambda \in k'$, then we conclude

$$\mathrm{Trace}_{A/k}(a) = \sum\nolimits_{i=0,\ldots,n-1} \mathrm{Trace}_k(a : I_i/I_{i-1} \to I_i/I_{i-1}) = n\mathrm{Trace}_{k'/k}(\lambda)$$

The proof of the lemma is finished by applying Fields, Lemma 9.20.7. □

49.5. Finite morphisms

0FKW In this section we collect some observations about the constructions in the previous sections for finite morphisms. Let $f : Y \to X$ be a finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 49.2.11.

The first remark is that

$$f_*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$$

as sheaves of $f_*\mathcal{O}_Y$-modules. Since f is affine, this formula uniquely characterizes $\omega_{Y/X}$, see Morphisms, Lemma 29.11.6. The formula holds because for $\mathrm{Spec}(A) =$

$U \subset X$ affine open, the inverse image $V = f^{-1}(U)$ is the spectrum of a finite A-algebra B and hence

$$H^0(U, f_*\omega_{Y/X}) = H^0(V, \omega_{Y/X}) = \omega_{B/A} = \mathrm{Hom}_A(B, A) = H^0(U, \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X))$$

by construction. In particular, we obtain a canonical evaluation map

$$f_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

which is given by evaluation at 1 if we think of $f_*\omega_{Y/X}$ as the sheaf $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$.

The second remark is that using the evaluation map we obtain canonical identifications

$$\mathrm{Hom}_Y(\mathcal{F}, f^*\mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathrm{Hom}_X(f_*\mathcal{F}, \mathcal{G})$$

functorially in the quasi-coherent module $\mathcal{F}$ on Y and the finite locally free module $\mathcal{G}$ on X. If $\mathcal{G} = \mathcal{O}_X$ this follows immediately from the above and Algebra, Lemma 10.14.4. For general $\mathcal{G}$ we can use the same lemma and the isomorphisms

$$f_*(f^*\mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{G})$$

of $f_*\mathcal{O}_Y$-modules where the first equality is the projection formula (Cohomology, Lemma 20.54.2). An alternative is to prove the formula affine locally by direct computation.

The third remark is that if f is in addition flat, then the composition

$$f_*\mathcal{O}_Y \xrightarrow{f_*\tau_{Y/X}} f_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

is equal to the trace map Trace_f discussed in Section 49.3. This follows immediately by looking over affine opens.

The fourth remark is that if f is flat and X Noetherian, then we obtain

$$\mathrm{Hom}_Y(K, Lf^*M \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathrm{Hom}_X(Rf_*K, M)$$

for any K in $D_{QCoh}(\mathcal{O}_Y)$ and M in $D_{QCoh}(\mathcal{O}_X)$. This follows from the material in Duality for Schemes, Section 48.12, but can be proven directly in this case as follows. First, if X is affine, then it holds by Dualizing Complexes, Lemmas 47.13.1 and 47.13.9[3] and Derived Categories of Schemes, Lemma 36.3.5. Then we can use the induction principle (Cohomology of Schemes, Lemma 30.4.1) and Mayer-Vietoris (in the form of Cohomology, Lemma 20.33.3) to finish the proof.

49.6. The Noether different

0BVK There are many different differents available in the literature. We list some of them in this and the next sections; for more information we suggest the reader consult [Kun86].

Let $A \to B$ be a ring map. Denote

$$\mu : B \otimes_A B \longrightarrow B, \quad b \otimes b' \longmapsto bb'$$

the multiplication map. Let $I = \mathrm{Ker}(\mu)$. It is clear that I is generated by the elements $b \otimes 1 - 1 \otimes b$ for $b \in B$. Hence the annihilator $J \subset B \otimes_A B$ of I is a B-module in a canonical manner. The Noether different of B over A is the image

[3]There is a simpler proof of this lemma in our case.

of J under the map $\mu : B \otimes_A B \to B$. Equivalently, the Noether different is the image of the map

$$J = \mathrm{Hom}_{B \otimes_A B}(B, B \otimes_A B) \longrightarrow B, \quad \varphi \longmapsto \mu(\varphi(1))$$

We begin with some obligatory lemmas.

0BVL Lemma 49.6.1. Let $A \to B_i$, $i = 1, 2$ be ring maps. Set $B = B_1 \times B_2$.

(1) The annihilator J of $\mathrm{Ker}(B \otimes_A B \to B)$ is $J_1 \times J_2$ where J_i is the annihilator of $\mathrm{Ker}(B_i \otimes_A B_i \to B_i)$.
(2) The Noether different $\mathfrak{D}$ of B over A is $\mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_i$ is the Noether different of B_i over A.

Proof. Omitted. □

0BVM Lemma 49.6.2. Let $A \to B$ be a finite type ring map. Let $A \to A'$ be a flat ring map. Set $B' = B \otimes_A A'$.

(1) The annihilator J' of $\mathrm{Ker}(B' \otimes_{A'} B' \to B')$ is $J \otimes_A A'$ where J is the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$.
(2) The Noether different $\mathfrak{D}'$ of B' over A' is $\mathfrak{D}B'$, where $\mathfrak{D}$ is the Noether different of B over A.

Proof. Choose generators $b_1, \ldots, b_n$ of B as an A-algebra. Then

$$J = \mathrm{Ker}(B \otimes_A B \xrightarrow{b_i \otimes 1 - 1 \otimes b_i} (B \otimes_A B)^{\oplus n})$$

Hence we see that the formation of J commutes with flat base change. The result on the Noether different follows immediately from this. □

0BVN Lemma 49.6.3. Let $A \to B' \to B$ be ring maps with $A \to B'$ of finite type and $B' \to B$ inducing an open immersion of spectra.

(1) The annihilator J of $\mathrm{Ker}(B \otimes_A B \to B)$ is $J' \otimes_{B'} B$ where J' is the annihilator of $\mathrm{Ker}(B' \otimes_A B' \to B')$.
(2) The Noether different $\mathfrak{D}$ of B over A is $\mathfrak{D}'B$, where $\mathfrak{D}'$ is the Noether different of B' over A.

Proof. Write $I = \mathrm{Ker}(B \otimes_A B \to B)$ and $I' = \mathrm{Ker}(B' \otimes_A B' \to B')$. As $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion, it follows that $B = (B \otimes_A B) \otimes_{B' \otimes_A B'} B'$. Thus we see that $I = I'(B \otimes_A B)$. Since I' is finitely generated and $B' \otimes_A B' \to B \otimes_A B$ is flat, we conclude that $J = J'(B \otimes_A B)$, see Algebra, Lemma 10.40.4. Since the $B' \otimes_A B'$-module structure of J' factors through $B' \otimes_A B' \to B'$ we conclude that (1) is true. Part (2) is a consequence of (1). □

0BVP Remark 49.6.4. Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Let J be the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$. There is a canonical B-bilinear pairing

0BVQ (49.6.4.1)
$$\omega_{B/A} \times J \longrightarrow B$$

defined as follows. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. Let J' be the annihilator of $\mathrm{Ker}(B' \otimes_A B' \to B')$. We first define

$$\mathrm{Hom}_A(B', A) \times J' \longrightarrow B', \quad (\lambda, \sum b_i \otimes c_i) \longmapsto \sum \lambda(b_i)c_i$$

This is B'-bilinear exactly because for $\xi \in J'$ and $b \in B'$ we have $(b \otimes 1)\xi = (1 \otimes b)\xi$. By Lemma 49.6.3 and the fact that $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$ we can extend this to a B-bilinear pairing as displayed above.

0BVR Lemma 49.6.5. Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings.

(1) If $A \to A'$ is a flat map of Noetherian rings, then

$$\begin{tikzcd} \omega_{B/A} \times J \arrow[r] \arrow[d] & B \arrow[d] \\ \omega_{B'/A'} \times J' \arrow[r] & B' \end{tikzcd}$$

is commutative where notation as in Lemma 49.6.2 and horizontal arrows are given by (49.6.4.1).

(2) If $B = B_1 \times B_2$, then

$$\begin{tikzcd} \omega_{B/A} \times J \arrow[r] \arrow[d] & B \arrow[d] \\ \omega_{B_i/A} \times J_i \arrow[r] & B_i \end{tikzcd}$$

is commutative for $i = 1, 2$ where notation as in Lemma 49.6.1 and horizontal arrows are given by (49.6.4.1).

Proof. Because of the construction of the pairing in Remark 49.6.4 both (1) and (2) reduce to the case where $A \to B$ is finite. Then (1) follows from the fact that the contraction map $\mathrm{Hom}_A(M, A) \otimes_A M \otimes_A M \to M$, $\lambda \otimes m \otimes m' \mapsto \lambda(m)m'$ commuted with base change. To see (2) use that $J = J_1 \times J_2$ is contained in the summands $B_1 \otimes_A B_1$ and $B_2 \otimes_A B_2$ of $B \otimes_A B$. $\square$

0BVS Lemma 49.6.6. Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The pairing of Remark 49.6.4 induces an isomorphism $J \to \mathrm{Hom}_B(\omega_{B/A}, B)$.

Proof. We first prove this when $A \to B$ is finite and flat. In this case we can localize on A and assume B is finite free as an A-module. Let $b_1, \ldots, b_n$ be a basis of B as an A-module and denote $b_1^\vee, \ldots, b_n^\vee$ the dual basis of $\omega_{B/A}$. Note that $\sum b_i \otimes c_i \in J$ maps to the element of $\mathrm{Hom}_B(\omega_{B/A}, B)$ which sends $b_i^\vee$ to c_i. Suppose $\varphi : \omega_{B/A} \to B$ is B-linear. Then we claim that $\xi = \sum b_i \otimes \varphi(b_i^\vee)$ is an element of J. Namely, the B-linearity of φ exactly implies that $(b \otimes 1)\xi = (1 \otimes b)\xi$ for all $b \in B$. Thus our map has an inverse and it is an isomorphism.

Let $\mathfrak q \subset B$ be a prime lying over $\mathfrak p \subset A$. We will show that the localization

$$J_{\mathfrak q} \longrightarrow \mathrm{Hom}_B(\omega_B/A, B)_{\mathfrak q}$$

is an isomorphism. This suffices by Algebra, Lemma 10.23.1. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak p' \subset A'$ lying over $\mathfrak p$ such that $\kappa(\mathfrak p') = \kappa(\mathfrak p)$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak q'$ of $B \otimes_A A'$ lying over $\mathfrak q$ and $\mathfrak p'$ corresponds to a prime of C. Let J' be the annihilator of $\mathrm{Ker}(B' \otimes_{A'} B' \to B')$. By Lemmas 49.2.5, 49.6.2, and 49.6.5 the map $J' \to \mathrm{Hom}_{B'}(\omega_{B'/A'}, B')$ is gotten by applying the functor $- \otimes_B B'$ to the map $J \to \mathrm{Hom}_B(\omega_{B/A}, B)$. Since $B_{\mathfrak q} \to B'_{\mathfrak q'}$

is faithfully flat it suffices to prove the result for $(A' \to B', \mathfrak{q}')$. By Lemmas 49.2.7, 49.6.1, and 49.6.5 this reduces us to the case proved in the first paragraph of the proof. $\square$

0BVT Lemma 49.6.7. Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The diagram

$$\begin{tikzcd} J \arrow[rr] \arrow[dr, "\mu"'] & & \mathrm{Hom}_B(\omega_{B/A}, B) \arrow[dl, "\varphi \mapsto \varphi(\tau_{B/A})"] \\ & B & \end{tikzcd}$$

commutes where the horizontal arrow is the isomorphism of Lemma 49.6.6. Hence the Noether different of B over A is the image of the map $\mathrm{Hom}_B(\omega_{B/A}, B) \to B$.

Proof. Exactly as in the proof of Lemma 49.6.6 this reduces to the case of a finite free map $A \to B$. In this case $\tau_{B/A} = \mathrm{Trace}_{B/A}$. Choose a basis $b_1, \ldots, b_n$ of B as an A-module. Let $\xi = \sum b_i \otimes c_i \in J$. Then $\mu(\xi) = \sum b_i c_i$. On the other hand, the image of ξ in $\mathrm{Hom}_B(\omega_{B/A}, B)$ sends $\mathrm{Trace}_{B/A}$ to $\sum \mathrm{Trace}_{B/A}(b_i)c_i$. Thus we have to show

$$\sum b_i c_i = \sum \mathrm{Trace}_{B/A}(b_i)c_i$$

when $\xi = \sum b_i \otimes c_i \in J$. Write $b_i b_j = \sum_k a_{ij}^k b_k$ for some $a_{ij}^k \in A$. Then the right hand side is $\sum_{i,j} a_{ij}^j c_i$. On the other hand, $\xi \in J$ implies

$$(b_j \otimes 1)(\sum\nolimits_i b_i \otimes c_i) = (1 \otimes b_j)(\sum\nolimits_i b_i \otimes c_i)$$

which implies that $b_j c_i = \sum_k a_{jk}^i c_k$. Thus the left hand side is $\sum_{i,j} a_{ij}^i c_j$. Since $a_{ij}^k = a_{ji}^k$ the equality holds. $\square$

0BVU Lemma 49.6.8. Let $A \to B$ be a finite type ring map. Let $\mathfrak{D} \subset B$ be the Noether different. Then $V(\mathfrak{D})$ is the set of primes $\mathfrak{q} \subset B$ such that $A \to B$ is not unramified at $\mathfrak{q}$.

Proof. Assume $A \to B$ is unramified at $\mathfrak{q}$. After replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ we may assume $A \to B$ is unramified (Algebra, Definition 10.151.1 and Lemma 49.6.3). In this case $\Omega_{B/A} = 0$. Hence if $I = \mathrm{Ker}(B \otimes_A B \to B)$, then $I/I^2 = 0$ by Algebra, Lemma 10.131.13. Since $A \to B$ is of finite type, we see that I is finitely generated. Hence by Nakayama's lemma (Algebra, Lemma 10.20.1) there exists an element of the form $1 + i$ annihilating I. It follows that $\mathfrak{D} = B$.

Conversely, assume that $\mathfrak{D} \not\subset \mathfrak{q}$. Then after replacing B by a principal localization as above we may assume $\mathfrak{D} = B$. This means there exists an element of the form $1 + i$ in the annihilator of I. Conversely this implies that $I/I^2 = \Omega_{B/A}$ is zero and we conclude. $\square$

49.7. The Kähler different

0BVV Let $A \to B$ be a finite type ring map. The Kähler different is the zeroth fitting ideal of $\Omega_{B/A}$ as a B-module. We globalize the definition as follows.

0BVW Definition 49.7.1. Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. The Kähler different is the 0th fitting ideal of $\Omega_{Y/X}$.

The Kähler different is a quasi-coherent sheaf of ideals on Y.

0BVX **Lemma 49.7.2.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with f locally of finite type. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the Kähler different of f, resp. f'. Then $Y' \to Y$ induces an isomorphism $R' \to R \times_Y Y'$.

Proof. This is true because $\Omega_{Y'/X'}$ is the pullback of $\Omega_{Y/X}$ (Morphisms, Lemma 29.32.10) and then we can apply More on Algebra, Lemma 15.8.4. $\square$

0BVY **Lemma 49.7.3.** Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. Let $R \subset Y$ be the closed subscheme defined by the Kähler different. Then $R \subset Y$ is exactly the set of points where f is not unramified.

Proof. This is a copy of Divisors, Lemma 31.10.2. $\square$

0BVZ **Lemma 49.7.4.** Let A be a ring. Let $n \geq 1$ and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. The Kähler different of B over A is the ideal of B generated by $\det(\partial f_i/\partial x_j)$.

Proof. This is true because $\Omega_{B/A}$ has a presentation

$$\bigoplus\nolimits_{i=1,\ldots,n} Bf_i \xrightarrow{\text{d}} \bigoplus\nolimits_{j=1,\ldots,n} B\text{d}x_j \to \Omega_{B/A} \to 0$$

by Algebra, Lemma 10.131.9. $\square$

49.8. The Dedekind different

0BW0 Let $A \to B$ be a ring map. We say the *Dedekind different is defined* if A is Noetherian, $A \to B$ is finite, any nonzerodivisor on A is a nonzerodivisor on B, and $K \to L$ is étale where $K = Q(A)$ and $L = B \otimes_A K$. Then $K \subset L$ is finite étale and

$$\mathcal{L}_{B/A} = \{x \in L \mid \text{Trace}_{L/K}(bx) \in A \text{ for all } b \in B\}$$

is the *Dedekind complementary module*. In this situation the *Dedekind different* is

$$\mathfrak{D}_{B/A} = \{x \in L \mid x\mathcal{L}_{B/A} \subset B\}$$

viewed as a B-submodule of L. By Lemma 49.8.1 the Dedekind different is an ideal of B either if A is normal or if B is flat over A.

0BW1 **Lemma 49.8.1.** Assume the Dedekind different of $A \to B$ is defined. Consider the statements

(1) $A \to B$ is flat,
(2) A is a normal ring,
(3) $\text{Trace}_{L/K}(B) \subset A$,
(4) $1 \in \mathcal{L}_{B/A}$, and
(5) the Dedekind different $\mathfrak{D}_{B/A}$ is an ideal of B.

Then we have (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (4), and (4) $\Rightarrow$ (5).

Proof. The equivalence of (3) and (4) and the implication (4) $\Rightarrow$ (5) are immediate.

If $A \to B$ is flat, then we see that $\text{Trace}_{B/A} : B \to A$ is defined and that $\text{Trace}_{L/K}$ is the base change. Hence (3) holds.

If A is normal, then A is a finite product of normal domains, hence we reduce to the case of a normal domain. Then K is the fraction field of A and $L = \prod L_i$ is a finite product of finite separable field extensions of K. Then $\text{Trace}_{L/K}(b) = \sum \text{Trace}_{L_i/K}(b_i)$ where $b_i \in L_i$ is the image of b. Since b is integral over A as B is finite over A, these traces are in A. This is true because the minimal polynomial of b_i over K has coefficients in A (Algebra, Lemma 10.38.6) and because $\text{Trace}_{L_i/K}(b_i)$ is an integer multiple of one of these coefficients (Fields, Lemma 9.20.3). $\square$

0BW2 **Lemma 49.8.2.** *If the Dedekind different of $A \to B$ is defined, then there is a canonical isomorphism $\mathcal{L}_{B/A} \to \omega_{B/A}$.*

Proof. Recall that $\omega_{B/A} = \text{Hom}_A(B, A)$ as $A \to B$ is finite. We send $x \in \mathcal{L}_{B/A}$ to the map $b \mapsto \text{Trace}_{L/K}(bx)$. Conversely, given an A-linear map $\varphi : B \to A$ we obtain a K-linear map $\varphi_K : L \to K$. Since $K \to L$ is finite étale, we see that the trace pairing is nondegenerate (Lemma 49.3.1) and hence there exists a $x \in L$ such that $\varphi_K(y) = \text{Trace}_{L/K}(xy)$ for all $y \in L$. Then $x \in \mathcal{L}_{B/A}$ maps to φ in $\omega_{B/A}$. $\square$

0BW3 **Lemma 49.8.3.** *If the Dedekind different of $A \to B$ is defined and $A \to B$ is flat, then*

(1) *the canonical isomorphism $\mathcal{L}_{B/A} \to \omega_{B/A}$ sends $1 \in \mathcal{L}_{B/A}$ to the trace element $\tau_{B/A} \in \omega_{B/A}$, and*
(2) *the Dedekind different is $\mathfrak{D}_{B/A} = \{b \in B \mid b\omega_{B/A} \subset B\tau_{B/A}\}$.*

Proof. The first assertion follows from the proof of Lemma 49.8.1 and Lemma 49.4.3. The second assertion is immediate from the first and the definitions. $\square$

49.9. The different

0BTC The motivation for the following definition is that it recovers the Dedekind different in the finite flat case as we will see below.

0BW4 **Definition 49.9.1.** Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be the relative dualizing module and let $\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$ be the trace element (Remarks 49.2.11 and 49.4.7). The annihilator of

$$\text{Coker}(\mathcal{O}_Y \xrightarrow{\tau_{Y/X}} \omega_{Y/X})$$

is the *different* of Y/X. It is a coherent ideal $\mathfrak{D}_f \subset \mathcal{O}_Y$.

We will generalize this in Remark 49.14.2 below. Observe that $\mathfrak{D}_f$ is locally generated by one element if $\omega_{Y/X}$ is an invertible $\mathcal{O}_Y$-module. We first state the agreement with the Dedekind different.

0BW5 **Lemma 49.9.2.** *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ be affine open subschemes with $f(V) \subset U$. If the Dedekind different of $A \to B$ is defined, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on V.

Proof. This is clear from Lemmas 49.8.1 and 49.8.3. ☐

0BW6 Lemma 49.9.3. Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ be affine open subschemes with $f(V) \subset U$. If $\omega_{Y/X}|_V$ is invertible, i.e., if $\omega_{B/A}$ is an invertible B-module, then

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}}$$

as coherent ideal sheaves on V where $\mathfrak{D} \subset B$ is the Noether different of B over A.

Proof. Consider the map

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_{Y/X}, \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y, \quad \varphi \longmapsto \varphi(\tau_{Y/X})$$

The image of this map corresponds to the Noether different on affine opens, see Lemma 49.6.7. Hence the result follows from the elementary fact that given an invertible module ω and a global section τ the image of $\tau : \mathcal{H}om(\omega, \mathcal{O}) = \omega^{\otimes -1} \to \mathcal{O}$ is the same as the annihilator of $\mathrm{Coker}(\tau : \mathcal{O} \to \omega)$. ☐

0BW7 Lemma 49.9.4. Consider a cartesian diagram of Noetherian schemes

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with f flat and quasi-finite. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the different $\mathfrak{D}_f$, resp. $\mathfrak{D}_{f'}$. Then $Y' \to Y$ induces a bijective closed immersion $R' \to R \times_Y Y'$. If g is flat or if $\omega_{Y/X}$ is invertible, then $R' = R \times_Y Y'$.

Proof. There is an immediate reduction to the case where X, X', Y, Y' are affine. In other words, we have a cocartesian diagram of Noetherian rings

$$\begin{tikzcd} B' & B \arrow[l] \\ A' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

with $A \to B$ flat and quasi-finite. The base change map $\omega_{B/A} \otimes_B B' \to \omega_{B'/A'}$ is an isomorphism (Lemma 49.2.10) and maps the trace element $\tau_{B/A}$ to the trace element $\tau_{B'/A'}$ (Lemma 49.4.4). Hence the finite B-module $Q = \mathrm{Coker}(\tau_{B/A} : B \to \omega_{B/A})$ satisfies $Q \otimes_B B' = \mathrm{Coker}(\tau_{B'/A'} : B' \to \omega_{B'/A'})$. Thus $\mathfrak{D}_{B/A} B' \subset \mathfrak{D}_{B'/A'}$ which means we obtain the closed immersion $R' \to R \times_Y Y'$. Since $R = \mathrm{Supp}(Q)$ and $R' = \mathrm{Supp}(Q \otimes_B B')$ (Algebra, Lemma 10.40.5) we see that $R' \to R \times_Y Y'$ is bijective by Algebra, Lemma 10.40.6. The equality $\mathfrak{D}_{B/A} B' = \mathfrak{D}_{B'/A'}$ holds if $B \to B'$ is flat, e.g., if $A \to A'$ is flat, see Algebra, Lemma 10.40.4. Finally, if $\omega_{B/A}$ is invertible, then we can localize and assume $\omega_{B/A} = B\lambda$. Writing $\tau_{B/A} = b\lambda$ we see that $Q = B/bB$ and $\mathfrak{D}_{B/A} = bB$. The same reasoning over B' gives $\mathfrak{D}_{B'/A'} = bB'$ and the lemma is proved. ☐

0BW8 Lemma 49.9.5. Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. Then $\mathrm{Norm}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ maps $f_*\mathfrak{D}_f$ into the ideal sheaf of the discriminant D_f.

Proof. The norm map is constructed in Divisors, Lemma 31.17.6 and the discriminant of f in Section 49.3. The question is affine local, hence we may assume $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and f given by a finite locally free ring map $A \to B$. Localizing further we may assume B is finite free as an A-module. Choose a basis $b_1, \ldots, b_n \in B$ for B as an A-module. Denote $b_1^\vee, \ldots, b_n^\vee$ the dual basis of $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as an A-module. Since the norm of b is the determinant of $b : B \to B$ as an A-linear map, we see that $\mathrm{Norm}_{B/A}(b) = \det(b_i^\vee(bb_j))$. The discriminant is the principal closed subscheme of $\mathrm{Spec}(A)$ defined by $\det(\mathrm{Trace}_{B/A}(b_ib_j))$. If $b \in \mathfrak{D}_{B/A}$ then there exist $c_i \in B$ such that $b \cdot b_i^\vee = c_i \cdot \mathrm{Trace}_{B/A}$ where we use a dot to indicate the B-module structure on $\omega_{B/A}$. Write $c_i = \sum a_{il}b_l$. We have

$$
\begin{aligned}
\mathrm{Norm}_{B/A}(b) &= \det(b_i^\vee(bb_j)) \\
&= \det((b \cdot b_i^\vee)(b_j)) \\
&= \det((c_i \cdot \mathrm{Trace}_{B/A})(b_j)) \\
&= \det(\mathrm{Trace}_{B/A}(c_ib_j)) \\
&= \det(a_{il})\det(\mathrm{Trace}_{B/A}(b_lb_j))
\end{aligned}
$$

which proves the lemma. □

0BW9 Lemma 49.9.6. Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. The closed subscheme $R \subset Y$ defined by the different $\mathfrak{D}_f$ is exactly the set of points where f is not étale (equivalently not unramified).

Proof. Since f is of finite presentation and flat, we see that it is étale at a point if and only if it is unramified at that point. Moreover, the formation of the locus of ramified points commutes with base change. See Morphisms, Section 29.36 and especially Morphisms, Lemma 29.36.17. By Lemma 49.9.4 the formation of R commutes set theoretically with base change. Hence it suffices to prove the lemma when X is the spectrum of a field. On the other hand, the construction of $(\omega_{Y/X}, \tau_{Y/X})$ is local on Y. Since Y is a finite discrete space (being quasi-finite over a field), we may assume Y has a unique point.

Say $X = \mathrm{Spec}(k)$ and $Y = \mathrm{Spec}(B)$ where k is a field and B is a finite local k-algebra. If $Y \to X$ is étale, then B is a finite separable extension of k, and the trace element $\mathrm{Trace}_{B/k}$ is a basis element of $\omega_{B/k}$ by Fields, Lemma 9.20.7. Thus $\mathfrak{D}_{B/k} = B$ in this case. Conversely, if $\mathfrak{D}_{B/k} = B$, then we see from Lemma 49.9.5 and the fact that the norm of 1 equals 1 that the discriminant is empty. Hence $Y \to X$ is étale by Lemma 49.3.1. □

0BWA Lemma 49.9.7. Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $R \subset Y$ be the closed subscheme defined by $\mathfrak{D}_f$.

(1) If $\omega_{Y/X}$ is invertible, then R is a locally principal closed subscheme of Y.
(2) If $\omega_{Y/X}$ is invertible and f is finite, then the norm of R is the discriminant D_f of f.
(3) If $\omega_{Y/X}$ is invertible and f is étale at the associated points of Y, then R is an effective Cartier divisor and there is an isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$.

Proof. Proof of (1). We may work locally on Y, hence we may assume $\omega_{Y/X}$ is free of rank 1. Say $\omega_{Y/X} = \mathcal{O}_Y\lambda$. Then we can write $\tau_{Y/X} = h\lambda$ and then we see that R is defined by h, i.e., R is locally principal.

Proof of (2). We may assume $Y \to X$ is given by a finite free ring map $A \to B$ and that $\omega_{B/A}$ is free of rank 1 as B-module. Choose a B-basis element λ for $\omega_{B/A}$ and write $\text{Trace}_{B/A} = b \cdot \lambda$ for some $b \in B$. Then $\mathfrak{D}_{B/A} = (b)$ and D_f is cut out by $\det(\text{Trace}_{B/A}(b_i b_j))$ where $b_1, \ldots, b_n$ is a basis of B as an A-module. Let $b_1^\vee, \ldots, b_n^\vee$ be the dual basis. Writing $b_i^\vee = c_i \cdot \lambda$ we see that $c_1, \ldots, c_n$ is a basis of B as well. Hence with $c_i = \sum a_{il} b_l$ we see that $\det(a_{il})$ is a unit in A. Clearly, $b \cdot b_i^\vee = c_i \cdot \text{Trace}_{B/A}$ hence we conclude from the computation in the proof of Lemma 49.9.5 that $\text{Norm}_{B/A}(b)$ is a unit times $\det(\text{Trace}_{B/A}(b_i b_j))$.

Proof of (3). In the notation above we see from Lemma 49.9.6 and the assumption that h does not vanish in the associated points of Y, which implies that h is a nonzerodivisor. The canonical isomorphism sends 1 to $\tau_{Y/X}$, see Divisors, Lemma 31.14.10. □

49.10. Quasi-finite syntomic morphisms

0DWJ This section discusses the fact that a quasi-finite syntomic morphism has an invertible relative dualizing module.

0BWE **Lemma 49.10.1.** Let $f : Y \to X$ be a morphism of schemes. The following are equivalent

(1) f is locally quasi-finite and syntomic,
(2) f is locally quasi-finite, flat, and a local complete intersection morphism,
(3) f is locally quasi-finite, flat, locally of finite presentation, and the fibres of f are local complete intersections,
(4) f is locally quasi-finite and for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ an integer n and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$,
(5) for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$,
(6) f is locally quasi-finite, flat, locally of finite presentation, and $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and
(7) f is flat, locally of finite presentation, $NL_{X/Y}$ is perfect of rank 0 with tor-amplitude in $[-1, 0]$,

Proof. The equivalence of (1) and (2) is More on Morphisms, Lemma 37.62.8. The equivalence of (1) and (3) is Morphisms, Lemma 29.30.11.

If $A \to B$ is as in (4), then $B = A[x, x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n]$ is a relative global complete intersection by see Algebra, Definition 10.136.5. Thus (4) implies (5). It is clear that (5) implies (4).

Condition (5) implies (1): by Algebra, Lemma 10.136.13 a relative global complete intersection is syntomic and the definition of a relative global complete intersection guarantees that a relative global complete intersection on n variables with n equations is quasi-finite, see Algebra, Definition 10.136.5 and Lemma 10.122.2.

Either Algebra, Lemma 10.136.15 or Morphisms, Lemma 29.30.10 shows that (1) implies (5).

More on Morphisms, Lemma 37.62.17 shows that (6) is equivalent to (1). If the equivalent conditions (1) – (6) hold, then we see that affine locally $Y \to X$ is given by a relative global complete intersection $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ with the same number of variables as the number of equations. Using this presentation we see that

$$NL_{B/A} = \Big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B\mathrm{d}x_i\Big)$$

By Algebra, Lemma 10.136.12 the module $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the congruence classes of the elements $f_1, \ldots, f_n$. Thus $NL_{B/A}$ has rank 0 and so does $NL_{Y/X}$. In this way we see that (1) – (6) imply (7).

Finally, assume (7). By More on Morphisms, Lemma 37.62.17 we see that f is syntomic. Thus on suitable affine opens f is given by a relative global complete intersection $A \to B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Morphisms, Lemma 29.30.10. Exactly as above we see that $NL_{B/A}$ is a perfect complex of rank $n - m$. Thus $n = m$ and we see that (5) holds. This finishes the proof. □

0DWK Lemma 49.10.2. Invertibility of the relative dualizing module.

(1) If $A \to B$ is a quasi-finite flat homomorphism of Noetherian rings, then $\omega_{B/A}$ is an invertible B-module if and only if $\omega_{B \otimes_A \kappa(\mathfrak p)/\kappa(\mathfrak p)}$ is an invertible $B \otimes_A \kappa(\mathfrak p)$-module for all primes $\mathfrak p \subset A$.
(2) If $Y \to X$ is a quasi-finite flat morphism of Noetherian schemes, then $\omega_{Y/X}$ is invertible if and only if $\omega_{Y_x/x}$ is invertible for all $x \in X$.

Proof. Proof of (1). As $A \to B$ is flat, the module $\omega_{B/A}$ is A-flat, see Lemma 49.2.9. Thus $\omega_{B/A}$ is an invertible B-module if and only if $\omega_{B/A} \otimes_A \kappa(\mathfrak p)$ is an invertible $B \otimes_A \kappa(\mathfrak p)$-module for every prime $\mathfrak p \subset A$, see More on Morphisms, Lemma 37.16.7. Still using that $A \to B$ is flat, we have that formation of $\omega_{B/A}$ commutes with base change, see Lemma 49.2.10. Thus we see that invertibility of the relative dualizing module, in the presence of flatness, is equivalent to invertibility of the relative dualizing module for the maps $\kappa(\mathfrak p) \to B \otimes_A \kappa(\mathfrak p)$.

Part (2) follows from (1) and the fact that affine locally the dualizing modules are given by their algebraic counterparts, see Remark 49.2.11. □

0DWL Lemma 49.10.3. Let k be a field. Let $B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be a global complete intersection over k of dimension 0. Then $\omega_{B/k}$ is invertible.

Proof. By Noether normalization, see Algebra, Lemma 10.115.4 we see that there exists a finite injection $k \to B$, i.e., $\dim_k(B) < \infty$. Hence $\omega_{B/k} = \mathrm{Hom}_k(B, k)$ as a B-module. By Dualizing Complexes, Lemma 47.15.8 we see that $R\,\mathrm{Hom}(B, k)$ is a dualizing complex for B and by Dualizing Complexes, Lemma 47.13.3 we see that $R\,\mathrm{Hom}(B, k)$ is equal to $\omega_{B/k}$ placed in degree 0. Thus it suffices to show that B is Gorenstein (Dualizing Complexes, Lemma 47.21.4). This is true by Dualizing Complexes, Lemma 47.21.7. □

0BWF Lemma 49.10.4. Let $f : Y \to X$ be a morphism of locally Noetherian schemes. If f satisfies the equivalent conditions of Lemma 49.10.1 then $\omega_{Y/X}$ is an invertible $\mathcal{O}_Y$-module.

Proof. We may assume $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ and we have to show $\omega_{B/A}$ is invertible. This follows in combining Lemmas 49.10.2 and 49.10.3. □

0FK8 Example 49.10.5. Let $n \geq 1$ and $d \geq 1$ be integers. Let T be the set of multi-indices $E = (e_1, \ldots, e_n)$ with $e_i \geq 0$ and $\sum e_i \leq d$. Consider the ring

$$A = \mathbf{Z}[a_{i,E}; 1 \leq i \leq n, E \in T]$$

In $A[x_1, \ldots, x_n]$ consider the elements $f_i = \sum_{E \in T} a_{i,E} x^E$ where $x^E = x_1^{e_1} \ldots x_n^{e_n}$ as is customary. Consider the A-algebra

$$B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Denote $X_{n,d} = \mathrm{Spec}(A)$ and let $Y_{n,d} \subset \mathrm{Spec}(B)$ be the maximal open subscheme such that the restriction of the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A) = X_{n,d}$ is quasi-finite, see Algebra, Lemma 10.123.13.

0FK9 Lemma 49.10.6. With notation as in Example 49.10.5 the schemes $X_{n,d}$ and $Y_{n,d}$ are regular and irreducible, the morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite and syntomic, and there is a dense open subscheme $V \subset Y_{n,d}$ such that $Y_{n,d} \to X_{n,d}$ restricts to an étale morphism $V \to X_{n,d}$.

Proof. The scheme $X_{n,d}$ is the spectrum of the polynomial ring A. Hence $X_{n,d}$ is regular and irreducible. Since we can write

$$f_i = a_{i,(0,\ldots,0)} + \sum\nolimits_{E \in T, E \neq (0,\ldots,0)} a_{i,E} x^E$$

we see that the ring B is isomorphic to the polynomial ring on $x_1, \ldots, x_n$ and the elements $a_{i,E}$ with $E \neq (0, \ldots, 0)$. Hence $\mathrm{Spec}(B)$ is an irreducible and regular scheme and so is the open $Y_{n,d}$. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite and syntomic by Lemma 49.10.1. To find V it suffices to find a single point where $Y_{n,d} \to X_{n,d}$ is étale (the locus of points where a morphism is étale is open by definition). Thus it suffices to find a point of $X_{n,d}$ where the fibre of $Y_{n,d} \to X_{n,d}$ is nonempty and étale, see Morphisms, Lemma 29.36.15. We choose the point corresponding to the ring map $\chi : A \to \mathbf{Q}$ sending f_i to $1 + x_i^d$. Then

$$B \otimes_{A,\chi} \mathbf{Q} = \mathbf{Q}[x_1, \ldots, x_n]/(x_1^d - 1, \ldots, x_n^d - 1)$$

which is a nonzero étale algebra over $\mathbf{Q}$. $\square$

0FKA Lemma 49.10.7. Let $f : Y \to X$ be a morphism of schemes. If f satisfies the equivalent conditions of Lemma 49.10.1 then for every $y \in Y$ there exist n, d and a commutative diagram

$$\begin{tikzcd}
Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & Y_{n,d} \arrow[d] \\
X & U \arrow[l] \arrow[r] & X_{n,d}
\end{tikzcd}$$

where $U \subset X$ and $V \subset Y$ are open, where $Y_{n,d} \to X_{n,d}$ is as in Example 49.10.5, and where the square on the right hand side is cartesian.

Proof. By Lemma 49.10.1 we can choose U and V affine so that $U = \mathrm{Spec}(R)$ and $V = \mathrm{Spec}(S)$ with $S = R[y_1, \ldots, y_n]/(g_1, \ldots, g_n)$. With notation as in Example 49.10.5 if we pick d large enough, then we can write each g_i as $g_i = \sum_{E \in T} g_{i,E} y^E$ with $g_{i,E} \in R$. Then the map $A \to R$ sending $a_{i,E}$ to $g_{i,E}$ and the map $B \to S$

sending $x_i \to y_i$ give a cocartesian diagram of rings

$$\begin{tikzcd} S & B \arrow[l] \\ R \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

which proves the lemma. □

49.11. Finite syntomic morphisms

0FKX This section is the analogue of Section 49.10 for finite syntomic morphisms.

0FKY Lemma 49.11.1. Let $f : Y \to X$ be a morphism of schemes. The following are equivalent

(1) f is finite and syntomic,
(2) f is finite, flat, and a local complete intersection morphism,
(3) f is finite, flat, locally of finite presentation, and the fibres of f are local complete intersections,
(4) f is finite and for every $x \in X$ there is an affine open $x \in U = \mathrm{Spec}(A) \subset X$ an integer n and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $f^{-1}(U)$ is isomorphic to the spectrum of $A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$,
(5) f is finite, flat, locally of finite presentation, and $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and
(6) f is finite, flat, locally of finite presentation, and $NL_{X/Y}$ is perfect of rank 0 with tor-amplitude in $[-1, 0]$,

Proof. The equivalence of (1), (2), (3), (5), and (6) and the implication (4) ⇒ (1) follow immediately from Lemma 49.10.1. Assume the equivalent conditions (1), (2), (3), (5), (6) hold. Choose a point $x \in X$ and an affine open $U = \mathrm{Spec}(A)$ of x in X and say x corresponds to the prime ideal $\mathfrak{p} \subset A$. Write $f^{-1}(U) = \mathrm{Spec}(B)$. Write $B = A[x_1, \ldots, x_n]/I$. Since $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$ by (6) we see that I/I^2 is a finite locally free B-module of rank n. Since $B_\mathfrak{p}$ is semi-local we see that $(I/I^2)_\mathfrak{p}$ is free of rank n, see Algebra, Lemma 10.78.7. Thus after replacing A by a principal localization at an element not in $\mathfrak{p}$ we may assume I/I^2 is a free B-module of rank n. Thus by Algebra, Lemma 10.136.6 we can find a presentation of B over A with the same number of variables as equations. In other words, we may assume $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. This proves (4). □

0FKZ Example 49.11.2. Let $d \geq 1$ be an integer. Consider variables a_{ij}^l for $1 \leq i, j, l \leq d$ and denote

$$A_d = \mathbf{Z}[a_{ij}^k]/J$$

where J is the ideal generated by the elements

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m - \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k - a_{ji}^k & \forall i, j, k \\ a_{i1}^j - \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} indices the Kronecker delta function. We define an A_d-algebra B_d as follows: as an A_d-module we set

$$B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$$

The algebra structure is given by $A_d \to B_d$ mapping 1 to e_1. The multiplication on B_d is the A_d-bilinar map

$$m : B_d \times B_d \longrightarrow B_d, \quad m(e_i, e_j) = \sum a_{ij}^k e_k$$

It is straightforward to check that the relations given above exactly force this to be an A_d-algebra structure. The morphism

$$\pi_d : Y_d = \mathrm{Spec}(B_d) \longrightarrow \mathrm{Spec}(A_d) = X_d$$

is the "universal" finite free morphism of rank d.

0FL0 Lemma 49.11.3. With notation as in Example 49.11.2 there is an open subscheme $U_d \subset X_d$ with the following property: a morphism of schemes $X \to X_d$ factors through U_d if and only if $Y_d \times_{X_d} X \to X$ is syntomic.

Proof. Recall that being syntomic is the same thing as being flat and a local complete intersection morphism, see More on Morphisms, Lemma 37.62.8. The set $W_d \subset Y_d$ of points where π_d is Koszul is open in Y_d and its formation commutes with arbitrary base change, see More on Morphisms, Lemma 37.62.21. Since π_d is finite and hence closed, we see that $Z = \pi_d(Y_d \setminus W_d)$ is closed. Since clearly $U_d = X_d \setminus Z$ and since its formation commutes with base change we find that the lemma is true. □

0FL1 Lemma 49.11.4. With notation as in Example 49.11.2 and U_d as in Lemma 49.11.3 then U_d is smooth over $\mathrm{Spec}(\mathbf{Z})$.

Proof. Let us use More on Morphisms, Lemma 37.12.1 to show that $U_d \to \mathrm{Spec}(\mathbf{Z})$ is smooth. Namely, suppose that $\mathrm{Spec}(A) \to U_d$ is a morphism and $A' \to A$ is a small extension. Then $B = A \otimes_{A_d} B_d$ is a finite free A-algebra which is syntomic over A (by construction of U_d). By Smoothing Ring Maps, Proposition 16.3.2 there exists a syntomic ring map $A' \to B'$ such that $B \cong B' \otimes_{A'} A$. Set $e'_1 = 1 \in B'$. For $1 < i \leq d$ choose lifts $e'_i \in B'$ of the elements $1 \otimes e_i \in A \otimes_{A_d} B_d = B$. Then $e'_1, \ldots, e'_d$ is a basis for B' over A' (for example see Algebra, Lemma 10.101.1). Thus we can write $e'_i e'_j = \sum \alpha_{ij}^l e'_l$ for unique elements $\alpha_{ij}^l \in A'$ which satisfy the relations $\sum_l \alpha_{ij}^l \alpha_{lk}^m = \sum_l \alpha_{il}^m \alpha_{jk}^l$ and $\alpha_{ij}^k = \alpha_{ji}^k$ and $\alpha_{i1}^j - \delta_{ij}$ in A'. This determines a morphism $\mathrm{Spec}(A') \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A'$. This morphism agrees with the given morphism $\mathrm{Spec}(A) \to U_d$. Since $\mathrm{Spec}(A')$ and $\mathrm{Spec}(A)$ have the same underlying topological space, we see that we obtain the desired lift $\mathrm{Spec}(A') \to U_d$ and we conclude that U_d is smooth over $\mathbf{Z}$. □

0FL2 Lemma 49.11.5. With notation as in Example 49.11.2 consider the open subscheme $U'_d \subset X_d$ over which π_d is étale. Then U'_d is a dense subset of the open U_d of Lemma 49.11.3.

Proof. By exactly the same reasoning as in the proof of Lemma 49.11.3, using Morphisms, Lemma 29.36.17, there is a maximal open $U'_d \subset X_d$ over which π_d is étale. Moreover, since an étale morphism is syntomic, we see that $U'_d \subset U_d$. To finish the proof we have to show that $U'_d \subset U_d$ is dense. Let $u : \mathrm{Spec}(k) \to U_d$ be a morphism where k is a field. Let $B = k \otimes_{A_d} B_d$ as in the proof of Lemma 49.11.4. We will show there is a local domain A' with residue field k and a finite syntomic A' algebra B' with $B = k \otimes_{A'} B'$ whose generic fibre is étale. Exactly as in the previous paragraph this will determine a morphism $\mathrm{Spec}(A') \to U_d$ which will map the generic point into U'_d and the closed point to u, thereby finishing the proof.

By Lemma 49.11.1 part (4) we can choose a presentation $B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Let d' be the maximum total degree of the polynomials $f_1, \ldots, f_n$. Let $Y_{n,d'} \to X_{n,d'}$ be as in Example 49.10.5. By construction there is a morphism $u' : \mathrm{Spec}(k) \to X_{n,d'}$ such that

$$\mathrm{Spec}(B) \cong Y_{n,d'} \times_{X_{n,d'},u'} \mathrm{Spec}(k)$$

Denote $A = \mathcal{O}^h_{X_{n,d'},u'}$ the henselization of the local ring of $X_{n,d'}$ at the image of u'. Then we can write

$$Y_{n,d'} \times_{X_{n,d'}} \mathrm{Spec}(A) = Z \amalg W$$

with $Z \to \mathrm{Spec}(A)$ finite and $W \to \mathrm{Spec}(A)$ having empty closed fibre, see Algebra, Lemma 10.153.3 part (13) or the discussion in More on Morphisms, Section 37.41. By Lemma 49.10.6 the local ring A is regular (here we also use More on Algebra, Lemma 15.45.10) and the morphism $Z \to \mathrm{Spec}(A)$ is étale over the generic point of $\mathrm{Spec}(A)$ (because it is mapped to the generic point of $X_{d,n'}$). By construction $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong \mathrm{Spec}(B)$. This proves what we want except that the map from residue field of A to k may not be an isomorphism. By Algebra, Lemma 10.159.1 there exists a flat local ring map $A \to A'$ such that the residue field of A' is k. If A' isn't a domain, then we choose a minimal prime $\mathfrak{p} \subset A'$ (which lies over the unique minimal prime of A by flatness) and we replace A' by $A'/\mathfrak{p}$. Set B' equal to the unique A'-algebra such that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') = \mathrm{Spec}(B')$. This finishes the proof. □

0FL3 Remark 49.11.6. Let $\pi_d : Y_d \to X_d$ be as in Example 49.11.2. Let $U_d \subset X_d$ be the maximal open over which $V_d = \pi_d^{-1}(U_d)$ is finite syntomic as in Lemma 49.11.3. Then it is also true that V_d is smooth over $\mathbf{Z}$. (Of course the morphism $V_d \to U_d$ is not smooth when $d \geq 2$.) Arguing as in the proof of Lemma 49.11.4 this corresponds to the following deformation problem: given a small extension $C' \to C$ and a finite syntomic C-algebra B with a section $B \to C$, find a finite syntomic C'-algebra B' and a section $B' \to C'$ whose tensor product with C recovers $B \to C$. By Lemma 49.11.1 we may write $B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ as a relative global complete intersection. After a change of coordinates we may assume $x_1, \ldots, x_n$ are in the kernel of $B \to C$. Then the polynomials f_i have vanishing constant terms. Choose any lifts $f'_i \in C'[x_1, \ldots, x_n]$ of f_i with vanishing constant terms. Then $B' = C'[x_1, \ldots, x_n]/(f'_1, \ldots, f'_n)$ with section $B' \to C'$ sending x_i to zero works.

0FL4 Lemma 49.11.7. Let $f : Y \to X$ be a morphism of schemes. If f satisfies the equivalent conditions of Lemma 49.11.1 then for every $x \in X$ there exist a d and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & V_d \arrow[r] \arrow[d] & Y_d \arrow[d, "\pi_d"] \\ X & U \arrow[l] \arrow[r] & U_d \arrow[r] & X_d \end{tikzcd}$$

with the following properties

(1) $U \subset X$ is open, $x \in U$, and $V = f^{-1}(U)$,
(2) $\pi_d : Y_d \to X_d$ is as in Example 49.11.2,
(3) $U_d \subset X_d$ is as in Lemma 49.11.3 and $V_d = \pi_d^{-1}(U_d) \subset Y_d$,
(4) where the middle square is cartesian.

Proof. Choose an affine open neighbourhood $U = \mathrm{Spec}(A) \subset X$ of x. Write $V = f^{-1}(U) = \mathrm{Spec}(B)$. Then B is a finite locally free A-module and the inclusion $A \subset B$ is a locally direct summand. Thus after shrinking U we can choose a basis $1 = e_1, e_2, \ldots, e_d$ of B as an A-module. Write $e_i e_j = \sum \alpha_{ij}^l e_l$ for unique elements $\alpha_{ij}^l \in A$ which satisfy the relations $\sum_l \alpha_{ij}^l \alpha_{lk}^m = \sum_l \alpha_{il}^m \alpha_{jk}^l$ and $\alpha_{ij}^k = \alpha_{ji}^k$ and $\alpha_{i1}^j - \delta_{ij}$ in A. This determines a morphism $\mathrm{Spec}(A) \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A$. By construction $V \cong \mathrm{Spec}(A) \times_{X_d} Y_d$. By the definition of U_d we see that $\mathrm{Spec}(A) \to X_d$ factors through U_d. This finishes the proof. □

49.12. A formula for the different

0BWB In this section we discuss the material in [MR70, Appendix A] due to Tate. In our language, this will show that the different is equal to the Kähler different in the case of a flat, quasi-finite, local complete intersection morphism. First we compute the Noether different in a special case.

0BWC Lemma 49.12.1. Let $A \to P$ be a ring map. Let $f_1, \ldots, f_n \in P$ be a Koszul regular sequence. Assume $B = P/(f_1, \ldots, f_n)$ is flat over A. Let $g_1, \ldots, g_n \in P \otimes_A B$ be a Koszul regular sequence generating the kernel of the multiplication map $P \otimes_A B \to B$. Write $f_i \otimes 1 = \sum g_{ij} g_j$. Then the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$ is a principal ideal generated by the image of $\det(g_{ij})$. [MR70, Appendix]

Proof. The Koszul complex $K_\bullet = K(P, f_1, \ldots, f_n)$ is a resolution of B by finite free P-modules. The Koszul complex $M_\bullet = K(P \otimes_A B, g_1, \ldots, g_n)$ is a resolution of B by finite free $P \otimes_A B$-modules. There is a map of complexes

$$K_\bullet \longrightarrow M_\bullet$$

which in degree 1 is given by the matrix (g_{ij}) and in degree n by $\det(g_{ij})$. See More on Algebra, Lemma 15.28.3. As B is a flat A-module, we can view $M_\bullet$ as a complex of flat P-modules (via $P \to P \otimes_A B$, $p \mapsto p \otimes 1$). Thus we may use both complexes to compute $\mathrm{Tor}_*^P(B, B)$ and it follows that the displayed map defines a quasi-isomorphism after tensoring with B. It is clear that $H_n(K_\bullet \otimes_P B) = B$. On the other hand, $H_n(M_\bullet \otimes_P B)$ is the kernel of

$$B \otimes_A B \xrightarrow{g_1, \ldots, g_n} (B \otimes_A B)^{\oplus n}$$

Since $g_1, \ldots, g_n$ generate the kernel of $B \otimes_A B \to B$ this proves the lemma. □

0BWD Lemma 49.12.2. Let A be a ring. Let $n \geq 1$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$. Assume that B is quasi-finite over A. Then

(1) B is flat over A and $A \to B$ is a relative local complete intersection,
(2) the annihilator J of $I = \mathrm{Ker}(B \otimes_A B \to B)$ is free of rank 1 over B,
(3) the Noether different of B over A is generated by $\det(\partial f_i/\partial x_j)$ in B.

Proof. Note that $B = A[x, x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n)$ is a relative global complete intersection over A, see Algebra, Definition 10.136.5. By Algebra, Lemma 10.136.13 we see that B is flat over A.

Write $P' = A[x, x_1, \ldots, x_n]$ and $P = P'/(xh - 1) = A[x_1, \ldots, x_n, 1/g]$. Then we have $P' \to P \to B$. By More on Algebra, Lemma 15.33.4 we see that $xh - 1, f_1, \ldots, f_n$ is a Koszul regular sequence in P'. Since $xh - 1$ is a Koszul regular sequence of length one in P' (by the same lemma for example) we conclude that $f_1, \ldots, f_n$ is a Koszul regular sequence in P by More on Algebra, Lemma 15.30.14.

Let $g_i \in P \otimes_A B$ be the image of $x_i \otimes 1 - 1 \otimes x_i$. Let us use the short hand $y_i = x_i \otimes 1$ and $z_i = 1 \otimes x_i$ in $A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n]$ so that g_i is the image of $y_i - z_i$. For a polynomial $f \in A[x_1, \ldots, x_n]$ we write $f(y) = f \otimes 1$ and $f(z) = 1 \otimes f$ in the above tensor product. Then we have

$$P \otimes_A B/(g_1, \ldots, g_n) = \frac{A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]}{(f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n)}$$

which is clearly isomorphic to B. Hence by the same arguments as above we find that $f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n$ is a Koszul regular sequence in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]$. The sequence $f_1(z), \ldots, f_n(z)$ is a Koszul regular in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]$ by flatness of the map

$$P \longrightarrow A[y_1, \ldots, y_n, z_1, \ldots, z_n, \tfrac{1}{h(y)h(z)}], \quad x_i \longmapsto z_i$$

and More on Algebra, Lemma 15.30.5. By More on Algebra, Lemma 15.30.14 we conclude that $g_1, \ldots, g_n$ is a regular sequence in $P \otimes_A B$.

At this point we have verified all the assumptions of Lemma 49.12.1 above with P, $f_1, \ldots, f_n$, and $g_i \in P \otimes_A B$ as above. In particular the annihilator J of I is freely generated by one element δ over B. Set $f_{ij} = \partial f_i / \partial x_j \in A[x_1, \ldots, x_n]$. An elementary computation shows that we can write

$$f_i(y) = f_i(z_1 + g_1, \ldots, z_n + g_n) = f_i(z) + \sum\nolimits_j f_{ij}(z) g_j + \sum\nolimits_{j,j'} F_{ijj'} g_j g_{j'}$$

for some $F_{ijj'} \in A[y_1, \ldots, y_n, z_1, \ldots, z_n]$. Taking the image in $P \otimes_A B$ the terms $f_i(z)$ map to zero and we obtain

$$f_i \otimes 1 = \sum\nolimits_j \left(1 \otimes f_{ij} + \sum\nolimits_{j'} F_{ijj'} g_{j'}\right) g_j$$

Thus we conclude from Lemma 49.12.1 that $\delta = \det(g_{ij})$ with $g_{ij} = 1 \otimes f_{ij} + \sum_{j'} F_{ijj'} g_{j'}$. Since $g_{j'}$ maps to zero in B, we conclude that the image of $\det(\partial f_i / \partial x_j)$ in B generates the Noether different of B over A. $\square$

0BWG Lemma 49.12.3. Let $f : Y \to X$ be a morphism of Noetherian schemes. If f satisfies the equivalent conditions of Lemma 49.10.1 then the different $\mathfrak{D}_f$ of f is the Kähler different of f.

Proof. By Lemmas 49.9.3 and 49.10.4 the different of f affine locally is the same as the Noether different. Then the lemma follows from the computation of the Noether different and the Kähler different on standard affine pieces done in Lemmas 49.7.4 and 49.12.2. $\square$

0BWH Lemma 49.12.4. Let A be a ring. Let $n \geq 1$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$. Assume that B is quasi-finite over A. Then there is an isomorphism $B \to \omega_{B/A}$ mapping $\det(\partial f_i / \partial x_j)$ to $\tau_{B/A}$.

Proof. Let J be the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$. By Lemma 49.12.2 the map $A \to B$ is flat and J is a free B-module with generator ξ mapping to $\det(\partial f_i / \partial x_j)$ in B. Thus the lemma follows from Lemma 49.6.7 and the fact (Lemma 49.10.4) that $\omega_{B/A}$ is an invertible B-module. (Warning: it is necessary to prove $\omega_{B/A}$ is invertible because a finite B-module M such that $\mathrm{Hom}_B(M, B) \cong B$ need not be free.) $\square$

0BWI Example 49.12.5. Let A be a Noetherian ring. Let $f, h \in A[x]$ such that

$$B = (A[x]/(f))_h = A[x, 1/h]/(f)$$

is quasi-finite over A. Let $f' \in A[x]$ be the derivative of f with respect to x. The ideal $\mathfrak{D} = (f') \subset B$ is the Noether different of B over A, is the Kähler different of B over A, and is the ideal whose associated quasi-coherent sheaf of ideals is the different of $\text{Spec}(B)$ over $\text{Spec}(A)$.

0BWJ Lemma 49.12.6. Let S be a Noetherian scheme. Let X, Y be smooth schemes of relative dimension n over S. Let $f : Y \to X$ be a locally quasi-finite morphism over S. Then f is flat and the closed subscheme $R \subset Y$ cut out by the different of f is the locally principal closed subscheme cut out by

$$\wedge^n(\text{d}f) \in \Gamma(Y, (f^*\Omega^n_{X/S})^{\otimes -1} \otimes_{\mathcal{O}_Y} \Omega^n_{Y/S})$$

If f is étale at the associated points of Y, then R is an effective Cartier divisor and

$$f^*\Omega^n_{X/S} \otimes_{\mathcal{O}_Y} \mathcal{O}(R) = \Omega^n_{Y/S}$$

as invertible sheaves on Y.

Proof. To prove that f is flat, it suffices to prove $Y_s \to X_s$ is flat for all $s \in S$ (More on Morphisms, Lemma 37.16.3). Flatness of $Y_s \to X_s$ follows from Algebra, Lemma 10.128.1. By More on Morphisms, Lemma 37.62.10 the morphism f is a local complete intersection morphism. Thus the statement on the different follows from the corresponding statement on the Kähler different by Lemma 49.12.3. Finally, since we have the exact sequence

$$f^*\Omega_{X/S} \xrightarrow{\text{d}f} \Omega_{Y/S} \to \Omega_{Y/X} \to 0$$

by Morphisms, Lemma 29.32.9 and since $\Omega_{X/S}$ and $\Omega_{Y/S}$ are finite locally free of rank n (Morphisms, Lemma 29.34.12), the statement for the Kähler different is clear from the definition of the zeroth fitting ideal. If f is étale at the associated points of Y, then $\wedge^n\text{d}f$ does not vanish in the associated points of Y, which implies that the local equation of R is a nonzerodivisor. Hence R is an effective Cartier divisor. The canonical isomorphism sends 1 to $\wedge^n\text{d}f$, see Divisors, Lemma 31.14.10. □

49.13. The Tate map

0FKB In this section we produce an isomorphism between the determinant of the relative cotangent complex and the relative dualizing module for a locally quasi-finite syntomic morphism of locally Noetherian schemes. Following [Gar84, 1.4.4] we dub the isomorphism the Tate map. Our approach is to avoid doing local calculations as much as is possible.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. We will use all the equivalent conditions for this notion given in Lemma 49.10.1 without further mention in this section. In particular, we see that $NL_{Y/X}$ is a perfect object of $D(\mathcal{O}_Y)$ with tor-amplitude in $[-1, 0]$. Thus we have a canonical invertible module $\det(NL_{Y/X})$ on Y and a global section

$$\delta(NL_{Y/X}) \in \Gamma(Y, \det(NL_{Y/X}))$$

See Derived Categories of Schemes, Lemma 36.39.1. Suppose given a commutative diagram of schemes

$$\begin{tikzcd} Y' \arrow[r, "b"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of Y' with an open of $X' \times_X Y$. Then the canonical map

$$Lb^* NL_{Y/X} \longrightarrow NL_{Y'/X'}$$

is a quasi-isomorphism by More on Morphisms, Lemma 37.13.16. Thus we get a canonical isomorphism $b^* \det(NL_{Y/X}) \to \det(NL_{Y'/X'})$ which sends the canonical section $\delta(NL_{Y/X})$ to $\delta(NL_{Y'/X'})$, see Derived Categories of Schemes, Remark 36.39.2.

0FKC Remark 49.13.1. Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. What does the pair $(\det(NL_{Y/X}), \delta(NL_{Y/X}))$ look like locally? Choose affine opens $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ with $f(V) \subset U$ and an integer n and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Then

$$NL_{B/A} = \Big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B\mathrm{d}x_i\Big)$$

and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\overline{f}_i$. See proof of Lemma 49.10.1. Thus $\det(L_{B/A})$ is free on the generator

$$\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \otimes (\overline{f}_1 \wedge \ldots \wedge \overline{f}_n)^{\otimes -1}$$

and the section $\delta(NL_{B/A})$ is the element

$$\delta(NL_{B/A}) = \det(\partial f_j/\partial x_i) \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \otimes (\overline{f}_1 \wedge \ldots \wedge \overline{f}_n)^{\otimes -1}$$

by definition.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of locally Noetherian schemes. By Remarks 49.2.11 and 49.4.7 we have a coherent $\mathcal{O}_Y$-module $\omega_{Y/X}$ and a canonical global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

which affine locally recovers the pair $\omega_{B/A}, \tau_{B/A}$. By Lemma 49.10.4 the module $\omega_{Y/X}$ is invertible. Suppose given a commutative diagram of locally Noetherian schemes

$$\begin{tikzcd} Y' \arrow[r, "b"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of Y' with an open of $X' \times_X Y$. Then there is a canonical base change map

$$b^*\omega_{Y/X} \longrightarrow \omega_{Y'/X'}$$

which is an isomorphism mapping $\tau_{Y/X}$ to $\tau_{Y'/X'}$. Namely, the base change map in the affine setting is (49.2.3.1), it is an isomorphism by Lemma 49.2.10, and it maps $\tau_{Y/X}$ to $\tau_{Y'/X'}$ by Lemma 49.4.4 part (1).

0FKD Proposition 49.13.2. There exists a unique rule that to every locally quasi-finite syntomic morphism of locally Noetherian schemes $Y \to X$ assigns an isomorphism

$$c_{Y/X} : \det(NL_{Y/X}) \longrightarrow \omega_{Y/X}$$

satisfying the following two properties

1. the section $\delta(NL_{Y/X})$ is mapped to $\tau_{Y/X}$, and
2. the rule is compatible with restriction to opens and with base change.

Proof. Let us reformulate the statement of the proposition. Consider the category $\mathcal{C}$ whose objects, denoted Y/X, are locally quasi-finite syntomic morphism $Y \to X$ of locally Noetherian schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$\begin{tikzcd} Y' \arrow[r, "b"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "a"] & X \end{tikzcd}$$

which induce an isomorphism of Y' with an open subscheme of $X' \times_X Y$. The proposition means that for every object Y/X of $\mathcal{C}$ we have an isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$ and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^*c_{Y/X} = c_{Y'/X'}$ via the identifications $b^* \det(NL_{Y/X}) = \det(NL_{Y'/X'})$ and $b^*\omega_{Y/X} = \omega_{Y'/X'}$ described above.

Given Y/X in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some isomorphism

$$\det(NL_{Y/X})|_V \longrightarrow \omega_{Y/X}|_V$$

mapping $\delta(NL_{Y/X})|_V$ to $\tau_{Y/X}|_V$. This follows from picking affine opens as in Lemma 49.10.1 part (5), the affine local description of $\delta(NL_{Y/X})$ in Remark 49.13.1, and Lemma 49.12.4. If the annihilator of the section $\tau_{Y/X}$ is zero, then these local maps are unique and automatically glue. Hence if the annihilator of $\tau_{Y/X}$ is zero, then there is a unique isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$ and the annihilator of $\tau_{Y'/X'}$ is zero as well, then $b^*c_{Y/X}$ is the unique isomorphism $c_{Y'/X'} : \det(NL_{Y'/X'}) \to \omega_{Y'/X'}$ with $c_{Y'/X'}(\delta(NL_{Y'/X'})) = \tau_{Y'/X'}$. This follows formally from the fact that $b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ and $b^*\tau_{Y/X} = \tau_{Y'/X'}$.

We can summarize the results of the previous paragraph as follows. Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of Y/X such that the annihilator of $\tau_{Y/X}$ is zero. Then we have solved the problem on $\mathcal{C}_{nice}$. For Y/X in $\mathcal{C}_{nice}$ we continue to denote $c_{Y/X}$ the solution we've just found.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that Y_1/X_1 and Y_2/X_2 are objects of $\mathcal{C}_{nice}$. Claim. $b_1^*c_{Y_1/X_1} = b_2^*c_{Y_2/X_2}$. We will first show that the claim implies the proposition and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Example 49.10.5. Then $Y_{n,d}$ is an irreducible regular scheme and the morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Lemma 49.10.6. Thus $\tau_{Y_{n,d}/X_{n,d}}$ is nonzero for example by Lemma 49.9.6. Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let Y/X be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Lemma 49.10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c_{Y_{n,d}/X_{n,d}}$ constructed above by b to V. The claim guarantees these local isomorphisms glue! Thus we get a well defined global isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c_{Y'/X'}$ is the pullback by b of the locally constructed map $c_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of y. Thus we may replace Y_1, Y_2 by open neighbourhoods of the image of y in Y_1 and Y_2. Thus we may assume there are morphisms

$$Y_{n_1,d_1}/X_{n_1,d_1} \leftarrow Y_1/X_1 \quad \text{and} \quad Y_2/X_2 \to Y_{n_2,d_2}/X_{n_2,d_2}$$

These are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities. Thus we may replace Y_1/X_1 by $Y_{n_1,d_1}/X_{n_1,d_1}$ and Y_2/X_2 by $Y_{n_2,d_2}/X_{n_2,d_2}$. This reduces us to the case that Y_1, X_1, Y_2, X_2 are of finite type over $\mathbf{Z}$. (The astute reader will realize that this step wouldn't have been necessary if we'd defined $\mathcal{C}_{nice}$ to consist only of those objects Y/X with Y and X of finite type over $\mathbf{Z}$.)

Assume Y_1, X_1, Y_2, X_2 are of finite type over $\mathbf{Z}$. After replacing Y, X, Y_1, X_1, Y_2, X_2 by suitable open neighbourhoods of the image of y we may assume Y, X, Y_1, X_1, Y_2, X_2 are affine. We may write $X = \lim X_\lambda$ as a cofiltered limit of affine schemes of finite type over $X_1 \times X_2$. For each λ we get

$$Y_1 \times_{X_1} X_\lambda \quad \text{and} \quad X_\lambda \times_{X_2} Y_2$$

If we take limits we obtain

$$\lim Y_1 \times_{X_1} X_\lambda = Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2 = \lim X_\lambda \times_{X_2} Y_2$$

By Limits, Lemma 32.4.11 we can find a λ and opens $V_{1,\lambda} \subset Y_1 \times_{X_1} X_\lambda$ and $V_{2,\lambda} \subset X_\lambda \times_{X_2} Y_2$ whose base change to X recovers Y (on both sides). After increasing λ we may assume there is an isomorphism $V_{1,\lambda} \to V_{2,\lambda}$ whose base change to X is the identity on Y, see Limits, Lemma 32.10.1. Then we have the commutative diagram

$$\begin{tikzcd} & Y/X \arrow[ld, "b_1/a_1"'] \arrow[d] \arrow[rd, "b_2/a_2"] & \\ Y_1/X_1 & V_{1,\lambda}/X_\lambda \arrow[l] \arrow[r] & Y_2/X_2 \end{tikzcd}$$

Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$. The ring map $A_1 \to A$ corresponding to $X \to X_1$ is of finite type and hence we may choose a surjection $A_1[x_1, \ldots, x_n] \to A$. Similarly, we may choose a surjection $A_2[y_1, \ldots, y_m] \to A$. Set $X_1' = \mathrm{Spec}(A_1[x_1, \ldots, x_n])$ and $X_2' = \mathrm{Spec}(A_2[y_1, \ldots, y_m])$. Set $Y_1' = Y_1 \times_{X_1} X_1'$ and $Y_2' = Y_2 \times_{X_2} X_2'$. We get the following diagram

$$Y_1/X_1 \leftarrow Y_1'/X_1' \leftarrow Y/X \to Y_2'/X_2' \to Y_2/X_2$$

Since $X_1' \to X_1$ and $X_2' \to X_2$ are flat, the same is true for $Y_1' \to Y_1$ and $Y_2' \to Y_2$. It follows easily that the annihilators of $\tau_{Y_1'/X_1'}$ and $\tau_{Y_2'/X_2'}$ are zero. Hence Y_1'/X_1' and Y_2'/X_2' are in $\mathcal{C}_{nice}$. Thus the outer morphisms in the displayed diagram are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities. Thus it suffices to prove the claim for $Y_1'/X_1' \leftarrow Y/X \to Y_2'/X_2'$. This reduces us to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions. Consider the open embeddings $Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2$. There is an open neighbourhood $V \subset Y$ of y which is a standard open of both $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$. This follows from Schemes, Lemma 26.11.5 applied to the scheme obtained by glueing $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$ along Y; details omitted. Since $X \times_{X_2} Y_2$ is a closed subscheme of Y_2 we can find a standard open $V_2 \subset Y_2$ such that $V_2 \times_{X_2} X = V$. Similarly, we can find a standard open $V_1 \subset Y_1$ such that $V_1 \times_{X_1} X = V$. After replacing Y, Y_1, Y_2 by V, V_1, V_2 we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions and $Y_1 \times_{X_1} X = Y = X \times_{X_2} Y_2$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$, $Y = \mathrm{Spec}(B)$, $Y_i = \mathrm{Spec}(B_i)$. Then we can consider the affine schemes

$$X' = \mathrm{Spec}(A_1 \times_A A_2) = \mathrm{Spec}(A') \quad \text{and} \quad Y' = \mathrm{Spec}(B_1 \times_B B_2) = \mathrm{Spec}(B')$$

Observe that $X' = X_1 \amalg_X X_2$ and $Y' = Y_1 \amalg_Y Y_2$, see More on Morphisms, Lemma 37.14.1. By More on Algebra, Lemma 15.5.1 the rings A' and B' are of finite type over $\mathbf{Z}$. By More on Algebra, Lemma 15.6.4 we have $B' \otimes_A A_1 = B_1$ and $B' \times_A A_2 = B_2$. In particular a fibre of $Y' \to X'$ over a point of $X' = X_1 \amalg_X X_2$ is always equal to either a fibre of $Y_1 \to X_1$ or a fibre of $Y_2 \to X_2$. By More on Algebra, Lemma 15.6.8 the ring map $A' \to B'$ is flat. Thus by Lemma 49.10.1 part (3) we conclude that Y'/X' is an object of $\mathcal{C}$. Consider now the commutative diagram

$$\begin{tikzcd} & Y/X \arrow[ld, "b_1/a_1"'] \arrow[rd, "b_2/a_2"] & \\ Y_1/X_1 \arrow[rd] & & Y_2/X_2 \arrow[ld] \\ & Y'/X' & \end{tikzcd}$$

Now we would be done if Y'/X' is an object of $\mathcal{C}_{nice}$. Namely, then pulling back $c_{Y'/X'}$ around the two sides of the square, we would obtain the desired conclusion. Now, in fact, it is true that Y'/X' is an object of $\mathcal{C}_{nice}$[4]. But it is amusing to note that we don't even need this. Namely, the arguments above show that, after

[4]Namely, the structure sheaf $\mathcal{O}_{Y'}$ is a subsheaf of $(Y_1 \to Y')_*\mathcal{O}_{Y_1} \times (Y_2 \to Y')_*\mathcal{O}_{Y_2}$.

possibly shrinking all of the schemes $X, Y, X_1, Y_1, X_2, Y_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

$$\begin{tikzcd}
 & Y/X \arrow[dl, "b_1/a_1"'] \arrow[dr, "b_2/a_2"] & \\
Y_1/X_1 \arrow[dr] & & Y_2/X_2 \arrow[dl] \\
 & Y'/X' \arrow[d] & \\
 & Y_{n,d}/X_{n,d} &
\end{tikzcd}$$

and then we can use the already given argument by pulling back from $c_{Y_{n,d}/X_{n,d}}$. This finishes the proof. □

49.14. A generalization of the different

0BWK In this section we generalize Definition 49.9.1 to take into account all cases of ring maps $A \to B$ where the Dedekind different is defined and $1 \in \mathcal{L}_{B/A}$. First we explain the condition "$A \to B$ maps nonzerodivisors to nonzerodivisors and induces a flat map $Q(A) \to Q(A) \otimes_A B$".

0BWL **Lemma 49.14.1.** Let $A \to B$ be a map of Noetherian rings. Consider the conditions

(1) nonzerodivisors of A map to nonzerodivisors of B,
(2) (1) holds and $Q(A) \to Q(A) \otimes_A B$ is flat,
(3) $A \to B_\mathfrak{q}$ is flat for every $\mathfrak{q} \in \text{Ass}(B)$,
(4) (3) holds and $A \to B_\mathfrak{q}$ is flat for every $\mathfrak{q}$ lying over an element in $\text{Ass}(A)$.

Then we have the following implications

$$\begin{tikzcd}
(1) & (2) \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
(3) \arrow[u, Rightarrow] & (4) \arrow[l, Rightarrow]
\end{tikzcd}$$

If going up holds for $A \to B$ then (2) and (4) are equivalent.

Proof. The horizontal implications in the diagram are trivial. Let $S \subset A$ be the set of nonzerodivisors so that $Q(A) = S^{-1}A$ and $Q(A) \otimes_A B = S^{-1}B$. Recall that $S = A \setminus \bigcup_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p}$ by Algebra, Lemma 10.63.9. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$.

Assume (2). If $\mathfrak{q} \in \text{Ass}(B)$ then $\mathfrak{q}$ consists of zerodivisors, hence (1) implies the same is true for $\mathfrak{p}$. Hence $\mathfrak{p}$ corresponds to a prime of $S^{-1}A$. Hence $A \to B_\mathfrak{q}$ is flat by our assumption (2). If $\mathfrak{q}$ lies over an associated prime $\mathfrak{p}$ of A, then certainly $\mathfrak{p} \in \text{Spec}(S^{-1}A)$ and the same argument works.

Assume (3). Let $f \in A$ be a nonzerodivisor. If f were a zerodivisor on B, then f is contained in an associated prime $\mathfrak{q}$ of B. Since $A \to B_\mathfrak{q}$ is flat by assumption, we conclude that $\mathfrak{p}$ is an associated prime of A by Algebra, Lemma 10.65.3. This would imply that f is a zerodivisor on A, a contradiction.

Assume (4) and going up for $A \to B$. We already know (1) holds. If $\mathfrak q$ corresponds to a prime of $S^{-1}B$ then $\mathfrak p$ is contained in an associated prime $\mathfrak p'$ of A. By going up there exists a prime $\mathfrak q'$ containing $\mathfrak q$ and lying over $\mathfrak p$. Then $A \to B_{\mathfrak q'}$ is flat by (4). Hence $A \to B_{\mathfrak q}$ is flat as a localization. Thus $A \to S^{-1}B$ is flat and so is $S^{-1}A \to S^{-1}B$, see Algebra, Lemma 10.39.18. ☐

0BWM Remark 49.14.2. We can generalize Definition 49.9.1. Suppose that $f : Y \to X$ is a quasi-finite morphism of Noetherian schemes with the following properties

(1) the open $V \subset Y$ where f is flat contains $\text{Ass}(\mathcal{O}_Y)$ and $f^{-1}(\text{Ass}(\mathcal{O}_X))$,
(2) the trace element $\tau_{V/X}$ comes from a section $\tau \in \Gamma(Y, \omega_{Y/X})$.

Condition (1) implies that V contains the associated points of $\omega_{Y/X}$ by Lemma 49.2.8. In particular, τ is unique if it exists (Divisors, Lemma 31.2.8). Given τ we can define the different $\mathfrak{D}_f$ as the annihilator of $\text{Coker}(\tau : \mathcal{O}_Y \to \omega_{Y/X})$. This agrees with the Dedekind different in many cases (Lemma 49.14.3). However, for non-flat maps between non-normal rings, this generalization no longer measures ramification of the morphism, see Example 49.14.4.

0BWN Lemma 49.14.3. Assume the Dedekind different is defined for $A \to B$. Set $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. The generalization of Remark 49.14.2 applies to the morphism $f : Y \to X$ if and only if $1 \in \mathcal{L}_{B/A}$ (e.g., if A is normal, see Lemma 49.8.1). In this case $\mathfrak{D}_{B/A}$ is an ideal of B and we have

$$\mathfrak{D}_f = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on Y.

Proof. As the Dedekind different for $A \to B$ is defined we can apply Lemma 49.14.1 to see that $Y \to X$ satisfies condition (1) of Remark 49.14.2. Recall that there is a canonical isomorphism $c : \mathcal{L}_{B/A} \to \omega_{B/A}$, see Lemma 49.8.2. Let $K = Q(A)$ and $L = K \otimes_A B$ as above. By construction the map c fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{L}_{B/A} \arrow[r] \arrow[d, "c"'] & L \arrow[d] \\
\omega_{B/A} \arrow[r] & \text{Hom}_K(L, K)
\end{tikzcd}$$

where the right vertical arrow sends $x \in L$ to the map $y \mapsto \text{Trace}_{L/K}(xy)$ and the lower horizontal arrow is the base change map (49.2.3.1) for $\omega_{B/A}$. We can factor the lower horizontal map as

$$\omega_{B/A} = \Gamma(Y, \omega_{Y/X}) \to \Gamma(V, \omega_{V/X}) \to \text{Hom}_K(L, K)$$

Since all associated points of $\omega_{V/X}$ map to associated primes of A (Lemma 49.2.8) we see that the second map is injective. The element $\tau_{V/X}$ maps to $\text{Trace}_{L/K}$ in $\text{Hom}_K(L, K)$ by the very definition of trace elements (Definition 49.4.1). Thus τ as in condition (2) of Remark 49.14.2 exists if and only if $1 \in \mathcal{L}_{B/A}$ and then $\tau = c(1)$. In this case, by Lemma 49.8.1 we see that $\mathfrak{D}_{B/A} \subset B$. Finally, the agreement of $\mathfrak{D}_f$ with $\mathfrak{D}_{B/A}$ is immediate from the definitions and the fact $\tau = c(1)$ seen above. ☐

0BWP Example 49.14.4. Let k be a field. Let $A = k[x, y]/(xy)$ and $B = k[u, v]/(uv)$ and let $A \to B$ be given by $x \mapsto u^n$ and $y \mapsto v^m$ for some $n, m \in \mathbf{N}$ prime to the characteristic of k. Then $A_{x+y} \to B_{x+y}$ is (finite) étale hence we are in the situation where the Dedekind different is defined. A computation shows that

$$\mathrm{Trace}_{L/K}(1) = (nx + my)/(x + y), \quad \mathrm{Trace}_{L/K}(u^i) = 0, \quad \mathrm{Trace}_{L/K}(v^j) = 0$$

for $1 \leq i < n$ and $1 \leq j < m$. We conclude that $1 \in \mathcal{L}_{B/A}$ if and only if $n = m$. Moreover, a computation shows that if $n = m$, then $\mathcal{L}_{B/A} = B$ and the Dedekind different is B as well. In other words, we find that the different of Remark 49.14.2 is defined for $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ if and only if $n = m$, and in this case the different is the unit ideal. Thus we see that in nonflat cases the nonvanishing of the different does not guarantee the morphism is étale or unramified.

49.15. Comparison with duality theory

0DWM In this section we compare the elementary algebraic constructions above with the constructions in the chapter on duality theory for schemes.

0BUL Lemma 49.15.1. Let $f : Y \to X$ be a quasi-finite separated morphism of Noetherian schemes. For every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ there is an isomorphism

$$H^0(V, f^!\mathcal{O}_Y) = \omega_{B/A}$$

where $f^!$ is as in Duality for Schemes, Section 48.16. These isomorphisms are compatible with restriction maps and define a canonical isomorphism $H^0(f^!\mathcal{O}_X) = \omega_{Y/X}$ with $\omega_{Y/X}$ as in Remark 49.2.11. Similarly, if $f : Y \to X$ is a quasi-finite morphism of schemes of finite type over a Noetherian base S endowed with a dualizing complex $\omega_S^\bullet$, then $H^0(f^!_{new}\mathcal{O}_X) = \omega_{Y/X}$.

Proof. By Zariski's main theorem we can choose a factorization $f = f' \circ j$ where $j : Y \to Y'$ is an open immersion and $f' : Y' \to X$ is a finite morphism, see More on Morphisms, Lemma 37.43.3. By our construction in Duality for Schemes, Lemma 48.16.2 we have $f^! = j^* \circ a'$ where $a' : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{Y'})$ is the right adjoint to Rf'_* of Duality for Schemes, Lemma 48.3.1. By Duality for Schemes, Lemma 48.11.4 we see that $\Phi(a'(\mathcal{O}_X)) = R\mathcal{H}om(f'_*\mathcal{O}_{Y'}, \mathcal{O}_X)$ in $D^+_{QCoh}(f'_*\mathcal{O}_{Y'})$. In particular $a'(\mathcal{O}_X)$ has vanishing cohomology sheaves in degrees < 0. The zeroth cohomology sheaf is determined by the isomorphism

$$f'_*H^0(a'(\mathcal{O}_X)) = \mathcal{H}om_{\mathcal{O}_X}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_X)$$

as $f'_*\mathcal{O}_{Y'}$-modules via the equivalence of Morphisms, Lemma 29.11.6. Writing $(f')^{-1}U = V' = \mathrm{Spec}(B')$, we obtain

$$H^0(V', a'(\mathcal{O}_X)) = \mathrm{Hom}_A(B', A).$$

As the zeroth cohomology sheaf of $a'(\mathcal{O}_X)$ is a quasi-coherent module we find that the restriction to V is given by $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$ as desired.

The statement about restriction maps signifies that the restriction mappings of the quasi-coherent $\mathcal{O}_{Y'}$-module $H^0(a'(\mathcal{O}_X))$ for opens in Y' agrees with the maps defined in Lemma 49.2.3 for the modules $\omega_{B/A}$ via the isomorphisms given above. This is clear.

Let $f : Y \to X$ be a quasi-finite morphism of schemes of finite type over a Noetherian base S endowed with a dualizing complex $\omega_S^\bullet$. Consider opens $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ and V and U separated over S. Denote $f|_V : V \to U$ the restriction of f. By the discussion above and Duality for Schemes, Lemma 48.20.9 there are canonical isomorphisms

$$H^0(f_{new}^!\mathcal{O}_X)|_V = H^0((f|_V)^!\mathcal{O}_U) = \omega_{V/U} = \omega_{Y/X}|_V$$

We omit the verification that these isomorphisms glue to a global isomorphism $H^0(f_{new}^!\mathcal{O}_X) \to \omega_{Y/X}$. □

0BVI Lemma 49.15.2. Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. The map

$$\mathrm{Trace}_f : f_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

of Section 49.3 corresponds to a map $\mathcal{O}_Y \to f^!\mathcal{O}_X$ (see proof). Denote $\tau_{Y/X} \in H^0(Y, f^!\mathcal{O}_X)$ the image of 1. Via the isomorphism $H^0(f^!\mathcal{O}_X) = \omega_{X/Y}$ of Lemma 49.15.1 this agrees with the construction in Remark 49.4.7.

Proof. The functor $f^!$ is defined in Duality for Schemes, Section 48.16. Since f is finite (and hence proper), we see that $f^!$ is given by the right adjoint to pushforward for f. In Duality for Schemes, Section 48.11 we have made this adjoint explicit. In particular, the object $f^!\mathcal{O}_X$ consists of a single cohomology sheaf placed in degree 0 and for this sheaf we have

$$f_*f^!\mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$$

To see this we use also that $f_*\mathcal{O}_Y$ is finite locally free as f is a finite flat morphism of Noetherian schemes and hence all higher Ext sheaves are zero. Some details omitted. Thus finally

$$\mathrm{Trace}_f \in \mathrm{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) = \Gamma(X, f_*f^!\mathcal{O}_X) = \Gamma(Y, f^!\mathcal{O}_X)$$

On the other hand, we have $f^!\mathcal{O}_X = \omega_{Y/X}$ by the identification of Lemma 49.15.1. Thus we now have two elements, namely Trace_f and $\tau_{Y/X}$ from Remark 49.4.7 in

$$\Gamma(Y, f^!\mathcal{O}_X) = \Gamma(Y, \omega_{Y/X})$$

and the lemma says these elements are the same.

Let $U = \mathrm{Spec}(A) \subset X$ be an affine open with inverse image $V = \mathrm{Spec}(B) \subset Y$. Since f is finite, we see that $A \to B$ is finite and hence the $\omega_{Y/X}(V) = \mathrm{Hom}_A(B, A)$ by construction and this isomorphism agrees with the identification of $f_*f^!\mathcal{O}_Y$ with $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$ discussed above. Hence the agreement of Trace_f and $\tau_{Y/X}$ follows from the fact that $\tau_{B/A} = \mathrm{Trace}_{B/A}$ by Lemma 49.4.3. □

49.16. Quasi-finite Gorenstein morphisms

0C14 This section discusses quasi-finite Gorenstein morphisms.

0C16 Lemma 49.16.1. Let $f : Y \to X$ be a quasi-finite morphism of Noetherian schemes. The following are equivalent

(1) f is Gorenstein,
(2) f is flat and the fibres of f are Gorenstein,
(3) f is flat and $\omega_{Y/X}$ is invertible (Remark 49.2.11),
(4) for every $y \in Y$ there are affine opens $y \in V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is flat and $\omega_{B/A}$ is an invertible B-module.

Proof. Parts (1) and (2) are equivalent by definition. Parts (3) and (4) are equivalent by the construction of $\omega_{Y/X}$ in Remark 49.2.11. Thus we have to show that (1)-(2) is equivalent to (3)-(4).

First proof. Working affine locally we can assume f is a separated morphism and apply Lemma 49.15.1 to see that $\omega_{Y/X}$ is the zeroth cohomology sheaf of $f^!\mathcal{O}_X$. Under both assumptions f is flat and quasi-finite, hence $f^!\mathcal{O}_X$ is isomorphic to $\omega_{Y/X}[0]$, see Duality for Schemes, Lemma 48.21.6. Hence the equivalence follows from Duality for Schemes, Lemma 48.25.10.

Second proof. By Lemma 49.10.2, we see that it suffices to prove the equivalence of (2) and (3) when X is the spectrum of a field k. Then $Y = \mathrm{Spec}(B)$ where B is a finite k-algebra. In this case $\omega_{B/A} = \omega_{B/k} = \mathrm{Hom}_k(B, k)$ placed in degree 0 is a dualizing complex for B, see Dualizing Complexes, Lemma 47.15.8. Thus the equivalence follows from Dualizing Complexes, Lemma 47.21.4. □

0C17 Remark 49.16.2. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of Noetherian schemes. Let $\mathfrak{D}_f \subset \mathcal{O}_Y$ be the different and let $R \subset Y$ be the closed subscheme cut out by $\mathfrak{D}_f$. Then we have

1. $\mathfrak{D}_f$ is a locally principal ideal,
2. R is a locally principal closed subscheme,
3. $\mathfrak{D}_f$ is affine locally the same as the Noether different,
4. formation of R commutes with base change,
5. if f is finite, then the norm of R is the discriminant of f, and
6. if f is étale in the associated points of Y, then R is an effective Cartier divisor and $\omega_{Y/X} = \mathcal{O}_Y(R)$.

This follows from Lemmas 49.9.3, 49.9.4, and 49.9.7.

0C18 Remark 49.16.3. Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of compactifyable schemes over S. Assume moreover Y and X Cohen-Macaulay and f étale at the generic points of Y. Then we can combine Duality for Schemes, Remark 48.23.4 and Remark 49.16.2 to see that we have a canonical isomorphism

$$\omega_Y = f^*\omega_X \otimes_{\mathcal{O}_Y} \omega_{Y/X} = f^*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R)$$

of $\mathcal{O}_Y$-modules. If further f is finite, then the isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$ comes from the global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which corresponds via duality to the map $\mathrm{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Lemma 49.15.2.

49.17. Other chapters

CHAPTER 50

de Rham Cohomology

0FK4
50.1. Introduction

0FK5 In this chapter we start with a discussion of the de Rham complex of a morphism of schemes and we end with a proof that de Rham cohomology defines a Weil cohomology theory when the base field has characteristic zero.

50.2. The de Rham complex

07HX Let $p : X \to S$ be a morphism of schemes. There is a complex

$$\Omega^\bullet_{X/S} = \mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

of $p^{-1}\mathcal{O}_S$-modules with $\Omega^i_{X/S} = \wedge^i(\Omega_{X/S})$ placed in degree i and differential determined by the rule $\text{d}(g_0\text{d}g_1 \wedge \ldots \wedge \text{d}g_p) = \text{d}g_0 \wedge \text{d}g_1 \wedge \ldots \wedge \text{d}g_p$ on local sections. See Modules, Section 17.30.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, there are canonical maps of complexes $f^{-1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X'/S'}$ and $\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'}$. See Modules, Section 17.30. Linearizing, for every p we obtain a linear map $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

In particular, if $f : Y \to X$ be a morphism of schemes over a base scheme S, then there is a map of complexes

$$\Omega^\bullet_{X/S} \longrightarrow f_*\Omega^\bullet_{Y/S}$$

Linearizing, we see that for every $p \geq 0$ we obtain a canonical map

$$\Omega^p_{X/S} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \longrightarrow f_*\Omega^p_{Y/S}$$

0FL5 **Lemma 50.2.1.** Let

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Then the maps discussed above induce isomorphisms $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

Proof. Combine Morphisms, Lemma 29.32.10 with the fact that formation of exterior power commutes with base change. $\square$

0FLV Lemma 50.2.2. Consider a commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

If $X' \to X$ and $S' \to S$ are étale, then the maps discussed above induce isomorphisms $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

Proof. We have $\Omega_{S'/S} = 0$ and $\Omega_{X'/X} = 0$, see for example Morphisms, Lemma 29.36.15. Then by the short exact sequences of Morphisms, Lemmas 29.32.9 and 29.34.16 we see that $\Omega_{X'/S'} = \Omega_{X'/S} = f^*\Omega_{X/S}$. Taking exterior powers we conclude. □

50.3. de Rham cohomology

0FL6 Let $p : X \to S$ be a morphism of schemes. We define the de Rham cohomology of X over S to be the cohomology groups

$$H^i_{dR}(X/S) = H^i(R\Gamma(X, \Omega^\bullet_{X/S}))$$

Since $\Omega^\bullet_{X/S}$ is a complex of $p^{-1}\mathcal{O}_S$-modules, these cohomology groups are naturally modules over $H^0(S, \mathcal{O}_S)$.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, using the canonical maps of Section 50.2 we obtain pullback maps

$$f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \longrightarrow R\Gamma(X', \Omega^\bullet_{X'/S'})$$

and

$$f^* : H^i_{dR}(X/S) \longrightarrow H^i_{dR}(X'/S')$$

These pullbacks satisfy an obvious composition law. In particular, if we work over a fixed base scheme S, then de Rham cohomology is a contravariant functor on the category of schemes over S.

0FLW Lemma 50.3.1. Let $X \to S$ be a morphism of affine schemes given by the ring map $R \to A$. Then $R\Gamma(X, \Omega^\bullet_{X/S}) = \Omega^\bullet_{A/R}$ in $D(R)$ and $H^i_{dR}(X/S) = H^i(\Omega^\bullet_{A/R})$.

Proof. This follows from Cohomology of Schemes, Lemma 30.2.2 and Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7). □

0FLX Lemma 50.3.2. Let $p : X \to S$ be a morphism of schemes. If p is quasi-compact and quasi-separated, then $Rp_*\Omega^\bullet_{X/S}$ is an object of $D_{QCoh}(\mathcal{O}_S)$.

Proof. There is a spectral sequence with first page $E_1^{a,b} = R^bp_*\Omega^a_{X/S}$ converging to the cohomology of $Rp_*\Omega^\bullet_{X/S}$ (see Derived Categories, Lemma 13.21.3). Hence by Homology, Lemma 12.25.3 it suffices to show that $R^bp_*\Omega^a_{X/S}$ is quasi-coherent. This follows from Cohomology of Schemes, Lemma 30.4.5. □

0FLY Lemma 50.3.3. Let $p : X \to S$ be a proper morphism of schemes with S locally Noetherian. Then $Rp_*\Omega^\bullet_{X/S}$ is an object of $D_{\text{Coh}}(\mathcal{O}_S)$.

Proof. In this case by Morphisms, Lemma 29.32.12 the modules $\Omega^i_{X/S}$ are coherent. Hence we can use exactly the same argument as in the proof of Lemma 50.3.2 using Cohomology of Schemes, Proposition 30.19.1. □

0FLZ Lemma 50.3.4. Let A be a Noetherian ring. Let X be a proper scheme over $S = \text{Spec}(A)$. Then $H^i_{dR}(X/S)$ is a finite A-module for all i.

Proof. This is a special case of Lemma 50.3.3. □

0FM0 Lemma 50.3.5. Let $f : X \to S$ be a proper smooth morphism of schemes. Then $Rf_*\Omega^p_{X/S}$, $p \geq 0$ and $Rf_*\Omega^\bullet_{X/S}$ are perfect objects of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary change of base.

Proof. Since f is smooth the modules $\Omega^p_{X/S}$ are finite locally free $\mathcal{O}_X$-modules, see Morphisms, Lemma 29.34.12. Their formation commutes with arbitrary change of base by Lemma 50.2.1. Hence $Rf_*\Omega^p_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$ whose formation commutes with abitrary base change, see Derived Categories of Schemes, Lemma 36.30.4. This proves the first assertion of the lemma.

To prove that $Rf_*\Omega^\bullet_{X/S}$ is perfect on S we may work locally on S. Thus we may assume S is quasi-compact. This means we may assume that $\Omega^n_{X/S}$ is zero for n large enough. For every $p \geq 0$ we claim that $Rf_*\sigma_{\geq p}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary change of base. By the above we see that this is true for $p \gg 0$. Suppose the claim holds for p and consider the distinguished triangle

$$\sigma_{\geq p}\Omega^\bullet_{X/S} \to \sigma_{\geq p-1}\Omega^\bullet_{X/S} \to \Omega^{p-1}_{X/S}[-(p-1)] \to (\sigma_{\geq p}\Omega^\bullet_{X/S})[1]$$

in $D(f^{-1}\mathcal{O}_S)$. Applying the exact functor Rf_* we obtain a distinguished triangle in $D(\mathcal{O}_S)$. Since we have the 2-out-of-3 property for being perfect (Cohomology, Lemma 20.49.7) we conclude $Rf_*\sigma_{\geq p-1}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$. Similarly for the commutation with arbitrary base change. □

50.4. Cup product

0FM1 Consider the maps $\Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ given by $(\omega, \eta) \longmapsto \omega \wedge \eta$. Using the formula for d given in Section 50.2 and the Leibniz rule for $\text{d} : \mathcal{O}_X \to \Omega_{X/S}$ we see that $\text{d}(\omega \wedge \eta) = \text{d}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge \text{d}(\eta)$. This means that $\wedge$ defines a morphism

0FM2 (50.4.0.1)
$$\wedge : \text{Tot}(\Omega^\bullet_{X/S} \otimes_{p^{-1}\mathcal{O}_S} \Omega^\bullet_{X/S}) \longrightarrow \Omega^\bullet_{X/S}$$

of complexes of $p^{-1}\mathcal{O}_S$-modules.

Combining the cup product of Cohomology, Section 20.31 with (50.4.0.1) we find a $H^0(S, \mathcal{O}_S)$-bilinear cup product map

$$\cup : H^i_{dR}(X/S) \times H^j_{dR}(X/S) \longrightarrow H^{i+j}_{dR}(X/S)$$

For example, if $\omega \in \Gamma(X, \Omega^i_{X/S})$ and $\eta \in \Gamma(X, \Omega^j_{X/S})$ are closed, then the cup product of the de Rham cohomology classes of ω and η is the de Rham cohomology class of $\omega \wedge \eta$, see discussion in Cohomology, Section 20.31.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, the pullback maps $f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \to R\Gamma(X', \Omega^\bullet_{X'/S'})$ and $f^* : H^i_{dR}(X/S) \longrightarrow H^i_{dR}(X'/S')$ are compatible with the cup product defined above.

0FM3 Lemma 50.4.1. Let $p : X \to S$ be a morphism of schemes. The cup product on $H^*_{dR}(X/S)$ is associative and graded commutative.

Proof. This follows from Cohomology, Lemmas 20.31.5 and 20.31.6 and the fact that $\wedge$ is associative and graded commutative. □

0FU6 Remark 50.4.2. Let $p : X \to S$ be a morphism of schemes. Then we can think of $\Omega^\bullet_{X/S}$ as a sheaf of differential graded $p^{-1}\mathcal{O}_S$-algebras, see Differential Graded Sheaves, Definition 24.12.1. In particular, the discussion in Differential Graded Sheaves, Section 24.32 applies. For example, this means that for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ S \arrow[r, "h"] & T \end{tikzcd}$$

of schemes there is a canonical relative cup product

$$\mu : Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

in $D(Y, q^{-1}\mathcal{O}_T)$ which is associative and which on cohomology reproduces the cup product discussed above.

0FU7 Remark 50.4.3. Let $f : X \to S$ be a morphism of schemes. Let $\xi \in H^n_{dR}(X/S)$. According to the discussion Differential Graded Sheaves, Section 24.32 there exists a canonical morphism

$$\xi' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[n]$$

in $D(f^{-1}\mathcal{O}_S)$ uniquely characterized by (1) and (2) of the following list of properties:

(1) ξ' can be lifted to a map in the derived category of right differential graded $\Omega^\bullet_{X/S}$-modules, and
(2) $\xi'(1) = \xi$ in $H^0(X, \Omega^\bullet_{X/S}[n]) = H^n_{dR}(X/S)$,
(3) the map ξ' sends $\eta \in H^m_{dR}(X/S)$ to $\xi \cup \eta$ in $H^{n+m}_{dR}(X/S)$,
(4) the construction of ξ' commutes with restrictions to opens: for $U \subset X$ open the restriction $\xi'|_U$ is the map corresponding to the image $\xi|_U \in H^n_{dR}(U/S)$,
(5) for any diagram as in Remark 50.4.2 we obtain a commutative diagram

$$\begin{tikzcd} Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \arrow[r, "\mu"] \arrow[d, "\xi' \otimes \mathrm{id}"'] & Rf_*\Omega^\bullet_{X/S} \arrow[d, "\xi'"] \\ Rf_*\Omega^\bullet_{X/S}[n] \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \arrow[r, "\mu"] & Rf_*\Omega^\bullet_{X/S}[n] \end{tikzcd}$$

in $D(Y, q^{-1}\mathcal{O}_T)$.

50.5. Hodge cohomology

0FM4 Let $p : X \to S$ be a morphism of schemes. We define the Hodge cohomology of X over S to be the cohomology groups

$$H^n_{Hodge}(X/S) = \bigoplus\nolimits_{n=p+q} H^q(X, \Omega^p_{X/S})$$

viewed as a graded $H^0(X, \mathcal{O}_X)$-module. The wedge product of forms combined with the cup product of Cohomology, Section 20.31 defines a $H^0(X, \mathcal{O}_X)$-bilinear cup product

$$\cup : H^i_{Hodge}(X/S) \times H^j_{Hodge}(X/S) \longrightarrow H^{i+j}_{Hodge}(X/S)$$

Of course if $\xi \in H^q(X, \Omega^p_{X/S})$ and $\xi' \in H^{q'}(X, \Omega^{p'}_{X/S})$ then $\xi \cup \xi' \in H^{q+q'}(X, \Omega^{p+p'}_{X/S})$.

0FM5 Lemma 50.5.1. Let $p : X \to S$ be a morphism of schemes. The cup product on $H^*_{Hodge}(X/S)$ is associative and graded commutative.

Proof. The proof is identical to the proof of Lemma 50.4.1. $\square$

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, there are pullback maps $f^* : H^i_{Hodge}(X/S) \longrightarrow H^i_{Hodge}(X'/S')$ compatible with gradings and with the cup product defined above.

50.6. Two spectral sequences

0FM6 Let $p : X \to S$ be a morphism of schemes. Since the category of $p^{-1}\mathcal{O}_S$-modules on X has enough injectives there exist a Cartan-Eilenberg resolution for $\Omega^\bullet_{X/S}$. See Derived Categories, Lemma 13.21.2. Hence we can apply Derived Categories, Lemma 13.21.3 to get two spectral sequences both converging to the de Rham cohomology of X over S.

The first is customarily called the Hodge-to-de Rham spectral sequence. The first page of this spectral sequence has

$$E_1^{p,q} = H^q(X, \Omega^p_{X/S})$$

which are the Hodge cohomology groups of X/S (whence the name). The differential d_1 on this page is given by the maps $d_1^{p,q} : H^q(X, \Omega^p_{X/S}) \to H^q(X. \Omega^{p+1}_{X/S})$ induced by the differential $\text{d} : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$. Here is a picture

$$\begin{tikzcd} H^2(X, \mathcal{O}_X) \arrow[r] \arrow[ddrrr, dotted] \arrow[drr, dashed] & H^2(X, \Omega^1_{X/S}) \arrow[r] \arrow[drr, dashed] & H^2(X, \Omega^2_{X/S}) \arrow[r] & H^2(X, \Omega^3_{X/S}) \\ H^1(X, \mathcal{O}_X) \arrow[r] \arrow[drr, dashed] & H^1(X, \Omega^1_{X/S}) \arrow[r] \arrow[drr, dashed] & H^1(X, \Omega^2_{X/S}) \arrow[r] & H^1(X, \Omega^3_{X/S}) \\ H^0(X, \mathcal{O}_X) \arrow[r] & H^0(X, \Omega^1_{X/S}) \arrow[r] & H^0(X, \Omega^2_{X/S}) \arrow[r] & H^0(X, \Omega^3_{X/S}) \end{tikzcd}$$

where we have drawn striped arrows to indicate the source and target of the differentials on the E_2 page and a dotted arrow for a differential on the E_3 page. Looking in degree 0 we conclude that

$$H^0_{dR}(X/S) = \mathrm{Ker}(\mathrm{d} : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/S}))$$

Of course, this is also immediately clear from the fact that the de Rham complex starts in degree 0 with $\mathcal{O}_X \to \Omega^1_{X/S}$.

The second spectral sequence is usually called the conjugate spectral sequence. The second page of this spectral sequence has

$$E_2^{p,q} = H^p(X, H^q(\Omega^\bullet_{X/S})) = H^p(X, \mathcal{H}^q)$$

where $\mathcal{H}^q = H^q(\Omega^\bullet_{X/S})$ is the qth cohomology sheaf of the de Rham complex of X/S. The differentials on this page are given by $E_2^{p,q} \to E_2^{p+2,q-1}$. Here is a picture

$$\begin{tikzcd}
H^0(X, \mathcal{H}^2) \arrow[rrd] \arrow[rrrdd, dotted] & H^1(X, \mathcal{H}^2) \arrow[rrd] & H^2(X, \mathcal{H}^2) & H^3(X, \mathcal{H}^2) \\
H^0(X, \mathcal{H}^1) \arrow[rrd] & H^1(X, \mathcal{H}^1) \arrow[rrd] & H^2(X, \mathcal{H}^1) & H^3(X, \mathcal{H}^1) \\
H^0(X, \mathcal{H}^0) & H^1(X, \mathcal{H}^0) & H^2(X, \mathcal{H}^0) & H^3(X, \mathcal{H}^0)
\end{tikzcd}$$

Looking in degree 0 we conclude that

$$H^0_{dR}(X/S) = H^0(X, \mathcal{H}^0)$$

which is obvious if you think about it. In degree 1 we get an exact sequence

$$0 \to H^1(X, \mathcal{H}^0) \to H^1_{dR}(X/S) \to H^0(X, \mathcal{H}^1) \to H^2(X, \mathcal{H}^0) \to H^2_{dR}(X/S)$$

It turns out that if $X \to S$ is smooth and S lives in characteristic p, then the sheaves $\mathcal{H}^q$ are computable (in terms of a certain sheaves of differentials) and the conjugate spectral sequence is a valuable tool (insert future reference here).

50.7. The Hodge filtration

0FM7 Let $X \to S$ be a morphism of schemes. The Hodge filtration on $H^n_{dR}(X/S)$ is the filtration induced by the Hodge-to-de Rham spectral sequence (Homology, Definition 12.24.5). To avoid misunderstanding, we explicitly define it as follows.

0FM8 **Definition 50.7.1.** Let $X \to S$ be a morphism of schemes. The Hodge filtration on $H^n_{dR}(X/S)$ is the filtration with terms

$$F^p H^n_{dR}(X/S) = \mathrm{Im}\left(H^n(X, \sigma_{\geq p}\Omega^\bullet_{X/S}) \longrightarrow H^n_{dR}(X/S)\right)$$

where $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is as in Homology, Section 12.15.

Of course $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is a subcomplex of the relative de Rham complex and we obtain a filtration

$$\Omega^\bullet_{X/S} = \sigma_{\geq 0}\Omega^\bullet_{X/S} \supset \sigma_{\geq 1}\Omega^\bullet_{X/S} \supset \sigma_{\geq 2}\Omega^\bullet_{X/S} \supset \sigma_{\geq 3}\Omega^\bullet_{X/S} \supset \ldots$$

of the relative de Rham complex with $\mathrm{gr}^p(\Omega^\bullet_{X/S}) = \Omega^p_{X/S}[-p]$. The spectral sequence constructed in Cohomology, Lemma 20.29.1 for $\Omega^\bullet_{X/S}$ viewed as a filtered complex of sheaves is the same as the Hodge-to-de Rham spectral sequence constructed in Section 50.6 by Cohomology, Example 20.29.4. Further the wedge product (50.4.0.1) sends $\mathrm{Tot}(\sigma_{\geq i}\Omega^\bullet_{X/S} \otimes_{p^{-1}\mathcal{O}_S} \sigma_{\geq j}\Omega^\bullet_{X/S})$ into $\sigma_{\geq i+j}\Omega^\bullet_{X/S}$. Hence we get commutative diagrams

$$\begin{tikzcd} H^n(X, \sigma_{\geq j}\Omega^\bullet_{X/S})) \times H^m(X, \sigma_{\geq j}\Omega^\bullet_{X/S})) \arrow[r] \arrow[d] & H^{n+m}(X, \sigma_{\geq i+j}\Omega^\bullet_{X/S})) \arrow[d] \\ H^n_{dR}(X/S) \times H^m_{dR}(X/S) \arrow[r, "\cup"] & H^{n+m}_{dR}(X/S) \end{tikzcd}$$

In particular we find that

$$F^iH^n_{dR}(X/S) \cup F^jH^m_{dR}(X/S) \subset F^{i+j}H^{n+m}_{dR}(X/S)$$

50.8. Künneth formula

0FM9 An important feature of de Rham cohomology is that there is a Künneth formula.

Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms and $f = a \circ p = b \circ q$. Here is a picture

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p"'] \arrow[dd, "f"] \arrow[dr, "q"] & \\ X \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\ & S & \end{tikzcd}$$

In this section, given an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$$

The bifunctor $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}$ on quasi-coherent modules extends to a bifunctor on quasi-coherent modules and differential operators of finite over over S, see Morphisms, Remark 29.33.3. Since the differentials of the de Rham complexes $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Y/S}$ are differential operators of order 1 over S by Modules, Lemma 17.30.5. Thus it makes sense to consider the complex

$$\mathrm{Tot}(\Omega^\bullet_{X/S} \boxtimes \Omega^\bullet_{Y/S})$$

Please see the discussion in Derived Categories of Schemes, Section 36.24.

0FMA **Lemma 50.8.1.** In the situation above there is a canonical isomorphism

$$\mathrm{Tot}(\Omega^\bullet_{X/S} \boxtimes \Omega^\bullet_{Y/S}) \longrightarrow \Omega^\bullet_{X \times_S Y/S}$$

of complexes of $f^{-1}\mathcal{O}_S$-modules.

Proof. We know that $\Omega_{X \times_S Y/S} = p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S}$ by Morphisms, Lemma 29.32.11. Taking exterior powers we obtain

$$\Omega^n_{X \times_S Y/S} = \bigoplus\nolimits_{i+j=n} p^*\Omega^i_{X/S} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\Omega^j_{Y/S} = \bigoplus\nolimits_{i+j=n} \Omega^i_{X/S} \boxtimes \Omega^j_{Y/S}$$

by elementary properties of exterior powers. These identifications determine isomorphisms between the terms of the complexes on the left and the right of the arrow in the lemma. We omit the verification that these maps are compatible with differentials. □

Set $A = \Gamma(S, \mathcal{O}_S)$. Combining the result of Lemma 50.8.1 with the map Derived Categories of Schemes, Equation (36.24.0.2) we obtain a cup product

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \Omega^\bullet_{Y/S}) \longrightarrow R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

On the level of cohomology, using the discussion in More on Algebra, Section 15.63, we obtain a canonical map

$$H^i_{dR}(X/S) \otimes_A H^j_{dR}(Y/S) \longrightarrow H^{i+j}_{dR}(X \times_S Y/S), \quad (\xi, \zeta) \longmapsto p^*\xi \cup q^*\zeta$$

We note that the construction above indeed proceeds by first pulling back and then taking the cup product.

0FMB **Lemma 50.8.2.** Assume X and Y are smooth, quasi-compact, with affine diagonal over $S = \mathrm{Spec}(A)$. Then the map

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \Omega^\bullet_{Y/S}) \longrightarrow R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

is an isomorphism in $D(A)$.

Proof. By Morphisms, Lemma 29.34.12 the sheaves $\Omega^n_{X/S}$ and $\Omega^m_{Y/S}$ are finite locally free $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules. On the other hand, X and Y are flat over S (Morphisms, Lemma 29.34.9) and hence we find that $\Omega^n_{X/S}$ and $\Omega^m_{Y/S}$ are flat over S. Also, observe that $\Omega^\bullet_{X/S}$ is a locally bounded. Thus the result by Lemma 50.8.1 and Derived Categories of Schemes, Lemma 36.24.1. □

There is a relative version of the cup product, namely a map

$$Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \longrightarrow Rf_*\Omega^\bullet_{X \times_S Y/S}$$

in $D(\mathcal{O}_S)$. The construction combines Lemma 50.8.1 with the map Derived Categories of Schemes, Equation (36.24.0.1). The construction shows that this map is given by the diagram

$$\begin{tikzcd}
Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \arrow[d, "\text{units of adjunction}"] & \\
Rf_*(p^{-1}\Omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(q^{-1}\Omega^\bullet_{Y/S}) \arrow[r] \arrow[d, "\text{relative cup product}"] & Rf_*(\Omega^\bullet_{X \times_S Y/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(\Omega^\bullet_{X \times_S Y/S}) \arrow[d, "\text{relative cup product}"] \\
Rf_*(p^{-1}\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) \arrow[r] \arrow[d, "\text{from derived to usual}"] & Rf_*(\Omega^\bullet_{X \times_S Y/S} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} \Omega^\bullet_{X \times_S Y/S}) \arrow[d, "\text{from derived to usual}"] \\
Rf_*\mathrm{Tot}(p^{-1}\Omega^\bullet_{X/S} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) \arrow[r] \arrow[d, "\text{canonical map}"] & Rf_*\mathrm{Tot}(\Omega^\bullet_{X \times_S Y/S} \otimes_{f^{-1}\mathcal{O}_S} \Omega^\bullet_{X \times_S Y/S}) \arrow[d, "\eta\otimes\omega\mapsto\eta\wedge\omega"] \\
Rf_*\mathrm{Tot}(\Omega^\bullet_{X/S} \boxtimes \Omega^\bullet_{Y/S}) \arrow[r, equal] & Rf_*\Omega^\bullet_{X \times_S Y/S}
\end{tikzcd}$$

Here the first arrow uses the units $\mathrm{id} \to Rp_*p^{-1}$ and $\mathrm{id} \to Rq_*q^{-1}$ of adjunction as well as the identifications $Rf_*p^{-1} = Ra_*Rp_*p^{-1}$ and $Rf_*q^{-1} = Rb_*Rq_*q^{-1}$. The second arrow is the relative cup product of Cohomology, Remark 20.28.7. The third arrow is the map sending a derived tensor product of complexes to the totalization of the tensor product of complexes. The final equality is Lemma 50.8.1. This construction recovers on global section the construction given earlier.

0FMC Lemma 50.8.3. Assume $X \to S$ and $Y \to S$ are smooth and quasi-compact and the morphisms $X \to X \times_S X$ and $Y \to Y \times_S Y$ are affine. Then the relative cup product

$$Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \longrightarrow Rf_*\Omega^\bullet_{X \times_S Y/S}$$

is an isomorphism in $D(\mathcal{O}_S)$.

Proof. Immediate consequence of Lemma 50.8.2. □

50.9. First Chern class in de Rham cohomology

0FLE Let $X \to S$ be a morphism of schemes. There is a map of complexes

$$\mathrm{d}\log : \mathcal{O}_X^*[-1] \longrightarrow \Omega^\bullet_{X/S}$$

which sends the section $g \in \mathcal{O}_X^*(U)$ to the section $\mathrm{d}\log(g) = g^{-1}\mathrm{d}g$ of $\Omega^1_{X/S}(U)$. Thus we can consider the map

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) = H^2(X, \mathcal{O}_X^*[-1]) \longrightarrow H^2_{dR}(X/S)$$

where the first equality is Cohomology, Lemma 20.6.1. The image of the isomorphism class of the invertible module $\mathcal{L}$ is denoted $c_1^{dR}(\mathcal{L}) \in H^2_{dR}(X/S)$.

We can also use the map $\mathrm{d}\log : \mathcal{O}_X^* \to \Omega^1_{X/S}$ to define a Chern class in Hodge cohomology

$$c_1^{Hodge} : \mathrm{Pic}(X) \longrightarrow H^1(X, \Omega^1_{X/S}) \subset H^2_{Hodge}(X/S)$$

These constructions are compatible with pullbacks.

0FMD Lemma 50.9.1. Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes the diagrams

$$\begin{tikzcd} \mathrm{Pic}(X') \arrow[d, "c_1^{dR}"'] & \mathrm{Pic}(X) \arrow[l, "f^*"'] \arrow[d, "c_1^{dR}"] \\ H^2_{dR}(X'/S') & H^2_{dR}(X/S) \arrow[l, "f^*"'] \end{tikzcd} \quad \begin{tikzcd} \mathrm{Pic}(X') \arrow[d, "c_1^{Hodge}"'] & \mathrm{Pic}(X) \arrow[l, "f^*"'] \arrow[d, "c_1^{Hodge}"] \\ H^1(X', \Omega^1_{X'/S'}) & H^1(X, \Omega^1_{X/S}) \arrow[l, "f^*"'] \end{tikzcd}$$

commute.

Proof. Omitted. □

Let us "compute" the element $c_1^{dR}(\mathcal{L})$ in Čech cohomology (with sign rules for Čech differentials as in Cohomology, Section 20.25). Namely, choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that we have a trivializing section s_i of $\mathcal{L}|_{U_i}$ for all i.

On the overlaps $U_{i_0i_1} = U_{i_0} \cap U_{i_1}$ we have an invertible function $f_{i_0i_1}$ such that $f_{i_0i_1} = s_{i_1}|_{U_{i_0i_1}} s_{i_0}|_{U_{i_0i_1}}^{-1}$ [1]. Of course we have

$$f_{i_1i_2}|_{U_{i_0i_1i_2}} f_{i_0i_2}^{-1}|_{U_{i_0i_1i_2}} f_{i_0i_1}|_{U_{i_0i_1i_2}} = 1$$

The cohomology class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$ is the image of the Čech cohomology class of the cocycle $\{f_{i_0i_1}\}$ in $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{O}_X^*)$. Therefore we see that $c_1^{dR}(\mathcal{L})$ is the image of the cohomology class associated to the Čech cocycle $\{\alpha_{i_0\ldots i_p}\}$ in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}))$ of degree 2 given by

(1) $\alpha_{i_0} = 0$ in $\Omega^2_{X/S}(U_{i_0})$,
(2) $\alpha_{i_0i_1} = f_{i_0i_1}^{-1}\text{d}f_{i_0i_1}$ in $\Omega^1_{X/S}(U_{i_0i_1})$, and
(3) $\alpha_{i_0i_1i_2} = 0$ in $\mathcal{O}_{X/S}(U_{i_0i_1i_2})$.

Suppose we have invertible modules $\mathcal{L}_k$, $k = 1, \ldots, a$ each trivialized over U_i for all $i \in I$ giving rise to cocycles f_{k,i_0i_1} and $\alpha_k = \{\alpha_{k,i_0\ldots i_p}\}$ as above. Using the rule in Cohomology, Section 20.25 we can compute

$$\beta = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_a$$

to be given by the cocycle $\beta = \{\beta_{i_0\ldots i_p}\}$ described as follows

(1) $\beta_{i_0\ldots i_p} = 0$ in $\Omega^{2a-p}_{X/S}(U_{i_0\ldots i_p})$ unless $p = a$, and
(2) $\beta_{i_0\ldots i_a} = (-1)^{a(a-1)/2}\alpha_{1,i_0i_1} \wedge \alpha_{2,i_1i_2} \wedge \ldots \wedge \alpha_{a,i_{a-1}i_a}$ in $\Omega^a_{X/S}(U_{i_0\ldots i_a})$.

Thus this is a cocycle representing $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ Of course, the same computation shows that the cocycle $\{\beta_{i_0\ldots i_a}\}$ in $\check{\mathcal{C}}^a(\mathcal{U}, \Omega^a_{X/S}))$ represents the cohomology class $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_a)$

0FME Remark 50.9.2. Here is a reformulation of the calculations above in more abstract terms. Let $p : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If we view $\text{d}\log$ as a map

$$\mathcal{O}_X^*[-1] \to \sigma_{\geq 1}\Omega^\bullet_{X/S}$$

then using $\text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ as above we find a cohomology class

$$\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1}\Omega^\bullet_{X/S})$$

The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$ recovers $c_1^{dR}(\mathcal{L})$. In particular we see that $c_1^{dR}(\mathcal{L}) \in F^1H^2_{dR}(X/S)$, see Section 50.7. The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1}\Omega^\bullet_{X/S} \to \Omega^1_{X/S}[-1]$ recovers $c_1^{Hodge}(\mathcal{L})$. Taking the cup product (see Section 50.7) we obtain

$$\xi = \gamma_1(\mathcal{L}_1) \cup \ldots \cup \gamma_1(\mathcal{L}_a) \in H^{2a}(X, \sigma_{\geq a}\Omega^\bullet_{X/S})$$

The commutative diagrams in Section 50.7 show that ξ is mapped to $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ in $H^{2a}_{dR}(X/S)$ by the map $\sigma_{\geq a}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$. Also, it follows $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ is contained in $F^aH^{2a}_{dR}(X/S)$. Similarly, the map $\sigma_{\geq a}\Omega^\bullet_{X/S} \to \Omega^a_{X/S}[-a]$ sends ξ to $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_a)$ in $H^a(X, \Omega^a_{X/S})$.

[1] The Čech differential of a 0-cycle $\{a_{i_0}\}$ has $a_{i_1} - a_{i_0}$ over $U_{i_0i_1}$.

0FMF Remark 50.9.3. Let $p : X \to S$ be a morphism of schemes. For $i > 0$ denote $\Omega^i_{X/S,log} \subset \Omega^i_{X/S}$ the abelian subsheaf generated by local sections of the form

$$\mathrm{d}\log(u_1) \wedge \ldots \wedge \mathrm{d}\log(u_i)$$

where $u_1, \ldots, u_n$ are invertible local sections of $\mathcal{O}_X$. For $i = 0$ the subsheaf $\Omega^0_{X/S,log} \subset \mathcal{O}_X$ is the image of $\mathbf{Z} \to \mathcal{O}_X$. For every $i \geq 0$ we have a map of complexes

$$\Omega^i_{X/S,log}[-i] \longrightarrow \Omega^\bullet_{X/S}$$

because the derivative of a logarithmic form is zero. Moreover, wedging logarithmic forms gives another, hence we find bilinear maps

$$\wedge : \Omega^i_{X/S,log} \times \Omega^j_{X/S,log} \longrightarrow \Omega^{i+j}_{X/S,log}$$

compatible with (50.4.0.1) and the maps above. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Using the map of abelian sheaves $\mathrm{d}\log : \mathcal{O}_X^* \to \Omega^1_{X/S,log}$ and the identification $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ we find a canonical cohomology class

$$\tilde{\gamma}_1(\mathcal{L}) \in H^1(X, \Omega^1_{X/S,log})$$

These classes have the following properties

1. the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log}[-1] \to \sigma_{\geq 1}\Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to the class $\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1}\Omega^\bullet_{X/S})$ of Remark 50.9.2,
2. the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log}[-1] \to \Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{dR}(\mathcal{L})$ in $H^2_{dR}(X/S)$,
3. the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log} \to \Omega^1_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{Hodge}(\mathcal{L})$ in $H^1(X, \Omega^1_{X/S})$,
4. the construction of these classes is compatible with pullbacks,
5. add more here.

50.10. de Rham cohomology of a line bundle

0FU8 A line bundle is a special case of a vector bundle, which in turn is a cone endowed with some extra structure. To intelligently talk about the de Rham complex of these, it makes sense to discuss the de Rham complex of a graded ring.

0FU9 Remark 50.10.1 (de Rham complex of a graded ring). Let G be an abelian monoid written additively with neutral element 0. Let $R \to A$ be a ring map and assume A comes with a grading $A = \bigoplus_{g \in G} A_g$ by R-modules such that R maps into A_0 and $A_g \cdot A_{g'} \subset A_{g+g'}$. Then the module of differentials comes with a grading

$$\Omega_{A/R} = \bigoplus\nolimits_{g \in G} \Omega_{A/R,g}$$

where $\Omega_{A/R,g}$ is the R-submodule of $\Omega_{A/R}$ generated by $a_0 \mathrm{d}a_1$ with $a_i \in A_{g_i}$ such that $g = g_0 + g_1$. Similarly, we obtain

$$\Omega^p_{A/R} = \bigoplus\nolimits_{g \in G} \Omega^p_{A/R,g}$$

where $\Omega^p_{A/R,g}$ is the R-submodule of $\Omega^p_{A/R}$ generated by $a_0 \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_p$ with $a_i \in A_{g_i}$ such that $g = g_0 + g_1 + \ldots + g_p$. Of course the differentials preserve the grading and the wedge product is compatible with the gradings in the obvious manner.

Let $f : X \to S$ be a morphism of schemes. Let $\pi : C \to X$ be a cone, see Constructions, Definition 27.7.2. Recall that this means π is affine and we have a grading $\pi_*\mathcal{O}_C = \bigoplus_{n \geq 0} \mathcal{A}_n$ with $\mathcal{A}_0 = \mathcal{O}_X$. Using the discussion in Remark 50.10.1 over affine opens we find that[2]

$$\pi_*(\Omega^\bullet_{C/S}) = \bigoplus\nolimits_{n \geq 0} \Omega^\bullet_{C/S,n}$$

is canonically a direct sum of subcomplexes. Moreover, we have a factorization

$$\Omega^\bullet_{X/S} \to \Omega^\bullet_{C/S,0} \to \pi_*(\Omega^\bullet_{C/S})$$

and we know that $\omega \wedge \eta \in \Omega^{p+q}_{C/S,n+m}$ if $\omega \in \Omega^p_{C/S,n}$ and $\eta \in \Omega^q_{C/S,m}$.

Let $f : X \to S$ be a morphism of schemes. Let $\pi : L \to X$ be the line bundle associated to the invertible $\mathcal{O}_X$-module $\mathcal{L}$. This means that π is the unique affine morphism such that

$$\pi_*\mathcal{O}_L = \bigoplus\nolimits_{n \geq 0} \mathcal{L}^{\otimes n}$$

as $\mathcal{O}_X$-algebras. Thus L is a cone over X. By the discussion above we find a canonical direct sum decomposition

$$\pi_*(\Omega^\bullet_{L/S}) = \bigoplus\nolimits_{n \geq 0} \Omega^\bullet_{L/S,n}$$

compatible with wedge product, compatible with the decomposition of $\pi_*\mathcal{O}_L$ above, and such that $\Omega_{X/S}$ maps into the part $\Omega_{L/S,0}$ of degree 0.

There is another case which will be useful to us. Namely, consider the complement[3] $L^\star \subset L$ of the zero section $o : X \to L$ in our line bundle L. A local computation shows we have a canonical isomorphism

$$(L^\star \to X)_*\mathcal{O}_{L^\star} = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$$

of $\mathcal{O}_X$-algebras. The right hand side is a $\mathbf{Z}$-graded quasi-coherent $\mathcal{O}_X$-algebra. Using the discussion in Remark 50.10.1 over affine opens we find that

$$(L^\star \to X)_*(\Omega^\bullet_{L^\star/S}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Omega^\bullet_{L^\star/S,n}$$

compatible with wedge product, compatible with the decomposition of $(L^\star \to X)_*\mathcal{O}_{L^\star}$ above, and such that $\Omega_{X/S}$ maps into the part $\Omega_{L^\star/S,0}$ of degree 0. The complex $\Omega^\bullet_{L^\star/S,0}$ will be of particular interest to us.

0FUF Lemma 50.10.2. With notation as above, there is a short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0} \to \Omega^\bullet_{X/S}[-1] \to 0$$

Proof. We have constructed the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0}$ above.

Construction of $\text{Res} : \Omega^\bullet_{L^\star/S,0} \to \Omega^\bullet_{X/S}[-1]$. Let $U \subset X$ be an open and let $s \in \mathcal{L}(U)$ and $s' \in \mathcal{L}^{\otimes -1}(U)$ be sections such that $s's = 1$. Then s gives an invertible section of the sheaf of algebras $(L^\star \to X)_*\mathcal{O}_{L^\star}$ over U with inverse $s' = s^{-1}$. Then we can consider the 1-form $\text{d}\log(s) = s'\text{d}(s)$ which is an element of $\Omega^1_{L^\star/S,0}(U)$ by our construction of the grading on $\Omega^1_{L^\star/S}$. Our computations on affines given below will

[2]With excuses for the notation!

[3]The scheme $L^\star$ is the $\mathbf{G}_m$-torsor over X associated to L. This is why the grading we get below is a $\mathbf{Z}$-grading, compare with Groupoids, Example 39.12.3 and Lemmas 39.12.4 and 39.12.5.

show that 1 and $\mathrm{d}\log(s)$ freely generate $\Omega^\bullet_{L^\star/S,0}|_U$ as a right module over $\Omega^\bullet_{X/S}|_U$. Thus we can define Res over U by the rule

$$\mathrm{Res}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \omega$$

for all $\omega', \omega \in \Omega^\bullet_{X/S}(U)$. This map is independent of the choice of local generator s and hence glues to give a global map. Namely, another choice of s would be of the form gs for some invertible $g \in \mathcal{O}_X(U)$ and we would get $\mathrm{d}\log(gs) = g^{-1}\mathrm{d}(g) + \mathrm{d}\log(s)$ from which the independence easily follows. Finally, observe that our rule for Res is compatible with differentials as $\mathrm{d}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \mathrm{d}(\omega') - \mathrm{d}\log(s) \wedge \mathrm{d}(\omega)$ and because the differential on $\Omega^\bullet_{X/S}[-1]$ sends ω' to $-\mathrm{d}(\omega')$ by our sign convention in Homology, Definition 12.14.7.

Local computation. We can cover X by affine opens $U \subset X$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$ which moreover map into an affine open $V \subset S$. Write $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$ and choose a generator s of $\mathcal{L}$. We find that we have

$$L^\star \times_X U = \mathrm{Spec}(A[s, s^{-1}])$$

Computing differentials we see that

$$\Omega^1_{A[s,s^{-1}]/R} = A[s, s^{-1}] \otimes_A \Omega^1_{A/R} \oplus A[s, s^{-1}]\mathrm{d}\log(s)$$

and therefore taking exterior powers we obtain

$$\Omega^p_{A[s,s^{-1}]/R} = A[s, s^{-1}] \otimes_A \Omega^p_{A/R} \oplus A[s, s^{-1}]\mathrm{d}\log(s) \otimes_A \Omega^{p-1}_{A/R}$$

Taking degree 0 parts we find

$$\Omega^p_{A[s,s^{-1}]/R,0} = \Omega^p_{A/R} \oplus \mathrm{d}\log(s) \otimes_A \Omega^{p-1}_{A/R}$$

and the proof of the lemma is complete. ☐

0FUG Lemma 50.10.3. The "boundary" map $\delta : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$ in $D(X, f^{-1}\mathcal{O}_S)$ coming from the short exact sequence in Lemma 50.10.2 is the map of Remark 50.4.3 for $\xi = c_1^{dR}(\mathcal{L})$.

Proof. To be precise we consider the shift

$$0 \to \Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{L^\star/S,0}[1] \to \Omega^\bullet_{X/S} \to 0$$

of the short exact sequence of Lemma 50.10.2. As the degree zero part of a grading on $(L^\star \to X)_*\Omega^\bullet_{L^\star/S}$ we see that $\Omega^\bullet_{L^\star/S,0}$ is a differential graded $\mathcal{O}_X$-algebra and that the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0}$ is a homomorphism of differential graded $\mathcal{O}_X$-algebras. Hence we may view $\Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{L^\star/S,0}[1]$ as a map of right differential graded $\Omega^\bullet_{X/S}$-modules on X. The map $\mathrm{Res} : \Omega^\bullet_{L^\star/S,0}[1] \to \Omega^\bullet_{X/S}$ is a map of right differential graded $\Omega^\bullet_{X/S}$-modules since it is locally defined by the rule $\mathrm{Res}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \omega$, see proof of Lemma 50.10.2. Thus by the discussion in Differential Graded Sheaves, Section 24.32 we see that δ comes from a map $\delta' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$ in the derived category $D(\Omega^\bullet_{X/S}, \mathrm{d})$ of right differential graded modules over the de Rham complex. The uniqueness averted in Remark 50.4.3 shows it suffices to prove that $\delta(1) = c_1^{dR}(\mathcal{L})$.

We claim that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d, "\mathrm{d}\log"'] & E \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] & \Omega^\bullet_{L^\star/S,0}[1] \arrow[r] & \Omega^\bullet_{X/S} \arrow[r] & 0 \end{tikzcd}$$

where the top row is a short exact sequence of abelian sheaves whose boundary map sends 1 to the class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$. It suffices to prove the claim by the compatibility of boundary maps with maps between short exact sequences. We define E as the sheafification of the rule

$$U \longmapsto \{(s, n) \mid n \in \mathbf{Z},\ s \in \mathcal{L}^{\otimes n}(U) \text{ generator}\}$$

with group structure given by $(s, n) \cdot (t, m) = (s \otimes t, n + m)$. The middle vertical map sends (s, n) to $\mathrm{d}\log(s)$. This produces a map of short exact sequences because the map $Res : \Omega^1_{L^\star/S,0} \to \mathcal{O}_X$ constructed in the proof of Lemma 50.10.2 sends $\mathrm{d}\log(s)$ to 1 if s is a local generator of $\mathcal{L}$. To calculate the boundary of 1 in the top row, choose local trivializations s_i of $\mathcal{L}$ over opens U_i as in Section 50.9. On the overlaps $U_{i_0i_1} = U_{i_0} \cap U_{i_1}$ we have an invertible function $f_{i_0i_1}$ such that $f_{i_0i_1} = s_{i_1}|_{U_{i_0i_1}} s_{i_0}|_{U_{i_0i_1}}^{-1}$ and the cohomology class of $\mathcal{L}$ is given by the Čech cocycle $\{f_{i_0i_1}\}$. Then of course we have

$$(f_{i_0i_1}, 0) = (s_{i_1}, 1)|_{U_{i_0i_1}} \cdot (s_{i_0}, 1)|_{U_{i_0i_1}}^{-1}$$

as sections of E which finishes the proof. □

0FUH Lemma 50.10.4. With notation as above we have

(1) $\Omega^p_{L^\star/S,n} = \Omega^p_{L^\star/S,0} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for all $n \in \mathbf{Z}$ as quasi-coherent $\mathcal{O}_X$-modules,
(2) $\Omega^\bullet_{X/S} = \Omega^\bullet_{L/X,0}$ as complexes, and
(3) for $n > 0$ and $p \geq 0$ we have $\Omega^p_{L/X,n} = \Omega^p_{L^\star/S,n}$.

Proof. In each case there is a globally defined canonical map which is an isomorphism by local calculations which we omit. □

0FUI Lemma 50.10.5. In the situation above, assume there is a morphism $S \to \mathrm{Spec}(\mathbf{Q})$. Then $\Omega^\bullet_{X/S} \to \pi_*\Omega^\bullet_{L/S}$ is a quasi-isomorphism and $H^*_{dR}(X/S) = H^*_{dR}(L/S)$.

Proof. Let R be a $\mathbf{Q}$-algebra. Let A be an R-algebra. The affine local statement is that the map

$$\Omega^\bullet_{A/R} \longrightarrow \Omega^\bullet_{A[t]/R}$$

is a quasi-isomorphism of complexes of R-modules. In fact it is a homotopy equivalence with homotopy inverse given by the map sending $g\omega + g'\mathrm{d}t \wedge \omega'$ to $g(0)\omega$ for $g, g' \in A[t]$ and $\omega, \omega' \in \Omega^\bullet_{A/R}$. The homotopy sends $g\omega + g'\mathrm{d}t \wedge \omega'$ to $(\int g')\omega'$ were $\int g' \in A[t]$ is the polynomial with vanishing constant term whose derivative with respect to t is g'. Of course, here we use that R contains $\mathbf{Q}$ as $\int t^n = (1/n)t^{n+1}$. □

0FUJ Example 50.10.6. Lemma 50.10.5 is false in positive characteristic. The de Rham complex of $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ over a field k looks like a direct sum

$$k \oplus \bigoplus\nolimits_{n \geq 1} (k \cdot t^n \xrightarrow{n} k \cdot t^{n-1}\mathrm{d}t)$$

Hence if the characteristic of k is $p > 0$, then we see that both $H^0_{dR}(\mathbf{A}^1_k/k)$ and $H^1_{dR}(\mathbf{A}^1_k/k)$ are infinite dimensional over k.

50.11. de Rham cohomology of projective space

0FMG Let A be a ring. Let $n \geq 1$. The structure morphism $\mathbf{P}^n_A \to \mathrm{Spec}(A)$ is a proper smooth of relative dimension n. It is smooth of relative dimension n and of finite type as $\mathbf{P}^n_A$ has a finite affine open covering by schemes each isomorphic to $\mathbf{A}^n_A$, see Constructions, Lemma 27.13.3. It is proper because it is also separated and universally closed by Constructions, Lemma 27.13.4. Let us denote $\mathcal{O}$ and $\mathcal{O}(d)$ the structure sheaf $\mathcal{O}_{\mathbf{P}^n_A}$ and the Serre twists $\mathcal{O}_{\mathbf{P}^n_A}(d)$. Let us denote $\Omega = \Omega_{\mathbf{P}^n_A/A}$ the sheaf of relative differentials and Ω^p its exterior powers.

0FMH **Lemma 50.11.1.** There exists a short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

Proof. To explain this, we recall that $\mathbf{P}^n_A = \mathrm{Proj}(A[T_0, \ldots, T_n])$, and we write symbolically

$$\mathcal{O}(-1)^{\oplus n+1} = \bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$$

The first arrow

$$\Omega \to \bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$$

in the short exact sequence above is given on each of the standard opens $D_+(T_i) = \mathrm{Spec}(A[T_0/T_i, \ldots, T_n/T_i])$ mentioned above by the rule

$$\sum\nolimits_{j \neq i} g_j \mathrm{d}(T_j/T_i) \longmapsto \sum\nolimits_{j \neq i} g_j/T_i \mathrm{d}T_j - (\sum\nolimits_{j \neq i} g_j T_j/T_i^2)\mathrm{d}T_i$$

This makes sense because $1/T_i$ is a section of $\mathcal{O}(-1)$ over $D_+(T_i)$. The map

$$\bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j \to \mathcal{O}$$

is given by sending $\mathrm{d}T_j$ to T_j, more precisely, on $D_+(T_i)$ we send the section $\sum g_j \mathrm{d}T_j$ to $\sum T_j g_j$. We omit the verification that this produces a short exact sequence. $\square$

Given an integer $k \in \mathbf{Z}$ and a quasi-coherent $\mathcal{O}_{\mathbf{P}^n_A}$-module $\mathcal{F}$ denote as usual $\mathcal{F}(k)$ the kth Serre twist of $\mathcal{F}$. See Constructions, Definition 27.10.1.

0FUK **Lemma 50.11.2.** In the situation above we have the following cohomology groups

(1) $H^q(\mathbf{P}^n_A, \Omega^p) = 0$ unless $0 \leq p = q \leq n$,
(2) for $0 \leq p \leq n$ the A-module $H^p(\mathbf{P}^n_A, \Omega^p)$ free of rank 1.
(3) for $q > 0$, $k > 0$, and p arbitrary we have $H^q(\mathbf{P}^n_A, \Omega^p(k)) = 0$, and
(4) add more here.

Proof. We are going to use the results of Cohomology of Schemes, Lemma 30.8.1 without further mention. In particular, the statements are true for $H^q(\mathbf{P}^n_A, \mathcal{O}(k))$.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 50.11.1. Since $\mathcal{O}(-1)$ has vanishing cohomology in all degrees, this gives that $H^q(\mathbf{P}^n_A, \Omega)$ is zero except in degree 1 where it is freely generated by the boundary of 1 in $H^0(\mathbf{P}^n_A, \mathcal{O})$.

Assume $p > 1$. Let us think of the short exact sequence above as defining a 2 step filtration on $\mathcal{O}(-1)^{\oplus n+1}$. The induced filtration on $\wedge^p \mathcal{O}(-1)^{\oplus n+1}$ looks like this

$$0 \to \Omega^p \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right) \to \Omega^{p-1} \to 0$$

Observe that $\wedge^p \mathcal{O}(-1)^{\oplus n+1}$ is isomorphic to a direct sum of $n+1$ choose p copies of $\mathcal{O}(-p)$ and hence has vanishing cohomology in all degrees. By induction hypothesis, this shows that $H^q(\mathbf{P}^n_A, \Omega^p)$ is zero unless $q = p$ and $H^p(\mathbf{P}^n_A, \Omega^p)$ is free of rank 1 with generator the boundary of the generator in $H^{p-1}(\mathbf{P}^n_A, \Omega^{p-1})$.

Let $k > 0$. Observe that $\Omega^n = \mathcal{O}(-n-1)$ for example by the short exact sequence above for $p = n + 1$. Hence $\Omega^n(k)$ has vanishing cohomology in positive degrees. Using the short exact sequences

$$0 \to \Omega^p(k) \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right)(k) \to \Omega^{p-1}(k) \to 0$$

and descending induction on p we get the vanishing of cohomology of $\Omega^p(k)$ in positive degrees for all p. □

0FMI **Lemma 50.11.3.** We have $H^q(\mathbf{P}^n_A, \Omega^p) = 0$ unless $0 \leq p = q \leq n$. For $0 \leq p \leq n$ the A-module $H^p(\mathbf{P}^n_A, \Omega^p)$ free of rank 1 with basis element $c_1^{Hodge}(\mathcal{O}(1))^p$.

Proof. We have the vanishing and and freeness by Lemma 50.11.2. For $p = 0$ it is certainly true that $1 \in H^0(\mathbf{P}^n_A, \mathcal{O})$ is a generator.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 50.11.1. In the proof of Lemma 50.11.2 we have seen that the generator of $H^1(\mathbf{P}^n_A, \Omega)$ is the boundary ξ of $1 \in H^0(\mathbf{P}^n_A, \mathcal{O})$. As in the proof of Lemma 50.11.1 we will identify $\mathcal{O}(-1)^{\oplus n+1}$ with $\bigoplus_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$. Consider the open covering

$$\mathcal{U} : \mathbf{P}^n_A = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

We can lift the restriction of the global section 1 of $\mathcal{O}$ to $U_i = D_+(T_i)$ by the section $T_i^{-1}\mathrm{d}T_i$ of $\bigoplus \mathcal{O}(-1)\mathrm{d}T_j$ over U_i. Thus the cocyle representing ξ is given by

$$T_{i_1}^{-1}\mathrm{d}T_{i_1} - T_{i_0}^{-1}\mathrm{d}T_{i_0} = \mathrm{d}\log(T_{i_1}/T_{i_0}) \in \Omega(U_{i_0 i_1})$$

On the other hand, for each i the section T_i is a trivializing section of $\mathcal{O}(1)$ over U_i. Hence we see that $f_{i_0 i_1} = T_{i_1}/T_{i_0} \in \mathcal{O}^*(U_{i_0 i_1})$ is the cocycle representing $\mathcal{O}(1)$ in $\mathrm{Pic}(\mathbf{P}^n_A)$, see Section 50.9. Hence $c_1^{Hodge}(\mathcal{O}(1))$ is given by the cocycle $\mathrm{d}\log(T_{i_1}/T_{i_0})$ which agrees with what we got for ξ above.

Proof for general p by induction. The base cases $p = 0, 1$ were handled above. Assume $p > 1$. In the proof of Lemma 50.11.2 we have seen that the generator of $H^p(\mathbf{P}^n_A, \Omega^p)$ is the boundary of $c_1^{Hodge}(\mathcal{O}(1))^{p-1}$ in the long exact cohomology sequence associated to

$$0 \to \Omega^p \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right) \to \Omega^{p-1} \to 0$$

By the calculation in Section 50.9 the cohomology class $c_1^{Hodge}(\mathcal{O}(1))^{p-1}$ is, up to a sign, represented by the cocycle with terms

$$\beta_{i_0 \ldots i_{p-1}} = \mathrm{d}\log(T_{i_1}/T_{i_0}) \wedge \mathrm{d}\log(T_{i_2}/T_{i_1}) \wedge \ldots \wedge \mathrm{d}\log(T_{i_{p-1}}/T_{i_{p-2}})$$

in $\Omega^{p-1}(U_{i_0\ldots i_{p-1}})$. These $\beta_{i_0\ldots i_{p-1}}$ can be lifted to the sections $\tilde{\beta}_{i_0\ldots i_{p-1}} = T_{i_0}^{-1}\mathrm{d}T_{i_0} \wedge \beta_{i_0\ldots i_{p-1}}$ of $\wedge^p(\bigoplus \mathcal{O}(-1)\mathrm{d}T_j)$ over $U_{i_0\ldots i_{p-1}}$. We conclude that the generator of $H^p(\mathbf{P}^n_A, \Omega^p)$ is given by the cocycle whose components are

$$\begin{aligned}\sum\nolimits_{a=0}^{p}(-1)^a\tilde{\beta}_{i_0\ldots\hat{i_a}\ldots i_p} &= T_{i_1}^{-1}\mathrm{d}T_{i_1} \wedge \beta_{i_1\ldots i_p} + \sum\nolimits_{a=1}^{p}(-1)^a T_{i_0}^{-1}\mathrm{d}T_{i_0} \wedge \beta_{i_0\ldots\hat{i_a}\ldots i_p} \\ &= (T_{i_1}^{-1}\mathrm{d}T_{i_1} - T_{i_0}^{-1}\mathrm{d}T_{i_0}) \wedge \beta_{i_1\ldots i_p} + T_{i_0}^{-1}\mathrm{d}T_{i_0} \wedge \mathrm{d}(\beta)_{i_0\ldots i_p} \\ &= \mathrm{d}\log(T_{i_1}/T_{i_0}) \wedge \beta_{i_1\ldots i_p}\end{aligned}$$

viewed as a section of Ω^p over $U_{i_0\ldots i_p}$. This is up to sign the same as the cocycle representing $c_1^{Hodge}(\mathcal{O}(1))^p$ and the proof is complete. □

0FMJ Lemma 50.11.4. For $0 \leq i \leq n$ the de Rham cohomology $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ is a free A-module of rank 1 with basis element $c_1^{dR}(\mathcal{O}(1))^i$. In all other degrees the de Rham cohomology of $\mathbf{P}^n_A$ over A is zero.

Proof. Consider the Hodge-to-de Rham spectral sequence of Section 50.6. By the computation of the Hodge cohomology of $\mathbf{P}^n_A$ over A done in Lemma 50.11.3 we see that the spectral sequence degenerates on the E_1 page. In this way we see that $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ is a free A-module of rank 1 for $0 \leq i \leq n$ and zero else. Observe that $c_1^{dR}(\mathcal{O}(1))^i \in H^{2i}_{dR}(\mathbf{P}^n_A/A)$ for $i = 0, \ldots, n$ and that for $i = n$ this element is the image of $c_1^{Hodge}(\mathcal{L})^n$ by the map of complexes

$$\Omega^n_{\mathbf{P}^n_A/A}[-n] \longrightarrow \Omega^\bullet_{\mathbf{P}^n_A/A}$$

This follows for example from the discussion in Remark 50.9.2 or from the explicit description of cocycles representing these classes in Section 50.9. The spectral sequence shows that the induced map

$$H^n(\mathbf{P}^n_A, \Omega^n_{\mathbf{P}^n_A/A}) \longrightarrow H^{2n}_{dR}(\mathbf{P}^n_A/A)$$

is an isomorphism and since $c_1^{Hodge}(\mathcal{L})^n$ is a generator of of the source (Lemma 50.11.3), we conclude that $c_1^{dR}(\mathcal{L})^n$ is a generator of the target. By the A-bilinearity of the cup products, it follows that also $c_1^{dR}(\mathcal{L})^i$ is a generator of $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ for $0 \leq i \leq n$. □

50.12. The spectral sequence for a smooth morphism

0FMK Consider a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where f is a smooth morphism. Then we obtain a locally split short exact sequence

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

by Morphisms, Lemma 29.34.16. Let us think of this as a descending filtration F on $\Omega_{X/S}$ with $F^0\Omega_{X/S} = \Omega_{X/S}$, $F^1\Omega_{X/S} = f^*\Omega_{Y/S}$, and $F^2\Omega_{X/S} = 0$. Applying the functor $\wedge^p$ we obtain for every p an induced filtration

$$\Omega^p_{X/S} = F^0\Omega^p_{X/S} \supset F^1\Omega^p_{X/S} \supset F^2\Omega^p_{X/S} \supset \ldots \supset F^{p+1}\Omega^p_{X/S} = 0$$

whose succesive quotients are

$$\mathrm{gr}^k\Omega^p_{X/S} = F^k\Omega^p_{X/S}/F^{k+1}\Omega^p_{X/S} = f^*\Omega^k_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{p-k}_{X/Y} = f^{-1}\Omega^k_{Y/S} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^{p-k}_{X/Y}$$

for $k = 0, \ldots, p$. In fact, the reader can check using the Leibniz rule that $F^k\Omega^\bullet_{X/S}$ is a subcomplex of $\Omega^\bullet_{X/S}$. In this way $\Omega^\bullet_{X/S}$ has the structure of a filtered complex. We can also see this by observing that

$$F^k\Omega^\bullet_{X/S} = \mathrm{Im}\left(\wedge : \mathrm{Tot}(f^{-1}\sigma_{\geq k}\Omega^\bullet_{Y/S} \otimes_{p^{-1}\mathcal{O}_S} \Omega^\bullet_{X/S}) \longrightarrow \Omega^\bullet_{X/S}\right)$$

is the image of a map of complexes on X. The filtered complex

$$\Omega^\bullet_{X/S} = F^0\Omega^\bullet_{X/S} \supset F^1\Omega^\bullet_{X/S} \supset F^2\Omega^\bullet_{X/S} \supset \ldots$$

has the following associated graded parts

$$\mathrm{gr}^k\Omega^\bullet_{X/S} = f^{-1}\Omega^k_{Y/S}[-k] \otimes_{f^{-1}\mathcal{O}_Y} \Omega^\bullet_{X/Y}$$

by what was said above.

0FMM Lemma 50.12.1. Let $f : X \to Y$ be a quasi-compact, quasi-separated, and smooth morphism of schemes over a base scheme S. There is a bounded spectral sequence with first page

$$E_1^{p,q} = H^q(\Omega^p_{Y/S} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\Omega^\bullet_{X/Y})$$

converging to $R^{p+q}f_*\Omega^\bullet_{X/S}$.

Proof. Consider $\Omega^\bullet_{X/S}$ as a filtered complex with the filtration introduced above. The spectral sequence is the spectral sequence of Cohomology, Lemma 20.29.5. By Derived Categories of Schemes, Lemma 36.23.2 we have

$$Rf_*\mathrm{gr}^k\Omega^\bullet_{X/S} = \Omega^k_{Y/S}[-k] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\Omega^\bullet_{X/Y}$$

and thus we conclude. $\square$

0FMN Remark 50.12.2. In Lemma 50.12.1 consider the cohomology sheaves

$$\mathcal{H}^q_{dR}(X/Y) = H^q(Rf_*\Omega^\bullet_{X/Y}))$$

If f is proper in addition to being smooth and S is a scheme over $\mathbf{Q}$ then $\mathcal{H}^q_{dR}(X/Y)$ is finite locally free (insert future reference here). If we only assume $\mathcal{H}^q_{dR}(X/Y)$ are flat $\mathcal{O}_Y$-modules, then we obtain (tiny argument omitted)

$$E_1^{p,q} = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$$

and the differentials in the spectral sequence are maps

$$d_1^{p,q} : \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \longrightarrow \Omega^{p+1}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$$

In particular, for $p = 0$ we obtain a map $d_1^{0,q} : \mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$ which turns out to be an integrable connection ∇ (insert future reference here) and the complex

$$\mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \to \Omega^2_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \to \ldots$$

with differentials given by $d_1^{\bullet,q}$ is the de Rham complex of ∇. The connection ∇ is known as the Gauss-Manin connection.

50.13. Leray-Hirsch type theorems

0FUL In this section we prove that for a smooth proper morphism one can sometimes express the de Rham cohomology upstairs in terms of the de Rham cohomology downstairs.

0FMP **Lemma 50.13.1.** Let $f : X \to Y$ be a smooth proper morphism of schemes. Let N and $n_1, \ldots, n_N \geq 0$ be integers and let $\xi_i \in H^{n_i}_{dR}(X/Y)$, $1 \leq i \leq N$. Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$. Then the map

$$\bigoplus\nolimits_{i=1}^N \mathcal{O}_Y[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/Y}$$

associated to $\xi_1, \ldots, \xi_N$ is an isomorphism.

Proof. By Lemma 50.3.5 $Rf_*\Omega^\bullet_{X/Y}$ is a perfect object of $D(\mathcal{O}_Y)$ whose formation commutes with arbitrary base change. Thus the map of the lemma is a map $a : K \to L$ between perfect objects of $D(\mathcal{O}_Y)$ whose derived restriction to any point is an isomorphism by our assumption on fibres. Then the cone C on a is a perfect object of $D(\mathcal{O}_Y)$ (Cohomology, Lemma 20.49.7) whose derived restriction to any point is zero. It follows that C is zero by More on Algebra, Lemma 15.75.6 and a is an isomorphism. (This also uses Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7 to translate into algebra.) □

We first prove the main result of this section in the following special case.

0FUM **Lemma 50.13.2.** Let $f : X \to Y$ be a smooth proper morphism of schemes over a base S. Assume

(1) Y and S are affine, and
(2) there exist integers N and $n_1, \ldots, n_N \geq 0$ and $\xi_i \in H^{n_i}_{dR}(X/S)$, $1 \leq i \leq N$ such that for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$.

Then the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y/S) \longrightarrow H^*_{dR}(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i \cup f^*a_i$$

is an isomorphism.

Proof. Say $Y = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. In this case $\Omega^\bullet_{A/R}$ computes $R\Gamma(Y, \Omega^\bullet_{Y/S})$ by Lemma 50.3.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Consider the complex

$$K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}))$$

as in Cohomology, Section 20.25. Let us collect some facts about this complex most of which can be found in the reference just given:

(1) $K^\bullet$ is a complex of R-modules whose terms are A-modules,
(2) $K^\bullet$ represents $R\Gamma(X, \Omega^\bullet_{X/S})$ in $D(R)$ (Cohomology of Schemes, Lemma 30.2.2 and Cohomology, Lemma 20.25.2),
(3) there is a natural map $\Omega^\bullet_{A/R} \to K^\bullet$ of complexes of R-modules which is A-linear on terms and induces the pullback map $H^*_{dR}(Y/S) \to H^*_{dR}(X/S)$ on cohomology,
(4) $K^\bullet$ has a multiplication denoted $\wedge$ which turns it into a differential graded R-algebra,

(5) the multiplication on $K^\bullet$ induces the cup product on $H^*_{dR}(X/S)$ (Cohomology, Section 20.31),
(6) the filtration F on $\Omega^*_{X/S}$ induces a filtration

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \ldots$$

by subcomplexes on $K^\bullet$ such that
  (a) $F^kK^n \subset K^n$ is an A-submmodule,
  (b) $F^kK^\bullet \wedge F^lK^\bullet \subset F^{k+l}K^\bullet$,
  (c) $\mathrm{gr}^kK^\bullet$ is a complex of A-modules,
  (d) $\mathrm{gr}^0K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/Y}))$ and represents $R\Gamma(X, \Omega^\bullet_{X/Y})$ in $D(A)$,
  (e) multiplication induces an isomorphism $\Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0K^\bullet \to \mathrm{gr}^kK^\bullet$

We omit the detailed proofs of these statements; please see discussion leading up to the construction of the spectral sequence in Lemma 50.12.1.

For every $i = 1, \ldots, N$ we choose a cocycle $x_i \in K^{n_i}$ representing ξ_i. Next, we look at the map of complexes

$$\tilde{x} : M^\bullet = \bigoplus\nolimits_{i=1,\ldots,N} \Omega^\bullet_{A/R}[-n_i] \longrightarrow K^\bullet$$

which sends ω in the ith summand to $x_i \wedge \omega$. All that remains is to show that this map is a quasi-isomorphism. We endow $M^\bullet$ with the structure of a filtered complex by the rule

$$F^kM^\bullet = \bigoplus\nolimits_{i=1,\ldots,N} (\sigma_{\geq k}\Omega^\bullet_{A/R})[-n_i]$$

With this choice the map $\tilde{x}$ is a morphism of filtered complexes. Observe that $\mathrm{gr}^0M^\bullet = \bigoplus A[-n_i]$ and multiplication induces an isomorphism $\Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0M^\bullet \to \mathrm{gr}^kM^\bullet$. By construction and Lemma 50.13.1 we see that

$$\mathrm{gr}^0\tilde{x} : \mathrm{gr}^0M^\bullet \longrightarrow \mathrm{gr}^0K^\bullet$$

is an isomorphism in $D(A)$. It follows that for all $k \geq 0$ we obtain isomorphisms

$$\mathrm{gr}^k\tilde{x} : \mathrm{gr}^kM^\bullet = \Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0M^\bullet \longrightarrow \Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0K^\bullet = \mathrm{gr}^kK^\bullet$$

in $D(A)$. Namely, the complex $\mathrm{gr}^0K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/Y}))$ is K-flat as a complex of A-modules by Derived Categories of Schemes, Lemma 36.23.3. Hence the tensor product on the right hand side is the derived tensor product as is true by inspection on the left hand side. Finally, taking the derived tensor product $\Omega^k_{A/R}[-k] \otimes^{\mathbf{L}}_A -$ is a functor on $D(A)$ and therefore sends isomorphisms to isomorphisms. Arguing by induction on k we deduce that

$$\tilde{x} : M^\bullet/F^kM^\bullet \to K^\bullet/F^kK^\bullet$$

is an isomorphism in $D(R)$ since we have the short exact sequences

$$0 \to F^kM^\bullet/F^{k+1}M^\bullet \to M^\bullet/F^{k+1}M^\bullet \to \mathrm{gr}^kM^\bullet \to 0$$

and similarly for $K^\bullet$. This proves that $\tilde{x}$ is a quasi-isomorphism as the filtrations are finite in any given degree. □

0FMR Proposition 50.13.3. Let $f : X \to Y$ be a smooth proper morphism of schemes over a base S. Let N and $n_1, \ldots, n_N \geq 0$ be integers and let $\xi_i \in H^{n_i}_{dR}(X/S)$, $1 \leq i \leq N$.

Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$. The map

$$\tilde\xi = \bigoplus \tilde\xi_i[-n_i] : \bigoplus \Omega^\bullet_{Y/S}[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

(see proof) is an isomorphism in $D(Y, (Y \to S)^{-1}\mathcal{O}_S)$ and correspondingly the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y/S) \longrightarrow H^*_{dR}(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i \cup f^*a_i$$

is an isomorphism.

Proof. Denote $p : X \to S$ and $q : Y \to S$ be the structure morphisms. Let $\xi_i' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[n_i]$ be the map of Remark 50.4.3 corresponding to ξ_i. Denote

$$\tilde\xi_i : \Omega^\bullet_{Y/S} \to Rf_*\Omega^\bullet_{X/S}[n_i]$$

the composition of ξ_i' with the canonical map $\Omega^\bullet_{Y/S} \to Rf_*\Omega^\bullet_{X/S}$. Using

$$R\Gamma(Y, Rf_*\Omega^\bullet_{X/S}) = R\Gamma(X, \Omega^\bullet_{X/S})$$

on cohomology $\tilde\xi_i$ is the map $\eta \mapsto \xi_i \cup f^*\eta$ from $H^m_{dR}(Y/S)$ to $H^{m+n}_{dR}(X/S)$. Further, since the formation of ξ_i' commutes with restrictions to opens, so does the formation of $\tilde\xi_i$ commute with restriction to opens.

Thus we can consider the map

$$\tilde\xi = \bigoplus \tilde\xi_i[-n_i] : \bigoplus \Omega^\bullet_{Y/S}[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

To prove the lemma it suffices to show that this is an isomorphism in $D(Y, q^{-1}\mathcal{O}_S)$. If we could show $\tilde\xi$ comes from a map of filtered complexes (with suitable filtrations), then we could appeal to the spectral sequence of Lemma 50.12.1 to finish the proof. This takes more work than is necessary and instead our approach will be to reduce to the affine case (whose proof does in some sense use the spectral sequence).

Indeed, if $Y' \subset Y$ is is any open with inverse image $X' \subset X$, then $\tilde\xi|_{X'}$ induces the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y'/S) \longrightarrow H^*_{dR}(X'/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i|_{X'} \cup f^*a_i$$

on cohomology over Y', see discussion above. Thus it suffices to find a basis for the topology on Y such that the proposition holds for the members of the basis (in particular we can forget about the map $\tilde\xi$ when we do this). This reduces us to the case where Y and S are affine which is handled by Lemma 50.13.2 and the proof is complete. □

50.14. Projective space bundle formula

0FMS The title says it all.

0FMT Proposition 50.14.1. Let $X \to S$ be a morphism of schemes. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank r. Consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$. Then the map

$$\bigoplus\nolimits_{i=0,\ldots,r-1} H^*_{dR}(X/S) \longrightarrow H^*_{dR}(P/S)$$

given by the rule

$$(a_0, \ldots, a_{r-1}) \longmapsto \sum\nolimits_{i=0,\ldots,r-1} c_1^{dR}(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

is an isomorphism.

Proof. Choose an affine open $\text{Spec}(A) \subset X$ such that $\mathcal{E}$ restricts to the trivial locally free module $\mathcal{O}_{\text{Spec}(A)}^{\oplus r}$. Then $P \times_X \text{Spec}(A) = \mathbf{P}^{r-1}_A$. Thus we see that p is proper and smooth, see Section 50.11. Moreover, the classes $c_1^{dR}(\mathcal{O}_P(1))^i$, $i = 0, 1, \ldots, r-1$ restricted to a fibre $X_y = \mathbf{P}^{r-1}_y$ freely generate the de Rham cohomology $H^*_{dR}(X_y/y)$ over $\kappa(y)$, see Lemma 50.11.4. Thus we've verified the conditions of Proposition 50.13.3 and we win. $\square$

0FUN Remark 50.14.2. In the situation of Proposition 50.14.1 we get moreover that the map

$$\tilde{\xi} : \bigoplus\nolimits_{t=0,\ldots,r-1} \Omega^\bullet_{X/S}[-2t] \longrightarrow Rp_*\Omega^\bullet_{P/S}$$

is an isomorphism in $D(X, (X \to S)^{-1}\mathcal{O}_X)$ as follows immediately from the application of Proposition 50.13.3. Note that the arrow for $t = 0$ is simply the canonical map $c_{P/X} : \Omega^\bullet_{X/S} \to Rp_*\Omega^\bullet_{P/S}$ of Section 50.2. In fact, we can pin down this map further in this particular case. Namely, consider the canonical map

$$\xi' : \Omega^\bullet_{P/S} \to \Omega^\bullet_{P/S}[2]$$

of Remark 50.4.3 corresponding to $c_1^{dR}(\mathcal{O}_P(1))$. Then

$$\xi'[2(t-1)] \circ \ldots \circ \xi'[2] \circ \xi' : \Omega^\bullet_{P/S} \to \Omega^\bullet_{P/S}[2t]$$

is the map of Remark 50.4.3 corresponding to $c_1^{dR}(\mathcal{O}_P(1))^t$. Tracing through the choices made in the proof of Proposition 50.13.3 we find the value

$$\tilde{\xi}|_{\Omega^\bullet_{X/S}[-2t]} = Rp_*\xi'[-2] \circ \ldots \circ Rp_*\xi'[-2(t-1)] \circ Rp_*\xi'[-2t] \circ c_{P/X}[-2t]$$

for the restriction of our isomorphism to the summand $\Omega^\bullet_{X/S}[-2t]$. This has the following simple consequence we will use below: let

$$M = \bigoplus\nolimits_{t=1,\ldots,r-1} \Omega^\bullet_{X/S}[-2t] \quad \text{and} \quad K = \bigoplus\nolimits_{t=0,\ldots,r-2} \Omega^\bullet_{X/S}[-2t]$$

viewed as subcomplexes of the source of the arrow $\tilde{\xi}$. It follows formally from the discussion above that

$$c_{P/X} \oplus \tilde{\xi}|_M : \Omega^\bullet_{X/S} \oplus M \longrightarrow Rp_*\Omega^\bullet_{P/S}$$

is an isomorphism and that the diagram

$$\begin{tikzcd} K \arrow[r, "\text{id}"'] \arrow[d, "\tilde{\xi}|_K"'] & M[2] \arrow[d, "(\tilde{\xi}|_M)[2]"] \\ Rp_*\Omega^\bullet_{P/S} \arrow[r, "Rp_*\xi'"] & Rp_*\Omega^\bullet_{P/S}[2] \end{tikzcd}$$

commutes where $\text{id} : K \to M[2]$ identifies the summand corresponding to t in the deomposition of K to the summand corresponding to $t+1$ in the decomposition of M.

50.15. Log poles along a divisor

0FMU Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. If X étale locally along Y looks like $Y \times \mathbf{A}^1$, then there is a canonical short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

having many good properties we will discuss in this section. There is a variant of this construction where one starts with a normal crossings divisor (Étale Morphisms, Definition 41.21.1) which we will discuss elsewhere (insert future reference here).

0FMV Definition 50.15.1. Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. We say the de Rham complex of log poles is defined for $Y \subset X$ over S if for all $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of Y we have

(1) $\mathcal{O}_{X,y} \to \Omega_{X/S,y}$, $g \mapsto g\text{d}f$ is a split injection, and
(2) $\Omega^p_{X/S,y}$ is f-torsion free for all p.

An easy local calculation shows that it suffices for every $y \in Y$ to find one local equation f for which conditions (1) and (2) hold.

0FMW Lemma 50.15.2. Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. There is a canonical short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

Proof. Our assumption is that for every $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of Y we have

$$\Omega_{X/S,y} = \mathcal{O}_{X,y}\text{d}f \oplus M \quad \text{and} \quad \Omega^p_{X/S,y} = \wedge^{p-1}(M)\text{d}f \oplus \wedge^p(M)$$

for some module M with f-torsion free exterior powers $\wedge^p(M)$. It follows that

$$\Omega^p_{Y/S,y} = \wedge^p(M/fM) = \wedge^p(M)/f\wedge^p(M)$$

Below we will tacitly use these facts. In particular the sheaves $\Omega^p_{X/S}$ have no nonzero local sections supported on Y and we have a canonical inclusion

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(Y)$$

see More on Flatness, Section 38.42. Let $U = \text{Spec}(A)$ be an affine open subscheme such that $Y \cap U = V(f)$ for some nonzerodivisor $f \in A$. Let us consider the $\mathcal{O}_U$-submodule of $\Omega^p_{X/S}(Y)|_U$ generated by $\Omega^p_{X/S}|_U$ and $\text{d}\log(f) \wedge \Omega^{p-1}_{X/S}$ where $\text{d}\log(f) = f^{-1}\text{d}(f)$. This is independent of the choice of f as another generator of the ideal of Y on U is equal to uf for a unit $u \in A$ and we get

$$\text{d}\log(uf) - \text{d}\log(f) = \text{d}\log(u) = u^{-1}\text{d}u$$

which is a section of $\Omega_{X/S}$ over U. These local sheaves glue to give a quasi-coherent submodule

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(\log Y) \subset \Omega^p_{X/S}(Y)$$

Let us agree to think of $\Omega^p_{Y/S}$ as a quasi-coherent $\mathcal{O}_X$-module. There is a unique surjective $\mathcal{O}_X$-linear map

$$\text{Res} : \Omega^p_{X/S}(\log Y) \to \Omega^{p-1}_{Y/S}$$

defined by the rule

$$\mathrm{Res}(\eta' + \mathrm{d}\log(f) \wedge \eta) = \eta|_{Y \cap U}$$

for all opens U as above and all $\eta' \in \Omega^p_{X/S}(U)$ and $\eta \in \Omega^{p-1}_{X/S}(U)$. If a form η over U restricts to zero on $Y \cap U$, then $\eta = \mathrm{d}f \wedge \eta' + f\eta''$ for some forms η' and η'' over U. We conclude that we have a short exact sequence

$$0 \to \Omega^p_{X/S} \to \Omega^p_{X/S}(\log Y) \to \Omega^{p-1}_{Y/S} \to 0$$

for all p. We still have to define the differentials $\Omega^p_{X/S}(\log Y) \to \Omega^{p+1}_{X/S}(\log Y)$. On the subsheaf $\Omega^p_{X/S}$ we use the differential of the de Rham complex of X over S. Finally, we define $\mathrm{d}(\mathrm{d}\log(f) \wedge \eta) = -\mathrm{d}\log(f) \wedge \mathrm{d}\eta$. The sign is forced on us by the Leibniz rule (on $\Omega^\bullet_{X/S}$) and it is compatible with the differential on $\Omega^\bullet_{Y/S}[-1]$ which is after all $-\mathrm{d}_{Y/S}$ by our sign convention in Homology, Definition 12.14.7. In this way we obtain a short exact sequence of complexes as stated in the lemma. $\square$

0FUA Definition 50.15.3. Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Then the complex

$$\Omega^\bullet_{X/S}(\log Y)$$

constructed in Lemma 50.15.2 is the de Rham complex of log poles for $Y \subset X$ over S.

This complex has many good properties.

0FUP Lemma 50.15.4. Let $p : X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S.

(1) The maps $\wedge : \Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ extend uniquely to $\mathcal{O}_X$-bilinear maps
$$\wedge : \Omega^p_{X/S}(\log Y) \times \Omega^q_{X/S}(\log Y) \to \Omega^{p+q}_{X/S}(\log Y)$$
satisfying the Leibniz rule $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge \mathrm{d}(\eta)$,
(2) with multiplication as in (1) the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log(Y)$ is a homomorphism of differential graded $\mathcal{O}_S$-algebras,
(3) via the maps in (1) we have $\Omega^p_{X/S}(\log Y) = \wedge^p(\Omega^1_{X/S}(\log Y))$, and
(4) the map $\mathrm{Res} : \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1]$ satisfies
$$\mathrm{Res}(\omega \wedge \eta) = \mathrm{Res}(\omega) \wedge \eta|_Y$$
for ω a local section of $\Omega^p_{X/S}(\log Y)$ and η a local section of $\Omega^q_{X/S}$.

Proof. This follows by direct calcuation from the local construction of the complex in the proof of Lemma 50.15.2. Details omitted. $\square$

Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes. Let $Y \subset X$ be an effective Cartier divisor whose pullback $Y' = f^*Y$ is defined (Divisors, Definition 31.13.12). Assume the de Rham complex of log poles is defined for $Y \subset X$ over S and the de Rham complex of log poles is defined for $Y' \subset X'$ over S'. In this case we obtain a map of short exact sequences of complexes

$$\begin{tikzcd} 0 \arrow[r] & f^{-1}\Omega^\bullet_{X/S} \arrow[r] \arrow[d] & f^{-1}\Omega^\bullet_{X/S}(\log Y) \arrow[r] \arrow[d] & f^{-1}\Omega^\bullet_{Y/S}[-1] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^\bullet_{X'/S'} \arrow[r] & \Omega^\bullet_{X'/S'}(\log Y') \arrow[r] & \Omega^\bullet_{Y'/S'}[-1] \arrow[r] & 0 \end{tikzcd}$$

Linearizing, for every p we obtain a linear map $f^*\Omega^p_{X/S}(\log Y) \to \Omega^p_{X'/S'}(\log Y')$.

0FUQ Lemma 50.15.5. Let $f : X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Denote

$$\delta : \Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}[2]$$

in $D(X, f^{-1}\mathcal{O}_S)$ the "boundary" map coming from the short exact sequence in Lemma 50.15.2. Denote

$$\xi' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$$

in $D(X, f^{-1}\mathcal{O}_S)$ the map of Remark 50.4.3 corresponding to $\xi = c_1^{dR}(\mathcal{O}_X(-Y))$. Denote

$$\zeta' : \Omega^\bullet_{Y/S} \to \Omega^\bullet_{Y/S}[2]$$

in $D(Y, f|_Y^{-1}\mathcal{O}_S)$ the map of Remark 50.4.3 corresponding to $\zeta = c_1^{dR}(\mathcal{O}_X(-Y)|_Y)$. Then the diagram

$$\begin{tikzcd} \Omega^\bullet_{X/S} \arrow[r] \arrow[d, "\xi'"'] & \Omega^\bullet_{Y/S} \arrow[d, "\zeta'"] \arrow[dl, "\delta"'] \\ \Omega^\bullet_{X/S}[2] \arrow[r] & \Omega^\bullet_{Y/S}[2] \end{tikzcd}$$

is commutative in $D(X, f^{-1}\mathcal{O}_S)$.

Proof. More precisely, we define δ as the boundary map corresponding to the shifted short exact sequence

$$0 \to \Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{X/S}(\log Y)[1] \to \Omega^\bullet_{Y/S} \to 0$$

It suffices to prove each triangle commutes. Set $\mathcal{L} = \mathcal{O}_X(-Y)$. Denote $\pi : L \to X$ the line bundle with $\pi_*\mathcal{O}_L = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$.

Commutativity of the upper left triangle. By Lemma 50.10.3 the map ξ' is the boundary map of the triangle given in Lemma 50.10.2. By functoriality it suffices to prove there exists a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] \arrow[d] & \Omega^\bullet_{L^\star/S,0}[1] \arrow[r] \arrow[d] & \Omega^\bullet_{X/S} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] & \Omega^\bullet_{X/S}(\log Y)[1] \arrow[r] & \Omega^\bullet_{Y/S} \arrow[r] & 0 \end{tikzcd}$$

where the left and right vertical arrows are the obvious ones. We can define the middle vertical arrow by the rule

$$\omega' + \mathrm{d}\log(s) \wedge \omega \longmapsto \omega' + \mathrm{d}\log(f) \wedge \omega$$

where ω', ω are local sections of $\Omega^\bullet_{X/S}$ and where s is a local generator of $\mathcal{L}$ and $f \in \mathcal{O}_X(-Y)$ is the corresponding section of the ideal sheaf of Y in X. Since the constructions of the maps in Lemmas 50.10.2 and 50.15.2 match exactly, this works.

Commutativity of the lower right triangle. Denote $\overline{L}$ the restriction of L to Y. By Lemma 50.10.3 the map ζ' is the boundary map of the triangle given in Lemma 50.10.2 using the line bundle $\overline{L}$ on Y. By functoriality it suffices to prove there exists a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] \arrow[d] & \Omega^\bullet_{X/S}(\log Y)[1] \arrow[r] \arrow[d] & \Omega^\bullet_{Y/S} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^\bullet_{Y/S}[1] \arrow[r] & \Omega^\bullet_{\overline{L}^\star/S,0}[1] \arrow[r] & \Omega^\bullet_{Y/S} \arrow[r] & 0 \end{tikzcd}$$

where the left and right vertical arrows are the obvious ones. We can define the middle vertical arrow by the rule

$$\omega' + \mathrm{d}\log(f) \wedge \omega \longmapsto \omega'|_Y + \mathrm{d}\log(\overline{s}) \wedge \omega|_Y$$

where ω', ω are local sections of $\Omega^\bullet_{X/S}$ and where f is a local generator of $\mathcal{O}_X(-Y)$ viewed as a function on X and where $\overline{s}$ is $f|_Y$ viewed as a section of $\mathcal{L}|_Y = \mathcal{O}_X(-Y)|_Y$. Since the constructions of the maps in Lemmas 50.10.2 and 50.15.2 match exactly, this works. $\square$

0FMX Lemma 50.15.6. Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Let $b \in H^m_{dR}(X/S)$ be a de Rham cohomology class whose restriction to Y is zero. Then $c_1^{dR}(\mathcal{O}_X(Y)) \cup b = 0$ in $H^{m+2}_{dR}(X/S)$.

Proof. This follows immediately from Lemma 50.15.5. Namely, we have

$$c_1^{dR}(\mathcal{O}_X(Y)) \cup b = -c_1^{dR}(\mathcal{O}_X(-Y)) \cup b = -\xi'(b) = -\delta(b|_Y) = 0$$

as desired. For the second equality, see Remark 50.4.3. $\square$

0FMY Lemma 50.15.7. Let $X \to T \to S$ be morphisms of schemes. Let $Y \subset X$ be an effective Cartier divisor. If both $X \to T$ and $Y \to T$ are smooth, then the de Rham complex of log poles is defined for $Y \subset X$ over S.

Proof. Let $y \in Y$ be a point. By More on Morphisms, Lemma 37.17.1 there exists an integer $0 \geq m$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & \mathbf{A}^m_T \arrow[d, "(a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0)"] \\ X & U \arrow[l] \arrow[r, "\pi"] & \mathbf{A}^{m+1}_T \end{tikzcd}$$

where $U \subset X$ is open, $V = Y \cap U$, π is étale, $V = \pi^{-1}(\mathbf{A}^m_T)$, and $y \in V$. Denote $z \in \mathbf{A}^m_T$ the image of y. Then we have

$$\Omega^p_{X/S,y} = \Omega^p_{\mathbf{A}^{m+1}_T/S,z} \otimes_{\mathcal{O}_{\mathbf{A}^{m+1}_T,z}} \mathcal{O}_{X,x}$$

by Lemma 50.2.2. Denote $x_1, \ldots, x_{m+1}$ the coordinate functions on $\mathbf{A}^{m+1}_T$. Since the conditions (1) and (2) in Definition 50.15.1 do not depend on the choice of the local coordinate, it suffices to check the conditions (1) and (2) when f is the image of x_{m+1} by the flat local ring homomorphism $\mathcal{O}_{\mathbf{A}^{m+1}_T, z} \to \mathcal{O}_{X,x}$. In this way we see that it suffices to check conditions (1) and (2) for $\mathbf{A}^m_T \subset \mathbf{A}^{m+1}_T$ and the point z. To prove this case we may assume $S = \text{Spec}(A)$ and $T = \text{Spec}(B)$ are affine. Let $A \to B$ be the ring map corresponding to the morphism $T \to S$ and set $P = B[x_1, \ldots, x_{m+1}]$ so that $\mathbf{A}^{m+1}_T = \text{Spec}(B)$. We have

$$\Omega_{P/A} = \Omega_{B/A} \otimes_B P \oplus \bigoplus\nolimits_{j=1,\ldots,m} P\text{d}x_j \oplus P\text{d}x_{m+1}$$

Hence the map $P \to \Omega_{P/A}$, $g \mapsto g\text{d}x_{m+1}$ is a split injection and x_{m+1} is a nonzerodivisor on $\Omega^p_{P/A}$ for all $p \geq 0$. Localizing at the prime ideal corresponding to z finishes the proof. $\square$

0FMZ Remark 50.15.8. Let S be a locally Noetherian scheme. Let X be locally of finite type over S. Let $Y \subset X$ be an effective Cartier divisor. If the map

$$\mathcal{O}^\wedge_{X,y} \longrightarrow \mathcal{O}^\wedge_{Y,y}$$

has a section for all $y \in Y$, then the de Rham complex of log poles is defined for $Y \subset X$ over S. If we ever need this result we will formulate a precise statement and add a proof here.

0FN0 Remark 50.15.9. Let S be a locally Noetherian scheme. Let X be locally of finite type over S. Let $Y \subset X$ be an effective Cartier divisor. If for every $y \in Y$ we can find a diagram of schemes over S

$$X \xleftarrow{\varphi} U \xrightarrow{\psi} V$$

with φ étale and $\psi|_{\varphi^{-1}(Y)} : \varphi^{-1}(Y) \to V$ étale, then the de Rham complex of log poles is defined for $Y \subset X$ over S. A special case is when the pair (X, Y) étale locally looks like $(V \times \mathbf{A}^1, V \times \{0\})$. If we ever need this result we will formulate a precise statement and add a proof here.

50.16. Calculations

0FUB In this section we calculate some Hodge and de Rham cohomology groups for a standard blowing up.

We fix a ring R and we set $S = \text{Spec}(R)$. Fix integers $0 \leq m$ and $1 \leq n$. Consider the closed immersion

$$Z = \mathbf{A}^m_S \longrightarrow \mathbf{A}^{m+n}_S = X, \quad (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots 0).$$

We are going to consider the blowing up L of X along the closed subscheme Z. Write

$$X = \mathbf{A}^{m+n}_S = \text{Spec}(A) \quad \text{with} \quad A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

We will consider $A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$ as a graded R-algebra by setting $\deg(x_i) = 0$ and $\deg(y_j) = 1$. With this grading we have

$$P = \text{Proj}(A) = \mathbf{A}^m_S \times_S \mathbf{P}^{n-1}_S = Z \times_S \mathbf{P}^{n-1}_S = \mathbf{P}^{n-1}_Z$$

Observe that the ideal cutting out Z in X is the ideal A_+. Hence L is the Proj of the Rees algebra

$$A \oplus A_+ \oplus (A_+)^2 \oplus \ldots = \bigoplus\nolimits_{d \geq 0} A_{\geq d}$$

Hence L is an example of the phenomenon studied in more generality in More on Morphisms, Section 37.51; we will use the observations we made there without further mention. In particular, we have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "0"'] \arrow[d, "p"'] & L \arrow[r, "\pi"'] \arrow[d, "b"] & P \arrow[d, "p"] \\ Z \arrow[r, "i"] & X \arrow[r] & Z \end{tikzcd}$$

such that $\pi : L \to P$ is a line bundle over $P = Z \times_S \mathbf{P}^{n-1}_S$ with zero section 0 whose image $E = 0(P) \subset L$ is the exceptional divisor of the blowup b.

0FUR Lemma 50.16.1. For $a \geq 0$ we have

(1) the map $\Omega^a_{X/S} \to b_*\Omega^a_{L/S}$ is an isomorphism,
(2) the map $\Omega^a_{Z/S} \to p_*\Omega^a_{P/S}$ is an isomorphism, and
(3) the map $Rb_*\Omega^a_{L/S} \to i_*Rp_*\Omega^a_{P/S}$ is an isomorphism on cohomology sheaves in degree ≥ 1.

Proof. Let us first prove part (2). Since $P = Z \times_S \mathbf{P}^{n-1}_S$ we see that

$$\Omega^a_{P/S} = \bigoplus\nolimits_{a=r+s} \text{pr}_1^*\Omega^r_{Z/S} \otimes \text{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}_S/S}$$

Recalling that $p = \text{pr}_1$ by the projection formula (Cohomology, Lemma 20.54.2) we obtain

$$p_*\Omega^a_{P/S} = \bigoplus\nolimits_{a=r+s} \Omega^r_{Z/S} \otimes \text{pr}_{1,*}\text{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}_S/S}$$

By the calculations in Section 50.11 and in particular in the proof of Lemma 50.11.3 we have $\text{pr}_{1,*}\text{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}_S/S} = 0$ except if $s = 0$ in which case we get $\text{pr}_{1,*}\mathcal{O}_P = \mathcal{O}_Z$. This proves (2).

By the material in Section 50.10 and in particular Lemma 50.10.4 we have $\pi_*\Omega^a_{L/S} = \Omega^a_{P/S} \oplus \bigoplus_{k \geq 1} \Omega^a_{L/S,k}$. Since the composition $\pi \circ 0$ in the diagram above is the identity morphism on P to prove part (3) it suffices to show that $\Omega^a_{L/S,k}$ has vanishing higher cohomology for $k > 0$. By Lemmas 50.10.2 and 50.10.4 there are short exact sequences

$$0 \to \Omega^a_{P/S} \otimes \mathcal{O}_P(k) \to \Omega^a_{L/S,k} \to \Omega^{a-1}_{P/S} \otimes \mathcal{O}_P(k) \to 0$$

where $\Omega^{a-1}_{P/S} = 0$ if $a = 0$. Since $P = Z \times_S \mathbf{P}^{n-1}_S$ we have

$$\Omega^a_{P/S} = \bigoplus\nolimits_{i+j=a} \Omega^i_{Z/S} \boxtimes \Omega^j_{\mathbf{P}^{n-1}_S/S}$$

by Lemma 50.8.1. Since $\Omega^i_{Z/S}$ is free of finite rank we see that it suffices to show that the higher cohomology of $\mathcal{O}_Z \boxtimes \Omega^j_{\mathbf{P}^{n-1}_S/S}(k)$ is zero for $k > 0$. This follows from Lemma 50.11.2 applied to $P = Z \times_S \mathbf{P}^{n-1}_S = \mathbf{P}^{n-1}_Z$ and the proof of (3) is complete.

We still have to prove (1). If $n = 1$, then we are blowing up an effective Cartier divisor and b is an isomorphism and we have (1). If $n > 1$, then the composition

$$\Gamma(X, \Omega^a_{X/S}) \to \Gamma(L, \Omega^a_{L/S}) \to \Gamma(L \setminus E, \Omega^a_{L/S}) = \Gamma(X \setminus Z, \Omega^a_{X/S})$$

is an isomorphism as $\Omega^a_{X/S}$ is finite free (small detail omitted). Thus the only way (1) can fail is if there are nonzero elements of $\Gamma(L, \Omega^a_{L/S})$ which vanish outside of

$E = 0(P)$. Since L is a line bundle over P with zero section $0 : P \to L$, it suffices to show that on a line bundle there are no nonzero sections of a sheaf of differentials which vanish identically outside the zero section. The reader sees this is true either (preferably) by a local caculation or by using that $\Omega_{L/S,k} \subset \Omega_{L^\star/S,k}$ (see references above). $\square$

We suggest the reader skip to the next section at this point.

0G5G Lemma 50.16.2. For $a \geq 0$ there are canonical maps

$$b^*\Omega^a_{X/S} \longrightarrow \Omega^a_{L/S} \longrightarrow b^*\Omega^a_{X/S} \otimes_{\mathcal{O}_L} \mathcal{O}_L((n-1)E)$$

whose composition is induced by the inclusion $\mathcal{O}_L \subset \mathcal{O}_L((n-1)E)$.

Proof. The first arrow in the displayed formula is discussed in Section 50.2. To get the second arrow we have to show that if we view a local section of $\Omega^a_{L/S}$ as a "meromorphic section" of $b^*\Omega^a_{X/S}$, then it has a pole of order at most $n-1$ along E. To see this we work on affine local charts on L. Namely, recall that L is covered by the spectra of the affine blowup algebras $A[\frac{I}{y_i}]$ where $I = A_+$ is the ideal generated by $y_1, \ldots, y_n$. See Algebra, Section 10.70 and Divisors, Lemma 31.32.2. By symmetry it is enough to work on the chart corresponding to $i = 1$. Then

$$A[\frac{I}{y_1}] = R[x_1, \ldots, x_m, y_1, t_2, \ldots, t_n]$$

where $t_i = y_i/y_1$, see More on Algebra, Lemma 15.31.2. Thus the module $\Omega^1_{L/S}$ is over the corresponding affine open freely generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$, $\mathrm{d}y_1$, and $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$. Of course, the first $m+1$ of these generators come from $b^*\Omega^1_{X/S}$ and for the remaining $n-1$ we have

$$\mathrm{d}t_j = \mathrm{d}\frac{y_j}{y_1} = \frac{1}{y_1}\mathrm{d}y_j - \frac{y_j}{y_1^2}\mathrm{d}y_1 = \frac{\mathrm{d}y_j - t_j\mathrm{d}y_1}{y_1}$$

which has a pole of order 1 along E since E is cut out by y_1 on this chart. Since the wedges of a of these elements give a basis of $\Omega^a_{L/S}$ over this chart, and since there are at most $n-1$ of the $\mathrm{d}t_j$ involved this finishes the proof. $\square$

0G5H Lemma 50.16.3. Let $E = 0(P)$ be the exceptional divisor of the blowing up b. For any locally free $\mathcal{O}_X$-module $\mathcal{E}$ and $0 \leq i \leq n-1$ the map

$$\mathcal{E} \longrightarrow Rb_*(b^*\mathcal{E} \otimes_{\mathcal{O}_L} \mathcal{O}_L(iE))$$

is an isomorphism in $D(\mathcal{O}_X)$.

Proof. By the projection formula it is enough to show this for $\mathcal{E} = \mathcal{O}_X$, see Cohomology, Lemma 20.54.2. Since X is affine it suffices to show that the maps

$$H^0(X, \mathcal{O}_X) \to H^0(L, \mathcal{O}_L) \to H^0(L, \mathcal{O}_L(iE))$$

are isomorphisms and that $H^j(X, \mathcal{O}_L(iE)) = 0$ for $j > 0$ and $0 \leq i \leq n-1$, see Cohomology of Schemes, Lemma 30.4.6. Since π is affine, we can compute global sections and cohomology after taking π_*, see Cohomology of Schemes, Lemma 30.2.4. If $n = 1$, then $L \to X$ is an isomorphism and $i = 0$ hence the first statement holds. If $n > 1$, then we consider the composition

$$H^0(X, \mathcal{O}_X) \to H^0(L, \mathcal{O}_L) \to H^0(L, \mathcal{O}_L(iE)) \to H^0(L \setminus E, \mathcal{O}_L) = H^0(X \setminus Z, \mathcal{O}_X)$$

Since $H^0(X \setminus Z, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ in this case as Z has codimension $n \geq 2$ in X (details omitted) we conclude the first statement holds. For the second, recall that $\mathcal{O}_L(E) = \mathcal{O}_L(-1)$, see Divisors, Lemma 31.32.4. Hence we have

$$\pi_*\mathcal{O}_L(iE) = \pi_*\mathcal{O}_L(-i) = \bigoplus\nolimits_{k \geq -i} \mathcal{O}_P(k)$$

as discussed in More on Morphisms, Section 37.51. Thus we conclude by the vanishing of the cohomology of twists of the structure sheaf on $P = \mathbf{P}^{n-1}_Z$ shown in Cohomology of Schemes, Lemma 30.8.1. $\square$

50.17. Blowing up and de Rham cohomology

0FUC Fix a base scheme S, a smooth morphism $X \to S$, and a closed subscheme $Z \subset X$ which is also smooth over S. Denote $b : X' \to X$ the blowing up of X along Z. Denote $E \subset X'$ the exceptional divisor. Picture

0FUS (50.17.0.1)
$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Our goal in this section is to prove that the map $b^* : H^*_{dR}(X/S) \longrightarrow H^*_{dR}(X'/S)$ is injective (although a lot more can be said).

0FUU **Lemma 50.17.1.** With notation as in More on Morphisms, Lemma 37.17.3 for $a \geq 0$ we have

(1) the map $\Omega^a_{X/S} \to b_*\Omega^a_{X'/S}$ is an isomorphism,
(2) the map $\Omega^a_{Z/S} \to p_*\Omega^a_{E/S}$ is an isomorphism,
(3) the map $Rb_*\Omega^a_{X'/S} \to i_*Rp_*\Omega^a_{E/S}$ is an isomorphism on cohomology sheaves in degree ≥ 1.

Proof. Let $\epsilon : X_1 \to X$ be a surjective étale morphism. Denote $i_1 : Z_1 \to X_1$, $b_1 : X_1' \to X_1$, $E_1 \subset X_1'$, and $p_1 : E_1 \to Z_1$ the base changes of the objects considered in More on Morphisms, Lemma 37.17.3. Observe that i_1 is a closed immersion of schemes smooth over S and that b_1 is the blowing up with center Z_1 by Divisors, Lemma 31.32.3. Suppose that we can prove (1), (2), and (3) for the morphisms b_1, p_1, and i_1. Then by Lemma 50.2.2 we obtain that the pullback by ϵ of the maps in (1), (2), and (3) are isomorphisms. As ϵ is a surjective flat morphism we conclude. Thus working étale locally, by More on Morphisms, Lemma 37.17.1, we may assume we are in the situation discussed in Section 50.16. In this case the lemma is the same as Lemma 50.16.1. $\square$

0FUV **Lemma 50.17.2.** With notation as in More on Morphisms, Lemma 37.17.3 and denoting $f : X \to S$ the structure morphism there is a canonical distinguished triangle

$$\Omega^\bullet_{X/S} \to Rb_*(\Omega^\bullet_{X'/S}) \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*(\Omega^\bullet_{E/S}) \to \Omega^\bullet_{X/S}[1]$$

in $D(X, f^{-1}\mathcal{O}_S)$ where the four maps

$$\begin{matrix} \Omega^\bullet_{X/S} & \to & Rb_*(\Omega^\bullet_{X'/S}), \\ \Omega^\bullet_{X/S} & \to & i_*\Omega^\bullet_{Z/S}, \\ Rb_*(\Omega^\bullet_{X'/S}) & \to & i_*Rp_*(\Omega^\bullet_{E/S}), \\ i_*\Omega^\bullet_{Z/S} & \to & i_*Rp_*(\Omega^\bullet_{E/S}) \end{matrix}$$

are the canonical ones (Section 50.2), except with sign reversed for one of them.

Proof. Choose a distinguished triangle

$$C \to Rb_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*\Omega^\bullet_{E/S} \to C[1]$$

in $D(X, f^{-1}\mathcal{O}_S)$. It suffices to show that $\Omega^\bullet_{X/S}$ is isomorphic to C in a manner compatible with the canonical maps. By the axioms of triangulated categories there exists a map of distinguished triangles

$$\begin{tikzcd}
C' \arrow[r] \arrow[d] & b_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \arrow[r] \arrow[d] & i_*p_*\Omega^\bullet_{E/S} \arrow[r] \arrow[d] & C'[1] \arrow[d] \\
C \arrow[r] & Rb_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \arrow[r] & i_*Rp_*\Omega^\bullet_{E/S} \arrow[r] & C[1]
\end{tikzcd}$$

By Lemma 50.17.1 part (3) and Derived Categories, Proposition 13.4.23 we conclude that $C' \to C$ is an isomorphism. By Lemma 50.17.1 part (2) the map $i_*\Omega^\bullet_{Z/S} \to i_*p_*\Omega^\bullet_{E/S}$ is an isomorphism. Thus $C' = b_*\Omega^\bullet_{X'/S}$ in the derived category. Finally we use Lemma 50.17.1 part (1) tells us this is equal to $\Omega^\bullet_{X/S}$. We omit the verification this is compatible with the canonical maps. $\square$

0FUW Proposition 50.17.3. With notation as in More on Morphisms, Lemma 37.17.3 the map $\Omega^\bullet_{X/S} \to Rb_*\Omega^\bullet_{X'/S}$ has a splitting in $D(X, (X \to S)^{-1}\mathcal{O}_S)$.

Proof. Consider the triangle constructed in Lemma 50.17.2. We claim that the map

$$Rb_*(\Omega^\bullet_{X'/S}) \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*(\Omega^\bullet_{E/S})$$

has a splitting whose image contains the summand $i_*\Omega^\bullet_{Z/S}$. By Derived Categories, Lemma 13.4.11 this will show that the first arrow of the triangle has a splitting which vanishes on the summand $i_*\Omega^\bullet_{Z/S}$ which proves the lemma. We will prove the claim by decomposing $Rp_*\Omega^\bullet_{E/S}$ into a direct sum where the first piece corresponds to $\Omega^\bullet_{Z/S}$ and the second piece can be lifted through $Rb_*\Omega^\bullet_{X'/S}$.

Proof of the claim. We may decompose X into open and closed subschemes having fixed relative dimension to S, see Morphisms, Lemma 29.34.12. Since the derived category $D(X, f^{-1}\mathcal{O})_S)$ correspondingly decomposes as a product of categories, we may assume X has fixed relative dimension N over S. We may decompose $Z = \coprod Z_m$ into open and closed subschemes of relative dimension $m \geq 0$ over S. The restriction $i_m : Z_m \to X$ of i to Z_m is a regular immersion of codimension $N - m$, see Divisors, Lemma 31.22.11. Let $E = \coprod E_m$ be the corresponding decomposition, i.e., we set $E_m = p^{-1}(Z_m)$. If $p_m : E_m \to Z_m$ denotes the restriction of p to E_m, then we have a canonical isomorphism

$$\tilde{\xi}_m : \bigoplus\nolimits_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t] \longrightarrow Rp_{m,*}\Omega^\bullet_{E_m/S}$$

in $D(Z_m, (Z_m \to S)^{-1}\mathcal{O}_S)$ where in degree 0 we have the canonical map $\Omega^\bullet_{Z_m/S} \to Rp_{m,*}\Omega^\bullet_{E_m/S}$. See Remark 50.14.2. Thus we have an isomorphism

$$\tilde{\xi} : \bigoplus\nolimits_m \bigoplus\nolimits_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t] \longrightarrow Rp_*(\Omega^\bullet_{E/S})$$

in $D(Z, (Z \to S)^{-1}\mathcal{O}_S)$ whose restriction to the summand $\Omega^\bullet_{Z/S} = \bigoplus \Omega^\bullet_{Z_m/S}$ of the source is the canonical map $\Omega^\bullet_{Z/S} \to Rp_*(\Omega^\bullet_{E/S})$. Consider the subcomplexes M_m

and K_m of the complex $\bigoplus_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t]$ introduced in Remark 50.14.2. We set

$$M = \bigoplus M_m \quad \text{and} \quad K = \bigoplus K_m$$

We have $M = K[-2]$ and by construction the map

$$c_{E/Z} \oplus \tilde{\xi}|_M : \Omega^\bullet_{Z/S} \oplus M \longrightarrow Rp_*(\Omega^\bullet_{E/S})$$

is an isomorphism (see remark referenced above).

Consider the map

$$\delta : \Omega^\bullet_{E/S}[-2] \longrightarrow \Omega^\bullet_{X'/S}$$

in $D(X', (X' \to S)^{-1}\mathcal{O}_S)$ of Lemma 50.15.5 with the property that the composition

$$\Omega^\bullet_{E/S}[-2] \longrightarrow \Omega^\bullet_{X'/S} \longrightarrow \Omega^\bullet_{E/S}$$

is the map θ' of Remark 50.4.3 for $c_1^{dR}(\mathcal{O}_{X'}(-E))|_E) = c_1^{dR}(\mathcal{O}_E(1))$. The final assertion of Remark 50.14.2 tells us that the diagram

$$\begin{tikzcd} K[-2] \arrow[r, "\mathrm{id}"'] \arrow[d, "(\tilde{\xi}|_K)[-2]"'] & M \arrow[d, "\tilde{x}|_M"] \\ Rp_*\Omega^\bullet_{E/S}[-2] \arrow[r, "Rp_*\theta'"] & Rp_*\Omega^\bullet_{E/S} \end{tikzcd}$$

commutes. Thus we see that we can obtain the desired splitting of the claim as the map

$$\begin{aligned} Rp_*(\Omega^\bullet_{E/S}) &\xrightarrow{(c_{E/Z} \oplus \tilde{\xi}|_M)^{-1}} \Omega^\bullet_{Z/S} \oplus M \\ &\xrightarrow{\mathrm{id} \oplus \mathrm{id}^{-1}} \Omega^\bullet_{Z/S} \oplus K[-2] \\ &\xrightarrow{\mathrm{id} \oplus (\tilde{\xi}|_K)[-2]} \Omega^\bullet_{Z/S} \oplus Rp_*\Omega^\bullet_{E/S}[-2] \\ &\xrightarrow{\mathrm{id} \oplus Rb_*\delta} \Omega^\bullet_{Z/S} \oplus Rb_*\Omega^\bullet_{X'/S} \end{aligned}$$

The relationship between θ' and δ stated above together with the commutative diagram involving θ', $\tilde{\xi}|_K$, and $\tilde{\xi}|_M$ above are exactly what's needed to show that this is a section to the canonical map $\Omega^\bullet_{Z/S} \oplus Rb_*(\Omega^\bullet_{X'/S}) \to Rp_*(\Omega^\bullet_{E/S})$ and the proof of the claim is complete. $\square$

Lemma 50.17.5 shows that producing the splitting on Hodge cohomology is a good deal easier than the result of Proposition 50.17.3. We urge the reader to skip ahead to the next section.

0G5I **Lemma 50.17.4.** Let $i : Z \to X$ be a closed immersion of schemes which is regular of codimension c. Then $\mathrm{Ext}^q_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}) = 0$ for $q < c$ for $\mathcal{E}$ locally free on X and $\mathcal{F}$ any $\mathcal{O}_Z$-module.

Proof. By the local to global spectral sequence of Ext it suffices to prove this affine locally on X. See Cohomology, Section 20.43. Thus we may assume $X = \mathrm{Spec}(A)$ and there exists a regular sequence $f_1, \ldots, f_c$ in A such that $Z = \mathrm{Spec}(A/(f_1, \ldots, f_c))$. We may assume $c \geq 1$. Then we see that $f_1 : \mathcal{E} \to \mathcal{E}$ is injective. Since $i_*\mathcal{F}$ is annihilated by f_1 this shows that the lemma holds for $i = 0$ and that we have a surjection

$$\mathrm{Ext}^{q-1}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}/f_1\mathcal{E}) \longrightarrow \mathrm{Ext}^q_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E})$$

Thus it suffices to show that the source of this arrow is zero. Next we repeat this argument: if $c \geq 2$ the map $f_2 : \mathcal{E}/f_1\mathcal{E} \to \mathcal{E}/f_1\mathcal{E}$ is injective. Since $i_*\mathcal{F}$ is annihilated by f_2 this shows that the lemma holds for $q = 1$ and that we have a surjection

$$\mathrm{Ext}^{q-2}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}/f_1\mathcal{E} + f_2\mathcal{E}) \longrightarrow \mathrm{Ext}^{q-1}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}/f_1\mathcal{E})$$

Continuing in this fashion the lemma is proved. □

0G5J Lemma 50.17.5. With notation as in More on Morphisms, Lemma 37.17.3 for $a \geq 0$ there is a unique arrow $Rb_*\Omega^a_{X'/S} \to \Omega^a_{X/S}$ in $D(\mathcal{O}_X)$ whose composition with $\Omega^a_{X/S} \to Rb_*\Omega^a_{X'/S}$ is the identity on $\Omega^a_{X/S}$.

Proof. We may decompose X into open and closed subschemes having fixed relative dimension to S, see Morphisms, Lemma 29.34.12. Since the derived category $D(X, f^{-1}\mathcal{O})_S)$ correspondingly decomposes as a product of categories, we may assume X has fixed relative dimension N over S. We may decompose $Z = \coprod Z_m$ into open and closed subschemes of relative dimension $m \geq 0$ over S. The restriction $i_m : Z_m \to X$ of i to Z_m is a regular immersion of codimension $N - m$, see Divisors, Lemma 31.22.11. Let $E = \coprod E_m$ be the corresponding decomposition, i.e., we set $E_m = p^{-1}(Z_m)$. We claim that there are natural maps

$$b^*\Omega^a_{X/S} \to \Omega^a_{X'/S} \to b^*\Omega^a_{X/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(\sum (N - m - 1)E_m)$$

whose composition is induced by the inclusion $\mathcal{O}_{X'} \to \mathcal{O}_{X'}(\sum(N - m - 1)E_m)$. Namely, in order to prove this, it suffices to show that the cokernel of the first arrow is locally on X' annihilated by a local equation of the effective Cartier divisor $\sum(N - m - 1)E_m$. To see this in turn we can work étale locally on X as in the proof of Lemma 50.17.1 and apply Lemma 50.16.2. Computing étale locally using Lemma 50.16.3 we see that the induced composition

$$\Omega^a_{X/S} \to Rb_*\Omega^a_{X'/S} \to Rb_*\left(b^*\Omega^a_{X/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(\sum (N - m - 1)E_m)\right)$$

is an isomorphism in $D(\mathcal{O}_X)$ which is how we obtain the existence of the map in the lemma.

For uniqueness, it suffices to show that there are no nonzero maps from $\tau_{\geq 1}Rb_*\Omega_{X'/S}$ to $\Omega^a_{X/S}$ in $D(\mathcal{O}_X)$. For this it suffices in turn to show that there are no nonzero maps from $R^qb_*\Omega_{X'/s}[-q]$ to $\Omega^a_{X/S}$ in $D(\mathcal{O}_X)$ for $q \geq 1$ (details omitted). By Lemma 50.17.1 we see that $R^qb_*\Omega_{X'/s} \cong i_*R^qp_*\Omega^a_{E/S}$ is the pushforward of a module on $Z = \coprod Z_m$. Moreover, observe that the restriction of $R^qp_*\Omega^a_{E/S}$ to Z_m is nonzero only for $q < N - m$. Namely, the fibres of $E_m \to Z_m$ have dimension $N - m - 1$ and we can apply Limits, Lemma 32.19.2. Thus the desired vanishing follows from Lemma 50.17.4. □

50.18. Comparing sheaves of differential forms

0FL7 The goal of this section is to compare the sheaves $\Omega^p_{X/\mathbf{Z}}$ and $\Omega^p_{Y/\mathbf{Z}}$ when given a locally quasi-finite syntomic morphism of schemes $f : Y \to X$. The result will be applied in Section 50.19 to the construction of the trace map on de Rham complexes if f is finite.

0FL8 **Lemma 50.18.1.** Let R be a ring and consider a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K^0 \arrow[r] & L^0 \arrow[r] & M^0 \arrow[r] & 0 \\ & & L^{-1} \arrow[u, "\partial"] \arrow[r, equal] & M^{-1} \arrow[u] \end{tikzcd}$$

of R-modules with exact top row and M^0 and M^{-1} finite free of the same rank. Then there are canonical maps

$$\wedge^i(H^0(L^\bullet)) \longrightarrow \wedge^i(K^0) \otimes_R \det(M^\bullet)$$

whose composition with $\wedge^i(K^0) \to \wedge^i(H^0(L^\bullet))$ is equal to multiplication with $\delta(M^\bullet)$.

Proof. Say M^0 and M^{-1} are free of rank n. For every $i \geq 0$ there is a canonical surjection

$$\pi_i : \wedge^{n+i}(L^0) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0)$$

whose kernel is the submodule generated by wedges $l_1 \wedge \ldots \wedge l_{n+i}$ such that $> i$ of the l_j are in K^0. On the other hand, the exact sequence

$$L^{-1} \to L^0 \to H^0(L^\bullet) \to 0$$

similarly produces canonical maps

$$\wedge^i(H^0(L^\bullet)) \otimes \wedge^n(L^{-1}) \longrightarrow \wedge^{n+i}(L^0)$$

by sending $\eta \otimes \theta$ to $\tilde{\eta} \wedge \partial(\theta)$ where $\tilde{\eta} \in \wedge^i(L^0)$ is a lift of η. The composition of these two maps, combined with the identification $\wedge^n(L^{-1}) = \wedge^n(M^{-1})$ gives a map

$$\wedge^i(H^0(L^\bullet)) \otimes \wedge^n(M^{-1}) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0)$$

Since $\det(M^\bullet) = \wedge^n(M^0) \otimes (\wedge^n(M^{-1}))^{\otimes -1}$ this produces a map as in the statement of the lemma. If η is the image of $\omega \in \wedge^i(K^0)$, then we see that $\theta \otimes \eta$ is mapped to $\pi_i(\omega \wedge \partial(\theta)) = \omega \otimes \overline{\theta}$ in $\wedge^i(K^0) \otimes \wedge^n(M^0)$ where $\overline{\theta}$ is the image of θ in $\wedge^n(M^0)$. Since $\delta(M^\bullet)$ is simply the determinant of the map $M^{-1} \to M^0$ this proves the last statement. $\square$

0FL9 **Remark 50.18.2.** Let A be a ring. Let $P = A[x_1, \ldots, x_n]$. Let $f_1, \ldots, f_n \in P$ and set $B = P/(f_1, \ldots, f_n)$. Assume $A \to B$ is quasi-finite. Then B is a relative global complete intersection over A (Algebra, Definition 10.136.5) and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\overline{f}_i$ by Algebra, Lemma 10.136.12. Consider the following diagram

$$\begin{tikzcd} \Omega_{A/\mathbf{Z}} \otimes_A B \arrow[r] & \Omega_{P/\mathbf{Z}} \otimes_P B \arrow[r] & \Omega_{P/A} \otimes_P B \\ & (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \arrow[u] \arrow[r, equal] & (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \arrow[u] \end{tikzcd}$$

The right column represents $NL_{B/A}$ in $D(B)$ hence has cohomology $\Omega_{B/A}$ in degree 0. The top row is the split short exact sequence $0 \to \Omega_{A/\mathbf{Z}} \otimes_A B \to \Omega_{P/\mathbf{Z}} \otimes_P B \to \Omega_{P/A} \otimes_P B \to 0$. The middle column has cohomology $\Omega_{B/\mathbf{Z}}$ in degree 0 by Algebra, Lemma 10.131.9. Thus by Lemma 50.18.1 we obtain canonical B-module maps

$$\Omega^p_{B/\mathbf{Z}} \longrightarrow \Omega^p_{A/\mathbf{Z}} \otimes_A \det(NL_{B/A})$$

whose composition with $\Omega^p_{A/\mathbf{Z}} \to \Omega^p_{B/\mathbf{Z}}$ is multiplication by $\delta(NL_{B/A})$.

0FLA **Lemma 50.18.3.** There exists a unique rule that to every locally quasi-finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_Y$-module maps

$$c^p_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \longrightarrow f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

satisfying the following two properties

1. the composition with $f^*\Omega^p_{X/\mathbf{Z}} \to \Omega^p_{Y/\mathbf{Z}}$ is multiplication by $\delta(NL_{Y/X})$, and
2. the rule is compatible with restriction to opens and with base change.

Proof. This proof is very similar to the proof of Discriminants, Proposition 49.13.2 and we suggest the reader look at that proof first. We fix $p \geq 0$ throughout the proof.

Let us reformulate the statement. Consider the category $\mathcal{C}$ whose objects, denoted Y/X, are locally quasi-finite syntomic morphism $f : Y \to X$ of schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$\begin{tikzcd} Y' \arrow[r, "b"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "a"] & X \end{tikzcd}$$

which induce an isomorphism of Y' with an open subscheme of $X' \times_X Y$. The lemma means that for every object Y/X of $\mathcal{C}$ we have maps $c^p_{Y/X}$ with property (1) and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^*c^p_{Y/X} = c^p_{Y'/X'}$ via the identifications $b^*\det(NL_{Y/X}) = \det(NL_{Y'/X'})$ (Discriminants, Section 49.13) and $b^*\Omega^p_{Y/X} = \Omega^p_{Y'/X'}$ (Lemma 50.2.1).

Given Y/X in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some maps

$$\Omega^p_{Y/\mathbf{Z}}|_V \longrightarrow \left(f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})\right)|_V$$

with property (1). This follows from picking affine opens as in Discriminants, Lemma 49.10.1 part (5) and Remark 50.18.2. If $\Omega^p_{X/\mathbf{Z}}$ is finite locally free and annihilator of the section $\delta(NL_{Y/X})$ is zero, then these local maps are unique and automatically glue!

Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of Y/X such that

1. X is of finite type over $\mathbf{Z}$,
2. $\Omega_{X/\mathbf{Z}}$ is locally free, and
3. the annihilator of $\delta(NL_{Y/X})$ is zero.

By the remarks in the previous paragraph, we see that for any object Y/X of $\mathcal{C}_{nice}$ we have a unique map $c^p_{Y/X}$ satisfying condition (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}_{nice}$, then $b^*c^p_{Y/X}$ is equal to $c^p_{Y'/X'}$ because $b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ (see Discriminants, Section 49.13). In other words, we have solved the problem on the full subcategory $\mathcal{C}_{nice}$. For Y/X in $\mathcal{C}_{nice}$ we continue to denote $c^p_{Y/X}$ the solution we've just found.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that Y_1/X_1 and Y_2/X_2 are objects of $\mathcal{C}_{nice}$. Claim. $b_1^*c^p_{Y_1/X_1} = b_2^*c^p_{Y_2/X_2}$. We will first show that the claim implies the lemma and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Discriminants, Example 49.10.5. Then $Y_{n,d}$ and $Y_{n,d}$ are irreducible schemes of finite type and smooth over $\mathbf{Z}$. Namely, $X_{n,d}$ is a spectrum of a polynomial ring over $\mathbf{Z}$ and $Y_{n,d}$ is an open subscheme of such. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Discriminants, Lemma 49.10.6. Thus $\delta(NL_{Y_{n,d}/X_{n,d}})$ is nonzero: for example we have the local description of $\delta(NL_{Y/X})$ in Discriminants, Remark 49.13.1 and we have the local description of étale morphisms in Morphisms, Lemma 29.36.15 part (8). Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let Y/X be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Discriminants, Lemma 49.10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c^p_{Y_{n,d}/X_{n,d}}$ constructed above by b to V. The claim guarantees these local isomorphisms glue! Thus we get a well defined global maps $c^p_{Y/X}$ with property (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c^p_{Y'/X'}$ is the pullback by b of the locally constructed map $c^p_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of y. Thus we may replace Y_1, Y_2 by open neighbourhoods of the image of y in Y_1 and Y_2. Thus we may assume Y, X, Y_1, X_1, Y_2, X_2 are affine. We may write $X = \lim X_\lambda$ as a cofiltered limit of affine schemes of finite type over $X_1 \times X_2$. For each λ we get

$$Y_1 \times_{X_1} X_\lambda \quad \text{and} \quad X_\lambda \times_{X_2} Y_2$$

If we take limits we obtain

$$\lim Y_1 \times_{X_1} X_\lambda = Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2 = \lim X_\lambda \times_{X_2} Y_2$$

By Limits, Lemma 32.4.11 we can find a λ and opens $V_{1,\lambda} \subset Y_1 \times_{X_1} X_\lambda$ and $V_{2,\lambda} \subset X_\lambda \times_{X_2} Y_2$ whose base change to X recovers Y (on both sides). After increasing λ we may assume there is an isomorphism $V_{1,\lambda} \to V_{2,\lambda}$ whose base change to X is the identity on Y, see Limits, Lemma 32.10.1. Then we have the commutative diagram

$$\begin{tikzcd} & Y/X \arrow[ld, "b_1/a_1"'] \arrow[d] \arrow[rd, "b_2/a_2"] & \\ Y_1/X_1 & V_{1,\lambda}/X_\lambda \arrow[l] \arrow[r] & Y_2/X_2 \end{tikzcd}$$

Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$. The ring map $A_1 \to A$ corresponding to $X \to X_1$ is of finite type and hence we may choose a surjection $A_1[x_1, \ldots, x_n] \to A$. Similarly, we may choose a surjection $A_2[y_1, \ldots, y_m] \to A$. Set $X_1' = \mathrm{Spec}(A_1[x_1, \ldots, x_n])$ and $X_2' = \mathrm{Spec}(A_2[y_1, \ldots, y_m])$. Observe that $\Omega_{X_1'/\mathbf{Z}}$ is the direct sum of the pullback of $\Omega_{X_1/\mathbf{Z}}$ and a finite free module. Similarly for X_2'. Set $Y_1' = Y_1 \times_{X_1} X_1'$ and $Y_2' = Y_2 \times_{X_2} X_2'$. We get the following diagram

$$Y_1/X_1 \leftarrow Y_1'/X_1' \leftarrow Y/X \to Y_2'/X_2' \to Y_2/X_2$$

Since $X_1' \to X_1$ and $X_2' \to X_2$ are flat, the same is true for $Y_1' \to Y_1$ and $Y_2' \to Y_2$. It follows easily that the annihilators of $\delta(NL_{Y_1'/X_1'})$ and $\delta(NL_{Y_2'/X_2'})$ are zero. Hence Y_1'/X_1' and Y_2'/X_2' are in $\mathcal{C}_{nice}$. Thus the outer morphisms in the displayed diagram are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities. Thus it suffices to prove the claim for $Y_1'/X_1' \leftarrow Y/X \to Y_2'/X_2'$. This reduces us to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions. Consider the open embeddings $Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2$. There is an open neighbourhood $V \subset Y$ of y which is a standard open of both $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$. This follows from Schemes, Lemma 26.11.5 applied to the scheme obtained by glueing $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$ along Y; details omitted. Since $X \times_{X_2} Y_2$ is a closed subscheme of Y_2 we can find a standard open $V_2 \subset Y_2$ such that $V_2 \times_{X_2} X = V$. Similarly, we can find a standard open $V_1 \subset Y_1$ such that $V_1 \times_{X_1} X = V$. After replacing Y, Y_1, Y_2 by V, V_1, V_2 we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions and $Y_1 \times_{X_1} X = Y = X \times_{X_2} Y_2$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$, $Y = \mathrm{Spec}(B)$, $Y_i = \mathrm{Spec}(B_i)$. Then we can consider the affine schemes

$$X' = \mathrm{Spec}(A_1 \times_A A_2) = \mathrm{Spec}(A') \quad \text{and} \quad Y' = \mathrm{Spec}(B_1 \times_B B_2) = \mathrm{Spec}(B')$$

Observe that $X' = X_1 \amalg_X X_2$ and $Y' = Y_1 \amalg_Y Y_2$, see More on Morphisms, Lemma 37.14.1. By More on Algebra, Lemma 15.5.1 the rings A' and B' are of finite type over $\mathbf{Z}$. By More on Algebra, Lemma 15.6.4 we have $B' \otimes_A A_1 = B_1$ and $B' \times_A A_2 = B_2$. In particular a fibre of $Y' \to X'$ over a point of $X' = X_1 \amalg_X X_2$ is always equal to either a fibre of $Y_1 \to X_1$ or a fibre of $Y_2 \to X_2$. By More on Algebra, Lemma 15.6.8 the ring map $A' \to B'$ is flat. Thus by Discriminants, Lemma 49.10.1 part (3) we conclude that Y'/X' is an object of $\mathcal{C}$. Consider now the commutative diagram

$$\begin{tikzcd} & Y/X \arrow[ld, "b_1/a_1"'] \arrow[rd, "b_2/a_2"] & \\ Y_1/X_1 \arrow[rd] & & Y_2/X_2 \arrow[ld] \\ & Y'/X' & \end{tikzcd}$$

Now we would be done if Y'/X' is an object of $\mathcal{C}_{nice}$, but this is almost never the case. Namely, then pulling back $c^p_{Y'/X'}$ around the two sides of the square, we would obtain the desired conclusion. To get around the problem that Y'/X' is not in $\mathcal{C}_{nice}$ we note the arguments above show that, after possibly shrinking all of the schemes $X, Y, X_1, Y_1, X_2, Y_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

$$\begin{tikzcd} & Y/X \arrow[dl, "b_1/a_1"'] \arrow[dr, "b_2/a_2"] & \\ Y_1/X_1 \arrow[dr] & & Y_2/X_2 \arrow[dl] \\ & Y'/X' \arrow[d] & \\ & Y_{n,d}/X_{n,d} & \end{tikzcd}$$

and then we can use the already given argument by pulling back from $c^p_{Y_{n,d}/X_{n,d}}$. This finishes the proof. $\square$

50.19. Trace maps on de Rham complexes

0FK6 A reference for some of the material in this section is [Gar84]. Let S be a scheme. Let $f : Y \to X$ be a finite locally free morphism of schemes over S. Then there is a trace map $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 49.3. In this situation a trace map on de Rham complexes is a map of complexes

$$\Theta_{Y/X} : f_*\Omega^\bullet_{Y/S} \longrightarrow \Omega^\bullet_{X/S}$$

such that $\Theta_{Y/X}$ is equal to Trace_f in degree 0 and satisfies

$$\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$$

for local sections ω of $\Omega^\bullet_{X/S}$ and η of $f_*\Omega^\bullet_{Y/S}$. It is not clear to us whether such a trace map $\Theta_{Y/X}$ exists for every finite locally free morphism $Y \to X$; please email stacks.project@gmail.com if you have a counterexample or a proof.

0FK7 **Example 50.19.1.** Here is an example where we do not have a trace map on de Rham complexes. For example, consider the $\mathbf{C}$-algebra $B = \mathbf{C}[x, y]$ with action of $G = \{\pm 1\}$ given by $x \mapsto -x$ and $y \mapsto -y$. The invariants $A = B^G$ form a normal domain of finite type over $\mathbf{C}$ generated by x^2, xy, y^2. We claim that for the inclusion $A \subset B$ there is no reasonable trace map $\Omega_{B/\mathbf{C}} \to \Omega_{A/\mathbf{C}}$ on 1-forms. Namely, consider the element $\omega = x\text{d}y \in \Omega_{B/\mathbf{C}}$. Since ω is invariant under the action of G if a "reasonable" trace map exists, then 2ω should be in the image of $\Omega_{A/\mathbf{C}} \to \Omega_{B/\mathbf{C}}$. This is not the case: there is no way to write 2ω as a linear combination of $\text{d}(x^2)$, $\text{d}(xy)$, and $\text{d}(y^2)$ even with coefficients in B. This example contradicts the main theorem in [Zan99].

0FLB **Lemma 50.19.2.** *There exists a unique rule that to every finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_X$-module maps*

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/\mathbf{Z}} \longrightarrow \Omega^p_{X/\mathbf{Z}}$$

satisfying the following properties

(1) *the composition with $\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbf{Z}}$ is equal to $\text{id} \otimes \text{Trace}_f$ where $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ is the map from Discriminants, Section 49.3,*

(2) the rule is compatible with base change.

Proof. First, assume that X is locally Noetherian. By Lemma 50.18.3 we have a canonical map

$$c^p_{Y/X} : \Omega^p_{Y/S} \longrightarrow f^*\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

By Discriminants, Proposition 49.13.2 we have a canonical isomorphism

$$c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$$

mapping $\delta(NL_{Y/X})$ to $\tau_{Y/X}$. Combined these maps give

$$c^p_{Y/X} \otimes c_{Y/X} : \Omega^p_{Y/S} \longrightarrow f^*\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

By Discriminants, Section 49.5 this is the same thing as a map

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/S} \longrightarrow \Omega^p_{X/S}$$

Recall that the relationship between $c^p_{Y/X} \otimes c_{Y/X}$ and $\Theta^p_{Y/X}$ uses the evaluation map $f_*\omega_{Y/X} \to \mathcal{O}_X$ which sends $\tau_{Y/X}$ to $\text{Trace}_f(1)$, see Discriminants, Section 49.5. Hence property (1) holds. Property (2) holds for base changes by $X' \to X$ with X' locally Noetherian because both $c^p_{Y/X}$ and $c_{Y/X}$ are compatible with such base changes. For $f : Y \to X$ finite syntomic and X locally Noetherian, we will continue to denote $\Theta^p_{Y/X}$ the solution we've just found.

Uniqueness. Suppose that we have a finite syntomic morphism $f : Y \to X$ such that X is smooth over $\text{Spec}(\mathbf{Z})$ and f is étale over a dense open of X. We claim that in this case $\Theta^p_{Y/X}$ is uniquely determined by property (1). Namely, consider the maps

$$\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbf{Z}} \to \Omega^p_{X/\mathbf{Z}}$$

The sheaf $\Omega^p_{X/\mathbf{Z}}$ is torsion free (by the assumed smoothness), hence it suffices to check that the restriction of $\Theta^p_{Y/X}$ is uniquely determined over the dense open over which f is étale, i.e., we may assume f is étale. However, if f is étale, then $f^*\Omega_{X/\mathbf{Z}} = \Omega_{Y/\mathbf{Z}}$ hence the first arrow in the displayed equation is an isomorphism. Since we've pinned down the composition, this guarantees uniqueness.

Let $f : Y \to X$ be a finite syntomic morphism of locally Noetherian schemes. Let $x \in X$. By Discriminants, Lemma 49.11.7 we can find $d \geq 1$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & V_d \arrow[d] \\ X & U \arrow[l] \arrow[r] & U_d \end{tikzcd}$$

such that $x \in U \subset X$ is open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Thus $\Theta^p_{Y/X}|_V$ is the pullback of the map $\Theta^p_{V_d/U_d}$. However, by the discussion on uniqueness above and Discriminants, Lemmas 49.11.4 and 49.11.5 the map $\Theta^p_{V_d/U_d}$ is uniquely determined by the requirement (1). Hence uniqueness holds.

At this point we know that we have existence and uniqueness for all finite syntomic morphisms $Y \to X$ with X locally Noetherian. We could now give an argument similar to the proof of Lemma 50.18.3 to extend to general X. However, instead it possible to directly use absolute Noetherian approximation to finish the proof. Namely, to construct $\Theta^p_{Y/X}$ it suffices to do so Zariski locally on X (provided we also show the uniqueness). Hence we may assume X is affine (small detail omitted). Then we can write $X = \lim_{i \in I} X_i$ as the limit over a directed set I of Noetherian affine schemes. By Algebra, Lemma 10.127.8 we can find $0 \in I$ and a finitely presented morphism of affines $f_0 : Y_0 \to X_0$ whose base change to X is $Y \to X$. After increasing 0 we may assume $Y_0 \to X_0$ is finite and syntomic, see Algebra, Lemma 10.168.9 and 10.168.3. For $i \geq 0$ also the base change $f_i : Y_i = Y_0 \times_{X_0} X_i \to X_i$ is finite syntomic. Then

$$\Gamma(X, f_*\Omega^p_{Y/\mathbf{Z}}) = \Gamma(Y, \Omega^p_{Y/\mathbf{Z}}) = \mathrm{colim}_{i \geq 0} \Gamma(Y_i, \Omega^p_{Y_i/\mathbf{Z}}) = \mathrm{colim}_{i \geq 0} \Gamma(X_i, f_{i,*}\Omega^p_{Y_i/\mathbf{Z}})$$

Hence we can (and are forced to) define $\Theta^p_{Y/X}$ as the colimit of the maps $\Theta^p_{Y_i/X_i}$. This map is compatible with any cartesian diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

with X' affine as we know this for the case of Noetherian affine schemes by the arguments given above (small detail omitted; hint: if we also write $X' = \lim_{j \in J} X'_j$ then for every $i \in I$ there is a $j \in J$ and a morphism $X'_j \to X_i$ compatible with the morphism $X' \to X$). This finishes the proof. □

0FLC Proposition 50.19.3. Let $f : Y \to X$ be a finite syntomic morphism of schemes. The maps $\Theta^p_{Y/X}$ of Lemma 50.19.2 define a map of complexes [Gar84]

$$\Theta_{Y/X} : f_*\Omega^\bullet_{Y/\mathbf{Z}} \longrightarrow \Omega^\bullet_{X/\mathbf{Z}}$$

with the following properties

(1) in degree 0 we get $\mathrm{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 49.3,
(2) we have $\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$ for ω in $\Omega^\bullet_{X/\mathbf{Z}}$ and η in $f_*\Omega^\bullet_{Y/\mathbf{Z}}$,
(3) if f is a morphism over a base scheme S, then $\Theta_{Y/X}$ induces a map of complexes $f_*\Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}$.

Proof. By Discriminants, Lemma 49.11.7 for every $x \in X$ we can find $d \geq 1$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & V_d \arrow[r] \arrow[d] & Y_d = \mathrm{Spec}(B_d) \arrow[d] \\ X & U \arrow[l] \arrow[r] & U_d \arrow[r] & X_d = \mathrm{Spec}(A_d) \end{tikzcd}$$

such that $x \in U \subset X$ is affine open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ and observe that $B = A \otimes_{A_d} B_d$ and recall that $B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$. Suppose we have $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_s \in B$. We may write $b_j = \sum a_{j,l} e_d$ with $a_{j,l} \in A$. Set $N = r + sd$ and consider the factorizations

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V' = \mathbf{A}^N \times V_d \arrow[r] \arrow[d] & V_d \arrow[d] \\ U \arrow[r] & U' = \mathbf{A}^N \times U_d \arrow[r] & U_d \end{tikzcd}$$

Here the horizontal lower right arrow is given by the morphism $U \to U_d$ (from the earlier diagram) and the morphism $U \to \mathbf{A}^N$ given by $a_1, \ldots, a_r, a_{1,1}, \ldots, a_{s,d}$. Then we see that the functions $a_1, \ldots, a_r$ are in the image of $\Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ and the functions $b_1, \ldots, b_s$ are in the image of $\Gamma(V', \mathcal{O}_{V'}) \to \Gamma(V, \mathcal{O}_V)$. In this way we see that for any finite collection of elements[4] of the groups

$$\Gamma(V, \Omega^i_{Y/\mathbf{Z}}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U, \Omega^j_{X/\mathbf{Z}}), \quad j = 0, 1, 2, \ldots$$

we can find a factorizations $V \to V' \to V_d$ and $U \to U' \to U_d$ with $V' = \mathbf{A}^N \times V_d$ and $U' = \mathbf{A}^N \times U_d$ as above such that these sections are the pullbacks of sections from

$$\Gamma(V', \Omega^i_{V'/\mathbf{Z}}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U', \Omega^j_{U'/\mathbf{Z}}), \quad j = 0, 1, 2, \ldots$$

The upshot of this is that to check $\mathrm{d} \circ \Theta_{Y/X} = \Theta_{Y/X} \circ \mathrm{d}$ it suffices to check this is true for $\Theta_{V'/U'}$. Similarly, for property (2) of the lemma.

By Discriminants, Lemmas 49.11.4 and 49.11.5 the scheme U_d is smooth and the morphism $V_d \to U_d$ is étale over a dense open of U_d. Hence the same is true for the morphism $V' \to U'$. Since $\Omega_{U'/\mathbf{Z}}$ is locally free and hence $\Omega^p_{U'/\mathbf{Z}}$ is torsion free, it suffices to check the desired relations after restricting to the open over which V' is finite étale. Then we may check the relations after a surjective étale base change. Hence we may split the finite étale cover and assume we are looking at a morphism of the form

$$\coprod\nolimits_{i=1,\ldots,d} W \longrightarrow W$$

with W smooth over $\mathbf{Z}$. In this case any local properties of our construction are trivial to check (provided they are true). This finishes the proof of (1) and (2).

Finally, we observe that (3) follows from (2) because $\Omega_{Y/S}$ is the quotient of $\Omega_{Y/\mathbf{Z}}$ by the submodule generated by pullbacks of local sections of $\Omega_{S/\mathbf{Z}}$. $\square$

0FLD Example 50.19.4. Let A be a ring. Let $f = x^d + \sum_{0 \leq i < d} a_{d-i} x^i \in A[x]$. Let $B = A[x]/(f)$. By Proposition 50.19.3 we have a morphism of complexes

$$\Theta_{B/A} : \Omega^\bullet_B \longrightarrow \Omega^\bullet_A$$

In particular, if $t \in B$ denotes the image of $x \in A[x]$ we can consider the elements

$$\Theta_{B/A}(t^i \mathrm{d}t) \in \Omega^1_A, \quad i = 0, \ldots, d-1$$

What are these elements? By the same principle as used in the proof of Proposition 50.19.3 it suffices to compute this in the universal case, i.e., when $A = \mathbf{Z}[a_1, \ldots, a_d]$ or even when A is replaced by the fraction field $\mathbf{Q}(a_1, \ldots, a_d)$. Writing symbolically

$$f = \prod\nolimits_{i=1,\ldots,d} (x - \alpha_i)$$

we see that over $\mathbf{Q}(\alpha_1, \ldots, \alpha_d)$ the algebra B becomes split:

$$\mathbf{Q}(a_0, \ldots, a_{d-1})[x]/(f) \longrightarrow \prod\nolimits_{i=1,\ldots,d} \mathbf{Q}(\alpha_1, \ldots, \alpha_d), \quad t \longmapsto (\alpha_1, \ldots, \alpha_d)$$

Thus for example

$$\Theta(\mathrm{d}t) = \sum \mathrm{d}\alpha_i = -\mathrm{d}a_1$$

[4]After all these elements will be finite sums of elements of the form $a_0 \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_i$ with $a_0, \ldots, a_i \in A$ or finite sums of elements of the form $b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_j$ with $b_0, \ldots, b_j \in B$.

Next, we have

$$\Theta(t\mathrm{d}t) = \sum \alpha_i \mathrm{d}\alpha_i = a_1\mathrm{d}a_1 - \mathrm{d}a_2$$

Next, we have

$$\Theta(t^2\mathrm{d}t) = \sum \alpha_i^2 \mathrm{d}\alpha_i = -a_1^2\mathrm{d}a_1 + a_1\mathrm{d}a_2 + a_2\mathrm{d}a_1 - \mathrm{d}a_3$$

(modulo calculation error), and so on. This suggests that if $f(x) = x^d - a$ then

$$\Theta_{B/A}(t^i\mathrm{d}t) = \begin{cases} 0 & \text{if} \quad i = 0, \ldots, d-2 \\ \mathrm{d}a & \text{if} \quad i = d-1 \end{cases}$$

in Ω_A. This is true for in this particular case one can do the calculation for the extension $\mathbf{Q}(a)[x]/(x^d - a)$ to verify this directly.

0FW2 Lemma 50.19.5. Let p be a prime number. Let $X \to S$ be a smooth morphism of relative dimension d of schemes in characteristic p. The relative Frobenius $F_{X/S} : X \to X^{(p)}$ of X/S (Varieties, Definition 33.36.4) is finite syntomic and the corresponding map

$$\Theta_{X/X^{(p)}} : F_{X/S,*}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X^{(p)}/S}$$

is zero in all degrees except in degree d where it defines a surjection.

Proof. Observe that $F_{X/S}$ is a finite morphism by Varieties, Lemma 33.36.8. To prove that $F_{X/S}$ is flat, it suffices to show that the morphism $F_{X/S,s} : X_s \to X_s^{(p)}$ between fibres is flat for all $s \in S$, see More on Morphisms, Theorem 37.16.2. Flatness of $X_s \to X_s^{(p)}$ follows from Algebra, Lemma 10.128.1 (and the finiteness already shown). By More on Morphisms, Lemma 37.62.10 the morphism $F_{X/S}$ is a local complete intersection morphism. Hence $F_{X/S}$ is finite syntomic (see More on Morphisms, Lemma 37.62.8).

For every point $x \in X$ we may choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d, "\pi"'] \\ S & \mathbf{A}^d_S \arrow[l] \end{tikzcd}$$

where π is étale and $x \in U$ is open in X, see Morphisms, Lemma 29.36.20. Observe that $\mathbf{A}^d_S \to \mathbf{A}^d_S$, $(x_1, \ldots, x_d) \mapsto (x_1^p, \ldots, x_d^p)$ is the relative Frobenius for $\mathcal{A}^d_S$ over S. The commutative diagram

$$\begin{tikzcd} U \arrow[r, "F_{X/S}"'] \arrow[d, "\pi"'] & U^{(p)} \arrow[d, "\pi^{(p)}"] \\ \mathbf{A}^d_S \arrow[r, "x_i \mapsto x_i^p"] & \mathbf{A}^d_S \end{tikzcd}$$

of Varieties, Lemma 33.36.5 for $\pi : U \to \mathbf{A}^d_S$ is cartesian by Étale Morphisms, Lemma 41.14.3. Since the construction of Θ is compatible with base change and since $\Omega_{U/S} = \pi^*\Omega_{\mathbf{A}^d_S/S}$ (Lemma 50.2.2) we conclude that it suffices to show the lemma for $\mathbf{A}^d_S$.

Let A be a ring of characteristic p. Consider the unique A-algebra homomorphism $A[y_1, \ldots, y_d] \to A[x_1, \ldots, x_d]$ sending y_i to x_i^p. The arguments above reduce us to computing the map

$$\Theta^i : \Omega^i_{A[x_1,\ldots,x_d]/A} \to \Omega^i_{A[y_1,\ldots,y_d]/A}$$

We urge the reader to do the computation in this case for themselves. As in Example 50.19.4 we may reduce this to computing a formula for Θ^i in the universal case

$$\mathbf{Z}[y_1, \ldots, y_d] \to \mathbf{Z}[x_1, \ldots, x_d], \quad y_i \mapsto x_i^p$$

In turn, we can find the formula for Θ^i by computing in the complex case, i.e., for the $\mathbf{C}$-algebra map

$$\mathbf{C}[y_1, \ldots, y_d] \to \mathbf{C}[x_1, \ldots, x_d], \quad y_i \mapsto x_i^p$$

We may even invert $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$. In this case, we have $\mathrm{d}x_i = p^{-1}x_i^{-p+1}\mathrm{d}y_i$. Hence we see that

$$\begin{aligned}\Theta^i(x_1^{e_1} \ldots x_d^{e_d}\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_i) &= p^{-i}\Theta^i(x_1^{e_1-p+1} \ldots x_i^{e_i-p+1}x_{i+1}^{e_{i+1}} \ldots x_d^{e_d}\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i) \\ &= p^{-i}\mathrm{Trace}(x_1^{e_1-p+1} \ldots x_i^{e_i-p+1}x_{i+1}^{e_{i+1}} \ldots x_d^{e_d})\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i\end{aligned}$$

by the properties of Θ^i. An elementary computation shows that the trace in the expression above is zero unless $e_1, \ldots, e_i$ are congruent to -1 modulo p and $e_{i+1}, \ldots, e_d$ are divisible by p. Moreover, in this case we obtain

$$p^{d-i}y_1^{(e_1-p+1)/p} \ldots y_i^{(e_i-p+1)/p}y_{i+1}^{e_{i+1}/p} \ldots y_d^{e_d/p}\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i$$

We conclude that we get zero in characteristic p unless $d = i$ and in this case we get every possible d-form. □

50.20. Poincaré duality

0FW3 In this section we prove Poincar'e duality for the de Rham cohomology of a proper smooth scheme over a field. Let us first explain how this works for Hodge cohomology.

0FW4 **Lemma 50.20.1.** Let k be a field. Let X be a nonempty smooth proper scheme over k equidimensional of dimension d. There exists a k-linear map

$$t : H^d(X, \Omega^d_{X/k}) \longrightarrow k$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all p, q the pairing

$$H^q(X, \Omega^p_{X/k}) \times H^{d-q}(X, \Omega^{d-p}_{X/k}) \longrightarrow k, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is perfect.

Proof. By Duality for Schemes, Lemma 48.27.1 we have $\omega_X^\bullet = \Omega^d_{X/k}[d]$. Since $\Omega_{X/k}$ is locally free of rank d (Morphisms, Lemma 29.34.12) we have

$$\Omega^d_{X/k} \otimes_{\mathcal{O}_X} (\Omega^p_{X/k})^\vee \cong \Omega^{d-p}_{X/k}$$

Thus we obtain a k-linear map $t : H^d(X, \Omega^d_{X/k}) \to k$ such that the statement is true by Duality for Schemes, Lemma 48.27.4. In particular the pairing $H^0(X, \mathcal{O}_X) \times H^d(X, \Omega^d_{X/k}) \to k$ is perfect, which implies that any k-linear map $t' : H^d(X, \Omega^d_{X/k}) \to k$ is of the form $\xi \mapsto t(g\xi)$ for some $g \in H^0(X, \mathcal{O}_X)$. Of course, in order for t' to still produce a duality between $H^0(X, \mathcal{O}_X)$ and $H^d(X, \Omega^d_{X/k})$ we need g to be a unit.

Denote $\langle -, - \rangle_{p,q}$ the pairing constructed using t and denote $\langle -, - \rangle'_{p,q}$ the pairing constructed using t'. Clearly we have

$$\langle \xi, \xi' \rangle'_{p,q} = \langle g\xi, \xi' \rangle_{p,q}$$

for $\xi \in H^q(X, \Omega^p_{X/k})$ and $\xi' \in H^{d-q}(X, \Omega^{d-p}_{X/k})$. Since g is a unit, i.e., invertible, we see that using t' instead of t we still get perfect pairings for all p, q. □

0FW5 Lemma 50.20.2. Let k be a field. Let X be a smooth proper scheme over k. The map

$$\mathrm{d} : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/k})$$

is zero.

Proof. Since X is smooth over k it is geometrically reduced over k, see Varieties, Lemma 33.25.4. Hence $H^0(X, \mathcal{O}_X) = \prod k_i$ is a finite product of finite separable field extensions k_i/k, see Varieties, Lemma 33.9.3. It follows that $\Omega_{H^0(X,\mathcal{O}_X)/k} = \prod \Omega_{k_i/k} = 0$ (see for example Algebra, Lemma 10.158.1). Since the map of the lemma factors as

$$H^0(X, \mathcal{O}_X) \to \Omega_{H^0(X,\mathcal{O}_X)/k} \to H^0(X, \Omega_{X/k})$$

by functoriality of the de Rham complex (see Section 50.2), we conclude. □

0FW6 Lemma 50.20.3. Let k be a field. Let X be a smooth proper scheme over k equidimensional of dimension d. The map

$$\mathrm{d} : H^d(X, \Omega^{d-1}_{X/k}) \to H^d(X, \Omega^d_{X/k})$$

is zero.

Proof. It is tempting to think this follows from a combination of Lemmas 50.20.2 and 50.20.1. However this doesn't work because the maps $\mathcal{O}_X \to \Omega^1_{X/k}$ and $\Omega^{d-1}_{X/k} \to \Omega^d_{X/k}$ are not $\mathcal{O}_X$-linear and hence we cannot use the functoriality discussed in Duality for Schemes, Remark 48.27.3 to conclude the map in Lemma 50.20.2 is dual to the one in this lemma.

We may replace X by a connected component of X. Hence we may assume X is irreducible. By Varieties, Lemmas 33.25.4 and 33.9.3 we see that $k' = H^0(X, \mathcal{O}_X)$ is a finite separable extension k'/k. Since $\Omega_{k'/k} = 0$ (see for example Algebra, Lemma 10.158.1) we see that $\Omega_{X/k} = \Omega_{X/k'}$ (see Morphisms, Lemma 29.32.9). Thus we may replace k by k' and assume that $H^0(X, \mathcal{O}_X) = k$.

Assume $H^0(X, \mathcal{O}_X) = k$. We conclude that $\dim H^d(X, \Omega^d_{X/k}) = 1$ by Lemma 50.20.1. Assume first that the characteristic of k is a prime number p. Denote $F_{X/k} : X \to X^{(p)}$ the relative Frobenius of X over k; please keep in mind the facts proved about this morphism in Lemma 50.19.5. Consider the commutative diagram

$$\begin{tikzcd}
H^d(X, \Omega^{d-1}_{X/k}) \arrow[r] \arrow[d] & H^d(X^{(p)}, F_{X/k,*}\Omega^{d-1}_{X/k}) \arrow[r, "\Theta^{d-1}"'] \arrow[d] & H^d(X^{(p)}, \Omega^{d-1}_{X^{(p)}/k}) \arrow[d] \\
H^d(X, \Omega^d_{X/k}) \arrow[r] & H^d(X^{(p)}, F_{X/k,*}\Omega^d_{X/k}) \arrow[r, "\Theta^d"] & H^d(X^{(p)}, \Omega^d_{X^{(p)}/k})
\end{tikzcd}$$

The left two horizontal arrows are isomorphisms as $F_{X/k}$ is finite, see Cohomology of Schemes, Lemma 30.2.4. The right square commutes as $\Theta_{X^{(p)}/X}$ is a morphism of complexes and Θ^{d-1} is zero. Thus it suffices to show that Θ^d is nonzero (because the dimension of the source of the map Θ^d is 1 by the discussion above). However, we know that

$$\Theta^d : F_{X/k,*}\Omega^d_{X/k} \to \Omega^d_{X^{(p)}/k}$$

is surjective and hence surjective after applying the right exact functor $H^d(X^{(p)}, -)$ (right exactness by the vanishing of cohomology beyond d as follows from Cohomology, Proposition 20.20.7). Finally, $H^d(X^{(d)}, \Omega^d_{X^{(d)}/k})$ is nonzero for example because it is dual to $H^0(X^{(d)}, \mathcal{O}_{X^{(p)}})$ by Lemma 50.20.1 applied to $X^{(p)}$ over k. This finishes the proof in this case.

Finally, assume the characteristic of k is 0. We can write k as the filtered colimit of its finite type $\mathbf{Z}$-subalgebras R. For one of these we can find a cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

such that $Y \to \mathrm{Spec}(R)$ is smooth of relative dimension d and proper. See Limits, Lemmas 32.10.1, 32.8.9, 32.18.4, and 32.13.1. The modules $M^{i,j} = H^j(Y, \Omega^i_{Y/R})$ are finite R-modules, see Cohomology of Schemes, Lemma 30.19.2. Thus after replacing R by a localization we may assume all of these modules are finite free. We have $M^{i,j} \otimes_R k = H^j(X, \Omega^i_{X/k})$ by flat base change (Cohomology of Schemes, Lemma 30.5.2). Thus it suffices to show that $M^{d-1,d} \to M^{d,d}$ is zero. This is a map of finite free modules over a domain, hence it suffices to find a dense set of primes $\mathfrak{p} \subset R$ such that after tensoring with $\kappa(\mathfrak{p})$ we get zero. Since R is of finite type over $\mathbf{Z}$, we can take the collection of primes $\mathfrak{p}$ whose residue field has positive characteristic (details omitted). Observe that

$$M^{d-1,d} \otimes_R \kappa(\mathfrak{p}) = H^d(Y_{\kappa(\mathfrak{p})}, \Omega^{d-1}_{Y_{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p})})$$

for example by Limits, Lemma 32.19.2. Similarly for $M^{d,d}$. Thus we see that $M^{d-1,d} \otimes_R \kappa(\mathfrak{p}) \to M^{d,d} \otimes_R \kappa(\mathfrak{p})$ is zero by the case of positive characteristic handled above. $\square$

0FW7 Proposition 50.20.4. Let k be a field. Let X be a nonempty smooth proper scheme over k equidimensional of dimension d. There exists a k-linear map

$$t : H^{2d}_{dR}(X/k) \longrightarrow k$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all i the pairing

$$H^i_{dR}(X/k) \times H^{2d-i}_{dR}(X/k) \longrightarrow k, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is perfect.

Proof. By the Hodge-to-de Rham spectral sequence (Section 50.6), the vanishing of $\Omega^i_{X/k}$ for $i > d$, the vanishing in Cohomology, Proposition 20.20.7 and the results of Lemmas 50.20.2 and 50.20.3 we see that $H^0_{dR}(X/k) = H^0(X, \mathcal{O}_X)$ and

$H^d(X, \Omega^d_{X/k}) = H^{2d}_{dR}(X/k)$. More precisely, these identifications come from the maps of complexes

$$\Omega^\bullet_{X/k} \to \mathcal{O}_X[0] \quad \text{and} \quad \Omega^d_{X/k}[-d] \to \Omega^\bullet_{X/k}$$

Let us choose $t : H^{2d}_{dR}(X/k) \to k$ which via this identification corresponds to a t as in Lemma 50.20.1. Then in any case we see that the pairing displayed in the lemma is perfect for $i = 0$.

Denote $\underline{k}$ the constant sheaf with value k on X. Let us abbreviate $\Omega^\bullet = \Omega^\bullet_{X/k}$. Consider the map (50.4.0.1) which in our situation reads

$$\wedge : \text{Tot}(\Omega^\bullet \otimes_{\underline{k}} \Omega^\bullet) \longrightarrow \Omega^\bullet$$

For every integer $p = 0, 1, \ldots, d$ this map annihilates the subcomplex $\text{Tot}(\sigma_{>p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet)$ for degree reasons. Hence we find that the restriction of $\wedge$ to the subcomplex $\text{Tot}(\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet)$ factors through a map of complexes

$$\gamma_p : \text{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \longrightarrow \Omega^\bullet$$

Using the same procedure as in Section 50.4 we obtain cup products

$$H^i(X, \sigma_{\leq p}\Omega^\bullet) \times H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet) \longrightarrow H^{2d}_{dR}(X, \Omega^\bullet)$$

We will prove by induction on p that these cup products via t induce perfect pairings between $H^i(X, \sigma_{\leq p}\Omega^\bullet)$ and $H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet)$. For $p = d$ this is the assertion of the proposition.

The base case is $p = 0$. In this case we simply obtain the pairing between $H^i(X, \mathcal{O}_X)$ and $H^{d-i}(X, \Omega^d)$ of Lemma 50.20.1 and the result is true.

Induction step. Say we know the result is true for p. Then we consider the distinguished triangle

$$\Omega^{p+1}[-p-1] \to \sigma_{\leq p+1}\Omega^\bullet \to \sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$$

and the distinguished triangle

$$\sigma_{\geq d-p}\Omega^\bullet \to \sigma_{\geq d-p-1}\Omega^\bullet \to \Omega^{d-p-1}[-d+p+1] \to (\sigma_{\geq d-p}\Omega^\bullet)[1]$$

Observe that both are distinguished triangles in the homotopy category of complexes of sheaves of $\underline{k}$-modules; in particular the maps $\sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$ and $\Omega^{d-p-1}[-d+p+1] \to (\sigma_{\geq d-p}\Omega^\bullet)[1]$ are given by actual maps of complexes, namely using the differential $\Omega^p \to \Omega^{p+1}$ and the differential $\Omega^{d-p-1} \to \Omega^{d-p}$. Consider the long exact cohomology sequences associated to these distinguished triangles

$$
\begin{tikzcd}
H^{i-1}(X, \sigma_{\leq p}\Omega^\bullet) \arrow[d, "a"] & H^{2d-i+1}(X, \sigma_{\geq d-p}\Omega^\bullet) \\
H^i(X, \Omega^{p+1}[-p-1]) \arrow[d, "b"] & H^{2d-i}(X, \Omega^{d-p-1}[-d+p+1]) \arrow[u, "a'"] \\
H^i(X, \sigma_{\leq p+1}\Omega^\bullet) \arrow[d, "c"] & H^{2d-i}(X, \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[u, "b'"] \\
H^i(X, \sigma_{\leq p}\Omega^\bullet) \arrow[d, "d"] & H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet) \arrow[u, "c'"] \\
H^{i+1}(X, \Omega^{p+1}[-p-1]) & H^{2d-i-1}(X, \Omega^{d-p-1}[-d+p+1]) \arrow[u, "d'"]
\end{tikzcd}
$$

By induction and Lemma 50.20.1 we know that the pairings constructed above between the k-vectorspaces on the first, second, fourth, and fifth rows are perfect. By the 5-lemma, in order to show that the pairing between the cohomology groups in the middle row is perfect, it suffices to show that the pairs (a, a'), (b, b'), (c, c'), and (d, d') are compatible with the given pairings (see below).

Let us prove this for the pair (c, c'). Here we observe simply that we have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \arrow[d, "\gamma_p"] & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \arrow[l] \arrow[d] \\
\Omega^\bullet & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[l, "\gamma_{p+1}"']
\end{tikzcd}
$$

Hence if we have $\alpha \in H^i(X, \sigma_{\leq p+1}\Omega^\bullet)$ and $\beta \in H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet)$ then we get $\gamma_p(\alpha \cup c'(\beta)) = \gamma_{p+1}(c(\alpha) \cup \beta)$ by functoriality of the cup product.

Similarly for the pair (b, b') we use the commutative diagram

$$
\begin{tikzcd}
\mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[d, "\gamma_{p+1}"] & \mathrm{Tot}(\Omega^{p+1}[-p-1] \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[l] \arrow[d] \\
\Omega^\bullet & \Omega^{p+1}[-p-1] \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p+1] \arrow[l, "\wedge"']
\end{tikzcd}
$$

and argue in the same manner.

For the pair (d, d') we use the commutative diagram

$$
\begin{tikzcd}
\Omega^{p+1}[-p] \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p] \arrow[d] & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p]) \arrow[l] \arrow[d] \\
\Omega^\bullet & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \arrow[l]
\end{tikzcd}
$$

and we look at cohomology classes in $H^i(X, \sigma_{\leq p}\Omega^\bullet)$ and $H^{2d-i}(X, \Omega^{d-p-1}[-d+p])$. Changing i to $i-1$ we get the result for the pair (a, a') thereby finishing the proof that our pairings are perfect.

We omit the argument showing the uniqueness of t up to precomposing by multiplication by a unit in $H^0(X, \mathcal{O}_X)$. $\square$

50.21. Chern classes

0FW8 The results proved so far suffice to use the discussion in Weil Cohomology Theories, Section 45.12 to produce Chern classes in de Rham cohomology.

0FW9 Lemma 50.21.1. There is a unique rule which assigns to every quasi-compact and quasi-separated scheme X a total Chern class

$$c^{dR} : K_0(\text{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Z})$$

with the following properties

(1) we have $c^{dR}(\alpha + \beta) = c^{dR}(\alpha)c^{dR}(\beta)$ for $\alpha, \beta \in K_0(\text{Vect}(X))$,
(2) if $f : X \to X'$ is a morphism of quasi-compact and quasi-separated schemes, then $c^{dR}(f^*\alpha) = f^*c^{dR}(\alpha)$,
(3) given $\mathcal{L} \in \text{Pic}(X)$ we have $c^{dR}([\mathcal{L}]) = 1 + c^{dR}_1(\mathcal{L})$

The construction can easily be extended to all schemes, but to do so one needs to slightly upgrade the discussion in Weil Cohomology Theories, Section 45.12.

Proof. We will apply Weil Cohomology Theories, Proposition 45.12.1 to get this.

Let $\mathcal{C}$ be the category of all quasi-compact and quasi-separated schemes. This certainly satisfies conditions (1), (2), and (3) (a), (b), and (c) of Weil Cohomology Theories, Section 45.12.

As our contravariant functor A from $\mathcal{C}$ to the category of graded algebras will send X to $A(X) = \bigoplus_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Z})$ endowed with its cup product. Functoriality is discussed in Section 50.3 and the cup product in Section 50.4. For the additive maps c^A_1 we take c^{dR}_1 constructed in Section 50.9.

In fact, we obtain commutative algebras by Lemma 50.4.1 which shows we have axiom (1) for A.

To check axiom (2) for A it suffices to check that $H^*_{dR}(X \coprod Y/\mathbf{Z}) = H^*_{dR}(X/\mathbf{Z}) \times H^*_{dR}(Y/\mathbf{Z})$. This is a consequence of the fact that de Rham cohomology is constructed by taking the cohomology of a sheaf of differential graded algebras (in the Zariski topology).

Axiom (3) for A is just the statement that taking first Chern classes of invertible modules is compatible with pullbacks. This follows from the more general Lemma 50.9.1.

Axiom (4) for A is the projective space bundle formula which we proved in Proposition 50.14.1.

Axiom (5). Let X be a quasi-compact and quasi-separated scheme and let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free $\mathcal{O}_X$-modules of ranks $r+1$ and r. Denote $i : P' = \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E}) = P$ the corresponding incusion morphism. This is a morphism of smooth projective schemes over X which exhibits P' as an effective Cartier divisor on P. Thus by Lemma 50.15.7 the complex of log poles for $P' \subset P$

over $\mathbf{Z}$ is defined. Hence for $a \in A(P)$ with $i^*a = 0$ we have $a \cup c_1^A(\mathcal{O}_P(P')) = 0$ by Lemma 50.15.6. This finishes the proof. $\square$

0FWA Remark 50.21.2. The analogues of Weil Cohomology Theories, Lemmas 45.12.2 (splitting principle) and 45.12.3 (chern classes of tensor products) hold for de Rham Chern classes on quasi-compact and quasi-separated schemes. This is clear as we've shown in the proof of Lemma 50.21.1 that all the axioms of Weil Cohomology Theories, Section 45.12 are satisfied.

Working with schemes over $\mathbf{Q}$ we can construct a Chern character.

0FWB Lemma 50.21.3. There is a unique rule which assigns to every quasi-compact and quasi-separated scheme X over $\mathbf{Q}$ a "chern character"

$$ch^{dR} : K_0(\text{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Q})$$

with the following properties

(1) ch^{dR} is a ring map for all X,
(2) if $f : X' \to X$ is a morphism of quasi-compact and quasi-separated schemes over $\mathbf{Q}$, then $f^* \circ ch^{dR} = ch^{dR} \circ f^*$, and
(3) given $\mathcal{L} \in \text{Pic}(X)$ we have $ch^{dR}([\mathcal{L}]) = \exp(c_1^{dR}(\mathcal{L}))$.

The construction can easily be extended to all schemes over $\mathbf{Q}$, but to do so one needs to slightly upgrade the discussion in Weil Cohomology Theories, Section 45.12.

Proof. Exactly as in the proof of Lemma 50.21.1 one shows that the category of quasi-compact and quasi-separated schemes over $\mathbf{Q}$ together with the functor $A^*(X) = \bigoplus_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Q})$ satisfy the axioms of Weil Cohomology Theories, Section 45.12. Moreover, in this case $A(X)$ is a $\mathbf{Q}$-algebra for all X. Hence the lemma follows from Weil Cohomology Theories, Proposition 45.12.4. $\square$

50.22. A Weil cohomology theory

0FWC Let k be a field of characteristic 0. In this section we prove that the functor

$$X \longmapsto H^*_{dR}(X/k)$$

defines a Weil cohomology theory over k with coefficients in k as defined in Weil Cohomology Theories, Definition 45.11.4. We will proceed by checking the constructions earlier in this chapter provide us with data (D0), (D1), and (D2') satisfying axioms (A1) – (A9) of Weil Cohomology Theories, Section 45.14.

Throughout the rest of this section we fix the field k of characteristic 0 and we set $F = k$. Next, we take the following data

(D0) For our 1-dimensional F vector space $F(1)$ we take $F(1) = F = k$.
(D1) For our functor H^* we take the functor sending a smooth projective scheme X over k to $H^*_{dR}(X/k)$. Functoriality is discussed in Section 50.3 and the cup product in Section 50.4. We obtain graded commutative F-algebras by Lemma 50.4.1.
(D2') For the maps $c_1^H : \text{Pic}(X) \to H^2(X)(1)$ we use the de Rham first Chern class introduced in Section 50.9.

We are going to show axioms (A1) – (A9) hold.

In this paragraph, we are going to reduce the checking of the axioms to the case where k is algebraically closed by using Weil Cohomology Theories, Lemma 45.14.18. Denote k' the algebraic closure of k. Set $F' = k'$. We obtain data (D0), (D1), (D2') over k' with coefficient field F' in exactly the same way as above. By Lemma 50.3.5 there are functorial isomorphisms

$$H^{2d}_{dR}(X/k) \otimes_k k' \longrightarrow H^{2d}_{dR}(X_{k'}/k')$$

for X smooth and projective over k. Moreover, the diagrams

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "c_1^{dR}"'] \arrow[d] & H^2_{dR}(X/k) \arrow[d] \\ \mathrm{Pic}(X_{k'}) \arrow[r, "c_1^{dR}"] & H^2_{dR}(X_{k'}/k') \end{tikzcd}$$

commute by Lemma 50.9.1. This finishes the proof of the reduction.

Assume k is algebraically closed field of characteristic zero. We will show axioms (A1) – (A9) for the data (D0), (D1), and (D2') given above.

Axiom (A1). Here we have to check that $H^*_{dR}(X \coprod Y/k) = H^*_{dR}(X/k) \times H^*_{dR}(Y/k)$. This is a consequence of the fact that de Rham cohomology is constructed by taking the cohomology of a sheaf of differential graded algebras (in the Zariski topology).

Axiom (A2). This is just the statement that taking first Chern classes of invertible modules is compatible with pullbacks. This follows from the more general Lemma 50.9.1.

Axiom (A3). This follows from the more general Proposition 50.14.1.

Axiom (A4). This follows from the more general Lemma 50.15.6.

Already at this point, using Weil Cohomology Theories, Lemmas 45.14.1 and 45.14.2, we obtain a Chern character and cycle class maps

$$\gamma : \mathrm{CH}^*(X) \longrightarrow \bigoplus\nolimits_{i \geq 0} H^{2i}_{dR}(X/k)$$

for X smooth projective over k which are graded ring homomorphisms compatible with pullbacks between morphisms $f : X \to Y$ of smooth projective schemes over k.

Axiom (A5). We have $H^*_{dR}(\mathrm{Spec}(k)/k) = k = F$ in degree 0. We have the Künneth formula for the product of two smooth projective k-schemes by Lemma 50.8.2 (observe that the derived tensor products in the statement are harmless as we are tensoring over the field k).

Axiom (A7). This follows from Proposition 50.17.3.

Axiom (A8). Let X be a smooth projective scheme over k. By the explanatory text to this axiom in Weil Cohomology Theories, Section 45.14 we see that $k' = H^0(X, \mathcal{O}_X)$ is a finite separable k-algebra. It follows that $H^*_{dR}(\mathrm{Spec}(k')/k) = k'$ sitting in degree 0 because $\Omega_{k'/k} = 0$. By Lemma 50.20.2 we also have $H^0_{dR}(X, \mathcal{O}_X) = k'$ and we get the axiom.

Axiom (A6). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. Denote $\Delta : X \to X \times_{\mathrm{Spec}(k)} X$ the diagonal morphism of X over k. We have to show that there exists a k-linear map

$$\lambda : H^{2d}_{dR}(X/k) \longrightarrow k$$

such that $(1 \otimes \lambda)\gamma([\Delta]) = 1$ in $H^0_{dR}(X/k)$. Let us write

$$\gamma = \gamma([\Delta]) = \gamma_0 + \ldots + \gamma_{2d}$$

with $\gamma_i \in H^i_{dR}(X/k) \otimes_k H^{2d-i}_{dR}(X/k)$ the Künneth components. Our problem is to show that there is a linear map $\lambda : H^{2d}_{dR}(X/k) \to k$ such that $(1 \otimes \lambda)\gamma_0 = 1$ in $H^0_{dR}(X/k)$.

Let $X = \coprod X_i$ be the decomposition of X into connected and hence irreducible components. Then we have correspondingly $\Delta = \coprod \Delta_i$ with $\Delta_i \subset X_i \times X_i$. It follows that

$$\gamma([\Delta]) = \sum \gamma([\Delta_i])$$

and moreover $\gamma([\Delta_i])$ corresponds to the class of $\Delta_i \subset X_i \times X_i$ via the decomposition

$$H^*_{dR}(X \times X) = \prod\nolimits_{i,j} H^*_{dR}(X_i \times X_j)$$

We omit the details; one way to show this is to use that in $\mathrm{CH}^0(X \times X)$ we have idempotents $e_{i,j}$ corresponding to the open and closed subschemes $X_i \times X_j$ and to use that γ is a ring map which sends $e_{i,j}$ to the corresponding idempotent in the displayed product decomposition of cohomology. If we can find $\lambda_i : H^{2d}_{dR}(X_i/k) \to k$ with $(1 \otimes \lambda_i)\gamma([\Delta_i]) = 1$ in $H^0_{dR}(X_i/k)$ then taking $\lambda = \sum \lambda_i$ will solve the problem for X. Thus we may and do assume X is irreducible.

Proof of Axiom (A6) for X irreducible. Since k is algebraically closed we have $H^0_{dR}(X/k) = k$ because $H^0(X, \mathcal{O}_X) = k$ as X is a projective variety over an algebraically closed field (see Varieties, Lemma 33.9.3 for example). Let $x \in X$ be any closed point. Consider the cartesian diagram

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \arrow[r, "x \times \mathrm{id}"] & X \times_{\mathrm{Spec}(k)} X
\end{tikzcd}$$

Compatibility of γ with pullbacks implies that $\gamma([\Delta])$ maps to $\gamma([x])$ in $H^{2d}_{dR}(X/k)$, in other words, we have $\gamma_0 = 1 \otimes \gamma([x])$. We conclude two things from this: (a) the class $\gamma([x])$ is independent of x, (b) it suffices to show the class $\gamma([x])$ is nonzero, and hence (c) it suffices to find any zero cycle α on X such that $\gamma(\alpha) \neq 0$. To do this we choose a finite morphism

$$f : X \longrightarrow \mathbf{P}^d_k$$

To see such a morphism exist, see Intersection Theory, Section 43.23 and in particular Lemma 43.23.1. Observe that f is finite syntomic (local complete intersection morphism by More on Morphisms, Lemma 37.62.10 and flat by Algebra, Lemma 10.128.1). By Proposition 50.19.3 we have a trace map

$$\Theta_f : f_*\Omega^\bullet_{X/k} \longrightarrow \Omega^\bullet_{\mathbf{P}^d_k/k}$$

whose composition with the canonical map

$$\Omega^\bullet_{\mathbf{P}^d_k/k} \longrightarrow f_*\Omega^\bullet_{X/k}$$

is multiplication by the degree of f. Hence we see that we get a map

$$\Theta : H^{2d}_{dR}(X/k) \to H^{2d}_{dR}(\mathbf{P}^d_k/k)$$

such that $\Theta \circ f^*$ is multiplication by a positive integer. Hence if we can find a zero cycle on $\mathbf{P}^d_k$ whose class is nonzero, then we conclude by the compatibility of γ with pullbacks. This is true by Lemma 50.11.4 and this finishes the proof of axiom (A6).

Below we will use the following without further mention. First, by Weil Cohomology Theories, Remark 45.14.6 the map $\lambda_X : H^{2d}_{dR}(X/k) \to k$ is unique. Second, in the proof of axiom (A6) we have seen that $\lambda_X(\gamma([x])) = 1$ when X is irreducible, i.e., the composition of the cycle class map $\gamma : \mathrm{CH}^d(X) \to H^{2d}_{dR}(X/k)$ with λ_X is the degree map.

Axiom (A9). Let $Y \subset X$ be a nonempty smooth divisor on a nonempty smooth equidimensional projective scheme X over k of dimension d. We have to show that the diagram

$$\begin{tikzcd}
H^{2d-2}_{dR}(X/k) \arrow[r, "c^{dR}_1(\mathcal{O}_X(Y))\cap -"'] \arrow[d, "restriction"'] & H^{2d}_{dR}(X) \arrow[d, "\lambda_X"] \\
H^{2d-2}_{dR}(Y/k) \arrow[r, "\lambda_Y"] & k
\end{tikzcd}$$

commutes where λ_X and λ_Y are as in axiom (A6). Above we have seen that if we decompose $X = \coprod X_i$ into connected (equivalently irreducible) components, then we have correspondingly $\lambda_X = \sum \lambda_{X_i}$. Similarly, if we decompoese $Y = \coprod Y_j$ into connected (equivalently irreducible) components, then we have $\lambda_Y = \sum \lambda_{Y_j}$. Moreover, in this case we have $\mathcal{O}_X(Y) = \otimes_j \mathcal{O}_X(Y_j)$ and hence

$$c^{dR}_1(\mathcal{O}_X(Y)) = \sum\nolimits_j c^{dR}_1(\mathcal{O}_X(Y_j))$$

in $H^2_{dR}(X/k)$. A straightforward diagram chase shows that it suffices to prove the commutativity of the diagram in case X and Y are both irreducible. Then $H^{2d-2}_{dR}(Y/k)$ is 1-dimensional as we have Poincar'e duality for Y by Weil Cohomology Theories, Lemma 45.14.5. By axiom (A4) the kernel of restriction (left vertical arrow) is contained in the kernel of cupping with $c^{dR}_1(\mathcal{O}_X(Y))$. This means it suffices to find one cohomology class $a \in H^{2d-2}_{dR}(X)$ whose restriction to Y is nonzero such that we have commutativity in the diagram for a. Take any ample invertible module $\mathcal{L}$ and set

$$a = c^{dR}_1(\mathcal{L})^{d-1}$$

Then we know that $a|_Y = c^{dR}_1(\mathcal{L}|_Y)^{d-1}$ and hence

$$\lambda_Y(a|_Y) = \deg(c_1(\mathcal{L}|_Y)^{d-1} \cap [Y])$$

by our description of λ_Y above. This is a positive integer by Chow Homology, Lemma 42.41.4 combined with Varieties, Lemma 33.45.9. Similarly, we find

$$\lambda_X(c^{dR}_1(\mathcal{O}_X(Y)) \cap a) = \deg(c_1(\mathcal{O}_X(Y)) \cap c_1(\mathcal{L})^{d-1} \cap [X])$$

Since we know that $c_1(\mathcal{O}_X(Y)) \cap [X] = [Y]$ more or less by definition we have an equality of zero cycles

$$(Y \to X)_*\left(c_1(\mathcal{L}|_Y)^{d-1} \cap [Y]\right) = c_1(\mathcal{O}_X(Y)) \cap c_1(\mathcal{L})^{d-1} \cap [X]$$

on X. Thus these cycles have the same degree and the proof is complete.

0FWD Proposition 50.22.1. Let k be a field of characteristic zero. The functor that sends a smooth projective scheme X over k to $H^*_{dR}(X/k)$ is a Weil cohomology theory in the sense of Weil Cohomology Theories, Definition 45.11.4.

Proof. In the discussion above we showed that our data (D0), (D1), (D2') satisfies axioms (A1) – (A9) of Weil Cohomology Theories, Section 45.14. Hence we conclude by Weil Cohomology Theories, Proposition 45.14.17.

Please don't read what follows. In the proof of the assertions we also used Lemmas 50.3.5, 50.9.1, 50.15.6, 50.8.2, 50.20.2, and 50.11.4, Propositions 50.14.1, 50.17.3, and 50.19.3, Weil Cohomology Theories, Lemmas 45.14.18, 45.14.1, 45.14.2, and 45.14.5, Weil Cohomology Theories, Remark 45.14.6, Varieties, Lemmas 33.9.3 and 33.45.9, Intersection Theory, Section 43.23 and Lemma 43.23.1, More on Morphisms, Lemma 37.62.10, Algebra, Lemma 10.128.1, and Chow Homology, Lemma 42.41.4. ☐

0FWE Remark 50.22.2. In exactly the same manner as above one can show that Hodge cohomology $X \mapsto H^*_{Hodge}(X/k)$ equipped with c_1^{Hodge} determines a Weil cohomology theory. If we ever need this, we will precisely formulate and prove this here. This leads to the following amusing consequence: If the betti numbers of a Weil cohomology theory are independent of the chosen Weil cohomology theory (over our field k of characteristic 0), then the Hodge-to-de Rham spectral sequence degenerates at E_1! Of course, the degeneration of the Hodge-to-de Rham spectral sequence is known (see for example [DI87] for a marvelous algebraic proof), but it is by no means an easy result! This suggests that proving the independence of betti numbers is a hard problem as well and as far as we know is still an open problem. See Weil Cohomology Theories, Remark 45.11.5 for a related question.

50.23. Gysin maps for closed immersions

0G82 In this section we define the gysin map for closed immersions.

0G83 Remark 50.23.1. Let $X \to S$ be a morphism of schemes. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Let $Z \subset X$ be the closed subscheme cut out by $f_1, \ldots, f_c$. Below we will study the gysin map

0G84 (50.23.1.1)
$$\gamma^p_{f_1,\ldots,f_c} : \Omega^p_{Z/S} \longrightarrow \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$$

defined as follows. Given a local section ω of $\Omega^p_{Z/S}$ which is the restriction of a section $\tilde{\omega}$ of $\Omega^p_{X/S}$ we set

$$\gamma^p_{f_1,\ldots,f_c}(\omega) = c_{f_1,\ldots,f_c}(\tilde{\omega}|_Z) \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c$$

where $c_{f_1,\ldots,f_c} : \Omega^p_{X/S} \otimes \mathcal{O}_Z \to \mathcal{H}^c_Z(\Omega^p_{X/S})$ is the map constructed in Derived Categories of Schemes, Remark 36.6.10. This is well defined: given ω we can change our choice of $\tilde{\omega}$ by elements of the form $\sum f_i\omega'_i + \sum \mathrm{d}(f_i) \wedge \omega''_i$ which are mapped to zero by the construction.

0G85 Lemma 50.23.2. The gysin map (50.23.1.1) is compatible with the de Rham differentials on $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Z/S}$.

Proof. This follows from an almost trivial calculation once we correctly interpret this. First, we recall that the functor $\mathcal{H}^c_Z$ computed on the category of $\mathcal{O}_X$-modules agrees with the similarly defined functor on the category of abelian sheaves on X, see Cohomology, Lemma 20.34.8. Hence, the differential $\text{d} : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$ induces a map $\mathcal{H}^c_Z(\Omega^p_{X/S}) \to \mathcal{H}^c_Z(\Omega^{p+1}_{X/S})$. Moreover, the formation of the extended alternating Čech complex in Derived Categories of Schemes, Remark 36.6.4 works on the category of abelian sheaves. The map

$$\text{Coker}\left(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c}\right) \longrightarrow i_*\mathcal{H}^c_Z(\mathcal{F})$$

used in the construction of $c_{f_1,\ldots,f_c}$ in Derived Categories of Schemes, Remark 36.6.10 is well defined and functorial on the category of all abelian sheaves on X. Hence we see that the lemma follows from the equality

$$\text{d}\left(\frac{\tilde{\omega} \wedge \text{d}f_1 \wedge \ldots \wedge \text{d}f_c}{f_1 \ldots f_c}\right) = \frac{\text{d}(\tilde{\omega}) \wedge \text{d}f_1 \wedge \ldots \wedge \text{d}f_c}{f_1 \ldots f_c}$$

which is clear. □

0G86 Lemma 50.23.3. Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a closed immersion of finite presentation whose conormal sheaf $\mathcal{C}_{Z/X}$ is locally free of rank c. Then there is a canonical map

$$\gamma^p : \Omega^p_{Z/S} \to \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$$

which is locally given by the maps $\gamma^p_{f_1,\ldots,f_c}$ of Remark 50.23.1.

Proof. The assumptions imply that given $x \in Z \subset X$ there exists an open neighbourhood U of x such that Z is cut out by c elements $f_1, \ldots, f_c \in \mathcal{O}_X(U)$. Thus it suffices to show that given $f_1, \ldots, f_c$ and $g_1, \ldots, g_c$ in $\mathcal{O}_X(U)$ cutting out $Z \cap U$, the maps $\gamma^p_{f_1,\ldots,f_c}$ and $\gamma^p_{g_1,\ldots,g_c}$ are the same. To do this, after shrinking U we may assume $g_j = \sum a_{ji} f_i$ for some $a_{ji} \in \mathcal{O}_X(U)$. Then we have $c_{f_1,\ldots,f_c} = \det(a_{ji}) c_{g_1,\ldots,g_c}$ by Derived Categories of Schemes, Lemma 36.6.12. On the other hand we have

$$\text{d}(g_1) \wedge \ldots \wedge \text{d}(g_c) \equiv \det(a_{ji})\text{d}(f_1) \wedge \ldots \wedge \text{d}(f_c) \bmod (f_1, \ldots, f_c)\Omega^c_{X/S}$$

Combining these relations, a straightforward calculation gives the desired equality. □

0G87 Lemma 50.23.4. Let $X \to S$ and $i : Z \to X$ be as in Lemma 50.23.3. The gysin map γ^p is compatible with the de Rham differentials on $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Z/S}$.

Proof. We may check this locally and then it follows from Lemma 50.23.2. □

0G88 Lemma 50.23.5. Let $X \to S$ and $i : Z \to X$ be as in Lemma 50.23.3. Given $\alpha \in H^q(X, \Omega^p_{X/S})$ we have $\gamma^p(\alpha|_Z) = i^{-1}\alpha \wedge \gamma^0(1)$ in $H^q(Z, \mathcal{H}^c_Z(\Omega^{p+c}_{X/S}))$. Please see proof for notation.

Proof. The restriction $\alpha|_Z$ is the element of $H^q(Z, \Omega^p_{Z/S})$ given by functoriality for Hodge cohomology. Applying functoriality for cohomology using $\gamma^p : \Omega^p_{Z/S} \to \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$ we get get $\gamma^p(\alpha|_Z)$ in $H^q(Z, \mathcal{H}^c_Z(\Omega^{p+c}_{X/S}))$. This explains the left hand side of the formula.

To explain the right hand side, we first pullback by the map of ringed spaces $i : (Z, i^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$ to get the element $i^{-1}\alpha \in H^q(Z, i^{-1}\Omega^p_{X/S})$. Let $\gamma^0(1) \in$

$H^0(Z, \mathcal{H}^c_Z(\Omega^c_{X/S}))$ be the image of $1 \in H^0(Z, \mathcal{O}_Z) = H^0(Z, \Omega^0_{Z/S})$ by γ^0. Using cup product we obtain an element

$$i^{-1}\alpha \cup \gamma^0(1) \in H^{q+c}(Z, i^{-1}\Omega^p_{X/S} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{H}^c_Z(\Omega^c_{X/S}))$$

Using Cohomology, Remark 20.34.9 and wedge product there are canonical maps

$$i^{-1}\Omega^p_{X/S} \otimes^{\mathbf{L}}_{i^{-1}\mathcal{O}_X} R\mathcal{H}_Z(\Omega^c_{X/S}) \to R\mathcal{H}_Z(\Omega^p_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \Omega^c_{X/S}) \to R\mathcal{H}_Z(\Omega^{p+c}_{X/S})$$

By Derived Categories of Schemes, Lemma 36.6.8 the objects $R\mathcal{H}_Z(\Omega^j_{X/S})$ have vanishing cohomology sheaves in degrees $> c$. Hence on cohomology sheaves in degree c we obtain a map

$$i^{-1}\Omega^p_{X/S} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{H}^c_Z(\Omega^c_{X/S}) \longrightarrow \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$$

The expression $i^{-1}\alpha \wedge \gamma^0(1)$ is the image of the cup product $i^{-1}\alpha \cup \gamma^0(1)$ by the functoriality of cohomology.

Having explained the content of the formula in this manner, by general properties of cup products (Cohomology, Section 20.31), it now suffices to prove that the diagram

$$\begin{tikzcd}
i^{-1}\Omega^p_X \otimes \Omega^0_Z \arrow[rr, "\mathrm{id}\otimes\gamma^0"'] \arrow[d] & & i^{-1}\Omega^p_X \otimes \mathcal{H}^c_Z(\Omega^c_X) \arrow[d, "\wedge"] \\
\Omega^p_Z \otimes \Omega^0_Z \arrow[r, "\wedge"] & \Omega^p_Z \arrow[r, "\gamma^p"] & \mathcal{H}^c_Z(\Omega^{p+c}_X)
\end{tikzcd}$$

is commutative in the category of sheaves on Z (with obvious abuse of notation). This boils down to a simple computation for the maps $\gamma^j_{f_1,\ldots,f_c}$ which we omit; in fact these maps are chosen exactly such that this works and such that 1 maps to $\frac{\mathrm{d}f_1\wedge\ldots\wedge\mathrm{d}f_c}{f_1\ldots f_c}$. $\square$

0G89 **Lemma 50.23.6.** Let $c \geq 0$ be a integer. Let

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d, "h"'] & X' \arrow[r] \arrow[d, "g"'] & S' \arrow[d] \\
Z \arrow[r] & X \arrow[r] & S
\end{tikzcd}$$

be a commutative diagram of schemes. Assume

1. $Z \to X$ and $Z' \to X'$ satisfy the assumptions of Lemma 50.23.3,
2. the left square in the diagram is cartesian, and
3. $h^*\mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'}$ (Morphisms, Lemma 29.31.3) is an isomorphism.

Then the diagram

$$\begin{tikzcd}
h^*\Omega^p_{Z/S} \arrow[r, "h^{-1}\gamma^p"'] \arrow[d] & \mathcal{O}_{X'}|_{Z'} \otimes_{h^{-1}\mathcal{O}_X|_Z} h^{-1}\mathcal{H}^c_Z(\Omega^{p+c}_{X/S}) \arrow[d] \\
\Omega^p_{Z'/S'} \arrow[r, "\gamma^p"] & \mathcal{H}^c_{Z'}(\Omega^{p+c}_{X'/S'})
\end{tikzcd}$$

is commutative. The left vertical arrow is functoriality of modules of differentials and the right vertical arrow uses Cohomology, Remark 20.34.12.

Proof. More precisely, consider the composition

$$\begin{aligned}\mathcal{O}_{X'}|_{Z'} \otimes^{\mathbf{L}}_{h^{-1}\mathcal{O}_X|_Z} h^{-1}R\mathcal{H}_Z(\Omega^{p+c}_{X/S}) &\to R\mathcal{H}_{Z'}(Lg^*\Omega^{p+c}_{X/S}) \\ &\to R\mathcal{H}_{Z'}(g^*\Omega^{p+c}_{X/S}) \\ &\to R\mathcal{H}_{Z'}(\Omega^{p+c}_{X'/S'})\end{aligned}$$

where the first arrow is given by Cohomology, Remark 20.34.12 and the last one by functoriality of differentials. Since we have the vanishing of cohomology sheaves in degrees $> c$ by Derived Categories of Schemes, Lemma 36.6.8 this induces the right vertical arrow. We can check the commutativity locally. Thus we may assume Z is cut out by $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Then Z' is cut out by $f_i' = g^\sharp(f_i)$. The maps $c_{f_1,\ldots,f_c}$ and $c_{f_1',\ldots,f_c'}$ fit into the commutative diagram

$$\begin{tikzcd} h^*i^*\Omega^p_{X/S} \arrow[r, "h^{-1}c_{f_1,\ldots,f_c}"'] \arrow[d] & \mathcal{O}_{X'}|_{Z'} \otimes_{h^{-1}\mathcal{O}_X|_Z} h^{-1}\mathcal{H}^c_Z(\Omega^p_{X/S}) \arrow[d] \\ (i')^*\Omega^p_{X'/S'} \arrow[r, "c_{f_1',\ldots,f_c'}"] & \mathcal{H}^c_{Z'}(\Omega^p_{X'/S'}) \end{tikzcd}$$

See Derived Categories of Schemes, Remark 36.6.14. Recall given a p-form ω on Z we define $\gamma^p(\omega)$ by choosing (locally on X and Z) a p-form $\tilde{\omega}$ on X lifting ω and taking $\gamma^p(\omega) = c_{f_1,\ldots,f_c}(\tilde{\omega}) \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c$. Since the form $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c$ pulls back to $\mathrm{d}f_1' \wedge \ldots \wedge \mathrm{d}f_c'$ we conclude. □

0G8A **Remark 50.23.7.** Let $X \to S$, $i : Z \to X$, and $c \geq 0$ be as in Lemma 50.23.3. Let $p \geq 0$ and assume that $\mathcal{H}^i_Z(\Omega^{p+c}_{X/S}) = 0$ for $i = 0, \ldots, c-1$. This vanishing holds if $X \to S$ is smooth and $Z \to X$ is a Koszul regular immersion, see Derived Categories of Schemes, Lemma 36.6.9. Then we obtain a map

$$\gamma^{p,q} : H^q(Z, \Omega^p_{Z/S}) \longrightarrow H^{q+c}(X, \Omega^{p+c}_{X/S})$$

by first using $\gamma^p : \Omega^p_{Z/S} \to \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$ to map into

$$H^q(Z, \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})) = H^q(Z, R\mathcal{H}_Z(\Omega^{p+c}_{X/S})[c]) = H^q(X, i_*R\mathcal{H}_Z(\Omega^{p+c}_{X/S})[c])$$

and then using the adjunction map $i_*R\mathcal{H}_Z(\Omega^{p+c}_{X/S}) \to \Omega^{p+c}_{X/S}$ to continue on to the desired Hodge cohomology module.

0G8B **Lemma 50.23.8.** *Let $X \to S$ and $i : Z \to X$ be as in Lemma 50.23.3. Assume $X \to S$ is smooth and $Z \to X$ Koszul regular. The gysin maps $\gamma^{p,q}$ are compatible with the de Rham differentials on $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Z/S}$.*

Proof. This follows immediately from Lemma 50.23.4. □

0G8C **Lemma 50.23.9.** *Let $X \to S$, $i : Z \to X$, and $c \geq 0$ be as in Lemma 50.23.3. Assume $X \to S$ smooth and $Z \to X$ Koszul regular. Given $\alpha \in H^q(X, \Omega^p_{X/S})$ we have $\gamma^{p,q}(\alpha|_Z) = \alpha \cup \gamma^{0,0}(1)$ in $H^{q+c}(X, \Omega^{p+c}_{X/S})$ with $\gamma^{a,b}$ as in Remark 50.23.7.*

Proof. This lemma follows from Lemma 50.23.5 and Cohomology, Lemma 20.34.11. We suggest the reader skip over the more detailed discussion below.

We will use without further mention that $R\mathcal{H}_Z(\Omega^j_{X/S}) = \mathcal{H}^c_Z(\Omega^j_{X/S})[-c]$ for all j as pointed out in Remark 50.23.7. We will also silently use the identifications

$H^{q+c}_Z(X, \Omega^j_{X/S}) = H^{q+c}(Z, R\mathcal{H}_Z(\Omega^j_{X/S}) = H^q(Z, \mathcal{H}^c_Z(\Omega^j_{X/S}))$, see Cohomology, Lemma 20.34.4 for the first one. With these identifications

(1) $\gamma^0(1) \in H^c_Z(X, \Omega^c_{X/S})$ maps to $\gamma^{0,0}(1)$ in $H^c(X, \Omega^c_{X/S})$,
(2) the right hand side $i^{-1}\alpha \wedge \gamma^0(1)$ of the equality in Lemma 50.23.5 is the (image by wedge product of the) cup product of Cohomology, Remark 20.34.10 of the elements α and $\gamma^0(1)$, in other words, the constructions in the proof of Lemma 50.23.5 and in Cohomology, Remark 20.34.10 match,
(3) by Cohomology, Lemma 20.34.11 this maps to $\alpha \cup \gamma^{0,0}(1)$ in $H^{q+c}(X, \Omega^p_{X/S} \otimes \Omega^c_{X/S})$, and
(4) the left hand side $\gamma^p(\alpha|_Z)$ of the equality in Lemma 50.23.5 maps to $\gamma^{p,q}(\alpha|_Z)$.

This finishes the proof. □

0G8D Lemma 50.23.10. Let $c \geq 0$ and

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "h"'] & X' \arrow[r] \arrow[d, "g"'] & S' \arrow[d] \\ Z \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

satisfy the assumptions of Lemma 50.23.6 and assume in addition that $X \to S$ and $X' \to S'$ are smooth and that $Z \to X$ and $Z' \to X'$ are Koszul regular immersions. Then the diagram

$$\begin{tikzcd} H^q(Z, \Omega^p_{Z/S}) \arrow[r, "\gamma^{p,q}"'] \arrow[d] & H^{q+c}(X, \Omega^{p+c}_{X/S}) \arrow[d] \\ H^q(Z', \Omega^p_{Z'/S'}) \arrow[r, "\gamma^{p,q}"] & H^{q+c}(X', \Omega^{p+c}_{X'/S'}) \end{tikzcd}$$

is commutative where $\gamma^{p,q}$ is as in Remark 50.23.7.

Proof. This follows on combining Lemma 50.23.6 and Cohomology, Lemma 20.34.13. □

0G8E Lemma 50.23.11. Let k be a field. Let X be an irreducible smooth proper scheme over k of dimension d. Let $Z \subset X$ be the reduced closed subscheme consisting of a single k-rational point x. Then the image of $1 \in k = H^0(Z, \mathcal{O}_Z) = H^0(Z, \Omega^0_{Z/k})$ by the map $H^0(Z, \Omega^0_{Z/k}) \to H^d(X, \Omega^d_{X/k})$ of Remark 50.23.7 is nonzero.

Proof. The map $\gamma^0 : \mathcal{O}_Z \to \mathcal{H}^d_Z(\Omega^d_{X/k}) = R\mathcal{H}_Z(\Omega^d_{X/k})[d]$ is adjoint to a map

$$g^0 : i_*\mathcal{O}_Z \longrightarrow \Omega^d_{X/k}[d]$$

in $D(\mathcal{O}_X)$. Recall that $\Omega^d_{X/k} = \omega_X$ is a dualizing sheaf for X/k, see Duality for Schemes, Lemma 48.27.1. Hence the k-linear dual of the map in the statement of the lemma is the map

$$H^0(X, \mathcal{O}_X) \to \text{Ext}^d_X(i_*\mathcal{O}_Z, \omega_X)$$

which sends 1 to g^0. Thus it suffices to show that g^0 is nonzero. This we may do in any neighbourhood U of the point x. Choose U such that there exist $f_1, \ldots, f_d \in \mathcal{O}_X(U)$ vanishing only at x and generating the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$. We may assume assume $U = \mathrm{Spec}(R)$ is affine. Looking over the construction of γ^0 we find that our extension is given by

$$k \to (R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r})[d] \to R[d]$$

where 1 maps to $1/f_1 \ldots f_c$ under the first map. This is nonzero because $1/f_1 \ldots f_c$ is a nonzero element of local cohomology group $H^d_{(f_1,\ldots,f_d)}(R)$ in this case, $\square$

50.24. Relative Poincaré duality

0G8F In this section we prove Poincar'e duality for the relative de Rham cohomology of a proper smooth scheme over a base. We strongly urge the reader to look at Section 50.20 first.

0G8G Situation 50.24.1. Here S is a quasi-compact and quasi-separated scheme and $f : X \to S$ is a proper smooth morphism of schemes all of whose fibres are nonempty and equidimensional of dimension n.

0G8H Lemma 50.24.2. In Situation 50.24.1 the psuhforward $f_*\mathcal{O}_X$ is a finite étale $\mathcal{O}_S$-algebra and locally on S we have $Rf_*\mathcal{O}_X = f_*\mathcal{O}_X \oplus P$ in $D(\mathcal{O}_S)$ with P perfect of tor amplitude in $[1, \infty)$. The map $\mathrm{d} : f_*\mathcal{O}_X \to f_*\Omega_{X/S}$ is zero.

Proof. The first part of the statement follows from Derived Categories of Schemes, Lemma 36.32.8. Setting $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$ we get a factorization $X \to S' \to S$ (this is the Stein factorization, see More on Morphisms, Section 37.53, although we don't need this) and we see that $\Omega_{X/S} = \Omega_{X/S'}$ for example by Morphisms, Lemma 29.32.9 and 29.36.15. This of course implies that $\mathrm{d} : f_*\mathcal{O}_X \to f_*\Omega_{X/S}$ is zero. $\square$

0G8I Lemma 50.24.3. In Situation 50.24.1 there exists an $\mathcal{O}_S$-module map

$$t : Rf_*\Omega^n_{X/S}[n] \longrightarrow \mathcal{O}_S$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all p the pairing

$$Rf_*\Omega^p_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\Omega^{n-p}_{X/S}[n] \longrightarrow \mathcal{O}_S$$

given by the relative cup product composed with t is a perfect pairing of perfect complexes on S.

Proof. Let $\omega^\bullet_{X/S}$ be the relative dualizing complex of X over S as in Duality for Schemes, Remark 48.12.5 and let $Rf_*\omega^\bullet_{X/S} \to \mathcal{O}_S$ be its trace map. By Duality for Schemes, Lemma 48.15.7 there exists an isomorphism $\omega^\bullet_{X/S} \cong \Omega^n_{X/S}[n]$ and using this isomorphism we obtain t. The complexes $Rf_*\Omega^p_{X/S}$ are perfect by Lemma 50.3.5. Since $\Omega^p_{X/S}$ is locally free and since $\Omega^p_{X/S} \otimes_{\mathcal{O}_X} \Omega^{n-p}_{X/S} \to \Omega^n_{X/S}$ exhibits an isomorphism $\Omega^p_{X/S} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^{n-p}_{X/S}, \Omega^n_{X/S})$ we see that the pairing induced by the relative cup product is perfect by Duality for Schemes, Remark 48.12.6.

Uniqueness of t. Choose a distinguished triangle $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X \to P \to f_*\mathcal{O}_X[1]$. By Lemma 50.24.2 the object P is perfect of tor amplitude in $[1, \infty)$ and the triangle is locally on S split. Thus $R\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X)$ is perfect of tor amplitude in $(-\infty, -1]$. Hence duality (above) shows that locally on S we have

$$Rf_*\Omega^n_{X/S}[n] \cong R\mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X, \mathcal{O}_S) \oplus R\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X)$$

This shows that $R^n f_*\Omega^n_{X/S}$ is finite locally free and that we obtain a perfect $\mathcal{O}_S$-bilinear pairing

$$f_*\mathcal{O}_X \times R^n f_*\Omega^n_{X/S} \longrightarrow \mathcal{O}_S$$

using t. This implies that any $\mathcal{O}_S$-linear map $t' : R^n f_*\Omega^n_{X/S} \to \mathcal{O}_S$ is of the form $t' = t \circ g$ for some $g \in \Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. In order for t' to still determine a perfect pairing g will have to be a unit. This finishes the proof. □

0G8J Lemma 50.24.4. In Situation 50.24.1 the map $\mathrm{d} : R^n f_*\Omega^{n-1}_{X/S} \to R^n f_*\Omega^n_{X/S}$ is zero.

As we mentioned in the proof of Lemma 50.20.3 this lemma is not an easy consequence of Lemmas 50.24.3 and 50.24.2.

Proof in case S is reduced. Assume S is reduced. Observe that $\mathrm{d} : R^n f_*\Omega^{n-1}_{X/S} \to R^n f_*\Omega^n_{X/S}$ is an $\mathcal{O}_S$-linear map of (quasi-coherent) $\mathcal{O}_S$-modules. The $\mathcal{O}_S$-module $R^n f_*\Omega^n_{X/S}$ is finite locally free (as the dual of the finite locally free $\mathcal{O}_S$-module $f_*\mathcal{O}_X$ by Lemmas 50.24.3 and 50.24.2). Since S is reduced it suffices to show that the stalk of d in every generic point $\eta \in S$ is zero; this follows by looking at sections over affine opens, using that the target of d is locally free, and Algebra, Lemma 10.25.2 part (2). Since S is reduced we have $\mathcal{O}_{S,\eta} = \kappa(\eta)$, see Algebra, Lemma 10.25.1. Thus d_η is identified with the map

$$\mathrm{d} : H^n(X_\eta, \Omega^{n-1}_{X_\eta/\kappa(\eta)}) \longrightarrow H^n(X_\eta, \Omega^n_{X_\eta/\kappa(\eta)})$$

which is zero by Lemma 50.20.3. □

Proof in the general case. Observe that the question is flat local on S: if $S' \to S$ is a surjective flat morphism of schemes and the map is zero after pullback to S', then the map is zero. Also, formation of the map commutes with base change by flat morphisms by flat base change (Cohomology of Schemes, Lemma 30.5.2).

Consider the Stein factorization $X \to S' \to S$ as in More on Morphisms, Theorem 37.53.5. By Lemma 50.24.2 the morphism $\pi : S' \to S$ is finite étale. The morphism $f : X \to S'$ is proper (by the theorem), smooth (by More on Morphisms, Lemma 37.13.12) with geometrically connected fibres by the theorem on Stein factorization. In the proof of Lemma 50.24.2 we saw that $\Omega_{X/S} = \Omega_{X/S'}$ because $S' \to S$ is étale. Hence $\Omega^\bullet_{X/S} = \Omega^\bullet_{X/S'}$. We have

$$R^q f_*\Omega^p_{X/S} = \pi_* R^q f'_*\Omega^p_{X/S'}$$

for all p, q by the Leray spectral sequence (Cohomology, Lemma 20.13.8), the fact that π is finite hence affine, and Cohomology of Schemes, Lemma 30.2.3 (of course we also use that $R^q f'_*\Omega^p_{X'/S}$ is quasi-coherent). Thus the map of the lemma is π_* applied to $\mathrm{d} : R^n f'_*\Omega^{n-1}_{X/S'} \to R^n f'_*\Omega^n_{X/S'}$. In other words, in order to prove the lemma we may replace $f : X \to S$ by $f' : X \to S'$ to reduce to the case discussed in the next pargraph.

Assume f has geometrically connected fibres and $f_*\mathcal{O}_X = \mathcal{O}_S$. For every $s \in S$ we can choose an étale neighbourhood $(S', s') \to (S, s)$ such that the base change $X' \to S'$ of S has a section. See More on Morphisms, Lemma 37.38.6. By the initial remarks of the proof this reduces us to the case discussed in the next paragraph.

Assume f has geometrically connected fibres, $f_*\mathcal{O}_X = \mathcal{O}_S$, and we have a section $s : S \to X$ of f. We may and do assume $S = \mathrm{Spec}(A)$ is affine. The map

$s^* : R\Gamma(X, \mathcal{O}_X) \to R\Gamma(S, \mathcal{O}_S) = A$ is a splitting of the map $A \to R\Gamma(X, \mathcal{O}_X)$. Thus we can write

$$R\Gamma(X, \mathcal{O}_X) = A \oplus P$$

where P is the "kernel" of s^*. By Lemma 50.24.2 the object P of $D(A)$ is perfect of tor amplitude in $[1, n]$. As in the proof of Lemma 50.24.3 we see that $H^n(X, \Omega^n_{X/S})$ is a locally free A-module of rank 1 (and in fact dual to A so free of rank 1 – we will soon choose a generator but we don't want to check it is the same generator nor will it be necessary to do so).

Denote $Z \subset X$ the image of s which is a closed subscheme of X by Schemes, Lemma 26.21.11. Observe that $Z \to X$ is a regular (and a fortiori Koszul regular by Divisors, Lemma 31.21.2) closed immersion by Divisors, Lemma 31.22.8. Of course $Z \to X$ has codimension n. Thus by Remark 50.23.7 we can consider the map

$$\gamma^{0,0} : H^0(Z, \Omega^0_{Z/S}) \longrightarrow H^n(X, \Omega^n_{X/S})$$

and we set $\xi = \gamma^{0,0}(1) \in H^n(X, \Omega^n_{X/S})$.

We claim ξ is a basis element. Namely, since we have base change in top degree (see for example Limits, Lemma 32.19.2) we see that $H^n(X, \Omega^n_{X/S}) \otimes_A k = H^n(X_k, \Omega^n_{X_k/k})$ for any ring map $A \to k$. By the compatibility of the construction of ξ with base change, see Lemma 50.23.10, we see that the image of ξ in $H^n(X_k, \Omega^n_{X_k/k})$ is nonzero by Lemma 50.23.11 if k is a field. Thus ξ is a nowhere vanishing section of an invertible module and hence a generator.

Let $\theta \in H^n(X, \Omega^{n-1}_{X/S})$. We have to show that $\mathrm{d}(\theta)$ is zero in $H^n(X, \Omega^n_{X/S})$. We may write $\mathrm{d}(\theta) = a\xi$ for some $a \in A$ as ξ is a basis element. Then we have to show $a = 0$.

Consider the closed immersion

$$\Delta : X \to X \times_S X$$

This is also a section of a smooth morphism (namely either projection) and hence a regular and Koszul immersion of codimension n as well. Thus we can consider the maps

$$\gamma^{p,q} : H^q(X, \Omega^p_{X/S}) \longrightarrow H^{q+n}(X \times_S X, \Omega^{p+n}_{X \times_S X/S})$$

of Remark 50.23.7. Consider the image

$$\gamma^{n-1,n}(\theta) \in H^{2n}(X \times_S X, \Omega^{2n-1}_{X \times_S X})$$

By Lemma 50.8.1 we have

$$\Omega^{2n-1}_{X \times_S X} = \Omega^{n-1}_{X/S} \boxtimes \Omega^n_{X/S} \oplus \Omega^n_{X/S} \boxtimes \Omega^{n-1}_{X/S}$$

By the Künneth formula (either Derived Categories of Schemes, Lemma 36.23.1 or Derived Categories of Schemes, Lemma 36.23.4) we see that

$$H^{2n}(X \times_S X, \Omega^{n-1}_{X/S} \boxtimes \Omega^n_{X/S}) = H^n(X, \Omega^{n-1}_{X/S}) \otimes_A H^n(X, \Omega^n_{X/S})$$

and

$$H^{2n}(X \times_S X, \Omega^n_{X/S} \boxtimes \Omega^{n-1}_{X/S}) = H^n(X, \Omega^n_{X/S}) \otimes_A H^n(X, \Omega^{n-1}_{X/S})$$

Namely, since we are looking in top degree there no higher tor groups that intervene. Combined with the fact that ξ is a generator this means we can write

$$\gamma^{n-1,n}(\theta) = \theta_1 \otimes \xi + \xi \otimes \theta_2$$

with $\theta_1, \theta_2 \in H^n(X, \Omega^{n-1}_{X/S})$. Arguing in exactly the same manner we can write

$$\gamma^{n,n}(\xi) = b\xi \otimes \xi$$

in $H^{2n}(X \times_S X, \Omega^{2n}_{X \times_S X/S}) = H^n(X, \Omega^n_{X/S}) \otimes_A H^n(X, \Omega^n_{X/S})$ for some $b \in H^0(S, \mathcal{O}_S)$.

Claim: $\theta_1 = \theta$, $\theta_2 = \theta$, and $b = 1$. Let us show that the claim implies the desired result $a = 0$. Namely, by Lemma 50.23.8 we have

$$\gamma^{n,n}(\mathrm{d}(\theta)) = \mathrm{d}(\gamma^{n-1,n}(\theta))$$

By our choices above this gives

$$a\xi \otimes \xi = \gamma^{n,n}(a\xi) = \mathrm{d}(\theta \otimes \xi + \xi \otimes \theta) = a\xi \otimes \xi + (-1)^n a\xi \otimes \xi$$

The right most equality comes from the fact that the map $\mathrm{d} : \Omega^{2n-1}_{X \otimes_S X/S} \to \Omega^{2n}_{X \times_S X/S}$ by Lemma 50.8.1 is the sum of the differential $\mathrm{d} \boxtimes 1 : \Omega^{n-1}_{X/S} \boxtimes \Omega^n_{X/S} \to \Omega^n_{X/S} \boxtimes \Omega^n_{X/S}$ and the differential $(-1)^n 1 \boxtimes \mathrm{d} : \Omega^n_{X/S} \boxtimes \Omega^{n-1}_{X/S} \to \Omega^n_{X/S} \boxtimes \Omega^n_{X/S}$. Please see discussion in Section 50.8 and Derived Categories of Schemes, Section 36.24 for more information. Since $\xi \otimes \xi$ is a basis for the rank 1 free A-module $H^n(X, \Omega^n_{X/S}) \otimes_A H^n(X, \Omega^n_{X/S})$ we conclude

$$a = a + (-1)^n a \Rightarrow a = 0$$

as desired.

In the rest of the proof we prove the claim above. Let us denote $\eta = \gamma^{0,0}(1) \in H^n(X \times_S X, \Omega^n_{X \times_S X/S})$. Since $\Omega^n_{X \times_S X/S} = \bigoplus_{p+p'=n} \Omega^p_{X/S} \boxtimes \Omega^{p'}_{X/S}$ we may write

$$\eta = \eta_0 + \eta_1 + \ldots + \eta_n$$

where η_p is in $H^n(X \times_S X, \Omega^p_{X/S} \boxtimes \Omega^{n-p}_{X/S})$. For $p = 0$ we can write

$$\begin{aligned} H^n(X \times_S X, \mathcal{O}_X \boxtimes \Omega^n_{X/S}) &= H^n(R\Gamma(X, \mathcal{O}_X) \otimes_A^{\mathbf{L}} R\Gamma(X, \Omega^n_{X/S})) \\ &= A \otimes_A H^n(X, \Omega^n_{X/S}) \oplus H^n(P \otimes_A^{\mathbf{L}} R\Gamma(X, \Omega^n_{X/S})) \end{aligned}$$

by our previously given decomposition $R\Gamma(X, \mathcal{O}_X) = A \oplus P$. Consider the morphism $(s, \mathrm{id}) : X \to X \times_S X$. Then $(s, \mathrm{id})^{-1}(\Delta) = Z$ scheme theoretically. Hence we see that $(s, \mathrm{id})^*\eta = \xi$ by Lemma 50.23.10. This means that

$$\xi = (s, \mathrm{id})^*\eta = (s^* \otimes \mathrm{id})(\eta_0)$$

This means exactly that the first component of η_0 in the direct sum decomposition above is ξ. In other words, we can write

$$\eta_0 = 1 \otimes \xi + \eta_0'$$

with $\eta_0' \in H^n(P \otimes_A^{\mathbf{L}} R\Gamma(X, \Omega^n_{X/S}))$. In exactly the same manner for $p = n$ we can write

$$\begin{aligned} H^n(X \times_S X, \Omega^n_{X/S} \boxtimes \mathcal{O}_X) &= H^n(R\Gamma(X, \Omega^n_{X/S}) \otimes_A^{\mathbf{L}} R\Gamma(X, \mathcal{O}_X)) \\ &= H^n(X, \Omega^n_{X/S}) \otimes_A A \oplus H^n(R\Gamma(X, \Omega^n_{X/S}) \otimes_A^{\mathbf{L}} P) \end{aligned}$$

and we can write

$$\eta_n = \xi \otimes 1 + \eta_n'$$

with $\eta_n' \in H^n(R\Gamma(X, \Omega^n_{X/S}) \otimes_A^{\mathbf{L}} P)$.

Observe that $\mathrm{pr}_1^*\theta = \theta \otimes 1$ and $\mathrm{pr}_2^*\theta = 1 \otimes \theta$ are Hodge cohomology classes on $X \times_S X$ which pull back to θ by Δ. Hence by Lemma 50.23.9 we have

$$\theta_1 \otimes \xi + \xi \otimes \theta_2 = \gamma^{n-1,n}(\theta) = (\theta \otimes 1) \cup \eta = (1 \otimes \theta) \cup \eta$$

in the Hodge cohomology ring of $X \times_S X$ over S. In terms of the direct sum decomposition on the modules of differentials of $X \times_S X/S$ we obtain

$$\theta_1 \otimes \xi = (\theta \otimes 1) \cup \eta_0 \quad \text{and} \quad \xi \otimes \theta_2 = (1 \otimes \theta) \cup \eta_n$$

Looking at the formula $\eta_0 = 1 \otimes \xi + \eta_0'$ we found above, we see that to show that $\theta_1 = \theta$ it suffices to prove that

$$(\theta \otimes 1) \cup \eta_0' = 0$$

To do this, observe that cupping with $\theta \otimes 1$ is given by the action on cohomology of the map

$$(P \otimes_A^{\mathbf{L}} R\Gamma(X, \Omega^n_{X/S}))[-n] \xrightarrow{\theta \otimes 1} R\Gamma(X, \Omega^{n-1}_{X/S}) \otimes_A^{\mathbf{L}} R\Gamma(X, \Omega^n_{X/S})$$

in the derived category, see Cohomology, Remark 20.31.2. This map is the derived tensor product of the two maps

$$\theta : P[-n] \to R\Gamma(X, \Omega^{n-1}_{X/S}) \quad \text{and} \quad 1 : R\Gamma(X, \Omega^n_{X/S}) \to R\Gamma(X, \Omega^n_{X/S})$$

by Derived Categories of Schemes, Remark 36.23.5. However, the first of these is zero in $D(A)$ because it is a map from a perfect complex of tor amplitude in $[n+1, 2n]$ to a complex with cohomology only in degrees $0, 1, \ldots, n$, see More on Algebra, Lemma 15.76.1. A similar argument works to show the vanishing of $(1 \otimes \theta) \cup \eta_n'$. Finally, in exactly the same manner we obtain

$$b\xi \otimes \xi = \gamma^{n,n}(\xi) = (\xi \otimes 1) \cup \eta_0$$

and we conclude as before by showing that $(\xi \otimes 1) \cup \eta_0' = 0$ in the same manner as above. This finishes the proof. □

0G8K Proposition 50.24.5. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a proper smooth morphism of schemes all of whose fibres are nonempty and equidimensional of dimension n. There exists an $\mathcal{O}_S$-module map

$$t : R^{2n} f_*\Omega^\bullet_{X/S} \longrightarrow \mathcal{O}_S$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: the pairing

$$Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\Omega^\bullet_{X/S}[2n] \longrightarrow \mathcal{O}_S, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is a perfect pairing of perfect complexes on S.

Proof. The proof is exactly the same as the proof of Proposition 50.20.4.

By the relative Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = R^q f_*\Omega^p_{X/S} \Rightarrow R^{p+q} f_*\Omega^\bullet_{X/S}$$

(Section 50.6), the vanishing of $\Omega^i_{X/S}$ for $i > n$, the vanishing in for example Limits, Lemma 32.19.2 and the results of Lemmas 50.24.2 and 50.24.4 we see that $R^0 f_*\Omega_{X/S} = R^0 f_*\mathcal{O}_X$ and $R^n f_*\Omega^n_{X/S} = R^{2n} f_*\Omega^\bullet_{X/S}$. More precisely, these identifications come from the maps of complexes

$$\Omega^\bullet_{X/S} \to \mathcal{O}_X[0] \quad \text{and} \quad \Omega^n_{X/S}[-n] \to \Omega^\bullet_{X/S}$$

Let us choose $t : R^{2n}f_*\Omega_{X/S} \to \mathcal{O}_S$ which via this identification corresponds to a t as in Lemma 50.24.3.

Let us abbreviate $\Omega^\bullet = \Omega^\bullet_{X/S}$. Consider the map (50.4.0.1) which in our situation reads

$$\wedge : \text{Tot}(\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \Omega^\bullet) \longrightarrow \Omega^\bullet$$

For every integer $p = 0, 1, \ldots, n$ this map annihilates the subcomplex $\text{Tot}(\sigma_{>p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet)$ for degree reasons. Hence we find that the restriction of $\wedge$ to the subcomplex $\text{Tot}(\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \geq_{n-p} \Omega^\bullet)$ factors through a map of complexes

$$\gamma_p : \text{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \longrightarrow \Omega^\bullet$$

Using the same procedure as in Section 50.4 we obtain relative cup products

$$Rf_*\sigma_{\leq p}\Omega^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\sigma_{\geq n-p}\Omega^\bullet \longrightarrow Rf_*\Omega^\bullet$$

We will prove by induction on p that these cup products via t induce perfect pairings between $Rf_*\sigma_{\leq p}\Omega^\bullet$ and $Rf_*\sigma_{\geq n-p}\Omega^\bullet[2n]$. For $p = n$ this is the assertion of the proposition.

The base case is $p = 0$. In this case we have

$$Rf_*\sigma_{\leq p}\Omega^\bullet = Rf_*\mathcal{O}_X \quad \text{and} \quad Rf_*\sigma_{\geq n-p}\Omega^\bullet[2n] = Rf_*(\Omega^n[-n])[2n] = Rf_*\Omega^n[n]$$

In this case we simply obtain the pairing between $Rf_*\mathcal{O}_X$ and $Rf_*\Omega^n[n]$ of Lemma 50.24.3 and the result is true.

Induction step. Say we know the result is true for p. Then we consider the distinguished triangle

$$\Omega^{p+1}[-p-1] \to \sigma_{\leq p+1}\Omega^\bullet \to \sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$$

and the distinguished triangle

$$\sigma_{\geq n-p}\Omega^\bullet \to \sigma_{\geq n-p-1}\Omega^\bullet \to \Omega^{n-p-1}[-n+p+1] \to (\sigma_{\geq n-p}\Omega^\bullet)[1]$$

Observe that both are distinguished triangles in the homotopy category of complexes of sheaves of $f^{-1}\mathcal{O}_S$-modules; in particular the maps $\sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$ and $\Omega^{n-p-1}[-d+p+1] \to (\sigma_{\geq n-p}\Omega^\bullet)[1]$ are given by actual maps of complexes, namely using the differential $\Omega^p \to \Omega^{p+1}$ and the differential $\Omega^{n-p-1} \to \Omega^{n-p}$. Consider the distinguished triangles associated gotten from these distinguished triangles by applying Rf_*

$$\begin{tikzcd}
Rf_*\sigma_{\leq p}\Omega^\bullet \arrow[d, "a"'] & Rf_*\sigma_{\geq n-p}\Omega^\bullet \\
Rf_*\Omega^{p+1}[-p-1] \arrow[d, "b"'] & Rf_*\Omega^{n-p-1}[-n+p+1] \arrow[u, "a'"'] \\
Rf_*\sigma_{\leq p+1}\Omega^\bullet \arrow[d, "c"'] & Rf_*\sigma_{\geq n-p-1}\Omega^\bullet \arrow[u, "b'"'] \\
Rf_*\sigma_{\leq p}\Omega^\bullet \arrow[d, "d"'] & Rf_*\sigma_{\geq n-p}\Omega^\bullet \arrow[u, "c'"'] \\
Rf_*\Omega^{p+1}[-p-1] & Rf_*\Omega^{n-p-1}[-n+p+1] \arrow[u, "d'"']
\end{tikzcd}$$

We will show below that the pairs (a, a'), (b, b'), (c, c'), and (d, d') are compatible with the given pairings. This means we obtain a map from the distinguished triangle on the left to the distuiguished triangle obtained by applying $R\mathcal{H}om(-, \mathcal{O}_S)$ to the distinguished triangle on the right. By induction and Lemma 50.20.1 we know that the pairings constructed above between the complexes on the first, second, fourth, and fifth rows are perfect, i.e., determine isomorphisms after taking duals. By Derived Categories, Lemma 13.4.3 we conclude the pairing between the complexes in the middle row is perfect as desired.

Let $e : K \to K'$ and $e' : M' \to M$ be maps of objects of $D(\mathcal{O}_S)$ and let $K \otimes^{\mathbf{L}}_{\mathcal{O}_S} M \to \mathcal{O}_S$ and $K' \otimes^{\mathbf{L}}_{\mathcal{O}_S} M' \to \mathcal{O}_S$ be pairings. Then we say these pairings are compatible if the diagram

$$\begin{tikzcd} K' \otimes^{\mathbf{L}}_{\mathcal{O}_S} M' \arrow[d] & K \otimes^{\mathbf{L}}_{\mathcal{O}_S} M' \arrow[l, "e \otimes 1"'] \arrow[d, "1 \otimes e'"] \\ \mathcal{O}_S & K \otimes^{\mathbf{L}}_{\mathcal{O}_S} M \arrow[l] \end{tikzcd}$$

commutes. This indeed means that the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "e"'] & R\mathcal{H}om(M, \mathcal{O}_S) \arrow[d, "R\mathcal{H}om(e', -)"] \\ K' \arrow[r] & R\mathcal{H}om(M', \mathcal{O}_S) \end{tikzcd}$$

commutes and hence is sufficient for our purposes.

Let us prove this for the pair (c, c'). Here we observe simply that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \arrow[d, "\gamma_p"'] & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \arrow[l] \arrow[d] \\ \Omega^\bullet & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p-1}\Omega^\bullet) \arrow[l, "\gamma_{p+1}"'] \end{tikzcd}$$

By functoriality of the cup product we obtain commutativity of the desired diagram.

Similarly for the pair (b, b') we use the commutative diagram

$$\begin{tikzcd} \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p-1}\Omega^\bullet) \arrow[d, "\gamma_{p+1}"'] & \mathrm{Tot}(\Omega^{p+1}[-p-1] \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p-1}\Omega^\bullet) \arrow[l] \arrow[d] \\ \Omega^\bullet & \Omega^{p+1}[-p-1] \otimes_{f^{-1}\mathcal{O}_S} \Omega^{n-p-1}[-n+p+1] \arrow[l, "\wedge"'] \end{tikzcd}$$

For the pairs (d, d') and (a, a') we use the commutative diagram

$$\begin{tikzcd} \Omega^{p+1}[-p] \otimes_{f^{-1}\mathcal{O}_S} \Omega^{n-p-1}[-n+p] \arrow[d] & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \Omega^{n-p-1}[-n+p]) \arrow[l] \arrow[d] \\ \Omega^\bullet & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \arrow[l] \end{tikzcd}$$

We omit the argument showing the uniqueness of t up to precomposing by multiplication by a unit in $H^0(X, \mathcal{O}_X)$. $\square$

50.25. Other chapters

CHAPTER 51

Local Cohomology

0DWN

51.1. Introduction

0DWP This chapter continues the study of local cohomology. A reference is [Gro68]. The definition of local cohomology can be found in Dualizing Complexes, Section 47.9. For Noetherian rings taking local cohomology is the same as deriving a suitable torsion functor as is shown in Dualizing Complexes, Section 47.10. The relationship with depth can be found in Dualizing Complexes, Section 47.11.

We discuss finiteness properties of local cohomology leading to a proof of a fairly general version of Grothendieck's finiteness theorem, see Theorem 51.11.6 and Lemma 51.12.1 (higher direct images of coherent modules under open immersions). Our methods incorporate a few very slick arguments the reader can find in papers of Faltings, see [Fal78b] and [Fal81].

As applications we offer a discussion of Hartshorne-Lichtenbaum vanishing. We also discuss the action of Frobenius and of differential operators on local cohomology.

51.2. Generalities

0DWQ The following lemma tells us that the functor $R\Gamma_Z$ is related to cohomology with supports.

0A6T Lemma 51.2.1. Let A be a ring and let I be a finitely generated ideal. Set $Z = V(I) \subset X = \mathrm{Spec}(A)$. For $K \in D(A)$ corresponding to $\widetilde{K} \in D_{QCoh}(\mathcal{O}_X)$ via Derived Categories of Schemes, Lemma 36.3.5 there is a functorial isomorphism

$$R\Gamma_Z(K) = R\Gamma_Z(X, \widetilde{K})$$

where on the left we have Dualizing Complexes, Equation (47.9.0.1) and on the right we have the functor of Cohomology, Section 20.34.

Proof. By Cohomology, Lemma 20.34.5 there exists a distinguished triangle

$$R\Gamma_Z(X, \widetilde{K}) \to R\Gamma(X, \widetilde{K}) \to R\Gamma(U, \widetilde{K}) \to R\Gamma_Z(X, \widetilde{K})[1]$$

where $U = X \setminus Z$. We know that $R\Gamma(X, \widetilde{K}) = K$ by Derived Categories of Schemes, Lemma 36.3.5. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. By Derived Categories of Schemes, Lemma 36.9.4 the alternating Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \widetilde{K^\bullet}))$ computes $R\Gamma(U, \widetilde{K})$ where $K^\bullet$ is any complex of A-modules representing K. Working through the definitions we find

$$R\Gamma(U, \widetilde{K}) = \mathrm{Tot}\left(K^\bullet \otimes_A (\prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

It is clear that $K^\bullet = R\Gamma(X, \widetilde{K^\bullet}) \to R\Gamma(U, \widetilde{K^\bullet})$ is induced by the diagonal map from A into $\prod A_{f_i}$. Hence we conclude that

$$R\Gamma_Z(X, \mathcal{F}^\bullet) = \text{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

By Dualizing Complexes, Lemma 47.9.1 this complex computes $R\Gamma_Z(K)$ and we see the lemma holds. $\square$

0DWR Lemma 51.2.2. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $X = \text{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. Then

(1) there exists an A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,
(2) given M there is an exact sequence

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,
(3) we may take $M = H^0(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.

Proof. The existence of M follows from Properties, Lemma 28.22.1 and the fact that quasi-coherent sheaves on X correspond to A-modules (Schemes, Lemma 26.7.5). Then we look at the distinguished triangle

$$R\Gamma_Z(X, \widetilde{M}) \to R\Gamma(X, \widetilde{M}) \to R\Gamma(U, \widetilde{M}|_U) \to R\Gamma_Z(X, \widetilde{M})[1]$$

of Cohomology, Lemma 20.34.5. Since X is affine we have $R\Gamma(X, \widetilde{M}) = M$ by Cohomology of Schemes, Lemma 30.2.2. By our choice of M we have $\mathcal{F} = \widetilde{M}|_U$ and hence this produces an exact sequence

$$0 \to H^0_Z(X, \widetilde{M}) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(X, \widetilde{M}) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(X, \widetilde{M})$ for $p \geq 1$. By Lemma 51.2.1 we have $H^i_Z(M) = H^i_Z(X, \widetilde{M})$ for all i. Thus (1) and (2) do hold. Finally, setting $M' = H^0(U, \mathcal{F})$ we see that the kernel and cokernel of $M \to M'$ are I-power torsion. Therefore $\widetilde{M}|_U \to \widetilde{M'}|_U$ is an isomorphism and we can indeed use M' as predicted in (3). It goes without saying that we obtain zero for both $H^0_Z(M')$ and $H^0_Z(M')$. $\square$

0DWS Lemma 51.2.3. Let $I, J \subset A$ be finitely generated ideals of a ring A. If M is an I-power torsion module, then the canonical map

$$H^i_{V(I) \cap V(J)}(M) \to H^i_{V(J)}(M)$$

is an isomorphism for all i.

Proof. Use the spectral sequence of Dualizing Complexes, Lemma 47.9.6 to reduce to the statement $R\Gamma_I(M) = M$ which is immediate from the construction of local cohomology in Dualizing Complexes, Section 47.9. $\square$

0DWT Lemma 51.2.4. Let $S \subset A$ be a multiplicative set of a ring A. Let M be an A-module with $S^{-1}M = 0$. Then $\text{colim}_{f \in S} H^0_{V(f)}(M) = M$ and $\text{colim}_{f \in S} H^1_{V(f)}(M) = 0$.

Proof. The statement on H^0 follows directly from the definitions. To see the statement on H^1 observe that $R\Gamma_{V(f)}$ and $H^1_{V(f)}$ commute with colimits. Hence we may assume M is annihilated by some $f \in S$. Then $H^1_{V(ff')}(M) = 0$ for all $f' \in S$ (for example by Lemma 51.2.3). $\square$

0DWU Lemma 51.2.5. Let $I \subset A$ be a finitely generated ideal of a ring A. Let $\mathfrak p$ be a prime ideal. Let M be an A-module. Let $i \geq 0$ be an integer and consider the map

$$\Psi : \mathrm{colim}_{f \in A, f \not\in \mathfrak p} H^i_{V((I,f))}(M) \longrightarrow H^i_{V(I)}(M)$$

Then

(1) $\mathrm{Im}(\Psi)$ is the set of elements which map to zero in $H^i_{V(I)}(M)_\mathfrak p$,
(2) if $H^{i-1}_{V(I)}(M)_\mathfrak p = 0$, then Ψ is injective,
(3) if $H^{i-1}_{V(I)}(M)_\mathfrak p = H^i_{V(I)}(M)_\mathfrak p = 0$, then Ψ is an isomorphism.

Proof. For $f \in A$, $f \not\in \mathfrak p$ the spectral sequence of Dualizing Complexes, Lemma 47.9.6 degenerates to give short exact sequences

$$0 \to H^1_{V(f)}(H^{i-1}_{V(I)}(M)) \to H^i_{V((I,f))}(M) \to H^0_{V(f)}(H^i_{V(I)}(M)) \to 0$$

This proves (1) and part (2) follows from this and Lemma 51.2.4. Part (3) is a formal consequence. $\square$

0DWV Lemma 51.2.6. Let $I \subset I' \subset A$ be finitely generated ideals of a Noetherian ring A. Let M be an A-module. Let $i \geq 0$ be an integer. Consider the map

$$\Psi : H^i_{V(I')}(M) \to H^i_{V(I)}(M)$$

The following are true:

(1) if $H^i_{\mathfrak pA_\mathfrak p}(M_\mathfrak p) = 0$ for all $\mathfrak p \in V(I) \setminus V(I')$, then Ψ is surjective,
(2) if $H^{i-1}_{\mathfrak pA_\mathfrak p}(M_\mathfrak p) = 0$ for all $\mathfrak p \in V(I) \setminus V(I')$, then Ψ is injective,
(3) if $H^i_{\mathfrak pA_\mathfrak p}(M_\mathfrak p) = H^{i-1}_{\mathfrak pA_\mathfrak p}(M_\mathfrak p) = 0$ for all $\mathfrak p \in V(I) \setminus V(I')$, then Ψ is an isomorphism.

Proof. Proof of (1). Let $\xi \in H^i_{V(I)}(M)$. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ is the image of some $\xi'' \in H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, choose a generic point $\mathfrak p \in V(I'')$ not in $V(I')$. Then we have $H^i_{V(I'')}(M)_\mathfrak p = H^i_{\mathfrak pA_\mathfrak p}(M_\mathfrak p) = 0$ by assumption. By Lemma 51.2.5 we can increase I'' which contradicts maximality.

Proof of (2). Let $\xi' \in H^i_{V(I')}(M)$ be in the kernel of Ψ. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ' maps to zero in $H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, then choose a generic point $\mathfrak p \in V(I'')$ not in $V(I')$. Then we have $H^{i-1}_{V(I'')}(M)_\mathfrak p = H^{i-1}_{\mathfrak pA_\mathfrak p}(M_\mathfrak p) = 0$ by assumption. By Lemma 51.2.5 we can increase I'' which contradicts maximality.

Part (3) is formal from parts (1) and (2). $\square$

51.3. Hartshorne's connectedness lemma

0FIV The title of this section refers to the following result.

0BLR **Lemma 51.3.1.** Let A be a Noetherian local ring of depth ≥ 2. Then the punctured spectra of A, A^h, and A^{sh} are connected. [Har62, Proposition 2.1]

Proof. Let U be the punctured spectrum of A. If U is disconnected then we see that $\Gamma(U, \mathcal{O}_U)$ has a nontrivial idempotent. But A, being local, does not have a nontrivial idempotent. Hence $A \to \Gamma(U, \mathcal{O}_U)$ is not an isomorphism. By Lemma 51.2.2 we conclude that either $H^0_\mathfrak{m}(A)$ or $H^1_\mathfrak{m}(A)$ is nonzero. Thus $\text{depth}(A) \leq 1$ by Dualizing Complexes, Lemma 47.11.1. To see the result for A^h and A^{sh} use More on Algebra, Lemma 15.45.8. $\square$

0FIW **Lemma 51.3.2.** Let A be a Noetherian local ring which is catenary and (S_2). Then $\text{Spec}(A)$ is equidimensional. [DG67, Corollary 5.10.9]

Proof. Set $X = \text{Spec}(A)$. Say $d = \dim(A) = \dim(X)$. Inside X consider the union X_1 of the irreducible components of dimension d and the union X_2 of the irreducible components of dimension $< d$. Of course $X = X_1 \cup X_2$. If $X_2 = \emptyset$, then the lemma holds. If not, then $Z = X_1 \cap X_2$ is a nonempty closed subset of X because it contains at least the closed point of X. Hence we can choose a generic point $z \in Z$ of an irreducible component of Z. Recall that the spectrum of $\mathcal{O}_{Z,z}$ is the set of points of X specializing to z. Since z is both contained in an irreducible component of dimension d and in an irreducible component of dimension $< d$ we obtain nontrivial specializations $x_1 \rightsquigarrow z$ and $x_2 \rightsquigarrow z$ such that the closures of x_1 and x_2 have different dimensions. Since X is catenary, this can only happen if at least one of the specializations $x_1 \rightsquigarrow z$ and $x_2 \rightsquigarrow z$ is not immediate! Thus $\dim(\mathcal{O}_{Z,z}) \geq 2$. Therefore $\text{depth}(\mathcal{O}_{Z,z}) \geq 2$ because A is (S_2). However, the punctured spectrum U of $\mathcal{O}_{Z,z}$ is disconnected because the closed subsets $U \cap X_1$ and $U \cap X_2$ are disjoint (by our choice of z) and cover U. This is a contradiction with Lemma 51.3.1 and the proof is complete. $\square$

51.4. Cohomological dimension

0DX6 A quick section about cohomological dimension.

0DX7 **Lemma 51.4.1.** Let $I \subset A$ be a finitely generated ideal of a ring A. Set $Y = V(I) \subset X = \text{Spec}(A)$. Let $d \geq -1$ be an integer. The following are equivalent

(1) $H^i_Y(A) = 0$ for $i > d$,
(2) $H^i_Y(M) = 0$ for $i > d$ for every A-module M, and
(3) if $d = -1$, then $Y = \emptyset$, if $d = 0$, then Y is open and closed in X, and if $d > 0$ then $H^i(X \setminus Y, \mathcal{F}) = 0$ for $i \geq d$ for every quasi-coherent $\mathcal{O}_{X \setminus Y}$-module $\mathcal{F}$.

Proof. Observe that $R\Gamma_Y(-)$ has finite cohomological dimension by Dualizing Complexes, Lemma 47.9.1 for example. Hence there exists an integer i_0 such that $H^i_Y(M) = 0$ for all A-modules M and $i \geq i_0$.

Let us prove that (1) and (2) are equivalent. It is immediate that (2) implies (1). Assume (1). By descending induction on $i > d$ we will show that $H^i_Y(M) = 0$ for all A-modules M. For $i \geq i_0$ we have seen this above. To do the induction step, let $i_0 > i > d$. Choose any A-module M and fit it into a short exact sequence

$0 \to N \to F \to M \to 0$ where F is a free A-module. Since $R\Gamma_Y$ is a right adjoint, we see that $H^i_Y(-)$ commutes with direct sums. Hence $H^i_Y(F) = 0$ as $i > d$ by assumption (1). Then we see that $H^i_Y(M) = H^{i+1}_Y(N) = 0$ as desired.

Assume $d = -1$ and (2) holds. Then $0 = H^0_Y(A/I) = A/I \Rightarrow A = I \Rightarrow Y = \emptyset$. Thus (3) holds. We omit the proof of the converse.

Assume $d = 0$ and (2) holds. Set $J = H^0_I(A) = \{x \in A \mid I^n x = 0 \text{ for some } n > 0\}$. Then

$$H^1_Y(A) = \text{Coker}(A \to \Gamma(X\setminus Y, \mathcal{O}_{X\setminus Y})) \quad \text{and} \quad H^1_Y(I) = \text{Coker}(I \to \Gamma(X\setminus Y, \mathcal{O}_{X\setminus Y}))$$

and the kernel of the first map is equal to J. See Lemma 51.2.2. We conclude from (2) that $I(A/J) = A/J$. Thus we may pick $f \in I$ mapping to 1 in A/J. Then $1 - f \in J$ so $I^n(1-f) = 0$ for some $n > 0$. Hence $f^n = f^{n+1}$. Then $e = f^n \in I$ is an idempotent. Consider the complementary idempotent $e' = 1 - f^n \in J$. For any element $g \in I$ we have $g^m e' = 0$ for some $m > 0$. Thus I is contained in the radical of ideal $(e) \subset I$. This means $Y = V(I) = V(e)$ is open and closed in X as predicted in (3). Conversely, if $Y = V(I)$ is open and closed, then the functor $H^0_Y(-)$ is exact and has vanshing higher derived functors.

If $d > 0$, then we see immediately from Lemma 51.2.2 that (2) is equivalent to (3). □

0DX8 Definition 51.4.2. Let $I \subset A$ be a finitely generated ideal of a ring A. The smallest integer $d \geq -1$ satisfying the equivalent conditions of Lemma 51.4.1 is called the cohomological dimension of I in A and is denoted $\text{cd}(A, I)$.

Thus we have $\text{cd}(A, I) = -1$ if $I = A$ and $\text{cd}(A, I) = 0$ if I is locally nilpotent or generated by an idempotent. Observe that $\text{cd}(A, I)$ exists by the following lemma.

0DX9 Lemma 51.4.3. Let $I \subset A$ be a finitely generated ideal of a ring A. Then

(1) $\text{cd}(A, I)$ is at most equal to the number of generators of I,
(2) $\text{cd}(A, I) \leq r$ if there exist $f_1, \ldots, f_r \in A$ such that $V(f_1, \ldots, f_r) = V(I)$,
(3) $\text{cd}(A, I) \leq c$ if $\text{Spec}(A) \setminus V(I)$ can be covered by c affine opens.

Proof. The explicit description for $R\Gamma_Y(-)$ given in Dualizing Complexes, Lemma 47.9.1 shows that (1) is true. We can deduce (2) from (1) using the fact that $R\Gamma_Z$ depends only on the closed subset Z and not on the choice of the finitely generated ideal $I \subset A$ with $V(I) = Z$. This follows either from the construction of local cohomology in Dualizing Complexes, Section 47.9 combined with More on Algebra, Lemma 15.88.6 or it follows from Lemma 51.2.1. To see (3) we use Lemma 51.4.1 and the vanishing result of Cohomology of Schemes, Lemma 30.4.2. □

0ECP Lemma 51.4.4. Let $I, J \subset A$ be finitely generated ideals of a ring A. Then $\text{cd}(A, I + J) \leq \text{cd}(A, I) + \text{cd}(A, J)$.

Proof. Use the definition and Dualizing Complexes, Lemma 47.9.6. □

0DXA Lemma 51.4.5. Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. Then $\text{cd}(B, IB) \leq \text{cd}(A, I)$. If $A \to B$ is faithfully flat, then equality holds.

Proof. Use the definition and Dualizing Complexes, Lemma 47.9.3. □

0DXB Lemma 51.4.6. Let $I \subset A$ be a finitely generated ideal of a ring A. Then $\text{cd}(A, I) = \max \text{cd}(A_\mathfrak{p}, I_\mathfrak{p})$.

Proof. Let $Y = V(I)$ and $Y' = V(I_\mathfrak{p}) \subset \mathrm{Spec}(A_\mathfrak{p})$. Recall that $R\Gamma_Y(A) \otimes_A A_\mathfrak{p} = R\Gamma_{Y'}(A_\mathfrak{p})$ by Dualizing Complexes, Lemma 47.9.3. Thus we conclude by Algebra, Lemma 10.23.1. $\square$

0DXC Lemma 51.4.7. Let $I \subset A$ be a finitely generated ideal of a ring A. If M is a finite A-module, then $H^i_{V(I)}(M) = 0$ for $i > \dim(\mathrm{Supp}(M))$. In particular, we have $\mathrm{cd}(A, I) \leq \dim(A)$.

Proof. We first prove the second statement. Recall that $\dim(A)$ denotes the Krull dimension. By Lemma 51.4.6 we may assume A is local. If $V(I) = \emptyset$, then the result is true. If $V(I) \neq \emptyset$, then $\dim(\mathrm{Spec}(A) \setminus V(I)) < \dim(A)$ because the closed point is missing. Observe that $U = \mathrm{Spec}(A) \setminus V(I)$ is a quasi-compact open of the spectral space $\mathrm{Spec}(A)$, hence a spectral space itself. See Algebra, Lemma 10.26.2 and Topology, Lemma 5.23.5. Thus Cohomology, Proposition 20.22.4 implies $H^i(U, \mathcal{F}) = 0$ for $i \geq \dim(A)$ which implies what we want by Lemma 51.4.1. In the Noetherian case the reader may use Grothendieck's Cohomology, Proposition 20.20.7.

We will deduce the first statement from the second. Let $\mathfrak{a}$ be the annihilator of the finite A-module M. Set $B = A/\mathfrak{a}$. Recall that $\mathrm{Spec}(B) = \mathrm{Supp}(M)$, see Algebra, Lemma 10.40.5. Set $J = IB$. Then M is a B-module and $H^i_{V(I)}(M) = H^i_{V(J)}(M)$, see Dualizing Complexes, Lemma 47.9.2. Since $\mathrm{cd}(B, J) \leq \dim(B) = \dim(\mathrm{Supp}(M))$ by the first part we conclude. $\square$

0DXD Lemma 51.4.8. Let $I \subset A$ be a finitely generated ideal of a ring A. If $\mathrm{cd}(A, I) = 1$ then $\mathrm{Spec}(A) \setminus V(I)$ is nonempty affine.

Proof. This follows from Lemma 51.4.1 and Cohomology of Schemes, Lemma 30.3.1. $\square$

0DXE Lemma 51.4.9. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then $H^d_\mathfrak{m}(A)$ is nonzero and $\mathrm{cd}(A, \mathfrak{m}) = d$.

Proof. By one of the characterizations of dimension, there exists an ideal of definition for A generated by d elements, see Algebra, Proposition 10.60.9. Hence $\mathrm{cd}(A, \mathfrak{m}) \leq d$ by Lemma 51.4.3. Thus $H^d_\mathfrak{m}(A)$ is nonzero if and only if $\mathrm{cd}(A, \mathfrak{m}) = d$ if and only if $\mathrm{cd}(A, \mathfrak{m}) \geq d$.

Let $A \to A^\wedge$ be the map from A to its completion. Observe that $A^\wedge$ is a Noetherian local ring of the same dimension as A with maximal ideal $\mathfrak{m}A^\wedge$. See Algebra, Lemmas 10.97.6, 10.97.4, and 10.97.3 and More on Algebra, Lemma 15.43.1. By Lemma 51.4.5 it suffices to prove the lemma for $A^\wedge$.

By the previous paragraph we may assume that A is a complete local ring. Then A has a normalized dualizing complex $\omega_A^\bullet$ (Dualizing Complexes, Lemma 47.22.4). The local duality theorem (in the form of Dualizing Complexes, Lemma 47.18.4) tells us $H^d_\mathfrak{m}(A)$ is Matlis dual to $\mathrm{Ext}^{-d}(A, \omega_A^\bullet) = H^{-d}(\omega_A^\bullet)$ which is nonzero for example by Dualizing Complexes, Lemma 47.16.11. $\square$

0DXF Lemma 51.4.10. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be a proper ideal. Let $\mathfrak{p} \subset A$ be a prime ideal such that $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$. Then $\dim(A/\mathfrak{p}) \leq \mathrm{cd}(A, I)$.

Proof. By Lemma 51.4.5 we have $\text{cd}(A, I) \geq \text{cd}(A/\mathfrak{p}, I(A/\mathfrak{p}))$. Since $V(I) \cap V(\mathfrak{p}) = \{\mathfrak{m}\}$ we have $\text{cd}(A/\mathfrak{p}, I(A/\mathfrak{p})) = \text{cd}(A/\mathfrak{p}, \mathfrak{m}/\mathfrak{p})$. By Lemma 51.4.9 this is equal to $\dim(A/\mathfrak{p})$. $\square$

0EHU Lemma 51.4.11. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $b : X' \to X = \text{Spec}(A)$ be the blowing up of I. If the fibres of b have dimension $\leq d - 1$, then $\text{cd}(A, I) \leq d$.

Proof. Set $U = X \setminus V(I)$. Denote $j : U \to X'$ the canonical open immersion, see Divisors, Section 31.32. Since the exceptional divisor is an effective Cartier divisor (Divisors, Lemma 31.32.4) we see that j is affine, see Divisors, Lemma 31.13.3. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. Then $R^pj_*\mathcal{F} = 0$ for $p > 0$, see Cohomology of Schemes, Lemma 30.2.3. On the other hand, we have $R^qb_*(j_*\mathcal{F}) = 0$ for $q \geq d$ by Limits, Lemma 32.19.2. Thus by the Leray spectral sequence (Cohomology, Lemma 20.13.8) we conclude that $R^n(b \circ j)_*\mathcal{F} = 0$ for $n \geq d$. Thus $H^n(U, \mathcal{F}) = 0$ for $n \geq d$ (by Cohomology, Lemma 20.13.6). This means that $\text{cd}(A, I) \leq d$ by definition. $\square$

51.5. More general supports

0EEY Let A be a Noetherian ring. Let M be an A-module. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization (Topology, Definition 5.19.1). Let us define

$$H^0_T(M) = \text{colim}_{Z \subset T} H^0_Z(M)$$

where the colimit is over the directed partially ordered set of closed subsets Z of $\text{Spec}(A)$ contained in T[1]. In other words, an element m of M is in $H^0_T(M) \subset M$ if and only if the support $V(\text{Ann}_R(m))$ of m is contained in T.

0EEZ Lemma 51.5.1. Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. For an A-module M the following are equivalent

(1) $H^0_T(M) = M$, and
(2) $\text{Supp}(M) \subset T$.

The category of such A-modules is a Serre subcategory of the category A-modules closed under direct sums.

Proof. The equivalence holds because the support of an element of M is contained in the support of M and conversely the support of M is the union of the supports of its elements. The category of these modules is a Serre subcategory (Homology, Definition 12.10.1) of Mod_A by Algebra, Lemma 10.40.9. We omit the proof of the statement on direct sums. $\square$

Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let us denote $\text{Mod}_{A,T} \subset \text{Mod}_A$ the Serre subcategory described in Lemma 51.5.1. Let us denote $D_T(A) \subset D(A)$ the strictly full saturated triangulated subcategory of $D(A)$ (Derived Categories, Lemma 13.17.1) consisting of complexes of A-modules whose cohomology modules are in $\text{Mod}_{A,T}$. We obtain functors

$$D(\text{Mod}_{A,T}) \to D_T(A) \to D(A)$$

See discussion in Derived Categories, Section 13.17. Denote $RH^0_T : D(A) \to D(\text{Mod}_{A,T})$ the right derived extension of H^0_T. We will denote

$$R\Gamma_T : D^+(A) \to D^+_T(A),$$

[1]Since T is stable under specialization we have $T = \bigcup_{Z \subset T} Z$, see Topology, Lemma 5.19.3.

the composition of $RH^0_T : D^+(A) \to D^+(\text{Mod}_{A,T})$ with $D^+(\text{Mod}_{A,T}) \to D^+_T(A)$. If the dimension of A is finite[2], then we will denote

$$R\Gamma_T : D(A) \to D_T(A)$$

the composition of RH^0_T with $D(\text{Mod}_{A,T}) \to D_T(A)$.

0EF0 Lemma 51.5.2. Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. The functor RH^0_T is the right adjoint to the functor $D(\text{Mod}_{A,T}) \to D(A)$.

Proof. This follows from the fact that the functor $H^0_T(-)$ is the right adjoint to the inclusion functor $\text{Mod}_{A,T} \to \text{Mod}_A$, see Derived Categories, Lemma 13.30.3. □

0EF1 Lemma 51.5.3. Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. For any object K of $D(A)$ we have

$$H^i(RH^0_T(K)) = \text{colim}_{Z\subset T \text{ closed}} H^i_Z(K)$$

Proof. Let $J^\bullet$ be a K-injective complex representing K. By definition RH^0_T is represented by the complex

$$H^0_T(J^\bullet) = \text{colim}\, H^0_Z(J^\bullet)$$

where the equality follows from our definition of H^0_T. Since filtered colimits are exact the cohomology of this complex in degree i is $\text{colim}\, H^i(H^0_Z(J^\bullet)) = \text{colim}\, H^i_Z(K)$ as desired. □

0EF2 Lemma 51.5.4. Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. The functor $D^+(\text{Mod}_{A,T}) \to D^+_T(A)$ is an equivalence.

Proof. Let M be an object of $\text{Mod}_{A,T}$. Choose an embedding $M \to J$ into an injective A-module. By Dualizing Complexes, Proposition 47.5.9 the module J is a direct sum of injective hulls of residue fields. Let E be an injective hull of the residue field of $\mathfrak{p}$. Since E is $\mathfrak{p}$-power torsion we see that $H^0_T(E) = 0$ if $\mathfrak{p} \not\in T$ and $H^0_T(E) = E$ if $\mathfrak{p} \in T$. Thus $H^0_T(J)$ is injective as a direct sum of injective hulls (by the proposition) and we have an embedding $M \to H^0_T(J)$. Thus every object M of $\text{Mod}_{A,T}$ has an injective resolution $M \to J^\bullet$ with J^n also in $\text{Mod}_{A,T}$. It follows that $RH^0_T(M) = M$.

Next, suppose that $K \in D^+_T(A)$. Then the spectral sequence

$$R^qH^0_T(H^p(K)) \Rightarrow R^{p+q}H^0_T(K)$$

(Derived Categories, Lemma 13.21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $RH^0_T(K) \to K$ is an isomorphism. Thus the functor $D^+(\text{Mod}_{A,T}) \to D^+_T(A)$ is an equivalence with quasi-inverse given by RH^0_T. □

0EF3 Lemma 51.5.5. Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. If $\dim(A) < \infty$, then functor $D(\text{Mod}_{A,T}) \to D_T(A)$ is an equivalence.

[2] If $\dim(A) = \infty$ the construction may have unexpected properties on unbounded complexes.

Proof. Say $\dim(A) = d$. Then we see that $H^i_Z(M) = 0$ for $i > d$ for every closed subset Z of $\mathrm{Spec}(A)$, see Lemma 51.4.7. By Lemma 51.5.3 we find that H^0_T has bounded cohomological dimension.

Let $K \in D_T(A)$. We claim that $RH^0_T(K) \to K$ is an isomorphism. We know this is true when K is bounded below, see Lemma 51.5.4. However, since H^0_T has bounded cohomological dimension, we see that the ith cohomology of $RH^0_T(K)$ only depends on $\tau_{\geq -d+i}K$ and we conclude. Thus $D(\mathrm{Mod}_{A,T}) \to D_T(A)$ is an equivalence with quasi-inverse RH^0_T. $\square$

0EF4 Remark 51.5.6. Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. The upshot of the discussion above is that $R\Gamma_T : D^+(A) \to D^+_T(A)$ is the right adjoint to the inclusion functor $D^+_T(A) \to D^+(A)$. If $\dim(A) < \infty$, then $R\Gamma_T : D(A) \to D_T(A)$ is the right adjoint to the inclusion functor $D_T(A) \to D(A)$. In both cases we have

$$H^i_T(K) = H^i(R\Gamma_T(K)) = R^iH^0_T(K) = \mathrm{colim}_{Z \subset T \text{ closed}} H^i_Z(K)$$

This follows by combining Lemmas 51.5.2, 51.5.3, 51.5.4, and 51.5.5.

0EF5 Lemma 51.5.7. Let $A \to B$ be a flat homomorphism of Noetherian rings. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $T' \subset \mathrm{Spec}(B)$ be the inverse image of T. Then the canonical map

$$R\Gamma_T(K) \otimes_A^{\mathbf{L}} B \longrightarrow R\Gamma_{T'}(K \otimes_A^{\mathbf{L}} B)$$

is an isomorphism for $K \in D^+(A)$. If A and B have finite dimension, then this is true for $K \in D(A)$.

Proof. From the map $R\Gamma_T(K) \to K$ we get a map $R\Gamma_T(K) \otimes_A^{\mathbf{L}} B \to K \otimes_A^{\mathbf{L}} B$. The cohomology modules of $R\Gamma_T(K) \otimes_A^{\mathbf{L}} B$ are supported on T' and hence we get the arrow of the lemma. This arrow is an isomorphism if T is a closed subset of $\mathrm{Spec}(A)$ by Dualizing Complexes, Lemma 47.9.3. Recall that $H^i_T(K)$ is the colimit of $H^i_Z(K)$ where Z runs over the (directed set of) closed subsets of T, see Lemma 51.5.3. Correspondingly $H^i_{T'}(K \otimes_A^{\mathbf{L}} B) = \mathrm{colim}\, H^i_{Z'}(K \otimes_A^{\mathbf{L}} B)$ where Z' is the inverse image of Z. Thus the result because $\otimes_A B$ commutes with filtered colimits and there are no higher Tors. $\square$

0EF6 Lemma 51.5.8. Let A be a ring and let $T, T' \subset \mathrm{Spec}(A)$ subsets stable under specialization. For $K \in D^+(A)$ there is a spectral sequence

$$E_2^{p,q} = H^p_T(H^p_{T'}(K)) \Rightarrow H^{p+q}_{T \cap T'}(K)$$

as in Derived Categories, Lemma 13.22.2.

Proof. Let E be an object of $D_{T \cap T'}(A)$. Then we have

$$\mathrm{Hom}(E, R\Gamma_T(R\Gamma_{T'}(K))) = \mathrm{Hom}(E, R\Gamma_{T'}(K)) = \mathrm{Hom}(E, K)$$

The first equality by the adjointness property of $R\Gamma_T$ and the second by the adjointness property of $R\Gamma_{T'}$. On the other hand, if $J^\bullet$ is a bounded below complex of injectives representing K, then $H^0_{T'}(J^\bullet)$ is a complex of injective A-modules representing $R\Gamma_{T'}(K)$ and hence $H^0_T(H^0_{T'}(J^\bullet))$ is a complex representing $R\Gamma_T(R\Gamma_{T'}(K))$. Thus $R\Gamma_T(R\Gamma_{T'}(K))$ is an object of $D^+_{T \cap T'}(A)$. Combining these two facts we find that $R\Gamma_{T \cap T'} = R\Gamma_T \circ R\Gamma_{T'}$. This produces the spectral sequence by the lemma referenced in the statement. $\square$

0EF7 Lemma 51.5.9. Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Assume A has finite dimension. Then

$$R\Gamma_T(K) = R\Gamma_T(A) \otimes_A^{\mathbf{L}} K$$

for $K \in D(A)$. For $K, L \in D(A)$ we have

$$R\Gamma_T(K \otimes_A^{\mathbf{L}} L) = K \otimes_A^{\mathbf{L}} R\Gamma_T(L) = R\Gamma_T(K) \otimes_A^{\mathbf{L}} L = R\Gamma_T(K) \otimes_A^{\mathbf{L}} R\Gamma_T(L)$$

If K or L is in $D_T(A)$ then so is $K \otimes_A^{\mathbf{L}} L$.

Proof. By construction we may represent $R\Gamma_T(A)$ by a complex $J^\bullet$ in $\mathrm{Mod}_{A,T}$. Thus if we represent K by a K-flat complex $K^\bullet$ then we see that $R\Gamma_T(A) \otimes_A^{\mathbf{L}} K$ is represented by the complex $\mathrm{Tot}(J^\bullet \otimes_A K^\bullet)$ in $\mathrm{Mod}_{A,T}$. Using the map $R\Gamma_T(A) \to A$ we obtain a map $R\Gamma_T(A) \otimes_A^{\mathbf{L}} K \to K$. Thus by the adjointness property of $R\Gamma_T$ we obtain a canonical map

$$R\Gamma_T(A) \otimes_A^{\mathbf{L}} K \longrightarrow R\Gamma_T(K)$$

factoring the just constructed map. Observe that $R\Gamma_T$ commutes with direct sums in $D(A)$ for example by Lemma 51.5.3, the fact that directed colimits commute with direct sums, and the fact that usual local cohomology commutes with direct sums (for example by Dualizing Complexes, Lemma 47.9.1). Thus by More on Algebra, Remark 15.59.11 it suffices to check the map is an isomorphism for $K = A[k]$ where $k \in \mathbf{Z}$. This is clear.

The final statements follow from the result we've just shown by transitivity of derived tensor products. $\square$

51.6. Filtrations on local cohomology

0EHV Some tricks related to the spectral sequence of Lemma 51.5.8.

0EF8 Lemma 51.6.1. Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and for every A-module M there is an exact sequence

$$0 \to \mathrm{colim}_{Z,f}\, H^1_f(H^{i-1}_Z(M)) \to H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak p \in T \setminus T'} H^i_{\mathfrak p A_{\mathfrak p}}(M_{\mathfrak p})$$

where the colimit is over closed subsets $Z \subset T$ and $f \in A$ with $V(f) \cap Z \subset T'$.

Proof. For every Z and f the spectral sequence of Dualizing Complexes, Lemma 47.9.6 degenerates to give short exact sequences

$$0 \to H^1_f(H^{i-1}_Z(M)) \to H^i_{Z \cap V(f)}(M) \to H^0_f(H^i_Z(M)) \to 0$$

We will use this without further mention below.

Let $\xi \in H^i_T(M)$ map to zero in the direct sum. Then we first write ξ as the image of some $\xi' \in H^i_Z(M)$ for some closed subset $Z \subset T$, see Lemma 51.5.3. Then ξ' maps to zero in $H^i_{\mathfrak p A_{\mathfrak p}}(M_{\mathfrak p})$ for every $\mathfrak p \in Z$, $\mathfrak p \not\in T'$. Since there are finitely many of these primes, we may choose $f \in A$ not contained in any of these such that f annihilates ξ'. Then ξ' is the image of some $\xi'' \in H^i_{Z'}(M)$ where $Z' = Z \cap V(f)$. By our choice of f we have $Z' \subset T'$ and we get exactness at the penultimate spot.

Let $\xi \in H^i_{T'}(M)$ map to zero in $H^i_T(M)$. Choose closed subsets $Z' \subset Z$ with $Z' \subset T'$ and $Z \subset T$ such that ξ comes from $\xi' \in H^i_{Z'}(M)$ and maps to zero in $H^i_Z(M)$. Then we can find $f \in A$ with $V(f) \cap Z = Z'$ and we conclude. $\square$

0EF9 **Lemma 51.6.2.** Let A be a Noetherian ring of finite dimension. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $\{M_n\}_{n \geq 0}$ be an inverse system of A-modules. Let $i \geq 0$ be an integer. Assume that for every m there exists an integer $m'(m) \geq m$ such that for all $\mathfrak{p} \in T$ the induced map

$$H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \longrightarrow H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$$

is zero for $k \geq m'(m)$. Let $m'' : \mathbf{N} \to \mathbf{N}$ be the $2^{\dim(T)}$-fold self-composition of m'. Then the map $H^i_T(M_k) \to H^i_T(M_m)$ is zero for all $k \geq m''(m)$.

Proof. We first make a general remark: suppose we have an exact sequence

$$(A_n) \to (B_n) \to (C_n)$$

of inverse systems of abelian groups. Suppose that for every m there exists an integer $m'(m) \geq m$ such that

$$A_k \to A_m \quad \text{and} \quad C_k \to C_m$$

are zero for $k \geq m'(m)$. Then for $k \geq m'(m'(m))$ the map $B_k \to B_m$ is zero.

We will prove the lemma by induction on $\dim(T)$ which is finite because $\dim(A)$ is finite. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\text{Spec}(A)$ stable under specialization and the hypotheses of the lemma apply to T'. Since $\dim(T') < \dim(T)$ we know the lemma holds for T'. For every A-module M there is an exact sequence

$$H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak{p} \in T \setminus T'} H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$$

by Lemma 51.6.1. Thus we conclude by the initial remark of the proof. $\square$

0EFA **Lemma 51.6.3.** Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $\{M_n\}_{n \geq 0}$ be an inverse system of A-modules. Let $i \geq 0$ be an integer. Assume the dimension of A is finite and that for every m there exists an integer $m'(m) \geq m$ such that for all $\mathfrak{p} \in T$ we have

(1) $H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \to H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$ is zero for $k \geq m'(m)$, and
(2) $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \to H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$ has image $G(\mathfrak{p}, m)$ independent of $k \geq m'(m)$ and moreover $G(\mathfrak{p}, m)$ maps injectively into $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{0,\mathfrak{p}})$.

Then there exists an integer m_0 such that for every $m \geq m_0$ there exists an integer $m''(m) \geq m$ such that for $k \geq m''(m)$ the image of $H^i_T(M_k) \to H^i_T(M_m)$ maps injectively into $H^i_T(M_{m_0})$.

Proof. We first make a general remark: suppose we have an exact sequence

$$(A_n) \to (B_n) \to (C_n) \to (D_n)$$

of inverse systems of abelian groups. Suppose that there exists an integer m_0 such that for every $m \geq m_0$ there exists an integer $m'(m) \geq m$ such that the maps

$$\text{Im}(B_k \to B_m) \longrightarrow B_{m_0} \quad \text{and} \quad \text{Im}(D_k \to D_m) \longrightarrow D_{m_0}$$

are injective for $k \geq m'(m)$ and $A_k \to A_m$ is zero for $k \geq m'(m)$. Then for $m \geq m'(m_0)$ and $k \geq m'(m'(m))$ the map

$$\text{Im}(C_k \to C_m) \to C_{m'(m_0)}$$

is injective. Namely, let $c_0 \in C_m$ be the image of $c_3 \in C_k$ and say c_0 maps to zero in $C_{m'(m_0)}$. Picture

$$C_k \to C_{m'(m'(m))} \to C_{m'(m)} \to C_m \to C_{m'(m_0)}, \quad c_3 \mapsto c_2 \mapsto c_1 \mapsto c_0 \mapsto 0$$

We have to show $c_0 = 0$. The image d_3 of c_3 maps to zero in C_{m_0} and hence we see that the image $d_1 \in D_{m'(m)}$ is zero. Thus we can choose $b_1 \in B_{m'(m)}$ mapping to the image c_1. Since c_3 maps to zero in $C_{m'(m_0)}$ we find an element $a_{-1} \in A_{m'(m_0)}$ which maps to the image $b_{-1} \in B_{m'(m_0)}$ of b_1. Since a_{-1} maps to zero in A_{m_0} we conclude that b_1 maps to zero in B_{m_0}. Thus the image $b_0 \in B_m$ is zero which of course implies $c_0 = 0$ as desired.

We will prove the lemma by induction on $\dim(T)$ which is finite because $\dim(A)$ is finite. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and the hypotheses of the lemma apply to T'. Since $\dim(T') < \dim(T)$ we know the lemma holds for T'. For every A-module M there is an exact sequence

$$0 \to \mathrm{colim}_{Z,f}\, H^1_f(H^{i-1}_Z(M)) \to H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak p \in T \setminus T'} H^i_{\mathfrak p A_{\mathfrak p}}(M_{\mathfrak p})$$

by Lemma 51.6.1. Thus we conclude by the initial remark of the proof and the fact that we've seen the system of groups

$$\left\{\mathrm{colim}_{Z,f}\, H^1_f(H^{i-1}_Z(M_n))\right\}_{n \geq 0}$$

is pro-zero in Lemma 51.6.2; this uses that the function $m''(m)$ in that lemma for $H^{i-1}_Z(M)$ is independent of Z. ☐

51.7. Finiteness of local cohomology, I

0AW7 We will follow Faltings approach to finiteness of local cohomology modules, see [Fal78b] and [Fal81]. Here is a lemma which shows that it suffices to prove local cohomology modules have an annihilator in order to prove that they are finite modules.

0AW8 **Lemma 51.7.1.** Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let M be a finite A-module. Let $n \geq 0$. The following are equivalent [Fal78b, Lemma 3]

(1) $H^i_T(M)$ is finite for $i \leq n$,
(2) there exists an ideal $J \subset A$ with $V(J) \subset T$ such that J annihilates $H^i_T(M)$ for $i \leq n$.

If $T = V(I) = Z$ for an ideal $I \subset A$, then these are also equivalent to

(3) there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(M)$ for $i \leq n$.

Proof. We prove the equivalence of (1) and (2) by induction on n. For $n = 0$ we have $H^0_T(M) \subset M$ is finite. Hence (1) is true. Since $H^0_T(M) = \mathrm{colim}\, H^0_{V(J)}(M)$ with J as in (2) we see that (2) is true. Assume that $n > 0$.

Assume (1) is true. Recall that $H^i_J(M) = H^i_{V(J)}(M)$, see Dualizing Complexes, Lemma 47.10.1. Thus $H^i_T(M) = \mathrm{colim}\, H^i_J(M)$ where the colimit is over ideals $J \subset A$ with $V(J) \subset T$, see Lemma 51.5.3. Since $H^i_T(M)$ is finitely generated for $i \leq n$ we can find a $J \subset A$ as in (2) such that $H^i_J(M) \to H^i_T(M)$ is surjective for $i \leq n$. Thus the finite list of generators are J-power torsion elements and we see that (2) holds with J replaced by some power.

Assume we have J as in (2). Let $N = H^0_T(M)$ and $M' = M/N$. By construction of $R\Gamma_T$ we find that $H^i_T(N) = 0$ for $i > 0$ and $H^0_T(N) = N$, see Remark 51.5.6. Thus we find that $H^0_T(M') = 0$ and $H^i_T(M') = H^i_T(M)$ for $i > 0$. We conclude that we may replace M by M'. Thus we may assume that $H^0_T(M) = 0$. This means that the finite set of associated primes of M are not in T. By prime avoidance (Algebra, Lemma 10.15.2) we can find $f \in J$ not contained in any of the associated primes of M. Then the long exact local cohomology sequence associated to the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

turns into short exact sequences

$$0 \to H^i_T(M) \to H^i_T(M/fM) \to H^{i+1}_T(M) \to 0$$

for $i < n$. We conclude that J^2 annihilates $H^i_T(M/fM)$ for $i < n$. By induction hypothesis we see that $H^i_T(M/fM)$ is finite for $i < n$. Using the short exact sequence once more we see that $H^{i+1}_T(M)$ is finite for $i < n$ as desired.

We omit the proof of the equivalence of (2) and (3) in case $T = V(I)$. □

The following result of Faltings allows us to prove finiteness of local cohomology at the level of local rings.

0AW9 Lemma 51.7.2. Let A be a Noetherian ring, $I \subset A$ an ideal, M a finite A-module, and $n \geq 0$ an integer. Let $Z = V(I)$. The following are equivalent

(1) the modules $H^i_Z(M)$ are finite for $i \leq n$, and
(2) for all $\mathfrak p \in \text{Spec}(A)$ the modules $H^i_Z(M)_{\mathfrak p}$, $i \leq n$ are finite $A_{\mathfrak p}$-modules.

This is a special case of [Fal81, Satz 1].

Proof. The implication (1) ⇒ (2) is immediate. We prove the converse by induction on n. The case $n = 0$ is clear because both (1) and (2) are always true in that case.

Assume $n > 0$ and that (2) is true. Let $N = H^0_Z(M)$ and $M' = M/N$. By Dualizing Complexes, Lemma 47.11.6 we may replace M by M'. Thus we may assume that $H^0_Z(M) = 0$. This means that $\text{depth}_I(M) > 0$ (Dualizing Complexes, Lemma 47.11.1). Pick $f \in I$ a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

which produces a long exact sequence

$$0 \to H^0_Z(M/fM) \to H^1_Z(M) \to H^1_Z(M) \to H^1_Z(M/fM) \to H^2_Z(M) \to \ldots$$

and similarly after localization. Thus assumption (2) implies that the modules $H^i_Z(M/fM)_{\mathfrak p}$ are finite for $i < n$. Hence by induction assumption $H^i_Z(M/fM)$ are finite for $i < n$.

Let $\mathfrak p$ be a prime of A which is associated to $H^i_Z(M)$ for some $i \leq n$. Say $\mathfrak p$ is the annihilator of the element $x \in H^i_Z(M)$. Then $\mathfrak p \in Z$, hence $f \in \mathfrak p$. Thus $fx = 0$ and hence x comes from an element of $H^{i-1}_Z(M/fM)$ by the boundary map δ in the long exact sequence above. It follows that $\mathfrak p$ is an associated prime of the finite module $\text{Im}(\delta)$. We conclude that $\text{Ass}(H^i_Z(M))$ is finite for $i \leq n$, see Algebra, Lemma 10.63.5.

Recall that

$$H^i_Z(M) \subset \prod\nolimits_{\mathfrak p \in \text{Ass}(H^i_Z(M))} H^i_Z(M)_{\mathfrak p}$$

by Algebra, Lemma 10.63.19. Since by assumption the modules on the right hand side are finite and I-power torsion, we can find integers $e_{\mathfrak p, i} \geq 0$, $i \leq n$, $\mathfrak p \in \text{Ass}(H^i_Z(M))$ such that $I^{e_{\mathfrak p, i}}$ annihilates $H^i_Z(M)_{\mathfrak p}$. We conclude that I^e with $e = \max\{e_{\mathfrak p, i}\}$ annihilates $H^i_Z(M)$ for $i \leq n$. By Lemma 51.7.1 we see that $H^i_Z(M)$ is finite for $i \leq n$. $\square$

0BPX Lemma 51.7.3. Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $i \geq 0$ be an integer. Set $Z = V(I)$. If $H^i_Z(A)$ is annihilated by J^n for some n, then $H^i_Z(M)$ annihilated by J^m for some $m = m(M)$ for every finitely presented A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.

Proof. Consider the annihilator $\mathfrak a$ of $H^i_Z(M)$. Let $\mathfrak p \subset A$ with $\mathfrak p \notin Z$. By assumption there exists an $f \in I$, $f \notin \mathfrak p$ and an isomorphism $\varphi : A_f^{\oplus r} \to M_f$ of A_f-modules. Clearing denominators (and using that M is of finite presentation) we find maps

$$a : A^{\oplus r} \longrightarrow M \quad \text{and} \quad b : M \longrightarrow A^{\oplus r}$$

with $a_f = f^N\varphi$ and $b_f = f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. Thus we see that $H^i_Z(M)$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak a$.

As $U = \text{Spec}(A) \setminus Z$ is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_j)$ and $f_j^{2N_j}J^n \subset \mathfrak a$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak a$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. $\square$

0BPY Lemma 51.7.4. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $Z = V(I)$. Let $n \geq 0$ be an integer. If $H^i_Z(A)$ is finite for $0 \leq i \leq n$, then the same is true for $H^i_Z(M)$, $0 \leq i \leq n$ for any finite A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.

Proof. The assumption that $H^i_Z(A)$ is finite for $0 \leq i \leq n$ implies there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(A)$ for $0 \leq i \leq n$, see Lemma 51.7.1. Then Lemma 51.7.3 implies that $H^i_Z(M)$, $0 \leq i \leq n$ is annihilated by I^m for some $m = m(M, i)$. We may take the same m for all $0 \leq i \leq n$. Then Lemma 51.7.1 implies that $H^i_Z(M)$ is finite for $0 \leq i \leq n$ as desired. $\square$

51.8. Finiteness of pushforwards, I

0BL8 In this section we discuss the easiest nontrivial case of the finiteness theorem, namely, the finiteness of the first local cohomology or what is equivalent, finiteness of $j_*\mathcal F$ where $j : U \to X$ is an open immersion, X is locally Noetherian, and $\mathcal F$ is a coherent sheaf on U. Following a method of Kollár ([Kol16b] and [Kol15]) we find a necessary and sufficient condition, see Proposition 51.8.7. The reader who is interested in higher direct images or higher local cohomology groups should skip ahead to Section 51.12 or Section 51.11 (which are developed independently of the rest of this section).

0BJZ Lemma 51.8.1. Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. For $x \in U$ let $i_x : W_x \to U$ be the integral closed subscheme with generic point x. Let $\mathcal F$ be a coherent $\mathcal O_U$-module. The following are equivalent

(1) for all $x \in \text{Ass}(\mathcal{F})$ the $\mathcal{O}_X$-module $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent,
(2) $j_*\mathcal{F}$ is coherent.

Proof. We first prove that (1) implies (2). Assume (1) holds. The statement is local on X, hence we may assume X is affine. Then U is quasi-compact, hence $\text{Ass}(\mathcal{F})$ is finite (Divisors, Lemma 31.2.5). Thus we may argue by induction on the number of associated points. Let $x \in U$ be a generic point of an irreducible component of the support of $\mathcal{F}$. By Divisors, Lemma 31.2.5 we have $x \in \text{Ass}(\mathcal{F})$. By our choice of x we have $\dim(\mathcal{F}_x) = 0$ as $\mathcal{O}_{X,x}$-module. Hence $\mathcal{F}_x$ has finite length as an $\mathcal{O}_{X,x}$-module (Algebra, Lemma 10.62.3). Thus we may use induction on this length.

Set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. This is a coherent $\mathcal{O}_X$-module by assumption. We have $\mathcal{G}_x = \kappa(x)$. Choose a nonzero map $\varphi_x : \mathcal{F}_x \to \kappa(x) = \mathcal{G}_x$. By Cohomology of Schemes, Lemma 30.9.6 there is an open $x \in V \subset U$ and a map $\varphi_V : \mathcal{F}|_V \to \mathcal{G}|_V$ whose stalk at x is φ_x. Choose $f \in \Gamma(X, \mathcal{O}_X)$ which does not vanish at x such that $D(f) \subset V$. By Cohomology of Schemes, Lemma 30.10.5 (for example) we see that φ_V extends to $f^n\mathcal{F} \to \mathcal{G}|_U$ for some n. Precomposing with multiplication by f^n we obtain a map $\mathcal{F} \to \mathcal{G}|_U$ whose stalk at x is nonzero. Let $\mathcal{F}' \subset \mathcal{F}$ be the kernel. Note that $\text{Ass}(\mathcal{F}') \subset \text{Ass}(\mathcal{F})$, see Divisors, Lemma 31.2.4. Since $\text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) - 1$ we may apply the induction hypothesis to conclude $j_*\mathcal{F}'$ is coherent. Since $\mathcal{G} = j_*(\mathcal{G}|_U) = j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent, we can consider the exact sequence

$$0 \to j_*\mathcal{F}' \to j_*\mathcal{F} \to \mathcal{G}$$

By Schemes, Lemma 26.24.1 the sheaf $j_*\mathcal{F}$ is quasi-coherent. Hence the image of $j_*\mathcal{F}$ in $j_*(\mathcal{G}|_U)$ is coherent by Cohomology of Schemes, Lemma 30.9.3. Finally, $j_*\mathcal{F}$ is coherent by Cohomology of Schemes, Lemma 30.9.2.

Assume (2) holds. Exactly in the same manner as above we reduce to the case X affine. We pick $x \in \text{Ass}(\mathcal{F})$ and we set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. Then we choose a nonzero map $\varphi_x : \mathcal{G}_x = \kappa(x) \to \mathcal{F}_x$ which exists exactly because x is an associated point of $\mathcal{F}$. Arguing exactly as above we may assume φ_x extends to an $\mathcal{O}_U$-module map $\varphi : \mathcal{G}|_U \to \mathcal{F}$. Then φ is injective (for example by Divisors, Lemma 31.2.10) and we find an injective map $\mathcal{G} = j_*(\mathcal{G}|_V) \to j_*\mathcal{F}$. Thus (1) holds. $\square$

0BK0 Lemma 51.8.2. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \text{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then

(1) there exists a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,
(2) given M there is an exact sequence
$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$
and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,
(3) given M and $p \geq 0$ the following are equivalent
  (a) $R^pj_*\mathcal{F}$ is coherent,
  (b) $H^p(U, \mathcal{F})$ is a finite A-module,
  (c) $H^{p+1}_Z(M)$ is a finite A-module,
(4) if the equivalent conditions in (3) hold for $p = 0$, we may take $M = \Gamma(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.

Proof. By Properties, Lemma 28.22.5 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ whose restriction to U is isomorphic to $\mathcal{F}$. Say $\mathcal{F}'$ corresponds to the finite A-module M as in (1). Note that $R^p j_*\mathcal{F}$ is quasi-coherent (Cohomology of Schemes, Lemma 30.4.5) and corresponds to the A-module $H^p(U, \mathcal{F})$. By Lemma 51.2.1 and the discussion in Cohomology, Sections 20.21 and 20.34 we obtain an exact sequence

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$. Here we use that $H^j(X, \mathcal{F}') = 0$ for $j > 0$ as X is affine and $\mathcal{F}'$ is quasi-coherent (Cohomology of Schemes, Lemma 30.2.2). This proves (2). Parts (3) and (4) are straightforward from (2); see also Lemma 51.2.2. $\square$

0AWA Lemma 51.8.3. Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume

(1) X is Nagata,
(2) X is universally catenary, and
(3) for $x \in \text{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}}, z}) \geq 2$.

Then $j_*\mathcal{F}$ is coherent.

Proof. By Lemma 51.8.1 it suffices to prove $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent for $x \in \text{Ass}(\mathcal{F})$. Let $\pi : Y \to X$ be the normalization of X in $\text{Spec}(\kappa(x))$, see Morphisms, Section 29.54. By Morphisms, Lemma 29.53.14 the morphism π is finite. Since π is finite $\mathcal{G} = \pi_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module by Cohomology of Schemes, Lemma 30.9.9. Observe that $W_x = U \cap \pi(Y)$. Thus $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ factors through $i_x : W_x \to U$ and we obtain a canonical map

$$i_{x,*}\mathcal{O}_{W_x} \longrightarrow (\pi|_{\pi^{-1}(U)})_*(\mathcal{O}_{\pi^{-1}(U)}) = (\pi_*\mathcal{O}_Y)|_U = \mathcal{G}|_U$$

This map is injective (for example by Divisors, Lemma 31.2.10). Hence $j_*i_{x,*}\mathcal{O}_{W_x} \subset j_*\mathcal{G}|_U$ and it suffices to show that $j_*\mathcal{G}|_U$ is coherent.

It remains to prove that $j_*(\mathcal{G}|_U)$ is coherent. We claim Divisors, Lemma 31.5.11 applies to

$$\mathcal{G} \longrightarrow j_*(\mathcal{G}|_U)$$

which finishes the proof. It suffices to show that $\text{depth}(\mathcal{G}_z) \geq 2$ for $z \in Z$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to z. By Algebra, Lemma 10.72.11 it suffices to show that $\text{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. If not, then by Properties, Lemma 28.12.5 we see that $\dim(\mathcal{O}_{Y,y_i}) = 1$ for some i. This is impossible by the dimension formula (Morphisms, Lemma 29.52.1) for $\pi : Y \to \overline{\{x\}}$ and assumption (3). $\square$

0BK1 Lemma 51.8.4. Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be the inclusion of a nonempty open subscheme with complement Z. Assume that for all $z \in Z$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{X,z}^\wedge$ we have $\dim(\mathcal{O}_{X,z}^\wedge/\mathfrak{p}) \geq 2$. Then $j_*\mathcal{O}_U$ is coherent.

Proof. We may assume X is affine. Using Lemmas 51.7.2 and 51.8.2 we reduce to $X = \text{Spec}(A)$ where $(A, \mathfrak{m})$ is a Noetherian local domain and $\mathfrak{m} \in Z$. Then we can use induction on $d = \dim(A)$. (The base case is $d = 0, 1$ which do not happen by our assumption on the local rings.) Set $V = \text{Spec}(A) \setminus \{\mathfrak{m}\}$. Observe that the local rings of V have dimension strictly smaller than d. Repeating the arguments for $j' : U \to V$ we and using induction we conclude that $j'_*\mathcal{O}_U$ is a coherent $\mathcal{O}_V$-module. Pick a nonzero $f \in A$ which vanishes on Z. Since $D(f) \cap V \subset U$ we find an n such that multiplication by f^n on U extends to a map $f^n : j'_*\mathcal{O}_U \to \mathcal{O}_V$ over V (for example by Cohomology of Schemes, Lemma 30.10.5). This map is injective hence there is an injective map

$$j_*\mathcal{O}_U = j''_*j'_*\mathcal{O}_U \to j''_*\mathcal{O}_V$$

on X where $j'' : V \to X$ is the inclusion morphism. Hence it suffices to show that $j''_*\mathcal{O}_V$ is coherent. In other words, we may assume that X is the spectrum of a local Noetherian domain and that Z consists of the closed point.

Assume $X = \text{Spec}(A)$ with $(A, \mathfrak{m})$ local and $Z = \{\mathfrak{m}\}$. Let $A^\wedge$ be the completion of A. Set $X^\wedge = \text{Spec}(A^\wedge)$, $Z^\wedge = \{\mathfrak{m}^\wedge\}$, $U^\wedge = X^\wedge \setminus Z^\wedge$, and $\mathcal{F}^\wedge = \mathcal{O}_{U^\wedge}$. The ring $A^\wedge$ is universally catenary and Nagata (Algebra, Remark 10.160.9 and Lemma 10.162.8). Moreover, condition (3) of Lemma 51.8.3 for $X^\wedge, Z^\wedge, U^\wedge, \mathcal{F}^\wedge$ holds by assumption! Thus we see that $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ is coherent. Since the morphism $c : X^\wedge \to X$ is flat we conclude that the pullback of $j_*\mathcal{O}_U$ is $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ (Cohomology of Schemes, Lemma 30.5.2). Finally, since c is faithfully flat we conclude that $j_*\mathcal{O}_U$ is coherent by Descent, Lemma 35.7.1. □

0BK2 Remark 51.8.5. Let $j : U \to X$ be an open immersion of locally Noetherian schemes. Let $x \in U$. Let $i_x : W_x \to U$ be the integral closed subscheme with generic point x and let $\overline{\{x\}}$ be the closure in X. Then we have a commutative diagram

$$\begin{tikzcd} W_x \arrow[r, "j'"'] \arrow[d, "i_x"'] & \overline{\{x\}} \arrow[d, "i"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

We have $j_*i_{x,*}\mathcal{O}_{W_x} = i_*j'_*\mathcal{O}_{W_x}$. As the left vertical arrow is a closed immersion we see that $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent if and only if $j'_*\mathcal{O}_{W_x}$ is coherent.

0AWC Remark 51.8.6. Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If there exists an $x \in \text{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ such that $\dim(\mathcal{O}_{\overline{\{x\}},z}) \leq 1$, then $j_*\mathcal{F}$ is not coherent. To prove this we can do a flat base change to the spectrum of $\mathcal{O}_{X,z}$. Let $X' = \overline{\{x\}}$. The assumption implies $\mathcal{O}_{X' \cap U} \subset \mathcal{F}$. Thus it suffices to see that $j_*\mathcal{O}_{X' \cap U}$ is not coherent. This is clear because $X' = \{x, z\}$, hence $j_*\mathcal{O}_{X' \cap U}$ corresponds to $\kappa(x)$ as an $\mathcal{O}_{X,z}$-module which cannot be finite as x is not a closed point.

In fact, the converse of Lemma 51.8.4 holds true: given an open immersion $j : U \to X$ of integral Noetherian schemes and there exists a $z \in X \setminus U$ and an associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{X,z}^\wedge$ with $\dim(\mathcal{O}_{X,z}^\wedge/\mathfrak{p}) = 1$, then $j_*\mathcal{O}_U$ is not coherent. Namely, you can pass to the local ring, you can enlarge U to the punctured spectrum, you can pass to the completion, and then the argument above gives the nonfiniteness.

0BK3 Proposition 51.8.7 (Kollár). Let $j : U \to X$ be an open immersion of locally Noetherian schemes with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent

See [Kol17] and see [DG67, IV, Proposition 7.2.2] for a special case.

(1) $j_*\mathcal{F}$ is coherent,
(2) for $x \in \text{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}^\wedge_{\overline{\{x\}},z}$ we have $\dim(\mathcal{O}^\wedge_{\overline{\{x\}},z}/\mathfrak{p}) \geq 2$.

Proof. If (2) holds we get (1) by a combination of Lemmas 51.8.1, Remark 51.8.5, and Lemma 51.8.4. If (2) does not hold, then $j_*i_{x,*}\mathcal{O}_{W_x}$ is not finite for some $x \in \text{Ass}(\mathcal{F})$ by the discussion in Remark 51.8.6 (and Remark 51.8.5). Thus $j_*\mathcal{F}$ is not coherent by Lemma 51.8.1. $\square$

0BL9 Lemma 51.8.8. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I)$. Let M be a finite A-module. The following are equivalent

(1) $H^1_Z(M)$ is a finite A-module, and
(2) for all $\mathfrak{p} \in \text{Ass}(M)$, $\mathfrak{p} \notin Z$ and all $\mathfrak{q} \in V(\mathfrak{p} + I)$ the completion of $(A/\mathfrak{p})_\mathfrak{q}$ does not have associated primes of dimension 1.

Proof. Follows immediately from Proposition 51.8.7 via Lemma 51.8.2. $\square$

The formulation in the following lemma has the advantage that conditions (1) and (2) are inherited by schemes of finite type over X. Moreover, this is the form of finiteness which we will generalize to higher direct images in Section 51.12.

0AWB Lemma 51.8.9. Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume

(1) X is universally catenary,
(2) for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_1).

In this situation the following are equivalent

(a) for $x \in \text{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$, and
(b) $j_*\mathcal{F}$ is coherent.

Proof. Let $x \in \text{Ass}(\mathcal{F})$. By Proposition 51.8.7 it suffices to check that $A = \mathcal{O}_{\overline{\{x\}},z}$ satisfies the condition of the proposition on associated primes of its completion if and only if $\dim(A) \geq 2$. Observe that A is universally catenary (this is clear) and that its formal fibres are (S_1) as follows from More on Algebra, Lemma 15.51.10 and Proposition 15.51.5. Let $\mathfrak{p}' \subset A^\wedge$ be an associated prime. As $A \to A^\wedge$ is flat, by Algebra, Lemma 10.65.3, we find that $\mathfrak{p}'$ lies over $(0) \subset A$. The formal fibre $A^\wedge \otimes_A F$ is (S_1) where F is the fraction field of A. We conclude that $\mathfrak{p}'$ is a minimal prime, see Algebra, Lemma 10.157.2. Since A is universally catenary it is formally catenary by More on Algebra, Proposition 15.109.5. Hence $\dim(A^\wedge/\mathfrak{p}') = \dim(A)$ which proves the equivalence. $\square$

51.9. Depth and dimension

0DWW Some helper lemmas.

0DWX Lemma 51.9.1. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let $\mathfrak{p} \in V(I)$ be a prime ideal. Assume $e = \text{depth}_{IA_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak{p})$ such that $\text{depth}_{IA_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for all $\mathfrak{q} \in U$.

Proof. By definition of depth we have $IM_{\mathfrak p} \neq M_{\mathfrak p}$ and there exists an $M_{\mathfrak p}$-regular sequence $f_1, \ldots, f_e \in IA_{\mathfrak p}$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in I$ form an M-regular sequence, see Algebra, Lemma 10.68.6. Consider the module $M' = M/IM$. Since $\mathfrak p \in \text{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak p) \subset \text{Supp}(M')$. Thus for $\mathfrak q \in V(\mathfrak p)$ we get $IM_{\mathfrak q} \neq M_{\mathfrak q}$. Hence, using that localization is exact, we see that $\text{depth}_{IA_{\mathfrak q}}(M_{\mathfrak q}) \geq e$ for any $\mathfrak q \in V(I)$ by definition of depth. $\square$

0DWY Lemma 51.9.2. Let A be a Noetherian ring. Let M be a finite A-module. Let $\mathfrak p$ be a prime ideal. Assume $e = \text{depth}_{A_{\mathfrak p}}(M_{\mathfrak p}) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak p)$ such that $\text{depth}_{A_{\mathfrak q}}(M_{\mathfrak q}) \geq e$ for all $\mathfrak q \in U$ and for all but finitely many $\mathfrak q \in U$ we have $\text{depth}_{A_{\mathfrak q}}(M_{\mathfrak q}) > e$.

Proof. By definition of depth we have $\mathfrak pM_{\mathfrak p} \neq M_{\mathfrak p}$ and there exists an $M_{\mathfrak p}$-regular sequence $f_1, \ldots, f_e \in \mathfrak pA_{\mathfrak p}$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in \mathfrak p$ form an M-regular sequence, see Algebra, Lemma 10.68.6. Consider the module $M' = M/(f_1, \ldots, f_e)M$. Since $\mathfrak p \in \text{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak p) \subset \text{Supp}(M')$. Thus for $\mathfrak q \in V(\mathfrak p)$ we get $\mathfrak qM_{\mathfrak q} \neq M_{\mathfrak q}$. Hence, using that localization is exact, we see that $\text{depth}_{A_{\mathfrak q}}(M_{\mathfrak q}) \geq e$ for any $\mathfrak q \in V(I)$ by definition of depth. Moreover, as soon as $\mathfrak q$ is not an associated prime of the module M', then the depth goes up. Thus we see that the final statement holds by Algebra, Lemma 10.63.5. $\square$

0ECN Lemma 51.9.3. Let X be a Noetherian scheme with dualizing complex $\omega_X^\bullet$. Let $\mathcal F$ be a coherent $\mathcal O_X$-module. Let $k \geq 0$ be an integer. Assume $\mathcal F$ is (S_k). Then there is a finite number of points $x \in X$ such that

$$\text{depth}(\mathcal F_x) = k \quad \text{and} \quad \dim(\text{Supp}(\mathcal F_x)) > k$$

Proof. We will prove this lemma by induction on k. The base case $k = 0$ says that $\mathcal F$ has a finite number of embedded associated points, which follows from Divisors, Lemma 31.2.5.

Assume $k > 0$ and the result holds for all smaller k. We can cover X by finitely many affine opens, hence we may assume $X = \text{Spec}(A)$ is affine. Then $\mathcal F$ is the coherent $\mathcal O_X$-module associated to a finite A-module M which satisfies (S_k). We will use Algebra, Lemmas 10.63.10 and 10.72.7 without further mention.

Let $f \in A$ be a nonzerodivisor on M. Then M/fM has (S_{k-1}). By induction we see that there are finitely many primes $\mathfrak p \in V(f)$ with $\text{depth}((M/fM)_{\mathfrak p}) = k - 1$ and $\dim(\text{Supp}((M/fM)_{\mathfrak p})) > k - 1$. These are exactly the primes $\mathfrak p \in V(f)$ with $\text{depth}(M_{\mathfrak p}) = k$ and $\dim(\text{Supp}(M_{\mathfrak p})) > k$. Thus we may replace A by A_f and M by M_f in trying to prove the finiteness statement.

Since M satisfies (S_k) and $k > 0$ we see that M has no embedded associated primes (Algebra, Lemma 10.157.2). Thus $\text{Ass}(M)$ is the set of generic points of the support of M. Thus Dualizing Complexes, Lemma 47.20.4 shows the set $U = \{\mathfrak q \mid M_{\mathfrak q} \text{ is Cohen-Macaulay}\}$ is an open containing $\text{Ass}(M)$. By prime avoidance (Algebra, Lemma 10.15.2) we can pick $f \in A$ with $f \not\in \mathfrak p$ for $\mathfrak p \in \text{Ass}(M)$ such that $D(f) \subset U$. Then f is a nonzerodivisor on M (Algebra, Lemma 10.63.9). After replacing A by A_f and M by M_f (see above) we find that M is Cohen-Macaulay. Thus for all $\mathfrak q \subset A$ we have $\dim(M_{\mathfrak q}) = \text{depth}(M_{\mathfrak q})$ and hence the set described in the lemma is empty and a fortiori finite. $\square$

0DWZ Lemma 51.9.4. Let $(A, \mathfrak m)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Set $E^i = \text{Ext}_A^{-i}(M, \omega_A^\bullet)$. Then

(1) E^i is a finite A-module nonzero only for $0 \leq i \leq \dim(\text{Supp}(M))$,
(2) $\dim(\text{Supp}(E^i)) \leq i$,
(3) $\text{depth}(M)$ is the smallest integer $\delta \geq 0$ such that $E^\delta \neq 0$,
(4) $\mathfrak p \in \text{Supp}(E^0 \oplus \ldots \oplus E^i) \Leftrightarrow \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) \leq i$,
(5) the annihilator of E^i is equal to the annihilator of $H^i_\mathfrak m(M)$.

Proof. Parts (1), (2), and (3) are copies of the statements in Dualizing Complexes, Lemma 47.16.5. For a prime $\mathfrak p$ of A we have that $(\omega_A^\bullet)_\mathfrak p[-\dim(A/\mathfrak p)]$ is a normalized dualzing complex for $A_\mathfrak p$. See Dualizing Complexes, Lemma 47.17.3. Thus

$$E^i_\mathfrak p = \text{Ext}_A^{-i}(M, \omega_A^\bullet)_\mathfrak p = \text{Ext}_{A_\mathfrak p}^{-i+\dim(A/\mathfrak p)}(M_\mathfrak p, (\omega_A^\bullet)_\mathfrak p[-\dim(A/\mathfrak p)])$$

is zero for $i - \dim(A/\mathfrak p) < \text{depth}_{A_\mathfrak p}(M_\mathfrak p)$ and nonzero for $i = \dim(A/\mathfrak p) + \text{depth}_{A_\mathfrak p}(M_\mathfrak p)$ by part (3) over $A_\mathfrak p$. This proves part (4). If E is an injective hull of the residue field of A, then we have

$$\text{Hom}_A(H^i_\mathfrak m(M), E) = \text{Ext}_A^{-i}(M, \omega_A^\bullet)^\wedge = (E^i)^\wedge = E^i \otimes_A A^\wedge$$

by the local duality theorem (in the form of Dualizing Complexes, Lemma 47.18.4). Since $A \to A^\wedge$ is faithfully flat, we find (5) is true by Matlis duality (Dualizing Complexes, Proposition 47.7.8). $\square$

51.10. Annihilators of local cohomology, I

0EFB This section discusses a result due to Faltings, see [Fal78b].

0EFC Proposition 51.10.1. Let A be a Noetherian ring which has a dualizing complex. Let $T \subset T' \subset \text{Spec}(A)$ be subsets stable under specialization. Let $s \geq 0$ an integer. Let M be a finite A-module. The following are equivalent [Fal78b].

(1) there exists an ideal $J \subset A$ with $V(J) \subset T'$ such that J annihilates $H^i_T(M)$ for $i \leq s$, and
(2) for all $\mathfrak p \not\in T'$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have
$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

Proof. Let $\omega_A^\bullet$ be a dualizing complex. Let δ be its dimension function, see Dualizing Complexes, Section 47.17. An important role will be played by the finite A-modules

$$E^i = \text{Ext}^i_A(M, \omega_A^\bullet)$$

For $\mathfrak p \subset A$ we will write $H^i_\mathfrak p$ to denote the local cohomology of an $A_\mathfrak p$-module with respect to $\mathfrak pA_\mathfrak p$. Then we see that the $\mathfrak pA_\mathfrak p$-adic completion of

$$(E^i)_\mathfrak p = \text{Ext}^{\delta(\mathfrak p)+i}_{A_\mathfrak p}(M_\mathfrak p, (\omega_A^\bullet)_\mathfrak p[-\delta(\mathfrak p)])$$

is Matlis dual to

$$H^{-\delta(\mathfrak p)-i}_\mathfrak p(M_\mathfrak p)$$

by Dualizing Complexes, Lemma 47.18.4. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_\mathfrak p$ if and only if J annihilates $H^{-\delta(\mathfrak p)-i}_\mathfrak p(M_\mathfrak p)$.

Set $T_n = \{\mathfrak p \in T \mid \delta(\mathfrak p) \leq n\}$. As δ is a bounded function, we see that $T_a = \emptyset$ for $a \ll 0$ and $T_b = T$ for $b \gg 0$.

Assume (2). Let us prove the existence of J as in (1). We will use a double induction to do this. For $i \leq s$ consider the induction hypothesis IH_i: $H^a_T(M)$ is annihilated by some $J \subset A$ with $V(J) \subset T'$ for $0 \leq a \leq i$. The case IH_0 is trivial because $H^0_T(M)$ is a submodule of M and hence finite and hence is annihilated by some ideal J with $V(J) \subset T$.

Induction step. Assume IH_{i-1} holds for some $0 < i \leq s$. Pick J' with $V(J') \subset T'$ annihilating $H^a_T(M)$ for $0 \leq a \leq i-1$ (the induction hypothesis guarantees we can do this). We will show by descending induction on n that there exists an ideal J with $V(J) \subset T'$ such that the associated primes of $JH^i_T(M)$ are in T_n. For $n \ll 0$ this implies $JH^i_T(M) = 0$ (Algebra, Lemma 10.63.7) and hence IH_i will hold. The base case $n \gg 0$ is trivial because $T = T_n$ in this case and all associated primes of $H^i_T(M)$ are in T.

Thus we assume given J with the property for n. Let $\mathfrak q \in T_n$. Let $T_{\mathfrak q} \subset \text{Spec}(A_{\mathfrak q})$ be the inverse image of T. We have $H^j_T(M)_{\mathfrak q} = H^j_{T_{\mathfrak q}}(M_{\mathfrak q})$ by Lemma 51.5.7. Consider the spectral sequence

$$H^p_{\mathfrak q}(H^q_{T_{\mathfrak q}}(M_{\mathfrak q})) \Rightarrow H^{p+q}_{\mathfrak q}(M_{\mathfrak q})$$

of Lemma 51.5.8. Below we will find an ideal $J'' \subset A$ with $V(J'') \subset T'$ such that $H^i_{\mathfrak q}(M_{\mathfrak q})$ is annihilated by J'' for all $\mathfrak q \in T_n \setminus T_{n-1}$. Claim: $J(J')^iJ''$ will work for $n-1$. Namely, let $\mathfrak q \in T_n \setminus T_{n-1}$. The spectral sequence above defines a filtration

$$E^{0,i}_\infty = E^{0,i}_{i+2} \subset \ldots \subset E^{0,i}_3 \subset E^{0,i}_2 = H^0_{\mathfrak q}(H^i_{T_{\mathfrak q}}(M_{\mathfrak q}))$$

The module $E^{0,i}_\infty$ is annihilated by J''. The subquotients $E^{0,i}_j/E^{0,i}_{j+1}$ for $i+1 \geq j \geq 2$ are annihilated by J' because the target of $d^{0,i}_j$ is a subquotient of

$$H^j_{\mathfrak q}(H^{i-j+1}_{T_{\mathfrak q}}(M_{\mathfrak q})) = H^j_{\mathfrak q}(H^{i-j+1}_T(M)_{\mathfrak q})$$

and $H^{i-j+1}_T(M)_{\mathfrak q}$ is annihilated by J' by choice of J'. Finally, by our choice of J we have $JH^i_T(M)_{\mathfrak q} \subset H^0_{\mathfrak q}(H^i_T(M)_{\mathfrak q})$ since the non-closed points of $\text{Spec}(A_{\mathfrak q})$ have higher δ values. Thus $\mathfrak q$ cannot be an associated prime of $J(J')^iJ''H^i_T(M)$ as desired.

By our initial remarks we see that J'' should annihilate

$$(E^{-\delta(\mathfrak q)-i})_{\mathfrak q} = (E^{-n-i})_{\mathfrak q}$$

for all $\mathfrak q \in T_n \setminus T_{n-1}$. But if J'' works for one $\mathfrak q$, then it works for all $\mathfrak q$ in an open neighbourhood of $\mathfrak q$ as the modules E^{-n-i} are finite. Since every subset of $\text{Spec}(A)$ is Noetherian with the induced topology (Topology, Lemma 5.9.2), we conclude that it suffices to prove the existence of J'' for one $\mathfrak q$.

Since the ext modules are finite the existence of J'' is equivalent to

$$\text{Supp}(E^{-n-i}) \cap \text{Spec}(A_{\mathfrak q}) \subset T'.$$

This is equivalent to showing the localization of E^{-n-i} at every $\mathfrak p \subset \mathfrak q$, $\mathfrak p \not\in T'$ is zero. Using local duality over $A_{\mathfrak p}$ we find that we need to prove that

$$H^{i+n-\delta(\mathfrak p)}_{\mathfrak p}(M_{\mathfrak p}) = H^{i-\dim((A/\mathfrak p)_{\mathfrak q})}_{\mathfrak p}(M_{\mathfrak p})$$

is zero (this uses that δ is a dimension function). This vanishes by the assumption in the lemma and $i \leq s$ and Dualizing Complexes, Lemma 47.11.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a $\mathfrak q \in T$, $\mathfrak p \subset \mathfrak q$ with $\mathfrak p \not\in T'$ such that

$$i = \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) \leq s$$

is minimal. Then $H_\mathfrak p^{i - \dim((A/\mathfrak p)_\mathfrak q)}(M_\mathfrak p)$ is nonzero by the nonvanishing in Dualizing Complexes, Lemma 47.11.1. Set $n = \delta(\mathfrak q)$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset T'$ such that $J(E^{-n-i})_\mathfrak q = 0$. Thus $H^i_\mathfrak q(M_\mathfrak q)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. By minimality of i it follows from the spectral sequence displayed above that the module $H^i_T(M)_\mathfrak q$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. Thus $H^i_T(M)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. This finishes the proof of the proposition. □

0EFE Lemma 51.10.2. Let I be an ideal of a Noetherian ring A. Let M be a finite A-module, let $\mathfrak p \subset A$ be a prime ideal, and let $s \geq 0$ be an integer. Assume

(1) A has a dualizing complex,
(2) $\mathfrak p \not\in V(I)$, and
(3) for all primes $\mathfrak p' \subset \mathfrak p$ and $\mathfrak q \in V(I)$ with $\mathfrak p' \subset \mathfrak q$ we have

$$\text{depth}_{A_{\mathfrak p'}}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_\mathfrak q) > s$$

Then there exists an $f \in A$, $f \not\in \mathfrak p$ which annihilates $H^i_{V(I)}(M)$ for $i \leq s$.

Proof. Consider the sets

$$T = V(I) \quad \text{and} \quad T' = \bigcup\nolimits_{f \in A, f \not\in \mathfrak p} V(f)$$

These are subsets of $\text{Spec}(A)$ stable under specialization. Observe that $T \subset T'$ and $\mathfrak p \not\in T'$. Assumption (3) says that hypothesis (2) of Proposition 51.10.1 holds. Hence we can find $J \subset A$ with $V(J) \subset T'$ such that $JH^i_{V(I)}(M) = 0$ for $i \leq s$. Choose $f \in A$, $f \not\in \mathfrak p$ with $V(J) \subset V(f)$. A power of f annihilates $H^i_{V(I)}(M)$ for $i \leq s$. □

51.11. Finiteness of local cohomology, II

0BJQ We continue the discussion of finiteness of local cohomology started in Section 51.7. Using Faltings Annihilator Theorem we easily prove the following fundamental result.

0EFD Proposition 51.11.1. Let A be a Noetherian ring which has a dualizing complex. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $s \geq 0$ an integer. Let M be a finite A-module. The following are equivalent [Fal78b].

(1) $H^i_T(M)$ is a finite A-module for $i \leq s$, and
(2) for all $\mathfrak p \not\in T$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

Proof. Formal consequence of Proposition 51.10.1 and Lemma 51.7.1. □

Besides some lemmas for later use, the rest of this section is concerned with the question to what extend the condition in Proposition 51.11.1 that A has a dualizing complex can be weakened. The answer is roughly that one has to assume the formal fibres of A are (S_n) for sufficiently large n.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \mathrm{Spec}(A)$ and $Z = V(I) \subset X$. Let M be a finite A-module. We define

0BJR (51.11.1.1) $s_{A,I}(M) = \min\{\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \mid \mathfrak{p} \in X \setminus Z, \mathfrak{q} \in Z, \mathfrak{p} \subset \mathfrak{q}\}$

Our conventions on depth are that the depth of 0 is ∞ thus we only need to consider primes $\mathfrak{p}$ in the support of M. It will turn out that $s_{A,I}(M)$ is an important invariant of the situation.

0BJS Lemma 51.11.2. Let $A \to B$ be a finite homomorphism of Noetherian rings. Let $I \subset A$ be an ideal and set $J = IB$. Let M be a finite B-module. If A is universally catenary, then $s_{B,J}(M) = s_{A,I}(M)$.

Proof. Let $\mathfrak{p} \subset \mathfrak{q} \subset A$ be primes with $I \subset \mathfrak{q}$ and $I \not\subset \mathfrak{p}$. Since $A \to B$ is finite there are finitely many primes $\mathfrak{p}_i$ lying over $\mathfrak{p}$. By Algebra, Lemma 10.72.11 we have

$$\mathrm{depth}(M_\mathfrak{p}) = \min \mathrm{depth}(M_{\mathfrak{p}_i})$$

Let $\mathfrak{p}_i \subset \mathfrak{q}_{ij}$ be primes lying over $\mathfrak{q}$. By going up for $A \to B$ (Algebra, Lemma 10.36.22) there is at least one $\mathfrak{q}_{ij}$ for each i. Then we see that

$$\dim((B/\mathfrak{p}_i)_{\mathfrak{q}_{ij}}) = \dim((A/\mathfrak{p})_\mathfrak{q})$$

by the dimension formula, see Algebra, Lemma 10.113.1. This implies that the minimum of the quantities used to define $s_{B,J}(M)$ for the pairs $(\mathfrak{p}_i, \mathfrak{q}_{ij})$ is equal to the quantity for the pair $(\mathfrak{p}, \mathfrak{q})$. This proves the lemma. $\square$

0EHW Lemma 51.11.3. Let A be a Noetherian ring which has a dualizing complex. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let A', M' be the I-adic completions of A, M. Let $\mathfrak{p}' \subset \mathfrak{q}'$ be prime ideals of A' with $\mathfrak{q}' \in V(IA')$ lying over $\mathfrak{p} \subset \mathfrak{q}$ in A. Then

$$\mathrm{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) \geq \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$$

and

$$\mathrm{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$$

Proof. We have

$$\mathrm{depth}(M'_{\mathfrak{p}'}) = \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) \geq \mathrm{depth}(M_\mathfrak{p})$$

by flatness of $A \to A'$, see Algebra, Lemma 10.163.1. Since the fibres of $A \to A'$ are Cohen-Macaulay (Dualizing Complexes, Lemma 47.23.2 and More on Algebra, Section 15.51) we see that $\mathrm{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) = \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'})$. Thus we obtain

$$\begin{aligned} \mathrm{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) &= \mathrm{depth}(M_\mathfrak{p}) + \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) \\ &= \mathrm{depth}(M_\mathfrak{p}) + \dim((A'/\mathfrak{p}A')_{\mathfrak{q}'}) \\ &= \mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \end{aligned}$$

Second equality because A' is catenary and third equality by More on Algebra, Lemma 15.43.1 as $(A/\mathfrak{p})_\mathfrak{q}$ and $(A'/\mathfrak{p}A')_{\mathfrak{q}'}$ have the same I-adic completions. $\square$

0BJT Lemma 51.11.4. Let A be a universally catenary Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then

$$s_{A,I}(M) \geq s_{A^\wedge, I^\wedge}(M^\wedge)$$

If the formal fibres of A are (S_n), then $\min(n+1, s_{A,I}(M)) \leq s_{A^\wedge, I^\wedge}(M^\wedge)$.

Proof. Write $X = \mathrm{Spec}(A)$, $X^\wedge = \mathrm{Spec}(A^\wedge)$, $Z = V(I) \subset X$, and $Z^\wedge = V(I^\wedge)$. Let $\mathfrak p' \subset \mathfrak q' \subset A^\wedge$ be primes with $\mathfrak p' \not\in Z^\wedge$ and $\mathfrak q' \in Z^\wedge$. Let $\mathfrak p \subset \mathfrak q$ be the corresponding primes of A. Then $\mathfrak p \not\in Z$ and $\mathfrak q \in Z$. Picture

$$\begin{tikzcd} \mathfrak p' \arrow[r] & \mathfrak q' \arrow[r] & A^\wedge \\ \mathfrak p \arrow[r] \arrow[u, dash] & \mathfrak q \arrow[r] \arrow[u, dash] & A \arrow[u] \end{tikzcd}$$

Let us write

$$\begin{aligned} a &= \dim(A/\mathfrak p) = \dim(A^\wedge/\mathfrak p A^\wedge), \\ b &= \dim(A/\mathfrak q) = \dim(A^\wedge/\mathfrak q A^\wedge), \\ a' &= \dim(A^\wedge/\mathfrak p'), \\ b' &= \dim(A^\wedge/\mathfrak q') \end{aligned}$$

Equalities by More on Algebra, Lemma 15.43.1. We also write

$$\begin{aligned} p &= \dim(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) = \dim((A^\wedge/\mathfrak p A^\wedge)_{\mathfrak p'}) \\ q &= \dim(A^\wedge_{\mathfrak q'}/\mathfrak p A^\wedge_{\mathfrak q'}) = \dim((A^\wedge/\mathfrak q A^\wedge)_{\mathfrak q'}) \end{aligned}$$

Since A is universally catenary we see that $A^\wedge/\mathfrak p A^\wedge = (A/\mathfrak p)^\wedge$ is equidimensional of dimension a (More on Algebra, Proposition 15.109.5). Hence $a = a' + p$. Similarly $b = b' + q$. By Algebra, Lemma 10.163.1 applied to the flat local ring map $A_{\mathfrak p} \to A^\wedge_{\mathfrak p'}$ we have

$$\mathrm{depth}(M^\wedge_{\mathfrak p'}) = \mathrm{depth}(M_{\mathfrak p}) + \mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'})$$

The quantity we are minimizing for $s_{A,I}(M)$ is

$$s(\mathfrak p, \mathfrak q) = \mathrm{depth}(M_{\mathfrak p}) + \dim((A/\mathfrak p)_{\mathfrak q}) = \mathrm{depth}(M_{\mathfrak p}) + a - b$$

(last equality as A is catenary). The quantity we are minimizing for $s_{A^\wedge, I^\wedge}(M^\wedge)$ is

$$s(\mathfrak p', \mathfrak q') = \mathrm{depth}(M^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'}) = \mathrm{depth}(M^\wedge_{\mathfrak p'}) + a' - b'$$

(last equality as $A^\wedge$ is catenary). Now we have enough notation in place to start the proof.

Let $\mathfrak p \subset \mathfrak q \subset A$ be primes with $\mathfrak p \not\in Z$ and $\mathfrak q \in Z$ such that $s_{A,I}(M) = s(\mathfrak p, \mathfrak q)$. Then we can pick $\mathfrak q'$ minimal over $\mathfrak q A^\wedge$ and $\mathfrak p' \subset \mathfrak q'$ minimal over $\mathfrak p A^\wedge$ (using going down for $A \to A^\wedge$). Then we have four primes as above with $p = 0$ and $q = 0$. Moreover, we have $\mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) = 0$ also because $p = 0$. This means that $s(\mathfrak p', \mathfrak q') = s(\mathfrak p, \mathfrak q)$. Thus we get the first inequality.

Assume that the formal fibres of A are (S_n). Then $\mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) \geq \min(n, p)$. Hence

$$s(\mathfrak p', \mathfrak q') \geq s(\mathfrak p, \mathfrak q) + q + \min(n, p) - p \geq s_{A,I}(M) + q + \min(n, p) - p$$

Thus the only way we can get in trouble is if $p > n$. If this happens then

$$\begin{aligned} s(\mathfrak p', \mathfrak q') &= \mathrm{depth}(M^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'}) \\ &= \mathrm{depth}(M_{\mathfrak p}) + \mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'}) \\ &\geq 0 + n + 1 \end{aligned}$$

because $(A^\wedge/\mathfrak p')_{\mathfrak q'}$ has at least two primes. This proves the second inequality. $\square$

The method of proof of the following lemma works more generally, but the stronger results one gets will be subsumed in Theorem 51.11.6 below.

0BJU Lemma 51.11.5. Let A be a Gorenstein Noetherian local ring. Let $I \subset A$ be an ideal and set $Z = V(I) \subset \text{Spec}(A)$. Let M be a finite A-module. Let $s = s_{A,I}(M)$ as in (51.11.1.1). Then $H^i_Z(M)$ is finite for $i < s$, but $H^s_Z(M)$ is not finite.

This is a special case of [Fal78b, Satz 1].

Proof. Since a Gorenstein local ring has a dualizing complex, this is a special case of Proposition 51.11.1. It would be helpful to have a short proof of this special case, which will be used in the proof of a general finiteness theorem below. $\square$

Observe that the hypotheses of the following theorem are satisfied by excellent Noetherian rings (by definition), by Noetherian rings which have a dualizing complex (Dualizing Complexes, Lemma 47.17.4 and Dualizing Complexes, Lemma 47.23.2), and by quotients of regular Noetherian rings.

0BJV Theorem 51.11.6. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I) \subset \text{Spec}(A)$. Let M be a finite A-module. Set $s = s_{A,I}(M)$ as in (51.11.1.1). Assume that

(1) A is universally catenary,
(2) the formal fibres of the local rings of A are Cohen-Macaulay.

Then $H^i_Z(M)$ is finite for $0 \leq i < s$ and $H^s_Z(M)$ is not finite.

This is a special case of [Fal81, Satz 2].

Proof. By Lemma 51.7.2 we may assume that A is a local ring.

If A is a Noetherian complete local ring, then we can write A as the quotient of a regular complete local ring B by Cohen's structure theorem (Algebra, Theorem 10.160.8). Using Lemma 51.11.2 and Dualizing Complexes, Lemma 47.9.2 we reduce to the case of a regular local ring which is a consequence of Lemma 51.11.5 because a regular local ring is Gorenstein (Dualizing Complexes, Lemma 47.21.3).

Let A be a Noetherian local ring. Let $\mathfrak{m}$ be the maximal ideal. We may assume $I \subset \mathfrak{m}$, otherwise the lemma is trivial. Let $A^\wedge$ be the completion of A, let $Z^\wedge = V(IA^\wedge)$, and let $M^\wedge = M \otimes_A A^\wedge$ be the completion of M (Algebra, Lemma 10.97.1). Then $H^i_Z(M) \otimes_A A^\wedge = H^i_{Z^\wedge}(M^\wedge)$ by Dualizing Complexes, Lemma 47.9.3 and flatness of $A \to A^\wedge$ (Algebra, Lemma 10.97.2). Hence it suffices to show that $H^i_{Z^\wedge}(M^\wedge)$ is finite for $i < s$ and not finite for $i = s$, see Algebra, Lemma 10.83.2. Since we know the result is true for $A^\wedge$ it suffices to show that $s_{A,I}(M) = s_{A^\wedge,I^\wedge}(M^\wedge)$. This follows from Lemma 51.11.4. $\square$

0BJW Remark 51.11.7. The astute reader will have realized that we can get away with a slightly weaker condition on the formal fibres of the local rings of A. Namely, in the situation of Theorem 51.11.6 assume A is universally catenary but make no assumptions on the formal fibres. Suppose we have an n and we want to prove that $H^i_Z(M)$ are finite for $i \leq n$. Then the exact same proof shows that it suffices that $s_{A,I}(M) > n$ and that the formal fibres of local rings of A are (S_n). On the other hand, if we want to show that $H^s_Z(M)$ is not finite where $s = s_{A,I}(M)$, then our arguments prove this if the formal fibres are (S_{s-1}).

51.12. Finiteness of pushforwards, II

0BJX This section is the continuation of Section 51.8. In this section we reap the fruits of the labor done in Section 51.11.

0BJY **Lemma 51.12.1.** Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $n \geq 0$ be an integer. Assume

(1) X is universally catenary,
(2) for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_n).

In this situation the following are equivalent

(a) for $x \in \text{Supp}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > n$,
(b) $R^pj_*\mathcal{F}$ is coherent for $0 \leq p < n$.

Proof. The statement is local on X, hence we may assume X is affine. Say $X = \text{Spec}(A)$ and $Z = V(I)$. Let M be a finite A-module whose associated coherent $\mathcal{O}_X$-module restricts to $\mathcal{F}$ over U, see Lemma 51.8.2. This lemma also tells us that $R^pj_*\mathcal{F}$ is coherent if and only if $H^{p+1}_Z(M)$ is a finite A-module. Observe that the minimum of the expressions $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z})$ is the number $s_{A,I}(M)$ of (51.11.1.1). Having said this the lemma follows from Theorem 51.11.6 as elucidated by Remark 51.11.7. $\square$

0BLT **Lemma 51.12.2.** Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $n \geq 0$ be an integer. If $R^pj_*\mathcal{O}_U$ is coherent for $0 \leq p < n$, then the same is true for $R^pj_*\mathcal{F}$, $0 \leq p < n$ for any finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.

Proof. The question is local on X, hence we may assume X is affine. Say $X = \text{Spec}(A)$ and $Z = V(I)$. Via Lemma 51.8.2 our lemma follows from Lemma 51.7.4. $\square$

0BM5 **Lemma 51.12.3.** Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $p \geq 0$ be an integer. Set $U = \text{Spec}(A) \setminus V(I)$. If $H^p(U, \mathcal{O}_U)$ is annihilated by J^n for some n, then $H^p(U, \mathcal{F})$ annihilated by J^m for some $m = m(\mathcal{F})$ for every finite locally free $\mathcal{O}_U$-module $\mathcal{F}$. [BdJ14, Lemma 1.9]

Proof. Consider the annihilator $\mathfrak{a}$ of $H^p(U, \mathcal{F})$. Let $u \in U$. There exists an open neighbourhood $u \in U' \subset U$ and an isomorphism $\varphi : \mathcal{O}_{U'}^{\oplus r} \to \mathcal{F}|_{U'}$. Pick $f \in A$ such that $u \in D(f) \subset U'$. There exist maps

$$a : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F} \quad \text{and} \quad b : \mathcal{F} \longrightarrow \mathcal{O}_U^{\oplus r}$$

whose restriction to $D(f)$ are equal to $f^N\varphi$ and $f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. This follows from Properties, Lemma 28.17.3 since U is quasi-compact (I is finitely generated), separated, and $\mathcal{F}$ and $\mathcal{O}_U^{\oplus r}$ are finitely presented. Thus we see that $H^p(U, \mathcal{F})$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak{a}$.

As U is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_i)$ and $f_i^{2N_i}J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. $\square$

51.13. Annihilators of local cohomology, II

0EHX We extend the discussion of annihilators of local cohomology in Section 51.10 to bounded below complexes with finite cohomology modules.

0EHY Definition 51.13.1. Let I be an ideal of a Noetherian ring A. Let $K \in D^+_{Coh}(A)$. We define the I-depth of K, denoted $\text{depth}_I(K)$, to be the maximal $m \in \mathbf{Z} \cup \{\infty\}$ such that $H^i_I(K) = 0$ for all $i < m$. If A is local with maximal ideal $\mathfrak m$ then we call $\text{depth}_{\mathfrak m}(K)$ simply the depth of K.

This definition does not conflict with Algebra, Definition 10.72.1 by Dualizing Complexes, Lemma 47.11.1.

0EHZ Proposition 51.13.2. Let A be a Noetherian ring which has a dualizing complex. Let $T \subset T' \subset \text{Spec}(A)$ be subsets stable under specialization. Let $s \in \mathbf{Z}$. Let K be an object of $D^+_{Coh}(A)$. The following are equivalent

(1) there exists an ideal $J \subset A$ with $V(J) \subset T'$ such that J annihilates $H^i_T(K)$ for $i \leq s$, and
(2) for all $\mathfrak p \not\in T'$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have

$$\text{depth}_{A_\mathfrak p}(K_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

Proof. This lemma is the natural generalization of Proposition 51.10.1 whose proof the reader should read first. Let $\omega_A^\bullet$ be a dualizing complex. Let δ be its dimension function, see Dualizing Complexes, Section 47.17. An important role will be played by the finite A-modules

$$E^i = \text{Ext}^i_A(K, \omega_A^\bullet)$$

For $\mathfrak p \subset A$ we will write $H^i_\mathfrak p$ to denote the local cohomology of an object of $D(A_\mathfrak p)$ with respect to $\mathfrak pA_\mathfrak p$. Then we see that the $\mathfrak pA_\mathfrak p$-adic completion of

$$(E^i)_\mathfrak p = \text{Ext}^{\delta(\mathfrak p)+i}_{A_\mathfrak p}(K_\mathfrak p, (\omega_A^\bullet)_\mathfrak p[-\delta(\mathfrak p)])$$

is Matlis dual to

$$H^{-\delta(\mathfrak p)-i}_\mathfrak p(K_\mathfrak p)$$

by Dualizing Complexes, Lemma 47.18.4. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_\mathfrak p$ if and only if J annihilates $H^{-\delta(\mathfrak p)-i}_\mathfrak p(K_\mathfrak p)$.

Set $T_n = \{\mathfrak p \in T \mid \delta(\mathfrak p) \leq n\}$. As δ is a bounded function, we see that $T_a = \emptyset$ for $a \ll 0$ and $T_b = T$ for $b \gg 0$.

Assume (2). Let us prove the existence of J as in (1). We will use a double induction to do this. For $i \leq s$ consider the induction hypothesis IH_i: $H^a_T(K)$ is annihilated by some $J \subset A$ with $V(J) \subset T'$ for $a \leq i$. The case IH_i is trivial for i small enough because K is bounded below.

Induction step. Assume IH_{i-1} holds for some $i \leq s$. Pick J' with $V(J') \subset T'$ annihilating $H^a_T(K)$ for $a \leq i-1$ (the induction hypothesis guarantees we can do this). We will show by descending induction on n that there exists an ideal J with $V(J) \subset T'$ such that the associated primes of $JH^i_T(K)$ are in T_n. For $n \ll 0$ this implies $JH^i_T(K) = 0$ (Algebra, Lemma 10.63.7) and hence IH_i will hold. The base case $n \gg 0$ is trivial because $T = T_n$ in this case and all associated primes of $H^i_T(K)$ are in T.

Thus we assume given J with the property for n. Let $\mathfrak q \in T_n$. Let $T_\mathfrak q \subset \text{Spec}(A_\mathfrak q)$ be the inverse image of T. We have $H^j_T(K)_\mathfrak q = H^j_{T_\mathfrak q}(K_\mathfrak q)$ by Lemma 51.5.7. Consider the spectral sequence

$$H^p_\mathfrak q(H^q_{T_\mathfrak q}(K_\mathfrak q)) \Rightarrow H^{p+q}_\mathfrak q(K_\mathfrak q)$$

of Lemma 51.5.8. Below we will find an ideal $J'' \subset A$ with $V(J'') \subset T'$ such that $H^i_{\mathfrak q}(K_{\mathfrak q})$ is annihilated by J'' for all $\mathfrak q \in T_n \setminus T_{n-1}$. Claim: $J(J')^iJ''$ will work for $n-1$. Namely, let $\mathfrak q \in T_n \setminus T_{n-1}$. The spectral sequence above defines a filtration

$$E_\infty^{0,i} = E_{i+2}^{0,i} \subset \ldots \subset E_3^{0,i} \subset E_2^{0,i} = H^0_{\mathfrak q}(H^i_{T_{\mathfrak q}}(K_{\mathfrak q}))$$

The module $E_\infty^{0,i}$ is annihilated by J''. The subquotients $E_j^{0,i}/E_{j+1}^{0,i}$ for $i+1 \geq j \geq 2$ are annihilated by J' because the target of $d_j^{0,i}$ is a subquotient of

$$H^j_{\mathfrak q}(H^{i-j+1}_{T_{\mathfrak q}}(K_{\mathfrak q})) = H^j_{\mathfrak q}(H^{i-j+1}_T(K)_{\mathfrak q})$$

and $H^{i-j+1}_T(K)_{\mathfrak q}$ is annihilated by J' by choice of J'. Finally, by our choice of J we have $JH^i_T(K)_{\mathfrak q} \subset H^0_{\mathfrak q}(H^i_T(K)_{\mathfrak q})$ since the non-closed points of $\text{Spec}(A_{\mathfrak q})$ have higher δ values. Thus $\mathfrak q$ cannot be an associated prime of $J(J')^iJ''H^i_T(K)$ as desired.

By our initial remarks we see that J'' should annihilate

$$(E^{-\delta(\mathfrak q)-i})_{\mathfrak q} = (E^{-n-i})_{\mathfrak q}$$

for all $\mathfrak q \in T_n \setminus T_{n-1}$. But if J'' works for one $\mathfrak q$, then it works for all $\mathfrak q$ in an open neighbourhood of $\mathfrak q$ as the modules E^{-n-i} are finite. Since every subset of $\text{Spec}(A)$ is Noetherian with the induced topology (Topology, Lemma 5.9.2), we conclude that it suffices to prove the existence of J'' for one $\mathfrak q$.

Since the ext modules are finite the existence of J'' is equivalent to

$$\text{Supp}(E^{-n-i}) \cap \text{Spec}(A_{\mathfrak q}) \subset T'.$$

This is equivalent to showing the localization of E^{-n-i} at every $\mathfrak p \subset \mathfrak q$, $\mathfrak p \not\in T'$ is zero. Using local duality over $A_{\mathfrak p}$ we find that we need to prove that

$$H^{i+n-\delta(\mathfrak p)}_{\mathfrak p}(K_{\mathfrak p}) = H^{i-\dim((A/\mathfrak p)_{\mathfrak q})}_{\mathfrak p}(K_{\mathfrak p})$$

is zero (this uses that δ is a dimension function). This vanishes by the assumption in the lemma and $i \leq s$ and our definition of depth in Definition 51.13.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a $\mathfrak q \in T$, $\mathfrak p \subset \mathfrak q$ with $\mathfrak p \not\in T'$ such that

$$i = \text{depth}_{A_{\mathfrak p}}(K_{\mathfrak p}) + \dim((A/\mathfrak p)_{\mathfrak q}) \leq s$$

is minimal. Then $H^{i-\dim((A/\mathfrak p)_{\mathfrak q})}_{\mathfrak p}(K_{\mathfrak p})$ is nonzero by the our definition of depth in Definition 51.13.1. Set $n = \delta(\mathfrak q)$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset T'$ such that $J(E^{-n-i})_{\mathfrak q} = 0$. Thus $H^i_{\mathfrak q}(K_{\mathfrak q})$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. By minimality of i it follows from the spectral sequence displayed above that the module $H^i_T(K)_{\mathfrak q}$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. Thus $H^i_T(K)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. This finishes the proof of the proposition. □

51.14. Finiteness of local cohomology, III

0EI0 We extend the discussion of finiteness of local cohomology in Sections 51.7 and 51.11 to bounded below complexes with finite cohomology modules.

0EI1 **Lemma 51.14.1.** Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let K be an object of $D^+_{Coh}(A)$. Let $n \in \mathbf{Z}$. The following are equivalent

(1) $H^i_T(K)$ is finite for $i \leq n$,
(2) there exists an ideal $J \subset A$ with $V(J) \subset T$ such that J annihilates $H^i_T(K)$ for $i \leq n$.

If $T = V(I) = Z$ for an ideal $I \subset A$, then these are also equivalent to

(3) there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(K)$ for $i \leq n$.

Proof. This lemma is the natural generalization of Lemma 51.7.1 whose proof the reader should read first. Assume (1) is true. Recall that $H^i_J(K) = H^i_{V(J)}(K)$, see Dualizing Complexes, Lemma 47.10.1. Thus $H^i_T(K) = \operatorname{colim} H^i_J(K)$ where the colimit is over ideals $J \subset A$ with $V(J) \subset T$, see Lemma 51.5.3. Since $H^i_T(K)$ is finitely generated for $i \leq n$ we can find a $J \subset A$ as in (2) such that $H^i_J(K) \to H^i_T(K)$ is surjective for $i \leq n$. Thus the finite list of generators are J-power torsion elements and we see that (2) holds with J replaced by some power.

Let $a \in \mathbf{Z}$ be an integer such that $H^i(K) = 0$ for $i < a$. We prove (2) $\Rightarrow$ (1) by descending induction on a. If $a > n$, then we have $H^i_T(K) = 0$ for $i \leq n$ hence both (1) and (2) are true and there is nothing to prove.

Assume we have J as in (2). Observe that $N = H^a_T(K) = H^0_T(H^a(K))$ is finite as a submodule of the finite A-module $H^a(K)$. If $n = a$ we are done; so assume $a < n$ from now on. By construction of $R\Gamma_T$ we find that $H^i_T(N) = 0$ for $i > 0$ and $H^0_T(N) = N$, see Remark 51.5.6. Choose a distinguished triangle

$$N[-a] \to K \to K' \to N[-a+1]$$

Then we see that $H^a_T(K') = 0$ and $H^i_T(K) = H^i_T(K')$ for $i > a$. We conclude that we may replace K by K'. Thus we may assume that $H^a_T(K) = 0$. This means that the finite set of associated primes of $H^a(K)$ are not in T. By prime avoidance (Algebra, Lemma 10.15.2) we can find $f \in J$ not contained in any of the associated primes of $H^a(K)$. Choose a distinguished triangle

$$L \to K \xrightarrow{f} K \to L[1]$$

By construction we see that $H^i(L) = 0$ for $i \leq a$. On the other hand we have a long exact cohomology sequence

$$0 \to H^{a+1}_T(L) \to H^{a+1}_T(K) \xrightarrow{f} H^{a+1}_T(K) \to H^{a+2}_T(L) \to H^{a+2}_T(K) \xrightarrow{f} \ldots$$

which breaks into the identification $H^{a+1}_T(L) = H^{a+1}_T(K)$ and short exact sequences

$$0 \to H^{i-1}_T(K) \to H^i_T(L) \to H^i_T(K) \to 0$$

for $i \leq n$ since $f \in J$. We conclude that J^2 annihilates $H^i_T(L)$ for $i \leq n$. By induction hypothesis applied to L we see that $H^i_T(L)$ is finite for $i \leq n$. Using the short exact sequence once more we see that $H^i_T(K)$ is finite for $i \leq n$ as desired.

We omit the proof of the equivalence of (2) and (3) in case $T = V(I)$. $\square$

0EI2 Proposition 51.14.2. Let A be a Noetherian ring which has a dualizing complex. Let $T \subset \operatorname{Spec}(A)$ be a subset stable under specialization. Let $s \in \mathbf{Z}$. Let $K \in D^+_{Coh}(A)$. The following are equivalent

(1) $H^i_T(K)$ is a finite A-module for $i \leq s$, and
(2) for all $\mathfrak{p} \notin T$, $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$ we have

$$\text{depth}_{A_\mathfrak{p}}(K_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Proof. Formal consequence of Proposition 51.13.2 and Lemma 51.14.1. $\square$

51.15. Improving coherent modules

0DX2 Similar constructions can be found in [DG67] and more recently in [Kol15] and [Kol16b].

0DX3 Lemma 51.15.1. Let X be a Noetherian scheme. Let $T \subset X$ be a subset stable under specialization. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is a unique map $\mathcal{F} \to \mathcal{F}'$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F} \to \mathcal{F}'$ is surjective,
(2) $\mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for $x \notin T$,
(3) $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 1$ for $x \in T$.

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding quotient for $f^{-1}(T) \subset Y$ and $f^*\mathcal{F}$.

Proof. Condition (3) just means that $\text{Ass}(\mathcal{F}') \cap T = \emptyset$. Thus $\mathcal{F} \to \mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections whose support is contained in T. This proves uniqueness. The statement on pullbacks follows from Divisors, Lemma 31.3.1 and the uniqueness.

Existence of $\mathcal{F} \to \mathcal{F}'$. By the uniqueness it suffices to prove the existence and uniqueness locally on X; small detail omitted. Thus we may assume $X = \text{Spec}(A)$ is affine and $\mathcal{F}$ is the coherent module associated to the finite A-module M. Set $M' = M/H^0_T(M)$ with $H^0_T(M)$ as in Section 51.5. Then $M_\mathfrak{p} = M'_\mathfrak{p}$ for $\mathfrak{p} \notin T$ which proves (1). On the other hand, we have $H^0_T(M) = \text{colim}\, H^0_Z(M)$ where Z runs over the closed subsets of X contained in T. Thus by Dualizing Complexes, Lemmas 47.11.6 we have $H^0_T(M') = 0$, i.e., no associated prime of M' is in T. Therefore $\text{depth}(M'_\mathfrak{p}) \geq 1$ for $\mathfrak{p} \in T$. □

0DX4 Lemma 51.15.2. Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}' = j_*(\mathcal{F}|_U)$ is coherent. Then $\mathcal{F} \to \mathcal{F}'$ is the unique map of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}|_U \to \mathcal{F}'|_U$ is an isomorphism,
(2) $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$ for $x \in X$, $x \notin U$.

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding map for $f^{-1}(U) \subset Y$.

Proof. We have $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$ by Divisors, Lemma 31.6.6 part (3). The uniqueness of $\mathcal{F} \to \mathcal{F}'$ follows from Divisors, Lemma 31.5.11. The compatibility with flat pullbacks follows from flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

0DX5 Lemma 51.15.3. Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume X is universally catenary and the formal fibres of local rings have (S_1). Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \in X \setminus Z$,
(2) $\mathcal{F}_x \to \mathcal{F}''_x$ is surjective and $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) = 1$ for $x \in Z$ such that there exists an immediate specialization $x' \rightsquigarrow x$ with $x' \notin Z$ and $x' \in \text{Ass}(\mathcal{F})$,
(3) $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for the remaining $x \in Z$.

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(Z) \subset Y$.

Proof. Let $\mathcal{F} \to \mathcal{F}'$ be the map constructed in Lemma 51.15.1 for the subset Z of X. Recall that $\mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections supported on Z.

We first prove uniqueness. Let $\mathcal{F} \to \mathcal{F}''$ be as in the lemma. We get a factorization $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ since $\text{Ass}(\mathcal{F}'') \cap Z = \emptyset$ by conditions (2) and (3). Let $U \subset X$ be a maximal open subscheme such that $\mathcal{F}'|_U \to \mathcal{F}''|_U$ is an isomorphism. We see that U contains all the points as in (2). Then by Divisors, Lemma 31.5.11 we conclude that $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$. In this way we get uniqueness (small detail: if we have two of these $\mathcal{F}''$ then we take the intersection of the opens U we get from either).

Proof of existence. Recall that $\text{Ass}(\mathcal{F}') = \{x_1, \ldots, x_n\}$ is finite and $x_i \notin Z$. Let Y_i be the closure of $\{x_i\}$. Let $Z_{i,j}$ be the irreducible components of $Z \cap Y_i$. Observe that $\text{Supp}(\mathcal{F}') \cap Z = \bigcup Z_{i,j}$. Let $z_{i,j} \in Z_{i,j}$ be the generic point. Let

$$d_{i,j} = \dim(\mathcal{O}_{\overline{\{x_i\}}, z_{i,j}})$$

If $d_{i,j} = 1$, then $z_{i,j}$ is one of the points as in (2). Thus we do not need to modify $\mathcal{F}'$ at these points. Furthermore, still assuming $d_{i,j} = 1$, using Lemma 51.9.2 we can find an open neighbourhood $z_{i,j} \in V_{i,j} \subset X$ such that $\text{depth}_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$ for $z \in Z_{i,j} \cap V_{i,j}$, $z \neq z_{i,j}$. Set

$$Z' = X \setminus \left(X \setminus Z \cup \bigcup\nolimits_{d_{i,j}=1} V_{i,j})\right)$$

Denote $j' : X \setminus Z' \to X$. By our choice of Z' the assumptions of Lemma 51.8.9 are satisfied. We conclude by setting $\mathcal{F}'' = j'_*(\mathcal{F}'|_{X \setminus Z'})$ and applying Lemma 51.15.2.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 10.163.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. □

0EI3 Lemma 51.15.4. Let X be a Noetherian scheme which locally has a dualizing complex. Let $T' \subset X$ be a subset stable under specialization. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that if $x \rightsquigarrow x'$ is an immediate specialization of points in X with $x' \in T'$ and $x \notin T'$, then $\text{depth}(\mathcal{F}_x) \geq 1$. Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that

1. $\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \notin T'$,
2. $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for $x \in T'$.

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(T') \subset Y$.

Proof. Let $\mathcal{F} \to \mathcal{F}'$ be the quotient of $\mathcal{F}$ constructed in Lemma 51.15.1 using T'. Recall that $\mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections supported on T'.

Proof of uniqueness. Let $\mathcal{F} \to \mathcal{F}''$ be as in the lemma. We get a factorization $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ since $\text{Ass}(\mathcal{F}'') \cap T' = \emptyset$ by condition (2). Let $U \subset X$ be a maximal open subscheme such that $\mathcal{F}'|_U \to \mathcal{F}''|_U$ is an isomorphism. We see that U contains all the points of T'. Then by Divisors, Lemma 31.5.11 we conclude that $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$. In this way we get uniqueness (small detail: if we have two of these $\mathcal{F}''$ then we take the intersection of the opens U we get from either).

Proof of existence. We will define

$$\mathcal{F}'' = \text{colim}\, j_*(\mathcal{F}'|_V)$$

where $j : V \to X$ runs over the open subschemes such that $X \setminus V \subset T'$. Observe that the colimit is filtered as T' is stable under specialization. Each of the maps $\mathcal{F}' \to j_*(\mathcal{F}'|_V)$ is injective as $\text{Ass}(\mathcal{F}')$ is disjoint from T'. Thus $\mathcal{F}' \to \mathcal{F}''$ is injective.

Suppose $X = \text{Spec}(A)$ is affine and $\mathcal{F}$ corresponds to the finite A-module M. Then $\mathcal{F}'$ corresponds to $M' = M/H^0_{T'}(M)$, see proof of Lemma 51.15.1. Applying Lemmas 51.2.2 and 51.5.3 we see that $\mathcal{F}''$ corresponds to an A-module M'' which fits into the short exact sequence

$$0 \to M' \to M'' \to H^1_{T'}(M') \to 0$$

By Proposition 51.11.1 and our condition on immediate specializations in the statement of the lemma we see that M'' is a finite A-module. In this way we see that $\mathcal{F}''$ is coherent.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 10.163.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. □

0EI4 Lemma 51.15.5. Let X be a Noetherian scheme which locally has a dualizing complex. Let $T' \subset T \subset X$ be subsets stable under specialization such that if $x \rightsquigarrow x'$ is an immediate specialization of points in X and $x' \in T'$, then $x \in T$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \notin T$,
(2) $\mathcal{F}_x \to \mathcal{F}''_x$ is surjective and $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 1$ for $x \in T$, $x \notin T'$, and
(3) $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for $x \in T'$.

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(T') \subset f^{-1}(T) \subset Y$.

Proof. First, let $\mathcal{F} \to \mathcal{F}'$ be the quotient of $\mathcal{F}$ constructed in Lemma 51.15.1 using T. Second, let $\mathcal{F}' \to \mathcal{F}''$ be the unique map of coherent modules construction in Lemma 51.15.4 using T'. Then $\mathcal{F} \to \mathcal{F}''$ is as desired. □

51.16. Hartshorne-Lichtenbaum vanishing

0EB0 This vanishing result is the local analogue of Lichtenbaum's theorem that the reader can find in Duality for Schemes, Section 48.34. This and much else besides can be found in [Har68].

0EB1 Lemma 51.16.1. Let A be a Noetherian ring of dimension d. Let $I \subset I' \subset A$ be ideals. If I' is contained in the Jacobson radical of A and $\text{cd}(A, I') < d$, then $\text{cd}(A, I) < d$.

Proof. By Lemma 51.4.7 we know $\text{cd}(A, I) \leq d$. We will use Lemma 51.2.6 to show

$$H^d_{V(I')}(A) \to H^d_{V(I)}(A)$$

is surjective which will finish the proof. Pick $\mathfrak{p} \in V(I) \setminus V(I')$. By our assumption on I' we see that $\mathfrak{p}$ is not a maximal ideal of A. Hence $\dim(A_\mathfrak{p}) < d$. Then $H^d_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) = 0$ by Lemma 51.4.7. □

0EB2 Lemma 51.16.2. Let A be a Noetherian ring of dimension d. Let $I \subset A$ be an ideal. If $H^d_{V(I)}(M) = 0$ for some finite A-module whose support contains all the irreducible components of dimension d, then $\text{cd}(A, I) < d$.

Proof. By Lemma 51.4.7 we know $\text{cd}(A, I) \leq d$. Thus for any finite A-module N we have $H^i_{V(I)}(N) = 0$ for $i > d$. Let us say property $\mathcal{P}$ holds for the finite A-module N if $H^d_{V(I)}(N) = 0$. One of our assumptions is that $\mathcal{P}(M)$ holds. Observe that $\mathcal{P}(N_1 \oplus N_2) \Leftrightarrow (\mathcal{P}(N_1) \wedge \mathcal{P}(N_2))$. Observe that if $N \to N'$ is surjective, then $\mathcal{P}(N) \Rightarrow \mathcal{P}(N')$ as we have the vanishing of $H^{d+1}_{V(I)}$ (see above). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of A with $\dim(A/\mathfrak{p}_i) = d$. Observe that $\mathcal{P}(N)$ holds if the support of N is disjoint from $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for dimension reasons, see Lemma 51.4.7. For each i set $M_i = M/\mathfrak{p}_i M$. This is a finite A-module annihilated by $\mathfrak{p}_i$ whose support is equal to $V(\mathfrak{p}_i)$ (here we use the assumption on the support of M). Finally, if $J \subset A$ is an ideal, then we have $\mathcal{P}(JM_i)$ as JM_i is a quotient of a direct sum of copies of M. Thus it follows from Cohomology of Schemes, Lemma 30.12.8 that $\mathcal{P}$ holds for every finite A-module. $\square$

0EB3 Lemma 51.16.3. Let A be a Noetherian local ring of dimension d. Let $f \in A$ be an element which is not contained in any minimal prime of dimension d. Then $f : H^d_{V(I)}(M) \to H^d_{V(I)}(M)$ is surjective for any finite A-module M and any ideal $I \subset A$.

Proof. The support of M/fM has dimension $< d$ by our assumption on f. Thus $H^d_{V(I)}(M/fM) = 0$ by Lemma 51.4.7. Thus $H^d_{V(I)}(fM) \to H^d_{V(I)}(M)$ is surjective. Since by Lemma 51.4.7 we know $\text{cd}(A, I) \leq d$ we also see that the surjection $M \to fM$, $x \mapsto fx$ induces a surjection $H^d_{V(I)}(M) \to H^d_{V(I)}(fM)$. $\square$

0EB4 Lemma 51.16.4. Let A be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $I \subset A$ be an ideal. If $H^0_{V(I)}(\omega_A^\bullet) = 0$, then $\text{cd}(A, I) < \dim(A)$.

Proof. Set $d = \dim(A)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal primes of dimension d. Recall that the finite A-module $H^{-i}(\omega_A^\bullet)$ is nonzero only for $i \in \{0, \ldots, d\}$ and that the support of $H^{-i}(\omega_A^\bullet)$ has dimension $\leq i$, see Lemma 51.9.4. Set $\omega_A = H^{-d}(\omega_A^\bullet)$. By prime avoidence (Algebra, Lemma 10.15.2) we can find $f \in A$, $f \notin \mathfrak{p}_i$ which annihilates $H^{-i}(\omega_A^\bullet)$ for $i < d$. Consider the distinguished triangle

$$\omega_A[d] \to \omega_A^\bullet \to \tau_{\geq -d+1}\omega_A^\bullet \to \omega_A[d+1]$$

See Derived Categories, Remark 13.12.4. By Derived Categories, Lemma 13.12.5 we see that f^d induces the zero endomorphism of $\tau_{\geq -d+1}\omega_A^\bullet$. Using the axioms of a triangulated category, we find a map

$$\omega_A^\bullet \to \omega_A[d]$$

whose composition with $\omega_A[d] \to \omega_A^\bullet$ is multiplication by f^d on $\omega_A[d]$. Thus we conclude that f^d annihilates $H^d_{V(I)}(\omega_A)$. By Lemma 51.16.3 we conlude $H^d_{V(I)}(\omega_A) = 0$. Then we conclude by Lemma 51.16.2 and the fact that $(\omega_A)_{\mathfrak{p}_i}$ is nonzero (see for example Dualizing Complexes, Lemma 47.16.11). $\square$

0EB5 Lemma 51.16.5. Let $(A, \mathfrak{m})$ be a complete Noetherian local domain. Let $\mathfrak{p} \subset A$ be a prime ideal of dimension 1. For every $n \geq 1$ there is an $m \geq n$ such that $\mathfrak{p}^{(m)} \subset \mathfrak{p}^n$.

Proof. Recall that the symbolic power $\mathfrak{p}^{(m)}$ is defined as the kernel of $A \to A_\mathfrak{p}/\mathfrak{p}^m A_\mathfrak{p}$. Since localization is exact we conclude that in the short exact sequence

$$0 \to \mathfrak{a}_n \to A/\mathfrak{p}^n \to A/\mathfrak{p}^{(n)} \to 0$$

the support of $\mathfrak{a}_n$ is contained in $\{\mathfrak{m}\}$. In particular, the inverse system $(\mathfrak{a}_n)$ is Mittag-Leffler as each $\mathfrak{a}_n$ is an Artinian A-module. We conclude that the lemma is equivalent to the requirement that $\lim \mathfrak{a}_n = 0$. Let $f \in \lim \mathfrak{a}_n$. Then f is an element of $A = \lim A/\mathfrak{p}^n$ (here we use that A is complete) which maps to zero in the completion $A_\mathfrak{p}^\wedge$ of $A_\mathfrak{p}$. Since $A_\mathfrak{p} \to A_\mathfrak{p}^\wedge$ is faithfully flat, we see that f maps to zero in $A_\mathfrak{p}$. Since A is a domain we see that f is zero as desired. □

0EB6 Proposition 51.16.6. Let A be a Noetherian local ring with completion $A^\wedge$. Let $I \subset A$ be an ideal such that [Har68, Theorem 3.1]

$$\dim V(IA^\wedge + \mathfrak{p}) \geq 1$$

for every minimal prime $\mathfrak{p} \subset A^\wedge$ of dimension $\dim(A)$. Then $\mathrm{cd}(A, I) < \dim(A)$.

Proof. Since $A \to A^\wedge$ is faithfully flat we have $H^d_{V(I)}(A) \otimes_A A^\wedge = H^d_{V(IA^\wedge)}(A^\wedge)$ by Dualizing Complexes, Lemma 47.9.3. Thus we may assume A is complete.

Assume A is complete. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal primes of dimension d. Consider the complete local ring $A_i = A/\mathfrak{p}_i$. We have $H^d_{V(I)}(A_i) = H^d_{V(IA_i)}(A_i)$ by Dualizing Complexes, Lemma 47.9.2. By Lemma 51.16.2 it suffices to prove the lemma for (A_i, IA_i). Thus we may assume A is a complete local domain.

Assume A is a complete local domain. We can choose a prime ideal $\mathfrak{p} \supset I$ with $\dim(A/\mathfrak{p}) = 1$. By Lemma 51.16.1 it suffices to prove the lemma for $\mathfrak{p}$.

By Lemma 51.16.4 it suffices to show that $H^0_{V(\mathfrak{p})}(\omega_A^\bullet) = 0$. Recall that

$$H^0_{V(\mathfrak{p})}(\omega_A^\bullet) = \mathrm{colim}\, \mathrm{Ext}^0_A(A/\mathfrak{p}^n, \omega_A^\bullet)$$

By Lemma 51.16.5 we see that the colimit is the same as

$$\mathrm{colim}\, \mathrm{Ext}^0_A(A/\mathfrak{p}^{(n)}, \omega_A^\bullet)$$

Since $\mathrm{depth}(A/\mathfrak{p}^{(n)}) = 1$ we see that these ext groups are zero by Lemma 51.9.4 as desired. □

0EB7 Lemma 51.16.7. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Assume A is excellent, normal, and $\dim V(I) \geq 1$. Then $\mathrm{cd}(A, I) < \dim(A)$. In particular, if $\dim(A) = 2$, then $\mathrm{Spec}(A) \setminus V(I)$ is affine.

Proof. By More on Algebra, Lemma 15.52.6 the completion $A^\wedge$ is normal and hence a domain. Thus the assumption of Proposition 51.16.6 holds and we conclude. The statement on affineness follows from Lemma 51.4.8. □

51.17. Frobenius action

0EBU Let p be a prime number. Let A be a ring with $p = 0$ in A. The Frobenius endomorphism of A is the map

$$F : A \longrightarrow A, \quad a \longmapsto a^p$$

In this section we prove lemmas on modules which have Frobenius actions.

0EBV Lemma 51.17.1. Let p be a prime number. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with $p = 0$ in A. Let M be a finite A-module such that $M \otimes_{A,F} A \cong M$. Then M is finite free.

Proof. Choose a presentation $A^{\oplus m} \to A^{\oplus n} \to M$ which induces an isomorphism $\kappa^{\oplus n} \to M/\mathfrak{m}M$. Let $T = (a_{ij})$ be the matrix of the map $A^{\oplus m} \to A^{\oplus n}$. Observe that $a_{ij} \in \mathfrak{m}$. Applying base change by F, using right exactness of base change, we get a presentation $A^{\oplus m} \to A^{\oplus n} \to M$ where the matrix is $T = (a_{ij}^p)$. Thus we have a presentation with $a_{ij} \in \mathfrak{m}^p$. Repeating this construction we find that for each $e \geq 1$ there exists a presentation with $a_{ij} \in \mathfrak{m}^e$. This implies the fitting ideals (More on Algebra, Definition 15.8.3) $\text{Fit}_k(M)$ for $k < n$ are contained in $\bigcap_{e \geq 1} \mathfrak{m}^e$. Since this is zero by Krull's intersection theorem (Algebra, Lemma 10.51.4) we conclude that M is free of rank n by More on Algebra, Lemma 15.8.7. $\square$

In this section, we say elements $f_1, \ldots, f_r$ of a ring A are independent if $\sum a_i f_i = 0$ implies $a_i \in (f_1, \ldots, f_r)$. In other words, with $I = (f_1, \ldots, f_r)$ we have I/I^2 is free over A/I with basis $f_1, \ldots, f_r$.

0EBW Lemma 51.17.2. Let A be a ring. If $f_1, \ldots, f_{r-1}, f_r g_r$ are independent, then $f_1, \ldots, f_r$ are independent. See [Lec64] and [Mat70a, Lemma 1 page 299].

Proof. Say $\sum a_i f_i = 0$. Then $\sum a_i g_r f_i = 0$. Hence $a_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$. Write $a_r = \sum_{i<r} b_i f_i + b f_r g_r$. Then $0 = \sum_{i<r}(a_i + b_i f_r) f_i + b f_r^2 g_r$. Thus $a_i + b_i f_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$ which implies $a_i \in (f_1, \ldots, f_r)$ as desired. $\square$

0EBX Lemma 51.17.3. Let A be a ring. If $f_1, \ldots, f_{r-1}, f_r g_r$ are independent and if the A-module $A/(f_1, \ldots, f_{r-1}, f_r g_r)$ has finite length, then See [Lec64] and [Mat70a, Lemma 2 page 300].

$$\begin{aligned} &\text{length}_A(A/(f_1, \ldots, f_{r-1}, f_r g_r)) \\ &= \text{length}_A(A/(f_1, \ldots, f_{r-1}, f_r)) + \text{length}_A(A/(f_1, \ldots, f_{r-1}, g_r)) \end{aligned}$$

Proof. We claim there is an exact sequence

$$0 \to A/(f_1, \ldots, f_{r-1}, g_r) \xrightarrow{f_r} A/(f_1, \ldots, f_{r-1}, f_r g_r) \to A/(f_1, \ldots, f_{r-1}, f_r) \to 0$$

Namely, if $a f_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$, then $\sum_{i<r} a_i f_i + (a + b g_r) f_r = 0$ for some $b, a_i \in A$. Hence $\sum_{i<r} a_i g_r f_i + (a + b g_r) g_r f_r = 0$ which implies $a + b g_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$ which means that a maps to zero in $A/(f_1, \ldots, f_{r-1}, g_r)$. This proves the claim. To finish use additivity of lengths (Algebra, Lemma 10.52.3). $\square$

0EBY Lemma 51.17.4. Let $(A, \mathfrak{m})$ be a local ring. If $\mathfrak{m} = (x_1, \ldots, x_r)$ and $x_1^{e_1}, \ldots, x_r^{e_r}$ are independent for some $e_i > 0$, then $\text{length}_A(A/(x_1^{e_1}, \ldots, x_r^{e_r})) = e_1 \ldots e_r$. See [Lec64] and [Mat70a, Lemma 3 page 300].

Proof. Use Lemmas 51.17.2 and 51.17.3 and induction. $\square$

0EBZ Lemma 51.17.5. Let $\varphi : A \to B$ be a flat ring map. If $f_1, \ldots, f_r \in A$ are independent, then $\varphi(f_1), \ldots, \varphi(f_r) \in B$ are independent.

Proof. Let $I = (f_1, \ldots, f_r)$ and $J = \varphi(I)B$. By flatness we have $I/I^2 \otimes_A B = J/J^2$. Hence freeness of I/I^2 over A/I implies freeness of J/J^2 over B/J. $\square$

0EC0 Lemma 51.17.6 (Kunz). Let p be a prime number. Let A be a Noetherian ring with $p = 0$. The following are equivalent [Kun69]

(1) A is regular, and
(2) $F : A \to A$, $a \mapsto a^p$ is flat.

Proof. Observe that $\text{Spec}(F) : \text{Spec}(A) \to \text{Spec}(A)$ is the identity map. Being regular is defined in terms of the local rings and being flat is something about local rings, see Algebra, Lemma 10.39.18. Thus we may and do assume A is a Noetherian local ring with maximal ideal $\mathfrak{m}$.

Assume A is regular. Let $x_1, \ldots, x_d$ be a system of parameters for A. Applying F we find $F(x_1), \ldots, F(x_d) = x_1^p, \ldots, x_d^p$, which is a system of parameters for A. Hence F is flat, see Algebra, Lemmas 10.128.1 and 10.106.3.

Conversely, assume F is flat. Write $\mathfrak{m} = (x_1, \ldots, x_r)$ with r minimal. Then $x_1, \ldots, x_r$ are independent in the sense defined above. Since F is flat, we see that $x_1^p, \ldots, x_r^p$ are independent, see Lemma 51.17.5. Hence $\text{length}_A(A/(x_1^p, \ldots, x_r^p)) = p^r$ by Lemma 51.17.4. Let $\chi(n) = \text{length}_A(A/\mathfrak{m}^n)$ and recall that this is a numerical polynomial of degree $\dim(A)$, see Algebra, Proposition 10.60.9. Choose $n \gg 0$. Observe that

$$\mathfrak{m}^{pn+pr} \subset F(\mathfrak{m}^n)A \subset \mathfrak{m}^{pn}$$

as can be seen by looking at monomials in $x_1, \ldots, x_r$. We have

$$A/F(\mathfrak{m}^n)A = A/\mathfrak{m}^n \otimes_{A,F} A$$

By flatness of F this has length $\chi(n)\text{length}_A(A/F(\mathfrak{m})A)$ (Algebra, Lemma 10.52.13) which is equal to $p^r\chi(n)$ by the above. We conclude

$$\chi(pn + pr) \geq p^r\chi(n) \geq \chi(pn)$$

Looking at the leading terms this implies $r = \dim(A)$, i.e., A is regular. $\square$

51.18. Structure of certain modules

0EC1 Some results on the structure of certain types of modules over regular local rings. These types of results and much more can be found in [HS93], [Lyu93], [Lyu97].

0EC2 **Lemma 51.18.1.** Let k be a field of characteristic 0. Let $d \geq 1$. Let $A = k[[x_1, \ldots, x_d]]$ with maximal ideal $\mathfrak{m}$. Let M be an $\mathfrak{m}$-power torsion A-module endowed with additive operators $D_1, \ldots, D_d$ satisfying the leibniz rule Special case of [Lyu93, Theorem 2.4]

$$D_i(fz) = \partial_i(f)z + fD_i(z)$$

for $f \in A$ and $z \in M$. Here ∂_i is differentiation with respect to x_i. Then M is isomorphic to a direct sum of copies of the injective hull E of k.

Proof. Choose a set J and an isomorphism $M[\mathfrak{m}] \to \bigoplus_{j\in J} k$. Since $\bigoplus_{j\in J} E$ is injective (Dualizing Complexes, Lemma 47.3.7) we can extend this isomorphism to an A-module homomorphism $\varphi : M \to \bigoplus_{j\in J} E$. We claim that φ is an isomorphism, i.e., bijective.

Injective. Let $z \in M$ be nonzero. Since M is $\mathfrak{m}$-power torsion we can choose an element $f \in A$ such that $fz \in M[\mathfrak{m}]$ and $fz \neq 0$. Then $\varphi(fz) = f\varphi(z)$ is nonzero, hence $\varphi(z)$ is nonzero.

Surjective. Let $z \in M$. Then $x_1^n z = 0$ for some $n \geq 0$. We will prove that $z \in x_1M$ by induction on n. If $n = 0$, then $z = 0$ and the result is true. If $n > 0$, then applying D_1 we find $0 = nx_1^{n-1}z + x_1^nD_1(z)$. Hence $x_1^{n-1}(nz + x_1D_1(z)) = 0$. By induction we get $nz + x_1D_1(z) \in x_1M$. Since n is invertible, we conclude $z \in x_1M$. Thus we see that M is x_1-divisible. If φ is not surjective, then we can choose $e \in \bigoplus_{j\in J} E$ not in M. Arguing as above we may assume $\mathfrak{m}e \subset M$, in particular $x_1e \in M$. There exists an element $z_1 \in M$ with $x_1z_1 = x_1e$. Hence $x_1(z_1 - e) = 0$.

Replacing e by $e - z_1$ we may assume e is annihilated by x_1. Thus it suffices to prove that

$$\varphi[x_1] : M[x_1] \longrightarrow \left(\bigoplus\nolimits_{j \in J} E\right)[x_1] = \bigoplus\nolimits_{j \in J} E[x_1]$$

is surjective. If $d = 1$, this is true by construction of φ. If $d > 1$, then we observe that $E[x_1]$ is the injective hull of the residue field of $k[[x_2, \ldots, x_d]]$, see Dualizing Complexes, Lemma 47.7.1. Observe that $M[x_1]$ as a module over $k[[x_2, \ldots, x_d]]$ is $\mathfrak{m}/(x_1)$-power torsion and comes equipped with operators $D_2, \ldots, D_d$ satisfying the displayed Leibniz rule. Thus by induction on d we conclude that $\varphi[x_1]$ is surjective as desired. $\square$

0EC3 Lemma 51.18.2. Let p be a prime number. Let $(A, \mathfrak{m}, k)$ be a regular local ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ be the Frobenius endomorphism. Let M be a $\mathfrak{m}$-power torsion module such that $M \otimes_{A,F} A \cong M$. Then M is isomorphic to a direct sum of copies of the injective hull E of k.

Follows from [HS93, Corollary 3.6] with a little bit of work. Also follows directly from [Lyu97, Theorem 1.4].

Proof. Choose a set J and an A-module homorphism $\varphi : M \to \bigoplus_{j \in J} E$ which maps $M[\mathfrak{m}]$ isomorphically onto $(\bigoplus_{j \in J} E)[\mathfrak{m}] = \bigoplus_{j \in J} k$. We claim that φ is an isomorphism, i.e., bijective.

Injective. Let $z \in M$ be nonzero. Since M is $\mathfrak{m}$-power torsion we can choose an element $f \in A$ such that $fz \in M[\mathfrak{m}]$ and $fz \neq 0$. Then $\varphi(fz) = f\varphi(z)$ is nonzero, hence $\varphi(z)$ is nonzero.

Surjective. Recall that F is flat, see Lemma 51.17.6. Let $x_1, \ldots, x_d$ be a minimal system of generators of $\mathfrak{m}$. Denote

$$M_n = M[x_1^{p^n}, \ldots, x_d^{p^n}]$$

the submodule of M consisting of elements killed by $x_1^{p^n}, \ldots, x_d^{p^n}$. So $M_0 = M[\mathfrak{m}]$ is a vector space over k. Also $M = \bigcup M_n$ by our assumption that M is $\mathfrak{m}$-power torsion. Since F^n is flat and $F^n(x_i) = x_i^{p^n}$ we have

$$M_n \cong (M \otimes_{A,F^n} A)[x_1^{p^n}, \ldots, x_d^{p^n}] = M[x_1, \ldots, x_d] \otimes_{A,F^n} A = M_0 \otimes_k A/(x_1^{p^n}, \ldots, x_d^{p^n})$$

Thus M_n is free over $A/(x_1^{p^n}, \ldots, x_d^{p^n})$. A computation shows that every element of $A/(x_1^{p^n}, \ldots, x_d^{p^n})$ annihilated by $x_1^{p^n - 1}$ is divisible by x_1; for example you can use that $A/(x_1^{p^n}, \ldots, x_d^{p^n}) \cong k[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})$ by Algebra, Lemma 10.160.10. Thus the same is true for every element of M_n. Since every element of M is in M_n for all $n \gg 0$ and since every element of M is killed by some power of x_1, we conclude that M is x_1-divisible.

Let $x = x_1$. Above we have seen that M is x-divisible. If φ is not surjective, then we can choose $e \in \bigoplus_{j \in J} E$ not in M. Arguing as above we may assume $\mathfrak{m}e \subset M$, in particular $xe \in M$. There exists an element $z_1 \in M$ with $xz_1 = xe$. Hence $x(z_1 - e) = 0$. Replacing e by $e - z_1$ we may assume e is annihilated by x. Thus it suffices to prove that

$$\varphi[x] : M[x] \longrightarrow \left(\bigoplus\nolimits_{j \in J} E\right)[x] = \bigoplus\nolimits_{j \in J} E[x]$$

is surjective. If $d = 1$, this is true by construction of φ. If $d > 1$, then we observe that $E[x]$ is the injective hull of the residue field of the regular ring A/xA, see

Dualizing Complexes, Lemma 47.7.1. Observe that $M[x]$ as a module over A/xA is $\mathfrak{m}/(x)$-power torsion and we have

$$\begin{aligned} M[x] \otimes_{A/xA,F} A/xA &= M[x] \otimes_{A,F} A \otimes_A A/xA \\ &= (M \otimes_{A,F} A)[x^p] \otimes_A A/xA \\ &\cong M[x^p] \otimes_A A/xA \end{aligned}$$

Argue using flatness of F as before. We claim that $M[x^p] \otimes_A A/xA \to M[x]$, $z \otimes 1 \mapsto x^{p-1}z$ is an isomorphism. This can be seen by proving it for each of the modules M_n, $n > 0$ defined above where it follows by the same result for $A/(x_1^{p^n}, \ldots, x_d^{p^n})$ and $x = x_1$. Thus by induction on $\dim(A)$ we conclude that $\varphi[x]$ is surjective as desired. □

51.19. Additional structure on local cohomology

0EC4 Here is a sample result.

0EC5 **Lemma 51.19.1.** *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Set $Z = V(I)$. For each derivation $\theta : A \to A$ there exists a canonical additive operator D on the local cohomology modules $H^i_Z(A)$ satisfying the Leibniz rule with respect to θ.*

Proof. Let $f_1, \ldots, f_r$ be elements generating I. Recall that $R\Gamma_Z(A)$ is computed by the complex

$$A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}$$

See Dualizing Complexes, Lemma 47.9.1. Since θ extends uniquely to an additive operator on any localization of A satisfying the Leibniz rule with respect to θ, the lemma is clear. □

0EC6 **Lemma 51.19.2.** *Let p be a prime number. Let A be a ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ the Frobenius endomorphism. Let $I \subset A$ be a finitely generated ideal. Set $Z = V(I)$. There exists an isomorphism $R\Gamma_Z(A) \otimes^{\mathbf{L}}_{A,F} A \cong R\Gamma_Z(A)$.*

Proof. Follows from Dualizing Complexes, Lemma 47.9.3 and the fact that $Z = V(f_1^p, \ldots, f_r^p)$ if $I = (f_1, \ldots, f_r)$. □

0EC7 **Lemma 51.19.3.** *Let A be a ring. Let $V \to \text{Spec}(A)$ be quasi-compact, quasi-separated, and étale. For each derivation $\theta : A \to A$ there exists a canonical additive operator D on $H^i(V, \mathcal{O}_V)$ satisfying the Leibniz rule with respect to θ.*

Proof. If V is separated, then we can argue using an affine open covering $V = \bigcup_{j=1,\ldots m} V_j$. Namely, because V is separated we may write $V_{j_0 \ldots j_p} = \text{Spec}(B_{j_0 \ldots j_p})$. See Schemes, Lemma 26.21.7. Then we find that the A-module $H^i(V, \mathcal{O}_V)$ is the ith cohomology group of the Čech complex

$$\prod B_{j_0} \to \prod B_{j_0 j_1} \to \prod B_{j_0 j_1 j_2} \to \ldots$$

See Cohomology of Schemes, Lemma 30.2.6. Each $B = B_{j_0 \ldots j_p}$ is an étale A-algebra. Hence $\Omega_B = \Omega_A \otimes_A B$ and we conclude θ extends uniquely to a derivation $\theta_B : B \to B$. These maps define an endomorphism of the Čech complex and define the desired operators on the cohomology groups.

In the general case we use a hypercovering of V by affine opens, exactly as in the first part of the proof of Cohomology of Schemes, Lemma 30.7.3. We omit the details. $\square$

0EC8 Remark 51.19.4. We can upgrade Lemmas 51.19.1 and 51.19.3 to include higher order differential operators. If we ever need this we will state and prove a precise lemma here.

0EC9 Lemma 51.19.5. Let p be a prime number. Let A be a ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ the Frobenius endomorphism. If $V \to \mathrm{Spec}(A)$ is quasi-compact, quasi-separated, and étale, then there exists an isomorphism $R\Gamma(V, \mathcal{O}_V) \otimes^{\mathbf{L}}_{A,F} A \cong R\Gamma(V, \mathcal{O}_V)$.

Proof. Observe that the relative Frobenius morphism

$$V \longrightarrow V \times_{\mathrm{Spec}(A), \mathrm{Spec}(F)} \mathrm{Spec}(A)$$

of V over A is an isomorphism, see Étale Morphisms, Lemma 41.14.3. Thus the lemma follows from cohomology and base change, see Derived Categories of Schemes, Lemma 36.22.5. Observe that since V is étale over A, it is flat over A. $\square$

51.20. A bit of uniformity, I

0G9S The main task of this section is to formulate and prove Lemma 51.20.2.

0G9T Lemma 51.20.1. Let R be a ring. Let $M \to M'$ be a map of R-modules with M of finite presentation such that $\mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M', N)$ is zero for all R-modules N. Then $M \to M'$ factors through a free R-module.

Proof. We may choose a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & R^{\oplus r} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] & \bigoplus_{i \in I} R \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

whose right vertical arrow is the given map. We can factor this map through the short exact sequence

0G9U (51.20.1.1) $$0 \to K' \to E \to M \to 0$$

which is the pushout of the first short exact sequence by $K \to K'$. By a diagram chase we see that the assumption in the lemma implies that the boundary map $\mathrm{Tor}_1^R(M, N) \to K' \otimes_R N$ induced by (51.20.1.1) is zero, i.e., the sequence (51.20.1.1) is universally exact. This implies by Algebra, Lemma 10.82.4 that (51.20.1.1) is split (this is where we use that M is of finite presentation). Hence the map $M \to M'$ factors through $\bigoplus_{i \in I} R$ and we win. $\square$

0G9V Lemma 51.20.2. Let R be a ring. Let $\alpha : M \to M'$ be a map of R-modules. Let $P_\bullet \to M$ and $P'_\bullet \to M'$ be resolutions by projective R-modules. Let $e \geq 0$ be an integer. Consider the following conditions

(1) We can find a map of complexes $a_\bullet : P_\bullet \to P'_\bullet$ inducing α on cohomology with $a_i = 0$ for $i > e$.

(2) We can find a map of complexes $a_\bullet : P_\bullet \to P'_\bullet$ inducing α on cohomology with $a_{e+1} = 0$.
(3) The map $\text{Ext}^i_R(M', N) \to \text{Ext}^i_R(M, N)$ is zero for all R-modules N and $i > e$.
(4) The map $\text{Ext}^{e+1}_R(M', N) \to \text{Ext}^{e+1}_R(M, N)$ is zero for all R-modules N.
(5) Let $N = \text{Im}(P'_{e+1} \to P'_e)$ and denote $\xi \in \text{Ext}^{e+1}_R(M', N)$ the canonical element (see proof). Then ξ maps to zero in $\text{Ext}^{e+1}_R(M, N)$.
(6) The map $\text{Tor}^R_i(M, N) \to \text{Tor}^R_i(M', N)$ is zero for all R-modules N and $i > e$.
(7) The map $\text{Tor}^R_{e+1}(M, N) \to \text{Tor}^R_{e+1}(M', N)$ is zero for all R-modules N.

Then we always have the implications

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Leftrightarrow (7)$$

If M is $(-e-1)$-pseudo-coherent (for example if R is Noetherian and M is a finite R-module), then all conditions are equivalent.

Proof. It is clear that (2) implies (1). If $a_\bullet$ is as in (1), then we can consider the map of complexes $a'_\bullet : P_\bullet \to P'_\bullet$ with $a'_i = a_i$ for $i \leq e+1$ and $a'_i = 0$ for $i \geq e+1$ to get a map of complexes as in (2). Thus (1) is equivalent to (2).

By the construction of the Ext and Tor functors using resolutions (Algebra, Sections 10.71 and 10.75) we see that (1) and (2) imply all of the other conditions.

It is clear that (3) implies (4) implies (5). Let N be as in (5). The canonical map $\tilde{\xi} : P'_{e+1} \to N$ precomposed with $P'_{e+2} \to P'_{e+1}$ is zero. Hence we may consider the class ξ of $\tilde{\xi}$ in

$$\text{Ext}^{e+1}_R(M', N) = \frac{\text{Ker}(\text{Hom}(P'_{e+1}, N \to \text{Hom}(P'_{e+2}, N)}{\text{Im}(\text{Hom}(P'_e, N \to \text{Hom}(P'_{e+1}, N)}$$

Choose a map of complexes $a_\bullet : P_\bullet \to P'_\bullet$ lifting α, see Derived Categories, Lemma 13.19.6. If ξ maps to zero in $\text{Ext}^{e+1}_R(M', N)$, then we find a map $\varphi : P_e \to N$ such that $\tilde{\xi} \circ a_{e+1} = \varphi \circ d$. Thus we obtain a map of complexes

$$\begin{tikzcd}
\ldots \arrow[r] & P_{e+1} \arrow[r] \arrow[d, "0"] & P_e \arrow[r] \arrow[d, "a_e - \varphi"] & P_{e-1} \arrow[r] \arrow[d, "a_{e-1}"] & \ldots \\
\ldots \arrow[r] & P'_{e+1} \arrow[r] & P'_e \arrow[r] & P'_{e-1} \arrow[r] & \ldots
\end{tikzcd}$$

as in (2). Hence (1) – (5) are equivalent.

The equivalence of (6) and (7) follows from dimension shifting; we omit the details.

Assume M is $(-e-1)$-pseudo-coherent. (The parenthetical statement in the lemma follows from More on Algebra, Lemma 15.64.17.) We will show that (7) implies (4) which finishes the proof. We will use induction on e. The base case is $e = 0$. Then M is of finite presentation by More on Algebra, Lemma 15.64.4 and we conclude from Lemma 51.20.1 that $M \to M'$ factors through a free module. Of course if $M \to M'$ factors through a free module, then $\text{Ext}^i_R(M', N) \to \text{Ext}^i_R(M, N)$ is zero for all $i > 0$ as desired. Assume $e > 0$. We may choose a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & R^{\oplus r} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K' \arrow[r] & \bigoplus_{i \in I} R \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

whose right vertical arrow is the given map. We obtain $\text{Tor}_{i+1}^R(M, N) = \text{Tor}_i^R(K, N)$ and $\text{Ext}^{i+1}_R(M, N) = \text{Ext}^i_R(K, N)$ for $i \geq 1$ and all R-modules N and similarly for M', K'. Hence we see that $\text{Tor}_e^R(K, N) \to \text{Tor}_e^R(K', N)$ is zero for all R-modules N. By More on Algebra, Lemma 15.64.2 we see that K is $(-e)$-pseudo-coherent. By induction we conclude that $\text{Ext}^e(K', N) \to \text{Ext}^e(K, N)$ is zero for all R-modules N, which gives what we want. $\square$

0EH1 Lemma 51.20.3. Let I be an ideal of a Noetherian ring A. For all $n \geq 1$ there exists an $m > n$ such that the map $A/I^m \to A/I^n$ satisfies the equivalent conditions of Lemma 51.20.2 with $e = \text{cd}(A, I)$.

Proof. Let $\xi \in \text{Ext}^{e+1}_A(A/I^n, N)$ be the element constructed in Lemma 51.20.2 part (5). Since $e = \text{cd}(A, I)$ we have $0 = H^{e+1}_Z(N) = H^{e+1}_I(N) = \text{colim}\,\text{Ext}^{e+1}(A/I^m, N)$ by Dualizing Complexes, Lemmas 47.10.1 and 47.8.2. Thus we may pick $m \geq n$ such that ξ maps to zero in $\text{Ext}^{e+1}_A(A/I^m, N)$ as desired. $\square$

51.21. A bit of uniformity, II

0G9W Let I be an ideal of a Noetherian ring A. Let M be a finite A-module. Let $i > 0$. By More on Algebra, Lemma 15.27.3 there exists a $c = c(A, I, M, i)$ such that $\text{Tor}_i^A(M, A/I^n) \to \text{Tor}_i^A(M, A/I^{n-c})$ is zero for all $n \geq c$. In this section, we discuss some results which show that one sometimes can choose a constant c which works for all A-modules M simultaneously (and for a range of indices i). This material is related to uniform Artin-Rees as discussed in [Hun92] and [AHS15].

In Remark 51.21.9 we will apply this to show that various pro-systems related to derived completion are (or are not) strictly pro-isomorphic.

The following lemma can be significantly strengthened.

0G9X Lemma 51.21.1. Let I be an ideal of a Noetherian ring A. For every $m \geq 0$ and $i > 0$ there exist a $c = c(A, I, m, i) \geq 0$ such that for every A-module M annihilated by I^m the map

$$\text{Tor}_i^A(M, A/I^n) \to \text{Tor}_i^A(M, A/I^{n-c})$$

is zero for all $n \geq c$.

Proof. By induction on i. Base case $i = 1$. The short exact sequence $0 \to I^n \to A \to A/I^n \to 0$ determines an injection $\text{Tor}_1^A(M, A/I^n) \subset I^n \otimes_A M$, see Algebra, Remark 10.75.9. As M is annihilated by I^m we see that the map $I^n \otimes_A M \to I^{n-m} \otimes_A M$ is zero for $n \geq m$. Hence the result holds with $c = m$.

Induction step. Let $i > 1$ and assume c works for $i-1$. By More on Algebra, Lemma 15.27.3 applied to $M = A/I^m$ we can choose $c' \geq 0$ such that $\text{Tor}_i(A/I^m, A/I^n) \to \text{Tor}_i(A/I^m, A/I^{n-c'})$ is zero for $n \geq c'$. Let M be annihilated by I^m. Choose a short exact sequence

$$0 \to S \to \bigoplus\nolimits_{i \in I} A/I^m \to M \to 0$$

The corresponding long exact sequence of tors gives an exact sequence

$$\mathrm{Tor}_i^A(\bigoplus\nolimits_{i\in I} A/I^m, A/I^n) \to \mathrm{Tor}_i^A(M, A/I^n) \to \mathrm{Tor}_{i-1}^A(S, A/I^n)$$

for all integers $n \geq 0$. If $n \geq c+c'$, then the map $\mathrm{Tor}_{i-1}^A(S, A/I^n) \to \mathrm{Tor}_{i-1}^A(S, A/I^{n-c})$ is zero and the map $\mathrm{Tor}_i^A(A/I^m, A/I^{n-c}) \to \mathrm{Tor}_i^A(A/I^m, A/I^{n-c-c'})$ is zero. Combined with the short exact sequences this implies the result holds for i with constant $c + c'$. □

0G9Y **Lemma 51.21.2.** Let $I = (a_1, \ldots, a_t)$ be an ideal of a Noetherian ring A. Set $a = a_1$ and denote $B = A[\frac{I}{a}]$ the affine blowup algebra. There exists a $c > 0$ such that $\mathrm{Tor}_i^A(B, M)$ is annihilated by I^c for all A-modules M and $i \geq t$.

Proof. Recall that B is the quotient of $A[x_2, \ldots, x_t]/(a_1x_2 - a_2, \ldots, a_1x_t - a_t)$ by its a_1-torsion, see Algebra, Lemma 10.70.6. Let

$$B_\bullet = \text{Koszul complex on } a_1x_2 - a_2, \ldots, a_1x_t - a_t \text{ over } A[x_2, \ldots, x_t]$$

viewed as a chain complex sitting in degrees $(t-1), \ldots, 0$. The complex $B_\bullet[1/a_1]$ is isomorphic to the Koszul complex on $x_2 - a_2/a_1, \ldots, x_t - a_t/a_1$ which is a regular sequence in $A[1/a_1][x_2, \ldots, x_t]$. Since regular sequences are Koszul regular, we conclude that the augmentation

$$\epsilon : B_\bullet \longrightarrow B$$

is a quasi-isomorphism after inverting a_1. Since the homology modules of the cone $C_\bullet$ on ϵ are finite $A[x_2, \ldots, x_n]$-modules and since $C_\bullet$ is bounded, we conclude that there exists a $c \geq 0$ such that a_1^c annihilates all of these. By Derived Categories, Lemma 13.12.5 this implies that, after possibly replacing c by a larger integer, that a_1^c is zero on $C_\bullet$ in $D(A)$. The proof is finished once the reader contemplates the distinguished triangle

$$B_\bullet \otimes_A^{\mathbf{L}} M \to B \otimes_A^{\mathbf{L}} M \to C_\bullet \otimes_A^{\mathbf{L}} M$$

Namely, the first term is represented by $B_\bullet \otimes_A M$ which is sitting in homological degrees $(t-1), \ldots, 0$ in view of the fact that the terms in the Koszul complex $B_\bullet$ are free (and hence flat) A-modules. Whence $\mathrm{Tor}_i^A(B, M) = H_i(C_\bullet \otimes_A^{\mathbf{L}} M)$ for $i > t-1$ and this is annihilated by a_1^c. Since $a_1^cB = I^cB$ and since the tor module is a module over B we conclude. □

For the rest of the discussion in this section we fix a Noetherian ring A and an ideal $I \subset A$. We denote

$$p : X \to \mathrm{Spec}(A)$$

the blowing up of $\mathrm{Spec}(A)$ in the ideal I. In other words, X is the Proj of the Rees algebra $\bigoplus_{n\geq 0} I^n$. By Cohomology of Schemes, Lemmas 30.14.2 and 30.14.3 we can choose an integer $q(A, I) \geq 0$ such that for all $q \geq q(A, I)$ we have $H^i(X, \mathcal{O}_X(q)) = 0$ for $i > 0$ and $H^0(X, \mathcal{O}_X(q)) = I^q$.

0G9Z **Lemma 51.21.3.** In the situation above, for $q \geq q(A, I)$ and any A-module M we have

$$R\Gamma(X, Lp^*\widetilde{M}(q)) \cong M \otimes_A^{\mathbf{L}} I^q$$

in $D(A)$.

Proof. Choose a free resolution $F_\bullet \to M$. Then $\widetilde{F}_\bullet$ is a flat resolution of $\widetilde{M}$. Hence $Lp^*\widetilde{M}$ is given by the complex $p^*\widetilde{F}_\bullet$. Thus $Lp^*\widetilde{M}(q)$ is given by the complex $p^*\widetilde{F}_\bullet(q)$. Since $p^*\widetilde{F}_i(q)$ are right acyclic for $\Gamma(X, -)$ by our choice of $q \geq q(A, I)$ and since we have $\Gamma(X, p^*\widetilde{F}_i(q)) = I^qF_i$ by our choice of $q \geq q(A, I)$, we get that $R\Gamma(X, Lp^*\widetilde{M}(q))$ is given by the complex with terms I^qF_i by Derived Categories of Schemes, Lemma 36.4.3. The result follows as the complex $I^qF_\bullet$ computes $M \otimes_A^{\mathbf{L}} I^q$ by definition. □

0GA0 Lemma 51.21.4. In the situation above, let t be an upper bound on the number of generators for I. There exists an integer $c = c(A, I) \geq 0$ such that for any A-module M the cohomology sheaves $H^j(Lp^*\widetilde{M})$ are annihilated by I^c for $j \leq -t$.

Proof. Say $I = (a_1, \ldots, a_t)$. The question is affine local on X. For $1 \leq i \leq t$ let $B_i = A[\frac{I}{a_i}]$ be the affine blowup algebra. Then X has an affine open covering by the spectra of the rings B_i, see Divisors, Lemma 31.32.2. By the description of derived pullback given in Derived Categories of Schemes, Lemma 36.3.8 we conclude it suffices to prove that for each i there exists a $c \geq 0$ such that

$$\mathrm{Tor}_j^A(B_i, M)$$

is annihilated by I^c for $j \geq t$. This is Lemma 51.21.2. □

0GA1 Lemma 51.21.5. In the situation above, let t be an upper bound on the number of generators for I. There exists an integer $c = c(A, I) \geq 0$ such that for any A-module M the tor modules $\mathrm{Tor}_i^A(M, A/I^q)$ are annihilated by I^c for $i > t$ and all $q \geq 0$.

Proof. Let $q(A, I)$ be as above. For $q \geq q(A, I)$ we have

$$R\Gamma(X, Lp^*\widetilde{M}(q)) = M \otimes_A^{\mathbf{L}} I^q$$

by Lemma 51.21.3. We have a bounded and convergent spectral sequence

$$H^a(X, H^b(Lp^*\widetilde{M}(q))) \Rightarrow \mathrm{Tor}_{-a-b}^A(M, I^q)$$

by Derived Categories of Schemes, Lemma 36.4.4. Let d be an integer as in Cohomology of Schemes, Lemma 30.4.4 (actually we can take $d = t$, see Cohomology of Schemes, Lemma 30.4.2). Then we see that $H^{-i}(X, Lp^*\widetilde{M}(q)) = \mathrm{Tor}_i^A(M, I^q)$ has a finite filtration with at most d steps whose graded are subquotients of the modules

$$H^a(X, H^{-i-a}(Lp^*\widetilde{M})(q)), \quad a = 0, 1, \ldots, d-1$$

If $i \geq t$ then all of these modules are annihilated by I^c where $c = c(A, I)$ is as in Lemma 51.21.4 because the cohomology sheaves $H^{-i-a}(Lp^*\widetilde{M})$ are all annihilated by I^c by the lemma. Hence we see that $\mathrm{Tor}_i^A(M, I^q)$ is annihilated by I^{dc} for $q \geq q(A, I)$ and $i \geq t$. Using the short exact sequence $0 \to I^q \to A \to A/I^q \to 0$ we find that $\mathrm{Tor}_i(M, A/I^q)$ is annihilated by I^{dc} for $q \geq q(A, I)$ and $i > t$. We conclude that I^m with $m = \max(dc, q(A, I) - 1)$ annihilates $\mathrm{Tor}_i^A(M, A/I^q)$ for all $q \geq 0$ and $i > t$ as desired. □

0GA2 Lemma 51.21.6. Let I be an ideal of a Noetherian ring A. Let $t \geq 0$ be an upper bound on the number of generators of I. There exist $N, c \geq 0$ such that the maps

$$\mathrm{Tor}_{t+1}^A(M, A/I^n) \to \mathrm{Tor}_{t+1}^A(M, A/I^{n-c})$$

are zero for any A-module M and all $n \geq N$.

Proof. Let c_1 be the constant found in Lemma 51.21.5. Please keep in mind that this constant c_1 works for Tor_i for all $i > t$ simultaneously.

Say $I = (a_1, \ldots, a_t)$. For an A-module M we set

$$\ell(M) = \#\{i \mid 1 \leq i \leq t,\ a_i^{c_1} \text{ is zero on } M\}$$

This is an element of $\{0, 1, \ldots, t\}$. We will prove by descending induction on $0 \leq s \leq t$ the following statement H_s: there exist $N, c \geq 0$ such that for every module M with $\ell(M) \geq s$ the maps

$$\text{Tor}^A_{t+1+i}(M, A/I^n) \to \text{Tor}^A_{t+1+i}(M, A/I^{n-c})$$

are zero for $i = 0, \ldots, s$ for all $n \geq N$.

Base case: $s = t$. If $\ell(M) = t$, then M is annihilated by $(a_1^{c_1}, \ldots, a_t^{c_1}\}$ and hence by $I^{t(c_1-1)+1}$. We conclude from Lemma 51.21.1 that H_t holds by taking $c = N$ to be the maximum of the integers $c(A, I, t(c_1-1)+1, t+1), \ldots, c(A, I, t(c_1-1)+1, 2t+1)$ found in the lemma.

Induction step. Say $0 \leq s < t$ we have N, c as in H_{s+1}. Consider a module M with $\ell(M) = s$. Then we can choose an i such that $a_i^{c_1}$ is nonzero on M. It follows that $\ell(M[a_i^c]) \geq s + 1$ and $\ell(M/a_i^{c_1}M) \geq s + 1$ and the induction hypothesis applies to them. Consider the exact sequence

$$0 \to M[a_i^{c_1}] \to M \xrightarrow{a_i^{c_1}} M \to M/a_i^{c_1}M \to 0$$

Denote $E \subset M$ the image of the middle arrow. Consider the corresponding diagram of Tor modules

$$\begin{tikzcd}
 & & \text{Tor}_{i+1}(M/a_i^{c_1}M, A/I^q) \arrow[d] \\
\text{Tor}_i(M[a_i^{c_1}], A/I^q) \arrow[r] & \text{Tor}_i(M, A/I^q) \arrow[r] \arrow[dr, "0"] & \text{Tor}_i(E, A/I^q) \arrow[d] \\
 & & \text{Tor}_i(M, A/I^q)
\end{tikzcd}$$

with exact rows and columns (for every q). The south-east arrow is zero by our choice of c_1. We conclude that the module $\text{Tor}_i(M, A/I^q)$ is sandwiched between a quotient module of $\text{Tor}_i(M[a_i^{c_1}], A/I^q)$ and a submodule of $\text{Tor}_{i+1}(M/a_i^{c_1}M, A/I^q)$. Hence we conclude H_s holds with N replaced by $N + c$ and c replaced by $2c$. Some details omitted. ☐

0GA3 **Proposition 51.21.7.** *Let I be an ideal of a Noetherian ring A. Let $t \geq 0$ be an upper bound on the number of generators of I. There exist $N, c \geq 0$ such that for $n \geq N$ the maps*

$$A/I^n \to A/I^{n-c}$$

satisfy the equivalent conditions of Lemma 51.20.2 with $e = t$.

Proof. Immediate consequence of Lemmas 51.21.6 and 51.20.2. ☐

0GA4 **Remark 51.21.8.** The paper [AHS15] shows, besides many other things, that if A is local, then Proposition 51.21.7 also holds with $e = t$ replaced by $e = \dim(A)$. Looking at Lemma 51.20.3 it is natural to ask whether Proposition 51.21.7 holds with $e = t$ replaced with $e = \text{cd}(A, I)$. We don't know.

0GA5 Remark 51.21.9. Let I be an ideal of a Noetherian ring A. Say $I = (f_1, \ldots, f_r)$. Denote $K_n^\bullet$ the Koszul complex on $f_1^n, \ldots, f_r^n$ as in More on Algebra, Situation 15.91.15 and denote $K_n \in D(A)$ the corresponding object. Let $M^\bullet$ be a bounded complex of finite A-modules and denote $M \in D(A)$ the corresponding object. Consider the following inverse systems in $D(A)$:

(1) $M^\bullet/I^nM^\bullet$, i.e., the complex whose terms are M^i/I^nM^i,
(2) $M \otimes_A^{\mathbf{L}} A/I^n$,
(3) $M \otimes_A^{\mathbf{L}} K_n$, and
(4) $M \otimes_P^{\mathbf{L}} P/J^n$ (see below).

All of these inverse systems are isomorphic as pro-objects: the isomorphism between (2) and (3) follows from More on Algebra, Lemma 15.94.1. The isomorphism between (1) and (2) is given in More on Algebra, Lemma 15.100.3. For the last one, see below.

However, we can ask if these isomorphisms of pro-systems are "strict"; this terminology and question is related to the discussion in [Qui, pages 61, 62]. Namely, given a category $\mathcal{C}$ we can define a "strict pro-category" whose objects are inverse systems (X_n) and whose morphisms $(X_n) \to (Y_n)$ are given by tuples (c, φ_n) consisting of a $c \geq 0$ and morphisms $\varphi_n : X_n \to Y_{n-c}$ for all $n \geq c$ satisfying an obvious compatibility condition and up to a certain equivalence (given essentially by increasing c). Then we ask whether the above inverse systems are isomorphic in this strict pro-category.

This clearly cannot be the case for (1) and (3) even when $M = A[0]$. Namely, the system $H^0(K_n) = A/(f_1^n, \ldots, f_r^n)$ is not strictly pro-isomorphic in the category of modules to the system A/I^n in general. For example, if we take $A = \mathbf{Z}[x_1, \ldots, x_r]$ and $f_i = x_i$, then $H^0(K_n)$ is not annihilated by $I^{r(n-1)}$.[3]

It turns out that the results above show that the natural map from (2) to (1) discussed in More on Algebra, Lemma 15.100.3 is a strict pro-isomorphism. We will sketch the proof. Using standard arguments involving stupid truncations, we first reduce to the case where $M^\bullet$ is given by a single finite A-module M placed in degree 0. Pick $N, c \geq 0$ as in Proposition 51.21.7. The proposition implies that for $n \geq N$ we get factorizations

$$M \otimes_A^{\mathbf{L}} A/I^n \to \tau_{\geq -t}(M \otimes_A^{\mathbf{L}} A/I^n) \to M \otimes_A^{\mathbf{L}} A/I^{n-c}$$

of the transition maps in the system (2). On the other hand, by More on Algebra, Lemma 15.27.3, we can find another constant $c' = c'(M) \geq 0$ such that the maps $\text{Tor}_i^A(M, A/I^{n'}) \to \text{Tor}_i(M, A/I^{n'-c'})$ are zero for $i = 1, 2, \ldots, t$ and $n' \geq c'$. Then it follows from Derived Categories, Lemma 13.12.5 that the map

$$\tau_{\geq -t}(M \otimes_A^{\mathbf{L}} A/I^{n+tc'}) \to \tau_{\geq -t}(M \otimes_A^{\mathbf{L}} A/I^n)$$

factors through $M \otimes_A^{\mathbf{L}} A/I^{n+tc'} \to M/I^{n+tc'}M$. Combined with the previous result we obtain a factorization

$$M \otimes_A^{\mathbf{L}} A/I^{n+tc'} \to M/I^{n+tc'}M \to M \otimes_A^{\mathbf{L}} A/I^{n-c}$$

[3] Of course, we can ask whether these pro-systems are isomorphic in a category whose objects are inverse systems and where maps are given by tuples (r, c, φ_n) consisting of $r \geq 1$, $c \geq 0$ and maps $\varphi_n : X_{rn} \to Y_{n-c}$ for $n \geq c$.

which gives us what we want. If we ever need this result, we will carefully state it and provide a detailed proof.

For number (4) suppose we have a Noetherian ring P, a ring homomorphism $P \to A$, and an ideal $J \subset P$ such that $I = JA$. By More on Algebra, Section 15.60 we get a functor $M \otimes_P^{\mathbf{L}} - : D(P) \to D(A)$ and we get an inverse system $M \otimes_P^{\mathbf{L}} P/J^n$ in $D(A)$ as in (4). If P is Noetherian, then the system in (4) is pro-isomorphic to the system in (1) because we can compare with Koszul complexes. If $P \to A$ is finite, then the system (4) is strictly pro-isomorphic to the system (2) because the inverse system $A \otimes_P^{\mathbf{L}} P/J^n$ is strictly pro-isomorphic to the inverse system A/I^n (by the discussion above) and because we have

$$M \otimes_P^{\mathbf{L}} P/J^n = M \otimes_A^{\mathbf{L}} (A \otimes_P^{\mathbf{L}} P/J^n)$$

by More on Algebra, Lemma 15.60.1.

A standard example in (4) is to take $P = \mathbf{Z}[x_1, \ldots, x_r]$, the map $P \to A$ sending x_i to f_i, and $J = (x_1, \ldots, x_r)$. In this case one shows that

$$M \otimes_P^{\mathbf{L}} P/J^n = M \otimes_{A[x_1,\ldots,x_r]}^{\mathbf{L}} A[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^n$$

and we reduce to one of the cases discussed above (although this case is strictly easier as $A[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^n$ has tor dimension at most r for all n and hence the step using Proposition 51.21.7 can be avoided). This case is discussed in the proof of [BS13, Proposition 3.5.1].

51.22. A bit of uniformity, III

0GA6 In this section we fix a Noetherian ring A and an ideal $I \subset A$. Our goal is to prove Lemma 51.22.7 which we will use in a later chapter to solve a lifting problem, see Algebraization of Formal Spaces, Lemma 88.5.3.

Throughout this section we denote

$$p : X \to \mathrm{Spec}(A)$$

the blowing up of $\mathrm{Spec}(A)$ in the ideal I. In other words, X is the Proj of the Rees algebra $\bigoplus_{n \geq 0} I^n$. We also consider the fibre product

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{Spec}(A/I) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Then Y is the exceptional divisor of the blowup and hence an effective Cartier divisor on X such that $\mathcal{O}_X(-1) = \mathcal{O}_X(Y)$. Since taking Proj commutes with base change we have

$$Y = \mathrm{Proj}(\bigoplus\nolimits_{n \geq 0} I^n/I^{n+1}) = \mathrm{Proj}(S)$$

where $S = \mathrm{Gr}_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.

We denote $d = d(S) = d(\mathrm{Gr}_I(A)) = d(\bigoplus_{n \geq 0} I^n/I^{n+1})$ the maximum of the dimensions of the fibres of p (and we set it equal to 0 if $X = \emptyset$). This is well defined. In fact, we have

(1) $d \leq t - 1$ if $I = (a_1, \ldots, a_t)$ since then $X \subset \mathbf{P}_A^{t-1}$, and

(2) d is also the maximal dimension of the fibres of $\text{Proj}(S) \to \text{Spec}(S_0)$ provided that Y is nonempty and $d = 0$ if $Y = \emptyset$ (equivalently $S = 0$, equivalently $I = A$).

Hence d only depends on the isomorphism class of $S = \text{Gr}_I(A)$. Observe that $H^i(X, \mathcal{F}) = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $i > d$ by Cohomology of Schemes, Lemmas 30.20.9 and 30.4.6. Of course the same is true for coherent modules on Y.

We denote $d = d(S) = d(\text{Gr}_I(A)) = d(\bigoplus_{n \geq 0} I^n/I^{n+1})$ the integer defined as follows. Note that the algebra $S = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Noetherian graded ring generated in degree 1 over degree 0. Hence by Cohomology of Schemes, Lemmas 30.14.2 and 30.14.3 we can define $q(S)$ as the smallest integer $q(S) \geq 0$ such that for all $q \geq q(S)$ we have $H^i(Y, \mathcal{O}_Y(q)) = 0$ for $1 \leq i \leq d$ and $H^0(Y, \mathcal{O}_Y(q)) = I^q/I^{q+1}$. (If $S = 0$, then $q(S) = 0$.)

For $n \geq 1$ we may consider the effective Cartier divisor nY which we will denote Y_n.

0GA7 **Lemma 51.22.1.** With $q_0 = q(S)$ and $d = d(S)$ as above, we have

(1) for $n \geq 1$, $q \geq q_0$, and $i > 0$ we have $H^i(X, \mathcal{O}_{Y_n}(q)) = 0$,
(2) for $n \geq 1$ and $q \geq q_0$ we have $H^0(X, \mathcal{O}_{Y_n}(q)) = I^q/I^{q+n}$,
(3) for $q \geq q_0$ and $i > 0$ we have $H^i(X, \mathcal{O}_X(q)) = 0$,
(4) for $q \geq q_0$ we have $H^0(X, \mathcal{O}_X(q)) = I^q$.

Proof. If $I = A$, then X is affine and the statements are trivial. Hence we may and do assume $I \neq A$. Thus Y and X are nonempty schemes.

Let us prove (1) and (2) by induction on n. The base case $n = 1$ is our definition of q_0 as $Y_1 = Y$. Recall that $\mathcal{O}_X(1) = \mathcal{O}_X(-Y)$. Hence we have a short exact sequence

$$0 \to \mathcal{O}_{Y_n}(1) \to \mathcal{O}_{Y_{n+1}} \to \mathcal{O}_Y \to 0$$

Hence for $i > 0$ we find

$$H^i(X, \mathcal{O}_{Y_n}(q+1)) \to H^i(X, \mathcal{O}_{Y_{n+1}}(q)) \to H^i(X, \mathcal{O}_Y(q))$$

and we obtain the desired vanishing of the middle term from the given vanishing of the outer terms. For $i = 0$ we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^{q+1}/I^{q+1+n} \arrow[r] \arrow[d] & I^q/I^{q+1+n} \arrow[r] \arrow[d] & I^q/I^{q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X, \mathcal{O}_{Y_n}(q+1)) \arrow[r] & H^0(X, \mathcal{O}_{Y_{n+1}}(q)) \arrow[r] & H^0(Y, \mathcal{O}_Y(q)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows for $q \geq q_0$ (for the bottom row observe that the next term in the long exact cohomology sequence vanishes for $q \geq q_0$). Since $q \geq q_0$ the left and right vertical arrows are isomorphisms and we conclude the middle one is too.

We omit the proofs of (3) and (4) which are similar. In fact, one can deduce (3) and (4) from (1) and (2) using the theorem on formal functors (but this would be overkill). $\square$

Let us introduce a notation: given $n \geq c \geq 0$ an (A, n, c)-module is a finite A-module M which is annihilated by I^n and which as an A/I^n-module is I^c/I^n-projective, see More on Algebra, Section 15.70.

We will use the following abuse of notation: given an A-module M we denote p^*M the quasi-coherent module gotten by pulling back by p the quasi-coherent module $\widetilde{M}$ on $\mathrm{Spec}(A)$ associated to M. For example we have $\mathcal{O}_{Y_n} = p^*(A/I^n)$. For a short exact sequence $0 \to K \to L \to M \to 0$ of A-modules we obtain an exact sequence

$$p^*K \to p^*L \to p^*M \to 0$$

as $\widetilde{\ }$ is an exact functor and p^* is a right exact functor.

0GA8 Lemma 51.22.2. Let $0 \to K \to L \to M \to 0$ be a short exact sequence of A-modules such that K and L are annihilated by I^n and M is an (A, n, c)-module. Then the kernel of $p^*K \to p^*L$ is scheme theoretically supported on Y_c.

Proof. Let $\mathrm{Spec}(B) \subset X$ be an affine open. The restriction of the exact sequence over $\mathrm{Spec}(B)$ corresponds to the sequence of B-modules

$$K \otimes_A B \to L \otimes_A B \to M \otimes_A B \to 0$$

which is isomorphismic to the sequence

$$K \otimes_{A/I^n} B/I^nB \to L \otimes_{A/I^n} B/I^nB \to M \otimes_{A/I^n} B/I^nB \to 0$$

Hence the kernel of the first map is the image of the module $\mathrm{Tor}_1^{A/I^n}(M, B/I^nB)$. Recall that the exceptional divisor Y is cut out by $I\mathcal{O}_X$. Hence it suffices to show that $\mathrm{Tor}_1^{A/I^n}(M, B/I^nB)$ is annihilated by I^c. Since multiplication by $a \in I^c$ on M factors through a finite free A/I^n-module, this is clear. $\square$

We have the canonical map $\mathcal{O}_X \to \mathcal{O}_X(1)$ which vanishes exactly along Y. Hence for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ we always have canonical maps $\mathcal{F}(q) \to \mathcal{F}(q+n)$ for any $q \in \mathbf{Z}$ and $n \geq 0$.

0GA9 Lemma 51.22.3. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is scheme theoretically supported on Y_c if and only if the canonical map $\mathcal{F} \to \mathcal{F}(c)$ is zero.

Proof. This is true because $\mathcal{O}_X \to \mathcal{O}_X(1)$ vanishes exactly along Y. $\square$

0GAA Lemma 51.22.4. With $q_0 = q(S)$ and $d = d(S)$ as above, suppose we have integers $n \geq c \geq 0$, an (A, n, c)-module M, an index $i \in \{0, 1, \ldots, d\}$, and an integer q. Then we distinguish the following cases

(1) In the case $i = d \geq 1$ and $q \geq q_0$ we have $H^d(X, p^*M(q)) = 0$.
(2) In the case $i = d - 1 \geq 1$ and $q \geq q_0$ we have $H^{d-1}(X, p^*M(q)) = 0$.
(3) In the case $d-1 > i > 0$ and $q \geq q_0+(d-1-i)c$ the map $H^i(X, p^*M(q)) \to H^i(X, p^*M(q - (d - 1 - i)c))$ is zero.
(4) In the case $i = 0$, $d \in \{0, 1\}$, and $q \geq q_0$, there is a surjection

$$I^qM \longrightarrow H^0(X, p^*M(q))$$

(5) In the case $i = 0$, $d > 1$, and $q \geq q_0 + (d - 1)c$ the map

$$H^0(X, p^*M(q)) \to H^0(X, p^*M(q - (d - 1)c))$$

has image contained in the image of the canonical map $I^{q-(d-1)c}M \to H^0(X, p^*M(q - (d - 1)c))$.

Proof. Let M be an (A, n, c)-module. Choose a short exact sequence

$$0 \to K \to (A/I^n)^{\oplus r} \to M \to 0$$

We will use below that K is an (A, n, c)-module, see More on Algebra, Lemma 15.70.6. Consider the corresponding exact sequence

$$p^*K \to (\mathcal{O}_{Y_n})^{\oplus r} \to p^*M \to 0$$

We split this into short exact sequences

$$0 \to \mathcal{F} \to p^*K \to \mathcal{G} \to 0 \quad \text{and} \quad 0 \to \mathcal{G} \to (\mathcal{O}_{Y_n})^{\oplus r} \to p^*M \to 0$$

By Lemma 51.22.2 the coherent module $\mathcal{F}$ is scheme theoretically supported on Y_c.

Proof of (1). Assume $d > 0$. We have to prove $H^d(X, p^*M(q)) = 0$ for $q \geq q_0$. By the vanishing of the cohomology of twists of $\mathcal{G}$ in degrees $> d$ and the long exact cohomology sequence associated to the second short exact sequence above, it suffices to prove that $H^d(X, \mathcal{O}_{Y_n}(q)) = 0$. This is true by Lemma 51.22.1.

Proof of (2). Assume $d > 1$. We have to prove $H^{d-1}(X, p^*M(q)) = 0$ for $q \geq q_0$. Arguing as in the previous paragraph, we see that it suffices to show that $H^d(X, \mathcal{G}(q)) = 0$. Using the first short exact sequence and the vanishing of the cohomology of twists of $\mathcal{F}$ in degrees $> d$ we see that it suffices to show $H^d(X, p^*K(q))$ is zero which is true by (1) and the fact that K is an (A, n, c)-module (see above).

Proof of (3). Let $0 < i < d-1$ and assume the statement holds for $i+1$ except in the case $i = d-2$ we have statement (2). Using the long exact sequence of cohomology associated to the second short exact sequence above we find an injection

$$H^i(X, p^*M(q - (d-1-i)c)) \subset H^{i+1}(X, \mathcal{G}(q - (d-1-i)c))$$

as $q - (d-1-i)c \geq q_0$ gives the vanishing of $H^i(X, \mathcal{O}_{Y_n}(q - (d-1-i)c))$ (see above). Thus it suffices to show that the map $H^{i+1}(X, \mathcal{G}(q)) \to H^{i+1}(X, \mathcal{G}(q - (d-1-i)c))$ is zero. To study this, we consider the maps of exact sequences

$$\begin{tikzcd}
H^{i+1}(X, p^*K(q)) \arrow[r] \arrow[d] & H^{i+1}(X, \mathcal{G}(q)) \arrow[r] \arrow[d] \arrow[dl, dotted] & H^{i+2}(X, \mathcal{F}(q)) \arrow[d] \\
H^{i+1}(X, p^*K(q-c)) \arrow[r] \arrow[d] & H^{i+1}(X, \mathcal{G}(q-c)) \arrow[r] \arrow[d] & H^{i+2}(X, \mathcal{F}(q-c)) \\
H^{i+1}(X, p^*K(q-(d-1-i)c)) \arrow[r] & H^{i+1}(X, \mathcal{G}(q-(d-1-i)c)) &
\end{tikzcd}$$

Since $\mathcal{F}$ is scheme theoretically supported on Y_c we see that the canonical map $\mathcal{G}(q) \to \mathcal{G}(q-c)$ factors through $p^*K(q-c)$ by Lemma 51.22.3. This gives the dotted arrow in the diagram. (In fact, for the proof it suffices to observe that the vertical arrow on the extreme right is zero in order to get the dotted arrow as a map of sets.) Thus it suffices to show that $H^{i+1}(X, p^*K(q-c)) \to H^{i+1}(X, p^*K(q-(d-1-i)c))$ is zero. If $i = d-2$, then the source of this arrow is zero by (2) as $q - c \geq q_0$ and K is an (A, n, c)-module. If $i < d-2$, then as K is an (A, n, c)-module, we get from the induction hypothesis that the map is indeed zero since $q - c - (q - (d-1-i)c) = (d-2-i)c = (d-1-(i+1))c$ and since $q - c \geq q_0 + (d-1-(i+1))c$. In this way we conclude the proof of (3).

Proof of (4). Assume $d \in \{0, 1\}$ and $q \geq q_0$. Then the first short exact sequence gives a surjection $H^1(X, p^*K(q)) \to H^1(X, \mathcal{G}(q))$ and the source of this arrow is zero by case (1). Hence for all $q \in \mathbf{Z}$ we see that the map

$$H^0(X, (\mathcal{O}_{Y_n})^{\oplus r}(q)) \longrightarrow H^0(X, p^*M(q))$$

is surjective. For $q \geq q_0$ the source is equal to $(I^q/I^{q+n})^{\oplus r}$ by Lemma 51.22.1 and this easily proves the statement.

Proof of (5). Assume $d > 1$. Arguing as in the proof of (4) we see that it suffices to show that the image of

$$H^0(X, p^*M(q)) \longrightarrow H^0(X, p^*M(q - (d-1)c))$$

is contained in the image of

$$H^0(X, (\mathcal{O}_{Y_n})^{\oplus r}(q - (d-1)c)) \longrightarrow H^0(X, p^*M(q - (d-1)c))$$

To show the inclusion above, it suffices to show that for $\sigma \in H^0(X, p^*M(q))$ with boundary $\xi \in H^1(X, \mathcal{G}(q))$ the image of ξ in $H^1(X, \mathcal{G}(q - (d-1)c))$ is zero. This follows by the exact same arguments as in the proof of (3). $\square$

0GAB Remark 51.22.5. Given a pair (M, n) consisting of an integer $n \geq 0$ and a finite A/I^n-module M we set $M^\vee = \mathrm{Hom}_{A/I^n}(M, A/I^n)$. Given a pair $(\mathcal{F}, n)$ consisting of an integer n and a coherent $\mathcal{O}_{Y_n}$-module $\mathcal{F}$ we set

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{Y_n}}(\mathcal{F}, \mathcal{O}_{Y_n})$$

Given (M, n) as above, there is a canonical map

$$can : p^*(M^\vee) \longrightarrow (p^*M)^\vee$$

Namely, if we choose a presentation $(A/I^n)^{\oplus s} \to (A/I^n)^{\oplus r} \to M \to 0$ then we obtain a presentation $\mathcal{O}_{Y_n}^{\oplus s} \to \mathcal{O}_{Y_n}^{\oplus r} \to p^*M \to 0$. Taking duals we obtain exact sequences

$$0 \to M^\vee \to (A/I^n)^{\oplus r} \to (A/I^n)^{\oplus s}$$

and

$$0 \to (p^*M)^\vee \to \mathcal{O}_{Y_n}^{\oplus r} \to \mathcal{O}_{Y_n}^{\oplus s}$$

Pulling back the first sequence by p we find the desired map can. The construction of this map is functorial in the finite A/I^n-module M. The kernel and cokernel of can are scheme theoretically supported on Y_c if M is an (A, n, c)-module. Namely, in that case for $a \in I^c$ the map $a : M \to M$ factors through a finite free A/I^n-module for which can is an isomorphism. Hence a annihilates the kernel and cokernel of can.

0GAC Lemma 51.22.6. With $q_0 = q(S)$ and $d = d(S)$ as above, let M be an (A, n, c)-module and let $\varphi : M \to I^n/I^{2n}$ be an A-linear map. Assume $n \geq \max(q_0 + (1 + d)c, (2 + d)c)$ and if $d = 0$ assume $n \geq q_0 + 2c$. Then the composition

$$M \xrightarrow{\varphi} I^n/I^{2n} \to I^{n-(1+d)c}/I^{2n-(1+d)c}$$

is of the form $\sum a_i\psi_i$ with $a_i \in I^c$ and $\psi_i : M \to I^{n-(2+d)c}/I^{2n-(2+d)c}$.

Proof. The case $d > 1$. Since we have a compatible system of maps $p^*(I^q) \to \mathcal{O}_X(q)$ for $q \geq 0$ there are canonical maps $p^*(I^q/I^{q+\nu}) \to \mathcal{O}_{Y_\nu}(q)$ for $\nu \geq 0$. Using this and pulling back φ we obtain a map

$$\chi : p^*M \longrightarrow \mathcal{O}_{Y_n}(n)$$

such that the composition $M \to H^0(X, p^*M) \to H^0(X, \mathcal{O}_{Y_n}(n))$ is the given homomorphism φ combined with the map $I^n/I^{2n} \to H^0(X, \mathcal{O}_{Y_n}(n))$. Since $\mathcal{O}_{Y_n}(n)$ is invertible on Y_n the linear map χ determines a section

$$\sigma \in \Gamma(X, (p^*M)^\vee(n))$$

with notation as in Remark 51.22.5. The discussion in Remark 51.22.5 shows the cokernel and kernel of $can : p^*(M^\vee) \to (p^*M)^\vee$ are scheme theoretically supported on Y_c. By Lemma 51.22.3 the map $(p^*M)^\vee(n) \to (p^*M)^\vee(n-2c)$ factors through $p^*(M^\vee)(n-2c)$; small detail omitted. Hence the image of σ in $\Gamma(X, (p^*M)^\vee(n-2c))$ comes from an element

$$\sigma' \in \Gamma(X, p^*(M^\vee)(n-2c))$$

By Lemma 51.22.4 part (5), the fact that $M^\vee$ is an (A, n, c)-module by More on Algebra, Lemma 15.70.7, and the fact that $n \geq q_0 + (1+d)c$ so $n-2c \geq q_0+(d-1)c$ we see that the image of σ' in $H^0(X, p^*M^\vee(n-(1+d)c))$ is the image of an element τ in $I^{n-(1+d)c}M^\vee$. Write $\tau = \sum a_i\tau_i$ with $\tau_i \in I^{n-(2+d)c}M^\vee$; this makes sense as $n-(2+d)c \geq 0$. Then τ_i determines a homomorphism of modules $\psi_i : M \to I^{n-(2+d)c}/I^{2n-(2+d)c}$ using the evaluation map $M \otimes M^\vee \to A/I^n$.

Let us prove that this works[4]. Pick $z \in M$ and let us show that $\varphi(z)$ and $\sum a_i\psi_i(z)$ have the same image in $I^{n-(1+d)c}/I^{2n-(1+d)c}$. First, the element z determines a map $p^*z : \mathcal{O}_{Y_n} \to p^*M$ whose composition with χ is equal to the map $\mathcal{O}_{Y_n} \to \mathcal{O}_{Y_n}(n)$ corresponding to $\varphi(z)$ via the map $I^n/I^{2n} \to \Gamma(\mathcal{O}_{Y_n}(n))$. Next z and p^*z determine evaluation maps $e_z : M^\vee \to A/I^n$ and $e_{p^*z} : (p^*M)^\vee \to \mathcal{O}_{Y_n}$. Since $\chi(p^*z)$ is the section corresponding to $\varphi(z)$ we see that $e_{p^*z}(\sigma)$ is the section corresponding to $\varphi(z)$. Here and below we abuse notation: for a map $a : \mathcal{F} \to \mathcal{G}$ of modules on X we also denote $a : \mathcal{F}(t) \to \mathcal{F}(t)$ the corresponding map of twisted modules. The diagram

$$\begin{tikzcd}
p^*(M^\vee) \arrow[r, "p^*e_z"'] \arrow[d, "can"'] & \mathcal{O}_{Y_n} \arrow[d, equal] \\
(p^*M)^\vee \arrow[r, "e_{p^*z}"] & \mathcal{O}_{Y_n}
\end{tikzcd}$$

commutes by functoriality of the construction can. Hence $(p^*e_z)(\sigma')$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-2c))$ is the section corresponding to the image of $\varphi(z)$ in I^{n-2c}/I^{2n-2c}. The next step is that σ' maps to the image of $\sum a_i\tau_i$ in $H^0(X, p^*M^\vee(n-(1+d)c))$. This implies that $(p^*e_z)(\sum a_i\tau_i) = \sum a_i p^*e_z(\tau_i)$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c))$ is the section corresponding to the image of $\varphi(z)$ in $I^{n-(1+d)c}/I^{2n-(1+d)c}$. Recall that ψ_i is defined from τ_i using an evaluation map. Hence if we denote

$$\chi_i : p^*M \longrightarrow \mathcal{O}_{Y_n}(n-(2+d)c)$$

the map we get from ψ_i, then we see by the same reasoning as above that the section corresponding to $\psi_i(z)$ is $\chi_i(p^*z) = e_{p^*z}(\chi_i) = p^*e_z(\tau_i)$. Hence we conclude that the image of $\varphi(z)$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c))$ is equal to the image of $\sum a_i\psi_i(z)$. Since $n-(1+d)c \geq q_0$ we have $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c)) = I^{n-(1+d)c}/I^{2n-(1+d)c}$ by Lemma 51.22.1 and we conclude the desired compatibility is true.

The case $d = 1$. Here we argue as above that we get

$$\chi : p^*M \longrightarrow \mathcal{O}_{Y_n}(n), \quad \sigma \in \Gamma(X, (p^*M)^\vee(n)), \quad \sigma' \in \Gamma(X, p^*(M^\vee)(n-2c)),$$

and then we use Lemma 51.22.4 part (4) to see that σ' is the image of some element $\tau \in I^{n-2c}M^\vee$. The rest of the argument is the same.

The case $d = 0$. Argument is exactly the same as in the case $d = 1$. □

0GAD **Lemma 51.22.7.** With $d = d(S)$ and $q_0 = q(S)$ as above. Then

[4]We hope some reader will suggest a less dirty proof of this fact.

(1) for integers $n \geq c \geq 0$ with $n \geq \max(q_0 + (1+d)c, (2+d)c)$,
(2) for K of $D(A/I^n)$ with $H^i(K) = 0$ for $i \neq -1, 0$ and $H^i(K)$ finite for $i = -1, 0$ such that $\text{Ext}^1_{A/I^c}(K, N)$ is annihilated by I^c for all finite A/I^n-modules N

the map

$$\text{Ext}^1_{A/I^n}(K, I^n/I^{2n}) \longrightarrow \text{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-2(1+d)c})$$

is zero.

Proof. The case $d > 0$. Let $K^{-1} \to K^0$ be a complex representing K as in More on Algebra, Lemma 15.84.5 part (5) with respect to the ideal I^c/I^n in the ring A/I^n. In particular K^{-1} is I^c/I^n-projective as multiplication by elements of I^c/I^n even factor through K^0. By More on Algebra, Lemma 15.84.4 part (1) we have

$$\text{Ext}^1_{A/I^n}(K, I^n/I^{2n}) = \text{Coker}(\text{Hom}_{A/I^n}(K^0, I^n/I^{2n}) \to \text{Hom}_{A/I^n}(K^{-1}, I^n/I^{2n}))$$

and similarly for other Ext groups. Hence any class ξ in $\text{Ext}^1_{A/I^n}(K, I^n/I^{2n})$ comes from an element $\varphi \in \text{Hom}_{A/I^n}(K^{-1}, I^n/I^{2n})$. Denote φ' the image of φ in $\text{Hom}_{A/I^n}(K^{-1}, I^{n-(1+d)c}/I^{2n-(1+d)c})$. By Lemma 51.22.6 we can write $\varphi' = \sum a_i\psi_i$ with $a_i \in I^c$ and $\psi_i \in \text{Hom}_{A/I^n}(M, I^{n-(2+d)c}/I^{2n-(2+d)c})$. Choose $h_i : K^0 \to K^{-1}$ such that $a_i\text{id}_{K^{-1}} = h_i \circ d_K^{-1}$. Set $\psi = \sum \psi_i \circ h_i : K^0 \to I^{n-(2+d)c}/I^{2n-(2+d)c}$. Then $\varphi' = \psi \circ \text{d}_K^{-1}$ and we conclude that ξ already maps to zero in $\text{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-(1+d)c})$ and a fortiori in $\text{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-2(1+d)c})$.

The case $d = 0$[5]. Let ξ and φ be as above. We consider the diagram

$$\begin{tikzcd}
K^0 & & \\
K^{-1} \arrow[u] \arrow[r, "\varphi"] & I^n/I^{2n} \arrow[r] & I^{n-c}/I^{2n-c}
\end{tikzcd}$$

Pulling back to X and using the map $p^*(I^n/I^{2n}) \to \mathcal{O}_{Y_n}(n)$ we find a solid diagram

$$\begin{tikzcd}
p^*K^0 \arrow[rrd, dotted] & & \\
p^*K^{-1} \arrow[u] \arrow[r] & \mathcal{O}_{Y_n}(n) \arrow[r] & \mathcal{O}_{Y_n}(n-c)
\end{tikzcd}$$

We can cover X by affine opens $U = \text{Spec}(B)$ such that there exists an $a \in I$ with the following property: $IB = aB$ and a is a nonzerodivisor on B. Namely, we can cover X by spectra of affine blowup algebras, see Divisors, Lemma 31.32.2. The restriction of $\mathcal{O}_{Y_n}(n) \to \mathcal{O}_{Y_n}(n-c)$ to U is isomorphic to the map of quasi-coherent $\mathcal{O}_U$-modules corresponding to the B-module map $a^c : B/a^nB \to B/a^nB$. Since $a^c : K^{-1} \to K^{-1}$ factors through K^0 we see that the dotted arrow exists over U. In other words, locally on X we can find the dotted arrow! Now the sheaf of dotted arrows fitting into the diagram is principal homogeneous under

$$\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\text{Coker}(p^*K^{-1} \to p^*K^0), \mathcal{O}_{Y_n}(n-c))$$

which is a coherent $\mathcal{O}_X$-module. Hence the obstruction for finding the dotted arrow is an element of $H^1(X, \mathcal{F})$. This cohomology group is zero as $1 > d = 0$,

[5]The argument given for $d > 0$ works but gives a slightly weaker result.

see discussion following the definition of $d = d(S)$. This proves that we can find a dotted arrow $\psi : p^*K^0 \to \mathcal{O}_{Y_n}(n-c)$ fitting into the diagram. Since $n - c \geq q_0$ we find that ψ induces a map $K^0 \to I^{n-c}/I^{2n-c}$. Chasing the diagram we conclude that $\varphi' = \psi \circ \text{d}_K^{-1}$ and the proof is finished as before. $\square$

51.23. Other chapters

CHAPTER 52

Algebraic and Formal Geometry

0EI5 52.1. Introduction

0EI6 This chapter continues the study of formal algebraic geometry and in particular the question of whether a formal object is the completion of an algebraic one. A fundamental reference is [Gro68]. Here is a list of results we have already discussed in the Stacks project:

(1) The theorem on formal functions, see Cohomology of Schemes, Section 30.20.
(2) Coherent formal modules, see Cohomology of Schemes, Section 30.23.
(3) Grothendieck's existence theorem, see Cohomology of Schemes, Sections 30.24, 30.25, and 30.27.
(4) Grothendieck's algebraization theorem, see Cohomology of Schemes, Section 30.28.
(5) Grothendieck's existence theorem more generally, see More on Flatness, Sections 38.28 and 38.29.

Let us give an overview of the contents of this chapter.

Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type quasi-coherent sheaf of ideals. Many questions in this chapter have to do with inverse systems $(\mathcal{F}_n)$ of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1}$. An important special case is where X is a scheme over a Noetherian ring A and $\mathcal{I} = I\mathcal{O}_X$ for some ideal $I \subset A$. In Cohomology, Sections 20.35, 20.36, and 20.39 we have some general results. In this chapter, Sections 52.2 and 52.3 contain results specific to schemes and quasi-coherent modules. In Section 52.4 we prove that the limit topology on $\lim H^p(X, \mathcal{F}_n)$ is I-adic in case $\text{cd}(A, I) = 1$. One of the themes of this chapter will be to show that results proven in the principal ideal case $I = (f)$ also hold when we only assume $\text{cd}(A, I) = 1$.

In Section 52.6 we discuss derived completion of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ with respect to a finite type sheaf of ideals $\mathcal{I}$. This section is the natural continuation of the theory of derived completion in commutative algebra as described in More on Algebra, Section 15.91. The first main result is that derived completion exists. The second main result is that for a morphism f if ringed sites derived completion commutes with derived pushforward:

$$(Rf_*K)^\wedge = Rf_*(K^\wedge)$$

if the ideal sheaf upstairs is locally generated by sections coming from the ideal downstairs, see Lemma 52.6.19. We stress that both main results are very elementary in case the ideals in question are globally finitely generated which will be true for all applications of this theory in this chapter. The displayed equality is the "correct" version of the theorem on formal functions, see discussion in Section 52.7.

Let A be a Noetherian ring and let I, J be two ideals of A. Let M be a finite A-module. The next topic in this chapter is the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge$$

from local cohomology of M into the derived I-adic completion of the same. It turns out that if we impose suitable depth conditions this map becomes an isomorphism on cohomology in a range of degrees. In Section 52.8 we work essentially in the generality just mentioned. In Section 52.9 we assume A is a local ring and $J = \mathfrak{m}$ is a maximal ideal. We encourage the reader to read this section before the other two in this part of the chapter. Finally, in Section 52.10 we bootstrap the local case to obtain stronger results back in the general case.

In the next part of this chapter we use the results on completion of local cohomology to get a nonexhaustive list of results on cohomology of the completion of coherent modules. More precisely, let A be a Noetherian ring, let $I \subset A$ be an ideal, and let $U \subset \text{Spec}(A)$ be an open subscheme. If $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module, then we may consider the maps

$$H^i(U, \mathcal{F}) \longrightarrow \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

and ask if we get an isomorphism in a certain range of degrees. In Section 52.11 we work out some examples where U is the punctured spectrum of a local ring. In Section 52.12 we discuss the general case. In Section 52.14 we apply some of the results obtained to questions of connectedness in algebraic geometry.

The remaining sections of this chapter are devoted to a discussion of algebraization of coherent formal modules. In other words, given an inverse system of coherent modules $(\mathcal{F}_n)$ on U as above with $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$ we ask whether there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $\mathcal{F}_n = \mathcal{F}/I^n\mathcal{F}$ for all n. We encourage the reader to read Section 52.16 for a precise statement of the question, a useful general result (Lemma 52.16.10), and a nontrivial application (Lemma 52.16.11). To prove a result going essentially beyond this case quite a bit more theory has to be developed. Please see Section 52.22 for the strongest results of this type obtained in this chapter.

52.2. Formal sections, I

0EH3 We suggest looking at Cohomology, Section 20.35 first.

0EI8 **Lemma 52.2.1.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1}$. Set $\mathcal{F} = \lim \mathcal{F}_n$. Then

(1) *$\mathcal{F} = R\lim \mathcal{F}_n$,*
(2) *for any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for $p > 0$, and*
(3) *for each p there is a short exact sequence $0 \to R^1 \lim H^{p-1}(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}) \to \lim H^p(X, \mathcal{F}_n) \to 0$.*

If moreover $\mathcal{I}$ is of finite type, then

(4) *$\mathcal{F}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$, and*
(5) *$\mathcal{I}^n\mathcal{F} = \lim_{m \geq n} \mathcal{I}^n\mathcal{F}_m$.*

Proof. Parts (1), (2), and (3) are general facts about inverse systems of quasi-coherent modules with surjective transition maps, see Derived Categories of Schemes, Lemma 36.3.2 and Cohomology, Lemma 20.37.1. Next, assume $\mathcal{I}$ is of finite type. Let $U \subset X$ be affine open. Say $U = \mathrm{Spec}(A)$ and $\mathcal{I}|_U$ corresponds to $I \subset A$. Observe that I is a finitely generated ideal. By the equivalence of categories between quasi-coherent $\mathcal{O}_U$-modules and A-modules (Schemes, Lemma 26.7.5) we find that $M_n = \mathcal{F}_n(U)$ is an inverse system of A-modules with $M_n = M_{n+1}/I^n M_{n+1}$. Thus

$$M = \mathcal{F}(U) = \lim \mathcal{F}_n(U) = \lim M_n$$

is an I-adically complete module with $M/I^nM = M_n$ by Algebra, Lemma 10.98.2. This proves (4). Part (5) translates into the statement that $\lim_{m \geq n} I^nM/I^mM = I^nM$. Since $I^mM = I^{m-n} \cdot I^nM$ this is just the statement that I^mM is I-adically complete. This follows from Algebra, Lemma 10.96.3 and the fact that M is complete. □

52.3. Formal sections, II

0BLA We suggest looking at Cohomology, Sections 20.36 and 20.39 first.

0EH9 **Lemma 52.3.1.** Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of quasi-coherent $\mathcal{O}_X$-modules. The following are equivalent

(1) for all $n \geq 1$ the map $f : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$,
(2) for all $n \geq 1$ the map $f^n : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_1$ to give a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$
(3) there exists an $\mathcal{O}_X$-module $\mathcal{G}$ which is f-divisible such that $\mathcal{F}_n = \mathcal{G}[f^n]$.
(4) there exists an $\mathcal{O}_X$-module $\mathcal{F}$ which is f-torsion free such that $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$.

Proof. The equivalence of (1), (2), (3) and the implication (4) ⇒ (1) are proven in Cohomology, Lemma 20.36.1. Assume (1) holds. Set $\mathcal{F} = \lim \mathcal{F}_n$. By Lemma 52.2.1 part (4) we have $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$. Let $U \subset X$ be open and $s = (s_n) \in \mathcal{F}(U) = \lim \mathcal{F}_n(U)$. Choose $n \geq 1$. If $fs = 0$, then s_{n+1} is in the kernel of $\mathcal{F}_{n+1} \to \mathcal{F}_n$ by condition (1). Hence $s_n = 0$. Since n was arbitrary, we see $s = 0$. Thus $\mathcal{F}$ is f-torsion free. □

0BLD **Lemma 52.3.2.** Let A be a ring and $f \in A$. Let X be a scheme over A. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathcal{F}[f^n] = \mathrm{Ker}(f^n : \mathcal{F} \to \mathcal{F})$ stabilizes. Then Slightly improved version of [BdJ14, Lemma 1.6]

$$R\Gamma(X, \lim \mathcal{F}/f^n\mathcal{F}) = R\Gamma(X, \mathcal{F})^\wedge$$

where the right hand side indicates the derived completion with respect to the ideal $(f) \subset A$. Consequently, for $p \in \mathbf{Z}$ we obtain a commutative diagram

$$
\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \widehat{H^p(X, \mathcal{F})} \arrow[u] \arrow[r] & \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) \arrow[u] \arrow[r] & T_f(H^{p+1}(X, \mathcal{F})) \arrow[r] & 0 \\
0 \arrow[r] & H^0(H^p(X, \mathcal{F})^\wedge) \arrow[u] \arrow[r] & H^p(X, \lim \mathcal{F}/f^n\mathcal{F}) \arrow[u] \arrow[r] & T_f(H^{p+1}(X, \mathcal{F})) \arrow[u, equal] \arrow[r] & 0 \\
 & R^1 \lim H^p(X, \mathcal{F})[f^n] \arrow[u] \arrow[r, "\cong"] & R^1 \lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

with exact rows and columns where $\widehat{H^p(X, \mathcal{F})} = \lim H^p(X, \mathcal{F})/f^nH^p(X, \mathcal{F})$ is the usual f-adic completion and $T_f(-)$ denotes the f-adic Tate module as in More on Algebra, Example 15.93.5.

Proof. By Lemma 52.2.1 we have $\lim \mathcal{F}/f^n\mathcal{F} = R\lim \mathcal{F}/f^n\mathcal{F}$. Everything else follows from Cohomology, Example 20.39.3. □

52.4. Formal sections, III

0EI9 In this section we prove Lemma 52.4.5 which (in the setting of Noetherian schemes and coherent modules) is the analogue of Cohomology, Lemma 20.36.2 in case the ideal I is not assumed principal but has the property that $\text{cd}(A, I) = 1$.

0EIA Lemma 52.4.1. Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A. If $\text{cd}(A, I) = 1$, then there exist $c \geq 1$ and maps $\varphi_j : I^c \to A$ such that $\sum f_j\varphi_j : I^c \to I$ is the inclusion map.

Proof. Since $\text{cd}(A, I) = 1$ the complement $U = \text{Spec}(A) \setminus V(I)$ is affine (Local Cohomology, Lemma 51.4.8). Say $U = \text{Spec}(B)$. Then $IB = B$ and we can write $1 = \sum_{j=1,\ldots,r} f_jb_j$ for some $b_j \in B$. By Cohomology of Schemes, Lemma 30.10.5 we can represent b_j by maps $\varphi_j : I^c \to A$ for some $c \geq 0$. Then $\sum f_j\varphi_j : I^c \to I \subset A$ is the canonical embedding, after possibly replacing c by a larger integer, by the same lemma. □

0EIB Lemma 52.4.2. Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $\text{cd}(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 52.4.1. Then there is a unique graded A-algebra map

$$\Phi : \bigoplus\nolimits_{n \geq 0} I^{nc} \to A[T_1, \ldots, T_r]$$

with $\Phi(g) = \sum \varphi_j(g)T_j$ for $g \in I^c$. Moreover, the composition of Φ with the map $A[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n$, $T_j \mapsto f_j$ is the inclusion map $\bigoplus_{n \geq 0} I^{nc} \to \bigoplus_{n \geq 0} I^n$.

Proof. For each j and $m \geq c$ the restriction of φ_j to I^m is a map $\varphi_j : I^m \to I^{m-c}$. Given $j_1, \ldots, j_n \in \{1, \ldots, r\}$ we claim that the composition

$$\varphi_{j_1} \ldots \varphi_{j_n} : I^{nc} \to I^{(n-1)c} \to \ldots \to I^c \to A$$

is independent of the order of the indices $j_1, \ldots, j_n$. Namely, if $g = g_1 \ldots g_n$ with $g_i \in I^c$, then we see that

$$(\varphi_{j_1} \ldots \varphi_{j_n})(g) = \varphi_{j_1}(g_1) \ldots \varphi_{j_n}(g_n)$$

is independent of the ordering as multiplication in A is commutative. Thus we can define Φ by sending $g \in I^{nc}$ to

$$\Phi(g) = \sum\nolimits_{e_1 + \ldots + e_r = n} (\varphi_1^{e_1} \circ \ldots \circ \varphi_r^{e_r})(g) T_1^{e_1} \ldots T_r^{e_r}$$

It is straightforward to prove that this is a graded A-algebra homomorphism with the desired property. Uniqueness is immediate as is the final property. This proves the lemma. □

0EIC **Lemma 52.4.3.** Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $\text{cd}(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 52.4.1. Let $A \to B$ be a ring map with B Noetherian and let N be a finite B-module. Then, after possibly increasing c and adjusting φ_j accordingly, there is a unique unique graded B-module map

$$\Phi_N : \bigoplus\nolimits_{n \geq 0} I^{nc} N \to N[T_1, \ldots, T_r]$$

with $\Phi_N(gx) = \Phi(g)x$ for $g \in I^{nc}$ and $x \in N$ where Φ is as in Lemma 52.4.2. The composition of Φ_N with the map $N[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n N$, $T_j \mapsto f_j$ is the inclusion map $\bigoplus_{n \geq 0} I^{nc} N \to \bigoplus_{n \geq 0} I^n N$.

Proof. The uniqueness is clear from the formula and the uniqueness of Φ in Lemma 52.4.2. Consider the Noetherian A-algebra $B' = B \oplus N$ where N is an ideal of square zero. To show the existence of Φ_N it is enough (via Lemma 52.4.1) to show that φ_j extends to a map $\varphi'_j : I^c B' \to B'$ after possibly increasing c to some c' (and replacing φ_j by the composition of the inclusion $I^{c'} \to I^c$ with φ_j). Recall that φ_j corresponds to a section

$$h_j \in \Gamma(\text{Spec}(A) \setminus V(I), \mathcal{O}_{\text{Spec}(A)})$$

see Cohomology of Schemes, Lemma 30.10.5. (This is in fact how we chose our φ_j in the proof of Lemma 52.4.1.) Let us use the same lemma to represent the pullback

$$h'_j \in \Gamma(\text{Spec}(B') \setminus V(IB'), \mathcal{O}_{\text{Spec}(B')})$$

of h_j by a B'-linear map $\varphi'_j : I^{c'} B' \to B'$ for some $c' \geq c$. The agreement with φ_j will hold for c' sufficiently large by a further application of the lemma: namely we can test agreement on a finite list of generators of $I^{c'}$. Small detail omitted. □

0EH6 **Lemma 52.4.4.** Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $\text{cd}(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 52.4.1. Let X be a Noetherian scheme over $\text{Spec}(A)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Set $\mathcal{F} = \lim \mathcal{F}_n$. Then, after possibly increasing c and adjusting φ_j accordingly, there exists a unique graded $\mathcal{O}_X$-module map

$$\Phi_{\mathcal{F}} : \bigoplus\nolimits_{n \geq 0} I^{nc}\mathcal{F} \longrightarrow \mathcal{F}[T_1, \ldots, T_r]$$

with $\Phi_{\mathcal{F}}(gs) = \Phi(g)s$ for $g \in I^{nc}$ and s a local section of $\mathcal{F}$ where Φ is as in Lemma 52.4.2. The composition of $\Phi_{\mathcal{F}}$ with the map $\mathcal{F}[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n\mathcal{F}$, $T_j \mapsto f_j$ is the canonical inclusion $\bigoplus_{n\geq 0} I^{nc}\mathcal{F} \to \bigoplus_{n \geq 0} I^n\mathcal{F}$.

Proof. The uniqueness is immediate from the $\mathcal{O}_X$-linearity and the requirement that $\Phi_{\mathcal{F}}(gs) = \Phi(g)s$ for $g \in I^{nc}$ and s a local section of $\mathcal{F}$. Thus we may assume $X = \mathrm{Spec}(B)$ is affine. Observe that $(\mathcal{F}_n)$ is an object of the category $\mathrm{Coh}(X, I\mathcal{O}_X)$ introduced in Cohomology of Schemes, Section 30.23. Let $B' = B^\wedge$ be the I-adic completion of B. By Cohomology of Schemes, Lemma 30.23.1 the object $(\mathcal{F}_n)$ corresponds to a finite B'-module N in the sense that $\mathcal{F}_n$ is the coherent module associated to the finite B-module N/I^nN. Applying Lemma 52.4.3 to $I \subset A \to B'$ and N we see that, after possibly increasing c and adjusting φ_j accordingly, we get unique maps

$$\Phi_N : \bigoplus\nolimits_{n\geq 0} I^{nc}N \to N[T_1, \ldots, T_r]$$

with the corresponding properties. Note that in degree n we obtain an inverse system of maps $N/I^mN \to \bigoplus_{e_1+\ldots+e_r=n} N/I^{m-nc}N \cdot T_1^{e_1} \ldots T_r^{e_r}$ for $m \geq nc$. Translating back into coherent sheaves we see that Φ_N corresponds to a system of maps

$$\Phi^n_m : I^{nc}\mathcal{F}_m \longrightarrow \bigoplus\nolimits_{e_1+\ldots+e_r=n} \mathcal{F}_{m-nc} \cdot T_1^{e_1} \ldots T_r^{e_r}$$

for varying $m \geq nc$ and $n \geq 1$. Taking the inverse limit of these maps over m we obtain $\Phi_{\mathcal{F}} = \bigoplus_n \lim_m \Phi^n_m$. Note that $\lim_m I^t\mathcal{F}_m = I^t\mathcal{F}$ as can be seen by evaluating on affines for example, but in fact we don't need this because it is clear there is a map $I^t\mathcal{F} \to \lim_m I^t\mathcal{F}_m$. □

0EH7 Lemma 52.4.5. Let I be an ideal of a Noetherian ring A. Let X be a Noetherian scheme over $\mathrm{Spec}(A)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. If $\mathrm{cd}(A, I) = 1$, then for all $p \in \mathbf{Z}$ the limit topology on $\lim H^p(X, \mathcal{F}_n)$ is I-adic.

Proof. First it is clear that $I^t \lim H^p(X, \mathcal{F}_n)$ maps to zero in $H^p(X, \mathcal{F}_t)$. Thus the I-adic topology is finer than the limit topology. For the converse we set $\mathcal{F} = \lim \mathcal{F}_n$, we pick generators $f_1, \ldots, f_r$ of I, we pick $c \geq 1$, and we choose $\Phi_{\mathcal{F}}$ as in Lemma 52.4.4. We will use the results of Lemma 52.2.1 without further mention. In particular we have a short exact sequence

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}) \to \lim H^p(X, \mathcal{F}_n) \to 0$$

Thus we can lift any element ξ of $\lim H^p(X, \mathcal{F}_n)$ to an element $\xi' \in H^p(X, \mathcal{F})$. Suppose ξ maps to zero in $H^p(X, \mathcal{F}_{nc})$ for some n, in other words, suppose ξ is "small" in the limit topology. We have a short exact sequence

$$0 \to I^{nc}\mathcal{F} \to \mathcal{F} \to \mathcal{F}_{nc} \to 0$$

and hence the assumption means we can lift ξ' to an element $\xi'' \in H^p(X, I^{nc}\mathcal{F})$. Applying $\Phi_\mathcal{F}$ we get

$$\Phi_\mathcal{F}(\xi'') = \sum\nolimits_{e_1+\ldots+e_r=n} \xi'_{e_1,\ldots,e_r} \cdot T_1^{e_1} \ldots T_r^{e_r}$$

for some $\xi'_{e_1,\ldots,e_r} \in H^p(X, \mathcal{F})$. Letting $\xi_{e_1,\ldots,e_r} \in \lim H^p(X, \mathcal{F}_n)$ be the images and using the final assertion of Lemma 52.4.4 we conclude that

$$\xi = \sum f_1^{e_1} \ldots f_r^{e_r} \xi_{e_1,\ldots,e_r}$$

is in $I^n \lim H^p(X, \mathcal{F}_n)$ as desired. □

0EH8 Example 52.4.6. Let k be a field. Let $A = k[x, y][[s, t]]/(xs - yt)$. Let $I = (s, t)$ and $\mathfrak{a} = (x, y, s, t)$. Let $X = \text{Spec}(A) - V(\mathfrak{a})$ and $\mathcal{F}_n = \mathcal{O}_X/I^n\mathcal{O}_X$. Observe that the rational function

$$g = \frac{t}{x} = \frac{s}{y}$$

is regular in an open neighbourhood $V \subset X$ of $V(I\mathcal{O}_X)$. Hence every power g^e determines a section $g^e \in M = \lim H^0(X, \mathcal{F}_n)$. Observe that $g^e \to 0$ as $e \to \infty$ in the limit topology on M since g^e maps to zero in $\mathcal{F}_e$. On the other hand, $g^e \notin IM$ for any e as the reader can see by computing $H^0(U, \mathcal{F}_n)$; computation omitted. Observe that $\text{cd}(A, I) = 2$. Thus the result of Lemma 52.4.5 is sharp.

52.5. Mittag-Leffler conditions

0EFN When taking local cohomology with respect to the maximal ideal of a local Noetherian ring, we often get the Mittag-Leffler condition for free. This implies the same thing is true for higher cohomology groups of an inverse system of coherent sheaves with surjective transition maps on the puncture spectrum.

0DX0 Lemma 52.5.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring.

(1) Let M be a finite A-module. Then the A-module $H^i_\mathfrak{m}(M)$ satisfies the descending chain condition for any i.
(2) Let $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then the A-module $H^i(U, \mathcal{F})$ satisfies the descending chain condition for $i > 0$.

Proof. We will prove part (1) by induction on the dimension of the support of M. The statement holds if $M = 0$, thus we may and do assume M is not zero.

Base case of the induction. If $\dim(\text{Supp}(M)) = 0$, then the support of M is $\{\mathfrak{m}\}$ and we see that $H^0_\mathfrak{m}(M) = M$ and $H^i_\mathfrak{m}(M) = 0$ for $i > 0$ as is clear from the construction of local cohomology, see Dualizing Complexes, Section 47.9. Since M has finite length (Algebra, Lemma 10.52.8) it has the descending chain condition.

Induction step. Assume $\dim(\text{Supp}(M)) > 0$. By the base case the finite module $H^0_\mathfrak{m}(M) \subset M$ has the descending chain condition. By Dualizing Complexes, Lemma 47.11.6 we may replace M by $M/H^0_\mathfrak{m}(M)$. Then $H^0_\mathfrak{m}(M) = 0$, i.e., M has depth ≥ 1, see Dualizing Complexes, Lemma 47.11.1. Choose $x \in \mathfrak{m}$ such that $x : M \to M$ is injective. By Algebra, Lemma 10.63.10 we have $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$ and the induction hypothesis applies. Pick an index i and consider the exact sequence

$$H^{i-1}_\mathfrak{m}(M/xM) \to H^i_\mathfrak{m}(M) \xrightarrow{x} H^i_\mathfrak{m}(M)$$

coming from the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. It follows that the x-torsion $H^i_{\mathfrak m}(M)[x]$ is a quotient of a module with the descending chain condition, and hence has the descending chain condition itself. Hence the $\mathfrak m$-torsion submodule $H^i_{\mathfrak m}(M)[\mathfrak m]$ has the descending chain condition (and hence is finite dimensional over $A/\mathfrak m$). Thus we conclude that the $\mathfrak m$-power torsion module $H^i_{\mathfrak m}(M)$ has the descending chain condition by Dualizing Complexes, Lemma 47.7.7.

Part (2) follows from (1) via Local Cohomology, Lemma 51.8.2. □

0DX1 Lemma 52.5.2. Let $(A, \mathfrak m)$ be a Noetherian local ring.

(1) Let (M_n) be an inverse system of finite A-modules. Then the inverse system $H^i_{\mathfrak m}(M_n)$ satisfies the Mittag-Leffler condition for any i.
(2) Let $U = \mathrm{Spec}(A) \setminus \{\mathfrak m\}$ be the punctured spectrum of A. Let $\mathcal F_n$ be an inverse system of coherent $\mathcal O_U$-modules. Then the inverse system $H^i(U, \mathcal F_n)$ satisfies the Mittag-Leffler condition for $i > 0$.

Proof. Follows immediately from Lemma 52.5.1. □

0EHB Lemma 52.5.3. Let $(A, \mathfrak m)$ be a Noetherian local ring. Let (M_n) be an inverse system of finite A-modules. Let $M \to \lim M_n$ be a map where M is a finite A-module such that for some i the map $H^i_{\mathfrak m}(M) \to \lim H^i_{\mathfrak m}(M_n)$ is an isomorphism. Then the inverse system $H^i_{\mathfrak m}(M_n)$ is essentially constant with value $H^i_{\mathfrak m}(M)$.

Proof. By Lemma 52.5.2 the inverse system $H^i_{\mathfrak m}(M_n)$ satisfies the Mittag-Leffler condition. Let $E_n \subset H^i_{\mathfrak m}(M_n)$ be the image of $H^i_{\mathfrak m}(M_{n'})$ for $n' \gg n$. Then (E_n) is an inverse system with surjective transition maps and $H^i_{\mathfrak m}(M) = \lim E_n$. Since $H^i_{\mathfrak m}(M)$ has the descending chain condition by Lemma 52.5.1 we find there can only be a finite number of nontrivial kernels of the surjections $H^i_{\mathfrak m}(M) \to E_n$. Thus $E_n \to E_{n-1}$ is an isomorphism for all $n \gg 0$ as desired. □

0DXJ Lemma 52.5.4. Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then

$$H^i(R\Gamma_{\mathfrak m}(M)^\wedge) = \lim H^i_{\mathfrak m}(M/I^nM)$$

for all i where $R\Gamma_{\mathfrak m}(M)^\wedge$ denotes the derived I-adic completion.

Proof. Apply Dualizing Complexes, Lemma 47.12.4 and Lemma 52.5.2 to see the vanishing of the $R^1 \lim$ terms. □

52.6. Derived completion on a ringed site

0995 We urge the reader to skip this section on a first reading.

The algebra version of this material can be found in More on Algebra, Section 15.91. Let $\mathcal O$ be a sheaf of rings on a site $\mathcal C$. Let f be a global section of $\mathcal O$. We denote $\mathcal O_f$ the sheaf associated to the presheaf of localizations $U \mapsto \mathcal O(U)_f$.

0996 Lemma 52.6.1. Let $(\mathcal C, \mathcal O)$ be a ringed site. Let f be a global section of $\mathcal O$.

(1) For $L, N \in D(\mathcal O_f)$ we have $R\mathcal{H}om_{\mathcal O}(L, N) = R\mathcal{H}om_{\mathcal O_f}(L, N)$. In particular the two $\mathcal O_f$-structures on $R\mathcal{H}om_{\mathcal O}(L, N)$ agree.
(2) For $K \in D(\mathcal O)$ and $L \in D(\mathcal O_f)$ we have
$$R\mathcal{H}om_{\mathcal O}(L, K) = R\mathcal{H}om_{\mathcal O_f}(L, R\mathcal{H}om_{\mathcal O}(\mathcal O_f, K))$$
In particular $R\mathcal{H}om_{\mathcal O}(\mathcal O_f, R\mathcal{H}om_{\mathcal O}(\mathcal O_f, K)) = R\mathcal{H}om_{\mathcal O}(\mathcal O_f, K)$.

(3) If g is a second global section of $\mathcal{O}$, then

$$R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, R\mathcal{H}om_\mathcal{O}(\mathcal{O}_g, K)) = R\mathcal{H}om_\mathcal{O}(\mathcal{O}_{gf}, K).$$

Proof. Proof of (1). Let $\mathcal{J}^\bullet$ be a K-injective complex of $\mathcal{O}_f$-modules representing N. By Cohomology on Sites, Lemma 21.20.10 it follows that $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules as well. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_\mathcal{O}(L, N) = R\mathcal{H}om_\mathcal{O}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$

by Modules on Sites, Lemma 18.11.4 because $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$ and of $\mathcal{O}_f$-modules.

Proof of (2). Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$ is represented by $\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet)$ which is a K-injective complex of $\mathcal{O}_f$-modules and of $\mathcal{O}$-modules by Cohomology on Sites, Lemmas 21.20.11 and 21.20.10. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_\mathcal{O}(L, K) = R\mathcal{H}om_\mathcal{O}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet))$$

by Modules on Sites, Lemma 18.27.8 and because $\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_f$-modules.

Proof of (3). This follows from the fact that $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_g, \mathcal{I}^\bullet)$ is K-injective as a complex of $\mathcal{O}$-modules and the fact that $\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{H}om_\mathcal{O}(\mathcal{O}_g, \mathcal{H})) = \mathcal{H}om_\mathcal{O}(\mathcal{O}_{gf}, \mathcal{H})$ for all sheaves of $\mathcal{O}$-modules $\mathcal{H}$. □

Let $K \in D(\mathcal{O})$. We denote $T(K, f)$ a derived limit (Derived Categories, Definition 13.34.1) of the inverse system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{O})$.

0997 Lemma 52.6.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$. Let $K \in D(\mathcal{O})$. The following are equivalent

(1) $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K) = 0$,
(2) $R\mathcal{H}om_\mathcal{O}(L, K) = 0$ for all L in $D(\mathcal{O}_f)$,
(3) $T(K, f) = 0$.

Proof. It is clear that (2) implies (1). The implication (1) ⇒ (2) follows from Lemma 52.6.1. A free resolution of the $\mathcal{O}$-module $\mathcal{O}_f$ is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \mathcal{O}_f \to 0$$

where the first map sends a local section $(x_0, x_1, \ldots)$ to $(x_0, x_1 - fx_0, x_2 - fx_1, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathcal{H}om_\mathcal{O}(-, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules representing K we get a short exact sequence of complexes

$$0 \to \mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet) \to \prod \mathcal{I}^\bullet \to \prod \mathcal{I}^\bullet \to 0$$

because $\mathcal{I}^n$ is an injective $\mathcal{O}$-module. The products are products in $D(\mathcal{O})$, see Injectives, Lemma 19.13.4. This means that the object $T(K, f)$ is a representative of $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$ in $D(\mathcal{O})$. Thus the equivalence of (1) and (3). □

0998 Lemma 52.6.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K \in D(\mathcal{O})$. The rule which associates to U the set $\mathcal{I}(U)$ of sections $f \in \mathcal{O}(U)$ such that $T(K|_U, f) = 0$ is a sheaf of ideals in $\mathcal{O}$.

Proof. We will use the results of Lemma 52.6.2 without further mention. If $f \in \mathcal{I}(U)$, and $g \in \mathcal{O}(U)$, then $\mathcal{O}_{U,gf}$ is an $\mathcal{O}_{U,f}$-module hence $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{U,gf}, K|_U) = 0$, hence $gf \in \mathcal{I}(U)$. Suppose $f, g \in \mathcal{O}(U)$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{U,f+g} \to \mathcal{O}_{U,f(f+g)} \oplus \mathcal{O}_{U,g(f+g)} \to \mathcal{O}_{U,gf(f+g)} \to 0$$

because f, g generate the unit ideal in $\mathcal{O}(U)_{f+g}$. This follows from Algebra, Lemma 10.24.2 and the easy fact that the last arrow is surjective. Because $R\mathcal{H}om_{\mathcal{O}}(-, K|_U)$ is an exact functor of triangulated categories the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f(f+g)}, K|_U)$, $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g(f+g)}, K|_U)$, and $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,gf(f+g)}, K|_U)$, implies the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f+g}, K|_U)$. We omit the verification of the sheaf condition. □

We can make the following definition for any ringed site.

0999 Definition 52.6.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let $K \in D(\mathcal{O})$. We say that K is derived complete with respect to $\mathcal{I}$ if for every object U of $\mathcal{C}$ and $f \in \mathcal{I}(U)$ the object $T(K|_U, f)$ of $D(\mathcal{O}_U)$ is zero.

It is clear that the full subcategory $D_{comp}(\mathcal{O}) = D_{comp}(\mathcal{O}, \mathcal{I}) \subset D(\mathcal{O})$ consisting of derived complete objects is a saturated triangulated subcategory, see Derived Categories, Definitions 13.3.4 and 13.6.1. This subcategory is preserved under products and homotopy limits in $D(\mathcal{O})$. But it is not preserved under countable direct sums in general.

099A Lemma 52.6.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. If $K \in D(\mathcal{O})$ and $L \in D_{comp}(\mathcal{O})$, then $R\mathcal{H}om_{\mathcal{O}}(K, L) \in D_{comp}(\mathcal{O})$.

Proof. Let U be an object of $\mathcal{C}$ and let $f \in \mathcal{I}(U)$. Recall that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{O}_{U,f}, L|_U)$$

by Cohomology on Sites, Lemma 21.35.2. The right hand side is zero by Lemma 52.6.2 and the relationship between internal hom and actual hom, see Cohomology on Sites, Lemma 21.35.1. The same vanishing holds for all U'/U. Thus the object $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ of $D(\mathcal{O}_U)$ has vanishing 0th cohomology sheaf (by locus citatus). Similarly for the other cohomology sheaves, i.e., $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ is zero in $D(\mathcal{O}_U)$. By Lemma 52.6.2 we conclude. □

099C Lemma 52.6.6. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. The inverse image of $D_{comp}(\mathcal{O}, \mathcal{I})$ under the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ is $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$.

Proof. Using Lemma 52.6.3 we see that $K' \in D(\mathcal{O}')$ is in $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ if and only if $T(K'|_U, f)$ is zero for every local section $f \in \mathcal{I}(U)$. Observe that the cohomology sheaves of $T(K'|_U, f)$ are computed in the category of abelian sheaves, so it doesn't matter whether we think of f as a section of $\mathcal{O}$ or take the image of f as a section of $\mathcal{O}'$. The lemma follows immediately from this and the definition of derived complete objects. □

099J Lemma 52.6.7. Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$.

Proof. We may assume f is given by a morphism of ringed sites corresponding to a continuous functor $\mathcal{C} \to \mathcal{D}$ (Modules on Sites, Lemma 18.7.2). Let U be an object of $\mathcal{C}$ and let g be a section of $\mathcal{I}$ over U. We have to show that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = 0$ whenever K is derived complete with respect to $\mathcal{I}'$. Namely, by Cohomology on Sites, Lemma 21.35.1 this, applied to all objects over U and all shifts of K, will imply that $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g}, Rf_*K|_U)$ is zero, which implies that $T(Rf_*K|_U, g)$ is zero (Lemma 52.6.2) which is what we have to show (Definition 52.6.4). Let V in $\mathcal{D}$ be the image of U. Then

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = \mathrm{Hom}_{D(\mathcal{O}'_V)}(\mathcal{O}'_{V,g'}, K|_V) = 0$$

where $g' = f^\sharp(g) \in \mathcal{I}'(V)$. The second equality because K is derived complete and the first equality because the derived pullback of $\mathcal{O}_{U,g}$ is $\mathcal{O}'_{V,g'}$ and Cohomology on Sites, Lemma 21.19.1. □

The following lemma is the simplest case where one has derived completion.

099B Lemma 52.6.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Then the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$ has a left adjoint, i.e., given any object K of $D(\mathcal{O})$ there exists a map $K \to K^\wedge$ with $K^\wedge$ in $D_{comp}(\mathcal{O})$ such that the map

$$\mathrm{Hom}_{D(\mathcal{O})}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K, E)$$

is bijective whenever E is in $D_{comp}(\mathcal{O})$. In fact we have

$$K^\wedge = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}, K)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}) \longrightarrow \mathcal{O}$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let f be a global section of $\mathcal{O}$. By Lemma 52.6.1 the object $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K^\wedge)$ is equal to

$$R\mathcal{H}om_{\mathcal{O}}((\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}), K)$$

If $f = f_i$ for some i, then $f_1, \ldots, f_r$ generate the unit ideal in $\mathcal{O}_f$, hence the extended alternating Čech complex

$$\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}$$

is zero (even homotopic to zero). In this way we see that $K^\wedge$ is derived complete.

If K is derived complete, then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(\mathcal{O})$. □

Next we explain why derived completion is a completion.

0A0E Lemma 52.6.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Let $K \in D(\mathcal{O})$. The derived completion $K^\wedge$ of Lemma 52.6.8 is given by the formula

$$K^\wedge = R\lim K \otimes_{\mathcal{O}}^{\mathbf{L}} K_n$$

where $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ is the Koszul complex on $f_1^n, \ldots, f_r^n$ over $\mathcal{O}$.

Proof. In More on Algebra, Lemma 15.29.6 we have seen that the extended alternating Čech complex

$$\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free $\mathcal{O}$-modules with dual $\mathcal{H}om_{\mathcal{O}}(K^n, \mathcal{O}) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). By Lemma 52.6.8 the functor $E \mapsto E^\wedge$ is gotten by taking $R\mathcal{H}om$ from the extended alternating Čech complex into E:

$$E^\wedge = R\mathcal{H}om(\operatorname{colim} K^n, E)$$

This is equal to $R\lim(E \otimes_{\mathcal{O}}^{\mathbf{L}} K_n)$ by Cohomology on Sites, Lemma 21.48.8. □

099D Lemma 52.6.10. There exist a way to construct

(1) for every pair (A, I) consisting of a ring A and a finitely generated ideal $I \subset A$ a complex $K(A, I)$ of A-modules,
(2) a map $K(A, I) \to A$ of complexes of A-modules,
(3) for every ring map $A \to B$ and finitely generated ideal $I \subset A$ a map of complexes $K(A, I) \to K(B, IB)$,

such that

(a) for $A \to B$ and $I \subset A$ finitely generated the diagram

$$\begin{tikzcd} K(A,I) \arrow[r] \arrow[d] & A \arrow[d] \\ K(B,IB) \arrow[r] & B \end{tikzcd}$$

commutes,
(b) for $A \to B \to C$ and $I \subset A$ finitely generated the composition of the maps $K(A, I) \to K(B, IB) \to K(C, IC)$ is the map $K(A, I) \to K(C, IC)$.
(c) for $A \to B$ and a finitely generated ideal $I \subset A$ the induced map $K(A, I) \otimes_A^{\mathbf{L}} B \to K(B, IB)$ is an isomorphism in $D(B)$, and
(d) if $I = (f_1, \ldots, f_r) \subset A$ then there is a commutative diagram

$$\begin{tikzcd} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \arrow[r] \arrow[d] & K(A,I) \arrow[d] \\ A \arrow[r, "1"] & A \end{tikzcd}$$

in $D(A)$ whose horizontal arrows are isomorphisms.

Proof. Let S be the set of rings A_0 of the form $A_0 = \mathbf{Z}[x_1, \ldots, x_n]/J$. Every finite type $\mathbf{Z}$-algebra is isomorphic to an element of S. Let $\mathcal{A}_0$ be the category whose objects are pairs (A_0, I_0) where $A_0 \in S$ and $I_0 \subset A_0$ is an ideal and whose morphisms $(A_0, I_0) \to (B_0, J_0)$ are ring maps $\varphi : A_0 \to B_0$ such that $J_0 = \varphi(I_0)B_0$.

Suppose we can construct $K(A_0, I_0) \to A_0$ functorially for objects of $\mathcal{A}_0$ having properties (a), (b), (c), and (d). Then we take

$$K(A, I) = \mathrm{colim}_{\varphi:(A_0,I_0)\to(A,I)} K(A_0, I_0)$$

where the colimit is over ring maps $\varphi : A_0 \to A$ such that $\varphi(I_0)A = I$ with (A_0, I_0) in $\mathcal{A}_0$. A morphism between $(A_0, I_0) \to (A, I)$ and $(A_0', I_0') \to (A, I)$ are given by maps $(A_0, I_0) \to (A_0', I_0')$ in $\mathcal{A}_0$ commuting with maps to A. The category of these $(A_0, I_0) \to (A, I)$ is filtered (details omitted). Moreover, $\mathrm{colim}_{\varphi:(A_0,I_0)\to(A,I)} A_0 = A$ so that $K(A, I)$ is a complex of A-modules. Finally, given $\varphi : A \to B$ and $I \subset A$ for every $(A_0, I_0) \to (A, I)$ in the colimit, the composition $(A_0, I_0) \to (B, IB)$ lives in the colimit for (B, IB). In this way we get a map on colimits. Properties (a), (b), (c), and (d) follow readily from this and the corresponding properties of the complexes $K(A_0, I_0)$.

Endow $\mathcal{C}_0 = \mathcal{A}_0^{opp}$ with the chaotic topology. We equip $\mathcal{C}_0$ with the sheaf of rings $\mathcal{O} : (A, I) \mapsto A$. The ideals I fit together to give a sheaf of ideals $\mathcal{I} \subset \mathcal{O}$. Choose an injective resolution $\mathcal{O} \to \mathcal{J}^\bullet$. Consider the object

$$\mathcal{F}^\bullet = \bigcup\nolimits_n \mathcal{J}^\bullet[\mathcal{I}^n]$$

Let $U = (A, I) \in \mathrm{Ob}(\mathcal{C}_0)$. Since the topology in $\mathcal{C}_0$ is chaotic, the value $\mathcal{J}^\bullet(U)$ is a resolution of A by injective A-modules. Hence the value $\mathcal{F}^\bullet(U)$ is an object of $D(A)$ representing the image of $R\Gamma_I(A)$ in $D(A)$, see Dualizing Complexes, Section 47.9. Choose a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ and a commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r] & \mathcal{J}^\bullet \\ \mathcal{K}^\bullet \arrow[u] \arrow[r] & \mathcal{F}^\bullet \arrow[u] \end{tikzcd}$$

where the horizontal arrows are quasi-isomorphisms. This is possible by the construction of the derived category $D(\mathcal{O})$. Set $K(A, I) = \mathcal{K}^\bullet(U)$ where $U = (A, I)$. Properties (a) and (b) are clear and properties (c) and (d) follow from Dualizing Complexes, Lemmas 47.10.2 and 47.10.3. □

099E Lemma 52.6.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a map $K \to \mathcal{O}$ in $D(\mathcal{O})$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{I}|_U$ is generated by $f_1, \ldots, f_r \in \mathcal{I}(U)$ there is an isomorphism

$$(\mathcal{O}_U \to \prod\nolimits_{i_0} \mathcal{O}_{U,f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{U,f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{U,f_1\ldots f_r}) \longrightarrow K|_U$$

compatible with maps to $\mathcal{O}_U$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects U such that $\mathcal{I}|_U$ is generated by finitely many sections. Then $\mathcal{C}' \to \mathcal{C}$ is a special cocontinuous functor (Sites, Definition 7.29.2). Hence it suffices to work with $\mathcal{C}'$, see Sites, Lemma 7.29.1. In other words we may assume that for every object U of $\mathcal{C}$ there exists a finitely generated ideal $I \subset \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathrm{Im}(I \otimes \mathcal{O}_U \to \mathcal{O}_U)$. We will say that I generates $\mathcal{I}|_U$. Warning: We do not know that $\mathcal{I}(U)$ is a finitely generated ideal in $\mathcal{O}(U)$.

Let U be an object and $I \subset \mathcal{O}(U)$ a finitely generated ideal which generates $\mathcal{I}|_U$. On the category $\mathcal{C}/U$ consider the complex of presheaves

$$K_{U,I}^\bullet : U'/U \longmapsto K(\mathcal{O}(U'), I\mathcal{O}(U'))$$

with $K(-,-)$ as in Lemma 52.6.10. We claim that the sheafification of this is independent of the choice of I. Indeed, if $I' \subset \mathcal{O}(U)$ is a finitely generated ideal which also generates $\mathcal{I}|_U$, then there exists a covering $\{U_j \to U\}$ such that $I\mathcal{O}(U_j) = I'\mathcal{O}(U_j)$. (Hint: this works because both I and I' are finitely generated and generate $\mathcal{I}|_U$.) Hence $K_{U,I}^\bullet$ and $K_{U,I'}^\bullet$ are the same for any object lying over one of the U_j. The statement on sheafifications follows. Denote $K_U^\bullet$ the common value.

The independence of choice of I also shows that $K_U^\bullet|_{\mathcal{C}/U'} = K_{U'}^\bullet$ whenever we are given a morphism $U' \to U$ and hence a localization morphism $\mathcal{C}/U' \to \mathcal{C}/U$. Thus the complexes $K_U^\bullet$ glue to give a single well defined complex $K^\bullet$ of $\mathcal{O}$-modules. The existence of the map $K^\bullet \to \mathcal{O}$ and the quasi-isomorphism of the lemma follow immediately from the corresponding properties of the complexes $K(-,-)$ in Lemma 52.6.10. □

099F Proposition 52.6.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a left adjoint to the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$.

Proof. Let $K \to \mathcal{O}$ in $D(\mathcal{O})$ be as constructed in Lemma 52.6.11. Let $E \in D(\mathcal{O})$. Then $E^\wedge = R\mathcal{H}om(K, E)$ together with the map $E \to E^\wedge$ will do the job. Namely, locally on the site $\mathcal{C}$ we recover the adjoint of Lemma 52.6.8. This shows that $E^\wedge$ is always derived complete and that $E \to E^\wedge$ is an isomorphism if E is derived complete. □

0CQH Remark 52.6.13 (Comparison with completion). Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 52.6.12. For any $n \geq 1$ the object $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$ is derived complete as it is annihilated by powers of local sections of $\mathcal{I}$. Hence there is a canonical factorization

$$K \to K^\wedge \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$$

of the canonical map $K \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$. These maps are compatible for varying n and we obtain a comparison map

$$K^\wedge \longrightarrow R\lim \left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n\right)$$

The right hand side is more recognizable as a kind of completion. In general this comparison map is not an isomorphism.

0A0F Remark 52.6.14 (Localization and derived completion). Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 52.6.12. It follows from the construction in the proof of the proposition that $K^\wedge|_U$ is the derived completion of $K|_U$ for any $U \in \text{Ob}(\mathcal{C})$. But we can also prove this as follows. From the definition of derived complete objects it follows that $K^\wedge|_U$ is derived complete. Thus we obtain a canonical map $a : (K|_U)^\wedge \to K^\wedge|_U$. On the other hand, if E is a derived complete object of $D(\mathcal{O}_U)$, then Rj_*E is a derived complete object of $D(\mathcal{O})$ by Lemma 52.6.7. Here j is the localization morphism (Modules on Sites, Section 18.19). Hence we also obtain a canonical map $b : K^\wedge \to Rj_*((K|_U)^\wedge)$. We omit the (formal) verification that the adjoint of b is the inverse of a.

099G Remark 52.6.15 (Completed tensor product). Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Denote $K \mapsto K^\wedge$ the adjoint of Proposition 52.6.12. Then we set

$$K \otimes_{\mathcal{O}}^\wedge L = (K \otimes_{\mathcal{O}}^{\mathbf{L}} L)^\wedge$$

This completed tensor product defines a functor $D_{comp}(\mathcal{O}) \times D_{comp}(\mathcal{O}) \to D_{comp}(\mathcal{O})$ such that we have

$$\mathrm{Hom}_{D_{comp}(\mathcal{O})}(K, R\mathcal{H}om_\mathcal{O}(L, M)) = \mathrm{Hom}_{D_{comp}(\mathcal{O})}(K \otimes^\wedge_\mathcal{O} L, M)$$

for $K, L, M \in D_{comp}(\mathcal{O})$. Note that $R\mathcal{H}om_\mathcal{O}(L, M) \in D_{comp}(\mathcal{O})$ by Lemma 52.6.5.

099H Lemma 52.6.16. Let $\mathcal{C}$ be a site. Assume $\varphi : \mathcal{O} \to \mathcal{O}'$ is a flat homomorphism of sheaves of rings. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$ such that $\mathcal{O}/(f_1, \ldots, f_r) \cong \mathcal{O}'/(f_1, \ldots, f_r)\mathcal{O}'$. Then the map of extended alternating Čech complexes

$$\begin{array}{c} \mathcal{O} \to \prod_{i_0} \mathcal{O}_{f_{i_0}} \to \prod_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r} \\ \downarrow \\ \mathcal{O}' \to \prod_{i_0} \mathcal{O}'_{f_{i_0}} \to \prod_{i_0 < i_1} \mathcal{O}'_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}'_{f_1 \ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. Observe that the second complex is the tensor product of the first complex with $\mathcal{O}'$. We can write the first extended alternating Čech complex as a colimit of the Koszul complexes $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$, see More on Algebra, Lemma 15.29.6. Hence it suffices to prove $K_n \to K_n \otimes_\mathcal{O} \mathcal{O}'$ is a quasi-isomorphism. Since $\mathcal{O} \to \mathcal{O}'$ is flat it suffices to show that $H^i \to H^i \otimes_\mathcal{O} \mathcal{O}'$ is an isomorphism where H^i is the ith cohomology sheaf $H^i = H^i(K_n)$. These sheaves are annihilated by $f_1^n, \ldots, f_r^n$, see More on Algebra, Lemma 15.28.6. Hence these sheaves are annihilated by $(f_1, \ldots, f_r)^m$ for some $m \gg 0$. Thus $H^i \to H^i \otimes_\mathcal{O} \mathcal{O}'$ is an isomorphism by Modules on Sites, Lemma 18.28.16. ☐

099I Lemma 52.6.17. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. If $\mathcal{O} \to \mathcal{O}'$ is flat and $\mathcal{O}/\mathcal{I} \cong \mathcal{O}'/\mathcal{I}\mathcal{O}'$, then the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ induces an equivalence $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}') \to D_{comp}(\mathcal{O}, \mathcal{I})$.

Proof. Lemma 52.6.7 implies restriction $r : D(\mathcal{O}') \to D(\mathcal{O})$ sends $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$. We will construct a quasi-inverse $E \mapsto E'$.

Let $K \to \mathcal{O}$ be the morphism of $D(\mathcal{O})$ constructed in Lemma 52.6.11. Set $K' = K \otimes^\mathbf{L}_\mathcal{O} \mathcal{O}'$ in $D(\mathcal{O}')$. Then $K' \to \mathcal{O}'$ is a map in $D(\mathcal{O}')$ which satisfies the conclusions of Lemma 52.6.11 with respect to $\mathcal{I}' = \mathcal{I}\mathcal{O}'$. The map $K \to r(K')$ is a quasi-isomorphism by Lemma 52.6.16. Now, for $E \in D_{comp}(\mathcal{O}, \mathcal{I})$ we set

$$E' = R\mathcal{H}om_\mathcal{O}(r(K'), E)$$

viewed as an object in $D(\mathcal{O}')$ using the $\mathcal{O}'$-module structure on K'. Since E is derived complete we have $E = R\mathcal{H}om_\mathcal{O}(K, E)$, see proof of Proposition 52.6.12. On the other hand, since $K \to r(K')$ is an isomorphism in we see that there is an isomorphism $E \to r(E')$ in $D(\mathcal{O})$. To finish the proof we have to show that, if $E = r(M')$ for an object M' of $D_{comp}(\mathcal{O}', \mathcal{I}')$, then $E' \cong M'$. To get a map we use

$$M' = R\mathcal{H}om_{\mathcal{O}'}(\mathcal{O}', M') \to R\mathcal{H}om_\mathcal{O}(r(\mathcal{O}'), r(M')) \to R\mathcal{H}om_\mathcal{O}(r(K'), r(M')) = E'$$

where the second arrow uses the map $K' \to \mathcal{O}'$. To see that this is an isomorphism, one shows that r applied to this arrow is the same as the isomorphism $E \to r(E')$ above. Details omitted. ☐

099K Lemma 52.6.18. Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be finite type sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion.

Proof. The first statement we have seen in Lemma 52.6.7. Note that the second statement makes sense as we have a derived completion functor $D(\mathcal{O}') \to D_{comp}(\mathcal{O}', \mathcal{I}')$ by Proposition 52.6.12. OK, so now let $K \in D_{comp}(\mathcal{O}, \mathcal{I})$ and $M \in D_{comp}(\mathcal{O}', \mathcal{I}')$. Then we have

$$\mathrm{Hom}(K, Rf_*M) = \mathrm{Hom}(Lf^*K, M) = \mathrm{Hom}(Lf^*_{comp}K, M)$$

by the universal property of derived completion. □

0A0G Lemma 52.6.19. Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $\mathcal{I}' \subset \mathcal{O}'$ be the ideal generated by $f^\sharp(f^{-1}\mathcal{I})$. Then Rf_* commutes with derived completion, i.e., $Rf_*(K^\wedge) = (Rf_*K)^\wedge$.

Generalization of [BS13, Lemma 6.5.9 (2)]. Compare with [HLP14, Theorem 6.5] in the setting of quasi-coherent modules and morphisms of (derived) algebraic stacks.

Proof. By Proposition 52.6.12 the derived completion functors exist. By Lemma 52.6.7 the object $Rf_*(K^\wedge)$ is derived complete, and hence we obtain a canonical map $(Rf_*K)^\wedge \to Rf_*(K^\wedge)$ by the universal property of derived completion. We may check this map is an isomorphism locally on $\mathcal{C}$. Thus, since derived completion commutes with localization (Remark 52.6.14) we may assume that $\mathcal{I}$ is generated by global sections $f_1, \ldots, f_r$. Then $\mathcal{I}'$ is generated by $g_i = f^\sharp(f_i)$. By Lemma 52.6.9 we have to prove that

$$R\lim \left(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n)\right) = Rf_* \left(R\lim K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

Because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.23.3) it suffices to prove that

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n) = Rf_* \left(K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

This follows from the projection formula (Cohomology on Sites, Lemma 21.50.1) and the fact that $Lf^*K(\mathcal{O}, f_1^n, \ldots, f_r^n) = K(\mathcal{O}', g_1^n, \ldots, g_r^n)$. □

0BLX Lemma 52.6.20. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $\mathcal{C}$ be a site and let $\mathcal{O}$ be a sheaf of A-algebras. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then we have

$$R\Gamma(\mathcal{C}, \mathcal{F})^\wedge = R\Gamma(\mathcal{C}, \mathcal{F}^\wedge)$$

in $D(A)$ where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $I\mathcal{O}$ and on the left hand wide we have the derived completion with respect to I. This produces two spectral sequences

$$E_2^{i,j} = H^i(H^j(\mathcal{C}, \mathcal{F})^\wedge) \quad \text{and} \quad E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{F}^\wedge))$$

both converging to $H^*(R\Gamma(\mathcal{C}, \mathcal{F})^\wedge) = H^*(\mathcal{C}, \mathcal{F}^\wedge)$

Proof. Apply Lemma 52.6.19 to the morphism of ringed topoi $(\mathcal{C}, \mathcal{O}) \to (pt, A)$ and take cohomology to get the first statement. The second spectral sequence is the second spectral sequence of Derived Categories, Lemma 13.21.3. The first spectral sequence is the spectral sequence of More on Algebra, Example 15.91.22 applied to $R\Gamma(\mathcal{C}, \mathcal{F})^\wedge$. □

0CQI Remark 52.6.21. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion of Proposition 52.6.12. Let $U \in \mathrm{Ob}(\mathcal{C})$ be an object such that $\mathcal{I}$ is generated as an ideal sheaf by $f_1, \ldots, f_r \in \mathcal{I}(U)$. Set $A = \mathcal{O}(U)$ and $I = (f_1, \ldots, f_r) \subset A$. Warning: it may not be the case that $I = \mathcal{I}(U)$. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

where the right hand side is the derived completion of the object $R\Gamma(U, K)$ of $D(A)$ with respect to I. This is true because derived completion commutes with localization (Remark 52.6.14) and Lemma 52.6.20.

52.7. The theorem on formal functions

0A0H We interrupt the flow of the exposition to talk a little bit about derived completion in the setting of quasi-coherent modules on schemes and to use this to give a somewhat different proof of the theorem on formal functions. We give some pointers to the literature in Remark 52.7.4.

Lemma 52.6.19 is a (very formal) derived version of the theorem on formal functions (Cohomology of Schemes, Theorem 30.20.5). To make this more explicit, suppose $f : X \to S$ is a morphism of schemes, $\mathcal{I} \subset \mathcal{O}_S$ is a quasi-coherent sheaf of ideals of finite type, and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then the lemma says that

0A0I (52.7.0.1) $$Rf_*(\mathcal{F}^\wedge) = (Rf_*\mathcal{F})^\wedge$$

where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ and the right hand side is the derived completion of $Rf_*\mathcal{F}$ with respect to $\mathcal{I}$. To see that this gives back the theorem on formal functions we have to do a bit of work.

0A0L Lemma 52.7.1. Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be a pseudo-coherent object of $D(\mathcal{O}_X)$ with derived completion $K^\wedge$. Then

$$H^p(U, K^\wedge) = \lim H^p(U, K)/I^n H^p(U, K) = H^p(U, K)^\wedge$$

for any affine open $U \subset X$ where $I = \mathcal{I}(U)$ and where on the right we have the derived completion with respect to I.

Proof. Write $U = \mathrm{Spec}(A)$. The ring A is Noetherian and hence $I \subset A$ is finitely generated. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

by Remark 52.6.21. Now $R\Gamma(U, K)$ is a pseudo-coherent complex of A-modules (Derived Categories of Schemes, Lemma 36.10.2). By More on Algebra, Lemma 15.94.4 we conclude that the pth cohomology module of $R\Gamma(U, K^\wedge)$ is equal to the I-adic completion of $H^p(U, K)$. This proves the first equality. The second (less important) equality follows immediately from a second application of the lemma just used. □

0A0K Lemma 52.7.2. Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be an object of $D(\mathcal{O}_X)$. Then

(1) the derived completion $K^\wedge$ is equal to $R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$.

Let K is a pseudo-coherent object of $D(\mathcal{O}_X)$. Then

(2) the cohomology sheaf $H^q(K^\wedge)$ is equal to $\lim H^q(K)/\mathcal{I}^n H^q(K)$.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module[1]. Then

(3) the derived completion $\mathcal{F}^\wedge$ is equal to $\lim \mathcal{F}/\mathcal{I}^n\mathcal{F}$,
(4) $\lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$,
(5) $H^p(U, \mathcal{F}^\wedge) = 0$ for $p \neq 0$ for all affine opens $U \subset X$.

Proof. Proof of (1). There is a canonical map

$$K \longrightarrow R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n),$$

see Remark 52.6.13. Derived completion commutes with passing to open subschemes (Remark 52.6.14). Formation of $R\lim$ commutes with passsing to open subschemes. It follows that to check our map is an isomorphism, we may work locally. Thus we may assume $X = U = \text{Spec}(A)$. Say $I = (f_1, \ldots, f_r)$. Let $K_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex. By More on Algebra, Lemma 15.94.1 we have seen that the pro-systems $\{K_n\}$ and $\{A/I^n\}$ of $D(A)$ are isomorphic. Using the equivalence $D(A) = D_{QCoh}(\mathcal{O}_X)$ of Derived Categories of Schemes, Lemma 36.3.5 we see that the pro-systems $\{K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\}$ and $\{\mathcal{O}_X/\mathcal{I}^n\}$ are isomorphic in $D(\mathcal{O}_X)$. This proves the second equality in

$$K^\wedge = R\lim\left(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\right) = R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$$

The first equality is Lemma 52.6.9.

Assume K is pseudo-coherent. For $U \subset X$ affine open we have $H^q(U, K^\wedge) = \lim H^q(U, K)/\mathcal{I}^n(U)H^q(U, K)$ by Lemma 52.7.1. As this is true for every U we see that $H^q(K^\wedge) = \lim H^q(K)/\mathcal{I}^nH^q(K)$ as sheaves. This proves (2).

Part (3) is a special case of (2). Parts (4) and (5) follow from Derived Categories of Schemes, Lemma 36.3.2. □

0A0M Lemma 52.7.3. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let X be a Noetherian scheme over A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $H^p(X, \mathcal{F})$ is a finite A-module for all p. Then there are short exact sequences

$$0 \to R^1\lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \mathcal{F})^\wedge \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules where $H^p(X, \mathcal{F})^\wedge$ is the usual I-adic completion. If f is proper, then the $R^1\lim$ term is zero.

Proof. Consider the two spectral sequences of Lemma 52.6.20. The first degenerates by More on Algebra, Lemma 15.94.4. We obtain $H^p(X, \mathcal{F})^\wedge$ in degree p. This is where we use the assumption that $H^p(X, \mathcal{F})$ is a finite A-module. The second degenerates because

$$\mathcal{F}^\wedge = \lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$$

is a sheaf by Lemma 52.7.2. We obtain $H^p(X, \lim \mathcal{F}/I^n\mathcal{F})$ in degree p. Since $R\Gamma(X, -)$ commutes with derived limits (Injectives, Lemma 19.13.6) we also get

$$R\Gamma(X, \lim \mathcal{F}/I^n\mathcal{F}) = R\Gamma(X, R\lim \mathcal{F}/I^n\mathcal{F}) = R\lim R\Gamma(X, \mathcal{F}/I^n\mathcal{F})$$

By More on Algebra, Remark 15.87.6 we obtain exact sequences

$$0 \to R^1\lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \lim \mathcal{F}/I^n\mathcal{F}) \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. Combining the above we get the first statement of the lemma. The vanishing of the $R^1\lim$ term follows from Cohomology of Schemes, Lemma 30.20.4. □

[1]For example $H^q(K)$ for K pseudo-coherent on our locally Noetherian X.

0AKL Remark 52.7.4. Here are some references to discussions of related material the literature. It seems that a "derived formal functions theorem" for proper maps goes back to [Lur04, Theorem 6.3.1]. There is the discussion in [Lur11], especially Chapter 4 which discusses the affine story, see More on Algebra, Section 15.91. In [GR13, Section 2.9] one finds a discussion of proper base change and derived completion using (ind) coherent modules. An analogue of (52.7.0.1) for complexes of quasi-coherent modules can be found as [HLP14, Theorem 6.5]

52.8. Algebraization of local cohomology, I

0EFF Let A be a Noetherian ring and let I and J be two ideals of A. Let M be a finite A-module. In this section we study the cohomology groups of the object

$$R\Gamma_J(M)^\wedge \quad \text{of} \quad D(A)$$

where $^\wedge$ denotes derived I-adic completion. Observe that in Dualizing Complexes, Lemma 47.12.5 we have shown, if A is complete with respect to I, that there is an isomorphism

$$\operatorname{colim} H^0_Z(M) \longrightarrow H^0(R\Gamma_J(M)^\wedge)$$

where the (directed) colimit is over the closed subsets $Z = V(J')$ with $J' \subset J$ and $V(J') \cap V(I) = V(J) \cap V(I)$. The union of these closed subsets is

0EFG (52.8.0.1) $$T = \{\mathfrak p \in \operatorname{Spec}(A) : V(\mathfrak p) \cap V(I) \subset V(J) \cap V(I)\}$$

This is a subset of $\operatorname{Spec}(A)$ stable under specialization. The result above becomes the statement that

$$H^0_T(M) \longrightarrow H^0(R\Gamma_J(M)^\wedge)$$

is an isomorphism provided A is complete with respect to I, see Local Cohomology, Lemma 51.5.3 and Remark 51.5.6. Our method to extend this isomorphism to higher cohomology groups rests on the following lemma.

0EFH Lemma 52.8.1. Let I, J be ideals of a Noetherian ring A. Let M be a finite A-module. Let $\mathfrak p \subset A$ be a prime. Let s and d be integers. Assume

(1) A has a dualizing complex,
(2) $\mathfrak p \notin V(J) \cap V(I)$,
(3) $\operatorname{cd}(A, I) \leq d$, and
(4) for all primes $\mathfrak p' \subset \mathfrak p$ we have $\operatorname{depth}_{A_{\mathfrak p'}}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q}) > d + s$ for all $\mathfrak q \in V(\mathfrak p') \cap V(J) \cap V(I)$.

Then there exists an $f \in A$, $f \notin \mathfrak p$ which annihilates $H^i(R\Gamma_J(M)^\wedge)$ for $i \leq s$ where $^\wedge$ indicates I-adic completion.

Proof. We will use that $R\Gamma_J = R\Gamma_{V(J)}$ and similarly for $I + J$, see Dualizing Complexes, Lemma 47.10.1. Observe that $R\Gamma_J(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_{I+J}(M)^\wedge$, see Dualizing Complexes, Lemmas 47.12.1 and 47.9.6. Thus we may replace J by $I + J$ and assume $I \subset J$ and $\mathfrak p \notin V(J)$. Recall that

$$R\Gamma_J(M)^\wedge = R\operatorname{Hom}_A(R\Gamma_I(A), R\Gamma_J(M))$$

by the description of derived completion in More on Algebra, Lemma 15.91.10 combined with the description of local cohomology in Dualizing Complexes, Lemma

47.10.2. Assumption (3) means that $R\Gamma_I(A)$ has nonzero cohomology only in degrees $\leq d$. Using the canonical truncations of $R\Gamma_I(A)$ we find it suffices to show that

$$\mathrm{Ext}^i(N, R\Gamma_J(M))$$

is annihilated by an $f \in A$, $f \notin \mathfrak{p}$ for $i \leq s + d$ and any A-module N. In turn using the canonical truncations for $R\Gamma_J(M)$ we see that it suffices to show $H^i_J(M)$ is annihilated by an $f \in A$, $f \notin \mathfrak{p}$ for $i \leq s + d$. This follows from Local Cohomology, Lemma 51.10.2. ☐

0EFI Lemma 52.8.2. Let I, J be ideals of a Noetherian ring. Let M be a finite A-module. Let s and d be integers. With T as in (52.8.0.1) assume

(1) A has a dualizing complex,
(2) if $\mathfrak{p} \in V(I)$, then no condition,
(3) if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \in T$, then $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq d$ for some $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$,
(4) if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T$, then

$$\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

for all $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.

Then there exists an ideal $J_0 \subset J$ with $V(J_0) \cap V(I) = V(J) \cap V(I)$ such that for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$ the map

$$R\Gamma_{J'}(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ and moreover these modules are annihilated by a power of $J_0 I$.

Proof. Let us consider the set

$$B = \{\mathfrak{p} \notin V(I),\ \mathfrak{p} \in T,\ \text{and } \mathrm{depth}(M_\mathfrak{p}) \leq s\}$$

Choose $J_0 \subset J$ such that $V(J_0)$ is the closure of $B \cup V(J)$.

Claim I: $V(J_0) \cap V(I) = V(J) \cap V(I)$.

Proof of Claim I. The inclusion $\supset$ holds by construction. Let $\mathfrak{p}$ be a minimal prime of $V(J_0)$. If $\mathfrak{p} \in B \cup V(J)$, then either $\mathfrak{p} \in T$ or $\mathfrak{p} \in V(J)$ and in both cases $V(\mathfrak{p}) \cap V(I) \subset V(J) \cap V(I)$ as desired. If $\mathfrak{p} \notin B \cup V(J)$, then $V(\mathfrak{p}) \cap B$ is dense, hence infinite, and we conclude that $\mathrm{depth}(M_\mathfrak{p}) < s$ by Local Cohomology, Lemma 51.9.2. In fact, let $V(\mathfrak{p}) \cap B = \{\mathfrak{p}_\lambda\}_{\lambda \in \Lambda}$. Pick $\mathfrak{q}_\lambda \in V(\mathfrak{p}_\lambda) \cap V(J) \cap V(I)$ as in (3). Let $\delta : \mathrm{Spec}(A) \to \mathbf{Z}$ be the dimension function associated to a dualizing complex $\omega_A^\bullet$ for A. Since Λ is infinite and δ is bounded, there exists an infinite subset $\Lambda' \subset \Lambda$ on which $\delta(\mathfrak{q}_\lambda)$ is constant. For $\lambda \in \Lambda'$ we have

$$\mathrm{depth}(M_{\mathfrak{p}_\lambda}) + \delta(\mathfrak{p}_\lambda) - \delta(\mathfrak{q}_\lambda) = \mathrm{depth}(M_{\mathfrak{p}_\lambda}) + \dim((A/\mathfrak{p}_\lambda)_{\mathfrak{q}_\lambda}) \leq d + s$$

by (3) and the definition of B. By the semi-continuity of the function depth $+\ \delta$ proved in Duality for Schemes, Lemma 48.2.8 we conclude that

$$\mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_{\mathfrak{q}_\lambda}) = \mathrm{depth}(M_\mathfrak{p}) + \delta(\mathfrak{p}) - \delta(\mathfrak{q}_\lambda) \leq d + s$$

Since also $\mathfrak{p} \notin V(I)$ we read off from (4) that $\mathfrak{p} \in T$, i.e., $V(\mathfrak{p}) \cap V(I) \subset V(J) \cap V(I)$. This finishes the proof of Claim I.

Claim II: $H^i_{J_0}(M) \to H^i_{J'}(M)$ is an isomorphism for $i \leq s$ and $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$.

Proof of claim II. Choose $\mathfrak p \in V(J')$ not in $V(J_0)$. It suffices to show that $H^i_{\mathfrak p A_\mathfrak p}(M_\mathfrak p) = 0$ for $i \leq s$, see Local Cohomology, Lemma 51.2.6. Observe that $\mathfrak p \in T$. Hence since $\mathfrak p$ is not in B we see that $\text{depth}(M_\mathfrak p) > s$ and the groups vanish by Dualizing Complexes, Lemma 47.11.1.

Claim III. The final statement of the lemma is true.

By Claim II for $i \leq s$ we have

$$H^i_T(M) = H^i_{J_0}(M) = H^i_{J'}(M)$$

for all ideals $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. See Local Cohomology, Lemma 51.5.3. Let us check the hypotheses of Local Cohomology, Proposition 51.10.1 for the subsets $T \subset T \cup V(I)$, the module M, and the integer s. We have to show that given $\mathfrak p \subset \mathfrak q$ with $\mathfrak p \not\in T \cup V(I)$ and $\mathfrak q \in T$ we have

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

If $\text{depth}(M_\mathfrak p) \geq s$, then this is true because the dimension of $(A/\mathfrak p)_\mathfrak q$ is at least 1. Thus we may assume $\text{depth}(M_\mathfrak p) < s$. If $\mathfrak q \in V(I)$, then $\mathfrak q \in V(J) \cap V(I)$ and the inequality holds by (4). If $\mathfrak q \not\in V(I)$, then we can use (3) to pick $\mathfrak q' \in V(\mathfrak q) \cap V(J) \cap V(I)$ with $\dim((A/\mathfrak q)_{\mathfrak q'}) \leq d$. Then assumption (4) gives

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_{\mathfrak q'}) > s + d$$

Since A is catenary this implies the inequality we want. Applying Local Cohomology, Proposition 51.10.1 we find $J'' \subset A$ with $V(J'') \subset T \cup V(I)$ such that J'' annihilates $H^i_T(M)$ for $i \leq s$. Then we can write $V(J'') \cup V(J_0) \cup V(I) = V(J'I)$ for some $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Replacing J_0 by J' the proof is complete. $\square$

0EFJ Lemma 52.8.3. In Lemma 52.8.2 if instead of the empty condition (2) we assume

(2') if $\mathfrak p \in V(I)$, $\mathfrak p \not\in V(J) \cap V(I)$, then $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$ for all $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$,

then the conditions also imply that $H^i_{J_0}(M)$ is a finite A-module for $i \leq s$.

Proof. Recall that $H^i_{J_0}(M) = H^i_T(M)$, see proof of Lemma 52.8.2. Thus it suffices to check that for $\mathfrak p \not\in T$ and $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$, see Local Cohomology, Proposition 51.11.1. Condition (2') tells us this is true for $\mathfrak p \in V(I)$. Since we know $H^i_T(M)$ is annihilated by a power of IJ_0 we know the condition holds if $\mathfrak p \not\in V(IJ_0)$ by Local Cohomology, Proposition 51.10.1. This covers all cases and the proof is complete. $\square$

0EFK Lemma 52.8.4. If in Lemma 52.8.2 we additionally assume

(6) if $\mathfrak p \not\in V(I)$, $\mathfrak p \in T$, then $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) > s$,

then $H^i_{J_0}(M) = H^i_J(M) = H^i_{J+I}(M)$ for $i \leq s$ and these modules are annihilated by a power of I.

Proof. Choose $\mathfrak p \in V(J)$ or $\mathfrak p \in V(J_0)$ but $\mathfrak p \not\in V(J + I) = V(J_0 + I)$. It suffices to show that $H^i_{\mathfrak p A_\mathfrak p}(M_\mathfrak p) = 0$ for $i \leq s$, see Local Cohomology, Lemma 51.2.6. These groups vanish by condition (6) and Dualizing Complexes, Lemma 47.11.1. The final statement follows from Local Cohomology, Proposition 51.10.1. $\square$

0EFL Lemma 52.8.5. Let I, J be ideals of a Noetherian ring A. Let M be a finite A-module. Let s and d be integers. With T as in (52.8.0.1) assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $\mathfrak p \in V(I)$ no condition,
(3) $\text{cd}(A, I) \leq d$,
(4) if $\mathfrak p \not\in V(I)$, $\mathfrak p \not\in T$ then
$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > d + s$$
for all $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$,
(5) if $\mathfrak p \not\in V(I)$, $\mathfrak p \not\in T$, $V(\mathfrak p) \cap V(J) \cap V(I) \neq \emptyset$, and $\text{depth}(M_\mathfrak p) < s$, then one of the following holds[2]:
  (a) $\dim(\text{Supp}(M_\mathfrak p)) < s + 2$[3], or
  (b) $\delta(\mathfrak p) > d + \delta_{max} - 1$ where δ is a dimension function and δ_{max} is the maximum of δ on $V(J) \cap V(I)$, or
  (c) $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > d + s + \delta_{max} - \delta_{min} - 2$ for all $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$.

Then there exists an ideal $J_0 \subset J$ with $V(J_0) \cap V(I) = V(J) \cap V(I)$ such that for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$ the map
$$R\Gamma_{J'}(M) \longrightarrow R\Gamma_J(M)^\wedge$$
induces an isomorphism on cohomology in degrees $\leq s$. Here $^\wedge$ denotes derived I-adic completion.

We encourage the reader to read the proof in the local case first (Lemma 52.9.5) as it explains the structure of the proof without having to deal with all the inequalities.

Proof. For an ideal $\mathfrak a \subset A$ we have $R\Gamma_\mathfrak a = R\Gamma_{V(\mathfrak a)}$, see Dualizing Complexes, Lemma 47.10.1. Next, we observe that
$$R\Gamma_J(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_{I+J}(M)^\wedge = R\Gamma_{I+J'}(M)^\wedge = R\Gamma_I(R\Gamma_{J'}(M))^\wedge = R\Gamma_{J'}(M)^\wedge$$
by Dualizing Complexes, Lemmas 47.9.6 and 47.12.1. This explains how we define the arrow in the statement of the lemma.

We claim that the hypotheses of Lemma 52.8.2 are implied by our current hypotheses on M. The only thing to verify is hypothesis (3). Thus let $\mathfrak p \not\in V(I)$, $\mathfrak p \in T$. Then $V(\mathfrak p) \cap V(I)$ is nonempty as I is contained in the Jacobson radical of A (Algebra, Lemma 10.96.6). Since $\mathfrak p \in T$ we have $V(\mathfrak p) \cap V(I) = V(\mathfrak p) \cap V(J) \cap V(I)$. Let $\mathfrak q \in V(\mathfrak p) \cap V(I)$ be the generic point of an irreducible component. We have $\text{cd}(A_\mathfrak q, I_\mathfrak q) \leq d$ by Local Cohomology, Lemma 51.4.6. We have $V(\mathfrak p A_\mathfrak q) \cap V(I_\mathfrak q) = \{\mathfrak q A_\mathfrak q\}$ by our choice of $\mathfrak q$ and we conclude $\dim((A/\mathfrak p)_\mathfrak q) \leq d$ by Local Cohomology, Lemma 51.4.10.

Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^i(R\Gamma_J(M)^\wedge)$ is zero for $i < 0$. However, this vanishing was established in Dualizing Complexes, Lemma 47.12.4. We will prove the lemma by induction for $s \geq 0$.

The lemma for $s = 0$ follows immediately from the conclusion of Lemma 52.8.2 and Dualizing Complexes, Lemma 47.12.5.

Assume $s > 0$ and the lemma has been shown for smaller values of s. Let $M' \subset M$ be the maximal submodule whose support is contained in $V(I) \cup T$. Then M' is a

[2]Our method forces this additional condition. We will return to this (insert future reference).
[3]For example if M satisfies Serre's condition (S_s) on the complement of $V(I) \cup T$.

finite A-module whose support is contained in $V(J') \cup V(I)$ for some ideal $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. We claim that

$$R\Gamma_{J'}(M') \to R\Gamma_J(M')^\wedge$$

is an isomorphism for any choice of J'. Namely, we can choose a short exact sequence $0 \to M_1 \oplus M_2 \to M' \to N \to 0$ with M_1 annihilated by a power of J', with M_2 annihilated by a power of I, and with N annihilated by a power of $I + J'$. Thus it suffices to show that the claim holds for M_1, M_2, and N. In the case of M_1 we see that $R\Gamma_{J'}(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. This proves the claim for M_1 by the initial remarks of the proof. In the case of M_2 we see that $H^i_J(M_2) = H^i_{I+J}(M) = H^i_{I+J'}(M) = H^i_{J'}(M_2)$ are annihilated by a power of I and hence derived complete. Thus the claim in this case also. For N we can use either of the arguments just given. Considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$ we see that it suffices to prove the lemma for M/M'. Thus we may assume $\text{Ass}(M) \cap (V(I) \cup T) = \emptyset$.

Let $\mathfrak p \in \text{Ass}(M)$ be such that $V(\mathfrak p) \cap V(J) \cap V(I) = \emptyset$. Since I is contained in the Jacobson radical of A this implies that $V(\mathfrak p) \cap V(J') = \emptyset$ for any $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Thus setting $N = H^0_{\mathfrak p}(M)$ we see that $R\Gamma_J(N) = R\Gamma_{J'}(N) = 0$ for all $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. In particular $R\Gamma_J(N)^\wedge = 0$. Thus we may replace M by M/N as this changes the structure of M only in primes which do not play a role in conditions (4) or (5). Repeating we may assume that $V(\mathfrak p) \cap V(J) \cap V(I) \neq \emptyset$ for all $\mathfrak p \in \text{Ass}(M)$.

Assume $\text{Ass}(M) \cap (V(I) \cup T) = \emptyset$ and that $V(\mathfrak p) \cap V(J) \cap V(I) \neq \emptyset$ for all $\mathfrak p \in \text{Ass}(M)$. Let $\mathfrak p \in \text{Ass}(M)$. We want to show that we may apply Lemma 52.8.1. It is in the verification of this that we will use the supplemental condition (5). Choose $\mathfrak p' \subset \mathfrak p$ and $\mathfrak q' \subset V(\mathfrak p) \cap V(J) \cap V(I)$.

(1) If $M_{\mathfrak p'} = 0$, then $\text{depth}(M_{\mathfrak p'}) = \infty$ and $\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) > d + s$.
(2) If $\text{depth}(M_{\mathfrak p'}) < s$, then $\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) > d + s$ by (4).

In the remaining cases we have $M_{\mathfrak p'} \neq 0$ and $\text{depth}(M_{\mathfrak p'}) \geq s$. In particular, we see that $\mathfrak p'$ is in the support of M and we can choose $\mathfrak p'' \subset \mathfrak p'$ with $\mathfrak p'' \in \text{Ass}(M)$.

(a) Observe that $\dim((A/\mathfrak p'')_{\mathfrak p'}) \geq \text{depth}(M_{\mathfrak p'})$ by Algebra, Lemma 10.72.9. If equality holds, then we have

$$\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) = \text{depth}(M_{\mathfrak p''}) + \dim((A/\mathfrak p'')_{\mathfrak q'}) > s + d$$

by (4) applied to $\mathfrak p''$ and we are done. This means we are only in trouble if $\dim((A/\mathfrak p'')_{\mathfrak p'}) > \text{depth}(M_{\mathfrak p'})$. This implies that $\dim(M_{\mathfrak p}) \geq s + 2$. Thus if (5)(a) holds, then this does not occur.

(b) If (5)(b) holds, then we get

$$\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) \geq s + \delta(\mathfrak p') - \delta(\mathfrak q') \geq s + 1 + \delta(\mathfrak p) - \delta_{max} > s + d$$

as desired.

(c) If (5)(c) holds, then we get

$$\begin{aligned}\mathrm{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) &\geq s + \delta(\mathfrak p') - \delta(\mathfrak q') \\ &\geq s + 1 + \delta(\mathfrak p) - \delta(\mathfrak q') \\ &= s + 1 + \delta(\mathfrak p) - \delta(\mathfrak q) + \delta(\mathfrak q) - \delta(\mathfrak q') \\ &> s + 1 + (s + d + \delta_{max} - \delta_{min} - 2) + \delta(\mathfrak q) - \delta(\mathfrak q') \\ &\geq 2s + d - 1 \geq s + d\end{aligned}$$

as desired. Observe that this argument works because we know that a prime $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$ exists.

Now we are ready to do the induction step.

Choose an ideal J_0 as in Lemma 52.8.2 and an integer $t > 0$ such that $(J_0I)^t$ annihilates $H^s_J(M)$. The assumptions of Lemma 52.8.1 are satisfied for every $\mathfrak p \in \mathrm{Ass}(M)$ (see previous paragraph). Thus the annihilator $\mathfrak a \subset A$ of $H^s(R\Gamma_J(M)^\wedge)$ is not contained in $\mathfrak p$ for $\mathfrak p \in \mathrm{Ass}(M)$. Thus we can find an $f \in \mathfrak a(J_0I)^t$ not in any associated prime of M which is an annihilator of both $H^s(R\Gamma_J(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{tikzcd} H^{s-1}_{J'}(M) \arrow[r] \arrow[d] & H^{s-1}_{J'}(M/fM) \arrow[r] \arrow[d] & H^s_{J'}(M) \arrow[r] \arrow[d] & 0 \\ H^{s-1}(R\Gamma_J(M)^\wedge) \arrow[r] & H^{s-1}(R\Gamma_J(M/fM)^\wedge) \arrow[r] & H^s(R\Gamma_J(M)^\wedge) \arrow[r] & 0 \end{tikzcd}$$

for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Thus if we choose J' such that it works for M and M/fM and $s - 1$ (possible by induction hypothesis – see next paragraph), then we conclude that the lemma is true.

To finish the proof we have to show that the module M/fM satisfies the hypotheses (4) and (5) for $s - 1$. Thus we let $\mathfrak p$ be a prime in the support of M/fM with $\mathrm{depth}((M/fM)_{\mathfrak p}) < s-1$ and with $V(\mathfrak p) \cap V(J) \cap V(I)$ nonempty. Then $\dim(M_{\mathfrak p}) = \dim((M/fM)_{\mathfrak p}) + 1$ and $\mathrm{depth}(M_{\mathfrak p}) = \mathrm{depth}((M/fM)_{\mathfrak p}) + 1$. In particular, we know (4) and (5) hold for $\mathfrak p$ and M with the original value s. The desired inequalities then follow by inspection. □

0EFM Example 52.8.6. In Lemma 52.8.5 we do not know that the inverse systems $H^i_J(M/I^nM)$ satisfy the Mittag-Leffler condition. For example, suppose that $A = \mathbf{Z}_p[[x, y]]$, $I = (p)$, $J = (p, x)$, and $M = A/(xy-p)$. Then the image of $H^0_J(M/p^nM) \to H^0_J(M/pM)$ is the ideal generated by y^n in $M/pM = A/(p, xy)$.

52.9. Algebraization of local cohomology, II

0EFP In this section we redo the arguments of Section 52.8 when $(A, \mathfrak m)$ is a local ring and we take local cohomology $R\Gamma_{\mathfrak m}$ with respect to $\mathfrak m$. As before our main tool is the following lemma.

0DXK Lemma 52.9.1. Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module and let $\mathfrak p \subset A$ be a prime. Let s and d be integers. Assume

(1) A has a dualizing complex,

(2) $\text{cd}(A, I) \leq d$, and
(3) $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$.

Then there exists an $f \in A \setminus \mathfrak{p}$ which annihilates $H^i(R\Gamma_\mathfrak{m}(M)^\wedge)$ for $i \leq s$ where $^\wedge$ indicates I-adic completion.

Proof. According to Local Cohomology, Lemma 51.9.4 the function

$$\mathfrak{p}' \longmapsto \text{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim(A/\mathfrak{p}')$$

is lower semi-continuous on $\text{Spec}(A)$. Thus the value of this function on $\mathfrak{p}' \subset \mathfrak{p}$ is $> s + d$. Thus our lemma is a special case of Lemma 52.8.1 provided that $\mathfrak{p} \neq \mathfrak{m}$. If $\mathfrak{p} = \mathfrak{m}$, then we have $H^i_\mathfrak{m}(M) = 0$ for $i \leq s + d$ by the relationship between depth and local cohomology (Dualizing Complexes, Lemma 47.11.1). Thus the argument given in the proof of Lemma 52.8.1 shows that $H^i(R\Gamma_\mathfrak{m}(M)^\wedge) = 0$ for $i \leq s$ in this (degenerate) case. □

0DXM Lemma 52.9.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume

(1) A has a dualizing complex,
(2) if $\mathfrak{p} \in V(I)$, then no condition,
(3) if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$, then $\dim(A/\mathfrak{p}) \leq d$,
(4) if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$$

Then there exists an ideal $J_0 \subset A$ with $V(J_0) \cap V(I) = \{\mathfrak{m}\}$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$ the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ and moreover these modules are annihilated by a power of $J_0 I$.

Proof. This is a special case of Lemma 52.8.2. □

0DXN Lemma 52.9.3. In Lemma 52.9.2 if instead of the empty condition (2) we assume

(2') if $\mathfrak{p} \in V(I)$ and $\mathfrak{p} \neq \mathfrak{m}$, then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > s$,

then the conditions also imply that $H^i_{J_0}(M)$ is a finite A-module for $i \leq s$.

Proof. This is a special case of Lemma 52.8.3. □

0EFQ Lemma 52.9.4. If in Lemma 52.9.2 we additionally assume

(6) if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$, then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$,

then $H^i_{J_0}(M) = H^i_J(M) = H^i_\mathfrak{m}(M)$ for $i \leq s$ and these modules are annihilated by a power of I.

Proof. This is a special case of Lemma 52.8.4. □

0DXP Lemma 52.9.5. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $\mathfrak{p} \in V(I)$, no condition,
(3) $\text{cd}(A, I) \leq d$,

(4) if $\mathfrak p \not\in V(I)$ and $V(\mathfrak p) \cap V(I) \neq \{\mathfrak m\}$ then

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > d + s$$

Then there exists an ideal $J_0 \subset A$ with $V(J_0) \cap V(I) = \{\mathfrak m\}$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak m\}$ the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge = R\Gamma_\mathfrak m(M)^\wedge$$

induces an isomorphism in cohomology in degrees $\leq s$. Here $^\wedge$ denotes derived I-adic completion.

Proof. This lemma is a special case of Lemma 52.8.5 since condition (5)(c) is implied by condition (4) as $\delta_{max} = \delta_{min} = \delta(\mathfrak m)$. We will give the proof of this important special case as it is somewhat easier (fewer things to check).

There is no difference between $R\Gamma_\mathfrak a$ and $R\Gamma_{V(\mathfrak a)}$ in our current situation, see Dualizing Complexes, Lemma 47.10.1. Next, we observe that

$$R\Gamma_\mathfrak m(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_J(M)^\wedge$$

by Dualizing Complexes, Lemmas 47.9.6 and 47.12.1 which explains the equality sign in the statement of the lemma.

Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^s(R\Gamma_\mathfrak m(M)^\wedge)$ is zero for negative s. However, this vanishing was established in Lemma 52.5.4. We will prove the lemma by induction for $s \geq 0$.

The assumptions of Lemma 52.9.2 are satisfied by Local Cohomology, Lemma 51.4.10. The lemma for $s = 0$ follows from Lemma 52.9.2 and Dualizing Complexes, Lemma 47.12.5.

Assume $s > 0$ and the lemma holds for smaller values of s. Let $M' \subset M$ be the submodule of elements whose support is condained in $V(I) \cup V(J)$ for some ideal J with $V(J) \cap V(I) = \{\mathfrak m\}$. Then M' is a finite A-module. We claim that

$$R\Gamma_J(M') \to R\Gamma_\mathfrak m(M')^\wedge$$

is an isomorphism for any choice of J. Namely, for any such module there is a short exact sequence $0 \to M_1 \oplus M_2 \to M' \to N \to 0$ with M_1 annihilated by a power of J, with M_2 annihilated by a power of I and with N annihilated by a power of $\mathfrak m$. In the case of M_1 we see that $R\Gamma_J(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. Thus the claim holds for M_1. In the case of M_2 we see that $H^i_J(M_2)$ is annihilated by a power of I and hence derived complete. Thus the claim for M_2. By the same arguments the claim holds for N and we conclude that the claim holds. Considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$ we see that it suffices to prove the lemma for M/M'. This we may assume $\mathfrak p \in \text{Ass}(M)$ implies $V(\mathfrak p) \cap V(I) \neq \{\mathfrak m\}$, i.e., $\mathfrak p$ is a prime as in (4).

Choose an ideal J_0 as in Lemma 52.9.2 and an integer $t > 0$ such that $(J_0 I)^t$ annihilates $H^s_J(M)$. Here J denotes an arbitrary ideal $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak m\}$. The assumptions of Lemma 52.9.1 are satisfied for every $\mathfrak p \in \text{Ass}(M)$ (see previous paragraph). Thus the annihilator $\mathfrak a \subset A$ of $H^s(R\Gamma_\mathfrak m(M)^\wedge)$ is not contained in $\mathfrak p$ for $\mathfrak p \in \text{Ass}(M)$. Thus we can find an $f \in \mathfrak a(J_0 I)^t$ not in any associated prime of M which is an annihilator of both $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{tikzcd} H^{s-1}_J(M) \arrow[r] \arrow[d] & H^{s-1}_J(M/fM) \arrow[r] \arrow[d] & H^s_J(M) \arrow[r] \arrow[d] & 0 \\ H^{s-1}(R\Gamma_\mathfrak{m}(M)^\wedge) \arrow[r] & H^{s-1}(R\Gamma_\mathfrak{m}(M/fM)^\wedge) \arrow[r] & H^s(R\Gamma_\mathfrak{m}(M)^\wedge) \arrow[r] & 0 \end{tikzcd}$$

for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Thus if we choose J such that it works for M and M/fM and $s-1$ (possible by induction hypothesis), then we conclude that the lemma is true. $\square$

52.10. Algebraization of local cohomology, III

0EFT In this section we bootstrap the material in Sections 52.8 and 52.9 to give a stronger result the following situation.

0EFU **Situation 52.10.1.** Here A is a Noetherian ring. We have an ideal $I \subset A$, a finite A-module M, and a subset $T \subset V(I)$ stable under specialization. We have integers s and d. We assume

(1) A has a dualizing complex,
(3) $\text{cd}(A, I) \leq d$,
(4) given primes $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{p} \not\in V(I)$, $\mathfrak{r} \in V(I) \setminus T$, $\mathfrak{q} \in T$ we have
$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$
(6) given $\mathfrak{q} \in T$ denoting $A', \mathfrak{m}', I', M'$ are the usual I-adic completions of $A_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, I_\mathfrak{q}, M_\mathfrak{q}$ we have
$$\text{depth}(M'_{\mathfrak{p}'}) > s$$
for all $\mathfrak{p}' \in \text{Spec}(A') \setminus V(I')$ with $V(\mathfrak{p}') \cap V(I') = \{\mathfrak{m}'\}$.

The following lemma explains why in Situation 52.10.1 it suffices to look at triples $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ of primes in (4) even though the actual assumption only involves $\mathfrak{p}$ and $\mathfrak{q}$.

0EID **Lemma 52.10.2.** *In Situation 52.10.1 let $\mathfrak{p} \subset \mathfrak{q}$ be primes of A with $\mathfrak{p} \not\in V(I)$ and $\mathfrak{q} \in T$. If there does not exist an $\mathfrak{r} \in V(I) \setminus T$ with $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ then $\text{depth}(M_\mathfrak{p}) > s$.*

Proof. Choose $\mathfrak{q}' \in T$ with $\mathfrak{p} \subset \mathfrak{q}' \subset \mathfrak{q}$ such that there is no prime in T strictly in between $\mathfrak{p}$ and $\mathfrak{q}'$. To prove the lemma we may and do replace $\mathfrak{q}$ by $\mathfrak{q}'$. Next, let $\mathfrak{p}' \subset A_\mathfrak{q}$ be the prime corresponding to $\mathfrak{p}$. After doing this we obtain that $V(\mathfrak{p}') \cap V(IA_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ because of the nonexistence of a prime $\mathfrak{r}$ as in the lemma. Let $A', I', \mathfrak{m}', M'$ be the I-adic completions of $A_\mathfrak{q}, I_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}$. Since $A_\mathfrak{q} \to A'$ is faithfully flat (Algebra, Lemma 10.97.3) we can choose $\mathfrak{p}'' \subset A'$ lying over $\mathfrak{p}'$ with $\dim(A'_{\mathfrak{p}''}/\mathfrak{p}'A'_{\mathfrak{p}''}) = 0$. Then we see that

$$\text{depth}(M'_{\mathfrak{p}''}) = \text{depth}((M_\mathfrak{q} \otimes_{A_\mathfrak{q}} A')_{\mathfrak{p}''}) = \text{depth}(M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A'_{\mathfrak{p}''}) = \text{depth}(M_\mathfrak{p})$$

by flatness of $A \to A'$ and our choice of $\mathfrak{p}''$, see Algebra, Lemma 10.163.1. Since $\mathfrak{p}''$ lies over $\mathfrak{p}'$ we have $V(\mathfrak{p}'') \cap V(I') = \{\mathfrak{m}'\}$. Thus condition (6) in Situation 52.10.1 implies $\text{depth}(M'_{\mathfrak{p}''}) > s$ which finishes the proof. $\square$

The following tedious lemma explains the relationships between various collections of conditions one might impose.

0EFV Lemma 52.10.3. In Situation 52.10.1 we have

(E) if $T' \subset T$ is a smaller specialization stable subset, then A, I, T', M satisfies the assumptions of Situation 52.10.1,
(F) if $S \subset A$ is a multiplicative subset, then $S^{-1}A, S^{-1}I, T', S^{-1}M$ satisfies the assumptions of Situation 52.10.1 where $T' \subset V(S^{-1}I)$ is the inverse image of T,
(G) the quadruple A', I', T', M' satisfies the assumptions of Situation 52.10.1 where A', I', M' are the usual I-adic completions of A, I, M and $T' \subset V(I')$ is the inverse image of T.

Let $I \subset \mathfrak{a} \subset A$ be an ideal such that $V(\mathfrak{a}) \subset T$. Then

(A) if I is contained in the Jacobson radical of A, then all hypotheses of Lemmas 52.8.2 and 52.8.4 are satisfied for $A, I, \mathfrak{a}, M$,
(B) if A is complete with respect to I, then all hypotheses except for possibly (5) of Lemma 52.8.5 are satisfied for $A, I, \mathfrak{a}, M$,
(C) if A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$, then all hypotheses of Lemmas 52.9.2 and 52.9.4 hold for $A, \mathfrak{m}, I, M$,
(D) if A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$ and I-adically complete, then all hypotheses of Lemma 52.9.5 hold for $A, \mathfrak{m}, I, M$,

Proof. Proof of (E). We have to prove assumptions (1), (3), (4), (6) of Situation 52.10.1 hold for A, I, T, M. Shrinking T to T' weakens assumption (6) and strengthens assumption (4). However, if we have $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{p} \not\in V(I)$, $\mathfrak{r} \in V(I) \setminus T'$, $\mathfrak{q} \in T'$ as in assumption (4) for A, I, T', M, then either we can pick $\mathfrak{r} \in V(I) \setminus T$ and condition (4) for A, I, T, M kicks in or we cannot find such an $\mathfrak{r}$ in which case we get $\text{depth}(M_\mathfrak{p}) > s$ by Lemma 52.10.2. This proves (4) holds for A, I, T', M as desired.

Proof of (F). This is straightforward and we omit the details.

Proof of (G). We have to prove assumptions (1), (3), (4), (6) of Situation 52.10.1 hold for the I-adic completions A', I', T', M'. Please keep in mind that $\text{Spec}(A') \to \text{Spec}(A)$ induces an isomorphism $V(I') \to V(I)$.

Assumption (1): The ring A' has a dualizing complex, see Dualizing Complexes, Lemma 47.22.4.

Assumption (3): Since $I' = IA'$ this follows from Local Cohomology, Lemma 51.4.5.

Assumption (4): If we have primes $\mathfrak{p}' \subset \mathfrak{r}' \subset \mathfrak{q}'$ in A' with $\mathfrak{p}' \not\in V(I')$, $\mathfrak{r}' \in V(I') \setminus T'$, $\mathfrak{q}' \in T'$ then their images $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ in the spectrum of A satisfy $\mathfrak{p} \not\in V(I)$, $\mathfrak{r} \in V(I) \setminus T$, $\mathfrak{q} \in T$. Then we have

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

by assumption (4) for A, I, T, M. We have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p})$ and $\text{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$ by Local Cohomology, Lemma 51.11.3. Thus assumption (4) holds for A', I', T', M'.

Assumption (6): Let $\mathfrak{q}' \in T'$ lying over the prime $\mathfrak{q} \in T$. Then $A'_{\mathfrak{q}'}$ and $A_\mathfrak{q}$ have isomorphic I-adic completions and similarly for $M_\mathfrak{q}$ and $M'_{\mathfrak{q}'}$. Thus assumption (6) for A', I', T', M' is equivalent to assumption (6) for A, I, T, M.

Proof of (A). We have to check conditions (1), (2), (3), (4), and (6) of Lemmas 52.8.2 and 52.8.4 for $(A, I, \mathfrak{a}, M)$. Warning: the set T in the statement of these lemmas is not the same as the set T above.

Condition (1): This holds because we have assumed A has a dualizing complex in Situation 52.10.1.

Condition (2): This is empty.

Condition (3): Let $\mathfrak{p} \subset A$ with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. Since I is contained in the Jacobson radical of A we see that $V(\mathfrak{p}) \cap V(I) \neq \emptyset$. Let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I)$ be a generic point. Since $\text{cd}(A_\mathfrak{q}, I_\mathfrak{q}) \leq d$ (Local Cohomology, Lemma 51.4.6) and since $V(\mathfrak{p}A_\mathfrak{q}) \cap V(I_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ we get $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq d$ by Local Cohomology, Lemma 51.4.10 which proves (3).

Condition (4): Suppose $\mathfrak{p} \not\in V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. It suffices to show

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

If there exists a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$, then this follows immediately from assumption (4) in Situation 52.10.1. If not, then $\text{depth}(M_\mathfrak{p}) > s$ by Lemma 52.10.2.

Condition (6): Let $\mathfrak{p} \not\in V(I)$ with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. Since I is contained in the Jacobson radical of A we see that $V(\mathfrak{p}) \cap V(I) \neq \emptyset$. Choose $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. It is clear there does not exist a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$. By Lemma 52.10.2 we have $\text{depth}(M_\mathfrak{p}) > s$ which proves (6).

Proof of (B). We have to check conditions (1), (2), (3), (4) of Lemma 52.8.5. Warning: the set T in the statement of this lemma is not the same as the set T above.

Condition (1): This holds because A is complete and has a dualizing complex.

Condition (2): This is empty.

Condition (3): This is the same as assumption (3) in Situation 52.10.1.

Condition (4): This is the same as assumption (4) in Lemma 52.8.2 which we proved in (A).

Proof of (C). This is true because the assumptions in Lemmas 52.9.2 and 52.9.4 are the same as the assumptions in Lemmas 52.8.2 and 52.8.4 in the local case and we proved these hold in (A).

Proof of (D). This is true because the assumptions in Lemma 52.9.5 are the same as the assumptions (1), (2), (3), (4) in Lemma 52.8.5 and we proved these hold in (B). □

0EFR Lemma 52.10.4. In Situation 52.10.1 assume A is local with maximal ideal $\mathfrak{m}$ and $T = \{\mathfrak{m}\}$. Then $H^i_\mathfrak{m}(M) \to \lim H^i_\mathfrak{m}(M/I^nM)$ is an isomorphism for $i \leq s$ and these modules are annihilated by a power of I.

Proof. Let $A', I', \mathfrak{m}', M'$ be the usual I-adic completions of $A, I, \mathfrak{m}, M$. Recall that we have $H^i_\mathfrak{m}(M) \otimes_A A' = H^i_{\mathfrak{m}'}(M')$ by flatness of $A \to A'$ and Dualizing Complexes, Lemma 47.9.3. Since $H^i_\mathfrak{m}(M)$ is $\mathfrak{m}$-power torsion we have $H^i_\mathfrak{m}(M) = H^i_\mathfrak{m}(M) \otimes_A A'$, see More on Algebra, Lemma 15.89.3. We conclude that $H^i_\mathfrak{m}(M) = H^i_{\mathfrak{m}'}(M')$. The exact same arguments will show that $H^i_\mathfrak{m}(M/I^nM) = H^i_{\mathfrak{m}'}(M'/(I')^nM')$ for all n and i.

Lemmas 52.9.5, 52.9.2, and 52.9.4 apply to $A', \mathfrak{m}', I', M'$ by Lemma 52.10.3 parts (C) and (D). Thus we get an isomorphism

$$H^i_{\mathfrak{m}'}(M') \longrightarrow H^i(R\Gamma_{\mathfrak{m}'}(M')^\wedge)$$

for $i \leq s$ where $^\wedge$ is derived I'-adic completion and these modules are annihilated by a power of I'. By Lemma 52.5.4 we obtain isomorphisms

$$H^i_{\mathfrak{m}'}(M') \longrightarrow \lim H^i_{\mathfrak{m}'}(M'/(I')^nM'))$$

for $i \leq s$. Combined with the already established comparison with local cohomology over A we conclude the lemma is true. $\square$

0EFW Lemma 52.10.5. Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let M be a finite A-module. Let s and d be integers. If we assume

(a) A has a dualizing complex,
(b) $\text{cd}(A, I) \leq d$,
(c) if $\mathfrak{p} \notin V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$ or $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$.

Then $A, I, V(\mathfrak{a}), M, s, d$ are as in Situation 52.10.1.

Proof. We have to show that assumptions (1), (3), (4), and (6) of Situation 52.10.1 hold. It is clear that (a) $\Rightarrow$ (1), (b) $\Rightarrow$ (3), and (c) $\Rightarrow$ (4). To finish the proof in the next paragraph we show (6) holds.

Let $\mathfrak{q} \in V(\mathfrak{a})$. Denote $A', I', \mathfrak{m}', M'$ the I-adic completions of $A_\mathfrak{q}, I_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}$. Let $\mathfrak{p}' \subset A'$ be a nonmaximal prime with $V(\mathfrak{p}') \cap V(I') = \{\mathfrak{m}'\}$. Observe that this implies $\dim(A'/\mathfrak{p}') \leq d$ by Local Cohomology, Lemma 51.4.10. Denote $\mathfrak{p} \subset A$ the image of $\mathfrak{p}'$. We have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p})$ and $\text{depth}(M'_{\mathfrak{p}'}) + \dim(A'/\mathfrak{p}') = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$ by Local Cohomology, Lemma 51.11.3. By assumption (c) either we have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p}) > s$ and we're done or we have $\text{depth}(M'_{\mathfrak{p}'}) + \dim(A'/\mathfrak{p}') > s + d$ which implies $\text{depth}(M'_{\mathfrak{p}'}) > s$ because of the already shown inequality $\dim(A'/\mathfrak{p}') \leq d$. In both cases we obtain what we want. $\square$

0EFX Lemma 52.10.6. In Situation 52.10.1 the inverse systems $\{H^i_T(I^nM)\}_{n\geq 0}$ are pro-zero for $i \leq s$. Moreover, there exists an integer m_0 such that for all $m \geq m_0$ there exists an integer $m'(m) \geq m$ such that for $k \geq m'(m)$ the image of $H^{s+1}_T(I^kM) \to H^{s+1}_T(I^mM)$ maps injectively to $H^{s+1}_T(I^{m_0}M)$.

Proof. Fix m. Let $\mathfrak{q} \in T$. By Lemmas 52.10.3 and 52.10.4 we see that

$$H^i_\mathfrak{q}(M_\mathfrak{q}) \longrightarrow \lim H^i_\mathfrak{q}(M_\mathfrak{q}/I^nM_\mathfrak{q})$$

is an isomorphism for $i \leq s$. The inverse systems $\{H^i_\mathfrak{q}(I^nM_\mathfrak{q})\}_{n\geq 0}$ and $\{H^i_\mathfrak{q}(M/I^nM)\}_{n\geq 0}$ satisfy the Mittag-Leffler condition for all i, see Lemma 52.5.2. Thus looking at the inverse system of long exact sequences

$$0 \to H^0_\mathfrak{q}(I^nM_\mathfrak{q}) \to H^0_\mathfrak{q}(M_\mathfrak{q}) \to H^0_\mathfrak{q}(M_\mathfrak{q}/I^nM_\mathfrak{q}) \to H^1_\mathfrak{q}(I^nM_\mathfrak{q}) \to H^1_\mathfrak{q}(M_\mathfrak{q}) \to \ldots$$

we conclude (some details omitted) that there exists an integer $m'(m, \mathfrak{q}) \geq m$ such that for all $k \geq m'(m, \mathfrak{q})$ the map $H^i_\mathfrak{q}(I^kM_\mathfrak{q}) \to H^i_\mathfrak{q}(I^mM_\mathfrak{q})$ is zero for $i \leq s$ and the image of $H^{s+1}_\mathfrak{q}(I^kM_\mathfrak{q}) \to H^{s+1}_\mathfrak{q}(I^mM_\mathfrak{q})$ is independent of $k \geq m'(m, \mathfrak{q})$ and maps injectively into $H^{s+1}_\mathfrak{q}(M_\mathfrak{q})$.

Suppose we can show that $m'(m, \mathfrak{q})$ can be chosen independently of $\mathfrak{q} \in T$. Then the lemma follows immediately from Local Cohomology, Lemmas 51.6.2 and 51.6.3.

Let $\omega_A^\bullet$ be a dualizing complex. Let $\delta : \text{Spec}(A) \to \mathbf{Z}$ be the corresponding dimension function. Recall that δ attains only a finite number of values, see Dualizing Complexes, Lemma 47.17.4. Claim: for each $d \in \mathbf{Z}$ the integer $m'(m, \mathfrak q)$ can be chosen independently of $\mathfrak q \in T$ with $\delta(\mathfrak q) = d$. Clearly the claim implies the lemma by what we said above.

Pick $\mathfrak q \in T$ with $\delta(\mathfrak q) = d$. Consider the ext modules

$$E(n, j) = \text{Ext}^j_A(I^n M, \omega_A^\bullet)$$

A key feature we will use is that these are finite A-modules. Recall that $(\omega_A^\bullet)_\mathfrak q[-d]$ is a normalized dualizing complex for $A_\mathfrak q$ by definition of the dimension function associated to a dualizing complex, see Dualizing Complexes, Section 47.17. The local duality theorem (Dualizing Complexes, Lemma 47.18.4) tells us that the $\mathfrak q A_\mathfrak q$-adic completion of $E(n, -d-i)_\mathfrak q$ is Matlis dual to $H^i_\mathfrak q(I^n M_\mathfrak q)$. Thus the choice of $m'(m, \mathfrak q)$ for $i \leq s$ in the first paragraph tells us that for $k \geq m'(m, \mathfrak q)$ and $j \geq -d-s$ the map

$$E(m, j)_\mathfrak q \to E(k, j)_\mathfrak q$$

is zero. Since these modules are finite and nonzero only for a finite number of possible j (small detail omitted), we can find an open neighbourhood $W \subset \text{Spec}(A)$ of $\mathfrak q$ such that

$$E(m, j)_{\mathfrak q'} \to E(m'(m, \mathfrak q), j)_{\mathfrak q'}$$

is zero for $j \geq -d-s$ for all $\mathfrak q' \in W$. Then of course the maps $E(m, j)_{\mathfrak q'} \to E(k, j)_{\mathfrak q'}$ for $k \geq m'(m, \mathfrak q)$ are zero as well.

For $i = s+1$ corresponding to $j = -d-s-1$ we obtain from local duality and the results of the first paragraph that

$$K_{k,\mathfrak q} = \text{Ker}(E(m, -d-s-1)_\mathfrak q \to E(k, -d-s-1)_\mathfrak q)$$

is independent of $k \geq m'(m, \mathfrak q)$ and that

$$E(0, -d-s-1)_\mathfrak q \to E(m, -d-s-1)_\mathfrak q / K_{m'(m,\mathfrak q),\mathfrak q}$$

is surjective. For $k \geq m'(m, \mathfrak q)$ set

$$K_k = \text{Ker}(E(m, -d-s-1) \to E(k, -d-s-1))$$

Since K_k is an increasing sequence of submodules of the finite module $E(m, -d-s-1)$ we see that, at the cost of increasing $m'(m, \mathfrak q)$ a little bit, we may assume $K_{m'(m,\mathfrak q)} = K_k$ for $k \geq m'(m, \mathfrak q)$. After shrinking W further if necessary, we may also assume that

$$E(0, -d-s-1)_{\mathfrak q'} \to E(m, -d-s-1)_{\mathfrak q'} / K_{m'(m,\mathfrak q),\mathfrak q'}$$

is surjective for all $\mathfrak q' \in W$ (as before use that these modules are finite and that the map is surjective after localization at $\mathfrak q$).

Any subset, in particular $T_d = \{\mathfrak q \in T \text{ with } \delta(\mathfrak q) = d\}$, of the Noetherian topological space $\text{Spec}(A)$ with the endowed topology is Noetherian and hence quasi-compact. Above we have seen that for every $\mathfrak q \in T_d$ there is an open neighbourhood W where $m'(m, \mathfrak q)$ works for all $\mathfrak q' \in T_d \cap W$. We conclude that we can find an integer $m'(m, d)$ such that for all $\mathfrak q \in T_d$ we have

$$E(m, j)_\mathfrak q \to E(m'(m, d), j)_\mathfrak q$$

is zero for $j \geq -d-s$ and with $K_{m'(m,d)} = \text{Ker}(E(m,-d-s-1) \to E(m'(m,d),-d-s-1))$ we have

$$K_{m'(m,d),\mathfrak q} = \text{Ker}(E(m,-d-s-1)_\mathfrak q \to E(k,-d-s-1)_\mathfrak q)$$

for all $k \geq m'(m,d)$ and the map

$$E(0,-d-s-1)_\mathfrak q \to E(m,-d-s-1)_\mathfrak q/K_{m'(m,d),\mathfrak q}$$

is surjective. Using the local duality theorem again (in the opposite direction) we conclude that the claim is correct. This finishes the proof. $\square$

0EFY Lemma 52.10.7. In Situation 52.10.1 there exists an integer $m_0 \geq 0$ such that

(1) $\{H^i_T(M/I^nM)\}_{n\geq 0}$ satisfies the Mittag-Leffler condition for $i < s$.
(2) $\{H^i_T(I^{m_0}M/I^nM)\}_{n\geq m_0}$ satisfies the Mittag-Leffler condition for $i \leq s$,
(3) $H^i_T(M) \to \lim H^i_T(M/I^nM)$ is an isomorphism for $i < s$,
(4) $H^s_T(I^{m_0}M) \to \lim H^s_T(I^{m_0}M/I^nM)$ is an isomorphism for $i \leq s$,
(5) $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is injective with cokernel killed by I^{m_0}, and
(6) $R^1\lim H^s_T(M/I^nM)$ is killed by I^{m_0}.

Proof. Consider the long exact sequences

$$0 \to H^0_T(I^nM) \to H^0_T(M) \to H^0_T(M/I^nM) \to H^1_T(I^nM) \to H^1_T(M) \to \ldots$$

Parts (1) and (3) follows easily from this and Lemma 52.10.6.

Let m_0 and $m'(-)$ be as in Lemma 52.10.6. For $m \geq m_0$ consider the long exact sequence

$$H^s_T(I^mM) \to H^s_T(I^{m_0}M) \to H^s_T(I^{m_0}M/I^mM) \to H^{s+1}_T(I^mM) \to H^1_T(I^{m_0}M)$$

Then for $k \geq m'(m)$ the image of $H^{s+1}_T(I^kM) \to H^{s+1}_T(I^mM)$ maps injectively to $H^{s+1}_T(I^{m_0}M)$. Hence the image of $H^s_T(I^{m_0}M/I^kM) \to H^s_T(I^{m_0}M/I^mM)$ maps to zero in $H^{s+1}_T(I^mM)$ for all $k \geq m'(m)$. We conclude that (2) and (4) hold.

Consider the short exact sequences $0 \to I^{m_0}M \to M \to M/I^{m_0}M \to 0$ and $0 \to I^{m_0}M/I^nM \to M/I^nM \to M/I^{m_0}M \to 0$. We obtain a diagram

$$\begin{tikzcd}
H^{s-1}_T(M/I^{m_0}M) \arrow[r] & \lim H^s_T(I^{m_0}M/I^nM) \arrow[r] & \lim H^s_T(M/I^nM) \arrow[r] & H^s_T(M/I^{m_0}M) \\
H^{s-1}_T(M/I^{m_0}M) \arrow[r] \arrow[u, equal] & H^s_T(I^{m_0}M) \arrow[r] \arrow[u, "\cong"'] & H^s_T(M) \arrow[r] \arrow[u] & H^s_T(M/I^{m_0}M) \arrow[u, equal]
\end{tikzcd}$$

whose lower row is exact. The top row is also exact (at the middle two spots) by Homology, Lemma 12.31.4. Part (5) follows.

Write $B_n = H^s_T(M/I^nM)$. Let $A_n \subset B_n$ be the image of $H^s_T(I^{m_0}M/I^nM) \to H^s_T(M/I^nM)$. Then (A_n) satisfies the Mittag-Leffler condition by (2) and Homology, Lemma 12.31.3. Also $C_n = B_n/A_n$ is killed by I^{m_0}. Thus $R^1\lim B_n \cong R^1\lim C_n$ is killed by I^{m_0} and we get (6). $\square$

0EIE Theorem 52.10.8. In Situation 52.10.1 the inverse system $\{H^i_T(M/I^nM)\}_{n\geq 0}$ satisfies the Mittag-Leffler condition for $i \leq s$, the map

$$H^i_T(M) \longrightarrow \lim H^i_T(M/I^nM)$$

is an isomorphism for $i \leq s$, and $H^i_T(M)$ is annihilated by a power of I for $i \leq s$.

Proof. To prove the final assertion of the theorem we apply Local Cohomology, Proposition 51.10.1 with $T \subset V(I) \subset \mathrm{Spec}(A)$. Namely, suppose that $\mathfrak p \not\in V(I)$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$. Then either there exists a prime $\mathfrak p \subset \mathfrak r \subset \mathfrak q$ with $\mathfrak r \in V(I) \setminus T$ and we get

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > d + s$$

by (4) in Situation 52.10.1 or there does not exist an $\mathfrak r$ and we get $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) > s$ by Lemma 52.10.2. In all three cases we see that $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$. Thus Local Cohomology, Proposition 51.10.1 (2) holds and we find that a power of I annihilates $H^i_T(M)$ for $i \leq s$.

We already know the other two assertions of the theorem hold for $i < s$ by Lemma 52.10.7 and for the module $I^{m_0}M$ for $i = s$ and m_0 large enough. To finish of the proof we will show that in fact these assertions for $i = s$ holds for M.

Let $M' = H^0_I(M)$ and $M'' = M/M'$ so that we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

and M'' has $H^0_I(M') = 0$ by Dualizing Complexes, Lemma 47.11.6. By Artin-Rees (Algebra, Lemma 10.51.2) we get short exact sequences

$$0 \to M' \to M/I^nM \to M''/I^nM'' \to 0$$

for n large enough. Consider the long exact sequences

$$H^s_T(M') \to H^s_T(M/I^nM) \to H^s_T(M''/I^nM'') \to H^{s+1}_T(M')$$

Now it is a simple matter to see that if we have Mittag-Leffler for the inverse system $\{H^s_T(M''/I^nM'')\}_{n \geq 0}$ then we have Mittag-Leffler for the inverse system $\{H^s_T(M/I^nM)\}_{n\geq 0}$. (Note that the ML condition for an inverse system of groups G_n only depends on the values of the inverse system for sufficiently large n.) Moreover the sequence

$$H^s_T(M') \to \lim H^s_T(M/I^nM) \to \lim H^s_T(M''/I^nM'') \to H^{s+1}_T(M')$$

is exact because we have ML in the required spots, see Homology, Lemma 12.31.4. Hence, if $H^s_T(M'') \to \lim H^s_T(M''/I^nM'')$ is an isomorphism, then $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is an isomorphism too by the five lemma (Homology, Lemma 12.5.20). This reduces us to the case discussed in the next paragraph.

Assume that $H^0_I(M) = 0$. Choose generators $f_1, \ldots, f_r$ of I^{m_0} where m_0 is the integer found for M in Lemma 52.10.7. Then we consider the exact sequence

$$0 \to M \xrightarrow{f_1, \ldots, f_r} (I^{m_0}M)^{\oplus r} \to Q \to 0$$

defining Q. Some observations: the first map is injective exactly because $H^0_I(M) = 0$. The cokernel Q of this injection is a finite A-module such that for every $1 \leq j \leq r$ we have $Q_{f_j} \cong (M_{f_j})^{\oplus r-1}$. In particular, for a prime $\mathfrak p \subset A$ with $\mathfrak p \not\in V(I)$ we have $Q_\mathfrak p \cong (M_\mathfrak p)^{\oplus r-1}$. Similarly, given $\mathfrak q \in T$ and $\mathfrak p' \subset A' = (A_\mathfrak q)^\wedge$ not contained in $V(IA')$, we have $Q'_{\mathfrak p'} \cong (M'_{\mathfrak p'})^{\oplus r-1}$ where $Q' = (Q_\mathfrak q)^\wedge$ and $M' = (M_\mathfrak q)^\wedge$. Thus the conditions in Situation 52.10.1 hold for A, I, T, Q. (Observe that Q may have nonvanishing $H^0_I(Q)$ but this won't matter.)

For any $n \geq 0$ we set $F^nM = M \cap I^n(I^{m_0}M)^{\oplus r}$ so that we get short exact sequences

$$0 \to F^nM \to I^n(I^{m_0}M)^{\oplus r} \to I^nQ \to 0$$

By Artin-Rees (Algebra, Lemma 10.51.2) there exists a $c \geq 0$ such that $I^n M \subset F^n M \subset I^{n-c} M$ for all $n \geq c$. Let m_0 be the integer and let $m'(m)$ be the function defined for $m \geq m_0$ found in Lemma 52.10.6 applied to M. Note that the integer m_0 is the same as our integer m_0 chosen above (you don't need to check this: you can just take the maximum of the two integers if you like). Finally, by Lemma 52.10.6 applied to Q for every integer m there exists an integer $m''(m) \geq m$ such that $H^s_T(I^k Q) \to H^s_T(I^m Q)$ is zero for all $k \geq m''(m)$.

Fix $m \geq m_0$. Choose $k \geq m'(m''(m+c))$. Choose $\xi \in H^{s+1}_T(I^k M)$ which maps to zero in $H^{s+1}_T(M)$. We want to show that ξ maps to zero in $H^{s+1}_T(I^m M)$. Namely, this will show that $\{H^s_T(M/I^n M)\}_{n \geq 0}$ is Mittag-Leffler exactly as in the proof of Lemma 52.10.7. Picture to help vizualize the argument:

$$\begin{tikzcd}
 & H^{s+1}_T(I^k M) \arrow[r] \arrow[d] & H^{s+1}_T(I^k (I^{m_0} M)^{\oplus r}) \arrow[d] \\
H^s_T(I^{m''(m+c)} Q) \arrow[r, "\delta"'] \arrow[d] & H^{s+1}_T(F^{m''(m+c)} M) \arrow[r] \arrow[d] & H^{s+1}_T(I^{m''(m+c)} (I^{m_0} M)^{\oplus r}) \\
H^s_T(I^{m+c} Q) \arrow[r] & H^{s+1}_T(F^{m+c} M) \arrow[d] & \\
 & H^{s+1}_T(I^m M) &
\end{tikzcd}$$

The image of ξ in $H^{s+1}_T(I^k (I^{m_0} M)^{\oplus r})$ maps to zero in $H^{s+1}_T((I^{m_0} M)^{\oplus r})$ and hence maps to zero in $H^{s+1}_T(I^{m''(m+c)} (I^{m_0} M)^{\oplus r})$ by choice of $m'(-)$. Thus the image $\xi' \in H^{s+1}_T(F^{m''(m+c)} M)$ maps to zero in $H^{s+1}_T(I^{m''(m+c)} (I^{m_0} M)^{\oplus r})$ and hence $\xi' = \delta(\eta)$ for some $\eta \in H^s_T(I^{m''(m+c)} Q)$. By our choice of $m''(-)$ we find that η maps to zero in $H^s_T(I^{m+c} Q)$. This in turn means that ξ' maps to zero in $H^{s+1}_T(F^{m+c} M)$. Since $F^{m+c} M \subset I^m M$ we conclude.

Finally, we prove the statement on limits. Consider the short exact sequences

$$0 \to M/F^n M \to (I^{m_0} M)^{\oplus r}/I^n (I^{m_0} M)^{\oplus r} \to Q/I^n Q \to 0$$

We have $\lim H^s_T(M/I^n M) = \lim H^s_T(M/F^n M)$ as these inverse systems are pro-isomorphic. We obtain a commutative diagram

$$\begin{tikzcd}
H^{s-1}_T(Q) \arrow[r] \arrow[d] & \lim H^{s-1}_T(Q/I^n Q) \arrow[d] \\
H^s_T(M) \arrow[r] \arrow[d] & \lim H^s_T(M/I^n M) \arrow[d] \\
H^s_T((I^{m_0} M)^{\oplus r}) \arrow[r] \arrow[d] & \lim H^s_T((I^{m_0} M)^{\oplus r}/I^n (I^{m_0} M)^{\oplus r}) \arrow[d] \\
H^s_T(Q) \arrow[r] & \lim H^s_T(Q/I^n Q)
\end{tikzcd}$$

The right column is exact because we have ML in the required spots, see Homology, Lemma 12.31.4. The lowest horizontal arrow is injective (!) by part (5) of Lemma 52.10.7. The horizontal arrow above it is bijective by part (4) of Lemma 52.10.7. The arrows in cohomological degrees $\leq s-1$ are isomorphisms. Thus we conclude $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is an isomorphism by the five lemma (Homology, Lemma 12.5.20). This finishes the proof of the theorem. □

0EG0 Lemma 52.10.9. Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A and let M be a finite A-module. Let s and d be integers. Suppose that

(1) $A, I, V(\mathfrak{a}), M$ satisfy the assumptions of Situation 52.10.1 for s and d, and
(2) $A, I, \mathfrak{a}, M$ satisfy the conditions of Lemma 52.8.5 for $s+1$ and d with $J = \mathfrak{a}$.

Then there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the map

$$H^{s+1}_J(M) \longrightarrow \lim H^{s+1}_{\mathfrak{a}}(M/I^nM)$$

is an isomorphism.

Proof. Namely, we have the existence of J_0 and the isomorphism $H^{s+1}_J(M) = H^{s+1}(R\Gamma_{\mathfrak{a}}(M)^\wedge)$ by Lemma 52.8.5, we have a short exact sequence

$$0 \to R^1 \lim H^s_{\mathfrak{a}}(M/I^nM) \to H^{s+1}(R\Gamma_{\mathfrak{a}}(M)^\wedge) \to \lim H^{s+1}_{\mathfrak{a}}(M/I^nM) \to 0$$

by Dualizing Complexes, Lemma 47.12.4, and the module $R^1 \lim H^s_{\mathfrak{a}}(M/I^nM)$ is zero because $\{H^s_{\mathfrak{a}}(M/I^nM)\}_{n\geq 0}$ has Mittag-Leffler by Theorem 52.10.8. □

52.11. Algebraization of formal sections, I

0DXH In this section we study the problem of algebraization of formal sections in the local case. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let

$$X = \mathrm{Spec}(A) \supset U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$$

and denote $Y = V(I)$ the closed subscheme corresponding to I. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. In this section we consider the limits

$$\lim_n H^i(U, \mathcal{F}/I^n\mathcal{F})$$

This is closely related to the cohomology of the pullback of $\mathcal{F}$ to the formal completion of U along Y; however, since we have not yet introduced formal schemes, we cannot use this terminology here.

0DXI Lemma 52.11.1. Let U be the punctured spectrum of a Noetherian local ring A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $I \subset A$ be an ideal. Then

$$H^i(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

for all i where $R\Gamma(U, \mathcal{F})^\wedge$ denotes the derived I-adic completion.

Proof. By Lemmas 52.6.20 and 52.7.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

Thus we obtain short exact sequences

$$0 \to R^1 \lim H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) \to H^i(R\Gamma(U, \mathcal{F})^\wedge) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F}) \to 0$$

by Cohomology, Lemma 20.37.1. The $R^1 \lim$ terms vanish because the inverse systems of groups $H^i(U, \mathcal{F}/I^n\mathcal{F})$ satisfy the Mittag-Leffler condition by Lemma 52.5.2. $\square$

0DXQ Theorem 52.11.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume

(1) $\mathrm{cd}(A, I) \leq d$, i.e., $H^i(X \setminus Y, \mathcal{G}) = 0$ for $i \geq d$ and quasi-coherent $\mathcal{G}$ on X,
(2) for any $x \in X \setminus Y$ whose closure $\overline{\{x\}}$ in X meets $U \cap Y$ we have

$$\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq s \quad \text{or} \quad \mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > d + s$$

Then there exists an open $V_0 \subset U$ containing $U \cap Y$ such that for any open $V \subset V_0$ containing $U \cap Y$ the map

$$H^i(V, \mathcal{F}) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism for $i < s$. If in addition $\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > s$ for all $x \in U \cap Y$, then these cohomology groups are finite A-modules.

The method of proof follows roughly the method of proof of [Fal79, Theorem 1] and [Fal80b, Satz 2]. The result is almost the same as [Ray74, Theorem 1.1] (affine complement case) and [Ray75, Theorem 3.9] (complement is union of few affines).

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent $\mathcal{O}_X$-module associated to M, see Local Cohomology, Lemma 51.8.2. Then the assumptions of Lemma 52.9.5 are satisfied. Pick J_0 as in that lemma and set $V_0 = X \setminus V(J_0)$. Then opens $V \subset V_0$ containing $U \cap Y$ correspond 1-to-1 with ideals $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Moreover, for such a choice we have a distinguished triangle

$$R\Gamma_J(M) \to M \to R\Gamma(V, \mathcal{F}) \to R\Gamma_J(M)[1]$$

We similarly have a distinguished triangle

$$R\Gamma_{\mathfrak{m}}(M)^\wedge \to M \to R\Gamma(U, \mathcal{F})^\wedge \to R\Gamma_{\mathfrak{m}}(M)^\wedge[1]$$

involving derived I-adic completions. The cohomology groups of $R\Gamma(U, \mathcal{F})^\wedge$ are equal to the limits in the statement of the theorem by Lemma 52.11.1. The canonical map between these triangles and some easy arguments show that our theorem follows from the main Lemma 52.9.5 (note that we have $i < s$ here whereas we have $i \leq s$ in the lemma; this is because of the shift). The finiteness of the cohomology groups (under the additional assumption) follows from Lemma 52.9.3. $\square$

0DXR Lemma 52.11.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume for any associated point $x \in U$ of $\mathcal{F}$ we have $\dim(\overline{\{x\}}) > \mathrm{cd}(A, I) + 1$ where $\overline{\{x\}}$ is the closure in X. Then the map

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism of finite A-modules where the colimit is over opens $V \subset U$ containing $U \cap Y$.

Proof. Apply Theorem 52.11.2 with $s = 1$ (we get finiteness too). $\square$

52.12. Algebraization of formal sections, II

0EG1 It is a bit difficult to succintly state all possible consequences of the results in Sections 52.8 and 52.10 for cohomology of coherent sheaves on quasi-affine schemes and their completion with respect to an ideal. This section gives a nonexhaustive list of applications to H^0. The next section contains applications to higher cohomology.

0H48 **Lemma 52.12.1.** Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of coherent modules on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes $V \subset U$ containing $U \cap V(I)$ ordered by reverse inclusion. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_\mathcal{V} H^0(V, \mathcal{F}') \arrow[r] \arrow[d] & \mathrm{colim}_\mathcal{V} H^0(V, \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{colim}_\mathcal{V} H^0(V, \mathcal{F}'') \arrow[d] \\
\lim H^0(U, \mathcal{F}'/I^n\mathcal{F}') \arrow[r] & \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}) \arrow[r] & \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}'')
\end{tikzcd}$$

If the left and right downarrows are isomorphisms so is the middle. If the middle and left downarrows are isomorphisms, so is the left.

Proof. The sequences in the diagram are exact in the middle and the first arrow is injective. Thus the final statement follows from an easy diagram chase. For the rest of the proof we assume the left and right downward arrows are isomorphisms. A diagram chase shows that the middle downward arrow is injective. All that remains is to show that it is surjective.

We may choose finite A-modules M and M' such that $\mathcal{F}$ and $\mathcal{F}'$ are the restriction of $\widetilde{M}$ and $\widetilde{M'}$ to U, see Local Cohomology, Lemma 51.8.2. After replacing M' by $\mathfrak{a}^n M'$ for some $n \geq 0$ we may assume that $\mathcal{F}' \to \mathcal{F}$ corresponds to a module map $M' \to M$, see Cohomology of Schemes, Lemma 30.10.5. After replacing M' by the image of $M' \to M$ and seting $M'' = M/M'$ we see that our short exact sequence corresponds to the restriction of the short exact sequence of coherent modules associated to the short exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules.

Let $\hat{s} \in \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ with image $\hat{s}'' \in \lim H^0(U, \mathcal{F}''/I^n\mathcal{F}'')$. By assumption we find $V \in \mathcal{V}$ and a section $s'' \in \mathcal{F}''(V)$ mapping to $\hat{s}''$. Let $J \subset A$ be an ideal such that $V(J) = \mathrm{Spec}(A) \setminus V$. By Cohomology of Schemes, Lemma 30.10.5 after replacing J by a power, we may assume there is an A-linear map $\varphi : J \to M''$ corresponding to s''. We fix this choice of J; in the rest of the proof we will replace V by a smaller V in $\mathcal{V}$, i.e, we will have $V \cap V(J) = \emptyset$.

Choose a presentation $A^{\oplus m} \to A^{\oplus n} \to J \to 0$. Denote $g_1, \ldots, g_n \in J$ the images of the basis vectors of $A^{\oplus n}$, so that $J = (g_1, \ldots, g_n)$. Let $A^{\oplus m} \to A^{\oplus n}$ be given by the matric (a_{ji}) so that $\sum a_{ji}g_i = 0$, $j = 1, \ldots, m$. Since $M \to M''$ is surjective, for each i we can choose $m_i \in M$ mapping to $\varphi(g_i) \in M''$. Then the element $g_i\hat{s} - m_i$ of $\lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ lies in the submodule $\lim H^0(U, \mathcal{F}'/I^n\mathcal{F}')$. By assumption after shrinking V we may assume there are $s'_i \in \mathcal{F}'(V)$, $i = 1, \ldots, n$ with s'_i mapping to $g_i\hat{s} - m_i$. Set $s_i = s'_i + m_i$ in $\mathcal{F}(V)$. Note that $\sum a_{ji}s_i$ maps to $\sum a_{ji}g_i\hat{s} = 0$ by the map

$$\mathrm{colim}_\mathcal{V} \mathcal{F}(V') \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

Since this map is injective (see above), we may after shrinking V assume that $\sum a_{ji}s_i = 0$ in $\mathcal{F}(V)$ for all $j = 1, \ldots, m$. Then it follows that we obtain an A-module map $J \to \mathcal{F}(V)$ sending g_i to s_i. By the universal property of $\widetilde{J}$ this A-module map corresponds to an $\mathcal{O}_V$-module map $\widetilde{J}|_V \to \mathcal{F}$. However, since $V(J) \cap V = \emptyset$ we have $\widetilde{J}|_V = \mathcal{O}_V$. Thus we have produced a section $s \in \mathcal{F}(V)$. We omit the computation that shows that s maps to $\hat{s}$ by the map displayed above. $\square$

The following lemma will be superceded by Proposition 52.12.3.

0EIF Lemma 52.12.2. Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $x \in \mathrm{Ass}(\mathcal{F})$, $x \not\in V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$, and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > \mathrm{cd}(A, I) + 1$,
(3) one of the following holds:
   (a) the restriction of $\mathcal{F}$ to $U \setminus V(I)$ is (S_1)
   (b) the dimension of $V(\mathfrak{a})$ is at most 2[4].

Then we obtain an isomorphism

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent module associated to M, see Local Cohomology, Lemma 51.8.2. Set $d = \mathrm{cd}(A, I)$. Let $\mathfrak{p}$ be a prime of A not contained in $V(I)$ and let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. Then either $\mathfrak{p}$ is not an associated prime of M and hence $\mathrm{depth}(M_\mathfrak{p}) \geq 1$ or we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > d + 1$ by (2). Thus the hypotheses of Lemma 52.8.5 are satisfied for $s = 1$ and d; here we use condition (3). Thus we find there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the maps

$$H^i_J(M) \longrightarrow H^i(R\Gamma_\mathfrak{a}(M)^\wedge)$$

are isomorphisms for $i = 0, 1$. Consider the morphisms of exact triangles

$$\begin{tikzcd}
R\Gamma_J(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & R\Gamma(V, \mathcal{F}) \arrow[r] \arrow[d] & R\Gamma_J(M)[1] \arrow[d] \\
R\Gamma_J(M)^\wedge \arrow[r] & M \arrow[r] & R\Gamma(V, \mathcal{F})^\wedge \arrow[r] & R\Gamma_J(M)^\wedge[1] \\
R\Gamma_\mathfrak{a}(M)^\wedge \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & R\Gamma(U, \mathcal{F})^\wedge \arrow[r] \arrow[u] & R\Gamma_\mathfrak{a}(M)^\wedge[1] \arrow[u]
\end{tikzcd}$$

where $V = \mathrm{Spec}(A) \setminus V(J)$. Recall that $R\Gamma_\mathfrak{a}(M)^\wedge \to R\Gamma_J(M)^\wedge$ is an isomorphism (because $\mathfrak{a}$, $\mathfrak{a}+I$, and $J+I$ cut out the same closed subscheme, for example see proof of Lemma 52.8.5). Hence $R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(V, \mathcal{F})^\wedge$. This produces a commutative

[4] In the sense that the difference of the maximal and minimal values on $V(\mathfrak{a})$ of a dimension function on $\mathrm{Spec}(A)$ is at most 2.

diagram

$$
\begin{tikzcd}
0 \arrow[r] & H^0_J(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \Gamma(V, \mathcal{F}) \arrow[r] \arrow[d] & H^1_J(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(R\Gamma_J(M)^\wedge) \arrow[r] & M \arrow[r] & H^0(R\Gamma(V, \mathcal{F})^\wedge) \arrow[r] & H^1(R\Gamma_J(M)^\wedge) \arrow[r] & 0 \\
0 \arrow[r] & H^0(R\Gamma_\mathfrak{a}(M)^\wedge) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & H^0(R\Gamma(U, \mathcal{F})^\wedge) \arrow[r] \arrow[u] & H^1(R\Gamma_\mathfrak{a}(M)^\wedge) \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

with exact rows and isomorphisms for the lower vertical arrows. Hence we obtain an isomorphism $\Gamma(V, \mathcal{F}) \to H^0(R\Gamma(U, \mathcal{F})^\wedge)$. By Lemmas 52.6.20 and 52.7.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

and we find $H^0(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ by Cohomology, Lemma 20.37.1. $\square$

Now we bootstrap the preceding lemma to get rid of condition (3).

0EG2 Proposition 52.12.3. Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $x \in \mathrm{Ass}(\mathcal{F})$, $x \not\in V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$, and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}}, z}) > \mathrm{cd}(A, I) + 1$.

Then we obtain an isomorphism

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. Let $T \subset U$ be the set of points x with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$. Let $\mathcal{F} \to \mathcal{F}'$ be the surjection of coherent modules on U constructed in Local Cohomology, Lemma 51.15.1. Since $\mathcal{F} \to \mathcal{F}'$ is an isomorphism over an open $V \subset U$ containing $U \cap V(I)$ it suffices to prove the lemma with $\mathcal{F}$ replaced by $\mathcal{F}'$. Hence we may and do assume for $x \in U$ with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ we have $\mathrm{depth}(\mathcal{F}_x) \geq 1$.

Let $\mathcal{V}$ be the set of open subschemes $V \subset U$ containing $U \cap V(I)$ ordered by reverse inclusion. This is a directed set. We first claim that

$$\mathcal{F}(V) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is injective for any $V \in \mathcal{F}$ (and in particular the map of the lemma is injective). Namely, an associated point x of $\mathcal{F}$ must have $\overline{\{x\}} \cap U \cap Y \neq \emptyset$ by the previous paragraph. If $y \in \overline{\{x\}} \cap U \cap Y$ then $\mathcal{F}_x$ is a localization of $\mathcal{F}_y$ and $\mathcal{F}_y \subset \lim \mathcal{F}_y/I^n\mathcal{F}_y$ by Krull's intersection theorem (Algebra, Lemma 10.51.4). This proves the claim as a section $s \in \mathcal{F}(V)$ in the kernel would have to have empty support, hence would have to be zero.

Choose a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U, see Local Cohomology, Lemma 51.8.2. We may and do assume that $H^0_\mathfrak{a}(M) = 0$. Let $\mathrm{Ass}(M) \setminus V(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. We will prove the lemma by induction on n. After reordering we may assume that $\mathfrak{p}_n$ is a minimal element of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ with respect to inclusion, i.e, $\mathfrak{p}_n$ is a generic point of the support of M. Set

$$M' = H^0_{\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} I}(M)$$

and $M'' = M/M'$. Let $\mathcal{F}'$ and $\mathcal{F}''$ be the coherent $\mathcal{O}_U$-modules corresponding to M' and M''. Dualizing Complexes, Lemma 47.11.6 implies that M'' has only one associated prime, namely $\mathfrak{p}_n$. Hence $\mathcal{F}''$ has only one associated point and we see that condition (3)(a) of Lemma 52.12.2 holds; thus the map $\operatorname{colim} H^0(V, \mathcal{F}'') \to \lim H^0(U, \mathcal{F}''/I^n\mathcal{F}'')$ is an isomorphism. On the other hand, since $\mathfrak{p}_n \not\in V(\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} I)$ we see that $\mathfrak{p}_n$ is not an associated prime of M'. Hence the induction hypothesis applies to M'; note that since $\mathcal{F}' \subset \mathcal{F}$ the condition $\text{depth}(\mathcal{F}'_x) \geq 1$ at points x with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ holds, see Algebra, Lemma 10.72.6. Thus the map $\operatorname{colim} H^0(V, \mathcal{F}') \to \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}')$ is an isomorphism too. We conclude by Lemma 52.12.1. $\square$

0EIG Lemma 52.12.4. Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $x \in \text{Ass}(\mathcal{F})$, $x \not\in V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$, and $z \in V(\mathfrak{a}) \cap \overline{\{x\}}$, then $\dim(\mathcal{O}_{\overline{\{x\}}, z}) > \text{cd}(A, I) + 1$,
(3) for $x \in U$ with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ we have $\text{depth}(\mathcal{F}_x) \geq 2$,

Then we obtain an isomorphism

$$H^0(U, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

Proof. Let $\hat{s} \in \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$. By Proposition 52.12.3 we find that $\hat{s}$ is the image of an element $s \in \mathcal{F}(V)$ for some $V \subset U$ open containing $U \cap V(I)$. However, condition (3) shows that $\text{depth}(\mathcal{F}_x) \geq 2$ for all $x \in U \setminus V$ and hence we find that $\mathcal{F}(V) = \mathcal{F}(U)$ by Divisors, Lemma 31.5.11 and the proof is complete. $\square$

0EIH Lemma 52.12.5. Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor on M,
(3) $H^1_{\mathfrak{a}}(M/fM)$ is a finite A-module.

Then with $U = \text{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$\operatorname{colim}_V \Gamma(V, \widetilde{M}) \longrightarrow \lim \Gamma(U, \widetilde{M/f^nM})$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap V(f)$.

Proof. Set $\mathcal{F} = \widetilde{M}|_U$. The finiteness of $H^1_{\mathfrak{a}}(M/fM)$ implies that $H^0(U, \mathcal{F}/f\mathcal{F})$ is finite, see Local Cohomology, Lemma 51.8.2. By Cohomology, Lemma 20.36.3 (which applies as f is a nonzerodivisor on $\mathcal{F}$) we see that $N = \lim H^0(U, \mathcal{F}/f^n\mathcal{F})$ is a finite A-module, is f-torsion free, and $N/fN \subset H^0(U, \mathcal{F}/f\mathcal{F})$. On the other hand, we have a map $M \to N$ and a compatible map

$$M/fM \longrightarrow H^0(U, \mathcal{F}/f\mathcal{F})$$

For $g \in \mathfrak{a}$ we see that $(M/fM)_g$ maps isomorphically to $H^0(U \cap D(f), \mathcal{F}/f\mathcal{F})$ since $\mathcal{F}/f\mathcal{F}$ is the restriction of $\widetilde{M/fM}$ to U. We conclude that $M_g \to N_g$ induces an isomorphism

$$M_g/fM_g = (M/fM)_g \to (N/fN)_g = N_g/fN_g$$

Since f is a nonzerodivisor on both N and M we conclude that $M_g \to N_g$ induces an isomorphism on f-adic completions which in turn implies $M_g \to N_g$ is an isomorphism in an open neightbourhood of $V(f) \cap D(g)$. Since $g \in \mathfrak a$ was arbitrary, we conclude that M and N determine isomorphic coherent modules over an open V as in the statement of the lemma. This finishes the proof. $\square$

0H49 Proposition 52.12.6. Let A be a Noetherian ring. Let $f \in \mathfrak a \subset A$ be an element of an ideal of A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak a)$. Assume

(1) A is f-adically complete and has a dualizing complex,
(2) if $x \in \mathrm{Ass}(\mathcal{F})$, $x \not\in V(f)$, $\overline{\{x\}} \cap V(f) \not\subset V(\mathfrak a)$, and $z \in \overline{\{x\}} \cap V(\mathfrak a)$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > 2$.

Then the map

$$\mathrm{colim}_V\, \Gamma(V, \mathcal{F}) \longrightarrow \lim \Gamma(U, \mathcal{F}/f^n\mathcal{F})$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap V(f)$.

First proof. Recall that A is universally catenary and with Gorenstein formal fibres, see Dualizing Complexes, Lemmas 47.23.2 and 47.17.4. Thus we may consider the map $\mathcal{F} \to \mathcal{F}'$ constructed in Local Cohomology, Lemma 51.15.3 for the closed subset $V(f) \cap U$ of U. Observe that

(1) The kernel and cokernel of $\mathcal{F} \to \mathcal{F}'$ are supported on $V(f) \cap U$.
(2) The module $\mathcal{F}'$ is f-torsion free as its stalks have depth ≥ 1 for all points of $V(f) \cap U$, i.e., $\mathcal{F}'$ has no associated points in $V(f) \cap U$.
(3) If $y \in V(f) \cap U$ is an associated point of $\mathcal{F}'/f\mathcal{F}'$, then $\mathrm{depth}(\mathcal{F}'_y) = 1$ and hence (by the construction of $\mathcal{F}'$) there is an immediate specialization $x \rightsquigarrow y$ with $x \not\in V(f)$ an associated point of $\mathcal{F}$. It follows that y cannot have an immediate specialization in $\mathrm{Spec}(A)$ to a point $z \in V(\mathfrak a)$ by our assumption (2).
(4) It follows from (3) that $H^0(U, \mathcal{F}'/f\mathcal{F}')$ is a finite A-module, see Local Cohomology, Lemma 51.12.1.

These observations will allow us to finish the proof.

First, we claim the lemma holds for $\mathcal{F}'$. Namely, choose a finite A-module M' such that $\mathcal{F}'$ is the restriction to U of the coherent module associated to M', see Local Cohomology, Lemma 51.8.2. Since $\mathcal{F}'$ is f-torsion free, we may assume M' is f-torsion free as well. Observation (4) above shows that $H^1_{\mathfrak a}(M')$ is a finite A-module, see Local Cohomology, Lemma 51.8.2. Thus the claim by Lemma 52.12.5.

Second, we observe that the lemma holds trivially for any coherent $\mathcal{O}_U$-module supported on $V(f) \cap U$. Let $\mathcal{K}$, resp. $\mathcal{G}$, resp. $\mathcal{Q}$ be the kernel, resp. image, resp. cokernel of the map $\mathcal{F} \to \mathcal{F}'$. The short exact sequence $0 \to \mathcal{G} \to \mathcal{F}' \to \mathcal{Q} \to 0$ and Lemma 52.12.1 show that the result holds for $\mathcal{G}$. Then we do this again with the short exact sequence $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0$ to finish the proof. $\square$

Second proof. The proposition is a special case of Proposition 52.12.3. $\square$

0EII Lemma 52.12.7. Let A be a Noetherian ring. Let $f \in \mathfrak a \subset A$ be an element of an ideal of A. Let M be a finite A-module. Assume

(1) A is f-adically complete,

(2) $H^1_\mathfrak{a}(M)$ and $H^2_\mathfrak{a}(M)$ are annihilated by a power of f.

Then with $U = \text{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$\Gamma(U, \widetilde{M}) \longrightarrow \lim \Gamma(U, \widetilde{M/f^nM})$$

is an isomorphism.

Proof. We may apply Lemma 52.3.2 to U and $\mathcal{F} = \widetilde{M}|_U$ because $\mathcal{F}$ is a Noetherian object in the category of coherent $\mathcal{O}_U$-modules. Since $H^1(U, \mathcal{F}) = H^2_\mathfrak{a}(M)$ (Local Cohomology, Lemma 51.8.2) is annihilated by a power of f, we see that its f-adic Tate module is zero. Hence the lemma shows $\lim H^0(U, \mathcal{F}/f^n\mathcal{F})$ is equal to the usual f-adic completion of $H^0(U, \mathcal{F})$. Consider the short exact sequence

$$0 \to M/H^0_\mathfrak{a}(M) \to H^0(U, \mathcal{F}) \to H^1_\mathfrak{a}(M) \to 0$$

of Local Cohomology, Lemma 51.8.2. Since $M/H^0_\mathfrak{a}(M)$ is a finite A-module, it is complete, see Algebra, Lemma 10.97.1. Since $H^1_\mathfrak{a}(M)$ is killed by a power of f, we conclude from Algebra, Lemma 10.96.4 that $H^0(U, \mathcal{F})$ is complete as well. This finishes the proof. ☐

52.13. Algebraization of formal sections, III

0EIJ The next section contains a nonexhaustive list of applications of the material on completion of local cohomology to higher cohomology of coherent modules on quasi-affine schemes and their completion with respect to an ideal.

0EG4 Proposition 52.13.1. Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Let $s \geq 0$. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $x \in U \setminus V(I)$ then $\text{depth}(\mathcal{F}_x) > s$ or
$$\text{depth}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > \text{cd}(A, I) + s + 1$$
for all $z \in V(\mathfrak{a}) \cap \overline{\{x\}}$,
(3) one of the following conditions holds:
  (a) the restriction of $\mathcal{F}$ to $U \setminus V(I)$ is (S_{s+1}), or
  (b) the dimension of $V(\mathfrak{a})$ is at most 2[5].

Then the maps
$$H^i(U, \mathcal{F}) \longrightarrow \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$
are isomorphisms for $i < s$. Moreover we have an isomorphism
$$\text{colim } H^s(V, \mathcal{F}) \longrightarrow \lim H^s(U, \mathcal{F}/I^n\mathcal{F})$$
where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. We may assume $s > 0$ as the case $s = 0$ was done in Proposition 52.12.3.

Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent module associated to M, see Local Cohomology, Lemma 51.8.2. Set $d = \text{cd}(A, I)$. Let $\mathfrak{p}$ be a prime of A not contained in $V(I)$ and let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. Then either $\text{depth}(M_\mathfrak{p}) \geq s + 1 > s$ or we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > d + s + 1$ by (2). By Lemma 52.10.5 we conclude that the assumptions of Situation 52.10.1 are satisfied

[5] In the sense that the difference of the maximal and minimal values on $V(\mathfrak{a})$ of a dimension function on $\text{Spec}(A)$ is at most 2.

for $A, I, V(\mathfrak a), M, s, d$. On the other hand, the hypotheses of Lemma 52.8.5 are satisfied for $s+1$ and d; this is where condition (3) is used.

Applying Lemma 52.8.5 we find there exists an ideal $J_0 \subset \mathfrak a$ with $V(J_0) \cap V(I) = V(\mathfrak a)$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak a)$ the maps

$$H^i_J(M) \longrightarrow H^i(R\Gamma_{\mathfrak a}(M)^\wedge)$$

is an isomorphism for $i \leq s+1$.

For $i \leq s$ the map $H^i_{\mathfrak a}(M) \to H^i_J(M)$ is an isomorphism by Lemmas 52.10.3 and 52.8.4. Using the comparison of cohomology and local cohomology (Local Cohomology, Lemma 51.2.2) we deduce $H^i(U, \mathcal F) \to H^i(V, \mathcal F)$ is an isomorphism for $V = \mathrm{Spec}(A) \setminus V(J)$ and $i < s$.

By Theorem 52.10.8 we have $H^i_{\mathfrak a}(M) = \lim H^i_{\mathfrak a}(M/I^nM)$ for $i \leq s$. By Lemma 52.10.9 we have $H^{s+1}_{\mathfrak a}(M) = \lim H^{s+1}_{\mathfrak a}(M/I^nM)$.

The isomorphism $H^0(U, \mathcal F) = H^0(V, \mathcal F) = \lim H^0(U, \mathcal F/I^n\mathcal F)$ follows from the above and Proposition 52.12.3. For $0 < i < s$ we get the desired isomorphisms $H^i(U, \mathcal F) = H^i(V, \mathcal F) = \lim H^i(U, \mathcal F/I^n\mathcal F)$ in the same manner using the relation between local cohomology and cohomology; it is easier than the case $i = 0$ because for $i > 0$ we have

$$H^i(U, \mathcal F) = H^{i+1}_{\mathfrak a}(M), \quad H^i(V, \mathcal F) = H^{i+1}_J(M), \quad H^i(R\Gamma(U, \mathcal F)^\wedge) = H^{i+1}(R\Gamma_{\mathfrak a}(M)^\wedge)$$

Similarly for the final statement. □

0EKM Lemma 52.13.2. Let A be a Noetherian ring. Let $f \in \mathfrak a \subset A$ be an element of an ideal of A. Let M be a finite A-module. Let $s \geq 0$. Assume

(1) A is f-adically complete,
(2) $H^i_{\mathfrak a}(M)$ is annihilated by a power of f for $i \leq s+1$.

Then with $U = \mathrm{Spec}(A) \setminus V(\mathfrak a)$ the map

$$H^i(U, \widetilde{M}) \longrightarrow \lim H^i(U, \widetilde{M/f^nM})$$

is an isomorphism for $i < s$.

Proof. By induction on s. If $s = 0$, the assertion is empty. If $s = 1$, then the result is Lemma 52.12.7. Assume $s > 1$. By induction it suffices to prove the result for $i = s - 1 \geq 1$. We may apply Lemma 52.3.2 to U and $\mathcal F = \widetilde{M}|_U$ because $\mathcal F$ is a Noetherian object in the category of coherent $\mathcal O_U$-modules. Observe that $H^j(U, \mathcal F) = H^{j+1}_{\mathfrak a}(M)$ for all j by Local Cohomology, Lemma 51.8.2. Thus for $j = s = (s-1)+1$ this is annihilated by a power of f by assumption. Thus it follows from Lemma 52.3.2 that $\lim H^{s-1}(U, \mathcal F/f^n\mathcal F)$ is the usual f-adic completion of $H^{s-1}(U, \mathcal F)$. Then again using that this module is killed by a power of f we see that the completion is simply equal to $H^{s-1}(U, \mathcal F)$ as desired. □

52.14. Application to connectedness

0ECQ In this section we discuss Grothendieck's connectedness theorem and variants; the original version can be found as [Gro68, Exposee XIII, Theorem 2.1]. There is a version called Faltings' connectedness theorem in the literature; our guess is that this refers to [Fal80a, Theorem 6]. Let us state and prove the optimal version for complete local rings given in [Var09, Theorem 1.6].

0ECR **Lemma 52.14.1.** Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. Let I be a proper ideal of A. Set $X = \mathrm{Spec}(A)$ and $Y = V(I)$. Denote [Var09, Theorem 1.6]

(1) d the minimal dimension of an irreducible component of X, and
(2) c the minimal dimension of a closed subset $Z \subset X$ such that $X \setminus Z$ is disconnected.

Then for $Z \subset Y$ closed we have $Y \setminus Z$ is connected if $\dim(Z) < \min(c, d-1) - \mathrm{cd}(A, I)$. In particular, the punctured spectrum of A/I is connected if $\mathrm{cd}(A, I) < \min(c, d-1)$.

Proof. Let us first prove the final assertion. As a first case, if the punctured spectrum of A/I is empty, then Local Cohomology, Lemma 51.4.10 shows every irreducible component of X has dimension $\leq \mathrm{cd}(A, I)$ and we get $\min(c, d-1) - \mathrm{cd}(A, I) < 0$ which implies the lemma holds in this case. Thus we may assume $U \cap Y$ is nonempty where $U = X \setminus \{\mathfrak{m}\}$ is the punctured spectrum of A. We may replace A by its reduction. Observe that A has a dualizing complex (Dualizing Complexes, Lemma 47.22.4) and that A is complete with respect to I (Algebra, Lemma 10.96.8). If we assume $d-1 > \mathrm{cd}(A, I)$, then we may apply Lemma 52.11.3 to see that

$$\mathrm{colim}\, H^0(V, \mathcal{O}_V) \longrightarrow \lim H^0(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap Y$. If $U \cap Y$ is disconnected, then its nth infinitesimal neighbourhood in U is disconnected for all n and we find the right hand side has a nontrivial idempotent (here we use that $U \cap Y$ is nonempty). Thus we can find a V which is disconnected. Set $Z = X \setminus V$. By Local Cohomology, Lemma 51.4.10 we see that every irreducible component of Z has dimension $\leq \mathrm{cd}(A, I)$. Hence $c \leq \mathrm{cd}(A, I)$ and this indeed proves the final statement.

We can deduce the statement of the lemma from what we just proved as follows. Suppose that $Z \subset Y$ closed and $Y \setminus Z$ is disconnected and $\dim(Z) = e$. Recall that a connected space is nonempty by convention. Hence we conclude either (a) $Y = Z$ or (b) $Y \setminus Z = W_1 \amalg W_2$ with W_i nonempty, open, and closed in $Y \setminus Z$. In case (b) we may pick points $w_i \in W_i$ which are closed in U, see Morphisms, Lemma 29.16.10. Then we can find $f_1, \ldots, f_e \in \mathfrak{m}$ such that $V(f_1, \ldots, f_e) \cap Z = \{\mathfrak{m}\}$ and in case (b) we may assume $w_i \in V(f_1, \ldots, f_e)$. Namely, we can inductively using prime avoidance choose f_i such that $\dim V(f_1, \ldots, f_i) \cap Z = e - i$ and such that in case (b) we have $w_1, w_2 \in V(f_i)$. It follows that the punctured spectrum of $A/I + (f_1, \ldots, f_e)$ is disconnected (small detail omitted). Since $\mathrm{cd}(A, I + (f_1, \ldots, f_e)) \leq \mathrm{cd}(A, I) + e$ by Local Cohomology, Lemmas 51.4.4 and 51.4.3 we conclude that

$$\mathrm{cd}(A, I) + e \geq \min(c, d-1)$$

by the first part of the proof. This implies $e \geq \min(c, d-1) - \mathrm{cd}(A, I)$ which is what we had to show. $\square$

0EG5 **Lemma 52.14.2.** Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) if $\mathfrak{p} \subset A$ is a minimal prime not contained in $V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$, then $\dim((A/\mathfrak{p})_\mathfrak{q}) > \mathrm{cd}(A, I) + 1$,

(3) any nonempty open $V \subset \text{Spec}(A)$ which contains $V(I) \setminus V(\mathfrak{a})$ is connected[6].

Then $V(I) \setminus V(\mathfrak{a})$ is either empty or connected.

Proof. We may replace A by its reduction. Then we have the inequality in (2) for all associated primes of A. By Proposition 52.12.3 we see that

$$\operatorname{colim} H^0(V, \mathcal{O}_V) = \lim H^0(T_n, \mathcal{O}_{T_n})$$

where the colimit is over the opens V as in (3) and T_n is the nth infinitesimal neighbourhood of $T = V(I) \setminus V(\mathfrak{a})$ in $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Thus T is either empty or connected, since if not, then the right hand side would have a nontrivial idempotent and we've assumed the left hand side does not. Some details omitted. □

0EG3 Lemma 52.14.3. Let A be a Noetherian domain which has a dualizing complex and which is complete with respect to a nonzero $f \in A$. Let $f \in \mathfrak{a} \subset A$ be an ideal. Assume every irreducible component of $Z = V(\mathfrak{a})$ has codimension > 2 in $X = \text{Spec}(A)$, i.e., assume every irreducible component of Z has codimension > 1 in $Y = V(f)$. Then $Y \setminus Z$ is connected.

Proof. This is a special case of Lemma 52.14.2 (whose proof relies on Proposition 52.12.3). Below we prove it using the easier Proposition 52.12.6.

Set $U = X \setminus Z$. By Proposition 52.12.6 we have an isomorphism

$$\operatorname{colim} \Gamma(V, \mathcal{O}_V) \to \lim_n \Gamma(U, \mathcal{O}_U/f^n\mathcal{O}_U)$$

where the colimit is over open $V \subset U$ containing $U \cap Y$. Hence if $U \cap Y$ is disconnected, then for some V there exists a nontrivial idempotent in $\Gamma(V, \mathcal{O}_V)$. This is impossible as V is an integral scheme as X is the spectrum of a domain. □

52.15. The completion functor

0EKN Let X be a Noetherian scheme. Let $Y \subset X$ be a closed subscheme with quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. In this section we consider inverse systems of coherent $\mathcal{O}_X$-modules $(\mathcal{F}_n)$ with $\mathcal{F}_n$ annihilated by I^n such that the transition maps induce isomorphisms $\mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1} \to \mathcal{F}_n$. The category of these inverse systems was denoted

$$\textit{Coh}(X, \mathcal{I})$$

in Cohomology of Schemes, Section 30.23. This category is equivalent to the category of coherent modules on the formal completion of X along Y; however, since we have not yet introduced formal schemes or coherent modules on them, we cannot use this terminology here. We are particularly interested in the completion functor

$$\textit{Coh}(\mathcal{O}_X) \longrightarrow \textit{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

See Cohomology of Schemes, Equation (30.23.3.1).

0EKP Lemma 52.15.1. Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Consider the following conditions

(1) X is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism,
(2) X has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^{\otimes m}) \to \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$ is an isomorphism for all $m \gg 0$,

[6] For example if A is a domain.

(3) for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism, and
(4) the completion functor $\mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(X, \mathcal{I})$ is fully faithful on the full subcategory of finite locally free objects.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3).

Proof. Proof of (3) $\Rightarrow$ (4). If $\mathcal{F}$ and $\mathcal{G}$ are finite locally free on X, then considering $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ and using Cohomology of Schemes, Lemma 30.23.5 we see that (3) implies (4).

Proof of (2) $\to$ (3). Namely, let $\mathcal{L}$ be ample on X and suppose that $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module. We claim we can find a universally exact sequence

$$0 \to \mathcal{E} \to (\mathcal{L}^{\otimes p})^{\oplus r} \to (\mathcal{L}^{\otimes q})^{\oplus s}$$

for some $r, s \geq 0$ and $0 \ll p \ll q$. If this holds, then using the exact sequence

$$0 \to \lim \Gamma(\mathcal{E}|_{Y_n}) \to \lim \Gamma((\mathcal{L}^{\otimes p})^{\oplus r}|_{Y_n}) \to \lim \Gamma((\mathcal{L}^{\otimes q})^{\oplus s}|_{Y_n})$$

and the isomorphisms in (2) we get the isomorphism in (3). To prove the claim, consider the dual locally free module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ and apply Properties, Proposition 28.26.13 to find a surjection

$$(\mathcal{L}^{\otimes -p})^{\oplus r} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

Taking duals we obtain the first map in the exact sequence (it is universally injective because being a surjection is universal). Repeat with the cokernel to get the second. Some details omitted.

Proof of (1) $\Rightarrow$ (2). This is true because if X is quasi-affine then $\mathcal{O}_X$ is an ample invertible module, see Properties, Lemma 28.27.1.

We omit the proof of (4) $\Rightarrow$ (3). $\square$

Given a Noetherian scheme and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we will say an object $(\mathcal{F}_n)$ of $\mathrm{Coh}(X, \mathcal{I})$ is finite locally free if each $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_X/\mathcal{I}^n$-module.

0EK2 Lemma 52.15.2. Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion.

(1) X is quasi-affine and
$$\mathrm{colim}_\mathcal{V}\, \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$$
is an isomorphism,
(2) X has an ample invertible module $\mathcal{L}$ and
$$\mathrm{colim}_\mathcal{V}\, \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$
is an isomorphism for all $m \gg 0$,
(3) for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map
$$\mathrm{colim}_{V' \geq V}\, \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism, and

(4) the completion functor
$$\operatorname{colim}_{\mathcal{V}} \operatorname{Coh}(\mathcal{O}_V) \longrightarrow \operatorname{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$
is fully faithful on the full subcategory of finite locally free objects (see explanation above).

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3).

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 4.19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 52.15.1; we urge the reader to skip it.

Proof of (3) $\Rightarrow$ (4). If $\mathcal{F}$ and $\mathcal{G}$ are finite locally free on $V \in \mathcal{V}$, then considering $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{G}, \mathcal{F})$ and using Cohomology of Schemes, Lemma 30.23.5 we see that (3) implies (4).

Proof of (2) $\Rightarrow$ (3). Let $\mathcal{L}$ be ample on X and suppose that $\mathcal{E}$ is a finite locally free $\mathcal{O}_V$-module for some $V \in \mathcal{V}$. We claim we can find a universally exact sequence
$$0 \to \mathcal{E} \to (\mathcal{L}^{\otimes p})^{\oplus r}|_V \to (\mathcal{L}^{\otimes q})^{\oplus s}|_V$$
for some $r, s \geq 0$ and $0 \ll p \ll q$. If this is true, then the isomorphism in (2) will imply the isomorphism in (3). To prove the claim, recall that $\mathcal{L}|_V$ is ample, see Properties, Lemma 28.26.14. Consider the dual locally free module $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}, \mathcal{O}_V)$ and apply Properties, Proposition 28.26.13 to find a surjection
$$(\mathcal{L}^{\otimes -p})^{\oplus r}|_V \longrightarrow \mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}, \mathcal{O}_V)$$
(it is universally injective because being a surjection is universal). Taking duals we obtain the first map in the exact sequence. Repeat with the cokernel to get the second. Some details omitted.

Proof of (1) $\Rightarrow$ (2). This is true because if X is quasi-affine then $\mathcal{O}_X$ is an ample invertible module, see Properties, Lemma 28.27.1.

We omit the proof of (4) $\Rightarrow$ (3). $\square$

0EIQ Lemma 52.15.3. Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The functor
$$\operatorname{Coh}(X, \mathcal{I}) \longrightarrow \text{Pro-}QCoh(\mathcal{O}_X)$$
is fully faithful, see Categories, Remark 4.22.5.

Proof. Let $(\mathcal{F}_n)$ and $(\mathcal{G}_n)$ be objects of $\operatorname{Coh}(X, \mathcal{I})$. A morphism of pro-objects α from $(\mathcal{F}_n)$ to $(\mathcal{G}_n)$ is given by a system of maps $\alpha_n : \mathcal{F}_{n'(n)} \to \mathcal{G}_n$ where $\mathbf{N} \to \mathbf{N}$, $n \mapsto n'(n)$ is an increasing function. Since $\mathcal{F}_n = \mathcal{F}_{n'(n)}/\mathcal{I}^n\mathcal{F}_{n'(n)}$ and since $\mathcal{G}_n$ is annihilated by $\mathcal{I}^n$ we see that α_n induces a map $\mathcal{F}_n \to \mathcal{G}_n$. $\square$

Next we add some examples of the kind of fully faithfulness result we will be able to prove using the work done earlier in this chapter.

0EKQ Lemma 52.15.4. Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $U = \operatorname{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) for any associated prime $\mathfrak{p} \subset A$ with $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \not\subset V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > \operatorname{cd}(A, I) + 1$,
(3) for $\mathfrak{p} \subset A$ with $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$ we have $\operatorname{depth}(A_\mathfrak{p}) \geq 2$.

Then the completion functor

$$\mathrm{Coh}(\mathcal{O}_U) \longrightarrow \mathrm{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.1 it suffices to show that

$$\Gamma(U, \mathcal{O}_U) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 52.12.4. □

0EKS Lemma 52.15.5. Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A is f-adically complete,
(2) $H^1_\mathfrak{a}(A)$ and $H^2_\mathfrak{a}(A)$ are annihilated by a power of f.

Then the completion functor

$$\mathrm{Coh}(\mathcal{O}_U) \longrightarrow \mathrm{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.1 it suffices to show that

$$\Gamma(U, \mathcal{O}_U) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 52.12.7. □

0EKT Lemma 52.15.6. Let A be a Noetherian ring. Let $f \in \mathfrak{a}$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A has a dualizing complex and is complete with respect to f,
(2) for every prime $\mathfrak{p} \subset A$, $f \not\in \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\mathrm{depth}(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > 2$.

Then the completion functor

$$\mathrm{Coh}(\mathcal{O}_U) \longrightarrow \mathrm{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. Follows from Lemma 52.15.5 and Local Cohomology, Proposition 51.10.1. □

0EKU Lemma 52.15.7. Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(I)$ ordered by reverse inclusion. Assume

(1) A is I-adically complete and has a dualizing complex,
(2) for any associated prime $\mathfrak{p} \subset A$ with $I \not\subset \mathfrak{p}$ and $V(\mathfrak{p}) \cap V(I) \not\subset V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > \mathrm{cd}(A, I) + 1$.

Then the completion functor

$$\mathrm{colim}_\mathcal{V} \mathrm{Coh}(\mathcal{O}_V) \longrightarrow \mathrm{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.2 it suffices to show that

$$\mathrm{colim}_\mathcal{V} \Gamma(V, \mathcal{O}_V) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Proposition 52.12.3. □

0EKV **Lemma 52.15.8.** Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(f)$ ordered by reverse inclusion. Assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor,
(3) $H^1_{\mathfrak{a}}(A/fA)$ is a finite A-module.

Then the completion functor

$$\mathrm{colim}_{\mathcal{V}} \mathrm{Coh}(\mathcal{O}_V) \longrightarrow \mathrm{Coh}(U, f\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.2 it suffices to show that

$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{O}_V) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 52.12.5. $\square$

0EIV **Lemma 52.15.9.** Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(I)$ ordered by reverse inclusion. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_V$-modules for some $V \in \mathcal{V}$. The map

$$\mathrm{colim}_{V' \geq V} \mathrm{Hom}_V(\mathcal{G}|_{V'}, \mathcal{F}|_{V'}) \longrightarrow \mathrm{Hom}_{\mathrm{Coh}(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective if the following assumptions hold:

(1) A is I-adically complete and has a dualizing complex,
(2) if $x \in \mathrm{Ass}(\mathcal{F})$, $x \not\in V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$ and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}}, z}) > \mathrm{cd}(A, I) + 1$.

Proof. We may choose coherent $\mathcal{O}_U$-modules $\mathcal{F}'$ and $\mathcal{G}'$ whose restriction to V is $\mathcal{F}$ and $\mathcal{G}$, see Properties, Lemma 28.22.5. We may modify our choice of $\mathcal{F}'$ to ensure that $\mathrm{Ass}(\mathcal{F}') \subset V$, see for example Local Cohomology, Lemma 51.15.1. Thus we may and do replace V by U and $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}'$ and $\mathcal{G}'$. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_U$-module. We have

$$\mathrm{Hom}_V(\mathcal{G}|_V, \mathcal{F}|_V) = H^0(V, \mathcal{H}) \quad \text{and} \quad \lim H^0(U, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{\mathrm{Coh}(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

See Cohomology of Schemes, Lemma 30.23.5. Thus if we can show that the assumptions of Proposition 52.12.3 hold for $\mathcal{H}$, then the proof is complete. This holds because $\mathrm{Ass}(\mathcal{H}) \subset \mathrm{Ass}(\mathcal{F})$. See Cohomology of Schemes, Lemma 30.11.2. $\square$

52.16. Algebraization of coherent formal modules, I

0DXS The essential surjectivity of the completion functor (see below) was studied systematically in [Gro68], [Ray75], and [Ray74]. We work in the following affine situation.

0EHC **Situation 52.16.1.** Here A is a Noetherian ring and $I \subset \mathfrak{a} \subset A$ are ideals. We set $X = \mathrm{Spec}(A)$, $Y = V(I) = \mathrm{Spec}(A/I)$, and $Z = V(\mathfrak{a}) = \mathrm{Spec}(A/\mathfrak{a})$. Furthermore $U = X \setminus Z$.

In this section we try to find conditions that guarantee an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$ is in the image of the completion functor $\mathrm{Coh}(\mathcal{O}_U) \to \mathrm{Coh}(U, I\mathcal{O}_U)$. See Cohomology of Schemes, Section 30.23 and Section 52.15.

0DXT **Lemma 52.16.2.** In Situation 52.16.1. Consider an inverse system (M_n) of A-modules such that

(1) M_n is a finite A-module,
(2) M_n is annihilated by I^n,
(3) the kernel and cokernel of $M_{n+1}/I^n M_{n+1} \to M_n$ are $\mathfrak{a}$-power torsion.

Then $(\widetilde{M_n}|_U)$ is in $\text{Coh}(U, I\mathcal{O}_U)$. Conversely, every object of $\text{Coh}(U, I\mathcal{O}_U)$ arises in this manner.

Proof. We omit the verification that $(\widetilde{M_n}|_U)$ is in $\text{Coh}(U, I\mathcal{O}_U)$. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. By Local Cohomology, Lemma 51.8.2 we see that $\mathcal{F}_n = \widetilde{M_n}$ for some finite A/I^n-module M_n. After dividing M_n by $H^0_{\mathfrak{a}}(M_n)$ we may assume $M_n \subset H^0(U, \mathcal{F}_n)$, see Dualizing Complexes, Lemma 47.11.6 and the already referenced lemma. After replacing inductively M_{n+1} by the inverse image of M_n under the map $M_{n+1} \to H^0(U, \mathcal{F}_{n+1}) \to H^0(U, \mathcal{F}_n)$, we may assume M_{n+1} maps into M_n. This gives a inverse system (M_n) satisfying (1) and (2) such that $\mathcal{F}_n = \widetilde{M_n}$. To see that (3) holds, use that $M_{n+1}/I^n M_{n+1} \to M_n$ is a map of finite A-modules which induces an isomorphism after applying $\widetilde{\ }$ and restriction to U (here we use the first referenced lemma one more time). $\square$

In Situation 52.16.1 we can study the completion functor Cohomology of Schemes, Equation (30.23.3.1)

0EIK (52.16.2.1)
$$\text{Coh}(\mathcal{O}_U) \longrightarrow \text{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

If A is I-adically complete, then this functor is fully faithful on suitable subcategories by our earlier work on algebraization of formal sections, see Section 52.15 and Lemma 52.19.6 for some sample results. Next, let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Still assuming A is I-adically complete, we can ask: When is $(\mathcal{F}_n)$ in the essential image of the completion functor displayed above?

0EIL **Lemma 52.16.3.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Consider the following conditions:

(1) $(\mathcal{F}_n)$ is in the essential image of the functor (52.16.2.1),
(2) $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module,
(3) $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_V$-module for $U \cap Y \subset V \subset U$ open,
(4) $(\mathcal{F}_n)$ is the completion of the restriction to U of a coherent $\mathcal{O}_X$-module,
(5) $(\mathcal{F}_n)$ is the restriction to U of the completion of a coherent $\mathcal{O}_X$-module,
(6) there exists an object $(\mathcal{G}_n)$ of $\text{Coh}(X, I\mathcal{O}_X)$ whose restriction to U is $(\mathcal{F}_n)$.

Then conditions (1), (2), (3), (4), and (5) are equivalent and imply (6). If A is I-adically complete then condition (6) implies the others.

Proof. Parts (1) and (2) are equivalent, because the completion of a coherent $\mathcal{O}_U$-module $\mathcal{F}$ is by definition the image of $\mathcal{F}$ under the functor (52.16.2.1). If $V \subset U$ is an open subscheme containing $U \cap Y$, then we have

$$\text{Coh}(V, I\mathcal{O}_V) = \text{Coh}(U, I\mathcal{O}_U)$$

since the category of coherent $\mathcal{O}_V$-modules supported on $V \cap Y$ is the same as the category of coherent $\mathcal{O}_U$-modules supported on $U \cap Y$. Thus the completion of a coherent $\mathcal{O}_V$-module is an object of $\text{Coh}(U, I\mathcal{O}_U)$. Having said this the equivalence of (2), (3), (4), and (5) holds because the functors $\text{Coh}(\mathcal{O}_X) \to \text{Coh}(\mathcal{O}_U) \to \text{Coh}(\mathcal{O}_V)$ are essentially surjective. See Properties, Lemma 28.22.5.

It is always the case that (5) implies (6). Assume A is I-adically complete. Then any object of $\text{Coh}(X, I\mathcal{O}_X)$ corresponds to a finite A-module by Cohomology of Schemes, Lemma 30.23.1. Thus we see that (6) implies (5) in this case. $\square$

0EHE Example 52.16.4. Let k be a field. Let $A = k[x, y][[t]]$ with $I = (t)$ and $\mathfrak{a} = (x, y, t)$. Let us use notation as in Situation 52.16.1. Observe that $U \cap Y = (D(x) \cap Y) \cup (D(y) \cap Y)$ is an affine open covering. For $n \geq 1$ consider the invertible module $\mathcal{L}_n$ of $\mathcal{O}_U/t^n\mathcal{O}_U$ given by glueing A_x/t^nA_x and A_y/t^nA_y via the invertible element of A_{xy}/t^nA_{xy} which is the image of any power series of the form

$$u = 1 + \frac{t}{xy} + \sum_{n \geq 2} a_n \frac{t^n}{(xy)^{\varphi(n)}}$$

with $a_n \in k[x, y]$ and $\varphi(n) \in \mathbf{N}$. Then $(\mathcal{L}_n)$ is an invertible object of $\text{Coh}(U, I\mathcal{O}_U)$ which is not the completion of a coherent $\mathcal{O}_U$-module $\mathcal{L}$. We only sketch the argument and we omit most of the details. Let $y \in U \cap Y$. Then the completion of the stalk $\mathcal{L}_y$ would be an invertible module hence $\mathcal{L}_y$ is invertible. Thus there would exist an open $V \subset U$ containing $U \cap Y$ such that $\mathcal{L}|_V$ is invertible. By Divisors, Lemma 31.28.3 we find an invertible A-module M with $\widetilde{M}|_V \cong \mathcal{L}|_V$. However the ring A is a UFD hence we see $M \cong A$ which would imply $\mathcal{L}_n \cong \mathcal{O}_U/I^n\mathcal{O}_U$. Since $\mathcal{L}_2 \not\cong \mathcal{O}_U/I^2\mathcal{O}_U$ by construction we get a contradiction as desired.

Note that if we take $a_n = 0$ for $n \geq 2$, then we see that $\lim H^0(U, \mathcal{L}_n)$ is nonzero: in this case we the function x on $D(x)$ and the function $x + t/y$ on $D(y)$ glue. On the other hand, if we take $a_n = 1$ and $\varphi(n) = 2^n$ or even $\varphi(n) = n^2$ then the reader can show that $\lim H^0(U, \mathcal{L}_n)$ is zero; this gives another proof that $(\mathcal{L}_n)$ is not algebraizable in this case.

If in Situation 52.16.1 the ring A is not I-adically complete, then Lemma 52.16.3 suggests the correct thing is to ask whether $(\mathcal{F}_n)$ is in the essential image of the restriction functor

$$\text{Coh}(X, I\mathcal{O}_X) \longrightarrow \text{Coh}(U, I\mathcal{O}_U)$$

However, we can no longer say that this means $(\mathcal{F}_n)$ is algebraizable. Thus we introduce the following terminology.

0EIM Definition 52.16.5. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. We say $(\mathcal{F}_n)$ extends to X if there exists an object $(\mathcal{G}_n)$ of $\text{Coh}(X, I\mathcal{O}_X)$ whose restriction to U is isomorphic to $(\mathcal{F}_n)$.

This notion is equivalent to being algebraizable over the completion.

0EIN Lemma 52.16.6. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Let $A', I', \mathfrak{a}'$ be the I-adic completions of $A, I, \mathfrak{a}$. Set $X' = \text{Spec}(A')$ and $U' = X' \setminus V(\mathfrak{a}')$. The following are equivalent

(1) $(\mathcal{F}_n)$ extends to X, and
(2) the pullback of $(\mathcal{F}_n)$ to U' is the completion of a coherent $\mathcal{O}_{U'}$-module.

Proof. Recall that $A \to A'$ is a flat ring map which induces an isomorphism $A/I \to A'/I'$. See Algebra, Lemmas 10.97.2 and 10.97.4. Thus $X' \to X$ is a flat morphism inducing an isomorphism $Y' \to Y$. Thus $U' \to U$ is a flat morphism which induces an isomorphism $U' \cap Y' \to U \cap Y$. This implies that in the commutative diagram

$$\begin{tikzcd} \mathrm{Coh}(X', I\mathcal{O}_{X'}) \arrow[r] & \mathrm{Coh}(U', I\mathcal{O}_{U'}) \\ \mathrm{Coh}(X, I\mathcal{O}_X) \arrow[r] \arrow[u] & \mathrm{Coh}(U, I\mathcal{O}_U) \arrow[u] \end{tikzcd}$$

the vertical functors are equivalences. See Cohomology of Schemes, Lemma 30.23.10. The lemma follows formally from this and the results of Lemma 52.16.3. $\square$

In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. To figure out if $(\mathcal{F}_n)$ extends to X it makes sense to look at the A-module

0EHD (52.16.6.1)
$$M = \lim H^0(U, \mathcal{F}_n)$$

Observe that M has a limit topology which is (a priori) coarser than the I-adic topology since $M \to H^0(U, \mathcal{F}_n)$ annihilates I^nM. There are canonical maps

$$\widetilde{M}|_U \to \widetilde{M/I^nM}|_U \to \widetilde{H^0(U, \mathcal{F}_n)}|_U \to \mathcal{F}_n$$

One could hope that $\widetilde{M}$ restricts to a coherent module on U and that $(\mathcal{F}_n)$ is the completion of this module. This is naive because this has almost no chance of being true if A is not complete. But even if A is I-adically complete this notion is very difficult to work with. A less naive approach is to consider the following requirement.

0EIP **Definition 52.16.7.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. We say $(\mathcal{F}_n)$ *canonically extends to* X if the the inverse system

$$\{\widetilde{H^0(U, \mathcal{F}_n)}\}_{n \geq 1}$$

in $QCoh(\mathcal{O}_X)$ is pro-isomorphic to an object $(\mathcal{G}_n)$ of $\mathrm{Coh}(X, I\mathcal{O}_X)$.

We will see in Lemma 52.16.8 that the condition in Definition 52.16.7 is stronger than the condition of Definition 52.16.5.

0EIR **Lemma 52.16.8.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. If $(\mathcal{F}_n)$ canonically extends to X, then

1. $(\widetilde{H^0(U, \mathcal{F}_n)})$ is pro-isomorphic to an object $(\mathcal{G}_n)$ of $\mathrm{Coh}(X, I\mathcal{O}_X)$ unique up to unique isomorphism,
2. the restriction of $(\mathcal{G}_n)$ to U is isomorphic to $(\mathcal{F}_n)$, i.e., $(\mathcal{F}_n)$ extends to X,
3. the inverse system $\{H^0(U, \mathcal{F}_n)\}$ satisfies the Mittag-Leffler condition, and
4. the module M in (52.16.6.1) is finite over the I-adic completion of A and the limit topology on M is the I-adic topology.

Proof. The existence of $(\mathcal{G}_n)$ in (1) follows from Definition 52.16.7. The uniqueness of $(\mathcal{G}_n)$ in (1) follows from Lemma 52.15.3. Write $\mathcal{G}_n = \widetilde{M_n}$. Then $\{M_n\}$ is an inverse system of finite A-modules with $M_n = M_{n+1}/I^nM_{n+1}$. By Definition 52.16.7 the inverse system $\{H^0(U, \mathcal{F}_n)\}$ is pro-isomorphic to $\{M_n\}$. Hence we see that the inverse system $\{H^0(U, \mathcal{F}_n)\}$ satisfies the Mittag-Leffler condition and that $M = \lim M_n$ (as topological modules). Thus the properties of M in (4) follow from Algebra, Lemmas 10.98.2, 10.96.12, and 10.96.3. Since U is quasi-affine the canonical maps

$$\widetilde{H^0(U, \mathcal{F}_n)}|_U \to \mathcal{F}_n$$

are isomorphisms (Properties, Lemma 28.18.2). We conclude that $(\mathcal{G}_n|_U)$ and $(\mathcal{F}_n)$ are pro-isomorphic and hence isomorphic by Lemma 52.15.3. □

0EIS Lemma 52.16.9. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Let $A \to A'$ be a flat ring map. Set $X' = \mathrm{Spec}(A')$, let $U' \subset X'$ be the inverse image of U, and denote $g : U' \to U$ the induced morphism. Set $(\mathcal{F}'_n) = (g^*\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 30.23.9. If $(\mathcal{F}_n)$ canonically extends to X, then $(\mathcal{F}'_n)$ canonically extends to X'. Moreover, the extension found in Lemma 52.16.8 for $(\mathcal{F}_n)$ pulls back to the extension for $(\mathcal{F}'_n)$.

Proof. Let $f : X' \to X$ be the induced morphism. We have $H^0(U', \mathcal{F}'_n) = H^0(U, \mathcal{F}_n) \otimes_A A'$ by flat base change, see Cohomology of Schemes, Lemma 30.5.2. Thus if $(\mathcal{G}_n)$ in $\mathrm{Coh}(X, I\mathcal{O}_X)$ is pro-isomorphic to $(\widetilde{H^0(U, \mathcal{F}_n)})$, then $(f^*\mathcal{G}_n)$ is pro-isomorphic to

$$(f^*\widetilde{H^0(U, \mathcal{F}_n)}) = (\widetilde{H^0(U, \mathcal{F}_n) \otimes_A A'}) = (\widetilde{H^0(U', \mathcal{F}'_n)})$$

This finishes the proof. □

0EHH Lemma 52.16.10. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Let M be as in (52.16.6.1). Assume

(a) the inverse system $H^0(U, \mathcal{F}_n)$ has Mittag-Leffler,
(b) the limit topology on M agrees with the I-adic topology, and
(c) the image of $M \to H^0(U, \mathcal{F}_n)$ is a finite A-module for all n.

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. Since $H^0(U, \mathcal{F}_n)$ has the Mittag-Leffler condition and since the limit topology on M is the I-adic topology we see that $\{M/I^nM\}$ and $\{H^0(U, \mathcal{F}_n)\}$ are pro-isomorphic inverse systems of A-modules. Thus if we set

$$\mathcal{G}_n = \widetilde{M/I^nM}$$

then we see that to verify the condition in Definition 52.16.7 it suffices to show that M is a finite module over the I-adic completion of A. This follows from the fact that M/I^nM is finite by condition (c) and the above and Algebra, Lemma 10.96.12. □

The following is in some sense the most straightforward possible application Lemma 52.16.10 above.

0DXW Lemma 52.16.11. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Assume

(1) $I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak{a}$,
(2) $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/f^n\mathcal{O}_U$-module,
(3) $H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$ are finite A-modules.

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 52.16.10.

Since $\mathcal{F}_n$ is locally free over $\mathcal{O}_U/f^n\mathcal{O}_U$ we see that we have short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$ for all n. Thus condition (b) holds by Cohomology, Lemma 20.36.2.

As f is a nonzerodivisor we obtain short exact sequences

$$0 \to A/f^nA \xrightarrow{f} A/f^{n+1}A \to A/fA \to 0$$

and we have corresponding short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$. We will use Local Cohomology, Lemma 51.8.2 without further mention. Our assumptions imply that $H^0(U, \mathcal{O}_U/f\mathcal{O}_U)$ and $H^1(U, \mathcal{O}_U/f\mathcal{O}_U)$ are finite A-modules. Hence the same thing is true for $\mathcal{F}_1$, see Local Cohomology, Lemma 51.12.2. Using induction and the short exact sequences we find that $H^0(U, \mathcal{F}_n)$ are finite A-modules for all n. In this way we see hypothesis (c) is satisfied.

Finally, as $H^1(U, \mathcal{F}_1)$ is a finite A-module we can apply Cohomology, Lemma 20.36.4 to see hypothesis (a) holds. □

0EHI Remark 52.16.12. In Lemma 52.16.11 if A is universally catenary with Cohen-Macaulay formal fibres (for example if A has a dualizing complex), then the condition that $H^1_{\mathfrak{a}}(A/fA)$ and $H^2_{\mathfrak{a}}(A/fA)$ are finite A-modules, is equivalent with

$$\text{depth}((A/f)_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) > 2$$

for all $\mathfrak{p} \in V(f) \setminus V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ by Local Cohomology, Theorem 51.11.6.

For example, if A/fA is (S_2) and if every irreducible component of $Z = V(\mathfrak{a})$ has codimension ≥ 3 in $Y = \text{Spec}(A/fA)$, then we get the finiteness of $H^1_{\mathfrak{a}}(A/fA)$ and $H^2_{\mathfrak{a}}(A/fA)$. This should be contrasted with the slightly weaker conditions found in Lemma 52.20.1 (see also Remark 52.20.2).

52.17. Algebraization of coherent formal modules, II

0EIT We continue the discussion started in Section 52.16. This section can be skipped on a first reading.

0EIU Lemma 52.17.1. In Situation 52.16.1. Let $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ be a morphism of $\text{Coh}(U, I\mathcal{O}_U)$ whose kernel and cokernel are annihilated by a power of I. Then

(1) $(\mathcal{F}_n)$ extends to X if and only if $(\mathcal{F}'_n)$ extends to X, and
(2) $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module if and only if $(\mathcal{F}'_n)$ is.

Proof. Part (2) follows immediately from Cohomology of Schemes, Lemma 30.23.6. To see part (1), we first use Lemma 52.16.6 to reduce to the case where A is I-adically complete. However, in that case (1) reduces to (2) by Lemma 52.16.3. □

The following two lemmas where originally used in the proof of Lemma 52.16.10. We keep them here for the reader who is interested to know what intermediate results one can obtain.

0EHF Lemma 52.17.2. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. If the inverse system $H^0(U, \mathcal{F}_n)$ has Mittag-Leffler, then the canonical maps

$$\widetilde{M/I^nM}|_U \to \mathcal{F}_n$$

are surjective for all n where M is as in (52.16.6.1).

Proof. Surjectivity may be checked on the stalk at some point $y \in Y \setminus Z$. If y corresponds to the prime $\mathfrak{q} \subset A$, then we can choose $f \in \mathfrak{a}$, $f \notin \mathfrak{q}$. Then it suffices to show

$$M_f \longrightarrow H^0(U, \mathcal{F}_n)_f = H^0(D(f), \mathcal{F}_n)$$

is surjective as $D(f)$ is affine (equality holds by Properties, Lemma 28.17.1). Since we have the Mittag-Leffler property, we find that

$$\mathrm{Im}(M \to H^0(U, \mathcal{F}_n)) = \mathrm{Im}(H^0(U, \mathcal{F}_m) \to H^0(U, \mathcal{F}_n))$$

for some $m \geq n$. Using the long exact sequence of cohomology we see that

$$\mathrm{Coker}(H^0(U, \mathcal{F}_m) \to H^0(U, \mathcal{F}_n)) \subset H^1(U, \mathrm{Ker}(\mathcal{F}_m \to \mathcal{F}_n))$$

Since $U = X \setminus V(\mathfrak{a})$ this H^1 is $\mathfrak{a}$-power torsion. Hence after inverting f the cokernel becomes zero. $\square$

0EHG Lemma 52.17.3. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Let M be as in (52.16.6.1). Set

$$\mathcal{G}_n = \widetilde{M/I^nM}.$$

If the limit topology on M agrees with the I-adic topology, then $\mathcal{G}_n|_U$ is a coherent $\mathcal{O}_U$-module and the map of inverse systems

$$(\mathcal{G}_n|_U) \longrightarrow (\mathcal{F}_n)$$

is injective in the abelian category $\mathrm{Coh}(U, I\mathcal{O}_U)$.

Proof. Observe that $\mathcal{G}_n$ is a quasi-coherent $\mathcal{O}_X$-module annihilated by I^n and that $\mathcal{G}_{n+1}/I^n\mathcal{G}_{n+1} = \mathcal{G}_n$. Consider

$$M_n = \mathrm{Im}(M \longrightarrow H^0(U, \mathcal{F}_n))$$

The assumption says that the inverse systems (M_n) and (M/I^nM) are isomorphic as pro-objects of Mod_A. Pick $f \in \mathfrak{a}$ so $D(f) \subset U$ is an affine open. Then we have

$$(M_n)_f \subset H^0(U, \mathcal{F}_n)_f = H^0(D(f), \mathcal{F}_n)$$

Equality holds by Properties, Lemma 28.17.1. Thus $\widetilde{M_n}|_U \to \mathcal{F}_n$ is injective. It follows that $\widetilde{M_n}|_U$ is a coherent module (Cohomology of Schemes, Lemma 30.9.3). Since $M \to M/I^nM$ is surjective and factors as $M_{n'} \to M/I^nM$ for some $n' \geq n$ we find that $\mathcal{G}_n|_U$ is coherent as the quotient of a coherent module. Combined with the initical remarks of the proof we conclude that $(\mathcal{G}_n|_U)$ indeed forms an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Finally, to show the injectivity of the map it suffices to show that

$$\lim (M/I^nM)_f = \lim H^0(D(f), \mathcal{G}_n) \to \lim H^0(D(f), \mathcal{F}_n)$$

is injective, see Cohomology of Schemes, Lemmas 30.23.2 and 30.23.1. The injectivity of $\lim (M_n)_f \to \lim H^0(D(f), \mathcal{F}_n)$ is clear (see above) and by our remark on pro-systems we have $\lim (M_n)_f = \lim (M/I^nM)_f$. This finishes the proof. $\square$

52.18. A distance function

0EIW Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset. We define a function

0EIX
$$\delta_Z^Y = \delta_Z : Y \longrightarrow \mathbf{Z}_{\geq 0} \cup \{\infty\} \quad (52.18.0.1)$$

which measures the "distance" of a point of Y from Z. For an informal discussion, please see Remark 52.18.3. Let $y \in Y$. We set $\delta_Z(y) = \infty$ if y is contained in a connected component of Y which does not meet Z. If y is contained in a connected component of Y which meets Z, then we can find $k \geq 0$ and a system

$$V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \supset V_k \subset W_k$$

of integral closed subschemes of Y such that $V_0 \subset Z$ and $y \in V_k$ is the generic point. Set $c_i = \text{codim}(V_i, W_i)$ for $i = 0, \ldots, k$ and $b_i = \text{codim}(V_{i+1}, W_i)$ for $i = 0, \ldots, k-1$. For such a system we set

$$\delta(V_0, W_0, V_1, \ldots, W_k) = k + \max_{i=0,1,\ldots,k} (c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1})$$

This is $\geq k$ as we can take $i = k$ and we have $c_k \geq 0$. Finally, we set

$$\delta_Z(y) = \min \delta(V_0, W_0, V_1, \ldots, W_k)$$

where the minimum is over all systems of integral closed subschemes of Y as above.

0EIY **Lemma 52.18.1.** Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset.

(1) For $y \in Y$ we have $\delta_Z(y) = 0 \Leftrightarrow y \in Z$.
(2) The subsets $\{y \in Y \mid \delta_Z(y) \leq k\}$ are stable under specialization.
(3) For $y \in Y$ and $z \in \overline{\{y\}} \cap Z$ we have $\dim(\mathcal{O}_{\overline{\{y\}},z}) \geq \delta_Z(y)$.
(4) If δ is a dimension function on Y, then $\delta(y) \leq \delta_Z(y) + \delta_{max}$ where δ_{max} is the maximum value of δ on Z.
(5) If $Y = \text{Spec}(A)$ is the spectrum of a catenary Noetherian local ring with maximal ideal $\mathfrak{m}$ and $Z = \{\mathfrak{m}\}$, then $\delta_Z(y) = \dim(\overline{\{y\}})$.
(6) Given a pattern of specializations

$$\begin{tikzcd}[column sep=small] & y'_0 \arrow[dl, squiggly] \arrow[dr, squiggly] & & y'_1 \arrow[dl, squiggly] & \ldots & y'_{k-1} \arrow[dr, squiggly] & \\ y_0 & & y_1 & & \ldots & & y_k = y \end{tikzcd}$$

between points of Y with $y_0 \in Z$ and $y'_i \rightsquigarrow y_i$ an immediate specialization, then $\delta_Z(y_k) \leq k$.
(7) If $Y' \subset Y$ is an open subscheme, then $\delta^{Y'}_{Y' \cap Z}(y') \geq \delta^Y_Z(y')$ for $y' \in Y'$.

Proof. Part (1) is essentially true by definition. Namely, if $y \in Z$, then we can take $k = 0$ and $V_0 = W_0 = \overline{\{y\}}$.

Proof of (2). Let $y \rightsquigarrow y'$ be a nontrivial specialization and let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ is a system for y. Here there are two cases. Case I: $V_k = W_k$, i.e., $c_k = 0$. In this case we can set $V'_k = W'_k = \overline{\{y'\}}$. An easy computation shows that $\delta(V_0, W_0, \ldots, V'_k, W'_k) \leq \delta(V_0, W_0, \ldots, V_k, W_k)$ because only b_{k-1} is changed into a bigger integer. Case II: $V_k \neq W_k$, i.e., $c_k > 0$. Observe that in this case $\max_{i=0,1,\ldots,k}(c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1}) > 0$. Hence if we set

$V'_{k+1} = W_{k+1} = \overline{\{y'\}}$, then although k is replaced by $k+1$, the maximum now looks like

$$\max_{i=0,1,\ldots,k+1}(c_i + c_{i+1} + \ldots + c_k + c_{k+1} - b_i - b_{i+1} - \ldots - b_{k-1} - b_k)$$

with $c_{k+1} = 0$ and $b_k = \text{codim}(V_{k+1}, W_k) > 0$. This is strictly smaller than $\max_{i=0,1,\ldots,k}(c_i+c_{i+1}+\ldots+c_k-b_i-b_{i+1}-\ldots-b_{k-1})$ and hence $\delta(V_0, W_0, \ldots, V'_{k+1}, W'_{k+1}) \leq \delta(V_0, W_0, \ldots, V_k, W_k)$ as desired.

Proof of (3). Given $y \in Y$ and $z \in \overline{\{y\}} \cap Z$ we get the system

$$V_0 = \overline{\{z\}} \subset W_0 = \overline{\{y\}}$$

and $c_0 = \text{codim}(V_0, W_0) = \dim(\mathcal{O}_{\overline{\{y\}},z})$ by Properties, Lemma 28.10.3. Thus we see that $\delta(V_0, W_0) = 0 + c_0 = c_0$ which proves what we want.

Proof of (4). Let δ be a dimension function on Y. Let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ be a system for y. Let $y'_i \in W_i$ and $y_i \in V_i$ be the generic points, so $y_0 \in Z$ and $y_k = y$. Then we see that

$$\delta(y_i) - \delta(y_{i-1}) = \delta(y'_{i-1}) - \delta(y_{i-1}) - \delta(y'_{i-1}) + \delta(y_i) = c_{i-1} - b_{i-1}$$

Finally, we have $\delta(y'_k) - \delta(y_{k-1}) = c_k$. Thus we see that

$$\delta(y) - \delta(y_0) = c_0 + \ldots + c_k - b_0 - \ldots - b_{k-1}$$

We conclude $\delta(V_0, W_0, \ldots, W_k) \geq k + \delta(y) - \delta(y_0)$ which proves what we want.

Proof of (5). The function $\delta(y) = \dim(\overline{\{y\}})$ is a dimension function. Hence $\delta(y) \leq \delta_Z(y)$ by part (4). By part (3) we have $\delta_Z(y) \leq \delta(y)$ and we are done.

Proof of (6). Given such a sequence of points, we may assume all the specializations $y'_i \rightsquigarrow y_{i+1}$ are nontrivial (otherwise we can shorten the chain of specializations). Then we set $V_i = \overline{\{y_i\}}$ and $W_i = \overline{\{y'_i\}}$ and we compute $\delta(V_0, W_1, V_1, \ldots, W_{k-1}) = k$ because all the codimensions c_i of $V_i \subset W_i$ are 1 and all $b_i > 0$. This implies $\delta_Z(y'_{k-1}) \leq k$ as y'_{k-1} is the generic point of W_k. Then $\delta_Z(y) \leq k$ by part (2) as y is a specialization of y_{k-1}.

Proof of (7). This is clear as their are fewer systems to consider in the computation of $\delta^{Y'}_{Y' \cap Z}$. ☐

0EIZ Lemma 52.18.2. Let Y be a universally catenary Noetherian scheme. Let $Z \subset Y$ be a closed subscheme. Let $f : Y' \to Y$ be a finite type morphism all of whose fibres have dimension $\leq e$. Set $Z' = f^{-1}(Z)$. Then

$$\delta_Z(y) \leq \delta_{Z'}(y') + e - \text{trdeg}_{\kappa(y)}(\kappa(y'))$$

for $y' \in Y'$ with image $y \in Y$.

Proof. If $\delta_{Z'}(y') = \infty$, then there is nothing to prove. If $\delta_{Z'}(y') < \infty$, then we choose a system of integral closed subschemes

$$V'_0 \subset W'_0 \supset V'_1 \subset W'_1 \supset \ldots \subset W'_k$$

of Y' with $V'_0 \subset Z'$ and y' the generic point of W'_k such that $\delta_{Z'}(y') = \delta(V'_0, W'_0, \ldots, W'_k)$. Denote

$$V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$$

the scheme theoretic images of the above schemes in Y. Observe that y is the generic point of W_k and that $V_0 \subset Z$. For each i we look at the diagram

$$\begin{tikzcd} V_i' \arrow[r] \arrow[d] & W_i' \arrow[d] & V_{i+1}' \arrow[l] \arrow[d] \\ V_i \arrow[r] & W_i & V_{i+1} \arrow[l] \end{tikzcd}$$

Denote n_i the relative dimension of V_i'/V_i and m_i the relative dimension of W_i'/W_i; more precisely these are the transcendence degrees of the corresponding extensions of the function fields. Set $c_i = \text{codim}(V_i, W_i)$, $c_i' = \text{codim}(V_i', W_i')$, $b_i = \text{codim}(V_{i+1}, W_i)$, and $b_i' = \text{codim}(V_{i+1}', W_i')$. By the dimension formula we have

$$c_i = c_i' + n_i - m_i \quad \text{and} \quad b_i = b_i' + n_{i+1} - m_i$$

See Morphisms, Lemma 29.52.1. Hence $c_i - b_i = c_i' - b_i' + n_i - n_{i+1}$. Thus we see that

$$\begin{aligned} & c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1} \\ & = c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i - n_k + c_k - c_k' \\ & = c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i - m_k \end{aligned}$$

Thus we see that

$$\begin{aligned} & \max_{i=0,\ldots,k} (c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1}) \\ & \quad = \max_{i=0,\ldots,k} (c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i - m_k) \\ & \quad = \max_{i=0,\ldots,k} (c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i) - m_k \\ & \quad \leq \max_{i=0,\ldots,k} (c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}') + e - m_k \end{aligned}$$

Since $m_k = \text{trdeg}_{\kappa(y)}(\kappa(y'))$ we conclude that

$$\delta(V_0, W_0, \ldots, W_k) \leq \delta(V_0', W_0', \ldots, W_k') + e - \text{trdeg}_{\kappa(y)}(\kappa(y'))$$

as desired. $\square$

0EJ0 Remark 52.18.3. Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset. By Lemma 52.18.1 we have

$$\delta_Z(y) \leq \min \left\{ k \middle| \begin{array}{c} \text{there exist specializations in } Y \\ y_0 \leftarrow y_0' \rightarrow y_1 \leftarrow y_1' \rightarrow \ldots \leftarrow y_{k-1}' \rightarrow y_k = y \\ \text{with } y_0 \in Z \text{ and } y_i' \rightsquigarrow y_i \text{ immediate} \end{array} \right\}$$

We claim that if Y is of finite type over a field, then equality holds. If we ever need this result we will formulate a precise result and prove it here. However, in general if we define δ_Z by the right hand side of this inequality, then we don't know if Lemma 52.18.2 remains true.

0EJ1 Example 52.18.4. Let k be a field and $Y = \mathbf{A}^n_k$. Denote $\delta : Y \to \mathbf{Z}_{\geq 0}$ the usual dimension function.

(1) If $Z = \{z\}$ for some closed point z, then
  (a) $\delta_Z(y) = \delta(y)$ if $y \rightsquigarrow z$ and
  (b) $\delta_Z(y) = \delta(y) + 1$ if $y \not\rightsquigarrow z$.
(2) If Z is a closed subvariety and $W = \overline{\{y\}}$, then

(a) $\delta_Z(y) = 0$ if $W \subset Z$,
(b) $\delta_Z(y) = \dim(W) - \dim(Z)$ if Z is contained in W,
(c) $\delta_Z(y) = 1$ if $\dim(W) \leq \dim(Z)$ and $W \not\subset Z$,
(d) $\delta_Z(y) = \dim(W) - \dim(Z) + 1$ if $\dim(W) > \dim(Z)$ and $Z \not\subset W$.

A generalization of case (1) is if Y is of finite type over a field and $Z = \{z\}$ is a closed point. Then $\delta_Z(y) = \delta(y) + t$ where t is the minimum length of a chain of curves connecting z to a closed point of $\overline{\{y\}}$.

52.19. Algebraization of coherent formal modules, III

0EJ2 We continue the discussion started in Sections 52.16 and 52.17. We will use the distance function of Section 52.18 to formulate a some natural conditions on coherent formal modules in Situation 52.16.1.

In Situation 52.16.1 given a point $y \in U \cap Y$ we can consider the I-adic completion

$$\mathcal{O}_{X,y}^\wedge = \lim \mathcal{O}_{X,y}/I^n\mathcal{O}_{X,y}$$

This is a Noetherian local ring complete with respect to $I\mathcal{O}_{X,y}^\wedge$ with maximal ideal $\mathfrak{m}_y^\wedge$, see Algebra, Section 10.97. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Let us define the "stalk" of $(\mathcal{F}_n)$ at y by the formula

$$\mathcal{F}_y^\wedge = \lim \mathcal{F}_{n,y}$$

This is a finite module over $\mathcal{O}_{X,y}^\wedge$. See Algebra, Lemmas 10.98.2 and 10.96.12.

0EJ3 Definition 52.19.1. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Let a, b be integers. Let δ_Z^Y be as in (52.18.0.1). We say $(\mathcal{F}_n)$ satisfies the (a, b)-inequalities if for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$

(1) if $V(\mathfrak{p}) \cap V(I\mathcal{O}_{X,y}^\wedge) \neq \{\mathfrak{m}_y^\wedge\}$, then

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \delta_Z^Y(y) \geq a \quad \text{or} \quad \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > b$$

(2) if $V(\mathfrak{p}) \cap V(I\mathcal{O}_{X,y}^\wedge) = \{\mathfrak{m}_y^\wedge\}$, then

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \delta_Z^Y(y) > a$$

We say $(\mathcal{F}_n)$ satisfies the strict (a, b)-inequalities if for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$ we have

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \delta_Z^Y(y) > a \quad \text{or} \quad \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > b$$

Here are some elementary observations.

0EJ4 Lemma 52.19.2. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Let a, b be integers.

(1) If $(\mathcal{F}_n)$ is annihilated by a power of I, then $(\mathcal{F}_n)$ satisfies the (a, b)-inequalities for any a, b.
(2) If $(\mathcal{F}_n)$ satisfies the $(a + 1, b)$-inequalities, then $(\mathcal{F}_n)$ satisfies the strict (a, b)-inequalities.

If $\text{cd}(A, I) \leq d$ and A has a dualizing complex, then

(3) $(\mathcal{F}_n)$ satisfies the $(s, s + d)$-inequalities if and only if for all $y \in U \cap Y$ the tuple $\mathcal{O}_{X,y}^\wedge, I\mathcal{O}_{X,y}^\wedge, \{\mathfrak{m}_y^\wedge\}, \mathcal{F}_y^\wedge, s - \delta_Z^Y(y), d$ is as in Situation 52.10.1.
(4) If $(\mathcal{F}_n)$ satisfies the strict $(s, s + d)$-inequalities, then $(\mathcal{F}_n)$ satisfies the $(s, s + d)$-inequalities.

Proof. Immediate except for part (4) which is a consequence of Lemma 52.10.5 and the translation in (3). $\square$

0EKW Lemma 52.19.3. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. If $\text{cd}(A, I) = 1$, then $\mathcal{F}$ satisfies the $(2,3)$-inequalities if and only if

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > 3$$

for all $y \in U \cap Y$ and $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$.

Proof. Observe that for a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$, $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$ we have $V(\mathfrak{p}) \cap V(I\mathcal{O}_{X,y}^\wedge) = \{\mathfrak{m}_y^\wedge\} \Leftrightarrow \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) = 1$ as $\text{cd}(A, I) = 1$. See Local Cohomology, Lemmas 51.4.5 and 51.4.10. OK, consider the three numbers $\alpha = \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) \geq 0$, $\beta = \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) \geq 1$, and $\gamma = \delta_Z^Y(y) \geq 1$. Then we see Definition 52.19.1 requires

(1) if $\beta > 1$, then $\alpha + \gamma \geq 2$ or $\alpha + \beta + \gamma > 3$, and
(2) if $\beta = 1$, then $\alpha + \gamma > 2$.

It is trivial to see that this is equivalent to $\alpha + \beta + \gamma > 3$. $\square$

In the rest of this section, which we suggest the reader skip on a first reading, we will show that, when A is I-adically complete, the category of $(\mathcal{F}_n)$ of $\textit{Coh}(U, I\mathcal{O}_U)$ which extend to X and satisfy the strict $(1, 1 + \text{cd}(A, I))$-inequalities is equivalent to a full subcategory of the category of coherent $\mathcal{O}_U$-modules.

0EJ5 Lemma 52.19.4. In Situation 52.16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume

(1) A is I-adically complete, has a dualizing complex, and $\text{cd}(A, I) \leq d$,
(2) the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities.

Let $x \in X$ be a point. Let $W = \overline{\{x\}}$. If $W \cap Y$ has an irreducible component contained in Z and one which is not, then $\text{depth}(\mathcal{F}_x) \geq 1$.

Proof. Let $W \cap Y = W_1 \cup \ldots \cup W_n$ be the decomposition into irreducible components. By assumption, after renumbering, we can find $0 < m < n$ such that $W_1, \ldots, W_m \subset Z$ and $W_{m+1}, \ldots, W_n \not\subset Z$. We conclude that

$$W \cap Y \setminus ((W_1 \cup \ldots \cup W_m) \cap (W_{m+1} \cup \ldots \cup W_n))$$

is disconnected. By Lemma 52.14.2 we can find $1 \leq i \leq m < j \leq n$ and $z \in W_i \cap W_j$ such that $\dim(\mathcal{O}_{W,z}) \leq d + 1$. Choose an immediate specialization $y \rightsquigarrow z$ with $y \in W_j$, $y \not\in Z$; existence of y follows from Properties, Lemma 28.6.4. Observe that $\delta_Z^Y(y) = 1$ and $\dim(\mathcal{O}_{W,y}) \leq d$. Let $\mathfrak{p} \subset \mathcal{O}_{X,y}$ be the prime corresponding to x. Let $\mathfrak{p}' \subset \mathcal{O}_{X,y}^\wedge$ be a minimal prime over $\mathfrak{p}\mathcal{O}_{X,y}^\wedge$. Then we have

$$\text{depth}(\mathcal{F}_x) = \text{depth}((\mathcal{F}_y^\wedge)_{\mathfrak{p}'}) \quad \text{and} \quad \dim(\mathcal{O}_{W,y}) = \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}')$$

See Algebra, Lemma 10.163.1 and Local Cohomology, Lemma 51.11.3. Now we read off the conclusion from the inequalities given to us. $\square$

0EJ6 Lemma 52.19.5. In Situation 52.16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume

(1) A is I-adically complete, has a dualizing complex, and $\text{cd}(A, I) \leq d$,
(2) the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities, and
(3) for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $\text{depth}(\mathcal{F}_x) \geq 2$.

Then $H^0(U, \mathcal{F}) \to \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ is an isomorphism.

Proof. We will prove this by showing that Lemma 52.12.4 applies. Thus we let $x \in \text{Ass}(\mathcal{F})$ with $x \notin Y$. Set $W = \overline{\{x\}}$. By condition (3) we see that $W \cap Y \not\subset Z$. By Lemma 52.19.4 we see that no irreducible component of $W \cap Y$ is contained in Z. Thus if $z \in W \cap Z$, then there is an immediate specialization $y \rightsquigarrow z$, $y \in W \cap Y$, $y \notin Z$. For existence of y use Properties, Lemma 28.6.4. Then $\delta^Y_Z(y) = 1$ and the assumption implies that $\dim(\mathcal{O}_{W,y}) > d$. Hence $\dim(\mathcal{O}_{W,z}) > 1 + d$ and we win. $\square$

0EJ7 Lemma 52.19.6. In Situation 52.16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume

(1) A is I-adically complete, has a dualizing complex, and $\text{cd}(A, I) \leq d$,
(2) the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities, and
(3) for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $\text{depth}(\mathcal{F}_x) \geq 2$.

Then the map

$$\text{Hom}_U(\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_{\textit{Coh}(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective for every coherent $\mathcal{O}_U$-module $\mathcal{G}$.

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F})$. Using Cohomology of Schemes, Lemma 30.11.2 or More on Algebra, Lemma 15.23.10 we see that the completion of $\mathcal{H}$ satisfies the strict $(1, 1 + d)$-inequalities and that for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $\text{depth}(\mathcal{H}_x) \geq 2$. Details omitted. Thus by Lemma 52.19.5 we have

$$\text{Hom}_U(\mathcal{G}, \mathcal{F}) = H^0(U, \mathcal{H}) = \lim H^0(U, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \text{Mor}_{\textit{Coh}(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

See Cohomology of Schemes, Lemma 30.23.5 for the final equality. $\square$

0EJ8 Lemma 52.19.7. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$ and $d \geq 1$. Assume

(1) A is I-adically complete, has a dualizing complex, and $\text{cd}(A, I) \leq d$,
(2) $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module,
(3) $(\mathcal{F}_n)$ satisfies the strict $(1, 1 + d)$-inequalities.

Then there exists a unique coherent $\mathcal{O}_U$-module $\mathcal{F}$ whose completion is $(\mathcal{F}_n)$ such that for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $\text{depth}(\mathcal{F}_x) \geq 2$.

Proof. Choose a coherent $\mathcal{O}_U$-module $\mathcal{F}$ whose completion is $(\mathcal{F}_n)$. Let $T = \{x \in U \mid \overline{\{x\}} \cap Y \subset Z\}$. We will construct $\mathcal{F}$ by applying Local Cohomology, Lemma 51.15.4 with $\mathcal{F}$ and T. Then uniqueness will follow from the mapping property of Lemma 52.19.6.

Since T is stable under specialization in U the only thing to check is the following. If $x' \rightsquigarrow x$ is an immediate specialization of points of U with $x \in T$ and $x' \notin T$, then $\text{depth}(\mathcal{F}_{x'}) \geq 1$. Set $W = \overline{\{x\}}$ and $W' = \overline{\{x'\}}$. Since $x' \notin T$ we see that $W' \cap Y$ is not contained in Z. If $W' \cap Y$ contains an irreducible component contained in Z, then we are done by Lemma 52.19.4. If not, we choose an irreducible component W_1 of $W \cap Y$ and an irreducible component W'_1 of $W' \cap Y$ with $W_1 \subset W'_1$. Let $z \in W_1$ be the generic point. Let $y \rightsquigarrow z$, $y \in W'_1$ be an immediate specialization with $y \notin Z$; existence of y follows from $W'_1 \not\subset Z$ (see above) and Properties, Lemma 28.6.4. Then we have the following $z \in Z$, $x \rightsquigarrow z$, $x' \rightsquigarrow y \rightsquigarrow z$, $y \in Y \setminus Z$, and $\delta^Y_Z(y) = 1$. By Local Cohomology, Lemma 51.4.10 and the fact that z is a generic point of $W \cap Y$ we have $\dim(\mathcal{O}_{W,z}) \leq d$. Since $x' \rightsquigarrow x$ is an immediate specialization we have $\dim(\mathcal{O}_{W',z}) \leq d + 1$. Since $y \neq z$ we conclude $\dim(\mathcal{O}_{W',y}) \leq d$. If $\text{depth}(\mathcal{F}_{x'}) = 0$

then we would get a contradiction with assumption (3); details about passage from $\mathcal{O}_{X,y}$ to its completion omitted. This finishes the proof. □

52.20. Algebraization of coherent formal modules, IV

0EHJ In this section we prove two stronger versions of Lemma 52.16.11 in the local case, namely, Lemmas 52.20.1 and 52.20.4. Although these lemmas will be obsoleted by the more general Proposition 52.22.2, their proofs are significantly easier.

0DXU **Lemma 52.20.1.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

1. A is local and $\mathfrak{a} = \mathfrak{m}$ is the maximal ideal,
2. A has a dualizing complex,
3. $I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak{m}$,
4. $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/f^n\mathcal{O}_U$-module,
5. if $\mathfrak{p} \in V(f) \setminus \{\mathfrak{m}\}$, then $\text{depth}((A/f)_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 1$, and
6. if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 52.16.10.

Since $\mathcal{F}_n$ is locally free over $\mathcal{O}_U/f^n\mathcal{O}_U$ we see that we have short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$ for all n. Thus condition (b) holds by Cohomology, Lemma 20.36.2.

By induction on n and the short exact sequences $0 \to A/f^n \to A/f^{n+1} \to A/f \to 0$ we see that the associated primes of A/f^nA agree with the associated primes of A/fA. Since the associated points of $\mathcal{F}_n$ correspond to the associated primes of A/f^nA not equal to $\mathfrak{m}$ by assumption (3), we conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a finite A-module by (5) and Local Cohomology, Proposition 51.8.7. Thus hypothesis (c) holds.

To finish the proof it suffices to show that there exists an $n > 1$ such that the image of

$$H^1(U, \mathcal{F}_n) \longrightarrow H^1(U, \mathcal{F}_1)$$

has finite length as an A-module. Namely, this will imply hypothesis (a) by Cohomology, Lemma 20.36.5. The image is independent of n for n large enough by Lemma 52.5.2. Let $\omega_A^\bullet$ be a normalized dualizing complex for A. By the local duality theorem and Matlis duality (Dualizing Complexes, Lemma 47.18.4 and Proposition 47.7.8) our claim is equivalent to: the image of

$$\text{Ext}^{-2}_A(M_1, \omega_A^\bullet) \to \text{Ext}^{-2}_A(M_n, \omega_A^\bullet)$$

has finite length for $n \gg 1$. The modules in question are finite A-modules supported at $V(f)$. Thus it suffices to show that this map is zero after localization at a prime $\mathfrak{q}$ containing f and different from $\mathfrak{m}$. Let $\omega_{A_\mathfrak{q}}^\bullet$ be a normalized dualizing complex on $A_\mathfrak{q}$ and recall that $\omega_{A_\mathfrak{q}}^\bullet = (\omega_A^\bullet)_\mathfrak{q}[\dim(A/\mathfrak{q})]$ by Dualizing Complexes, Lemma 47.17.3. Using the local structure of $\mathcal{F}_n$ given in (4) we find that it suffices to show the vanishing of

$$\text{Ext}^{-2+\dim(A/\mathfrak{q})}_{A_\mathfrak{q}}(A_\mathfrak{q}/f, \omega_{A_\mathfrak{q}}^\bullet) \to \text{Ext}^{-2+\dim(A/\mathfrak{q})}_{A_\mathfrak{q}}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet)$$

for n large enough. If $\dim(A/\mathfrak q) > 3$, then this is immediate from Local Cohomology, Lemma 51.9.4. For the other cases we will use the long exact sequence

$$\ldots \xrightarrow{f^n} H^{-1}(\omega_{A_\mathfrak q}^\bullet) \to \mathrm{Ext}^{-1}_{A_\mathfrak q}(A_\mathfrak q/f^n, \omega_{A_\mathfrak q}^\bullet) \to H^0(\omega_{A_\mathfrak q}^\bullet) \xrightarrow{f^n} H^0(\omega_{A_\mathfrak q}^\bullet) \to \mathrm{Ext}^0_{A_\mathfrak q}(A_\mathfrak q/f^n, \omega_{A_\mathfrak q}^\bullet) \to 0$$

If $\dim(A/\mathfrak q) = 2$, then $H^0(\omega_{A_\mathfrak q}^\bullet) = 0$ because $\mathrm{depth}(A_\mathfrak q) \geq 1$ as f is a nonzerodivisor. Thus the long exact sequence shows the condition is that

$$f^{n-1} : H^{-1}(\omega_{A_\mathfrak q}^\bullet)/f \to H^{-1}(\omega_{A_\mathfrak q}^\bullet)/f^n$$

is zero. Now $H^{-1}(\omega_\mathfrak q^\bullet)$ is a finite module supported in the primes $\mathfrak p \subset A_\mathfrak q$ such that $\mathrm{depth}(A_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) \leq 1$. Since $\dim((A/\mathfrak p)_\mathfrak q) = \dim(A/\mathfrak p) - 2$ condition (6) tells us these primes are contained in $V(f)$. Thus the desired vanishing for n large enough. Finally, if $\dim(A/\mathfrak q) = 1$, then condition (5) combined with the fact that f is a nonzerodivisor insures that $A_\mathfrak q$ has depth at least 2. Hence $H^0(\omega_{A_\mathfrak q}^\bullet) = H^{-1}(\omega_{A_\mathfrak q}^\bullet) = 0$ and the long exact sequence shows the claim is equivalent to the vanishing of

$$f^{n-1} : H^{-2}(\omega_{A_\mathfrak q}^\bullet)/f \to H^{-2}(\omega_{A_\mathfrak q}^\bullet)/f^n$$

Now $H^{-2}(\omega_\mathfrak q^\bullet)$ is a finite module supported in the primes $\mathfrak p \subset A_\mathfrak q$ such that $\mathrm{depth}(A_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) \leq 2$. By condition (6) all of these primes are contained in $V(f)$. Thus the desired vanishing for n large enough. □

0DXV Remark 52.20.2. Let $(A, \mathfrak m)$ be a complete Noetherian normal local domain of dimension ≥ 4 and let $f \in \mathfrak m$ be nonzero. Then assumptions (1), (2), (3), (5), and (6) of Lemma 52.20.1 are satisfied. Thus vectorbundles on the formal completion of U along $U \cap V(f)$ can be algebraized. In Lemma 52.20.4 we will generalize this to more general coherent formal modules; please also compare with Remark 52.20.7.

0EHK Lemma 52.20.3. In Situation 52.16.1 let (M_n) be an inverse system of A-modules as in Lemma 52.16.2 and let $(\mathcal F_n)$ be the corresponding object of $\mathrm{Coh}(U, I\mathcal O_U)$. Let $d \geq \mathrm{cd}(A, I)$ and $s \geq 0$ be integers. With notation as above assume

(1) A is local with maximal ideal $\mathfrak m = \mathfrak a$,
(2) A has a dualizing complex, and
(3) $(\mathcal F_n)$ satisfies the $(s, s + d)$-inequalities (Definition 52.19.1).

Let E be an injective hull of the residue field of A. Then for $i \leq s$ there exists a finite A-module N annihilated by a power of I and for $n \gg 0$ compatible maps

$$H^i_\mathfrak m(M_n) \to \mathrm{Hom}_A(N, E)$$

whose cokernels are finite length A-modules and whose kernels K_n form an inverse system such that $\mathrm{Im}(K_{n''} \to K_{n'})$ has finite length for $n'' \gg n' \gg 0$.

Proof. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\delta^Y_Z = \delta$ is the dimension function associated with this dualizing complex. Observe that $\mathrm{Ext}^{-i}_A(M_n, \omega_A^\bullet)$ is a finite A-module annihilated by I^n. Fix $0 \leq i \leq s$. Below we will find $n_1 > n_0 > 0$ such that if we set

$$N = \mathrm{Im}(\mathrm{Ext}^{-i}_A(M_{n_0}, \omega_A^\bullet) \to \mathrm{Ext}^{-i}_A(M_{n_1}, \omega_A^\bullet))$$

then the kernels of the maps

$$N \to \mathrm{Ext}^{-i}_A(M_n, \omega_A^\bullet), \quad n \geq n_1$$

are finite length A-modules and the cokernels Q_n form a system such that $\mathrm{Im}(Q_{n'} \to Q_{n''})$ has finite length for $n'' \gg n' \gg n_1$. This is equivalent to the statement that the system $\{\text{Ext}^{-i}_A(M_n, \omega_A^\bullet)\}_{n \geq 1}$ is essentially constant in the quotient of the category of finite A-modules modulo the Serre subcategory of finite length A-modules. By the local duality theorem (Dualizing Complexes, Lemma 47.18.4) and Matlis duality (Dualizing Complexes, Proposition 47.7.8) we conclude that there are maps

$$H^i_\mathfrak{m}(M_n) \to \text{Hom}_A(N, E), \quad n \geq n_1$$

as in the statement of the lemma.

Pick $f \in \mathfrak{m}$. Let $B = A_f^\wedge$ be the I-adic completion of the localization A_f. Recall that $\omega_{A_f}^\bullet = \omega_A^\bullet \otimes_A A_f$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$ are dualizing complexes (Dualizing Complexes, Lemma 47.15.6 and 47.22.3). Let M be the finite B-module $\lim M_{n,f}$ (compare with discussion in Cohomology of Schemes, Lemma 30.23.1). Then

$$\text{Ext}^{-i}_A(M_n, \omega_A^\bullet)_f = \text{Ext}^{-i}_{A_f}(M_{n,f}, \omega_{A_f}^\bullet) = \text{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)$$

Since $\mathfrak{m}$ can be generated by finitely many $f \in \mathfrak{m}$ it suffices to show that for each f the system

$$\{\text{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)\}_{n \geq 1}$$

is essentially constant. Some details omitted.

Let $\mathfrak{q} \subset IB$ be a prime ideal. Then $\mathfrak{q}$ corresponds to a point $y \in U \cap Y$. Observe that $\delta(\mathfrak{q}) = \dim(\overline{\{y\}})$ is also the value of the dimension function associated to $\omega_B^\bullet$ (we omit the details; use that $\omega_B^\bullet$ is gotten from $\omega_A^\bullet$ by tensoring up with B). Assumption (3) guarantees via Lemma 52.19.2 that Lemma 52.10.4 applies to $B_\mathfrak{q}, IB_\mathfrak{q}, \mathfrak{q}B_\mathfrak{q}, M_\mathfrak{q}$ with s replaced by $s - \delta(y)$. We obtain that

$$H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_\mathfrak{q}}(M_\mathfrak{q}) = \lim H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_\mathfrak{q}}((M/I^nM)_\mathfrak{q})$$

and this module is annihilated by a power of I. By Lemma 52.5.3 we find that the inverse systems $H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_\mathfrak{q}}((M/I^nM)_\mathfrak{q})$ are essentially constant with value $H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_\mathfrak{q}}(M_\mathfrak{q})$. Since $(\omega_B^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})]$ is a normalized dualizing complex on $B_\mathfrak{q}$ the local duality theorem shows that the system

$$\text{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)_\mathfrak{q}$$

is essentially constant with value $\text{Ext}^{-i}_B(M, \omega_B^\bullet)_\mathfrak{q}$.

To finish the proof we globalize as in the proof of Lemma 52.10.6; the argument here is easier because we know the value of our system already. Namely, consider the maps

$$\alpha_n : \text{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet) \longrightarrow \text{Ext}^{-i}_B(M, \omega_B^\bullet)$$

for varying n. By the above, for every $\mathfrak{q}$ we can find an n such that α_n is surjective after localization at $\mathfrak{q}$. Since B is Noetherian and $\text{Ext}^{-i}_B(M, \omega_B^\bullet)$ a finite module, we can find an n such that α_n is surjective. For any n such that α_n is surjective, given a prime $\mathfrak{q} \in V(IB)$ we can find an $n' > n$ such that $\text{Ker}(\alpha_n)$ maps to zero in $\text{Ext}^{-i}(M/I^{n'}M, \omega_B^\bullet)$ at least after localizing at $\mathfrak{q}$. Since $\text{Ker}(\alpha_n)$ is a finite A-module and since supports of sections are quasi-compact, we can find an n' such that $\text{Ker}(\alpha_n)$ maps to zero in $\text{Ext}^{-i}(M/I^{n'}M, \omega_B^\bullet)$. In this way we see that $\text{Ext}^{-i}(M/I^nM, \omega_B^\bullet)$ is essentially constant with value $\text{Ext}^{-i}(M, \omega_B^\bullet)$. This finishes the proof. $\square$

Here is a more general version of Lemma 52.20.1.

0EJ9 **Lemma 52.20.4.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A is local and $\mathfrak{a} = \mathfrak{m}$ is the maximal ideal,
(2) A has a dualizing complex,
(3) $I = (f)$ is a principal ideal,
(4) $(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities.

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. Recall that $\text{Coh}(U, I\mathcal{O}_U)$ is an abelian category, see Cohomology of Schemes, Lemma 30.23.2. Over affine opens of U the object $(\mathcal{F}_n)$ corresponds to a finite module over a Noetherian ring (Cohomology of Schemes, Lemma 30.23.1). Thus the kernels of the maps $f^N : (\mathcal{F}_n) \to (\mathcal{F}_n)$ stabilize for N large enough. By Lemmas 52.17.1 and 52.16.3 in order to prove the lemma we may replace $(\mathcal{F}_n)$ by the image of such a map. Thus we may assume f is injective on $(\mathcal{F}_n)$. After this replacement the equivalent conditions of Lemma 52.3.1 hold for the inverse system $(\mathcal{F}_n)$ on U. We will use this without further mention in the rest of the proof.

We will check hypotheses (a), (b), and (c) of Lemma 52.16.10. Hypothesis (b) holds by Cohomology, Lemma 20.36.2.

Pick a inverse system of modules $\{M_n\}$ as in Lemma 52.16.2. We may assume $H^0_{\mathfrak{m}}(M_n) = 0$ by replacing M_n by $M_n/H^0_{\mathfrak{m}}(M_n)$ if necessary. Then we obtain short exact sequences

$$0 \to M_n \to H^0(U, \mathcal{F}_n) \to H^1_{\mathfrak{m}}(M_n) \to 0$$

for all n. Let E be an injective hull of the residue field of A. By Lemma 52.20.3 and our current assumption (4) we can choose, an integer $m \geq 0$, finite A-modules N_1 and N_2 annihilated by f^c for some $c \geq 0$ and compatible systems of maps

$$H^i_{\mathfrak{m}}(M_n) \to \text{Hom}_A(N_i, E), \quad i = 1, 2$$

for $n \geq m$ with the properties stated in the lemma.

We know that $M = \lim H^0(U, \mathcal{F}_n)$ is an A-module whose limit topology is the f-adic topology. Thus, given n, the module M/f^nM is a subquotient of $H^0(U, \mathcal{F}_N)$ for some $N \gg n$. Looking at the information obtained above we see that f^cM/f^nM is a finite A-module. Since f is a nonzerodivisor on M we conclude that $M/f^{n-c}M$ is a finite A-module. In this way we see that hypothesis (c) of Lemma 52.16.10 holds.

Next, we study the module

$$Ob = \lim H^1(U, \mathcal{F}_n) = \lim H^2_{\mathfrak{m}}(M_n)$$

For $n \geq m$ let K_n be the kernel of the map $H^2_{\mathfrak{m}}(M_n) \to \text{Hom}_A(N_2, E)$. Set $K = \lim K_n$. We obtain an exact sequence

$$0 \to K \to Ob \to \text{Hom}_A(N_2, E)$$

By the above the limit topology on $Ob = \lim H^2_{\mathfrak{m}}(M_n)$ is the f-adic topology. Since N_2 is annihilated by f^c we conclude the same is true for the limit topology on $K = \lim K_n$. Thus K/fK is a subquotient of K_n for $n \gg 1$. However, since $\{K_n\}$ is pro-isomorphic to a inverse system of finite length A-modules (by the conclusion of Lemma 52.20.3) we conclude that K/fK is a subquotient of a finite length A-module. It follows that K is a finite A-module, see Algebra, Lemma 10.96.12. (In fact, we even see that $\dim(\text{Supp}(K)) = 1$ but we will not need this.)

Given $n \geq 1$ consider the boundary map

$$\delta_n : H^0(U, \mathcal{F}_n) \longrightarrow \lim_N H^1(U, f^n\mathcal{F}_N) \xrightarrow{f^{-n}} Ob$$

(the second map is an isomorphism) coming from the short exact sequences

$$0 \to f^n\mathcal{F}_N \to \mathcal{F}_N \to \mathcal{F}_n \to 0$$

For each n set

$$P_n = \mathrm{Im}(H^0(U, \mathcal{F}_{n+m}) \to H^0(U, \mathcal{F}_n))$$

where m is as above. Observe that $\{P_n\}$ is an inverse system and that the map $f : \mathcal{F}_n \to \mathcal{F}_{n+1}$ on global sections maps P_n into P_{n+1}. If $p \in P_n$, then $\delta_n(p) \in K \subset Ob$ because $\delta_n(p)$ maps to zero in $H^1(U, f^n\mathcal{F}_{n+m}) = H^2_\mathfrak{m}(M_m)$ and the composition of δ_n and $Ob \to \mathrm{Hom}_A(N_2, E)$ factors through $H^2_\mathfrak{m}(M_m)$ by our choice of m. Hence

$$\bigoplus\nolimits_{n \geq 0} \mathrm{Im}(P_n \to Ob)$$

is a finite graded $A[T]$-module where T acts via multiplication by f. Namely, it is a graded submodule of $K[T]$ and K is finite over A. Arguing as in the proof of Cohomology, Lemma 20.35.1[7] we find that the inverse system $\{P_n\}$ satisfies ML. Since $\{P_n\}$ is pro-isomorphic to $\{H^0(U, \mathcal{F}_n)\}$ we conclude that $\{H^0(U, \mathcal{F}_n)\}$ has ML. Thus hypothesis (a) of Lemma 52.16.10 holds and the proof is complete. $\square$

We can unwind condition of Lemma 52.20.4 as follows.

0EJA Lemma 52.20.5. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,
(2) $\mathrm{cd}(A, I) = 1$.

Then $(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities if and only if for all $y \in U \cap Y$ with $\dim(\{y\}) = 1$ and every prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$, $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$ we have

$$\mathrm{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) > 2$$

Proof. We will use Lemma 52.19.3 without further mention. In particular, we see the condition is necessary. Conversely, suppose the condition is true. Note that $\delta^Y_Z(y) = \dim(\overline{\{y\}})$ by Lemma 52.18.1. Let us write δ for this function. Let $y \in U \cap Y$. If $\delta(y) > 2$, then the inequality of Lemma 52.19.3 holds. Finally, suppose $\delta(y) = 2$. We have to show that

$$\mathrm{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) > 1$$

Choose a specialization $y \rightsquigarrow y'$ with $\delta(y') = 1$. Then there is a ring map $\mathcal{O}_{X,y'}^\wedge \to \mathcal{O}_{X,y}^\wedge$ which identifies the target with the completion of the localization of $\mathcal{O}_{X,y'}^\wedge$ at a prime $\mathfrak{q}$ with $\dim(\mathcal{O}_{X,y'}^\wedge/\mathfrak{q}) = 1$. Moreover, we then obtain

$$\mathcal{F}_y^\wedge = \mathcal{F}_{y'}^\wedge \otimes_{\mathcal{O}_{X,y'}^\wedge} \mathcal{O}_{X,y}^\wedge$$

Let $\mathfrak{p}' \subset \mathcal{O}_{X,y'}^\wedge$ be the image of $\mathfrak{p}$. By Local Cohomology, Lemma 51.11.3 we have

$$\begin{aligned} \mathrm{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) &= \mathrm{depth}((\mathcal{F}_{y'}^\wedge)_{\mathfrak{p}'}) + \dim((\mathcal{O}_{X,y}^\wedge/\mathfrak{p})_{\mathfrak{p}'}) \\ &= \mathrm{depth}((\mathcal{F}_{y'}^\wedge)_{\mathfrak{p}'}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}') - 1 \end{aligned}$$

[7]Choose homogeneous generators of the form $\delta_{n_j}(p_j)$ for the displayed module. Then if $k = \max(n_j)$ we find that for $n \geq k$ and any $p \in P_n$ we can find $a_j \in A$ such that $p - \sum a_j f^{n-n_j} p_j$ is in the kernel of δ_n and hence in the image of $P_{n'}$ for all $n' \geq n$. Thus $\mathrm{Im}(P_n \to P_{n-k}) = \mathrm{Im}(P_{n'} \to P_{n-k})$ for all $n' \geq n$.

the last equality because the specialization is immediate. Thus the lemma is prove by the assumed inequality for $y', \mathfrak{p}'$. $\square$

0EJB Lemma 52.20.6. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,
(2) A has a dualizing complex,
(3) $\text{cd}(A, I) = 1$,
(4) for $y \in U \cap Y$ the module $\mathcal{F}_y^\wedge$ is finite locally free outside $V(I\mathcal{O}_{X,y}^\wedge)$, for example if $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/I^n\mathcal{O}_U$-module, and
(5) one of the following is true
    (a) A_f is (S_2) and every irreducible component of X not contained in Y has dimension ≥ 4, or
    (b) if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.

Then $(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities.

Proof. We will use the criterion of Lemma 52.20.5. Let $y \in U \cap Y$ with $\dim(\overline{\{y\}} = 1$ and let $\mathfrak{p}$ be a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$. Condition (4) shows that $\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) = \text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p})$. Thus we have to prove

$$\text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) > 2$$

Let $\mathfrak{p}_0 \subset A$ be the image of $\mathfrak{p}$. Let $\mathfrak{q} \subset A$ be the prime corresponding to y. By Local Cohomology, Lemma 51.11.3 we have

$$\begin{aligned} \text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) &= \text{depth}(A_{\mathfrak{p}_0}) + \dim((A/\mathfrak{p}_0)_\mathfrak{q}) \\ &= \text{depth}(A_{\mathfrak{p}_0}) + \dim(A/\mathfrak{p}_0) - 1 \end{aligned}$$

If (5)(a) holds, then we get that this is

$$\geq \min(2, \dim(A_{\mathfrak{p}_0})) + \dim(A/\mathfrak{p}_0) - 1$$

Note that in any case $\dim(A/\mathfrak{p}_0) \geq 2$. Hence if we get 2 for the minimum, then we are done. If not we get

$$\dim(A_{\mathfrak{p}_0}) + \dim(A/\mathfrak{p}_0) - 1 \geq 4 - 1$$

because every component of $\text{Spec}(A)$ passing through $\mathfrak{p}_0$ has dimension ≥ 4. If (5)(b) holds, then we win immediately. $\square$

0EJC Remark 52.20.7. Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to $f \in \mathfrak{m}$. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, f\mathcal{O}_U)$ where U is the punctured spectrum of A. Set $Y = V(f) \subset X = \text{Spec}(A)$. If for $y \in U \cap V(f)$ closed in U, i.e., with $\dim(\overline{\{y\}}) = 1$, we assume the $\mathcal{O}_{X,y}^\wedge$-module $\mathcal{F}_y^\wedge$ satisfies the following two conditions

(1) $\mathcal{F}_y^\wedge[1/f]$ is (S_2) as a $\mathcal{O}_{X,y}^\wedge[1/f]$-module, and
(2) for $\mathfrak{p} \in \text{Ass}(\mathcal{F}_y^\wedge[1/f])$ we have $\dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) \geq 3$.

Then $(\mathcal{F}_n)$ is the completion of a coherent module on U. This follows from Lemmas 52.20.4 and 52.20.5.

52.21. Improving coherent formal modules

0EJD Let X be a Noetherian scheme. Let $Y \subset X$ be a closed subscheme with quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}(X, \mathcal{I})$. In this section we construct maps $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ similar to the maps constructed in Local Cohomology, Section 51.15 for coherent modules. For a point $y \in Y$ we set

$$\mathcal{O}_{X,y}^\wedge = \lim \mathcal{O}_{X,y}/\mathcal{I}_y^n, \quad \mathcal{I}_y^\wedge = \lim \mathcal{I}_y/\mathcal{I}_y^n \quad \text{and} \quad \mathfrak{m}_y^\wedge = \lim \mathfrak{m}_y/\mathcal{I}_y^n$$

Then $\mathcal{O}_{X,y}^\wedge$ is a Noetherian local ring with maximal ideal $\mathfrak{m}_y^\wedge$ complete with respect to $\mathcal{I}_y^\wedge = \mathcal{I}_y \mathcal{O}_{X,y}^\wedge$. We also set

$$\mathcal{F}_y^\wedge = \lim \mathcal{F}_{n,y}$$

Then $\mathcal{F}_y^\wedge$ is a finite module over $\mathcal{O}_{X,y}^\wedge$ with $\mathcal{F}_y^\wedge/(\mathcal{I}_y^\wedge)^n\mathcal{F}_y^\wedge = \mathcal{F}_{n,y}$ for all n, see Algebra, Lemmas 10.98.2 and 10.96.12.

0EJE **Lemma 52.21.1.** *In the situation above assume X locally has a dualizing complex. Let $T \subset Y$ be a subset stable under specialization. Assume for $y \in T$ and for a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $V(\mathfrak{p}) \cap V(\mathcal{I}_y^\wedge) = \{\mathfrak{m}_y^\wedge\}$ we have*

$$\text{depth}_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}_y^\wedge)_\mathfrak{p}) > 0$$

Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules with the following properties

1. *for $y \in T$ we have $\text{depth}(\mathcal{F}'_{n,y}) \geq 1$,*
2. *$(\mathcal{F}'_n)$ is isomorphic as a pro-system to an object $(\mathcal{G}_n)$ of $\text{Coh}(X, \mathcal{I})$,*
3. *the induced morphism $(\mathcal{F}_n) \to (\mathcal{G}_n)$ of $\text{Coh}(X, \mathcal{I})$ is surjective with kernel annihilated by a power of $\mathcal{I}$.*

Proof. For every n we let $\mathcal{F}_n \to \mathcal{F}'_n$ be the surjection constructed in Local Cohomology, Lemma 51.15.1. Since this is the quotient of $\mathcal{F}_n$ by the subsheaf of sections supported on T we see that we get canonical maps $\mathcal{F}'_{n+1} \to \mathcal{F}'_n$ such that we obtain a map $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules. Property (1) holds by construction.

To prove properties (2) and (3) we may assume that $X = \text{Spec}(A_0)$ is affine and A_0 has a dualizing complex. Let $I_0 \subset A_0$ be the ideal corresponding to Y. Let A, I be the I-adic completions of A_0, I_0. For later use we observe that A has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Let M be the finite A-module corresponding to $(\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 30.23.1. Then $\mathcal{F}_n$ corresponds to $M_n = M/I^nM$. Recall that $\mathcal{F}'_n$ corresponds to the quotient $M'_n = M_n/H^0_T(M_n)$, see Local Cohomology, Lemma 51.15.1 and its proof.

Set $s = 0$ and $d = \text{cd}(A, I)$. We claim that A, I, T, M, s, d satisfy assumptions (1), (3), (4), (6) of Situation 52.10.1. Namely, (1) and (3) are immediate from the above, (4) is the empty condition as $s = 0$, and (6) is the assumption we made in the statement of the lemma.

By Theorem 52.10.8 we see that $\{H^0_T(M_n)\}$ is Mittag-Leffler, that $\lim H^0_T(M_n) = H^0_T(M)$, and that $H^0_T(M)$ is killed by a power of I. Thus the limit of the short exact sequences $0 \to H^0_T(M_n) \to M_n \to M'_n \to 0$ is the short exact sequence

$$0 \to H^0_T(M) \to M \to \lim M'_n \to 0$$

Setting $M' = \lim M'_n$ we find that $\mathcal{G}_n$ corresponds to the finite A_0-module M'/I^nM'. To finish the prove we have to show that the canonical map $\{M'/I^nM'\} \to \{M'_n\}$ is a pro-isomorphism. This is equivalent to saying that $\{H^0_T(M)+I^nM\} \to \{\ker(M \to M'_n)\}$ is a pro-isomorphism. Which in turn says that $\{H^0_T(M)/H^0_T(M) \cap I^nM\} \to \{H^0_T(M_n)\}$ is a pro-isomorphism. This is true because $\{H^0_T(M_n)\}$ is Mittag-Leffler, $\lim H^0_T(M_n) = H^0_T(M)$, and $H^0_T(M)$ is killed by a power of I (so that Artin-Rees tells us that $H^0_T(M) \cap I^nM = 0$ for n large enough). □

0EJF Lemma 52.21.2. In the situation above assume X locally has a dualizing complex. Let $T' \subset T \subset Y$ be subsets stable under specialization. Let $d \geq 0$ be an integer. Assume

(a) affine locally we have $X = \mathrm{Spec}(A_0)$ and $Y = V(I_0)$ and $\mathrm{cd}(A_0, I_0) \leq d$,
(b) for $y \in T$ and a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\}$ we have
$$\mathrm{depth}_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) > 0$$
(c) for $y \in T'$ and for a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(\mathcal{I}^\wedge_y)$ and $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) \neq \{\mathfrak{m}^\wedge_y\}$ we have
$$\mathrm{depth}_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) \geq 1 \quad \text{or} \quad \mathrm{depth}_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) > 1 + d$$
(d) for $y \in T'$ and a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\}$ we have
$$\mathrm{depth}_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) > 1$$
(e) if $y \rightsquigarrow y'$ is an immediate specialization and $y' \in T'$, then $y \in T$.

Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}''_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules with the following properties

(1) for $y \in T$ we have $\mathrm{depth}(\mathcal{F}''_{n,y}) \geq 1$,
(2) for $y' \in T'$ we have $\mathrm{depth}(\mathcal{F}''_{n,y'}) \geq 2$,
(3) $(\mathcal{F}''_n)$ is isomorphic as a pro-system to an object $(\mathcal{H}_n)$ of $\mathrm{Coh}(X, \mathcal{I})$,
(4) the induced morphism $(\mathcal{F}_n) \to (\mathcal{H}_n)$ of $\mathrm{Coh}(X, \mathcal{I})$ has kernel and cokernel annihilated by a power of $\mathcal{I}$.

Proof. As in Lemma 52.21.1 and its proof for every n we let $\mathcal{F}_n \to \mathcal{F}'_n$ be the surjection constructed in Local Cohomology, Lemma 51.15.1. Next, we let $\mathcal{F}'_n \to \mathcal{F}''_n$ be the injection constructed in Local Cohomology, Lemma 51.15.5 and its proof. The constructions show that we get canonical maps $\mathcal{F}''_{n+1} \to \mathcal{F}''_n$ such that we obtain maps
$$(\mathcal{F}_n) \longrightarrow (\mathcal{F}'_n) \longrightarrow (\mathcal{F}''_n)$$
of inverse systems of coherent $\mathcal{O}_X$-modules. Properties (1) and (2) hold by construction.

To prove properties (3) and (4) we may assume that $X = \mathrm{Spec}(A_0)$ is affine and A_0 has a dualizing complex. Let $I_0 \subset A_0$ be the ideal corresponding to Y. Let A, I be the I-adic completions of A_0, I_0. For later use we observe that A has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Let M be the finite A-module corresponding to $(\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 30.23.1. Then $\mathcal{F}_n$ corresponds to $M_n = M/I^nM$. Recall that $\mathcal{F}'_n$ corresponds to the quotient $M'_n = M_n/H^0_T(M_n)$. Also, recall that $M' = \lim M'_n$ is the quotient of M by $H^0_T(M)$ and that $\{M'_n\}$ and $\{M'/I^nM'\}$ are isomorphic as pro-systems. Finally, we see that $\mathcal{F}''_n$ corresponds to an extension
$$0 \to M'_n \to M''_n \to H^1_{T'}(M'_n) \to 0$$

see proof of Local Cohomology, Lemma 51.15.5.

Set $s = 1$. We claim that A, I, T', M', s, d satisfy assumptions (1), (3), (4), (6) of Situation 52.10.1. Namely, (1) and (3) are immediate, (4) is implied by (c), and (6) follows from (d). We omit the details of the verification (c) $\Rightarrow$ (4).

By Theorem 52.10.8 we see that $\{H^1_{T'}(M'/I^nM')\}$ is Mittag-Leffler, that $H^1_{T'}(M') = \lim H^1_{T'}(M'/I^nM')$, and that $H^1_{T'}(M')$ is killed by a power of I. We deduce $\{H^1_{T'}(M'_n)\}$ is Mittag-Leffler and $H^1_{T'}(M') = \lim H^1_{T'}(M'_n)$. Thus the limit of the short exact sequences displayed above is the short exact sequence

$$0 \to M' \to \lim M''_n \to H^1_{T'}(M') \to 0$$

Set $M'' = \lim M''_n$. It follows from Local Cohomology, Proposition 51.11.1 that $H^1_{T'}(M')$ and hence M'' are finite A-modules. Thus we find that $\mathcal{H}_n$ corresponds to the finite A_0-module M''/I^nM''. To finish the prove we have to show that the canonical map $\{M''/I^nM''\} \to \{M''_n\}$ is a pro-isomorphism. Since we already know that $\{M'/I^nM'\}$ is pro-isomorphic to $\{M'_n\}$ the reader verifies (omitted) this is equivalent to asking $\{H^1_{T'}(M')/I^nH^1_{T'}(M')\} \to \{H^1_{T'}(M'_n)\}$ to be a pro-isomorphism. This is true because $\{H^1_{T'}(M'_n)\}$ is Mittag-Leffler, $H^1_{T'}(M') = \lim H^1_{T'}(M'_n)$, and $H^1_{T'}(M')$ is killed by a power of I. □

0EJG Lemma 52.21.3. In Situation 52.16.1 assume that A has a dualizing complex. Let $d \geq \text{cd}(A, I)$. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume $(\mathcal{F}_n)$ satisfies the $(2, 2+d)$-inequalities, see Definition 52.19.1. Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}''_n)$ of inverse systems of coherent $\mathcal{O}_U$-modules with the following properties

(1) if $\text{depth}(\mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq 3$ for all $y \in U \cap Y$,
(2) $(\mathcal{F}''_n)$ is isomorphic as a pro-system to an object $(\mathcal{H}_n)$ of $\text{Coh}(U, I\mathcal{O}_U)$,
(3) the induced morphism $(\mathcal{F}_n) \to (\mathcal{H}_n)$ of $\text{Coh}(U, I\mathcal{O}_U)$ has kernel and cokernel annihilated by a power of I,
(4) the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ are finite A-modules for all n.

Proof. The existence and properties (2), (3), (4) follow immediately from Lemma 52.21.2 applied to U, $U \cap Y$, $T = \{y \in U \cap Y : \delta^Y_Z(y) \leq 2\}$, $T' = \{y \in U \cap Y : \delta^Y_Z(y) \leq 1\}$, and $(\mathcal{F}_n)$. The finiteness of the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ follows from Local Cohomology, Lemma 51.12.1 and the elementary properties of the function $\delta^Y_Z(-)$ proved in Lemma 52.18.1. □

52.22. Algebraization of coherent formal modules, V

0EJH In this section we prove our most general results on algebraization of coherent formal modules. We first prove it in case the ideal has cohomological dimension 1. Then we apply this to a blowup to prove a more general result.

0EJI Lemma 52.22.1. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A has a dualizing complex and $\text{cd}(A, I) = 1$,
(2) $(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}''_n)$ of coherent $\mathcal{O}_U$-modules such that $\text{depth}(\mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq 3$ for all $y \in U \cap Y$.

Then $(\mathcal{F}_n)$ extends canonically to X, see Definition 52.16.7.

Proof. We will check hypotheses (a), (b), and (c) of Lemma 52.16.10. Before we start, let us point out that the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ are finite A-modules for all n by Local Cohomology, Lemma 51.12.1.

Observe that for each $p \geq 0$ the limit topology on $\lim H^p(U, \mathcal{F}_n)$ is the I-adic topology by Lemma 52.4.5. In particular, hypothesis (b) holds.

We know that $M = \lim H^0(U, \mathcal{F}_n)$ is an A-module whose limit topology is the I-adic topology. Thus, given n, the module M/I^nM is a subquotient of $H^0(U, \mathcal{F}_N)$ for some $N \gg n$. Since the inverse system $\{H^0(U, \mathcal{F}_N)\}$ is pro-isomorphic to an inverse system of finite A-modules, namely $\{H^0(U, \mathcal{F}''_N)\}$, we conclude that M/I^nM is finite. It follows that M is finite, see Algebra, Lemma 10.96.12. In particular hypothesis (c) holds.

For each $n \geq 0$ let us write $Ob_n = \lim_N H^1(U, I^n\mathcal{F}_N)$. A special case is $Ob = Ob_0 = \lim_N H^1(U, \mathcal{F}_N)$. Arguing exactly as in the previous paragraph we find that Ob is a finite A-module. (In fact, we also know that Ob/IOb is annihilated by a power of $\mathfrak{a}$, but it seems somewhat difficult to use this.)

We set $\mathcal{F} = \lim \mathcal{F}_n$, we pick generators $f_1, \ldots, f_r$ of I, we pick $c \geq 1$, and we choose $\Phi_\mathcal{F}$ as in Lemma 52.4.4. We will use the results of Lemma 52.2.1 without further mention. In particular, for each $n \geq 1$ there are maps

$$\delta_n : H^0(U, \mathcal{F}_n) \longrightarrow H^1(U, I^n\mathcal{F}) \longrightarrow Ob_n$$

The first comes from the short exact sequence $0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}_n \to 0$ and the second from $I^n\mathcal{F} = \lim I^n\mathcal{F}_N$. We will later use that if $\delta_n(s) = 0$ for $s \in H^0(U, \mathcal{F}_n)$ then we can for each $n' \geq n$ find $s' \in H^0(U, \mathcal{F}_{n'})$ mapping to s. Observe that there are commutative diagrams

$$\begin{tikzcd}
H^0(U, \mathcal{F}_{nc}) \arrow[r] \arrow[dd] & H^1(U, I^{nc}\mathcal{F}) \arrow[dd] \arrow[dr, "\Phi_\mathcal{F}"] & \\
& & \bigoplus_{e_1+\ldots+e_r=n} H^1(U, \mathcal{F}) \cdot T_1^{e_1} \ldots T_r^{e_r} \arrow[dl] \\
H^0(U, \mathcal{F}_n) \arrow[r] & H^1(U, I^n\mathcal{F}) &
\end{tikzcd}$$

We conclude that the obstruction map $H^0(U, \mathcal{F}_n) \to Ob_n$ sends the image of $H^0(U, \mathcal{F}_{nc}) \to H^0(U, \mathcal{F}_n)$ into the submodule

$$Ob'_n = \text{Im}\left(\bigoplus\nolimits_{e_1+\ldots+e_r=n} Ob \cdot T_1^{e_1} \ldots T_r^{e_r} \to Ob_n\right)$$

where on the summand $Ob \cdot T_1^{e_1} \ldots T_r^{e_r}$ we use the map on cohomology coming from the reductions modulo powers of I of the multiplication map $f_1^{e_1} \ldots f_r^{e_r} : \mathcal{F} \to I^n\mathcal{F}$. By construction

$$\bigoplus\nolimits_{n \geq 0} Ob'_n$$

is a finite graded module over the Rees algebra $\bigoplus_{n \geq 0} I^n$. For each n we set

$$M_n = \{s \in H^0(U, \mathcal{F}_n) \mid \delta_n(s) \in Ob'_n\}$$

Observe that $\{M_n\}$ is an inverse system and that $f_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$ on global sections maps M_n into M_{n+1}. By exactly the same argument as in the proof of Cohomology, Lemma 20.35.1 we find that $\{M_n\}$ is ML. Namely, because the Rees algebra is Noetherian we can choose a finite number of homogeneous generators of the form $\delta_{n_j}(z_j)$ with $z_j \in M_{n_j}$ for the graded submodule $\bigoplus_{n \geq 0} \mathrm{Im}(M_n \to Ob'_n)$. Then if $k = \max(n_j)$ we find that for $n \geq k$ and any $z \in M_n$ we can find $a_j \in I^{n-n_j}$ such that $z - \sum a_j z_j$ is in the kernel of δ_n and hence in the image of $M_{n'}$ for all $n' \geq n$ (because the vanishing of δ_n means that we can lift $z - \sum a_j z_j$ to an element $z' \in H^0(U, \mathcal{F}_{n'c})$ for all $n' \geq n$ and then the image of z' in $H^0(U, \mathcal{F}_{n'})$ is in $M_{n'}$ by what we proved above). Thus $\mathrm{Im}(M_n \to M_{n-k}) = \mathrm{Im}(M_{n'} \to M_{n-k})$ for all $n' \geq n$.

Choose n. By the Mittag-Leffler property of $\{M_n\}$ we just established we can find an $n' \geq n$ such that the image of $M_{n'} \to M_n$ is the same as the image of $M' \to M_n$. By the above we see that the image of $M' \to M_n$ contains the image of $H^0(U, \mathcal{F}_{n'c}) \to H^0(U, \mathcal{F}_n)$. Thus we see that $\{M_n\}$ and $\{H^0(U, \mathcal{F}_n)\}$ are pro-isomorphic. Therefore $\{H^0(U, \mathcal{F}_n)\}$ has ML and we finally conclude that hypothesis (a) holds. This concludes the proof. □

0EJJ Proposition 52.22.2 (Algebraization in cohomological dimension 1). In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A has a dualizing complex and $\mathrm{cd}(A, I) = 1$,
(2) $(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities, see Definition 52.19.1.

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

The local case of this result is [Ray75, IV Corollaire 2.9].

Proof. By Lemma 52.17.1 we may replace $(\mathcal{F}_n)$ by the object $(\mathcal{H}_n)$ of $\mathrm{Coh}(U, I\mathcal{O}_U)$ found in Lemma 52.21.3. Thus we may assume that $(\mathcal{F}_n)$ is pro-isomorphic to a inverse system $(\mathcal{F}''_n)$ with the properties mentioned in Lemma 52.21.3. In Lemma 52.22.1 we proved that $(\mathcal{F}_n)$ canonically extends to X. The final statement follows from Lemma 52.16.8. □

0EJK Lemma 52.22.3. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A has a dualizing complex,
(2) all fibres of the blowing up $b : X' \to X$ of I have dimension $\leq d - 1$,
(3) one of the following is true
    (a) $(\mathcal{F}_n)$ satisfies the $(d + 1, d + 2)$-inequalities (Definition 52.19.1), or
    (b) for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \not\in V(I\mathcal{O}^\wedge_{X,y})$ we have

$$\mathrm{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > d + 2$$

Then $(\mathcal{F}_n)$ extends to X.

Proof. Let $Y' \subset X'$ be the exceptional divisor. Let $Z' \subset Y'$ be the inverse image of $Z \subset Y$. Then $U' = X' \setminus Z'$ is the inverse image of U. With $\delta^{Y'}_{Z'}$ as in (52.18.0.1) we set

$$T' = \{y' \in Y' \mid \delta^{Y'}_{Z'}(y') = 1 \text{ or } 2\} \subset T = \{y' \in Y' \mid \delta^{Y'}_{Z'}(y') = 1\}$$

These are specialization stable subsets of $U' \cap Y' = Y' \setminus Z'$. Consider the object $(b|^*_{U'}\mathcal{F}_n)$ of $\mathrm{Coh}(U', I\mathcal{O}_{U'})$, see Cohomology of Schemes, Lemma 30.23.9. For $y' \in U' \cap Y'$ let us denote

$$\mathcal{F}^\wedge_{y'} = \lim (b|^*_{U'}\mathcal{F}_n)_{y'}$$

the "stalk" of this pullback at y'. We claim that conditions (a), (b), (c), (d), and (e) of Lemma 52.21.2 hold for the object $(b|^*_{U'}\mathcal{F}_n)$ on U' with d replaced by 1 and the subsets $T' \subset T \subset U' \cap Y'$. Condition (a) holds because Y' is an effective Cartier divisor and hence locally cut out by 1 equation. Condition (e) holds by Lemma 52.18.1 parts (1) and (2). To prove (b), (c), and (d) we need some preparation.

Let $y' \in U' \cap Y'$ and let $\mathfrak p' \subset \mathcal{O}^\wedge_{X',y'}$ be a prime ideal not contained in $V(I\mathcal{O}^\wedge_{X',y'})$. Denote $y = b(y') \in U \cap Y$. Choose $f \in I$ such that y' is contained in the spectrum of the affine blowup algebra $A[\frac{I}{f}]$, see Divisors, Lemma 31.32.2. For any A-algebra B denote $B' = B[\frac{IB}{f}]$ the corresponding affine blowup algebra. Denote I-adic completion by $^\wedge$. By our choice of f we get a ring map $(\mathcal{O}^\wedge_{X,y})' \to \mathcal{O}^\wedge_{X',y'}$. If we let $\mathfrak q' \subset (\mathcal{O}^\wedge_{X,y})'$ be the inverse image of $\mathfrak m^\wedge_{y'}$, then we see that $((\mathcal{O}^\wedge_{X,y})'_{\mathfrak q'})^\wedge = \mathcal{O}^\wedge_{X',y'}$. Let $\mathfrak p \subset \mathcal{O}^\wedge_{X,y}$ be the corresponding prime. At this point we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}^\wedge_{X,y} \arrow[r] \arrow[d] & (\mathcal{O}^\wedge_{X,y})' \arrow[r] \arrow[d, "\alpha"'] & (\mathcal{O}^\wedge_{X,y})'_{\mathfrak q'} \arrow[r, "\beta"'] \arrow[d] & \mathcal{O}^\wedge_{X',y'} \arrow[d] \\
\mathcal{O}^\wedge_{X,y}/\mathfrak p \arrow[r] & (\mathcal{O}^\wedge_{X,y}/\mathfrak p)' \arrow[r] & (\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'} \arrow[r, "\gamma"] & ((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'})^\wedge \arrow[d] \\
 & & & \mathcal{O}^\wedge_{X',y'}/\mathfrak p'
\end{tikzcd}$$

whose vertical arrows are surjective. By More on Algebra, Lemma 15.43.1 and the dimension formula (Algebra, Lemma 10.113.1) we have

$$\dim(((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'})^\wedge) = \dim((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'}) = \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) - \text{trdeg}(\kappa(y')/\kappa(y))$$

Tracing through the definitions of pullbacks, stalks, localizations, and completions we find

$$(\mathcal{F}^\wedge_y)_{\mathfrak p} \otimes_{(\mathcal{O}^\wedge_{X,y})_{\mathfrak p}} (\mathcal{O}^\wedge_{X',y'})_{\mathfrak p'} = (\mathcal{F}^\wedge_{y'})_{\mathfrak p'}$$

Details omitted. The ring maps β and γ in the diagram are flat with Gorenstein (hence Cohen-Macaulay) fibres, as these are completions of rings having a dualizing complex. See Dualizing Complexes, Lemmas 47.23.1 and 47.23.2 and the discussion in More on Algebra, Section 15.51. Observe that $(\mathcal{O}^\wedge_{X,y})_{\mathfrak p} = (\mathcal{O}^\wedge_{X,y})'_{\tilde{\mathfrak p}}$ where $\tilde{\mathfrak p}$ is the kernel of α in the diagram. On the other hand, $(\mathcal{O}^\wedge_{X,y})'_{\tilde{\mathfrak p}} \to (\mathcal{O}^\wedge_{X',y'})_{\mathfrak p'}$ is flat with CM fibres by the above. Whence $(\mathcal{O}^\wedge_{X,y})_{\mathfrak p} \to (\mathcal{O}^\wedge_{X',y'})_{\mathfrak p'}$ is flat with CM fibres. Using Algebra, Lemma 10.163.1 we see that

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) = \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(F_{\mathfrak r})$$

where F is the generic formal fibre of $(\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'}$ and $\mathfrak r$ is the prime corresponding to $\mathfrak p'$. Since $(\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'}$ is a universally catenary local domain, its I-adic completion is equidimensional and (universally) catenary by Ratliff's theorem (More on Algebra, Proposition 15.109.5). It then follows that

$$\dim(((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'})^\wedge) = \dim(F_{\mathfrak r}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p')$$

Combined with Lemma 52.18.2 we get

0EJL (52.22.3.1)
$$\begin{aligned}
& \text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \delta^{Y'}_{Z'}(y') \\
& = \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(F_{\mathfrak r}) + \delta^{Y'}_{Z'}(y') \\
& \geq \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \delta^Y_Z(y) + \dim(F_{\mathfrak r}) + \text{trdeg}(\kappa(y')/\kappa(y)) - (d-1) \\
& = \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \delta^Y_Z(y) - (d-1) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) - \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p')
\end{aligned}$$

Please keep in mind that $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) \geq \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p')$. Rewriting this we get

0EJM (52.22.3.2)
$$\begin{aligned} & \text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p') + \delta^{Y'}_{Z'}(y') \\ & \geq \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) + \delta^Y_Z(y) - (d-1) \end{aligned}$$

This inequality will allow us to check the remaning conditions.

Conditions (b) and (d) of Lemma 52.21.2. Assume $V(\mathfrak p') \cap V(I\mathcal{O}^\wedge_{X',y'}) = \{\mathfrak m^\wedge_{y'}\}$. This implies that $\dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p') = 1$ because Z' is an effective Cartier divisor. The combination of (b) and (d) is equivalent with

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \delta^{Y'}_{Z'}(y') > 2$$

If $(\mathcal{F}_n)$ satisfies the inequalities in (3)(b) then we immediately conclude this is true by applying (52.22.3.2). If $(\mathcal{F}_n)$ satisfies (3)(a), i.e., the $(d+1, d+2)$-inequalities, then we see that in any case

$$\text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \delta^Y_Z(y) \geq d+1 \quad \text{or} \quad \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) + \delta^Y_Z(y) > d+2$$

Looking at (52.22.3.1) and (52.22.3.2) above this gives what we want except possibly if $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) = 1$. However, if $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) = 1$, then we have $V(\mathfrak p) \cap V(I\mathcal{O}^\wedge_{X,y}) = \{\mathfrak m^\wedge_y\}$ and we see that actually

$$\text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \delta^Y_Z(y) > d+1$$

as $(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities and we conclude again.

Condition (c) of Lemma 52.21.2. Assume $V(\mathfrak p') \cap V(I\mathcal{O}^\wedge_{X',y'}) \neq \{\mathfrak m^\wedge_{y'}\}$. Then condition (c) is equivalent to

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \delta^{Y'}_{Z'}(y') \geq 2 \quad \text{or} \quad \text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p') + \delta^{Y'}_{Z'}(y') > 3$$

If $(\mathcal{F}_n)$ satisfies the inequalities in (3)(b) then we see the second of the two displayed inequalities holds true by applying (52.22.3.2). If $(\mathcal{F}_n)$ satisfies (3)(a), i.e., the $(d+1, d+2)$-inequalities, then this follows immediately from (52.22.3.1) and (52.22.3.2). This finishes the proof of our claim.

Choose $(b|_{U'}^*\mathcal{F}_n) \to (\mathcal{F}''_n)$ and $(\mathcal{H}_n)$ in $\text{Coh}(U', I\mathcal{O}_{U'})$ as in Lemma 52.21.2. For any affine open $W \subset X'$ observe that $\delta^{W\cap Y'}_{W\cap Z'}(y') \geq \delta^{Y'}_{Z'}(y')$ by Lemma 52.18.1 part (7). Hence we see that $(\mathcal{H}_n|_W)$ satisfies the assumptions of Lemma 52.22.1. Thus $(\mathcal{H}_n|_W)$ extends canonically to W. Let $(\mathcal{G}_{W,n})$ in $\text{Coh}(W, I\mathcal{O}_W)$ be the canonical extension as in Lemma 52.16.8. By Lemma 52.16.9 we see that for $W' \subset W$ there is a unique isomorphism

$$(\mathcal{G}_{W,n}|_{W'}) \longrightarrow (\mathcal{G}_{W',n})$$

compatible with the given isomorphisms $(\mathcal{G}_{W,n}|_{W\cap U}) \cong (\mathcal{H}_n|_{W\cap U})$. We conclude that there exists an object $(\mathcal{G}_n)$ of $\text{Coh}(X', I\mathcal{O}_{X'})$ whose restriction to U is isomorphic to $(\mathcal{H}_n)$.

If A is I-adically complete we can finish the proof as follows. By Grothedieck's existence theorem (Cohomology of Schemes, Lemma 30.24.3) we see that $(\mathcal{G}_n)$ is the completion of a coherent $\mathcal{O}_{X'}$-module. Then by Cohomology of Schemes, Lemma 30.23.6 we see that $(b|_{U'}^*\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_{U'}$-module $\mathcal{F}'$. By Cohomology of Schemes, Lemma 30.25.3 we see that there is a map

$$(\mathcal{F}_n) \longrightarrow ((b|_{U'})_*\mathcal{F}')^\wedge$$

whose kernel and cokernel is annihilated by a power of I. Then finally, we win by applying Lemma 52.17.1.

If A is not complete, then, before starting the proof, we may replace A by its completion, see Lemma 52.16.6. After completion the assumptions still hold: this is immediate for condition (3), follows from Dualizing Complexes, Lemma 47.22.4 for condition (1), and from Divisors, Lemma 31.32.3 for condition (2). Thus the complete case implies the general case. □

0EJN Proposition 52.22.4 (Algebraization for ideals with few generators). In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A has a dualizing complex,
(2) $V(I) = V(f_1, \ldots, f_d)$ for some $d \geq 1$ and $f_1, \ldots, f_d \in A$,
(3) one of the following is true
  (a) $(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities (Definition 52.19.1), or
  (b) for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$ we have

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > d + 2$$

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We may assume $I = (f_1, \ldots, f_d)$, see Cohomology of Schemes, Lemma 30.23.11. Then we see that all fibres of the blowup of X in I have dimension at most $d-1$. Thus we get the extension from Lemma 52.22.3. The final statement follows from Lemma 52.16.3. □

Please compare the next lemma with Remarks 52.16.12, 52.20.2, 52.20.7, and 52.23.2.

0EJP Lemma 52.22.5. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A is a local ring which has a dualizing complex,
(2) all irreducible components of X have the same dimension,
(3) the scheme $X \setminus Y$ is Cohen-Macaulay,
(4) I is generated by d elements,
(5) $\dim(X) - \dim(Z) > d + 2$, and
(6) for $y \in U \cap Y$ the module $\mathcal{F}_y^\wedge$ is finite locally free outside $V(I\mathcal{O}_{X,y}^\wedge)$, for example if $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/I^n\mathcal{O}_U$-module.

Then $(\mathcal{F}_n)$ extends to X. In particular if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will show that the hypotheses (1), (2), (3)(b) of Proposition 52.22.4 are satisfied. This is clear for (1) and (2).

Let $y \in U \cap Y$ and let $\mathfrak{p}$ be a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$. The last condition shows that $\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) = \text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p})$. Since $X \setminus Y$ is Cohen-Macaulay we see that $(\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}$ is Cohen-Macaulay. Thus we see that

$$\begin{aligned} & \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) \\ & = \dim((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) \\ & = \dim(\mathcal{O}_{X,y}^\wedge) + \delta_Z^Y(y) \end{aligned}$$

The final equality because $\mathcal{O}_{X,y}$ is equidimensional by the second condition. Let $\delta(y) = \dim(\overline{\{y\}})$. This is a dimension function as A is a catenary local ring. By Lemma 52.18.1 we have $\delta^Y_Z(y) \geq \delta(y) - \dim(Z)$. Since X is equidimensional we get

$$\dim(\mathcal{O}^\wedge_{X,y}) + \delta^Y_Z(y) \geq \dim(\mathcal{O}^\wedge_{X,y}) + \delta(y) - \dim(Z) = \dim(X) - \dim(Z)$$

Thus we get the desired inequality and we win. □

0EJQ Remark 52.22.6. We are unable to prove or disprove the analogue of Proposition 52.22.4 where the assumption that I has d generators is replaced with the assumption $\text{cd}(A, I) \leq d$. If you know a proof or have a counter example, please email stacks.project@gmail.com. Another obvious question is to what extend the conditions in Proposition 52.22.4 are necessary.

52.23. Algebraization of coherent formal modules, VI

0EJR In this section we add a few more easier to prove cases.

0EJS Proposition 52.23.1. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) there exist $f_1, \ldots, f_d \in I$ such that for $y \in U \cap Y$ the ideal $I\mathcal{O}_{X,y}$ is generated by $f_1, \ldots, f_d$ and $f_1, \ldots, f_d$ form a $\mathcal{F}^\wedge_y$-regular sequence,
(2) $H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite A-modules.

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 52.16.10. For every n we have a short exact sequence

$$0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$$

Since $f_1, \ldots, f_d$ forms a regular sequence (and hence quasi-regular, see Algebra, Lemma 10.69.2) on each of the "stalks" $\mathcal{F}^\wedge_y$ and since we have $I\mathcal{F}_n = (f_1, \ldots, f_d)\mathcal{F}_n$ for all n, we find that

$$I^n\mathcal{F}_{n+1} = \bigoplus\nolimits_{e_1+\ldots+e_d=n} \mathcal{F}_1 \cdot f_1^{e_1} \ldots f_d^{e_d}$$

by checking on stalks. Using the assumption of finiteness of $H^0(U, \mathcal{F}_1)$ and induction, we first conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a finite A-module for all n. In this way we see that condition (c) of Lemma 52.16.10 holds. We also see that

$$\bigoplus\nolimits_{n \geq 0} H^1(U, I^n\mathcal{F}_{n+1})$$

is a finite graded $R = \bigoplus I^n/I^{n+1}$-module. By Cohomology, Lemma 20.35.1 we conclude that condition (a) of Lemma 52.16.10 is satisfied. Finally, condition (b) of Lemma 52.16.10 is satisfied because $\bigoplus H^0(U, I^n\mathcal{F}_{n+1})$ is a finite graded R-module and we can apply Cohomology, Lemma 20.35.3. □

0EJT Remark 52.23.2. In the situation of Proposition 52.23.1 if we assume A has a dualizing complex, then the condition that $H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite is equivalent to

$$\text{depth}(\mathcal{F}_{1,y}) + \dim(\mathcal{O}_{\overline{\{y\}},z}) > 2$$

for all $y \in U \cap Y$ and $z \in Z \cap \overline{\{y\}}$. See Local Cohomology, Lemma 51.12.1. This holds for example if $\mathcal{F}_1$ is a finite locally free $\mathcal{O}_{U \cap Y}$-module, Y is (S_2), and

$\text{codim}(Z', Y') \geq 3$ for every pair of irreducible components Y' of Y, Z' of Z with $Z' \subset Y'$.

0EJU Proposition 52.23.3. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume there is Noetherian local ring $(R, \mathfrak{m})$ and a ring map $R \to A$ such that

(1) $I = \mathfrak{m}A$,
(2) for $y \in U \cap Y$ the stalk $\mathcal{F}_y^\wedge$ is R-flat,
(3) $H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite A-modules.

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. The proof is exactly the same as the proof of Proposition 52.23.1. Namely, if $\kappa = R/\mathfrak{m}$ then for $n \geq 0$ there is an isomorphism

$$I^n\mathcal{F}_{n+1} \cong \mathcal{F}_1 \otimes_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

and the right hand side is a finite direct sum of copies of $\mathcal{F}_1$. This can be checked by looking at stalks. Everything else is exactly the same. $\square$

0EJV Remark 52.23.4. Proposition 52.23.3 is a local version of [Bar10, Theorem 2.10 (i)]. It is straightforward to deduce the global results from the local one; we will sketch the argument. Namely, suppose $(R, \mathfrak{m})$ is a complete Noetherian local ring and $X \to \text{Spec}(R)$ is a proper morphism. For $n \geq 1$ set $X_n = X \times_{\text{Spec}(R)} \text{Spec}(R/\mathfrak{m}^n)$. Let $Z \subset X_1$ be a closed subset of the special fibre. Set $U = X \setminus Z$ and denote $j : U \to X$ the inclusion morphism. Suppose given an object

$$(\mathcal{F}_n) \text{ of } \text{Coh}(U, \mathfrak{m}\mathcal{O}_U)$$

which is flat over R in the sense that $\mathcal{F}_n$ is flat over $R/\mathfrak{m}^n$ for all n. Assume that $j_*\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1$ are coherent modules. Then affine locally on X we get a canonical extension of $(\mathcal{F}_n)$ by Proposition 52.23.3 and formation of this extension commutes with localization (by Lemma 52.16.11). Thus we get a canonical global object $(\mathcal{G}_n)$ of $\text{Coh}(X, \mathfrak{m}\mathcal{O}_X)$ whose restriction of U is $(\mathcal{F}_n)$. By Grothendieck's existence theorem (Cohomology of Schemes, Proposition 30.25.4) we see there exists a coherent $\mathcal{O}_X$-module $\mathcal{G}$ whose completion is $(\mathcal{G}_n)$. In this way we see that $(\mathcal{F}_n)$ is algebraizable, i.e., it is the completion of a coherent $\mathcal{O}_U$-module.

We add that the coherence of $j_*\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1$ is a condition on the special fibre. Namely, if we denote $j_1 : U_1 \to X_1$ the special fibre of $j : U \to X$, then we can think of $\mathcal{F}_1$ as a coherent sheaf on U_1 and we have $j_*\mathcal{F}_1 = j_{1,*}\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1 = R^1j_{1,*}\mathcal{F}_1$. Hence for example if X_1 is (S_2) and irreducible, we have $\dim(X_1) - \dim(Z) \geq 3$, and $\mathcal{F}_1$ is a locally free $\mathcal{O}_{U_1}$-module, then $j_{1,*}\mathcal{F}_1$ and $R^1j_{1,*}\mathcal{F}_1$ are coherent modules.

52.24. Application to the completion functor

0EKX In this section we just combine some already obtained results in order to conveniently reference them. There are many (stronger) results we could state here.

0EKY Lemma 52.24.1. In Situation 52.16.1 assume

(1) A has a dualizing complex and is I-adically complete,
(2) $I = (f)$ generated by a single element,
(3) A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,
(4) one of the following is true
   (a) A_f is (S_2) and for $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ minimal we have $\dim(A/\mathfrak{p}) \geq 4$, or

(b) if $\mathfrak{p} \not\in V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.

Then with $U_0 = U \cap V(f)$ the completion functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{Coh}(\mathcal{O}_{U'}) \longrightarrow \text{Coh}(U, f\mathcal{O}_U)$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. It follows from Lemma 52.15.7 that the functor is fully faithful (details omitted). Let us prove essential surjectivity. Let $(\mathcal{F}_n)$ be a finite locally free object of $\text{Coh}(U, f\mathcal{O}_U)$. By either Lemma 52.20.4 or Proposition 52.22.2 there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $(\mathcal{F}_n)$ is the completion of $\mathcal{F}$. Namely, for the application of either result the only thing to check is that $(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities. This is done in Lemma 52.20.6. If $y \in U_0$, then the f-adic completion of the stalk $\mathcal{F}_y$ is isomorphic to a finite free module over the f-adic completion of $\mathcal{O}_{U,y}$. Hence $\mathcal{F}$ is finite locally free in an open neighbourhood U' of U_0. This finishes the proof. ☐

0EKZ Lemma 52.24.2. In Situation 52.16.1 assume

(1) $I = (f)$ is principal,
(2) A is f-adically complete,
(3) f is a nonzerodivisor,
(4) $H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$ are finite A-modules.

Then with $U_0 = U \cap V(f)$ the completion functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{Coh}(\mathcal{O}_{U'}) \longrightarrow \text{Coh}(U, f\mathcal{O}_U)$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. The functor is fully faithful by Lemma 52.15.8. Essential surjectivity follows from Lemma 52.16.11. ☐

52.25. Coherent triples

0F22 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $X = \text{Spec}(A)$, $X_0 = \text{Spec}(A/fA)$, $U = X \setminus V(\mathfrak{m})$, and $U_0 = U \cap X_0$. We say $(\mathcal{F}, \mathcal{F}_0, \alpha)$ is a coherent triple if we have

(1) $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module such that $f : \mathcal{F} \to \mathcal{F}$ is injective,
(2) $\mathcal{F}_0$ is a coherent $\mathcal{O}_{X_0}$-module,
(3) $\alpha : \mathcal{F}/f\mathcal{F} \to \mathcal{F}_0|_{U_0}$ is an isomorphism.

There is an obvious notion of a morphism of coherent triples which turns the collection of all coherent triples into a category.

The category of coherent triples is additive but not abelian. However, it is clear what a short exact sequence of coherent triples is.

Given two coherent triples $(\mathcal{F}, \mathcal{F}_0, \alpha)$ and $(\mathcal{G}, \mathcal{G}_0, \beta)$ it may not be the case that $(\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{G}, \mathcal{F}_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{G}_0, \alpha \otimes \beta)$ is a coherent triple[8]. However, if the stalks $\mathcal{G}_x$ are free for all $x \in U_0$, then this does hold.

We will say the coherent triple $(\mathcal{G}, \mathcal{G}_0, \beta)$ is locally free, resp. invertible if $\mathcal{G}$ and $\mathcal{G}_0$ are locally free, resp. invertible modules. In this case tensoring with $(\mathcal{G}, \mathcal{G}_0, \beta)$ makes sense (see above) and turns short exact sequences of coherent triples into short exact sequences of coherent triples.

[8]Namely, it isn't necessarily the case that f is injective on $\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{G}$.

0F23 Lemma 52.25.1. For any coherent triple $(\mathcal{F}, \mathcal{F}_0, \alpha)$ there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ such that $f : \mathcal{F}' \to \mathcal{F}'$ is injective, an isomorphism $\alpha' : \mathcal{F}'|_U \to \mathcal{F}$, and a map $\alpha'_0 : \mathcal{F}'/f\mathcal{F}' \to \mathcal{F}_0$ such that $\alpha \circ (\alpha' \bmod f) = \alpha'_0|_{U_0}$.

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent $\mathcal{O}_X$-module associated to M, see Local Cohomology, Lemma 51.8.2. Since $\mathcal{F}$ is f-torsion free, we may replace M by its quotient by f-power torsion. On the other hand, let $M_0 = \Gamma(X_0, \mathcal{F}_0)$ so that $\mathcal{F}_0$ is the coherent $\mathcal{O}_{X_0}$-module associated to the finite A/fA-module M_0. By Cohomology of Schemes, Lemma 30.10.5 there exists an n such that the isomorphism α_0 corresponds to an A/fA-module homomorphism $\mathfrak{m}^n M/fM \to M_0$ (whose kernel and cokernel are annihilated by a power of $\mathfrak{m}$, but we don't need this). Thus if we take $M' = \mathfrak{m}^n M$ and we let $\mathcal{F}'$ be the coherent $\mathcal{O}_X$-module associated to M', then the lemma is clear. $\square$

Let $(\mathcal{F}, \mathcal{F}_0, \alpha)$ be a coherent triple. Choose $\mathcal{F}', \alpha', \alpha'_0$ as in Lemma 52.25.1. Set

0F24 (52.25.1.1) $$\chi(\mathcal{F}, \mathcal{F}_0, \alpha) = \text{length}_A(\text{Coker}(\alpha'_0)) - \text{length}_A(\text{Ker}(\alpha'_0))$$

The expression on the right makes sense as α'_0 is an isomorphism over U_0 and hence its kernel and coherent are coherent modules supported on $\{\mathfrak{m}\}$ which therefore have finite length (Algebra, Lemma 10.62.3).

0F25 Lemma 52.25.2. The quantity $\chi(\mathcal{F}, \mathcal{F}_0, \alpha)$ in (52.25.1.1) does not depend on the choice of $\mathcal{F}', \alpha', \alpha'_0$ as in Lemma 52.25.1.

Proof. Let $\mathcal{F}', \alpha', \alpha'_0$ and $\mathcal{F}'', \alpha'', \alpha''_0$ be two such choices. For $n > 0$ set $\mathcal{F}'_n = \mathfrak{m}^n\mathcal{F}'$. By Cohomology of Schemes, Lemma 30.10.5 for some n there exists an $\mathcal{O}_X$-module map $\mathcal{F}'_n \to \mathcal{F}''$ agreeing with the identification $\mathcal{F}''|_U = \mathcal{F}'|_U$ determined by α' and α''. Then the diagram

$$\begin{tikzcd} \mathcal{F}'_n/f\mathcal{F}'_n \arrow[r] \arrow[d] & \mathcal{F}'/f\mathcal{F}' \arrow[d, "\alpha'_0"] \\ \mathcal{F}''/f\mathcal{F}'' \arrow[r, "\alpha''_0"] & \mathcal{F}_0 \end{tikzcd}$$

is commutative after restricting to U_0. Hence by Cohomology of Schemes, Lemma 30.10.5 it is commutative after restricting to $\mathfrak{m}^l(\mathcal{F}'_n/f\mathcal{F}'_n)$ for some $l > 0$. Since $\mathcal{F}'_{n+l}/f\mathcal{F}'_{n+l} \to \mathcal{F}'_n/f\mathcal{F}'_n$ factors through $\mathfrak{m}^l(\mathcal{F}'_n/f\mathcal{F}'_n)$ we see that after replacing n by $n+l$ the diagram is commutative. In other words, we have found a third choice $\mathcal{F}''', \alpha''', \alpha'''_0$ such that there are maps $\mathcal{F}''' \to \mathcal{F}''$ and $\mathcal{F}''' \to \mathcal{F}'$ over X compatible with the maps over U and X_0. This reduces us to the case discussed in the next paragraph.

Assume we have a map $\mathcal{F}'' \to \mathcal{F}'$ over X compatible with α', α'' over U and with α'_0, α''_0 over X_0. Observe that $\mathcal{F}'' \to \mathcal{F}'$ is injective as it is an isomorphism over U and since $f : \mathcal{F}'' \to \mathcal{F}''$ is injective. Clearly $\mathcal{F}'/\mathcal{F}''$ is supported on $\{\mathfrak{m}\}$ hence has finite length. We have the maps of coherent $\mathcal{O}_{X_0}$-modules

$$\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}' \xrightarrow{\alpha'_0} \mathcal{F}_0$$

whose composition is α''_0 and which are isomorphisms over U_0. Elementary homological algebra gives a 6-term exact sequence

$$\begin{aligned} 0 \to \text{Ker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}') \to \text{Ker}(\alpha''_0) \to \text{Ker}(\alpha'_0) \to \\ \text{Coker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}') \to \text{Coker}(\alpha''_0) \to \text{Coker}(\alpha'_0) \to 0 \end{aligned}$$

By additivity of lengths (Algebra, Lemma 10.52.3) we find that it suffices to show that

$$\text{length}_A(\text{Coker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}')) - \text{length}_A(\text{Ker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}')) = 0$$

This follows from applying the snake lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'' \arrow[r, "f"'] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & \mathcal{F}''/f\mathcal{F}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r, "f"] & \mathcal{F}' \arrow[r] & \mathcal{F}'/f\mathcal{F}' \arrow[r] & 0 \end{tikzcd}$$

and the fact that $\mathcal{F}'/\mathcal{F}''$ has finite length. □

0F26 **Lemma 52.25.3.** We have $\chi(\mathcal{G}, \mathcal{G}_0, \beta) = \chi(\mathcal{F}, \mathcal{F}_0, \alpha) + \chi(\mathcal{H}, \mathcal{H}_0, \gamma)$ if

$$0 \to (\mathcal{F}, \mathcal{F}_0, \alpha) \to (\mathcal{G}, \mathcal{G}_0, \beta) \to (\mathcal{H}, \mathcal{H}_0, \gamma) \to 0$$

is a short exact sequence of coherent triples.

Proof. Choose $\mathcal{G}', \beta', \beta'_0$ as in Lemma 52.25.1 for the triple $(\mathcal{G}, \mathcal{G}_0, \beta)$. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}' \subset \mathcal{G}'$ be the kernel of the composition

$$\mathcal{G}' \xrightarrow{\beta'} j_*\mathcal{G} \to j_*\mathcal{H}$$

Observe that $\mathcal{H}' = \mathcal{G}'/\mathcal{F}'$ is a coherent subsheaf of $j_*\mathcal{H}$ and hence $f : \mathcal{H}' \to \mathcal{H}'$ is injective. Hence by the snake lemma we obtain a short exact sequence

$$0 \to \mathcal{F}'/f\mathcal{F}' \to \mathcal{G}'/f\mathcal{G}' \to \mathcal{H}'/f\mathcal{H}' \to 0$$

We have isomorphisms $\alpha' : \mathcal{F}'|_U \to \mathcal{F}$, $\beta' : \mathcal{G}'|_U \to \mathcal{G}$, and $\gamma' : \mathcal{H}'|_U \to \mathcal{H}$ by construction. To finish the proof we'll need to construct maps $\alpha'_0 : \mathcal{F}'/f\mathcal{F}' \to \mathcal{F}_0$ and $\gamma'_0 : \mathcal{H}'/f\mathcal{H}' \to \mathcal{H}_0$ as in Lemma 52.25.1 and fitting into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'/f\mathcal{F}' \arrow[r] \arrow[d, dotted, "\alpha'_0"] & \mathcal{G}'/f\mathcal{G}' \arrow[r] \arrow[d, "\beta'_0"] & \mathcal{H}'/f\mathcal{H}' \arrow[r] \arrow[d, dotted, "\gamma'_0"] & 0 \\ 0 \arrow[r] & \mathcal{F}_0 \arrow[r] & \mathcal{G}_0 \arrow[r] & \mathcal{H}_0 \arrow[r] & 0 \end{tikzcd}$$

However, this may not be possible with our initial choice of $\mathcal{G}'$. From the displayed diagram we see the obstruction is exactly the composition

$$\delta : \mathcal{F}'/f\mathcal{F}' \to \mathcal{G}'/f\mathcal{G}' \xrightarrow{\beta'_0} \mathcal{G}_0 \to \mathcal{H}_0$$

Note that the restriction of δ to U_0 is zero by our choice of $\mathcal{F}'$ and $\mathcal{H}'$. Hence by Cohomology of Schemes, Lemma 30.10.5 there exists an $k > 0$ such that δ vanishes on $\mathfrak{m}^k \cdot (\mathcal{F}'/f\mathcal{F}')$. For $n > k$ set $\mathcal{G}'_n = \mathfrak{m}^n\mathcal{G}'$, $\mathcal{F}'_n = \mathcal{G}'_n \cap \mathcal{F}'$, and $\mathcal{H}'_n = \mathcal{G}'_n/\mathcal{F}'_n$. Observe that β'_0 can be composed with $\mathcal{G}'_n/f\mathcal{G}'_n \to \mathcal{G}'/f\mathcal{G}'$ to give a map $\beta'_{n,0} : \mathcal{G}'_n/f\mathcal{G}'_n \to \mathcal{G}_0$ as in Lemma 52.25.1. By Artin-Rees (Algebra, Lemma 10.51.2) we may choose n such that $\mathcal{F}'_n \subset \mathfrak{m}^k\mathcal{F}'$. As above the maps $f : \mathcal{F}'_n \to \mathcal{F}'_n$, $f : \mathcal{G}'_n \to \mathcal{G}'_n$, and $f : \mathcal{H}'_n \to \mathcal{H}'_n$ are injective and as above using the snake lemma we obtain a short exact sequence

$$0 \to \mathcal{F}'_n/f\mathcal{F}'_n \to \mathcal{G}'_n/f\mathcal{G}'_n \to \mathcal{H}'_n/f\mathcal{H}'_n \to 0$$

As above we have isomorphisms $\alpha'_n : \mathcal{F}'_n|_U \to \mathcal{F}$, $\beta'_n : \mathcal{G}'_n|_U \to \mathcal{G}$, and $\gamma'_n : \mathcal{H}'_n|_U \to \mathcal{H}$. We consider the obstruction

$$\delta_n : \mathcal{F}'_n/f\mathcal{F}'_n \to \mathcal{G}'_n/f\mathcal{G}'_n \xrightarrow{\beta'_{n,0}} \mathcal{G}_0 \to \mathcal{H}_0$$

as before. However, the commutative diagram

$$\begin{tikzcd}
\mathcal{F}'_n/f\mathcal{F}'_n \arrow[r] \arrow[d] & \mathcal{G}'_n/f\mathcal{G}'_n \arrow[r, "\beta'_{n,0}"'] \arrow[d] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{H}_0 \arrow[d] \\
\mathcal{F}'/f\mathcal{F}' \arrow[r] & \mathcal{G}'/f\mathcal{G}' \arrow[r, "\beta'_0"] & \mathcal{G}_0 \arrow[r] & \mathcal{H}_0
\end{tikzcd}$$

our choice of n and our observation about δ show that $\delta_n = 0$. This produces the desired maps $\alpha'_{n,0} : \mathcal{F}'_n/f\mathcal{F}'_n \to \mathcal{F}_0$, and $\gamma'_{n,0} : \mathcal{H}'_n/f\mathcal{H}'_n \to \mathcal{H}_0$. OK, so we may use $\mathcal{F}'_n, \alpha'_n, \alpha'_{n,0}$, $\mathcal{G}'_n, \beta'_n, \beta'_{n,0}$, and $\mathcal{H}'_n, \gamma'_n, \gamma'_{n,0}$ to compute $\chi(\mathcal{F}, \mathcal{F}_0, \alpha)$, $\chi(\mathcal{G}, \mathcal{G}_0, \beta)$, and $\chi(\mathcal{H}, \mathcal{H}_0, \gamma)$. Now finally the lemma follows from an application of the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'_n/f\mathcal{F}'_n \arrow[r] \arrow[d] & \mathcal{G}'_n/f\mathcal{G}'_n \arrow[r] \arrow[d] & \mathcal{H}'_n/f\mathcal{H}'_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_0 \arrow[r] & \mathcal{G}_0 \arrow[r] & \mathcal{H}_0 \arrow[r] & 0
\end{tikzcd}$$

and additivity of lengths (Algebra, Lemma 10.52.3). □

0F27 Proposition 52.25.4. Let $(\mathcal{F}, \mathcal{F}_0, \alpha)$ be a coherent triple. Let $(\mathcal{L}, \mathcal{L}_0, \lambda)$ be an invertible coherent triple. Then the function

$$\mathbf{Z} \longrightarrow \mathbf{Z}, \quad n \longmapsto \chi((\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial of degree $\leq \dim(\text{Supp}(\mathcal{F}))$.

More precisely, if $\mathcal{F} = 0$, then the function is constant. If $\mathcal{F}$ has finite support in U, then the function is constant. If the support of $\mathcal{F}$ in U has dimension 1, i.e., the closure of the support of $\mathcal{F}$ in X has dimension 2, then the function is linear, etc.

Proof. We will prove this by induction on the dimension of the support of $\mathcal{F}$.

The base case is when $\mathcal{F} = 0$. Then either $\mathcal{F}_0$ is zero or its support is $\{\mathfrak{m}\}$. In this case we have

$$(\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n} = (0, \mathcal{F}_0 \otimes \mathcal{L}_0^{\otimes n}, 0) \cong (0, \mathcal{F}_0, 0)$$

Thus the function of the lemma is constant with value equal to the length of $\mathcal{F}_0$.

Induction step. Assume the support of $\mathcal{F}$ is nonempty. Let $\mathcal{G}_0 \subset \mathcal{F}_0$ denote the submodule of sections supported on $\{\mathfrak{m}\}$. Then we get a short exact sequence

$$0 \to (0, \mathcal{G}_0, 0) \to (\mathcal{F}, \mathcal{F}_0, \alpha) \to (\mathcal{F}, \mathcal{F}_0/\mathcal{G}_0, \alpha) \to 0$$

This sequence remains exact if we tensor by the invertible coherent triple $(\mathcal{L}, \mathcal{L}_0, \lambda)$, see discussion above. Thus by additivity of χ (Lemma 52.25.3) and the base case explained above, it suffices to prove the induction step for $(\mathcal{F}, \mathcal{F}_0/\mathcal{G}_0, \alpha)$. In this way we see that we may assume $\mathfrak{m}$ is not an associated point of $\mathcal{F}_0$.

Let $T = \text{Ass}(\mathcal{F}) \cup \text{Ass}(\mathcal{F}/f\mathcal{F})$. Since U is quasi-affine, we can find $s \in \Gamma(U, \mathcal{L})$ which does not vanish at any $u \in T$, see Properties, Lemma 28.29.7. After multiplying s by a suitable element of $\mathfrak{m}$ we may assume $\lambda(s \bmod f) = s_0|_{U_0}$ for some $s_0 \in \Gamma(X_0, \mathcal{L}_0)$; details omitted. We obtain a morphism

$$(s, s_0) : (\mathcal{O}_U, \mathcal{O}_{X_0}, 1) \longrightarrow (\mathcal{L}, \mathcal{L}_0, \lambda)$$

in the category of coherent triples. Let $\mathcal{G} = \text{Coker}(s : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L})$ and $\mathcal{G}_0 = \text{Coker}(s_0 : \mathcal{F}_0 \to \mathcal{F}_0 \otimes \mathcal{L}_0)$. Observe that $s_0 : \mathcal{F}_0 \to \mathcal{F}_0 \otimes \mathcal{L}_0$ is injective as it is injective on U_0 by our choice of s and as $\mathfrak{m}$ isn't an associated point of $\mathcal{F}_0$. It follows that there exists an isomorphism $\beta : \mathcal{G}/f\mathcal{G} \to \mathcal{G}_0|_{U_0}$ such that we obtain a short exact sequence

$$0 \to (\mathcal{F}, \mathcal{F}_0, \alpha) \to (\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda) \to (\mathcal{G}, \mathcal{G}_0, \beta) \to 0$$

By induction on the dimension of the support we know the proposition holds for the coherent triple $(\mathcal{G}, \mathcal{G}_0, \beta)$. Using the additivity of Lemma 52.25.3 we see that

$$n \longmapsto \chi((\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n+1}) - \chi((\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial. We conclude by a variant of Algebra, Lemma 10.58.5 for functions defined for all integers (details omitted). □

0F28 Lemma 52.25.5. Assume $\text{depth}(A) \geq 3$ or equivalently $\text{depth}(A/fA) \geq 2$. Let $(\mathcal{L}, \mathcal{L}_0, \lambda)$ be an invertible coherent triple. Then

$$\chi(\mathcal{L}, \mathcal{L}_0, \lambda) = \text{length}_A \text{Coker}(\Gamma(U, \mathcal{L}) \to \Gamma(U_0, \mathcal{L}_0))$$

and in particular this is ≥ 0. Moreover, $\chi(\mathcal{L}, \mathcal{L}_0, \lambda) = 0$ if and only if $\mathcal{L} \cong \mathcal{O}_U$.

Proof. The equivalence of the depth conditions follows from Algebra, Lemma 10.72.7. By the depth condition we see that $\Gamma(U, \mathcal{O}_U) = A$ and $\Gamma(U_0, \mathcal{O}_{U_0}) = A/fA$, see Dualizing Complexes, Lemma 47.11.1 and Local Cohomology, Lemma 51.8.2. Using Local Cohomology, Lemma 51.12.2 we find that $M = \Gamma(U, \mathcal{L})$ is a finite A-module. This in turn implies $\text{depth}(M) \geq 2$ for example by part (4) of Local Cohomology, Lemma 51.8.2 or by Divisors, Lemma 31.6.6. Also, we have $\mathcal{L}_0 \cong \mathcal{O}_{X_0}$ as X_0 is a local scheme. Hence we also see that $M_0 = \Gamma(X_0, \mathcal{L}_0) = \Gamma(U_0, \mathcal{L}_0|_{U_0})$ and that this module is isomorphic to A/fA.

By the above $\mathcal{F}' = \widetilde{M}$ is a coherent $\mathcal{O}_X$-module whose restriction to U is isomorphic to $\mathcal{L}$. The isomorphism $\lambda : \mathcal{L}/f\mathcal{L} \to \mathcal{L}_0|_{U_0}$ determines a map $M/fM \to M_0$ on global sections which is an isomorphism over U_0. Since $\text{depth}(M) \geq 2$ we see that $H^0_\mathfrak{m}(M/fM) = 0$ and it follows that $M/fM \to M_0$ is injective. Thus by definition

$$\chi(\mathcal{L}, \mathcal{L}_0, \lambda) = \text{length}_A \text{Coker}(M/fM \to M_0)$$

which gives the first statement of the lemma.

Finally, if this length is 0, then $M \to M_0$ is surjective. Hence we can find $s \in M = \Gamma(U, \mathcal{L})$ mapping to a trivializing section of $\mathcal{L}_0$. Consider the finite A-modules K, Q defined by the exact sequence

$$0 \to K \to A \xrightarrow{s} M \to Q \to 0$$

The supports of K and Q do not meet U_0 because s is nonzero at points of U_0. Using Algebra, Lemma 10.72.6 we see that $\text{depth}(K) \geq 2$ (observe that $As \subset M$ has depth ≥ 1 as a submodule of M). Thus the support of K if nonempty has dimension ≥ 2 by Algebra, Lemma 10.72.3. This contradicts $\text{Supp}(M) \cap V(f) \subset \{\mathfrak{m}\}$ unless $K = 0$. When $K = 0$ we find that $\text{depth}(Q) \geq 2$ and we conclude $Q = 0$ as before. Hence $A \cong M$ and $\mathcal{L}$ is trivial. □

52.26. Invertible modules on punctured spectra, I

0F29 In this section we prove some local Lefschetz theorems for the Picard group. Some of the ideas are taken from [Kol13], [BdJ14], and [Kol16a].

0F2A **Lemma 52.26.1.** Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor and assume that $\text{depth}(A/fA) \geq 2$, or equivalently $\text{depth}(A) \geq 3$. Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. The map

$$\text{Pic}(U) \to \text{Pic}(U_0)$$

is injective on torsion.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. Observe that $\mathcal{L}$ maps to 0 in $\text{Pic}(U_0)$ if and only if we can extend $\mathcal{L}$ to an invertible coherent triple $(\mathcal{L}, \mathcal{L}_0, \lambda)$ as in Section 52.25. By Proposition 52.25.4 the function

$$n \longmapsto \chi((\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial. By Lemma 52.25.5 the value of this polynomial is zero if and only if $\mathcal{L}^{\otimes n}$ is trivial. Thus if $\mathcal{L}$ is torsion, then this polynomial has infinitely many zeros, hence is identically zero, hence $\mathcal{L}$ is trivial. $\square$

0F2B **Proposition 52.26.2** (Kollár). Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume [Kol16a, Theorem 1.9]

(1) A has a dualizing complex,
(2) f is a nonzerodivisor,
(3) $\text{depth}(A/fA) \geq 2$, or equivalently $\text{depth}(A) \geq 3$,
(4) if $f \in \mathfrak{p} \subset A$ is a prime ideal with $\dim(A/\mathfrak{p}) = 2$, then $\text{depth}(A_\mathfrak{p}) \geq 2$.

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. The map

$$\text{Pic}(U) \to \text{Pic}(U_0)$$

is injective. Finally, if (1), (2), (3), A is (S_2), and $\dim(A) \geq 4$, then (4) holds.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. Observe that $\mathcal{L}$ maps to 0 in $\text{Pic}(U_0)$ if and only if we can extend $\mathcal{L}$ to an invertible coherent triple $(\mathcal{L}, \mathcal{L}_0, \lambda)$ as in Section 52.25. By Proposition 52.25.4 the function

$$n \longmapsto \chi((\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial P. By Lemma 52.25.5 we have $P(n) \geq 0$ for all $n \in \mathbf{Z}$ with equality if and only if $\mathcal{L}^{\otimes n}$ is trivial. In particular $P(0) = 0$ and P is either identically zero and we win or P has even degree ≥ 2.

Set $M = \Gamma(U, \mathcal{L})$ and $M_0 = \Gamma(X_0, \mathcal{L}_0) = \Gamma(U_0, \mathcal{L}_0)$. Then M is a finite A-module of depth ≥ 2 and $M_0 \cong A/fA$, see proof of Lemma 52.25.5. Note that $H^2_\mathfrak{m}(M)$ is finite A-module by Local Cohomology, Lemma 51.7.4 and the fact that $H^i_\mathfrak{m}(A) = 0$ for $i = 0, 1, 2$ since $\text{depth}(A) \geq 3$. Consider the short exact sequence

$$0 \to M/fM \to M_0 \to Q \to 0$$

Lemma 52.25.5 tells us Q has finite length equal to $\chi(\mathcal{L}, \mathcal{L}_0, \lambda)$. We obtain $Q = H^1_\mathfrak{m}(M/fM)$ and $H^i_\mathfrak{m}(M/fM) = H^i_\mathfrak{m}(M_0) \cong H^i_\mathfrak{m}(A/fA)$ for $i > 1$ from the long exact sequence of local cohomology associated to the displayed short exact sequence. Consider the long exact sequence of local cohomology associated to the sequence $0 \to M \to M \to M/fM \to 0$. It starts with

$$0 \to Q \to H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(A/fA)$$

Using additivity of lengths we see that $\chi(\mathcal{L}, \mathcal{L}_0, \lambda)$ is equal to the length of the image of $H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(A/fA)$.

Let prove the lemma in a special case to elucidate the rest of the proof. Namely, assume for a moment that $H^2_\mathfrak{m}(A/fA)$ is a finite length module. Then we would have $P(1) \leq \text{length}_A H^2_\mathfrak{m}(A/fA)$. The exact same argument applied to $\mathcal{L}^{\otimes n}$ shows that $P(n) \leq \text{length}_A H^2_\mathfrak{m}(A/fA)$ for all n. Thus P cannot have positive degree and we win. In the rest of the proof we will modify this argument to give a linear upper bound for $P(n)$ which suffices.

Let us study the map $H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(M_0) \cong H^2_\mathfrak{m}(A/fA)$. Choose a normalized dualizing complex $\omega_A^\bullet$ for A. By local duality (Dualizing Complexes, Lemma 47.18.4) this map is Matlis dual to the map

$$\text{Ext}^{-2}_A(M, \omega_A^\bullet) \longleftarrow \text{Ext}^{-2}_A(M_0, \omega_A^\bullet)$$

whose image therefore has the same (finite) length. The support (if nonempty) of the finite A-module $\text{Ext}^{-2}_A(M_0, \omega_A^\bullet)$ consists of $\mathfrak{m}$ and a finite number of primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ containing f with $\dim(A/\mathfrak{p}_i) = 1$. Namely, by Local Cohomology, Lemma 51.9.4 the support is contained in the set of primes $\mathfrak{p} \subset A$ with $\text{depth}_{A_\mathfrak{p}}(M_{0,\mathfrak{p}}) + \dim(A/\mathfrak{p}) \leq 2$. Thus it suffices to show there is no prime $\mathfrak{p}$ containing f with $\dim(A/\mathfrak{p}) = 2$ and $\text{depth}_{A_\mathfrak{p}}(M_{0,\mathfrak{p}}) = 0$. However, because $M_{0,\mathfrak{p}} \cong (A/fA)_\mathfrak{p}$ this would give $\text{depth}(A_\mathfrak{p}) = 1$ which contradicts assumption (4). Choose a section $t \in \Gamma(U, \mathcal{L}^{\otimes -1})$ which does not vanish in the points $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, see Properties, Lemma 28.29.7. Multiplication by t on global sections determines a map $t : M \to A$ which defines an isomorphism $M_{\mathfrak{p}_i} \to A_{\mathfrak{p}_i}$ for $i = 1, \ldots, r$. Denote $t_0 = t|_{U_0}$ the corresponding section of $\Gamma(U_0, \mathcal{L}_0^{\otimes -1})$ which similarly determines a map $t_0 : M_0 \to A/fA$ compatible with t. We conclude that there is a commutative diagram

$$\begin{tikzcd}
\text{Ext}^{-2}_A(M, \omega_A^\bullet) & \text{Ext}^{-2}_A(M_0, \omega_A^\bullet) \arrow[l] \\
\text{Ext}^{-2}_A(A, \omega_A^\bullet) \arrow[u, "t"] & \text{Ext}^{-2}_A(A/fA, \omega_A^\bullet) \arrow[l] \arrow[u, "t_0"']
\end{tikzcd}$$

It follows that the length of the image of the top horizontal map is at most the length of $\text{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ plus the length of the cokernel of t_0.

However, if we replace $\mathcal{L}$ by $\mathcal{L}^n$ for $n > 1$, then we can use

$$t^n : M_n = \Gamma(U, \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(U, \mathcal{O}_U) = A$$

instead of t. This replaces t_0 by its nth power. Thus the length of the image of the map $\text{Ext}^{-2}_A(M_n, \omega_A^\bullet) \leftarrow \text{Ext}^{-2}_A(M_{n,0}, \omega_A^\bullet)$ is at most the length of $\text{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ plus the length of the cokernel of

$$t_0^n : \text{Ext}^{-2}_A(A/fA, \omega_A^\bullet) \longrightarrow \text{Ext}^{-2}_A(M_{n,0}, \omega_A^\bullet)$$

Via the isomorphism $M_0 \cong A/fA$ the map t_0 becomes $g : A/fA \to A/fA$ for some $g \in A/fA$ and via the corresponding isomorphisms $M_{n,0} \cong A/fA$ the map t_0^n becomes $g^n : A/fA \to A/fA$. Thus the length of the cokernel above is the length of the quotient of $\text{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ by g^n. Since $\text{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ is a finite A-module with support T of dimension 1 and since $V(g) \cap T$ consists of the closed point by our choice of t this length grows linearly in n by Algebra, Lemma 10.62.6.

To finish the proof we prove the final assertion. Assume $f \in \mathfrak{m} \subset A$ satisfies (1), (2), (3), A is (S_2), and $\dim(A) \geq 4$. Condition (1) implies A is catenary, see Dualizing Complexes, Lemma 47.17.4. Then $\text{Spec}(A)$ is equidimensional by Local Cohomology, Lemma 51.3.2. Thus $\dim(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) \geq 4$ for every prime $\mathfrak{p}$ of A. Then $\text{depth}(A_\mathfrak{p}) \geq \min(2, \dim(A_\mathfrak{p})) \geq \min(2, 4 - \dim(A/\mathfrak{p}))$ and hence (4) holds. □

0FIX Remark 52.26.3. In SGA2 we find the following result. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume A is a quotient of a regular ring, the element f is a nonzerodivisor, and

(a) if $\mathfrak{p} \subset A$ is a prime ideal with $\dim(A/\mathfrak{p}) = 1$, then $\text{depth}(A_\mathfrak{p}) \geq 2$, and
(b) $\text{depth}(A/fA) \geq 3$, or equivalently $\text{depth}(A) \geq 4$.

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Then the map

$$\text{Pic}(U) \to \text{Pic}(U_0)$$

is injective. This is [Gro68, Exposee XI, Lemma 3.16][9]. This result from SGA2 follows from Proposition 52.26.2 because

(1) a quotient of a regular ring has a dualizing complex (see Dualizing Complexes, Lemma 47.21.3 and Proposition 47.15.11), and
(2) if $\text{depth}(A) \geq 4$ then $\text{depth}(A_\mathfrak{p}) \geq 2$ for all primes $\mathfrak{p}$ with $\dim(A/\mathfrak{p}) = 2$, see Algebra, Lemma 10.72.10.

52.27. Invertible modules on punctured spectra, II

0F2C Next we turn to surjectivity in local Lefschetz for the Picard group. First to extend an invertible module on U_0 to an open neighbourhood we have the following simple criterion.

0F2D Lemma 52.27.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor,
(3) $H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules, and
(4) $H^3_\mathfrak{m}(A/fA) = 0$[10].

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Then

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{Pic}(U') \longrightarrow \text{Pic}(U_0)$$

is surjective.

Proof. Let $U_0 \subset U_n \subset U$ be the nth infinitesimal neighbourhood of U_0. Observe that the ideal sheaf of U_n in U_{n+1} is isomorphic to $\mathcal{O}_{U_0}$ as $U_0 \subset U$ is the principal closed subscheme cut out by the nonzerodivisor f. Hence we have an exact sequence of abelian groups

$$\text{Pic}(U_{n+1}) \to \text{Pic}(U_n) \to H^2(U_0, \mathcal{O}_{U_0}) = H^3_\mathfrak{m}(A/fA) = 0$$

see More on Morphisms, Lemma 37.4.1. Thus every invertible $\mathcal{O}_{U_0}$-module is the restriction of an invertible coherent formal module, i.e., an invertible object of $\text{Coh}(U, f\mathcal{O}_U)$. We conclude by applying Lemma 52.24.2. □

[9] Condition (a) follows from condition (b), see Algebra, Lemma 10.72.10.
[10] Observe that (3) and (4) hold if $\text{depth}(A/fA) \geq 4$, or equivalently $\text{depth}(A) \geq 5$.

0F2E Remark 52.27.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. The conclusion of Lemma 52.27.1 holds if we assume

(1) A has a dualizing complex,
(2) A is f-adically complete,
(3) f is a nonzerodivisor,
(4) one of the following is true
   (a) A_f is (S_2) and for $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ minimal we have $\dim(A/\mathfrak{p}) \geq 4$, or
   (b) if $\mathfrak{p} \not\in V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.
(5) $H^3_\mathfrak{m}(A/fA) = 0$.

The proof is exactly the same as the proof of Lemma 52.27.1 using Lemma 52.24.1 instead of Lemma 52.24.2. Two points need to be made here: (a) it seems hard to find examples where one knows $H^3_\mathfrak{m}(A/fA) = 0$ without assuming $\text{depth}(A/fA) \geq 4$, and (b) the proof of Lemma 52.24.1 is a good deal harder than the proof of Lemma 52.24.2.

0F2F Lemma 52.27.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Assume

(1) the conditions of Lemma 52.27.1 hold, and
(2) for every maximal ideal $\mathfrak{p} \subset A_f$ the punctured spectrum of $(A_f)_\mathfrak{p}$ has trivial Picard group.

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Then

$$\text{Pic}(U) \longrightarrow \text{Pic}(U_0)$$

is surjective.

Proof. Let $\mathcal{L}_0 \in \text{Pic}(U_0)$. By Lemma 52.27.1 there exists an open $U_0 \subset U' \subset U$ and $\mathcal{L}' \in \text{Pic}(U')$ whose restriction to U_0 is $\mathcal{L}_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ as in (2). By assumption we can find invertible modules $\mathcal{L}'_i$ on $\text{Spec}(A_{\mathfrak{p}_i})$ agreeing with $\mathcal{L}'$ over the punctured spectrum $U' \times_U \text{Spec}(A_{\mathfrak{p}_i})$ since trivial invertible modules always extend. By Limits, Lemma 32.20.2 applied n times we see that $\mathcal{L}'$ extends to an invertible module on U. $\square$

0F2G Lemma 52.27.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $A^\wedge$ be its completion. Let U, resp. $U^\wedge$ be the punctured spectrum of A, resp. $A^\wedge$. Then $\text{Pic}(U) \to \text{Pic}(U^\wedge)$ is injective.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module with pullback $\mathcal{L}^\wedge$ on $U^\wedge$. We have $H^0(U, \mathcal{O}_U) = A$ by our assumption on depth and Dualizing Complexes, Lemma 47.11.1 and Local Cohomology, Lemma 51.8.2. Thus $\mathcal{L}$ is trivial if and only if $M = H^0(U, \mathcal{L})$ is isomorphic to A as an A-module. (Details omitted.) Since $A \to A^\wedge$ is flat we have $M \otimes_A A^\wedge = \Gamma(U^\wedge, \mathcal{L}^\wedge)$ by flat base change, see Cohomology of Schemes, Lemma 30.5.2. Finally, it is easy to see that $M \cong A$ if and only if $M \otimes_A A^\wedge \cong A^\wedge$. $\square$

0F2H Lemma 52.27.5. Let $(A, \mathfrak{m})$ be a regular local ring. Then the Picard group of the punctured spectrum of A is trivial.

Proof. Combine Divisors, Lemma 31.28.3 with More on Algebra, Lemma 15.121.2. $\square$

Now we can bootstrap the earlier results to prove that Picard groups are trivial for punctured spectra of complete intersections of dimension ≥ 4. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 23.8.

0F2I Proposition 52.27.6 (Grothendieck). Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is a complete intersection of dimension ≥ 4, then the Picard group of the punctured spectrum of A is trivial.

Proof. By Lemma 52.27.4 we may assume that A is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where B is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on r. The base case is $r = 0$ which follows from Lemma 52.27.5.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 52.27.3 to $f_r \in A'$. This is permissible:

(1) condition (1) of Lemma 52.27.1 holds because our local rings are complete,
(2) condition (2) of Lemma 52.27.1 holds holds as $f_1, \ldots, f_r$ is a regular sequence,
(3) condition (3) and (4) of Lemma 52.27.1 hold as $A = A'/f_rA'$ is Cohen-Macaulay of dimension $\dim(A) \geq 4$,
(4) condition (2) of Lemma 52.27.3 holds by induction hypothesis as $\dim((A'_{f_r})_\mathfrak{p}) \geq 4$ for a maximal prime $\mathfrak{p}$ of A'_{f_r} and as $(A'_{f_r})_\mathfrak{p} = B_\mathfrak{q}/(f_1, \ldots, f_{r-1})$ for some prime ideal $\mathfrak{q} \subset B$ and $B_\mathfrak{q}$ is regular.

This finishes the proof. □

0F2J Example 52.27.7. The dimension bound in Proposition 52.27.6 is sharp. For example the Picard group of the punctured spectrum of $A = k[[x, y, z, w]]/(xy - zw)$ is nontrivial. Namely, the ideal $I = (x, z)$ cuts out an effective Cartier divisor D on the punctured spectrum U of A as it is easy to see that I_x, I_y, I_z, I_w are invertible ideals in A_x, A_y, A_z, A_w. But on the other hand, A/I has depth ≥ 1 (in fact 2), hence I has depth ≥ 2 (in fact 3), hence $I = \Gamma(U, \mathcal{O}_U(-D))$. Thus if $\mathcal{O}_U(-D)$ were trivial, then we'd have $I \cong \Gamma(U, \mathcal{O}_U) = A$ which isn't true as I isn't generated by 1 element.

0F9L Example 52.27.8. Proposition 52.27.6 cannot be extended to quotients

$$A = B/(f_1, \ldots, f_r)$$

where B is regular and $\dim(B) - r \geq 4$. In other words, the condition that $f_1, \ldots, f_r$ be a regular sequence is (in general) needed for vanishing of the Picard group of the punctured spectrum of A. Namely, let k be a field and set

$$A = k[[a, b, x, y, z, u, v, w]]/(a^3, b^3, xa^2 + yab + zb^2, w^2)$$

Observe that $A = A_0[w]/(w^2)$ with $A_0 = k[[a, b, x, y, z, u, v]]/(a^3, b^3, xa^2 + yab + zb^2)$. We will show below that A_0 has depth 2. Denote U the punctured spectrum of A and U_0 the punctured spectrum of A_0. Observe there is a short exact sequence $0 \to A_0 \to A \to A_0 \to 0$ where the first arrow is given by multiplication by w. By More on Morphisms, Lemma 37.4.1 we find that there is an exact sequence

$$H^0(U, \mathcal{O}_U^*) \to H^0(U_0, \mathcal{O}_{U_0}^*) \to H^1(U_0, \mathcal{O}_{U_0}) \to \mathrm{Pic}(U)$$

Since the depth of A_0 and hence A is 2 we see that $H^0(U_0, \mathcal{O}_{U_0}) = A_0$ and $H^0(U, \mathcal{O}_U) = A$ and that $H^1(U_0, \mathcal{O}_{U_0})$ is nonzero, see Dualizing Complexes, Lemma 47.11.1 and Local Cohomology, Lemma 51.2.2. Thus the last arrow displayed above is nonzero and we conclude that $\text{Pic}(U)$ is nonzero.

To show that A_0 has depth 2 it suffices to show that $A_1 = k[[a, b, x, y, z]]/(a^3, b^3, xa^2 + yab + zb^2)$ has depth 0. This is true because a^2b^2 maps to a nonzero element of A_1 which is annihilated by each of the variables a, b, x, y, z. For example $ya^2b^2 = (yab)(ab) = -(xa^2 + zb^2)(ab) = -xa^3b - yab^3 = 0$ in A_1. The other cases are similar.

52.28. Application to Lefschetz theorems

0EL0 In this section we discuss the relation between coherent sheaves on a projective scheme P and coherent modules on formal completion along an ample divisor Q.

Let k be a field. Let P be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Let $s \in \Gamma(P, \mathcal{L})$ be a section[11] and let $Q = Z(s)$ be the zero scheme, see Divisors, Definition 31.14.8. For all $n \geq 1$ we denote $Q_n = Z(s^n)$ the nth infinitesimal neighbourhood of Q. If $\mathcal{F}$ is a coherent $\mathcal{O}_P$-module, then we denote $\mathcal{F}_n = \mathcal{F}|_{Q_n}$ the restriction, i.e., the pullback of $\mathcal{F}$ by the closed immersion $Q_n \to P$.

0EL1 Proposition 52.28.1. In the situation above assume for all points $p \in P \setminus Q$ we have

$$\text{depth}(\mathcal{F}_p) + \dim(\overline{\{p\}}) > s$$

Then the map

$$H^i(P, \mathcal{F}) \longrightarrow \lim H^i(Q_n, \mathcal{F}_n)$$

is an isomorphism for $0 \leq i < s$.

Proof. We will use More on Morphisms, Lemma 37.51.1 and we will use the notation used and results found More on Morphisms, Section 37.51 without further mention; this proof will not make sense without at least understanding the statement of the lemma. Observe that in our case $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$ is a finite type k-algebra all of whose graded parts are finite dimensional k-vector spaces, see Cohomology of Schemes, Lemma 30.16.1.

We may and do think of s as an element $f \in A_1 \subset A$, i.e., a homogeneous element of degree 1 of A. Denote $Y = V(f) \subset X$ the closed subscheme defined by f. Then $U \cap Y = (\pi|_U)^{-1}(Q)$ scheme theoretically. Recall the notation $\mathcal{F}_U = \pi^*\mathcal{F}|_U = (\pi|_U)^*\mathcal{F}$. This is a coherent $\mathcal{O}_U$-module. Choose a finite A-module M such that $\mathcal{F}_U = \widetilde{M}|_U$ (for existence see Local Cohomology, Lemma 51.8.2). We claim that $H^i_Z(M)$ is annihilated by a power of f for $i \leq s + 1$.

To prove the claim we will apply Local Cohomology, Proposition 51.10.1. Translating into geometry we see that it suffices to prove for $u \in U$, $u \notin Y$ and $z \in \overline{\{u\}} \cap Z$ that

$$\text{depth}(\mathcal{F}_{U,u}) + \dim(\mathcal{O}_{\overline{\{u\}},z}) > s + 1$$

This requires only a small amount of thought.

[11] We do not require s to be a regular section. Correspondingly, Q is only a locally principal closed subscheme of P and not necessarily an effective Cartier divisor.

Observe that $Z = \mathrm{Spec}(A_0)$ is a finite set of closed points of X because A_0 is a finite dimensional k-algebra. (The reader who would like Z to be a singleton can replace the finite k-algebra A_0 by k; it won't affect anything else in the proof.)

The morphism $\pi : L \to P$ and its restriction $\pi|_U : U \to P$ are smooth of relative dimension 1. Let $u \in U$, $u \not\in Y$ and $z \in \overline{\{u\}} \cap Z$. Let $p = \pi(u) \in P \setminus Q$ be its image. Then either u is a generic point of the fibre of π over p or a closed point of the fibre. If u is a generic point of the fibre, then $\mathrm{depth}(\mathcal{F}_{U,u}) = \mathrm{depth}(\mathcal{F}_p)$ and $\dim(\overline{\{u\}}) = \dim(\overline{\{p\}}) + 1$. If u is a closed point of the fibre, then $\mathrm{depth}(\mathcal{F}_{U,u}) = \mathrm{depth}(\mathcal{F}_p) + 1$ and $\dim(\overline{\{u\}}) = \dim(\overline{\{p\}})$. In both cases we have $\dim(\overline{\{u\}}) = \dim(\mathcal{O}_{\overline{\{u\}},z})$ because every point of Z is closed. Thus the desired inequality follows from the assumption in the statement of the lemma.

Let A' be the f-adic completion of A. So $A \to A'$ is flat by Algebra, Lemma 10.97.2. Denote $U' \subset X' = \mathrm{Spec}(A')$ the inverse image of U and similarly for Y' and Z'. Let $\mathcal{F}'$ on U' be the pullback of $\mathcal{F}_U$ and let $M' = M \otimes_A A'$. By flat base change for local cohomology (Local Cohomology, Lemma 51.5.7) we have

$$H^i_{Z'}(M') = H^i_Z(M) \otimes_A A'$$

and we find that for $i \leq s + 1$ these are annihilated by a power of f. Consider the diagram

$$\begin{tikzcd}
 & H^i(U, \mathcal{F}_U) \arrow[r] \arrow[dl] \arrow[d] & \lim H^i(U, \mathcal{F}_U/f^n\mathcal{F}_U) \arrow[d, equal] \\
H^i(U, \mathcal{F}_U) \otimes_A A' \arrow[r, equal] & H^i(U', \mathcal{F}') \arrow[r] & \lim H^i(U', \mathcal{F}'/f^n\mathcal{F}')
\end{tikzcd}$$

The lower horizontal arrow is an isomorphism for $i < s$ by Lemma 52.13.2 and the torsion property we just proved. The horizontal equal sign is flat base change (Cohomology of Schemes, Lemma 30.5.2) and the vertical equal sign is because $U \cap Y$ and $U' \cap Y'$ as well as their nth infinitesimal neighbourhoods are mapped isomorphically onto each other (as we are completing with respect to f).

Applying More on Morphisms, Equation (37.51.0.2) we have compatible direct sum decompositions

$$\lim H^i(U, \mathcal{F}_U/f^n\mathcal{F}_U) = \lim \left(\bigoplus\nolimits_{m \in \mathbf{Z}} H^i(Q_n, \mathcal{F}_n \otimes \mathcal{L}^{\otimes m})\right)$$

and

$$H^i(U, \mathcal{F}_U) = \bigoplus\nolimits_{m \in \mathbf{Z}} H^i(P, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$$

Thus we conclude by Algebra, Lemma 10.98.4. $\square$

0EL2 Lemma 52.28.2. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s with nth infinitesimal neighbourhood $Y_n = Z(s^n)$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that for all $x \in X \setminus Y$ we have

$$\mathrm{depth}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > 1$$

Then $\Gamma(V, \mathcal{F}) \to \lim \Gamma(Y_n, \mathcal{F}|_{Y_n})$ is an isomorphism for any open subscheme $V \subset X$ containing Y.

Proof. By Proposition 52.28.1 this is true for $V = X$. Thus it suffices to show that the map $\Gamma(V, \mathcal{F}) \to \lim \Gamma(Y_n, \mathcal{F}|_{Y_n})$ is injective. If $\sigma \in \Gamma(V, \mathcal{F})$ maps to zero, then its support is disjoint from Y (details omitted; hint: use Krull's intersection theorem). Then the closure $T \subset X$ of $\text{Supp}(\sigma)$ is disjoint from Y. Whence T is proper over k (being closed in X) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 29.43.18) and hence finite over k (Morphisms, Lemma 29.44.11). Thus T is a finite set of closed points of X. Thus $\text{depth}(\mathcal{F}_x) \geq 2$ is at least 1 for $x \in T$ by our assumption. We conclude that $\Gamma(V, \mathcal{F}) \to \Gamma(V \setminus T, \mathcal{F})$ is injective and $\sigma = 0$ as desired. $\square$

0EL3 Example 52.28.3. Let k be a field and let X be a proper variety over k. Let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y)$ is ample and denote Y_n its nth infinitesimal neighbourhood. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Here are some special cases of Proposition 52.28.1.

(1) If X is a curve, we don't learn anything.
(2) If X is a Cohen-Macaulay (for example normal) surface, then
$$H^0(X, \mathcal{E}) \to \lim H^0(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism.
(3) If X is a Cohen-Macaulay threefold, then
$$H^0(X, \mathcal{E}) \to \lim H^0(Y_n, \mathcal{E}|_{Y_n}) \quad \text{and} \quad H^1(X, \mathcal{E}) \to \lim H^1(Y_n, \mathcal{E}|_{Y_n})$$
are isomorphisms.

Presumably the pattern is clear. If X is a normal threefold, then we can conclude the result for H^0 but not for H^1.

Before we prove the next main result, we need a lemma.

0EL4 Lemma 52.28.4. In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $\text{Coh}(U, I\mathcal{O}_U)$. Assume

(1) A is a graded ring, $\mathfrak{a} = A_+$, and I is a homogeneous ideal,
(2) $(\mathcal{F}_n) = (\widetilde{M_n}|_U)$ where (M_n) is an inverse system of graded A-modules, and
(3) $(\mathcal{F}_n)$ extends canonically to X.

Then there is a finite graded A-module N such that

(a) the inverse systems (N/I^nN) and (M_n) are pro-isomorphic in the category of graded A-modules modulo A_+-power torsion modules, and
(b) $(\mathcal{F}_n)$ is the completion of of the coherent module associated to N.

Proof. Let $(\mathcal{G}_n)$ be the canonical extension as in Lemma 52.16.8. The grading on A and M_n determines an action
$$a : \mathbf{G}_m \times X \longrightarrow X$$
of the group scheme $\mathbf{G}_m$ on X such that $(\widetilde{M_n})$ becomes an inverse system of $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-modules, see Groupoids, Example 39.12.3. Since $\mathfrak{a}$ and I are homogeneous ideals the closed subschemes Z, Y and the open subscheme U are $\mathbf{G}_m$-invariant closed and open subschemes. The restriction $(\mathcal{F}_n)$ of $(\widetilde{M_n})$ is an inverse system of $\mathbf{G}_m$-equivariant coherent $\mathcal{O}_U$-modules. In other words, $(\mathcal{F}_n)$ is a $\mathbf{G}_m$-equivariant coherent formal module, in the sense that there is an isomorphism
$$\alpha : (a^*\mathcal{F}_n) \longrightarrow (p^*\mathcal{F}_n)$$

over $\mathbf{G}_m \times U$ satisfying a suitable cocycle condition. Since a and p are flat morphisms of affine schemes, by Lemma 52.16.9 we conclude that there exists a unique isomorphism

$$\beta : (a^*\mathcal{G}_n) \longrightarrow (p^*\mathcal{G}_n)$$

over $\mathbf{G}_m \times X$ restricting to α on $\mathbf{G}_m \times U$. The uniqueness guarantees that β satisfies the corresponding cocycle condition. In this way each $\mathcal{G}_n$ becomes a $\mathbf{G}_m$-equivariant coherent $\mathcal{O}_X$-module in a manner compatible with transition maps.

By Groupoids, Lemma 39.12.5 we see that $\mathcal{G}_n$ with its $\mathbf{G}_m$-equivariant structure corresponds to a graded A-module N_n. The transition maps $N_{n+1} \to N_n$ are graded module maps. Note that N_n is a finite A-module and $N_n = N_{n+1}/I^nN_{n+1}$ because $(\mathcal{G}_n)$ is an object of $\text{Coh}(X, I\mathcal{O}_X)$. Let N be the finite graded A-module foud in Algebra, Lemma 10.98.3. Then $N_n = N/I^nN$, whence $(\mathcal{G}_n)$ is the completion of the coherent module associated to N, and a fortiori we see that (b) is true.

To see (a) we have to unwind the situation described above a bit more. First, observe that the kernel and cokernel of $M_n \to H^0(U, \mathcal{F}_n)$ is A_+-power torsion (Local Cohomology, Lemma 51.8.2). Observe that $H^0(U, \mathcal{F}_n)$ comes with a natural grading such that these maps and the transition maps of the system are graded A-module map; for example we can use that $(U \to X)_*\mathcal{F}_n$ is a $\mathbf{G}_m$-equivariant module on X and use Groupoids, Lemma 39.12.5. Next, recall that (N_n) and $(H^0(U, \mathcal{F}_n))$ are pro-isomorphic by Definition 52.16.7 and Lemma 52.16.8. We omit the verification that the maps defining this pro-isomorphism are graded module maps. Thus (N_n) and (M_n) are pro-isomorphic in the category of graded A-modules modulo A_+-power torsion modules. $\square$

Let k be a field. Let P be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Let $s \in \Gamma(P, \mathcal{L})$ be a section and let $Q = Z(s)$ be the zero scheme, see Divisors, Definition 31.14.8. Let $\mathcal{I} \subset \mathcal{O}_P$ be the ideal sheaf of Q. We will use $\text{Coh}(P, \mathcal{I})$ to denote the category of coherent formal modules introduced in Cohomology of Schemes, Section 30.23.

0EL5 Proposition 52.28.5. In the situation above let $(\mathcal{F}_n)$ be an object of $\text{Coh}(P, \mathcal{I})$. Assume for all $q \in Q$ and for all primes $\mathfrak{p} \in \mathcal{O}_{P,q}^\wedge$, $\mathfrak{p} \notin V(\mathcal{I}_q^\wedge)$ we have

$$\text{depth}((\mathcal{F}_q^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{P,q}^\wedge/\mathfrak{p}) + \dim(\overline{\{q\}}) > 2$$

Then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_P$-module.

Proof. By Cohomology of Schemes, Lemma 30.23.6 to prove the lemma, we may replace $(\mathcal{F}_n)$ by an object differing from it by $\mathcal{I}$-torsion (see below for more precision). Let $T' = \{q \in Q \mid \dim(\overline{\{q\}}) = 0\}$ and $T = \{q \in Q \mid \dim(\overline{\{q\}}) \leq 1\}$. The assumption in the proposition is exactly that $Q \subset P$, $(\mathcal{F}_n)$, and $T' \subset T \subset Q$ satisfy the conditions of Lemma 52.21.2 with $d = 1$; besides trivial manipulations of inequalities, use that $V(\mathfrak{p}) \cap V(\mathcal{I}_y^\wedge) = \{\mathfrak{m}_y^\wedge\} \Leftrightarrow \dim(\mathcal{O}_{P,q}^\wedge/\mathfrak{p}) = 1$ as $\mathcal{I}_y^\wedge$ is generated by 1 element. Combining these two remarks, we may replace $(\mathcal{F}_n)$ by the object $(\mathcal{H}_n)$ of $\text{Coh}(P, \mathcal{I})$ found in Lemma 52.21.2. Thus we may and do assume $(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}_n'')$ of coherent $\mathcal{O}_P$-modules such that $\text{depth}(\mathcal{F}_{n,q}'') + \dim(\overline{\{q\}}) \geq 2$ for all $q \in Q$.

We will use More on Morphisms, Lemma 37.51.1 and we will use the notation used and results found More on Morphisms, Section 37.51 without further mention; this proof will not make sense without at least understanding the statement of the lemma. Observe that in our case $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$ is a finite type k-algebra all of whose graded parts are finite dimensional k-vector spaces, see Cohomology of Schemes, Lemma 30.16.1.

By Cohomology of Schemes, Lemma 30.23.9 the pull back by $\pi|_U : U \to P$ is an object $(\pi|_U^*\mathcal{F}_n)$ of $\text{Coh}(U, f\mathcal{O}_U)$ which is pro-isomorphic to the inverse system $(\pi|_U^*\mathcal{F}_n'')$ of coherent $\mathcal{O}_U$-modules. We claim

$$\text{depth}(\pi|_U^*\mathcal{F}_{n,y}'') + \delta_Z^Y(y) \geq 3$$

for all $y \in U \cap Y$. Since all the points of Z are closed, we see that $\delta^Y_Z(y) \geq \dim(\overline{\{y\}})$ for all $y \in U \cap Y$, see Lemma 52.18.1. Let $q \in Q$ be the image of y. Since the morphism $\pi : U \to P$ is smooth of relative dimension 1 we see that either y is a closed point of a fibre of π or a generic point. Thus we see that

$$\text{depth}(\pi^*\mathcal{F}_{n,y}'') + \delta_Z^Y(y) \geq \text{depth}(\pi^*\mathcal{F}_{n,y}'') + \dim(\overline{\{y\}}) = \text{depth}(\mathcal{F}_{n,q}'') + \dim(\overline{\{q\}}) + 1$$

because either the depth goes up by 1 or the dimension. This proves the claim.

By Lemma 52.22.1 we conclude that $(\pi|_U^*\mathcal{F}_n)$ canonically extends to X. Observe that

$$M_n = \Gamma(U, \pi|_U^*\mathcal{F}_n) = \bigoplus_{m \in \mathbf{Z}} \Gamma(P, \mathcal{F}_n \otimes_{\mathcal{O}_P} \mathcal{L}^{\otimes m})$$

is canonically a graded A-module, see More on Morphisms, Equation (37.51.0.2). By Properties, Lemma 28.18.2 we have $\pi|_U^*\mathcal{F}_n = \widetilde{M_n}|_U$. Thus we may apply Lemma 52.28.4 to find a finite graded A-module N such that (M_n) and (N/I^nN) are pro-isomorphic in the category of graded A-modules modulo A_+-torsion modules. Let $\mathcal{F}$ be the coherent $\mathcal{O}_P$-module associated to N, see Cohomology of Schemes, Proposition 30.15.3. The same proposition tells us that $(\mathcal{F}/\mathcal{I}^n\mathcal{F})$ is pro-isomorphic to $(\mathcal{F}_n)$. Since both are objects of $\text{Coh}(P, \mathcal{I})$ we win by Lemma 52.15.3. $\square$

0EL6 Example 52.28.6. Let k be a field and let X be a proper variety over k. Let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y)$ is ample and denote $\mathcal{I} \subset \mathcal{O}_X$ the corresponding sheaf of ideals. Let $(\mathcal{E}_n)$ an object of $\text{Coh}(X, \mathcal{I})$ with $\mathcal{E}_n$ finite locally free. Here are some special cases of Proposition 52.28.5.

(1) If X is a curve or a surface, we don't learn anything.
(2) If X is a Cohen-Macaulay threefold, then $(\mathcal{E}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{E}$.
(3) More generally, if $\dim(X) \geq 3$ and X is (S_3), then $(\mathcal{E}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{E}$.

Of course, if $\mathcal{E}$ exists, then $\mathcal{E}$ is finite locally free in an open neighbourhood of Y.

0EL7 Proposition 52.28.7. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s and denote $\mathcal{I} \subset \mathcal{O}_X$ the corresponding sheaf of ideals. Let $\mathcal{V}$ be the set of open subschemes of X containing Y ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have

$$\text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

Then the completion functor

$$\text{colim}_\mathcal{V} \text{Coh}(\mathcal{O}_V) \longrightarrow \text{Coh}(X, \mathcal{I})$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. To prove fully faithfulness it suffices to prove that

$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$

is an isomorphism for all m, see Lemma 52.15.2. This follows from Lemma 52.28.2.

Essential surjectivity. Let $(\mathcal{F}_n)$ be a finite locally free object of $\mathrm{Coh}(X, \mathcal{I})$. Then for $y \in Y$ we have $\mathcal{F}_y^\wedge = \lim \mathcal{F}_{n,y}$ is is a finite free $\mathcal{O}_{X,y}^\wedge$-module. Let $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ be a prime with $\mathfrak{p} \notin V(\mathcal{I}_y^\wedge)$. Then $\mathfrak{p}$ lies over a prime $\mathfrak{p}_0 \subset \mathcal{O}_{X,y}$ which corresponds to a specialization $x \rightsquigarrow y$ with $x \notin Y$. By Local Cohomology, Lemma 51.11.3 and some dimension theory (see Varieties, Section 33.20) we have

$$\mathrm{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) = \mathrm{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) - \dim(\overline{\{y\}})$$

Thus our assumptions imply the assumptions of Proposition 52.28.5 are satisfied and we find that $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{F}$. It then follows that $\mathcal{F}_y$ is finite free for all $y \in Y$ and hence $\mathcal{F}$ is finite locally free in an open neighbourhood V of Y. This finishes the proof. $\square$

52.29. Other chapters

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CHAPTER 53

Algebraic Curves

0BRV 53.1. Introduction

0BRW In this chapter we develop some of the theory of algebraic curves. A reference covering algebraic curves over the complex numbers is the book [ACGH85].

What we already know. Besides general algebraic geometry, we have already proved some specific results on algebraic curves. Here is a list.

(1) We have discussed affine opens of and ample invertible sheaves on 1 dimensional Noetherian schemes in Varieties, Section 33.38.
(2) We have seen a curve is either affine or projective in Varieties, Section 33.43.
(3) We have discussed degrees of locally free modules on proper curves in Varieties, Section 33.44.
(4) We have discussed the Picard scheme of a nonsingular projective curve over an algebraically closed field in Picard Schemes of Curves, Section 44.1.

53.2. Curves and function fields

0BXX In this section we elaborate on the results of Varieties, Section 33.4 in the case of curves.

0BXY Lemma 53.2.1. Let k be a field. Let X be a curve and Y a proper variety. Let $U \subset X$ be a nonempty open and let $f : U \to Y$ be a morphism. If $x \in X$ is a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring, then there exist an open $U \subset U' \subset X$ containing x and a morphism of varieties $f' : U' \to Y$ extending f.

Proof. This is a special case of Morphisms, Lemma 29.42.5. $\square$

0BXZ Lemma 53.2.2. Let k be a field. Let X be a normal curve and Y a proper variety. The set of rational maps from X to Y is the same as the set of morphisms $X \to Y$.

Proof. A rational map from X to Y can be extended to a morphism $X \to Y$ by Lemma 53.2.1 as every local ring is a discrete valuation ring (for example by Varieties, Lemma 33.43.8). Conversely, if two morphisms $f, g : X \to Y$ are equivalent as rational maps, then $f = g$ by Morphisms, Lemma 29.7.10. $\square$

0CCK Lemma 53.2.3. Let k be a field. Let $f : X \to Y$ be a nonconstant morphism of curves over k. If Y is normal, then f is flat.

Proof. Pick $x \in X$ mapping to $y \in Y$. Then $\mathcal{O}_{Y,y}$ is either a field or a discrete valuation ring (Varieties, Lemma 33.43.8). Since f is nonconstant it is dominant (as it must map the generic point of X to the generic point of Y). This implies that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective (Morphisms, Lemma 29.8.7). Hence $\mathcal{O}_{X,x}$ is torsion free as a $\mathcal{O}_{Y,y}$-module and therefore $\mathcal{O}_{X,x}$ is flat as a $\mathcal{O}_{Y,y}$-module by More on Algebra, Lemma 15.22.10. $\square$

0CCL Lemma 53.2.4. Let k be a field. Let $f : X \to Y$ be a morphism of schemes over k. Assume

(1) Y is separated over k,
(2) X is proper of dimension ≤ 1 over k,
(3) $f(Z)$ has at least two points for every irreducible component $Z \subset X$ of dimension 1.

Then f is finite.

Proof. The morphism f is proper by Morphisms, Lemma 29.41.7. Thus $f(X)$ is closed and images of closed points are closed. Let $y \in Y$ be the image of a closed point in X. Then $f^{-1}(\{y\})$ is a closed subset of X not containing any of the generic points of irreducible components of dimension 1 by condition (3). It follows that $f^{-1}(\{y\})$ is finite. Hence f is finite over an open neighbourhood of y by More on Morphisms, Lemma 37.44.2 (if Y is Noetherian, then you can use the easier Cohomology of Schemes, Lemma 30.21.2). Since we've seen above that there are enough of these points y, the proof is complete. $\square$

0BY0 Lemma 53.2.5. Let k be a field. Let $X \to Y$ be a morphism of varieties with Y proper and X a curve. There exists a factorization $X \to \overline{X} \to Y$ where $X \to \overline{X}$ is an open immersion and $\overline{X}$ is a projective curve.

Proof. This is clear from Lemma 53.2.1 and Varieties, Lemma 33.43.6. $\square$

Here is the main theorem of this section. We will say a morphism $f : X \to Y$ of varieties is constant if the image $f(X)$ consists of a single point y of Y. If this happens then y is a closed point of Y (since the image of a closed point of X will be a closed point of Y).

0BY1 Theorem 53.2.6. Let k be a field. The following categories are canonically equivalent

(1) The category of finitely generated field extensions K/k of transcendence degree 1.
(2) The category of curves and dominant rational maps.
(3) The category of normal projective curves and nonconstant morphisms.
(4) The category of nonsingular projective curves and nonconstant morphisms.
(5) The category of regular projective curves and nonconstant morphisms.
(6) The category of normal proper curves and nonconstant morphisms.

Proof. The equivalence between categories (1) and (2) is the restriction of the equivalence of Varieties, Theorem 33.4.1. Namely, a variety is a curve if and only if its function field has transcendence degree 1, see for example Varieties, Lemma 33.20.3.

The categories in (3), (4), (5), and (6) are the same. First of all, the terms "regular" and "nonsingular" are synonyms, see Properties, Definition 28.9.1. Being normal and regular are the same thing for Noetherian 1-dimensional schemes (Properties, Lemmas 28.9.4 and 28.12.6). See Varieties, Lemma 33.43.8 for the case of curves. Thus (3) is the same as (5). Finally, (6) is the same as (3) by Varieties, Lemma 33.43.4.

If $f : X \to Y$ is a nonconstant morphism of nonsingular projective curves, then f sends the generic point η of X to the generic point ξ of Y. Hence we obtain a morphism $k(Y) = \mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\eta} = k(X)$ in the category (1). If two morphisms $f, g : X \to Y$ gives the same morphism $k(Y) \to k(X)$, then by the equivalence between (1) and (2), f and g are equivalent as rational maps, so $f = g$ by Lemma 53.2.2. Conversely, suppose that we have a map $k(Y) \to k(X)$ in the category (1). Then we obtain a morphism $U \to Y$ for some nonempty open $U \subset X$. By Lemma 53.2.1 this extends to all of X and we obtain a morphism in the category (5). Thus we see that there is a fully faithful functor (5)$\to$(1).

To finish the proof we have to show that every K/k in (1) is the function field of a normal projective curve. We already know that $K = k(X)$ for some curve X. After replacing X by its normalization (which is a variety birational to X) we may assume X is normal (Varieties, Lemma 33.27.1). Then we choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ as in Varieties, Lemma 33.43.6. Since X is normal and since each of the local rings $\mathcal{O}_{\overline{X},x_i}$ is normal we conclude that $\overline{X}$ is a normal projective curve as desired. (Remark: We can also first compactify using Varieties, Lemma 33.43.5 and then normalize using Varieties, Lemma 33.27.1. Doing it this way we avoid using the somewhat tricky Morphisms, Lemma 29.53.16.) □

0BY2 Definition 53.2.7. Let k be a field. Let X be a curve. A nonsingular projective model of X is a pair (Y, φ) where Y is a nonsingular projective curve and $\varphi : k(X) \to k(Y)$ is an isomorphism of function fields.

A nonsingular projective model is determined up to unique isomorphism by Theorem 53.2.6. Thus we often say "the nonsingular projective model". We usually drop φ from the notation. Warning: it needn't be the case that Y is smooth over k but Lemma 53.2.8 shows this can only happen in positive characteristic.

0BY3 Lemma 53.2.8. Let k be a field. Let X be a curve and let Y be the nonsingular projective model of X. If k is perfect, then Y is a smooth projective curve.

Proof. See Varieties, Lemma 33.43.8 for example. □

0BY4 Lemma 53.2.9. Let k be a field. Let X be a geometrically irreducible curve over k. For a field extension K/k denote Y_K a nonsingular projective model of $(X_K)_{red}$.

(1) If X is proper, then Y_K is the normalization of X_K.
(2) There exists K/k finite purely inseparable such that Y_K is smooth.
(3) Whenever Y_K is smooth[1] we have $H^0(Y_K, \mathcal{O}_{Y_K}) = K$.
(4) Given a commutative diagram

$$\begin{tikzcd} \Omega & K' \arrow[l] \\ K \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

of fields such that Y_K and $Y_{K'}$ are smooth, then $Y_\Omega = (Y_K)_\Omega = (Y_{K'})_\Omega$.

Proof. Let X' be a nonsingular projective model of X. Then X' and X have isomorphic nonempty open subschemes. In particular X' is geometrically irreducible as X is (some details omitted). Thus we may assume that X is projective.

[1] Or even geometrically reduced.

Assume X is proper. Then X_K is proper and hence the normalization $(X_K)^\nu$ is proper as a scheme finite over a proper scheme (Varieties, Lemma 33.27.1 and Morphisms, Lemmas 29.44.11 and 29.41.4). On the other hand, X_K is irreducible as X is geometrically irreducible. Hence X_K^ν is proper, normal, irreducible, and birational to $(X_K)_{red}$. This proves (1) because a proper curve is projective (Varieties, Lemma 33.43.4).

Proof of (2). As X is proper and we have (1), we can apply Varieties, Lemma 33.27.4 to find K/k finite purely inseparable such that Y_K is geometrically normal. Then Y_K is geometrically regular as normal and regular are the same for curves (Properties, Lemma 28.12.6). Then Y is a smooth variety by Varieties, Lemma 33.12.6.

If Y_K is geometrically reduced, then Y_K is geometrically integral (Varieties, Lemma 33.9.2) and we see that $H^0(Y_K, \mathcal{O}_{Y_K}) = K$ by Varieties, Lemma 33.26.2. This proves (3) because a smooth variety is geometrically reduced (even geometrically regular, see Varieties, Lemma 33.12.6).

If Y_K is smooth, then for every extension Ω/K the base change $(Y_K)_\Omega$ is smooth over Ω (Morphisms, Lemma 29.34.5). Hence it is clear that $Y_\Omega = (Y_K)_\Omega$. This proves (4). $\square$

53.3. Linear series

0CCM We deviate from the classical story (see Remark 53.3.6) by defining linear series in the following manner.

0CCN Definition 53.3.1. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $d \geq 0$ and $r \geq 0$. A linear series of degree d and dimension r is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module of degree d (Varieties, Definition 33.44.1) and $V \subset H^0(X, \mathcal{L})$ is a k-subvector space of dimension $r+1$. We will abbreviate this by saying $(\mathcal{L}, V)$ is a $\mathfrak{g}^r_d$ on X.

We will mostly use this when X is a nonsingular proper curve. In fact, the definition above is just one way to generalize the classical definition of a $\mathfrak{g}^r_d$. For example, if X is a proper curve, then one can generalize linear series by allowing $\mathcal{L}$ to be a torsion free coherent $\mathcal{O}_X$-module of rank 1. On a nonsingular curve every torsion free coherent module is locally free, so this agrees with our notion for nonsingular proper curves.

The following lemma explains the geometric meaning of linear series for proper nonsingular curves.

0CCP Lemma 53.3.2. Let k be a field. Let X be a nonsingular proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ on X. Then there exists a morphism

$$\varphi : X \longrightarrow \mathbf{P}^r_k = \text{Proj}(k[T_0, \ldots, T_r])$$

of varieties over k and a map $\alpha : \varphi^*\mathcal{O}_{\mathbf{P}^r_k}(1) \to \mathcal{L}$ such that $\varphi^*T_0, \ldots, \varphi^*T_r$ are sent to a basis of V by α.

Proof. Let $s_0, \ldots, s_r \in V$ be a k-basis. Since X is nonsingular the image $\mathcal{L}' \subset \mathcal{L}$ of the map $s_0, \ldots, s_r : \mathcal{O}_X^{\oplus r+1} \to \mathcal{L}$ is an invertible $\mathcal{O}_X$-module for example by Divisors, Lemma 31.11.11. Then we use Constructions, Lemma 27.13.1 to get a morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k$$

as in the statement of the lemma. □

0CCQ Lemma 53.3.3. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. If X has a $\mathfrak{g}^r_d$, then X has a $\mathfrak{g}^s_d$ for all $0 \leq s \leq r$.

Proof. This is true because a vector space V of dimension $r+1$ over k has a linear subspace of dimension $s+1$ for all $0 \leq s \leq r$. □

0CCR Lemma 53.3.4. Let k be a field. Let X be a nonsingular proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^1_d$ on X. Then the morphism $\varphi : X \to \mathbf{P}^1_k$ of Lemma 53.3.2 either

(1) is nonconstant and has degree $\leq d$, or
(2) factors through a closed point of $\mathbf{P}^1_k$ and in this case $H^0(X, \mathcal{O}_X) \neq k$.

Proof. By Lemma 53.3.2 we see that $\mathcal{L}' = \varphi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ has a nonzero map $\mathcal{L}' \to \mathcal{L}$. Hence by Varieties, Lemma 33.44.12 we see that $0 \leq \deg(\mathcal{L}') \leq d$. If $\deg(\mathcal{L}') = 0$, then the same lemma tells us $\mathcal{L}' \cong \mathcal{O}_X$ and since we have two linearly independent sections we find we are in case (2). If $\deg(\mathcal{L}') > 0$ then φ is nonconstant (since the pullback of an invertible module by a constant morphism is trivial). Hence

$$\deg(\mathcal{L}') = \deg(X/\mathbf{P}^1_k)\deg(\mathcal{O}_{\mathbf{P}^1_k}(1))$$

by Varieties, Lemma 33.44.11. This finishes the proof as the degree of $\mathcal{O}_{\mathbf{P}^1_k}(1)$ is 1. □

0CCS Lemma 53.3.5. Let k be a field. Let X be a proper curve over k with $H^0(X, \mathcal{O}_X) = k$. If X has a $\mathfrak{g}^r_d$, then $r \leq d$. If equality holds, then $H^1(X, \mathcal{O}_X) = 0$, i.e., the genus of X (Definition 53.8.1) is 0.

Proof. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$. Since this will only increase r, we may assume $V = H^0(X, \mathcal{L})$. Choose a nonzero element $s \in V$. Then the zero scheme of s is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$, and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 31.14.10 and Remark 31.14.11. By Varieties, Lemma 33.44.9 we have $\deg(D) = \deg(\mathcal{L}) = d$. Since D is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[2]. Thus

$$\dim_k H^0(D, \mathcal{L}|_D) = \dim_k H^0(D, \mathcal{O}_D) = \deg(D) = d$$

On the other hand, by assumption $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim H^0(X, \mathcal{L}) = r+1$. We conclude that $r + 1 \leq 1 + d$, i.e., $r \leq d$ as in the lemma.

Assume equality holds. Then $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. If we knew that $H^1(X, \mathcal{L})$ was zero, then we would conclude that $H^1(X, \mathcal{O}_X)$ is zero by the long exact cohomology sequence and the proof would be complete. Our strategy will be to replace $\mathcal{L}$ by a large power which has vanishing. As $\mathcal{L}|_D$ is the trivial invertible module (see above), we can find a section t of $\mathcal{L}$ whose restriction of D generates $\mathcal{L}|_D$. Consider the multiplication map

$$\mu : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 2})$$

and consider the short exact sequence

$$0 \to \mathcal{L} \xrightarrow{s} \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}|_D \to 0$$

[2] In our case this follows from Divisors, Lemma 31.17.1 as $D \to \mathrm{Spec}(k)$ is finite.

Since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since t maps to a trivialization of $\mathcal{L}|_D$ we see that $\mu(H^0(X, \mathcal{L}) \otimes t)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes 2})$ surjecting onto the global sections of $\mathcal{L}^{\otimes 2}|_D$. Thus we see that

$$\dim H^0(X, \mathcal{L}^{\otimes 2}) = r + 1 + d = 2r + 1 = \deg(\mathcal{L}^{\otimes 2}) + 1$$

Ok, so $\mathcal{L}^{\otimes 2}$ has the same property as $\mathcal{L}$, i.e., that the dimension of the space of global sections is equal to the degree plus one. Since $\mathcal{L}$ is ample (Varieties, Lemma 33.44.14) there exists some n_0 such that $\mathcal{L}^{\otimes n}$ has vanishing H^1 for all $n \geq n_0$ (Cohomology of Schemes, Lemma 30.16.1). Thus applying the argument above to $\mathcal{L}^{\otimes n}$ with $n = 2^m$ for some sufficiently large m we conclude the lemma is true. □

0CCT Remark 53.3.6 (Classical definition). Let X be a smooth projective curve over an algebraically closed field k. We say two effective Cartier divisors $D, D' \subset X$ are linearly equivalent if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules. Since $\text{Pic}(X) = \text{Cl}(X)$ (Divisors, Lemma 31.27.7) we see that D and D' are linearly equivalent if and only if the Weil divisors associated to D and D' define the same element of $\text{Cl}(X)$. Given an effective Cartier divisor $D \subset X$ of degree d the complete linear system or complete linear series $|D|$ of D is the set of effective Cartier divisors $E \subset X$ which are linearly equivalent to D. Another way to say it is that $|D|$ is the set of closed points of the fibre of the morphism

$$\gamma_d : \underline{\text{Hilb}}^d_{X/k} \longrightarrow \underline{\text{Pic}}^d_{X/k}$$

(Picard Schemes of Curves, Lemma 44.6.7) over the closed point corresponding to $\mathcal{O}_X(D)$. This gives $|D|$ a natural scheme structure and it turns out that $|D| \cong \mathbf{P}^m_k$ with $m + 1 = h^0(\mathcal{O}_X(D))$. In fact, more canonically we have

$$|D| = \mathbf{P}(H^0(X, \mathcal{O}_X(D))^\vee)$$

where $(-)^\vee$ indicates k-linear dual and $\mathbf{P}$ is as in Constructions, Example 27.21.2. In this language a linear system or a linear series on X is a closed subvariety $L \subset |D|$ which can be cut out by linear equations. If L has dimension r, then $L = \mathbf{P}(V^\vee)$ where $V \subset H^0(X, \mathcal{O}_X(D))$ is a linear subspace of dimension $r+1$. Thus the classical linear series $L \subset |D|$ corresponds to the linear series $(\mathcal{O}_X(D), V)$ as defined above.

53.4. Duality

0E31 In this section we work out the consequences of the very general material on dualizing complexes and duality for proper 1-dimensional schemes over fields. If you are interested in the analogous discussion for higher dimension proper schemes over fields, see Duality for Schemes, Section 48.27.

0BS2 Lemma 53.4.1. Let X be a proper scheme of dimension ≤ 1 over a field k. There exists a dualizing complex $\omega_X^\bullet$ with the following properties

(1) $H^i(\omega_X^\bullet)$ is nonzero only for $i = -1, 0$,
(2) $\omega_X = H^{-1}(\omega_X^\bullet)$ is a coherent Cohen-Macaulay module whose support is the irreducible components of dimension 1,
(3) for $x \in X$ closed, the module $H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if either
  (a) $\dim(\mathcal{O}_{X,x}) = 0$ or
  (b) $\dim(\mathcal{O}_{X,x}) = 1$ and $\mathcal{O}_{X,x}$ is not Cohen-Macaulay,

(4) for $K \in D_{QCoh}(\mathcal{O}_X)$ there are functorial isomorphisms[3]

$$\mathrm{Ext}^i_X(K, \omega_X^\bullet) = \mathrm{Hom}_k(H^{-i}(X, K), k)$$

compatible with shifts and distinguished triangles,

(5) there are functorial isomorphisms $\mathrm{Hom}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X,

(6) if $X \to \mathrm{Spec}(k)$ is smooth of relative dimension 1, then $\omega_X \cong \Omega_{X/k}$.

Proof. Denote $f : X \to \mathrm{Spec}(k)$ the structure morphism. We start with the relative dualizing complex

$$\omega_X^\bullet = \omega_{X/k}^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$$

as described in Duality for Schemes, Remark 48.12.5. Then property (4) holds by construction as a is the right adjoint for $f_* : D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_{\mathrm{Spec}(k)})$. Since f is proper we have $f^!(\mathcal{O}_{\mathrm{Spec}(k)}) = a(\mathcal{O}_{\mathrm{Spec}(k)})$ by definition, see Duality for Schemes, Section 48.16. Hence $\omega_X^\bullet$ and ω_X are as in Duality for Schemes, Example 48.22.1 and as in Duality for Schemes, Example 48.22.2. Parts (1) and (2) follow from Duality for Schemes, Lemma 48.22.4. For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Duality for Schemes, Lemma 48.21.1. Assertion (3) then follows from Dualizing Complexes, Lemma 47.20.2. Assertion (5) follows from Duality for Schemes, Lemma 48.22.5 for coherent $\mathcal{F}$ and in general by unwinding (4) for $K = \mathcal{F}[0]$ and $i = -1$. Assertion (6) follows from Duality for Schemes, Lemma 48.15.7. □

0BS3 Lemma 53.4.2. Let X be a proper scheme over a field k which is Cohen-Macaulay and equidimensional of dimension 1. The module ω_X of Lemma 53.4.1 has the following properties

(1) ω_X is a dualizing module on X (Duality for Schemes, Section 48.22),
(2) ω_X is a coherent Cohen-Macaulay module whose support is X,
(3) there are functorial isomorphisms $\mathrm{Ext}^i_X(K, \omega_X[1]) = \mathrm{Hom}_k(H^{-i}(X, K), k)$ compatible with shifts for $K \in D_{QCoh}(X)$,
(4) there are functorial isomorphisms $\mathrm{Ext}^{1+i}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^{-i}(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X.

Proof. Recall from the proof of Lemma 53.4.1 that ω_X is as in Duality for Schemes, Example 48.22.1 and hence is a dualizing module. The other statements follow from Lemma 53.4.1 and the fact that $\omega_X^\bullet = \omega_X[1]$ as X is Cohen-Macaulay (Duality for Schemes, Lemma 48.23.1). □

0BS4 Remark 53.4.3. Let X be a proper scheme of dimension ≤ 1 over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 53.4.1. If $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$ then we have canonical isomorphisms

$$\mathrm{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^i(X, \mathcal{E}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_X^\bullet)$$

This follows from the lemma and Cohomology, Lemma 20.50.5. If X is Cohen-Macaulay and equidimensional of dimension 1, then we have canonical isomorphisms

$$\mathrm{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^{1+i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)$$

[3]This property characterizes $\omega_X^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$ up to unique isomorphism by the Yoneda lemma. Since $\omega_X^\bullet$ is in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ in fact it suffices to consider $K \in D^b_{\mathrm{Coh}}(\mathcal{O}_X)$.

by Lemma 53.4.2. In particular if $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we have

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

and

$$\dim_k H^1(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

Here is a sanity check for the dualizing complex.

0E32 Lemma 53.4.4. Let X be a proper scheme of dimension ≤ 1 over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 53.4.1.

(1) If $X \to \text{Spec}(k)$ factors as $X \to \text{Spec}(k') \to \text{Spec}(k)$ for some field k', then $\omega_X^\bullet$ and ω_X satisfy properties (4), (5), (6) with k replaced with k'.
(2) If K/k is a field extension, then the pullback of $\omega_X^\bullet$ and ω_X to the base change X_K are as in Lemma 53.4.1 for the morphism $X_K \to \text{Spec}(K)$.

Proof. Denote $f : X \to \text{Spec}(k)$ the structure morphism. Assertion (1) really means that $\omega_X^\bullet$ and ω_X are as in Lemma 53.4.1 for the morphism $f' : X \to \text{Spec}(k')$. In the proof of Lemma 53.4.1 we took $\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$ where a be is the right adjoint of Duality for Schemes, Lemma 48.3.1 for f. Thus we have to show $a(\mathcal{O}_{\text{Spec}(k)}) \cong a'(\mathcal{O}_{\text{Spec}(k)})$ where a' be is the right adjoint of Duality for Schemes, Lemma 48.3.1 for f'. Since $k' \subset H^0(X, \mathcal{O}_X)$ we see that k'/k is a finite extension (Cohomology of Schemes, Lemma 30.19.2). By uniqueness of adjoints we have $a = a' \circ b$ where b is the right adjoint of Duality for Schemes, Lemma 48.3.1 for $g : \text{Spec}(k') \to \text{Spec}(k)$. Another way to say this: we have $f^! = (f')^! \circ g^!$. Thus it suffices to show that $\text{Hom}_k(k', k) \cong k'$ as k'-modules, see Duality for Schemes, Example 48.3.2. This holds because these are k'-vector spaces of the same dimension (namely dimension 1).

Proof of (2). This holds because we have base change for a by Duality for Schemes, Lemma 48.6.2. See discussion in Duality for Schemes, Remark 48.12.5. □

0E33 Lemma 53.4.5. Let X be a proper scheme of dimension ≤ 1 over a field k. Let $i : Y \to X$ be a closed immersion. Let $\omega_X^\bullet$, ω_X, $\omega_Y^\bullet$, ω_Y be as in Lemma 53.4.1. Then

(1) $\omega_Y^\bullet = R\mathcal{H}om(\mathcal{O}_Y, \omega_X^\bullet)$,
(2) $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ and $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$.

Proof. Denote $g : Y \to \text{Spec}(k)$ and $f : X \to \text{Spec}(k)$ the structure morphisms. Then $g = f \circ i$. Denote a, b, c the right adjoint of Duality for Schemes, Lemma 48.3.1 for f, g, i. Then $b = c \circ a$ by uniqueness of right adjoints and because $Rg_* = Rf_* \circ Ri_*$. In the proof of Lemma 53.4.1 we set $\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$ and $\omega_Y^\bullet = b(\mathcal{O}_{\text{Spec}(k)})$. Hence $\omega_Y^\bullet = c(\omega_X^\bullet)$ which implies (1) by Duality for Schemes, Lemma 48.9.7. Since $\omega_X = H^{-1}(\omega_X^\bullet)$ and $\omega_Y = H^{-1}(\omega_Y^\bullet)$ we conclude that $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$. This implies $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$ by Duality for Schemes, Lemma 48.9.3. □

0E34 Lemma 53.4.6. Let X be a proper scheme over a field k which is Gorenstein, reduced, and equidimensional of dimension 1. Let $i : Y \to X$ be a reduced closed subscheme equidimensional of dimension 1. Let $j : Z \to X$ be the scheme theoretic closure of $X \setminus Y$. Then

(1) Y and Z are Cohen-Macaulay,

(2) if $\mathcal{I} \subset \mathcal{O}_X$, resp. $\mathcal{J} \subset \mathcal{O}_X$ is the ideal sheaf of Y, resp. Z in X, then

$$\mathcal{I} = i_*\mathcal{I}' \quad \text{and} \quad \mathcal{J} = j_*\mathcal{J}'$$

where $\mathcal{I}' \subset \mathcal{O}_Z$, resp. $\mathcal{J}' \subset \mathcal{O}_Y$ is the ideal sheaf of $Y \cap Z$ in Z, resp. Y,

(3) $\omega_Y = \mathcal{J}'(i^*\omega_X)$ and $i_*(\omega_Y) = \mathcal{J}\omega_X$,

(4) $\omega_Z = \mathcal{I}'(i^*\omega_X)$ and $i_*(\omega_Z) = \mathcal{I}\omega_X$,

(5) we have the following short exact sequences

$$\begin{aligned} 0 \to \omega_X \to i_*i^*\omega_X \oplus j_*j^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to i_*\omega_Y \to \omega_X \to j_*j^*\omega_X \to 0 \\ 0 \to j_*\omega_Z \to \omega_X \to i_*i^*\omega_X \to 0 \\ 0 \to i_*\omega_Y \oplus j_*\omega_Z \to \omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \omega_Y \to i^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \omega_Z \to j^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \end{aligned}$$

Here ω_X, ω_Y, ω_Z are as in Lemma 53.4.1.

Proof. A reduced 1-dimensional Noetherian scheme is Cohen-Macaulay, so (1) is true. Since X is reduced, we see that $X = Y \cup Z$ scheme theoretically. With notation as in Morphisms, Lemma 29.4.6 and by the statement of that lemma we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0$$

Since $\mathcal{J} = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Z)$, $\mathcal{J}' = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{Y\cap Z})$, $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Y)$, and $\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_Z \to \mathcal{O}_{Y\cap Z})$ a diagram chase implies (2). Observe that $\mathcal{I}+\mathcal{J}$ is the ideal sheaf of $Y \cap Z$ and that $\mathcal{I} \cap \mathcal{J} = 0$. Hence we have the following exact sequences

$$\begin{aligned} 0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \\ 0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \\ 0 \to \mathcal{J} \oplus \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \mathcal{J}' \to \mathcal{O}_Y \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \mathcal{I}' \to \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0 \end{aligned}$$

Since X is Gorenstein ω_X is an invertible $\mathcal{O}_X$-module (Duality for Schemes, Lemma 48.24.4). Since $Y \cap Z$ has dimension 0 we have $\omega_X|_{Y\cap Z} \cong \mathcal{O}_{Y\cap Z}$. Thus if we prove (3) and (4), then we obtain the short exact sequences of the lemma by tensoring the above short exact sequence with the invertible module ω_X. By symmetry it suffices to prove (3) and by (2) it suffices to prove $i_*(\omega_Y) = \mathcal{J}\omega_X$.

We have $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$ by Lemma 53.4.5. Again using that ω_X is invertible we finally conclude that it suffices to show $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X)$ maps isomorphically to $\mathcal{J}$ by evaluation at 1. In other words, that $\mathcal{J}$ is the annihilator of $\mathcal{I}$. This follows from the above. $\square$

53.5. Riemann-Roch

0B5B

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. In Varieties, Section 33.44 we have defined the degree of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank by the formula

0BRX (53.5.0.1)
$$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X)$$

see Varieties, Definition 33.44.1. In the chapter on Chow Homology we defined the first Chern class of $\mathcal{E}$ as an operation on cycles (Chow Homology, Section 42.38) and we proved that

0BRY (53.5.0.2)
$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

see Chow Homology, Lemma 42.41.3. Combining (53.5.0.1) and (53.5.0.2) we obtain our first version of the Riemann-Roch formula

0BRZ (53.5.0.3)
$$\chi(X, \mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1) + \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X)$$

If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we can also consider the numerical intersection $(\mathcal{L} \cdot X)$ as defined in Varieties, Definition 33.45.3. However, this does not give anything new as

0BS0 (53.5.0.4)
$$(\mathcal{L} \cdot X) = \deg(\mathcal{L})$$

by Varieties, Lemma 33.45.12. If $\mathcal{L}$ is ample, then this integer is positive and is called the degree

0BS1 (53.5.0.5)
$$\deg_{\mathcal{L}}(X) = (\mathcal{L} \cdot X) = \deg(\mathcal{L})$$

of X with respect to $\mathcal{L}$, see Varieties, Definition 33.45.10.

To obtain a true Riemann-Roch theorem we would like to write $\chi(X, \mathcal{O}_X)$ as the degree of a canonical zero cycle on X. We refer to [Ful98] for a fully general version of this. We will use duality to get a formula in the case where X is Gorenstein; however, in some sense this is a cheat (for example because this method cannot work in higher dimension).

We first use Lemmas 53.4.1 and 53.4.2 to get a relation between the euler characteristic of $\mathcal{O}_X$ and the euler characteristic of the dualizing complex or the dualizing module.

0BS5 **Lemma 53.5.1.** Let X be a proper scheme of dimension ≤ 1 over a field k. With $\omega_X^\bullet$ and ω_X as in Lemma 53.4.1 we have

$$\chi(X, \mathcal{O}_X) = \chi(X, \omega_X^\bullet)$$

If X is Cohen-Macaulay and equidimensional of dimension 1, then

$$\chi(X, \mathcal{O}_X) = -\chi(X, \omega_X)$$

Proof. We define the right hand side of the first formula as follows:

$$\chi(X, \omega_X^\bullet) = \sum\nolimits_{i \in \mathbf{Z}} (-1)^i \dim_k H^i(X, \omega_X^\bullet)$$

This is well defined because $\omega_X^\bullet$ is in $D^b_{\text{Coh}}(\mathcal{O}_X)$, but also because

$$H^i(X, \omega_X^\bullet) = \text{Ext}^i(\mathcal{O}_X, \omega_X^\bullet) = H^{-i}(X, \mathcal{O}_X)$$

which is always finite dimensional and nonzero only if $i = 0, -1$. This of course also proves the first formula. The second is a consequence of the first because $\omega_X^\bullet = \omega_X[1]$ in the CM case, see Lemma 53.4.2. $\square$

We will use Lemma 53.5.1 to get the desired formula for $\chi(X, \mathcal{O}_X)$ in the case that ω_X is invertible, i.e., that X is Gorenstein. The statement is that $-1/2$ of the first Chern class of ω_X capped with the cycle $[X]_1$ associated to X is a natural zero cycle on X with half-integer coefficients whose degree is $\chi(X, \mathcal{O}_X)$. The occurence of fractions in the statement of Riemann-Roch cannot be avoided.

0BS6 Lemma 53.5.2 (Riemann-Roch). Let X be a proper scheme over a field k which is Gorenstein and equidimensional of dimension 1. Let ω_X be as in Lemma 53.4.1. Then

(1) ω_X is an invertible $\mathcal{O}_X$-module,
(2) $\deg(\omega_X) = -2\chi(X, \mathcal{O}_X)$,
(3) for a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank we have

$$\chi(X, \mathcal{E}) = \deg(\mathcal{E}) - \tfrac{1}{2}\text{rank}(\mathcal{E}) \deg(\omega_X)$$

and $\dim_k(H^i(X, \mathcal{E})) = \dim_k(H^{1-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X))$ for all $i \in \mathbf{Z}$.

Nonsingular (normal) curves are Gorenstein, see Duality for Schemes, Lemma 48.24.3.

Proof. Recall that Gorenstein schemes are Cohen-Macaulay (Duality for Schemes, Lemma 48.24.2) and hence ω_X is a dualizing module on X, see Lemma 53.4.2. It follows more or less from the definition of the Gorenstein property that the dualizing sheaf is invertible, see Duality for Schemes, Section 48.24. By (53.5.0.3) applied to ω_X we have

$$\chi(X, \omega_X) = \deg(c_1(\omega_X) \cap [X]_1) + \chi(X, \mathcal{O}_X)$$

Combined with Lemma 53.5.1 this gives

$$2\chi(X, \mathcal{O}_X) = -\deg(c_1(\omega_X) \cap [X]_1) = -\deg(\omega_X)$$

the second equality by (53.5.0.2). Putting this back into (53.5.0.3) for $\mathcal{E}$ gives the displayed formula of the lemma. The symmetry in dimensions is a consequence of duality for X, see Remark 53.4.3. $\square$

53.6. Some vanishing results

0B5C This section contains some very weak vanishing results. Please see Section 53.21 for a few more and more interesting results.

0BY5 Lemma 53.6.1. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then X is connected, Cohen-Macaulay, and equidimensional of dimension 1.

Proof. Since $\Gamma(X, \mathcal{O}_X) = k$ has no nontrivial idempotents, we see that X is connected. This already shows that X is equidimensional of dimension 1 (any irreducible component of dimension 0 would be a connected component). Let $\mathcal{I} \subset \mathcal{O}_X$ be the maximal coherent submodule supported in closed points. Then $\mathcal{I}$ exists (Divisors, Lemma 31.4.6) and is globally generated (Varieties, Lemma 33.33.3). Since $1 \in \Gamma(X, \mathcal{O}_X)$ is not a section of $\mathcal{I}$ we conclude that $\mathcal{I} = 0$. Thus X does not have embedded points (Divisors, Lemma 31.4.6). Thus X has (S_1) by Divisors, Lemma 31.4.3. Hence X is Cohen-Macaulay. $\square$

In this section we work in the following situation.

0B5D Situation 53.6.2. Here k is a field, X is a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$.

By Lemma 53.6.1 the scheme X is Cohen-Macaulay and equidimensional of dimension 1. The dualizing module ω_X discussed in Lemmas 53.4.1 and 53.4.2 has nonvanishing H^1 because in fact $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$. It turns out that anything slightly more "positive" than ω_X has vanishing H^1.

0B5E Lemma 53.6.3. In Situation 53.6.2. Given an exact sequence

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

of coherent $\mathcal{O}_X$-modules with $H^1(X, \mathcal{Q}) = 0$ (for example if $\dim(\text{Supp}(\mathcal{Q})) = 0$), then either $H^1(X, \mathcal{F}) = 0$ or $\mathcal{F} = \omega_X \oplus \mathcal{Q}$.

Proof. (The parenthetical statement follows from Cohomology of Schemes, Lemma 30.9.10.) Since $H^0(X, \mathcal{O}_X) = k$ is dual to $H^1(X, \omega_X)$ (see Section 53.5) we see that $\dim H^1(X, \omega_X) = 1$. The sheaf ω_X represents the functor $\mathcal{F} \mapsto \text{Hom}_k(H^1(X, \mathcal{F}), k)$ on the category of coherent $\mathcal{O}_X$-modules (Duality for Schemes, Lemma 48.22.5). Consider an exact sequence as in the statement of the lemma and assume that $H^1(X, \mathcal{F}) \neq 0$. Since $H^1(X, \mathcal{Q}) = 0$ we see that $H^1(X, \omega_X) \to H^1(X, \mathcal{F})$ is an isomorphism. By the universal property of ω_X stated above, we conclude there is a map $\mathcal{F} \to \omega_X$ whose action on H^1 is the inverse of this isomorphism. The composition $\omega_X \to \mathcal{F} \to \omega_X$ is the identity (by the universal property) and the lemma is proved. $\square$

0B62 Lemma 53.6.4. In Situation 53.6.2. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is globally generated and not isomorphic to $\mathcal{O}_X$. Then $H^1(X, \omega_X \otimes \mathcal{L}) = 0$.

Proof. By duality as discussed in Section 53.5 we have to show that $H^0(X, \mathcal{L}^{\otimes -1}) = 0$. If not, then we can choose a global section t of $\mathcal{L}^{\otimes -1}$ and a global section s of $\mathcal{L}$ such that $st \neq 0$. However, then st is a constant multiple of 1, by our assumption that $H^0(X, \mathcal{O}_X) = k$. It follows that $\mathcal{L} \cong \mathcal{O}_X$, which is a contradiction. $\square$

0B5F Lemma 53.6.5. In Situation 53.6.2. Given an exact sequence

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

of coherent $\mathcal{O}_X$-modules with $\dim(\text{Supp}(\mathcal{Q})) = 0$ and $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and such that there is no nonzero submodule $\mathcal{Q}' \subset \mathcal{F}$ such that $\mathcal{Q}' \to \mathcal{Q}$ is injective. Then the submodule of $\mathcal{F}$ generated by global sections surjects onto $\mathcal{Q}$.

Proof. Let $\mathcal{F}' \subset \mathcal{F}$ be the submodule generated by global sections and the image of $\omega_X \to \mathcal{F}$. Since $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$, we see that $\mathcal{F}' \to \mathcal{Q}$ is not zero and $\omega_X \to \mathcal{F}'$ is not an isomorphism. Hence $H^1(X, \mathcal{F}') = 0$ by Lemma 53.6.3 and our assumption on $\mathcal{F}$. Consider the short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q}/\operatorname{Im}(\mathcal{F}' \to \mathcal{Q}) \to 0$$

If the quotient on the right is nonzero, then we obtain a contradiction because then $H^0(X, \mathcal{F})$ is bigger than $H^0(X, \mathcal{F}')$. $\square$

Here is an example global generation statement.

0B5G Lemma 53.6.6. In Situation 53.6.2 assume that X is integral. Let $0 \to \omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules with $\mathcal{F}$ torsion free, $\dim(\text{Supp}(\mathcal{Q})) = 0$, and $\dim_k H^0(X, \mathcal{Q}) \geq 2$. Then $\mathcal{F}$ is globally generated.

Proof. Consider the submodule $\mathcal{F}'$ generated by the global sections. By Lemma 53.6.5 we see that $\mathcal{F}' \to \mathcal{Q}$ is surjective, in particular $\mathcal{F}' \neq 0$. Since X is a curve, we see that $\mathcal{F}' \subset \mathcal{F}$ is an inclusion of rank 1 sheaves, hence $\mathcal{Q}' = \mathcal{F}/\mathcal{F}'$ is supported in finitely many points. To get a contradiction, assume that $\mathcal{Q}'$ is nonzero. Then we see that $H^1(X, \mathcal{F}') \neq 0$. Then we get a nonzero map $\mathcal{F}' \to \omega_X$ by the universal property (Duality for Schemes, Lemma 48.22.5). The image of the composition $\mathcal{F}' \to \omega_X \to \mathcal{F}$ is generated by global sections, hence is inside of $\mathcal{F}'$. Thus we get a nonzero self map $\mathcal{F}' \to \mathcal{F}'$. Since $\mathcal{F}'$ is torsion free of rank 1 on a proper curve this has to be an automorphism (details omitted). But then this implies that $\mathcal{F}'$ is contained in $\omega_X \subset \mathcal{F}$ contradicting the surjectivity of $\mathcal{F}' \to \mathcal{Q}$. □

0B5H Lemma 53.6.7. In Situation 53.6.2. Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module with $\deg(\mathcal{L}) \geq 2$. Then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is globally generated.

Proof. Assume k is algebraically closed. Let $x \in X$ be a closed point. Let $C_i \subset X$ be the irreducible components and for each i let $x_i \in C_i$ be the generic point. By Varieties, Lemma 33.22.2 we can choose a section $s \in H^0(X, \mathcal{L})$ such that s vanishes at x but not at x_i for all i. The corresponding module map $s : \mathcal{O}_X \to \mathcal{L}$ is injective with cokernel $\mathcal{Q}$ supported in finitely many points and with $H^0(X, \mathcal{Q}) \geq 2$. Consider the corresponding exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{L} \to \omega_X \otimes \mathcal{Q} \to 0$$

By Lemma 53.6.5 we see that the module generated by global sections surjects onto $\omega_X \otimes \mathcal{Q}$. Since x was arbitrary this proves the lemma. Some details omitted.

We will reduce the case where k is not algebraically closed, to the algebraically closed field case. We suggest the reader skip the rest of the proof. Choose an algebraic closure $\overline{k}$ of k and consider the base change $X_{\overline{k}}$. Let us check that $X_{\overline{k}} \to \mathrm{Spec}(\overline{k})$ is an example of Situation 53.6.2. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we see that $H^0(X_{\overline{k}}, \mathcal{O}) = \overline{k}$. The scheme $X_{\overline{k}}$ is proper over $\overline{k}$ (Morphisms, Lemma 29.41.5) and equidimensional of dimension 1 (Morphisms, Lemma 29.28.3). The pullback of ω_X to $X_{\overline{k}}$ is the dualizing module of $X_{\overline{k}}$ by Lemma 53.4.4. The pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is very ample (Morphisms, Lemma 29.38.8). The degree of the pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is equal to the degree of $\mathcal{L}$ on X (Varieties, Lemma 33.44.2). Finally, we see that $\omega_X \otimes \mathcal{L}$ is globally generated if and only if its base change is so (Varieties, Lemma 33.22.1). In this way we see that the result follows from the result in the case of an algebraically closed ground field. □

53.7. Very ample invertible sheaves

0E8U An often used criterion for very ampleness of an invertible module $\mathcal{L}$ on a scheme X of finite type over an algebraically closed field is: sections of $\mathcal{L}$ separate points and tangent vectors (Varieties, Section 33.23). Here is another criterion for curves; please compare with Varieties, Subsection 33.35.6.

0E8V Lemma 53.7.1. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume

(1) $\mathcal{L}$ has a regular global section,
(2) $H^1(X, \mathcal{L}) = 0$, and
(3) $\mathcal{L}$ is ample.

Then $\mathcal{L}^{\otimes 6}$ is very ample on X over k.

Proof. Let s be a regular global section of $\mathcal{L}$. Let $i : Z = Z(s) \to X$ be the zero scheme of s, see Divisors, Section 31.14. By condition (3) we see that $Z \neq \emptyset$ (small detail omitted). Consider the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{L} \to i_*(\mathcal{L}|_Z) \to 0$$

Tensoring with $\mathcal{L}$ we obtain

$$0 \to \mathcal{L} \to \mathcal{L}^{\otimes 2} \to i_*(\mathcal{L}^{\otimes 2}|_Z) \to 0$$

Observe that Z has dimension 0 (Divisors, Lemma 31.13.5) and hence is the spectrum of an Artinian ring (Varieties, Lemma 33.20.2) hence $\mathcal{L}|_Z \cong \mathcal{O}_Z$ (Algebra, Lemma 10.78.7). The short exact sequence also shows that $H^1(X, \mathcal{L}^{\otimes 2}) = 0$ (for example using Varieties, Lemma 33.33.3 to see vanishing in the spot on the right). Using induction on $n \geq 1$ and the sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to i_*(\mathcal{L}^{\otimes n+1}|_Z) \to 0$$

we see that $H^1(X, \mathcal{L}^{\otimes n}) = 0$ for $n > 0$ and that there exists a global section t_{n+1} of $\mathcal{L}^{\otimes n+1}$ which gives a trivialization of $\mathcal{L}^{\otimes n+1}|_Z \cong \mathcal{O}_Z$.

Consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \oplus H^0(X, \mathcal{L}^{\otimes 2}) \otimes_k H^0(X, \mathcal{L}^{\otimes n-1}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective for $n \geq 3$. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to i_*(\mathcal{L}^{\otimes n+1}|_Z) \to 0$$

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of μ_n. On the other hand, since $H^0(\mathcal{L}^{\otimes 2}) \to H^0(\mathcal{L}^{\otimes 2}|_Z)$ is surjective (see above) and since t_{n-1} maps to a trivialization of $\mathcal{L}^{\otimes n-1}|_Z$ we see that $\mu_n(H^0(X, \mathcal{L}^{\otimes 2}) \otimes t_{n-1})$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_Z$. This proves the claim.

From the claim in the previous paragraph we conclude that the graded k-algebra

$$S = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is generated in degrees $0, 1, 2, 3$ over k. Recall that $X = \text{Proj}(S)$, see Morphisms, Lemma 29.43.17. Thus $S^{(6)} = \bigoplus_n S_{6n}$ is generated in degree 1. This means that $\mathcal{L}^{\otimes 6}$ is very ample as desired. $\square$

0E8W Lemma 53.7.2. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume

(1) $\mathcal{L}$ is globally generated,
(2) $H^1(X, \mathcal{L}) = 0$, and
(3) $\mathcal{L}$ is ample.

Then $\mathcal{L}^{\otimes 2}$ is very ample on X over k.

Proof. Choose basis $s_0, \ldots, s_n$ of $H^0(X, \mathcal{L}^{\otimes 2})$ over k. By property (1) we see that $\mathcal{L}^{\otimes 2}$ is globally generated and we get a morphism

$$\varphi_{\mathcal{L}^{\otimes 2}, (s_0, \ldots, s_n)} : X \longrightarrow \mathbf{P}^n_k$$

See Constructions, Section 27.13. The lemma asserts that this morphism is a closed immersion. To check this we may replace k by its algebraic closure, see Descent, Lemma 35.23.19. Thus we may assume k is algebraically closed.

Assume k is algebraically closed. For each generic point $\eta_i \in X$ let $V_i \subset H^0(X, \mathcal{L})$ be the k-subvector space of sections vanishing at η_i. Since $\mathcal{L}$ is globally generated, we see that $V_i \neq H^0(X, \mathcal{L})$. Since X has only a finite number of irreducible components and k is infinite, we can find $s \in H^0(X, \mathcal{L})$ nonvanishing at η_i for all i. Then s is a regular section of $\mathcal{L}$ (because X is Cohen-Macaulay by Lemma 53.6.1 and hence $\mathcal{L}$ has no embedded associated points).

In particular, all of the statements given in the proof of Lemma 53.7.1 hold with this s. Moreover, as $\mathcal{L}$ is globally generated, we can find a global section $t \in H^0(X, \mathcal{L})$ such that $t|_Z$ is nonvanishing (argue as above using the finite number of points of Z). Then in the proof of Lemma 53.7.1 we can use t to see that additionally the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes 2}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 3})$$

is surjective. Thus

$$S = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is generated in degrees $0, 1, 2$ over k. Arguing as in the proof of Lemma 53.7.1 we find that $S^{(2)} = \bigoplus_n S_{2n}$ is generated in degree 1. This means that $\mathcal{L}^{\otimes 2}$ is very ample as desired. Some details omitted. □

53.8. The genus of a curve

0BY6 If X is a smooth projective geometrically irreducible curve over a field k, then we've previously defined the genus of X as the dimension of $H^1(X, \mathcal{O}_X)$, see Picard Schemes of Curves, Definition 44.6.3. Observe that $H^0(X, \mathcal{O}_X) = k$ in this case, see Varieties, Lemma 33.26.2. Let us generalize this as follows.

0BY7 Definition 53.8.1. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then the genus of X is $g = \dim_k H^1(X, \mathcal{O}_X)$.

This is sometimes called the arithmetic genus of X. In the literature the arithmetic genus of a proper curve X over k is sometimes defined as

$$p_a(X) = 1 - \chi(X, \mathcal{O}_X) = 1 - \dim_k H^0(X, \mathcal{O}_X) + \dim_k H^1(X, \mathcal{O}_X)$$

This agrees with our definition when it applies because we assume $H^0(X, \mathcal{O}_X) = k$. But note that

(1) $p_a(X)$ can be negative, and
(2) $p_a(X)$ depends on the base field k and should be written $p_a(X/k)$.

For example if $k = \mathbf{Q}$ and $X = \mathbf{P}^1_{\mathbf{Q}(i)}$ then $p_a(X/\mathbf{Q}) = -1$ and $p_a(X/\mathbf{Q}(i)) = 0$.

The assumption that $H^0(X, \mathcal{O}_X) = k$ in our definition has two consequences. On the one hand, it means there is no confusion about the base field. On the other hand, it implies the scheme X is Cohen-Macaulay and equidimensional of dimension 1 (Lemma 53.6.1). If ω_X denotes the dualizing module as in Lemmas 53.4.1 and 53.4.2 we see that

0BY8 (53.8.1.1)
$$g = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$$

by duality, see Remark 53.4.3.

If X is proper over k of dimension ≤ 1 and $H^0(X, \mathcal{O}_X)$ is not equal to the ground field k, instead of using the arithmetic genus $p_a(X)$ given by the displayed formula above we shall use the invariant $\chi(X, \mathcal{O}_X)$. In fact, it is advocated in [Ser55b, page 276] and [Hir95, Introduction] that we should call $\chi(X, \mathcal{O}_X)$ the arithmetic genus.

0BY9 Lemma 53.8.2. Let k'/k be a field extension. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then $X_{k'}$ is a proper scheme over k' having dimension 1 and $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$. Moreover the genus of $X_{k'}$ is equal to the genus of X.

Proof. The dimension of $X_{k'}$ is 1 for example by Morphisms, Lemma 29.28.3. The morphism $X_{k'} \to \mathrm{Spec}(k')$ is proper by Morphisms, Lemma 29.41.5. The equality $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ follows from Cohomology of Schemes, Lemma 30.5.2. The equality of the genus follows from the same lemma. □

0C19 Lemma 53.8.3. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. If X is Gorenstein, then

$$\deg(\omega_X) = 2g - 2$$

where g is the genus of X and ω_X is as in Lemma 53.4.1.

Proof. Immediate from Lemma 53.5.2. □

0C1A Lemma 53.8.4. Let X be a smooth proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Then

$$\dim_k H^0(X, \Omega_{X/k}) = g \quad \text{and} \quad \deg(\Omega_{X/k}) = 2g - 2$$

where g is the genus of X.

Proof. By Lemma 53.4.1 we have $\Omega_{X/k} = \omega_X$. Hence the formulas hold by (53.8.1.1) and Lemma 53.8.3. □

53.9. Plane curves

0BYA Let k be a field. A plane curve will be a curve X which is isomorphic to a closed subscheme of $\mathbf{P}^2_k$. Often the embedding $X \to \mathbf{P}^2_k$ will be considered given. By Divisors, Example 31.31.2 a curve is determined by the corresponding homogeneous ideal

$$I(X) = \mathrm{Ker}\left(k[T_0, T_2, T_2] \longrightarrow \bigoplus \Gamma(X, \mathcal{O}_X(n))\right)$$

Recall that in this situation we have

$$X = \mathrm{Proj}(k[T_0, T_2, T_2]/I)$$

as closed subschemes of $\mathbf{P}^2_k$. For more general information on these constructions we refer the reader to Divisors, Example 31.31.2 and the references therein. It turns out that $I(X) = (F)$ for some homogeneous polynomial $F \in k[T_0, T_1, T_2]$, see Lemma 53.9.1. Since X is irreducible, it follows that F is irreducible, see Lemma 53.9.2. Moreover, looking at the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \xrightarrow{F} \mathcal{O}_{\mathbf{P}^2_k} \to \mathcal{O}_X \to 0$$

where $d = \deg(F)$ we find that $H^0(X, \mathcal{O}_X) = k$ and that X has genus $(d-1)(d-2)/2$, see proof of Lemma 53.9.3.

To find smooth plane curves it is easiest to write explicit equations. Let p denote the characteristic of k. If p does not divide d, then we can take

$$F = T_0^d + T_1^d + T_2^d$$

The corresponding curve $X = V_+(F)$ is called the Fermat curve of degree d. It is smooth because on each standard affine piece $D_+(T_i)$ we obtain a curve isomorphic to the affine curve

$$\mathrm{Spec}(k[x, y]/(x^d + y^d + 1))$$

The ring map $k \to k[x, y]/(x^d + y^d + 1)$ is smooth by Algebra, Lemma 10.137.16 as dx^{d-1} and dy^{d-1} generate the unit ideal in $k[x, y]/(x^d + y^d + 1)$. If $p|d$ but $p \neq 3$ then you can use the equation

$$F = T_0^{d-1}T_1 + T_1^{d-1}T_2 + T_2^{d-1}T_0$$

Namely, on the affine pieces you get $x + x^{d-1}y + y^{d-1}$ with derivatives $1 - x^{d-2}y$ and $x^{d-1} - y^{d-2}$ whose common zero set (of all three) is empty[4]. We leave it to the reader to make examples in characteristic 3.

More generally for any field k and any n and d there exists a smooth hypersurface of degree d in $\mathbf{P}^n_k$, see for example [Poo05].

Of course, in this way we only find smooth curves whose genus is a triangular number. To get smooth curves of an arbitrary genus one can look for smooth curves lying on $\mathbf{P}^1 \times \mathbf{P}^1$ (insert future reference here).

0BYB Lemma 53.9.1. Let $Z \subset \mathbf{P}^2_k$ be a closed subscheme which is equidimensional of dimension 1 and has no embedded points (equivalently Z is Cohen-Macaulay). Then the ideal $I(Z) \subset k[T_0, T_1, T_2]$ corresponding to Z is principal.

Proof. This is a special case of Divisors, Lemma 31.31.3 (see also Varieties, Lemma 33.34.4). The parenthetical statement follows from the fact that a 1 dimensional Noetherian scheme is Cohen-Macaulay if and only if it has no embedded points, see Divisors, Lemma 31.4.4. □

0BYC Lemma 53.9.2. Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 53.9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then Z is a curve if and only if F is irreducible.

Proof. If F is reducible, say $F = F'F''$ then let Z' be the closed subscheme of $\mathbf{P}^2_k$ defined by F'. It is clear that $Z' \subset Z$ and that $Z' \neq Z$. Since Z' has dimension 1 as well, we conclude that either Z is not reduced, or that Z is not irreducible. Conversely, write $Z = \sum a_i D_i$ where D_i are the irreducible components of Z, see Divisors, Lemmas 31.15.8 and 31.15.9. Let $F_i \in k[T_0, T_1, T_2]$ be the homogeneous polynomial generating the ideal of D_i. Then it is clear that F and $\prod F_i^{a_i}$ cut out the same closed subscheme of $\mathbf{P}^2_k$. Hence $F = \lambda \prod F_i^{a_i}$ for some $\lambda \in k^*$ because both generate the ideal of Z. Thus we see that if F is irreducible, then Z is a prime divisor, i.e., a curve. □

0BYD Lemma 53.9.3. Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 53.9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then $H^0(Z, \mathcal{O}_Z) = k$ and the genus of Z is $(d-1)(d-2)/2$ where $d = \deg(F)$.

Proof. Let $S = k[T_0, T_1, T_2]$. There is an exact sequence of graded modules

$$0 \to S(-d) \xrightarrow{F} S \to S/(F) \to 0$$

[4]Namely, as $x^{d-1} = y^{d-2}$, then $0 = x + x^{d-1}y + y^{d-1} = x + 2x^{d-1}y$. Since $x \neq 0$ because $1 = x^{d-2}y$ we get $0 = 1 + 2x^{d-2}y = 3$ which is absurd unless $3 = 0$.

Denote $i : Z \to \mathbf{P}^2_k$ the given closed immersion. Applying the exact functor $\widetilde{\ }$ (Constructions, Lemma 27.8.4) we obtain

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_Z \to 0$$

because F generates the ideal of Z. Note that the cohomology groups of $\mathcal{O}_{\mathbf{P}^2_k}(-d)$ and $\mathcal{O}_{\mathbf{P}^2_k}$ are given in Cohomology of Schemes, Lemma 30.8.1. On the other hand, we have $H^q(Z, \mathcal{O}_Z) = H^q(\mathbf{P}^2_k, i_*\mathcal{O}_Z)$ by Cohomology of Schemes, Lemma 30.2.4. Applying the long exact cohomology sequence we first obtain that

$$k = H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}) \longrightarrow H^0(Z, \mathcal{O}_Z)$$

is an isomorphism and next that the boundary map

$$H^1(Z, \mathcal{O}_Z) \longrightarrow H^2(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-d)) \cong k[T_0, T_1, T_2]_{d-3}$$

is an isomorphism. Since it is easy to see that the dimension of this is $(d-1)(d-2)/2$ the proof is finished. $\square$

0CCU Lemma 53.9.4. Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 53.9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. If $Z \to \mathrm{Spec}(k)$ is smooth in at least one point and k is infinite, then there exists a closed point $z \in Z$ contained in the smooth locus such that $\kappa(z)/k$ is finite separable of degree at most d.

Proof. Suppose that $z' \in Z$ is a point where $Z \to \mathrm{Spec}(k)$ is smooth. After renumbering the coordinates if necessary we may assume z' is contained in $D_+(T_0)$. Set $f = F(1, x, y) \in k[x, y]$. Then $Z \cap D_+(X_0)$ is isomorphic to the spectrum of $k[x, y]/(f)$. Let f_x, f_y be the partial derivatives of f with respect to x, y. Since z' is a smooth point of Z/k we see that either f_x or f_y is nonzero in z' (see discussion in Algebra, Section 10.137). After renumbering the coordinates we may assume f_y is not zero at z'. Hence there is a nonempty open subscheme $V \subset Z \cap D_+(X_0)$ such that the projection

$$p : V \longrightarrow \mathrm{Spec}(k[x])$$

is étale. Because the degree of f as a polynomial in y is at most d, we see that the degrees of the fibres of the projection p are at most d (see discussion in Morphisms, Section 29.57). Moreover, as p is étale the image of p is an open $U \subset \mathrm{Spec}(k[x])$. Finally, since k is infinite, the set of k-rational points $U(k)$ of U is infinite, in particular not empty. Pick any $t \in U(k)$ and let $z \in V$ be a point mapping to t. Then z works. $\square$

53.10. Curves of genus zero

0C6L Later we will need to know what a proper genus zero curve looks like. It turns out that a Gorenstein proper genus zero curve is a plane curve of degree 2, i.e., a conic, see Lemma 53.10.3. A general proper genus zero curve is obtained from a nonsingular one (over a bigger field) by a pushout procedure, see Lemma 53.10.5. Since a nonsingular curve is Gorenstein, these two results cover all possible cases.

0C6M Lemma 53.10.1. Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. If X has genus 0, then every invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree 0 is trivial.

Proof. Namely, we have $\dim_k H^0(X, \mathcal{L}) \geq 0 + 1 - 0 = 1$ by Riemann-Roch (Lemma 53.5.2), hence $\mathcal{L}$ has a nonzero section, hence $\mathcal{L} \cong \mathcal{O}_X$ by Varieties, Lemma 33.44.12. $\square$

0C6T **Lemma 53.10.2.** Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Assume X has genus 0. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module of degree $d > 0$. Then we have

(1) $\dim_k H^0(X, \mathcal{L}) = d + 1$ and $\dim_k H^1(X, \mathcal{L}) = 0$,
(2) $\mathcal{L}$ is very ample and defines a closed immersion into $\mathbf{P}^d_k$.

Proof. By definition of degree and genus we have

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = d + 1$$

Let s be a nonzero section of $\mathcal{L}$. Then the zero scheme of s is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$ and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 31.14.10 and Remark 31.14.11. Since $H^1(X, \mathcal{O}_X) = 0$ by assumption, we see that $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. As $\mathcal{L}|_D$ is generated by global sections (because $\dim(D) = 0$, see Varieties, Lemma 33.33.3) we conclude that the invertible module $\mathcal{L}$ is generated by global sections. In fact, since D is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[5] and hence we can find a section t of $\mathcal{L}$ whose restriction of D generates $\mathcal{L}|_D$. The short exact sequence also shows that $H^1(X, \mathcal{L}) = 0$.

For $n \geq 1$ consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to \mathcal{L}^{\otimes n+1}|_D \to 0$$

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of μ_n. On the other hand, since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since t^n maps to a trivialization of $\mathcal{L}^{\otimes n}|_D$ we see that $\mu_n(H^0(X, \mathcal{L}) \otimes t^n)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_D$. This proves the claim.

Observe that $\mathcal{L}$ is ample by Varieties, Lemma 33.44.14. Hence Morphisms, Lemma 29.43.17 gives an isomorphism

$$X \longrightarrow \text{Proj}\left(\bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})\right)$$

Since the maps μ_n are surjective for all $n \geq 1$ we see that the graded algebra on the right hand side is a quotient of the symmetric algebra on $H^0(X, \mathcal{L})$. Choosing a k-basis $s_0, \ldots, s_d$ of $H^0(X, \mathcal{L})$ we see that it is a quotient of a polynomial algebra in $d + 1$ variables. Since quotients of graded rings correspond to closed immersions of Proj (Constructions, Lemma 27.11.5) we find a closed immersion $X \to \mathbf{P}^d_k$. We omit the verification that this morphism is the morphism of Constructions, Lemma 27.13.1 associated to the sections $s_0, \ldots, s_d$ of $\mathcal{L}$. $\square$

0C6N **Lemma 53.10.3.** Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. If X is Gorenstein and has genus 0, then X is isomorphic to a plane curve of degree 2.

[5]In our case this follows from Divisors, Lemma 31.17.1 as $D \to \text{Spec}(k)$ is finite.

Proof. Consider the invertible sheaf $\mathcal{L} = \omega_X^{\otimes -1}$ where ω_X is as in Lemma 53.4.1. Then $\deg(\omega_X) = -2$ by Lemma 53.8.3 and hence $\deg(\mathcal{L}) = 2$. By Lemma 53.10.2 we conclude that choosing a basis s_0, s_1, s_2 of the k-vector space of global sections of $\mathcal{L}$ we obtain a closed immersion

$$\varphi_{(\mathcal{L},(s_0,s_1,s_2))} : X \longrightarrow \mathbf{P}^2_k$$

Thus X is a plane curve of some degree d. Let $F \in k[T_0, T_1, T_2]_d$ be its equation (Lemma 53.9.1). Because the genus of X is 0 we see that d is 1 or 2 (Lemma 53.9.3). Observe that F restricts to the zero section on $\varphi(X)$ and hence $F(s_0, s_1, s_2)$ is the zero section of $\mathcal{L}^{\otimes 2}$. Because s_0, s_1, s_2 are linearly independent we see that F cannot be linear, i.e., $d = \deg(F) \geq 2$. Thus $d = 2$ and the proof is complete. $\square$

0C6U Proposition 53.10.4 (Characterization of the projective line). Let k be a field. Let X be a proper curve over k. The following are equivalent

(1) $X \cong \mathbf{P}^1_k$,
(2) X is smooth and geometrically irreducible over k, X has genus 0, and X has an invertible module of odd degree,
(3) X is geometrically integral over k, X has genus 0, X is Gorenstein, and X has an invertible sheaf of odd degree,
(4) $H^0(X, \mathcal{O}_X) = k$, X has genus 0, X is Gorenstein, and X has an invertible sheaf of odd degree,
(5) X is geometrically integral over k, X has genus 0, and X has an invertible $\mathcal{O}_X$-module of degree 1,
(6) $H^0(X, \mathcal{O}_X) = k$, X has genus 0, and X has an invertible $\mathcal{O}_X$-module of degree 1,
(7) $H^1(X, \mathcal{O}_X) = 0$ and X has an invertible $\mathcal{O}_X$-module of degree 1,
(8) $H^1(X, \mathcal{O}_X) = 0$ and X has closed points $x_1, \ldots, x_n$ such that $\mathcal{O}_{X,x_i}$ is normal and $\gcd([\kappa(x_i) : k]) = 1$, and
(9) add more here.

Proof. We will prove that each condition (2) – (8) implies (1) and we omit the verification that (1) implies (2) – (8).

Assume (2). A smooth scheme over k is geometrically reduced (Varieties, Lemma 33.25.4) and regular (Varieties, Lemma 33.25.3). Hence X is Gorenstein (Duality for Schemes, Lemma 48.24.3). Thus we reduce to (3).

Assume (3). Since X is geometrically integral over k we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 33.26.2. and we reduce to (4).

Assume (4). Since X is Gorenstein the dualizing module ω_X as in Lemma 53.4.1 has degree $\deg(\omega_X) = -2$ by Lemma 53.8.3. Combined with the assumed existence of an odd degree invertible module, we conclude there exists an invertible module of degree 1. In this way we reduce to (6).

Assume (5). Since X is geometrically integral over k we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 33.26.2. and we reduce to (6).

Assume (6). Then $X \cong \mathbf{P}^1_k$ by Lemma 53.10.2.

Assume (7). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over k by Varieties, Lemma 33.26.2. If $d = [\kappa : k] > 1$, then every invertible sheaf has degree divisible by d and there cannot be an invertible sheaf of degree 1. Hence $d = 1$ and we reduce to case (6).

Assume (8). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over k by Varieties, Lemma 33.26.2. Since $\kappa \subset \kappa(x_i)$ we see that $k = \kappa$ by the assumption on the gcd of the degrees. The same condition allows us to find integers a_i such that $1 = \sum a_i[\kappa(x_i) : k]$. Because x_i defines an effective Cartier divisor on X by Varieties, Lemma 33.43.8 we can consider the invertible module $\mathcal{O}_X(\sum a_i x_i)$. By our choice of a_i the degree of $\mathcal{L}$ is 1. Thus $X \cong \mathbf{P}^1_k$ by Lemma 53.10.2. $\square$

0DJB Lemma 53.10.5. Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Assume X is singular and has genus 0. Then there exists a diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\nu"] & \mathrm{Spec}(k') \arrow[d] \\ x \arrow[r] & X \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

where

(1) k'/k is a nontrivial finite extension,
(2) $X' \cong \mathbf{P}^1_{k'}$,
(3) x' is a k'-rational point of X',
(4) x is a k-rational point of X,
(5) $X' \setminus \{x'\} \to X \setminus \{x\}$ is an isomorphism,
(6) $0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to k'/k \to 0$ is a short exact sequence where $k'/k = \kappa(x')/\kappa(x)$ indicates the skyscraper sheaf on the point x.

Proof. Let $\nu : X' \to X$ be the normalization of X, see Varieties, Sections 33.27 and 33.41. Since X is singular ν is not an isomorphism. Then $k' = H^0(X', \mathcal{O}_{X'})$ is a finite extension of k (Varieties, Lemma 33.26.2). The short exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

and the fact that $\mathcal{Q}$ is supported in finitely many closed points give us that

(1) $H^1(X', \mathcal{O}_{X'}) = 0$, i.e., X' has genus 0 as a curve over k',
(2) there is a short exact sequence $0 \to k \to k' \to H^0(X, \mathcal{Q}) \to 0$.

In particular k'/k is a nontrivial extension.

Next, we consider what is often called the conductor ideal

$$\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$$

This is a quasi-coherent $\mathcal{O}_X$-module. We view $\mathcal{I}$ as an ideal in $\mathcal{O}_X$ via the map $\varphi \mapsto \varphi(1)$. Thus $\mathcal{I}(U)$ is the set of $f \in \mathcal{O}_X(U)$ such that $f(\nu_*\mathcal{O}_{X'}(U)) \subset \mathcal{O}_X(U)$. In other words, the condition is that f annihilates $\mathcal{Q}$. In other words, there is a defining exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{Q})$$

Let $U \subset X$ be an affine open containing the support of $\mathcal{Q}$. Then $V = \mathcal{Q}(U) = H^0(X, \mathcal{Q})$ is a k-vector space of dimension $n-1$. The image of $\mathcal{O}_X(U) \to \mathrm{Hom}_k(V, V)$ is a commutative subalgebra, hence has dimension $\leq n-1$ over k (this is a property of commutative subalgebras of matrix algebras; details omitted). We conclude that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{A} \to 0$$

where $\text{Supp}(\mathcal{A}) = \text{Supp}(\mathcal{Q})$ and $\dim_k H^0(X, \mathcal{A}) \leq n - 1$. On the other hand, the description $\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$ provides $\mathcal{I}$ with a $\nu_*\mathcal{O}_{X'}$-module structure such that the inclusion map $\mathcal{I} \to \nu_*\mathcal{O}_{X'}$ is a $\nu_*\mathcal{O}_{X'}$-module map. We conclude that $\mathcal{I} = \nu_*\mathcal{I}'$ for some quasi-coherent sheaf of ideals $\mathcal{I}' \subset \mathcal{O}_{X'}$, see Morphisms, Lemma 29.11.6. Define $\mathcal{A}'$ as the cokernel:

$$0 \to \mathcal{I}' \to \mathcal{O}_{X'} \to \mathcal{A}' \to 0$$

Combining the exact sequences so far we obtain a short exact sequence $0 \to \mathcal{A} \to \nu_*\mathcal{A}' \to \mathcal{Q} \to 0$. Using the estimate above, combined with $\dim_k H^0(X, \mathcal{Q}) = n - 1$, gives

$$\dim_k H^0(X', \mathcal{A}') = \dim_k H^0(X, \mathcal{A}) + \dim_k H^0(X, \mathcal{Q}) \leq 2n - 2$$

However, since X' is a curve over k' we see that the left hand side is divisible by n (Varieties, Lemma 33.44.10). As $\mathcal{A}$ and $\mathcal{A}'$ cannot be zero, we conclude that $\dim_k H^0(X', \mathcal{A}') = n$ which means that $\mathcal{I}'$ is the ideal sheaf of a k'-rational point x'. By Proposition 53.10.4 we find $X' \cong \mathbf{P}^1_{k'}$. Going back to the equalities above, we conclude that $\dim_k H^0(X, \mathcal{A}) = 1$. This means that $\mathcal{I}$ is the ideal sheaf of a k-rational point x. Then $\mathcal{A} = \kappa(x) = k$ and $\mathcal{A}' = \kappa(x') = k'$ as skyscraper sheaves. Comparing the exact sequences given above, this immediately implies the result on structure sheaves as stated in the lemma. $\square$

0DJC Example 53.10.6. In fact, the situation described in Lemma 53.10.5 occurs for any nontrivial finite extension k'/k. Namely, we can consider

$$A = \{f \in k'[x] \mid f(0) \in k\}$$

The spectrum of A is an affine curve, which we can glue to the spectrum of $B = k'[y]$ using the isomorphism $A_x \cong B_y$ sending x^{-1} to y. The result is a proper curve X with $H^0(X, \mathcal{O}_X) = k$ and singular point x corresponding to the maximal ideal $A \cap (x)$. The normalization of X is $\mathbf{P}^1_{k'}$ exactly as in the lemma.

53.11. Geometric genus

0BYE If X is a proper and smooth curve over k with $H^0(X, \mathcal{O}_X) = k$, then

$$p_g(X) = \dim_k H^0(X, \Omega_{X/k})$$

is called the geometric genus of X. By Lemma 53.8.4 the geometric genus of X agrees with the (arithmetic) genus. However, in higher dimensions there is a difference between the geometric genus and the arithmetic genus, see Remark 53.11.2.

For singular curves, we will define the geometric genus as follows.

0BYF Definition 53.11.1. Let k be a field. Let X be a geometrically irreducible curve over k. The geometric genus of X is the genus of a smooth projective model of X possibly defined over an extension field of k as in Lemma 53.2.9.

If k is perfect, then the nonsingular projective model Y of X is smooth (Lemma 53.2.8) and the geometric genus of X is just the genus of Y. But if k is not perfect, this may not be true. In this case we choose an extension K/k such that the nonsingular projective model Y_K of $(X_K)_{red}$ is a smooth projective curve and we define the geometric genus of X to be the genus of Y_K. This is well defined by Lemmas 53.2.9 and 53.8.2.

0BYG Remark 53.11.2. Suppose that X is a d-dimensional proper smooth variety over an algebraically closed field k. Then the arithmetic genus is often defined as $p_a(X) = (-1)^d(\chi(X, \mathcal{O}_X) - 1)$ and the geometric genus as $p_g(X) = \dim_k H^0(X, \Omega^d_{X/k})$. In this situation the arithmetic genus and the geometric genus no longer agree even though it is still true that $\omega_X \cong \Omega^d_{X/k}$. For example, if $d = 2$, then we have

$$\begin{aligned} p_a(X) - p_g(X) &= h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - 1 - h^0(X, \Omega^2_{X/k}) \\ &= -h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - h^0(X, \omega_X) \\ &= -h^1(X, \mathcal{O}_X) \end{aligned}$$

where $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$ and where the last equality follows from duality. Hence for a surface the difference $p_g(X) - p_a(X)$ is always nonnegative; it is sometimes called the irregularity of the surface. If $X = C_1 \times C_2$ is a product of smooth projective curves of genus g_1 and g_2, then the irregularity is $g_1 + g_2$.

53.12. Riemann-Hurwitz

0C1B Let k be a field. Let $f : X \to Y$ be a morphism of smooth curves over k. Then we obtain a canonical exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{\text{d}f} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

by Morphisms, Lemma 29.32.9. Since X and Y are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 29.34.12. Assume the first map is nonzero, i.e., assume f is generically étale, see Lemma 53.12.1. Let $R \subset X$ be the closed subscheme cut out by the different $\mathfrak{D}_f$ of f. By Discriminants, Lemma 49.12.6 this is the same as the vanishing locus of $\text{d}f$, it is an effective Cartier divisor, and we get

$$f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R) = \Omega_{X/k}$$

In particular, if X, Y are projective with $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and X, Y have genus g_X, g_Y, then we get the Riemann-Hurwitz formula

$$\begin{aligned} 2g_X - 2 &= \deg(\Omega_{X/k}) \\ &= \deg(f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R)) \\ &= \deg(f)\deg(\Omega_{Y/k}) + \deg(R) \\ &= \deg(f)(2g_Y - 2) + \deg(R) \end{aligned}$$

The first and last equality by Lemma 53.8.4. The second equality by the isomorphism of invertible sheaves given above. The third equality by additivity of degrees (Varieties, Lemma 33.44.7), the formula for the degree of a pullback (Varieties, Lemma 33.44.11), and finally the formula for the degree of $\mathcal{O}_X(R)$ (Varieties, Lemma 33.44.9).

To use the Riemann-Hurwitz formula we need to compute $\deg(R) = \dim_k \Gamma(R, \mathcal{O}_R)$. By the structure of zero dimensional schemes over k (see for example Varieties, Lemma 33.20.2), we see that R is a finite disjoint union of spectra of Artinian local rings $R = \coprod_{x \in R} \text{Spec}(\mathcal{O}_{R,x})$ with each $\mathcal{O}_{R,x}$ of finite dimension over k. Thus

$$\deg(R) = \sum\nolimits_{x \in R} \dim_k \mathcal{O}_{R,x} = \sum\nolimits_{x \in R} d_x[\kappa(x) : k]$$

with

$$d_x = \text{length}_{\mathcal{O}_{R,x}} \mathcal{O}_{R,x} = \text{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{R,x}$$

the multiplicity of x in R (see Algebra, Lemma 10.52.12). Let $x \in X$ be a closed point with image $y \in Y$. Looking at stalks we obtain an exact sequence

$$\Omega_{Y/k,y} \to \Omega_{X/k,x} \to \Omega_{X/Y,x} \to 0$$

Choosing local generators η_x and η_y of the (free rank 1) modules $\Omega_{X/k,x}$ and $\Omega_{Y/k,y}$ we see that $\eta_y \mapsto h\eta_x$ for some nonzero $h \in \mathcal{O}_{X,x}$. By the exact sequence we see that $\Omega_{X/Y,x} \cong \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules. Since the divisor R is cut out by h (see above) we have $\mathcal{O}_{R,x} = \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$. Thus we find the following equalities

$$\begin{aligned} d_x &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{R,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/h\mathcal{O}_{X,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x}) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(h) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(\text{“}\eta_y/\eta_x\text{”}) \end{aligned}$$

The first equality by our definition of d_x. The second and third we saw above. The fourth equality is the definition of ord, see Algebra, Definition 10.121.2. Note that since $\mathcal{O}_{X,x}$ is a discrete valuation ring, the integer $\text{ord}_{\mathcal{O}_{X,x}}(h)$ just the valuation of h. The fifth equality is a mnemonic.

Here is a case where one can “calculate” the multiplicity d_x in terms of other invariants. Namely, if $\kappa(x)$ is separable over k, then we may choose $\eta_x = \text{d}s$ and $\eta_y = \text{d}t$ where s and t are uniformizers in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ (Lemma 53.12.3). Then $t \mapsto us^{e_x}$ for some unit $u \in \mathcal{O}_{X,x}$ where e_x is the ramification index of the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$. Hence we get

$$\eta_y = \text{d}t = \text{d}(us^{e_x}) = es^{e_x-1}u\text{d}s + s^{e_x}\text{d}u$$

Writing $\text{d}u = w\text{d}s$ for some $w \in \mathcal{O}_{X,x}$ we see that

$$\text{“}\eta_y/\eta_x\text{”} = es^{e_x-1}u + s^{e_x}w = (e_xu + sw)s^{e_x-1}$$

We conclude that the order of vanishing of this is $e_x - 1$ unless the characteristic of $\kappa(x)$ is $p > 0$ and p divides e_x in which case the order of vanishing is $> e_x - 1$.

Combining all of the above we find that if k has characteristic zero, then

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum\nolimits_{x \in X}(e_x - 1)[\kappa(x) : k]$$

where e_x is the ramification index of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,f(x)}$. This precise formula will hold if and only if all the ramification is tame, i.e., when the residue field extensions $\kappa(x)/\kappa(y)$ are separable and e_x is prime to the characteristic of k, although the arguments above are insufficient to prove this. We refer the reader to Lemma 53.12.4 and its proof.

0C1C Lemma 53.12.1. Let k be a field. Let $f : X \to Y$ be a morphism of smooth curves over k. The following are equivalent

(1) $\text{d}f : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero,
(2) $\Omega_{X/Y}$ is supported on a proper closed subset of X,
(3) there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is unramified,
(4) there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is étale,
(5) the extension $k(X)/k(Y)$ of function fields is finite separable.

Proof. Since X and Y are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 29.34.12. Using the exact sequence

$$f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

of Morphisms, Lemma 29.32.9 we see that (1) and (2) are equivalent and equivalent to the condition that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero in the generic point. The equivalence of (2) and (3) follows from Morphisms, Lemma 29.35.2. The equivalence between (3) and (4) follows from Morphisms, Lemma 29.36.16 and the fact that flatness is automatic (Lemma 53.2.3). To see the equivalence of (5) and (4) use Algebra, Lemma 10.140.9. Some details omitted. □

0C1D Lemma 53.12.2. Let $f : X \to Y$ be a morphism of smooth proper curves over a field k which satisfies the equivalent conditions of Lemma 53.12.1. If $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and X and Y have genus g_X and g_Y, then

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \deg(R)$$

where $R \subset X$ is the effective Cartier divisor cut out by the different of f.

Proof. See discussion above; we used Discriminants, Lemma 49.12.6, Lemma 53.8.4, and Varieties, Lemmas 33.44.7 and 33.44.11. □

0C1E Lemma 53.12.3. Let $X \to \mathrm{Spec}(k)$ be smooth of relative dimension 1 at a closed point $x \in X$. If $\kappa(x)$ is separable over k, then for any uniformizer s in the discrete valuation ring $\mathcal{O}_{X,x}$ the element $\mathrm{d}s$ freely generates $\Omega_{X/k,x}$ over $\mathcal{O}_{X,x}$.

Proof. The ring $\mathcal{O}_{X,x}$ is a discrete valuation ring by Algebra, Lemma 10.140.3. Since x is closed $\kappa(x)$ is finite over k. Hence if $\kappa(x)/k$ is separable, then any uniformizer s maps to a nonzero element of $\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ by Algebra, Lemma 10.140.4. Since $\Omega_{X/k,x}$ is free of rank 1 over $\mathcal{O}_{X,x}$ the result follows. □

0C1F Lemma 53.12.4. Notation and assumptions as in Lemma 53.12.2. For a closed point $x \in X$ let d_x be the multiplicity of x in R. Then

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum d_x[\kappa(x) : k]$$

Moreover, we have the following results

(1) $d_x = \mathrm{length}_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x})$,
(2) $d_x \geq e_x - 1$ where e_x is the ramification index of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,y}$,
(3) $d_x = e_x - 1$ if and only if $\mathcal{O}_{X,x}$ is tamely ramified over $\mathcal{O}_{Y,y}$.

Proof. By Lemma 53.12.2 and the discussion above (which used Varieties, Lemma 33.20.2 and Algebra, Lemma 10.52.12) it suffices to prove the results on the multiplicity d_x of x in R. Part (1) was proved in the discussion above. In the discussion above we proved (2) and (3) only in the case where $\kappa(x)$ is separable over k. In the rest of the proof we give a uniform treatment of (2) and (3) using material on differents of quasi-finite Gorenstein morphisms.

First, observe that f is a quasi-finite Gorenstein morphism. This is true for example because f is a flat quasi-finite morphism and X is Gorenstein (see Duality for Schemes, Lemma 48.25.7) or because it was shown in the proof of Discriminants, Lemma 49.12.6 (which we used above). Thus $\omega_{X/Y}$ is invertible by Discriminants, Lemma 49.16.1 and the same remains true after replacing X by opens and after performing a base change by some $Y' \to Y$. We will use this below without further mention.

Choose affine opens $U \subset X$ and $V \subset Y$ such that $x \in U$, $y \in V$, $f(U) \subset V$, and x is the only point of U lying over y. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Then $R \cap U$ is the different of $f|_U : U \to V$. By Discriminants, Lemma 49.9.4 formation of the different commutes with arbitrary base change in our case. By our choice of U and V we have

$$A \otimes_B \kappa(y) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = \mathcal{O}_{X,x}/(s^{e_x})$$

where e_x is the ramification index as in the statement of the lemma. Let $C = \mathcal{O}_{X,x}/(s^{e_x})$ viewed as a finite algebra over $\kappa(y)$. Let $\mathfrak{D}_{C/\kappa(y)}$ be the different of C over $\kappa(y)$ in the sense of Discriminants, Definition 49.9.1. It suffices to show: $\mathfrak{D}_{C/\kappa(y)}$ is nonzero if and only if the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$ is tamely ramified and in the tamely ramified case $\mathfrak{D}_{C/\kappa(y)}$ is equal to the ideal generated by $s^{e_x - 1}$ in C. Recall that tame ramification means exactly that $\kappa(x)/\kappa(y)$ is separable and that the characteristic of $\kappa(y)$ does not divide e_x. On the other hand, the different of $C/\kappa(y)$ is nonzero if and only if $\tau_{C/\kappa(y)} \in \omega_{C/\kappa(y)}$ is nonzero. Namely, since $\omega_{C/\kappa(y)}$ is an invertible C-module (as the base change of $\omega_{A/B}$) it is free of rank 1, say with generator λ. Write $\tau_{C/\kappa(y)} = h\lambda$ for some $h \in C$. Then $\mathfrak{D}_{C/\kappa(y)} = (h) \subset C$ whence the claim. By Discriminants, Lemma 49.4.8 we have $\tau_{C/\kappa(y)} \neq 0$ if and only if $\kappa(x)/\kappa(y)$ is separable and e_x is prime to the characteristic. Finally, even if $\tau_{C/\kappa(y)}$ is nonzero, then it is still the case that $s\tau_{C/\kappa(y)} = 0$ because $s\tau_{C/\kappa(y)} : C \to \kappa(y)$ sends c to the trace of the nilpotent operator sc which is zero. Hence $sh = 0$, hence $h \in (s^{e_x - 1})$ which proves that $\mathfrak{D}_{C/\kappa(y)} \subset (s^{e_x - 1})$ always. Since $(s^{e_x - 1}) \subset C$ is the smallest nonzero ideal, we have proved the final assertion. $\square$

53.13. Inseparable maps

0CCV Some remarks on the behaviour of the genus under inseparable maps.

0CCW **Lemma 53.13.1.** *Let k be a field. Let $f : X \to Y$ be a surjective morphism of curves over k. If X is smooth over k and Y is normal, then Y is smooth over k.*

Proof. Let $y \in Y$. Pick $x \in X$ mapping to y. By Varieties, Lemma 33.25.9 it suffices to show that f is flat at x. This follows from Lemma 53.2.3. $\square$

0CCX **Lemma 53.13.2.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If the extension $k(X)/k(Y)$ of function fields is purely inseparable, then there exists a factorization*

$$X = X_0 \to X_1 \to \ldots \to X_n = Y$$

such that each X_i is a proper nonsingular curve and $X_i \to X_{i+1}$ is a degree p morphism with $k(X_{i+1}) \subset k(X_i)$ inseparable.

Proof. This follows from Theorem 53.2.6 and the fact that a finite purely inseparable extension of fields can always be gotten as a sequence of (inseparable) extensions of degree p, see Fields, Lemma 9.14.5. $\square$

0CCY **Lemma 53.13.3.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If X is smooth and $k(Y) \subset k(X)$ is inseparable of degree p, then there is a unique isomorphism $Y = X^{(p)}$ such that f is $F_{X/k}$.*

Proof. The relative frobenius morphism $F_{X/k} : X \to X^{(p)}$ is constructed in Varieties, Section 33.36. Observe that $X^{(p)}$ is a smooth proper curve over k as a base change of X. The morphism $F_{X/k}$ has degree p by Varieties, Lemma 33.36.10. Thus $k(X^{(p)})$ and $k(Y)$ are both subfields of $k(X)$ with $[k(X) : k(Y)] = [k(X) : k(X^{(p)})] = p$. To prove the lemma it suffices to show that $k(Y) = k(X^{(p)})$ inside $k(X)$. See Theorem 53.2.6.

Write $K = k(X)$. Consider the map $\mathrm{d} : K \to \Omega_{K/k}$. It follows from Lemma 53.12.1 that both $k(Y)$ is contained in the kernel of d. By Varieties, Lemma 33.36.7 we see that $k(X^{(p)})$ is in the kernel of d. Since X is a smooth curve we know that $\Omega_{K/k}$ is a vector space of dimension 1 over K. Then More on Algebra, Lemma 15.46.2. implies that $\mathrm{Ker}(\mathrm{d}) = kK^p$ and that $[K : kK^p] = p$. Thus $k(Y) = kK^p = k(X^{(p)})$ for reasons of degree. $\square$

0CCZ Lemma 53.13.4. Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If X is smooth and $k(Y) \subset k(X)$ is purely inseparable, then there is a unique $n \geq 0$ and a unique isomorphism $Y = X^{(p^n)}$ such that f is the n-fold relative Frobenius of X/k.

Proof. The n-fold relative Frobenius of X/k is defined in Varieties, Remark 33.36.11. The lemma follows by combining Lemmas 53.13.3 and 53.13.2. $\square$

0CD0 Lemma 53.13.5. Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. Assume

(1) X is smooth,
(2) $H^0(X, \mathcal{O}_X) = k$,
(3) $k(X)/k(Y)$ is purely inseparable.

Then Y is smooth, $H^0(Y, \mathcal{O}_Y) = k$, and the genus of Y is equal to the genus of X.

Proof. By Lemma 53.13.4 we see that $Y = X^{(p^n)}$ is the base change of X by $F^n_{\mathrm{Spec}(k)}$. Thus Y is smooth and the result on the cohomology and genus follows from Lemma 53.8.2. $\square$

0CD1 Example 53.13.6. This example will show that the genus can change under a purely inseparable morphism of nonsingular projective curves. Let k be a field of characteristic 3. Assume there exists an element $a \in k$ which is not a 3rd power. For example $k = \mathbf{F}_3(a)$ would work. Let X be the plane curve with homogeneous equation

$$F = T_1^2T_0 - T_2^3 + aT_0^3$$

as in Section 53.9. On the affine piece $D_+(T_0)$ using coordinates $x = T_1/T_0$ and $y = T_2/T_0$ we obtain $x^2 - y^3 + a = 0$ which defines a nonsingular affine curve. Moreover, the point at infinity $(0 : 1 : 0)$ is a smooth point. Hence X is a nonsingular projective curve of genus 1 (Lemma 53.9.3). On the other hand, consider the morphism $f : X \to \mathbf{P}^1_k$ which on $D_+(T_0)$ sends (x, y) to $x \in \mathbf{A}^1_k \subset \mathbf{P}^1_k$. Then f is a morphism of proper nonsingular curves over k inducing an inseparable function field extension of degree $p = 3$ but the genus of X is 1 and the genus of $\mathbf{P}^1_k$ is 0.

0CD2 Proposition 53.13.7. Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper smooth curves over k. Then we can factor f as

$$X \longrightarrow X^{(p^n)} \longrightarrow Y$$

where $X^{(p^n)} \to Y$ is a nonconstant morphism of proper smooth curves inducing a separable field extension $k(X^{(p^n)})/k(Y)$, we have

$$X^{(p^n)} = X \times_{\mathrm{Spec}(k), F^n_{\mathrm{Spec}(k)}} \mathrm{Spec}(k),$$

and $X \to X^{(p^n)}$ is the n-fold relative frobenius of X.

Proof. By Fields, Lemma 9.14.6 there is a subextension $k(X)/E/k(Y)$ such that $k(X)/E$ is purely inseparable and $E/k(Y)$ is separable. By Theorem 53.2.6 this corresponds to a factorization $X \to Z \to Y$ of f with Z a nonsingular proper curve. Apply Lemma 53.13.4 to the morphism $X \to Z$ to conclude. □

0CD3 Lemma 53.13.8. Let k be a field of characteristic $p > 0$. Let X be a smooth proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ with $r \geq 1$. Then one of the following two is true

(1) there exists a $\mathfrak{g}^1_d$ whose corresponding morphism $X \to \mathbf{P}^1_k$ (Lemma 53.3.2) is generically étale (i.e., is as in Lemma 53.12.1), or
(2) there exists a $\mathfrak{g}^r_{d'}$ on $X^{(p)}$ where $d' \leq d/p$.

Proof. Pick two k-linearly independent elements $s, t \in V$. Then $f = s/t$ is the rational function defining the morphism $X \to \mathbf{P}^1_k$ corresponding to the linear series $(\mathcal{L}, ks + kt)$. If this morphism is not generically étale, then $f \in k(X^{(p)})$ by Proposition 53.13.7. Now choose a basis $s_0, \ldots, s_r$ of V and let $\mathcal{L}' \subset \mathcal{L}$ be the invertible sheaf generated by $s_0, \ldots, s_r$. Set $f_i = s_i/s_0$ in $k(X)$. If for each pair (s_0, s_i) we have $f_i \in k(X^{(p)})$, then the morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k = \mathrm{Proj}(k[T_0, \ldots, T_r])$$

factors through $X^{(p)}$ as this is true over the affine open $D_+(T_0)$ and we can extend the morphism over the affine part to the whole of the smooth curve $X^{(p)}$ by Lemma 53.2.2. Introducing notation, say we have the factorization

$$X \xrightarrow{F_{X/k}} X^{(p)} \xrightarrow{\psi} \mathbf{P}^r_k$$

of φ. Then $\mathcal{N} = \psi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ is an invertible $\mathcal{O}_{X^{(p)}}$-module with $\mathcal{L}' = F^*_{X/k}\mathcal{N}$ and with $\psi^*T_0, \ldots, \psi^*T_r$ k-linearly independent (as they pullback to $s_0, \ldots, s_r$ on X). Finally, we have

$$d = \deg(\mathcal{L}) \geq \deg(\mathcal{L}') = \deg(F_{X/k}) \deg(\mathcal{N}) = p \deg(\mathcal{N})$$

as desired. Here we used Varieties, Lemmas 33.44.12, 33.44.11, and 33.36.10. □

0CD4 Lemma 53.13.9. Let k be a field. Let X be a smooth proper curve over k with $H^0(X, \mathcal{O}_X) = k$ and genus $g \geq 2$. Then there exists a closed point $x \in X$ with $\kappa(x)/k$ separable of degree $\leq 2g - 2$.

Proof. Set $\omega = \Omega_{X/k}$. By Lemma 53.8.4 this has degree $2g - 2$ and has g global sections. Thus we have a $\mathfrak{g}^{g-1}_{2g-2}$. By the trivial Lemma 53.3.3 there exists a $\mathfrak{g}^1_{2g-2}$ and by Lemma 53.3.4 we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

of some degree $d \leq 2g-2$. Since φ is flat (Lemma 53.2.3) and finite (Lemma 53.2.4) it is finite locally free of degree d (Morphisms, Lemma 29.48.2). Pick any rational point $t \in \mathbf{P}^1_k$ and any point $x \in X$ with $\varphi(x) = t$. Then

$$d \geq [\kappa(x) : \kappa(t)] = [\kappa(x) : k]$$

for example by Morphisms, Lemmas 29.57.3 and 29.57.2. Thus if k is perfect (for example has characteristic zero or is finite) then the lemma is proved. Thus we reduce to the case discussed in the next paragraph.

Assume that k is an infinite field of characteristic $p > 0$. As above we will use that X has a $\mathfrak{g}^{g-1}_{2g-2}$. The smooth proper curve $X^{(p)}$ has the same genus as X. Hence its genus is > 0. We conclude that $X^{(p)}$ does not have a $\mathfrak{g}^{g-1}_d$ for any $d \leq g-1$ by Lemma 53.3.5. Applying Lemma 53.13.8 to our $\mathfrak{g}^{g-1}_{2g-2}$ (and noting that $2g-2/p \leq g-1$) we conclude that possibility (2) does not occur. Hence we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

which is generically étale (in the sense of the lemma) and has degree $\leq 2g-2$. Let $U \subset X$ be the nonempty open subscheme where φ is étale. Then $\varphi(U) \subset \mathbf{P}^1_k$ is a nonempty Zariski open and we can pick a k-rational point $t \in \varphi(U)$ as k is infinite. Let $u \in U$ be a point with $\varphi(u) = t$. Then $\kappa(u)/\kappa(t)$ is separable (Morphisms, Lemma 29.36.7), $\kappa(t) = k$, and $[\kappa(u) : k] \leq 2g-2$ as before. □

The following lemma does not really belong in this section but we don't know a good place for it elsewhere.

0C1G Lemma 53.13.10. Let X be a smooth curve over a field k. Let $\overline{x} \in X_{\overline{k}}$ be a closed point with image $x \in X$. The ramification index of $\mathcal{O}_{X,x} \subset \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is the inseparable degree of $\kappa(x)/k$.

Proof. After shrinking X we may assume there is an étale morphism $\pi : X \to \mathbf{A}^1_k$, see Morphisms, Lemma 29.36.20. Then we can consider the diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X_{\overline{k}},\overline{x}} & \mathcal{O}_{\mathbf{A}^1_{\overline{k}},\pi(\overline{x})} \arrow[l] \\ \mathcal{O}_{X,x} \arrow[u] & \mathcal{O}_{\mathbf{A}^1_k,\pi(x)} \arrow[l] \arrow[u] \end{tikzcd}$$

The horizontal arrows have ramification index 1 as they correspond to étale morphisms. Moreover, the extension $\kappa(x)/\kappa(\pi(x))$ is separable hence $\kappa(x)$ and $\kappa(\pi(x))$ have the same inseparable degree over k. By multiplicativity of ramification indices it suffices to prove the result when x is a point of the affine line.

Assume $X = \mathbf{A}^1_k$. In this case, the local ring of X at x looks like

$$\mathcal{O}_{X,x} = k[t]_{(P)}$$

where P is an irreducible monic polynomial over k. Then $P(t) = Q(t^q)$ for some separable polynomial $Q \in k[t]$, see Fields, Lemma 9.12.1. Observe that $\kappa(x) = k[t]/(P)$ has inseparable degree q over k. On the other hand, over $\overline{k}$ we can factor $Q(t) = \prod(t - \alpha_i)$ with α_i pairwise distinct. Write $\alpha_i = \beta_i^q$ for some unique $\beta_i \in \overline{k}$. Then our point $\overline{x}$ corresponds to one of the β_i and we conclude because the ramification index of

$$k[t]_{(P)} \longrightarrow \overline{k}[t]_{(t-\beta_i)}$$

is indeed equal to q as the uniformizer P maps to $(t-\beta_i)^q$ times a unit. □

53.14. Pushouts

0E35 Let k be a field. Consider a solid diagram

$$\begin{tikzcd} Z' \arrow[r, "i'"'] \arrow[d] & X' \arrow[d, dotted, "a"] \\ Z \arrow[r, dotted, "i"] & X \end{tikzcd}$$

of schemes over k satisfying

(a) X' is separated of finite type over k of dimension ≤ 1,
(b) $i : Z' \to X'$ is a closed immersion,
(c) Z' and Z are finite over $\mathrm{Spec}(k)$, and
(d) $Z' \to Z$ is surjective.

In this situation every finite set of points of X' are contained in an affine open, see Varieties, Proposition 33.42.7. Thus the assumptions of More on Morphisms, Proposition 37.67.3 are satisfied and we obtain the following

(1) the pushout $X = Z \amalg_{Z'} X'$ exists in the category of schemes,
(2) $i : Z \to X$ is a closed immersion,
(3) $a : X' \to X$ is integral surjective,
(4) $X \to \mathrm{Spec}(k)$ is separated by More on Morphisms, Lemma 37.67.4
(5) $X \to \mathrm{Spec}(k)$ is of finite type by More on Morphisms, Lemmas 37.67.5,
(6) thus $a : X' \to X$ is finite by Morphisms, Lemmas 29.44.4 and 29.15.8,
(7) if $X' \to \mathrm{Spec}(k)$ is proper, then $X \to \mathrm{Spec}(k)$ is proper by Morphisms, Lemma 29.41.9.

The following lemma can be generalized significantly.

0E36 **Lemma 53.14.1.** In the situation above, let $Z = \mathrm{Spec}(k')$ where k' is a field and $Z' = \mathrm{Spec}(k'_1 \times \ldots \times k'_n)$ with k'_i/k' finite extensions of fields. Let $x \in X$ be the image of $Z \to X$ and $x'_i \in X'$ the image of $\mathrm{Spec}(k'_i) \to X'$. Then we have a fibre product diagram

$$\begin{tikzcd} \prod_{i=1,\ldots,n} k'_i & \prod_{i=1,\ldots,n} \mathcal{O}^\wedge_{X',x'_i} \arrow[l] \\ k' \arrow[u] & \mathcal{O}^\wedge_{X,x} \arrow[l] \arrow[u] \end{tikzcd}$$

where the horizontal arrows are given by the maps to the residue fields.

Proof. Choose an affine open neighbourhood $\mathrm{Spec}(A)$ of x in X. Let $\mathrm{Spec}(A') \subset X'$ be the inverse image. By construction we have a fibre product diagram

$$\begin{tikzcd} \prod_{i=1,\ldots,n} k'_i & A' \arrow[l] \\ k' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Since everything is finite over A we see that the diagram remains a fibre product diagram after completion with respect to the maximal ideal $\mathfrak{m} \subset A$ corresponding to x (Algebra, Lemma 10.97.2). Finally, apply Algebra, Lemma 10.97.8 to identify the completion of A'. $\square$

53.15. Glueing and squishing

0C1H Below we will indicate $k[\epsilon]$ the algebra of dual numbers over k as defined in Varieties, Definition 33.16.1.

0C1I **Lemma 53.15.1.** *Let k be an algebraically closed field. Let $k \subset A$ be a ring extension such that A has exactly two k-sub algebras, then either $A = k \times k$ or $A = k[\epsilon]$.*

Proof. The assumption means $k \neq A$ and any subring $k \subset C \subset A$ is equal to either k or A. Let $t \in A$, $t \notin k$. Then A is generated by t over k. Hence $A = k[x]/I$ for some ideal I. If $I = (0)$, then we have the subalgebra $k[x^2]$ which is not allowed. Otherwise I is generated by a monic polynomial P. Write $P = \prod_{i=1}^{d}(t - a_i)$. If $d > 2$, then the subalgebra generated by $(t-a_1)(t-a_2)$ gives a contradiction. Thus $d = 2$. If $a_1 \neq a_2$, then $A = k \times k$, if $a_1 = a_2$, then $A = k[\epsilon]$. $\square$

0C1J **Example 53.15.2** (Glueing points)**.** Let k be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic k-schemes. We say X is obtained by glueing a and b in X' if the following are true:

(1) $a, b \in X'(k)$ are distinct points which map to the same point $x \in X(k)$,
(2) f is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{a-b} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}$ to the difference $h(a) - h(b) \in k$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}^*$ to the multiplicative difference $h(a)h(b)^{-1} \in k^*$.

0C1K **Example 53.15.3** (Squishing a tangent vector)**.** Let k be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic k-schemes. We say X is obtained by squishing the tangent vector ϑ in X' if the following are true:

(1) $\vartheta : \mathrm{Spec}(k[\epsilon]) \to X'$ is a closed immersion over k such that $f \circ \vartheta$ factors through a point $x \in X(k)$,
(2) f is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}$ to the coefficient of ϵ in $\vartheta^\sharp(h) \in k[\epsilon]$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}^*$ to $\mathrm{d}\log(\vartheta^\sharp(h))$ where $\mathrm{d}\log : k[\epsilon]^* \to k$ is the homomorphism of abelian groups sending $a + b\epsilon$ to $b/a \in k$.

0C1L Lemma 53.15.4. Let k be an algebraically closed field. Let $f : X' \to X$ be a finite morphism algebraic k-schemes such that $\mathcal{O}_X \subset f_*\mathcal{O}_{X'}$ and such that f is an isomorphism away from a finite set of points. Then there is a factorization

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

such that each $X_i \to X_{i-1}$ is either the glueing of two points or the squishing of a tangent vector (see Examples 53.15.2 and 53.15.3).

Proof. Let $U \subset X$ be the maximal open set over which f is an isomorphism. Then $X \setminus U = \{x_1, \ldots, x_n\}$ with $x_i \in X(k)$. We will consider factorizations $X' \to Y \to X$ of f such that both morphisms are finite and

$$\mathcal{O}_X \subset g_*\mathcal{O}_Y \subset f_*\mathcal{O}_{X'}$$

where $g : Y \to X$ is the given morphism. By assumption $\mathcal{O}_{X,x} \to (f_*\mathcal{O}_{X'})_x$ is an isomorphism onless $x = x_i$ for some i. Hence the cokernel

$$f_*\mathcal{O}_{X'}/\mathcal{O}_X = \bigoplus \mathcal{Q}_i$$

is a direct sum of skyscraper sheaves $\mathcal{Q}_i$ supported at $x_1, \ldots, x_n$. Because the displayed quotient is a coherent $\mathcal{O}_X$-module, we conclude that $\mathcal{Q}_i$ has finite length over $\mathcal{O}_{X,x_i}$. Hence we can argue by induction on the sum of these lengths, i.e., the length of the whole cokernel.

If $n > 1$, then we can define an $\mathcal{O}_X$-subalgebra $\mathcal{A} \subset f_*\mathcal{O}_{X'}$ by taking the inverse image of $\mathcal{Q}_1$. This will give a nontrivial factorization and we win by induction.

Assume $n = 1$. We abbreviate $x = x_1$. Consider the finite k-algebra extension

$$A = \mathcal{O}_{X,x} \subset (f_*\mathcal{O}_{X'})_x = B$$

Note that $\mathcal{Q} = \mathcal{Q}_1$ is the skyscraper sheaf with value B/A. We have a k-subalgebra $A \subset A + \mathfrak{m}_A B \subset B$. If both inclusions are strict, then we obtain a nontrivial factorization and we win by induction as above. If $A + \mathfrak{m}_A B = B$, then $A = B$ by Nakayama, then f is an isomorphism and there is nothing to prove. We conclude that we may assume $B = A + \mathfrak{m}_A B$. Set $C = B/\mathfrak{m}_A B$. If C has more than 2 k-subalgebras, then we obtain a subalgebra between A and B by taking the inverse image in B. Thus we may assume C has exactly 2 k-subalgebras. Thus $C = k \times k$ or $C = k[\epsilon]$ by Lemma 53.15.1. In this case f is correspondingly the glueing two points or the squishing of a tangent vector. □

0C1M Lemma 53.15.5. Let k be an algebraically closed field. If $f : X' \to X$ is the glueing of two points a, b as in Example 53.15.2, then there is an exact sequence

$$k^* \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

The first map is zero if a and b are on different connected components of X' and injective if X' is proper and a and b are on the same connected component of X'.

Proof. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 33.38.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{ab^{-1}} k^* \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}_{X'}^*) \subset H^1(X', \mathcal{O}_{X'}^*)$ (for example by the Leray spectral sequence, see Cohomology, Lemma 20.13.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of $ab^{-1} : H^0(X', \mathcal{O}_{X'}^*) \to k^*$. If a and b are on different connected components of X', then ab^{-1} is surjective. Because k is algebraically closed any regular function on a reduced connected proper scheme over k comes from an element of k, see Varieties, Lemma 33.9.3. Thus ab^{-1} is zero if X' is proper and a and b are on the same connected component. □

0C1N Lemma 53.15.6. Let k be an algebraically closed field. If $f : X' \to X$ is the squishing of a tangent vector ϑ as in Example 53.15.3, then there is an exact sequence

$$(k, +) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

and the first map is injective if X' is proper and reduced.

Proof. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 33.38.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

of Example 53.15.3 we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{\vartheta} k \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}_{X'}^*) \subset H^1(X', \mathcal{O}_{X'}^*)$ (for example by the Leray spectral sequence, see Cohomology, Lemma 20.13.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of the map $\vartheta : H^0(X', \mathcal{O}_{X'}^*) \to k$. Because k is algebraically closed any regular function on a reduced connected proper scheme over k comes from an element of k, see Varieties, Lemma 33.9.3. Thus the final statement of the lemma. □

53.16. Multicross and nodal singularities

0C1P In this section we discuss the simplest possible curve singularities.

Let k be a field. Consider the complete local k-algebra

0C1U (53.16.0.1) $$A = \{(f_1, \ldots, f_n) \in k[[t]] \times \ldots \times k[[t]] \mid f_1(0) = \ldots = f_n(0)\}$$

In the language introduced in Varieties, Definition 33.40.4 we see that A is a wedge of n copies of the power series ring in 1 variable over k. Observe that $k[[t]]\times\ldots\times k[[t]]$ is the integral closure of A in its total ring of fractions. Hence the δ-invariant of A is $n-1$. There is an isomorphism

$$k[[x_1, \ldots, x_n]]/(\{x_ix_j\}_{i\neq j}) \longrightarrow A$$

obtained by sending x_i to $(0, \ldots, 0, t, 0, \ldots, 0)$ in A. It follows that $\dim(A) = 1$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. In particular, A is regular if and only if $n = 1$.

0C1V Lemma 53.16.1. Let k be a separably closed field. Let A be a 1-dimensional reduced Nagata local k-algebra with residue field k. Then

$$\delta\text{-invariant } A \geq \text{number of branches of } A - 1$$

If equality holds, then $A^\wedge$ is as in (53.16.0.1).

Proof. Since the residue field of A is separably closed, the number of branches of A is equal to the number of geometric branches of A, see More on Algebra, Definition 15.106.6. The inequality holds by Varieties, Lemma 33.40.6. Assume equality holds. We may replace A by the completion of A; this does not change the number of branches or the δ-invariant, see More on Algebra, Lemma 15.108.7 and Varieties, Lemma 33.39.6. Then A is strictly henselian, see Algebra, Lemma 10.153.9. By Varieties, Lemma 33.40.5 we see that A is a wedge of complete discrete valuation rings. Each of these is isomorphic to $k[[t]]$ by Algebra, Lemma 10.160.10. Hence A is as in (53.16.0.1). □

0C1W Definition 53.16.2. Let k be an algebraically closed field. Let X be an algebraic 1-dimensional k-scheme. Let $x \in X$ be a closed point. We say x defines a multicross singularity if the completion $\mathcal{O}_{X,x}^\wedge$ is isomorphic to (53.16.0.1) for some $n \geq 2$. We say x is a node, or an ordinary double point, or defines a nodal singularity if $n = 2$.

These singularities are in some sense the simplest kind of singularities one can have on a curve over an algebraically closed field.

0C1X Lemma 53.16.3. Let k be an algebraically closed field. Let X be a reduced algebraic 1-dimensional k-scheme. Let $x \in X$. The following are equivalent

(1) x defines a multicross singularity,
(2) the δ-invariant of X at x is the number of branches of X at x minus 1,
(3) there is a sequence of morphisms $U_n \to U_{n-1} \to \ldots \to U_0 = U \subset X$ where U is an open neighbourhood of x, where U_n is nonsingular, and where each $U_i \to U_{i-1}$ is the glueing of two points as in Example 53.15.2.

Proof. The equivalence of (1) and (2) is Lemma 53.16.1.

Assume (3). We will argue by descending induction on i that all singularities of U_i are multicross. This is true for U_n as U_n has no singular points. If U_i is gotten from U_{i+1} by glueing $a, b \in U_{i+1}$ to a point $c \in U_i$, then we see that

$$\mathcal{O}_{U_i,c}^\wedge \subset \mathcal{O}_{U_{i+1},a}^\wedge \times \mathcal{O}_{U_{i+1},b}^\wedge$$

is the set of elements having the same residue classes in k. Thus the number of branches at c is the sum of the number of branches at a and b, and the δ-invariant at c is the sum of the δ-invariants at a and b plus 1 (because the displayed inclusion has codimension 1). This proves that (2) holds as desired.

Assume the equivalent conditions (1) and (2). We may choose an open $U \subset X$ such that x is the only singular point of U. Then we apply Lemma 53.15.4 to the normalization morphism

$$U^\nu = U_n \to U_{n-1} \to \ldots \to U_1 \to U_0 = U$$

All we have to do is show that in none of the steps we are squishing a tangent vector. Suppose $U_{i+1} \to U_i$ is the smallest i such that this is the squishing of a tangent vector θ at $u' \in U_{i+1}$ lying over $u \in U_i$. Arguing as above, we see that u_i is a multicross singularity (because the maps $U_i \to \ldots \to U_0$ are glueing of pairs of points). But now the number of branches at u' and u is the same and the δ-invariant of U_i at u is 1 bigger than the δ-invariant of U_{i+1} at u'. By Lemma 53.16.1 this implies that u cannot be a multicross singularity which is a contradiction. □

0CDZ Lemma 53.16.4. Let k be an algebraically closed field. Let X be a reduced algebraic 1-dimensional k-scheme. Let $x \in X$ be a multicross singularity (Definition 53.16.2). If X is Gorenstein, then x is a node.

Proof. The map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^\wedge$ is flat and unramified in the sense that $\kappa(x) = \mathcal{O}_{X,x}^\wedge/\mathfrak{m}_x\mathcal{O}_{X,x}^\wedge$. (See More on Algebra, Section 15.43.) Thus X is Gorenstein implies $\mathcal{O}_{X,x}$ is Gorenstein, implies $\mathcal{O}_{X,x}^\wedge$ is Gorenstein by Dualizing Complexes, Lemma 47.21.8. Thus it suffices to show that the ring A in (53.16.0.1) with $n \geq 2$ is Gorenstein if and only if $n = 2$.

If $n = 2$, then $A = k[[x, y]]/(xy)$ is a complete intersection and hence Gorenstein. For example this follows from Duality for Schemes, Lemma 48.24.5 applied to $k[[x, y]] \to A$ and the fact that the regular local ring $k[[x, y]]$ is Gorenstein by Dualizing Complexes, Lemma 47.21.3.

Assume $n > 2$. If A where Gorenstein, then A would be a dualizing complex over A (Duality for Schemes, Definition 48.24.1). Then $R\operatorname{Hom}(k, A)$ would be equal to $k[n]$ for some $n \in \mathbf{Z}$, see Dualizing Complexes, Lemma 47.15.12. It would follow that $\operatorname{Ext}^1_A(k, A) \cong k$ or $\operatorname{Ext}^1_A(k, A) = 0$ (depending on the value of n; in fact n has to be -1 but it doesn't matter to us here). Using the exact sequence

$$0 \to \mathfrak{m}_A \to A \to k \to 0$$

we find that

$$\operatorname{Ext}^1_A(k, A) = \operatorname{Hom}_A(\mathfrak{m}_A, A)/A$$

where $A \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$ is given by $a \mapsto (a' \mapsto aa')$. Let $e_i \in \operatorname{Hom}_A(\mathfrak{m}_A, A)$ be the element that sends $(f_1, \ldots, f_n) \in \mathfrak{m}_A$ to $(0, \ldots, 0, f_i, 0, \ldots, 0)$. The reader verifies easily that $e_1, \ldots, e_{n-1}$ are k-linearly independent in $\operatorname{Hom}_A(\mathfrak{m}_A, A)/A$. Thus $\dim_k \operatorname{Ext}^1_A(k, A) \geq n - 1 \geq 2$ which finishes the proof. (Observe that $e_1 + \ldots + e_n$ is the image of 1 under the map $A \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$.) □

53.17. Torsion in the Picard group

0C1Y In this section we bound the torsion in the Picard group of a 1-dimensional proper scheme over a field. We will use this in our study of semistable reduction for curves.

There does not seem to be an elementary way to obtain the result of Lemma 53.17.1. Analyzing the proof there are two key ingredients: (1) there is an abelian variety classifying degree zero invertible sheaves on a smooth projective curve and (2) the structure of torsion points on an abelian variety can be determined.

0C1Z Lemma 53.17.1. Let k be an algebraically closed field. Let X be a smooth projective curve of genus g over k.

(1) If $n \geq 1$ is invertible in k, then $\operatorname{Pic}(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.
(2) If the characteristic of k is $p > 0$, then there exists an integer $0 \leq f \leq g$ such that $\operatorname{Pic}(X)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$ for all $m \geq 1$.

Proof. Let $\operatorname{Pic}^0(X) \subset \operatorname{Pic}(X)$ denote the subgroup of invertible sheaves of degree 0. In other words, there is a short exact sequence

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The group $\operatorname{Pic}^0(X)$ is the k-points of the group scheme $\underline{\operatorname{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 44.6.7. The same lemma tells us that $\underline{\operatorname{Pic}}^0_{X/k}$ is a g-dimensional abelian variety over k as defined in Groupoids, Definition 39.9.1. Thus we conclude by the results of Groupoids, Proposition 39.9.11. □

0CDU Lemma 53.17.2. Let k be a field. Let n be prime to the characteristic of k. Let X be a smooth proper curve over k with $H^0(X, \mathcal{O}_X) = k$ and of genus g.

(1) If $g = 1$ then there exists a finite separable extension k'/k such that $X_{k'}$ has a k'-rational point and $\text{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$.
(2) If $g \geq 2$ then there exists a finite separable extension k'/k with $[k' : k] \leq (2g-2)(n^{2g})!$ such that $X_{k'}$ has a k'-rational point and $\text{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.

Proof. Assume $g \geq 2$. First we may choose a finite separable extension of degree at most $2g - 2$ such that X acquires a rational point, see Lemma 53.13.9. Thus we may assume X has a k-rational point $x \in X(k)$ but now we have to prove the lemma with $[k' : k] \leq (n^{2g})!$. Let $k \subset k^{sep} \subset \overline{k}$ be a separable algebraic closure inside an algebraic closure. By Lemma 53.17.1 we have

$$\text{Pic}(X_{\overline{k}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 44.7.2 we conclude that

$$\text{Pic}(X_{k^{sep}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 44.7.2 there is a continuous action

$$\text{Gal}(k^{sep}/k) \longrightarrow \text{Aut}(\text{Pic}(X_{k^{sep}})[n]$$

and the lemma is true for the fixed field k' of the kernel of this map. The kernel is open because the action is continuous which implies that k'/k is finite. By Galois theory $\text{Gal}(k'/k)$ is the image of the displayed arrow. Since the permutation group of a set of cardinality n^{2g} has cardinality $(n^{2g})!$ we conclude by Galois theory that $[k' : k] \leq (n^{2g})!$. (Of course this proves the lemma with the bound $|\text{GL}_{2g}(\mathbf{Z}/n\mathbf{Z})|$, but all we want here is that there is some bound.)

If the genus is 1, then there is no upper bound on the degree of a finite separable field extension over which X acquires a rational point (details omitted). Still, there is such an extension for example by Varieties, Lemma 33.25.6. The rest of the proof is the same as in the case of $g \geq 2$. □

0C20 Proposition 53.17.3. Let k be an algebraically closed field. Let X be a proper scheme over k which is reduced, connected, and has dimension 1. Let g be the genus of X and let g_{geom} be the sum of the geometric genera of the irreducible components of X. For any prime ℓ different from the characteristic of k we have

$$\dim_{\mathbf{F}_\ell} \text{Pic}(X)[\ell] \leq g + g_{geom}$$

and equality holds if and only if all the singularities of X are multicross.

Proof. Let $\nu : X^\nu \to X$ be the normalization (Varieties, Lemma 33.41.2). Choose a factorization

$$X^\nu = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

as in Lemma 53.15.4. Let us denote $h_i^0 = \dim_k H^0(X_i, \mathcal{O}_{X_i})$ and $h_i^1 = \dim_k H^1(X_i, \mathcal{O}_{X_i})$. By Lemmas 53.15.5 and 53.15.6 for each $n > i \geq 0$ we have one of the following there possibilities

(1) X_i is obtained by glueing $a, b \in X_{i+1}$ which are on different connected components: in this case $\text{Pic}(X_i) = \text{Pic}(X_{i+1})$, $h_{i+1}^0 = h_i^0 + 1$, $h_{i+1}^1 = h_i^1$,

(2) X_i is obtained by glueing $a, b \in X_{i+1}$ which are on the same connected component: in this case there is a short exact sequence
$$0 \to k^* \to \mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{i+1}) \to 0,$$
and $h^0_{i+1} = h^0_i$, $h^1_{i+1} = h^1_i - 1$,

(3) X_i is obtained by squishing a tangent vector in X_{i+1}: in this case there is a short exact sequence
$$0 \to (k, +) \to \mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{i+1}) \to 0,$$
and $h^0_{i+1} = h^0_i$, $h^1_{i+1} = h^1_i - 1$.

To prove the statements on dimensions of cohomology groups of the structure sheaf, use the exact sequences in Examples 53.15.2 and 53.15.3. Since k is algebraically closed of characteristic prime to ℓ we see that $(k, +)$ and k^* are ℓ-divisible and with ℓ-torsion $(k, +)[\ell] = 0$ and $k^*[\ell] \cong \mathbf{F}_\ell$. Hence
$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_{i+1})[\ell] - \dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_i)[\ell]$$
is zero, except in case (2) where it is equal to -1. At the end of this process we get the normalization $X^\nu = X_n$ which is a disjoint union of smooth projective curves over k. Hence we have

(1) $h^1_n = g_{geom}$ and
(2) $\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_n)[\ell] = 2g_{geom}$.

The last equality by Lemma 53.17.1. Since $g = h^1_0$ we see that the number of steps of type (2) and (3) is at most $h^1_0 - h^1_n = g - g_{geom}$. By our comptation of the differences in ranks we conclude that
$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X)[\ell] \leq g - g_{geom} + 2g_{geom} = g + g_{geom}$$
and equality holds if and only if no steps of type (3) occur. This indeed means that all singularities of X are multicross by Lemma 53.16.3. Conversely, if all the singularities are multicross, then Lemma 53.16.3 guarantees that we can find a sequence $X^\nu = X_n \to \ldots \to X_0 = X$ as above such that no steps of type (3) occur in the sequence and we find equality holds in the lemma (just glue the local sequences for each point to find one that works for all singular points of x; some details omitted). □

53.18. Genus versus geometric genus

0CE0 Let k be a field with algebraic closure $\overline{k}$. Let X be a proper scheme of dimension ≤ 1 over k. We define $g_{geom}(X/k)$ to be the sum of the geometric genera of the irreducible components of $X_{\overline{k}}$ which have dimension 1.

0CE1 Lemma 53.18.1. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Then
$$g_{geom}(X/k) = \sum\nolimits_{C \subset X} g_{geom}(C/k)$$
where the sum is over irreducible components $C \subset X$ of dimension 1.

Proof. This is immediate from the definition and the fact that an irreducible component $\overline{Z}$ of $X_{\overline{k}}$ maps onto an irreducible component Z of X (Varieties, Lemma 33.8.10) of the same dimension (Morphisms, Lemma 29.28.3 applied to the generic point of $\overline{Z}$). □

0CE2 Lemma 53.18.2. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Then

(1) We have $g_{geom}(X/k) = g_{geom}(X_{red}/k)$.
(2) If $X' \to X$ is a birational proper morphism, then $g_{geom}(X'/k) = g_{geom}(X/k)$.
(3) If $X^\nu \to X$ is the normalization morphism, then $g_{geom}(X^\nu/k) = g_{geom}(X/k)$.

Proof. Part (1) is immediate from Lemma 53.18.1. If $X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 33.17.2 and 33.17.3). Hence $X'_{\overline{k}} \to X_{\overline{k}}$ is an isomorphism over a dense open. Thus the irreducible components of $X'_{\overline{k}}$ and $X_{\overline{k}}$ are in bijective correspondence and the corresponding components have isomorphic function fields. In particular these components have isomorphic nonsingular projective models and hence have the same geometric genera. This proves (2). Part (3) follows from (1) and (2) and the fact that $X^\nu \to X_{red}$ is birational (Morphisms, Lemma 29.54.7). □

0CE3 Lemma 53.18.3. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $f : Y \to X$ be a finite morphism such that there exists a dense open $U \subset X$ over which f is a closed immersion. Then

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(Y, \mathcal{O}_Y)$$

Proof. Consider the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to \mathcal{F} \to 0$$

of coherent sheaves on X. By assumption $\mathcal{F}$ is supported in finitely many closed points and hence has vanishing higher cohomology (Varieties, Lemma 33.33.3). On the other hand, we have $H^2(X, \mathcal{G}) = 0$ by Cohomology, Proposition 20.20.7. It follows formally that the induced map $H^1(X, \mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_Y)$ is surjective. Since $H^1(X, f_*\mathcal{O}_Y) = H^1(Y, \mathcal{O}_Y)$ (Cohomology of Schemes, Lemma 30.2.4) we conclude the lemma holds. □

0CE4 Lemma 53.18.4. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. If $X' \to X$ is a birational proper morphism, then

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(X', \mathcal{O}_{X'})$$

If X is reduced, $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ is surjective, and equality holds, then $X' = X$.

Proof. If $f : X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 33.17.2 and 33.17.3). Thus the inequality by Lemma 53.18.3. Assume X is reduced. Then $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective and we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{F} \to 0$$

Under the assumptions given in the second statement, we conclude from the long exact cohomology sequence that $H^0(X, \mathcal{F}) = 0$. Then $\mathcal{F} = 0$ because $\mathcal{F}$ is generated by global sections (Varieties, Lemma 33.33.3). and $\mathcal{O}_X = f_*\mathcal{O}_{X'}$. Since f is affine this implies $X = X'$. □

0CE5 Lemma 53.18.5. Let k be a field. Let C be a proper curve over k. Set $\kappa = H^0(C, \mathcal{O}_C)$. Then

$$[\kappa : k]_s \dim_\kappa H^1(C, \mathcal{O}_C) \geq g_{geom}(C/k)$$

Proof. Varieties, Lemma 33.26.2 implies κ is a field and a finite extension of k. By Fields, Lemma 9.14.8 we have $[\kappa : k]_s = |\operatorname{Mor}_k(\kappa, \overline{k})|$ and hence $\operatorname{Spec}(\kappa \otimes_k \overline{k})$ has $[\kappa : k]_s$ points each with residue field $\overline{k}$. Thus

$$C_{\overline{k}} = \bigcup\nolimits_{t \in \operatorname{Spec}(\kappa \otimes_k \overline{k})} C_t$$

(set theoretic union). Here $C_t = C \times_{\operatorname{Spec}(\kappa), t} \operatorname{Spec}(\overline{k})$ where we view t as a k-algebra map $t : \kappa \to \overline{k}$. The conclusion is that $g_{geom}(C/k) = \sum_t g_{geom}(C_t/\overline{k})$ and the sum is over an index set of size $[\kappa : k]_s$. We have

$$H^0(C_t, \mathcal{O}_{C_t}) = \overline{k} \quad \text{and} \quad \dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) = \dim_\kappa H^1(C, \mathcal{O}_C)$$

by cohomology and base change (Cohomology of Schemes, Lemma 30.5.2). Observe that the normalization C_t^ν is the disjoint union of the nonsingular projective models of the irreducible components of C_t (Morphisms, Lemma 29.54.6). Hence $\dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$ is equal to $g_{geom}(C_t/\overline{k})$. By Lemma 53.18.3 we have

$$\dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) \geq \dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$$

and this finishes the proof. □

0CE6 Lemma 53.18.6. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let ℓ be a prime number invertible in k. Then

$$\dim_{\mathbf{F}_\ell} \operatorname{Pic}(X)[\ell] \leq \dim_k H^1(X, \mathcal{O}_X) + g_{geom}(X/k)$$

where $g_{geom}(X/k)$ is as defined above.

Proof. The map $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\overline{k}})$ is injective by Varieties, Lemma 33.30.3. By Cohomology of Schemes, Lemma 30.5.2 $\dim_k H^1(X, \mathcal{O}_X)$ equals $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$. Hence we may assume k is algebraically closed.

Let X_{red} be the reduction of X. Then the surjection $\mathcal{O}_X \to \mathcal{O}_{X_{red}}$ induces a surjection $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_{X_{red}})$ because cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology, Proposition 20.20.7. Since $X_{red} \to X$ induces an isomorphism on irreducible components over $\overline{k}$ and an isomorphism on ℓ-torsion in Picard groups (Picard Schemes of Curves, Lemma 44.7.2) we may replace X by X_{red}. In this way we reduce to Proposition 53.17.3. □

53.19. Nodal curves

0C46 We have already defined ordinary double points over algebraically closed fields, see Definition 53.16.2. Namely, if $x \in X$ is a closed point of a 1-dimensional algebraic scheme over an algebraically closed field k, then x is an ordinary double point if and only if

$$\mathcal{O}_{X,x}^\wedge \cong k[[x, y]]/(xy)$$

See discussion following (53.16.0.1) in Section 53.16.

0C47 Definition 53.19.1. Let k be a field. Let X be a 1-dimensional locally algebraic k-scheme.

(1) We say a closed point $x \in X$ is a node, or an ordinary double point, or defines a nodal singularity if there exists an ordinary double point $\overline{x} \in X_{\overline{k}}$ mapping to x.

(2) We say the singularities of X are at-worst-nodal if all closed points of X are either in the smooth locus of the structure morphism $X \to \mathrm{Spec}(k)$ or are ordinary double points.

Often a 1-dimensional algebraic scheme X is called a nodal curve if the singularities of X are at worst nodal. Sometimes a nodal curve is required to be proper. Since a nodal curve so defined need not be irreducible, this conflicts with our earlier definition of a curve as a variety of dimension 1.

0C48 Lemma 53.19.2. Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $I \subset \mathfrak{m}$ be an ideal.

(1) If A/I is reduced, then $I = (0)$, $I = \mathfrak{m}$, or $I = (f)$ for some nonzero $f \in \mathfrak{m}$.
(2) If A/I has depth 1, then $I = (f)$ for some nonzero $f \in \mathfrak{m}$.

Proof. Assume $I \neq 0$. Write $I = (f_1, \ldots, f_r)$. As A is a UFD (More on Algebra, Lemma 15.121.2) we can write $f_i = fg_i$ where f is the gcd of $f_1, \ldots, f_r$. Thus the gcd of $g_1, \ldots, g_r$ is 1 which means that there is no height 1 prime ideal over $g_1, \ldots, g_r$. Then either $(g_1, \ldots, g_r) = A$ which implies $I = (f)$ or if not, then $\dim(A) = 2$ implies that $V(g_1, \ldots, g_r) = \{\mathfrak{m}\}$, i.e., $\mathfrak{m} = \sqrt{(g_1, \ldots, g_r)}$.

Assume A/I reduced, i.e., I radical. If f is a unit, then since I is radical we see that $I = \mathfrak{m}$. If $f \in \mathfrak{m}$, then we see that f^n maps to zero in A/I. Hence $f \in I$ by reducedness and we conclude $I = (f)$.

Assume A/I has depth 1. Then $\mathfrak{m}$ is not an associated prime of A/I. Since the class of f modulo I is annihilated by $g_1, \ldots, g_r$, this implies that the class of f is zero in A/I. Thus $I = (f)$ as desired. $\square$

Let κ be a field and let V be a vector space over κ. We will say $q \in \mathrm{Sym}^2_\kappa(V)$ is nondegenerate if the induced κ-linear map $V^\vee \to V$ is an isomorphism. If $q = \sum_{i \leq j} a_{ij}x_ix_j$ for some κ-basis $x_1, \ldots, x_n$ of V, then this means that the determinant of the matrix

$$\begin{pmatrix} 2a_{11} & a_{12} & \ldots \\ a_{12} & 2a_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

is nonzero. This is equivalent to the condition that the partial derivatives of q with respect to the x_i cut out 0 scheme theoretically.

0C49 Lemma 53.19.3. Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local k-algebra. The following are equivalent

(1) κ/k is separable, A is reduced, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$, and there exists a nondegenerate $q \in \mathrm{Sym}^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ which maps to zero in $\mathfrak{m}^2/\mathfrak{m}^3$,
(2) κ/k is separable, $\mathrm{depth}(A) = 1$, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$, and there exists a nondegenerate $q \in \mathrm{Sym}^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ which maps to zero in $\mathfrak{m}^2/\mathfrak{m}^3$,
(3) κ/k is separable, $A^\wedge \cong \kappa[[x, y]]/(ax^2 + bxy + cy^2)$ as a k-algebra where $ax^2 + bxy + cy^2$ is a nondegenerate quadratic form over κ.

Proof. Assume (3). Then $A^\wedge$ is reduced because $ax^2+bxy+cy^2$ is either irreducible or a product of two nonassociated prime elements. Hence $A \subset A^\wedge$ is reduced. It follows that (1) is true.

Assume (1). Then A cannot be Artinian, since it would not be reduced because $\mathfrak{m} \neq (0)$. Hence $\dim(A) \geq 1$, hence $\text{depth}(A) \geq 1$ by Algebra, Lemma 10.157.3. On the other hand $\dim(A) = 2$ implies A is regular which contradicts the existence of q by Algebra, Lemma 10.106.1. Thus $\dim(A) \leq 1$ and we conclude $\text{depth}(A) = 1$ by Algebra, Lemma 10.72.3. It follows that (2) is true.

Assume (2). Since the depth of A is the same as the depth of $A^\wedge$ (More on Algebra, Lemma 15.43.2) and since the other conditions are insensitive to completion, we may assume that A is complete. Choose $\kappa \to A$ as in More on Algebra, Lemma 15.38.3. Since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$ we can choose $x_0, y_0 \in \mathfrak{m}$ which map to a basis. We obtain a continuous κ-algebra map

$$\kappa[[x, y]] \longrightarrow A$$

by the rules $x \mapsto x_0$ and $y \mapsto y_0$. Let q be the class of $ax_0^2 + bx_0y_0 + cy_0^2$ in $\text{Sym}^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Write $Q(x, y) = ax^2 + bxy + cy^2$ viewed as a polynomial in two variables. Then we see that

$$Q(x_0, y_0) = ax_0^2 + bx_0y_0 + cy_0^2 = \sum\nolimits_{i+j=3} a_{ij}x_0^iy_0^j$$

for some a_{ij} in A. We want to prove that we can increase the order of vanishing by changing our choice of x_0, y_0. Suppose that $x_1, y_1 \in \mathfrak{m}^2$. Then

$$Q(x_0 + x_1, y_0 + y_1) = Q(x_0, y_0) + (2ax_0 + by_0)x_1 + (bx_0 + 2cy_0)y_1 \bmod \mathfrak{m}^4$$

Nondegeneracy of Q means exactly that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are a κ-basis for $\mathfrak{m}/\mathfrak{m}^2$, see discussion preceding the lemma. Hence we can certainly choose $x_1, y_1 \in \mathfrak{m}^2$ such that $Q(x_0 + x_1, y_0 + y_1) \in \mathfrak{m}^4$. Continuing in this fashion by induction we can find $x_i, y_i \in \mathfrak{m}^{i+1}$ such that

$$Q(x_0 + x_1 + \ldots + x_n, y_0 + y_1 + \ldots + y_n) \in \mathfrak{m}^{n+3}$$

Since A is complete we can set $x_\infty = \sum x_i$ and $y_\infty = \sum y_i$ and we can consider the map $\kappa[[x, y]] \longrightarrow A$ sending x to x_∞ and y to y_∞. This map induces a surjection $\kappa[[x, y]]/(Q) \longrightarrow A$ by Algebra, Lemma 10.96.1. By Lemma 53.19.2 the kernel of $k[[x, y]] \to A$ is principal. But the kernel cannot contain a proper divisor of Q as such a divisor would have degree 1 in x, y and this would contradict $\dim(\mathfrak{m}/\mathfrak{m}^2) = 2$. Hence Q generates the kernel as desired. □

0C4A Lemma 53.19.4. Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Nagata local k-algebra. The following are equivalent

(1) $k \to A$ is as in Lemma 53.19.3,
(2) κ/k is separable, A is reduced of dimension 1, the δ-invariant of A is 1, and A has 2 geometric branches.

If this holds, then the integral closure A' of A in its total ring of fractions has either 1 or 2 maximal ideals $\mathfrak{m}'$ and the extensions $\kappa(\mathfrak{m}')/k$ are separable.

Proof. In both cases A and $A^\wedge$ are reduced. In case (2) because the completion of a reduced local Nagata ring is reduced (More on Algebra, Lemma 15.43.6). In both cases A and $A^\wedge$ have dimension 1 (More on Algebra, Lemma 15.43.1). The δ-invariant and the number of geometric branches of A and $A^\wedge$ agree by Varieties, Lemma 33.39.6 and More on Algebra, Lemma 15.108.7. Let A' be the integral closure of A in its total ring of fractions as in Varieties, Lemma 33.39.2. By Varieties, Lemma 33.39.5 we see that $A' \otimes_A A^\wedge$ plays the same role for $A^\wedge$. Thus we may replace A by $A^\wedge$ and assume A is complete.

Assume (1) holds. It suffices to show that A has two geometric branches and δ-invariant 1. We may assume $A = \kappa[[x, y]]/(ax^2+bxy+cy^2)$ with $q = ax^2+bxy+cy^2$ nondegenerate. There are two cases.

Case I: q splits over κ. In this case we may after changing coordinates assume that $q = xy$. Then we see that

$$A' = \kappa[[x, y]]/(x) \times \kappa[[x, y]]/(y)$$

Case II: q does not split. In this case $c \neq 0$ and nondegenerate means $b^2 - 4ac \neq 0$. Hence $\kappa' = \kappa[t]/(a + bt + ct^2)$ is a degree 2 separable extension of κ. Then $t = y/x$ is integral over A and we conclude that

$$A' = \kappa'[[x]]$$

with y mapping to tx on the right hand side.

In both cases one verifies by hand that the δ-invariant is 1 and the number of geometric branches is 2. In this way we see that (1) implies (2). Moreover we conclude that the final statement of the lemma holds.

Assume (2) holds. More on Algebra, Lemma 15.106.7 implies A' either has two maximal ideals or A' has one maximal ideal and $[\kappa(\mathfrak{m}') : \kappa]_s = 2$.

Case I: A' has two maximal ideals $\mathfrak{m}'_1$, $\mathfrak{m}'_2$ with residue fields κ_1, κ_2. Since the δ-invariant is the length of A'/A and since there is a surjection $A'/A \to (\kappa_1 \times \kappa_2)/\kappa$ we see that $\kappa = \kappa_1 = \kappa_2$. Since A is complete (and henselian by Algebra, Lemma 10.153.9) and A' is finite over A we see that $A' = A_1 \times A_2$ (by Algebra, Lemma 10.153.4). Since A' is a normal ring it follows that A_1 and A_2 are discrete valuation rings. Hence A_1 and A_2 are isomorphic to $\kappa[[t]]$ (as k-algebras) by More on Algebra, Lemma 15.38.4. Since the δ-invariant is 1 we conclude that A is the wedge of A_1 and A_2 (Varieties, Definition 33.40.4). It follows easily that $A \cong \kappa[[x, y]]/(xy)$.

Case II: A' has a single maximal ideal $\mathfrak{m}'$ with residue field κ' and $[\kappa' : \kappa]_s = 2$. Arguing exactly as in Case I we see that $[\kappa' : \kappa] = 2$ and κ' is separable over κ. Since A' is normal we see that A' is isomorphic to $\kappa'[[t]]$ (see reference above). Since A'/A has length 1 we conclude that

$$A = \{f \in \kappa'[[t]] \mid f(0) \in \kappa\}$$

Then a simple computation shows that A as in case (1). $\square$

0C4B Lemma 53.19.5. Let k be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset A$ be an ideal. For any $r \geq 0$ the ideal in A/I generated by the $r \times r$-minors of the matrix $(\partial f_j/\partial x_i)$ is independent of the choice of the generators of I or the regular system of parameters $x_1, \ldots, x_n$ of A.

Proof. The "correct" proof of this lemma is to prove that this ideal is the $(n-r)$th Fitting ideal of a module of continuous differentials of A/I over k. Here is a direct proof. If $g_1, \ldots g_l$ is a second set of generators of I, then we can write $g_s = \sum a_{sj} f_j$ and we have the equality of matrices

$$(\partial g_s/\partial x_i) = (a_{sj})(\partial f_j/\partial x_i) + (\partial a_{sj}/\partial x_i f_j)$$

The final term is zero in A/I. By the Cauchy-Binet formula we see that the ideal of minors for the g_s is contained in the ideal for the f_j. By symmetry these ideals are the same. If $y_1, \ldots, y_n \in \mathfrak{m}_A$ is a second regular system of parameters, then the matrix $(\partial y_j/\partial x_i)$ is invertible and we can use the chain rule for differentiation. Some details omitted. $\square$

0C4C Lemma 53.19.6. Let k be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset \mathfrak{m}_A$ be an ideal. The following are equivalent

(1) $k \to A/I$ is as in Lemma 53.19.3,
(2) A/I is reduced and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$,
(3) $\text{depth}(A/I) = 1$ and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$.

Proof. By Lemma 53.19.5 we may change our system of coordinates and the choice of generators during the proof.

If (1) holds, then we may change coordinates such that $x_1, \ldots, x_{n-2}$ map to zero in A/I and $A/I = k[[x_{n-1}, x_n]]/(ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2)$ for some nondegenerate quadric $ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2$. Then we can explicitly compute to show that both (2) and (3) are true.

Assume the $(n-1)\times(n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I+\mathfrak{m}_A$. Suppose that for some i and j the partial derivative $\partial f_j/\partial x_i$ is a unit in A. Then we may use the system of parameters $f_j, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n$ and the generators $f_j, f_1, \ldots, f_{j-1}, \hat{f}_j, f_{j+1}, \ldots, f_m$ of I. Then we get a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, f_2, \ldots, f_m$ of I. Next, we look for an $i \geq 2$ and $j \geq 2$ such that $\partial f_j/\partial x_i$ is a unit in A. If such a pair exists, then we can make a replacement as above and assume that we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, x_2, f_3, \ldots, f_m$ of I. Continuing, in finitely many steps we reach the situation where we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, \ldots, x_t, f_{t+1}, \ldots, f_m$ of I such that $\partial f_j/\partial x_i \in \mathfrak{m}_A$ for all $i, j \geq t+1$.

In this case the matrix of partial derivatives has the following block shape

$$\begin{pmatrix} I_{t\times t} & * \\ 0 & \mathfrak{m}_A \end{pmatrix}$$

Hence every $(n-1) \times (n-1)$-minor is in $\mathfrak{m}_A^{n-1-t}$. Note that $I \neq \mathfrak{m}_A$ otherwise the ideal of minors would contain 1. It follows that $n - 1 - t \leq 1$ because there is an element of $\mathfrak{m}_A \setminus \mathfrak{m}_A^2 + I$ (otherwise $I = \mathfrak{m}_A$ by Nakayama). Thus $t \geq n-2$. We have seen that $t \neq n$ above and similarly if $t = n-1$, then there is an invertible $(n-1) \times (n-1)$-minor which is disallowed as well. Hence $t = n-2$. Then A/I is a quotient of $k[[x_{n-1}, x_n]]$ and Lemma 53.19.2 implies in both cases (2) and (3) that I is generated by $x_1, \ldots, x_{n-2}, f$ for some $f = f(x_{n-1}, x_n)$. In this case the condition on the minors exactly says that the quadratic term in f is nondegenerate, i.e., A/I is as in Lemma 53.19.3. $\square$

0C4D Lemma 53.19.7. Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. The following are equivalent

(1) x is a node,
(2) $k \to \mathcal{O}_{X,x}$ is as in Lemma 53.19.3,
(3) any $\overline{x} \in X_{\overline{k}}$ mapping to x defines a nodal singularity,
(4) $\kappa(x)/k$ is separable, $\mathcal{O}_{X,x}$ is reduced, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,

(5) $\kappa(x)/k$ is separable, $\text{depth}(\mathcal{O}_{X,x}) = 1$, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,
(6) $\kappa(x)/k$ is separable and $\mathcal{O}_{X,x}$ is reduced, has δ-invariant 1, and has 2 geometric branches.

Proof. First assume that k is algebraically closed. In this case the equivalence of (1) and (3) is trivial. The equivalence of (1) and (3) with (2) holds because the only nondegenerate quadric in two variables is xy up to change in coordinates. The equivalence of (1) and (6) is Lemma 53.16.1. After replacing X by an affine neighbourhood of x, we may assume there is a closed immersion $X \to \mathbf{A}^n_k$ mapping x to 0. Let $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be generators for the ideal I of X in $\mathbf{A}^n_k$. Then $\Omega_{X/k}$ corresponds to the $R = k[x_1, \ldots, x_n]/I$-module $\Omega_{R/k}$ which has a presentation

$$R^{\oplus m} \xrightarrow{(\partial f_j/\partial x_i)} R^{\oplus n} \to \Omega_{R/k} \to 0$$

(See Algebra, Sections 10.131 and 10.134.) The first Fitting ideal of $\Omega_{R/k}$ is thus the ideal generated by the $(n-1) \times (n-1)$-minors of the matrix $(\partial f_j/\partial x_i)$. Hence (2), (4), (5) are equivalent by Lemma 53.19.6 applied to the completion of $k[x_1, \ldots, x_n] \to R$ at the maximal ideal $(x_1, \ldots, x_n)$.

Now assume k is an arbitrary field. In cases (2), (4), (5), (6) the residue field $\kappa(x)$ is separable over k. Let us show this holds as well in cases (1) and (3). Namely, let $Z \subset X$ be the closed subscheme of X defined by the first Fitting ideal of $\Omega_{X/k}$. The formation of Z commutes with field extension (Divisors, Lemma 31.10.1). If (1) or (3) is true, then there exists a point $\overline{x}$ of $X_{\overline{k}}$ such that $\overline{x}$ is an isolated point of multiplicity 1 of $Z_{\overline{k}}$ (as we have the equivalence of the conditions of the lemma over $\overline{k}$). In particular $Z_{\overline{x}}$ is geometrically reduced at $\overline{x}$ (because $\overline{k}$ is algebraically closed). Hence Z is geometrically reduced at x (Varieties, Lemma 33.6.6). In particular, Z is reduced at x, hence $Z = \text{Spec}(\kappa(x))$ in a neighbourhood of x and $\kappa(x)$ is geometrically reduced over k. This means that $\kappa(x)/k$ is separable (Algebra, Lemma 10.44.1).

The argument of the previous paragraph shows that if (1) or (3) holds, then the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat and since $\mathcal{O}_{X_{\overline{k}},\overline{x}}$ is reduced and has depth 1, we see that (4) and (5) hold (use Algebra, Lemmas 10.164.2 and 10.163.2). Conversely, (4) implies (5) by Algebra, Lemma 10.157.3. If (5) holds, then Z is geometrically reduced at x (because $\kappa(x)/k$ separable and Z is x in a neighbourhood). Hence $Z_{\overline{k}}$ is reduced at any point $\overline{x}$ of $X_{\overline{k}}$ lying over x. In other words, the first fitting ideal of $\Omega_{X_{\overline{k}}/\overline{k}}$ generates $\mathfrak{m}_{\overline{x}}$ in $\mathcal{O}_{X_{\overline{k}},\overline{x}}$. Moreover, since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat we see that $\text{depth}(\mathcal{O}_{X_{\overline{k}},\overline{x}}) = 1$ (see reference above). Hence (5) holds for $\overline{x} \in X_{\overline{k}}$ and we conclude that (3) holds (because of the equivalence over algebraically closed fields). In this way we see that (1), (3), (4), (5) are equivalent.

The equivalence of (2) and (6) follows from Lemma 53.19.4.

Finally, we prove the equivalence of (2) = (6) with (1) = (3) = (4) = (5). First we note that the geometric number of branches of X at x and the geometric number of branches of $X_{\overline{k}}$ at $\overline{x}$ are equal by Varieties, Lemma 33.40.2. We conclude from the information available to us at this point that in all cases this number is equal to 2. On the other hand, in case (1) it is clear that X is geometrically reduced at x, and hence

$$\delta\text{-invariant of } X \text{ at } x \leq \delta\text{-invariant of } X_{\overline{k}} \text{ at } \overline{x}$$

by Varieties, Lemma 33.39.8. Since in case (1) the right hand side is 1, this forces the δ-invariant of X at x to be 1 (because if it were zero, then $\mathcal{O}_{X,x}$ would be a discrete valuation ring by Varieties, Lemma 33.39.4 which is unibranch, a contradiction). Thus (5) holds. Conversely, if (2) = (5) is true, then assumptions (a), (b), (c) of Varieties, Lemma 33.27.6 hold for $x \in X$ by Lemma 53.19.4. Thus Varieties, Lemma 33.39.9 applies and shows that we have equality in the above displayed inequality. We conclude that (5) holds for $\overline{x} \in X_{\overline{k}}$ and we are back in case (1) by the equivalence of the conditions over an algebraically closed field. □

0CBT Remark 53.19.8 (The quadratic extension associated to a node). Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local k-algebra. Assume that either $(A, \mathfrak{m}, \kappa)$ is as in Lemma 53.19.3, or A is Nagata as in Lemma 53.19.4, or A is complete and as in Lemma 53.19.6. Then A defines canonically a degree 2 separable κ-algebra κ' as follows

(1) let $q = ax^2 + bxy + cy^2$ be a nondegenerate quadric as in Lemma 53.19.3 with coordinates x, y chosen such that $a \neq 0$ and set $\kappa' = \kappa[x]/(ax^2 + bx + c)$,
(2) let $A' \supset A$ be the integral closure of A in its total ring of fractions and set $\kappa' = A'/\mathfrak{m}A'$, or
(3) let κ' be the κ-algebra such that $\text{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}) = \text{Spec}(\kappa')$.

The equivalence of (1) and (2) was shown in the proof of Lemma 53.19.4. We omit the equivalence of this with (3). If X is a locally Noetherian k-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then (3) shows that $\text{Spec}(\kappa') = X^\nu \times_X \text{Spec}(\kappa)$ where $\nu : X^\nu \to X$ is the normalization morphism.

0CBU Remark 53.19.9 (Trivial quadratic extension). Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be as in Remark 53.19.8 and let κ'/κ be the associated separable algebra of degree 2. Then the following are equivalent

(1) $\kappa' \cong \kappa \times \kappa$ as κ-algebra,
(2) the form q of Lemma 53.19.3 can be chosen to be xy,
(3) A has two branches,
(4) the extension A'/A of Lemma 53.19.4 has two maximal ideals, and
(5) $A^\wedge \cong \kappa[[x, y]]/(xy)$ as a k-algebra.

The equivalence between these conditions has been shown in the proof of Lemma 53.19.4. If X is a locally Noetherian k-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then this means exactly that there are two points x_1, x_2 of the normalization X^ν lying over x and that $\kappa(x) = \kappa(x_1) = \kappa(x_2)$.

0CBV Definition 53.19.10. Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. We say x is a split node if x is a node, $\kappa(x) = k$, and the equivalent assertions of Remark 53.19.9 hold for $A = \mathcal{O}_{X,x}$.

We formulate the obligatory lemma stating what we already know about this concept.

0CBW Lemma 53.19.11. Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. The following are equivalent

(1) x is a split node,
(2) x is a node and there are exactly two points x_1, x_2 of the normalization X^ν lying over x with $k = \kappa(x_1) = \kappa(x_2)$,

(3) $\mathcal{O}_{X,x}^\wedge \cong k[[x, y]]/(xy)$ as a k-algebra, and
(4) add more here.

Proof. This follows from the discussion in Remark 53.19.9 and Lemma 53.19.7. □

0C56 Lemma 53.19.12. Let K/k be an extension of fields. Let X be a locally algebraic k-scheme of dimension 1. Let $y \in X_K$ be a point with image $x \in X$. The following are equivalent

(1) x is a closed point of X and a node, and
(2) y is a closed point of Y and a node.

Proof. If x is a closed point of X, then y is too (look at residue fields). But conversely, this need not be the case, i.e., it can happen that a closed point of Y maps to a nonclosed point of X. However, in this case y cannot be a node. Namely, then X would be geometrically unibranch at x (because x would be a generic point of X and $\mathcal{O}_{X,x}$ would be Artinian and any Artinian local ring is geometrically unibranch), hence Y is geometrically unibranch at y (Varieties, Lemma 33.40.3), which means that y cannot be a node by Lemma 53.19.7. Thus we may and do assume that both x and y are closed points.

Choose algebraic closures $\overline{k}$, $\overline{K}$ and a map $\overline{k} \to \overline{K}$ extending the given map $k \to K$. Using the equivalence of (1) and (3) in Lemma 53.19.7 we reduce to the case where k and K are algebraically closed. In this case we can argue as in the proof of Lemma 53.19.7 that the geometric number of branches and δ-invariants of X at x and Y at y are the same. Another argument can be given by choosing an isomorphism $k[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}_{X,x}^\wedge$ of k-algebras as in Varieties, Lemma 33.21.1. By Varieties, Lemma 33.21.2 this gives an isomorphism $K[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}_{Y,y}^\wedge$ of K-algebras. By definition we have to show that

$$k[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \cong k[[s, t]]/(st)$$

if and only if

$$K[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \cong K[[s, t]]/(st)$$

We encourage the reader to prove this for themselves. Since k and K are algebraically closed fields, this is the same as asking these rings to be as in Lemma 53.19.3. Via Lemma 53.19.6 this translates into a statement about the $(n-1) \times (n-1)$-minors of the matrix $(\partial g_j/\partial x_i)$ which is clearly independent of the field used. We omit the details. □

0C57 Lemma 53.19.13. Let k be a field. Let X be a locally algebraic k-scheme of dimension 1. Let $Y \to X$ be an étale morphism. Let $y \in Y$ be a point with image $x \in X$. The following are equivalent

(1) x is a closed point of X and a node, and
(2) y is a closed point of Y and a node.

Proof. By Lemma 53.19.12 we may base change to the algebraic closure of k. Then the residue fields of x and y are k. Hence the map $\mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{Y,y}^\wedge$ is an isomorphism (for example by Étale Morphisms, Lemma 41.11.3 or More on Algebra, Lemma 15.43.9). Thus the lemma is clear. □

0CD6 Lemma 53.19.14. Let k'/k be a finite separable field extension. Let X be a locally algebraic k'-scheme of dimension 1. Let $x \in X$ be a closed point. The following are equivalent

(1) x is a node, and
(2) x is a node when X viewed as a locally algebraic k-scheme.

Proof. Follows immediately from the characterization of nodes in Lemma 53.19.7. □

0C4E Lemma 53.19.15. Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1. The following are equivalent

(1) the singularities of X are at-worst-nodal, and
(2) X is a local complete intersection over k and the closed subscheme $Z \subset X$ cut out by the first fitting ideal of $\Omega_{X/k}$ is unramified over k.

Proof. We urge the reader to find their own proof of this lemma; what follows is just putting together earlier results and may hide what is really going on.

Assume (2). Since $Z \to \text{Spec}(k)$ is quasi-finite (Morphisms, Lemma 29.35.10) we see that the residue fields of points $x \in Z$ are finite over k (as well as separable) by Morphisms, Lemma 29.20.5. Hence each $x \in Z$ is a closed point of X by Morphisms, Lemma 29.20.2. The local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay by Algebra, Lemma 10.135.3. Since $\dim(\mathcal{O}_{X,x}) = 1$ by dimension theory (Varieties, Section 33.20), we conclude that $\text{depth}(\mathcal{O}_{X,x})) = 1$. Thus x is a node by Lemma 53.19.7. If $x \in X$, $x \notin Z$, then $X \to \text{Spec}(k)$ is smooth at x by Divisors, Lemma 31.10.3.

Assume (1). Under this assumption X is geometrically reduced at every closed point (see Varieties, Lemma 33.6.6). Hence $X \to \text{Spec}(k)$ is smooth on a dense open by Varieties, Lemma 33.25.7. Thus Z is closed and consists of closed points. By Divisors, Lemma 31.10.3 the morphism $X \setminus Z \to \text{Spec}(k)$ is smooth. Hence $X \setminus Z$ is a local complete intersection by Morphisms, Lemma 29.34.7 and the definition of a local complete intersection in Morphisms, Definition 29.30.1. By Lemma 53.19.7 for every point $x \in Z$ the local ring $\mathcal{O}_{Z,x}$ is equal to $\kappa(x)$ and $\kappa(x)$ is separable over k. Thus $Z \to \text{Spec}(k)$ is unramified (Morphisms, Lemma 29.35.11). Finally, Lemma 53.19.7 via part (3) of Lemma 53.19.3, shows that $\mathcal{O}_{X,x}$ is a complete intersection in the sense of Divided Power Algebra, Definition 23.8.5. However, Divided Power Algebra, Lemma 23.8.8 and Morphisms, Lemma 29.30.9 show that this agrees with the notion used to define a local complete intersection scheme over a field and the proof is complete. □

0E37 Lemma 53.19.16. Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1 whose singularities are at-worst-nodal. Then X is Gorenstein and geometrically reduced.

Proof. The Gorenstein assertion follows from Lemma 53.19.15 and Duality for Schemes, Lemma 48.24.5. Or you can use that it suffices to check after passing to the algebraic closure (Duality for Schemes, Lemma 48.25.1), then use that a Noetherian local ring is Gorenstein if and only if its completion is so (by Dualizing Complexes, Lemma 47.21.8), and then prove that the local rings $k[[t]]$ and $k[[x, y]]/(xy)$ are Gorenstein by hand.

To see that X is geometrically reduced, it suffices to show that $X_{\overline{k}}$ is reduced (Varieties, Lemmas 33.6.3 and 33.6.4). But $X_{\overline{k}}$ is a nodal curve over an algebraically closed field. Thus the complete local rings of $X_{\overline{k}}$ are isomorphic to either $\overline{k}[[t]]$ or $\overline{k}[[x, y]]/(xy)$ which are reduced as desired. □

0E38 Lemma 53.19.17. Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1 whose singularities are at-worst-nodal. If $Y \subset X$ is a reduced closed subscheme equidimensional of dimension 1, then

(1) the singularities of Y are at-worst-nodal, and
(2) if $Z \subset X$ is the scheme theoretic closure of $X \setminus Y$, then
   (a) the scheme theoretic intersection $Y \cap Z$ is the disjoint union of spectra of finite separable extensions of k,
   (b) each point of $Y \cap Z$ is a node of X, and
   (c) $Y \to \mathrm{Spec}(k)$ is smooth at every point of $Y \cap Z$.

Proof. Since X and Y are reduced and equidimensional of dimension 1, we see that Y is the scheme theoretic union of a subset of the irreducible components of X (in a reduced ring (0) is the intersection of the minimal primes). Let $y \in Y$ be a closed point. If y is in the smooth locus of $X \to \mathrm{Spec}(k)$, then y is on a unique irreducible component of X and we see that Y and X agree in an open neighbourhood of y. Hence $Y \to \mathrm{Spec}(k)$ is smooth at y. If y is a node of X but still lies on a unique irreducible component of X, then y is a node on Y by the same argument. Suppose that y lies on more than 1 irreducible component of X. Since the number of geometric branches of X at y is 2 by Lemma 53.19.7, there can be at most 2 irreducible components passing through y by Properties, Lemma 28.15.5. If Y contains both of these, then again $Y = X$ in an open neighbourhood of y and y is a node of Y. Finally, assume Y contains only one of the irreducible components. After replacing X by an open neighbourhood of x we may assume Y is one of the two irreducble components and Z is the other. By Properties, Lemma 28.15.5 again we see that X has two branches at y, i.e., the local ring $\mathcal{O}_{X,y}$ has two branches and that these branches come from $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,y}$. Write $\mathcal{O}_{X,y}^\wedge \cong \kappa(y)[[u,v]]/(uv)$ as in Remark 53.19.9. The field $\kappa(y)$ is finite separable over k by Lemma 53.19.7 for example. Thus, after possibly switching the roles of u and v, the completion of the map $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,Y}$ corresponds to $\kappa(y)[[u,v]]/(uv) \to \kappa(y)[[u]]$ and the completion of the map $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,Y}$ corresponds to $\kappa(y)[[u,v]]/(uv) \to \kappa(y)[[v]]$. The scheme theoretic intersection of $Y \cap Z$ is cut out by the sum of their ideas which in the completion is (u,v), i.e., the maximal ideal. Thus (2)(a) and (2)(b) are clear. Finally, (2)(c) holds: the completion of $\mathcal{O}_{Y,y}$ is regular, hence $\mathcal{O}_{Y,y}$ is regular (More on Algebra, Lemma 15.43.4) and $\kappa(y)/k$ is separable, hence smoothness in an open neighbourhood by Algebra, Lemma 10.140.5. $\square$

53.20. Families of nodal curves

0C58 In the Stacks project curves are irreducible varieties of dimension 1, but in the literature a "semi-stable curve" or a "nodal curve" is usually not irreducible and often assumed to be proper, especially when used in a phrase such as "family of semistable curves" or "family of nodal curves", or "nodal family". Thus it is a bit difficult for us to choose a terminology which is consistent with the literature as well as internally consistent. Moreover, we really want to first study the notion introduced in the following lemma (which is local on the source).

0C59 Lemma 53.20.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) f is flat, locally of finite presentation, every nonempty fibre X_s is equidimensional of dimension 1, and X_s has at-worst-nodal singularities, and

(2) f is syntomic of relative dimension 1 and the closed subscheme $\text{Sing}(f) \subset X$ defined by the first Fitting ideal of $\Omega_{X/S}$ is unramified over S.

Proof. Recall that the formation of $\text{Sing}(f)$ commutes with base change, see Divisors, Lemma 31.10.1. Thus the lemma follows from Lemma 53.19.15, Morphisms, Lemma 29.30.11, and Morphisms, Lemma 29.35.12. (We also use the trivial Morphisms, Lemmas 29.30.6 and 29.30.7.) □

0C5A Definition 53.20.2. Let $f : X \to S$ be a morphism of schemes. We say f is at-worst-nodal of relative dimension 1 if f satisfies the equivalent conditions of Lemma 53.20.1.

Here are some reasons for the cumbersome terminology[6]. First, we want to make sure this notion is not confused with any of the other notions in the literature (see introduction to this section). Second, we can imagine several generalizations of this notion to morphisms of higher relative dimension (for example, one can ask for morphisms which are étale locally compositions of at-worst-nodal morphisms or one can ask for morphisms whose fibres are higher dimensional but have at worst ordinary double points).

0CD7 Lemma 53.20.3. A smooth morphism of relative dimension 1 is at-worst-nodal of relative dimension 1.

Proof. Omitted. □

0C5B Lemma 53.20.4. Let $f : X \to S$ be at-worst-nodal of relative dimension 1. Then the same is true for any base change of f.

Proof. This is true because the base change of a syntomic morphism is syntomic (Morphisms, Lemma 29.30.4), the base change of a morphism of relative dimension 1 has relative dimension 1 (Morphisms, Lemma 29.29.2), the formation of $\text{Sing}(f)$ commutes with base change (Divisors, Lemma 31.10.1), and the base change of an unramified morphism is unramified (Morphisms, Lemma 29.35.5). □

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

0DSC Lemma 53.20.5. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Then there is a maximal open subscheme $U \subset X$ such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of U commutes with arbitrary base change.

Proof. By Morphisms, Lemma 29.30.12 we find that there is such an open where f is syntomic. Hence we may assume that f is a syntomic morphism. In particular f is a Cohen-Macaulay morphism (Duality for Schemes, Lemmas 48.25.5 and 48.25.4). Thus X is a disjoint union of open and closed subschemes on which f has given relative dimension, see Morphisms, Lemma 29.29.4. This decomposition is preserved by arbitrary base change, see Morphisms, Lemma 29.29.2. Discarding all but one piece we may assume f is syntomic of relative dimension 1. Let $\text{Sing}(f) \subset X$ be the closed subscheem defined by the first fitting ideal of $\Omega_{X/S}$. There is a maximal open subscheme $W \subset \text{Sing}(f)$ such that $W \to S$ is unramified and its formation

[6]But please email the maintainer of the Stacks project if you have a better suggestion.

commutes with base change (Morphisms, Lemma 29.35.15). Since also formation of $\text{Sing}(f)$ commutes with base change (Divisors, Lemma 31.10.1), we see that

$$U = (X \setminus \text{Sing}(f)) \cup W$$

is the maximal open subscheme of X such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1 and that formation of U commutes with base change. $\square$

0C5C Lemma 53.20.6. Let $f : X \to S$ be at-worst-nodal of relative dimension 1. If $Y \to X$ is an étale morphism, then the composition $g : Y \to S$ is at-worst-nodal of relative dimension 1.

Proof. Observe that g is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 29.36.11, 29.36.12, 29.21.3, and 29.25.6). Thus it suffices to prove the fibres have at-worst-nodal singularities. This follows from Lemma 53.19.13 (and the fact that the composition of an étale morphism and a smooth morphism is smooth by Morphisms, Lemmas 29.36.5 and 29.34.4). $\square$

0CD8 Lemma 53.20.7. Let $S' \to S$ be an étale morphism of schemes. Let $f : X \to S'$ be at-worst-nodal of relative dimension 1. Then the composition $g : X \to S$ is at-worst-nodal of relative dimension 1.

Proof. Observe that g is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 29.36.11, 29.36.12, 29.21.3, and 29.25.6). Thus it suffices to prove the fibres of g have at-worst-nodal singularities. This follows from Lemma 53.19.14 and the analogous result for smooth points. $\square$

0C5D Lemma 53.20.8. Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to X\}$ be an étale covering. The following are equivalent

(1) f is at-worst-nodal of relative dimension 1,
(2) each $U_i \to S$ is at-worst-nodal of relative dimension 1.

In other words, being at-worst-nodal of relative dimension 1 is étale local on the source.

Proof. One direction we have seen in Lemma 53.20.6. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is étale local on the source (Descent, Lemmas 35.28.1, 35.27.1, and 35.33.8). Taking fibres we reduce to the case where S is the spectrum of a field. In this case the result follows from Lemma 53.19.13 (and the fact that being smooth is étale local on the source by Descent, Lemma 35.30.1). $\square$

0C5E Lemma 53.20.9. Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to S\}$ be an fpqc covering. The following are equivalent

(1) f is at-worst-nodal of relative dimension 1,
(2) each $X \times_S U_i \to U_i$ is at-worst-nodal of relative dimension 1.

In other words, being at-worst-nodal of relative dimension 1 is fpqc local on the target.

Proof. One direction we have seen in Lemma 53.20.4. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is fpqc local on the target (Descent, Lemmas 35.23.11, 35.23.15, and Morphisms, Lemma 29.28.3). Taking fibres we reduce to the case where S is the spectrum of a field. In this case the result follows from Lemma 53.19.12 (and the fact that being smooth is fpqc local on the target by Descent, Lemma 35.23.27). □

0C5F Lemma 53.20.10. Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S. If

(1) f is at-worst-nodal of relative dimension 1, and
(2) f_0 is locally of finite presentation,

then there exists an $i \geq 0$ such that f_i is at-worst-nodal of relative dimension 1.

Proof. By Limits, Lemma 32.8.16 there exists an i such that f_i is syntomic. Then $X_i = \coprod_{d \geq 0} X_{i,d}$ is a disjoint union of open and closed subschemes such that $X_{i,d} \to Y_i$ has relative dimension d, see Morphisms, Lemma 29.30.14. Because of the behaviour of dimensions of fibres under base change given in Morphisms, Lemma 29.28.3 we see that $X \to X_i$ maps into $X_{i,1}$. Then there exists an $i' \geq i$ such that $X_{i'} \to X_i$ maps into $X_{i,1}$, see Limits, Lemma 32.4.10. Thus $f_{i'} : X_{i'} \to Y_{i'}$ is syntomic of relative dimension 1 (by Morphisms, Lemma 29.28.3 again). Consider the morphism $\text{Sing}(f_{i'}) \to Y_{i'}$. We know that the base change to Y is an unramified morphism. Hence by Limits, Lemma 32.8.4 we see that after increasing i' the morphism $\text{Sing}(f_{i'}) \to Y_{i'}$ becomes unramified. This finishes the proof. □

0CBX Lemma 53.20.11. Let $f : T \to S$ be a morphism of schemes. Let $t \in T$ with image $s \in S$. Assume

(1) f is flat at t,
(2) $\mathcal{O}_{S,s}$ is Noetherian,
(3) f is locally of finite type,
(4) t is a split node of the fibre T_s.

Then there exists an $h \in \mathfrak{m}_s^\wedge$ and an isomorphism

$$\mathcal{O}_{T,t}^\wedge \cong \mathcal{O}_{S,s}^\wedge[[x, y]]/(xy - h)$$

of $\mathcal{O}_{S,s}^\wedge$-algebras.

Proof. We replace S by $\text{Spec}(\mathcal{O}_{S,s})$ and T by the base change to $\text{Spec}(\mathcal{O}_{S,s})$. Then T is locally Noetherian and hence $\mathcal{O}_{T,t}$ is Noetherian. Set $A = \mathcal{O}_{S,s}^\wedge$, $\mathfrak{m} = \mathfrak{m}_A$, and $B = \mathcal{O}_{T,t}^\wedge$. By More on Algebra, Lemma 15.43.8 we see that $A \to B$ is flat. Since $\mathcal{O}_{T,t}/\mathfrak{m}_s\mathcal{O}_{T,t} = \mathcal{O}_{T_s,t}$ we see that $B/\mathfrak{m}B = \mathcal{O}_{T_s,t}^\wedge$. By assumption (4) and Lemma 53.19.11 we conclude there exist $\overline{u}, \overline{v} \in B/\mathfrak{m}B$ such that the map

$$(A/\mathfrak{m})[[x, y]] \longrightarrow B/\mathfrak{m}B, \quad x \longmapsto \overline{u}, x \longmapsto \overline{v}$$

is surjective with kernel (xy).

Assume we have $n \geq 1$ and $u, v \in B$ mapping to $\overline{u}, \overline{v}$ such that

$$uv = h + \delta$$

for some $h \in A$ and $\delta \in \mathfrak{m}^n B$. We claim that there exist $u', v' \in B$ with $u - u', v - v' \in \mathfrak{m}^n B$ such that

$$u'v' = h' + \delta'$$

for some $h' \in A$ and $\delta' \in \mathfrak{m}^{n+1}B$. To see this, write $\delta = \sum f_i b_i$ with $f_i \in \mathfrak{m}^n$ and $b_i \in B$. Then write $b_i = a_i + ub_{i,1} + vb_{i,2} + \delta_i$ with $a_i \in A$, $b_{i,1}, b_{i,2} \in B$ and $\delta_i \in \mathfrak{m}B$. This is possible because the residue field of B agrees with the residue field of A and the images of u and v in $B/\mathfrak{m}B$ generate the maximal ideal. Then we set

$$u' = u - \sum b_{i,2} f_i, \quad v' = v - \sum b_{i,1} f_i$$

and we obtain

$$u'v' = h + \delta - \sum (b_{i,1}u + b_{i,2}v) f_i + \sum c_{ij} f_i f_j = h + \sum a_i f_i + \sum f_i \delta_i + \sum c_{ij} f_i f_j$$

for some $c_{i,j} \in B$. Thus we get a formula as above with $h' = h + \sum a_i f_i$ and $\delta' = \sum f_i \delta_i + \sum c_{ij} f_i f_j$.

Arguing by induction and starting with any lifts $u_1, v_1 \in B$ of $\overline{u}, \overline{v}$ the result of the previous paragraph shows that we find a sequence of elements $u_n, v_n \in B$ and $h_n \in A$ such that $u_n - u_{n+1} \in \mathfrak{m}^n B$, $v_n - v_{n+1} \in \mathfrak{m}^n B$, $h_n - h_{n+1} \in \mathfrak{m}^n$, and such that $u_n v_n - h_n \in \mathfrak{m}^n B$. Since A and B are complete we can set $u_\infty = \lim u_n$, $v_\infty = \lim v_n$, and $h_\infty = \lim h_n$, and then we obtain $u_\infty v_\infty = h_\infty$ in B. Thus we have an A-algebra map

$$A[[x, y]]/(xy - h_\infty) \longrightarrow B$$

sending x to u_∞ and v to v_∞. This is a map of flat A-algebras which is an isomorphism after dividing by $\mathfrak{m}$. It is surjective modulo $\mathfrak{m}$ and hence surjective by completeness and Algebra, Lemma 10.96.1. Then we can apply Algebra, Lemma 10.99.1 to conclude it is an isomorphism. □

Consider the morphism of schemes

$$\mathrm{Spec}(\mathbf{Z}[u, v, a]/(uv - a)) \longrightarrow \mathrm{Spec}(\mathbf{Z}[a])$$

The next lemma shows that this morphism is a model for the étale local structure of a nodal family of curves. If you know a proof of this lemma avoiding the use of Artin approximation, then please email stacks.project@gmail.com.

0CBY Lemma 53.20.12. Let $f : X \to S$ be a morphism of schemes. Assume that f is at-worst-nodal of relative dimension 1. Let $x \in X$ be a point which is a singular point of the fibre X_s. Then there exists a commutative diagram of schemes

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[rr] \arrow[dr] & & W \arrow[r] \arrow[dl] & \mathrm{Spec}(\mathbf{Z}[u, v, a]/(uv - a)) \arrow[d] \\
S & & V \arrow[ll] \arrow[rr] & & \mathrm{Spec}(\mathbf{Z}[a])
\end{tikzcd}$$

with $X \leftarrow U$, $S \leftarrow V$, and $U \to W$ étale morphisms, and with the right hand square cartesian, such that there exists a point $u \in U$ mapping to x in X.

Proof. We first use absolute Noetherian approximation to reduce to the case of schemes of finite type over $\mathbf{Z}$. The question is local on X and S. Hence we may assume that X and S are affine. Then we can write $S = \mathrm{Spec}(R)$ and write R as a filtered colimit $R = \mathrm{colim}\, R_i$ of finite type $\mathbf{Z}$-algebras. Using Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(R_i)$ whose base change to S is f. After increasing i we may assume that f_i is at-worst-nodal of relative dimension 1, see Lemma 53.20.10. The image $x_i \in X_i$ of x will be a singular point of its fibre, for example because the formation of $\mathrm{Sing}(f)$ commutes with base change

(Divisors, Lemma 31.10.1). If we can prove the lemma for $f_i : X_i \to S_i$ and x_i, then the lemma follows for $f : X \to S$ by base change. Thus we reduce to the case studied in the next paragraph.

Assume S is of finite type over $\mathbf{Z}$. Let $s \in S$ be the image of x. Recall that $\kappa(x)$ is a finite separable extension of $\kappa(s)$, for example because $\text{Sing}(f) \to S$ is unramified or because x is a node of the fibre X_s and we can apply Lemma 53.19.7. Furthermore, let $\kappa'/\kappa(x)$ be the degree 2 separable algebra associated to $\mathcal{O}_{X_s,x}$ in Remark 53.19.8. By More on Morphisms, Lemma 37.35.2 we can choose an étale neighbourhood $(V, v) \to (S, s)$ such that the extension $\kappa(v)/\kappa(s)$ realizes either the extension $\kappa(x)/\kappa(s)$ in case $\kappa' \cong \kappa(x) \times \kappa(x)$ or the extension $\kappa'/\kappa(s)$ if κ' is a field. After replacing X by $X \times_S V$ and S by V we reduce to the situation described in the next paragraph.

Assume S is of finite type over $\mathbf{Z}$ and $x \in X_s$ is a split node, see Definition 53.19.10. By Lemma 53.20.11 we see that there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{S,s}[[s, t]]/(st - h)$$

for some $h \in \mathfrak{m}^\wedge_s \subset \mathcal{O}^\wedge_{S,s}$. In other words, if we consider the homomorphism

$$\sigma : \mathbf{Z}[a] \longrightarrow \mathcal{O}^\wedge_{S,s}$$

sending a to h, then there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma,y_\sigma}$$

where

$$Y_\sigma = \text{Spec}(\mathbf{Z}[u, v, t]/(uv - a)) \times_{\text{Spec}(\mathbf{Z}[a]),\sigma} \text{Spec}(\mathcal{O}^\wedge_{S,s})$$

and y_σ is the point of Y_σ lying over the closed point of $\text{Spec}(\mathcal{O}^\wedge_{S,s})$ and having coordinates u, v equal to zero. Since $\mathcal{O}_{S,s}$ is a G-ring by More on Algebra, Proposition 15.50.12 we may apply More on Morphisms, Lemma 37.39.3 to conclude. $\square$

0GKA Lemma 53.20.13. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is proper,
(2) f is at-worst-nodal of relative dimension 1, and
(3) the geometric fibres of f are connected.

Then (a) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change, (b) $R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module whose formation commutes with any base change, and (c) $R^qf_*\mathcal{O}_X = 0$ for $q \geq 2$.

Proof. Part (a) follows from Derived Categories of Schemes, Lemma 36.32.6. By Derived Categories of Schemes, Lemma 36.32.5 locally on S we can write $Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$ where P is perfect of tor amplitude in $[1, \infty)$. Recall that formation of $Rf_*\mathcal{O}_X$ commutes with arbitrary base change (Derived Categories of Schemes, Lemma 36.30.4). Thus for $s \in S$ we have

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition 20.20.7. Then $P = H^1(P)[-1]$ and $H^1(P)$ is finite locally free for example by More on Algebra, Lemma 15.75.6. Since everything is compatible with base change we conclude. $\square$

53.21. More vanishing results

0E39 Continuation of Section 53.6.

0E3A **Lemma 53.21.1.** In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then

$$\deg(\mathcal{L}) \leq 2g - 2 + \deg(Z)$$

with strict inequality unless $\mathcal{I}\mathcal{L} \cong \omega_X$.

Proof. Any curve, e.g. X, is Cohen-Macaulay. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then there is a nonzero map $\mathcal{I}\mathcal{L} \to \omega_X$, see Lemma 53.4.2. Since $\mathcal{I}\mathcal{L}$ is torsion free, this map is injective. Since a field is Gorenstein and X is reduced, we find that the Gorenstein locus $U \subset X$ of X is nonempty, see Duality for Schemes, Lemma 48.24.4. This lemma also tells us that $\omega_X|_U$ is invertible. In this way we see we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \omega_X \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is zero dimensional. Hence we have

$$\begin{aligned} 0 &\leq \dim \Gamma(X, \mathcal{Q}) \\ &= \chi(\mathcal{Q}) \\ &= \chi(\omega_X) - \chi(\mathcal{I}\mathcal{L}) \\ &= \chi(\omega_X) - \deg(\mathcal{L}) - \chi(\mathcal{I}) \\ &= 2g - 2 - \deg(\mathcal{L}) + \deg(Z) \end{aligned}$$

by Lemmas 53.5.1 and 53.5.2, by (53.8.1.1), and by Varieties, Lemmas 33.33.3 and 33.44.5. We have also used that $\deg(Z) = \dim_k \Gamma(Z, \mathcal{O}_Z) = \chi(\mathcal{O}_Z)$ and the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. The lemma follows. $\square$

0E3B **Lemma 53.21.2.** In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $\deg(\mathcal{L}) > 2g - 2 + \deg(Z)$, then $H^1(X, \mathcal{I}\mathcal{L}) = 0$ and one of the following possibilities occurs [Lee05, Lemma 2]

(1) $H^0(X, \mathcal{I}\mathcal{L}) \neq 0$, or
(2) $g = 0$ and $\deg(\mathcal{L}) = \deg(Z) - 1$.

In case (2) if $Z = \emptyset$, then $X \cong \mathbf{P}^1_k$ and $\mathcal{L}$ corresponds to $\mathcal{O}_{\mathbf{P}^1}(-1)$.

Proof. The vanishing of $H^1(X, \mathcal{I}\mathcal{L})$ follows from Lemma 53.21.1. If $H^0(X, \mathcal{I}\mathcal{L}) = 0$, then $\chi(\mathcal{I}\mathcal{L}) = 0$. From the short exact sequence $0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_Z \to 0$ we conclude $\deg(\mathcal{L}) = g - 1 + \deg(Z)$. Thus $g - 1 + \deg(Z) > 2g - 2 + \deg(Z)$ which implies $g = 0$ hence (2) holds. If $Z = \emptyset$ in case (2), then $\mathcal{L}^{-1}$ is an invertible sheaf of degree 1. This implies there is an isomorphism $X \to \mathbf{P}^1_k$ and $\mathcal{L}^{-1}$ is the pullback of $\mathcal{O}_{\mathbf{P}^1}(1)$ by Lemma 53.10.2. $\square$

0E3C **Lemma 53.21.3.** In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}) \geq 2g$, then $\mathcal{L}$ is globally generated. [Lee05, Lemma 3]

Proof. Let $Z \subset X$ be the closed subscheme cut out by the global sections of $\mathcal{L}$. By Lemma 53.21.2 we see that $Z \neq X$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf cutting out Z. Consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_Z \to 0$$

If $Z \neq \emptyset$, then $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero as follows from the long exact sequence of cohomology. By Lemma 53.4.2 this gives a nonzero and hence injective map

$$\mathcal{I}\mathcal{L} \longrightarrow \omega_X$$

In particular, we find an injective map $H^0(X, \mathcal{L}) = H^0(X, \mathcal{I}\mathcal{L}) \to H^0(X, \omega_X)$. This is impossible as

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}) + \deg(\mathcal{L}) + 1 - g \geq g + 1$$

and $\dim H^0(X, \omega_X) = g$ by (53.8.1.1). $\square$

0E3D Lemma 53.21.4. In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a nonempty 0-dimensional closed subscheme. If $\deg(\mathcal{L}) \geq 2g - 1 + \deg(Z)$, then $\mathcal{L}$ is globally generated and $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_Z)$ is surjective.

Proof. Global generation by Lemma 53.21.3. If $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z, then $H^1(X, \mathcal{I}\mathcal{L}) = 0$ by Lemma 53.21.1. Hence surjectivity. $\square$

0H2V Lemma 53.21.5. In Situation 53.6.2, assume X is geometrically integral over k and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}) \geq 2g + 1$, then $\mathcal{L}$ is very ample.

Proof. By Lemma 53.21.3, $\mathcal{L}$ is globally generated, and so it determines a morphism $f : X \to \mathbf{P}^n_k$ where $n = h^0(X, \mathcal{L}) - 1$. To show that $\mathcal{L}$ is very ample means to show that f is a closed immersion. It suffices to check that the base change of f to an algebraic closure $\overline{k}$ of k is a closed immersion (Descent, Lemma 35.23.19). So we may assume that k is algebraically closed; X remains integral, by assumption. Lemma 53.21.4 gives that for every 0-dimensional closed subscheme $Z \subset X$ of degree 2, the restriction map $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_Z)$ is surjective. By Varieties, Lemma 33.23.2, $\mathcal{L}$ is very ample. $\square$

0E3E Lemma 53.21.6. Let k be a field. Let X be a proper scheme over k which is reduced, connected, and of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $H^1(X, \mathcal{I}\mathcal{L}) \neq 0$, then there exists a reduced connected closed subscheme $Y \subset X$ of dimension 1 such that Weak version of [Lee05, Lemma 4]

$$\deg(\mathcal{L}|_Y) \leq -2\chi(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

where $Z \cap Y$ is the scheme theoretic intersection.

Proof. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then there is a nonzero map $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$, see Lemma 53.4.2. Let $Y \subset X$ be the union of the irreducible components C of X such that φ is nonzero in the generic point of C. Then Y is a reduced closed subscheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Y. Since $\mathcal{J}\mathcal{I}\mathcal{L}$ has no embedded associated points (as a submodule of $\mathcal{L}$) and as φ is zero in the generic points of the support of $\mathcal{J}$ (by choice of Y and as X is reduced), we find that φ factors as

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_X$$

We can view $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L}$ as the pushforward of a coherent sheaf on Y which by abuse of notation we indicate with the same symbol. Since $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ by Lemma 53.4.5 we find a map

$$\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_Y$$

of $\mathcal{O}_Y$-modules which is injective in the generic points of Y. Let $\mathcal{I}' \subset \mathcal{O}_Y$ be the ideal sheaf of $Z \cap Y$. There is a map $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \mathcal{I}'\mathcal{L}|_Y$ whose kernel is supported in closed points. Since ω_Y is a Cohen-Macaulay module, the map above factors through an injective map $\mathcal{I}'\mathcal{L}|_Y \to \omega_Y$. We see that we get an exact sequence

$$0 \to \mathcal{I}'\mathcal{L}|_Y \to \omega_Y \to \mathcal{Q} \to 0$$

of coherent sheaves on Y where $\mathcal{Q}$ is supported in dimension 0 (this uses that ω_Y is an invertible module in the generic points of Y). We conclude that

$$0 \leq \dim \Gamma(Y, \mathcal{Q}) = \chi(\mathcal{Q}) = \chi(\omega_Y) - \chi(\mathcal{I}'\mathcal{L}) = -2\chi(\mathcal{O}_Y) - \deg(\mathcal{L}|_Y) + \deg(Z \cap Y)$$

by Lemma 53.5.1 and Varieties, Lemma 33.33.3. If Y is connected, then this proves the lemma. If not, then we repeat the last part of the argument for one of the connected components of Y. □

0E3F Lemma 53.21.7. Let k be a field. Let X be a proper scheme over k which is reduced, connected, and of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for every reduced connected closed subscheme $Y \subset X$ of dimension 1 we have

$$\deg(\mathcal{L}|_Y) \geq 2 \dim_k H^1(Y, \mathcal{O}_Y)$$

Then $\mathcal{L}$ is globally generated.

Proof. By induction on the number of irreducible components of X. If X is irreducible, then the lemma holds by Lemma 53.21.3 applied to X viewed as a scheme over the field $k' = H^0(X, \mathcal{O}_X)$. Assume X is not irreducible. Before we continue, if k is finite, then we replace k by a purely transcendental extension K. This is allowed by Varieties, Lemmas 33.22.1, 33.44.2, 33.6.7, and 33.8.4, Cohomology of Schemes, Lemma 30.5.2, Lemma 53.4.4 and the elementary fact that K is geometrically integral over k.

Assume that $\mathcal{L}$ is not globally generated to get a contradiction. Then we may choose a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $H^0(X, \mathcal{I}\mathcal{L}) = H^0(X, \mathcal{L})$ and such that $\mathcal{O}_X/\mathcal{I}$ is nonzero with support of dimension 0. For example, take $\mathcal{I}$ the ideal sheaf of any closed point in the common vanishing locus of the global sections of $\mathcal{L}$. We consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{L}/\mathcal{I}\mathcal{L} \to 0$$

Since the support of $\mathcal{L}/\mathcal{I}\mathcal{L}$ has dimension 0 we see that $\mathcal{L}/\mathcal{I}\mathcal{L}$ is generated by global sections (Varieties, Lemma 33.33.3). From the short exact sequence, and the fact that $H^0(X, \mathcal{I}\mathcal{L}) = H^0(X, \mathcal{L})$ we get an injection $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^1(X, \mathcal{I}\mathcal{L})$.

Recall that the k-vector space $H^1(X, \mathcal{I}\mathcal{L})$ is dual to $\text{Hom}(\mathcal{I}\mathcal{L}, \omega_X)$. Choose $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$. By Lemma 53.21.6 we have $H^1(X, \mathcal{L}) = 0$. Hence

$$\dim_k H^0(X, \mathcal{I}\mathcal{L}) = \dim_k H^0(X, \mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X) > \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$$

We conclude that φ is not injective on global sections, in particular φ is not injective. For every generic point $\eta \in X$ of an irreducible component of X denote $V_\eta \subset \text{Hom}(\mathcal{I}\mathcal{L}, \omega_X)$ the k-subvector space consisting of those φ which are zero at η. Since every associated point of $\mathcal{I}\mathcal{L}$ is a generic point of X, the above shows that $\text{Hom}(\mathcal{I}\mathcal{L}, \omega_X) = \bigcup V_\eta$. As X has finitely many generic points and k is infinite, we conclude $\text{Hom}(\mathcal{I}\mathcal{L}, \omega_X) = V_\eta$ for some η. Let $\eta \in C \subset X$ be the corresponding irreducible component. Let $Y \subset X$ be the union of the other irreducible components of X. Then Y is a nonempty reduced closed subscheme not equal to X. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Y. Please keep in mind that the support of $\mathcal{J}$ is C.

Let $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$ be arbitrary. Since $\mathcal{J}\mathcal{I}\mathcal{L}$ has no embedded associated points (as a submodule of $\mathcal{L}$) and as φ is zero in the generic point η of the support of $\mathcal{J}$, we find that φ factors as

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_X$$

We can view $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L}$ as the pushforward of a coherent sheaf on Y which by abuse of notation we indicate with the same symbol. Since $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ by Lemma 53.4.5 we find a factorization

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \xrightarrow{\varphi'} \omega_Y \to \omega_X$$

of φ. Let $\mathcal{I}' \subset \mathcal{O}_Y$ be the image of $\mathcal{I} \subset \mathcal{O}_X$. There is a surjective map $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \mathcal{I}'\mathcal{L}|_Y$ whose kernel is supported in closed points. Since ω_Y is a Cohen-Macaulay module on Y, the map φ' factors through a map $\varphi'' : \mathcal{I}'\mathcal{L}|_Y \to \omega_Y$. Thus we have commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}\mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{I}\mathcal{L} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}'\mathcal{L}|_Y \arrow[r] & \mathcal{L}|_Y \arrow[r] & \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{I}\mathcal{L} \arrow[r, "\varphi"'] \arrow[d] & \omega_X \\ \mathcal{I}'\mathcal{L}|_Y \arrow[r, "\varphi''"] & \omega_Y \arrow[u] \end{tikzcd}$$

Now we can finish the proof as follows: Since for every φ we have a φ'' and since $\omega_X \in \text{Coh}(\mathcal{O}_X)$ represents the functor $\mathcal{F} \mapsto \text{Hom}_k(H^1(X, \mathcal{F}), k)$, we find that $H^1(X, \mathcal{I}\mathcal{L}) \to H^1(Y, \mathcal{I}'\mathcal{L}|_Y)$ is injective. Since the boundary $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^1(X, \mathcal{I}\mathcal{L})$ is injective, we conclude the composition

$$H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y) \to H^1(X, \mathcal{I}'\mathcal{L}|_Y)$$

is injective. Since $\mathcal{L}/\mathcal{I}\mathcal{L} \to \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y$ is a surjective map of coherent modules whose supports have dimension 0, we see that the first map $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y)$ is surjective (and hence bijective). But by induction we have that $\mathcal{L}|_Y$ is globally generated (if Y is disconnected this still works of course) and hence the boundary map

$$H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y) \to H^1(X, \mathcal{I}'\mathcal{L}|_Y)$$

cannot be injective. This contradiction finishes the proof. $\square$

53.22. Contracting rational tails

0E3G In this section we discuss the simplest possible case of contracting a scheme to improve positivity properties of its canonical sheaf.

0E3H Example 53.22.1 (Contracting a rational tail). Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. A rational tail will be an irreducible component $C \subset X$ (viewed as an integral closed subscheme) with the following properties

1. $X' \neq \emptyset$ where $X' \subset X$ is the scheme theoretic closure of $X \setminus C$,
2. the scheme theoretic intersection $C \cap X'$ is a single reduced point x,
3. $H^0(C, \mathcal{O}_C)$ maps isomorphically to the residue field of x, and
4. C has genus zero.

Since there are at least two irreducible components of X passing through x, we conclude that x is a node. Set $k' = H^0(C, \mathcal{O}_C) = \kappa(x)$. Then k'/k is a finite separable extension of fields (Lemma 53.19.7). There is a canonical morphism

$$c : X \longrightarrow X'$$

inducing the identity on X' and mapping C to $x \in X'$ via the canonical morphism $C \to \mathrm{Spec}(k') = x$. This follows from Morphisms, Lemma 29.4.6 since X is the scheme theoretic union of C and X' (as X is reduced). Moreover, we claim that

$$c_*\mathcal{O}_X = \mathcal{O}_{X'} \quad \text{and} \quad R^1c_*\mathcal{O}_X = 0$$

To see this, denote $i_C : C \to X$, $i_{X'} : X' \to X$ and $i_x : x \to X$ the embeddings and use the exact sequence

$$0 \to \mathcal{O}_X \to i_{C,*}\mathcal{O}_C \oplus i_{X',*}\mathcal{O}_{X'} \to i_{x,*}\kappa(x) \to 0$$

of Morphisms, Lemma 29.4.6. Looking at the long exact sequence of higher direct images, it follows that it suffices to show $H^0(C, \mathcal{O}_C) = k'$ and $H^1(C, \mathcal{O}_C) = 0$ which follows from the assumptions. Observe that X' is also a proper scheme over k, of dimension 1 whose singularities are at-worst-nodal (Lemma 53.19.17) has $H^0(X', \mathcal{O}_{X'}) = k$, and X' has the same genus as X. We will say $c : X \to X'$ is the contraction of a rational tail.

0E63 Lemma 53.22.2. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational tail (Example 53.22.1). Then $\deg(\omega_X|_C) < 0$.

Proof. Let $X' \subset X$ be as in the example. Then we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemmas 53.4.6, 53.19.16, and 53.19.17. With k' as in the example we see that $\deg(\omega_C) = -2[k' : k]$ as $C \cong \mathbf{P}^1_{k'}$ by Proposition 53.10.4 and $\deg(C \cap X') = [k' : k]$. Hence $\deg(\omega_X|_C) = -[k' : k]$ which is negative. $\square$

0E3I Lemma 53.22.3. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational tail (Example 53.22.1). For any field extension K/k the base change $C_K \subset X_K$ is a finite disjoint union of rational tails.

Proof. Let $x \in C$ and $k' = \kappa(x)$ be as in the example. Observe that $C \cong \mathbf{P}^1_{k'}$ by Proposition 53.10.4. Since k'/k is finite separable, we see that $k' \otimes_k K = K'_1 \times \ldots \times K'_n$ is a finite product of finite separable extensions K'_i/K. Set $C_i = \mathbf{P}^1_{K'_i}$ and denote $x_i \in C_i$ the inverse image of x. Then $C_K = \coprod C_i$ and $X'_K \cap C_i = x_i$ as desired. $\square$

0E3J Lemma 53.22.4. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. If X does not have a rational tail (Example 53.22.1), then for every reduced connected closed subscheme $Y \subset X$, $Y \neq X$ of dimension 1 we have $\deg(\omega_X|_Y) \geq \dim_k H^1(Y, \mathcal{O}_Y)$.

Proof. Let $Y \subset X$ be as in the statement. Then $k' = H^0(Y, \mathcal{O}_Y)$ is a field and a finite extension of k and $[k' : k]$ divides all numerical invariants below associated to Y and coherent sheaves on Y, see Varieties, Lemma 33.44.10. Let $Z \subset X$ be as in

Lemma 53.4.6. We will use the results of this lemma and of Lemmas 53.19.16 and 53.19.17 without further mention. Then we get a short exact sequence

$$0 \to \omega_Y \to \omega_X|_Y \to \mathcal{O}_{Y \cap Z} \to 0$$

See Lemma 53.4.6. We conclude that

$$\deg(\omega_X|_Y) = \deg(Y \cap Z) + \deg(\omega_Y) = \deg(Y \cap Z) - 2\chi(Y, \mathcal{O}_Y)$$

Hence, if the lemma is false, then

$$2[k' : k] > \deg(Y \cap Z) + \dim_k H^1(Y, \mathcal{O}_Y)$$

Since $Y \cap Z$ is nonempty and by the divisiblity mentioned above, this can happen only if $Y \cap Z$ is a single k'-rational point of the smooth locus of Y and $H^1(Y, \mathcal{O}_Y) = 0$. If Y is irreducible, then this implies Y is a rational tail. If Y is reducible, then since $\deg(\omega_X|_Y) = -[k' : k]$ we find there is some irreducible component C of Y such that $\deg(\omega_X|_C) < 0$, see Varieties, Lemma 33.44.6. Then the analysis above applied to C gives that C is a rational tail. □

0E3K Lemma 53.22.5. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Assume X does not have a rational tail (Example 53.22.1). If

(1) the genus of X is 0, then X is isomorphic to an irreducible plane conic and $\omega_X^{\otimes -1}$ is very ample,
(2) the genus of X is 1, then $\omega_X \cong \mathcal{O}_X$,
(3) the genus of X is ≥ 2, then $\omega_X^{\otimes m}$ is globally generated for $m \geq 2$.

Proof. By Lemma 53.19.16 we find that X is Gorenstein, i.e., ω_X is an invertible $\mathcal{O}_X$-module.

If the genus of X is zero, then $\deg(\omega_X) < 0$, hence if X has more than one irreducible component, we get a contradiction with Lemma 53.22.4. In the irreducible case we see that X is isomorphic to an irreducible plane conic and $\omega_X^{\otimes -1}$ is very ample by Lemma 53.10.3.

If the genus of X is 1, then ω_X has a global section and $\deg(\omega_X|_C) = 0$ for all irreducible components. Namely, $\deg(\omega_X|_C) \geq 0$ for all irreducible components C by Lemma 53.22.4, the sum of these numbers is 0 by Lemma 53.8.3, and we can apply Varieties, Lemma 33.44.6. Then $\omega_X \cong \mathcal{O}_X$ by Varieties, Lemma 33.44.13.

Assume the genus g of X is greater than or equal to 2. If X is irreducible, then we are done by Lemma 53.21.3. Assume X reducible. By Lemma 53.22.4 the inequalities of Lemma 53.21.7 hold for every $Y \subset X$ as in the statement, except for $Y = X$. Analyzing the proof of Lemma 53.21.7 we see that (in the reducible case) the only inequality used for $Y = X$ are

$$\deg(\omega_X^{\otimes m}) > -2\chi(\mathcal{O}_X) \quad \text{and} \quad \deg(\omega_X^{\otimes m}) + \chi(\mathcal{O}_X) > \dim_k H^1(X, \mathcal{O}_X)$$

Since these both hold under the assumption $g \geq 2$ and $m \geq 2$ we win. □

0E3L Lemma 53.22.6. Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Consider a sequence

$$X = X_0 \to X_1 \to \ldots \to X_n = X'$$

of contractions of rational tails (Example 53.22.1) until none are left. Then

(1) if the genus of X is 0, then X' is an irreducible plane conic,
(2) if the genus of X is 1, then $\omega_{X'} \cong \mathcal{O}_X$,
(3) if the genus of X is > 1, then $\omega_{X'}^{\otimes m}$ is globally generated for $m \geq 2$.

If the genus of X is ≥ 1, then the morphism $X \to X'$ is independent of choices and formation of this morphism commutes with base field extensions.

Proof. We proceed by contracting rational tails until there are none left. Then we see that (1), (2), (3) hold by Lemma 53.22.5.

Uniqueness. To see that $f : X \to X'$ is independent of the choices made, it suffices to show: any rational tail $C \subset X$ is mapped to a point by $X \to X'$; some details omitted. If not, then we can find a section $s \in \Gamma(X', \omega_{X'}^{\otimes 2})$ which does not vanish in the generic point of the irreducible component $f(C)$. Since in each of the contractions $X_i \to X_{i+1}$ we have a section $X_{i+1} \to X_i$, there is a section $X' \to X$ of f. Then we have an exact sequence

$$0 \to \omega_{X'} \to \omega_X \to \omega_X|_{X''} \to 0$$

where $X'' \subset X$ is the union of the irreducible components contracted by f. See Lemma 53.4.6. Thus we get a map $\omega_{X'}^{\otimes 2} \to \omega_X^{\otimes 2}$ and we can take the image of s to get a section of $\omega_X^{\otimes 2}$ not vanishing in the generic point of C. This is a contradiction with the fact that the restriction of ω_X to a rational tail has negative degree (Lemma 53.22.2).

The statement on base field extensions follows from Lemma 53.22.3. Some details omitted. $\square$

53.23. Contracting rational bridges

0E7M In this section we discuss the next simplest possible case (after the case discussed in Section 53.22) of contracting a scheme to improve positivity properties of its canonical sheaf.

0E3M Example 53.23.1 (Contracting a rational bridge). Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. A rational bridge will be an irreducible component $C \subset X$ (viewed as an integral closed subscheme) with the following properties

(1) $X' \neq \emptyset$ where $X' \subset X$ is the scheme theoretic closure of $X \setminus C$,
(2) the scheme theoretic interesection $C \cap X'$ has degree 2 over $H^0(C, \mathcal{O}_C)$, and
(3) C has genus zero.

Set $k' = H^0(C, \mathcal{O}_C)$ and $k'' = H^0(C \cap X', \mathcal{O}_{C \cap X'})$. Then k' is a field (Varieties, Lemma 33.9.3) and $\dim_{k'}(k'') = 2$. Since there are at least two irreducible components of X passing through each point of $C \cap X'$, we conclude these points are nodes of X and smooth points on both C and X' (Lemma 53.19.17). Hence k'/k is a finite separable extension of fields and k''/k' is either a degree 2 separable extension of fields or $k'' = k' \times k'$ (Lemma 53.19.7). By Section 53.14 there exists a pushout

$$\begin{tikzcd} C \cap X' \arrow[r] \arrow[d] & X' \arrow[d, "a"] \\ \mathrm{Spec}(k') \arrow[r] & Y \end{tikzcd}$$

with many good properties (all of which we will use below without futher mention). Let $y \in Y$ be the image of $\mathrm{Spec}(k') \to Y$. Then

$$\mathcal{O}^\wedge_{Y,y} \cong k'[[s,t]]/(st) \quad \text{or} \quad \mathcal{O}^\wedge_{Y,y} \cong \{f \in k''[[s]] : f(0) \in k'\}$$

depending on whether $C \cap X'$ has 2 or 1 points. This follows from Lemma 53.14.1 and the fact that $\mathcal{O}_{X',p} \cong \kappa(p)[[t]]$ for $p \in C \cap X'$ by More on Algebra, Lemma 15.38.4. Thus we see that $y \in Y$ is a node, see Lemmas 53.19.7 and 53.19.4 and in particular the discussion of Case II in the proof of (2) $\Rightarrow$ (1) in Lemma 53.19.4. Thus the singularities of Y are at-worst-nodal.

We can extend the commutative diagram above to a diagram

$$\begin{tikzcd}
C \cap X' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "a"] & X \arrow[dl, "c"] & C \arrow[l] \arrow[dl] \\
\mathrm{Spec}(k') \arrow[r] & Y & \mathrm{Spec}(k') \arrow[l] &
\end{tikzcd}$$

where the two lower horizontal arrows are the same. Namely, X is the scheme theoretic union of X' and C (thus a pushout by Morphisms, Lemma 29.4.6) and the morphisms $C \to Y$ and $X' \to Y$ agree on $C \cap X'$. Finally, we claim that

$$c_*\mathcal{O}_X = \mathcal{O}_Y \quad \text{and} \quad R^1c_*\mathcal{O}_X = 0$$

To see this use the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_C \oplus \mathcal{O}_{X'} \to \mathcal{O}_{C\cap X'} \to 0$$

of Morphisms, Lemma 29.4.6. The long exact sequence of higher direct images is

$$0 \to c_*\mathcal{O}_X \to c_*\mathcal{O}_C \oplus c_*\mathcal{O}_{X'} \to c_*\mathcal{O}_{C\cap X'} \to R^1c_*\mathcal{O}_X \to R^1c_*\mathcal{O}_C \oplus R^1c_*\mathcal{O}_{X'}$$

Since $c|_{X'} = a$ is affine we see that $R^1c_*\mathcal{O}_{X'} = 0$. Since $c|_C$ factors as $C \to \mathrm{Spec}(k') \to X$ and since C has genus zero, we find that $R^1c_*\mathcal{O}_C = 0$. Since $\mathcal{O}_{X'} \to \mathcal{O}_{C\cap X'}$ is surjective and since $c|_{X'}$ is affine, we see that $c_*\mathcal{O}_{X'} \to c_*\mathcal{O}_{C\cap X'}$ is surjective. This proves that $R^1c_*\mathcal{O}_X = 0$. Finally, we have $\mathcal{O}_Y = c_*\mathcal{O}_X$ by the exact sequence and the description of the structure sheaf of the pushout in More on Morphisms, Proposition 37.67.3.

All of this means that Y is also a proper scheme over k having dimension 1 and $H^0(Y, \mathcal{O}_Y) = k$ whose singularities are at-worst-nodal (Lemma 53.19.17) and that Y has the same genus as X. We will say $c : X \to Y$ is the contraction of a rational bridge.

0E64 **Lemma 53.23.2.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational bridge (Example 53.23.1). Then $\deg(\omega_X|_C) = 0$.

Proof. Let $X' \subset X$ be as in the example. Then we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C\cap X'} \to 0$$

See Lemmas 53.4.6, 53.19.16, and 53.19.17. With $k''/k'/k$ as in the example we see that $\deg(\omega_C) = -2[k' : k]$ as C has genus 0 (Lemma 53.5.2) and $\deg(C \cap X') = [k'' : k] = 2[k' : k]$. Hence $\deg(\omega_X|_C) = 0$. $\square$

0E65 **Lemma 53.23.3.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational bridge (Example 53.23.1). For any field extension K/k the base change $C_K \subset X_K$ is a finite disjoint union of rational bridges.

Proof. Let $k''/k'/k$ be as in the example. Since k'/k is finite separable, we see that $k' \otimes_k K = K'_1 \times \ldots \times K'_n$ is a finite product of finite separable extensions K'_i/K. The corresponding product decomposition $k'' \otimes_k K = \prod K''_i$ gives degree 2 separable algebra extensions K''_i/K'_i. Set $C_i = C_{K'_i}$. Then $C_K = \coprod C_i$ and therefore each C_i has genus 0 (viewed as a curve over K'_i), because $H^1(C_K, \mathcal{O}_{C_K}) = 0$ by flat base change. Finally, we have $X'_K \cap C_i = \mathrm{Spec}(K''_i)$ has degree 2 over K'_i as desired. $\square$

0E3N **Lemma 53.23.4.** Let $c : X \to Y$ be the contraction of a rational bridge (Example 53.23.1). Then $c^*\omega_Y \cong \omega_X$.

Proof. You can prove this by direct computation, but we prefer to use the characterization of ω_X as the coherent $\mathcal{O}_X$-module which represents the functor $\mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Sets}$, $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k) = H^1(X, \mathcal{F})^\vee$, see Lemma 53.4.2 or Duality for Schemes, Lemma 48.22.5.

To be precise, denote $\mathcal{C}_Y$ the category whose objects are invertible $\mathcal{O}_Y$-modules and whose maps are $\mathcal{O}_Y$-module homomorphisms. Denote $\mathcal{C}_X$ the category whose objects are invertible $\mathcal{O}_X$-modules $\mathcal{L}$ with $\mathcal{L}|_C \cong \mathcal{O}_C$ and whose maps are $\mathcal{O}_Y$-module homomorphisms. We claim that the functor

$$c^* : \mathcal{C}_Y \to \mathcal{C}_X$$

is an equivalence of categories. Namely, by More on Morphisms, Lemma 37.72.8 it is essentially surjective. Then the projection formula (Cohomology, Lemma 20.54.2) shows $c_*c^*\mathcal{N} = \mathcal{N}$ and hence c^* is an equivalence with quasi-inverse given by c_*.

We claim ω_X is an object of $\mathcal{C}_X$. Namely, we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemma 53.4.6. Taking degrees we find $\deg(\omega_X|_C) = 0$ (small detail omitted). Thus $\omega_X|_C$ is trivial by Lemma 53.10.1 and ω_X is an object of $\mathcal{C}_X$.

Since $R^1c_*\mathcal{O}_X = 0$ the projection formula shows that $R^1c_*c^*\mathcal{N} = 0$ for $\mathcal{N} \in \mathrm{Ob}(\mathcal{C}_Y)$. Therefore the Leray spectral sequence (Cohomology, Lemma 20.13.6) the diagram

$$\begin{tikzcd} \mathcal{C}_Y \arrow[rr, "c^*"'] \arrow[dr, "H^1(Y,-)^\vee"'] & & \mathcal{C}_X \arrow[dl, "H^1(X,-)^\vee"] \\ & \mathrm{Sets} & \end{tikzcd}$$

of categories and functors is commutative. Since $\omega_Y \in \mathrm{Ob}(\mathcal{C}_Y)$ represents the south-east arrow and $\omega_X \in \mathrm{Ob}(\mathcal{C}_X)$ represents the south-east arrow we conclude by the Yoneda lemma (Categories, Lemma 4.3.5). $\square$

0E3P **Lemma 53.23.5.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume

(1) the singularities of X are at-worst-nodal,
(2) X does not have a rational tail (Example 53.22.1),
(3) X does not have a rational bridge (Example 53.23.1),

(4) the genus g of X is ≥ 2.

Then ω_X is ample.

Proof. It suffices to show that $\deg(\omega_X|_C) > 0$ for every irreducible component C of X, see Varieties, Lemma 33.44.15. If $X = C$ is irreducible, this follows from $g \geq 2$ and Lemma 53.8.3. Otherwise, set $k' = H^0(C, \mathcal{O}_C)$. This is a field and a finite extension of k and $[k' : k]$ divides all numerical invariants below associated to C and coherent sheaves on C, see Varieties, Lemma 33.44.10. Let $X' \subset X$ be the closure of $X \setminus C$ as in Lemma 53.4.6. We will use the results of this lemma and of Lemmas 53.19.16 and 53.19.17 without further mention. Then we get a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemma 53.4.6. We conclude that

$$\deg(\omega_X|_C) = \deg(C \cap X') + \deg(\omega_C) = \deg(C \cap X') - 2\chi(C, \mathcal{O}_C)$$

Hence, if the lemma is false, then

$$2[k' : k] \geq \deg(C \cap X') + 2\dim_k H^1(C, \mathcal{O}_C)$$

Since $C \cap X'$ is nonempty and by the divisiblity mentioned above, this can happen only if either

(a) $C \cap X'$ is a single k'-rational point of C and $H^1(C, \mathcal{O}_C) = 0$, and
(b) $C \cap X'$ has degree 2 over k' and $H^1(C, \mathcal{O}_C) = 0$.

The first possibility means C is a rational tail and the second that C is a rational bridge. Since both are excluded the proof is complete. □

0E3Q Lemma 53.23.6. Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal and that X has no rational tails. Consider a sequence

$$X = X_0 \to X_1 \to \ldots \to X_n = X'$$

of contractions of rational bridges (Example 53.23.1) until none are left. Then $\omega_{X'}$ ample. The morphism $X \to X'$ is independent of choices and formation of this morphism commutes with base field extensions.

Proof. We proceed by contracting rational bridges until there are none left. Then $\omega_{X'}$ is ample by Lemma 53.23.5.

Denote $f : X \to X'$ the composition. By Lemma 53.23.4 and induction we see that $f^*\omega_{X'} = \omega_X$. We have $f_*\mathcal{O}_X = \mathcal{O}_{X'}$ because this is true for contraction of a rational bridge. Thus the projection formula says that $f_*f^*\mathcal{L} = \mathcal{L}$ for all invertible $\mathcal{O}_{X'}$-modules $\mathcal{L}$. Hence

$$\Gamma(X', \omega_{X'}^{\otimes m}) = \Gamma(X, \omega_X^{\otimes m})$$

for all m. Since X' is the Proj of the direct sum of these by Morphisms, Lemma 29.43.17 we conclude that the morphism $X \to X'$ is completely canonical.

Let K/k be an extension of fields, then ω_{X_K} is the pullback of ω_X (Lemma 53.4.4) and we have $\Gamma(X, \omega_X^{\otimes m}) \otimes_k K$ is equal to $\Gamma(X_K, \omega_{X_K}^{\otimes m})$ by Cohomology of Schemes, Lemma 30.5.2. Thus formation of $f : X \to X'$ commutes with base change by K/k by the arguments given above. Some details omitted. □

53.24. Contracting to a stable curve

0E7N In this section we combine the contraction morphisms found in Sections 53.22 and 53.23. Namely, suppose that k is a field and let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal. Composing the morphism of Lemma 53.22.6 with the morphism of Lemma 53.23.6 we get a morphism

$$c : X \longrightarrow Y$$

such that Y also is a proper scheme over k of dimension 1 whose singularities are at worst nodal, with $k = H^0(Y, \mathcal{O}_Y)$ and having genus g, such that $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$, and such that ω_Y is ample on Y. Lemma 53.24.2 shows these conditions in fact characterize this morphism.

0E7P **Lemma 53.24.1.** Let k be a field. Let $c : X \to Y$ be a morphism of proper schemes over k Assume

(1) $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$,
(2) X and Y are reduced, Gorenstein, and have dimension 1,
(3) $\exists\ m \in \mathbf{Z}$ with $H^1(X, \omega_X^{\otimes m}) = 0$ and $\omega_X^{\otimes m}$ generated by global sections.

Then $c^*\omega_Y \cong \omega_X$.

Proof. The fibres of c are geometrically connected by More on Morphisms, Theorem 37.53.4. In particular c is surjective. There are finitely many closed points $y = y_1, \ldots, y_r$ of Y where X_y has dimension 1 and over $Y \setminus \{y_1, \ldots, y_r\}$ the morphism c is an isomorphism. Some details omitted; hint: outside of $\{y_1, \ldots, y_r\}$ the morphism c is finite, see Cohomology of Schemes, Lemma 30.21.1.

Let us carefully construct a map $b : c^*\omega_Y \to \omega_X$. Denote $f : X \to \mathrm{Spec}(k)$ and $g : Y \to \mathrm{Spec}(k)$ the structure morphisms. We have $f^!k = \omega_X[1]$ and $g^!k = \omega_Y[1]$, see Lemma 53.4.1 and its proof. Then $f^! = c^! \circ g^!$ and hence $c^!\omega_Y = \omega_X$. Thus there is a functorial isomorphism

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{F}, \omega_X) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rc_*\mathcal{F}, \omega_Y)$$

for coherent $\mathcal{O}_X$-modules $\mathcal{F}$ by definition of $c^!$[7]. This isomorphism is induced by a trace map $t : Rc_*\omega_X \to \omega_Y$ (the counit of the adjunction). By the projection formula (Cohomology, Lemma 20.54.2) the canonical map $a : \omega_Y \to Rc_*c^*\omega_Y$ is an isomorphism. Combining the above we see there is a canonical map $b : c^*\omega_Y \to \omega_X$ such that

$$t \circ Rc_*(b) = a^{-1}$$

In particular, if we restrict b to $c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ then it is an isomorphism (because it is a map between invertible modules whose composition with another gives the isomorphism a^{-1}).

Choose $m \in \mathbf{Z}$ as in (3) consider the map

$$b^{\otimes m} : \Gamma(Y, \omega_Y^{\otimes m}) \longrightarrow \Gamma(X, \omega_X^{\otimes m})$$

This map is injective because Y is reduced and by the last property of b mentioned in its construction. By Riemann-Roch (Lemma 53.5.2) we have $\chi(X, \omega_X^{\otimes m}) = \chi(Y, \omega_Y^{\otimes m})$. Thus

$$\dim_k \Gamma(Y, \omega_Y^{\otimes m}) \geq \dim_k \Gamma(X, \omega_X^{\otimes m}) = \chi(X, \omega_X^{\otimes m})$$

[7] As the restriction of the right adjoint of Duality for Schemes, Lemma 48.3.1 to $D^+_{QCoh}(\mathcal{O}_Y)$.

and we conclude $b^{\otimes m}$ induces an isomorphism on global sections. So $b^{\otimes m} : c^*\omega_Y^{\otimes m} \to \omega_X^{\otimes m}$ is surjective as generators of $\omega_X^{\otimes m}$ are in the image. Hence $b^{\otimes m}$ is an isomorphism. Thus b is an isomorphism. $\square$

0E7Q Lemma 53.24.2. Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal. There is a unique morphism (up to unique isomorphism)

$$c : X \longrightarrow Y$$

of schemes over k having the following properties:

(1) Y is proper over k, $\dim(Y) = 1$, the singularities of Y are at-worst-nodal,
(2) $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$, and
(3) ω_Y is ample on Y.

Proof. Existence: A morphism with all the properties listed exists by combining Lemmas 53.22.6 and 53.23.6 as discussed in the introduction to this section. Moreover, we see that it can be written as a composition

$$X \to X_1 \to X_2 \ldots \to X_n \to X_{n+1} \to \ldots \to X_{n+n'}$$

where the first n morphisms are contractions of rational tails and the last n' morphisms are contractions of rational bridges. Note that property (2) holds for each contraction of a rational tail (Example 53.22.1) and contraction of a rational bridge (Example 53.23.1). It is easy to see that this property is inherited by compositions of morphisms.

Uniqueness: Let $c : X \to Y$ be a morphism satisfying conditions (1), (2), and (3). We will show that there is a unique isomorphism $X_{n+n'} \to Y$ compatible with the morphisms $X \to X_{n+n'}$ and c.

Before we start the proof we make some observations about c. We first observe that the fibres of c are geometrically connected by More on Morphisms, Theorem 37.53.4. In particular c is surjective. For a closed point $y \in Y$ the fibre X_y satisfies

$$H^1(X_y, \mathcal{O}_{X_y}) = 0 \quad \text{and} \quad H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$$

The first equality by More on Morphisms, Lemma 37.72.1 and the second by More on Morphisms, Lemma 37.72.4. Thus either $X_y = x$ where x is the unique point of X mapping to y and has the same residue field as y, or X_y is a 1-dimensional proper scheme over $\kappa(y)$. Observe that in the second case X_y is Cohen-Macaulay (Lemma 53.6.1). However, since X is reduced, we see that X_y must be reduced at all of its generic points (details omitted), and hence X_y is reduced by Properties, Lemma 28.12.4. It follows that the singularities of X_y are at-worst-nodal (Lemma 53.19.17). Note that the genus of X_y is zero (see above). Finally, there are only a finite number of points y where the fibre X_y has dimension 1, say $\{y_1, \ldots, y_r\}$, and $c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ maps isomorphically to $Y \setminus \{y_1, \ldots, y_r\}$ by c. Some details omitted; hint: outside of $\{y_1, \ldots, y_r\}$ the morphism c is finite, see Cohomology of Schemes, Lemma 30.21.1.

Let $C \subset X$ be a rational tail. We claim that c maps C to a point. Assume that this is not the case to get a contradiction. Then the image of C is an irreducible component $D \subset Y$. Recall that $H^0(C, \mathcal{O}_C) = k'$ is a finite separable extension of k and that C has a k'-rational point x which is also the unique intersection of C with the "rest" of X. We conclude from the general discussion above that

$C \setminus \{x\} \subset c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ maps isomorphically to an open V of D. Let $y = c(x) \in D$. Observe that y is the only point of D meeting the "rest" of Y. If $y \notin \{y_1, \ldots, y_r\}$, then $C \cong D$ and it is clear that D is a rational tail of Y which is a contradiction with the ampleness of ω_Y (Lemma 53.22.2). Thus $y \in \{y_1, \ldots, y_r\}$ and $\dim(X_y) = 1$. Then $x \in X_y \cap C$ and x is a smooth point of X_y and C (Lemma 53.19.17). If $y \in D$ is a singular point of D, then y is a node and then $Y = D$ (because there cannot be another component of Y passing through y by Lemma 53.19.17). Then $X = X_y \cup C$ which means $g = 0$ because it is equal to the genus of X_y by the discussion in Example 53.22.1; a contradiction. If $y \in D$ is a smooth point of D, then $C \to D$ is an isomorphism (because the nonsingular projective model is unique and C and D are birational, see Section 53.2). Then D is a rational tail of Y which is a contradiction with ampleness of ω_Y.

Assume $n \geq 1$. If $C \subset X$ is the rational tail contracted by $X \to X_1$, then we see that C is mapped to a point of Y by the previous paragraph. Hence $c : X \to Y$ factors through $X \to X_1$ (because X is the pushout of C and X_1, see discussion in Example 53.22.1). After replacing X by X_1 we have decreased n. By induction we may assume $n = 0$, i.e., X does not have a rational tail.

Assume $n = 0$, i.e., X does not have any rational tails. Then $\omega_X^{\otimes 2}$ and $\omega_X^{\otimes 3}$ are globally generated by Lemma 53.22.5. It follows that $H^1(X, \omega_X^{\otimes 3}) = 0$ by Lemma 53.6.4. By Lemma 53.24.1 applied with $m = 3$ we find that $c^*\omega_Y \cong \omega_X$. We also have that $\omega_X = (X \to X_{n'})^*\omega_{X_{n'}}$ by Lemma 53.23.4 and induction. Applying the projection formula for both c and $X \to X_{n'}$ we conclude that

$$\Gamma(X_{n'}, \omega_{X_{n'}}^{\otimes m}) = \Gamma(X, \omega_X^{\otimes m}) = \Gamma(Y, \omega_Y^{\otimes m})$$

for all m. Since $X_{n'}$ and Y are the Proj of the direct sum of these by Morphisms, Lemma 29.43.17 we conclude that there is a canonical isomorphism $X_{n'} = Y$ as desired. We omit the verification that this is the unique isomorphism making the diagram commute. $\square$

0E8X Lemma 53.24.3. Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal and ω_X is ample. Then $\omega_X^{\otimes 3}$ is very ample and $H^1(X, \omega_X^{\otimes 3}) = 0$.

Proof. Combining Varieties, Lemma 33.44.15 and Lemmas 53.22.2 and 53.23.2 we see that X contains no rational tails or bridges. Then we see that $\omega_X^{\otimes 3}$ is globally generated by Lemma 53.22.6. Choose a k-basis $s_0, \ldots, s_n$ of $H^0(X, \omega_X^{\otimes 3})$. We get a morphism

$$\varphi_{\omega_X^{\otimes 3}, (s_0, \ldots, s_n)} : X \longrightarrow \mathbf{P}^n_k$$

See Constructions, Section 27.13. The lemma asserts that this morphism is a closed immersion. To check this we may replace k by its algebraic closure, see Descent, Lemma 35.23.19. Thus we may assume k is algebraically closed.

Assume k is algebraically closed. We will use Varieties, Lemma 33.23.2 to prove the lemma. Let $Z \subset X$ be a closed subscheme of degree 2 over Z with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. We have to show that

$$H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}|_Z)$$

is surjective. Thus it suffices to show that $H^1(X, \mathcal{I}\mathcal{L}) = 0$. To do this we will use Lemma 53.21.6. Thus it suffices to show that

$$3\deg(\omega_X|_Y) > -2\chi(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

for every reduced connected closed subscheme $Y \subset X$. Since k is algebraically closed and Y connected and reduced we have $H^0(Y, \mathcal{O}_Y) = k$ (Varieties, Lemma 33.9.3). Hence $\chi(Y, \mathcal{O}_Y) = 1 - \dim H^1(Y, \mathcal{O}_Y)$. Thus we have to show

$$3\deg(\omega_X|_Y) > -2 + 2\dim H^1(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

which is true by Lemma 53.22.4 except possibly if $Y = X$ or if $\deg(\omega_X|_Y) = 0$. Since ω_X is ample the second possibility does not occur (see first lemma cited in this proof). Finally, if $Y = X$ we can use Riemann-Roch (Lemma 53.5.2) and the fact that $g \geq 2$ to see that the inquality holds. The same argument with $Z = \emptyset$ shows that $H^1(X, \omega_X^{\otimes 3}) = 0$. $\square$

53.25. Vector fields

0E66 In this section we study the space of vector fields on a curve. Vector fields correspond to infinitesimal automorphisms, see More on Morphisms, Section 37.9, hence play an important role in moduli theory.

Let k be an algebraically closed field. Let X be a finite type scheme over k. Let $x \in X$ be a closed point. We will say an element $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ fixes x if $D(\mathcal{I}) \subset \mathcal{I}$ where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of x.

0E67 **Lemma 53.25.1.** *Let k be an algebraically closed field. Let X be a smooth, proper, connected curve over k. Let g be the genus of X.*

1. *If $g \geq 2$, then $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ is zero,*
2. *if $g = 1$ and $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D does not fix any closed point of X, and*
3. *if $g = 0$ and $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D fixes at most 2 closed points of X.*

Proof. Recall that we have a universal k-derivation $d : \mathcal{O}_X \to \Omega_{X/k}$ and hence $D = \theta \circ d$ for some $\mathcal{O}_X$-linear map $\theta : \Omega_{X/k} \to \mathcal{O}_X$. Recall that $\Omega_{X/k} \cong \omega_X$, see Lemma 53.4.1. By Riemann-Roch we have $\deg(\omega_X) = 2g - 2$ (Lemma 53.5.2). Thus we see that θ is forced to be zero if $g > 1$ by Varieties, Lemma 33.44.12. This proves part (1). If $g = 1$, then a nonzero θ does not vanish anywhere and if $g = 0$, then a nonzero θ vanishes in a divisor of degree 2. Thus parts (2) and (3) follow if we show that vanishing of θ at a closed point $x \in X$ is equivalent to the statement that D fixes x (as defined above). Let $z \in \mathcal{O}_{X,x}$ be a uniformizer. Then dz is a basis element for $\Omega_{X,x}$, see Lemma 53.12.3. Since $D(z) = \theta(dz)$ we conclude. $\square$

0E68 **Lemma 53.25.2.** *Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k. Let $\nu : X^\nu \to X$ be the normalization. Let $S \subset X^\nu$ be the set of points where ν is not an isomorphism. Then*

$$\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \{D' \in \text{Der}_k(\mathcal{O}_{X^\nu}, \mathcal{O}_{X^\nu}) \mid D' \text{ fixes every } x^\nu \in S\}$$

Proof. Let $x \in X$ be a node. Let $x', x'' \in X^\nu$ be the inverse images of x. (Every node is a split node since k is algebriacally closed, see Definition 53.19.10 and

Lemma 53.19.11.) Let $u \in \mathcal{O}_{X^\nu,x'}$ and $v \in \mathcal{O}_{X^\nu,x''}$ be uniformizers. Observe that we have an exact sequence

$$0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X^\nu,x'} \times \mathcal{O}_{X^\nu,x''} \to k \to 0$$

This follows from Lemma 53.16.3. Thus we can view u and v as elements of $\mathcal{O}_{X,x}$ with $uv = 0$.

Let $D \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Then $0 = D(uv) = vD(u) + uD(v)$. Since (u) is annihilator of v in $\mathcal{O}_{X,x}$ and vice versa, we see that $D(u) \in (u)$ and $D(v) \in (v)$. As $\mathcal{O}_{X^\nu,x'} = k + (u)$ we conclude that we can extend D to $\mathcal{O}_{X^\nu,x'}$ and moreover the extension fixes x'. This produces a D' in the right hand side of the equality. Conversely, given a D' fixing x' and x'' we find that D' preserves the subring $\mathcal{O}_{X,x} \subset \mathcal{O}_{X^\nu,x'} \times \mathcal{O}_{X^\nu,x''}$ and this is how we go from right to left in the equality. $\square$

0E69 **Lemma 53.25.3.** Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k. Assume the genus of X is at least 2 and that X has no rational tails or bridges. Then $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.

Proof. Let $D \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Let X^ν be the normalization of X. Let $D' \in \mathrm{Der}_k(\mathcal{O}_{X^\nu}, \mathcal{O}_{X^\nu})$ be the element corresponding to D via Lemma 53.25.2. Let $C \subset X^\nu$ be an irreducible component. If the genus of C is > 1, then $D'|_{\mathcal{O}_C} = 0$ by Lemma 53.25.1 part (1). If the genus of C is 1, then there is at least one closed point c of C which maps to a node on X (since otherwise $X \cong C$ would have genus 1). By the correspondence this means that $D'|_{\mathcal{O}_C}$ fixes c hence is zero by Lemma 53.25.1 part (2). Finally, if the genus of C is zero, then there are at least 3 pairwise distinct closed points $c_1, c_2, c_3 \in C$ mapping to nodes in X, since otherwise either X is C with two points glued (two points of C mapping to the same node), or C is a rational bridge (two points mapping to different nodes of X), or C is a rational tail (one point mapping to a node of X). These three possibilities are not permitted since C has genus ≥ 2 and has no rational bridges, or rational tails. Whence $D'|_{\mathcal{O}_C}$ fixes c_1, c_2, c_3 hence is zero by Lemma 53.25.1 part (3). $\square$

53.26. Other chapters

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CHAPTER 54

Resolution of Surfaces

0ADW

54.1. Introduction

0ADX This chapter discusses resolution of singularities of surfaces following Lipman [Lip78] and mostly following the exposition of Artin in [Art86]. The main result (Theorem 54.14.5) tells us that a Noetherian 2-dimensional scheme Y has a resolution of singularities when it has a finite normalization $Y^\nu \to Y$ with finitely many singular points $y_i \in Y^\nu$ and for each i the completion $\mathcal{O}^\wedge_{Y^\nu, y_i}$ is normal.

To be sure, if Y is a 2-dimensional scheme of finite type over a quasi-excellent base ring R (for example a field or a Dedekind domain with fraction field of characteristic 0 such as $\mathbf{Z}$) then the normalization of Y is finite, has finitely many singular points, and the completions of the local rings are normal. See the discussion in More on Algebra, Sections 15.47, 15.50, and 15.52 and More on Algebra, Lemma 15.42.2. Thus such a Y has a resolution of singularities.

A rough outline of the proof is as follows. Let A be a Noetherian local domain of dimension 2. The steps of the proof are as follows

- N replace A by its normalization,
- V prove Grauert-Riemenschneider,
- B show there is a maximum g of the lengths of $H^1(X, \mathcal{O}_X)$ over all normal modifications $X \to \text{Spec}(A)$ and reduce to the case $g = 0$,
- R we say A defines a rational singularity if $g = 0$ and in this case after a finite number of blowups we may assume A is Gorenstein and $g = 0$,
- D we say A defines a rational double point if $g = 0$ and A is Gorenstein and in this case we explicitly resolve singularities.

Each of these steps needs assumptions on the ring A. We will discuss each of these in turn.

Ad N: Here we need to assume that A has a finite normalization (this is not automatic). Throughout most of the chapter we will assume that our scheme is Nagata if we need to know some normalization is finite. However, being Nagata is a slightly stronger condition than is given to us in the statement of the theorem. A solution to this (slight) problem would have been to use that our ring A is formally unramified (i.e., its completion is reduced) and to use Lemma 54.11.5. However, the way our proof works, it turns out it is easier to use Lemma 54.11.6 to lift finiteness of the normalization over the completion to finiteness of the normalization over A.

Ad V: This is Proposition 54.7.8 and it roughly states that for a normal modification $f : X \to \text{Spec}(A)$ one has $R^1 f_*\omega_X = 0$ where ω_X is the dualizing module of X/A (Remark 54.7.7). In fact, by duality the result is equivalent to a statement (Lemma 54.7.6) about the object $Rf_*\mathcal{O}_X$ in the derived category $D(A)$. Having said this, the proof uses the standard fact that components of the special fibre have positive conormal sheaves (Lemma 54.7.4).

Ad B: This is in some sense the most subtle part of the proof. In the end we only need to use the output of this step when A is a complete Noetherian local ring, although the writeup is a bit more general. The terminology is set in Definition 54.8.3. If g (as defined above) is bounded, then a straightforward argument shows that we can find a normal modification $X \to \text{Spec}(A)$ such that all singular points of X are rational singularities, see Lemma 54.8.5. We show that given a finite extension $A \subset B$, then g is bounded for B if it is bounded for A in the following two cases: (1) if the fraction field extension is separable, see Lemma 54.8.5 and (2) if the fraction field extension has degree p, the characteristic is p, and A is regular and complete, see Lemma 54.8.10.

Ad R: Here we reduce the case $g = 0$ to the Gorenstein case. A marvellous fact, which makes everything work, is that the blowing up of a rational surface singularity is normal, see Lemma 54.9.4.

Ad D: The resolution of rational double points proceeds more or less by hand, see Section 54.12. A rational double point is a hypersurface singularity (this is true but we don't prove it as we don't need it). The local equation looks like

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = \sum a_{ijk}x_ix_jx_k$$

Using that the quadratic part cannot be zero because the multiplicity is 2 and remains 2 after any blowup and the fact that every blowup is normal one quickly achieves a resolution. One twist is that we do not have an invariant which decreases every blowup, but we rely on the material on formal arcs from Section 54.10 to demonstrate that the process stops.

To put everything together some additional work has to be done. The main kink is that we want to lift a resolution of the completion $A^\wedge$ to a resolution of $\text{Spec}(A)$. In order to do this we first show that if a resolution exists, then there is a resolution by normalized blowups (Lemma 54.14.3). A sequence of normalized blowups can be lifted from the completion by Lemma 54.11.7. We then use this even in the proof of resolution of complete local rings A because our strategy works by induction on the degree of a finite inclusion $A_0 \subset A$ with A_0 regular, see Lemma 54.14.4. With a stronger result in B (such as is proved in Lipman's paper) this step could be avoided.

54.2. A trace map in positive characteristic

0ADY Some of the results in this section can be deduced from the much more general discussion on traces on differential forms in de Rham Cohomology, Section 50.19. See Remark 54.2.3 for a discussion.

We fix a prime number p. Let R be an $\mathbf{F}_p$-algebra. Given an $a \in R$ set $S = R[x]/(x^p - a)$. Define an R-linear map

$$\text{Tr}_x : \Omega_{S/R} \longrightarrow \Omega_R$$

by the rule

$$x^i \text{d}x \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ \text{d}a & \text{if} \quad i = p-1 \end{cases}$$

This makes sense as $\Omega_{S/R}$ is a free R-module with basis $x^i\mathrm{d}x$, $0 \leq i \leq p-1$. The following lemma implies that the trace map is well defined, i.e., independent of the choice of the coordinate x.

0ADZ Lemma 54.2.1. Let $\varphi : R[x]/(x^p - a) \to R[y]/(y^p - b)$ be an R-algebra homomorphism. Then $\mathrm{Tr}_x = \mathrm{Tr}_y \circ \varphi$.

Proof. Say $\varphi(x) = \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$ with $\lambda_i \in R$. The condition that mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$ induces an R-algebra homomorphism $R[x]/(x^p - a) \to R[y]/(y^p - b)$ is equivalent to the condition that

$$a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$$

in the ring R. Consider the polynomial ring

$$R_{univ} = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$$

with the element $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ Consider the universal algebra map $\varphi_{univ} : R_{univ}[x]/(x^p - a) \to R_{univ}[y]/(y^p - b)$ given by mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$. We obtain a canonical map

$$R_{univ} \longrightarrow R$$

sending b, λ_i to b, λ_i. By construction we get a commutative diagram

$$\begin{tikzcd}
R_{univ}[x]/(x^p - a) \arrow[r] \arrow[d, "\varphi_{univ}"'] & R[x]/(x^p - a) \arrow[d, "\varphi"] \\
R_{univ}[y]/(y^p - b) \arrow[r] & R[y]/(y^p - b)
\end{tikzcd}$$

and the horizontal arrows are compatible with the trace maps. Hence it suffices to prove the lemma for the map φ_{univ}. Thus we may assume $R = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$ is a polynomial ring. We will check the lemma holds in this case by evaluating $\mathrm{Tr}_y(\varphi(x)^i\mathrm{d}\varphi(x))$ for $i = 0, \ldots, p-1$.

The case $0 \leq i \leq p-2$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1})^i(\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1}y^{p-2})$$

in the ring $R[y]/(y^p - b)$. We have to show that the coefficient of y^{p-1} is zero. For this it suffices to show that the expression above as a polynomial in y has vanishing coefficients in front of the powers y^{pk-1}. Then we write our polynomial as

$$\frac{\mathrm{d}}{(i+1)\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1})^{i+1}$$

and indeed the coefficients of y^{kp-1} are all zero.

The case $i = p-1$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1})^{p-1}(\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1}y^{p-2})$$

in the ring $R[y]/(y^p - b)$. To finish the proof we have to show that the coefficient of y^{p-1} times $\mathrm{d}b$ is $\mathrm{d}a$. Here we use that R is S/pS where $S = \mathbf{Z}[b, \lambda_0, \ldots, \lambda_{p-1}]$. Then the above, as a polynomial in y, is equal to

$$\frac{\mathrm{d}}{p\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1})^p$$

Since $\frac{\mathrm{d}}{\mathrm{d}y}(y^{pk}) = pky^{pk-1}$ it suffices to understand the coefficients of y^{pk} in the polynomial $(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1})^p$ modulo p. The sum of these terms gives

$$\lambda_0^p + \lambda_1^p y^p + \ldots + \lambda_{p-1}^p y^{p(p-1)} \bmod p$$

Whence we see that we obtain after applying the operator $\frac{\mathrm{d}}{p\mathrm{d}y}$ and after reducing modulo $y^p - b$ the value

$$\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}b^{p-2}$$

for the coefficient of y^{p-1} we wanted to compute. Now because $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ in R we obtain that

$$\mathrm{d}a = (\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}^p b^{p-2})\mathrm{d}b$$

in R. This proves that the coefficient of y^{p-1} is as desired. □

0AX5 Lemma 54.2.2. Let $\mathbf{F}_p \subset \Lambda \subset R \subset S$ be ring extensions and assume that S is isomorphic to $R[x]/(x^p - a)$ for some $a \in R$. Then there are canonical R-linear maps

$$\mathrm{Tr} : \Omega^{t+1}_{S/\Lambda} \longrightarrow \Omega^{t+1}_{R/\Lambda}$$

for $t \geq 0$ such that

$$\eta_1 \wedge \ldots \wedge \eta_t \wedge x^i \mathrm{d}x \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ \eta_1 \wedge \ldots \wedge \eta_t \wedge \mathrm{d}a & \text{if} \quad i = p-1 \end{cases}$$

for $\eta_i \in \Omega_{R/\Lambda}$ and such that Tr annihilates the image of $S \otimes_R \Omega^{t+1}_{R/\Lambda} \to \Omega^{t+1}_{S/\Lambda}$.

Proof. For $t = 0$ we use the composition

$$\Omega_{S/\Lambda} \to \Omega_{S/R} \to \Omega_R \to \Omega_{R/\Lambda}$$

where the second map is Lemma 54.2.1. There is an exact sequence

$$H_1(L_{S/R}) \xrightarrow{\delta} \Omega_{R/\Lambda} \otimes_R S \to \Omega_{S/\Lambda} \to \Omega_{S/R} \to 0$$

(Algebra, Lemma 10.134.4). The module $\Omega_{S/R}$ is free over S with basis $\mathrm{d}x$ and the module $H_1(L_{S/R})$ is free over S with basis $x^p - a$ which δ maps to $-\mathrm{d}a \otimes 1$ in $\Omega_{R/\Lambda} \otimes_R S$. In particular, if we set

$$M = \mathrm{Coker}(R \to \Omega_{R/\Lambda}, 1 \mapsto -\mathrm{d}a)$$

then we see that $\mathrm{Coker}(\delta) = M \otimes_R S$. We obtain a canonical map

$$\Omega^{t+1}_{S/\Lambda} \to \wedge^t_S(\mathrm{Coker}(\delta)) \otimes_S \Omega_{S/R} = \wedge^t_R(M) \otimes_R \Omega_{S/R}$$

Now, since the image of the map $\mathrm{Tr} : \Omega_{S/R} \to \Omega_{R/\Lambda}$ of Lemma 54.2.1 is contained in $R\mathrm{d}a$ we see that wedging with an element in the image annihilates $\mathrm{d}a$. Hence there is a canonical map

$$\wedge^t_R(M) \otimes_R \Omega_{S/R} \to \Omega^{t+1}_{R/\Lambda}$$

mapping $\overline{\eta}_1 \wedge \ldots \wedge \overline{\eta}_t \wedge \omega$ to $\eta_1 \wedge \ldots \wedge \eta_t \wedge \mathrm{Tr}(\omega)$. □

0FLF Remark 54.2.3. Let $\mathbf{F}_p \subset \Lambda \subset R \subset S$ and Tr be as in Lemma 54.2.2. By de Rham Cohomology, Proposition 50.19.3 there is a canonical map of complexes

$$\Theta_{S/R} : \Omega^\bullet_{S/\Lambda} \longrightarrow \Omega^\bullet_{R/\Lambda}$$

The computation in de Rham Cohomology, Example 50.19.4 shows that $\Theta_{S/R}(x^i \mathrm{d}x) = \mathrm{Tr}_x(x^i \mathrm{d}x)$ for all i. Since $\mathrm{Trace}_{S/R} = \Theta^0_{S/R}$ is identically zero and since

$$\Theta_{S/R}(a \wedge b) = a \wedge \Theta_{S/R}(b)$$

for $a \in \Omega^i_{R/\Lambda}$ and $b \in \Omega^j_{S/\Lambda}$ it follows that $\mathrm{Tr} = \Theta_{S/R}$. The advantage of using Tr is that it is a good deal more elementary to construct.

0AX6 Lemma 54.2.4. Let S be a scheme over $\mathbf{F}_p$. Let $f : Y \to X$ be a finite morphism of Noetherian normal integral schemes over S. Assume

(1) the extension of function fields is purely inseparable of degree p, and
(2) $\Omega_{X/S}$ is a coherent $\mathcal{O}_X$-module (for example if X is of finite type over S).

For $i \geq 1$ there is a canonical map

$$\mathrm{Tr} : f_*\Omega^i_{Y/S} \longrightarrow (\Omega^i_{X/S})^{**}$$

whose stalk in the generic point of X recovers the trace map of Lemma 54.2.2.

Proof. The exact sequence $f^*\Omega_{X/S} \to \Omega_{Y/S} \to \Omega_{Y/X} \to 0$ shows that $\Omega_{Y/S}$ and hence $f_*\Omega_{Y/S}$ are coherent modules as well. Thus it suffices to prove the trace map in the generic point extends to stalks at $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$, see Divisors, Lemma 31.12.14. Thus we reduce to the case discussed in the next paragraph.

Assume $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ with A a discrete valuation ring and B finite over A. Since the induced extension L/K of fraction fields is purely inseparable, we see that B is local too. Hence B is a discrete valuation ring too. Then either

(1) B/A has ramification index p and hence $B = A[x]/(x^p - a)$ where $a \in A$ is a uniformizer, or
(2) $\mathfrak{m}_B = \mathfrak{m}_A B$ and the residue field $B/\mathfrak{m}_A B$ is purely inseparable of degree p over $\kappa_A = A/\mathfrak{m}_A$. Choose any $x \in B$ whose residue class is not in κ_A and then we'll have $B = A[x]/(x^p - a)$ where $a \in A$ is a unit.

Let $\mathrm{Spec}(\Lambda) \subset S$ be an affine open such that X maps into $\mathrm{Spec}(\Lambda)$. Then we can apply Lemma 54.2.2 to see that the trace map extends to $\Omega^i_{B/\Lambda} \to \Omega^i_{A/\Lambda}$ for all $i \geq 1$. □

54.3. Quadratic transformations

0AGP In this section we study what happens when we blow up a nonsingular point on a surface. We hesitate the formally define such a morphism as a quadratic transformation as on the one hand often other names are used and on the other hand the phrase "quadratic transformation" is sometimes used with a different meaning.

0AGQ Lemma 54.3.1. Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$ wotj exceptional divisor E. There is a closed immersion

$$r : X \longrightarrow \mathbf{P}^1_S$$

over S such that

(1) $r|_E : E \to \mathbf{P}^1_\kappa$ is an isomorphism,
(2) $\mathcal{O}_X(E) = \mathcal{O}_X(-1) = r^*\mathcal{O}_{\mathbf{P}^1}(-1)$, and
(3) $\mathcal{C}_{E/X} = (r|_E)^*\mathcal{O}_{\mathbf{P}^1}(1)$ and $\mathcal{N}_{E/X} = (r|_E)^*\mathcal{O}_{\mathbf{P}^1}(-1)$.

Proof. As A is regular of dimension 2 we can write $\mathfrak m = (x, y)$. Then x and y placed in degree 1 generate the Rees algebra $\bigoplus_{n \geq 0} \mathfrak m^n$ over A. Recall that $X = \text{Proj}(\bigoplus_{n \geq 0} \mathfrak m^n)$, see Divisors, Lemma 31.32.2. Thus the surjection

$$A[T_0, T_1] \longrightarrow \bigoplus\nolimits_{n \geq 0} \mathfrak m^n, \quad T_0 \mapsto x,\ T_1 \mapsto y$$

of graded A-algebras induces a closed immersion $r : X \to \mathbf{P}^1_S = \text{Proj}(A[T_0, T_1])$ such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbf{P}^1_S}(1)$, see Constructions, Lemma 27.11.5. This proves (2) because $\mathcal{O}_X(E) = \mathcal{O}_X(-1)$ by Divisors, Lemma 31.32.4.

To prove (1) note that

$$\left(\bigoplus\nolimits_{n \geq 0} \mathfrak m^n\right) \otimes_A \kappa = \bigoplus\nolimits_{n \geq 0} \mathfrak m^n/\mathfrak m^{n+1} \cong \kappa[\overline{x}, \overline{y}]$$

a polynomial algebra, see Algebra, Lemma 10.106.1. This proves that the fibre of $X \to S$ over $\text{Spec}(\kappa)$ is equal to $\text{Proj}(\kappa[\overline{x}, \overline{y}]) = \mathbf{P}^1_\kappa$, see Constructions, Lemma 27.11.6. Recall that E is the closed subscheme of X defined by $\mathfrak m\mathcal{O}_X$, i.e., $E = X_\kappa$. By our choice of the morphism r we see that $r|_E$ in fact produces the identification of $E = X_\kappa$ with the special fibre of $\mathbf{P}^1_S \to S$.

Part (3) follows from (1) and (2) and Divisors, Lemma 31.14.2. $\square$

0AGR **Lemma 54.3.2.** Let $(A, \mathfrak m, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak m$. Then X is an irreducible regular scheme.

Proof. Observe that X is integral by Divisors, Lemma 31.32.9 and Algebra, Lemma 10.106.2. To see X is regular it suffices to check that $\mathcal{O}_{X,x}$ is regular for closed points $x \in X$, see Properties, Lemma 28.9.2. Let $x \in X$ be a closed point. Since f is proper x maps to $\mathfrak m$, i.e., x is a point of the exceptional divisor E. Then E is an effective Cartier divisor and $E \cong \mathbf{P}^1_\kappa$. Thus if $g \in \mathfrak m_x \subset \mathcal{O}_{X,x}$ is a local equation for E, then $\mathcal{O}_{X,x}/(g) \cong \mathcal{O}_{\mathbf{P}^1_\kappa, x}$. Since $\mathbf{P}^1_\kappa$ is covered by two affine opens which are the spectrum of a polynomial ring over κ, we see that $\mathcal{O}_{\mathbf{P}^1_\kappa, x}$ is regular by Algebra, Lemma 10.114.1. We conclude by Algebra, Lemma 10.106.7. $\square$

0C5G **Lemma 54.3.3.** Let $(A, \mathfrak m, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak m$. Then $\text{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_X(E)$.

Proof. Recall that $E = \mathbf{P}^1_\kappa$ has Picard group $\mathbf{Z}$ with generator $\mathcal{O}(1)$, see Divisors, Lemma 31.28.5. By Lemma 54.3.1 the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(E)$ restricts to $\mathcal{O}(-1)$. Hence $\mathcal{O}_X(E)$ generates an infinite cyclic group in $\text{Pic}(X)$. Since A is regular it is a UFD, see More on Algebra, Lemma 15.121.2. Then the punctured spectrum $U = S \setminus \{\mathfrak m\} = X \setminus E$ has trivial Picard group, see Divisors, Lemma 31.28.4. Hence for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ there is an isomorphism $s : \mathcal{O}_U \to \mathcal{L}|_U$. Then s is a regular meromorphic section of $\mathcal{L}$ and we see that $\text{div}_\mathcal{L}(s) = nE$ for some $n \in \mathbf{Z}$ (Divisors, Definition 31.27.4). By Divisors, Lemma 31.27.6 (and the fact that X is normal by Lemma 54.3.2) we conclude that $\mathcal{L} = \mathcal{O}_X(nE)$. $\square$

0AGS **Lemma 54.3.4.** Let $(A, \mathfrak m, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak m$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) $H^p(X, \mathcal{F}) = 0$ for $p \not\in \{0, 1\}$,
(2) $H^1(X, \mathcal{O}_X(n)) = 0$ for $n \geq -1$,
(3) $H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ or $\mathcal{F}(1)$ is globally generated,
(4) $H^0(X, \mathcal{O}_X(n)) = \mathfrak m^{\max(0,n)}$,

(5) $\text{length}_A H^1(X, \mathcal{O}_X(n)) = -n(-n-1)/2$ if $n < 0$.

Proof. If $\mathfrak{m} = (x, y)$, then X is covered by the spectra of the affine blowup algebras $A[\frac{\mathfrak{m}}{x}]$ and $A[\frac{\mathfrak{m}}{y}]$ because x and y placed in degree 1 generate the Rees algebra $\bigoplus \mathfrak{m}^n$ over A. See Divisors, Lemma 31.32.2 and Constructions, Lemma 27.8.9. Since X is separated by Constructions, Lemma 27.8.8 we see that cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology of Schemes, Lemma 30.4.2.

Let $i : E \to X$ be the exceptional divisor, see Divisors, Definition 31.32.1. Recall that $\mathcal{O}_X(-E) = \mathcal{O}_X(1)$ is f-relatively ample, see Divisors, Lemma 31.32.4. Hence we know that $H^1(X, \mathcal{O}_X(-nE)) = 0$ for some $n > 0$, see Cohomology of Schemes, Lemma 30.16.2. Consider the filtration

$$\mathcal{O}_X(-nE) \subset \mathcal{O}_X(-(n-1)E) \subset \ldots \subset \mathcal{O}_X(-E) \subset \mathcal{O}_X \subset \mathcal{O}_X(E)$$

The successive quotients are the sheaves

$$\mathcal{O}_X(-tE)/\mathcal{O}_X(-(t+1)E) = \mathcal{O}_X(t)/\mathcal{I}(t) = i_*\mathcal{O}_E(t)$$

where $\mathcal{I} = \mathcal{O}_X(-E)$ is the ideal sheaf of E. By Lemma 54.3.1 we have $E = \mathbf{P}^1_\kappa$ and $\mathcal{O}_E(1)$ indeed corresponds to the usual Serre twist of the structure sheaf on $\mathbf{P}^1$. Hence the cohomology of $\mathcal{O}_E(t)$ vanishes in degree 1 for $t \geq -1$, see Cohomology of Schemes, Lemma 30.8.1. Since this is equal to $H^1(X, i_*\mathcal{O}_E(t))$ (by Cohomology of Schemes, Lemma 30.2.4) we find that $H^1(X, \mathcal{O}_X(-(t+1)E)) \to H^1(X, \mathcal{O}_X(-tE))$ is surjective for $t \geq -1$. Hence

$$0 = H^1(X, \mathcal{O}_X(-nE)) \longrightarrow H^1(X, \mathcal{O}_X(-tE)) = H^1(X, \mathcal{O}_X(t))$$

is surjective for $t \geq -1$ which proves (2).

Let $\mathcal{F}$ be globally generated. This means there exists a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{i \in I} \mathcal{O}_X \to \mathcal{F} \to 0$$

Note that $H^1(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} H^1(X, \mathcal{O}_X)$ by Cohomology, Lemma 20.19.1. By part (2) we have $H^1(X, \mathcal{O}_X) = 0$. If $\mathcal{F}(1)$ is globally generated, then we can find a surjection $\bigoplus_{i \in I} \mathcal{O}_X(-1) \to \mathcal{F}$ and argue in a similar fashion. In other words, part (3) follows from part (2).

For part (4) we note that for all n large enough we have $\Gamma(X, \mathcal{O}_X(n)) = \mathfrak{m}^n$, see Cohomology of Schemes, Lemma 30.14.3. If $n \geq 0$, then we can use the short exact sequence

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

and the vanishing of H^1 for the sheaf on the left to get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^{\max(0,n)} \arrow[r] \arrow[d] & \mathfrak{m}^{\max(0,n-1)} \arrow[r] \arrow[d] & \mathfrak{m}^{\max(0,n)}/\mathfrak{m}^{\max(0,n-1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{O}_X(n)) \arrow[r] & \Gamma(X, \mathcal{O}_X(n-1)) \arrow[r] & \Gamma(E, \mathcal{O}_E(n-1)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. In fact, the rows are exact also for $n < 0$ because in this case the groups on the right are zero. In the proof of Lemma 54.3.1 we have seen that the right vertical arrow is an isomorphism (details omitted). Hence if the left vertical arrow is an isomorphism, so is the middle one. In this way we see that (4) holds by descending induction on n.

Finally, we prove (5) by descending induction on n and the sequences

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

Namely, for $n \geq -1$ we already know $H^1(X, \mathcal{O}_X(n)) = 0$. Since

$$H^1(X, i_*\mathcal{O}_E(-2)) = H^1(E, \mathcal{O}_E(-2)) = H^1(\mathbf{P}^1_\kappa, \mathcal{O}(-2)) \cong \kappa$$

by Cohomology of Schemes, Lemma 30.8.1 which has length 1 as an A-module, we conclude from the long exact cohomology sequence that (5) holds for $n = -2$. And so on and so forth. $\square$

0AGT Lemma 54.3.5. Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathfrak{m}^n \subset I \subset \mathfrak{m}$ be an ideal. Let $d \geq 0$ be the largest integer such that

$$I\mathcal{O}_X \subset \mathcal{O}_X(-dE)$$

where E is the exceptional divisor. Set $\mathcal{I}' = I\mathcal{O}_X(dE) \subset \mathcal{O}_X$. Then $d > 0$, the sheaf $\mathcal{O}_X/\mathcal{I}'$ is supported in finitely many closed points $x_1, \ldots, x_r$ of X, and

$$\begin{aligned} \text{length}_A(A/I) &> \text{length}_A \Gamma(X, \mathcal{O}_X/\mathcal{I}') \\ &\geq \sum\nolimits_{i=1,\ldots,r} \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \end{aligned}$$

Proof. Since $I \subset \mathfrak{m}$ we see that every element of I vanishes on E. Thus we see that $d \geq 1$. On the other hand, since $\mathfrak{m}^n \subset I$ we see that $d \leq n$. Consider the short exact sequence

$$0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to 0$$

Since $I\mathcal{O}_X$ is globally generated, we see that $H^1(X, I\mathcal{O}_X) = 0$ by Lemma 54.3.4. Hence we obtain a surjection $A/I \to \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-dE)/I\mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X(-dE) \to 0$$

By Divisors, Lemma 31.15.8 we see that $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ is supported in finitely many closed points of X. In particular, this coherent sheaf has vanishing higher cohomology groups (detail omitted). Thus in the following diagram

$$\begin{tikzcd} & & A/I \arrow[d] & & \\ 0 \arrow[r] & \Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) \arrow[r] & \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X) \arrow[r] & \Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE)) \arrow[r] & 0 \end{tikzcd}$$

the bottom row is exact and the vertical arrow surjective. We have

$$\text{length}_A \Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) < \text{length}_A(A/I)$$

since $\Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE))$ is nonzero. Namely, the image of $1 \in \Gamma(X, \mathcal{O}_X)$ is nonzero as $d > 0$.

To finish the proof we translate the results above into the statements of the lemma. Since $\mathcal{O}_X(dE)$ is invertible we have

$$\mathcal{O}_X/\mathcal{I}' = \mathcal{O}_X(-dE)/I\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(dE).$$

Thus $\mathcal{O}_X/\mathcal{I}'$ and $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ are supported in the same set of finitely many closed points, say $x_1, \ldots, x_r \in E \subset X$. Moreover we obtain

$$\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) = \bigoplus \mathcal{O}_X(-dE)_{x_i}/I\mathcal{O}_{X,x_i} \cong \bigoplus \mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i} = \Gamma(X, \mathcal{O}_X/\mathcal{I}')$$

because an invertible module over a local ring is trivial. Thus we obtain the strict inequality. We also get the second because

$$\text{length}_A(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \geq \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i})$$

as is immediate from the definition of length. □

0B4L Lemma 54.3.6. Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then $\Omega_{X/S} = i_*\Omega_{E/\kappa}$, where $i : E \to X$ is the immersion of the exceptional divisor.

Proof. Writing $\mathbf{P}^1 = \mathbf{P}^1_S$, let $r : X \to \mathbf{P}^1$ be as in Lemma 54.3.1. Then we have an exact sequence

$$\mathcal{C}_{X/\mathbf{P}^1} \to r^*\Omega_{\mathbf{P}^1/S} \to \Omega_{X/S} \to 0$$

see Morphisms, Lemma 29.32.15. Since $\Omega_{\mathbf{P}^1/S}|_E = \Omega_{E/\kappa}$ by Morphisms, Lemma 29.32.10 it suffices to see that the first arrow defines a surjection onto the kernel of the canonical map $r^*\Omega_{\mathbf{P}^1/S} \to i_*\Omega_{E/\kappa}$. This we can do locally. With notation as in the proof of Lemma 54.3.1 on an affine open of X the morphism f corresponds to the ring map

$$A \to A[t]/(xt - y)$$

where $x, y \in \mathfrak{m}$ are generators. Thus $\text{d}(xt - y) = x\text{d}t$ and $y\text{d}t = t \cdot x\text{d}t$ which proves what we want. □

54.4. Dominating by quadratic transformations

0BFS Using the result above we can prove that blowups in points dominate any modification of a regular 2 dimensional scheme.

Let X be a scheme. Let $x \in X$ be a closed point. As usual, we view $i : x = \text{Spec}(\kappa(x)) \to X$ as a closed subscheme. The blowing up $X' \to X$ of X at x is the blowing up of X in the closed subscheme $x \subset X$. Observe that if X is locally Noetherian, then $X' \to X$ is projective (in particular proper) by Divisors, Lemma 31.32.13.

0AHH Lemma 54.4.1. Let X be a Noetherian scheme. Let $T \subset X$ be a finite set of closed points x such that $\mathcal{O}_{X,x}$ is regular of dimension 2 for $x \in T$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Denote I_i the stalk of $\mathcal{I}$ at x_i. Set

$$n_i = \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/I_i)$$

This is finite as $\mathcal{O}_X/\mathcal{I}$ is supported on T and hence $\mathcal{O}_{X,x_i}/I_i$ has support equal to $\{\mathfrak{m}_{x_i}\}$ (see Algebra, Lemma 10.62.3). We are going to use induction on $\sum n_i$. If $n_i = 0$ for all i, then $\mathcal{I} = \mathcal{O}_X$ and we are done.

Suppose $n_i > 0$. Let $X' \to X$ be the blowing up of X in x_i (see discussion above the lemma). Since $\text{Spec}(\mathcal{O}_{X,x_i}) \to X$ is flat we see that $X' \times_X \text{Spec}(\mathcal{O}_{X,x_i})$ is the blowup of the ring $\mathcal{O}_{X,x_i}$ in the maximal ideal, see Divisors, Lemma 31.32.3. Hence the square in the commutative diagram

$$\begin{tikzcd} \text{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{x_i}^d) \arrow[r] \arrow[d] & X' \arrow[d] \\ \text{Spec}(\mathcal{O}_{X,x_i}) \arrow[r] & X \end{tikzcd}$$

is cartesian. Let $E \subset X'$ and $E' \subset \text{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{x_i}^d)$ be the exceptional divisors. Let $d \geq 1$ be the integer found in Lemma 54.3.5 for the ideal $\mathcal{I}_i \subset \mathcal{O}_{X,x_i}$. Since the horizontal arrows in the diagram are flat, since $E' \to E$ is surjective, and since E' is the pullback of E, we see that

$$\mathcal{I}\mathcal{O}_{X'} \subset \mathcal{O}_{X'}(-dE)$$

(some details omitted). Set $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}(dE) \subset \mathcal{O}_{X'}$. Then we see that $\mathcal{O}_{X'}/\mathcal{I}'$ is supported in finitely many closed points $T' \subset |X'|$ because this holds over $X \setminus \{x_i\}$ and for the pullback to $\text{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{x_i}^d)$. The final assertion of Lemma 54.3.5 tells us that the sum of the lengths of the stalks $\mathcal{O}_{X',x'}/\mathcal{I}'\mathcal{O}_{X',x'}$ for x' lying over x_i is $< n_i$. Hence the sum of the lengths has decreased.

By induction hypothesis, there exists a sequence

$$X'_n \to \ldots \to X'_1 \to X'$$

of blowups at closed points lying over T' such that $\mathcal{I}'\mathcal{O}_{X'_n}$ is invertible. Since $\mathcal{I}'\mathcal{O}_{X'}(-dE) = \mathcal{I}\mathcal{O}_{X'}$, we see that $\mathcal{I}\mathcal{O}_{X'_n} = \mathcal{I}'\mathcal{O}_{X'_n}(-d(f')^{-1}E)$ where $f' : X'_n \to X'$ is the composition. Note that $(f')^{-1}E$ is an effective Cartier divisor by Divisors, Lemma 31.32.11. Thus we are done by Divisors, Lemma 31.13.7. □

0AHI Lemma 54.4.2. Let X be a Noetherian scheme. Let $T \subset X$ be a finite set of closed points x such that $\mathcal{O}_{X,x}$ is a regular local ring of dimension 2. Let $f : Y \to X$ be a proper morphism of schemes which is an isomorphism over $U = X \setminus T$. Then there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Flatness, Lemma 38.31.4 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 54.4.1 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors, Lemma 31.32.5) we find the desired factorization. □

0C5H Lemma 54.4.3. Let S be a scheme. Let X be a scheme over S which is regular and has dimension 2. Let Y be a proper scheme over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

and an S-morphism $f_n : X_n \to Y$ such that $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point not lying over U and f_n and f agree.

Proof. We may assume U contains every point of codimension 1, see Morphisms, Lemma 29.42.5. Hence the complement $T \subset X$ of U is a finite set of closed points whose local rings are regular of dimension 2. Applying Divisors, Lemma 31.36.2 we find a proper morphism $p : X' \to X$ which is an isomorphism over U and a morphism $f' : X' \to Y$ agreeing with f over U. Apply Lemma 54.4.2 to the morphism $p : X' \to X$. The composition $X_n \to X' \to Y$ is the desired morphism. □

54.5. Dominating by normalized blowups

0BBR In this section we prove that a modification of a surface can be dominated by a sequence of normalized blowups in points.

0BBS Definition 54.5.1. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $x \in X$ be a closed point. The normalized blowup of X at x is the composition $X'' \to X' \to X$ where $X' \to X$ is the blowup of X in x and $X'' \to X'$ is the normalization of X'.

Here the normalization $X'' \to X'$ is defined as the scheme X' has an open covering by opens which have finitely many irreducible components by Divisors, Lemma 31.32.10. See Morphisms, Definition 29.54.1 for the definition of the normalization.

In general the normalized blowing up need not be proper even when X is Noetherian. Recall that a scheme is Nagata if it has an open covering by affines which are spectra of Nagata rings (Properties, Definition 28.13.1).

0BFT Lemma 54.5.2. In Definition 54.5.1 if X is Nagata, then the normalized blowing up of X at x is normal, Nagata, and proper over X.

Proof. The blowup morphism $X' \to X$ is proper (as X is locally Noetherian we may apply Divisors, Lemma 31.32.13). Thus X' is Nagata (Morphisms, Lemma 29.18.1). Therefore the normalization $X'' \to X'$ is finite (Morphisms, Lemma 29.54.10) and we conclude that $X'' \to X$ is proper as well (Morphisms, Lemmas 29.44.11 and 29.41.4). It follows that the normalized blowing up is a normal (Morphisms, Lemma 29.54.5) Nagata algebraic space. □

In the following lemma we need to assume X is Noetherian in order to make sure that it has finitely many irreducible components. Then the properness of $f : Y \to X$ assures that Y has finitely many irreducible components too and it makes sense to require f to be birational (Morphisms, Definition 29.50.1).

0BBT Lemma 54.5.3. Let X be a scheme which is Noetherian, Nagata, and has dimension 2. Let $f : Y \to X$ be a proper birational morphism. Then there exists a commutative diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] & \ldots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[d] \\ Y \arrow[rrrr] & & & & X \end{tikzcd}$$

where $X_0 \to X$ is the normalization and where $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point.

Proof. We will use the results of Morphisms, Sections 29.18, 29.52, and 29.54 without further mention. We may replace Y by its normalization. Let $X_0 \to X$ be the normalization. The morphism $Y \to X$ factors through X_0. Thus we may assume that both X and Y are normal.

Assume X and Y are normal. The morphism $f : Y \to X$ is an isomorphism over an open which contains every point of codimension 0 and 1 in Y and every point of Y over which the fibre is finite, see Varieties, Lemma 33.17.3. Hence there is a finite set of closed points $T \subset X$ such that f is an isomorphism over $X \setminus T$. For each $x \in T$ the fibre Y_x is a proper geometrically connected scheme of dimension 1 over $\kappa(x)$, see More on Morphisms, Lemma 37.53.6. Thus

$$BadCurves(f) = \{C \subset Y \text{ closed} \mid \dim(C) = 1, f(C) = \text{a point}\}$$

is a finite set. We will prove the lemma by induction on the number of elements of $BadCurves(f)$. The base case is the case where $BadCurves(f)$ is empty, and in that case f is an isomorphism.

Fix $x \in T$. Let $X' \to X$ be the normalized blowup of X at x and let Y' be the normalization of $Y \times_X X'$. Picture

$$\begin{tikzcd} Y' \arrow[r, "f'"'] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Let $x' \in X'$ be a closed point lying over x such that the fibre $Y'_{x'}$ has dimension ≥ 1. Let $C' \subset Y'$ be an irreducible component of $Y'_{x'}$, i.e., $C' \in BadCurves(f')$. Since $Y' \to Y \times_X X'$ is finite we see that C' must map to an irreducible component $C \subset Y_x$. If is clear that $C \in BadCurves(f)$. Since $Y' \to Y$ is birational and hence an isomorphism over points of codimension 1 in Y, we see that we obtain an injective map

$$BadCurves(f') \longrightarrow BadCurves(f)$$

Thus it suffices to show that after a finite number of these normalized blowups we get rid at of at least one of the bad curves, i.e., the displayed map is not surjective.

We will get rid of a bad curve using an argument due to Zariski. Pick $C \in BadCurves(f)$ lying over our x. Denote $\mathcal{O}_{Y,C}$ the local ring of Y at the generic point of C. Choose an element $u \in \mathcal{O}_{X,C}$ whose image in the residue field $R(C)$ is transcendental over $\kappa(x)$ (we can do this because $R(C)$ has transcendence degree 1 over $\kappa(x)$ by Varieties, Lemma 33.20.3). We can write $u = a/b$ with $a, b \in \mathcal{O}_{X,x}$ as $\mathcal{O}_{Y,C}$ and $\mathcal{O}_{X,x}$ have the same fraction fields. By our choice of u it must be the case that $a, b \in \mathfrak{m}_x$. Hence

$$N_{u,a,b} = \min\{\text{ord}_{\mathcal{O}_{Y,C}}(a), \text{ord}_{\mathcal{O}_{Y,C}}(b)\} > 0$$

Thus we can do descending induction on this integer. Let $X' \to X$ be the normalized blowing up of x and let Y' be the normalization of $X' \times_X Y$ as above. We will show that if C is the image of some bad curve $C' \subset Y'$ lying over $x' \in X'$, then there exists a choice of $a', b'\mathcal{O}_{X',x'}$ such that $N_{u,a',b'} < N_{u,a,b}$. This will finish the proof. Namely, since $X' \to X$ factors through the blowing up, we see that there exists a nonzero element $d \in \mathfrak{m}_{x'}$ such that $a = a'd$ and $b = b'd$ (namely, take d to be the local equation for the exceptional divisor of the blowup). Since $Y' \to Y$

is an isomorphism over an open containing the generic point of C (seen above) we see that $\mathcal{O}_{Y',C'} = \mathcal{O}_{Y,C}$. Hence

$$\text{ord}_{\mathcal{O}_{Y,C}}(a) = \text{ord}_{\mathcal{O}_{Y',C'}}(a'd) = \text{ord}_{\mathcal{O}_{Y',C'}}(a') + \text{ord}_{\mathcal{O}_{Y',C'}}(d) > \text{ord}_{\mathcal{O}_{Y',C'}}(a')$$

Similarly for b and the proof is complete. $\square$

0C5I Lemma 54.5.4. Let S be a scheme. Let X be a scheme over S which is Noetherian, Nagata, and has dimension 2. Let Y be a proper scheme over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 \to X$$

and an S-morphism $f_n : X_n \to Y$ such that $X_0 \to X$ is the normalization, $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point, and f_n and f agree.

Proof. Applying Divisors, Lemma 31.36.2 we find a proper morphism $p : X' \to X$ which is an isomorphism over U and a morphism $f' : X' \to Y$ agreeing with f over U. Apply Lemma 54.5.3 to the morphism $p : X' \to X$. The composition $X_n \to X' \to Y$ is the desired morphism. $\square$

54.6. Modifying over local rings

0AE1 Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points. Assume that the open embedding

$$U = S \setminus \{s_1, \ldots, s_n\} \longrightarrow S$$

is quasi-compact. Denote $FP_{S,\{s_1,\ldots,s_n\}}$ the category of morphisms $f : X \to S$ of finite presentation which induce an isomorphism $f^{-1}(U) \to U$. Morphisms are morphisms of schemes over S. For each i set $S_i = \text{Spec}(\mathcal{O}_{S,s_i})$ and let $V_i = S_i \setminus \{s_i\}$. Denote FP_{S_i,s_i} the category of morphisms $g_i : Y_i \to S_i$ of finite presentation which induce an isomorphism $g_i^{-1}(V_i) \to V_i$. Morphisms are morphisms over S_i. Base change defines an functor

0BFU (54.6.0.1)
$$F : FP_{S,\{s_1,\ldots,s_n\}} \longrightarrow FP_{S_1,s_1} \times \ldots \times FP_{S_n,s_n}$$

To reduce at least some of the problems in this chapter to the case of local rings we have the following lemma.

0BFV Lemma 54.6.1. The functor F (54.6.0.1) is an equivalence.

Proof. For $n = 1$ this is Limits, Lemma 32.21.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $g_i : Y_i \to S_i$ are objects of FP_{S_i,s_i}. Then by the case $n = 1$ we can find $f'_i : X'_i \to S$ of finite presentation which are isomorphisms over $S \setminus \{s_i\}$ and whose base change to S_i is g_i. Then we can set

$$f : X = X'_1 \times_S \ldots \times_S X'_n \to S$$

This is an object of $FP_{S,\{s_1,\ldots,s_n\}}$ whose base change by $S_i \to S$ recovers g_i. Thus the functor is essentially surjective. We omit the proof of fully faithfulness. $\square$

0BFW Lemma 54.6.2. Let S, s_i, S_i be as in (54.6.0.1). If $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F, then f is separated, proper, finite, if and only if g_i is so for $i = 1, \ldots, n$.

Proof. Follows from Limits, Lemma 32.21.2. $\square$

0BFX Lemma 54.6.3. Let S, s_i, S_i be as in (54.6.0.1). If $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F, then $X_{s_i} \cong (Y_i)_{s_i}$ as schemes over $\kappa(s_i)$.

Proof. This is clear. $\square$

0BFY Lemma 54.6.4. Let S, s_i, S_i be as in (54.6.0.1) and assume $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F. Then there exists a factorization

$$X = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$$

of f where $Z_{j+1} \to Z_j$ is the blowing up of Z_j at a closed point z_j lying over $\{s_1, \ldots, s_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over s_i.

Proof. Let's start with a sequence of blowups $Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$. The first morphism $Z_1 \to S$ is given by blowing up one of the s_i, say s_1. Applying F to $Z_1 \to S$ we find a blowup $Z_{1,1} \to S_1$ at s_1 is the blowing up at s_1 and otherwise $Z_{i,0} = S_i$ for $i > 1$. In the next step, we either blow up one of the s_i, $i \geq 2$ on Z_1 or we pick a closed point z_1 of the fibre of $Z_1 \to S$ over s_1. In the first case it is clear what to do and in the second case we use that $(Z_1)_{s_1} \cong (Z_{1,1})_{s_1}$ (Lemma 54.6.3) to get a closed point $z_{1,1} \in Z_{1,1}$ corresponding to z_1. Then we set $Z_{1,2} \to Z_{1,1}$ equal to the blowing up in $z_{1,1}$. Continuing in this manner we construct the factorizations of each g_i.

Conversely, given sequences of blowups $Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$ we construct the sequence of blowing ups of S in exactly the same manner. $\square$

Here is the analogue of Lemma 54.6.4 for normalized blowups.

0BFZ Lemma 54.6.5. Let S, s_i, S_i be as in (54.6.0.1) and assume $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F. Assume every quasi-compact open of S has finitely many irreducible components. Then there exists a factorization

$$X = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$$

of f where $Z_{j+1} \to Z_j$ is the normalized blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the normalized blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over s_i.

Proof. The assumption on S is used to assure us (successively) that the schemes we are normalizing have locally finitely many irreducible components so that the statement makes sense. Having said this the lemma follows by the exact same argument as used to prove Lemma 54.6.4. $\square$

54.7. Vanishing

0AX7 In this section we will often work in the following setting. Recall that a modification is a proper birational morphism between integral schemes (Morphisms, Definition 29.51.11).

0AX8 Situation 54.7.1. Here $(A, \mathfrak{m}, \kappa)$ be a local Noetherian normal domain of dimension 2. Let s be the closed point of $S = \mathrm{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to S$ be a modification. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre X_s of f.

By Varieties, Lemma 33.17.3 the morphism f defines an isomorphism $f^{-1}(U) \to U$. The special fibre X_s is proper over $\mathrm{Spec}(\kappa)$ and has dimension at most 1 by Varieties, Lemma 33.19.3. By Stein factorization (More on Morphisms, Lemma 37.53.6) we have $f_*\mathcal{O}_X = \mathcal{O}_S$ and the special fibre X_s is geometrically connected over κ. If X_s has dimension 0, then f is finite (More on Morphisms, Lemma 37.44.2) and hence an isomorphism (Morphisms, Lemma 29.54.8). We will discard this uninteresting case and we conclude that $\dim(C_i) = 1$ for $i = 1, \ldots, r$.

0B4M Lemma 54.7.2. In Situation 54.7.1 there exists a U-admissible blowup $X' \to S$ which dominates X.

Proof. This is a special case of More on Flatness, Lemma 38.31.4. □

0AX9 Lemma 54.7.3. In Situation 54.7.1 there exists a nonzero $f \in \mathfrak{m}$ such that for every $i = 1, \ldots, r$ there exist

(1) a closed point $x_i \in C_i$ with $x_i \notin C_j$ for $j \neq i$,
(2) a factorization $f = g_i f_i$ of f in $\mathcal{O}_{X,x_i}$ such that $g_i \in \mathfrak{m}_{x_i}$ maps to a nonzero element of $\mathcal{O}_{C_i,x_i}$.

Proof. We will use the observations made following Situation 54.7.1 without further mention. Pick a closed point $x_i \in C_i$ which is not in C_j for $j \neq i$. Pick $g_i \in \mathfrak{m}_{x_i}$ which maps to a nonzero element of $\mathcal{O}_{C_i,x_i}$. Since the fraction field of A is the fraction field of $\mathcal{O}_{X_i,x_i}$ we can write $g_i = a_i/b_i$ for some $a_i, b_i \in A$. Take $f = \prod a_i$. □

0AXA Lemma 54.7.4. In Situation 54.7.1 assume X is normal. Let $Z \subset X$ be a nonempty effective Cartier divisor such that $Z \subset X_s$ set theoretically. Then the conormal sheaf of Z is not trivial. More precisely, there exists an i such that $C_i \subset Z$ and $\deg(\mathcal{C}_{Z/X}|_{C_i}) > 0$.

Proof. We will use the observations made following Situation 54.7.1 without further mention. Let f be a function as in Lemma 54.7.3. Let $\xi_i \in C_i$ be the generic point. Let $\mathcal{O}_i$ be the local ring of X at ξ_i. Then $\mathcal{O}_i$ is a discrete valuation ring. Let e_i be the valuation of f in $\mathcal{O}_i$, so $e_i > 0$. Let $h_i \in \mathcal{O}_i$ be a local equation for Z and let d_i be its valuation. Then $d_i \geq 0$. Choose and fix i with d_i/e_i maximal (then $d_i > 0$ as Z is not empty). Replace f by f^{d_i} and Z by $e_i Z$. This is permissible, by the relation $\mathcal{O}_X(e_i Z) = \mathcal{O}_X(Z)^{\otimes e_i}$, the relation between the conormal sheaf and $\mathcal{O}_X(Z)$ (see Divisors, Lemmas 31.14.4 and 31.14.2, and since the degree gets multiplied by e_i, see Varieties, Lemma 33.44.7. Let $\mathcal{I}$ be the ideal sheaf of Z so that $\mathcal{C}_{Z/X} = \mathcal{I}|_Z$. Consider the image $\overline{f}$ of f in $\Gamma(Z, \mathcal{O}_Z)$. By our choices above we see that $\overline{f}$ vanishes in the generic points of irreducible components of Z (these are all generic points of C_j as Z is contained in the special fibre). On the other hand, Z is

(S_1) by Divisors, Lemma 31.15.6. Thus the scheme Z has no embedded associated points and we conclude that $\overline{f} = 0$ (Divisors, Lemmas 31.4.3 and 31.5.6). Hence f is a global section of $\mathcal{I}$ which generates $\mathcal{I}_{\xi_i}$ by construction. Thus the image s_i of f in $\Gamma(C_i, \mathcal{I}|_{C_i})$ is nonzero. However, our choice of f guarantees that s_i has a zero at x_i. Hence the degree of $\mathcal{I}|_{C_i}$ is > 0 by Varieties, Lemma 33.44.12. $\square$

0AXB Lemma 54.7.5. In Situation 54.7.1 assume X is normal and A Nagata. The map

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(f^{-1}(U), \mathcal{O}_X)$$

is injective.

Proof. Let $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0$ be the extension corresponding to a nontrivial element ξ of $H^1(X, \mathcal{O}_X)$ (Cohomology, Lemma 20.5.1). Let $\pi : P = \mathbf{P}(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. The surjection $\mathcal{E} \to \mathcal{O}_X$ defines a section $\sigma : X \to P$ whose conormal sheaf is isomorphic to $\mathcal{O}_X$ (Divisors, Lemma 31.31.6). If the restriction of ξ to $f^{-1}(U)$ is trivial, then we get a map $\mathcal{E}|_{f^{-1}(U)} \to \mathcal{O}_{f^{-1}(U)}$ splitting the injection $\mathcal{O}_X \to \mathcal{E}$. This defines a second section $\sigma' : f^{-1}(U) \to P$ disjoint from σ. Since ξ is nontrivial we conclude that σ' cannot extend to all of X and be disjoint from σ. Let $X' \subset P$ be the scheme theoretic image of σ' (Morphisms, Definition 29.6.2). Picture

$$\begin{tikzcd} & X' \arrow[r] \arrow[rd, "g"] & P \arrow[d, "\pi"'] \\ f^{-1}(U) \arrow[ru, "\sigma'"] \arrow[rr] & & X \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

The morphism $P \setminus \sigma(X) \to X$ is affine. If $X' \cap \sigma(X) = \emptyset$, then $X' \to X$ is both affine and proper, hence finite (Morphisms, Lemma 29.44.11), hence an isomorphism (as X is normal, see Morphisms, Lemma 29.54.8). This is impossible as mentioned above.

Let X^ν be the normalization of X'. Since A is Nagata, we see that $X^\nu \to X'$ is finite (Morphisms, Lemmas 29.54.10 and 29.18.2). Let $Z \subset X^\nu$ be the pullback of the effective Cartier divisor $\sigma(X) \subset P$. By the above we see that Z is not empty and is contained in the closed fibre of $X^\nu \to S$. Since $P \to X$ is smooth, we see that $\sigma(X)$ is an effective Cartier divisor (Divisors, Lemma 31.22.8). Hence $Z \subset X^\nu$ is an effective Cartier divisor too. Since the conormal sheaf of $\sigma(X)$ in P is $\mathcal{O}_X$, the conormal sheaf of Z in X^ν (which is a priori invertible) is $\mathcal{O}_Z$ by Morphisms, Lemma 29.31.4. This is impossible by Lemma 54.7.4 and the proof is complete. $\square$

0AXC Lemma 54.7.6. In Situation 54.7.1 assume X is normal and A Nagata. Then

$$\mathrm{Hom}_{D(A)}(\kappa[-1], Rf_*\mathcal{O}_X)$$

is zero. This uses $D(A) = D_{QCoh}(\mathcal{O}_S)$ to think of $Rf_*\mathcal{O}_X$ as an object of $D(A)$.

Proof. By adjointness of Rf_* and Lf^* such a map is the same thing as a map $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$. Note that

$$H^i(Lf^*\kappa[-1]) = \begin{cases} 0 & \text{if } i > 1 \\ \mathcal{O}_{X_s} & \text{if } i = 1 \\ \text{some } \mathcal{O}_{X_s}\text{-module} & \text{if } i \leq 0 \end{cases}$$

Since $\mathrm{Hom}(H^0(Lf^*\kappa[-1]), \mathcal{O}_X) = 0$ as $\mathcal{O}_X$ is torsion free, the spectral sequence for Ext (Cohomology on Sites, Example 21.32.1) implies that $\mathrm{Hom}_{D(\mathcal{O}_X)}(Lf^*\kappa[-1], \mathcal{O}_X)$

is equal to $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_s}, \mathcal{O}_X)$. We conclude that $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$ is given by an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_{X_s} \to 0$$

By Lemma 54.7.5 the pullback of this extension via the surjection $\mathcal{O}_X \to \mathcal{O}_{X_s}$ is zero (since this pullback is clearly split over $f^{-1}(U)$). Thus $1 \in \mathcal{O}_{X_s}$ lifts to a global section s of $\mathcal{E}$. Multiplying s by the ideal sheaf $\mathcal{I}$ of X_s we obtain an $\mathcal{O}_X$-module map $c_s : \mathcal{I} \to \mathcal{O}_X$. Applying f_* we obtain an A-linear map $f_*c_s : \mathfrak{m} \to A$. Since A is a Noetherian normal local domain this map is given by multiplication by an element $a \in A$. Changing s into $s - a$ we find that s is annihilated by $\mathcal{I}$ and the extension is trivial as desired. $\square$

0B4R Remark 54.7.7. Let X be an integral Noetherian normal scheme of dimension 2. In this case the following are equivalent

(1) X has a dualizing complex $\omega_X^\bullet$,
(2) there is a coherent $\mathcal{O}_X$-module ω_X such that $\omega_X[n]$ is a dualizing complex, where n can be any integer.

This follows from the fact that X is Cohen-Macaulay (Properties, Lemma 28.12.7) and Duality for Schemes, Lemma 48.23.1. In this situation we will say that ω_X is a dualizing module in accordance with Duality for Schemes, Section 48.22. In particular, when A is a Noetherian normal local domain of dimension 2, then we say A has a dualizing module ω_A if the above is true. In this case, if $X \to \mathrm{Spec}(A)$ is a normal modification, then X has a dualizing module too, see Duality for Schemes, Example 48.22.1. In this situation we always denote ω_X the dualizing module normalized with respect to ω_A, i.e., such that $\omega_X[2]$ is the dualizing complex normalized relative to $\omega_A[2]$. See Duality for Schemes, Section 48.20.

The Grauert-Riemenschneider vanishing of the next proposition is a formal consequence of Lemma 54.7.6 and the general theory of duality.

0AXD Proposition 54.7.8 (Grauert-Riemenschneider). In Situation 54.7.1 assume

(1) X is a normal scheme,
(2) A is Nagata and has a dualizing complex $\omega_A^\bullet$.

Let ω_X be the dualizing module of X (Remark 54.7.7). Then $R^1f_*\omega_X = 0$.

Proof. In this proof we will use the identification $D(A) = D_{QCoh}(\mathcal{O}_S)$ to identify quasi-coherent $\mathcal{O}_S$-modules with A-modules. Moreover, we may assume that $\omega_A^\bullet$ is normalized, see Dualizing Complexes, Section 47.16. Since X is a Noetherian normal 2-dimensional scheme it is Cohen-Macaulay (Properties, Lemma 28.12.7). Thus $\omega_X^\bullet = \omega_X[2]$ (Duality for Schemes, Lemma 48.23.1 and the normalization in Duality for Schemes, Example 48.22.1). If the proposition is false, then we can find a nonzero map $R^1f_*\omega_X \to \kappa$. In other words we obtain a nonzero map $\alpha : Rf_*\omega_X^\bullet \to \kappa[1]$. Applying $R\mathrm{Hom}_A(-, \omega_A^\bullet)$ we get a nonzero map

$$\beta : \kappa[-1] \longrightarrow Rf_*\mathcal{O}_X$$

which is impossible by Lemma 54.7.6. To see that $R\mathrm{Hom}_A(-, \omega_A^\bullet)$ does what we said, first note that

$$R\mathrm{Hom}_A(\kappa[1], \omega_A^\bullet) = R\mathrm{Hom}_A(\kappa, \omega_A^\bullet)[-1] = \kappa[-1]$$

as $\omega_A^\bullet$ is normalized and we have

$$R\mathrm{Hom}_A(Rf_*\omega_X^\bullet, \omega_A^\bullet) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\omega_X^\bullet, \omega_X^\bullet) = Rf_*\mathcal{O}_X$$

The first equality by Duality for Schemes, Example 48.3.9 and the fact that $\omega_X^\bullet = f^!\omega_A^\bullet$ by construction, and the second equality because $\omega_X^\bullet$ is a dualizing complex for X (which goes back to Duality for Schemes, Lemma 48.17.7). $\square$

54.8. Boundedness

0AXE In this section we begin the discussion which will lead to a reduction to the case of rational singularities for 2-dimensional schemes.

0AXF Lemma 54.8.1. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[rr, "g"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

where f and f' are modifications as in Situation 54.7.1 and X normal. Then we have a short exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}_{X'}) \to H^0(X, R^1g_*\mathcal{O}_{X'}) \to 0$$

Also $\dim(\mathrm{Supp}(R^1g_*\mathcal{O}_{X'})) = 0$ and $R^1g_*\mathcal{O}_{X'}$ is generated by global sections.

Proof. We will use the observations made following Situation 54.7.1 without further mention. As X is normal and g is dominant and birational, we have $g_*\mathcal{O}_{X'} = \mathcal{O}_X$, see for example More on Morphisms, Lemma 37.53.6. Since the fibres of g have dimension ≤ 1, we have $R^pg_*\mathcal{O}_{X'} = 0$ for $p > 1$, see for example Cohomology of Schemes, Lemma 30.20.9. The support of $R^1g_*\mathcal{O}_{X'}$ is contained in the set of points of X where the fibres of g' have dimension ≥ 1. Thus it is contained in the set of images of those irreducible components $C' \subset X'_s$ which map to points of X_s which is a finite set of closed points (recall that $X'_s \to X_s$ is a morphism of proper 1-dimensional schemes over κ). Then $R^1g_*\mathcal{O}_{X'}$ is globally generated by Cohomology of Schemes, Lemma 30.9.10. Using the morphism $f : X \to S$ and the references above we find that $H^p(X, \mathcal{F}) = 0$ for $p > 1$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence the short exact sequence of the lemma is a consequence of the Leray spectral sequence for g and $\mathcal{O}_{X'}$, see Cohomology, Lemma 20.13.4. $\square$

0AXJ Lemma 54.8.2. Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2. Let $a \in A$ be nonzero. There exists an integer N such that for every modification $f : X \to \mathrm{Spec}(A)$ with X normal the A-module

$$M_{X,a} = \mathrm{Coker}(A \longrightarrow H^0(Z, \mathcal{O}_Z))$$

where $Z \subset X$ is cut out by a has length bounded by N.

Proof. By the short exact sequence $0 \to \mathcal{O}_X \xrightarrow{a} \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we see that

0AXK (54.8.2.1) $$M_{X,a} = H^1(X, \mathcal{O}_X)[a]$$

Here $N[a] = \{n \in N \mid an = 0\}$ for an A-module N. Thus if a divides b, then $M_{X,a} \subset M_{X,b}$. Suppose that for some $c \in A$ the modules $M_{X,c}$ have bounded length. Then for every X we have an exact sequence

$$0 \to M_{X,c} \to M_{X,c^2} \to M_{X,c}$$

where the second arrow is given by multiplication by c. Hence we see that M_{X,c^2} has bounded length as well. Thus it suffices to find a $c \in A$ for which the lemma is true such that a divides c^n for some $n > 0$. By More on Algebra, Lemma 15.125.6 we may assume $A/(a)$ is a reduced ring.

Assume that $A/(a)$ is reduced. Let $A/(a) \subset B$ be the normalization of $A/(a)$ in its quotient ring. Because A is Nagata, we see that $\text{Coker}(A \to B)$ is finite. We claim the length of this finite module is a bound. To see this, consider $f : X \to \text{Spec}(A)$ as in the lemma and let $Z' \subset Z$ be the scheme theoretic closure of $Z \cap f^{-1}(U)$. Then $Z' \to \text{Spec}(A/(a))$ is finite for example by Varieties, Lemma 33.17.2. Hence $Z' = \text{Spec}(B')$ with $A/(a) \subset B' \subset B$. On the other hand, we claim the map

$$H^0(Z, \mathcal{O}_Z) \to H^0(Z', \mathcal{O}_{Z'})$$

is injective. Namely, if $s \in H^0(Z, \mathcal{O}_Z)$ is in the kernel, then the restriction of s to $f^{-1}(U) \cap Z$ is zero. Hence the image of s in $H^1(X, \mathcal{O}_X)$ vanishes in $H^1(f^{-1}(U), \mathcal{O}_X)$. By Lemma 54.7.5 we see that s comes from an element $\tilde{s}$ of A. But by assumption $\tilde{s}$ maps to zero in B' which implies that $s = 0$. Putting everything together we see that $M_{X,a}$ is a subquotient of B'/A, namely not every element of B' extends to a global section of $\mathcal{O}_Z$, but in any case the length of $M_{X,a}$ is bounded by the length of B/A. □

In some cases, resolution of singularities reduces to the case of rational singularities.

0B4N Definition 54.8.3. Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2.

(1) We say A defines a rational singularity if for every normal modification $X \to \text{Spec}(A)$ we have $H^1(X, \mathcal{O}_X) = 0$.
(2) We say that reduction to rational singularities is possible for A if the length of the A-modules

$$H^1(X, \mathcal{O}_X)$$

is bounded for all modifications $X \to \text{Spec}(A)$ with X normal.

The meaning of the language in (2) is explained by Lemma 54.8.5. The following lemma says roughly speaking that local rings of modifications of $\text{Spec}(A)$ with A defining a rational singularity also define rational singularities.

0BG0 Lemma 54.8.4. Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Let $A \subset B$ be a local extension of domains with the same fraction field which is essentially of finite type such that $\dim(B) = 2$ and B normal. Then B defines a rational singularity.

Proof. Choose a finite type A-algebra C such that $B = C_\mathfrak{q}$ for some prime $\mathfrak{q} \subset C$. After replacing C by the image of C in B we may assume that C is a domain with fraction field equal to the fraction field of A. Then we can choose a closed immersion $\text{Spec}(C) \to \mathbf{A}^n_A$ and take the closure in $\mathbf{P}^n_A$ to conclude that B is isomorphic to $\mathcal{O}_{X,x}$ for some closed point $x \in X$ of a projective modification $X \to \text{Spec}(A)$. (Morphisms, Lemma 29.52.1, shows that $\kappa(x)$ is finite over κ and then Morphisms, Lemma 29.20.2 shows that x is a closed point.) Let $\nu : X^\nu \to X$ be the normalization. Since A is Nagata the morphism ν is finite (Morphisms, Lemma 29.54.10). Thus X^ν is projective over A by More on Morphisms, Lemma 37.50.2. Since $B = \mathcal{O}_{X,x}$ is normal, we see that $\mathcal{O}_{X,x} = (\nu_*\mathcal{O}_{X^\nu})_x$. Hence there is a unique point $x^\nu \in X^\nu$ lying over x and $\mathcal{O}_{X^\nu,x^\nu} = \mathcal{O}_{X,x}$. Thus we may assume X is normal and projective over A. Let $Y \to \mathrm{Spec}(\mathcal{O}_{X,x}) = \mathrm{Spec}(B)$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Limits, Lemma 32.21.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$ is isomorphic to Y. Then g is a modification as it is proper by Limits, Lemma 32.21.2. The local ring of X' at a point of x' is either isomorphic to the local ring of X at $g(x')$ if $g(x') \neq x$ and if $g(x') = x$, then the local ring of X' at x' is isomorphic to the local ring of Y at the corresponding point. Hence we see that X' is normal as both X and Y are normal. Thus $H^1(X', \mathcal{O}_{X'}) = 0$ by our assumption on A. By Lemma 54.8.1 we have $R^1 g_* \mathcal{O}_{X'} = 0$. Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired. □

0B4P Lemma 54.8.5. Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2. If reduction to rational singularities is possible for A, then there exists a finite sequence of normalized blowups

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

in closed points such that for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ defines a rational singularity. In particular $X \to \mathrm{Spec}(A)$ is a modification and X is a normal scheme projective over A.

Proof. We choose a modification $X \to \mathrm{Spec}(A)$ with X normal which maximizes the length of $H^1(X, \mathcal{O}_X)$. By Lemma 54.8.1 for any further modification $g : X' \to X$ with X' normal we have $R^1 g_* \mathcal{O}_{X'} = 0$ and $H^1(X, \mathcal{O}_X) = H^1(X', \mathcal{O}_{X'})$.

Let $x \in X$ be a closed point. We will show that $\mathcal{O}_{X,x}$ defines a rational singularity. Let $Y \to \mathrm{Spec}(\mathcal{O}_{X,x})$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Limits, Lemma 32.21.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$ is isomorphic to Y. Then g is a modification as it is proper by Limits, Lemma 32.21.2. The local ring of X' at a point of x' is either isomorphic to the local ring of X at $g(x')$ if $g(x') \neq x$ and if $g(x') = x$, then the local ring of X' at x' is isomorphic to the local ring of Y at the corresponding point. Hence we see that X' is normal as both X and Y are normal. By maximality we have $R^1 g_* \mathcal{O}_{X'} = 0$ (see first paragraph). Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired.

The conclusion is that we've found one normal modification X of $\mathrm{Spec}(A)$ such that the local rings of X at closed points all define rational singularities. Then we choose a sequence of normalized blowups $X_n \to \ldots \to X_1 \to \mathrm{Spec}(A)$ such that X_n dominates X, see Lemma 54.5.3. For a closed point $x' \in X_n$ mapping to $x \in X$ we can apply Lemma 54.8.4 to the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_n,x'}$ to see that $\mathcal{O}_{X_n,x'}$ defines a rational singularity. □

0AXL Lemma 54.8.6. Let $A \to B$ be a finite injective local ring map of local normal Nagata domains of dimension 2. Assume that the induced extension of fraction fields is separable. If reduction to rational singularities is possible for A then it is possible for B.

Proof. Let n be the degree of the fraction field extension L/K. Let $\mathrm{Trace}_{L/K} : L \to K$ be the trace. Since the extension is finite separable the trace pairing $(h, g) \mapsto \mathrm{Trace}_{L/K}(fg)$ is a nondegenerate bilinear form on L over K. See Fields, Lemma 9.20.7. Pick $b_1, \ldots, b_n \in B$ which form a basis of L over K. By the above $d = \det(\mathrm{Trace}_{L/K}(b_i b_j)) \in A$ is nonzero.

Let $Y \to \mathrm{Spec}(B)$ be a modification with Y normal. We can find a U-admissible blowup X' of $\mathrm{Spec}(A)$ such that the strict transform Y' of Y is finite over X', see More on Flatness, Lemma 38.31.2. Picture

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[r] & \mathrm{Spec}(B) \arrow[d] \\ X' \arrow[rr] & & \mathrm{Spec}(A) \end{tikzcd}$$

After replacing X' and Y' by their normalizations we may assume that X' and Y' are normal modifications of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. In this way we reduce to the case where there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[d, "\pi"'] & \mathrm{Spec}(B) \arrow[d] \\ X \arrow[r, "f"] & \mathrm{Spec}(A) \end{tikzcd}$$

with X and Y normal modifications of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ and π finite.

The trace map on L over K extends to a map of $\mathcal{O}_X$-modules $\mathrm{Trace} : \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. Consider the map

$$\Phi : \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X^{\oplus n}, \quad s \longmapsto (\mathrm{Trace}(b_1 s), \ldots, \mathrm{Trace}(b_n s))$$

This map is injective (because it is injective in the generic point) and there is a map

$$\mathcal{O}_X^{\oplus n} \longrightarrow \pi_*\mathcal{O}_Y, \quad (s_1, \ldots, s_n) \longmapsto \sum b_i s_i$$

whose composition with Φ has matrix $\mathrm{Trace}(b_i b_j)$. Hence the cokernel of Φ is annihilated by d. Thus we see that we have an exact sequence

$$H^0(X, \mathrm{Coker}(\Phi)) \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)^{\oplus n}$$

Since the right hand side is bounded by assumption, it suffices to show that the d-torsion in $H^1(Y, \mathcal{O}_Y)$ is bounded. This is the content of Lemma 54.8.2 and (54.8.2.1). $\square$

0B4Q Lemma 54.8.7. Let A be a Nagata regular local ring of dimension 2. Then A defines a rational singularity.

Proof. (The assumption that A be Nagata is not necessary for this proof, but we've only defined the notion of rational singularity in the case of Nagata 2-dimensional normal local domains.) Let $X \to \mathrm{Spec}(A)$ be a modification with X normal. By Lemma 54.4.2 we can dominate X by a scheme X_n which is the last in a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

of blowing ups in closed points. By Lemma 54.3.2 the schemes X_i are regular, in particular normal (Algebra, Lemma 10.157.5). By Lemma 54.8.1 we have $H^1(X, \mathcal{O}_X) \subset H^1(X_n, \mathcal{O}_{X_n})$. Thus it suffices to prove $H^1(X_n, \mathcal{O}_{X_n}) = 0$. Using Lemma 54.8.1 again, we see that it suffices to prove $R^1(X_i \to X_{i-1})_*\mathcal{O}_{X_i} = 0$ for $i = 1, \ldots, n$. This follows from Lemma 54.3.4. $\square$

0B4S Lemma 54.8.8. Let A be a local normal Nagata domain of dimension 2 which has a dualizing complex $\omega_A^\bullet$. If there exists a nonzero $d \in A$ such that for all normal modifications $X \to \mathrm{Spec}(A)$ the cokernel of the trace map

$$\Gamma(X, \omega_X) \to \omega_A$$

is annihilated by d, then reduction to rational singularities is possible for A.

Proof. For $X \to \mathrm{Spec}(A)$ as in the statement we have to bound $H^1(X, \mathcal{O}_X)$. Let ω_X be the dualizing module of X as in the statement of Grauert-Riemenschneider (Proposition 54.7.8). The trace map is the map $Rf_*\omega_X \to \omega_A$ described in Duality for Schemes, Section 48.7. By Grauert-Riemenschneider we have $Rf_*\omega_X = f_*\omega_X$ thus the trace map indeed produces a map $\Gamma(X, \omega_X) \to \omega_A$. By duality we have $Rf_*\omega_X = R\,\mathrm{Hom}_A(Rf_*\mathcal{O}_X, \omega_A)$ (this uses that $\omega_X[2]$ is the dualizing complex on X normalized relative to $\omega_A[2]$, see Duality for Schemes, Lemma 48.20.9 or more directly Section 48.19 or even more directly Example 48.3.9). The distinguished triangle

$$A \to Rf_*\mathcal{O}_X \to R^1f_*\mathcal{O}_X[-1] \to A[1]$$

is transformed by $R\,\mathrm{Hom}_A(-, \omega_A)$ into the short exact sequence

$$0 \to f_*\omega_X \to \omega_A \to \mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) \to 0$$

(and $\mathrm{Ext}^i_A(R^1f_*\mathcal{O}_X, \omega_A) = 0$ for $i \neq 2$; this will follow from the discussion below as well). Since $R^1f_*\mathcal{O}_X$ is supported in $\{\mathfrak{m}\}$, the local duality theorem tells us that

$$\mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) = \mathrm{Ext}^0_A(R^1f_*\mathcal{O}_X, \omega_A[2]) = \mathrm{Hom}_A(R^1f_*\mathcal{O}_X, E)$$

is the Matlis dual of $R^1f_*\mathcal{O}_X$ (and the other ext groups are zero), see Dualizing Complexes, Lemma 47.18.4. By the equivalence of categories inherent in Matlis duality (Dualizing Complexes, Proposition 47.7.8), if $R^1f_*\mathcal{O}_X$ is not annihilated by d, then neither is the Ext^2 above. Hence we see that $H^1(X, \mathcal{O}_X)$ is annihilated by d. Thus the required boundedness follows from Lemma 54.8.2 and (54.8.2.1). □

0B4T Lemma 54.8.9. Let p be a prime number. Let A be a regular local ring of dimension 2 and characteristic p. Let $A_0 \subset A$ be a subring such that Ω_{A/A_0} is free of rank $r < \infty$. Set $\omega_A = \Omega^r_{A/A_0}$. If $X \to \mathrm{Spec}(A)$ is the result of a sequence of blowups in closed points, then there exists a map

$$\varphi_X : (\Omega^r_{X/\mathrm{Spec}(A_0)})^{**} \longrightarrow \omega_X$$

extending the given identification in the generic point.

Proof. Observe that A is Gorenstein (Dualizing Complexes, Lemma 47.21.3) and hence the invertible module ω_A does indeed serve as a dualizing module. Moreover, any X as in the lemma has an invertible dualizing module ω_X as X is regular (hence Gorenstein) and proper over A, see Remark 54.7.7 and Lemma 54.3.2. Suppose we have constructed the map $\varphi_X : (\Omega^r_{X/A_0})^{**} \to \omega_X$ and suppose that $b : X' \to X$ is a blowup in a closed point. Set $\Omega^r_X = (\Omega^r_{X/A_0})^{**}$ and $\Omega^r_{X'} = (\Omega^r_{X'/A_0})^{**}$. Since $\omega_{X'} = b^!(\omega_X)$ a map $\Omega^r_{X'} \to \omega_{X'}$ is the same thing as a map $Rb_*(\Omega^r_{X'}) \to \omega_X$. See discussion in Remark 54.7.7 and Duality for Schemes, Section 48.19. Thus in turn it suffices to produce a map

$$Rb_*(\Omega^r_{X'}) \longrightarrow \Omega^r_X$$

The sheaves $\Omega^r_{X'}$ and Ω^r_X are invertible, see Divisors, Lemma 31.12.15. Consider the exact sequence

$$b^*\Omega_{X/A_0} \to \Omega_{X'/A_0} \to \Omega_{X'/X} \to 0$$

A local calculation shows that $\Omega_{X'/X}$ is isomorphic to an invertible module on the exceptional divisor E, see Lemma 54.3.6. It follows that either

$$\Omega^r_{X'} \cong (b^*\Omega^r_X)(E) \quad \text{or} \quad \Omega^r_{X'} \cong b^*\Omega^r_X$$

see Divisors, Lemma 31.15.13. (The second possibility never happens in characteristic zero, but can happen in characteristic p.) In both cases we see that $R^1b_*(\Omega^r_{X'}) = 0$ and $b_*(\Omega^r_{X'}) = \Omega^r_X$ by Lemma 54.3.4. □

0B4U Lemma 54.8.10. Let p be a prime number. Let A be a complete regular local ring of dimension 2 and characteristic p. Let L/K be a degree p inseparable extension of the fraction field K of A. Let $B \subset L$ be the integral closure of A. Then reduction to rational singularities is possible for B.

Proof. We have $A = k[[x, y]]$. Write $L = K[x]/(x^p - f)$ for some $f \in A$ and denote $g \in B$ the congruence class of x, i.e., the element such that $g^p = f$. By Algebra, Lemma 10.158.2 we see that $\mathrm{d}f$ is nonzero in $\Omega_{K/\mathbf{F}_p}$. By More on Algebra, Lemma 15.46.5 there exists a subfield $k^p \subset k' \subset k$ with $p^e = [k : k'] < \infty$ such that $\mathrm{d}f$ is nonzero in Ω_{K/K_0} where K_0 is the fraction field of $A_0 = k'[[x^p, y^p]] \subset A$. Then

$$\Omega_{A/A_0} = A \otimes_k \Omega_{k/k'} \oplus A\mathrm{d}x \oplus A\mathrm{d}y$$

is finite free of rank $e + 2$. Set $\omega_A = \Omega^{e+2}_{A/A_0}$. Consider the canonical map

$$\mathrm{Tr} : \Omega^{e+2}_{B/A_0} \longrightarrow \Omega^{e+2}_{A/A_0} = \omega_A$$

of Lemma 54.2.4. By duality this determines a map

$$c : \Omega^{e+2}_{B/A_0} \to \omega_B = \mathrm{Hom}_A(B, \omega_A)$$

Claim: the cokernel of c is annihilated by a nonzero element of B.

Since $\mathrm{d}f$ is nonzero in Ω_{A/A_0} we can find $\eta_1, \ldots, \eta_{e+1} \in \Omega_{A/A_0}$ such that $\theta = \eta_1 \wedge \ldots \wedge \eta_{e+1} \wedge \mathrm{d}f$ is nonzero in $\omega_A = \Omega^{e+2}_{A/A_0}$. To prove the claim we will construct elements ω_i of Ω^{e+2}_{B/A_0}, $i = 0, \ldots, p-1$ which are mapped to $\varphi_i \in \omega_B = \mathrm{Hom}_A(B, \omega_A)$ with $\varphi_i(g^j) = \delta_{ij}\theta$ for $j = 0, \ldots, p-1$. Since $\{1, g, \ldots, g^{p-1}\}$ is a basis for L/K this proves the claim. We set $\eta = \eta_1 \wedge \ldots \wedge \eta_{e+1}$ so that $\theta = \eta \wedge \mathrm{d}f$. Set $\omega_i = \eta \wedge g^{p-1-i}\mathrm{d}g$. Then by construction we have

$$\varphi_i(g^j) = \mathrm{Tr}(g^j\eta \wedge g^{p-1-i}\mathrm{d}g) = \mathrm{Tr}(\eta \wedge g^{p-1-i+j}\mathrm{d}g) = \delta_{ij}\theta$$

by the explicit description of the trace map in Lemma 54.2.2.

Let $Y \to \mathrm{Spec}(B)$ be a normal modification. Exactly as in the proof of Lemma 54.8.6 we can reduce to the case where Y is finite over a modification X of $\mathrm{Spec}(A)$. By Lemma 54.4.2 we may even assume $X \to \mathrm{Spec}(A)$ is the result of a sequence of blowing ups in closed points. Picture:

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[d, "\pi"'] & \mathrm{Spec}(B) \arrow[d] \\ X \arrow[r, "f"] & \mathrm{Spec}(A) \end{tikzcd}$$

We may apply Lemma 54.2.4 to π and we obtain the first arrow in

$$\pi_*(\Omega^{e+2}_{Y/A_0}) \xrightarrow{\text{Tr}} (\Omega^{e+2}_{X/A_0})^{**} \xrightarrow{\varphi_X} \omega_X$$

and the second arrow is from Lemma 54.8.9 (because f is a sequence of blowups in closed points). By duality for the finite morphism π this corresponds to a map

$$c_Y : \Omega^{e+2}_{Y/A_0} \longrightarrow \omega_Y$$

extending the map c above. Hence we see that the image of $\Gamma(Y, \omega_Y) \to \omega_B$ contains the image of c. By our claim we see that the cokernel is annihilated by a fixed nonzero element of B. We conclude by Lemma 54.8.8. □

54.9. Rational singularities

0B4V In this section we reduce from rational singular points to Gorenstein rational singular points. See [Lip69] and [Mat70b].

0B4W Situation 54.9.1. Here $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Let s be the closed point of $S = \text{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to S$ be a modification with X normal. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre X_s of f.

0B4X Lemma 54.9.2. In Situation 54.9.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

(1) $H^p(X, \mathcal{F}) = 0$ for $p \not\in \{0, 1\}$, and
(2) $H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.

Proof. Part (1) follows from Cohomology of Schemes, Lemma 30.20.9. If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on H^1. Since $H^1(X, \mathcal{O}_X) = 0$ as S has a rational singularity, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 20.19.1) we conclude. □

0B4Y Lemma 54.9.3. In Situation 54.9.1 assume $E = X_s$ is an effective Cartier divisor. Let $\mathcal{I}$ be the ideal sheaf of E. Then $H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$ and $H^1(X, \mathcal{I}^n) = 0$.

Proof. We have $H^0(X, \mathcal{O}_X) = A$, see discussion following Situation 54.7.1. Then $\mathfrak{m} \subset H^0(X, \mathcal{I}) \subset H^0(X, \mathcal{O}_X)$. The second inclusion is not an equality as $X_s \neq \emptyset$. Thus $H^0(X, \mathcal{I}) = \mathfrak{m}$. As $\mathcal{I}^n = \mathfrak{m}^n\mathcal{O}_X$ our Lemma 54.9.2 shows that $H^1(X, \mathcal{I}^n) = 0$.

Choose generators $x_1, \ldots, x_{\mu+1}$ of $\mathfrak{m}$. These define global sections of $\mathcal{I}$ which generate it. Hence a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus \mu+1} \to \mathcal{I} \to 0$$

Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module of rank μ and $\mathcal{F} \otimes \mathcal{I}$ is globally generated by Constructions, Lemma 27.13.9. Hence $\mathcal{F} \otimes \mathcal{I}^n$ is globally generated for all $n \geq 1$. Thus for $n \geq 2$ we can consider the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{I}^{n-1} \to (\mathcal{I}^{n-1})^{\oplus \mu+1} \to \mathcal{I}^n \to 0$$

Applying the long exact sequence of cohomology using that $H^1(X, \mathcal{F} \otimes \mathcal{I}^{n-1}) = 0$ by Lemma 54.9.2 we obtain that every element of $H^0(X, \mathcal{I}^n)$ is of the form $\sum x_i a_i$ for some $a_i \in H^0(X, \mathcal{I}^{n-1})$. This shows that $H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$ by induction. □

0B4Z Lemma 54.9.4. In Situation 54.9.1 the blowup of $\text{Spec}(A)$ in $\mathfrak{m}$ is normal.

Proof. Let $X' \to \mathrm{Spec}(A)$ be the blowup, in other words

$$X' = \mathrm{Proj}(A \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \ldots).$$

is the Proj of the Rees algebra. This in particular shows that X' is integral and that $X' \to \mathrm{Spec}(A)$ is a projective modification. Let X be the normalization of X'. Since A is Nagata, we see that $\nu : X \to X'$ is finite (Morphisms, Lemma 29.54.10). Let $E' \subset X'$ be the exceptional divisor and let $E \subset X$ be the inverse image. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be their ideal sheaves. Recall that $\mathcal{I}' = \mathcal{O}_{X'}(1)$ (Divisors, Lemma 31.32.13). Observe that $\mathcal{I} = \nu^*\mathcal{I}'$ and that E is an effective Cartier divisor (Divisors, Lemma 31.13.13). We are trying to show that ν is an isomorphism. As ν is finite, it suffices to show that $\mathcal{O}_{X'} \to \nu_*\mathcal{O}_X$ is an isomorphism. If not, then we can find an $n \geq 0$ such that

$$H^0(X', (\mathcal{I}')^n) \neq H^0(X', (\nu_*\mathcal{O}_X) \otimes (\mathcal{I}')^n)$$

for example because we can recover quasi-coherent $\mathcal{O}_{X'}$-modules from their associated graded modules, see Properties, Lemma 28.28.3. By the projection formula we have

$$H^0(X', (\nu_*\mathcal{O}_X) \otimes (\mathcal{I}')^n) = H^0(X, \nu^*(\mathcal{I}')^n) = H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$$

the last equality by Lemma 54.9.3. On the other hand, there is clearly an injection $\mathfrak{m}^n \to H^0(X', (\mathcal{I}')^n)$. Since $H^0(X', (\mathcal{I}')^n)$ is torsion free we conclude equality holds for all n, hence $X = X'$. □

0B63 Lemma 54.9.5. In Situation 54.9.1. Let X be the blowup of $\mathrm{Spec}(A)$ in $\mathfrak{m}$. Let $E \subset X$ be the exceptional divisor. With $\mathcal{O}_X(1) = \mathcal{I}$ as usual and $\mathcal{O}_E(1) = \mathcal{O}_X(1)|_E$ we have

(1) E is a proper Cohen-Macaulay curve over κ.
(2) $\mathcal{O}_E(1)$ is very ample
(3) $\deg(\mathcal{O}_E(1)) \geq 1$ and equality holds only if A is a regular local ring,
(4) $H^1(E, \mathcal{O}_E(n)) = 0$ for $n \geq 0$, and
(5) $H^0(E, \mathcal{O}_E(n)) = \mathfrak{m}^n/\mathfrak{m}^{n+1}$ for $n \geq 0$.

Proof. Since $\mathcal{O}_X(1)$ is very ample by construction, we see that its restriction to the special fibre E is very ample as well. By Lemma 54.9.4 the scheme X is normal. Then E is Cohen-Macaulay by Divisors, Lemma 31.15.6. Lemma 54.9.3 applies and we obtain (4) and (5) from the exact sequences

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to i_*\mathcal{O}_E(n) \to 0$$

and the long exact cohomology sequence. In particular, we see that

$$\deg(\mathcal{O}_E(1)) = \chi(E, \mathcal{O}_E(1)) - \chi(E, \mathcal{O}_E) = \dim(\mathfrak{m}/\mathfrak{m}^2) - 1$$

by Varieties, Definition 33.44.1. Thus (3) follows as well. □

0BBU Lemma 54.9.6. In Situation 54.9.1 assume A has a dualizing complex $\omega_A^\bullet$. With ω_X the dualizing module of X, the trace map $H^0(X, \omega_X) \to \omega_A$ is an isomorphism and consequently there is a canonical map $f^*\omega_A \to \omega_X$.

Proof. By Grauert-Riemenschneider (Proposition 54.7.8) we see that $Rf_*\omega_X = f_*\omega_X$. By duality we have a short exact sequence

$$0 \to f_*\omega_X \to \omega_A \to \mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) \to 0$$

(for example see proof of Lemma 54.8.8) and since A defines a rational singularity we obtain $f_*\omega_X = \omega_A$. $\square$

0B64 Lemma 54.9.7. In Situation 54.9.1 assume A has a dualizing complex $\omega_A^\bullet$ and is not regular. Let X be the blowup of $\mathrm{Spec}(A)$ in $\mathfrak m$ with exceptional divisor $E \subset X$. Let ω_X be the dualizing module of X. Then

(1) $\omega_E = \omega_X|_E \otimes \mathcal{O}_E(-1)$,
(2) $H^1(X, \omega_X(n)) = 0$ for $n \geq 0$,
(3) the map $f^*\omega_A \to \omega_X$ of Lemma 54.9.6 is surjective.

Proof. We will use the results of Lemma 54.9.5 without further mention. Observe that $\omega_E = \omega_X|_E \otimes \mathcal{O}_E(-1)$ by Duality for Schemes, Lemmas 48.14.2 and 48.9.7. Thus $\omega_X|_E = \omega_E(1)$. Consider the short exact sequences

$$0 \to \omega_X(n+1) \to \omega_X(n) \to i_*\omega_E(n+1) \to 0$$

By Algebraic Curves, Lemma 53.6.4 we see that $H^1(E, \omega_E(n+1)) = 0$ for $n \geq 0$. Thus we see that the maps

$$\ldots \to H^1(X, \omega_X(2)) \to H^1(X, \omega_X(1)) \to H^1(X, \omega_X)$$

are surjective. Since $H^1(X, \omega_X(n))$ is zero for $n \gg 0$ (Cohomology of Schemes, Lemma 30.16.2) we conclude that (2) holds.

By Algebraic Curves, Lemma 53.6.7 we see that $\omega_X|_E = \omega_E \otimes \mathcal{O}_E(1)$ is globally generated. Since we seen above that $H^1(X, \omega_X(1)) = 0$ the map $H^0(X, \omega_X) \to H^0(E, \omega_X|_E)$ is surjective. We conclude that ω_X is globally generated hence (3) holds because $\Gamma(X, \omega_X) = \omega_A$ is used in Lemma 54.9.6 to define the map. $\square$

0BBV Lemma 54.9.8. Let $(A, \mathfrak m, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Assume A has a dualizing complex. Then there exists a finite sequence of blowups in singular closed points

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

such that X_i is normal for each i and such that the dualizing sheaf ω_X of X is an invertible $\mathcal{O}_X$-module.

Proof. The dualizing module ω_A is a finite A-module whose stalk at the generic point is invertible. Namely, $\omega_A \otimes_A K$ is a dualizing module for the fraction field K of A, hence has rank 1. Thus there exists a blowup $b : Y \to \mathrm{Spec}(A)$ such that the strict transform of ω_A with respect to b is an invertible $\mathcal{O}_Y$-module, see Divisors, Lemma 31.35.3. By Lemma 54.5.3 we can choose a sequence of normalized blowups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \mathrm{Spec}(A)$$

such that X_n dominates Y. By Lemma 54.9.4 and arguing by induction each $X_i \to X_{i-1}$ is simply a blowing up.

We claim that ω_{X_n} is invertible. Since ω_{X_n} is a coherent $\mathcal{O}_{X_n}$-module, it suffices to see its stalks are invertible modules. If $x \in X_n$ is a regular point, then this is clear from the fact that regular schemes are Gorenstein (Dualizing Complexes, Lemma 47.21.3). If x is a singular point of X_n, then each of the images $x_i \in X_i$ of x is a singular point (because the blowup of a regular point is regular by Lemma 54.3.2). Consider the canonical map $f_n^*\omega_A \to \omega_{X_n}$ of Lemma 54.9.6. For each i the morphism $X_{i+1} \to X_i$ is either a blowup of x_i or an isomorphism at x_i. Since x_i is always a singular point, it follows from Lemma 54.9.7 and induction that the maps $f_i^*\omega_A \to \omega_{X_i}$ is always surjective on stalks at x_i. Hence

$$(f_n^*\omega_A)_x \longrightarrow \omega_{X_n,x}$$

is surjective. On the other hand, by our choice of b the quotient of $f_n^*\omega_A$ by its torsion submodule is an invertible module $\mathcal{L}$. Moreover, the dualizing module is torsion free (Duality for Schemes, Lemma 48.22.3). It follows that $\mathcal{L}_x \cong \omega_{X_n,x}$ and the proof is complete. □

54.10. Formal arcs

0BG1 Let X be a locally Noetherian scheme. In this section we say that a formal arc in X is a morphism $a : T \to X$ where T is the spectrum of a complete discrete valuation ring R whose residue field κ is identified with the residue field of the image p of the closed point of $\mathrm{Spec}(R)$. Let us say that the formal arc a is centered at p in this case. We say the formal arc $T \to X$ is nonsingular if the induced map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathfrak{m}_R/\mathfrak{m}_R^2$ is surjective.

Let $a : T \to X$, $T = \mathrm{Spec}(R)$ be a nonsingular formal arc centered at a closed point p of X. Assume X is locally Noetherian. Let $b : X_1 \to X$ be the blowing up of X at x. Since a is nonsingular, we see that there is an element $f \in \mathfrak{m}_p$ which maps to a uniformizer in R. In particular, we find that the generic point of T maps to a point of X not equal to p. In other words, with K the fraction field of R, the restriction of a defines a morphism $\mathrm{Spec}(K) \to X \setminus \{p\}$. Since the morphism b is proper and an isomorphism over $X \setminus \{x\}$ we can apply the valuative criterion of properness to obtain a unique morphism a_1 making the following diagram commute

$$\begin{tikzcd} T \arrow[r, "a_1"'] \arrow[dr, "a"'] & X_1 \arrow[d, "b"] \\ & X \end{tikzcd}$$

Let $p_1 \in X_1$ be the image of the closed point of T. Observe that p_1 is a closed point as it is a $\kappa = \kappa(p)$-rational point on the fibre of $X_1 \to X$ over x. Since we have a factorization

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X_1,p_1} \to R$$

we see that a_1 is a nonsingular formal arc as well.

We can repeat the process and obtain a sequence of blowing ups

$$\begin{tikzcd} T \arrow[d, "a"'] \arrow[dr, "a_1"] \arrow[drr, "a_2"] \arrow[drrr, "a_3"] & & & & \\ (X,p) & (X_1,p_1) \arrow[l] & (X_2,p_2) \arrow[l] & (X_3,p_3) \arrow[l] & \ldots \arrow[l] \end{tikzcd}$$

This kind of sequence of blowups can be characterized as follows.

0BG2 **Lemma 54.10.1.** *Let X be a locally Noetherian scheme. Let*

$$(X,p) = (X_0,p_0) \leftarrow (X_1,p_1) \leftarrow (X_2,p_2) \leftarrow (X_3,p_3) \leftarrow \ldots$$

be a sequence of blowups such that

(1) *p_i is closed, maps to p_{i-1}, and $\kappa(p_i) = \kappa(p_{i-1})$,*

(2) there exists an $x_1 \in \mathfrak{m}_p$ whose image in $\mathfrak{m}_{p_i}$, $i > 0$ defines the exceptional divisor $E_i \subset X_i$.

Then the sequence is obtained from a nonsingular arc $a : T \to X$ as above.

Proof. Let us write $\mathcal{O}_n = \mathcal{O}_{X_n, p_n}$ and $\mathcal{O} = \mathcal{O}_{X,p}$. Denote $\mathfrak{m} \subset \mathcal{O}$ and $\mathfrak{m}_n \subset \mathcal{O}_n$ the maximal ideals.

We claim that $x_1^t \not\in \mathfrak{m}_n^{t+1}$. Namely, if this were the case, then in the local ring $\mathcal{O}_{n+1}$ the element x_1^t would be in the ideal of $(t+1)E_{n+1}$. This contradicts the assumption that x_1 defines E_{n+1}.

For every n choose generators $y_{n,1}, \ldots, y_{n,t_n}$ for $\mathfrak{m}_n$. As $\mathfrak{m}_n \mathcal{O}_{n+1} = x_1 \mathcal{O}_{n+1}$ by assumption (2), we can write $y_{n,i} = a_{n,i} x_1$ for some $a_{n,i} \in \mathcal{O}_{n+1}$. Since the map $\mathcal{O}_n \to \mathcal{O}_{n+1}$ defines an isomorphism on residue fields by (1) we can choose $c_{n,i} \in \mathcal{O}_n$ having the same residue class as $a_{n,i}$. Then we see that

$$\mathfrak{m}_n = (x_1, z_{n,1}, \ldots, z_{n,t_n}), \quad z_{n,i} = y_{n,i} - c_{n,i} x_1$$

and the elements $z_{n,i}$ map to elements of $\mathfrak{m}_{n+1}^2$ in $\mathcal{O}_{n+1}$.

Let us consider

$$J_n = \mathrm{Ker}(\mathcal{O} \to \mathcal{O}_n/\mathfrak{m}_n^{n+1})$$

We claim that $\mathcal{O}/J_n$ has length $n+1$ and that $\mathcal{O}/(x_1) + J_n$ equals the residue field. For $n = 0$ this is immediate. Assume the statement holds for n. Let $f \in J_n$. Then in $\mathcal{O}_n$ we have

$$f = a x_1^{n+1} + x_1^n A_1(z_{n,i}) + x_1^{n-1} A_2(z_{n,i}) + \ldots + A_{n+1}(z_{n,i})$$

for some $a \in \mathcal{O}_n$ and some A_i homogeneous of degree i with coefficients in $\mathcal{O}_n$. Since $\mathcal{O} \to \mathcal{O}_n$ identifies residue fields, we may choose $a \in \mathcal{O}$ (argue as in the construction of $z_{n,i}$ above). Taking the image in $\mathcal{O}_{n+1}$ we see that f and $a x_1^{n+1}$ have the same image modulo $\mathfrak{m}_{n+1}^{n+2}$. Since $x_n^{n+1} \not\in \mathfrak{m}_{n+1}^{n+2}$ it follows that J_n/J_{n+1} has length 1 and the claim is true.

Consider $R = \lim \mathcal{O}/J_n$. This is a quotient of the $\mathfrak{m}$-adic completion of $\mathcal{O}$ hence it is a complete Noetherian local ring. On the other hand, it is not finite length and x_1 generates the maximal ideal. Thus R is a complete discrete valuation ring. The map $\mathcal{O} \to R$ lifts to a local homomorphism $\mathcal{O}_n \to R$ for every n. There are two ways to show this: (1) for every n one can use a similar procedure to construct $\mathcal{O}_n \to R_n$ and then one can show that $\mathcal{O} \to \mathcal{O}_n \to R_n$ factors through an isomorphism $R \to R_n$, or (2) one can use Divisors, Lemma 31.32.6 to show that $\mathcal{O}_n$ is a localization of a repeated affine blowup algebra to explicitly construct a map $\mathcal{O}_n \to R$. Having said this it is clear that our sequence of blowups comes from the nonsingular arc $a : T = \mathrm{Spec}(R) \to X$. □

The following lemma is a kind of Néron desingularization lemma.

0BG3 Lemma 54.10.2. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain of dimension 2. Let $A \to R$ be a surjection onto a complete discrete valuation ring. This defines a nonsingular arc $a : T = \mathrm{Spec}(R) \to \mathrm{Spec}(A)$. Let

$$\mathrm{Spec}(A) = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

be the sequence of blowing ups constructed from a. If $A_\mathfrak{p}$ is a regular local ring where $\mathfrak{p} = \mathrm{Ker}(A \to R)$, then for some i the scheme X_i is regular at x_i.

Proof. Let $x_1 \in \mathfrak m$ map to a uniformizer of R. Observe that $\kappa(\mathfrak p) = K$ is the fraction field of R. Write $\mathfrak p = (x_2, \ldots, x_r)$ with r minimal. If $r = 2$, then $\mathfrak m = (x_1, x_2)$ and A is regular and the lemma is true. Assume $r > 2$. After renumbering if necessary, we may assume that x_2 maps to a uniformizer of $A_\mathfrak p$. Then $\mathfrak p/\mathfrak p^2 + (x_2)$ is annihilated by a power of x_1. For $i > 2$ we can find $n_i \geq 0$ and $a_i \in A$ such that

$$x_1^{n_i} x_i - a_i x_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k$$

for some $a_{jk} \in A$. If $n_i = 0$ for some i, then we can remove x_i from the list of generators of $\mathfrak p$ and we win by induction on r. If for some i the element a_i is a unit, then we can remove x_2 from the list of generators of $\mathfrak p$ and we win in the same manner. Thus either $a_i \in \mathfrak p$ or $a_i = u_i x_1^{m_1} \bmod \mathfrak p$ for some $m_1 > 0$ and unit $u_i \in A$. Thus we have either

$$x_1^{n_i} x_i = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k \quad \text{or} \quad x_1^{n_i} x_i - u_i x_1^{m_i} x_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k$$

We will prove that after blowing up the integers n_i, m_i decrease which will finish the proof.

Let us see what happens with these equations on the affine blowup algebra $A' = A[\mathfrak m/x_1]$. As $\mathfrak m = (x_1, \ldots, x_r)$ we see that A' is generated over R by $y_i = x_i/x_1$ for $i \geq 2$. Clearly $A \to R$ extends to $A' \to R$ with kernel $(y_2, \ldots, y_r)$. Then we see that either

$$x_1^{n_i - 1} y_i = \sum\nolimits_{2 \leq j \leq k} a_{jk} y_j y_k \quad \text{or} \quad x_1^{n_i - 1} y_i - u_i x_1^{m_1 - 1} y_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} y_j y_k$$

and the proof is complete. ☐

54.11. Base change to the completion

0BG4 The following simple lemma will turn out to be a useful tool in what follows.

0BG5 Lemma 54.11.1. Let $(A, \mathfrak m, \kappa)$ be a local ring with finitely generated maximal ideal $\mathfrak m$. Let X be a scheme over A. Let $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ where $A^\wedge$ is the $\mathfrak m$-adic completion of A. For a point $q \in Y$ with image $p \in X$ lying over the closed point of $\mathrm{Spec}(A)$ the local ring map $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ induces an isomorphism on completions.

Proof. We may assume X is affine. Then we may write $X = \mathrm{Spec}(B)$. Let $\mathfrak q \subset B' = B \otimes_A A^\wedge$ be the prime corresponding to q and let $\mathfrak p \subset B$ be the prime ideal corresponding to p. By Algebra, Lemma 10.96.3 we have

$$B'/(\mathfrak m^\wedge)^n B' = A^\wedge/(\mathfrak m^\wedge)^n \otimes_A B = A/\mathfrak m^n \otimes_A B = B/\mathfrak m^n B$$

for all n. Since $\mathfrak m B \subset \mathfrak p$ and $\mathfrak m^\wedge B' \subset \mathfrak q$ we see that $B/\mathfrak p^n$ and $B'/\mathfrak q^n$ are both quotients of the ring displayed above by the nth power of the same prime ideal. The lemma follows. ☐

0BG6 Lemma 54.11.2. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $y \in Y$ with image $x \in X$. Then

(1) if $\mathcal{O}_{Y,y}$ is regular, then $\mathcal{O}_{X,x}$ is regular,
(2) if y is in the closed fibre, then $\mathcal{O}_{Y,y}$ is regular $\Leftrightarrow \mathcal{O}_{X,x}$ is regular, and
(3) If X is proper over A, then X is regular if and only if Y is regular.

Proof. Since $A \to A^\wedge$ is faithfully flat (Algebra, Lemma 10.97.3), we see that $Y \to X$ is flat. Hence (1) by Algebra, Lemma 10.164.4. Lemma 54.11.1 shows the morphism $Y \to X$ induces an isomorphism on complete local rings at points of the special fibres. Thus (2) by More on Algebra, Lemma 15.43.4. If X is proper over A, then Y is proper over $A^\wedge$ (Morphisms, Lemma 29.41.5) and we see every closed point of X and Y lies in the closed fibre. Thus we see that Y is a regular scheme if and only if X is so by Properties, Lemma 28.9.2. □

0AFK Lemma 54.11.3. Let $(A, \mathfrak m)$ be a Noetherian local ring with completion $A^\wedge$. Let $U \subset \text{Spec}(A)$ and $U^\wedge \subset \text{Spec}(A^\wedge)$ be the punctured spectra. If $Y \to \text{Spec}(A^\wedge)$ is a $U^\wedge$-admissible blowup, then there exists a U-admissible blowup $X \to \text{Spec}(A)$ such that $Y = X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge)$.

Proof. By definition there exists an ideal $J \subset A^\wedge$ such that $V(J) = \{\mathfrak m A^\wedge\}$ and such that Y is the blowup of $S^\wedge$ in the closed subscheme defined by J, see Divisors, Definition 31.34.1. Since $A^\wedge$ is Noetherian this implies $\mathfrak m^n A^\wedge \subset J$ for some n. Since $A^\wedge/\mathfrak m^n A^\wedge = A/\mathfrak m^n$ we find an ideal $\mathfrak m^n \subset I \subset A$ such that $J = IA^\wedge$. Let $X \to S$ be the blowup in I. Since $A \to A^\wedge$ is flat we conclude that the base change of X is Y by Divisors, Lemma 31.32.3. □

0BG7 Lemma 54.11.4. Let $(A, \mathfrak m, \kappa)$ be a Nagata local normal domain of dimension 2. Assume A defines a rational singularity and that the completion $A^\wedge$ of A is normal. Then

(1) $A^\wedge$ defines a rational singularity, and
(2) if $X \to \text{Spec}(A)$ is the blowing up in $\mathfrak m$, then for a closed point $x \in X$ the completion $\mathcal{O}_{X,x}$ is normal.

Proof. Let $Y \to \text{Spec}(A^\wedge)$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Varieties, Lemma 33.17.3 $Y \to \text{Spec}(A^\wedge)$ is an isomorphism over the punctured spectrum $U^\wedge = \text{Spec}(A^\wedge) \setminus \{\mathfrak m^\wedge\}$. By Lemma 54.7.2 there exists a $U^\wedge$-admissible blowup $Y' \to \text{Spec}(A^\wedge)$ dominating Y. By Lemma 54.11.3 we find there exists a U-admissible blowup $X \to \text{Spec}(A)$ whose base change to $A^\wedge$ dominates Y. Since A is Nagata, we can replace X by its normalization after which $X \to \text{Spec}(A)$ is a normal modification (but possibly no longer a U-admissible blowup). Then $H^1(X, \mathcal{O}_X) = 0$ as A defines a rational singularity. It follows that $H^1(X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge), \mathcal{O}_{X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge)}) = 0$ by flat base change (Cohomology of Schemes, Lemma 30.5.2 and flatness of $A \to A^\wedge$ by Algebra, Lemma 10.97.2). We find that $H^1(Y, \mathcal{O}_Y) = 0$ by Lemma 54.8.1.

Finally, let $X \to \text{Spec}(A)$ be the blowing up of $\text{Spec}(A)$ in $\mathfrak m$. Then $Y = X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge)$ is the blowing up of $\text{Spec}(A^\wedge)$ in $\mathfrak m^\wedge$. By Lemma 54.9.4 we see that both Y and X are normal. On the other hand, $A^\wedge$ is excellent (More on Algebra, Proposition 15.52.3) hence every affine open in Y is the spectrum of an excellent normal domain (More on Algebra, Lemma 15.52.2). Thus for $y \in Y$ the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}^\wedge$ is regular and by More on Algebra, Lemma 15.42.2 we find that $\mathcal{O}_{Y,y}^\wedge$ is normal. If $x \in X$ is a closed point of the special fibre, then there is a unique closed point $y \in Y$ lying over x. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induces an isomorphism on completions (Lemma 54.11.1) we conclude. □

0BG8 Lemma 54.11.5. Let $(A, \mathfrak m)$ be a local Noetherian ring. Let X be a scheme over A. Assume

(1) A is analytically unramified (Algebra, Definition 10.162.9),
(2) X is locally of finite type over A, and
(3) $X \to \mathrm{Spec}(A)$ is étale at the generic points of irreducible components of X.

Then the normalization of X is finite over X.

Proof. Since A is analytically unramified it is reduced by Algebra, Lemma 10.162.10. Since the normalization of X depends only on the reduction of X, we may replace X by its reduction X_{red}; note that $X_{red} \to X$ is an isomorphism over the open U where $X \to \mathrm{Spec}(A)$ is étale because U is reduced (Descent, Lemma 35.18.1) hence condition (3) remains true after this replacement. In addition we may and do assume that $X = \mathrm{Spec}(B)$ is affine.

The map

$$K = \prod\nolimits_{\mathfrak p \subset A \text{ minimal}} \kappa(\mathfrak p) \longrightarrow K^\wedge = \prod\nolimits_{\mathfrak p^\wedge \subset A^\wedge \text{ minimal}} \kappa(\mathfrak p^\wedge)$$

is injective because $A \to A^\wedge$ is faithfully flat (Algebra, Lemma 10.97.3) hence induces a surjective map between sets of minimal primes (by going down for flat ring maps, see Algebra, Section 10.41). Both sides are finite products of fields as our rings are Noetherian. Let $L = \prod_{\mathfrak q \subset B \text{ minimal}} \kappa(\mathfrak q)$. Our assumption (3) implies that $L = B \otimes_A K$ and that $K \to L$ is a finite étale ring map (this is true because $A \to B$ is generically finite, for example use Algebra, Lemma 10.122.10 or the more detailed results in Morphisms, Section 29.51). Since B is reduced we see that $B \subset L$. This implies that

$$C = B \otimes_A A^\wedge \subset L \otimes_A A^\wedge = L \otimes_K K^\wedge = M$$

Then M is the total ring of fractions of C and is a finite product of fields as a finite separable algebra over $K^\wedge$. It follows that C is reduced and that its normalization C' is the integral closure of C in M. The normalization B' of B is the integral closure of B in L. By flatness of $A \to A^\wedge$ we obtain an injective map $B' \otimes_A A^\wedge \to M$ whose image is contained in C'. Picture

$$B' \otimes_A A^\wedge \longrightarrow C'$$

As $A^\wedge$ is Nagata (by Algebra, Lemma 10.162.8), we see that C' is finite over $C = B \otimes_A A^\wedge$ (see Algebra, Lemmas 10.162.8 and 10.162.2). As C is Noetherian, we conclude that $B' \otimes_A A^\wedge$ is finite over $C = B \otimes_A A^\wedge$. Therefore by faithfully flat descent (Algebra, Lemma 10.83.2) we see that B' is finite over B which is what we had to show. $\square$

0BG9 Lemma 54.11.6. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. If the complement of the special fibre in Y is normal, then the normalization $X^\nu \to X$ is finite and the base change of X^ν to $\mathrm{Spec}(A^\wedge)$ recovers the normalization of Y.

Proof. There is an immediate reduction to the case where $X = \mathrm{Spec}(B)$ is affine with B a finite type A-algebra. Set $C = B \otimes_A A^\wedge$ so that $Y = \mathrm{Spec}(C)$. Since $A \to A^\wedge$ is faithfully flat, for any prime $\mathfrak q \subset B$ there exists a prime $\mathfrak r \subset C$ lying over $\mathfrak q$. Then $B_{\mathfrak q} \to C_{\mathfrak r}$ is faithfully flat. Hence if $\mathfrak q$ does not lie over $\mathfrak m$, then $C_{\mathfrak r}$ is normal by assumption on Y and we conclude that $B_{\mathfrak q}$ is normal by Algebra, Lemma 10.164.3. In this way we see that X is normal away from the special fibre.

Recall that the complete Noetherian local ring $A^\wedge$ is Nagata (Algebra, Lemma 10.162.8). Hence the normalization $Y^\nu \to Y$ is finite (Morphisms, Lemma 29.54.10) and an isomorphism away from the special fibre. Say $Y^\nu = \text{Spec}(C')$. Then $C \to C'$ is finite and an isomorphism away from $V(\mathfrak m C)$. Since $B \to C$ is flat and induces an isomorphism $B/\mathfrak m B \to C/\mathfrak m C$ there exists a finite ring map $B \to B'$ whose base change to C recovers $C \to C'$. See More on Algebra, Lemma 15.89.16 and Remark 15.89.19. Thus we find a finite morphism $X' \to X$ which is an isomorphism away from the special fibre and whose base change recovers $Y^\nu \to Y$. By the discussion in the first paragraph we see that X' is normal at points not on the special fibre. For a point $x \in X'$ on the special fibre we have a corresponding point $y \in Y^\nu$ and a flat map $\mathcal{O}_{X',x} \to \mathcal{O}_{Y^\nu,y}$. Since $\mathcal{O}_{Y^\nu,y}$ is normal, so is $\mathcal{O}_{X',x}$, see Algebra, Lemma 10.164.3. Thus X' is normal and it follows that it is the normalization of X. □

0BGA Lemma 54.11.7. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local domain whose completion $A^\wedge$ is normal. Then given any sequence

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to \text{Spec}(A^\wedge)$$

of normalized blowups, there exists a sequence of (proper) normalized blowups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \text{Spec}(A)$$

whose base change to $A^\wedge$ recovers the given sequence.

Proof. Given the sequence $Y_n \to \ldots \to Y_1 \to Y_0 = \text{Spec}(A^\wedge)$ we inductively construct $X_n \to \ldots \to X_1 \to X_0 = \text{Spec}(A)$. The base case is $i = 0$. Given X_i whose base change is Y_i, let $Y_i' \to Y_i$ be the blowing up in the closed point $y_i \in Y_i$ such that Y_{i+1} is the normalization of Y_i. Since the closed fibres of Y_i and X_i are isomorphic, the point y_i corresponds to a closed point x_i on the special fibre of X_i. Let $X_i' \to X_i$ be the blowup of X_i in x_i. Then the base change of X_i' to $\text{Spec}(A^\wedge)$ is isomorphic to Y_i'. By Lemma 54.11.6 the normalization $X_{i+1} \to X_i'$ is finite and its base change to $\text{Spec}(A^\wedge)$ is isomorphic to Y_{i+1}. □

54.12. Rational double points

0BGB In Section 54.9 we argued that resolution of 2-dimensional rational singularities reduces to the Gorenstein case. A Gorenstein rational surface singularity is a rational double point. We will resolve them by explicit computations.

According to the discussion in Examples, Section 110.19 there exists a normal Noetherian local domain A whose completion is isomorphic to $\mathbf{C}[[x, y, z]]/(z^2)$. In this case one could say that A has a rational double point singularity, but on the other hand, $\text{Spec}(A)$ does not have a resolution of singularities. This kind of behaviour cannot occur if A is a Nagata ring, see Algebra, Lemma 10.162.13.

However, it gets worse as there exists a local normal Nagata domain A whose completion is $\mathbf{C}[[x, y, z]]/(yz)$ and another whose completion is $\mathbf{C}[[x, y, z]]/(y^2 - z^3)$. This is Example 2.5 of [Nis12]. This is why we need to assume the completion of our ring is normal in this section.

0BGC Situation 54.12.1. Here $(A, \mathfrak m, \kappa)$ be a Nagata local normal domain of dimension 2 which defines a rational singularity, whose completion is normal, and which is Gorenstein. We assume A is not regular.

The arguments in this section will show that repeatedly blowing up singular points resolves $\text{Spec}(A)$ in this situation. We will need the following lemma in the course of the proof.

0BGD **Lemma 54.12.2.** Let κ be a field. Let $I \subset \kappa[x, y]$ be an ideal. Let

$$a + bx + cy + dx^2 + exy + fy^2 \in I^2$$

for some $a, b, c, d, e, f \in k$ not all zero. If the colength of I in $\kappa[x, y]$ is > 1, then $a + bx + cy + dx^2 + exy + fy^2 = j(g + hx + iy)^2$ for some $j, g, h, i \in \kappa$.

Proof. Consider the partial derivatives $b + 2dx + ey$ and $c + ex + 2fy$. By the Leibniz rules these are contained in I. If one of these is nonzero, then after a linear change of coordinates, i.e., of the form $x \mapsto \alpha + \beta x + \gamma y$ and $y \mapsto \delta + \epsilon x + \zeta y$, we may assume that $x \in I$. Then we see that $I = (x)$ or $I = (x, F)$ with F a monic polynomial of degree ≥ 2 in y. In the first case the statement is clear. In the second case observe that we can write any element in I^2 in the form

$$A(x, y)x^2 + B(y)xF + C(y)F^2$$

for some $A(x, y) \in \kappa[x, y]$ and $B, C \in \kappa[y]$. Thus

$$a + bx + cy + dx^2 + exy + fy^2 = A(x, y)x^2 + B(y)xF + C(y)F^2$$

and by degree reasons we see that $B = C = 0$ and A is a constant.

To finish the proof we need to deal with the case that both partial derivatives are zero. This can only happen in characteristic 2 and then we get

$$a + dx^2 + fy^2 \in I^2$$

We may assume f is nonzero (if not, then switch the roles of x and y). After dividing by f we obtain the case where the characteristic of κ is 2 and

$$a + dx^2 + y^2 \in I^2$$

If a and d are squares in κ, then we are done. If not, then there exists a derivation $\theta : \kappa \to \kappa$ with $\theta(a) \neq 0$ or $\theta(d) \neq 0$, see Algebra, Lemma 10.158.2. We can extend this to a derivation of $\kappa[x, y]$ by setting $\theta(x) = \theta(y) = 0$. Then we find that

$$\theta(a) + \theta(d)x^2 \in I$$

The case $\theta(d) = 0$ is absurd. Thus we may assume that $\alpha + x^2 \in I$ for some $\alpha \in \kappa$. Combining with the above we find that $a + \alpha d + y^2 \in I$. Hence

$$J = (\alpha + x^2, a + \alpha d + y^2) \subset I$$

with codimension at most 2. Observe that J/J^2 is free over $\kappa[x, y]/J$ with basis $\alpha + x^2$ and $a + \alpha d + y^2$. Thus $a + dx^2 + y^2 = 1 \cdot (a + \alpha d + y^2) + d \cdot (\alpha + x^2) \in I^2$ implies that the inclusion $J \subset I$ is strict. Thus we find a nonzero element of the form $g + hx + iy + jxy$ in I. If $j = 0$, then I contains a linear form and we can conclude as in the first paragraph. Thus $j \neq 0$ and $\dim_\kappa(I/J) = 1$ (otherwise we could find an element as above in I with $j = 0$). We conclude that I has the form $(\alpha + x^2, \beta + y^2, g + hx + iy + jxy)$ with $j \neq 0$ and has colength 3. In this case $a + dx^2 + y^2 \in I^2$ is impossible. This can be shown by a direct computation, but we prefer to argue as follows. Namely, to prove this statement we may assume that κ is algebraically closed. Then we can do a coordinate change $x \mapsto \sqrt{\alpha} + x$ and $y \mapsto \sqrt{\beta} + y$ and assume that $I = (x^2, y^2, g' + h'x + i'y + jxy)$ with the same j. Then

$g' = h' = i' = 0$ otherwise the colength of I is not 3. Thus we get $I = (x^2, y^2, xy)$ and the result is clear. □

Let $(A, \mathfrak{m}, \kappa)$ be as in Situation 54.12.1. Let $X \to \text{Spec}(A)$ be the blowing up of $\mathfrak{m}$ in $\text{Spec}(A)$. By Lemma 54.9.4 we see that X is normal. All singularities of X are rational singularities by Lemma 54.8.4. Since $\omega_A = A$ we see from Lemma 54.9.7 that $\omega_X \cong \mathcal{O}_X$ (see discussion in Remark 54.7.7 for conventions). Thus all singularities of X are Gorenstein. Moreover, the local rings of X at closed point have normal completions by Lemma 54.11.4. In other words, by blowing up $\text{Spec}(A)$ we obtain a normal surface X whose singular points are as in Situation 54.12.1. We will use this below without further mention. (Note: we will see in the course of the discussion below that there are finitely many of these singular points.)

Let $E \subset X$ be the exceptional divisor. We have $\omega_E = \mathcal{O}_E(-1)$ by Lemma 54.9.7. By Lemma 54.9.5 we have $\kappa = H^0(E, \mathcal{O}_E)$. Thus E is a Gorenstein curve and by Riemann-Roch as discussed in Algebraic Curves, Section 53.5 we have

$$\chi(E, \mathcal{O}_E) = 1 - g = -(1/2)\deg(\omega_E) = (1/2)\deg(\mathcal{O}_E(1))$$

where $g = \dim_\kappa H^1(E, \mathcal{O}_E) \geq 0$. Since $\deg(\mathcal{O}_E(1))$ is positive by Varieties, Lemma 33.44.15 we find that $g = 0$ and $\deg(\mathcal{O}_E(1)) = 2$. It follows that we have

$$\dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 2n + 1$$

by Lemma 54.9.5 and Riemann-Roch on E.

Choose $x_1, x_2, x_3 \in \mathfrak{m}$ which map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Because $\dim_\kappa(\mathfrak{m}^2/\mathfrak{m}^3) = 5$ the images of x_ix_j, $i \geq j$ in this κ-vector space satisfy a relation. In other words, we can find $a_{ij} \in A$, $i \geq j$, not all contained in $\mathfrak{m}$, such that

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = \sum a_{ijk}x_ix_jx_k$$

for some $a_{ijk} \in A$ where $i \leq j \leq k$. Denote $a \mapsto \overline{a}$ the map $A \to \kappa$. The quadratic form $q = \sum \overline{a}_{ij}t_it_j \in \kappa[t_1, t_2, t_3]$ is well defined up to multiplication by an element of κ^* by our choices. If during the course of our arguments we find that $\overline{a}_{ij} = 0$ in κ, then we can subsume the term $a_{ij}x_ix_j$ in the right hand side and assume $a_{ij} = 0$; this operation changes the a_{ijk} but not the other $a_{i'j'}$.

The blowing up is covered by 3 affine charts corresponding to the "variables" x_1, x_2, x_3. By symmetry it suffices to study one of the charts. To do this let

$$A' = A[\mathfrak{m}/x_1]$$

be the affine blowup algebra (as in Algebra, Section 10.70). Since x_1, x_2, x_3 generate $\mathfrak{m}$ we see that A' is generated by $y_2 = x_2/x_1$ and $y_3 = x_3/x_1$ over A. We will occasionally use $y_1 = 1$ to simplify formulas. Moreover, looking at our relation above we find that

$$a_{11} + a_{12}y_2 + a_{13}y_3 + a_{22}y_2^2 + a_{23}y_2y_3 + a_{33}y_3^2 = x_1(\sum a_{ijk}y_iy_jy_k)$$

in A'. Recall that $x_1 \in A'$ defines the exceptional divisor E on our affine open of X which is therefore scheme theoretically given by

$$\kappa[y_2, y_3]/(\overline{a}_{11} + \overline{a}_{12}y_2 + \overline{a}_{13}y_3 + \overline{a}_{22}y_2^2 + \overline{a}_{23}y_2y_3 + \overline{a}_{33}y_3^2)$$

In other words, $E \subset \mathbf{P}^2_\kappa = \text{Proj}(\kappa[t_1, t_2, t_3])$ is the zero scheme of the quadratic form q introduced above.

The quadratic form q is an important invariant of the singularity defined by A. Let us say we are in case II if q is a square of a linear form times an element of κ^* and in case I otherwise. Observe that we are in case II exactly if, after changing our choice of x_1, x_2, x_3, we have

$$x_3^2 = \sum a_{ijk} x_i x_j x_k$$

in the local ring A.

Let $\mathfrak m' \subset A'$ be a maximal ideal lying over $\mathfrak m$ with residue field κ'. In other words, $\mathfrak m'$ corresponds to a closed point $p \in E$ of the exceptional divisor. Recall that the surjection

$$\kappa[y_2, y_3] \to \kappa'$$

has kernel generated by two elements $f_2, f_3 \in \kappa[y_2, y_3]$ (see for example Algebra, Example 10.27.3 or the proof of Algebra, Lemma 10.114.1). Let $z_2, z_3 \in A'$ map to f_2, f_3 in $\kappa[y_2, y_3]$. Then we see that $\mathfrak m' = (x_1, z_2, z_3)$ because x_2 and x_3 become divisible by x_1 in A'.

Claim. If X is singular at p, then $\kappa' = \kappa$ or we are in case II. Namely, if $A'_{\mathfrak m'}$ is singular, then $\dim_{\kappa'} \mathfrak m'/(\mathfrak m')^2 = 3$ which implies that $\dim_{\kappa'} \overline{\mathfrak m}'/(\overline{\mathfrak m}')^2 = 2$ where $\overline{m}'$ is the maximal ideal of $\mathcal{O}_{E,p} = \mathcal{O}_{X,p}/x_1\mathcal{O}_{X,p}$. This implies that

$$q(1, y_2, y_3) = \overline{a}_{11} + \overline{a}_{12}y_2 + \overline{a}_{13}y_3 + \overline{a}_{22}y_2^2 + \overline{a}_{23}y_2y_3 + \overline{a}_{33}y_3^2 \in (f_2, f_3)^2$$

otherwise there would be a relation between the classes of z_2 and z_3 in $\overline{\mathfrak m}'/(\overline{\mathfrak m}')^2$. The claim now follows from Lemma 54.12.2.

Resolution in case I. By the claim any singular point of X is κ-rational. Pick such a singular point p. We may choose our $x_1, x_2, x_3 \in \mathfrak m$ such that p lies on the chart described above and has coordinates $y_2 = y_3 = 0$. Since it is a singular point arguing as in the proof of the claim we find that $q(1, y_2, y_3) \in (y_2, y_3)^2$. Thus we can choose $a_{11} = a_{12} = a_{13} = 0$ and $q(t_1, t_2, t_3) = q(t_2, t_3)$. It follows that

$$E = V(q) \subset \mathbf{P}^1_\kappa$$

either is the union of two distinct lines meeting at p or is a degree 2 curve with a unique κ-rational point (small detail omitted; use that q is not a square of a linear form up to a scalar). In both cases we conclude that X has a unique singular point p which is κ-rational. We need a bit more information in this case. First, looking at higher terms in the expression above, we find that $\overline{a}_{111} = 0$ because p is singular. Then we can write $a_{111} = b_{111}x_1 \bmod (x_2, x_3)$ for some $b_{111} \in A$. Then the quadratic form at p for the generators x_1, y_2, y_3 of $\mathfrak m'$ is

$$q' = \overline{b}_{111}t_1^2 + \overline{a}_{112}t_1t_2 + \overline{a}_{113}t_1t_3 + \overline{a}_{22}t_2^2 + \overline{a}_{23}t_2t_3 + \overline{a}_{33}t_3^2$$

We see that $E' = V(q')$ intersects the line $t_1 = 0$ in either two points or one point of degree 2. We conclude that p lies in case I.

Suppose that the blowing up $X' \to X$ of X at p again has a singular point p'. Then we see that p' is a κ-rational point and we can blow up to get $X'' \to X'$. If this process does not stop we get a sequence of blowings up

$$\mathrm{Spec}(A) \leftarrow X \leftarrow X' \leftarrow X'' \leftarrow \ldots$$

We want to show that Lemma 54.10.1 applies to this situation. To do this we have to say something about the choice of the element x_1 of $\mathfrak m$. Suppose that A is in case I and that X has a singular point. Then we will say that $x_1 \in \mathfrak m$ is a good coordinate if for any (equivalently some) choice of x_2, x_3 the quadratic form $q(t_1, t_2, t_3)$ has the property that $q(0, t_2, t_3)$ is not a scalar times a square. We have seen above that a good coordinate exists. If x_1 is a good coordinate, then the singular point $p \in E$ of X does not lie on the hypersurface $t_1 = 0$ because either this does not have a rational point or if it does, then it is not singular on X. Observe that this is equivalent to the statement that the image of x_1 in $\mathcal{O}_{X,p}$ cuts out the exceptional divisor E. Now the computations above show that if x_1 is a good coordinate for A, then $x_1 \in \mathfrak{m}'\mathcal{O}_{X,p}$ is a good coordinate for p. This of course uses that the notion of good coordinate does not depend on the choice of x_2, x_3 used to do the computation. Hence x_1 maps to a good coordinate at p', p'', etc. Thus Lemma 54.10.1 applies and our sequence of blowing ups comes from a nonsingular arc $A \to R$. Then the map $A^\wedge \to R$ is a surjection. Since the completion of A is normal, we conclude by Lemma 54.10.2 that after a finite number of blowups

$$\mathrm{Spec}(A^\wedge) \leftarrow X^\wedge \leftarrow (X')^\wedge \leftarrow \ldots$$

the resulting scheme $(X^{(n)})^\wedge$ is regular. Since $(X^{(n)})^\wedge \to X^{(n)}$ induces isomorphisms on complete local rings (Lemma 54.11.1) we conclude that the same is true for $X^{(n)}$.

Resolution in case II. Here we have

$$x_3^2 = \sum a_{ijk} x_i x_j x_k$$

in A for some choice of generators x_1, x_2, x_3 of $\mathfrak{m}$. Then $q = t_3^2$ and $E = 2C$ where C is a line. Recall that in A' we get

$$y_3^2 = x_1 (\sum a_{ijk} y_i y_j y_k)$$

Since we know that X is normal, we get a discrete valuation ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of C. The element $y_3 \in A'$ maps to a uniformizer of $\mathcal{O}_{X,\xi}$. Since x_1 scheme theoretically cuts out E which is C with multiplicity 2, we see that x_1 is a unit times y_3^2 in $\mathcal{O}_{X,\xi}$. Looking at our equality above we conclude that

$$h(y_2) = \overline{a}_{111} + \overline{a}_{112} y_2 + \overline{a}_{122} y_2^2 + \overline{a}_{222} y_2^3$$

must be nonzero in the residue field of ξ. Now, suppose that $p \in C$ defines a singular point. Then y_3 is zero at p and p must correspond to a zero of h by the reasoning used in proving the claim above. If h does not have a double zero at p, then the quadratic form q' at p is not a square and we conclude that p falls in case I which we have treated above[1]. Since the degree of h is 3 we get at most one singular point $p \in C$ falling into case II which is moreover κ-rational. After changing our choice of x_1, x_2, x_3 we may assume this is the point $y_2 = y_3 = 0$. Then $h = \overline{a}_{122} y_2^2 + \overline{a}_{222} y_2^3$. Moreover, it still has to be the case that $\overline{a}_{113} = 0$ for the quadratic form q' to have the right shape. Thus the local ring $\mathcal{O}_{X,p}$ defines a singularity as in the next paragraph.

[1]The maximal ideal at p in A' is generated by y_3, x_1 and a third element g whose image in $\kappa[y_2]$ is the prime divisor of h corresponding to p. If this prime divisor doesn't divide h twice, then we see that the quadratic form at p looks like

$$y_3^2 - x_1((\mathit{something})x_1 + (\mathit{something})y_3 + (\mathit{unit})g)$$

and this can never be a square in $\kappa[y_3, x_1, g]$.

The final case we treat is the case where we can choose our generators x_1, x_2, x_3 of $\mathfrak m$ such that

$$x_3^2 + x_1(ax_2^2 + bx_2x_3 + cx_3^2) \in \mathfrak m^4$$

for some $a, b, c \in A$. This is a subclass of case II. If $\overline{a} = 0$, then we can write $a = a_1x_1 + a_2x_2 + a_3x_3$ and we get after blowing up

$$y_3^2 + x_1(a_1x_1y_2^2 + a_2x_1y_2^3 + a_3x_1y_2^2y_3 + by_2y_3 + cy_3^2) = x_1^2(\sum a_{ijkl}y_iy_jy_ky_l)$$

This means that X is not normal[2] a contradiction. By the result of the previous paragraph, if the blowup X has a singular point p which falls in case II, then there is only one and it is κ-rational. Computing the affine blowup algebras $A[\frac{\mathfrak m}{x_2}]$ and $A[\frac{\mathfrak m}{x_3}]$ the reader easily sees that p cannot be contained the corresponding opens of X. Thus p is in the spectrum of $A[\frac{\mathfrak m}{x_1}]$. Doing the blowing up as before we see that p must be the point with coordinates $y_2 = y_3 = 0$ and the new equation looks like

$$y_3^2 + x_1(ay_2^2 + by_2y_3 + cy_3^2) \in (\mathfrak m')^4$$

which has the same shape as before and has the property that x_1 defines the exceptional divisor. Thus if the process does not stop we get an infinite sequence of blowups and on each of these x_1 defines the exceptional divisor in the local ring of the singular point. Thus we can finish the proof using Lemmas 54.10.1 and 54.10.2 and the same reasoning as before.

0BGE Lemma 54.12.3. Let $(A, \mathfrak m, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity, whose completion is normal, and which is Gorenstein. Then there exists a finite sequence of blowups in singular closed points

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

such that X_n is regular and such that each intervening schemes X_i is normal with finitely many singular points of the same type.

Proof. This is exactly what was proved in the discussion above. □

54.13. Implied properties

0BGF In this section we prove that for a Noetherian integral scheme the existence of a regular alteration has quite a few consequences. This section should be skipped by those not interested in "bad" Noetherian rings.

0BGG Lemma 54.13.1. Let Y be a Noetherian integral scheme. Assume there exists an alteration $f : X \to Y$ with X regular. Then the normalization $Y^\nu \to Y$ is finite and Y has a dense open which is regular.

Proof. It suffices to prove this when $Y = \mathrm{Spec}(A)$ where A is a Noetherian domain. Let B be the integral closure of A in its fraction field. Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Schemes, Lemma 30.19.2 we see that C is a finite A-module. As X is normal (Properties, Lemma 28.9.4) we see that C is normal domain (Properties, Lemma 28.7.9). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian.

There exists a nonempty open $V \subset Y$ such that $f^{-1}V \to V$ is finite, see Morphisms, Definition 29.51.12. After shrinking V we may assume that $f^{-1}V \to V$ is flat

[2] Namely, the equation shows that you get something singular along the 1-dimensional locus $x_1 = y_3 = 0$ which cannot happen for a normal surface.

(Morphisms, Proposition 29.27.1). Thus $f^{-1}V \to V$ is faithfully flat. Then V is regular by Algebra, Lemma 10.164.4. $\square$

0BGH Lemma 54.13.2. Let $(A, \mathfrak m)$ be a local Noetherian ring. Let $B \subset C$ be finite A-algebras. Assume that (a) B is a normal ring, and (b) the $\mathfrak m$-adic completion $C^\wedge$ is a normal ring. Then $B^\wedge$ is a normal ring.

Proof. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ B^\wedge \arrow[r] & C^\wedge \end{tikzcd}$$

Recall that $\mathfrak m$-adic completion on the category of finite A-modules is exact because it is given by tensoring with the flat A-algebra $A^\wedge$ (Algebra, Lemma 10.97.2). We will use Serre's criterion (Algebra, Lemma 10.157.4) to prove that the Noetherian ring $B^\wedge$ is normal. Let $\mathfrak q \subset B^\wedge$ be a prime lying over $\mathfrak p \subset B$. If $\dim(B_\mathfrak p) \geq 2$, then $\text{depth}(B_\mathfrak p) \geq 2$ and since $B_\mathfrak p \to B^\wedge_\mathfrak q$ is flat we find that $\text{depth}(B^\wedge_\mathfrak q) \geq 2$ (Algebra, Lemma 10.163.2). If $\dim(B_\mathfrak p) \leq 1$, then $B_\mathfrak p$ is either a discrete valuation ring or a field. In that case $C_\mathfrak p$ is faithfully flat over $B_\mathfrak p$ (because it is finite and torsion free). Hence $B^\wedge_\mathfrak p \to C^\wedge_\mathfrak p$ is faithfully flat and the same holds after localizing at $\mathfrak q$. As $C^\wedge$ and hence any localization is (S_2) we conclude that $B^\wedge_\mathfrak p$ is (S_2) by Algebra, Lemma 10.164.5. All in all we find that (S_2) holds for $B^\wedge$. To prove that $B^\wedge$ is (R_1) we only have to consider primes $\mathfrak q \subset B^\wedge$ with $\dim(B^\wedge_\mathfrak q) \leq 1$. Since $\dim(B^\wedge_\mathfrak q) = \dim(B_\mathfrak p) + \dim(B^\wedge_\mathfrak q/\mathfrak p B^\wedge_\mathfrak q)$ by Algebra, Lemma 10.112.6 we find that $\dim(B_\mathfrak p) \leq 1$ and we see that $B^\wedge_\mathfrak q \to C^\wedge_\mathfrak q$ is faithfully flat as before. We conclude using Algebra, Lemma 10.164.6. $\square$

0BGI Lemma 54.13.3. Let $(A, \mathfrak m, \kappa)$ be a local Noetherian domain. Assume there exists an alteration $f : X \to \text{Spec}(A)$ with X regular. Then

(1) there exists a nonzero $f \in A$ such that A_f is regular,
(2) the integral closure B of A in its fraction field is finite over A,
(3) the $\mathfrak m$-adic completion of B is a normal ring, i.e., the completions of B at its maximal ideals are normal domains, and
(4) the generic formal fibre of A is regular.

Proof. Parts (1) and (2) follow from Lemma 54.13.1. We have to redo part of the proof of that lemma in order to set up notation for the proof of (3). Set $C = \Gamma(X, \mathcal O_X)$. By Cohomology of Schemes, Lemma 30.19.2 we see that C is a finite A-module. As X is normal (Properties, Lemma 28.9.4) we see that C is normal domain (Properties, Lemma 28.7.9). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian. By Lemma 54.13.2 in order to prove (3) it suffices to show that the $\mathfrak m$-adic completion $C^\wedge$ is normal.

By Algebra, Lemma 10.97.8 the completion $C^\wedge$ is the product of the completions of C at the prime ideals of C lying over $\mathfrak m$. There are finitely many of these and these are the maximal ideals $\mathfrak m_1, \ldots, \mathfrak m_r$ of C. (The corresponding result for B explains the final statement of the lemma.) Thus replacing A by $C_{\mathfrak m_i}$ and X by $X_i = X \times_{\text{Spec}(C)} \text{Spec}(C_{\mathfrak m_i})$ we reduce to the case discussed in the next paragraph. (Note that $\Gamma(X_i, \mathcal O) = C_{\mathfrak m_i}$ by Cohomology of Schemes, Lemma 30.5.2.)

Here A is a Noetherian local normal domain and $f : X \to \mathrm{Spec}(A)$ is a regular alteration with $\Gamma(X, \mathcal{O}_X) = A$. We have to show that the completion $A^\wedge$ of A is a normal domain. By Lemma 54.11.2 $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular. Since $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ by Cohomology of Schemes, Lemma 30.5.2, we conclude that $A^\wedge$ is normal as before. Namely, Y is normal by Properties, Lemma 28.9.4. It is connected because $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is local. Hence Y is normal and integral (as connected and normal implies integral for Noetherian schemes). Thus $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is a normal domain by Properties, Lemma 28.7.9. This proves (3).

Proof of (4). Let $\eta \in \mathrm{Spec}(A)$ denote the generic point and denote by a subscript η the base change to η. Since f is an alteration, the scheme X_η is finite and faithfully flat over η. Since $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular by Lemma 54.11.2 we see that Y_η is regular (as a limit of opens in Y). Then $Y_\eta \to \mathrm{Spec}(A^\wedge \otimes_A \kappa(\eta))$ is finite faithfully flat onto the generic formal fibre. We conclude by Algebra, Lemma 10.164.4. □

54.14. Resolution

0BGJ Here is a definition.

0BGK **Definition 54.14.1.** Let Y be a Noetherian integral scheme. A *resolution of singularities* of Y is a modification $f : X \to Y$ such that X is regular.

In the case of surfaces we sometimes want a bit more information.

0BGL **Definition 54.14.2.** Let Y be a 2-dimensional Noetherian integral scheme. We say Y has a *resolution of singularities by normalized blowups* if there exists a sequence

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0 \to Y$$

where

(1) Y_i is proper over Y for $i = 0, \ldots, n$,
(2) $Y_0 \to Y$ is the normalization,
(3) $Y_i \to Y_{i-1}$ is a normalized blowup for $i = 1, \ldots, n$, and
(4) Y_n is regular.

Observe that condition (1) implies that the normalization Y_0 of Y is finite over Y and that the normalizations used in the normalized blowing ups are finite as well.

0BGM **Lemma 54.14.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Assume A is normal and has dimension 2. If $\mathrm{Spec}(A)$ has a resolution of singularities, then $\mathrm{Spec}(A)$ has a resolution by normalized blowups.*

Proof. By Lemma 54.13.3 the completion $A^\wedge$ of A is normal. By Lemma 54.11.2 we see that $\mathrm{Spec}(A^\wedge)$ has a resolution. By Lemma 54.11.7 any sequence $Y_n \to Y_{n-1} \to \ldots \to \mathrm{Spec}(A^\wedge)$ of normalized blowups of comes from a sequence of normalized blowups $X_n \to \ldots \to \mathrm{Spec}(A)$. Moreover if Y_n is regular, then X_n is regular by Lemma 54.11.2. Thus it suffices to prove the lemma in case A is complete.

Assume in addition A is a complete. We will use that A is Nagata (Algebra, Proposition 10.162.16), excellent (More on Algebra, Proposition 15.52.3), and has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Moreover, the same is true for any ring essentially of finite type over A. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on

Algebra, Lemma 15.42.2 applies). We will use this without further mention in the rest of the proof.

Let $X \to \mathrm{Spec}(A)$ be a resolution of singularities. Choose a sequence of normalized blowing ups

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to \mathrm{Spec}(A)$$

dominating X (Lemma 54.5.3). The morphism $Y_n \to X$ is an isomorphism away from finitely many points of X. Hence we can apply Lemma 54.4.2 to find a sequence of blowing ups

$$X_m \to X_{m-1} \to \ldots \to X$$

in closed points such that X_m dominates Y_n. Diagram

$$\begin{tikzcd} & Y_n \arrow[rr] \arrow[dr] & & \mathrm{Spec}(A) \\ X_m \arrow[ur] \arrow[rr] & & X \arrow[ur] & \end{tikzcd}$$

To prove the lemma it suffices to show that a finite number of normalized blowups of Y_n produce a regular scheme. By our diagram above we see that Y_n has a resolution (namely X_m). As Y_n is a normal surface this implies that Y_n has at most finitely many singularities $y_1, \ldots, y_t$ (because $X_m \to Y_n$ is an isomorphism away from the fibres of dimension 1, see Varieties, Lemma 33.17.3).

Let $x_a \in X$ be the image of y_a. Then $\mathcal{O}_{X,x_a}$ is regular and hence defines a rational singularity (Lemma 54.8.7). Apply Lemma 54.8.4 to $\mathcal{O}_{X,x_a} \to \mathcal{O}_{Y_n,y_a}$ to see that $\mathcal{O}_{Y_n,y_a}$ defines a rational singularity. By Lemma 54.9.8 there exists a finite sequence of blowups in singular closed points

$$Y_{a,n_a} \to Y_{a,n_a-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}_{Y_n,y_a})$$

such that Y_{a,n_a} is Gorenstein, i.e., has an invertible dualizing module. By (the essentially trivial) Lemma 54.6.4 with $n' = \sum n_a$ these sequences correspond to a sequence of blowups

$$Y_{n+n'} \to Y_{n+n'-1} \to \ldots \to Y_n$$

such that $Y_{n+n'}$ is normal and the local rings of $Y_{n+n'}$ are Gorenstein. Using the references given above we can dominate $Y_{n+n'}$ by a sequence of blowups $X_{m+m'} \to \ldots \to X_m$ dominating $Y_{n+n'}$ as in the following

$$\begin{tikzcd} & Y_{n+n'} \arrow[rr] & & Y_n \arrow[rr] \arrow[dr] & & \mathrm{Spec}(A) \\ X_{m+m'} \arrow[ur] \arrow[rr] & & X_m \arrow[ur] \arrow[rr] & & X \arrow[ur] & \end{tikzcd}$$

Thus again $Y_{n+n'}$ has a finite number of singular points $y'_1, \ldots, y'_s$, but this time the singularities are rational double points, more precisely, the local rings $\mathcal{O}_{Y_{n+n'},y'_b}$ are as in Lemma 54.12.3. Arguing exactly as above we conclude that the lemma is true. $\square$

0BGN Lemma 54.14.4. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. Assume A is a normal domain of dimension 2. Then $\mathrm{Spec}(A)$ has a resolution of singularities.

Proof. A Noetherian complete local ring is J-2 (More on Algebra, Proposition 15.48.7), Nagata (Algebra, Proposition 10.162.16), excellent (More on Algebra, Proposition 15.52.3), and has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Moreover, the same is true for any ring essentially of finite type over A. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 15.42.2 applies). In other words, the local rings which we encounter in the rest of the proof will have the required "excellency" properties required of them.

Choose $A_0 \subset A$ with A_0 a regular complete local ring and $A_0 \to A$ finite, see Algebra, Lemma 10.160.11. This induces a finite extension of fraction fields K/K_0. We will argue by induction on $[K : K_0]$. The base case is when the degree is 1 in which case $A_0 = A$ and the result is true.

Suppose there is an intermediate field $K_0 \subset L \subset K$, $K_0 \neq L \neq K$. Let $B \subset A$ be the integral closure of A_0 in L. By induction we choose a resolution of singularities $Y \to \text{Spec}(B)$. Let X be the normalization of $Y \times_{\text{Spec}(B)} \text{Spec}(A)$. Picture:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \text{Spec}(A) \arrow[d] \\ Y \arrow[r] & \text{Spec}(B) \end{tikzcd}$$

Since A is J-2 the regular locus of X is open. Since X is a normal surface we conclude that X has at worst finitely many singular points $x_1, \ldots, x_n$ which are closed points with $\dim(\mathcal{O}_{X,x_i}) = 2$. For each i let $y_i \in Y$ be the image. Since $\mathcal{O}^\wedge_{Y,y_i} \to \mathcal{O}^\wedge_{X,x_i}$ is finite of smaller degree than before we conclude by induction hypothesis that $\mathcal{O}^\wedge_{X,x_i}$ has resolution of singularities. By Lemma 54.14.3 there is a sequence

$$Z^\wedge_{i,n_i} \to \ldots \to Z^\wedge_{i,1} \to \text{Spec}(\mathcal{O}^\wedge_{X,x_i})$$

of normalized blowups with $Z^\wedge_{i,n_i}$ regular. By Lemma 54.11.7 there is a corresponding sequence of normalized blowing ups

$$Z_{i,n_i} \to \ldots \to Z_{i,1} \to \text{Spec}(\mathcal{O}_{X,x_i})$$

Then Z_{i,n_i} is a regular scheme by Lemma 54.11.2. By Lemma 54.6.5 we can fit these normalized blowing ups into a corresponding sequence

$$Z_n \to Z_{n-1} \to \ldots \to Z_1 \to X$$

and of course Z_n is regular too (look at the local rings). This proves the induction step.

Assume there is no intermediate field $K_0 \subset L \subset K$ with $K_0 \neq L \neq K$. Then either K/K_0 is separable or the characteristic to K is p and $[K : K_0] = p$. Then either Lemma 54.8.6 or 54.8.10 implies that reduction to rational singularities is possible. By Lemma 54.8.5 we conclude that there exists a normal modification $X \to \text{Spec}(A)$ such that for every singular point x of X the local ring $\mathcal{O}_{X,x}$ defines a rational singularity. Since A is J-2 we find that X has finitely many singular points $x_1, \ldots, x_n$. By Lemma 54.9.8 there exists a finite sequence of blowups in singular closed points

$$X_{i,n_i} \to X_{i,n_i-1} \to \ldots \to \text{Spec}(\mathcal{O}_{X,x_i})$$

such that X_{i,n_i} is Gorenstein, i.e., has an invertible dualizing module. By (the essentially trivial) Lemma 54.6.4 with $n = \sum n_a$ these sequences correspond to a sequence of blowups

$$X_n \to X_{n-1} \to \ldots \to X$$

such that X_n is normal and the local rings of X_n are Gorenstein. Again X_n has a finite number of singular points $x'_1, \ldots, x'_s$, but this time the singularities are rational double points, more precisely, the local rings $\mathcal{O}_{X_n,x'_i}$ are as in Lemma 54.12.3. Arguing exactly as above we conclude that the lemma is true. □

We finally come to the main theorem of this chapter.

0BGP Theorem 54.14.5 (Lipman). Let Y be a two dimensional integral Noetherian scheme. The following are equivalent [Lip78, Theorem on page 151]

(1) there exists an alteration $X \to Y$ with X regular,
(2) there exists a resolution of singularities of Y,
(3) Y has a resolution of singularities by normalized blowups,
(4) the normalization $Y^\nu \to Y$ is finite, Y^ν has finitely many singular points $y_1, \ldots, y_m$, and for each y_i the completion of $\mathcal{O}_{Y^\nu,y_i}$ is normal.

Proof. The implications (3) ⇒ (2) ⇒ (1) are immediate.

Let $X \to Y$ be an alteration with X regular. Then $Y^\nu \to Y$ is finite by Lemma 54.13.1. Consider the factorization $f : X \to Y^\nu$ from Morphisms, Lemma 29.54.5. The morphism f is finite over an open $V \subset Y^\nu$ containing every point of codimension ≤ 1 in Y^ν by Varieties, Lemma 33.17.2. Then f is flat over V by Algebra, Lemma 10.128.1 and the fact that a normal local ring of dimension ≤ 2 is Cohen-Macaulay by Serre's criterion (Algebra, Lemma 10.157.4). Then V is regular by Algebra, Lemma 10.164.4. As Y^ν is Noetherian we conclude that $Y^\nu \setminus V = \{y_1, \ldots, y_m\}$ is finite. By Lemma 54.13.3 the completion of $\mathcal{O}_{Y^\nu,y_i}$ is normal. In this way we see that (1) ⇒ (4).

Assume (4). We have to prove (3). We may immediately replace Y by its normalization. Let $y_1, \ldots, y_m \in Y$ be the singular points. Applying Lemmas 54.14.4 and 54.14.3 we find there exists a finite sequence of normalized blowups

$$Y_{i,n_i} \to Y_{i,n_i-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}_{Y,y_i}^\wedge)$$

such that Y_{i,n_i} is regular. By Lemma 54.11.7 there is a corresponding sequence of normalized blowing ups

$$X_{i,n_i} \to \ldots \to X_{i,1} \to \mathrm{Spec}(\mathcal{O}_{Y,y_i})$$

Then X_{i,n_i} is a regular scheme by Lemma 54.11.2. By Lemma 54.6.5 we can fit these normalized blowing ups into a corresponding sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to Y$$

and of course X_n is regular too (look at the local rings). This completes the proof. □

54.15. Embedded resolution

0BI3 Given a curve on a surface there is a blowing up which turns the curve into a strict normal crossings divisor. In this section we will use that a one dimensional locally Noetherian scheme is normal if and only if it is regular (Algebra, Lemma 10.119.7). We will also use that any point on a locally Noetherian scheme specializes to a closed point (Properties, Lemma 28.5.9).

0BI4 **Lemma 54.15.1.** *Let Y be a one dimensional integral Noetherian scheme. The following are equivalent*

1. *there exists an alteration $X \to Y$ with X regular,*
2. *there exists a resolution of singularities of Y,*
3. *there exists a finite sequence $Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y$ of blowups in closed points with Y_n regular, and*
4. *the normalization $Y^\nu \to Y$ is finite.*

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate. The implication (1) $\Rightarrow$ (4) follows from Lemma 54.13.1. Observe that a normal one dimensional scheme is regular hence the implication (4) $\Rightarrow$ (2) is clear as well. Thus it remains to show that the equivalent conditions (1), (2), and (4) imply (3).

Let $f : X \to Y$ be a resolution of singularities. Since the dimension of Y is one we see that f is finite by Varieties, Lemma 33.17.2. We will construct factorizations

$$X \to \ldots \to Y_2 \to Y_1 \to Y$$

where $Y_i \to Y_{i-1}$ is a blowing up of a closed point and not an isomorphism as long as Y_{i-1} is not regular. Each of these morphisms will be finite (by the same reason as above) and we will get a corresponding system

$$f_*\mathcal{O}_X \supset \ldots \supset f_{2,*}\mathcal{O}_{Y_2} \supset f_{1,*}\mathcal{O}_{Y_1} \supset \mathcal{O}_Y$$

where $f_i : Y_i \to Y$ is the structure morphism. Since Y is Noetherian, this increasing sequence of coherent submodules must stabilize (Cohomology of Schemes, Lemma 30.10.1) which proves that for some n the scheme Y_n is regular as desired. To construct Y_i given Y_{i-1} we pick a singular closed point $y_{i-1} \in Y_{i-1}$ and we let $Y_i \to Y_{i-1}$ be the corresponding blowup. Since X is regular of dimension 1 (and hence the local rings at closed points are discrete valuation rings and in particular PIDs), the ideal sheaf $\mathfrak{m}_{y_{i-1}} \cdot \mathcal{O}_X$ is invertible. By the universal property of blowing up (Divisors, Lemma 31.32.5) this gives us a factorization $X \to Y_i$. Finally, $Y_i \to Y_{i-1}$ is not an isomorphism as $\mathfrak{m}_{y_{i-1}}$ is not an invertible ideal. $\square$

0BI5 **Lemma 54.15.2.** *Let X be a Noetherian scheme. Let $Y \subset X$ be an integral closed subscheme of dimension 1 satisfying the equivalent conditions of Lemma 54.15.1. Then there exists a finite sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X$$

of blowups in closed points such that the strict transform of Y in X_n is a regular curve.

Proof. Let $Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y$ be the sequence of blowups given to us by Lemma 54.15.1. Let $X_n \to X_{n-1} \to \ldots \to X_1 \to X$ be the corresponding sequence of blowups of X. This works because the strict transform is the blowup by Divisors, Lemma 31.33.2. $\square$

Let X be a locally Noetherian scheme. Let $Y, Z \subset X$ be closed subschemes. Let $p \in Y \cap Z$ be a closed point. Assume that Y is integral of dimension 1 and that the generic point of Y is not contained in Z. In this situation we can consider the invariant

0BI6 (54.15.2.1)
$$m_p(Y \cap Z) = \text{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{Y \cap Z,p})$$

This is an integer ≥ 1. Namely, if $I, J \subset \mathcal{O}_{X,p}$ are the ideals corresponding to Y, Z, then we see that $\mathcal{O}_{Y \cap Z,p} = \mathcal{O}_{X,p}/I + J$ has support equal to $\{\mathfrak{m}_p\}$ because we assumed that $Y \cap Z$ does not contain the unique point of Y specializing to p. Hence the length is finite by Algebra, Lemma 10.62.3.

0BI7 Lemma 54.15.3. In the situation above let $X' \to X$ be the blowing up of X in p. Let $Y', Z' \subset X'$ be the strict transforms of Y, Z. If $\mathcal{O}_{Y,p}$ is regular, then

(1) $Y' \to Y$ is an isomorphism,
(2) Y' meets the exceptional fibre $E \subset X'$ in one point q and $m_q(Y \cap E) = 1$,
(3) if $q \in Z'$ too, then $m_q(Y \cap Z') < m_p(Y \cap Z)$.

Proof. Since $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective and $\mathcal{O}_{Y,p}$ is a discrete valuation ring, we can pick an element $x_1 \in \mathfrak{m}_p$ mapping to a uniformizer in $\mathcal{O}_{Y,p}$. Choose an affine open $U = \text{Spec}(A)$ containing p such that $x_1 \in A$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to p. Let $I, J \subset A$ be the ideals defining Y, Z in $\text{Spec}(A)$. After shrinking U we may assume that $\mathfrak{m} = I + (x_1)$, in other words, that $V(x_1) \cap U \cap Y = \{p\}$ scheme theoretically. We conclude that p is an effective Cartier divisor on Y and since Y' is the blowing up of Y in p (Divisors, Lemma 31.33.2) we see that $Y' \to Y$ is an isomorphism by Divisors, Lemma 31.32.7. The relationship $\mathfrak{m} = I + (x_1)$ implies that $\mathfrak{m}^n \subset I + (x_1^n)$ hence we can define a map

$$\psi : A[\tfrac{\mathfrak{m}}{x_1}] \longrightarrow A/I$$

by sending $y/x_1^n \in A[\frac{\mathfrak{m}}{x_1}]$ to the class of a in A/I where a is chosen such that $y \equiv ax_1^n \bmod I$. Then ψ corresponds to the morphism of $Y \cap U$ into X' over U given by $Y' \cong Y$. Since the image of x_1 in $A[\frac{\mathfrak{m}}{x_1}]$ cuts out the exceptional divisor we conclude that $m_q(Y', E) = 1$. Finally, since $J \subset \mathfrak{m}$ implies that the ideal $J' \subset A[\frac{\mathfrak{m}}{x_1}]$ certainly contains the elements f/x_1 for $f \in J$. Thus if we choose $f \in J$ whose image $\overline{f}$ in A/I has minimal valuation equal to $m_p(Y \cap Z)$, then we see that $\psi(f/x_1) = \overline{f}/x_1$ in A/I has valuation one less proving the last part of the lemma. $\square$

0BI8 Lemma 54.15.4. Let X be a Noetherian scheme. Let $Y_i \subset X$, $i = 1, \ldots, n$ be an integral closed subschemes of dimension 1 each satisfying the equivalent conditions of Lemma 54.15.1. Then there exists a finite sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X$$

of blowups in closed points such that the strict transform $Y_i' \subset X_n$ of Y_i in X_n are pairwise disjoint regular curves.

Proof. It follows from Lemma 54.15.2 that we may assume Y_i is a regular curve for $i = 1, \ldots, n$. For every $i \neq j$ and $p \in Y_i \cap Y_j$ we have the invariant $m_p(Y_i \cap Y_j)$ (54.15.2.1). If the maximum of these numbers is > 1, then we can decrease it (Lemma 54.15.3) by blowing up in all the points p where the maximum is attained. If the maximum is 1 then we can separate the curves using the same lemma by blowing up in all these points p. $\square$

When our curve is contained on a regular surface we often want to turn it into a divisor with normal crossings.

0BIB Lemma 54.15.5. Let X be a regular scheme of dimension 2. Let $Z \subset X$ be a proper closed subscheme. There exists a sequence

$$X_n \to \ldots \to X_1 \to X$$

of blowing ups in closed points such that the inverse image Z_n of Z in X_n is an effective Cartier divisor.

Proof. Let $D \subset Z$ be the largest effective Cartier divisor contained in Z. Then $\mathcal{I}_Z \subset \mathcal{I}_D$ and the quotient is supported in closed points by Divisors, Lemma 31.15.8. Thus we can write $\mathcal{I}_Z = \mathcal{I}_{Z'}\mathcal{I}_D$ where $Z' \subset X$ is a closed subscheme which set theoretically consists of finitely many closed points. Applying Lemma 54.4.1 we find a sequence of blowups as in the statement of our lemma such that $\mathcal{I}_{Z'}\mathcal{O}_{X_n}$ is invertible. This proves the lemma. □

0BIC Lemma 54.15.6. Let X be a regular scheme of dimension 2. Let $Z \subset X$ be a proper closed subscheme such that every irreducible component $Y \subset Z$ of dimension 1 satisfies the equivalent conditions of Lemma 54.15.1. Then there exists a sequence

$$X_n \to \ldots \to X_1 \to X$$

of blowups in closed points such that the inverse image Z_n of Z in X_n is an effective Cartier divisor supported on a strict normal crossings divisor.

Proof. Let $X' \to X$ be a blowup in a closed point p. Then the inverse image $Z' \subset X'$ of Z is supported on the strict transform of Z and the exceptional divisor. The exceptional divisor is a regular curve (Lemma 54.3.1) and the strict transform Y' of each irreducible component Y is either equal to Y or the blowup of Y at p. Thus in this process we do not produce additional singular components of dimension 1. Thus it follows from Lemmas 54.15.5 and 54.15.4 that we may assume Z is an effective Cartier divisor and that all irreducible components Y of Z are regular. (Of course we cannot assume the irreducible components are pairwise disjoint because in each blowup of a point of Z we add a new irreducible component to Z, namely the exceptional divisor.)

Assume Z is an effective Cartier divisor whose irreducible components Y_i are regular. For every $i \neq j$ and $p \in Y_i \cap Y_j$ we have the invariant $m_p(Y_i \cap Y_j)$ (54.15.2.1). If the maximum of these numbers is > 1, then we can decrease it (Lemma 54.15.3) by blowing up in all the points p where the maximum is attained (note that the “new” invariants $m_{q_i}(Y'_i \cap E)$ are always 1). If the maximum is 1 then, if $p \in Y_1 \cap \ldots \cap Y_r$ for some $r > 2$ and not any of the others (for example), then after blowing up p we see that $Y'_1, \ldots, Y'_r$ do not meet in points above p and $m_{q_i}(Y'_i, E) = 1$ where $Y'_i \cap E = \{q_i\}$. Thus continuing to blowup points where more than 3 of the components of Z meet, we reach the situation where for every closed point $p \in X$ there is either (a) no curves Y_i passing through p, (b) exactly one curve Y_i passing through p and $\mathcal{O}_{Y_i,p}$ is regular, or (c) exactly two curves Y_i, Y_j passing through p, the local rings $\mathcal{O}_{Y_i,p}$, $\mathcal{O}_{Y_j,p}$ are regular and $m_p(Y_i \cap Y_j) = 1$. This means that $\sum Y_i$ is a strict normal crossings divisor on the regular surface X, see Étale Morphisms, Lemma 41.21.2. □

54.16. Contracting exceptional curves

0C2I Let X be a Noetherian scheme. Let $E \subset X$ be a closed subscheme with the following properties

(1) E is an effective Cartier divisor on X,
(2) there exists a field k and an isomorphism $\mathbf{P}^1_k \to E$ of schemes,
(3) the normal sheaf $\mathcal{N}_{E/X}$ pulls back to $\mathcal{O}_{\mathbf{P}^1}(-1)$.

Such a closed subscheme is called an exceptional curve of the first kind.

Let X' be a Noetherian scheme and let $x \in X'$ be a closed point such that $\mathcal{O}_{X',x}$ is regular of dimension 2. Let $b : X \to X'$ be the blowing up of X' at x. In this case the exceptional fibre $E \subset X$ is an exceptional curve of the first kind. This follows from Lemma 54.3.1.

Question: Is every exceptional curve of the first kind obtained as the fibre of a blowing up as above? In other words, does there always exist a proper morphism of schemes $X \to X'$ such that E maps to a closed point $x \in X'$, such that $\mathcal{O}_{X',x}$ is regular of dimension 2, and such that X is the blowing up of X' at x. If true we say there exists a contraction of E.

0C5J **Lemma 54.16.1.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If a contraction $X \to X'$ of E exists, then it has the following universal property: for every morphism $\varphi : X \to Y$ such that $\varphi(E)$ is a point, there is a unique factorization $X \to X' \to Y$ of φ.*

Proof. Let $b : X \to X'$ be a contraction of E. As a topological space X' is the quotient of X by the relation identifying all points of E to one point. Namely, b is proper (Divisors, Lemma 31.32.13 and Morphisms, Lemma 29.43.5) and surjective, hence defines a submersive map of topological spaces (Topology, Lemma 5.6.5). On the other hand, the canonical map $\mathcal{O}_{X'} \to b_*\mathcal{O}_X$ is an isomorphism. Namely, this is clear over the complement of the image point $x \in X'$ of E and on stalks at x the map is an isomorphism by part (4) of Lemma 54.3.4. Thus the pair $(X', \mathcal{O}_{X'})$ is constructed from X by taking the quotient as a topological space and endowing this with $b_*\mathcal{O}_X$ as structure sheaf.

Given φ we can let $\varphi' : X' \to Y$ be the unique map of topological spaces such that $\varphi = \varphi' \circ b$. Then the map

$$\varphi^\sharp : \varphi^{-1}\mathcal{O}_Y = b^{-1}((\varphi')^{-1}\mathcal{O}_Y) \to \mathcal{O}_X$$

is adjoint to a map

$$(\varphi')^\sharp : (\varphi')^{-1}\mathcal{O}_Y \to b_*\mathcal{O}_X = \mathcal{O}_{X'}$$

Then $(\varphi', (\varphi')^\sharp)$ is a morphism of ringed spaces from X' to Y such that we get the desired factorization. Since φ is a morphism of locally ringed spaces, it follows that φ' is too. Namely, the only thing to check is that the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X',x}$ is local, where $y \in Y$ is the image of E under φ. This is true because an element $f \in \mathfrak{m}_y$ pulls back to a function on X which is zero in every point of E hence the pull back of f to X' is a function defined on a neighbourhood of x in X' with the same property. Then it is clear that this function must vanish at x as desired. $\square$

0C5K **Lemma 54.16.2.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If there exists a contraction of E, then it is unique up to unique isomorphism.*

Proof. This is immediate from the universal property of Lemma 54.16.1. □

0C2K Lemma 54.16.3. Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. Let $E_n = nE$ and denote $\mathcal{O}_n$ its structure sheaf. Then

$$A = \lim H^0(E_n, \mathcal{O}_n)$$

is a complete local Noetherian regular local ring of dimension 2 and $\text{Ker}(A \to H^0(E_n, \mathcal{O}_n))$ is the nth power of its maximal ideal.

Proof. Recall that there exists an isomorphism $\mathbf{P}^1_k \to E$ such that the normal sheaf of E in X pulls back to $\mathcal{O}(-1)$. Then $H^0(E, \mathcal{O}_E) = k$. We will denote $\mathcal{O}_n(iE)$ the restriction of the invertible sheaf $\mathcal{O}_X(iE)$ to E_n for all $n \geq 1$ and $i \in \mathbf{Z}$. Recall that $\mathcal{O}_X(-nE)$ is the ideal sheaf of E_n. Hence for $d \geq 0$ we obtain a short exact sequence

$$0 \to \mathcal{O}_E(-(d+n)E) \to \mathcal{O}_{n+1}(-dE) \to \mathcal{O}_n(-dE) \to 0$$

Since $\mathcal{O}_E(-(d+n)E) = \mathcal{O}_{\mathbf{P}^1_k}(d+n)$ the first cohomology group vanishes for all $d \geq 0$ and $n \geq 1$. We conclude that the transition maps of the system $H^0(E_n, \mathcal{O}_n(-dE))$ are surjective. For $d = 0$ we get an inverse system of surjections of rings such that the kernel of each transition map is a nilpotent ideal. Hence $A = \lim H^0(E_n, \mathcal{O}_n)$ is a local ring with residue field k and maximal ideal

$$\lim \text{Ker}(H^0(E_n, \mathcal{O}_n) \to H^0(E, \mathcal{O}_E)) = \lim H^0(E_n, \mathcal{O}_n(-E))$$

Pick x, y in this kernel mapping to a k-basis of $H^0(E, \mathcal{O}_E(-E)) = H^0(\mathbf{P}^1_k, \mathcal{O}(1))$. Then $x^d, x^{d-1}y, \ldots, y^d$ are elements of $\lim H^0(E_n, \mathcal{O}_n(-dE))$ which map to a basis of $H^0(E, \mathcal{O}_E(-dE)) = H^0(\mathbf{P}^1_k, \mathcal{O}(d))$. In this way we see that A is separated and complete with respect to the linear topology defined by the kernels

$$I_n = \text{Ker}(A \longrightarrow H^0(E_n, \mathcal{O}_n))$$

We have $x, y \in I_1$, $I_d I_{d'} \subset I_{d+d'}$ and I_d/I_{d+1} is a free k-module on $x^d, x^{d-1}y, \ldots, y^d$. We will show that $I_d = (x, y)^d$. Namely, if $z_e \in I_e$ with $e \geq d$, then we can write

$$z_e = a_{e,0}x^d + a_{e,1}x^{d-1}y + \ldots + a_{e,d}y^d + z_{e+1}$$

where $a_{e,j} \in (x, y)^{e-d}$ and $z_{e+1} \in I_{e+1}$ by our description of I_d/I_{d+1}. Thus starting with some $z = z_d \in I_d$ we can do this inductively

$$z = \sum\nolimits_{e \geq d} \sum\nolimits_j a_{e,j} x^{d-j} y^j$$

with some $a_{e,j} \in (x, y)^{e-d}$. Then $a_j = \sum_{e \geq d} a_{e,j}$ exists (by completeness and the fact that $a_{e,j} \in I_{e-d}$) and we have $z = \sum a_{e,j} x^{d-j} y^j$. Hence $I_d = (x, y)^d$. Thus A is (x, y)-adically complete. Then A is Noetherian by Algebra, Lemma 10.97.5. It is clear that the dimension is 2 by the description of $(x, y)^d/(x, y)^{d+1}$ and Algebra, Proposition 10.60.9. Since the maximal ideal is generated by two elements it is regular. □

0C2L Lemma 54.16.4. Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If there exists a morphism $f : X \to Y$ such that

(1) Y is Noetherian,
(2) f is proper,
(3) f maps E to a point y of Y,
(4) f is quasi-finite at every point not in E,

Then there exists a contraction of E and it is the Stein factorization of f.

Proof. We apply More on Morphisms, Theorem 37.53.4 to get a Stein factorization $X \to X' \to Y$. Then $X \to X'$ satisfies all the hypotheses of the lemma (some details omitted). Thus after replacing Y by X' we may in addition assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that the fibres of f are geometrically connected.

Assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that the fibres of f are geometrically connected. Note that $y \in Y$ is a closed point as f is closed and E is closed. The restriction $f^{-1}(Y \setminus \{y\}) \to Y \setminus \{y\}$ of f is a finite morphism (More on Morphisms, Lemma 37.44.1). Hence this restriction is an isomorphism since $f_*\mathcal{O}_X = \mathcal{O}_Y$ since finite morphisms are affine. To prove that $\mathcal{O}_{Y,y}$ is regular of dimension 2 we consider the isomorphism

$$\mathcal{O}_{Y,y}^\wedge \longrightarrow \lim H^0(X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n), \mathcal{O})$$

of Cohomology of Schemes, Lemma 30.20.7. Let $E_n = nE$ as in Lemma 54.16.3. Observe that

$$E_n \subset X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$$

because $E \subset X_y = X \times_Y \mathrm{Spec}(\kappa(y))$. On the other hand, since $E = f^{-1}(\{y\})$ set theoretically (because the fibres of f are geometrically connected), we see that the scheme theoretic fibre X_y is scheme theoretically contained in E_n for some $n > 0$. Namely, apply Cohomology of Schemes, Lemma 30.10.2 to the coherent $\mathcal{O}_X$-module $\mathcal{F} = \mathcal{O}_{X_y}$ and the ideal sheaf $\mathcal{I}$ of E and use that $\mathcal{I}^n$ is the ideal sheaf of E_n. This shows that

$$X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^m) \subset E_{nm}$$

Thus the inverse limit displayed above is equal to $\lim H^0(E_n, \mathcal{O}_n)$ which is a regular two dimensional local ring by Lemma 54.16.3. Hence $\mathcal{O}_{Y,y}$ is a two dimensional regular local ring because its completion is so (More on Algebra, Lemma 15.43.4 and 15.43.1).

We still have to prove that $f : X \to Y$ is the blowup $b : Y' \to Y$ of Y at y. We encourage the reader to find her own proof. First, we note that Lemma 54.16.3 also implies that $X_y = E$ scheme theoretically. Since the ideal sheaf of E is invertible, this shows that $f^{-1}\mathfrak{m}_y \cdot \mathcal{O}_X$ is invertible. Hence we obtain a factorization

$$X \to Y' \to Y$$

of the morphism f by the universal property of blowing up, see Divisors, Lemma 31.32.5. Recall that the exceptional fibre of $E' \subset Y'$ is an exceptional curve of the first kind by Lemma 54.3.1. Let $g : E \to E'$ be the induced morphism. Because for both E' and E the conormal sheaf is generated by (pullbacks of) a and b, we see that the canonical map $g^*\mathcal{C}_{E'/Y'} \to \mathcal{C}_{E/X}$ (Morphisms, Lemma 29.31.3) is surjective. Since both are invertible, this map is an isomorphism. Since $\mathcal{C}_{E/X}$ has positive degree, it follows that g cannot be a constant morphism. Hence g has finite fibres. Hence g is a finite morphism (same reference as above). However, since Y' is regular (and hence normal) at all points of E' and since $X \to Y'$ is birational and an isomorphism away from E', we conclude that $X \to Y'$ is an isomorphism by Varieties, Lemma 33.17.3. □

0C5L Lemma 54.16.5. Let $b : X \to X'$ be the contraction of an exceptional curve of the first kind $E \subset X$. Then there is a short exact sequence

$$0 \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \to \mathbf{Z} \to 0$$

where the first map is pullback by b and the second map sends $\mathcal{L}$ to the degree of $\mathcal{L}$ on the exceptional curve E. The sequence is split by the map $n \mapsto \mathcal{O}_X(-nE)$.

Proof. Since $E = \mathbf{P}^1_k$ we see that the Picard group of E is $\mathbf{Z}$, see Divisors, Lemma 31.28.5. Hence we can think of the last map as $\mathcal{L} \mapsto \mathcal{L}|_E$. The degree of the restriction of $\mathcal{O}_X(E)$ to E is -1 by definition of exceptional curves of the first kind. Combining these remarks we see that it suffices to show that $\mathrm{Pic}(X') \to \mathrm{Pic}(X)$ is injective with image the invertible sheaves restricting to $\mathcal{O}_E$ on E.

Given an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ we claim the map $\mathcal{L}' \to b_*b^*\mathcal{L}'$ is an isomorphism. This is clear everywhere except possibly at the image point $x \in X'$ of E. To check it is an isomorphism on stalks at x we may replace X' by an open neighbourhood at x and assume $\mathcal{L}'$ is $\mathcal{O}_{X'}$. Then we have to show that the map $\mathcal{O}_{X'} \to b_*\mathcal{O}_X$ is an isomorphism. This follows from Lemma 54.3.4 part (4).

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with $\mathcal{L}|_E = \mathcal{O}_E$. Then we claim (1) $b_*\mathcal{L}$ is invertible and (2) $b^*b_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Statements (1) and (2) are clear over $X' \setminus \{x\}$. Thus it suffices to prove (1) and (2) after base change to $\mathrm{Spec}(\mathcal{O}_{X',x})$. Computing b_* commutes with flat base change (Cohomology of Schemes, Lemma 30.5.2) and similarly for b^* and formation of the adjunction map. But if X' is the spectrum of a regular local ring then $\mathcal{L}$ is trivial by the description of the Picard group in Lemma 54.3.3. Thus the claim is proved.

Combining the claims proved in the previous two paragraphs we see that the map $\mathcal{L} \mapsto b_*\mathcal{L}$ is an inverse to the map

$$\mathrm{Pic}(X') \longrightarrow \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(E))$$

and the lemma is proved. □

0C5M Remark 54.16.6. Let $b : X \to X'$ be the contraction of an exceptional curve of the first kind $E \subset X$. From Lemma 54.16.5 we obtain an identification

$$\mathrm{Pic}(X) = \mathrm{Pic}(X') \oplus \mathbf{Z}$$

where $\mathcal{L}$ corresponds to the pair $(\mathcal{L}', n)$ if and only if $\mathcal{L} = (b^*\mathcal{L}')(-nE)$, i.e., $\mathcal{L}(nE) = b^*\mathcal{L}'$. In fact the proof of Lemma 54.16.5 shows that $\mathcal{L}' = b_*\mathcal{L}(nE)$. Of course the assignment $\mathcal{L} \mapsto \mathcal{L}'$ is a group homomorphism.

0C2J Lemma 54.16.7. Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let n be the integer such that $\mathcal{L}|_E$ has degree n viewed as an invertible module on $\mathbf{P}^1$. Then

(1) If $H^1(X, \mathcal{L}) = 0$ and $n \geq 0$, then $H^1(X, \mathcal{L}(iE)) = 0$ for $0 \leq i \leq n+1$.
(2) If $n \leq 0$, then $H^1(X, \mathcal{L}) \subset H^1(X, \mathcal{L}(E))$.

Proof. Observe that $\mathcal{L}|_E = \mathcal{O}(n)$ by Divisors, Lemma 31.28.5. Use induction, the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(E) \to \mathcal{L}(E)|_E \to 0,$$

and use the fact that $H^1(\mathbf{P}^1, \mathcal{O}(d)) = 0$ for $d \geq -1$ and $H^0(\mathbf{P}^1, \mathcal{O}(d)) = 0$ for $d \leq -1$. Some details omitted. □

0C2M Lemma 54.16.8. Let $S = \mathrm{Spec}(R)$ be an affine Noetherian scheme. Let $X \to S$ be a proper morphism. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $E \subset X$ be an exceptional curve of the first kind. Then

(1) there exists a contraction $b : X \to X'$ of E,
(2) X' is proper over S, and
(3) the invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ is ample with $\mathcal{L}'$ as in Remark 54.16.6.

Proof. Let n be the degree of $\mathcal{L}|_E$ as in Lemma 54.16.7. Observe that $n > 0$ as $\mathcal{L}$ is ample on E (Varieties, Lemma 33.44.14 and Properties, Lemma 28.26.3). After replacing $\mathcal{L}$ by a power we may assume $H^i(X, \mathcal{L}^{\otimes e}) = 0$ for all $i > 0$ and $e > 0$, see Cohomology of Schemes, Lemma 30.17.1. Finally, after replacing $\mathcal{L}$ by another power we may assume there exist global sections $t_0, \ldots, t_n$ of $\mathcal{L}$ which define a closed immersion $\psi : X \to \mathbf{P}^n_S$, see Morphisms, Lemma 29.39.4.

Set $\mathcal{M} = \mathcal{L}(nE)$. Then $\mathcal{M}|_E \cong \mathcal{O}_E$. Since we have the short exact sequence

$$0 \to \mathcal{M}(-E) \to \mathcal{M} \to \mathcal{O}_E \to 0$$

and since $H^1(X, \mathcal{M}(-E))$ is zero (by Lemma 54.16.7 and the fact that $n > 0$) we can pick a section s_{n+1} of $\mathcal{M}$ which generates $\mathcal{M}|_E$. Finally, denote $s_0, \ldots, s_n$ the sections of $\mathcal{M}$ we get from the sections $t_0, \ldots, t_n$ of $\mathcal{L}$ chosen above via $\mathcal{L} \subset \mathcal{L}(nE) = \mathcal{M}$. Combined the sections $s_0, \ldots, s_n, s_{n+1}$ generate $\mathcal{M}$ in every point of X and therefore define a morphism

$$\varphi : X \longrightarrow \mathbf{P}^{n+1}_S$$

over S, see Constructions, Lemma 27.13.1.

Below we will check the conditions of Lemma 54.16.4. Once this is done we see that the Stein factorization $X \to X' \to \mathbf{P}^{n+1}_S$ of φ is the desired contraction which proves (1). Moreover, the morphism $X' \to \mathbf{P}^{n+1}_S$ is finite hence X' is proper over S (Morphisms, Lemmas 29.44.11 and 29.41.4). This proves (2). Observe that X' has an ample invertible sheaf. Namely the pullback $\mathcal{M}'$ of $\mathcal{O}_{\mathbf{P}^{n+1}_S}(1)$ is ample by Morphisms, Lemma 29.37.7. Observe that $\mathcal{M}'$ pulls back to $\mathcal{M}$ on X (by Constructions, Lemma 27.13.1). Finally, $\mathcal{M} = \mathcal{L}(nE)$. Since in the arguments above we have replaced the original $\mathcal{L}$ by a positive power we conclude that the invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ mentioned in (3) of the lemma is ample on X' by Properties, Lemma 28.26.2.

Easy observations: $\mathbf{P}^{n+1}_S$ is Noetherian and φ is proper. Details omitted.

Next, we observe that any point of $U = X \setminus E$ is mapped to the open subscheme W of $\mathbf{P}^{n+1}_S$ where one of the first $n+1$ homogeneous coordinates is nonzero. On the other hand, any point of E is mapped to a point where the first $n+1$ homogeneous coordinates are all zero, in particular into the complement of W. Moreover, it is clear that there is a factorization

$$U = \varphi^{-1}(W) \xrightarrow{\varphi|_U} W \xrightarrow{\text{pr}} \mathbf{P}^n_S$$

of $\psi|_U$ where pr is the projection using the first $n+1$ coordinates and $\psi : X \to \mathbf{P}^n_S$ is the embedding chosen above. It follows that $\varphi|_U : U \to W$ is quasi-finite.

Finally, we consider the map $\varphi|_E : E \to \mathbf{P}^{n+1}_S$. Observe that for any point $x \in E$ the image $\varphi(x)$ has its first $n+1$ coordinates equal to zero, i.e., the morphism $\varphi|_E$ factors through the closed subscheme $\mathbf{P}^0_S \cong S$. The morphism $E \to S = \mathrm{Spec}(R)$ factors as $E \to \mathrm{Spec}(H^0(E, \mathcal{O}_E)) \to \mathrm{Spec}(R)$ by Schemes, Lemma 26.6.4. Since by assumption $H^0(E, \mathcal{O}_E)$ is a field we conclude that E maps to a point in $S \subset \mathbf{P}^{n+1}_S$ which finishes the proof. $\square$

0C2N Lemma 54.16.9. Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. Let $E \subset X$ be an exceptional curve of the first kind which is in a fibre of f.

(1) If X is projective over S, then there exists a contraction $X \to X'$ of E and X' is projective over S.
(2) If X is quasi-projective over S, then there exists a contraction $X \to X'$ of E and X' is quasi-projective over S.

Proof. Both cases follow from Lemma 54.16.8 using standard results on ample invertible modules and (quasi-)projective morphisms.

Proof of (1). Projectivity of f means that f is proper and there exists an f-ample invertible module $\mathcal{L}$, see Morphisms, Lemma 29.43.13 and Definition 29.40.1. Let $U \subset S$ be an affine open containing the image of E. By Lemma 54.16.8 there exists a contraction $c : f^{-1}(U) \to V'$ of E and an ample invertible module $\mathcal{N}'$ on V' whose pullback to $f^{-1}(U)$ is equal to $\mathcal{L}(nE)|_{f^{-1}(U)}$. Let $v \in V'$ be the closed point such that c is the blowing up of v. Then we can glue V' and $X \setminus E$ along $f^{-1}(U) \setminus E = V' \setminus \{v\}$ to get a scheme X' over S. The morphisms c and $\text{id}_{X \setminus E}$ glue to a morphism $b : X \to X'$ which is the contraction of E. The inverse image of U in X' is proper over U. On the other hand, the restriction of $X' \to S$ to the complement of the image of v in S is isomorphic to the restriction of $X \to S$ to that open. Hence $X' \to S$ is proper (as being proper is local on the base by Morphisms, Lemma 29.41.3). Finally, $\mathcal{N}'$ and $\mathcal{L}|_{X \setminus E}$ restrict to isomorphic invertible modules over $f^{-1}(U) \setminus E = V' \setminus \{v\}$ and hence glue to an invertible module $\mathcal{L}'$ over X'. The restriction of $\mathcal{L}'$ to the inverse image of U in X' is ample because this is true for $\mathcal{N}'$. For affine opens of S avoiding the image of v, we see that the same is true because it holds for $\mathcal{L}$. Thus $\mathcal{L}'$ is $(X' \to S)$-relatively ample by Morphisms, Lemma 29.37.4 and (1) is proved.

Proof of (2). We can write X as an open subscheme of a scheme $\overline{X}$ projective over S by Morphisms, Lemma 29.43.12. By (1) there is a contraction $b : \overline{X} \to \overline{X}'$ and $\overline{X}'$ is projective over S. Then we let $X' \subset \overline{X}$ be the image of $X \to \overline{X}'$; this is an open as b is an isomorphism away from E. Then $X \to X'$ is the desired contraction. Note that X' is quasi-projective over S as it has an S-relatively ample invertible module by the construction in the proof of part (1). ☐

0C5N Lemma 54.16.10. Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type with X regular of dimension 2. Then X is quasi-projective over S.

Proof. By Chow's lemma (Cohomology of Schemes, Lemma 30.18.1) there exists a proper morphism $\pi : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that $X' \to S$ is H-quasi-projective. By Lemma 54.4.3 there exists a sequence of blowups in closed points

$$X_n \to \ldots \to X_1 \to X_0 = X$$

and an S-morphism $X_n \to X'$ extending the rational map $U \to X'$. Observe that $X_n \to X$ is projective by Divisors, Lemma 31.32.13 and Morphisms, Lemma 29.43.14. This implies that $X_n \to X'$ is projective by Morphisms, Lemma 29.43.15. Hence $X_n \to S$ is quasi-projective by Morphisms, Lemma 29.40.3 (and the fact that a projective morphism is quasi-projective, see Morphisms, Lemma 29.43.10).

By Lemma 54.16.9 (and uniqueness of contractions Lemma 54.16.2) we conclude that $X_{n-1}, \ldots, X_0 = X$ are quasi-projective over S as desired. ☐

0C5P Lemma 54.16.11. Let S be a Noetherian scheme. Let $f : X \to S$ be a proper morphism with X regular of dimension 2. Then X is projective over S.

Proof. This follows from Lemma 54.16.10 and Morphisms, Lemma 29.43.13. ☐

54.17. Factorization birational maps

0C5Q Proper birational morphisms between nonsingular surfaces are given by sequences of quadratic transforms.

0C5R Lemma 54.17.1. Let $f : X \to Y$ be a proper birational morphism between integral Noetherian schemes regular of dimension 2. Then f is a sequence of blowups in closed points.

Proof. Let $V \subset Y$ be the maximal open over which f is an isomorphism. Then V contains all codimension 1 points of V (Varieties, Lemma 33.17.3). Let $y \in Y$ be a closed point not contained in V. Then we want to show that f factors through the blowup $b : Y' \to Y$ of Y at y. Namely, if this is true, then at least one (and in fact exactly one) component of the fibre $f^{-1}(y)$ will map isomorphically onto the exceptional curve in Y' and the number of curves in fibres of $X \to Y'$ will be strictly less that the number of curves in fibres of $X \to Y$, so we conclude by induction. Some details omitted.

By Lemma 54.4.3 we know that there exists a sequence of blowing ups

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

in closed points lying over the fibre $f^{-1}(y)$ and a morphism $X' \to Y'$ such that

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is commutative. We want to show that the morphism $X' \to Y'$ factors through X and hence we can use induction on n to reduce to the case where $X' \to X$ is the blowup of X in a closed point $x \in X$ mapping to y.

Let $E \subset X'$ be the exceptional fibre of the blowing up $X' \to X$. If E maps to a point in Y', then we obtain the desired factorization by Lemma 54.16.1. We will prove that if this is not the case we obtain a contradiction. Namely, if $f'(E)$ is not a point, then $E' = f'(E)$ must be the exceptional curve in Y'. Picture

$$\begin{tikzcd} E \arrow[r] \arrow[d, "g"'] & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ E' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

Arguing as before f' is an isomorphism in an open neighbourhood of the generic point of E'. Hence $g : E \to E'$ is a finite birational morphism. Then the inverse of g (a rational map) is everywhere defined by Morphisms, Lemma 29.42.5 and g is an isomorphism. Consider the map

$$g^*\mathcal{C}_{E'/Y'} \longrightarrow \mathcal{C}_{E/X'}$$

of Morphisms, Lemma 29.31.3. Since the source and target are invertible modules of degree 1 on $E = E' = \mathbf{P}^1_\kappa$ and since the map is nonzero (as f' is an isomorphism in the generic point of E) we conclude it is an isomorphism. By Morphisms, Lemma 29.32.18 we conclude that $\Omega_{X'/Y'}|_E = 0$. This means that f' is unramified at every point of E (Morphisms, Lemma 29.35.14). Hence f' is quasi-finite at every point of E (Morphisms, Lemma 29.35.10). Hence the maximal open $V' \subset Y'$ over which f' is an isomorphism contains E' by Varieties, Lemma 33.17.3. This in turn implies that the inverse image of y in X' is E'. Hence the inverse image of y in X is x. Hence $x \in X$ is in the maximal open over which f is an isomorphism by Varieties, Lemma 33.17.3. This is a contradiction as we assumed that y is not in this open. □

0C5S Lemma 54.17.2. Let S be a Noetherian scheme. Let X and Y be proper integral schemes over S which are regular of dimension 2. Then X and Y are S-birational if and only if there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \rightarrow \ldots \rightarrow Y_1 \rightarrow Y_0 = Y$$

where each morphism is a blowup in a closed point.

Proof. Let $U \subset X$ be open and let $f : U \rightarrow Y$ be the given S-rational map (which is invertible as an S-rational map). By Lemma 54.4.3 we can factor f as $X_n \rightarrow \ldots \rightarrow X_1 \rightarrow X_0 = X$ and $f_n : X_n \rightarrow Y$. Since X_n is proper over S and Y separated over S the morphism f_n is proper. Clearly f_n is birational. Hence f_n is a composition of contractions by Lemma 54.17.1. We omit the proof of the converse. □

54.18. Other chapters

CHAPTER 55

Semistable Reduction

0C2P 55.1. Introduction

0C2Q In this chapter we prove the semistable reduction theorem for curves. We will use the method of Artin and Winters from their paper [AW71].

It turns out that one can prove the semistable reduction theorem for curves without any results on desingularization. Namely, there is a way to establish the existence and projectivity of moduli of semistable curves using Geometric Invariant Theory (GIT) as developed by Mumford, see [MFK94]. This method was championed by Gieseker who proved the full result in his lecture notes [Gie82][1]. This is quite an amazing feat: it seems somewhat counter intuitive that one can prove such a result without ever truly studying families of curves over a positive dimensional base.

Historically the first proof of the semistable reduction theorem for curves can be found in the paper [DM69] by Deligne and Mumford. It proves the theorem by reducing the problem to the case of Abelian varieties which was already known at the time thanks to Grothendieck and others, see [GRR72] and [DK73]). The semistable reduction theorem for abelian varieties uses the theory of Néron models which in turn rests on a treatment of birational group laws over a base.

The method in the paper by Artin and Winters relies on desingularization of singularities of surfaces to obtain regular models. Given the existence of regular models, the proof consists in analyzing the possibilities for the special fibre and concluding using an inequality for torsion in the Picard group of a 1-dimensional scheme over a field. A similar argument can be found in a paper [Sai87] of Saito who uses étale cohomology directly and who obtains a stronger result in that he can characterize semistable reduction in terms of the action of the inertia on ℓ-adic étale cohomology.

A different approach one can use to prove the theorem is to use rigid analytic geometry techniques. Here we refer the reader to [vdP84] and [AW12].

The paper [Tem10] by Temkin uses valuation theoretic techniques (and proves a lot more besides); also Appendix A of this paper gives a nice overview of the different proofs and the relationship with desingularizations of 2 dimensional schemes.

Another overview paper that the reader may wish to consult is [Abb00] written by Ahmed Abbes.

55.2. Linear algebra

0C5T A couple of lemmas we will use later on.

[1]Gieseker's lecture notes are written over an algebraically closed field, but the same method works over $\mathbf{Z}$.

0C5U **Lemma 55.2.1.** Let $A = (a_{ij})$ be a complex $n \times n$ matrix. [Tau49, Theorem I]

(1) If $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for each i, then $\det(A)$ is nonzero.
(2) If there exists a real vector $m = (m_1, \ldots, m_n)$ with $m_i > 0$ such that $|a_{ii} m_i| > \sum_{j \neq i} |a_{ij} m_j|$ for each i, then $\det(A)$ is nonzero.

Proof. If A is as in (1) and $\det(A) = 0$, then there is a nonzero vector z with $Az = 0$. Choose r with $|z_r|$ maximal. Then

$$|a_{rr} z_r| = |\sum\nolimits_{k \neq r} a_{rk} z_k| \leq \sum\nolimits_{k \neq r} |a_{rk}||z_k| \leq |z_r| \sum\nolimits_{k \neq r} |a_{rk}| < |a_{rr}||z_r|$$

which is a contradiction. To prove (2) apply (1) to the matrix $(a_{ij} m_j)$ whose determinant is $m_1 \ldots m_n \det(A)$. $\square$

0C5V **Lemma 55.2.2.** Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be a real vector with $m_i > 0$. For $I \subset \{1, \ldots, n\}$ let $x_I \in \mathbf{R}^n$ be the vector whose ith coordinate is m_i if $i \in I$ and 0 otherwise. If

0C5W (55.2.2.1)
$$-a_{ii} m_i \geq \sum\nolimits_{j \neq i} a_{ij} m_j$$

for each i, then $\text{Ker}(A)$ is the vector space spanned by the vectors x_I such that

(1) $a_{ij} = 0$ for $i \in I$, $j \notin I$, and
(2) equality holds in (55.2.2.1) for $i \in I$.

Proof. After replacing a_{ij} by $a_{ij} m_j$ we may assume $m_i = 1$ for all i. If $I \subset \{1, \ldots, n\}$ such that (1) and (2) are true, then a simple computation shows that x_I is in the kernel of A. Conversely, let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ be a nonzero vector in the kernel of A. We will show by induction on the number of nonzero coordinates of x that x is in the span of the vectors x_I satisfying (1) and (2). Let $I \subset \{1, \ldots, n\}$ be the set of indices r with $|x_r|$ maximal. For $r \in I$ we have

$$|a_{rr} x_r| = |\sum\nolimits_{k \neq r} a_{rk} x_k| \leq \sum\nolimits_{k \neq r} a_{rk} |x_k| \leq |x_r| \sum\nolimits_{k \neq r} a_{rk} \leq |a_{rr}||x_r|$$

Thus equality holds everywhere. In particular, we see that $a_{rk} = 0$ if $r \in I$, $k \notin I$ and equality holds in (55.2.2.1) for $r \in I$. Then we see that we can substract a suitable multiple of x_I from x to decrease the number of nonzero coordinates. $\square$

0C5X **Lemma 55.2.3.** Let $A = (a_{ij})$ be a symmetric real $n \times n$ matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be a real vector with $m_i > 0$. Assume

(1) $Am = 0$,
(2) there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$ and $j \notin I$.

Then $x^t A x \leq 0$ with equality if and only if $x = qm$ for some $q \in \mathbf{R}$.

First proof. After replacing a_{ij} by $a_{ij} m_i m_j$ we may assume $m_i = 1$ for all i. Condition (1) means $-a_{ii} = \sum_{j \neq i} a_{ij}$ for all i. Recall that $x^t A x = \sum_{i,j} x_i a_{ij} x_j$. Then

$$\begin{aligned}
\sum\nolimits_{i \neq j} -a_{ij}(x_j - x_i)^2 &= \sum\nolimits_{i \neq j} -a_{ij} x_j^2 + 2a_{ij} x_i x_i - a_{ij} x_i^2 \\
&= \sum\nolimits_{j} a_{jj} x_j^2 + \sum\nolimits_{i \neq j} 2a_{ij} x_i x_i + \sum\nolimits_{j} a_{jj} x_i^2 \\
&= 2x^t A x
\end{aligned}$$

This is clearly ≤ 0. If equality holds, then let I be the set of indices i with $x_i \neq x_1$. Then $a_{ij} = 0$ for $i \in I$ and $j \notin I$. Thus $I = \{1, \ldots, n\}$ by condition (2) and x is a multiple of $m = (1, \ldots, 1)$. $\square$

Second proof. The matrix A has real eigenvalues by the spectral theorem. We claim all the eigenvalues are ≤ 0. Namely, since property (1) means $-a_{ii}m_i = \sum_{j \neq i} a_{ij}m_j$ for all i, we find that the matrix $A' = A - \lambda I$ for $\lambda > 0$ satisfies $|a'_{ii}m_i| > \sum a'_{ij}m_j = \sum |a'_{ij}m_j|$ for all i. Hence A' is invertible by Lemma 55.2.1. This implies that the symmetric bilinear form x^tAy is semi-negative definite, i.e., $x^tAx \leq 0$ for all x. It follows that the kernel of A is equal to the set of vectors x with $x^tAx = 0$. The description of the kernel in Lemma 55.2.2 gives the final statement of the lemma. □

0C6V Lemma 55.2.4. Let L be a finite free $\mathbf{Z}$-module endowed with an integral symmetric bilinear positive definite form $\langle\ ,\ \rangle : L \times L \to \mathbf{Z}$. Let $A \subset L$ be a submodule with L/A torsion free. Set $B = \{b \in L \mid \langle a, b\rangle = 0,\ \forall a \in A\}$. Then we have injective maps

$$A^\#/A \leftarrow L/(A \oplus B) \to B^\#/B$$

whose cokernels are quotients of $L^\#/L$. Here $A^\# = \{a' \in A \otimes \mathbf{Q} \mid \langle a, a'\rangle \in \mathbf{Z},\ \forall a \in A\}$ and similarly for B and L.

Proof. Observe that $L \otimes \mathbf{Q} = A \otimes \mathbf{Q} \oplus B \otimes \mathbf{Q}$ because the form is nondegenerate on A (by positivity). We denote $\pi_B : L \otimes \mathbf{Q} \to B \otimes \mathbf{Q}$ the projection. Observe that $\pi_B(x) \in B^\#$ for $x \in L$ because the form is integral. This gives an exact sequence

$$0 \to A \to L \xrightarrow{\pi_B} B^\# \to Q \to 0$$

where Q is the cokernel of $L \to B^\#$. Observe that Q is a quotient of $L^\#/L$ as the map $L^\# \to B^\#$ is surjective since it is the $\mathbf{Z}$-linear dual to $B \to L$ which is split as a map of $\mathbf{Z}$-modules. Dividing by $A \oplus B$ we get a short exact sequence

$$0 \to L/(A \oplus B) \to B^\#/B \to Q \to 0$$

This proves the lemma. □

0C6W Lemma 55.2.5. Let L_0, L_1 be a finite free $\mathbf{Z}$-modules endowed with integral symmetric bilinear positive definite forms $\langle\ ,\ \rangle : L_i \times L_i \to \mathbf{Z}$. Let $\mathrm{d} : L_0 \to L_1$ and $\mathrm{d}^* : L_1 \to L_0$ be adjoint. If $\langle\ ,\ \rangle$ on L_0 is unimodular, then there is an isomorphism

$$\Phi : \mathrm{Coker}(\mathrm{d}^*\mathrm{d})_{torsion} \longrightarrow \mathrm{Im}(\mathrm{d})^\# / \mathrm{Im}(\mathrm{d})$$

with notation as in Lemma 55.2.4.

Proof. Let $x \in L_0$ be an element representing a torsion class in $\mathrm{Coker}(\mathrm{d}^*\mathrm{d})$. Then for some $a > 0$ we can write $ax = \mathrm{d}^*\mathrm{d}(y)$. For any $z \in \mathrm{Im}(\mathrm{d})$, say $z = \mathrm{d}(y')$, we have

$$\langle (1/a)\mathrm{d}(y), z\rangle = \langle (1/a)\mathrm{d}(y), \mathrm{d}(y')\rangle = \langle x, y'\rangle \in \mathbf{Z}$$

Hence $(1/a)\mathrm{d}(y) \in \mathrm{Im}(\mathrm{d})^\#$. We define $\Phi(x) = (1/a)\mathrm{d}(y) \bmod \mathrm{Im}(\mathrm{d})$. We omit the proof that Φ is well defined, additive, and injective.

To prove Φ is surjective, let $z \in \mathrm{Im}(\mathrm{d})^\#$. Then z defines a linear map $L_0 \to \mathbf{Z}$ by the rule $x \mapsto \langle z, \mathrm{d}(x)\rangle$. Since the pairing on L_0 is unimodular by assumption we can find an $x' \in L_0$ with $\langle x', x\rangle = \langle z, \mathrm{d}(x)\rangle$ for all $x \in L_0$. In particular, we see that x' pairs to zero with $\mathrm{Ker}(\mathrm{d})$. Since $\mathrm{Im}(\mathrm{d}^*\mathrm{d}) \otimes \mathbf{Q}$ is the orthogonal complement of $\mathrm{Ker}(\mathrm{d}) \otimes \mathbf{Q}$ this means that x' defines a torsion class in $\mathrm{Coker}(\mathrm{d}^*\mathrm{d})$. We claim that $\Phi(x') = z$. Namely, write $ax' = \mathrm{d}^*\mathrm{d}(y)$ for some $y \in L_0$ and $a > 0$. For any $x \in L_0$ we get

$$\langle z, \mathrm{d}(x)\rangle = \langle x', x\rangle = \langle (1/a)\mathrm{d}^*\mathrm{d}(y), x\rangle = \langle (1/a)\mathrm{d}(y), \mathrm{d}(x)\rangle$$

Hence $z = \Phi(x')$ and the proof is complete. $\square$

0C6X Lemma 55.2.6. Let $A = (a_{ij})$ be a symmetric $n \times n$ integer matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be an integer vector with $m_i > 0$. Assume

(1) $Am = 0$,
(2) there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$ and $j \notin I$.

Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. Then for ℓ a prime number coprime with all a_{ij} and m_i we have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Coker}(A)[\ell]) \leq 1 - n + e$$

Proof. By Lemma 55.2.3 the rank of A is $n - 1$. The composition

$$\mathbf{Z}^{\oplus n} \xrightarrow{\mathrm{diag}(m_1,\ldots,m_n)} \mathbf{Z}^{\oplus n} \xrightarrow{(a_{ij})} \mathbf{Z}^{\oplus n} \xrightarrow{\mathrm{diag}(m_1,\ldots,m_n)} \mathbf{Z}^{\oplus n}$$

has matrix $a_{ij}m_im_j$. Since the cokernel of the first and last maps are torsion of order prime to ℓ by our restriction on ℓ we see that it suffices to prove the lemma for the matrix with entries $a_{ij}m_im_j$. Thus we may assume $m = (1, \ldots, 1)$.

Assume $m = (1, \ldots, 1)$. Set $V = \{1, \ldots, n\}$ and $E = \{(i, j) \mid i < j \text{ and } a_{ij} > 0\}$. For $e = (i, j) \in E$ set $a_e = a_{ij}$. Define maps $s, t : E \to V$ by setting $s(i, j) = i$ and $t(i, j) = j$. Set $\mathbf{Z}(V) = \bigoplus_{i \in V} \mathbf{Z}i$ and $\mathbf{Z}(E) = \bigoplus_{e \in E} \mathbf{Z}e$. We define symmetric positive definite integer valued pairings on $\mathbf{Z}(V)$ and $\mathbf{Z}(E)$ by setting

$$\langle i, i\rangle = 1 \text{ for } i \in V, \quad \langle e, e\rangle = a_e \text{ for } e \in E$$

and all other pairings zero. Consider the maps

$$\mathrm{d} : \mathbf{Z}(V) \to \mathbf{Z}(E), \quad i \longmapsto \sum\nolimits_{e \in E,\ s(e) = i} e - \sum\nolimits_{e \in E,\ t(e) = i} e$$

and

$$\mathrm{d}^*(e) = a_e(s(e) - t(e))$$

A computation shows that

$$\langle d(x), y\rangle = \langle x, \mathrm{d}^*(y)\rangle$$

in other words, d and d^* are adjoint. Next we compute

$$\begin{aligned} \mathrm{d}^*\mathrm{d}(i) &= \mathrm{d}^*(\sum\nolimits_{e \in E,\ s(e) = i} e - \sum\nolimits_{e \in E,\ t(e) = i} e) \\ &= \sum\nolimits_{e \in E,\ s(e) = i} a_e(s(e) - t(e)) - \sum\nolimits_{e \in E,\ t(e) = i} a_e(s(e) - t(e)) \end{aligned}$$

The coefficient of i in $\mathrm{d}^*\mathrm{d}(i)$ is

$$\sum\nolimits_{e \in E,\ s(e) = i} a_e + \sum\nolimits_{e \in E,\ t(e) = i} a_e = -a_{ii}$$

because $\sum_j a_{ij} = 0$ and the coefficient of $j \neq i$ in $\mathrm{d}^*\mathrm{d}(i)$ is $-a_{ij}$. Hence $\mathrm{Coker}(A) = \mathrm{Coker}(\mathrm{d}^*\mathrm{d})$.

Consider the inclusion

$$\mathrm{Im}(\mathrm{d}) \oplus \mathrm{Ker}(\mathrm{d}^*) \subset \mathbf{Z}(E)$$

The left hand side is an orthogonal direct sum. Clearly $\mathbf{Z}(E)/\mathrm{Ker}(\mathrm{d}^*)$ is torsion free. We claim $\mathbf{Z}(E)/\mathrm{Im}(\mathrm{d})$ is torsion free as well. Namely, say $x = \sum x_e e \in \mathbf{Z}(E)$ and $a > 1$ are such that $ax = \mathrm{d}y$ for some $y = \sum y_i i \in \mathbf{Z}(V)$. Then $ax_e = y_{s(e)} - y_{t(e)}$. By property (2) we conclude that all y_i have the same congruence class modulo a. Hence we can write $y = ay' + (y_1, y_1, \ldots, y_1)$. Since $\mathrm{d}(y_1, y_1, \ldots, y_1) = 0$ we conclude that $x = \mathrm{d}(y')$ which is what we had to show.

Hence we may apply Lemma 55.2.4 to get injective maps

$$\mathrm{Im}(\mathrm{d})^{\#}/\,\mathrm{Im}(\mathrm{d}) \leftarrow \mathbf{Z}(E)/(\mathrm{Im}(\mathrm{d}) \oplus \mathrm{Ker}(\mathrm{d}^*)) \to \mathrm{Ker}(\mathrm{d}^*)^{\#}/\,\mathrm{Ker}(\mathrm{d}^*)$$

whose cokernels are annihilated by the product of the a_e (which is prime to ℓ). Since $\mathrm{Ker}(\mathrm{d}^*)$ is a lattice of rank $1 - n + e$ we see that the proof is complete if we prove that there exists an isomorphism

$$\Phi : M_{torsion} \longrightarrow \mathrm{Im}(\mathrm{d})^{\#}/\,\mathrm{Im}(\mathrm{d})$$

This is proved in Lemma 55.2.5. □

55.3. Numerical types

0C6Y Part of the arguments will involve the combinatorics of the following data structures.

0C6Z **Definition 55.3.1.** A numerical type T is given by

$$n, m_i, a_{ij}, w_i, g_i$$

where $n \geq 1$ is an integer and m_i, a_{ij}, w_i, g_i are integers for $1 \leq i, j \leq n$ subject to the following conditions

1. $m_i > 0$, $w_i > 0$, $g_i \geq 0$,
2. the matrix $A = (a_{ij})$ is symmetric and $a_{ij} \geq 0$ for $i \neq j$,
3. there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$, $j \notin I$,
4. for each i we have $\sum_j a_{ij} m_j = 0$, and
5. $w_i | a_{ij}$.

This is obviously a somewhat annoying type of structure to work with, but it is exactly what shows up in special fibres of proper regular models of smooth geometrically connected curves. Of course we only care about these types up to reordering the indices.

0C70 **Definition 55.3.2.** We say two numerical types n, m_i, a_{ij}, w_i, g_i and $n', m'_i, a'_{ij}, w'_i, g'_i$ are equivalent types if there exists a permutation σ of $\{1, \ldots, n\}$ such that $m_i = m'_{\sigma(i)}$, $a_{ij} = a'_{\sigma(i)\sigma(j)}$, $w_i = w'_{\sigma(i)}$, and $g_i = g'_{\sigma(i)}$.

A numerical type has a genus.

0C71 **Lemma 55.3.3.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type. Then the expression*

$$g = 1 + \sum m_i (w_i (g_i - 1) - \frac{1}{2} a_{ii})$$

is an integer.

Proof. To prove g is an integer we have to show that $\sum a_{ii}m_i$ is even. This we can see by computing modulo 2 as follows

$$\begin{aligned}\sum\nolimits_i a_{ii}m_i &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} a_{ii}m_i \\ &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} \sum\nolimits_{j \neq i} a_{ij}m_j \\ &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} \sum\nolimits_{j\neq i,\ m_j \text{ odd}} a_{ij}m_j \\ &\equiv \sum\nolimits_{i<j,\ m_i \text{ and } m_j \text{ odd}} a_{ij}(m_i+m_j) \\ &\equiv 0\end{aligned}$$

where we have used that $a_{ij} = a_{ji}$ and that $\sum_j a_{ij}m_j = 0$ for all i. $\square$

0C72 **Definition 55.3.4.** We say n, m_i, a_{ij}, w_i, g_i is a *numerical type of genus g* if $g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ is the integer from Lemma 55.3.3.

We will prove below (Lemma 55.3.14) that the genus is almost always ≥ 0. But you can have numerical types with negative genus.

0C73 **Lemma 55.3.5.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. If $n = 1$, then $a_{11} = 0$ and $g = 1 + m_1w_1(g_1 - 1)$. Moreover, we can classify all such numerical types as follows*

1. *If $g < 0$, then $g_1 = 0$ and there are finitely many possible numerical types of genus g with $n = 1$ corresponding to factorizations $m_1w_1 = 1 - g$.*
2. *If $g = 0$, then $m_1 = 1$, $w_1 = 1$, $g_1 = 0$ as in Lemma 55.6.1.*
3. *If $g = 1$, then we conclude $g_1 = 1$ but m_1, w_1 can be arbitrary positive integers; this is case (1) of Lemma 55.6.2.*
4. *If $g > 1$, then $g_1 > 1$ and there are finitely many possible numerical types of genus g with $n = 1$ corresponding to factorizations $m_1w_1(g_1-1) = g-1$.*

Proof. The lemma proves itself. $\square$

0C74 **Lemma 55.3.6.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. If $n > 1$, then $a_{ii} < 0$ for all i.*

Proof. Lemma 55.2.3 applies to the matrix A. $\square$

0C75 **Lemma 55.3.7.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Assume $n > 1$. If i is such that the contribution $m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ to the genus g is < 0, then $g_i = 0$ and $a_{ii} = -w_i$.*

Proof. Follows immediately from Lemma 55.3.6 and $w_i > 0$, $g_i \geq 0$, and $w_i | a_{ii}$. $\square$

0C76 **Definition 55.3.8.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type. We say i is a *(-1)-index* if $g_i = 0$ and $a_{ii} = -w_i$.

We can "contract" (-1)-indices.

0C77 **Lemma 55.3.9.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Assume n is a (-1)-index. Then there is a numerical type T' given by $n', m'_i, a'_{ij}, w'_i, g'_i$ with*

1. *$n' = n - 1$,*
2. *$m'_i = m_i$,*
3. *$a'_{ij} = a_{ij} - a_{in}a_{jn}/a_{nn}$,*

(4) $w'_i = w_i/2$ if a_{in}/w_n even and a_{in}/w_i odd and $w'_i = w_i$ else,
(5) $g'_i = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{a_{in}^2 - w_n a_{in}}{2w'_i w_n}$.

Moreover, we have $g = g'$.

Proof. Observe that $n > 1$ for example by Lemma 55.3.5 and hence $n' \geq 1$. We check conditions (1) – (5) of Definition 55.3.1 for $n', m'_i, a'_{ij}, w'_i, g'_i$.

Condition (1) is immediate.

Condition (2). Symmetry of $A' = (a'_{ij})$ is immediate and since $a_{nn} < 0$ by Lemma 55.3.6 we see that $a'_{ij} \geq a_{ij} \geq 0$ if $i \neq j$.

Condition (3). Suppose that $I \subset \{1, \ldots, n-1\}$ such that $a'_{ii'} = 0$ for $i \in I$ and $i' \in \{1, \ldots, n-1\} \setminus I$. Then we see that for each $i \in I$ and $i' \in I'$ we have $a_{in}a_{i'n} = 0$. Thus either $a_{in} = 0$ for all $i \in I$ and $I \subset \{1, \ldots, n\}$ is a contradiction for property (3) for T, or $a_{i'n} = 0$ for all $i' \in \{1, \ldots, n-1\} \setminus I$ and $I \cup \{n\} \subset \{1, \ldots, n\}$ is a contradiction for property (3) of T. Hence (3) holds for T'.

Condition (4). We compute

$$\sum\nolimits_{j=1}^{n-1} a'_{ij}m_j = \sum\nolimits_{j=1}^{n-1}(a_{ij}m_j - \frac{a_{in}a_{jn}m_j}{a_{nn}}) = -a_{in}m_n - \frac{a_{in}}{a_{nn}}(-a_{nn}m_n) = 0$$

as desired.

Condition (5). We have to show that w'_i divides $a_{in}a_{jn}/a_{nn}$. This is clear because $a_{nn} = -w_n$ and $w_n | a_{jn}$ and $w_i | a_{in}$.

To show that $g = g'$ we first write

$$\begin{aligned} g &= 1 + \sum\nolimits_{i=1}^{n} m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) \\ &= 1 + \sum\nolimits_{i=1}^{n-1} m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) - \frac{1}{2}m_n w_n \\ &= 1 + \sum\nolimits_{i=1}^{n-1} m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii} - \frac{1}{2}a_{in}) \end{aligned}$$

Comparing with the expression for g' we see that it suffices if

$$w'_i(g'_i - 1) - \frac{1}{2}a'_{ii} = w_i(g_i - 1) - \frac{1}{2}a_{in} - \frac{1}{2}a_{ii}$$

for $i \leq n - 1$. In other words, we have

$$g'_i = \frac{2w_i(g_i - 1) - a_{in} - a_{ii} + a'_{ii} + 2w'_i}{2w'_i} = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{a_{in}^2 - w_n a_{in}}{2w'_i w_n}$$

It is elementary to check that this is an integer ≥ 0 if we choose w'_i as in (4). $\square$

0C78 Lemma 55.3.10. Let n, m_i, a_{ij}, w_i, g_i be a numerical type. Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. Then the expression $g_{top} = 1 - n + e$ is ≥ 0.

Proof. If not, then $e < n - 1$ which means there exists an i such that $a_{ij} = 0$ for all $j \neq i$. This contradicts assumption (3) of Definition 55.3.1. $\square$

0C79 Definition 55.3.11. Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The topological genus of T is the nonnegative integer $g_{top} = 1 - n + e$ from Lemma 55.3.10.

We want to bound the genus by the topological genus. However, this will not always be the case, for example for numerical types with $n = 1$ as in Lemma 55.3.5. But it will be true for minimal numerical types which are defined as follows.

0C7A Definition 55.3.12. We say the numerical type n, m_i, a_{ij}, w_i, g_i of genus g is minimal if there does not exist an i with $g_i = 0$ and $a_{ii} = -w_i$, in other words, if there does not exist a (-1)-index.

We will prove that the genus g of a minimal type with $n > 1$ is greater than or equal to $\max(1, g_{top})$.

0C7B Lemma 55.3.13. If n, m_i, a_{ij}, w_i, g_i is a minimal numerical type with $n > 1$, then $g \geq 1$.

Proof. This is true because $g = 1 + \sum \Phi_i$ with $\Phi_i = m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ nonnegative by Lemma 55.3.7 and the definition of minimal types. $\square$

0C7C Lemma 55.3.14. If n, m_i, a_{ij}, w_i, g_i is a minimal numerical type with $n > 1$, then $g \geq g_{top}$.

Proof. The reader who is only interested in the case of numerical types associated to proper regular models can skip this proof as we will reprove this in the geometric situation later. We can write

$$g_{top} = 1 - n + \frac{1}{2}\sum\nolimits_{a_{ij} > 0} 1 = 1 + \sum\nolimits_i (-1 + \frac{1}{2}\sum\nolimits_{j \neq i,\ a_{ij} > 0} 1)$$

On the other hand, we have

$$\begin{aligned} g & = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) \\ & = 1 + \sum m_i w_i g_i - \sum m_i w_i + \frac{1}{2}\sum\nolimits_{i \neq j} a_{ij} m_j \\ & = 1 + \sum\nolimits_i m_i w_i (-1 + g_i + \frac{1}{2}\sum\nolimits_{j \neq i} \frac{a_{ij}}{w_i}) \end{aligned}$$

The first equality is the definition, the second equality uses that $\sum a_{ij} m_j = 0$, and the last equality uses that uses $a_{ij} = a_{ji}$ and switching order of summation. Comparing with the formula for g_{top} we conclude that the lemma holds if

$$\Psi_i = m_i w_i (-1 + g_i + \frac{1}{2}\sum\nolimits_{j \neq i} \frac{a_{ij}}{w_i}) - (-1 + \frac{1}{2}\sum\nolimits_{j \neq i,\ a_{ij} > 0} 1)$$

is ≥ 0 for each i. However, this may not be the case. Let us analyze for which indices we can have $\Psi_i < 0$. First, observe that

$$(-1 + g_i + \frac{1}{2}\sum\nolimits_{j \neq i} \frac{a_{ij}}{w_i}) \geq (-1 + \frac{1}{2}\sum\nolimits_{j \neq i,\ a_{ij} > 0} 1)$$

because a_{ij}/w_i is a nonnegative integer. Since $m_i w_i$ is a positive integer we conclude that $\Psi_i \geq 0$ as soon as either $m_i w_i = 1$ or the left hand side of the inequality is ≥ 0 which happens if $g_i > 0$, or $a_{ij} > 0$ for at least two indices j, or if there is a j with $a_{ij} > w_i$. Thus

$$P = \{i : \Psi_i < 0\}$$

is the set of indices i such that $m_i w_i > 1$, $g_i = 0$, $a_{ij} > 0$ for a unique j, and $a_{ij} = w_i$ for this j. Moreover

$$i \in P \Rightarrow \Psi_i = \frac{1}{2}(-m_i w_i + 1)$$

The strategy of proof is to show that given $i \in P$ we can borrow a bit from Ψ_j where j is the neighbour of i, i.e., $a_{ij} > 0$. However, this won't quite work because j may be an index with $\Psi_j = 0$.

Consider the set

$$Z = \{j : g_j = 0 \text{ and } j \text{ has exactly two neighbours } i, k \text{ with } a_{ij} = w_j = a_{jk}\}$$

For $j \in Z$ we have $\Psi_j = 0$. We will consider sequences $M = (i, j_1, \ldots, j_s)$ where $s \geq 0$, $i \in P$, $j_1, \ldots, j_s \in Z$, and $a_{ij_1} > 0, a_{j_1 j_2} > 0, \ldots, a_{j_{s-1} j_s} > 0$. If our numerical type consists of two indices which are in P or more generally if our numerical type consists of two indices which are in P and all other indices in Z, then $g_{top} = 0$ and we win by Lemma 55.3.13. We may and do discard these cases.

Let $M = (i, j_1, \ldots, j_s)$ be a maximal sequence and let k be the second neighbour of j_s. (If $s = 0$, then k is the unique neighbour of i.) By maximality $k \notin Z$ and by what we just said $k \notin P$. Observe that $w_i = a_{ij_1} = w_{j_1} = a_{j_1 j_2} = \ldots = w_{j_s} = a_{j_s k}$. Looking at the definition of a numerical type we see that

$$\begin{aligned} m_i a_{ii} + m_{j_1} w_i &= 0, \\ m_i w_i + m_{j_1} a_{j_1 j_1} + m_{j_2} w_i &= 0, \\ \ldots\ldots \\ m_{j_{s-1}} w_i + m_{j_s} a_{j_s j_s} + m_k w_i &= 0 \end{aligned}$$

The first equality implies $m_{j_1} \geq 2m_i$ because the numerical type is minimal. Then the second equality implies $m_{j_2} \geq 3m_i$, and so on. In any case, we conclude that $m_k \geq 2m_i$ (including when $s = 0$).

Let k be an index such that we have a $t > 0$ and pairwise distinct maximal sequences $M_1, \ldots, M_t$ as above, with $M_b = (i_b, j_{b,1}, \ldots, j_{b,s_b})$ such that k is a neighbour of j_{b,s_b} for $b = 1, \ldots, t$. We will show that $\Phi_j + \sum_{b=1,\ldots,t} \Phi_{i_b} \geq 0$. This will finish the proof of the lemma by what we said above. Let M be the union of the indices occurring in M_b, $b = 1, \ldots, t$. We write

$$\Psi_k = -\sum\nolimits_{b=1,\ldots,t} \Psi_{i_b} + \Psi'_k$$

where

$$\begin{aligned} \Psi'_k = m_k w_k & \left(-1 + g_k + \frac{1}{2}\sum\nolimits_{b=1,\ldots t}\left(\frac{a_{kj_{b,s_b}}}{w_k} - \frac{m_{i_b} w_{i_b}}{m_k w_k}\right) + \frac{1}{2}\sum\nolimits_{l \neq k,\ l \notin M} \frac{a_{kl}}{w_k}\right) \\ & - \left(-1 + \frac{1}{2}\sum\nolimits_{l \neq k,\ l \notin M,\ a_{kl} > 0} 1\right) \end{aligned}$$

Assume $\Psi'_k < 0$ to get a contradiction. If the set $\{l : l \neq k,\ l \notin M,\ a_{kl} > 0\}$ is empty, then $\{1, \ldots, n\} = M \cup \{k\}$ and $g_{top} = 0$ because $e = n - 1$ in this case and the result holds by Lemma 55.3.13. Thus we may assume there is at least one such l which contributes $(1/2)a_{kl}/w_k \geq 1/2$ to the sum inside the first brackets. For each $b = 1, \ldots, t$ we have

$$\frac{a_{kj_{b,s_b}}}{w_k} - \frac{m_{i_b} w_{i_b}}{m_k w_k} = \frac{w_{i_b}}{w_k}\left(1 - \frac{m_{i_b}}{m_k}\right)$$

This expression is $\geq \frac{1}{2}$ because $m_k \geq 2m_{i_b}$ by the previous paragraph and is ≥ 1 if $w_k < w_{i_b}$. It follows that $\Psi'_k < 0$ implies $g_k = 0$. If $t \geq 2$ or $t = 1$ and $w_k < w_{i_1}$, then $\Psi'_k \geq 0$ (here we use the existence of an l as shown above) which is a contradiction too. Thus $t = 1$ and $w_k = w_{i_1}$. If there at least two nonzero terms in the sum over l or if there is one such k and $a_{kl} > w_k$, then $\Psi'_k \geq 0$ as well. The final possibility is that $t = 1$ and there is one l with $a_{kl} = w_k$. This is disallowed as this would mean $k \in Z$ contradicting the maximality of M_1. □

0C7D Lemma 55.3.15. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Assume $n > 1$. If i is such that the contribution $m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ to the genus g is 0, then $g_i = 0$ and $a_{ii} = -2w_i$.

Proof. Follows immediately from Lemma 55.3.6 and $w_i > 0$, $g_i \geq 0$, and $w_i | a_{ii}$. □

It turns out that the indices satisfying this relation play an important role in the structure of minimal numerical types. Hence we give them a name.

0C7E Definition 55.3.16. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. We say i is a (-2)-index if $g_i = 0$ and $a_{ii} = -2w_i$.

Given a minimal numerical type of genus g the (-2)-indices are exactly the indices which do not contribute a positive number to the genus in the formula

$$g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$$

Thus it will be somewhat tricky to bound the quantities associated with (-2)-indices as we will see later.

0C7F Remark 55.3.17. Let n, m_i, a_{ij}, w_i, g_i be a minimal numerical type with $n > 1$. Equality $g = g_{top}$ can hold in Lemma 55.3.14. For example, if $m_i = w_i = 1$ and $g_i = 0$ for all i and $a_{ij} \in \{0, 1\}$ for $i < j$.

55.4. The Picard group of a numerical type

0C7G Here is the definition.

0C7H Definition 55.4.1. Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The Picard group of T is the cokernel of the matrix (a_{ij}/w_i), more precisely

$$\text{Pic}(T) = \text{Coker}\left(\mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n}, \quad e_i \mapsto \sum \frac{a_{ij}}{w_j} e_j\right)$$

where e_i denotes the ith standard basis vector for $\mathbf{Z}^{\oplus n}$.

0C7I Lemma 55.4.2. Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The Picard group of T is a finitely generated abelian group of rank 1.

Proof. If $n = 1$, then $A = (a_{ij})$ is the zero matrix and the result is clear. For $n > 1$ the matrix A has rank $n - 1$ by either Lemma 55.2.2 or Lemma 55.2.3. Of course the rank is not affected by scaling the rows by $1/w_i$. This proves the lemma. □

0CE7 Lemma 55.4.3. Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Then $\text{Pic}(T) \subset \text{Coker}(A)$ where $A = (a_{ij})$.

Proof. Since $\text{Pic}(T)$ is the cokernel of (a_{ij}/w_i) we see that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}^{\oplus n} \arrow[r, "A"'] & \mathbf{Z}^{\oplus n} \arrow[r] & \text{Coker}(A) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Z}^{\oplus n} \arrow[u, "\text{id}"'] \arrow[r, "(a_{ij}/w_i)"] & \mathbf{Z}^{\oplus n} \arrow[u, "\text{diag}(w_1, \ldots, w_n)"'] \arrow[r] & \text{Pic}(T) \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By the snake lemma we conclude that $\text{Pic}(T) \subset \text{Coker}(A)$. □

0C7J **Lemma 55.4.4.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Assume n is a (-1)-index. Let T' be the numerical type constructed in Lemma 55.3.9. There exists an injective map

$$\text{Pic}(T) \to \text{Pic}(T')$$

whose cokernel is an elementary abelian 2-group.

Proof. Recall that $n' = n-1$. Let e_i, resp., e_i' be the ith basis vector of $\mathbf{Z}^{\oplus n}$, resp. $\mathbf{Z}^{\oplus n-1}$. First we denote

$$q : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n-1}, \quad e_n \mapsto 0 \text{ and } e_i \mapsto e_i' \text{ for } i \leq n-1$$

and we set

$$p : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n-1}, \quad e_n \mapsto \sum\nolimits_{j=1}^{n-1} \frac{a_{nj}}{w_j'} e_j' \text{ and } e_i \mapsto \frac{w_i}{w_i'} e_i' \text{ for } i \leq n-1$$

A computation (which we omit) shows there is a commutative diagram

$$\begin{tikzcd} \mathbf{Z}^{\oplus n} \arrow[r, "(a_{ij}/w_i)"'] \arrow[d, "q"'] & \mathbf{Z}^{\oplus n} \arrow[d, "p"] \\ \mathbf{Z}^{\oplus n'} \arrow[r, "(a'_{ij}/w'_i)"] & \mathbf{Z}^{\oplus n'} \end{tikzcd}$$

Since the cokernel of the top arrow is $\text{Pic}(T)$ and the cokernel of the bottom arrow is $\text{Pic}(T')$, we obtain the desired homomorphism of Picard groups. Since $\frac{w_i}{w_i'} \in \{1, 2\}$ we see that the cokernel of $\text{Pic}(T) \to \text{Pic}(T')$ is annihilated by 2 (because $2e_i'$ is in the image of p for all $i \leq n-1$). Finally, we show $\text{Pic}(T) \to \text{Pic}(T')$ is injective. Let $L = (l_1, \ldots, l_n)$ be a representative of an element of $\text{Pic}(T)$ mapping to zero in $\text{Pic}(T')$. Since q is surjective, a diagram chase shows that we can assume L is in the kernel of p. This means that $l_n a_{ni}/w_i' + l_i w_i/w_i' = 0$, i.e., $l_i = -a_{ni}/w_i l_n$. Thus L is the image of $-l_n e_n$ under the map (a_{ij}/w_j) and the lemma is proved. $\square$

0C7K **Lemma 55.4.5.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. If the genus g of T is ≤ 0, then $\text{Pic}(T) = \mathbf{Z}$.

Proof. By induction on n. If $n = 1$, then the assertion is clear. If $n > 1$, then T is not minimal by Lemma 55.3.13. After replacing T by an equivalent type we may assume n is a (-1)-index. By Lemma 55.4.4 we find $\text{Pic}(T) \subset \text{Pic}(T')$. By Lemma 55.3.9 we see that the genus of T' is equal to the genus of T and we conclude by induction. $\square$

55.5. Classification of proper subgraphs

0C7L In this section we assume given a numerical type n, m_i, a_{ij}, w_i, g_i of genus g. We will find a complete list of possible "subgraphs" consisting entirely of (-2)-indices (Definition 55.3.16) and at the same time we classify all possible minimal numerical types of genus 1. In other words, in this section we prove Proposition 55.5.17 and Lemma 55.6.2

Our strategy will be as follows. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Let $I \subset \{1, \ldots, n\}$ be a subset consisting of (-2)-indices such that there does not exist a nonempty proper subset $J \subset I$ with $a_{jj'} = 0$ for $j \in J$, $j' \in I \setminus J$. We work by induction on the cardinality $|I|$ of I. If $I = \{i\}$ consists of 1 index, then the only constraints on m_i, a_{ii}, and w_i are $w_i | a_{ii}$ from Definition 55.3.1 and $a_{ii} < 0$ from Lemma 55.3.6 and this will serve as our base case. In the induction step we first apply the induction hypothesis to subsets $I' \subset I$ of size $|I'| < |I|$. This will put some constraints on the possible m_i, a_{ij}, w_i, $i, j \in I$. In particular, since $|I'| < |I| \leq n$ it will follow from $\sum a_{ij} m_j = 0$ and Lemma 55.2.3 that the sub matrices $(a_{ij})_{i,j \in I'}$ are negative definite and their determinant will have sign $(-1)^m$. For each possibility left over we compute the determinant of $(a_{ij})_{i,j \in I}$. If the determinant has sign $-(-1)^{|I|}$ then this case can be discarded because Sylvester's theorem tells us the matrix $(a_{ij})_{i,j \in I}$ is not negative semi-definite. If the determinant has sign $(-1)^{|I|}$, then $|I| < n$ and we (tentatively) conclude this case can occur as a possible proper subgraph and we list it in one of the lemmas in this section. If the determinant is 0, then we must have $|I| = n$ (by Lemma 55.2.3 again) and $g = 0$. In these cases we actually find all possible m_i, a_{ij}, w_i, $i, j \in I$ and list them in Lemma 55.6.2. After completing the argument we obtain all possible minimal numerical types of genus 1 with $n > 1$ because each of these necessarily consists entirely of (-2)-indices (and hence will show up in the induction process) by the formula for the genus and the remarks in the previous section. At the very end of the day the reader can go through the list of possibilities given in Lemma 55.6.2 to see that all configurations of proper subgraphs listed in this section as possible do in fact occur already for numerical types of genus 1.

Suppose that i and j are (-2)-indices with $a_{ij} > 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite by Lemma 55.2.3 we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} \\ a_{ij} & -2w_j \end{pmatrix}$$

is negative definite unless $n = 2$. The case $n = 2$ can happen: then the determinant $4w_1w_2 - a_{12}^2$ is zero. Using that $\text{lcm}(w_1, w_2)$ divides a_{12} the reader easily finds that the only possibilities are

$$(w_1, w_2, a_{12}) = (w, w, 2w), (w, 4w, 4w), \text{ or } (4w, w, 4w)$$

Observe that the case $(4w, w, 4w)$ is obtained from the case $(w, 4w, 4w)$ by switching the indices i, j. In these cases $g = 1$. This leads to cases (2) and (3) of Lemma 55.6.2. Assuming $n > 2$ we see that the determinant $4w_iw_j - a_{ij}^2$ of the displayed matrix is > 0 and we conclude that $a_{ij}^2/w_iw_j < 4$. On the other hand, we know that $\text{lcm}(w_i, w_j) | a_{ij}$ and hence a_{ij}^2/w_iw_j is an integer. Thus $a_{ij}^2/w_iw_j \in \{1, 2, 3\}$ and $w_i | w_j$ or vice versa. This leads to the following possibilities

$$(w_1, w_2, a_{12}) = (w, w, w), (w, 2w, 2w), (w, 3w, 3w), (2w, w, 2w), \text{ or } (3w, w, 3w)$$

Observe that the case $(2w, w, 2w)$ is obtained from the case $(w, 2w, 2w)$ by switching the indices i, j and similarly for the cases $(3w, w, 3w)$ and $(w, 3w, 3w)$. The first three solutions lead to cases (1), (2), and (3) of Lemma 55.5.1. In this lemma we wrote out the consequences for the integers m_i and m_j using that $\sum_l a_{kl} m_l = 0$ for each k in particular implies $a_{ii} m_i + a_{ij} m_j \leq 0$ for $k = i$ and $a_{ij} m_i + a_{jj} m_j \leq 0$ for $k = j$.

0C7M **Lemma 55.5.1.** Classification of proper subgraphs of the form

$$\bullet \text{———} \bullet$$

If $n > 2$, then given a pair i, j of (-2)-indices with $a_{ij} > 0$, then up to ordering we have the m's, a's, w's

0C7N (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & w \\ w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \end{pmatrix}$$

with w arbitrary and $2m_1 \geq m_2$ and $2m_2 \geq m_1$, or

0C7P (2) are given by

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w \\ 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \end{pmatrix}$$

with w arbitrary and $m_1 \geq m_2$ and $2m_2 \geq m_1$, or

0C7Q (3) are given by

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & 3w \\ 3w & -6w \end{pmatrix}, \quad \begin{pmatrix} w \\ 3w \end{pmatrix}$$

with w arbitrary and $2m_1 \geq 3m_2$ and $2m_2 \geq m_1$.

Proof. See discussion above. □

Suppose that i, j, and k are three (-2)-indices with $a_{ij} > 0$ and $a_{jk} > 0$. In other words, the index i "meets" j and j "meets" k. We will use without further mention that each pair (i, j), (i, k), and (j, k) is as listed in Lemma 55.5.1. Since the matrix $A = (a_{ij})$ is semi-negative definite by Lemma 55.2.3 we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & a_{ik} \\ a_{ij} & -2w_j & a_{jk} \\ a_{ik} & a_{jk} & -2w_k \end{pmatrix}$$

is negative definite unless $n = 3$. The case $n = 3$ can happen: then the determinant[2] of the matrix is zero and we obtain the equation

$$4 = \frac{a_{ij}^2}{w_i w_j} + \frac{a_{jk}^2}{w_j w_k} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{ij} a_{ik} a_{jk}}{w_i w_j w_k}$$

of integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{ij} a_{ik} a_{jk}}{w_i w_j w_k}\right)^2 = \frac{a_{ij}^2}{w_i w_j} \frac{a_{jk}^2}{w_j w_k} \frac{a_{ik}^2}{w_i w_k}$$

Since we have seen above that $\frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}$ are in $\{1, 2, 3\}$ and $\frac{a_{ik}^2}{w_i w_k}$ in $\{0, 1, 2, 3\}$, we conclude that the only possibilities are

$$\left(\frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{ik}^2}{w_i w_k}\right) = (1, 1, 1), (1, 3, 0), (2, 2, 0), \text{ or } (3, 1, 0)$$

Observe that the case $(3, 1, 0)$ is obtained from the case $(1, 3, 0)$ by reversing the order the indices i, j, k. In each of these cases $g = 1$; the reader can find these as cases (4), (5), (6), (7), (8), (9) of Lemma 55.6.2 with one case corresponding to $(1, 1, 1)$, two cases corresponding to $(1, 3, 0)$, and three cases corresponding to $(2, 2, 0)$. Assuming $n > 3$ we obtain the inequality

$$4 > \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{jk}^2}{w_j w_k} + \frac{a_{ij} a_{ik} a_{jk}}{w_i w_j w_k}$$

[2]It is $-8w_i w_j w_k + 2a_{ij}^2 w_k + 2a_{jk}^2 w_i + 2a_{ik}^2 w_j + 2a_{ij} a_{jk} a_{ik}$.

of integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{ik}^2}{w_iw_k}) = (1,1,0), (1,2,0), \text{ or } (2,1,0)$$

in particular $a_{ik} = 0$ (recall we are assuming $a_{ij} > 0$ and $a_{jk} > 0$). Observe that the case $(2,1,0)$ is obtained from the case $(1,2,0)$ by reversing the ordering of the indices i, j, k. The first two solutions lead to cases (1), (2), and (3) of Lemma 55.5.2 where we also wrote out the consequences for the integers m_i, m_j, and m_k.

0C7R **Lemma 55.5.2.** Classification of proper subgraphs of the form

$$\bullet \text{---} \bullet \text{---} \bullet$$

If $n > 3$, then given a triple i, j, k of (-2)-indices with at least two a_{ij}, a_{ik}, a_{jk} nonzero, then up to ordering we have the m's, a's, w's

0C7S (1) are given by
$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & w \\ 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$
with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2$, or

0C7T (2) are given by
$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & 2w \\ 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 2w \end{pmatrix}$$
with $2m_1 \geq m_2$, $2m_2 \geq m_1 + 2m_3$, $2m_3 \geq m_2$, or

0C7U (3) are given by
$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 \\ 2w & -4w & 2w \\ 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ w \end{pmatrix}$$
with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $m_3 \geq m_2$.

Proof. See discussion above. $\square$

Suppose that i, j, k, and l are four (-2)-indices with $a_{ij} > 0$, $a_{jk} > 0$, and $a_{kl} > 0$. In other words, the index i "meets" j, j "meets" k, and k "meets" l. Then we see from Lemma 55.5.2 that $a_{ik} = a_{jl} = 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & 0 & a_{il} \\ a_{ij} & -2w_j & a_{jk} & 0 \\ 0 & a_{jk} & -2w_k & a_{kl} \\ a_{il} & 0 & a_{kl} & -2w_l \end{pmatrix}$$

is negative definite unless $n = 4$. The case $n = 4$ can happen: then the determinant[3] of the matrix is zero and we obtain the equation

$$16 + \frac{a_{ij}^2}{w_iw_j}\frac{a_{kl}^2}{w_kw_l} + \frac{a_{jk}^2}{w_jw_k}\frac{a_{il}^2}{w_iw_l} = 4\frac{a_{ij}^2}{w_iw_j} + 4\frac{a_{jk}^2}{w_jw_k} + 4\frac{a_{kl}^2}{w_kw_l} + 4\frac{a_{il}^2}{w_iw_l} + 2\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}$$

[3] It is $16w_iw_jw_kw_l - 4a_{ij}^2w_kw_l - 4a_{jk}^2w_iw_l - 4a_{kl}^2w_iw_j - 4a_{il}^2w_jw_k + a_{ij}^2a_{kl}^2 + a_{jk}^2a_{il}^2 - 2a_{ij}a_{il}a_{jk}a_{kl}$.

of nonnegative integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}\right)^2 = \frac{a_{ij}^2}{w_iw_j}\frac{a_{jk}^2}{w_jw_k}\frac{a_{kl}^2}{w_kw_l}\frac{a_{il}^2}{w_iw_l}$$

Since we have seen above that $\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}$ are in $\{1, 2\}$ and $\frac{a_{il}^2}{w_iw_l}$ in $\{0, 1, 2\}$, we conclude that the only possible solutions are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}, \frac{a_{il}^2}{w_iw_l}) = (1, 1, 1, 1) \text{ or } (2, 1, 2, 0)$$

and case $g = 1$; the reader can find these as cases (10), (11), (12), and (13) of Lemma 55.6.2. Assuming $n > 4$ we obtain the inequality

$$16 + \frac{a_{ij}^2}{w_iw_j}\frac{a_{kl}^2}{w_kw_l} + \frac{a_{jk}^2}{w_jw_k}\frac{a_{il}^2}{w_iw_l} > 4\frac{a_{ij}^2}{w_iw_j} + 4\frac{a_{jk}^2}{w_jw_k} + 4\frac{a_{kl}^2}{w_kw_l} + 4\frac{a_{il}^2}{w_iw_l} + 2\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}$$

of nonnegative integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}, \frac{a_{il}^2}{w_iw_l}) = (1, 1, 1, 0), (1, 1, 2, 0), (1, 2, 1, 0), \text{ or } (2, 1, 1, 0)$$

in particular $a_{il} = 0$ (recall that we assumed the other three to be nonzero). Observe that the case $(2, 1, 1, 0)$ is obtained from the case $(1, 1, 2, 0)$ by reversing the ordering of the indices i, j, k, l. The first three solutions lead to cases (1), (2), (3), and (4) of Lemma 55.5.3 where we also wrote out the consequences for the integers m_i, m_j, m_k, and m_l.

0C7V **Lemma 55.5.3.** Classification of proper subgraphs of the form

$$\bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

If $n > 4$, then given four (-2)-indices i, j, k, l with a_{ij}, a_{jk}, a_{kl} nonzero, then up to ordering we have the m's, a's, w's

0C7W (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & w & 0 \\ 0 & w & -2w & w \\ 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, and $2m_4 \geq m_3$, or

0C7X (2) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & w & 0 \\ 0 & w & -2w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + 2m_4$, and $2m_4 \geq m_3$, or

0C7Y (3) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, and $m_4 \geq m_3$, or

0C7Z (4) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 2w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + 2m_3$, $2m_3 \geq m_2 + m_4$, and $2m_4 \geq m_3$.

Proof. See discussion above. $\square$

Suppose that i, j, k, and l are four (-2)-indices with $a_{ij} > 0$, $a_{ij} > 0$, and $a_{il} > 0$. In other words, the index i "meets" the indices j, k, l. Then we see from Lemma 55.5.2 that $a_{jk} = a_{jl} = a_{kl} = 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & a_{ik} & a_{il} \\ a_{ij} & -2w_j & 0 & 0 \\ a_{ik} & 0 & -2w_k & 0 \\ a_{il} & 0 & 0 & -2w_l \end{pmatrix}$$

is negative definite unless $n = 4$. The case $n = 4$ can happen: then the determinant[4] of the matrix is zero and we obtain the equation

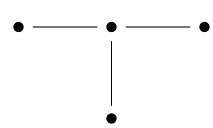
$$4 = \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{il}^2}{w_j w_l}$$

of nonnegative integers. Since we have seen above that $\frac{a_{ij}^2}{w_i w_j}, \frac{a_{ik}^2}{w_i w_k}, \frac{a_{il}^2}{w_i w_l}$ are in $\{1, 2\}$, we conclude that the only possibilities are up to reordering: $4 = 1 + 1 + 2$. In each of these cases $g = 1$; the reader can find these as cases (14) and (15) of Lemma 55.6.2. Assuming $n > 4$ we obtain the inequality

$$4 > \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{il}^2}{w_j w_l}$$

of nonnegative integers. This implies that $\frac{a_{ij}^2}{w_i w_j} = \frac{a_{ik}^2}{w_i w_k} = \frac{a_{il}^2}{w_j w_l} = 1$ and that $w_i = w_j = w_k = w_l$. This leads to case (1) of Lemma 55.5.4 where we also wrote out the consequences for the integers m_i, m_j, m_k, and m_l.

0C80 **Lemma 55.5.4.** Classification of proper subgraphs of the form



[4] It is $16 w_i w_j w_k w_l - 4a_{ij}^2 w_k w_l - 4a_{ik}^2 w_j w_l - 4a_{il}^2 w_j w_k$.

If $n > 4$, then given four (-2)-indices i, j, k, l with a_{ij}, a_{ik}, a_{il} nonzero, then up to ordering we have the m's, a's, w's

0C81 (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & w \\ w & -2w & 0 & 0 \\ w & 0 & -2w & 0 \\ w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2 + m_3 + m_4$, $2m_2 \geq m_1$, $2m_3 \geq m_1$, $2m_4 \geq m_1$. Observe that this implies $m_1 \geq \max(m_2, m_3, m_4)$.

Proof. See discussion above. $\square$

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{ij} > 0$, $a_{jk} > 0$, and $a_{kl} > 0$. In other words, the index h "meets" i, i "meets" j, j "meets" k, and k "meets" l. Then we can apply Lemmas 55.5.2 and 55.5.3 to see that $a_{hj} = a_{hk} = a_{ik} = a_{il} = a_{jl} = 0$ and that the fractions $\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}$ are in $\{1, 2\}$ and the fraction $\frac{a_{hl}^2}{w_h w_l} \in \{0, 1, 2\}$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_h & a_{hi} & 0 & 0 & a_{hl} \\ a_{hi} & -2w_i & a_{ij} & 0 & 0 \\ 0 & a_{ij} & -2w_j & a_{jk} & 0 \\ 0 & 0 & a_{jk} & -2w_k & a_{kl} \\ a_{hl} & 0 & 0 & a_{kl} & -2w_l \end{pmatrix}$$

is negative definite unless $n = 5$. The case $n = 5$ can happen: then the determinant[5] of the matrix is zero and we obtain the equation

$$\begin{aligned} 16 + \frac{a_{hi}^2}{w_h w_i}\frac{a_{jk}^2}{w_j w_k} + \frac{a_{hi}^2}{w_h w_i}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{ij}^2}{w_i w_j}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{ij}^2}{w_i w_j} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{jk}^2}{w_j w_k} \\ = 4\frac{a_{hi}^2}{w_h w_i} + 4\frac{a_{ij}^2}{w_i w_j} + 4\frac{a_{jk}^2}{w_j w_k} + 4\frac{a_{kl}^2}{w_k w_l} + 4\frac{a_{hl}^2}{w_h w_l} + \frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l} \end{aligned}$$

of nonnegative integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l}\right)^2 = \frac{a_{hi}^2}{w_h w_i}\frac{a_{ij}^2}{w_i w_j}\frac{a_{jk}^2}{w_j w_k}\frac{a_{kl}^2}{w_k w_l}\frac{a_{hl}^2}{w_h w_l}$$

We conclude the only possible solutions are

$$\left(\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}, \frac{a_{hl}^2}{w_h w_l}\right) = (1,1,1,1,1), (1,1,2,1,0), (1,2,1,1,0), \text{ or } (2,1,1,2,0)$$

Observe that the case $(1, 2, 1, 1, 0)$ is obtained from the case $(1, 1, 2, 1, 0)$ by reversing the order of the indices h, i, j, k, l. In these cases $g = 1$; the reader can find these as cases (16), (17), (18), (19), (20), and (21) of Lemma 55.6.2 with one case

[5] It is $-32w_h w_i w_j w_k w_l + 8a_{hi}^2 w_j w_k w_l + 8a_{ij}^2 w_h w_k w_l + 8a_{jk}^2 w_h w_i w_l + 8a_{kl}^2 w_h w_i w_j + 8a_{hl}^2 w_i w_j w_k - 2a_{hi}^2 a_{jk}^2 w_l - 2a_{hi}^2 a_{kl}^2 w_j - 2a_{ij}^2 a_{kl}^2 w_h - 2a_{hl}^2 a_{ij}^2 w_k - 2a_{hl}^2 a_{jk}^2 w_i + 2a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}$.

corresponding to $(1,1,1,1,1)$, two cases corresponding to $(1,1,2,1,0)$, and three cases corresponding to $(2,1,1,2,0)$. Assuming $n > 5$ we obtain the inequality

$$\begin{aligned} 16 + \frac{a_{hi}^2}{w_h w_i}\frac{a_{jk}^2}{w_j w_k} + \frac{a_{hi}^2}{w_h w_i}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{ij}^2}{w_i w_j}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{ij}^2}{w_i w_j} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{jk}^2}{w_j w_k} \\ > 4\frac{a_{hi}^2}{w_h w_i} + 4\frac{a_{ij}^2}{w_i w_j} + 4\frac{a_{jk}^2}{w_j w_k} + 4\frac{a_{kl}^2}{w_k w_l} + 4\frac{a_{hl}^2}{w_h w_l} + \frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l} \end{aligned}$$

of nonnegative integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$\left(\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}, \frac{a_{hl}^2}{w_h w_l}\right) = (1,1,1,1,0), (1,1,1,2,0), \text{ or } (2,1,1,1,0)$$

in particular $a_{hl} = 0$ (recall that we assumed the other four to be nonzero). Observe that the case $(1,1,1,2,0)$ is obtained from the case $(2,1,1,1,0)$ by reversing the order of the indices h, i, j, k, l. The first two solutions lead to cases (1), (2), and (3) of Lemma 55.5.5 where we also wrote out the consequences for the integers m_h, m_i, m_j, m_k, and m_l.

0C82 **Lemma 55.5.5.** Classification of proper subgraphs of the form

$$\bullet - \bullet - \bullet - \bullet - \bullet$$

If $n > 5$, then given five (-2)-indices h, i, j, k, l with $a_{hi}, a_{ij}, a_{jk}, a_{kl}$ nonzero, then up to ordering we have the m's, a's, w's

0C83 (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & w \\ 0 & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, and $2m_5 \geq m_4$, or

0C84 (2) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + 2m_4$, $2m_4 \geq m_3 + m_5$, and $2m_5 \geq m_4$, or

0C85 (3) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, and $m_4 \geq m_3$.

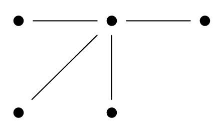
Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{hj} > 0$, $a_{hk} > 0$, and $a_{hl} > 0$. In other words, the index h "meets" the indices i, j, k, l. Then we see from Lemma 55.5.2 that $a_{ij} = a_{ik} = a_{il} = a_{jk} = a_{jl} = a_{kl} = 0$ and by Lemma 55.5.4 that $w_h = w_i = w_j = w_k = w_l = w$ for some integer $w > 0$ and $a_{hi} = a_{hj} = a_{hk} = a_{hl} = -2w$. The corresponding matrix

$$\begin{pmatrix} -2w & w & w & w & w \\ w & -2w & 0 & 0 & 0 \\ w & 0 & -2w & 0 & 0 \\ w & 0 & 0 & -2w & 0 \\ w & 0 & 0 & 0 & -2w \end{pmatrix}$$

is singular. Hence this can only happen if $n = 5$ and $g = 1$. The reader can find this as case (22) Lemma 55.6.2.

0C86 Lemma 55.5.6. Nonexistence of proper subgraphs of the form



If $n > 5$, there do not exist five (-2)-indices h, i, j, k with $a_{hi} > 0$, $a_{hj} > 0$, $a_{hk} > 0$, and $a_{hl} > 0$.

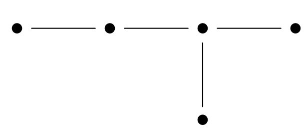
Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{ij} > 0$, $a_{jk} > 0$, and $a_{jl} > 0$. In other words, the index h "meets" i and the index j "meets" the indices i, k, l. Then we see from Lemma 55.5.4 that $a_{ik} = a_{il} = a_{kl} = 0$, $w_i = w_j = w_k = w_l = w$, and $a_{ij} = a_{jk} = a_{jl} = w$ for some integer w. Applying Lemma 55.5.3 to the four tuples h, i, j, k and h, i, j, l we see that $a_{hj} = a_{hk} = a_{hl} = 0$, that $w_h = \frac{1}{2}w$, w, or $2w$, and that correspondingly $a_{hi} = w$, w, or $2w$. Since A is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_h & a_{hi} & 0 & 0 & 0 \\ a_{hi} & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}$$

is negative definite unless $n = 5$. The reader computes that the determinant of the matrix is 0 when $w_h = \frac{1}{2}w$ or $2w$. This leads to cases (23) and (24) of Lemma 55.6.2. For $w_h = w$ we obtain case (1) of Lemma 55.5.7.

0C87 Lemma 55.5.7. Classification of proper subgraphs of the form



If $n > 5$, then given five (-2)-indices h, i, j, k, l with $a_{hi}, a_{ij}, a_{jk}, a_{jl}$ nonzero, then up to ordering we have the m's, a's, w's

0C88 (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4 + m_5$, $2m_4 \geq m_3$, and $2m_5 \geq m_3$.

Proof. See discussion above. $\square$

Suppose that $t > 5$ and $i_1, \ldots, i_t$ are t distinct (-2)-indices such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$. We will prove by induction on t that if $n = t$ this leads to possibilities (25), (26), (27), (28) of Lemma 55.6.2 and if $n > t$ to cases (1), (2), and (3) of Lemma 55.5.8. First, if $a_{i_1 i_t}$ is nonzero, then it is clear from the result of Lemma 55.5.5 that $w_{i_1} = \ldots = w_{i_t} = w$ and that $a_{i_j i_{j+1}} = w$ for $j = 1, \ldots, t-1$ and $a_{i_1 i_t} = w$. Then the vector $(1, \ldots, 1)$ is in the kernel of the corresponding $t \times t$ matrix. Thus we must have $n = t$ and we see that the genus is 1 and that we are in case (25) of Lemma 55.6.2. Thus we may assume $a_{i_1 i_t} = 0$. By induction hypothesis (or Lemma 55.5.5 if $t = 6$) we see that $a_{i_j i_k} = 0$ if $k > j + 1$. Moreover, we have $w_{i_1} = \ldots = w_{i_{t-1}} = w$ for some integer w and $w_{i_1}, w_{i_t} \in \{\frac{1}{2}w, w, 2w\}$. Moreover, the value of w_{i_1}, resp. w_{i_t} being $\frac{1}{2}w$, w, or $2w$ implies that the value of $a_{i_1 i_2}$, resp. $a_{i_{t-1} i_t}$ is w, w, or $2w$. This gives 9 possibilities. In each case it is easy to decide what happens:

(1) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, \frac{1}{2}w)$, then we are in case (27) of Lemma 55.6.2,
(2) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, w)$ or $(w, \frac{1}{2}w)$ then we are in case (3) of Lemma 55.5.8,
(3) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, 2w)$ or $(2w, \frac{1}{2}w)$ then we are in case (26) of Lemma 55.6.2,
(4) if $(w_{i_1}, w_{i_t}) = (w, w)$ then we are in case (1) of Lemma 55.5.8,
(5) if $(w_{i_1}, w_{i_t}) = (w, 2w)$ or $(2w, w)$ then we are in case (2) of Lemma 55.5.8, and
(6) if $(w_{i_1}, w_{i_t}) = (2w, 2w)$ then we are in case (28) of Lemma 55.6.2.

0C89 Lemma 55.5.8. Classification of proper subgraphs of the form

Let $t > 5$ and $n > t$. Then given t distinct (-2)-indices $i_1, \ldots, i_t$ such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$, then up to reversing the order of these indices we have the a's and w's

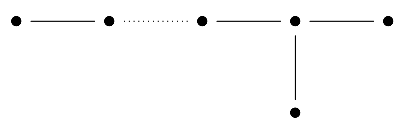
0C8A (1) are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_t} = w$, $a_{i_j i_{j+1}} = w$, and $a_{i_j i_k} = 0$ if $k > j + 1$, or
0C8B (2) are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t-1}} = w$, $w_{j_t} = 2w$, $a_{i_j i_{j+1}} = w$ for $j < t - 1$, $a_{i_{t-1} i_t} = 2w$, and $a_{i_j i_k} = 0$ if $k > j + 1$, or
0C8C (3) are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t-1}} = 2w$, $w_{j_t} = w$, $a_{i_j i_{j+1}} = 2w$, and $a_{i_{t-1} i_t} = 2w$, and $a_{i_j i_k} = 0$ if $k > j + 1$.

Proof. See discussion above. $\square$

Suppose that $t > 4$ and $i_1, \ldots, i_{t+1}$ are $t+1$ distinct (-2)-indices such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and such that $a_{j_{t-1} j_{t+1}} > 0$. See picture in Lemma 55.5.9. We will prove by induction on t that if $n = t+1$ this leads to possibilities (29) and (30) of Lemma 55.6.2 and if $n > t+1$ to case (1) of Lemma 55.5.9. By induction hypothesis (or Lemma 55.5.7 in case $t = 5$) we see that $a_{i_j i_k}$ is zero outside of the required nonvanishing ones for $j, k \geq 2$. Moreover, we see that $w_2 = \ldots = w_{t+1} = w$ for some integer w and that the nonvanishing $a_{i_j i_k}$ for $j, k \geq 2$ are equal to w. Applying Lemma 55.5.8 (or Lemma 55.5.5 if $t = 5$) to the sequence $i_1, \ldots, i_t$ and to the sequence $i_1, \ldots, i_{t-1}, i_{t+1}$ we conclude that $a_{i_1 i_j} = 0$ for $j \geq 3$ and that w_1 is equal to $\frac{1}{2}w$, w, or $2w$ and that correspondingly $a_{i_1 i_2}$ is $w, w, 2w$. This gives 3 possibilities. In each case it is easy to decide what happens:

(1) If $w_1 = \frac{1}{2}w$, then we are in case (30) of Lemma 55.6.2.
(2) If $w_1 = w$, then we are in case (1) of Lemma 55.5.9.
(3) If $w_1 = 2w$, then we are in case (29) of Lemma 55.6.2.

0C8D **Lemma 55.5.9.** Classification of proper subgraphs of the form



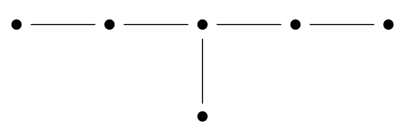
Let $t > 4$ and $n > t + 1$. Then given $t + 1$ distinct (-2)-indices $i_1, \ldots, i_{t+1}$ such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$ and $a_{i_{t-1} i_{t+1}}$ is nonzero, then we have the a's and w's

0C8E (1) *are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t+1}} = w$, $a_{i_j i_{j+1}} = w$ for $j = 1, \ldots, t-1$, $a_{i_{t-1} i_{t+1}} = w$ and $a_{i_j i_k} = 0$ for other pairs (j, k) with $j > k$.*

Proof. See discussion above. □

Suppose we are given 6 distinct (-2)-indices g, h, i, j, k, l such that $a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{il}$ are nonzero. See picture in Lemma 55.5.10. Then we can apply Lemma 55.5.7 to see that we must be in the situation of Lemma 55.5.10. Since the determinant is $3w^6 > 0$ we conclude that in this case it never happens that $n = 6$!

0C8F **Lemma 55.5.10.** Classification of proper subgraphs of the form



Let $n > 6$. Then given 6 distinct (-2)-indices $i_1, \ldots, i_6$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{36}$ are nonzero, then we have the m's, a's, and w's

0C8G (1) *are given by*

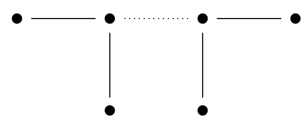
$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & w \\ 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4 + m_6$, $2m_4 \geq m_3 + m_5$, $2m_5 \geq m_3$, and $2m_6 \geq m_3$.

Proof. See discussion above. □

Suppose that $t \geq 4$ and $i_0, \ldots, i_{t+1}$ are $t+2$ distinct (-2)-indices such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and $a_{i_0 i_2} > 0$ and $a_{i_{t-1} i_{t+1}} > 0$. See picture in Lemma 55.5.11. Then we can apply Lemmas 55.5.7 and 55.5.9 to see that all other $a_{i_j i_k}$ for $j < k$ are zero and that $w_{i_0} = \ldots = w_{i_{t+1}} = w$ for some integer w and that the required nonzero off diagonal entries of A are equal to w. A computation shows that the determinant of the corresponding matrix is zero. Hence $n = t + 2$ and we are in case (31) of Lemma 55.6.2.

0C8H **Lemma 55.5.11.** Nonexistence of proper subgraphs of the form



Assume $t \geq 4$ and $n > t+2$. There do not exist $t+2$ distinct (-2)-indices $i_0, \ldots, i_{t+1}$ such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and $a_{i_0 i_2} > 0$ and $a_{i_{t-1} i_{t+1}} > 0$.

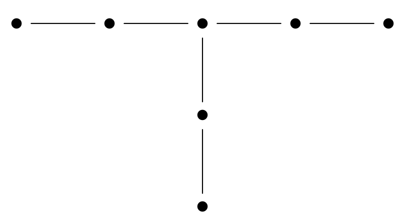
Proof. See discussion above. □

Suppose we are given 7 distinct (-2)-indices f, g, h, i, j, k, l such that the numbers $a_{fg}, a_{gh}, a_{ij}, a_{jh}, a_{kl}, a_{lh}$ are nonzero. See picture in Lemma 55.5.12. Then we can apply Lemma 55.5.7 to see that the corresponding matrix is

$$\begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & 0 & w & 0 & w \\ 0 & 0 & 0 & -2w & w & 0 & 0 \\ 0 & 0 & w & w & -2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2w & w \\ 0 & 0 & w & 0 & 0 & w & -2w \end{pmatrix}$$

Since the determinant is 0 we conclude that we must have $n = 7$ and $g = 1$ and we get case (32) of Lemma 55.6.2.

0C8I **Lemma 55.5.12.** Nonexistence of proper subgraphs of the form

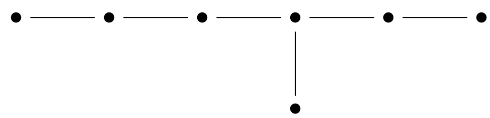


Assume $n > 7$. There do not exist 7 distinct (-2)-indices f, g, h, i, j, k, l such that $a_{fg}, a_{gh}, a_{ij}, a_{jh}, a_{kl}, a_{lh}$ are nonzero.

Proof. See discussion above. □

Suppose we are given 7 distinct (-2)-indices f, g, h, i, j, k, l such that the numbers $a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{il}$ are nonzero. See picture in Lemma 55.5.13. Then we can apply Lemmas 55.5.7 and 55.5.9 to see that we must be in the situation of Lemma 55.5.13. Since the determinant is $-8w^7 > 0$ we conclude that in this case it never happens that $n = 7$!

0C8J **Lemma 55.5.13.** Classification of proper subgraphs of the form



Let $n > 7$. Then given 7 distinct (-2)-indices $i_1, \ldots, i_7$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{47}$ are nonzero, then we have the m's, a's, and w's

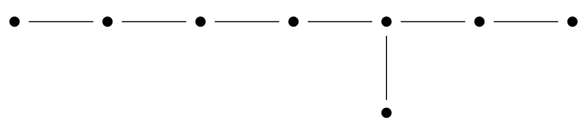
0C8K (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

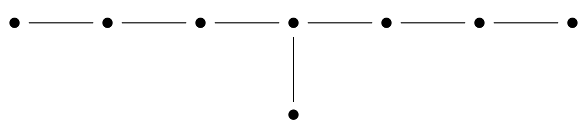
with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5 + m_7$, $2m_5 \geq m_4 + m_6$, $2m_6 \geq m_5$, and $2m_7 \geq m_4$.

Proof. See discussion above. $\square$

Suppose we are given 8 distinct (-2)-indices whose pattern of nonzero entries a_{ij} of the matrix A looks like

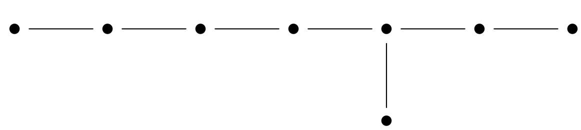


or like



Arguing exactly as in the proof of Lemma 55.5.13 we see that the first pattern leads to case (1) in Lemma 55.5.14 and does not lead to a new case in Lemma 55.6.2. Arguing exactly as in the proof of Lemma 55.5.12 we see that the second pattern does not occur if $n > 8$, but leads to case (33) in Lemma 55.6.2 when $n = 8$.

0C8L **Lemma 55.5.14.** Classification of proper subgraphs of the form



Let $n > 8$. Then given 8 distinct (-2)-indices $i_1, \ldots, i_8$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{65}, a_{57}$ are nonzero, then we have the m's, a's, and w's

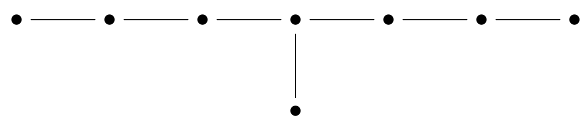
0C8M (1) are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{pmatrix}, \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, $2m_5 \geq m_4 + m_6 + m_8$, $2m_6 \geq m_5 + m_7$, $2m_7 \geq m_6$, and $2m_8 \geq m_5$.

Proof. See discussion above. $\square$

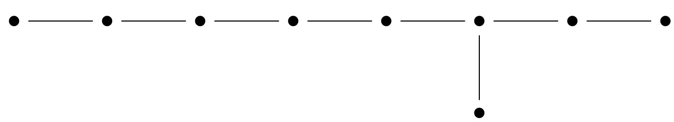
0C8N Lemma 55.5.15. Nonexistence of proper subgraphs of the form



Assume $n > 8$. There do not exist 8 distinct (-2)-indices e, f, g, h, i, j, k, l such that $a_{ef}, a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{lh}$ are nonzero.

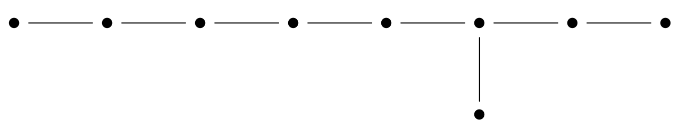
Proof. See discussion above. $\square$

Suppose we are given 9 distinct (-2)-indices whose pattern of nonzero entries a_{ij} of the matrix A looks like



Arguing exactly as in the proof of Lemma 55.5.12 we see that this pattern does not occur if $n > 9$, but leads to case (34) in Lemma 55.6.2 when $n = 9$.

0C8P Lemma 55.5.16. Nonexistence of proper subgraphs of the form



Assume $n > 9$. There do not exist 9 distinct (-2)-indices $d, e, f, g, h, i, j, k, l$ such that $a_{de}, a_{ef}, a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{lh}$ are nonzero.

Proof. See discussion above. $\square$

Collecting all the information together we find the following.

0C8Q Proposition 55.5.17. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Let $I \subset \{1, \ldots, n\}$ be a proper subset of cardinality ≥ 2 consisting of (-2)-indices such that there does not exist a nonempty proper subset $I' \subset I$ with $a_{i'i} = 0$ for $i' \in I$, $i \in I \setminus I'$. Then up to reordering the m_i's, a_{ij}'s, w_i's for $i, j \in I$ are as listed in Lemmas 55.5.1, 55.5.2, 55.5.3, 55.5.4, 55.5.5, 55.5.7, 55.5.8, 55.5.9, 55.5.10, 55.5.13, or 55.5.14.

Proof. This follows from the discussion above; see discussion at the start of Section 55.5. ☐

55.6. Classification of minimal type for genus zero and one

0C8R The title of the section explains it all.

0C8S **Lemma 55.6.1** (Genus zero). *The only minimal numerical type of genus zero is $n = 1$, $m_1 = 1$, $a_{11} = 0$, $w_1 = 1$, $g_1 = 0$.*

Proof. Follows from Lemmas 55.3.13 and 55.3.5. ☐

0C8T **Lemma 55.6.2** (Genus one). *The minimal numerical types of genus one are up to equivalence*

0C8U (1) $n = 1$, $a_{11} = 0$, $g_1 = 1$, $m_1, w_1 \geq 1$ arbitrary,

0C8V (2) $n = 2$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w \\ 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C8W (3) $n = 2$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 4w \\ 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C8X (4) $n = 3$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w \\ w & -2w & w \\ w & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C8Y (5) $n = 3$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & 3w \\ 0 & 3w & -6w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 3w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C8Z (6) $n = 3$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 3m \end{pmatrix}, \quad \begin{pmatrix} -6w & 3w & 0 \\ 3w & -6w & 3w \\ 0 & 3w & -2w \end{pmatrix}, \quad \begin{pmatrix} 3w \\ 3w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C90 (7) $n = 3$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 \\ 2w & -4w & 4w \\ 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C91 (8) $n = 3$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 \\ 2w & -4w & 2w \\ 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C92 (9) $n = 3$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 \\ 2w & -2w & 2w \\ 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C93 (10) $n = 4$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & w \\ w & -2w & w & 0 \\ 0 & w & -2w & w \\ w & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C94 (11) $n = 4$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 4w \\ 0 & 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C95 (12) $n = 4$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C96 (13) $n = 4$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 \\ 2w & -2w & w & 0 \\ 0 & w & -2w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C97 (14) $n = 4$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & 2w \\ w & -2w & 0 & 0 \\ w & 0 & -2w & 0 \\ 2w & 0 & 0 & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C98 (15) $n = 4$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 2w & 2w \\ 2w & -4w & 0 & 0 \\ 2w & 0 & -4w & 0 \\ 2w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C99 (16) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & w \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & w \\ w & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9A (17) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9B (18) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -2w & w \\ 0 & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9C (19) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 4w \\ 0 & 0 & 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9D (20) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9E (21) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9F (22) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & w & w \\ w & -2w & 0 & 0 & 0 \\ w & 0 & -2w & 0 & 0 \\ w & 0 & 0 & -2w & 0 \\ w & 0 & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9G (23) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 2m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9H (24) $n = 5$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 2w \\ 0 & 0 & 2w & -4w & 0 \\ 0 & 0 & 2w & 0 & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9I (25) $n \geq 6$ and we have an n-cycle generalizing (16):
   (a) $m_1 = \ldots = m_n = m$,
   (b) $a_{12} = \ldots = a_{(n-1)n} = w$, $a_{1n} = w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = \ldots = w_n = w$

with w and m arbitrary,

0C9J (26) $n \geq 6$ and we have a chain generalizing (19):
   (a) $m_1 = \ldots = m_{n-1} = 2m$, $m_n = m$,
   (b) $a_{12} = \ldots = a_{(n-2)(n-1)} = 2w$, $a_{(n-1)n} = 4w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = w$, $w_2 = \ldots = w_{n-1} = 2w$, $w_n = 4w$

with w and m arbitrary,

0C9K (27) $n \geq 6$ and we have a chain generalizing (20):
   (a) $m_1 = \ldots = m_n = m$,
   (b) $a_{12} = \ldots = a_{(n-1)n} = w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = w$, $w_2 = \ldots = w_{n-1} = 2w$, $w_n = w$

with w and m arbitrary,

0C9L (28) $n \geq 6$ and we have a chain generalizing (21):
   (a) $m_1 = w$, $w_2 = \ldots = m_{n-1} = 2m$, $m_n = m$,
   (b) $a_{12} = 2w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-1)n} = 2w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = 2w$, $w_2 = \ldots = w_{n-1} = w$, $w_n = 2w$

   with w and m arbitrary,

0C9M (29) $n \geq 6$ and we have a type generalizing (23):
   (a) $m_1 = m$, $m_2 = \ldots = m_{n-3} = 2m$, $m_{n-1} = m_n = m$,
   (b) $a_{12} = 2w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-2)n} = w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = 2w$, $w_2 = \ldots = w_n = w$

   with w and m arbitrary,

0C9N (30) $n \geq 6$ and we have a type generalizing (24):
   (a) $m_1 = \ldots = m_{n-3} = 2m$, $m_{n-1} = m_n = m$,
   (b) $a_{12} = \ldots = a_{(n-2)(n-1)} = 2w$, $a_{(n-2)n} = 2w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = w$, $w_2 = \ldots = w_n = 2w$

   with w and m arbitrary,

0C9P (31) $n \geq 6$ and we have a type generalizing (22):
   (a) $m_1 = m_2 = m$, $m_3 = \ldots = m_{n-2} = 2m$, $m_{n-1} = m_n = m$,
   (b) $a_{13} = w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-2)n} = w$, and for other $i < j$ we have $a_{ij} = 0$,
   (c) $w_1 = \ldots = w_n = w$,

   with w and m arbitrary,

0C9Q (32) $n = 7$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 3m \\ m \\ 2m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & 0 & w & 0 & w \\ 0 & 0 & 0 & -2w & w & 0 & 0 \\ 0 & 0 & w & w & -2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2w & w \\ 0 & 0 & w & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   with w and m arbitrary,

0C9R (33) $n = 8$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 3m \\ 2m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & w \\ 0 & 0 & 0 & w & -2w & w & 0 & 0 \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   with w and m arbitrary,

0C9S (34) $n = 9$, and m_i, a_{ij}, w_i, g_i given by

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 5m \\ 6m \\ 4m \\ 2m \\ 3m \end{pmatrix}, \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary.

Proof. This is proved in Section 55.5. See discussion at the start of Section 55.5. $\square$

55.7. Bounding invariants of numerical types

0C9T In our proof of semistable reduction for curves we'll use a bound on Picard groups of numerical types of genus g which we will prove in this section.

0C9U **Lemma 55.7.1.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Given i, j with $a_{ij} > 0$ we have $m_i a_{ij} \leq m_j|a_{jj}|$ and $m_i w_i \leq m_j|a_{jj}|$.

Proof. For every index j we have $m_j a_{jj} + \sum_{i \neq j} m_i a_{ij} = 0$. Thus if we have an upper bound on $|a_{jj}|$ and m_j, then we also get an upper bound on the nonzero (and hence positive) a_{ij} as well as m_i. Recalling that w_i divides a_{ij}, the reader easily sees the lemma is correct. $\square$

0C9V **Lemma 55.7.2.** Fix $g \geq 2$. For every minimal numerical type n, m_i, a_{ij}, w_i, g_i of genus g with $n > 1$ we have

(1) the set $J \subset \{1, \ldots, n\}$ of non-(-2)-indices has at most $2g - 2$ elements,
(2) for $j \in J$ we have $g_j < g$,
(3) for $j \in J$ we have $m_j|a_{jj}| \leq 6g - 6$, and
(4) for $j \in J$ and $i \in \{1, \ldots, n\}$ we have $m_i a_{ij} \leq 6g - 6$.

Proof. Recall that $g = 1 + \sum m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$. For $j \in J$ the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is > 0 and hence $\geq 1/2$. This uses Lemma 55.3.7, Definition 55.3.8, Definition 55.3.12, Lemma 55.3.15, and Definition 55.3.16; we will use these results without further mention in the following. Thus J has at most $2(g - 1)$ elements. This proves (1).

Recall that $-a_{ii} > 0$ for all i by Lemma 55.3.6. Hence for $j \in J$ the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is $> m_j w_j(g_j - 1)$. Thus

$$g - 1 > m_j w_j(g_j - 1) \Rightarrow g_j < (g - 1)/m_j w_j + 1$$

This indeed implies $g_j < g$ which proves (2).

For $j \in J$ if $g_j > 0$, then the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is $\geq -\frac{1}{2}m_j a_{jj}$ and we immediately conclude that $m_j|a_{jj}| \leq 2(g - 1)$. Otherwise $a_{jj} = -kw_j$ for some integer $k \geq 3$ (because $j \in J$) and we get

$$m_j w_j(-1 + \frac{k}{2}) \leq g - 1 \Rightarrow m_j w_j \leq \frac{2(g-1)}{k-2}$$

Plugging this back into $a_{jj} = -km_j w_j$ we obtain

$$m_j|a_{jj}| \leq 2(g-1)\frac{k}{k-2} \leq 6(g-1)$$

This proves (3).

Part (4) follows from Lemma 55.7.1 and (3). $\square$

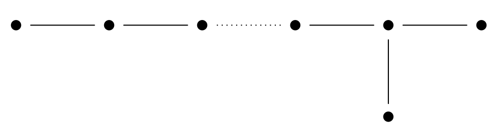
0C9W Lemma 55.7.3. Fix $g \geq 2$. For every minimal numerical type n, m_i, a_{ij}, w_i, g_i of genus g we have $m_i|a_{ij}| \leq 768g$.

Proof. By Lemma 55.7.1 it suffices to show $m_i|a_{ii}| \leq 768g$ for all i. Let $J \subset \{1, \ldots, n\}$ be the set of non-(-2)-indices as in Lemma 55.7.2. Observe that J is nonempty as $g \geq 2$. Also $m_j|a_{jj}| \leq 6g$ for $j \in J$ by the lemma.

Suppose we have $j \in J$ and a sequence $i_1, \ldots, i_7$ of (-2)-indices such that a_{ji_1} and $a_{i_1i_2}$, $a_{i_2i_3}$, $a_{i_3i_4}$, $a_{i_4i_5}$, $a_{i_5i_6}$, and $a_{i_6i_7}$ are nonzero. Then we see from Lemma 55.7.1 that $m_{i_1}w_{i_1} \leq 6g$ and $m_{i_1}a_{ji_1} \leq 6g$. Because i_1 is a (-2)-index, we have $a_{i_1i_1} = -2w_{i_1}$ and we conclude that $m_{i_1}|a_{i_1i_1}| \leq 12g$. Repeating the argument we conclude that $m_{i_2}w_{i_2} \leq 12g$ and $m_{i_2}a_{i_1i_2} \leq 12g$. Then $m_{i_2}|a_{i_2i_2}| \leq 24g$ and so on. Eventually we conclude that $m_{i_k}|a_{i_ki_k}| \leq 2^k(6g) \leq 768g$ for $k = 1, \ldots, 7$.

Let $I \subset \{1, \ldots, n\} \setminus J$ be a maximal connected subset. In other words, there does not exist a nonempty proper subset $I' \subset I$ such that $a_{i'i} = 0$ for $i' \in I'$ and $i \in I \setminus I'$ and I is maximal with this property. In particular, since a numerical type is connected by definition, we see that there exists a $j \in J$ and $i \in I$ with $a_{ij} > 0$. Looking at the classification of such I in Proposition 55.5.17 and using the result of the previous paragraph, we see that $w_i|a_{ii}| \leq 768g$ for all $i \in I$ unless I is as described in Lemma 55.5.8 or Lemma 55.5.9. Thus we may assume the nonvanishing of $a_{ii'}$, $i, i' \in I$ has either the shape

(which has 3 subcases as detailed in Lemma 55.5.8) or the shape



We will prove the bound holds for the first subcase of Lemma 55.5.8 and leave the other cases to reader (the argument is almost exactly the same in those cases).

After renumbering we may assume $I = \{1, \ldots, t\} \subset \{1, \ldots, n\}$ and there is an integer w such that

$$w = w_1 = \ldots = w_t = a_{12} = \ldots = a_{(t-1)t} = -\frac{1}{2}a_{i_1i_2} = \ldots = -\frac{1}{2}a_{(t-1)t}$$

The equalities $a_{ii}m_i + \sum_{j \neq i} a_{ij}m_j = 0$ imply that we have

$$2m_2 \geq m_1 + m_3, \ldots, 2m_{t-1} \geq m_{t-2} + m_t$$

Equality holds in $2m_i \geq m_{i-1} + m_{i+1}$ if and only if i does not "meet" any indices besides $i-1$ and $i+1$. And if i does meet another index, then this index is in J (by maximality of I). In particular, the map $\{1, \ldots, t\} \to \mathbf{Z}$, $i \mapsto m_i$ is concave.

Let $m = \max(m_i, i \in \{1, \ldots, t\})$. Then $m_i|a_{ii}| \leq 2mw$ for $i \leq t$ and our goal is to show that $2mw \leq 768g$. Let s, resp. s' in $\{1, \ldots, t\}$ be the smallest, resp. biggest index with $m_s = m = m_{s'}$. By concavity we see that $m_i = m$ for $s \leq i \leq s'$. If $s > 1$, then we do not have equality in $2m_s \geq m_{s-1} + m_{s+1}$ and we see that s meets an index from J. In this case $2mw \leq 12g$ by the result of the second paragraph of the proof. Similarly, if $s' < t$, then s' meets an index from J and we get $2mw \leq 12g$ as well. But if $s = 1$ and $s' = t$, then we conclude that $a_{ij} = 0$ for all $j \in J$ and $i \in \{2, \ldots, t-1\}$. But as we've seen that there must be a pair $(i, j) \in I \times J$ with $a_{ij} > 0$, we conclude that this happens either with $i = 1$ or with $i = t$ and we conclude $2mw \leq 12g$ in the same manner as before (as $m_1 = m = m_t$ in this case). □

0C9X Proposition 55.7.4. Let $g \geq 2$. For every numerical type T of genus g and prime number $\ell > 768g$ we have

$$\dim_{\mathbf{F}_\ell} \text{Pic}(T)[\ell] \leq g$$

where $\text{Pic}(T)$ is as in Definition 55.4.1. If T is minimal, then we even have

$$\dim_{\mathbf{F}_\ell} \text{Pic}(T)[\ell] \leq g_{top} \leq g$$

where g_{top} as in Definition 55.3.11.

Proof. Say T is given by n, m_i, a_{ij}, w_i, g_i. If T is not minimal, then there exists a (-1)-index. After replacing T by an equivalent type we may assume n is a (-1)-index. Applying Lemma 55.4.4 we find $\text{Pic}(T) \subset \text{Pic}(T')$ where T' is a numerical type of genus g (Lemma 55.3.9) with $n-1$ indices. Thus we conclude by induction on n provided we prove the lemma for minimal numerical types.

Assume that T is a minimal numerical type of genus ≥ 2. Observe that $g_{top} \leq g$ by Lemma 55.3.14. If $A = (a_{ij})$ then since $\text{Pic}(T) \subset \text{Coker}(A)$ by Lemma 55.4.3. Thus it suffices to prove the lemma for $\text{Coker}(A)$. By Lemma 55.7.3 we see that $m_i|a_{ij}| \leq 768g$ for all i, j. Hence the result by Lemma 55.2.6. □

55.8. Models

0C2R In this chapter R will be a discrete valuation ring and K will be its fraction field. If needed we will denote $\pi \in R$ a uniformizer and $k = R/(\pi)$ its residue field.

Let V be an algebraic K-scheme (Varieties, Definition 33.20.1). A model for V will mean a flat finite type[6] morphism $X \to \text{Spec}(R)$ endowed with an isomorphism $V \to X_K = X \times_{\text{Spec}(R)} \text{Spec}(K)$. We often will identify V and the generic fibre X_K of X and just write $V = X_K$. The special fibre is $X_k = X \times_{\text{Spec}(R)} \text{Spec}(k)$. A morphism of models $X \to X'$ for V is a morphism $X \to X'$ of schemes over R which induces the identity on V.

We will say X is a proper model of V if X is a model of V and the structure morphism $X \to \text{Spec}(R)$ is proper. Similarly for separated models, smooth models, and add more here. We will say X is a regular model of V if X is a model of V and X is a regular scheme. Similarly for normal models, reduced models, and add more here.

Let $R \subset R'$ be an extension of discrete valuation rings (More on Algebra, Definition 15.111.1). This induces an extension K'/K of fraction fields. Given an algebraic

[6]Occasionally it is useful to allow models to be locally of finite type over R, but we'll cross that bridge when we come to it.

scheme V over K, denote V' the base change $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$. Then there is a functor

$$\text{models for } V \text{ over } R \longrightarrow \text{models for } V' \text{ over } R'$$

sending X to $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$.

0C2S Lemma 55.8.1. Let $V_1 \to V_2$ be a closed immersion of algebraic schemes over K. If X_2 is a model for V_2, then the scheme theoretic image of $V_1 \to X_2$ is a model for V_1.

Proof. Using Morphisms, Lemma 29.6.3 and Example 29.6.4 this boils down to the following algebra statement. Let A_1 be a finite type R-algebra flat over R. Let $A_1 \otimes_R K \to B_2$ be a surjection. Then $A_2 = A_1/\mathrm{Ker}(A_1 \to B_2)$ is a finite type R-algebra flat over R such that $B_2 = A_2 \otimes_R K$. We omit the detailed proof; use More on Algebra, Lemma 15.22.11 to prove that A_2 is flat. $\square$

0C2T Lemma 55.8.2. Let X be a model of a geometrically normal variety V over K. Then the normalization $\nu : X^\nu \to X$ is finite and the base change of X^ν to the completion $R^\wedge$ is the normalization of the base change of X. Moreover, for each $x \in X^\nu$ the completion of $\mathcal{O}_{X^\nu,x}$ is normal.

Proof. Observe that $R^\wedge$ is a discrete valuation ring (More on Algebra, Lemma 15.43.5). Set $Y = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R^\wedge)$. Since $R^\wedge$ is a discrete valuation ring, we see that

$$Y \setminus Y_k = Y \times_{\mathrm{Spec}(R^\wedge)} \mathrm{Spec}(K^\wedge) = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^\wedge)$$

where $K^\wedge$ is the fraction field of $R^\wedge$. Since V is geometrically normal, we find that this is a normal scheme. Hence the first part of the lemma follows from Resolution of Surfaces, Lemma 54.11.6.

To prove the second part we may assume X and Y are normal (by the first part). If x is in the generic fibre, then $\mathcal{O}_{X,x} = \mathcal{O}_{V,x}$ is a normal local ring essentially of finite type over a field. Such a ring is excellent (More on Algebra, Proposition 15.52.3). If x is a point of the special fibre with image $y \in Y$, then $\mathcal{O}_{X,x}^\wedge = \mathcal{O}_{Y,y}^\wedge$ by Resolution of Surfaces, Lemma 54.11.1. In this case $\mathcal{O}_{Y,y}$ is a excellent normal local domain by the same reference as before as $R^\wedge$ is excellent. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 15.42.2 applies). This finishes the proof. $\square$

0C2U Lemma 55.8.3. Let X be a model of a smooth curve C over K. Then there exists a resolution of singularities of X and any resolution is a model of C.

Proof. We check condition (4) of Lipman's theorem (Resolution of Surfaces, Theorem 54.14.5) hold. This is clear from Lemma 55.8.2 except for the statement that X^ν has finitely many singular points. To see this we can use that R is J-2 by More on Algebra, Proposition 15.48.7 and hence the nonsingular locus is open in X^ν. Since X^ν is normal of dimension ≤ 2, the singular points are closed, hence closedness of the singular locus means there are finitely many of them (as X is quasi-compact). Observe that any resolution of X is a modification of X (Resolution of Surfaces, Definition 54.14.1). This will be an isomorphism over the normal locus of X by Varieties, Lemma 33.17.3. Since the set of normal points includes $C = X_K$ we conclude any resolution is a model of C. $\square$

0C2V Definition 55.8.4. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A minimal model will be a regular, proper model X for C such that X does not contain an exceptional curve of the first kind (Resolution of Surfaces, Section 54.16).

Really such a thing should be called a minimal regular proper model or even a relatively minimal regular projective model. But as long as we stick to models over discrete valuation rings (as we will in this chapter), no confusion should arise.

Minimal models always exist (Proposition 55.8.6) and are unique when the genus is > 0 (Lemma 55.10.1).

0CD9 Lemma 55.8.5. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If X is a regular proper model for C, then there exists a sequence of morphisms

$$X = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind, and such that X_0 is a minimal model.

Proof. By Resolution of Surfaces, Lemma 54.16.11 we see that X is projective over R. Hence X has an ample invertible sheaf by More on Morphisms, Lemma 37.50.1 (we will use this below). Let $E \subset X$ be an exceptional curve of the first kind. See Resolution of Surfaces, Section 54.16. By Resolution of Surfaces, Lemma 54.16.8 we can contract E by a morphism $X \to X'$ such that X' is regular and is projective over R. Clearly, the number of irreducible components of X'_k is exactly one less than the number of irreducible components of X_k. Thus we can only perform a finite number of these contractions until we obtain a minimal model. $\square$

0C2W Proposition 55.8.6. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A minimal model exists.

Proof. Choose a closed immersion $C \to \mathbf{P}^n_K$. Let X be the scheme theoretic image of $C \to \mathbf{P}^n_R$. Then $X \to \mathrm{Spec}(R)$ is a projective model of C by Lemma 55.8.1. By Lemma 55.8.3 there exists a resolution of singularities $X' \to X$ and X' is a model for C. Then $X' \to \mathrm{Spec}(R)$ is proper as a composition of proper morphisms. Then we may apply Lemma 55.8.5 to obtain a minimal model. $\square$

55.9. The geometry of a regular model

0C5Y In this section we describe the geometry of a proper regular model X of a smooth projective curve C over K with $H^0(C, \mathcal{O}_C) = K$.

0C5Z Lemma 55.9.1. Let X be a regular model of a smooth curve C over K.

(1) the special fibre X_k is an effective Cartier divisor on X,
(2) each irreducible component C_i of X_k is an effective Cartier divisor on X,
(3) $X_k = \sum m_i C_i$ (sum of effective Cartier divisors) where m_i is the multiplicity of C_i in X_k,
(4) $\mathcal{O}_X(X_k) \cong \mathcal{O}_X$.

Proof. Recall that R is a discrete valuation ring with uniformizer π and residue field $k = R/(\pi)$. Because $X \to \mathrm{Spec}(R)$ is flat, the element π is a nonzerodivisor affine locally on X (see More on Algebra, Lemma 15.22.11). Thus if $U = \mathrm{Spec}(A) \subset X$ is an affine open, then

$$X_K \cap U = U_k = \mathrm{Spec}(A \otimes_R k) = \mathrm{Spec}(A/\pi A)$$

and π is a nonzerodivisor in A. Hence $X_k = V(\pi)$ is an effective Cartier divisor by Divisors, Lemma 31.13.2. Hence (1) is true.

The discussion above shows that the pair $(\mathcal{O}_X(X_k), 1)$ is isomorphic to the pair $(\mathcal{O}_X, \pi)$ which proves (4).

By Divisors, Lemma 31.15.11 there exist pairwise distinct integral effective Cartier divisors $D_i \subset X$ and integers $a_i \geq 0$ such that $X_k = \sum a_i D_i$. We can throw out those divisors D_i such that $a_i = 0$. Then it is clear (from the definition of addition of effective Cartier divisors) that $X_k = \bigcup D_i$ set theoretically. Thus $C_i = D_i$ are the irreducible components of X_k which proves (2). Let ξ_i be the generic point of C_i. Then $\mathcal{O}_{X,\xi_i}$ is a discrete valuation ring (Divisors, Lemma 31.15.4). The uniformizer $\pi_i \in \mathcal{O}_{X,\xi_i}$ is a local equation for C_i and the image of π is a local equation for X_k. Since $X_k = \sum a_i C_i$ we see that π and $\pi_i^{a_i}$ generate the same ideal in $\mathcal{O}_{X,\xi_i}$. On the other hand, the multiplicity of C_i in X_k is

$$m_i = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X_k,\xi_i} = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X,\xi_i}/(\pi) = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X,\xi_i}/(\pi_i^{a_i}) = a_i$$

See Chow Homology, Definition 42.9.2. Thus $a_i = m_i$ and (3) is proved. □

0C60 Lemma 55.9.2. Let X be a regular model of a smooth curve C over K. Then

(1) $X \to \text{Spec}(R)$ is a Gorenstein morphism of relative dimension 1,
(2) each of the irreducible components C_i of X_k is Gorenstein.

Proof. Since $X \to \text{Spec}(R)$ is flat, to prove (1) it suffices to show that the fibres are Gorenstein (Duality for Schemes, Lemma 48.25.3). The generic fibre is a smooth curve, which is regular and hence Gorenstein (Duality for Schemes, Lemma 48.24.3). For the special fibre X_k we use that it is an effective Cartier divisor on a regular (hence Gorenstein) scheme and hence Gorenstein for example by Dualizing Complexes, Lemma 47.21.6. The curves C_i are Gorenstein by the same argument. □

0C61 Situation 55.9.3. Let R be a discrete valuation ring with fraction field K, residue field k, and uniformizer π. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Let X be a regular proper model of C. Let $C_1, \ldots, C_n$ be the irreducible components of the special fibre X_k. Write $X_k = \sum m_i C_i$ as in Lemma 55.9.1.

0C62 Lemma 55.9.4. In Situation 55.9.3 the special fibre X_k is connected.

Proof. Consequence of More on Morphisms, Lemma 37.53.6. □

0C63 Lemma 55.9.5. In Situation 55.9.3 there is an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}^{\oplus n} \to \text{Pic}(X) \to \text{Pic}(C) \to 0$$

where the first map sends 1 to $(m_1, \ldots, m_n)$ and the second maps sends the ith basis vector to $\mathcal{O}_X(C_i)$.

Proof. Observe that $C \subset X$ is an open subscheme. The restriction map $\text{Pic}(X) \to \text{Pic}(C)$ is surjective by Divisors, Lemma 31.28.3. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that there is an isomorphism $s : \mathcal{O}_C \to \mathcal{L}|_C$. Then s is a regular meromorphic section of $\mathcal{L}$ and we see that $\text{div}_\mathcal{L}(s) = \sum a_i C_i$ for some $a_i \in \mathbf{Z}$ (Divisors, Definition 31.27.4). By Divisors, Lemma 31.27.6 (and the fact that X is normal) we conclude that $\mathcal{L} = \mathcal{O}_X(\sum a_i C_i)$. Finally, suppose that $\mathcal{O}_X(\sum a_i C_i) \cong \mathcal{O}_X$. Then there exists an element g of the function field of X with $\text{div}_X(g) = \sum a_i C_i$. In particular the rational function g has no zeros or poles on the generic fibre C of X. Since C

is a normal scheme this implies $g \in H^0(C, \mathcal{O}_C) = K$. Thus $g = \pi^a u$ for some $a \in \mathbf{Z}$ and $u \in R^*$. We conclude that $\text{div}_X(g) = a \sum m_i C_i$ and the proof is complete. □

In Situation 55.9.3 for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ and every i we get an integer

$$\deg(\mathcal{L}|_{C_i}) = \chi(C_i, \mathcal{L}|_{C_i}) - \chi(C_i, \mathcal{O}_{C_i})$$

by taking the degree of the restriction of $\mathcal{L}$ to C_i relative to the ground field k[7] as in Varieties, Section 33.44.

0C64 Lemma 55.9.6. In Situation 55.9.3 given $\mathcal{L}$ an invertible $\mathcal{O}_X$-module and $a = (a_1, \ldots, a_n) \in \mathbf{Z}^{\oplus n}$ we define

$$\langle a, \mathcal{L} \rangle = \sum a_i \deg(\mathcal{L}|_{C_i})$$

Then $\langle , \rangle$ is bilinear and for $b = (b_1, \ldots, b_n) \in \mathbf{Z}^{\oplus n}$ we have

$$\left\langle a, \mathcal{O}_X(\sum b_i C_i) \right\rangle = \left\langle b, \mathcal{O}_X(\sum a_i C_i) \right\rangle$$

Proof. Bilinearity is immediate from the definition and Varieties, Lemma 33.44.7. To prove symmetry it suffices to assume a and b are standard basis vectors in $\mathbf{Z}^{\oplus n}$. Hence it suffices to prove that

$$\deg(\mathcal{O}_X(C_j)|_{C_i}) = \deg(\mathcal{O}_X(C_i)|_{C_j})$$

for all $1 \leq i, j \leq n$. If $i = j$ there is nothing to prove. If $i \neq j$, then the canonical section 1 of $\mathcal{O}_X(C_j)$ restricts to a nonzero (hence regular) section of $\mathcal{O}_X(C_j)|_{C_i}$ whose zero scheme is exactly $C_i \cap C_j$ (scheme theoretic intersection). In other words, $C_i \cap C_j$ is an effective Cartier divisor on C_i and

$$\deg(\mathcal{O}_X(C_j)|_{C_i}) = \deg(C_i \cap C_j)$$

by Varieties, Lemma 33.44.9. By symmetry we obtain the same (!) formula for the other side and the proof is complete. □

In Situation 55.9.3 it is often convenient to think of $\mathbf{Z}^{\oplus n}$ as the free abelian group on the set $\{C_1, \ldots, C_n\}$. We will indicate an element of this group as $\sum a_i C_i$; here we think of this as a formal sum although equivalently we may (and we sometimes do) think of such a sum as a Weil divisor on X supported on the special fibre X_k. Now Lemma 55.9.6 allows us to define a symmetric bilinear form $(\cdot)$ on this free abelian group by the rule

0C65 (55.9.6.1)
$$\left(\sum a_i C_i \cdot \sum b_j C_j\right) = \left\langle a, \mathcal{O}_X(\sum b_j C_j) \right\rangle = \left\langle b, \mathcal{O}_X(\sum a_i C_i) \right\rangle$$

We will prove some properties of this bilinear form.

0C66 Lemma 55.9.7. In Situation 55.9.3 the symmetric bilinear form (55.9.6.1) has the following properties

1. $(C_i \cdot C_j) \geq 0$ if $i \neq j$ with equality if and only if $C_i \cap C_j = \emptyset$,
2. $(\sum m_i C_i \cdot C_j) = 0$,
3. there is no nonempty proper subset $I \subset \{1, \ldots, n\}$ such that $(C_i \cdot C_j) = 0$ for $i \in I$, $j \notin I$.
4. $(\sum a_i C_i \cdot \sum a_i C_i) \leq 0$ with equality if and only if there exists a $q \in \mathbf{Q}$ such that $a_i = q m_i$ for $i = 1, \ldots, n$,

[7] Observe that it may happen that the field $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ is strictly bigger than k. In this case every invertible module on C_i has degree (as defined above) divisible by $[\kappa_i : k]$.

Proof. In the proof of Lemma 55.9.6 we saw that $(C_i \cdot C_j) = \deg(C_i \cap C_j)$ if $i \neq j$. This is ≥ 0 and > 0 if and only if $C_i \cap C_j \neq \emptyset$. This proves (1).

Proof of (2). This is true because by Lemma 55.9.1 the invertible sheaf associated to $\sum m_i C_i$ is trivial and the trivial sheaf has degree zero.

Proof of (3). This is expressing the fact that X_k is connected (Lemma 55.9.4) via the description of the intersection products given in the proof of (1).

Part (4) follows from (1), (2), and (3) by Lemma 55.2.3. $\square$

0C67 Lemma 55.9.8. In Situation 55.9.3 set $d = \gcd(m_1, \ldots, m_n)$ and let $D = \sum (m_i/d) C_i$ as an effective Cartier divisor. Then $\mathcal{O}_X(D)$ has order dividing d in $\text{Pic}(X)$ and $\mathcal{C}_{D/X}$ an invertible $\mathcal{O}_D$-module of order dividing d in $\text{Pic}(D)$.

Proof. We have

$$\mathcal{O}_X(D)^{\otimes d} = \mathcal{O}_X(dD) = \mathcal{O}_X(X_k) = \mathcal{O}_X$$

by Lemma 55.9.1. We conclude as $\mathcal{C}_{D/X}$ is the pullback of $\mathcal{O}_X(-D)$. $\square$

0C68 Lemma 55.9.9. In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$. Let $D = \sum (m_i/d) C_i$ as an effective Cartier divisor. Then there exists a sequence of effective Cartier divisors [AW71, Lemma 2.6]

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_m = D$$

such that $Z_j = Z_{j-1} + C_{i_j}$ for some $i_j \in \{1, \ldots, n\}$ for $j = 1, \ldots, m$ and such that $H^0(Z_j, \mathcal{O}_{Z_j})$ is a field finite over k for $j = 0, \ldots m$.

Proof. The reduction $D_{red} = (X_k)_{red} = \sum C_i$ is connected (Lemma 55.9.4) and proper over k. Hence $H^0(D_{red}, \mathcal{O})$ is a field and a finite extension of k by Varieties, Lemma 33.9.3. Thus the result for $Z_0 = D_{red} = (X_k)_{red}$ is true. Suppose that we have already constructed

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_t \subset D$$

with $Z_j = Z_{j-1} + C_{i_j}$ for some $i_j \in \{1, \ldots, n\}$ for $j = 1, \ldots, t$ and such that $H^0(Z_j, \mathcal{O}_{Z_j})$ is a field finite over k for $j = 0, \ldots, t$. Write $Z_t = \sum a_i C_i$ with $1 \leq a_i \leq m_i/d$. If $a_i = m_i/d$ for all i, then $Z_t = D$ and the lemma is proved. If not, then $a_i < m_i/d$ for some i and it follows that $(Z_t \cdot Z_t) < 0$ by Lemma 55.9.7. This means that $(D - Z_t \cdot Z_t) > 0$ because $(D \cdot Z_t) = 0$ by the lemma. Thus we can find an i with $a_i < m_i/d$ such that $(C_i \cdot Z_t) > 0$. Set $Z_{t+1} = Z_t + C_i$ and $i_{t+1} = i$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-Z_t)|_{C_i} \to \mathcal{O}_{Z_{t+1}} \to \mathcal{O}_{Z_t} \to 0$$

of Divisors, Lemma 31.14.3. By our choice of i we see that $\mathcal{O}_X(-Z_t)|_{C_i}$ is an invertible sheaf of negative degree on the proper curve C_i, hence it has no nonzero global sections (Varieties, Lemma 33.44.12). We conclude that $H^0(\mathcal{O}_{Z_{t+1}}) \subset H^0(\mathcal{O}_{Z_t})$ is a field (this is clear but also follows from Algebra, Lemma 10.36.18) and a finite extension of k. Thus we have extended the sequence. Since the process must stop, for example because $t \leq \sum (m_i/d - 1)$, this finishes the proof. $\square$

0C69 Lemma 55.9.10. In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$. Let $D = \sum (m_i/d) C_i$ as an effective Cartier divisor on X. Then [AW71, Lemma 2.6]

$$1 - g_C = d[\kappa : k](1 - g_D)$$

where g_C is the genus of C, g_D is the genus of D, and $\kappa = H^0(D, \mathcal{O}_D)$.

Proof. By Lemma 55.9.9 we see that κ is a field and a finite extension of k. Since also $H^0(C, \mathcal{O}_C) = K$ we see that the genus of C and D are defined (see Algebraic Curves, Definition 53.8.1) and we have $g_C = \dim_K H^1(C, \mathcal{O}_C)$ and $g_D = \dim_\kappa H^1(D, \mathcal{O}_D)$. By Derived Categories of Schemes, Lemma 36.32.2 we have

$$1 - g_C = \chi(C, \mathcal{O}_C) = \chi(X_k, \mathcal{O}_{X_k}) = \dim_k H^0(X_k, \mathcal{O}_{X_k}) - \dim_k H^1(X_k, \mathcal{O}_{X_k})$$

We claim that

$$\chi(X_k, \mathcal{O}_{X_k}) = d\chi(D, \mathcal{O}_D)$$

This will prove the lemma because

$$\chi(D, \mathcal{O}_D) = \dim_k H^0(D, \mathcal{O}_D) - \dim_k H^1(D, \mathcal{O}_D) = [\kappa : k](1 - g_D)$$

Observe that $X_k = dD$ as an effective Cartier divisor. To prove the claim we prove by induction on $1 \leq r \leq d$ that $\chi(rD, \mathcal{O}_{rD}) = r\chi(D, \mathcal{O}_D)$. The base case $r = 1$ is trivial. If $1 \leq r < d$, then we consider the short exact sequence

$$0 \to \mathcal{O}_X(rD)|_D \to \mathcal{O}_{(r+1)D} \to \mathcal{O}_{rD} \to 0$$

of Divisors, Lemma 31.14.3. By additivity of Euler characteristics (Varieties, Lemma 33.33.2) it suffices to prove that $\chi(D, \mathcal{O}_X(rD)|_D) = \chi(D, \mathcal{O}_D)$. This is true because $\mathcal{O}_X(rD)|_D$ is a torsion element of $\mathrm{Pic}(D)$ (Lemma 55.9.8) and because the degree of a line bundle is additive (Varieties, Lemma 33.44.7) hence zero for torsion invertible sheaves. $\square$

0C6A **Lemma 55.9.11.** In Situation 55.9.3 given a pair of indices i, j such that C_i and C_j are exceptional curves of the first kind and $C_i \cap C_j \neq \emptyset$, then $n = 2$, $m_1 = m_2 = 1$, $C_1 \cong \mathbf{P}^1_k$, $C_2 \cong \mathbf{P}^1_k$, C_1 and C_2 meet in a k-rational point, and C has genus 0.

Proof. Choose isomorphisms $C_i = \mathbf{P}^1_{\kappa_i}$ and $C_j = \mathbf{P}^1_{\kappa_j}$. The scheme $C_i \cap C_j$ is a nonempty effective Cartier divisor in both C_i and C_j. Hence

$$(C_i \cdot C_j) = \deg(C_i \cap C_j) \geq \max([\kappa_i : k], [\kappa_j : k])$$

The first equality was shown in the proof of Lemma 55.9.6. On the other hand, the self intersection $(C_i \cdot C_i)$ is equal to the degree of $\mathcal{O}_X(C_i)$ on C_i which is $-[\kappa_i : k]$ as C_i is an exceptional curve of the first kind. Similarly for C_j. By Lemma 55.9.7

$$0 \geq (C_i + C_j)^2 = -[\kappa_i : k] + 2(C_i \cdot C_j) - [\kappa_j : k]$$

This implies that $[\kappa_i : k] = \deg(C_i \cap C_j) = [\kappa_j : k]$ and that we have $(C_i + C_j)^2 = 0$. Looking at the lemma again we conclude that $n = 2$, $\{1, 2\} = \{i, j\}$, and $m_1 = m_2$. Moreover, the scheme theoretic intersection $C_i \cap C_j$ consists of a single point p with residue field κ and $\kappa_i \to \kappa \leftarrow \kappa_j$ are isomorphisms. Let $D = C_1 + C_2$ as effective Cartier divisor on X. Observe that D is the scheme theoretic union of C_1 and C_2 (Divisors, Lemma 31.13.10) hence we have a short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_p \to 0$$

by Morphisms, Lemma 29.4.6. Since we know the cohomology of $C_i \cong \mathbf{P}^1_\kappa$ (Cohomology of Schemes, Lemma 30.8.1) we conclude from the long exact cohomology sequence that $H^0(D, \mathcal{O}_D) = \kappa$ and $H^1(D, \mathcal{O}_D) = 0$. By Lemma 55.9.10 we conclude

$$1 - g_C = d[\kappa : k](1 - 0)$$

where $d = m_1 = m_2$. It follows that $g_C = 0$ and $d = m_1 = m_2 = 1$ and $\kappa = k$. $\square$

55.10. Uniqueness of the minimal model

0C9Y If the genus of the generic fibre is positive, then minimal models are unique (Lemma 55.10.1) and consequently have a suitable mapping property (Lemma 55.10.2).

0C6B **Lemma 55.10.1.** Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus > 0. There is a unique minimal model for C.

Proof. We have already proven the hard part of the lemma which is the existence of a minimal model (whose proof relies on resolution of surface singularities), see Proposition 55.8.6. To prove uniqueness, suppose that X and Y are two minimal models. By Resolution of Surfaces, Lemma 54.17.2 there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \rightarrow \ldots \rightarrow Y_1 \rightarrow Y_0 = Y$$

where each morphism is a blowup in a closed point. The exceptional fibre of the morphism $X_n \to X_{n-1}$ is an exceptional curve of the first kind E. We claim that E is contracted to a point under the morphism $X_n = Y_m \to Y$. If this is true, then $X_n \to Y$ factors through X_{n-1} by Resolution of Surfaces, Lemma 54.16.1. In this case the morphism $X_{n-1} \to Y$ is still a sequence of contractions of exceptional curves by Resolution of Surfaces, Lemma 54.17.1. Hence by induction on n we conclude. (The base case $n = 0$ means that there is a sequence of contractions $X = Y_m \to \ldots \to Y_1 \to Y_0 = Y$ ending with Y. However as X is a minimal model it contains no exceptional curves of the first kind, hence $m = 0$ and $X = Y$.)

Proof of the claim. We will show by induction on m that any exceptional curve of the first kind $E \subset Y_m$ is mapped to a point by the morphism $Y_m \to Y$. If $m = 0$ this is clear because Y is a minimal model. If $m > 0$, then either $Y_m \to Y_{m-1}$ contracts E (and we're done) or the exceptional fibre $E' \subset Y_m$ of $Y_m \to Y_{m-1}$ is a second exceptional curve of the first kind. Since both E and E' are irreducible components of the special fibre and since $g_C > 0$ by assumption, we conclude that $E \cap E' = \emptyset$ by Lemma 55.9.11. Then the image of E in Y_{m-1} is an exceptional curve of the first kind (this is clear because the morphism $Y_m \to Y_{m-1}$ is an isomorphism in a neighbourhood of E). By induction we see that $Y_{m-1} \to Y$ contracts this curve and the proof is complete. $\square$

0C9Z **Lemma 55.10.2.** Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus > 0. Let X be the minimal model for C (Lemma 55.10.1). Let Y be a regular proper model for C. Then there is a unique morphism of models $Y \to X$ which is a sequence of contractions of exceptional curves of the first kind.

Proof. The existence and properties of the morphism $X \to Y$ follows immediately from Lemma 55.8.5 and the uniqueness of the minimal model. The morphism $Y \to X$ is unique because $C \subset Y$ is scheme theoretically dense and X is separated (see Morphisms, Lemma 29.7.10). $\square$

0CA0 **Example 55.10.3.** If the genus of C is 0, then minimal models are indeed nonunique. Namely, consider the closed subscheme

$$X \subset \mathbf{P}^2_R$$

defined by $T_1T_2 - \pi T_0^2 = 0$. More precisely X is defined as $\text{Proj}(R[T_0, T_1, T_2]/(T_1T_2 - \pi T_0^2))$. Then the special fibre X_k is a union of two exceptional curves C_1, C_2 both isomorphic to $\mathbf{P}^1_k$ (exactly as in Lemma 55.9.11). Projection from $(0 : 1 : 0)$ defines a morphism $X \to \mathbf{P}^1_R$ contracting C_2 and inducing an isomorphism of C_1 with the special fiber of $\mathbf{P}^1_R$. Projection from $(0 : 0 : 1)$ defines a morphism $X \to \mathbf{P}^1_R$ contracting C_1 and inducing an isomorphism of C_2 with the special fiber of $\mathbf{P}^1_R$. More precisely, these morphisms correspond to the graded R-algebra maps

$$R[T_0, T_1] \longrightarrow R[T_0, T_1, T_2]/(T_1T_2 - \pi T_0^2) \longleftarrow R[T_0, T_2]$$

In Lemma 55.12.4 we will study this phenomenon.

55.11. A formula for the genus

0CA1 There is one more restriction on the combinatorial structure coming from a proper regular model.

0CA2 Lemma 55.11.1. In Situation 55.9.3 suppose we have an effective Cartier divisors $D, D' \subset X$ such that $D' = D + C_i$ for some $i \in \{1, \ldots, n\}$ and $D' \subset X_k$. Then

$$\chi(X_k, \mathcal{O}_{D'}) - \chi(X_k, \mathcal{O}_D) = \chi(X_k, \mathcal{O}_X(-D)|_{C_i}) = -(D \cdot C_i) + \chi(C_i, \mathcal{O}_{C_i})$$

Proof. The second equality follows from the definition of the bilinear form $(\cdot)$ in (55.9.6.1) and Lemma 55.9.6. To see the first equality we distinguish two cases. Namely, if $C_i \not\subset D$, then D' is the scheme theoretic union of D and C_i (by Divisors, Lemma 31.13.10) and we get a short exact sequence

$$0 \to \mathcal{O}_{D'} \to \mathcal{O}_D \times \mathcal{O}_{C_i} \to \mathcal{O}_{D \cap C_i} \to 0$$

by Morphisms, Lemma 29.4.6. Since we also have an exact sequence

$$0 \to \mathcal{O}_X(-D)|_{C_i} \to \mathcal{O}_{C_i} \to \mathcal{O}_{D \cap C_i} \to 0$$

(Divisors, Remark 31.14.11) we conclude that the claim holds by additivity of euler characteristics (Varieties, Lemma 33.33.2). On the other hand, if $C_i \subset D$ then we get an exact sequence

$$0 \to \mathcal{O}_X(-D)|_{C_i} \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

by Divisors, Lemma 31.14.3 and we immediately see the lemma holds. $\square$

0CA3 Lemma 55.11.2. In Situation 55.9.3 we have

$$g_C = 1 + \sum\nolimits_{i=1,\ldots,n} m_i \left([\kappa_i : k](g_i - 1) - \frac{1}{2}(C_i \cdot C_i) \right)$$

where $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$, g_i is the genus of C_i, and g_C is the genus of C.

Proof. Our basic tool will be Derived Categories of Schemes, Lemma 36.32.2 which shows that

$$1 - g_C = \chi(C, \mathcal{O}_C) = \chi(X_k, \mathcal{O}_{X_k})$$

Choose a sequence of effective Cartier divisors

$$X_k = D_m \supset D_{m-1} \supset \ldots \supset D_1 \supset D_0 = \emptyset$$

such that $D_{j+1} = D_j + C_{i_j}$ for each j. (It is clear that we can choose such a sequence by decreasing one nonzero multiplicity of D_{j+1} one step at a time.) Applying

Lemma 55.11.1 starting with $\chi(\mathcal{O}_{D_0}) = 0$ we get

$$\begin{aligned} 1 - g_C &= \chi(X_k, \mathcal{O}_{X_k}) \\ &= \sum\nolimits_j \Big(-(D_j \cdot C_{i_j}) + \chi(C_{i_j}, \mathcal{O}_{C_{i_j}})\Big) \\ &= -\sum\nolimits_j (C_{i_1} + C_{i_2} + \ldots + C_{i_{j-1}} \cdot C_{i_j}) + \sum\nolimits_j \chi(C_{i_j}, \mathcal{O}_{C_{i_j}}) \\ &= -\frac{1}{2}\sum\nolimits_{j \neq j'} (C_{i_{j'}} \cdot C_{i_j}) + \sum m_i \chi(C_i, \mathcal{O}_{C_i}) \\ &= \frac{1}{2}\sum m_i (C_i \cdot C_i) + \sum m_i \chi(C_i, \mathcal{O}_{C_i}) \end{aligned}$$

Perhaps the last equality deserves some explanation. Namely, since $\sum_j C_{i_j} = \sum m_i C_i$ we have $(\sum_j C_{i_j} \cdot \sum_j C_{i_j}) = 0$ by Lemma 55.9.7. Thus we see that

$$0 = \sum\nolimits_{j \neq j'} (C_{i_{j'}} \cdot C_{i_j}) + \sum m_i (C_i \cdot C_i)$$

by splitting this product into "nondiagonal" and "diagonal" terms. Note that κ_i is a field finite over k by Varieties, Lemma 33.26.2. Hence the genus of C_i is defined and we have $\chi(C_i, \mathcal{O}_{C_i}) = [\kappa_i : k](1 - g_i)$. Putting everything together and rearranging terms we get

$$g_C = -\frac{1}{2}\sum m_i (C_i \cdot C_i) + \sum m_i [\kappa_i : k](g_i - 1) + 1$$

which is what the lemma says too. □

0CA4 Lemma 55.11.3. In Situation 55.9.3 with $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and g_i the genus of C_i the data

$$n, m_i, (C_i \cdot C_j), [\kappa_i : k], g_i$$

is a numerical type of genus equal to the genus of C.

Proof. (In the proof of Lemma 55.11.2 we have seen that the quantities used in the statement of the lemma are well defined.) We have to verify the conditions (1) – (5) of Definition 55.3.1.

Condition (1) is immediate.

Condition (2). Symmetry of the matrix $(C_i \cdot C_j)$ follows from Equation (55.9.6.1) and Lemma 55.9.6. Nonnegativity of $(C_i \cdot C_j)$ for $i \neq j$ is part (1) of Lemma 55.9.7.

Condition (3) is part (3) of Lemma 55.9.7.

Condition (4) is part (2) of Lemma 55.9.7.

Condition (5) follows from the fact that $(C_i \cdot C_j)$ is the degree of an invertible module on C_i which is divisible by $[\kappa_i : k]$, see Varieties, Lemma 33.44.10.

The genus formula proved in Lemma 55.11.2 tells us that the numerical type has the genus as stated, see Definition 55.3.4. □

0CA5 Definition 55.11.4. In Situation 55.9.3 the numerical type associated to X is the numerical type described in Lemma 55.11.3.

Now we match minimality of the model with minimality of the type.

0CA6 Lemma 55.11.5. In Situation 55.9.3. The following are equivalent

(1) X is a minimal model, and

(2) the numerical type associated to X is minimal.

Proof. If the numerical type is minimal, then there is no i with $g_i = 0$ and $(C_i \cdot C_i) = -[\kappa_i : k]$, see Definition 55.3.12. Certainly, this implies that none of the curves C_i are exceptional curves of the first kind.

Conversely, suppose that the numerical type is not minimal. Then there exists an i such that $g_i = 0$ and $(C_i \cdot C_i) = -[\kappa_i : k]$. We claim this implies that C_i is an exceptional curve of the first kind. Namely, the invertible sheaf $\mathcal{O}_X(-C_i)|_{C_i}$ has degree $-(C_i \cdot C_i) = [\kappa_i : k]$ when C_i is viewed as a proper curve over k, hence has degree 1 when C_i is viewed as a proper curve over κ_i. Applying Algebraic Curves, Proposition 53.10.4 we conclude that $C_i \cong \mathbf{P}^1_{\kappa_i}$ as schemes over κ_i. Since the Picard group of $\mathbf{P}^1$ over a field is $\mathbf{Z}$, we see that the normal sheaf of C_i in X is isomorphic to $\mathcal{O}_{\mathbf{P}_{\kappa_i}}(-1)$ and the proof is complete. □

0CA7 Remark 55.11.6. Not every numerical type comes from a model for the silly reason that there exist numerical types whose genus is negative. There exist a minimal numerical types of positive genus which are not the numerical type associated to a model (over some dvr) of a smooth projective geometrically irreducible curve (over the fraction field of the dvr). A simple example is $n = 1$, $m_1 = 1$, $a_{11} = 0$, $w_1 = 6$, $g_1 = 1$. Namely, in this case the special fibre X_k would not be geometrically connected because it would live over an extension κ of k of degree 6. This is a contradiction with the fact that the generic fibre is geometrically connected (see More on Morphisms, Lemma 37.53.6). Similarly, $n = 2$, $m_1 = m_2 = 1$, $-a_{11} = -a_{22} = a_{12} = a_{21} = 6$, $w_1 = w_2 = 6$, $g_1 = g_2 = 1$ would be an example for the same reason (details omitted). But if the gcd of the w_i is 1 we do not have an example.

0CE8 Lemma 55.11.7. In Situation 55.9.3 assume C has a K-rational point. Then

(1) X_k has a k-rational point x which is a smooth point of X_k over k,
(2) if $x \in C_i$, then $H^0(C_i, \mathcal{O}_{C_i}) = k$ and $m_i = 1$, and
(3) $H^0(X_k, \mathcal{O}_{X_k}) = k$ and X_k has genus equal to the genus of C.

Proof. Since $X \to \text{Spec}(R)$ is proper, the K-rational point extends to a morphism $a : \text{Spec}(R) \to X$ by the valuative criterion of properness (Morphisms, Lemma 29.42.1). Let $x \in X$ be the image under a of the closed point of $\text{Spec}(R)$. Then a corresponds to an R-algebra homomorphism $\psi : \mathcal{O}_{X,x} \to R$ (see Schemes, Section 26.13). It follows that $\pi \notin \mathfrak{m}_x^2$ (since the image of π in R is not in $\mathfrak{m}_R^2$). Hence $\mathcal{O}_{X_k,x} = \mathcal{O}_{X,x}/\pi\mathcal{O}_{X,x}$ is regular (Algebra, Lemma 10.106.3). Then $X_k \to \text{Spec}(k)$ is smooth at x by Algebra, Lemma 10.140.5. It follows that x is contained in a unique irreducible component C_i of X_k, that $\mathcal{O}_{C_i,x} = \mathcal{O}_{X_k,x}$, and that $m_i = 1$. The fact that C_i has a k-rational point implies that the field $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ (Varieties, Lemma 33.26.2) is equal to k. This proves (1). We have $H^0(X_k, \mathcal{O}_{X_k}) = k$ because $H^0(X_k, \mathcal{O}_{X_k})$ is a field extension of k (Lemma 55.9.9) which maps to $H^0(C_i, \mathcal{O}_{C_i}) = k$. The genus equality follows from Lemma 55.9.10. □

0CE9 Lemma 55.11.8. In Situation 55.9.3 assume X is a minimal model, $\gcd(m_1, \ldots, m_n) = 1$, and $H^0((X_k)_{red}, \mathcal{O}) = k$. Then the map

$$H^1(X_k, \mathcal{O}_{X_k}) \to H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}})$$

is surjective and has a nontrivial kernel as soon as $(X_k)_{red} \neq X_k$.

Proof. By vanishing of cohomology in degrees ≥ 2 over X_k (Cohomology, Proposition 20.20.7) any surjection of abelian sheaves on X_k induces a surjection on H^1. Consider the sequence

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_m = X_k$$

of Lemma 55.9.9. Since the field maps $H^0(Z_j, \mathcal{O}_{Z_j}) \to H^0((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) = k$ are injective we conclude that $H^0(Z_j, \mathcal{O}_{Z_j}) = k$ for $j = 0, \ldots, m$. It follows that $H^0(X_k, \mathcal{O}_{X_k}) \to H^0(Z_{m-1}, \mathcal{O}_{Z_{m-1}})$ is surjective. Let $C = C_{i_m}$. Then $X_k = Z_{m-1} + C$. Let $\mathcal{L} = \mathcal{O}_X(-Z_{m-1})|_C$. Then $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module. As in the proof of Lemma 55.9.9 there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_{X_k} \to \mathcal{O}_{Z_{m-1}} \to 0$$

of coherent sheaves on X_k. We conclude that we get a short exact sequence

$$0 \to H^1(C, \mathcal{L}) \to H^1(X_k, \mathcal{O}_{X_k}) \to H^1(Z_{m-1}, \mathcal{O}_{Z_{m-1}}) \to 0$$

The degree of $\mathcal{L}$ on C over k is

$$(C \cdot -Z_{m-1}) = (C \cdot C - X_k) = (C \cdot C)$$

Set $\kappa = H^0(C, \mathcal{O}_C)$ and $w = [\kappa : k]$. By definition of the degree of an invertible sheaf we see that

$$\chi(C, \mathcal{L}) = \chi(C, \mathcal{O}_C) + (C \cdot C) = w(1 - g_C) + (C \cdot C)$$

where g_C is the genus of C. This expression is < 0 as X is minimal and hence C is not an exceptional curve of the first kind (see proof of Lemma 55.11.5). Thus $\dim_k H^1(C, \mathcal{L}) > 0$ which finishes the proof. $\square$

0CEA Lemma 55.11.9. In Situation 55.9.3 assume X_k has a k-rational point x which is a smooth point of $X_k \to \text{Spec}(k)$. Then

$$\dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \geq g_{top} + g_{geom}(X_k/k)$$

where g_{geom} is as in Algebraic Curves, Section 53.18 and g_{top} is the topological genus (Definition 55.3.11) of the numerical type associated to X_k (Definition 55.11.4).

Proof. We are going to prove the inequality

$$\dim_k H^1(D, \mathcal{O}_D) \geq g_{top}(D) + g_{geom}(D/k)$$

for all connected reduced effective Cartier divisors $D \subset (X_k)_{red}$ containing x by induction on the number of irreducible components of D. Here $g_{top}(D) = 1 - m + e$ where m is the number of irreducible components of D and e is the number of unordered pairs of components of D which meet.

Base case: D has one irreducible component. Then $D = C_i$ is the unique irreducible component containing x. In this case $\dim_k H^1(D, \mathcal{O}_D) = g_i$ and $g_{top}(D) = 0$. Since C_i has a k-rational smooth point it is geometrically integral (Varieties, Lemma 33.25.10). It follows that g_i is the genus of $C_{i,\overline{k}}$ (Algebraic Curves, Lemma 53.8.2). It also follows that $g_{geom}(D/k)$ is the genus of the normalization $C^\nu_{i,\overline{k}}$ of $C_{i,\overline{k}}$. Applying Algebraic Curves, Lemma 53.18.4 to the normalization morphism $C^\nu_{i,\overline{k}} \to C_{i,\overline{k}}$ we get

0CEB (55.11.9.1) $$\text{genus of } C_{i,\overline{k}} \geq \text{genus of } C^\nu_{i,\overline{k}}$$

Combining the above we conclude that $\dim_k H^1(D, \mathcal{O}_D) \geq g_{top}(D) + g_{geom}(D/k)$ in this case.

Induction step. Suppose we have D with more than 1 irreducible component. Then we can write $D = C_i + D'$ where $x \in D'$ and D' is still connected. This is an exercise in graph theory we leave to the reader (hint: let C_i be the component of D which is farthest from x). We compute how the invariants change. As $x \in D'$ we have $H^0(D, \mathcal{O}_D) = H^0(D', \mathcal{O}_{D'}) = k$. Looking at the short exact sequence of sheaves

$$0 \to \mathcal{O}_D \to \mathcal{O}_{C_i} \oplus \mathcal{O}_{D'} \to \mathcal{O}_{C_i \cap D'} \to 0$$

(Morphisms, Lemma 29.4.6) and using additivity of euler characteristics we find

$$\begin{aligned} \dim_k H^1(D, \mathcal{O}_D) - \dim_k H^1(D', \mathcal{O}_{D'}) &= -\chi(\mathcal{O}_{C_i}) + \chi(\mathcal{O}_{C_i \cap D'}) \\ &= w_i(g_i - 1) + \sum\nolimits_{C_j \subset D'} a_{ij} \end{aligned}$$

Here as in Lemma 55.11.3 we set $w_i = [\kappa_i : k]$, $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$, g_i is the genus of C_i, and $a_{ij} = (C_i \cdot C_j)$. We have

$$g_{top}(D) - g_{top}(D') = -1 + \sum\nolimits_{C_j \subset D' \text{ meeting } C_i} 1$$

We have

$$g_{geom}(D/k) - g_{geom}(D'/k) = g_{geom}(C_i/k)$$

by Algebraic Curves, Lemma 53.18.1. Combining these with our induction hypothesis, we conclude that it suffices to show that

$$w_i g_i - g_{geom}(C_i/k) + \sum\nolimits_{C_j \subset D' \text{ meets } C_i} (a_{ij} - 1) - (w_i - 1)$$

is nonnegative. In fact, we have

0CEC (55.11.9.2) $$w_i g_i \geq [\kappa_i : k]_s g_i \geq g_{geom}(C_i/k)$$

The second inequality by Algebraic Curves, Lemma 53.18.5. On the other hand, since w_i divides a_{ij} (Varieties, Lemma 33.44.10) it is clear that

0CED (55.11.9.3) $$\sum\nolimits_{C_j \subset D' \text{ meets } C_i} (a_{ij} - 1) - (w_i - 1) \geq 0$$

because there is at least one $C_j \subset D'$ which meets C_i. □

0CEE Lemma 55.11.10. If equality holds in Lemma 55.11.9 then

(1) the unique irreducible component of X_k containing x is a smooth projective geometrically irreducible curve over k,
(2) if $C \subset X_k$ is another irreducible component, then $\kappa = H^0(C, \mathcal{O}_C)$ is a finite separable extension of k, C has a κ-rational point, and C is smooth over κ

Proof. Looking over the proof of Lemma 55.11.9 we see that in order to get equality, the inequalities (55.11.9.1), (55.11.9.2), and (55.11.9.3) have to be equalities.

Let C_i be the irreducible component containing x. Equality in (55.11.9.1) shows via Algebraic Curves, Lemma 53.18.4 that $C^\nu_{i,\overline{k}} \to C_{i,\overline{k}}$ is an isomorphism. Hence $C_{i,\overline{k}}$ is smooth and part (1) holds.

Next, let $C_i \subset X_k$ be another irreducible component. Then we may assume we have $D = D' + C_i$ as in the induction step in the proof of Lemma 55.11.9. Equality in (55.11.9.2) immediately implies that κ_i/k is finite separable. Equality in (55.11.9.3) implies either $a_{ij} = 1$ for some j or that there is a unique $C_j \subset D'$ meeting C_i and $a_{ij} = w_i$. In both cases we find that C_i has a κ_i-rational point c and $c = C_i \cap C_j$ scheme theoretically. Since $\mathcal{O}_{X,c}$ is a regular local ring, this implies that the local equations of C_i and C_j form a regular system of parameters in the local ring $\mathcal{O}_{X,c}$. Then $\mathcal{O}_{C_i,c}$ is regular by (Algebra, Lemma 10.106.3). We conclude that $C_i \to \text{Spec}(\kappa_i)$ is smooth at c (Algebra, Lemma 10.140.5). It follows that C_i is geometrically integral over κ_i (Varieties, Lemma 33.25.10). To finish we have to show that C_i is smooth over κ_i. Observe that

$$C_{i,\overline{k}} = C_i \times_{\text{Spec}(k)} \text{Spec}(\overline{k}) = \coprod\nolimits_{\kappa_i \to \overline{k}} C_i \times_{\text{Spec}(\kappa_i)} \text{Spec}(\overline{k})$$

where there are $[\kappa_i : k]$-summands. Thus if C_i is not smooth over κ_i, then each of these curves is not smooth, then these curves are not normal and the normalization morphism drops the genus (Algebraic Curves, Lemma 53.18.4) which is disallowed because it would drop the geometric genus of C_i/k contradicting $[\kappa_i : k]g_i = g_{geom}(C_i/k)$. □

55.12. Blowing down exceptional curves

0CEF The following lemma tells us what happens with the intersection numbers when we contract an exceptional curve of the first kind in a regular proper model. We put this here mostly to compare with the numerical contractions introduced in Lemma 55.3.9. We will compare the geometric and numerical contractions in Remark 55.12.3.

0C6C **Lemma 55.12.1.** In Situation 55.9.3 assume that C_n is an exceptional curve of the first kind. Let $f : X \to X'$ be the contraction of C_n. Let $C'_i = f(C_i)$. Write $X'_k = \sum m'_i C'_i$. Then X', C'_i, $i = 1, \ldots, n' = n - 1$, and $m'_i = m_i$ is as in Situation 55.9.3 and we have

(1) for $i, j < n$ we have $(C'_i \cdot C'_j) = (C_i \cdot C_j) - (C_i \cdot C_n)(C_j \cdot C_n)/(C_n \cdot C_n)$,
(2) for $i < n$ if $C_i \cap C_n \neq \emptyset$, then there are maps $\kappa_i \leftarrow \kappa'_i \to \kappa_n$.

Here $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and $\kappa'_i = H^0(C'_i, \mathcal{O}_{C'_i})$.

Proof. By Resolution of Surfaces, Lemma 54.16.8 we can contract C_n by a morphism $f : X \to X'$ such that X' is regular and is projective over R. Thus we see that X' is as in Situation 55.9.3. Let $x \in X'$ be the image of C_n. Since f defines an isomorphism $X \setminus C_n \to X' \setminus \{x\}$ it is clear that $m'_i = m_i$ for $i < n$.

Part (2) of the lemma is immediately clear from the existence of the morphisms $C_i \to C'_i$ and $C_n \to x \to C'_i$.

By Divisors, Lemma 31.32.11 the pullback $f^{-1}C'_i$ is defined. By Divisors, Lemma 31.15.11 we see that $f^{-1}C'_i = C_i + e_i C_n$ for some $e_i \geq 0$. Since $\mathcal{O}_X(C_i + e_i C_n) = \mathcal{O}_X(f^{-1}C'_i) = f^*\mathcal{O}_{X'}(C'_i)$ (Divisors, Lemma 31.14.5) and since the pullback of an invertible sheaf restricts to the trivial invertible sheaf on C_n we see that

$$0 = \deg_{C_n}(\mathcal{O}_X(C_i + e_i C_n)) = (C_i + e_i C_n \cdot C_n) = (C_i \cdot C_n) + e_i(C_n \cdot C_n)$$

As $f_j = f|_{C_j} : C_j \to C_j$ is a proper birational morphism of proper curves over k, we see that $\deg_{C'_j}(\mathcal{O}_{X'}(C'_i)|_{C'_j})$ is the same as $\deg_{C_j}(f_j^*\mathcal{O}_{X'}(C'_i)|_{C'_j})$ (Varieties, Lemma 33.44.4). Looking at the commutative diagram

$$\begin{tikzcd} C_j \arrow[r] \arrow[d, "f_j"'] & X \arrow[d, "f"] \\ C'_j \arrow[r] & X' \end{tikzcd}$$

and using Divisors, Lemma 31.14.5 we see that

$$(C'_i \cdot C'_j) = \deg_{C'_j}(\mathcal{O}_{X'}(C'_i)|_{C'_j}) = \deg_{C_j}(\mathcal{O}_X(C_i + e_iC_n)) = (C_i + e_iC_n \cdot C_j)$$

Plugging in the formula for e_i found above we see that (1) holds. □

0CA8 Remark 55.12.2. In the situation of Lemma 55.12.1 we can also say exactly how the genus g_i of C_i and the genus g'_i of C'_i are related. The formula is

$$g'_i = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{(C_i \cdot C_n)^2 - w_n(C_i \cdot C_n)}{2w'_iw_n}$$

where $w_i = [\kappa_i : k]$, $w_n = [\kappa_n : k]$, and $w'_i = [\kappa'_i : k]$. To prove this we consider the short exact sequence

$$0 \to \mathcal{O}_{X'}(-C'_i) \to \mathcal{O}_{X'} \to \mathcal{O}_{C'_i} \to 0$$

and its pullback to X which reads

$$0 \to \mathcal{O}_X(-C'_i - e_iC_n) \to \mathcal{O}_X \to \mathcal{O}_{C_i+e_iC_n} \to 0$$

with e_i as in the proof of Lemma 55.12.1. Since $Rf_*f^*\mathcal{L} = \mathcal{L}$ for any invertible module $\mathcal{L}$ on X' (details omitted), we conclude that

$$Rf_*\mathcal{O}_{C_i+e_iC_n} = \mathcal{O}_{C'_i}$$

as complexes of coherent sheaves on X'_k. Hence both sides have the same Euler characteristic and this agrees with the Euler characteristic of $\mathcal{O}_{C_i+e_iC_n}$ on X_k. Using the exact sequence

$$0 \to \mathcal{O}_{C_i+e_iC_n} \to \mathcal{O}_{C_i} \oplus \mathcal{O}_{e_iC_n} \to \mathcal{O}_{C_i\cap e_iC_n} \to 0$$

and further filtering $\mathcal{O}_{e_iC_n}$ (details omitted) we find

$$\chi(\mathcal{O}_{C'_i}) = \chi(\mathcal{O}_{C_i}) - \binom{e_i+1}{2}(C_n \cdot C_n) - e_i(C_i \cdot C_n)$$

Since $e_i = -(C_i \cdot C_n)/(C_n \cdot C_n)$ and $(C_n \cdot C_n) = -w_n$ this leads to the formula stated at the start of this remark. If we ever need this we will formulate this as a lemma and provide a detailed proof.

0CA9 Remark 55.12.3. Let $f : X \to X'$ be as in Lemma 55.12.1. Let n, m_i, a_{ij}, w_i, g_i be the numerical type associated to X and let $n', m'_i, a'_{ij}, w'_i, g'_i$ be the numerical type associated to X'. It is clear from Lemma 55.12.1 and Remark 55.12.2 that this agrees with the contraction of numerical types in Lemma 55.3.9 except for the value of w'_i. In the geometric situation w'_i is some positive integer dividing both w_i and w_n. In the numerical case we chose w'_i to be the largest possible integer dividing w_i such that g'_i (as given by the formula) is an integer. This works well in the numerical setting in that it helps compare the Picard groups of the numerical types, see Lemma 55.4.4 (although only injectivity is every used in the following and this injectivity works as well for smaller w'_i).

0CDA **Lemma 55.12.4.** Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus 0. If there is more than one minimal model for C, then the special fibre of every minimal model is isomorphic to $\mathbf{P}^1_k$.

This lemma can be improved to say that the birational transformation between two nonisomorphic minimal models can be factored as a sequence of elementary transformations as in Example 55.10.3. If we ever need this, we will precisely formulate and prove this here.

Proof. Let X be some minimal model of C. The numerical type associated to X has genus 0 and is minimal (Definition 55.11.4 and Lemma 55.11.5). Hence by Lemma 55.6.1 we see that X_k is reduced, irreducible, has $H^0(X_k, \mathcal{O}_{X_k}) = k$, and has genus 0. Let Y be a second minimal model for C which is not isomorphic to X. By Resolution of Surfaces, Lemma 54.17.2 there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \to \ldots \to Y_1 \to Y_0 = Y$$

where each morphism is a blowup in a closed point. We will prove the lemma by induction on m. The base case is $m = 0$; it is true in this case because we assumed that Y is minimal hence this would mean $n = 0$, but X is not isomorphic to Y, so this does not happen, i.e., there is nothing to check.

Before we continue, note that $n + 1 = m + 1$ is equal to the number of irreducible components of the special fibre of $X_n = Y_m$ because both X_k and Y_k are irreducible. Another observation we will use below is that if $X' \to X''$ is a morphism of regular proper models for C, then $X' \to X''$ is an isomorphism over an open set of X'' whose complement is a finite set of closed points of the special fibre of X'', see Varieties, Lemma 33.17.3. In fact, any such $X' \to X''$ is a sequence of blowing ups in closed points (Resolution of Surfaces, Lemma 54.17.1) and the number of blowups is the difference in the number of irreducible components of the special fibres of X' and X''.

Let $E_i \subset Y_i$, $m \geq i \geq 1$ be the curve which is contracted by the morphism $Y_i \to Y_{i-1}$. Let i be the biggest index such that E_i has multiplicity > 1 in the special fibre of Y_i. Then the further blowups $Y_m \to \ldots \to Y_{i+1} \to Y_i$ are isomorphisms over E_i since otherwise E_j for some $j > i$ would have multiplicity > 1. Let $E \subset Y_m$ be the inverse image of E_i. By what we just said $E \subset Y_m$ is an exceptional curve of the first kind. Let $Y_m \to Y'$ be the contraction of E (which exists by Resolution of Surfaces, Lemma 54.16.9). The morphism $Y_m \to X$ has to contract E, because X_k is reduced. Hence there are morphisms $Y' \to Y$ and $Y' \to X$ (by Resolution of Surfaces, Lemma 54.16.1) which are compositions of at most $n - 1 = m - 1$ contractions of exceptional curves (see discussion above). We win by induction on m. Upshot: we may assume that the special fibres of all of the curves X_i and Y_i are reduced.

Since the fibres of X_i and Y_i are reduced, it has to be the case that the blowups $X_i \to X_{i-1}$ and $Y_i \to Y_{i-1}$ happen in closed points which are regular points of the special fibres. Namely, if X'' is a regular model for C and if $x \in X''$ is a closed point of the special fibre, and $\pi \in \mathfrak{m}_x^2$, then the exceptional fibre E of the blowup $X' \to X''$ at x has multiplicity at least 2 in the special fibre of X' (local computation omitted). Hence $\mathcal{O}_{X''_k, x} = \mathcal{O}_{X'', x}/\pi$ is regular (Algebra, Lemma 10.106.3) as claimed. In particular x is a Cartier divisor on the unique irreducible component Z' of X''_k it lies on (Varieties, Lemma 33.43.8). It follows that the strict transform $Z \subset X'$ of Z' maps isomorphically to Z' (use Divisors, Lemmas 31.33.2 and 31.32.7). In other words, if an irreducible component Z of X_i is not contracted under the map $X_i \to X_j$ $(i > j)$ then it maps isomorphically to its image.

Now we are ready to prove the lemma. Let $E \subset Y_m$ be the exceptional curve of the first kind which is contracted by the morphism $Y_m \to Y_{m-1}$. If E is contracted by the morphism $Y_m = X_n \to X$, then there is a factorization $Y_{m-1} \to X$ (Resolution of Surfaces, Lemma 54.16.1) and moreover $Y_{m-1} \to X$ is a sequence of blowups in closed points (Resolution of Surfaces, Lemma 54.17.1). In this case we lower m and we win by induction. Finally, assume that E is not contracted by the morphism $Y_m \to X$. Then $E \to X_k$ is surjective as X_k is irreducible and by the above this means it is an isomorphism. Hence X_k is isomorphic to a projective line as desired. $\square$

55.13. Picard groups of models

0CAA Assume $R, K, k, \pi, C, X, n, C_1, \ldots, C_n, m_1, \ldots, m_n$ are as in Situation 55.9.3. In Lemma 55.9.5 we found an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}^{\oplus n} \to \text{Pic}(X) \to \text{Pic}(C) \to 0$$

We want to use this sequence to study the ℓ-torsion in the Picard groups for suitable primes ℓ.

0CAB **Lemma 55.13.1.** In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$. If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which

(1) restricts to the trivial invertible module on C, and
(2) has degree 0 on each C_i,

then $\mathcal{L}^{\otimes d} \cong \mathcal{O}_X$.

Proof. By Lemma 55.9.5 we have $\mathcal{L} \cong \mathcal{O}_X(\sum a_i C_i)$ for some $a_i \in \mathbf{Z}$. The degree of $\mathcal{L}|_{C_j}$ is $\sum_j a_j (C_i \cdot C_j)$. In particular $(\sum a_i C_i \cdot \sum a_i C_i) = 0$. Hence we see from Lemma 55.9.7 that $(a_1, \ldots, a_n) = q(m_1, \ldots, m_n)$ for some $q \in \mathbf{Q}$. Thus $\mathcal{L} = \mathcal{O}_X(lD)$ for some $l \in \mathbf{Z}$ where $D = \sum (m_i/d) C_i$ is as in Lemma 55.9.8 and we conclude. $\square$

0CAC **Lemma 55.13.2.** In Situation 55.9.3 let T be the numerical type associated to X. There exists a canonical map

$$\text{Pic}(C) \to \text{Pic}(T)$$

whose kernel is exactly those invertible modules on C which are the restriction of invertible modules $\mathcal{L}$ on X with $\deg_{C_i}(\mathcal{L}|_{C_i}) = 0$ for $i = 1, \ldots, n$.

Proof. Recall that $w_i = [\kappa_i : k]$ where $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and recall that the degree of any invertible module on C_i is divisible by w_i (Varieties, Lemma 33.44.10). Thus we can consider the map

$$\frac{\deg}{w} : \text{Pic}(X) \to \mathbf{Z}^{\oplus n}, \quad \mathcal{L} \mapsto (\frac{\deg(\mathcal{L}|_{C_1})}{w_1}, \ldots, \frac{\deg(\mathcal{L}|_{C_n})}{w_n})$$

The image of $\mathcal{O}_X(C_j)$ under this map is

$$((C_j \cdot C_1)/w_1, \ldots, (C_j \cdot C_n)/w_n) = (a_{1j}/w_1, \ldots, a_{nj}/w_n)$$

which is exactly the image of the jth basis vector under the map $(a_{ij}/w_i) : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n}$ defining the Picard group of T, see Definition 55.4.1. Thus the canonical map of the lemma comes from the commutative diagram

$$\begin{tikzcd} \mathbf{Z}^{\oplus n} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "\frac{\deg}{w}"] & \mathrm{Pic}(C) \arrow[r] \arrow[d] & 0 \\ \mathbf{Z}^{\oplus n} \arrow[r, "(a_{ij}/w_i)"] & \mathbf{Z}^{\oplus n} \arrow[r] & \mathrm{Pic}(T) \arrow[r] & 0 \end{tikzcd}$$

with exact rows (top row by Lemma 55.9.5). The description of the kernel is clear. $\square$

0CAD Lemma 55.13.3. In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$ and let T be the numerical type associated to X. Let $h \geq 1$ be an integer prime to d. There exists an exact sequence

$$0 \to \mathrm{Pic}(X)[h] \to \mathrm{Pic}(C)[h] \to \mathrm{Pic}(T)[h]$$

Proof. Taking h-torsion in the exact sequence of Lemma 55.9.5 we obtain the exactness of $0 \to \mathrm{Pic}(X)[h] \to \mathrm{Pic}(C)[h]$ because h is prime to d. Using the map of Lemma 55.13.2 we get a map $\mathrm{Pic}(C)[h] \to \mathrm{Pic}(T)[h]$ which annihilates elements of $\mathrm{Pic}(X)[h]$. Conversely, if $\xi \in \mathrm{Pic}(C)[h]$ maps to zero in $\mathrm{Pic}(T)[h]$, then we can find an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\deg(\mathcal{L}|_{C_i}) = 0$ for all i whose restriction to C is ξ. Then $\mathcal{L}^{\otimes h}$ is d-torsion by Lemma 55.13.1. Let d' be an integer such that $dd' \equiv 1 \bmod h$. Such an integer exists because h and d are coprime. Then $\mathcal{L}^{\otimes dd'}$ is an h-torsion invertible sheaf on X whose restriction to C is ξ. $\square$

0CAE Lemma 55.13.4. In Situation 55.9.3 let h be an integer prime to the characteristic of k. Then the map

$$\mathrm{Pic}(X)[h] \longrightarrow \mathrm{Pic}((X_k)_{red})[h]$$

is injective.

Proof. Observe that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)$ is a finite order thickening of $(X_k)_{red}$ (this follows for example from Cohomology of Schemes, Lemma 30.10.2). Thus the canonical map $\mathrm{Pic}(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)) \to \mathrm{Pic}((X_k)_{red})$ identifies h torsion by More on Morphisms, Lemma 37.4.2 and our assumption on h. Thus if $\mathcal{L}$ is an h-torsion invertible sheaf on X which restricts to the trivial sheaf on $(X_k)_{red}$ then $\mathcal{L}$ restricts to the trivial sheaf on $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)$ for all n. We find

$$\begin{aligned} H^0(X, \mathcal{L})^\wedge &= \lim H^0(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n), \mathcal{L}|_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}) \\ &\cong \lim H^0(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n), \mathcal{O}_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}) \\ &= R^\wedge \end{aligned}$$

using the theorem on formal functions (Cohomology of Schemes, Theorem 30.20.5) for the first and last equality and for example More on Algebra, Lemma 15.100.5 for the middle isomorphism. Since $H^0(X, \mathcal{L})$ is a finite R-module and R is a discrete valuation ring, this means that $H^0(X, \mathcal{L})$ is free of rank 1 as an R-module. Let $s \in H^0(X, \mathcal{L})$ be a basis element. Then tracing back through the isomorphisms above we see that $s|_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}$ is a trivialization for all n. Since the vanishing locus of s is closed in X and $X \to \mathrm{Spec}(R)$ is proper we conclude that the vanishing locus of s is empty as desired. $\square$

55.14. Semistable reduction

0CDB In this section we carefully define what we mean by semistable reduction.

0CDC Example 55.14.1. Let R be a discrete valuation ring with uniformizer π. Given $n \geq 0$, consider the ring map

$$R \longrightarrow A = R[x, y]/(xy - \pi^n)$$

Set $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. If $n = 0$, then $X \to S$ is smooth. For all n the morphism $X \to S$ is at-worst-nodal of relative dimension 1 as defined in Algebraic Curves, Section 53.20. If $n = 1$, then X is regular, but if $n > 1$, then X is not regular as (x, y) no longer generate the maximal ideal $\mathfrak{m} = (\pi, x, y)$. To ameliorate the situation in case $n > 1$ we consider the blowup $b : X' \to X$ of X in $\mathfrak{m}$. See Divisors, Section 31.32. By construction X' is covered by three affine pieces corresponding to the blowup algebras $A[\frac{\mathfrak{m}}{\pi}]$, $A[\frac{\mathfrak{m}}{x}]$, and $A[\frac{\mathfrak{m}}{y}]$.

The algebra $A[\frac{\mathfrak{m}}{\pi}]$ has generators $x' = x/\pi$ and $y' = y/\pi$ and $x'y' = \pi^{n-2}$. Thus this part of X' is the spectrum of $R[x', y'](x'y' - \pi^{n-2})$.

The algebra $A[\frac{\mathfrak{m}}{x}]$ has generators x, $u = \pi/x$ subject to the relation $xu - \pi$. Note that this ring contains $y/x = \pi^n/x^2 = u^2\pi^{n-2}$. Thus this part of X' is regular.

By symmetry the case of the algebra $A[\frac{\mathfrak{m}}{y}]$ is the same as the case of $A[\frac{\mathfrak{m}}{x}]$.

Thus we see that $X' \to S$ is at-worst-nodal of relative dimension 1 and that X' is regular, except for one point which has an affine open neighbourhood exactly as above but with n replaced by $n - 2$. Using induction on n we conclude that there is a sequence of blowing ups in closed points

$$X_{\lfloor n/2 \rfloor} \to \ldots \to X_1 \to X_0 = X$$

such that $X_{\lfloor n/2 \rfloor} \to S$ is at-worst-nodal of relative dimension 1 and $X_{\lfloor n/2 \rfloor}$ is regular.

0CDD Lemma 55.14.2. Let R be a discrete valuation ring. Let X be a scheme which is at-worst-nodal of relative dimension 1 over R. Let $x \in X$ be a point of the special fibre of X over R. Then there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[dl] \\
\mathrm{Spec}(R) & \mathrm{Spec}(R') \arrow[l] &
\end{tikzcd}$$

where $R \subset R'$ is an étale extension of discrete valuation rings, the morphism $U \to X$ is étale, the morphism $U \to \mathrm{Spec}(A)$ is étale, there is a point $x' \in U$ mapping to x, and

$$A = R'[u, v]/(uv) \quad \text{or} \quad A = R'[u, v]/(uv - \pi^n)$$

where $n \geq 0$ and $\pi \in R'$ is a uniformizer.

Proof. We have already proved this lemma in much greater generality, see Algebraic Curves, Lemma 53.20.12. All we have to do here is to translate the statement given there into the statement given above.

First, if the morphism $X \to \mathrm{Spec}(R)$ is smooth at x, then we can find an étale morphism $U \to \mathbf{A}^1_R = \mathrm{Spec}(R[u])$ for some affine open neighbourhood $U \subset X$ of x.

This is Morphisms, Lemma 29.36.20. After replacing the coordinate u by $u+1$ if necessary, we may assume that x maps to a point in the standard open $D(u) \subset \mathbf{A}^1_R$. Then $D(u) = \mathrm{Spec}(A)$ with $A = R[u,v]/(uv-1)$ and we see that the result is true in this case.

Next, assume that x is a singular point of the fibre. Then we may apply Algebraic Curves, Lemma 53.20.12 to get a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r] \arrow[dr] & W \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \arrow[d] \\ \mathrm{Spec}(R) & & V \arrow[ll] \arrow[r] & \mathrm{Spec}(\mathbf{Z}[a]) \end{tikzcd}$$

with all the properties mentioned in the statement of the cited lemma. Let $x' \in U$ be the point mapping to x promised by the lemma. First we shrink V to an affine neighbourhood of the image of x'. Say $V = \mathrm{Spec}(R')$. Then $R \to R'$ is étale. Since R is a discrete valuation ring, we see that R' is a finite product of quasi-local Dedekind domains (use More on Algebra, Lemma 15.44.4). Hence (for example using prime avoidance) we find a standard open $D(f) \subset V = \mathrm{Spec}(R')$ containing the image of x' such that R'_f is a discrete valuation ring. Replacing R' by R'_f we reach the situation where $V = \mathrm{Spec}(R')$ with $R \subset R'$ an étale extension of discrete valuation rings (extensions of discrete valuation rings are defined in More on Algebra, Definition 15.111.1).

The morphism $V \to \mathrm{Spec}(\mathbf{Z}[a])$ is determined by the image h of a in R'. Then $W = \mathrm{Spec}(R'[u,v]/(uv-h))$. Thus the lemma holds with $A = R'[u,v]/(uv-h)$. If $h = 0$ then we clearly obtain the first case mentioned in the lemma. If $h \neq 0$ then we may write $h = \epsilon\pi^n$ for some $n \geq 0$ where ϵ is a unit of R'. Changing coordinates $u_{new} = \epsilon u$ and $v_{new} = v$ we obtain the second isomorphism type of A listed in the lemma. □

0CDE Lemma 55.14.3. Let R be a discrete valuation ring. Let X be a quasi-compact scheme which is at-worst-nodal of relative dimension 1 with smooth generic fibre over R. Then there exists $m \geq 0$ and a sequence

$$X_m \to \ldots \to X_1 \to X_0 = X$$

such that

(1) $X_{i+1} \to X_i$ is the blowing up of a closed point x_i where X_i is singular,
(2) $X_i \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1,
(3) X_m is regular.

A slightly stronger statement (also true) would be that no matter how you blow up in singular points you eventually end up with a resolution and all the intermediate blowups are at-worst-nodal of relative dimension 1 over R.

Proof. Since X is quasi-compact we see that the special fibre X_k is quasi-compact. Since the singularities of X_k are at-worst-nodal, we see that X_k has a finite number of nodes and is otherwise smooth over k. As $X \to \mathrm{Spec}(R)$ is flat with smooth generic fibre it follows that X is smooth over R except at the finite number of nodes of X_k (use Morphisms, Lemma 29.34.14). It follows that X is regular at every point except for possibly the nodes of its special fibre (see Algebra, Lemma 10.163.10). Let $x \in X$ be such a node. Choose a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[dl] \\ \mathrm{Spec}(R) & \mathrm{Spec}(R') \arrow[l] \end{tikzcd}$$

as in Lemma 55.14.2. Observe that the case $A = R'[u, v]/(uv)$ cannot occur, as this would mean that the generic fibre of X/R is singular (tiny detail omitted). Thus $A = R'[u, v]/(uv - \pi^n)$ for some $n \geq 0$. Since x is a singular point, we have $n \geq 2$, see discussion in Example 55.14.1.

After shrinking U we may assume there is a unique point $u \in U$ mapping to x. Let $w \in \mathrm{Spec}(A)$ be the image of u. We may also assume that u is the unique point of U mapping to w. Since the two horizontal arrows are étale we see that u, viewed as a closed subscheme of U, is the scheme theoretic inverse image of $x \in X$ and the scheme theoretic inverse image of $w \in \mathrm{Spec}(A)$. Since blowing up commutes with flat base change (Divisors, Lemma 31.32.3) we find a commutative diagram

$$\begin{tikzcd} X' \arrow[d] & U' \arrow[l] \arrow[r] \arrow[d] & W' \arrow[d] \\ X & U \arrow[l] \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

with cartesian squares where the vertical arrows are the blowing up of x, u, w in $X, U, \mathrm{Spec}(A)$. The scheme W' was described in Example 55.14.1. We saw there that W' at-worst-nodal of relative dimension 1 over R'. Thus W' is at-worst-nodal of relative dimension 1 over R (Algebraic Curves, Lemma 53.20.7). Hence U' is at-worst-nodal of relative dimension 1 over R (see Algebraic Curves, Lemma 53.20.8). Since $X' \to X$ is an isomorphism over the complement of x, we conclude the same thing is true of X'/R (by Algebraic Curves, Lemma 53.20.8 again).

Finally, we need to argue that after doing a finite number of these blowups we arrive at a regular model X_m. This is rather clear because the "invariant" n decreases by 2 under the blowup described above, see computation in Example 55.14.1. However, as we want to avoid precisely defining this invariant and establishing its properties, we in stead argue as follows. If $n = 2$, then W' is regular and hence X' is regular at all points lying over x and we have decreased the number of singular points of X by 1. If $n > 2$, then the unique singular point w' of W' lying over w has $\kappa(w) = \kappa(w')$. Hence U' has a unique singular point u' lying over u with $\kappa(u) = \kappa(u')$. Clearly, this implies that X' has a unique singular point x' lying over x, namely the image of u'. Thus we can argue exactly as above that we get a commutative diagram

$$\begin{tikzcd} X'' \arrow[d] & U'' \arrow[l] \arrow[r] \arrow[d] & W'' \arrow[d] \\ X' & U' \arrow[l] \arrow[r] & W' \end{tikzcd}$$

with cartesian squares where the vertical arrows are the blowing up of x', u', w' in X', U', W'. Continuing like this we get a compatible sequence of blowups which stops after $\lfloor n/2 \rfloor$ steps. At the completion of this process the scheme $X^{(\lfloor n/2 \rfloor)}$ will have one fewer singular point than X. Induction on the number of singular points completes the proof. $\square$

0CDF Lemma 55.14.4. Let R be a discrete valuation ring with fraction field K and residue field k. Assume $X \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1 over R. Let $X \to X'$ be the contraction of an exceptional curve $E \subset X$ of the first kind. Then X' is at-worst-nodal of relative dimension 1 over R.

Proof. Namely, let $x' \in X'$ be the image of E. Then the only issue is to see that $X' \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1 in a neighbourhood of x'. The closed fibre of $X \to \mathrm{Spec}(R)$ is reduced, hence $\pi \in R$ vanishes to order 1 on E. This immediately implies that π viewed as an element of $\mathfrak{m}_{x'} \subset \mathcal{O}_{X',x'}$ but is not in $\mathfrak{m}_{x'}^2$. Since $\mathcal{O}_{X',x'}$ is regular of dimension 2 (by definition of contractions in Resolution of Surfaces, Section 54.16), this implies that $\mathcal{O}_{X'_k,x'}$ is regular of dimension 1 (Algebra, Lemma 10.106.3). On the other hand, the curve E has to meet at least one other component, say C of the closed fibre X_k. Say $x \in E \cap C$. Then x is a node of the special fibre X_k and hence $\kappa(x)/k$ is finite separable, see Algebraic Curves, Lemma 53.19.7. Since $x \mapsto x'$ we conclude that $\kappa(x')/k$ is finite separable. By Algebra, Lemma 10.140.5 we conclude that $X'_k \to \mathrm{Spec}(k)$ is smooth in an open neighbourhood of x'. Combined with flatness, this proves that $X' \to \mathrm{Spec}(R)$ is smooth in a neighbourhood of x' (Morphisms, Lemma 29.34.14). This finishes the proof as a smooth morphism of relative dimension 1 is at-worst-nodal of relative dimension 1 (Algebraic Curves, Lemma 53.20.3). $\square$

0CDG Lemma 55.14.5. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. The following are equivalent

(1) there exists a proper model of C which is at-worst-nodal of relative dimension 1 over R,
(2) there exists a minimal model of C which is at-worst-nodal of relative dimension 1 over R, and
(3) any minimal model of C is at-worst-nodal of relative dimension 1 over R.

Proof. To make sense out of this statement, recall that a minimal model is defined as a regular proper model without exceptional curves of the first kind (Definition 55.8.4), that minimal models exist (Proposition 55.8.6), and that minimal models are unique if the genus of C is > 0 (Lemma 55.10.1). Keeping this in mind the implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are clear.

Assume (1). Let X be a proper model of C which is at-worst-nodal of relative dimension 1 over R. Applying Lemma 55.14.3 we see that we may assume X is regular as well. Let

$$X = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0$$

be as in Lemma 55.8.5. By Lemma 55.14.4 and induction this implies X_0 is at-worst-nodal of relative dimension 1 over R.

To finish the proof we have to show that (2) implies (3). This is clear if the genus of C is > 0, since then the minimal model is unique (see discussion above). On the other hand, if the minimal model is not unique, then the morphism $X \to \mathrm{Spec}(R)$ is smooth for any minimal model as its special fibre will be isomorphic to $\mathbf{P}^1_k$ by Lemma 55.12.4. $\square$

0CDH Definition 55.14.6. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. We say that C has semistable reduction if the equivalent conditions of Lemma 55.14.5 are satisfied.

0CDI Lemma 55.14.7. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. The following are equivalent

(1) there exists a proper smooth model for C,
(2) there exists a minimal model for C which is smooth over R,
(3) any minimal model is smooth over R.

Proof. If X is a smooth proper model, then the special fibre is connected (Lemma 55.9.4) and smooth, hence irreducible. This immediately implies that it is minimal. Thus (1) implies (2). To finish the proof we have to show that (2) implies (3). This is clear if the genus of C is > 0, since then the minimal model is unique (Lemma 55.10.1). On the other hand, if the minimal model is not unique, then the morphism $X \to \mathrm{Spec}(R)$ is smooth for any minimal model as its special fibre will be isomorphic to $\mathbf{P}^1_k$ by Lemma 55.12.4. $\square$

0CDJ Definition 55.14.8. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. We say that C has good reduction if the equivalent conditions of Lemma 55.14.7 are satisfied.

55.15. Semistable reduction in genus zero

0CDK In this section we prove the semistable reduction theorem (Theorem 55.18.1) for genus zero curves.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If the genus of C is 0, then C is isomorphic to a conic, see Algebraic Curves, Lemma 53.10.3. Thus there exists a finite separable extension K'/K of degree at most 2 such that $C(K') \neq \emptyset$, see Algebraic Curves, Lemma 53.9.4. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 15.111.6. We will show that $C_{K'}$ has semistable reduction over $R'_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of R' (of course in the current case there are at most two such ideals). After replacing R by $R'_\mathfrak{m}$ and C by $C_{K'}$ we reduce to the case discussed in the next paragraph.

In this paragraph R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, of genus 0, and C has a K-rational point. In this case $C \cong \mathbf{P}^1_K$ by Algebraic Curves, Proposition 53.10.4. Thus we can use $\mathbf{P}^1_R$ as a model and we see that C has both good and semistable reduction.

0CDL Example 55.15.1. Let $R = \mathbf{R}[[\pi]]$ and consider the scheme

$$X = V(T_1^2 + T_2^2 - \pi T_0^2) \subset \mathbf{P}^2_R$$

The base change of X to $\mathbf{C}[[\pi]]$ is isomorphic to the scheme defined in Example 55.10.3 because we have the factorization $T_1^2 + T_2^2 = (T_1 + iT_2)(T_1 - iT_2)$ over $\mathbf{C}$. Thus X is regular and its special fibre is irreducible yet singular, hence X is the unique minimal model of its generic fibre (use Lemma 55.12.4). It follows that an extension is needed even in genus 0.

55.16. Semistable reduction in genus one

0CEG In this section we prove the semistable reduction theorem (Theorem 55.18.1) for curves of genus one. We suggest the reader first read the proof in the case of genus ≥ 2 (Section 55.17). We are going to use as much as possible the classification of minimal numerical types of genus 1 given in Lemma 55.6.2.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Assume the genus of C is 1. Choose a prime $\ell \geq 7$ different from the characteristic of k. Choose a finite separable extension K'/K of such that $C(K') \neq \emptyset$ and such that $\text{Pic}(C_{K'})[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$. See Algebraic Curves, Lemma 53.17.2. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 15.111.6. We may replace R by $R'_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in R' and C by $C_{K'}$. This reduces us to the case discussed in the next paragraph.

In the rest of this section R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, with genus 1, having a K-rational point, and with $\text{Pic}(C)[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$ for some prime $\ell \geq 7$ different from the characteristic of k. We will prove that C has semistable reduction.

Let X be a minimal model for C, see Proposition 55.8.6. Let $T = (n, m_i, (a_{ij}), w_i, g_i)$ be the numerical type associated to X (Definition 55.11.4). Then T is a minimal numerical type (Lemma 55.11.5). As C has a rational point, there exists an i such that $m_i = w_i = 1$ by Lemma 55.11.7. Looking at the classification of minimal numerical types of genus 1 in Lemma 55.6.2 we see that $m = w = 1$ and that cases (3), (6), (7), (9), (11), (13), (15), (18), (19), (21), (24), (26), (28), (30) are disallowed (because there is no index where both w_i and m_i is equal to 1). Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. For the remaining cases we have

(A) $e = n - 1$ for cases (1), (2), (5), (8), (12), (14), (17), (20), (22), (23), (27), (29), (31), (32), (33), and (34), and
(B) $e = n$ for cases (4), (10), (16), and (25).

We will argue these cases separately.

Case (A). In this case $\text{Pic}(T)[\ell]$ is trivial (the Picard group of a numerical type is defined in Section 55.4). The vanishing follows as $\text{Pic}(T) \subset \text{Coker}(A)$ (Lemma 55.4.3) and $\text{Coker}(A)[\ell] = 0$ by Lemma 55.2.6 and the fact that ℓ was chosen relatively prime to a_{ij} and m_i. By Lemmas 55.13.3 and 55.13.4 we conclude that there is an embedding

$$(\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2} \subset \text{Pic}((X_k)_{red})[\ell].$$

By Algebraic Curves, Lemma 53.18.6 we obtain

$$2 \leq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) + g_{geom}((X_k)_{red}/k)$$

By Algebraic Curves, Lemmas 53.18.1 and 53.18.5 we see that $g_{geom}((X_k)_{red}/k) \leq \sum w_i g_i$. The assumptions of Lemma 55.11.8 hold by Lemma 55.11.7 and we conclude that we have $\dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \leq g = 1$. Combining these we see

$$2 \leq 1 + \sum w_i g_i$$

Looking at the list we conclude that the numerical type is given by $n = 1$, $w_1 = m_1 = g_1 = 1$. Because we have equality everywhere we see that $g_{geom}(C_1/k) = 1$.

On the other hand, we know that C_1 has a k-rational point x such that $C_1 \to \mathrm{Spec}(k)$ is smooth at x. It follows that C_1 is geometrically integral (Varieties, Lemma 33.25.10). Thus $g_{geom}(C_1/k) = 1$ is both equal to the genus of the normalization of $C_{1,\overline{k}}$ and the genus of $C_{1,\overline{k}}$. It follows that the normalization morphism $C^\nu_{1,\overline{k}} \to C_{1,\overline{k}}$ is an isomorphism (Algebraic Curves, Lemma 53.18.4). We conclude that C_1 is smooth over k as desired.

Case (B). Here we only conclude that there is an embedding

$$\mathbf{Z}/\ell\mathbf{Z} \subset \mathrm{Pic}(X_k)[\ell]$$

From the classification of types we see that $m_i = w_i = 1$ and $g_i = 0$ for each i. Thus each C_i is a genus zero curve over k. Moreover, for each i there is a j such that $C_i \cap C_j$ is a k-rational point. Then it follows that $C_i \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 53.10.4. In particular, since X_k is the scheme theoretic union of the C_i we see that $X_{\overline{k}}$ is the scheme theoretic union of the $C_{i,\overline{k}}$. Hence $X_{\overline{k}}$ is a reduced connected proper scheme of dimension 1 over $\overline{k}$ with $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = 1$. Also, by Varieties, Lemma 33.30.3 and the above we still have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Pic}(X_{\overline{k}}) \geq 1$$

By Algebraic Curves, Proposition 53.17.3 we see that $X_{\overline{k}}$ has at only multicross singularities. But since X_k is Gorenstein (Lemma 55.9.2), so is $X_{\overline{k}}$ (Duality for Schemes, Lemma 48.25.1). We conclude $X_{\overline{k}}$ is at-worst-nodal by Algebraic Curves, Lemma 53.16.4. This finishes the proof in case (B).

0CEH Example 55.16.1. Let k be an algebraically closed field. Let Z be a smooth projective curve over k of positive genus g. Let $n \geq 1$ be an integer prime to the characteristic of k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Z$-module of order n, see Algebraic Curves, Lemma 53.17.1. Pick an isomorphism $\varphi : \mathcal{L}^{\otimes n} \to \mathcal{O}_Z$. Set $R = k[[\pi]]$ with fraction field $K = k((\pi))$. Denote Z_R the base change of Z to R. Let $\mathcal{L}_R$ be the pullback of $\mathcal{L}$ to Z_R. Consider the finite flat morphism

$$p : X \longrightarrow Z_R$$

such that

$$p_*\mathcal{O}_X = \mathrm{Sym}^*_{\mathcal{O}_{Z_R}}(\mathcal{L}_R)/(\varphi - \pi) = \mathcal{O}_{Z_R} \oplus \mathcal{L}_R \oplus \mathcal{L}_R^{\otimes 2} \oplus \ldots \oplus \mathcal{L}_R^{\otimes n-1}$$

More precisely, if $U = \mathrm{Spec}(A) \subset Z$ is an affine open such that $\mathcal{L}|_U$ is trivialized by a section s with $\varphi(s^{\otimes n}) = f$ (with f a unit), then

$$p^{-1}(U_R) = \mathrm{Spec}\left((A \otimes_R R[[\pi]])[x]/(x^n - \pi f)\right)$$

The reader verifies that the morphism $X_K \to Z_K$ of generic fibres is finite étale. Looking at the description of the structure sheaf we see that $H^0(X, \mathcal{O}_X) = R$ and $H^0(X_K, \mathcal{O}_{X_K}) = K$. By Riemann-Hurwitz (Algebraic Curves, Lemma 53.12.4) the genus of X_K is $n(g-1)+1$. In particular X_K has genus 1, if Z has genus 1. On the other hand, the scheme X is regular by the local equation above and the special fibre X_k is n times the reduced special fibre as an effective Cartier divisor. It follows that any finite extension K'/K over which X_K attains semistable reduction has to ramify with ramification index at least n (some details omitted). Thus there does not exist a universal bound for the degree of an extension over which a genus 1 curve attains semistable reduction.

55.17. Semistable reduction in genus at least two

0CEI In this section we prove the semistable reduction theorem (Theorem 55.18.1) for curves of genus ≥ 2. Fix $g \geq 2$.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Assume the genus of C is g. Choose a prime $\ell > 768g$ different from the characteristic of k. Choose a finite separable extension K'/K of such that $C(K') \neq \emptyset$ and such that $\text{Pic}(C_{K'})[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g}$. See Algebraic Curves, Lemma 53.17.2. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 15.111.6. We may replace R by $R'_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in R' and C by $C_{K'}$. This reduces us to the case discussed in the next paragraph.

In the rest of this section R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, with genus g, having a K-rational point, and with $\text{Pic}(C)[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g}$ for some prime $\ell \geq 768g$ different from the characteristic of k. We will prove that C has semistable reduction.

In the rest of this section we will use without further mention that the conclusions of Lemma 55.11.7 are true.

Let X be a minimal model for C, see Proposition 55.8.6. Let $T = (n, m_i, (a_{ij}), w_i, g_i)$ be the numerical type associated to X (Definition 55.11.4). Then T is a minimal numerical type of genus g (Lemma 55.11.5). By Proposition 55.7.4 we have

$$\dim_{\mathbf{F}_\ell} \text{Pic}(T)[\ell] \leq g_{top}$$

By Lemmas 55.13.3 and 55.13.4 we conclude that there is an embedding

$$(\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g - g_{top}} \subset \text{Pic}((X_k)_{red})[\ell].$$

By Algebraic Curves, Lemma 53.18.6 we obtain

$$2g - g_{top} \leq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) + g_{geom}(X_k/k)$$

By Lemmas 55.11.8 and 55.11.9 we have

$$g \geq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \geq g_{top} + g_{geom}(X_k/k)$$

Elementary number theory tells us that the only way these 3 inequalities can hold is if they are all equalities. Looking at Lemma 55.11.8 we conclude that $m_i = 1$ for all i. Looking at Lemma 55.11.10 we conclude that every irreducible component of X_k is smooth over k.

In particular, since X_k is the scheme theoretic union of its irreducible components C_i we see that $X_{\overline{k}}$ is the scheme theoretic union of the $C_{i,\overline{k}}$. Hence $X_{\overline{k}}$ is a reduced connected proper scheme of dimension 1 over $\overline{k}$ with $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = g$. Also, by Varieties, Lemma 33.30.3 and the above we still have

$$\dim_{\mathbf{F}_\ell}(\text{Pic}(X_{\overline{k}})[\ell]) \geq 2g - g_{top} = \dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) + g_{geom}(X_{\overline{k}})$$

By Algebraic Curves, Proposition 53.17.3 we see that $X_{\overline{k}}$ has at only multicross singularities. But since X_k is Gorenstein (Lemma 55.9.2), so is $X_{\overline{k}}$ (Duality for Schemes, Lemma 48.25.1). We conclude $X_{\overline{k}}$ is at-worst-nodal by Algebraic Curves, Lemma 53.16.4. This finishes the proof.

55.18. Semistable reduction for curves

0CDM In this section we finish the proof of the theorem. For $g \geq 2$ let $768g < \ell' < \ell$ be the first two primes $> 768g$ and set

0CEJ (55.18.0.1)
$$B_g = (2g-2)(\ell^{2g})!$$

The precise form of B_g is unimportant; the point we are trying to make is that it depends only on g.

0CDN **Theorem 55.18.1.** Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Then there exists an extension of discrete valuation rings $R \subset R'$ which induces a finite separable extension of fraction fields K'/K such that $C_{K'}$ has semistable reduction. More precisely, we have the following [DM69, Corollary 2.7]

(1) If the genus of C is zero, then there exists a degree 2 separable extension K'/K such that $C_{K'} \cong \mathbf{P}^1_{K'}$ and hence $C_{K'}$ is isomorphic to the generic fibre of the smooth projective scheme $\mathbf{P}^1_{R'}$ over the integral closure R' of R in K'.
(2) If the genus of C is one, then there exists a finite separable extension K'/K such that $C_{K'}$ has semistable reduction over $R'_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of the integral closure R' of R in K'. Moreover, the special fibre of the (unique) minimal model of $C_{K'}$ over $R'_\mathfrak{m}$ is either a smooth genus one curve or a cycle of rational curves.
(3) If the genus g of C is greater than one, then there exists a finite separable extension K'/K of degree at most B_g (55.18.0.1) such that $C_{K'}$ has semistable reduction over $R'_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of the integral closure R' of R in K'.

Proof. For the case of genus zero, see Section 55.15. For the case of genus one, see Section 55.16. For the case of genus greater than one, see Section 55.17. To see that we have a bound on the degree $[K' : K]$ you can use the bound on the degree of the extension needed to make all ℓ or ℓ' torsion visible proved in Algebraic Curves, Lemma 53.17.2. (The reason for using ℓ and ℓ' is that we need to avoid the characteristic of the residue field k.) $\square$

0CEK **Remark 55.18.2** (Improving the bound). Results in the literature suggest that one can improve the bound given in the statement of Theorem 55.18.1. For example, in [DM69] it is shown that semistable reduction of C and its Jacobian are the same thing if the residue field is perfect and presumably this is true for general residue fields as well. For an abelian variety we have semistable reduction if the action of Galois on the ℓ-torsion is trivial for any $\ell \geq 3$ not equal to the residue characteristic. Thus we can presumably choose $\ell = 5$ in the formula (55.18.0.1) for B_g (but the proof would take a lot more work; if we ever need this we will make a precise statement and provide a proof here).

55.19. Other chapters

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Algebraic Stacks

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CHAPTER 56

Functors and Morphisms

0GNG

56.1. Introduction

0GNH Let X and Y be schemes. This chapter circles around the relationship between functors $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ and morphisms of schemes $X \to Y$. More broadly speaking we study the relationship between $QCoh(\mathcal{O}_X)$ and X or, if X is Noetherian, the relationship between $\text{Coh}(\mathcal{O}_X)$ and X. This relationship was studied in [Gab62].

56.2. Functors on module categories

0GNI For a ring A let us denote Mod_A^{fp} the category of finitely presented A-modules.

0GNJ **Lemma 56.2.1.** Let A be a ring. Let $\mathcal{B}$ be a category having filtered colimits. Let $F : \text{Mod}_A^{fp} \to \mathcal{B}$ be a functor. Then F extends uniquely to a functor $F' : \text{Mod}_A \to \mathcal{B}$ which commutes with filtered colimits.

Proof. This follows from Categories, Lemma 4.26.2. To see that the lemma applies observe that finitely presented A-modules are categorically compact objects of Mod_A by Algebra, Lemma 10.11.4. Also, every A-module is a filtered colimit of finitely presented A-modules by Algebra, Lemma 10.11.3. $\square$

If a category $\mathcal{B}$ is additive and has filtered colimits, then $\mathcal{B}$ has arbitrary direct sums: any direct sum can be written as a filtered colimit of finite direct sums.

0GNK **Lemma 56.2.2.** Let A, $\mathcal{B}$, F be as in Lemma 56.2.1. Assume $\mathcal{B}$ is additive and F is additive. Then F' is additive and commutes with arbitrary direct sums.

Proof. To show that F' is additive it suffices to show that $F'(M) \oplus F'(M') \to F'(M \oplus M')$ is an isomorphism for any A-modules M, M', see Homology, Lemma 12.7.1. Write $M = \text{colim}_i M_i$ and $M' = \text{colim}_j M'_j$ as filtered colimits of finitely presented A-modules M_i. Then $F'(M) = \text{colim}_i F(M_i)$, $F'(M') = \text{colim}_j F(M'_j)$, and

$$\begin{aligned} F'(M \oplus M') &= F'(\text{colim}_{i,j} M_i \oplus M'_j) \\ &= \text{colim}_{i,j} F(M_i \oplus M'_j) \\ &= \text{colim}_{i,j} F(M_i) \oplus F(M'_j) \\ &= F'(M) \oplus F'(M') \end{aligned}$$

as desired. To show that F' commutes with direct sums, assume we have $M = \bigoplus_{i \in I} M_i$. Then $M = \mathrm{colim}_{I' \subset I \text{ finite}} \bigoplus_{i \in I'} M_i$ is a filtered colimit. We obtain

$$\begin{aligned} F'(M) &= \mathrm{colim}_{I' \subset I \text{ finite}} F'(\bigoplus\nolimits_{i \in I'} M_i) \\ &= \mathrm{colim}_{I' \subset I \text{ finite}} \bigoplus\nolimits_{i \in I'} F'(M_i) \\ &= \bigoplus\nolimits_{i \in I} F'(M_i) \end{aligned}$$

The second equality holds by the additivity of F' already shown. □

If a category $\mathcal{B}$ is additive, has filtered colimits, and has cokernels, then $\mathcal{B}$ has arbitrary colimits, see discussion above and Categories, Lemma 4.14.12.

0GNL Lemma 56.2.3. Let A, $\mathcal{B}$, F be as in Lemma 56.2.1. Assume $\mathcal{B}$ is additive, has cokernels, and F is right exact. Then F' is additive, right exact, and commutes with arbitrary direct sums.

Proof. Since F is right exact, F commutes with coproducts of pairs, which are represented by direct sums. Hence F is additive by Homology, Lemma 12.7.1. Hence F' is additive and commutes with direct sums by Lemma 56.2.2. We urge the reader to prove that F' is right exact themselves instead of reading the proof below.

To show that F' is right exact, it suffices to show that F' commutes with coequalizers, see Categories, Lemma 4.23.3. Now, if $a, b : K \to L$ are maps of A-modules, then the coequalizer of a and b is the cokernel of $a - b : K \to L$. Thus let $K \to L \to M \to 0$ be an exact sequence of A-modules. We have to show that in

$$F'(K) \to F'(L) \to F'(M) \to 0$$

the second arrow is a cokernel for the first arrow in $\mathcal{B}$ (if $\mathcal{B}$ were abelian we would say that the displayed sequence is exact). Write $M = \mathrm{colim}_{i \in I} M_i$ as a filtered colimit of finitely presented A-modules, see Algebra, Lemma 10.11.3. Let $L_i = L \times_M M_i$. We obtain a system of exact sequences $K \to L_i \to M_i \to 0$ over I. Since colimits commute with colimits by Categories, Lemma 4.14.10 and since cokernels are a type of coequalizer, it suffices to show that $F'(L_i) \to F(M_i)$ is a cokernel of $F'(K) \to F'(L_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume M is finitely presented. Write $L = \mathrm{colim}_{i \in I} L_i$ as a filtered colimit of finitely presented A-modules with the property that each L_i surjects onto M. Let $K_i = K \times_L L_i$. We obtain a system of short exact sequences $K_i \to L_i \to M \to 0$ over I. Repeating the argument already given, we reduce to showing $F(L_i) \to F(M_i)$ is a cokernel of $F'(K) \to F(L_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume both L and M are finitely presented A-modules. In this case the module $\mathrm{Ker}(L \to M)$ is finite (Algebra, Lemma 10.5.3). Thus we can write $K = \mathrm{colim}_{i \in I} K_i$ as a filtered colimit of finitely presented A-modules each surjecting onto $\mathrm{Ker}(L \to M)$. We obtain a system of short exact sequences $K_i \to L \to M \to 0$ over I. Repeating the argument already given, we reduce to showing $F(L) \to F(M)$ is a cokernel of $F(K_i) \to F(L)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume K, L, and M are finitely presented A-modules. This final case follows from the assumption that F is right exact. □

If a category $\mathcal{B}$ is additive and has kernels, then $\mathcal{B}$ has finite limits. Namely, finite products are direct sums which exist and the equalizer of $a, b : L \to M$ is the kernel of $a - b : K \to L$ which exists. Thus all finite limits exist by Categories, Lemma 4.18.4.

0GNM Lemma 56.2.4. Let A, $\mathcal{B}$, F be as in Lemma 56.2.1. Assume A is a coherent ring (Algebra, Definition 10.90.1), $\mathcal{B}$ is additive, has kernels, filtered colimits commute with taking kernels, and F is left exact. Then F' is additive, left exact, and commutes with arbitrary direct sums.

Proof. Since A is coherent, the category Mod_A^{fp} is abelian with same kernels and cokernels as in Mod_A, see Algebra, Lemmas 10.90.4 and 10.90.3. Hence all finite limits exist in Mod_A^{fp} and Categories, Definition 4.23.1 applies. Since F is left exact, F commutes with products of pairs, which are represented by direct sums. Hence F is additive by Homology, Lemma 12.7.1. Hence F' is additive and commutes with direct sums by Lemma 56.2.2. We urge the reader to prove that F' is left exact themselves instead of reading the proof below.

To show that F' is left exact, it suffices to show that F' commutes with equalizers, see Categories, Lemma 4.23.2. Now, if $a, b : L \to M$ are maps of A-modules, then the equalizer of a and b is the kernel of $a - b : L \to M$. Thus let $0 \to K \to L \to M$ be an exact sequence of A-modules. We have to show that in

$$0 \to F'(K) \to F'(L) \to F'(M)$$

the arrow $F'(K) \to F'(L)$ is a kernel for $F'(L) \to F'(M)$ in $\mathcal{B}$ (if $\mathcal{B}$ were abelian we would say that the displayed sequence is exact). Write $M = \text{colim}_{i \in I} M_i$ as a filtered colimit of finitely presented A-modules, see Algebra, Lemma 10.11.3. Let $L_i = L \times_M M_i$. We obtain a system of exact sequences $0 \to K \to L_i \to M_i$ over I. Since filtered colimits commute with taking kernels in $\mathcal{B}$ by assumption, it suffices to show that $F'(K) \to F'(L_i)$ is a kernel of $F'(L_i) \to F(M_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume M is finitely presented. Write $L = \text{colim}_{i \in I} L_i$ as a filtered colimit of finitely presented A-modules. Let $K_i = K \times_L L_i$. We obtain a system of short exact sequences $0 \to K_i \to L_i \to M$ over I. Repeating the argument already given, we reduce to showing $F'(K_i) \to F(L_i)$ is a kernel of $F(L_i) \to F(M)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume both L and M are finitely presented A-modules. Since A is coherent, the A-module $K = \text{Ker}(L \to M)$ is of finite presentation as the category of finitely presented A-modules is abelian (see references given above). In other words, all three modules K, L, and M are finitely presented A-modules. This final case follows from the assumption that F is left exact. $\square$

If a category $\mathcal{B}$ is additive and has cokernels, then $\mathcal{B}$ has finite colimits. Namely, finite coproducts are direct sums which exist and the coequalizer of $a, b : K \to L$ is the cokernel of $a - b : K \to L$ which exists. Thus all finite colimits exist by Categories, Lemma 4.18.7.

0GNN Lemma 56.2.5. Let A be a ring. Let $\mathcal{B}$ be an additive category with cokernels. There is an equivalence of categories between

1. the category of functors $F : \text{Mod}_A^{fp} \to \mathcal{B}$ which are right exact, and
2. the category of pairs (K, κ) where $K \in \text{Ob}(\mathcal{B})$ and $\kappa : A \to \text{End}_{\mathcal{B}}(K)$ is a ring homomorphism

given by the rule sending F to $F(A)$ with its natural A-action.

Proof. Let (K, κ) be as in (2). We will construct a functor $F : \text{Mod}_A^{fp} \to \mathcal{B}$ such that $F(A) = K$ endowed with the given A-action κ. Namely, given an integer $n \geq 0$ let us set

$$F(A^{\oplus n}) = K^{\oplus n}$$

Given an A-linear map $\varphi : A^{\oplus m} \to A^{\oplus n}$ with matrix $(a_{ij}) \in \text{Mat}(n \times m, A)$ we define

$$F(\varphi) : F(A^{\oplus m}) = K^{\oplus m} \longrightarrow K^{\oplus n} = F(A^{\oplus n})$$

to be the map with matrix $(\kappa(a_{ij}))$. This defines an additive functor F from the full subcategory of Mod_A^{fp} with objects 0, A, $A^{\oplus 2}$, ... to $\mathcal{B}$; we omit the verification.

For each object M of Mod_A^{fp} choose a presentation

$$A^{\oplus m_M} \xrightarrow{\varphi_M} A^{\oplus n_M} \to M \to 0$$

of M as an A-module. Let us use the trivial presentation $0 \to A^{\oplus n} \xrightarrow{1} A^{\oplus n} \to 0$ if $M = A^{\oplus n}$ (this isn't necessary but simplifies the exposition). For each morphism $f : M \to N$ of Mod_A^{fp} we can choose a commutative diagram

0GNP (56.2.5.1)
$$\begin{tikzcd}
A^{\oplus m_M} \arrow[r, "\varphi_M"'] \arrow[d, "\psi_f"'] & A^{\oplus n_M} \arrow[r] \arrow[d, "\chi_f"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
A^{\oplus m_N} \arrow[r, "\varphi_N"] & A^{\oplus n_N} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Having made these choices we can define: for an object M of Mod_A^{fp} we set

$$F(M) = \text{Coker}(F(\varphi_M) : F(A^{\oplus m_M}) \to F(A^{\oplus n_M}))$$

and for a morphism $f : M \to N$ of Mod_A^{fp} we set

$$F(f) = \text{the map } F(M) \to F(N) \text{ induced by } F(\psi_f) \text{ and } F(\chi_f) \text{ on cokernels}$$

Note that this rule extends the given functor F on the full subcategory consisting of the free modules $A^{\oplus n}$. We still have to show that F is a functor, that F is additive, and that F is right exact.

Let $f : M \to N$ be a morphism Mod_A^{fp}. We claim that the map $F(f)$ defined above is independent of the choices of ψ_f and χ_f in (56.2.5.1). Namely, say

$$\begin{tikzcd}
A^{\oplus m_M} \arrow[r, "\varphi_M"'] \arrow[d, "\psi"'] & A^{\oplus n_M} \arrow[r] \arrow[d, "\chi"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
A^{\oplus m_N} \arrow[r, "\varphi_N"] & A^{\oplus n_N} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

is also commutative. Denote $F(f)' : F(M) \to F(N)$ the map induced by $F(\psi)$ and $F(\chi)$. Looking at the commutative diagrams, by elementary commutative algebra there exists a map $\omega : A^{\oplus n_M} \to A^{\oplus m_N}$ such that $\chi = \chi_f + \varphi_N \circ \omega$. Applying F we find that $F(\chi) = F(\chi_f) + F(\varphi_N) \circ F(\omega)$. As $F(N)$ is the cokernel of $F(\varphi_N)$ we find that the map $F(A^{\oplus n_M}) \to F(M)$ equalizes $F(f)$ and $F(f)'$. Since a cokernel is an epimorphism, we conclude that $F(f) = F(f)'$.

Let us prove F is a functor. First, observe that $F(\text{id}_M) = \text{id}_{F(M)}$ because we may pick the identities for ψ_f and χ_f in the diagram above in case $f = \text{id}_M$. Second, suppose we have $f : M \to N$ and $g : L \to M$. Then we see that $\psi = \psi_f \circ \psi_g$ and $\chi = \chi_f \circ \chi_g$ fit into (56.2.5.1) for $f \circ g$. Hence these induce the correct map which exactly says that $F(f) \circ F(g) = F(f \circ g)$.

Let us prove that F is additive. Namely, suppose we have $f, g : M \to N$. Then we see that $\psi = \psi_f + \psi_g$ and $\chi = \chi_f + \chi_g$ fit into (56.2.5.1) for $f + g$. Hence these induce the correct map which exactly says that $F(f) + F(g) = F(f + g)$.

Finally, let us prove that F is right exact. It suffices to show that F commutes with coequalizers, see Categories, Lemma 4.23.3. For this, it suffices to prove that F commutes with cokernels. Let $K \to L \to M \to 0$ be an exact sequence of A-modules with K, L, M finitely presented. Since F is an additive functor, this certainly gives a complex

$$F(K) \to F(L) \to F(M) \to 0$$

and we have to show that the second arrow is the cokernel of the first in $\mathcal{B}$. In any case, we obtain a map $\text{Coker}(F(K) \to F(L)) \to F(M)$. By elementary commutative algebra there exists a commutative diagram

$$\begin{tikzcd} A^{\oplus m_M} \arrow[r, "\varphi_M"'] \arrow[d, "\psi"'] & A^{\oplus n_M} \arrow[r] \arrow[d, "\chi"'] & M \arrow[r] \arrow[d, "1"'] & 0 \\ K \arrow[r] & L \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Applying F to this diagram and using the construction of $F(M)$ as the cokernel of $F(\varphi_M)$ we find there exists a map $F(M) \to \text{Coker}(F(K) \to F(L))$ which is a right inverse to the map $\text{Coker}(F(K) \to F(L)) \to F(M)$. This first implies that $F(L) \to F(M)$ is an epimorphism always. Next, the above shows we have

$$\text{Coker}(F(K) \to F(L)) = F(M) \oplus E$$

where the direct sum decomposition is compatible with both $F(M) \to \text{Coker}(F(K) \to F(L))$ and $\text{Coker}(F(K) \to F(L)) \to F(M)$. However, then the epimorphism $p : F(L) \to E$ becomes zero both after composition with $F(K) \to F(L)$ and after composition with $F(A^{n_M}) \to F(L)$. However, since $K \oplus A^{n_M} \to L$ is surjective (algebra argument omitted), we conclude that $F(K \oplus A^{n_M}) \to F(L)$ is an epimorphism (by the above) whence $E = 0$. This finishes the proof. □

0GNQ **Lemma 56.2.6.** Let A be a ring. Let $\mathcal{B}$ be an additive category with arbitrary direct sums and cokernels. There is an equivalence of categories between

(1) the category of functors $F : \text{Mod}_A \to \mathcal{B}$ which are right exact and commute with arbitrary direct sums, and
(2) the category of pairs (K, κ) where $K \in \text{Ob}(\mathcal{B})$ and $\kappa : A \to \text{End}_{\mathcal{B}}(K)$ is a ring homomorphism

given by the rule sending F to $F(A)$ with its natural A-action.

Proof. Combine Lemmas 56.2.5 and 56.2.3. □

56.3. Functors between categories of modules

0GNR The following lemma is archetypical of the results in this chapter.

0GNS **Lemma 56.3.1.** Let A and B be rings. Let $F : \text{Mod}_A \to \text{Mod}_B$ be a functor. The following are equivalent

(1) F is isomorphic to the functor $M \mapsto M \otimes_A K$ for some $A \otimes_{\mathbf{Z}} B$-module K,
(2) F is right exact and commutes with all direct sums,
(3) F commutes with all colimits,

(4) F has a right adjoint G.

Proof. If (1), then (4) as a right adjoint for $M \mapsto M \otimes_A K$ is $N \mapsto \mathrm{Hom}_B(K, N)$, see Differential Graded Algebra, Lemma 22.30.3. If (4), then (3) by Categories, Lemma 4.24.5. The implication (3) $\Rightarrow$ (2) is immediate from the definitions.

Assume (2). We will prove (1). By the discussion in Homology, Section 12.7 the functor F is additive. Hence F induces a ring map $A \to \mathrm{End}_B(F(M))$, $a \mapsto F(a \cdot \mathrm{id}_M)$ for every A-module M. We conclude that $F(M)$ is an $A \otimes_{\mathbf{Z}} B$-module functorially in M. Set $K = F(A)$. Define

$$M \otimes_A K = M \otimes_A F(A) \longrightarrow F(M), \quad m \otimes k \longmapsto F(\varphi_m)(k)$$

Here $\varphi_m : A \to M$ sends $a \to am$. The rule $(m, k) \mapsto F(\varphi_m)(k)$ is A-bilinear (and B-linear on the right) as required to obtain the displayed $A \otimes_{\mathbf{Z}} B$-linear map. This construction is functorial in M, hence defines a transformation of functors $- \otimes_A K \to F(-)$ which is an isomorphism when evaluated on A. For every A-module M we can choose an exact sequence

$$\bigoplus\nolimits_{j \in J} A \to \bigoplus\nolimits_{i \in I} A \to M \to 0$$

Using the maps constructed above we find a commutative diagram

$$\begin{tikzcd}
(\bigoplus_{j \in J} A) \otimes_A K \arrow[r] \arrow[d] & (\bigoplus_{i \in I} A) \otimes_A K \arrow[r] \arrow[d] & M \otimes_A K \arrow[r] \arrow[d] & 0 \\
F(\bigoplus_{j \in J} A) \arrow[r] & F(\bigoplus_{i \in I} A) \arrow[r] & F(M) \arrow[r] & 0
\end{tikzcd}$$

The lower row is exact as F is right exact. The upper row is exact as tensor product with K is right exact. Since F commutes with direct sums the left two vertical arrows are bijections. Hence we conclude. □

0GNT Example 56.3.2. Let R be a ring. Let A and B be R-algebras. Let K be a $A \otimes_R B$-module. Then we can consider the functor

0GNU (56.3.2.1)
$$F : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto M \otimes_A K$$

This functor is R-linear, right exact, commutes with arbitrary direct sums, commutes with all colimits, has a right adjoint (Lemma 56.3.1).

0GNV Lemma 56.3.3. Let R be a ring. Let A and B be R-algebras. There is an equivalence of categories between

(1) the category of R-linear functors $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ which are right exact and commute with arbitrary direct sums, and
(2) the category $\mathrm{Mod}_{A \otimes_R B}$.

given by sending K to the functor F in (56.3.2.1).

Proof. Let F be an object of the first category. By Lemma 56.3.1 we may assume $F(M) = M \otimes_A K$ functorially in M for some $A \otimes_{\mathbf{Z}} B$-module K. The R-linearity of F immediately implies that the $A \otimes_{\mathbf{Z}} B$-module structure on K comes from a (unique) $A \otimes_R B$-module structure on K. Thus we see that sending K to F as in (56.3.2.1) is essentially surjective.

To prove that our functor is fully faithful, we have to show that given $A \otimes_R B$-modules K and K' any transformation $t : F \to F'$ between the corresponding functors, comes from a unique $\varphi : K \to K'$. Since $K = F(A)$ and $K' = F'(A)$ we can take φ to be the value $t_A : F(A) \to F'(A)$ of t at A. This maps is $A \otimes_R B$-linear by the definition of the $A \otimes B$-module structure on $F(A)$ and $F'(A)$ given in the proof of Lemma 56.3.1. $\square$

0GNW Remark 56.3.4. Let R be a ring. Let A, B, C be R-algebras. Let $F : \text{Mod}_A \to \text{Mod}_B$ and $F' : \text{Mod}_B \to \text{Mod}_C$ be R-linear, right exact functors which commute with arbitrary direct sums. If by the equivalence of Lemma 56.3.3 the object K in $\text{Mod}_{A \otimes_R B}$ corresponds to F and the object K' in $\text{Mod}_{B \otimes_R C}$ corresponds to F', then $K \otimes_B K'$ viewed as an object of $\text{Mod}_{A \otimes_R C}$ corresponds to $F' \circ F$.

0GNX Remark 56.3.5. In the situation of Lemma 56.3.3 suppose that F corresponds to K. Then F is exact $\Leftrightarrow$ K is flat over A.

0GNY Remark 56.3.6. In the situation of Lemma 56.3.3 suppose that F corresponds to K. Then F sends finite A-modules to finite B-modules $\Leftrightarrow$ K is finite as a B-module.

0GNZ Remark 56.3.7. In the situation of Lemma 56.3.3 suppose that F corresponds to K. Then F sends finitely presented A-modules to finitely presented B-modules $\Leftrightarrow$ K is finitely presented as a B-module.

0GP0 Lemma 56.3.8. Let A and B be rings. If

$$F : \text{Mod}_A \longrightarrow \text{Mod}_B$$

is an equivalence of categories, then there exists an isomorphism $A \to B$ of rings and an invertible B-module L such that F is isomorphic to the functor $M \mapsto (M \otimes_A B) \otimes_B L$.

Proof. Since an equivalence commutes with all colimits, we see that Lemmas 56.3.1 applies. Let K be the $A \otimes_{\mathbf{Z}} B$-module such that F is isomorphic to the functor $M \mapsto M \otimes_A K$. Let K' be the $B \otimes_{\mathbf{Z}} A$-module such that a quasi-inverse of F is isomorphic to the functor $N \mapsto N \otimes_B K'$. By Remark 56.3.4 and Lemma 56.3.3 we have an isomorphism

$$\psi : K \otimes_B K' \longrightarrow A$$

of $A \otimes_{\mathbf{Z}} A$-modules. Similarly, we have an isomorphism

$$\psi' : K' \otimes_A K \longrightarrow B$$

of $B \otimes_{\mathbf{Z}} B$-modules. Choose an element $\xi = \sum_{i=1,\ldots,n} x_i \otimes y_i \in K \otimes_B K'$ such that $\psi(\xi) = 1$. Consider the isomorphisms

$$K \xrightarrow{\psi^{-1} \otimes \text{id}_K} K \otimes_B K' \otimes_A K \xrightarrow{\text{id}_K \otimes \psi'} K$$

The composition is an isomorphism and given by

$$k \longmapsto \sum x_i \psi'(y_i \otimes k)$$

We conclude this automorphism factors as

$$K \to B^{\oplus n} \to K$$

as a map of B-modules. It follows that K is finite projective as a B-module.

We claim that K is invertible as a B-module. This is equivalent to asking the rank of K as a B-module to have the constant value 1, see More on Algebra, Lemma 15.117.2 and Algebra, Lemma 10.78.2. If not, then there exists a maximal ideal $\mathfrak{m} \subset B$ such that either (a) $K \otimes_B B/\mathfrak{m} = 0$ or (b) there is a surjection $K \to (B/\mathfrak{m})^{\oplus 2}$

of B-modules. Case (a) is absurd as $K' \otimes_A K \otimes_B N = N$ for all B-modules N. Case (b) would imply we get a surjection

$$A = K \otimes_B K' \longrightarrow (B/\mathfrak m \otimes_B K')^{\oplus 2}$$

of (right) A-modules. This is impossible as the target is an A-module which needs at least two generators: $B/\mathfrak m \otimes_B K'$ is nonzero as the image of the nonzero module $B/\mathfrak m$ under the quasi-inverse of F.

Since K is invertible as a B-module we see that $\mathrm{Hom}_B(K, K) = B$. Since $K = F(A)$ the action of A on K defines a ring isomorphism $A \to B$. The lemma follows. $\square$

0GP1 Lemma 56.3.9. Let R be a ring. Let A and B be R-algebras. If

$$F : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$$

is an R-linear equivalence of categories, then there exists an isomorphism $A \to B$ of R-algebras and an invertible B-module L such that F is isomorphic to the functor $M \mapsto (M \otimes_A B) \otimes_B L$.

Proof. We get $A \to B$ and L from Lemma 56.3.8. To finish the proof, we need to show that the R-linearity of F forces $A \to B$ to be an R-algebra map. We omit the details. $\square$

0GP2 Remark 56.3.10. Let A and B be rings. Let us endow Mod_A and Mod_B with the usual monoidal structure given by tensor products of modules. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be a functor of monoidal categories, see Categories, Definition 4.43.2. Here are some comments:

(1) Since $F(A)$ is a unit (by our definitions) we have $F(A) = B$.
(2) We obtain a multiplicative map $\varphi : A \to B$ by sending $a \in A$ to its action on $F(A) = B$.
(3) Take $A = B$ and $F(M) = M \otimes_A M$. In this case $\varphi(a) = a^2$.
(4) If F is additive, then φ is a ring map.
(5) Take $A = B = \mathbf{Z}$ and $F(M) = M/\text{torsion}$. Then $\varphi = \mathrm{id}_\mathbf{Z}$ but F is not the identity functor.
(6) If F is right exact and commutes with direct sums, then $F(M) = M \otimes_{A,\varphi} B$ by Lemma 56.3.1.

In other words, ring maps $A \to B$ are in bijection with isomorphism classes of functors of monoidal categories $\mathrm{Mod}_A \to \mathrm{Mod}_B$ which commute with all colimits.

56.4. Extending functors on categories of modules

0GP3 For a ring A let us denote Mod^{fp}_A the category of finitely presented A-modules.

0GP4 Lemma 56.4.1. Let A and B be rings. Let $F : \mathrm{Mod}^{fp}_A \to \mathrm{Mod}^{fp}_B$ be a functor. Then F extends uniquely to a functor $F' : \mathrm{Mod}_A \to \mathrm{Mod}_B$ which commutes with filtered colimits.

Proof. Special case of Lemma 56.2.1. $\square$

0GP5 Remark 56.4.2. With A, B, F, and F' as in Lemma 56.4.1. Observe that the tensor product of two finitely presented modules is finitely presented, see Algebra, Lemma 10.12.14. Thus we may endow Mod^{fp}_A, Mod^{fp}_B, Mod_A, and Mod_B with the usual monoidal structure given by tensor products of modules. In this case, if F is a functor of monoidal categories, so is F'. This follows immediately from the fact that tensor products of modules commutes with filtered colimits.

0GP6 Lemma 56.4.3. With A, B, F, and F' as in Lemma 56.4.1.

(1) If F is additive, then F' is additive and commutes with arbitrary direct sums, and
(2) if F is right exact, then F' is right exact.

Proof. Follows from Lemmas 56.2.2 and 56.2.3. □

0GP7 Remark 56.4.4. Combining Remarks 56.3.10 and 56.4.2 and Lemma 56.4.3 we find the following. Given rings A and B the set of ring maps $A \to B$ is in bijection with the set of isomorphism classes of functors of monoidal categories $\text{Mod}_A^{fp} \to \text{Mod}_B^{fp}$ which are right exact.

0GP8 Lemma 56.4.5. With A, B, F, and F' as in Lemma 56.4.1. Assume A is a coherent ring (Algebra, Definition 10.90.1). If F is left exact, then F' is left exact.

Proof. Special case of Lemma 56.2.4. □

For a ring A let us denote Mod_A^{fg} the category of finitely generated A-modules (AKA finite A-modules).

0GP9 Lemma 56.4.6. Let A and B be Noetherian rings. Let $F : \text{Mod}_A^{fg} \to \text{Mod}_B^{fg}$ be a functor. Then F extends uniquely to a functor $F' : \text{Mod}_A \to \text{Mod}_B$ which commutes with filtered colimits. If F is additive, then F' is additive and commutes with arbitrary direct sums. If F is exact, left exact, or right exact, so is F'.

Proof. See Lemmas 56.4.3 and 56.4.5. Also, use the finite A-modules are finitely presented A-modules, see Algebra, Lemma 10.31.4, and use that Noetherian rings are coherent, see Algebra, Lemma 10.90.5. □

56.5. Functors between categories of quasi-coherent modules

0FZA In this section we briefly study functors between categories of quasi-coherent modules.

0FZB Example 56.5.1. Let R be a ring. Let X and Y be schemes over R with X quasi-compact and quasi-separated. Let $\mathcal{K}$ be a quasi-coherent $\mathcal{O}_{X \times_R Y}$-module. Then we can consider the functor

0FZC
$$F : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_Y), \quad \mathcal{F} \longmapsto \text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) \tag{56.5.1.1}$$

The morphism pr_2 is quasi-compact and quasi-separated (Schemes, Lemmas 26.19.3 and 26.21.12). Hence pushforward along this morphism preserves quasi-coherent modules, see Schemes, Lemma 26.24.1. Moreover, our functor is R-linear and commutes with arbitrary direct sums, see Cohomology of Schemes, Lemma 30.6.1.

The following lemma is a natural generalization of Lemma 56.3.3.

0FZD Lemma 56.5.2. Let R be a ring. Let X and Y be schemes over R with X affine. There is an equivalence of categories between

(1) the category of R-linear functors $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which are right exact and commute with arbitrary direct sums, and
(2) the category $QCoh(\mathcal{O}_{X \times_R Y})$

given by sending $\mathcal{K}$ to the functor F in (56.5.1.1).

Proof. Let $\mathcal{K}$ be an object of $QCoh(\mathcal{O}_{X\times_R Y})$ and $F_\mathcal{K}$ the functor (56.5.1.1). By the discussion in Example 56.5.1 we already know that F is R-linear and commutes with arbitrary direct sums. Since $\text{pr}_2 : X \times_R Y \to Y$ is affine (Morphisms, Lemma 29.11.8) the functor $\text{pr}_{2,*}$ is exact, see Cohomology of Schemes, Lemma 30.2.3. Hence F is right exact as well, in other words F is as in (1).

Let F be as in (1). Say $X = \text{Spec}(A)$. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G} = F(\mathcal{O}_X)$. The functor F induces an R-linear map $A \to \text{End}_{\mathcal{O}_Y}(\mathcal{G})$, $a \mapsto F(a\cdot\text{id})$. Thus $\mathcal{G}$ is a sheaf of modules over

$$A \otimes_R \mathcal{O}_Y = \text{pr}_{2,*}\mathcal{O}_{X\times_R Y}$$

By Morphisms, Lemma 29.11.6 we find that there is a unique quasi-coherent module $\mathcal{K}$ on $X \times_R Y$ such that $F(\mathcal{O}_X) = \mathcal{G} = \text{pr}_{2,*}\mathcal{K}$ compatible with action of A and $\mathcal{O}_Y$. Denote $F_\mathcal{K}$ the functor given by (56.5.1.1). There is an equivalence $\text{Mod}_A \to QCoh(\mathcal{O}_X)$ sending A to $\mathcal{O}_X$, see Schemes, Lemma 26.7.5. Hence we find an isomorphism $F \cong F_\mathcal{K}$ by Lemma 56.2.6 because we have an isomorphism $F(\mathcal{O}_X) \cong F_\mathcal{K}(\mathcal{O}_X)$ compatible with A-action by construction.

This shows that the functor sending $\mathcal{K}$ to $F_\mathcal{K}$ is essentially surjective. We omit the verification of fully faithfulness. $\square$

0FZE **Remark 56.5.3.** Below we will use that for an affine morphism $h : T \to S$ we have $h_*\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{H} = h_*(\mathcal{G} \otimes_{\mathcal{O}_T} h^*\mathcal{H})$ for $\mathcal{G} \in QCoh(\mathcal{O}_T)$ and $\mathcal{H} \in QCoh(\mathcal{O}_S)$. This follows immediately on translating into algebra.

0FZF **Lemma 56.5.4.** In Lemma 56.5.2 let F correspond to $\mathcal{K}$ in $QCoh(\mathcal{O}_{X\times_R Y})$. We have

(1) If $f : X' \to X$ is an affine morphism, then $F \circ f_*$ corresponds to $(f \times \text{id}_Y)^*\mathcal{K}$.
(2) If $g : Y' \to Y$ is a flat morphism, then $g^* \circ F$ corresponds to $(\text{id}_X \times g)^*\mathcal{K}$.
(3) If $j : V \to Y$ is an open immersion, then $j^* \circ F$ corresponds to $\mathcal{K}|_{X\times_R V}$.

Proof. Proof of (1). Consider the commutative diagram

$$\begin{tikzcd}
X' \times_R Y \arrow[dd, "\text{pr}'_1"'] \arrow[rd, "f\times\text{id}_Y"] \arrow[rrd, "\text{pr}'_2"] & & \\
& X \times_R Y \arrow[r, "\text{pr}_2"'] \arrow[d, "\text{pr}_1"] & Y \\
X' \arrow[r, "f"] & X &
\end{tikzcd}$$

Let $\mathcal{F}'$ be a quasi-coherent module on X'. We have

$$\begin{aligned}
\text{pr}_{2,*}(\text{pr}_1^* f_*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) &= \text{pr}_{2,*}((f\times\text{id}_Y)_*(\text{pr}'_1)^*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) \\
&= \text{pr}_{2,*}(f\times\text{id}_Y)_*\left((\text{pr}'_1)^*\mathcal{F}' \otimes_{\mathcal{O}_{X'\times_R Y}} (f\times\text{id}_Y)^*\mathcal{K})\right) \\
&= \text{pr}'_{2,*}((\text{pr}'_1)^*\mathcal{F}' \otimes_{\mathcal{O}_{X'\times_R Y}} (f\times\text{id}_Y)^*\mathcal{K})
\end{aligned}$$

Here the first equality is affine base change for the left hand square in the diagram, see Cohomology of Schemes, Lemma 30.5.1. The second equality hold by Remark

56.5.3. The third equality is functoriality of pushforwards for modules. This proves (1).

Proof of (2). Consider the commutative diagram

$$\begin{tikzcd}
X \times_R Y' \arrow[rr, "\mathrm{pr}'_2"] \arrow[dr, "\mathrm{id}_X \times g"] \arrow[ddr, "\mathrm{pr}'_1"'] & & Y' \arrow[d, "g"] \\
& X \times_R Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & Y \\
& X &
\end{tikzcd}$$

We have

$$\begin{aligned} g^*\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) &= \mathrm{pr}'_{2,*}((\mathrm{id}_X \times g)^*(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K})) \\ &= \mathrm{pr}'_{2,*}((\mathrm{pr}'_1)^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y'}} (\mathrm{id}_X \times g)^*\mathcal{K}) \end{aligned}$$

The first equality by flat base change for the square in the diagram, see Cohomology of Schemes, Lemma 30.5.2. The second equality by functoriality of pullback and the fact that a pullback of tensor products it the tensor product of the pullbacks.

Part (3) is a special case of (2). □

0GPA Lemma 56.5.5. Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. Let $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ be an R-linear, right exact functor which commutes with arbitrary direct sums. Then we can construct

(1) a quasi-coherent module $\mathcal{K}$ on $X \times_R Y$, and
(2) a natural transformation $t : F \to F_\mathcal{K}$ where $F_\mathcal{K}$ denotes the functor (56.5.1.1)

such that $t : F \circ f_* \to F_\mathcal{K} \circ f_*$ is an isomorphism for every morphism $f : X' \to X$ whose source is an affine scheme.

Proof. Consider a morphism $f' : X' \to X$ with X' affine. Since the diagonal of X is affine, we see that f' is an affine morphism (Morphisms, Lemma 29.11.11). Thus $f'_* : QCoh(\mathcal{O}_{X'}) \to QCoh(\mathcal{O}_X)$ is an R-linear exact functor (Cohomology of Schemes, Lemma 30.2.3) which commutes with direct sums (Cohomology of Schemes, Lemma 30.6.1). Thus $F \circ f'_*$ is an R-linear, right exact functor which commutes with arbitrary direct sums. Whence $F \circ f'_* = F_{\mathcal{K}'}$ for some $\mathcal{K}'$ on $X' \times_R Y$ by Lemma 56.5.2. Moreover, given a morphism $f'' : X'' \to X'$ with X'' affine we obtain a canonical identification $(f'' \times \mathrm{id}_Y)^*\mathcal{K}' = \mathcal{K}''$ by the references already given combined with Lemma 56.5.4. These identifications satisfy a cocycle condition given another morphism $f''' : X''' \to X''$ which we leave it to the reader to spell out.

Choose an affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Since the diagonal of X is affine, we see that the intersections $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ are affine. As above the inclusion morphisms $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ are affine. Denote $\mathcal{K}_{i_0 \ldots i_p}$ the quasi-coherent module on $U_{i_0 \ldots i_p} \times_R Y$ corresponding to $F \circ j_{i_0 \ldots i_p *}$ as above. By the above we obtain identifications

$$\mathcal{K}_{i_0 \ldots i_p} = \mathcal{K}_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0 \ldots i_p} \times_R Y}$$

which satisfy the usual compatibilites for glueing. In other words, we obtain a unique quasi-coherent module $\mathcal{K}$ on $X \times_R Y$ whose restriction to $U_{i_0 \ldots i_p} \times_R Y$ is $\mathcal{K}_{i_0 \ldots i_p}$ compatible with the displayed identifications.

Next, we construct the transformation t. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$ and denote $(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 \ldots i_p}$ the restriction of $\text{pr}_1^*\mathcal{F} \otimes \mathcal{K}$ to $U_{i_0 \ldots i_p} \times_R Y$. Observe that

$$\begin{aligned} F(j_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}) &= \text{pr}_{i_0 \ldots i_p, 2, *}(\text{pr}_{i_0 \ldots i_p, 1}^* \mathcal{F}_{i_0 \ldots i_p} \otimes \mathcal{K}_{i_0 \ldots i_p}) \\ &= \text{pr}_{i_0 \ldots i_p, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 \ldots i_p} \end{aligned}$$

where $\text{pr}_{i_0 \ldots i_p, 2} : U_{i_0 \ldots i_p} \times_R Y \to Y$ is the projection and similarly for the other projection. Moreover, these identifications are compatible with the displayed identifications in the previous paragraph. Recall, from Cohomology of Schemes, Lemma 30.7.1 that the relative Čech complex

$$\bigoplus \text{pr}_{i_0, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \to \bigoplus \text{pr}_{i_0 i_1, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \to \bigoplus \text{pr}_{i_0 i_1 i_2, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1 i_2} \to \ldots$$

computes $R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})$. Hence the cohomology sheaf in degree 0 is $F_\mathcal{K}(\mathcal{F})$. Thus we obtain the desired map $t : F(\mathcal{F}) \to F_\mathcal{K}(\mathcal{F})$ by contemplating the following commutative diagram

$$\begin{tikzcd} & F(\mathcal{F}) \arrow[r] \arrow[d, dotted] & \bigoplus F(j_{i_0 *}\mathcal{F}_{i_0}) \arrow[r] \arrow[d] & \bigoplus F(j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1}) \arrow[d] \\ 0 \arrow[r] & F_\mathcal{K}(\mathcal{F}) \arrow[r] & \bigoplus \text{pr}_{i_0, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \arrow[r] & \bigoplus \text{pr}_{i_0 i_1, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \end{tikzcd}$$

We obtain the top row by applying F to the (exact) complex $0 \to \mathcal{F} \to \bigoplus j_{i_0 *}\mathcal{F}_{i_0} \to \bigoplus j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1}$ (but since F is not exact, the top row is just a complex and not necessarily exact). The solid vertical arrows are the identifications above. This does indeed define the dotted arrow as desired. The arrow is functorial in $\mathcal{F}$; we omit the details.

We still have to prove the final assertion. Let $f : X' \to X$ be as in the statement of the lemma and let $\mathcal{K}'$ be the quasi-coherent module on $X' \times_R Y$ constructed in the first paragraph of the proof. If the morphism $f : X' \to X$ maps into one of the opens U_i, then the result follows from Lemma 56.5.4 because in this case we know that $\mathcal{K}_i = \mathcal{K}|_{U_i \times_R Y}$ pulls back to $\mathcal{K}$. In general, we obtain an affine open covering $X' = \bigcup U_i'$ with $U_i' = f^{-1}(U_i)$ and we obtain isomorphisms $\mathcal{K}'|_{U_i'} = f_i^*\mathcal{K}_i$ where $f_i : U_i' \to U_i$ is the induced morphism. These morphisms satisfy the compatibility conditions needed to glue to an isomorphism $\mathcal{K}' = f^*\mathcal{K}$ and we conclude. Some details omitted. $\square$

0FZG Lemma 56.5.6. In Lemma 56.5.2 or in Lemma 56.5.5 if F is an exact functor, then the corresponding object $\mathcal{K}$ of $QCoh(\mathcal{O}_{X \times_R Y})$ is flat over X.

Proof. We may assume X is affine, so we are in the case of Lemma 56.5.2. By Lemma 56.5.4 we may assume Y is affine. In the affine case the statement translates into Remark 56.3.5. $\square$

0FZH Lemma 56.5.7. Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. There is an equivalence of categories between

(1) the category of R-linear exact functors $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which commute with arbitrary direct sums, and
(2) the full subcategory of $QCoh(\mathcal{O}_{X \times_R Y})$ consisting of $\mathcal{K}$ such that
    (a) $\mathcal{K}$ is flat over X,
    (b) for $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have $R^q\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) = 0$ for $q > 0$.

given by sending $\mathcal{K}$ to the functor F in (56.5.1.1).

Proof. Let $\mathcal{K}$ be as in (2). The functor F in (56.5.1.1) commutes with direct sums. Since by (1) (a) the modules $\mathcal{K}$ is X-flat, we see that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we obtain a short exact sequence

$$0 \to \text{pr}_1^*\mathcal{F}_1 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \to \text{pr}_1^*\mathcal{F}_2 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \to \text{pr}_1^*\mathcal{F}_3 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \to 0$$

Since by (2)(b) the higher direct image $R^1\text{pr}_{2,*}$ on the first term is zero, we conclude that $0 \to F(\mathcal{F}_1) \to F(\mathcal{F}_2) \to F(\mathcal{F}_3) \to 0$ is exact and we see that F is as in (1).

Let F be as in (1). Let $\mathcal{K}$ and $t : F \to F_\mathcal{K}$ be as in Lemma 56.5.5. By Lemma 56.5.6 we see that $\mathcal{K}$ is flat over X. To finish the proof we have to show that t is an isomorphism and the statement on higher direct images. Both of these follow from the fact that the relative Čech complex

$$\bigoplus \text{pr}_{i_0,2,*}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \to \bigoplus \text{pr}_{i_0i_1,2,*}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0i_1} \to \bigoplus \text{pr}_{i_0i_1i_2,2,*}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0i_1i_2} \to \ldots$$

computes $R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})$. Please see proof of Lemma 56.5.5 for notation and for the reason why this is so. In the proof of Lemma 56.5.5 we also found that this complex is equal to F applied to the complex

$$\bigoplus j_{i_0*}\mathcal{F}_{i_0} \to \bigoplus j_{i_0i_1*}\mathcal{F}_{i_0i_1} \to \bigoplus j_{i_0i_1i_2*}\mathcal{F}_{i_0i_1i_2} \to \ldots$$

This complex is exact except in degree zero with cohomology sheaf equal to $\mathcal{F}$. Hence since F is an exact functor we conclude $F = F_\mathcal{K}$ and that (2)(b) holds.

We omit the proof that the construction that sends F to $\mathcal{K}$ is functorial and a quasi-inverse to the functor sending $\mathcal{K}$ to the functor $F_\mathcal{K}$ determined by (56.5.1.1). □

0GPB Remark 56.5.8. Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. Lemma 56.5.7 may be generalized as follows: the functors (56.5.1.1) associated to quasi-coherent modules on $X \times_R Y$ are exactly those $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which have the following properties

(1) F is R-linear and commutes with arbitrary direct sums,
(2) $F \circ j_*$ is right exact when $j : U \to X$ is the inclusion of an affine open, and
(3) $0 \to F(\mathcal{F}) \to F(\mathcal{G}) \to F(\mathcal{H})$ is exact whenever $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence such that for all $x \in X$ the sequence on stalks $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is a split short exact sequence.

Namely, these assumptions are enough to get construct a transformation $t : F \to F_\mathcal{K}$ as in Lemma 56.5.5 and to show that it is an isomorphism. Moreover, properties (1), (2), and (3) do hold for functors (56.5.1.1). If we ever need this we will carefully state and prove this here.

0GPC Lemma 56.5.9. Let R be a ring. Let X, Y, Z be schemes over R. Assume X and Y are quasi-compact and have affine diagonal. Let

$$F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y) \quad \text{and} \quad G : QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_Z)$$

be R-linear exact functors which commute with arbitrary direct sums. Let $\mathcal{K}$ in $QCoh(\mathcal{O}_{X \times_R Y})$ and $\mathcal{L}$ in $QCoh(\mathcal{O}_{Y \times_R Z})$ be the corresponding "kernels", see Lemma 56.5.7. Then $G \circ F$ corresponds to $\text{pr}_{13,*}(\text{pr}_{12}^*\mathcal{K} \otimes_{\mathcal{O}_{X \times_R Y \times_R Z}} \text{pr}_{23}^*\mathcal{L})$ in $QCoh(\mathcal{O}_{X \times_R Z})$.

Proof. Since $G \circ F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ is R-linear, exact, and commutes with arbitrary direct sums, we find by Lemma 56.5.7 that there exists an $\mathcal{M}$ in $QCoh(\mathcal{O}_{X \times_R Z})$ corresponding to $G \circ F$. On the other hand, denote $\mathcal{E} = \text{pr}_{13,*}(\text{pr}_{12}^*\mathcal{K} \otimes \text{pr}_{23}^*\mathcal{L})$. Here and in the rest of the proof we omit the subscript from the tensor products. Let $U \subset X$ and $W \subset Z$ be affine open subschemes. To prove the lemma, we will construct an isomorphism

$$\Gamma(U \times_R W, \mathcal{E}) \cong \Gamma(U \times_R W, \mathcal{M})$$

compatible with restriction mappings for varying U and W.

First, we observe that

$$\Gamma(U \times_R W, \mathcal{E}) = \Gamma(U \times_R Y \times_R W, \text{pr}_{12}^*\mathcal{K} \otimes \text{pr}_{23}^*\mathcal{L})$$

by construction. Thus we have to show that the same thing is true for $\mathcal{M}$.

Write $U = \text{Spec}(A)$ and denote $j : U \to X$ the inclusion morphism. Recall from the construction of $\mathcal{M}$ in the proof of Lemma 56.5.2 that

$$\Gamma(U \times_R W, \mathcal{M}) = \Gamma(W, G(F(j_*\mathcal{O}_U)))$$

where the A-module action on the right hand side is given by the action of A on $\mathcal{O}_U$. The correspondence between F and $\mathcal{K}$ tells us that $F(j_*\mathcal{O}_U) = b_*(a^*j_*\mathcal{O}_U \otimes \mathcal{K})$ where $a : X \times_R Y \to X$ and $b : X \times_R Y \to Y$ are the projection morphisms. Since j is an affine morphism, we have $a^*j_*\mathcal{O}_U = (j \times \text{id}_Y)_*\mathcal{O}_{U \times_R Y}$ by Cohomology of Schemes, Lemma 30.5.1. Next, we have $(j \times \text{id}_Y)_*\mathcal{O}_{U \times_R Y} \otimes \mathcal{K} = (j \times \text{id}_Y)_*\mathcal{K}|_{U \times_R Y}$ by Remark 56.5.3 for example. Putting what we have found together we find

$$F(j_*\mathcal{O}_U) = (U \times_R Y \to Y)_*\mathcal{K}|_{U \times_R Y}$$

with obvious A-action. (This formula is implicit in the proof of Lemma 56.5.2.) Applying the functor G we obtain

$$G(F(j_*\mathcal{O}_U)) = t_*(s^*((U \times_R Y \to Y)_*\mathcal{K}|_{U \times_R Y}) \otimes \mathcal{L})$$

where $s : Y \times_R Z \to Y$ and $t : Y \times_R Z \to Z$ are the projection morphisms. Again using affine base change (Cohomology of Schemes, Lemma 30.5.1) but this time for the square

$$\begin{tikzcd}
U \times_R Y \times_R Z \arrow[r] \arrow[d] & U \times_R Y \arrow[d] \\
Y \times_R Z \arrow[r] & Y
\end{tikzcd}$$

we obtain

$$s^*((U \times_R Y \to Y)_*\mathcal{K}|_{U \times_R Y}) = (U \times_R Y \times_R Z \to Y \times_R Z)_*\text{pr}_{12}^*\mathcal{K}|_{U \times_R Y \times_R Z}$$

Using Remark 56.5.3 again we find

$$\begin{aligned}
&(U \times_R Y \times_R Z \to Y \times_R Z)_*\text{pr}_{12}^*\mathcal{K}|_{U \times_R Y \times_R Z} \otimes \mathcal{L} \\
&= (U \times_R Y \times_R Z \to Y \times_R Z)_* \left(\text{pr}_{12}^*\mathcal{K} \otimes \text{pr}_{23}^*\mathcal{L}\right)|_{U \times_R Y \times_R Z}
\end{aligned}$$

Applying the functor $\Gamma(W, t_*(-)) = \Gamma(Y \times_R W, -)$ to this we obtain

$$\begin{aligned}\Gamma(U \times_R W, \mathcal{M}) &= \Gamma(W, G(F(j_*\mathcal{O}_U))) \\ &= \Gamma(Y \times_R W, (U \times_R Y \times_R Z \to Y \times_R Z)_*(\text{pr}_{12}^*\mathcal{K} \otimes \text{pr}_{23}^*\mathcal{L})|_{U \times_R Y \times_R Z}) \\ &= \Gamma(U \times_R Y \times_R W, \text{pr}_{12}^*\mathcal{K} \otimes \text{pr}_{23}^*\mathcal{L})\end{aligned}$$

as desired. We omit the verication that these isomorphisms are compatible with restriction mappings. $\square$

0FZI Lemma 56.5.10. Let R, X, Y, and $\mathcal{K}$ be as in Lemma 56.5.7 part (2). Then for any scheme T over R we have

$$R^q\text{pr}_{13,*}(\text{pr}_{12}^*\mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} \text{pr}_{23}^*\mathcal{K}) = 0$$

for $\mathcal{F}$ quasi-coherent on $T \times_R X$ and $q > 0$.

Proof. The question is local on T hence we may assume T is affine. In this case we can consider the diagram

$$\begin{tikzcd} T \times_R X \arrow[d] & T \times_R X \times_R Y \arrow[l] \arrow[r] \arrow[d] & T \times_R Y \arrow[d] \\ X & X \times_R Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

whose vertical arrows are affine. In particular the pushforward along $T \times_R Y \to Y$ is faithful and exact (Cohomology of Schemes, Lemma 30.2.3 and Morphisms, Lemma 29.11.6). Chasing around in the diagram using that higher direct images along affine morphisms vanish (see reference above) we see that it suffices to prove

$$R^q\text{pr}_{2,*}(\text{pr}_{23,*}(\text{pr}_{12}^*\mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} \text{pr}_{23}^*\mathcal{K})) = R^q\text{pr}_{2,*}(\text{pr}_{23,*}(\text{pr}_{12}^*\mathcal{F}) \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}))$$

is zero which is true by assumption on $\mathcal{K}$. The equality holds by Remark 56.5.3. $\square$

0FZJ Lemma 56.5.11. In Lemma 56.5.7 let F and $\mathcal{K}$ correspond. If X is separated and flat over R, then there is a surjection $\mathcal{O}_X \boxtimes F(\mathcal{O}_X) \to \mathcal{K}$.

Proof. Let $\Delta : X \to X \times_R X$ be the diagonal morphism and set $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$. Since Δ is a closed immersion have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X \times_R X} \to \mathcal{O}_\Delta \to 0$$

Since $\mathcal{K}$ is flat over X, the pullback $\text{pr}_{23}^*\mathcal{K}$ to $X \times_R X \times_R Y$ is flat over $X \times_R X$. We obtain a short exact sequence

$$0 \to \text{pr}_{12}^*\mathcal{I} \otimes \text{pr}_{23}^*\mathcal{K} \to \text{pr}_{23}^*\mathcal{K} \to \text{pr}_{12}^*\mathcal{O}_\Delta \otimes \text{pr}_{23}^*\mathcal{K} \to 0$$

on $X \times_R X \times_R Y$, see Modules, Lemma 17.20.4. Thus, by Lemma 56.5.10 we obtain a surjection

$$\text{pr}_{13,*}(\text{pr}_{23}^*\mathcal{K}) \to \text{pr}_{13,*}(\text{pr}_{12}^*\mathcal{O}_\Delta \otimes \text{pr}_{23}^*\mathcal{K})$$

By flat base change (Cohomology of Schemes, Lemma 30.5.2) the source of this arrow is equal to $\text{pr}_2^*\text{pr}_{2,*}\mathcal{K} = \mathcal{O}_X \boxtimes F(\mathcal{O}_X)$. On the other hand the target is equal to

$$\text{pr}_{13,*}(\text{pr}_{12}^*\mathcal{O}_\Delta \otimes \text{pr}_{23}^*\mathcal{K}) = \text{pr}_{13,*}(\Delta \times \text{id}_Y)_*\mathcal{K} = \mathcal{K}$$

which finishes the proof. The first equality holds for example by Cohomology, Lemma 20.54.4 and the fact that $\text{pr}_{12}^*\mathcal{O}_\Delta = (\Delta \times \text{id}_Y)_*\mathcal{O}_{X \times_R Y}$. $\square$

56.6. Gabriel-Rosenberg reconstruction

0GPD The title of this section refers to results like Proposition 56.6.6. Besides Gabriel's original paper [Gab62], please consult [Bra18] which has a proof of the result for quasi-separated schemes and discusses the literature. In this section we will only prove Gabriel-Rosenberg reconstruction for quasi-compact and quasi-separated schemes.

0GPE **Lemma 56.6.1.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a categorically compact object of $QCoh(\mathcal{O}_X)$ if and only if $\mathcal{F}$ is of finite presentation.*

Proof. See Categories, Definition 4.26.1 for our notion of categorically compact objects in a category. If $\mathcal{F}$ is of finite presentation then it is categorically compact by Modules, Lemma 17.22.8. Conversely, any quasi-coherent module $\mathcal{F}$ can be written as a filtered colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ of finitely presented (hence quasi-coherent) $\mathcal{O}_X$-modules, see Properties, Lemma 28.22.7. If $\mathcal{F}$ is categorically compact, then we find some i and a morphism $\mathcal{F} \to \mathcal{F}_i$ which is a right inverse to the given map $\mathcal{F}_i \to \mathcal{F}$. We conclude that $\mathcal{F}$ is a direct summand of a finitely presented module, and hence finitely presented itself. $\square$

0GPF **Lemma 56.6.2.** *Let X be an affine scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{E}$ be a nonzero quasi-coherent $\mathcal{O}_X$-module. If $\text{Supp}(\mathcal{E}) \subset \text{Supp}(\mathcal{F})$, then there exists a nonzero map $\mathcal{F} \to \mathcal{E}$.*

Proof. Let us translate the statement into algebra. Let A be a ring. Let M be a finitely presented A-module. Let N be a nonzero A-module. Assume $\text{Supp}(N) \subset \text{Supp}(M)$. To show: $\text{Hom}_A(M, N)$ is nonzero. We may assume $N = A/I$ is cyclic (replace N by any nonzero cyclic submodule). Choose a presentation

$$A^{\oplus m} \xrightarrow{T} A^{\oplus n} \to M \to 0$$

Recall that $\text{Supp}(M)$ is cut out by $\text{Fit}_0(M)$ which is the ideal generated by the $n \times n$ minors of the matrix T. See More on Algebra, Lemma 15.8.4. The assumption $\text{Supp}(N) \subset \text{Supp}(M)$ now means that the elements of $\text{Fit}_0(M)$ are nilpotent in A/I. Consider the exact sequence

$$0 \to \text{Hom}_A(M, A/I) \to (A/I)^{\oplus n} \xrightarrow{T^t} (A/I)^{\oplus m}$$

We have to show that T^t cannot be injective; we urge the reader to find their own proof of this using the nilpotency of elements of $\text{Fit}_0(M)$ in A/I. Here is our proof. Since $\text{Fit}_0(M)$ is finitely generated, the nilpotency means that the annihilator $J \subset A/I$ of $\text{Fit}_0(M)$ in A/I is nonzero. To show the non-injectivity of T^t we may localize at a prime. Choosing a suitable prime we may assume A is local and J is still nonzero. Then T^t has a nonzero kernel by More on Algebra, Lemma 15.15.6. $\square$

0GPG **Lemma 56.6.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. The following two subcategories of $QCoh(\mathcal{O}_X)$ are equal*

1. *the full subcategory $\mathcal{A} \subset QCoh(\mathcal{O}_X)$ whose objects are the quasi-coherent modules whose support is (set theoretically) contained in $\text{Supp}(\mathcal{F})$,*
2. *the smallest Serre subcategory $\mathcal{B} \subset QCoh(\mathcal{O}_X)$ containing $\mathcal{F}$ closed under extensions and arbitrary direct sums.*

Proof. Observe that the statement makes sense as finitely presented $\mathcal{O}_X$-modules are quasi-coherent. Since $\mathcal{A}$ is a Serre subcategory closed under extensions and direct sums and since $\mathcal{F}$ is an object of $\mathcal{A}$ we see that $\mathcal{B} \subset \mathcal{A}$. Thus it remains to show that $\mathcal{A}$ is contained in $\mathcal{B}$.

Let $\mathcal{E}$ be an object of $\mathcal{A}$. There exists a maximal submodule $\mathcal{E}' \subset \mathcal{E}$ which is in $\mathcal{B}$. Namely, suppose $\mathcal{E}_i \subset \mathcal{E}$, $i \in I$ is the set of subobjects which are objects of $\mathcal{B}$. Then $\bigoplus \mathcal{E}_i$ is in $\mathcal{B}$ and so is

$$\mathcal{E}' = \mathrm{Im}(\bigoplus \mathcal{E}_i \longrightarrow \mathcal{E})$$

This is clearly the maximal submodule we were looking for.

Now suppose that we have a nonzero map $\mathcal{G} \to \mathcal{E}/\mathcal{E}'$ with $\mathcal{G}$ in $\mathcal{B}$. Then $\mathcal{G}' = \mathcal{E} \times_{\mathcal{E}/\mathcal{E}'} \mathcal{G}$ is in $\mathcal{B}$ as an extension of $\mathcal{E}'$ and $\mathcal{G}$. Then the image $\mathcal{G}' \to \mathcal{E}$ would be strictly bigger than $\mathcal{E}'$, contradicting the maximality of $\mathcal{E}'$. Thus it suffices to show the claim in the following paragraph.

Let $\mathcal{E}$ be an nonzero object of $\mathcal{A}$. We claim that there is a nonzero map $\mathcal{G} \to \mathcal{E}$ with $\mathcal{G}$ in $\mathcal{B}$. We will prove this by induction on the minimal number n of affine opens U_i of X such that $\mathrm{Supp}(\mathcal{E}) \subset U_1 \cup \ldots \cup U_n$. Set $U = U_n$ and denote $j : U \to X$ the inclusion morphism. Denote $\mathcal{E}' = \mathrm{Im}(\mathcal{E} \to j_*\mathcal{E}|_U)$. Then the kernel $\mathcal{E}''$ of the surjection $\mathcal{E} \to \mathcal{E}'$ has support contained in $U_1 \cup \ldots \cup U_{n-1}$. Thus if $\mathcal{E}''$ is nonzero, then we win. In other words, we may assume that $\mathcal{E} \subset j_*\mathcal{E}|_U$. In particular, we see that $\mathcal{E}|_U$ is nonzero. By Lemma 56.6.2 there exists a nonzero map $\mathcal{F}|_U \to \mathcal{E}|_U$. This corresponds to a map

$$\varphi : \mathcal{F} \longrightarrow j_*(\mathcal{E}|_U)$$

whose restriction to U is nonzero. Setting $\mathcal{G} = \varphi^{-1}(\mathcal{E})$ we conclude. □

0GPH Lemma 56.6.4. Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subset such that $U = X \setminus Z$ is quasi-compact. Let $\mathcal{A} \subset QCoh(\mathcal{O}_X)$ be the full subcategory whose objects are the quasi-coherent modules supported on Z. Then the restriction functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ induces an equivalence $QCoh(\mathcal{O}_X)/\mathcal{A} \cong QCoh(\mathcal{O}_U)$.

Proof. By the universal property of the quotient construction (Homology, Lemma 12.10.6) we certainly obtain an induced functor $QCoh(\mathcal{O}_X)/\mathcal{A} \cong QCoh(\mathcal{O}_U)$. Denote $j : U \to X$ the inclusion morphism. Since j is quasi-compact and quasi-separated we obtain a functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$. The reader shows that this defines a quasi-inverse; details omitted. □

0GPI Lemma 56.6.5. Let X be a quasi-compact and quasi-separated scheme. If $QCoh(\mathcal{O}_X)$ is equivalent to the category of modules over a ring, then X is affine.

Proof. Say $F : \mathrm{Mod}_R \to QCoh(\mathcal{O}_X)$ is an equivalence. Then $\mathcal{F} = F(R)$ has the following properties:

(1) it is a finitely presented $\mathcal{O}_X$-module (Lemma 56.6.1),
(2) $\mathrm{Hom}_X(\mathcal{F}, -)$ is exact,
(3) $\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$ is a commutative ring,
(4) every object of $QCoh(\mathcal{O}_X)$ is a quotient of a direct sum of copies of $\mathcal{F}$.

Let $x \in X$ be a closed point. Consider the surjection

$$\mathcal{O}_X \to i_*\kappa(x)$$

where the target is the pushforward of $\kappa(x)$ by the inclusion morphism $i : x \to X$. We have

$$\mathrm{Hom}_X(\mathcal{F}, i_*\kappa(x)) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \kappa(x))$$

This first by (4) implies that $\mathcal{F}_x$ is nonzero. From (2) we deduce that every map $\mathcal{F}_x \to \kappa(x)$ lifts to a map $\mathcal{F}_x \to \mathcal{O}_{X,x}$ (as it even lifts to a global map $\mathcal{F} \to \mathcal{O}_X$). Since $\mathcal{F}_x$ is a finite $\mathcal{O}_{X,x}$-module, this implies that $\mathcal{F}_x$ is a (nonzero) finite free $\mathcal{O}_{X,x}$-module. Then since $\mathcal{F}$ is of finite presentation, this implies that $\mathcal{F}$ is finite free of positive rank in an open neighbourhood of x (Modules, Lemma 17.11.6). Since every closed subset of X contains a closed point (Topology, Lemma 5.12.8) this implies that $\mathcal{F}$ is finite locally free of positive rank. Similarly, the map

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}_X(\mathcal{F}, i_*i^*\mathcal{F}) = \mathrm{Hom}_{\kappa(x)}(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x, \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x)$$

is surjective. By property (3) we conclude that the rank $\mathcal{F}_x$ must be 1. Hence $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module. But then we conclude that the functor

$$\mathcal{H} \longmapsto \Gamma(X, \mathcal{H}) = \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{H}) = \mathrm{Hom}_X(\mathcal{F}, \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F})$$

on $QCoh(\mathcal{O}_X)$ is exact too. This implies that the first Ext group

$$\mathrm{Ext}^1_{QCoh(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{H}) = 0$$

computed in the abelian category $QCoh(\mathcal{O}_X)$ vanishes for all $\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. However, since $QCoh(\mathcal{O}_X) \subset \mathrm{Mod}(\mathcal{O}_X)$ is closed under extensions (Schemes, Section 26.24) we see that Ext^1 between quasi-coherent modules computed in $QCoh(\mathcal{O}_X)$ is the same as computed in $\mathrm{Mod}(\mathcal{O}_X)$. Hence we conclude that

$$H^1(X, \mathcal{H}) = \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{H}) = 0$$

for all $\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. This implies that X is affine for example by Cohomology of Schemes, Lemma 30.3.1. □

0GPJ Proposition 56.6.6. Let X and Y be quasi-compact and quasi-separated schemes. If $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is an equivalence, then there exists an isomorphism $f : Y \to X$ of schemes and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Special case of [Bra18, Theorem 1.2]

Proof. Of course F is additive, exact, commutes with all limits, commutes with all colimits, commutes with direct sums, etc. Let $U \subset X$ be an affine open subscheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type quasi-coherent sheaf of ideals such that $Z = V(\mathcal{I})$ is the complement of U in X, see Properties, Lemma 28.24.1. Then $\mathcal{O}_X/\mathcal{I}$ is a finitely presented $\mathcal{O}_X$-module. Hence $\mathcal{G} = F(\mathcal{O}_X/\mathcal{I})$ is a finitely presented $\mathcal{O}_Y$-module by Lemma 56.6.1. Denote $T \subset Y$ the support of $\mathcal{G}$ and set $V = Y \setminus T$. Since $\mathcal{G}$ is of finite presentation, the scheme V is a quasi-compact open of Y. By Lemma 56.6.3 we see that F induces an equivalence between

(1) the full subcategory of $QCoh(\mathcal{O}_X)$ consisting of modules supported on Z, and
(2) the full subcategory of $QCoh(\mathcal{O}_Y)$ consisting of modules supported on T.

By Lemma 56.6.4 we obtain a commutative diagram

$$\begin{tikzcd} QCoh(\mathcal{O}_X) \arrow[r, "F"'] \arrow[d] & QCoh(\mathcal{O}_Y) \arrow[d] \\ QCoh(\mathcal{O}_U) \arrow[r, "F_U"] & QCoh(\mathcal{O}_V) \end{tikzcd}$$

where the vertical arrows are the restruction functors and the horizontal arrows are equivalences. By Lemma 56.6.5 we conclude that V is affine. For the affine case we have Lemma 56.3.8. Thus we find that there is an isomorphism $f_U : V \to U$ and an invertible $\mathcal{O}_V$-module $\mathcal{L}_U$ such that F_U is the functor $\mathcal{F} \mapsto f_U^*\mathcal{F} \otimes \mathcal{L}_U$.

The proof can be finished by noticing that the diagrams above satisfy an obvious compatibility with regards to inclusions of affine open subschemes of X. Thus the morphisms f_U and the invertible modules $\mathcal{L}_U$ glue. We omit the details. □

56.7. Functors between categories of coherent modules

0FZK The following lemma guarantees that we can use the material on functors between categories of quasi-coherent modules when we are given a functor between categories of coherent modules.

0FZL Lemma 56.7.1. Let X and Y be Noetherian schemes. Let $F : \text{Coh}(\mathcal{O}_X) \to \text{Coh}(\mathcal{O}_Y)$ be a functor. Then F extends uniquely to a functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which commutes with filtered colimits. If F is additive, then its extension commutes with arbitrary direct sums. If F is exact, left exact, or right exact, so is its extension.

Proof. The existence and uniqueness of the extension is a general fact, see Categories, Lemma 4.26.2. To see that the lemma applies observe that coherent modules are of finite presentation (Modules, Lemma 17.12.2) and hence categorically compact objects of $\text{Mod}(\mathcal{O}_X)$ by Modules, Lemma 17.22.8. Finally, every quasi-coherent module is a filtered colimit of coherent ones for example by Properties, Lemma 28.22.3.

Assume F is additive. If $\mathcal{F} = \bigoplus_{j \in J} \mathcal{H}_j$ with $\mathcal{H}_j$ quasi-coherent, then $\mathcal{F} = \text{colim}_{J' \subset J \text{ finite}} \bigoplus_{j \in J'} \mathcal{H}_j$. Denoting the extension of F also by F we obtain

$$\begin{aligned} F(\mathcal{F}) &= \text{colim}_{J' \subset J \text{ finite}} F(\bigoplus\nolimits_{j \in J'} \mathcal{H}_j) \\ &= \text{colim}_{J' \subset J \text{ finite}} \bigoplus\nolimits_{j \in J'} F(\mathcal{H}_j) \\ &= \bigoplus\nolimits_{j \in J} F(\mathcal{H}_j) \end{aligned}$$

Thus F commutes with arbitrary direct sums.

Suppose $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then we write $\mathcal{F}' = \bigcup \mathcal{F}'_i$ as the union of its coherent submodules, see Properties, Lemma 28.22.3. Denote $\mathcal{F}''_i \subset \mathcal{F}''$ the image of $\mathcal{F}'_i$ and denote $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \text{Ker}(\mathcal{F}'_i \to \mathcal{F}''_i)$. Then it is clear that $\mathcal{F} = \bigcup \mathcal{F}_i$ and $\mathcal{F}'' = \bigcup \mathcal{F}''_i$ and that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{F}'_i \to \mathcal{F}''_i \to 0$$

Since the extension commutes with filtered colimits we have $F(\mathcal{F}) = \text{colim}_{i \in I} F(\mathcal{F}_i)$, $F(\mathcal{F}') = \text{colim}_{i \in I} F(\mathcal{F}'_i)$, and $F(\mathcal{F}'') = \text{colim}_{i \in I} F(\mathcal{F}''_i)$. Since filtered colimits are exact (Modules, Lemma 17.3.2) we conclude that exactness properties of F are inherited by its extension. □

0GPK Lemma 56.7.2. Let X and Y be Noetherian schemes. Let $F : \text{Coh}(\mathcal{O}_X) \to \text{Coh}(\mathcal{O}_Y)$ be an equivalence of categories. Then there is an isomorphism $f : Y \to X$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.

Proof. By Lemma 56.7.1 we obtain a unique functor $F' : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ extending F. The same is true for the quasi-inverse of F and by the uniqueness we conclude that F' is an equivalence. By Proposition 56.6.6 we find an isomorphism $f : Y \to X$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F'(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Then f and $\mathcal{L}$ work for F as well. $\square$

0GPL Remark 56.7.3. In Lemma 56.7.2 if X and Y are defined over a common base ring R and F is R-linear, then the isomorphism f will be a morphism of schemes over R.

0FZM Lemma 56.7.4. Let $f : V \to X$ be a quasi-finite separated morphism of Noetherian schemes. If there exists a coherent $\mathcal{O}_V$-module $\mathcal{K}$ whose support is V such that $f_*\mathcal{K}$ is coherent and $R^qf_*\mathcal{K} = 0$, then f is finite.

Proof. By Zariski's main theorem we can find an open immersion $j : V \to Y$ over X with $\pi : Y \to X$ finite, see More on Morphisms, Lemma 37.43.3. Since π is affine the functor π_* is exact and faithful on the category of coherent $\mathcal{O}_X$-modules. Hence we see that $j_*\mathcal{K}$ is coherent and that $R^qj_*\mathcal{K}$ is zero for $q > 0$. In other words, we reduce to the case discussed in the next paragraph.

Assume f is an open immersion. We may replace X by the scheme theoretic closure of V. Assume $X \setminus V$ is nonempty to get a contradiction. Choose a generic point $\xi \in X \setminus V$ of an irreducible component of $X \setminus V$. Looking at the situation after base change by $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to X$ using flat base change and using Local Cohomology, Lemma 51.8.2 we reduce to the algebra problem discussed in the next paragraph.

Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite A-module whose support is $\mathrm{Spec}(A)$. Then $H^i_\mathfrak{m}(M) \neq 0$ for some i. This is true by Dualizing Complexes, Lemma 47.11.1 and the fact that M is not zero hence has finite depth. $\square$

The next lemma can be generalized to the case where k is a Noetherian ring and X flat over k (all other assumptions stay the same).

0FZN Lemma 56.7.5. Let k be a field. Let X, Y be finite type schemes over k with X separated. There is an equivalence of categories between

(1) the category of k-linear exact functors $F : \mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(\mathcal{O}_Y)$, and
(2) the category of coherent $\mathcal{O}_{X\times Y}$-modules $\mathcal{K}$ which are flat over X and have support finite over Y

given by sending $\mathcal{K}$ to the restriction of the functor (56.5.1.1) to $\mathrm{Coh}(\mathcal{O}_X)$.

Proof. Let $\mathcal{K}$ be as in (2). By Lemma 56.5.7 the functor F given by (56.5.1.1) is exact and k-linear. Moreover, F sends $\mathrm{Coh}(\mathcal{O}_X)$ into $\mathrm{Coh}(\mathcal{O}_Y)$ for example by Cohomology of Schemes, Lemma 30.26.10.

Let us construct the quasi-inverse to the construction. Let F be as in (1). By Lemma 56.7.1 we can extend F to a k-linear exact functor on the categories of quasi-coherent modules which commutes with arbitrary direct sums. By Lemma 56.5.7 the extension corresponds to a unique quasi-coherent module $\mathcal{K}$, flat over X, such that $R^q\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{K}) = 0$ for $q > 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$. Since $F(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-module, we conclude from Lemma 56.5.11 that $\mathcal{K}$ is coherent.

For a closed point $x \in X$ denote $\mathcal{O}_x$ the skyscraper sheaf at x with value the residue field of x. We have

$$F(\mathcal{O}_x) = \mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{O}_x \otimes \mathcal{K}) = (x \times Y \to Y)_*(\mathcal{K}|_{x \times Y})$$

Since $x \times Y \to Y$ is finite, we see that the pushforward along this morphism is faithful. Hence if $y \in Y$ is in the image of the support of $\mathcal{K}|_{x \times Y}$, then y is in the support of $F(\mathcal{O}_x)$.

Let $Z \subset X \times Y$ be the scheme theoretic support Z of $\mathcal{K}$, see Morphisms, Definition 29.5.5. We first prove that $Z \to Y$ is quasi-finite, by proving that its fibres over closed points are finite. Namely, if the fibre of $Z \to Y$ over a closed point $y \in Y$ has dimension > 0, then we can find infinitely many pairwise distinct closed points $x_1, x_2, \ldots$ in the image of $Z_y \to X$. Since we have a surjection $\mathcal{O}_X \to \bigoplus_{i=1,\ldots,n} \mathcal{O}_{x_i}$ we obtain a surjection

$$F(\mathcal{O}_X) \to \bigoplus\nolimits_{i=1,\ldots,n} F(\mathcal{O}_{x_i})$$

By what we said above, the point y is in the support of each of the coherent modules $F(\mathcal{O}_{x_i})$. Since $F(\mathcal{O}_X)$ is a coherent module, this will lead to a contradiction because the stalk of $F(\mathcal{O}_X)$ at y will be generated by $< n$ elements if n is large enough. Hence $Z \to Y$ is quasi-finite. Since $\mathrm{pr}_{2,*}\mathcal{K}$ is coherent and $R^q\mathrm{pr}_{2,*}\mathcal{K} = 0$ for $q > 0$ we conclude that $Z \to Y$ is finite by Lemma 56.7.4. $\square$

0FZP Lemma 56.7.6. Let $f : X \to Y$ be a finite type separated morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent module on X with support finite over Y and with $\mathcal{L} = f_*\mathcal{F}$ an invertible $\mathcal{O}_X$-module. Then there exists a section $s : Y \to X$ such that $\mathcal{F} \cong s_*\mathcal{L}$.

Proof. Looking affine locally this translates into the following algebra problem. Let $A \to B$ be a ring map and let N be a B-module which is invertible as an A-module. Then the annihilator J of N in B has the property that $A \to B/J$ is an isomorphism. We omit the details. $\square$

0FZQ Lemma 56.7.7. Let $f : X \to Y$ be a finite type separated morphism of schemes with a section $s : Y \to X$. Let $\mathcal{F}$ be a finite type quasi-coherent module on X, set theoretically supported on $s(Y)$ with $\mathcal{L} = f_*\mathcal{F}$ an invertible $\mathcal{O}_X$-module. If Y is reduced, then $\mathcal{F} \cong s_*\mathcal{L}$.

Proof. By Lemma 56.7.6 there exists a section $s' : Y \to X$ such that $\mathcal{F} = s'_*\mathcal{L}$. Since $s'(Y)$ and $s(Y)$ have the same underlying closed subset and since both are reduced closed subschemes of X, they have to be equal. Hence $s = s'$ and the lemma holds. $\square$

0FZR Lemma 56.7.8. Let k be a field. Let X, Y be finite type schemes over k with X separated and Y reduced. If there is a k-linear equivalence $F : \mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(\mathcal{O}_Y)$ of categories, then there is an isomorphism $f : Y \to X$ over k and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.

Weak version of the result in [Gab62] stating that the category of quasi-coherent modules determines the isomorphism class of a scheme.

Proof using Gabriel-Rosenberg reconstruction. This lemma is a weak form of the results discussed in Lemma 56.7.2 and Remark 56.7.3. $\square$

Proof not relying on Gabriel-Rosenberg reconstruction. By Lemma 56.7.5 we obtain a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{K}$ which is flat over X with support finite over Y such that F is given by the restriction of the functor (56.5.1.1) to $\mathrm{Coh}(\mathcal{O}_X)$. If we can show that $F(\mathcal{O}_X)$ is an invertible $\mathcal{O}_Y$-module, then by Lemma 56.7.6 we see that $\mathcal{K} = s_*\mathcal{L}$ for some section $s : Y \to X \times Y$ of pr_2 and some invertible $\mathcal{O}_Y$-module $\mathcal{L}$. This will show that F has the form indicated with $f = \text{pr}_1 \circ s$. Some details omitted.

It remains to show that $F(\mathcal{O}_X)$ is invertible. We only sketch the proof and we omit some of the details. For a closed point $x \in X$ we denote $\mathcal{O}_x$ in $\text{Coh}(\mathcal{O}_X)$ the skyscraper sheaf at x with value $\kappa(x)$. First we observe that the only simple objects of the category $\text{Coh}(\mathcal{O}_X)$ are these skyscraper sheaves $\mathcal{O}_x$. The same is true for Y. Hence for every closed point $y \in Y$ there exists a closed point $x \in X$ such that $\mathcal{O}_y \cong F(\mathcal{O}_x)$. Moreover, looking at endomorphisms we find that $\kappa(x) \cong \kappa(y)$ as finite extensions of k. Then

$$\text{Hom}_Y(F(\mathcal{O}_X), \mathcal{O}_y) \cong \text{Hom}_Y(F(\mathcal{O}_X), F(\mathcal{O}_x)) \cong \text{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) \cong \kappa(x) \cong \kappa(y)$$

This implies that the stalk of the coherent $\mathcal{O}_Y$-module $F(\mathcal{O}_X)$ at $y \in Y$ can be generated by 1 generator (and no less) for each closed point $y \in Y$. It follows immediately that $F(\mathcal{O}_X)$ is locally generated by 1 element (and no less) and since Y is reduced this indeed tells us it is an invertible module. □

56.8. Other chapters

CHAPTER 57

Derived Categories of Varieties

0FY0 57.1. Introduction

0FY1 In this chapter we continue the discussion started in Derived Categories of Schemes, Section 36.1. We will discuss Fourier-Mukai transforms, first studied by Mukai in [Muk81]. We will prove Orlov's theorem on derived equivalences ([Orl97]). We also discuss the countability of derived equivalence classes proved by Anel and Toën in [AT09].

A good introduction to this material is the book [Huy06] by Daniel Huybrechts. Some other papers which helped popularize this topic are

(1) the paper by Bondal and Kapranov, see [BK89]
(2) the paper by Bondal and Orlov, see [BO01]
(3) the paper by Bondal and Van den Bergh, see [BV03]
(4) the papers by Beilinson, see [Bei78] and [Bei84]
(5) the paper by Orlov, see [Orl02]
(6) the paper by Orlov, see [Orl05]
(7) the paper by Rouquier, see [Rou08]
(8) there are many more we could mention here.

57.2. Conventions and notation

0FY2 Let k be a field. A k-linear triangulated category $\mathcal{T}$ is a triangulated category (Derived Categories, Section 13.3) which is endowed with a k-linear structure (Differential Graded Algebra, Section 22.24) such that the translation functors $[n] : \mathcal{T} \to \mathcal{T}$ are k-linear for all $n \in \mathbf{Z}$.

Let k be a field. We denote Vect_k the category of k-vector spaces. For a k-vector space V we denote $V^\vee$ the k-linear dual of V, i.e., $V^\vee = \text{Hom}_k(V, k)$.

Let X be a scheme. We denote $D_{perf}(\mathcal{O}_X)$ the full subcategory of $D(\mathcal{O}_X)$ consisting of perfect complexes (Cohomology, Section 20.49). If X is Noetherian then $D_{perf}(\mathcal{O}_X) \subset D^b_{\textit{Coh}}(\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 36.11.6. If X is Noetherian and regular, then $D_{perf}(\mathcal{O}_X) = D^b_{\textit{Coh}}(\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 36.11.8.

Let k be a field. Let X and Y be schemes over k. In this situation we will write $X \times Y$ instead of $X \times_{\text{Spec}(k)} Y$.

Let S be a scheme. Let X, Y be schemes over S. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module and let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. We set

$$\mathcal{F} \boxtimes \mathcal{G} = \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} \text{pr}_2^*\mathcal{G}$$

as $\mathcal{O}_{X\times_S Y}$-modules. If $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$ then we set

$$K \boxtimes M = L\mathrm{pr}_1^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} L\mathrm{pr}_2^*M$$

as an object of $D(\mathcal{O}_{X\times_S Y})$. Thus our notation is potentially ambiguous, but context should make it clear which of the two is meant.

57.3. Serre functors

0FY3 The material in this section is taken from [BK89].

0FY4 Lemma 57.3.1. Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$. The following are equivalent

1. there exists a k-linear equivalence $S : \mathcal{T} \to \mathcal{T}$ and k-linear isomorphisms $c_{X,Y} : \mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(Y, S(X))^\vee$ functorial in $X, Y \in \mathrm{Ob}(\mathcal{T})$,
2. for every $X \in \mathrm{Ob}(\mathcal{T})$ the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$ is representable and the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(Y, X)^\vee$ is corepresentable.

Proof. Condition (1) implies (2) since given (S, c) and $X \in \mathrm{Ob}(\mathcal{T})$ the object $S(X)$ represents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$ and the object $S^{-1}(X)$ corepresents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(Y, X)^\vee$.

Assume (2). We will repeatedly use the Yoneda lemma, see Categories, Lemma 4.3.5. For every X denote $S(X)$ the object representing the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$. Given $\varphi : X \to X'$, we obtain a unique arrow $S(\varphi) : S(X) \to S(X')$ determined by the corresponding transformation of functors $\mathrm{Hom}_{\mathcal{T}}(X, -)^\vee \to \mathrm{Hom}_{\mathcal{T}}(X', -)^\vee$. Thus S is a functor and we obtain the isomorphisms $c_{X,Y}$ by construction. It remains to show that S is an equivalence. For every X denote $S'(X)$ the object corepresenting the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(Y, X)^\vee$. Arguing as above we find that S' is a functor. We claim that S' is quasi-inverse to S. To see this observe that

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(Y, S(X))^\vee = \mathrm{Hom}_{\mathcal{T}}(S'(S(X)), Y)$$

bifunctorially, i.e., we find $S' \circ S \cong \mathrm{id}_{\mathcal{T}}$. Similarly, we have

$$\mathrm{Hom}_{\mathcal{T}}(Y, X) = \mathrm{Hom}_{\mathcal{T}}(S'(X), Y)^\vee = \mathrm{Hom}_{\mathcal{T}}(Y, S(S'(X)))$$

and we find $S \circ S' \cong \mathrm{id}_{\mathcal{T}}$. □

0FY5 Definition 57.3.2. Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$. We say a Serre functor exists if the equivalent conditions of Lemma 57.3.1 are satisfied. In this case a Serre functor is a k-linear equivalence $S : \mathcal{T} \to \mathcal{T}$ endowed with k-linear isomorphisms $c_{X,Y} : \mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(Y, S(X))^\vee$ functorial in $X, Y \in \mathrm{Ob}(\mathcal{T})$.

0FY6 Lemma 57.3.3. In the situation of Definition 57.3.2. If a Serre functor exists, then it is unique up to unique isomorphism and it is an exact functor of triangulated categories.

Proof. Given a Serre functor S the object $S(X)$ represents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$. Thus the object $S(X)$ together with the functorial identification $\mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee = \mathrm{Hom}_{\mathcal{T}}(Y, S(X))$ is determined up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5). Moreover, for $\varphi : X \to X'$, the arrow $S(\varphi) : S(X) \to S(X')$ is uniquely determined by the corresponding transformation of functors $\mathrm{Hom}_{\mathcal{T}}(X, -)^\vee \to \mathrm{Hom}_{\mathcal{T}}(X', -)^\vee$.

For objects X, Y of $\mathcal{T}$ we have

$$\begin{aligned}
\mathrm{Hom}(Y, S(X)[1])^\vee &= \mathrm{Hom}(Y[-1], S(X))^\vee \\
&= \mathrm{Hom}(X, Y[-1]) \\
&= \mathrm{Hom}(X[1], Y) \\
&= \mathrm{Hom}(Y, S(X[1]))^\vee
\end{aligned}$$

By the Yoneda lemma we conclude that there is a unique isomorphism $S(X[1]) \to S(X)[1]$ inducing the isomorphism from top left to bottom right. Since each of the isomorphisms above is functorial in both X and Y we find that this defines an isomorphism of functors $S \circ [1] \to [1] \circ S$.

Let (A, B, C, f, g, h) be a distinguished triangle in $\mathcal{T}$. We have to show that the triangle $(S(A), S(B), S(C), S(f), S(g), S(h))$ is distinguished. Here we use the canonical isomorphism $S(A[1]) \to S(A)[1]$ constructed above to identify the target $S(A[1])$ of $S(h)$ with $S(A)[1]$. We first observe that for any X in $\mathcal{T}$ the triangle $(S(A), S(B), S(C), S(f), S(g), S(h))$ induces a long exact sequence

$$\ldots \to \mathrm{Hom}(X, S(A)) \to \mathrm{Hom}(X, S(B)) \to \mathrm{Hom}(X, S(C)) \to \mathrm{Hom}(X, S(A)[1]) \to \ldots$$

of finite dimensional k-vector spaces. Namely, this sequence is k-linear dual of the sequence

$$\ldots \leftarrow \mathrm{Hom}(A, X) \leftarrow \mathrm{Hom}(B, X) \leftarrow \mathrm{Hom}(C, X) \leftarrow \mathrm{Hom}(A[1], X) \leftarrow \ldots$$

which is exact by Derived Categories, Lemma 13.4.2. Next, we choose a distinguished triangle $(S(A), E, S(C), i, p, S(h))$ which is possible by axioms TR1 and TR2. We want to construct the dotted arrow making following diagram commute

$$\begin{tikzcd}
S(C)[-1] \arrow[r, "S(h[-1])"'] \arrow[d, equal] & S(A) \arrow[r, "S(f)"'] \arrow[d, equal] & S(B) \arrow[r, "S(g)"'] & S(C) \arrow[r, "S(h)"'] \arrow[d, equal] & S(A)[1] \arrow[d, equal] \\
S(C)[-1] \arrow[r, "S(h[-1])"] & S(A) \arrow[r, "i"] & E \arrow[r, "p"] \arrow[u, dotted, "\varphi"] & S(C) \arrow[r, "S(h)"] & S(A)[1]
\end{tikzcd}$$

Namely, if we have φ, then we claim for any X the resulting map $\mathrm{Hom}(X, E) \to \mathrm{Hom}(X, S(B))$ will be an isomorphism of k-vector spaces. Namely, we will obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(X, S(C)[-1]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(X, S(A)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(X, S(B)) \arrow[r] & \mathrm{Hom}(X, S(C)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(X, S(A)[1]) \arrow[d, equal] \\
\mathrm{Hom}(X, S(C)[-1]) \arrow[r] & \mathrm{Hom}(X, S(A)) \arrow[r] & \mathrm{Hom}(X, E) \arrow[r] \arrow[u, "\varphi"] & \mathrm{Hom}(X, S(C)) \arrow[r] & \mathrm{Hom}(X, S(A)[1])
\end{tikzcd}$$

with exact rows (see above) and we can apply the 5 lemma (Homology, Lemma 12.5.20) to see that the middle arrow is an isomorphism. By the Yoneda lemma we conclude that φ is an isomorphism. To find φ consider the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(E, S(C)) \arrow[r] & \mathrm{Hom}(S(A), S(C)) \\
\mathrm{Hom}(E, S(B)) \arrow[r] \arrow[u] & \mathrm{Hom}(S(A), S(B)) \arrow[u]
\end{tikzcd}$$

The elements p and $S(f)$ in positions $(0, 1)$ and $(1, 0)$ define a cohomology class ξ in the total complex of this double complex. The existence of φ is equivalent to whether ξ is zero. If we take k-linear duals of this and we use the defining property of S we obtain

$$\begin{tikzcd} \mathrm{Hom}(C, E) \arrow[d] & \mathrm{Hom}(C, S(A)) \arrow[l] \arrow[d] \\ \mathrm{Hom}(B, E) & \mathrm{Hom}(B, S(A)) \arrow[l] \end{tikzcd}$$

Since both $A \to B \to C$ and $S(A) \to E \to S(C)$ are distinguished triangles, we know by TR3 that given elements $\alpha \in \mathrm{Hom}(C, E)$ and $\beta \in \mathrm{Hom}(B, S(A))$ mapping to the same element in $\mathrm{Hom}(B, E)$, there exists an element in $\mathrm{Hom}(C, S(A))$ mapping to both α and β. In other words, the cohomology of the total complex associated to this double complex is zero in degree 1, i.e., the degree corresponding to $\mathrm{Hom}(C, E) \oplus \mathrm{Hom}(B, S(A))$. Taking duals the same must be true for the previous one which concludes the proof. □

57.4. Examples of Serre functors

0FY7 The lemma below is the standard example.

0FY8 **Lemma 57.4.1.** Let k be a field. Let X be a proper scheme over k which is Gorenstein. Consider the complex $\omega_X^\bullet$ of Duality for Schemes, Lemmas 48.27.1. Then the functor

$$S : D_{perf}(\mathcal{O}_X) \longrightarrow D_{perf}(\mathcal{O}_X), \quad K \longmapsto S(K) = \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K$$

is a Serre functor.

Proof. The statement make sense because $\dim \mathrm{Hom}_X(K, L) < \infty$ for $K, L \in D_{perf}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.7. Since X is Gorenstein the dualizing complex $\omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$, see Duality for Schemes, Lemma 48.24.4. In particular, locally on X the complex $\omega_X^\bullet$ has one nonzero cohomology sheaf which is an invertible module, see Cohomology, Lemma 20.52.2. Thus $S(K)$ lies in $D_{perf}(\mathcal{O}_X)$. On the other hand, the invertibility of $\omega_X^\bullet$ clearly implies that S is a self-equivalence of $D_{perf}(\mathcal{O}_X)$. Finally, we have to find an isomorphism

$$c_{K,L} : \mathrm{Hom}_X(K, L) \longrightarrow \mathrm{Hom}_X(L, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K)^\vee$$

bifunctorially in K, L. To do this we use the canonical isomorphisms

$$\mathrm{Hom}_X(K, L) = H^0(X, L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee)$$

and

$$\mathrm{Hom}_X(L, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K) = H^0(X, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L^\vee)$$

given in Cohomology, Lemma 20.50.5. Since $(L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee)^\vee = (K^\vee)^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L^\vee$ and since there is a canonical isomorphism $K \to (K^\vee)^\vee$ we find these k-vector spaces are canonically dual by Duality for Schemes, Lemma 48.27.4. This produces the isomorphisms $c_{K,L}$. We omit the proof that these isomorphisms are functorial. □

57.5. Characterizing coherent modules

0FY9 This section is in some sense a continuation of the discussion in Derived Categories of Schemes, Section 36.34 and More on Morphisms, Section 37.69.

Before we can state the result we need some notation. Let k be a field. Let $n \geq 0$ be an integer. Let $S = k[X_0, \ldots, X_n]$. For an integer e denote $S_e \subset S$ the homogeneous polynomials of degree e. Consider the (noncommutative) k-algebra

$$R = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots & \ldots \\ 0 & S_0 & S_1 & \ldots & \ldots \\ 0 & 0 & S_0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & S_0 \end{pmatrix}$$

(with $n + 1$ rows and columns) with obvious multiplication and addition.

0FYA **Lemma 57.5.1.** With k, n, and R as above, for an object K of $D(R)$ the following are equivalent

(1) $\sum_{i \in \mathbf{Z}} \dim_k H^i(K) < \infty$, and
(2) K is a compact object.

Proof. If K is a compact object, then K can be represented by a complex $M^\bullet$ which is finite projective as a graded R-module, see Differential Graded Algebra, Lemma 22.36.6. Since $\dim_k R < \infty$ we conclude $\sum \dim_k M^i < \infty$ and a fortiori $\sum \dim_k H^i(M^\bullet) < \infty$. (One can also easily deduce this implication from the easier Differential Graded Algebra, Proposition 22.36.4.)

Assume K satisfies (1). Consider the distinguished triangle of trunctions $\tau_{\leq m}K \to K \to \tau_{\geq m+1}K$, see Derived Categories, Remark 13.12.4. It is clear that both $\tau_{\leq m}K$ and $\tau_{\geq m+1}K$ satisfy (1). If we can show both are compact, then so is K, see Derived Categories, Lemma 13.37.2. Hence, arguing on the number of nonzero cohomology modules of K we may assume $H^i(K)$ is nonzero only for one i. Shifting, we may assume K is given by the complex consisting of a single finite dimensional R-module M sitting in degree 0.

Since $\dim_k(M) < \infty$ we see that M is Artinian as an R-module. Thus it suffices to show that every simple R-module represents a compact object of $D(R)$. Observe that

$$I = \begin{pmatrix} 0 & S_1 & S_2 & \ldots & \ldots \\ 0 & 0 & S_1 & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

is a nilpotent two sided ideal of R and that R/I is a commutative k-algebra isomorphic to a product of $n + 1$ copies of k (placed along the diagonal in the matrix, i.e., R/I can be lifted to a k-subalgebra of R). It follows that R has exactly $n + 1$ isomorphism classes of simple modules $M_0, \ldots, M_n$ (sitting along the diagonal). Consider the right R-module P_i of row vectors

$$P_i = \begin{pmatrix} 0 & \ldots & 0 & S_0 & \ldots & S_{i-1} & S_i \end{pmatrix}$$

with obvious multiplication $P_i \times R \to P_i$. Then we see that $R \cong P_0 \oplus \ldots \oplus P_n$ as a right R-module. Since clearly R is a compact object of $D(R)$, we conclude each P_i is a compact object of $D(R)$. (We of course also conclude each P_i is projective as an R-module, but this isn't what we have to show in this proof.) Clearly, $P_0 = M_0$ is the first of our simple R-modules. For P_1 we have a short exact sequence

$$0 \to P_0^{\oplus n+1} \to P_1 \to M_1 \to 0$$

which proves that M_1 fits into a distinguished triangle whose other members are compact objects and hence M_1 is a compact object of $D(R)$. More generally, there exists a short exact sequence

$$0 \to C_i \to P_i \to M_i \to 0$$

where C_i is a finite dimensional R-module whose simple constituents are isomorphic to M_j for $j < i$. By induction, we first conclude that C_i determines a compact object of $D(R)$ whereupon we conclude that M_i does too as desired. □

0FYB Lemma 57.5.2. Let k be a field. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_k})$. The following are equivalent

(1) K is in $D^b_{\text{Coh}}(\mathcal{O}_{\mathbf{P}^n_k})$,
(2) $\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}^n_k, E \otimes^{\mathbf{L}} K) < \infty$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_k})$,
(3) $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}^i_{\mathbf{P}^n_k}(E, K) < \infty$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_k})$,
(4) $\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}^n_k, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_k}(d)) < \infty$ for $d = 0, 1, \ldots, n$.

Proof. Parts (2) and (3) are equivalent by Cohomology, Lemma 20.50.5. If (1) is true, then for E perfect the derived tensor product $E \otimes^{\mathbf{L}} K$ is in $D^b_{\text{Coh}}(\mathcal{O}_{\mathbf{P}^n_k})$ and we see that (2) holds by Derived Categories of Schemes, Lemma 36.11.3. It is clear that (2) implies (4) as $\mathcal{O}_{\mathbf{P}^n_k}(d)$ can be viewed as a perfect object of the derived category of $\mathbf{P}^n_k$. Thus it suffices to prove that (4) implies (1).

Assume (4). Let R be as in Lemma 57.5.1. Let $P = \bigoplus_{d=0,\ldots,n} \mathcal{O}_{\mathbf{P}^n_k}(-d)$. Recall that $R = \text{End}_{\mathbf{P}^n_k}(P)$ whereas all other self-Exts of P are zero and that P determines an equivalence $- \otimes^{\mathbf{L}} P : D(R) \to D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_k})$ by Derived Categories of Schemes, Lemma 36.20.1. Say K corresponds to L in $D(R)$. Then

$$\begin{aligned} H^i(L) &= \text{Ext}^i_{D(R)}(R, L) \\ &= \text{Ext}^i_{\mathbf{P}^n_k}(P, K) \\ &= H^i(\mathbf{P}^n_k, K \otimes P^\vee) \\ &= \bigoplus\nolimits_{d=0,\ldots,n} H^i(\mathbf{P}^n_k, K \otimes \mathcal{O}(d)) \end{aligned}$$

by Differential Graded Algebra, Lemma 22.35.4 (and the fact that $- \otimes^{\mathbf{L}} P$ is an equivalence) and Cohomology, Lemma 20.50.5. Thus our assumption (4) implies that L satisfies condition (2) of Lemma 57.5.1 and hence is a compact object of $D(R)$. Therefore K is a compact object of $D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_k})$. Thus K is perfect by Derived Categories of Schemes, Proposition 36.17.1. Since $D_{perf}(\mathcal{O}_{\mathbf{P}^n_k}) = D^b_{\text{Coh}}(\mathcal{O}_{\mathbf{P}^n_k})$ by Derived Categories of Schemes, Lemma 36.11.8 we conclude (1) holds. □

0FYC Lemma 57.5.3. Let X be a scheme proper over a field k. Let $K \in D^b_{\text{Coh}}(\mathcal{O}_X)$ and let E in $D(\mathcal{O}_X)$ be perfect. Then $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}^i_X(E, K) < \infty$.

Proof. This follows for example by combining Derived Categories of Schemes, Lemmas 36.11.7 and 36.18.2. Alternative proof: combine Derived Categories of Schemes, Lemmas 36.11.6 and 36.11.3. □

0FYD Lemma 57.5.4. Let X be a proper scheme over a field k. Let $K \in \text{Ob}(D_{QCoh}(\mathcal{O}_X))$. The following are equivalent In the projective case this is [Rou08, Lemma 7.46] and implicit in [BV03, Theorem A.1]

(1) $K \in D^b_{\text{Coh}}(\mathcal{O}_X)$, and
(2) $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}^i_X(E, K) < \infty$ for all perfect E in $D(\mathcal{O}_X)$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 57.5.3. The implication (2) $\Rightarrow$ (1) follows from More on Morphisms, Lemma 37.69.6 (see Derived Categories of Schemes, Example 36.35.2 for the meaning of a relatively perfect object over a field); the easier proof in the projective case is in the next paragraph.

Assume (2) and X projective over k. Choose a closed immersion $i : X \to \mathbf{P}^n_k$. It suffices to show that Ri_*K is in $D^b_{Coh}(\mathbf{P}^n_k)$ since a quasi-coherent module $\mathcal{F}$ on X is coherent, resp. zero if and only if $i_*\mathcal{F}$ is coherent, resp. zero. For a perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_k})$, Li^*E is a perfect object of $D(\mathcal{O}_X)$ and

$$\mathrm{Ext}^q_{\mathbf{P}^n_k}(E, Ri_*K) = \mathrm{Ext}^q_X(Li^*E, K)$$

Hence by our assumption we see that $\sum_{q \in \mathbf{Z}} \dim_k \mathrm{Ext}^q_{\mathbf{P}^n_k}(E, Ri_*K) < \infty$. We conclude by Lemma 57.5.2. $\square$

57.6. A representability theorem

0FYE The material in this section is taken from [BV03].

Let $\mathcal{T}$ be a k-linear triangulated category. In this section we consider k-linear cohomological functors H from $\mathcal{T}$ to the category of k-vector spaces. This will mean H is a functor

$$H : \mathcal{T}^{opp} \longrightarrow \mathrm{Vect}_k$$

which is k-linear such that for any distinguished triangle $X \to Y \to Z$ in $\mathcal{T}$ the sequence $H(Z) \to H(Y) \to H(X)$ is an exact sequence of k-vector spaces. See Derived Categories, Definition 13.3.5 and Differential Graded Algebra, Section 22.24.

0FYF Lemma 57.6.1. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Let $X \in \mathrm{Ob}(\mathcal{D})$. The category of arrows $E \to X$ with $E \in \mathrm{Ob}(\mathcal{D}')$ is filtered.

Proof. We check the conditions of Categories, Definition 4.19.1. The category is nonempty because it contains $0 \to X$. If $E_i \to X$, $i = 1, 2$ are objects, then $E_1 \oplus E_2 \to X$ is an object and there are morphisms $(E_i \to X) \to (E_1 \oplus E_2 \to X)$. Finally, suppose that $a, b : (E \to X) \to (E' \to X)$ are morphisms. Choose a distinguished triangle $E \xrightarrow{a-b} E' \to E''$ in $\mathcal{D}'$. By Axiom TR3 we obtain a morphism of triangles

$$\begin{tikzcd} E \arrow[r, "a-b"'] \arrow[d] & E' \arrow[r] \arrow[d] & E'' \arrow[d] \\ 0 \arrow[r] & X \arrow[r] & X \end{tikzcd}$$

and we find that the resulting arrow $(E' \to X) \to (E'' \to X)$ equalizes a and b. $\square$

0FYG Lemma 57.6.2. Let k be a field. Let $\mathcal{D}$ be a k-linear triangulated category which has direct sums and is compactly generated. Denote $\mathcal{D}_c$ the full subcategory of compact objects. Let $H : \mathcal{D}_c^{opp} \to \mathrm{Vect}_k$ be a k-linear cohomological functor such that $\dim_k H(X) < \infty$ for all $X \in \mathrm{Ob}(\mathcal{D}_c)$. Then H is isomorphic to the functor $X \mapsto \mathrm{Hom}(X, Y)$ for some $Y \in \mathrm{Ob}(\mathcal{D})$. [CKN01, Lemma 2.14]

Proof. We will use Derived Categories, Lemma 13.37.2 without further mention. Denote $G : \mathcal{D}_c \to \mathrm{Vect}_k$ the k-linear homological functor which sends X to $H(X)^\vee$. For any object Y of $\mathcal{D}$ we set

$$G'(Y) = \mathrm{colim}_{X \to Y, X \in \mathrm{Ob}(\mathcal{D}_c)} G(X)$$

The colimit is filtered by Lemma 57.6.1. We claim that G' is a k-linear homological functor, the restriction of G' to $\mathcal{D}_c$ is G, and G' sends direct sums to direct sums.

Namely, suppose that $Y_1 \to Y_2 \to Y_3$ is a distinguished triangle. Let $\xi \in G'(Y_2)$ map to zero in $G'(Y_3)$. Since the colimit is filtered ξ is represented by some $X \to Y_2$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$ and $g \in G(X)$. The fact that ξ maps to zero in $G'(Y_3)$ means the composition $X \to Y_2 \to Y_3$ factors as $X \to X' \to Y_3$ with $X' \in \mathcal{D}_c$ and g mapping to zero in $G(X')$. Choose a distinguished triangle $X'' \to X \to X'$. Then $X'' \in \mathrm{Ob}(\mathcal{D}_c)$. Since G is homological we find that g is the image of some $g'' \in G'(X'')$. By Axiom TR3 the maps $X \to Y_2$ and $X' \to Y_3$ fit into a morphism of distinguished triangles $(X'' \to X \to X') \to (Y_1 \to Y_2 \to Y_3)$ and we find that indeed ξ is the image of the element of $G'(Y_1)$ represented by $X'' \to Y_1$ and $g'' \in G(X'')$.

If $Y \in \mathrm{Ob}(\mathcal{D}_c)$, then $\mathrm{id} : Y \to Y$ is the final object in the category of arrows $X \to Y$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$. Hence we see that $G'(Y) = G(Y)$ in this case and the statement on restriction holds. Let $Y = \bigoplus_{i \in I} Y_i$ be a direct sum. Let $a : X \to Y$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$ and $g \in G(X)$ represent an element ξ of $G'(Y)$. The morphism $a : X \to Y$ can be uniquely written as a sum of morphisms $a_i : X \to Y_i$ almost all zero as X is a compact object of $\mathcal{D}$. Let $I' = \{i \in I \mid a_i \neq 0\}$. Then we can factor a as the composition

$$X \xrightarrow{(1,\ldots,1)} \bigoplus\nolimits_{i \in I'} X \xrightarrow{\bigoplus_{i \in I'} a_i} \bigoplus\nolimits_{i \in I} Y_i = Y$$

We conclude that $\xi = \sum_{i \in I'} \xi_i$ is the sum of the images of the elements $\xi_i \in G'(Y_i)$ corresponding to $a_i : X \to Y_i$ and $g \in G(X)$. Hence $\bigoplus G'(Y_i) \to G'(Y)$ is surjective. We omit the (trivial) verification that it is injective.

It follows that the functor $Y \mapsto G'(Y)^\vee$ is cohomological and sends direct sums to direct products. Hence by Brown representability, see Derived Categories, Proposition 13.38.2 we conclude that there exists a $Y \in \mathrm{Ob}(\mathcal{D})$ and an isomorphism $G'(Z)^\vee = \mathrm{Hom}(Z, Y)$ functorially in Z. For $X \in \mathrm{Ob}(\mathcal{D}_c)$ we have $G'(X)^\vee = G(X)^\vee = (H(X)^\vee)^\vee = H(X)$ because $\dim_k H(X) < \infty$ and the proof is complete. $\square$

0FYH Theorem 57.6.3. Let X be a proper scheme over a field k. Let $F : D_{perf}(\mathcal{O}_X)^{opp} \to \mathrm{Vect}_k$ be a k-linear cohomological functor such that

In the projective case this is [BV03, Theorem A.1]

$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then F is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(E, K)$ for some $K \in D^b_{\mathrm{Coh}}(\mathcal{O}_X)$.

Proof. The derived category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, is compactly generated, and $D_{perf}(\mathcal{O}_X)$ is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 36.3.1, Theorem 36.15.3, and Proposition 36.17.1. By Lemma 57.6.2 we may assume $F(E) = \mathrm{Hom}_X(E, K)$ for some $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. Then it follows that K is in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ by Lemma 57.5.4. $\square$

0H4A Lemma 57.6.4. Let X be a proper scheme over a field k which is regular. Let $G : D_{perf}(\mathcal{O}_X) \to \mathrm{Vect}_k$ be a k-linear homological functor such that

$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k G(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then G is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(K, E)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

Proof. Consider the contravariant functor $E \mapsto E^\vee$ on $D_{perf}(\mathcal{O}_X)$, see Cohomology, Lemma 20.50.5. This functor is an exact anti-self-equivalence of $D_{perf}(\mathcal{O}_X)$. Hence we may apply Theorem 57.6.3 to the functor $F(E) = G(E^\vee)$ to find $K \in D_{perf}(\mathcal{O}_X)$ such that $G(E^\vee) = \mathrm{Hom}_X(E, K)$. It follows that $G(E) = \mathrm{Hom}_X(E^\vee, K) = \mathrm{Hom}_X(K^\vee, E)$ and we conclude that taking $K^\vee$ works. $\square$

57.7. Existence of adjoints

0FYM As a consequence of the results in the paper of Bondal and van den Bergh we get the following automatic existence of adjoints.

0FYN Lemma 57.7.1. Let k be a field. Let X and Y be proper schemes over k. If X is regular, then any k-linear exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ has an exact right adjoint and an exact left adjoint.

Proof. If an adjoint exists it is an exact functor by the very general Derived Categories, Lemma 13.7.1.

Let us prove the existence of a right adjoint. To see existence, it suffices to show that for $M \in D_{perf}(\mathcal{O}_Y)$ the contravariant functor $K \mapsto \mathrm{Hom}_Y(F(K), M)$ is representable. This functor is contravariant, k-linear, and cohomological. Hence by Theorem 57.6.3 it suffices to show that

$$\sum\nolimits_{i \in \mathbf{Z}} \dim_k \mathrm{Ext}^i_Y(F(K), M) < \infty$$

This follows from Lemma 57.5.3.

For the existence of the left adjoint we argue in the same manner using Lemma 57.6.4 in stead of Theorem 57.6.3. $\square$

57.8. Fourier-Mukai functors

0FYP These functors were first introduced in [Muk81].

0FYQ Definition 57.8.1. Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X \times_S Y})$. The exact functor

$$\Phi_K : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y), \quad M \longmapsto R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K)$$

of triangulated categories is called a Fourier-Mukai functor and K is called a Fourier-Mukai kernel for this functor. Moreover,

(1) if Φ_K sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor,
(2) if Φ_K sends $D_{perf}(\mathcal{O}_X)$ into $D_{perf}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor, and
(3) if X and Y are Noetherian and Φ_K sends $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ into $D^b_{\mathrm{Coh}}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D^b_{\mathrm{Coh}}(\mathcal{O}_X) \to D^b_{\mathrm{Coh}}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor. Similarly for D_{Coh}, D^+_{Coh}, D^-_{Coh}.

0FYR Lemma 57.8.2. Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X \times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ if K is in $D_{QCoh}(\mathcal{O}_{X \times_S Y})$ and $X \to S$ is quasi-compact and quasi-separated.

Proof. This follows from the fact that derived pullback preserves D_{QCoh} (Derived Categories of Schemes, Lemma 36.3.8), derived tensor products preserve D_{QCoh} (Derived Categories of Schemes, Lemma 36.3.9), the projection $\text{pr}_2 : X \times_S Y \to Y$ is quasi-compact and quasi-separated (Schemes, Lemmas 26.19.3 and 26.21.12), and total direct image along a quasi-separated and quasi-compact morphism preserves D_{QCoh} (Derived Categories of Schemes, Lemma 36.4.1). □

0FYS Lemma 57.8.3. Let S be a scheme. Let X, Y, Z be schemes over S. Assume $X \to S$, $Y \to S$, and $Z \to S$ are quasi-compact and quasi-separated. Let $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$. Let $K' \in D_{QCoh}(\mathcal{O}_{Y \times_S Z})$. Consider the Fourier-Mukai functors $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ and $\Phi_{K'} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Z)$. If X and Z are tor independent over S and $Y \to S$ is flat, then

$$\Phi_{K'} \circ \Phi_K = \Phi_{K''} : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_Z)$$

where

$$K'' = R\text{pr}_{13,*}(L\text{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S Z}} L\text{pr}_{23}^*K')$$

in $D_{QCoh}(\mathcal{O}_{X \times_S Z})$.

Proof. The statement makes sense by Lemma 57.8.2. We are going to use Derived Categories of Schemes, Lemmas 36.3.8, 36.3.9, and 36.4.1 and Schemes, Lemmas 26.19.3 and 26.21.12 without further mention. By Derived Categories of Schemes, Lemma 36.22.4 we see that $X \times_S Y$ and $Y \times_S Z$ are tor independent over Y. This means that we have base change for the cartesian diagram

$$\begin{tikzcd} X \times_S Y \times_S Z \arrow[r] \arrow[d] & Y \times_S Z \arrow[d, "p^{YZ}_Y"] \\ X \times_S Y \arrow[r, "p^{XY}_Y"] & Y \end{tikzcd}$$

for complexes with quasi-coherent cohomology sheaves, see Derived Categories of Schemes, Lemma 36.22.5. Abbreviating $p^* = Lp^*$, $p_* = Rp_*$ and $\otimes = \otimes^{\mathbf{L}}$ we have for $M \in D_{QCoh}(\mathcal{O}_X)$ the sequence of equalities

$$\begin{aligned} \Phi_{K'}(\Phi_K(M)) &= p^{YZ}_{Z,*}(p^{YZ,*}_Y p^{XY}_{Y,*}(p^{XY,*}_X M \otimes K) \otimes K') \\ &= p^{YZ}_{Z,*}(\text{pr}_{23,*}\text{pr}_{12}^*(p^{XY,*}_X M \otimes K) \otimes K') \\ &= p^{YZ}_{Z,*}(\text{pr}_{23,*}(\text{pr}_1^*M \otimes \text{pr}_{12}^*K) \otimes K') \\ &= p^{YZ}_{Z,*}(\text{pr}_{23,*}(\text{pr}_1^*M \otimes \text{pr}_{12}^*K \otimes \text{pr}_{23}^*K')) \\ &= \text{pr}_{3,*}(\text{pr}_1^*M \otimes \text{pr}_{12}^*K \otimes \text{pr}_{23}^*K') \\ &= p^{XZ}_{Z,*}\text{pr}_{13,*}(\text{pr}_1^*M \otimes \text{pr}_{12}^*K \otimes \text{pr}_{23}^*K') \\ &= p^{XZ}_{Z,*}(p^{XZ,*}_X M \otimes \text{pr}_{13,*}(\text{pr}_{12}^*K \otimes \text{pr}_{23}^*K')) \end{aligned}$$

as desired. Here we have used the remark on base change in the second equality and we have use Derived Categories of Schemes, Lemma 36.22.1 in the 4th and last equality. □

0FYT Lemma 57.8.4. Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X \times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D_{perf}(\mathcal{O}_X)$ into $D_{perf}(\mathcal{O}_Y)$ if at least one of the following conditions is satisfied:

(1) S is Noetherian, $X \to S$ and $Y \to S$ are of finite type, $K \in D^b_{Coh}(\mathcal{O}_{X \times_S Y})$, the support of $H^i(K)$ is proper over Y for all i, and K has finite tor dimension as an object of $D(\text{pr}_2^{-1}\mathcal{O}_Y)$,
(2) $X \to S$ is of finite presentation and K can be represented by a bounded complex $\mathcal{K}^\bullet$ of finitely presented $\mathcal{O}_{X \times_S Y}$-modules, flat over Y, with support proper over Y,
(3) $X \to S$ is a proper flat morphism of finite presentation and K is perfect,
(4) S is Noetherian, $X \to S$ is flat and proper, and K is perfect
(5) $X \to S$ is a proper flat morphism of finite presentation and K is Y-perfect,
(6) S is Noetherian, $X \to S$ is flat and proper, and K is Y-perfect.

Proof. If M is perfect on X, then $L\text{pr}_1^*M$ is perfect on $X \times_S Y$, see Cohomology, Lemma 20.49.6. We will use this without further mention below. We will also use that if $X \to S$ is of finite type, or proper, or flat, or of finite presentation, then the same thing is true for the base change $\text{pr}_2 : X \times_S Y \to Y$, see Morphisms, Lemmas 29.15.4, 29.41.5, 29.25.8, and 29.21.4.

Part (1) follows from Derived Categories of Schemes, Lemma 36.27.1 combined with Derived Categories of Schemes, Lemma 36.11.6.

Part (2) follows from Derived Categories of Schemes, Lemma 36.30.1.

Part (3) follows from Derived Categories of Schemes, Lemma 36.30.4.

Part (4) follows from part (3) and the fact that a finite type morphism of Noetherian schemes is of finite presentation by Morphisms, Lemma 29.21.9.

Part (5) follows from Derived Categories of Schemes, Lemma 36.35.10 combined with Derived Categories of Schemes, Lemma 36.35.5.

Part (6) follows from part (5) in the same way that part (4) follows from part (3). □

0FYU Lemma 57.8.5. Let S be a Noetherian scheme. Let X and Y be schemes of finite type over S. Let $K \in D^b_{Coh}(\mathcal{O}_{X \times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ if at least one of the following conditions is satisfied:

(1) the support of $H^i(K)$ is proper over Y for all i, and K has finite tor dimension as an object of $D(\text{pr}_1^{-1}\mathcal{O}_X)$,
(2) K can be represented by a bounded complex $\mathcal{K}^\bullet$ of coherent $\mathcal{O}_{X \times_S Y}$-modules, flat over X, with support proper over Y,
(3) the support of $H^i(K)$ is proper over Y for all i and X is a regular scheme,
(4) K is perfect, the support of $H^i(K)$ is proper over Y for all i, and $Y \to S$ is flat.

Furthermore in each case the support condition is automatic if $X \to S$ is proper.

Proof. Let M be an object of $D^b_{Coh}(\mathcal{O}_X)$. In each case we will use Derived Categories of Schemes, Lemma 36.11.3 to show that

$$\Phi_K(M) = R\text{pr}_{2,*}(L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K)$$

is in $D^b_{Coh}(\mathcal{O}_Y)$. The derived tensor product $L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K$ is a pseudo-coherent object of $D(\mathcal{O}_{X \times_S Y})$ (by Cohomology, Lemma 20.47.3, Derived Categories of Schemes, Lemma 36.10.3, and Cohomology, Lemma 20.47.5) whence has coherent cohomology sheaves (by Derived Categories of Schemes, Lemma 36.10.3 again). In each case the supports of the cohomology sheaves $H^i(L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K)$ is proper over Y as these supports are contained in the union of the supports of the $H^i(K)$. Hence in each case it suffices to prove that this tensor product is bounded below.

Case (1). By Cohomology, Lemma 20.27.4 we have

$$L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} K \cong \text{pr}_1^{-1}M \otimes^{\mathbf{L}}_{\text{pr}_1^{-1}\mathcal{O}_X} K$$

with obvious notation. Hence the assumption on tor dimension and the fact that M has only a finite number of nonzero cohomology sheaves, implies the bound we want.

Case (2) follows because here the assumption implies that K has finite tor dimension as an object of $D(\text{pr}_1^{-1}\mathcal{O}_X)$ hence the argument in the previous paragraph applies.

In Case (3) it is also the case that K has finite tor dimension as an object of $D(\text{pr}_1^{-1}\mathcal{O}_X)$. Namely, choose affine opens $U = \text{Spec}(A)$ and $V = \text{Spec}(B)$ of X and Y mapping into the affine open $W = \text{Spec}(R)$ of S. Then $K|_{U \times V}$ is given by a bounded complex of finite $A \otimes_R B$-modules $M^\bullet$. Since A is a regular ring of finite dimension we see that each M^i has finite projective dimension as an A-module (Algebra, Lemma 10.110.8) and hence finite tor dimension as an A-module. Thus $M^\bullet$ has finite tor dimension as a complex of A-modules (More on Algebra, Lemma 15.66.8). Since $X \times Y$ is quasi-compact we conclude there exist $[a, b]$ such that for every point $z \in X \times Y$ the stalk K_z has tor amplitude in $[a, b]$ over $\mathcal{O}_{X,\text{pr}_1(z)}$. This implies K has bounded tor dimension as an object of $D(\text{pr}_1^{-1}\mathcal{O}_X)$, see Cohomology, Lemma 20.48.5. We conclude as in the previous to paragraphs.

Case (4). With notation as above, the ring map $R \to B$ is flat. Hence the ring map $A \to A \otimes_R B$ is flat. Hence any projective $A \otimes_R B$-module is A-flat. Thus any perfect complex of $A \otimes_R B$-modules has finite tor dimension as a complex of A-modules and we conclude as before. $\square$

0FYV Example 57.8.6. Let $X \to S$ be a separated morphism of schemes. Then the diagonal $\Delta : X \to X \times_S X$ is a closed immersion and hence $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X = R\Delta_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{X \times_S X}$-module of finite type which is flat over X (under either projection). The Fourier-Mukai functor $\Phi_{\mathcal{O}_\Delta}$ is equal to the identity in this case. Namely, for any $M \in D(\mathcal{O}_X)$ we have

$$\begin{aligned} L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S X}} \mathcal{O}_\Delta &= L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S X}} R\Delta_*\mathcal{O}_X \\ &= R\Delta_*(L\Delta^*L\text{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X) \\ &= R\Delta_*(M) \end{aligned}$$

The first equality we discussed above. The second equality is Cohomology, Lemma 20.54.4. The third because $\text{pr}_1 \circ \Delta = \text{id}_X$ and we have Cohomology, Lemma 20.27.2. If we push this to X using $R\text{pr}_{2,*}$ we obtain M by Cohomology, Lemma 20.28.2 and the fact that $\text{pr}_2 \circ \Delta = \text{id}_X$.

0FYW Lemma 57.8.7. Let $X \to S$ and $Y \to S$ be morphisms of quasi-compact and quasi-separated schemes. Let $\Phi : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ be a Fourier-Mukai functor with pseudo-coherent kernel $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$. Let $a : D_{QCoh}(\mathcal{O}_Y) \to$ Compare with discussion in [Riz17].

$D_{QCoh}(\mathcal{O}_{X \times_S Y})$ be the right adjoint to $R\text{pr}_{2,*}$, see Duality for Schemes, Lemma 48.3.1. Denote

$$K' = (Y \times_S X \to X \times_S Y)^* R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y)) \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$$

and denote $\Phi' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ the corresponding Fourier-Mukai transform. There is a canonical map

$$\text{Hom}_X(M, \Phi'(N)) \longrightarrow \text{Hom}_Y(\Phi(M), N)$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$ which is an isomorphism if

(1) N is perfect, or
(2) K is perfect and $X \to S$ is proper flat and of finite presentation.

Proof. By Lemma 57.8.2 we obtain a functor Φ as in the statement. Observe that $a(\mathcal{O}_Y)$ is in $D^+_{QCoh}(\mathcal{O}_{X \times_S Y})$ by Duality for Schemes, Lemma 48.3.5. Hence for K pseudo-coherent we have $K' \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$ by Derived Categories of Schemes, Lemma 36.10.8 we we obtain Φ' as indicated.

We abbreviate $\otimes^{\mathbf{L}} = \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}}$ and $\mathcal{H}om = R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}$. Let M be in $D_{QCoh}(\mathcal{O}_X)$ and let N be in $D_{QCoh}(\mathcal{O}_Y)$. We have

$$\begin{aligned} \text{Hom}_Y(\Phi(M), N) &= \text{Hom}_Y(R\text{pr}_{2,*}(L\text{pr}_1^*M \otimes^{\mathbf{L}} K), N) \\ &= \text{Hom}_{X \times_S Y}(L\text{pr}_1^*M \otimes^{\mathbf{L}} K, a(N)) \\ &= \text{Hom}_{X \times_S Y}(L\text{pr}_1^*M, R\mathcal{H}om(K, a(N))) \\ &= \text{Hom}_X(M, R\text{pr}_{1,*}R\mathcal{H}om(K, a(N))) \end{aligned}$$

where we have used Cohomology, Lemmas 20.42.2 and 20.28.1. There are canonical maps

$$L\text{pr}_2^*N \otimes^{\mathbf{L}} R\mathcal{H}om(K, a(\mathcal{O}_Y)) \xrightarrow{\alpha} R\mathcal{H}om(K, L\text{pr}_2^*N \otimes^{\mathbf{L}} a(\mathcal{O}_Y)) \xrightarrow{\beta} R\mathcal{H}om(K, a(N))$$

Here α is Cohomology, Lemma 20.42.6 and β is Duality for Schemes, Equation (48.8.0.1). Combining all of these arrows we obtain the functorial displayed arrow in the statement of the lemma.

The arrow α is an isomorphism by Derived Categories of Schemes, Lemma 36.10.9 as soon as either K or N is perfect. The arrow β is an isomorphism if N is perfect by Duality for Schemes, Lemma 48.8.1 or in general if $X \to S$ is flat proper of finite presentation by Duality for Schemes, Lemma 48.12.3. □

0FYX Lemma 57.8.8. Let S be a Noetherian scheme. Let $Y \to S$ be a flat proper Gorenstein morphism and let $X \to S$ be a finite type morphism. Denote $\omega^\bullet_{Y/S}$ the relative dualizing complex of Y over S. Let $\Phi : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ be a Fourier-Mukai functor with perfect kernel $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$. Denote

Compare with discussion in [Riz17].

$$K' = (Y \times_S X \to X \times_S Y)^*(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} L\text{pr}_2^*\omega^\bullet_{Y/S}) \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$$

and denote $\Phi' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ the corresponding Fourier-Mukai transform. There is a canonical isomorphism

$$\text{Hom}_Y(N, \Phi(M)) \longrightarrow \text{Hom}_X(\Phi'(N), M)$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 57.8.2 we obtain a functor Φ as in the statement.

Observe that formation of the relative dualizing complex commutes with base change in our setting, see Duality for Schemes, Remark 48.12.5. Thus $L\mathrm{pr}_2^*\omega_{Y/S}^\bullet = \omega_{X\times_S Y/X}^\bullet$. Moreover, we observe that $\omega_{Y/S}^\bullet$ is an invertible object of the derived category, see Duality for Schemes, Lemma 48.25.10, and a fortiori perfect.

To actually prove the lemma we're going to cheat. Namely, we will show that if we replace the roles of X and Y and K and K' then these are as in Lemma 57.8.7 and we get the result. It is clear that K' is perfect as a tensor product of perfect objects so that the discussion in Lemma 57.8.7 applies to it. To show that the procedure of Lemma 57.8.7 applied to K' on $Y \times_S X$ produces a complex isomorphic to K it suffices (details omitted) to show that

$$R\mathcal{H}om(R\mathcal{H}om(K, \omega_{X\times_S Y/X}^\bullet), \omega_{X\times_S Y/X}^\bullet) = K$$

This is clear because K is perfect and $\omega_{X\times_S Y/X}^\bullet$ is invertible; details omitted. Thus Lemma 57.8.7 produces a map

$$\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$ which is an isomorphism because K' is perfect. This finishes the proof. $\square$

0FYY Lemma 57.8.9. Let S be a Noetherian scheme.

(1) For X, Y proper and flat over S and K in $D_{perf}(\mathcal{O}_{X\times_S Y})$ we obtain a Fourier-Mukai functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.
(2) For X, Y, Z proper and flat over S, $K \in D_{perf}(\mathcal{O}_{X\times_S Y})$, $K' \in D_{perf}(\mathcal{O}_{Y\times_S Z})$ the composition $\Phi_{K'} \circ \Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Z)$ is equal to $\Phi_{K''}$ with $K'' \in D_{perf}(\mathcal{O}_{X\times_S Z})$ computed as in Lemma 57.8.3,
(3) For X, Y, K, Φ_K as in (1) if $X \to S$ is Gorenstein, then $\Phi_{K'} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_X)$ is a right adjoint to Φ_K where $K' \in D_{perf}(\mathcal{O}_{Y\times_S X})$ is the pullback of $L\mathrm{pr}_1^*\omega_{X/S}^\bullet \otimes_{\mathcal{O}_{X\times_S Y}}^{\mathbf{L}} K^\vee$ by $Y \times_S X \to X \times_S Y$.
(4) For X, Y, K, Φ_K as in (1) if $Y \to S$ is Gorenstein, then $\Phi_{K''} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_X)$ is a left adjoint to Φ_K where $K'' \in D_{perf}(\mathcal{O}_{Y\times_S X})$ is the pullback of $L\mathrm{pr}_2^*\omega_{Y/S}^\bullet \otimes_{\mathcal{O}_{X\times_S Y}}^{\mathbf{L}} K^\vee$ by $Y \times_S X \to X \times_S Y$.

Proof. Part (1) is immediate from Lemma 57.8.4 part (4).

Part (2) follows from Lemma 57.8.3 and the fact that $K'' = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes_{\mathcal{O}_{X\times_S Y\times_S Z}}^{\mathbf{L}} L\mathrm{pr}_{23}^*K')$ is perfect for example by Derived Categories of Schemes, Lemma 36.27.4.

The adjointness in part (3) on all complexes with quasi-coherent cohomology sheaves follows from Lemma 57.8.7 with K' equal to the pullback of $R\mathcal{H}om_{\mathcal{O}_{X\times_S Y}}(K, a(\mathcal{O}_Y))$ by $Y \times_S X \to X \times_S Y$ where a is the right adjoint to $R\mathrm{pr}_{2,*} : D_{QCoh}(\mathcal{O}_{X\times_S Y}) \to D_{QCoh}(\mathcal{O}_Y)$. Denote $f : X \to S$ the structure morphism of X. Since f is proper the functor $f^! : D_{QCoh}^+(\mathcal{O}_S) \to D_{QCoh}^+(\mathcal{O}_X)$ is the restriction to $D_{QCoh}^+(\mathcal{O}_S)$ of the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$, see Duality for Schemes, Section 48.16. Hence the relative dualizing complex $\omega_{X/S}^\bullet$ as defined in Duality for Schemes, Remark 48.12.5 is equal to $\omega_{X/S}^\bullet = f^!\mathcal{O}_S$. Since formation of the relative dualizing complex commutes with base change (see Duality for Schemes, Remark 48.12.5) we see that $a(\mathcal{O}_Y) = L\mathrm{pr}_1^*\omega_{X/S}^\bullet$. Thus

$$R\mathcal{H}om_{\mathcal{O}_{X\times_S Y}}(K, a(\mathcal{O}_Y)) \cong L\mathrm{pr}_1^*\omega_{X/S}^\bullet \otimes_{\mathcal{O}_{X\times_S Y}}^{\mathbf{L}} K^\vee$$

by Cohomology, Lemma 20.50.5. Finally, since $X \to S$ is assumed Gorenstein the relative dualizing complex is invertible: this follows from Duality for Schemes, Lemma 48.25.10. We conclude that $\omega^\bullet_{X/S}$ is perfect (Cohomology, Lemma 20.52.2) and hence K' is perfect. Therefore $\Phi_{K'}$ does indeed map $D_{perf}(\mathcal{O}_Y)$ into $D_{perf}(\mathcal{O}_X)$ which finishes the proof of (3).

The proof of (4) is the same as the proof of (3) except one uses Lemma 57.8.8 instead of Lemma 57.8.7. □

57.9. Resolutions and bounds

0FYZ The diagonal of a smooth proper scheme has a nice resolution.

0FZ0 Lemma 57.9.1. Let R be a Noetherian ring. Let X, Y be finite type schemes over R having the resolution property. For any coherent $\mathcal{O}_{X\times_R Y}$-module $\mathcal{F}$ there exist a surjection $\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{F}$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and $\mathcal{G}$ is a finite locally free $\mathcal{O}_Y$-module.

Proof. Let $U \subset X$ and $V \subset Y$ be affine open subschemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of the reduced induced closed subscheme structure on $X \setminus U$. Similarly, let $\mathcal{I}' \subset \mathcal{O}_Y$ be the ideal sheaf of the reduced induced closed subscheme structure on $Y \setminus V$. Then the ideal sheaf

$$\mathcal{J} = \mathrm{Im}(\mathrm{pr}_1^*\mathcal{I} \otimes_{\mathcal{O}_{X\times_R Y}} \mathrm{pr}_2^*\mathcal{I}' \to \mathcal{O}_{X\times_R Y})$$

satisfies $V(\mathcal{J}) = X \times_R Y \setminus U \times_R V$. For any section $s \in \mathcal{F}(U \times_R V)$ we can find an integer $n > 0$ and a map $\mathcal{J}^n \to \mathcal{F}$ whose restriction to $U \times_R V$ gives s, see Cohomology of Schemes, Lemma 30.10.5. By assumption we can choose surjections $\mathcal{E} \to \mathcal{I}$ and $\mathcal{G} \to \mathcal{I}'$. These produce corresponding surjections

$$\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{J} \quad \text{and} \quad \mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \to \mathcal{J}^n$$

and hence a map $\mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \to \mathcal{F}$ whose image contains the section s over $U \times_R V$. Since we can cover $X \times_R Y$ by a finite number of affine opens of the form $U \times_R V$ and since $\mathcal{F}|_{U\times_R V}$ is generated by finitely many sections (Properties, Lemma 28.16.1) we conclude that there exists a surjection

$$\bigoplus\nolimits_{j=1,\ldots,N} \mathcal{E}_j^{\otimes n_j} \boxtimes \mathcal{G}_j^{\otimes n_j} \to \mathcal{F}$$

where $\mathcal{E}_j$ is finite locally free on X and $\mathcal{G}_j$ is finite locally free on Y. Setting $\mathcal{E} = \bigoplus \mathcal{E}_j^{\otimes n_j}$ and $\mathcal{G} = \bigoplus \mathcal{G}_j^{\otimes n_j}$ we conclude that the lemma is true. □

0FZ1 Lemma 57.9.2. Let R be a ring. Let X, Y be quasi-compact and quasi-separated schemes over R having the resolution property. For any finite type quasi-coherent $\mathcal{O}_{X\times_R Y}$-module $\mathcal{F}$ there exist a surjection $\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{F}$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and $\mathcal{G}$ is a finite locally free $\mathcal{O}_Y$-module.

Proof. Follows from Lemma 57.9.1 by a limit argument. We urge the reader to skip the proof. Since $X \times_R Y$ is a closed subscheme of $X \times_{\mathbf{Z}} Y$ it is harmless if we replace R by $\mathbf{Z}$. We can write $\mathcal{F}$ as the quotient of a finitely presented $\mathcal{O}_{X\times_R Y}$-module by Properties, Lemma 28.22.8. Hence we may assume $\mathcal{F}$ is of finite presentation. Next we can write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$ and similarly $Y = \lim Y_j$, see Limits, Proposition 32.5.4. Then $\mathcal{F}$ will descend to $\mathcal{F}_{ij}$ on some $X_i \times_R Y_j$ (Limits, Lemma 32.10.2) and so does the property of having the resolution property (Derived Categories of Schemes, Lemma 36.36.9). Then we apply Lemma 57.9.1 to $\mathcal{F}_{ij}$ and we pullback. ☐

0FZ2 Lemma 57.9.3. Let R be a Noetherian ring. Let X be a separated finite type scheme over R which has the resolution property. Set $\mathcal{O}_\Delta = \Delta_*(\mathcal{O}_X)$ where $\Delta : X \to X \times_R X$ is the diagonal of X/k. There exists a resolution

$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_\Delta \to 0$$

where each $\mathcal{E}_i$ and $\mathcal{G}_i$ is a finite locally free $\mathcal{O}_X$-module.

Proof. Since X is separated, the diagonal morphism Δ is a closed immersion and hence $\mathcal{O}_\Delta$ is a coherent $\mathcal{O}_{X \times_R X}$-module (Cohomology of Schemes, Lemma 30.9.8). Thus the lemma follows immediately from Lemma 57.9.1. ☐

0FZ3 Lemma 57.9.4. Let X be a regular Noetherian scheme of dimension $d < \infty$. Then

(1) for $\mathcal{F}$, $\mathcal{G}$ coherent $\mathcal{O}_X$-modules we have $\mathrm{Ext}^n_X(\mathcal{F}, \mathcal{G}) = 0$ for $n > d$, and
(2) for $K, L \in D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$ if $H^i(K) = 0$ for $i < a+d$ and $H^i(L) = 0$ for $i \geq a$ then $\mathrm{Hom}_X(K, L) = 0$.

Proof. To prove (1) we use the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F}, \mathcal{G})$$

of Cohomology, Section 20.43. Let $x \in X$. We have

$$\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})_x = \mathcal{E}xt^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

see Cohomology, Lemma 20.51.4 (this also uses that $\mathcal{F}$ is pseudo-coherent by Derived Categories of Schemes, Lemma 36.10.3). Set $d_x = \dim(\mathcal{O}_{X,x})$. Since $\mathcal{O}_{X,x}$ is regular the ring $\mathcal{O}_{X,x}$ has global dimension d_x, see Algebra, Proposition 10.110.1. Thus $\mathcal{E}xt^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is zero for $q > d_x$. It follows that the modules $\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})$ have support of dimension at most $d - q$. Hence we have $H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) = 0$ for $p > d - q$ by Cohomology, Proposition 20.20.7. This proves (1).

Proof of (2). We may use induction on the number of nonzero cohomology sheaves of K and L. The case where these numbers are $0, 1$ follows from (1). If the number of nonzero cohomology sheaves of K is > 1, then we let $i \in \mathbf{Z}$ be minimal such that $H^i(K)$ is nonzero. We obtain a distinguished triangle

$$H^i(K)[-i] \to K \to \tau_{\geq i+1} K$$

(Derived Categories, Remark 13.12.4) and we get the vanishing of $\mathrm{Hom}(K, L)$ from the vanishing of $\mathrm{Hom}(H^i(K)[-i], L)$ and $\mathrm{Hom}(\tau_{\geq i+1}K, L)$ by Derived Categories, Lemma 13.4.2. Simlarly if L has more than one nonzero cohomology sheaf. ☐

0FZ4 Lemma 57.9.5. Let X be a regular Noetherian scheme of dimension $d < \infty$. Let $K \in D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. If $H^i(K) = 0$ for $a < i < a + d$, then $K = \tau_{\leq a} K \oplus \tau_{\geq a+d} K$.

Proof. We have $\tau_{\leq a} K = \tau_{\leq a+d-1} K$ by the assumed vanishing of cohomology sheaves. By Derived Categories, Remark 13.12.4 we have a distinguished triangle

$$\tau_{\leq a} K \to K \to \tau_{\geq a+d} K \xrightarrow{\delta} (\tau_{\leq a} K)[1]$$

By Derived Categories, Lemma 13.4.11 it suffices to show that the morphism δ is zero. This follows from Lemma 57.9.4. ☐

0FZ5 Lemma 57.9.6. Let k be a field. Let X be a quasi-compact separated smooth scheme over k. There exist finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{G}$ such that

$$\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$$

in $D(\mathcal{O}_{X \times X})$ where the notation is as in Derived Categories, Section 13.36.

Proof. Recall that X is regular by Varieties, Lemma 33.25.3. Hence X has the resolution property by Derived Categories of Schemes, Lemma 36.36.8. Hence we may choose a resolution as in Lemma 57.9.3. Say $\dim(X) = d$. Since $X \times X$ is smooth over k it is regular. Hence $X \times X$ is a regular Noetherian scheme with $\dim(X \times X) = 2d$. The object

$$K = (\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \to \ldots \to \mathcal{E}_0 \boxtimes \mathcal{G}_0)$$

of $D_{perf}(\mathcal{O}_{X \times X})$ has cohomology sheaves $\mathcal{O}_\Delta$ in degree 0 and $\text{Ker}(\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \to \mathcal{E}_{2d-1} \boxtimes \mathcal{G}_{2d-1})$ in degree $-2d$ and zero in all other degrees. Hence by Lemma 57.9.5 we see that $\mathcal{O}_\Delta$ is a summand of K in $D_{perf}(\mathcal{O}_{X \times X})$. Clearly, the object K is in

$$\left\langle \bigoplus\nolimits_{i=0,\ldots,2d} \mathcal{E}_i \boxtimes \mathcal{G}_i \right\rangle \subset \left\langle \left(\bigoplus\nolimits_{i=0,\ldots,2d} \mathcal{E}_i\right) \boxtimes \left(\bigoplus\nolimits_{i=0,\ldots,2d} \mathcal{G}_i\right)\right\rangle$$

which finishes the proof. (The reader may consult Derived Categories, Lemmas 13.36.1 and 13.35.7 to see that our object is contained in this category.) $\square$

0FZ6 Lemma 57.9.7. Let k be a field. Let X be a scheme proper and smooth over k. Then $D_{perf}(\mathcal{O}_X)$ has a strong generator.

Proof. Using Lemma 57.9.6 choose finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{G}$ such that $\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$ in $D(\mathcal{O}_{X \times X})$. We claim that $\mathcal{G}$ is a strong generator for $D_{perf}(\mathcal{O}_X)$. With notation as in Derived Categories, Section 13.35 choose $m, n \geq 1$ such that

$$\mathcal{O}_\Delta \in smd(add(\mathcal{E} \boxtimes \mathcal{G}[-m, m])^{\star n})$$

This is possible by Derived Categories, Lemma 13.36.2. Let K be an object of $D_{perf}(\mathcal{O}_X)$. Since $L\text{pr}_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} -$ is an exact functor and since

$$L\text{pr}_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} (\mathcal{E} \boxtimes \mathcal{G}) = (K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}) \boxtimes \mathcal{G}$$

we conclude from Derived Categories, Remark 13.35.5 that

$$L\text{pr}_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} \mathcal{O}_\Delta \in smd(add((K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}) \boxtimes \mathcal{G}[-m, m])^{\star n})$$

Applying the exact functor $R\text{pr}_{2,*}$ and observing that

$$R\text{pr}_{2,*}\left((K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}) \boxtimes \mathcal{G}\right) = R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}) \otimes_k \mathcal{G}$$

by Derived Categories of Schemes, Lemma 36.22.1 we conclude that

$$K = R\text{pr}_{2,*}(L\text{pr}_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} \mathcal{O}_\Delta) \in smd(add(R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}) \otimes_k \mathcal{G}[-m, m])^{\star n})$$

The equality follows from the discussion in Example 57.8.6. Since K is perfect, there exist $a \leq b$ such that $H^i(X, K)$ is nonzero only for $i \in [a, b]$. Since X is proper, each $H^i(X, K)$ is finite dimensional. We conclude that the right hand side is contained in $smd(add(\mathcal{G}[-m + a, m + b])^{\star n})$ which is itself contained in $\langle \mathcal{G} \rangle_n$ by one of the references given above. This finishes the proof. $\square$

0FZ7 Lemma 57.9.8. Let k be a field. Let X be a proper smooth scheme over k. There exists integers $m, n \geq 1$ and a finite locally free $\mathcal{O}_X$-module $\mathcal{G}$ such that every coherent $\mathcal{O}_X$-module is contained in $smd(add(\mathcal{G}[-m, m])^{\star n})$ with notation as in Derived Categories, Section 13.35.

Proof. In the proof of Lemma 57.9.7 we have shown that there exist $m', n \geq 1$ such that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathcal{F} \in smd(add(\mathcal{G}[-m' + a, m' + b])^{\star n})$$

for any $a \leq b$ such that $H^i(X, \mathcal{F})$ is nonzero only for $i \in [a, b]$. Thus we can take $a = 0$ and $b = \dim(X)$. Taking $m = \max(m', m' + b)$ finishes the proof. $\square$

The following lemma is the boundedness result referred to in the title of this section.

0FZ8 Lemma 57.9.9. Let k be a field. Let X be a smooth proper scheme over k. Let $\mathcal{A}$ be an abelian category. Let $H : D_{perf}(\mathcal{O}_X) \to \mathcal{A}$ be a homological functor (Derived Categories, Definition 13.3.5) such that for all K in $D_{perf}(\mathcal{O}_X)$ the object $H^i(K)$ is nonzero for only a finite number of $i \in \mathbf{Z}$. Then there exists an integer $m \geq 1$ such that $H^i(\mathcal{F}) = 0$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $i \notin [-m, m]$. Similarly for cohomological functors.

Proof. Combine Lemma 57.9.8 with Derived Categories, Lemma 13.35.8. $\square$

0FZ9 Lemma 57.9.10. Let k be a field. Let X, Y be finite type schemes over k. Let $K_0 \to K_1 \to K_2 \to \ldots$ be a system of objects of $D_{perf}(\mathcal{O}_{X\times Y})$ and $m \geq 0$ an integer such that

(1) $H^q(K_i)$ is nonzero only for $q \leq m$,
(2) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(\text{Supp}(\mathcal{F})) = 0$ the object
$$R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n)$$
has vanishing cohomology sheaves in degrees outside $[-m, m] \cup [-m - n, m - n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in degrees in $[-m, m]$.

Then K_n has vanishing cohomology sheaves in degrees outside $[-m, m] \cup [-m - n, m - n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in degrees in $[-m, m]$. Moreover, if X and Y are smooth over k, then for n large enough we find $K_n = K \oplus C_n$ in $D_{perf}(\mathcal{O}_{X\times Y})$ where K has cohomology only indegrees $[-m, m]$ and C_n only in degrees $[-m - n, m - n]$ and the transition maps define isomorphisms between various copies of K.

Proof. Let Z be the scheme theoretic support of an $\mathcal{F}$ as in (2). Then $Z \to \text{Spec}(k)$ is finite, hence $Z \times Y \to Y$ is finite. It follows that for an object M of $D_{QCoh}(\mathcal{O}_{X\times Y})$ with cohomology sheaves supported on $Z \times Y$ we have $H^i(R\text{pr}_{2,*}(M)) = \text{pr}_{2,*}H^i(M)$ and the functor $\text{pr}_{2,*}$ is faithful on quasi-coherent modules supported on $Z \times Y$; details omitted. Hence we see that the objects

$$\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n$$

in $D_{perf}(\mathcal{O}_{X\times Y})$ have vanishing cohomology sheaves outside $[-m, m] \cup [-m-n, m-n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in $[-m, m]$. Let $z \in X \times Y$ be a closed point mapping to the closed point $x \in X$. Then we know that

$$K_{n,z} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y,z}} \mathcal{O}_{X\times Y,z}/\mathfrak{m}_x^t\mathcal{O}_{X\times Y,z}$$

has nonzero cohomology only in the intervals $[-m, m] \cup [-m-n, m-n]$. We conclude by More on Algebra, Lemma 15.100.2 that $K_{n,z}$ only has nonzero cohomology in degrees $[-m, m] \cup [-m-n, m-n]$. Since this holds for all closed points of $X \times Y$, we conclude K_n only has nonzero cohomology sheaves in degrees $[-m, m] \cup [-m-n, m-$

$n]$. In exactly the same way we see that the maps $K_n \to K_{n+1}$ are isomorphisms on cohomology sheaves in degrees $[-m, m]$ for $n > 2m$.

If X and Y are smooth over k, then $X \times Y$ is smooth over k and hence regular by Varieties, Lemma 33.25.3. Thus we will obtain the direct sum decomposition of K_n as soon as $n > 2m + \dim(X \times Y)$ from Lemma 57.9.5. The final statement is clear from this. □

57.10. Sibling functors

0FZS In this section we prove some categorical result on the following notion.

0FZT Definition 57.10.1. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. We say two exact functors of triangulated categories

$$F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$$

are siblings, or we say F' is a sibling of F, if the following two conditions are satisfied

(1) the functors $F \circ i$ and $F' \circ i$ are isomorphic where $i : \mathcal{A} \to D^b(\mathcal{A})$ is the inclusion functor, and
(2) $F(K) \cong F'(K)$ for any K in $D^b(\mathcal{A})$.

Sometimes the second condition is a consequence of the first.

0FZU Lemma 57.10.2. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. Let $F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$ be exact functors of triangulated categories. Assume

(1) the functors $F \circ i$ and $F' \circ i$ are isomorphic where $i : \mathcal{A} \to D^b(\mathcal{A})$ is the inclusion functor, and
(2) for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ we have $\mathrm{Ext}^q_{\mathcal{D}}(F(X), F(Y)) = 0$ for $q < 0$ (for example if F is fully faithful).

Then F and F' are siblings.

Proof. Let $K \in D^b(\mathcal{A})$. We will show $F(K)$ is isomorphic to $F'(K)$. We can represent K by a bounded complex $A^\bullet$ of objects of $\mathcal{A}$. After replacing K by a translation we may assume $A^i = 0$ for $i > 0$. Choose $n \geq 0$ such that $A^{-i} = 0$ for $i > n$. The objects

$$M_i = (A^{-i} \to \ldots \to A^0)[-i], \quad i = 0, \ldots, n$$

form a Postnikov system in $D^b(\mathcal{A})$ for the complex $A^\bullet = A^{-n} \to \ldots \to A^0$ in $D^b(\mathcal{A})$. See Derived Categories, Example 13.41.2. Since both F and F' are exact functors of triangulated categories both

$$F(M_i) \quad \text{and} \quad F'(M_i)$$

form a Postnikov system in $\mathcal{D}$ for the complex

$$F(A^{-n}) \to \ldots \to F(A^0) = F'(A^{-n}) \to \ldots \to F'(A^0)$$

Since all negative Exts between these objects vanish by assumption we conclude by uniqueness of Postnikov systems (Derived Categories, Lemma 13.41.6) that $F(K) = F(M_n[n]) \cong F'(M_n[n]) = F'(K)$. □

0FZV Lemma 57.10.3. Let F and F' be siblings as in Definition 57.10.1. Then

(1) if F is essentially surjective, then F' is essentially surjective,
(2) if F is fully faithful, then F' is fully faithful.

Proof. Part (1) is immediate from property (2) for siblings.

Assume F is fully faithful. Denote $\mathcal{D}' \subset \mathcal{D}$ the essential image of F so that $F : D^b(\mathcal{A}) \to \mathcal{D}'$ is an equivalence. Since the functor F' factors through $\mathcal{D}'$ by property (2) for siblings, we can consider the functor $H = F^{-1} \circ F' : D^b(\mathcal{A}) \to D^b(\mathcal{A})$. Observe that H is a sibling of the identity functor. Since it suffices to prove that H is fully faithful, we reduce to the problem discussed in the next paragraph.

Set $\mathcal{D} = D^b(\mathcal{A})$. We have to show a sibling $F : \mathcal{D} \to \mathcal{D}$ of the identity functor is fully faithful. Denote $a_X : X \to F(X)$ the functorial isomorphism for $X \in \mathrm{Ob}(\mathcal{A})$ given to us by Definition 57.10.1. For any K in $\mathcal{D}$ and distinguished triangle $K_1 \to K_2 \to K_3$ of $\mathcal{D}$ if the maps

$$F : \mathrm{Hom}(K, K_i[n]) \to \mathrm{Hom}(F(K), F(K_i[n]))$$

are isomorphisms for all $n \in \mathbf{Z}$ and $i = 1, 3$, then the same is true for $i = 2$ and all $n \in \mathbf{Z}$. This uses the 5-lemma Homology, Lemma 12.5.20 and Derived Categories, Lemma 13.4.2; details omitted. Similarly, if the maps

$$F : \mathrm{Hom}(K_i[n], K) \to \mathrm{Hom}(F(K_i[n]), F(K))$$

are isomorphisms for all $n \in \mathbf{Z}$ and $i = 1, 3$, then the same is true for $i = 2$ and all $n \in \mathbf{Z}$. Using the canonical truncations and induction on the number of nonzero cohomology objects, we see that it is enough to show

$$F : \mathrm{Ext}^q(X, Y) \to \mathrm{Ext}^q(F(X), F(Y))$$

is bijective for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ and all $q \in \mathbf{Z}$. Since F is a sibling of id we have $F(X) \cong X$ and $F(Y) \cong Y$ hence the right hand side is zero for $q < 0$. The case $q = 0$ is OK by our assumption that F is a sibling of the identity functor. It remains to prove the cases $q > 0$.

The case $q = 1$: Injectivity. An element ξ of $\mathrm{Ext}^1(X, Y)$ gives rise to a distinguished triangle

$$Y \to E \to X \xrightarrow{\xi} Y[1]$$

Observe that $E \in \mathrm{Ob}(\mathcal{A})$. Since F is a sibling of the identity functor we obtain a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X \arrow[d] \\ F(E) \arrow[r] & F(X) \end{tikzcd}$$

whose vertical arrows are the isomorphisms a_E and a_X. By TR3 the distinguished triangle associated to ξ we started with is isomorphic to the distinguished triangle

$$F(Y) \to F(E) \to F(X) \xrightarrow{F(\xi)} F(Y[1]) = F(Y)[1]$$

Thus $\xi = 0$ if and only if $F(\xi)$ is zero, i.e., we see that $F : \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(F(X), F(Y))$ is injective.

The case $q = 1$: Surjectivity. Let θ be an element of $\mathrm{Ext}^1(F(X), F(Y))$. This defines an extension of $F(X)$ by $F(Y)$ in $\mathcal{A}$ which we may write as $F(E)$ as F is a sibling of the identity functor. We thus get a distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{\theta} F(Y[1]) = F(Y)[1]$$

for some morphisms $\alpha : Y \to E$ and $\beta : E \to X$. Since F is a sibling of the identity functor, the sequence $0 \to Y \to E \to X \to 0$ is a short exact sequence in $\mathcal{A}$! Hence we obtain a distinguished triangle

$$Y \xrightarrow{\alpha} E \xrightarrow{\beta} X \xrightarrow{\delta} Y[1]$$

for some morphism $\delta : X \to Y[1]$. Applying the exact functor F we obtain the distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{F(\delta)} F(Y)[1]$$

Arguing as above, we see that these triangles are isomorphic. Hence there exists a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(\delta)"'] \arrow[d, "\gamma"] & F(Y[1]) \arrow[d, "\epsilon"'] \\ F(X) \arrow[r, "\theta"] & F(Y[1]) \end{tikzcd}$$

for some isomorphisms γ, ϵ (we can say more but we won't need more information). We may write $\gamma = F(\gamma')$ and $\epsilon = F(\epsilon')$. Then we have $\theta = F(\epsilon' \circ \delta \circ (\gamma')^{-1})$ and we see the surjectivity holds.

The case $q > 1$: surjectivity. Using Yoneda extensions, see Derived Categories, Section 13.27, we find that for any element ξ in $\text{Ext}^q(F(X), F(Y))$ we can find $F(X) = B_0, B_1, \ldots, B_{q-1}, B_q = F(Y) \in \text{Ob}(\mathcal{A})$ and elements

$$\xi_i \in \text{Ext}^1(B_{i-1}, B_i)$$

such that ξ is the composition $\xi_q \circ \ldots \circ \xi_1$. Write $B_i = F(A_i)$ (of course we have $A_i = B_i$ but we don't need to use this) so that

$$\xi_i = F(\eta_i) \in \text{Ext}^1(F(A_{i-1}), F(A_i)) \quad \text{with} \quad \eta_i \in \text{Ext}^1(A_{i-1}, A_i)$$

by surjectivity for $q = 1$. Then $\eta = \eta_q \circ \ldots \circ \eta_1$ is an element of $\text{Ext}^q(X, Y)$ with $F(\eta) = \xi$.

The case $q > 1$: injectivity. An element ξ of $\text{Ext}^q(X, Y)$ gives rise to a distinguished triangle

$$Y[q-1] \to E \to X \xrightarrow{\xi} Y[q]$$

Applying F we obtain a distinguished triangle

$$F(Y)[q-1] \to F(E) \to F(X) \xrightarrow{F(\xi)} F(Y)[q]$$

If $F(\xi) = 0$, then $F(E) \cong F(Y)[q-1] \oplus F(X)$ in $\mathcal{D}$, see Derived Categories, Lemma 13.4.11. Since F is a sibling of the identity functor we have $E \cong F(E)$ and hence

$$E \cong F(E) \cong F(Y)[q-1] \oplus F(X) \cong Y[q-1] \oplus X$$

In other words, E is isomorphic to the direct sum of its cohomology objects. This implies that the initial distinguished triangle is split, i.e., $\xi = 0$. $\square$

Let us make a nonstandard definition. Let $\mathcal{A}$ be an abelian category. Let us say $\mathcal{A}$ has enough negative objects if given any $X \in \text{Ob}(\mathcal{A})$ there exists an object N such that

(1) there is a surjection $N \to X$ and
(2) $\text{Hom}(X, N) = 0$.

Let us prove a couple of lemmas about this notion in order to help with the proof of Proposition 57.10.6.

0GWF Lemma 57.10.4. Let $\mathcal{A}$ be an abelian category with enough negative objects. Let $X \in D^b(\mathcal{A})$. Let $b \in \mathbf{Z}$ with $H^i(X) = 0$ for $i > b$. Then there exists a map $N[-b] \to X$ such that the induced map $N \to H^b(X)$ is surjective and $\mathrm{Hom}(H^b(X), N) = 0$.

Proof. Using the truncation functors we can represent X by a complex $A^a \to A^{a+1} \to \ldots \to A^b$ of objects of $\mathcal{A}$. Choose N in $\mathcal{A}$ such that there exists a surjection $t : N \to A^b$ and such that $\mathrm{Hom}(A^b, N) = 0$. Then the surjection t defines a map $N[-b] \to X$ as desired. $\square$

0GWG Lemma 57.10.5. Let $\mathcal{A}$ be an abelian category with enough negative objects. Let $f : X \to X'$ be a morphism of $D^b(\mathcal{A})$. Let $b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > b$ and $H^i(X') = 0$ for $i \geq b$. Then there exists a map $N[-b] \to X$ such that the induced map $N \to H^b(X)$ is surjective, such that $\mathrm{Hom}(H^b(X), N) = 0$, and such that the composition $N[-b] \to X \to X'$ is zero.

Proof. We can represent f by a map $f^\bullet : A^\bullet \to B^\bullet$ of bounded complexes of objects of $\mathcal{A}$, see for example Derived Categories, Lemma 13.11.6. Consider the object

$$C = \mathrm{Ker}(A^b \to A^{b+1}) \times_{\mathrm{Ker}(B^b \to B^{b+1})} B^{b-1}$$

of $\mathcal{A}$. Since $H^b(B^\bullet) = 0$ we see that $C \to H^b(A^\bullet)$ is surjective. On the other hand, the map $C \to A^b \to B^b$ is the same as the map $C \to B^{b-1} \to B^b$ and hence the composition $C[-b] \to X \to X'$ is zero. Since $\mathcal{A}$ has enough negative objects, we can find an object N which has a surjection $N \to C \oplus H^b(X)$ such that $\mathrm{Hom}(C \oplus H^b(X), N) = 0$. Then N together with the map $N[-b] \to X$ is a solution to the problem posed by the lemma. $\square$

We encourage the reader to read the original [Orl97, Proposition 2.16] for the marvellous ideas that go into the proof of the following proposition.

0FZW Proposition 57.10.6. Let F and F' be siblings as in Definition 57.10.1. Assume that F is fully faithful and that $\mathcal{A}$ has enough negative objects (see above). Then F and F' are isomorphic functors.

[Orl97, Proposition 2.16]; the fact that we do not need to assume vanishing of $\mathrm{Ext}^q(N, X)$ for $q > 0$ in the definition of negative objects above is due to [CS14].

Proof. By part (2) of Definition 57.10.1 the image of the functor F' is contained in the essential image of the functor F. Hence the functor $H = F^{-1} \circ F'$ is a sibling of the identity functor. This reduces us to the case described in the next paragraph.

Let $\mathcal{D} = D^b(\mathcal{A})$. We have to show a sibling $F : \mathcal{D} \to \mathcal{D}$ of the identity functor is isomorphic to the identity functor. Given an object X of $\mathcal{D}$ let us say X has width $w = w(X)$ if $w \geq 0$ is minimal such that there exists an integer $a \in \mathbf{Z}$ with $H^i(X) = 0$ for $i \notin [a, a + w - 1]$. Since F is a sibling of the identity and since $F \circ [n] = [n] \circ F$ we are aready given isomorphisms

$$c_X : X \to F(X)$$

for $w(X) \leq 1$ compatible with shifts. Moreover, if $X = A[-a]$ and $X' = A'[-a]$ for some $A, A' \in \mathrm{Ob}(\mathcal{A})$ then for any morphism $f : X \to X'$ the diagram

0FZX (57.10.6.1)
$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "c_X"'] & X' \arrow[d, "c_{X'}"] \\ F(X) \arrow[r, "F(f)"] & F(X') \end{tikzcd}$$

is commutative.

Next, let us show that for any morphism $f : X \to X'$ with $w(X), w(X') \leq 1$ the diagram (57.10.6.1) commutes. If X or X' is zero, this is clear. If not then we can write $X = A[-a]$ and $X' = A'[-a']$ for unique A, A' in $\mathcal{A}$ and $a, a' \in \mathbf{Z}$. The case $a = a'$ was discussed above. If $a' > a$, then $f = 0$ (Derived Categories, Lemma 13.27.3) and the result is clear. If $a' < a$ then f corresponds to an element $\xi \in \text{Ext}^q(A, A')$ with $q = a - a'$. Using Yoneda extensions, see Derived Categories, Section 13.27, we can find $A = A_0, A_1, \ldots, A_{q-1}, A_q = A' \in \text{Ob}(\mathcal{A})$ and elements

$$\xi_i \in \text{Ext}^1(A_{i-1}, A_i)$$

such that ξ is the composition $\xi_q \circ \ldots \circ \xi_1$. In other words, setting $X_i = A_i[-a+i]$ we obtain morphisms

$$X = X_0 \xrightarrow{f_1} X_1 \to \ldots \to X_{q-1} \xrightarrow{f_q} X_q = X'$$

whose compostion is f. Since the commutativity of (57.10.6.1) for $f_1, \ldots, f_q$ implies it for f, this reduces us to the case $q = 1$. In this case after shifting we may assume we have a distinguished triangle

$$A' \to E \to A \xrightarrow{f} A'[1]$$

Observe that E is an object of $\mathcal{A}$. Consider the following diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "c_E"'] & A \arrow[r, "f"'] \arrow[d, "c_A"'] & A'[1] \arrow[r] \arrow[d, "c_{A'}[1]"] \arrow[d, dotted, shift right, "\gamma"'] \arrow[dl, dotted, "\epsilon"] & E[1] \arrow[d, "c_E[1]"] \\
F(E) \arrow[r] & F(A) \arrow[r, "F(f)"] & F(A')[1] \arrow[r] & F(E)[1]
\end{tikzcd}$$

whose rows are distinguished triangles. The square on the right commutes already but we don't yet know that the middle square does. By the axioms of a triangulated category we can find a morphism γ which does make the diagram commute. Then $\gamma - c_{A'}[1]$ composed with $F(A')[1] \to F(E)[1]$ is zero hence we can find $\epsilon : A'[1] \to F(A)$ such that $\gamma - c_{A'}[1] = F(f) \circ \epsilon$. However, any arrow $A'[1] \to F(A)$ is zero as it is a negative ext class between objects of $\mathcal{A}$. Hence $\gamma = c_{A'}[1]$ and we conclude the middle square commutes too which is what we wanted to show.

To finish the proof we are going to argue by induction on w that there exist isomorphisms $c_X : X \to F(X)$ for all X with $w(X) \leq w$ compatible with all morphisms between such objects. The base case $w = 1$ was shown above. Assume we know the result for some $w \geq 1$.

Let X be an object with $w(X) = w+1$. Pick $a \in \mathbf{Z}$ with $H^i(X) = 0$ for $i \notin [a, a+w]$. Set $b = a + w$ so that $H^b(X)$ is nonzero. Choose $N[-b] \to X$ as in Lemma 57.10.4. Choose a distinguished diagram

$$N[-b] \to X \to Y \to N[-b+1]$$

Computing the long exact cohomology sequence we find $w(Y) \leq w$. Hence by induction we find the solid arrows in the following diagram

$$\begin{tikzcd}
N[-b] \arrow[r] \arrow[d, "c_N{[-b]}"'] & X \arrow[r] \arrow[d, dotted, "c_{N[-b] \to X}"'] & Y \arrow[r] \arrow[d, "c_Y"] & N[-b+1] \arrow[d, "c_N{[-b+1]}"] \\
F(N)[-b] \arrow[r] & F(X) \arrow[r] & F(Y) \arrow[r] & F(N)[-b+1]
\end{tikzcd}$$

We obtain the dotted arrow $c_{N[-b]\to X}$. By Derived Categories, Lemma 13.4.8 the dotted arrow is unique because $\mathrm{Hom}(X, F(N)[-b]) \cong \mathrm{Hom}(X, N[-b]) = 0$ by our choice of N. In fact, $c_{N[-b]\to X}$ is the unique dotted arrow making the square with vertices $X, Y, F(X), F(Y)$ commute.

Let $N'[-b] \to X$ be another map as in Lemma 57.10.4 and let us prove that $c_{N[-b]\to X} = c_{N'[-b]\to X}$. Observe that the map $(N \oplus N')[-b] \to X$ also satisfies the conditions of Lemma 57.10.4. Thus we may assume $N'[-b] \to X$ factors as $N'[-b] \to N[-b] \to X$ for some morphism $N' \to N$. Choose distinguished triangles $N[-b] \to X \to Y \to N[-b+1]$ and $N'[-b] \to X \to Y' \to N'[-b+1]$. By axiom TR3 we can find a morphism $g : Y' \to Y$ which joint with id_X and $N' \to N$ forms a morphism of triangles. Since we have (57.10.6.1) for g we conclude that

$$(F(X) \to F(Y)) \circ c_{N'[-b]\to X} = (F(X) \to F(Y)) \circ c_{N[-b]\to X}$$

The uniqueness of $c_{N[-b]\to X}$ pointed out in the construction above now shows that $c_{N'[-b]\to X} = c_{N[-b]\to X}$.

Thus we can now define for X of width $w + 1$ the isomorphism $c_X : X \to F(X)$ as the common value of the maps $c_{N[-b]\to X}$ where $N[-b] \to X$ is as in Lemma 57.10.4. To finish the proof, we have to show that the diagrams (57.10.6.1) commute for all morphisms $f : X \to X'$ between objects with $w(X) \leq w + 1$ and $w(X') \leq w + 1$. Choose $a \leq b \leq a + w$ such that $H^i(X) = 0$ for $i \notin [a, b]$ and $a' \leq b' \leq a' + w$ such that $H^i(X') = 0$ for $i \notin [a', b']$. We will use induction on $(b' - a') + (b - a)$ to show the claim. (The base case is when this number is zero which is OK because $w \geq 1$.) We distinguish two cases.

Case I: $b' < b$. In this case, by Lemma 57.10.5 we may choose $N[-b] \to X$ as in Lemma 57.10.4 such that the composition $N[-b] \to X \to X'$ is zero. Choose a distuiguished triangle $N[-b] \to X \to Y \to N[-b+1]$. Since $N[-b] \to X'$ is zero, we find that f factors as $X \to Y \to X'$. Since $H^i(Y)$ is nonzero only for $i \in [a, b-1]$ we see by induction that (57.10.6.1) commutes for $Y \to X'$. The diagram (57.10.6.1) commutes for $X \to Y$ by construction if $w(X) = w + 1$ and by our first induction hypothesis if $w(X) \leq w$. Hence (57.10.6.1) commutes for f.

Case II: $b' \geq b$. In this case we choose $N'[-b'] \to X'$ as in Lemma 57.10.4. We may also assume that $\mathrm{Hom}(H^{b'}(X), N') = 0$ (this is relevant only if $b' = b$), for example because we can replace N' by an object N'' which surjects onto $N' \oplus H^{b'}(X)$ and such that $\mathrm{Hom}(N' \oplus H^{b'}(X), N'') = 0$. We choose a distinguished triangle $N'[-b'] \to X' \to Y' \to N'[-b'+1]$. Since $\mathrm{Hom}(X, X') \to \mathrm{Hom}(X, Y')$ is injective by our choice of N' (details omitted) the same is true for $\mathrm{Hom}(X, F(X')) \to \mathrm{Hom}(X, F(Y'))$. Hence it suffices in this case to check that (57.10.6.1) commutes for the composition $X \to Y'$ of the morphisms $X \to X' \to Y'$. Since $H^i(Y')$ is nonzero only for $i \in [a', b'-1]$ we conclude by induction hypothesis. □

57.11. Deducing fully faithfulness

0G23 It will be useful for us to know when a functor is fully faithful we offer the following variant of [Orl97, Lemma 2.15].

0G24 **Lemma 57.11.1.** Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Let $S \subset \mathrm{Ob}(\mathcal{D})$ be a set of objects. Assume Variant of [Orl97, Lemma 2.15]

(1) F has both right and left adjoints,

(2) for $K \in \mathcal{D}$ if $\mathrm{Hom}(E, K[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$ then $K = 0$,
(3) for $K \in \mathcal{D}$ if $\mathrm{Hom}(K, E[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$ then $K = 0$,
(4) the map $\mathrm{Hom}(E, E'[i]) \to \mathrm{Hom}(F(E), F(E')[i])$ induced by F is bijective for all $E, E' \in S$ and $i \in \mathbf{Z}$.

Then F is fully faithful.

Proof. Denote F_r and F_l the right and left adjoints of F. For $E \in S$ choose a distinguished triangle

$$E \to F_r(F(E)) \to C \to E[1]$$

where the first arrow is the unit of the adjunction. For $E' \in S$ we have

$$\mathrm{Hom}(E', F_r(F(E))[i]) = \mathrm{Hom}(F(E'), F(E)[i]) = \mathrm{Hom}(E', E[i])$$

The last equality holds by assumption (4). Hence applying the homological functor $\mathrm{Hom}(E', -)$ (Derived Categories, Lemma 13.4.2) to the distinguished triangle above we conclude that $\mathrm{Hom}(E', C[i]) = 0$ for all $i \in \mathbf{Z}$ and $E' \in S$. By assumption (2) we conclude that $C = 0$ and $E = F_r(F(E))$.

For $K \in \mathrm{Ob}(\mathcal{D})$ choose a distinguished triangle

$$F_l(F(K)) \to K \to C \to F_l(F(K))[1]$$

where the first arrow is the counit of the adjunction. For $E \in S$ we have

$$\mathrm{Hom}(F_l(F(K)), E[i]) = \mathrm{Hom}(F(K), F(E)[i]) = \mathrm{Hom}(K, F_r(F(E))[i]) = \mathrm{Hom}(K, E[i])$$

where the last equality holds by the result of the first paragraph. Thus we conclude as before that $\mathrm{Hom}(C, E[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$. Hence $C = 0$ by assumption (3). Thus F is fully faithful by Categories, Lemma 4.24.4. $\square$

0G02 Lemma 57.11.2. Let k be a field. Let X be a scheme of finite type over k which is regular. Let $x \in X$ be a closed point. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ supported at x choose a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ supported at x such that $\mathcal{F}_x$ and $\mathcal{F}'_x$ are Matlis dual. Then there is an isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, M) = H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}'[-d_x])$$

where $d_x = \dim(\mathcal{O}_{X,x})$ functorial in M in $D_{perf}(\mathcal{O}_X)$.

Proof. Since $\mathcal{F}$ is supported at x we have

$$\mathrm{Hom}_X(\mathcal{F}, M) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M_x)$$

and similarly we have

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}'[-d_x]) = \mathrm{Tor}^{\mathcal{O}_{X,x}}_{d_x}(M_x, \mathcal{F}'_x)$$

Thus it suffices to show that given a Noetherian regular local ring A of dimension d and a finite length A-module N, if N' is the Matlis dual to N, then there exists a functorial isomorphism

$$\mathrm{Hom}_A(N, K) = \mathrm{Tor}^A_d(K, N')$$

for K in $D_{perf}(A)$. We can write the left hand side as $H^0(R\,\mathrm{Hom}_A(N, A) \otimes^{\mathbf{L}}_A K)$ by More on Algebra, Lemma 15.74.15 and the fact that N determines a perfect object of $D(A)$. Hence the formula holds because

$$R\,\mathrm{Hom}_A(N, A) = R\,\mathrm{Hom}_A(N, A[d])[-d] = N'[-d]$$

by Dualizing Complexes, Lemma 47.16.4 and the fact that $A[d]$ is a normalized dualizing complex over A (A is Gorenstein by Dualizing Complexes, Lemma 47.21.3). □

0G03 **Lemma 57.11.3.** Let k be a field. Let X be a scheme of finite type over k which is regular. Let $x \in X$ be a closed point and denote $\mathcal{O}_x$ the skyscraper sheaf at x with value $\kappa(x)$. Let K in $D_{perf}(\mathcal{O}_X)$.

(1) If $\mathrm{Ext}^i_X(\mathcal{O}_x, K) = 0$ then there exists an open neighbourhood U of x such that $H^{i-d_x}(K)|_U = 0$ where $d_x = \dim(\mathcal{O}_{X,x})$.
(2) If $\mathrm{Hom}_X(\mathcal{O}_x, K[i]) = 0$ for all $i \in \mathbf{Z}$, then K is zero in an open neighbourhood of x.
(3) If $\mathrm{Ext}^i_X(K, \mathcal{O}_x) = 0$ then there exists an open neighbourhood U of x such that $H^i(K^\vee)|_U = 0$.
(4) If $\mathrm{Hom}_X(K, \mathcal{O}_x[i]) = 0$ for all $i \in \mathbf{Z}$, then K is zero in an open neighbourhood of x.
(5) If $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x) = 0$ then there exists an open neighbourhood U of x such that $H^i(K)|_U = 0$.
(6) If $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x) = 0$ for $i \in \mathbf{Z}$ then K is zero in an open neighbourhood of x.

Proof. Observe that $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x)$ is equal to $K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \kappa(x)$. Hence part (5) follows from More on Algebra, Lemma 15.76.4. Part (6) follows from part (5). Part (1) follows from part (5), Lemma 57.11.2, and the fact that the Matlis dual of $\kappa(x)$ is $\kappa(x)$. Part (2) follows from part (1). Part (3) follows from part (5) and the fact that $\mathrm{Ext}^i(K, \mathcal{O}_x) = H^i(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x)$ by Cohomology, Lemma 20.50.5. Part (4) follows from part (3) and the fact that $K \cong (K^\vee)^\vee$ by the lemma just cited. □

0GWZ **Lemma 57.11.4.** Let X be a Noetherian scheme. Let $x \in X$ be a closed point and denote $\mathcal{O}_x$ the skyscraper sheaf at x with value $\kappa(x)$. Let K in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$. Let $b \in \mathbf{Z}$. The following are equivalent

(1) $H^i(K)_x = 0$ for all $i > b$ and
(2) $\mathrm{Hom}_X(K, \mathcal{O}_x[-i]) = 0$ for all $i > b$.

Proof. Consider the complex K_x in $D^b_{\mathrm{Coh}}(\mathcal{O}_{X,x})$. There exist an integer $b_x \in \mathbf{Z}$ such that K_x can be represented by a bounded above complex

$$\ldots \to \mathcal{O}_{X,x}^{\oplus n_{b_x-2}} \to \mathcal{O}_{X,x}^{\oplus n_{b_x-1}} \to \mathcal{O}_{X,x}^{\oplus n_{b_x}} \to 0 \to \ldots$$

with $\mathcal{O}_{X,x}^{\oplus n_i}$ sitting in degree i where all the transition maps are given by matrices whose coefficients are in $\mathfrak{m}_x$. See More on Algebra, Lemma 15.75.5. The result follows easily from this (and the equivalent conditions hold if and only if $b \geq b_x$). □

0G25 **Lemma 57.11.5.** Let k be a field. Let X and Y be proper schemes over k. Assume X is regular. Then a k-linear exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is fully faithful if and only if for any closed points $x, x' \in X$ the maps

$$F : \mathrm{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \mathrm{Ext}^i_Y(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all $i \in \mathbf{Z}$. Here $\mathcal{O}_x$ is the skyscraper sheaf at x with value $\kappa(x)$.

Proof. By Lemma 57.7.1 the functor F has both a left and a right adjoint. Thus we may apply the criterion of Lemma 57.11.1 because assumptions (2) and (3) of that lemma follow from Lemma 57.11.3. $\square$

0G26 Lemma 57.11.6. Let k be a field. Let X be a proper scheme over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(\text{Supp}(\mathcal{F})) = 0$ there is an isomorphism $\mathcal{F} \cong F(\mathcal{F})$. Then F is fully faithful. Email from Noah Olander of Jun 9, 2020

Proof. By Lemma 57.11.5 it suffices to show that the maps

$$F : \text{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \text{Ext}^i_X(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all $i \in \mathbf{Z}$ and all closed points $x, x' \in X$. By assumption, the source and the target are isomorphic. If $x \neq x'$, then both sides are zero and the result is true. If $x = x'$, then it suffices to prove that the map is either injective or surjective. For $i < 0$ both sides are zero and the result is true. For $i = 0$ any nonzero map $\alpha : \mathcal{O}_x \to \mathcal{O}_x$ of $\mathcal{O}_X$-modules is an isomorphism. Hence $F(\alpha)$ is an isomorphism too and so $F(\alpha)$ is nonzero. Thus the result for $i = 0$. For $i = 1$ a nonzero element ξ in $\text{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$ corresponds to a nonsplit short exact sequence

$$0 \to \mathcal{O}_x \to \mathcal{F} \to \mathcal{O}_x \to 0$$

Since $F(\mathcal{F}) \cong \mathcal{F}$ we see that $F(\mathcal{F})$ is a nonsplit extension of $\mathcal{O}_x$ by $\mathcal{O}_x$ as well. Since $\mathcal{O}_x \cong F(\mathcal{O}_x)$ is a simple $\mathcal{O}_X$-module and $\mathcal{F} \cong F(\mathcal{F})$ has length 2, we see that in the distinguished triangle

$$F(\mathcal{O}_x) \to F(\mathcal{F}) \to F(\mathcal{O}_x) \xrightarrow{F(\xi)} F(\mathcal{O}_x)[1]$$

the first two arrows must form a short exact sequence which must be isomorphic to the above short exact sequence and hence is nonsplit. It follows that $F(\xi)$ is nonzero and we conclude for $i = 1$. For $i > 1$ composition of ext classes defines a surjection

$$\text{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \otimes \ldots \otimes \text{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \longrightarrow \text{Ext}^i(F(\mathcal{O}_x), F(\mathcal{O}_x))$$

See Duality for Schemes, Lemma 48.15.4. Hence surjectivity in degree 1 implies surjectivity for $i > 0$. This finishes the proof. $\square$

57.12. Special functors

0FZY In this section we prove some results on functors of a special type that we will use later in this chapter.

0FZZ Definition 57.12.1. Let k be a field. Let X, Y be finite type schemes over k. Recall that $D^b_{\text{Coh}}(\mathcal{O}_X) = D^b(\text{Coh}(\mathcal{O}_X))$ by Derived Categories of Schemes, Proposition 36.11.2. We say two k-linear exact functors

$$F, F' : D^b_{\text{Coh}}(\mathcal{O}_X) = D^b(\text{Coh}(\mathcal{O}_X)) \longrightarrow D^b_{\text{Coh}}(\mathcal{O}_Y)$$

are siblings, or we say F' is a sibling of F if F and F' are siblings in the sense of Definition 57.10.1 with abelian category being $\text{Coh}(\mathcal{O}_X)$. If X is regular then $D_{perf}(\mathcal{O}_X) = D^b_{\text{Coh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.6 and we use the same terminology for k-linear exact functors $F, F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

0G00 Lemma 57.12.2. Let k be a field. Let X, Y be finite type schemes over k with X separated. Let $F : D^b_{\text{Coh}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_Y)$ be a k-linear exact functor sending $\text{Coh}(\mathcal{O}_X) \subset D^b_{\text{Coh}}(\mathcal{O}_X)$ into $\text{Coh}(\mathcal{O}_Y) \subset D^b_{\text{Coh}}(\mathcal{O}_Y)$. Then there exists a Fourier-Mukai functor $F' : D^b_{\text{Coh}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_Y)$ whose kernel is a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{K}$ flat over X and with support finite over Y which is a sibling of F.

Proof. Denote $H : \text{Coh}(\mathcal{O}_X) \to \text{Coh}(\mathcal{O}_Y)$ the restriction of F. Since F is an exact functor of triangulated categories, we see that H is an exact functor of abelian categories. Of course H is k-linear as F is. By Functors and Morphisms, Lemma 56.7.5 we obtain a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{K}$ which is flat over X and has support finite over Y. Let F' be the Fourier-Mukai functor defined using $\mathcal{K}$ so that F' restricts to H on $\text{Coh}(\mathcal{O}_X)$. The functor F' sends $D^b_{\text{Coh}}(\mathcal{O}_X)$ into $D^b_{\text{Coh}}(\mathcal{O}_Y)$ by Lemma 57.8.5. Observe that F and F' satisfy the first and second condition of Lemma 57.10.2 and hence are siblings. □

0G01 Remark 57.12.3. If $F, F' : D^b_{\text{Coh}}(\mathcal{O}_X) \to \mathcal{D}$ are siblings, F is fully faithful, and X is reduced and projective over k then $F \cong F'$; this follows from Proposition 57.10.6 via the argument given in the proof of Theorem 57.13.3. However, in general we do not know whether siblings are isomorphic. Even in the situation of Lemma 57.12.2 it seems difficult to prove that the siblings F and F' are isomorphic functors. If X is smooth and proper over k and F is fully faithful, then $F \cong F'$ as is shown in [Ola20]. If you have a proof or a counter example in more general situations, please email stacks.project@gmail.com.

0GX0 Lemma 57.12.4. Let k be a field. Let X, Y be proper schemes over k. Assume X is regular. Let $F, G : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ be k-linear exact functors such that

(1) $F(\mathcal{F}) \cong G(\mathcal{F})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(\text{Supp}(\mathcal{F})) = 0$,
(2) F is fully faithful.

Then the essential image of G is contained in the essential image of F.

Proof. Recall that F and G have both adjoints, see Lemma 57.7.1. In particular the essential image $\mathcal{A} \subset D_{perf}(\mathcal{O}_Y)$ of F satisfies the equivalent conditions of Derived Categories, Lemma 13.40.7. We claim that G factors through $\mathcal{A}$. Since $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$ by Derived Categories, Lemma 13.40.7 it suffices to show that $\text{Hom}_Y(G(M), N) = 0$ for all M in $D_{perf}(\mathcal{O}_X)$ and $N \in \mathcal{A}^\perp$. We have

$$\text{Hom}_Y(G(M), N) = \text{Hom}_X(M, G_r(N))$$

where G_r is the right adjoint to G. Thus it suffices to prove that $G_r(N) = 0$. Since $G(\mathcal{F}) \cong F(\mathcal{F})$ for $\mathcal{F}$ as in (1) we see that

$$\text{Hom}_X(\mathcal{F}, G_r(N)) = \text{Hom}_Y(G(\mathcal{F}), N) = \text{Hom}_Y(F(\mathcal{F}), N) = 0$$

as N is in the right orthogonal to the essential image $\mathcal{A}$ of F. Of course, the same vanishing holds for $\text{Hom}_X(\mathcal{F}, G_r(N)[i])$ for any $i \in \mathbf{Z}$. Thus $G_r(N) = 0$ by Lemma 57.11.3 and we win. □

0G27 Lemma 57.12.5. Let k be a field. Let X be a proper scheme over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(\text{Supp}(\mathcal{F})) = 0$ there is an isomorphism $\mathcal{F} \cong F(\mathcal{F})$. Then there exists an automorphism $f : X \to X$ over k which induces

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the identity on the underlying topological space[1] and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that F and $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$ are siblings.

Proof. By Lemma 57.11.6 the functor F is fully faithful. By Lemma 57.12.4 the essential image of the identity functor is contained in the essential image of F, i.e., we see that F is essentially surjective. Thus F is an equivalence. Observe that the quasi-inverse F^{-1} satisfies the same assumptions as F.

Let $M \in D_{perf}(\mathcal{O}_X)$ and say $H^i(M) = 0$ for $i > b$. Since F is fully faithful, we see that

$$\mathrm{Hom}_X(M, \mathcal{O}_x[-i]) = \mathrm{Hom}_X(F(M), F(\mathcal{O}_x)[-i]) \cong \mathrm{Hom}_X(F(M), \mathcal{O}_x[-i])$$

for any $i \in \mathbf{Z}$ for any closed point x of X. Thus by Lemma 57.11.4 we see that $F(M)$ has vanishing cohomology sheaves in degrees $> b$.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By the above $F(\mathcal{F})$ has nonzero cohomology sheaves only in degrees ≤ 0. Set $\mathcal{G} = H^0(F(\mathcal{F}))$. Choose a distinguished triangle

$$K \to F(\mathcal{F}) \to \mathcal{G} \to K[1]$$

Then K has nonvanishing cohomology sheaves only in degrees ≤ -1. Applying F^{-1} we obtain a distinguished triangle

$$F^{-1}(K) \to \mathcal{F} \to F^{-1}(\mathcal{G}) \to F^{-1}(K')[1]$$

Since $F^{-1}(K)$ has nonvanishing cohomology sheaves only in degrees ≤ -1 (by the previous paragraph applied to F^{-1}) we see that the arrow $F^{-1}(K) \to \mathcal{F}$ is zero (Derived Categories, Lemma 13.27.3). Hence $K \to F(\mathcal{F})$ is zero, which implies that $F(\mathcal{F}) = \mathcal{G}$ by our choice of the first distinguished triangle.

From the preceding paragraph, we deduce that F preserves $\mathrm{Coh}(\mathcal{O}_X)$ and indeed defines an equivalence $H : \mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(\mathcal{O}_X)$. By Functors and Morphisms, Lemma 56.7.8 we get an automorphism $f : X \to X$ over k and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that $H(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Set $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$. Using Lemma 57.10.2 we see that F and F' are siblings. To see that f is the identity on the underlying topological space of X, we use that $F(\mathcal{O}_x) \cong \mathcal{O}_x$ and that the support of $\mathcal{O}_x$ is $\{x\}$. This finishes the proof. □

0G06 Lemma 57.12.6. Let k be a field. Let X, Y be proper schemes over k. Assume X regular. Let $F, G : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ be k-linear exact functors such that

(1) $F(\mathcal{F}) \cong G(\mathcal{F})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(\mathrm{Supp}(\mathcal{F})) = 0$,
(2) F is fully faithful, and
(3) G is a Fourier-Mukai functor whose kernel is in $D_{perf}(\mathcal{O}_{X \times Y})$.

Then there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X \times Y})$ such that F and F' are siblings.

Proof. The essential image of G is contained in the essential image of F by Lemma 57.12.4. Consider the functor $H = F^{-1} \circ G$ which makes sense as F is fully faithful. By Lemma 57.12.5 we obtain an automorphism $f : X \to X$ and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that the functor $H' : K \mapsto f^*K \otimes \mathcal{L}$ is a sibling of H. In particular H is an auto-equivalence by Lemma 57.10.3 and H induces an auto-equivalence of $\mathrm{Coh}(\mathcal{O}_X)$ (as this is true for its sibling functor H'). Thus the quasi-inverses H^{-1} and $(H')^{-1}$ exist, are siblings (small detail omitted), and $(H')^{-1}$ sends M to

[1]This often forces f to be the identity, see Varieties, Lemma 33.32.1.

$(f^{-1})^*(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ which is a Fourier-Mukai functor (details omitted). Then of course $F = G \circ H^{-1}$ is a sibling of $G \circ (H')^{-1}$. Since compositions of Fourier-Mukai functors are Fourier-Mukai by Lemma 57.8.3 we conclude. □

57.13. Fully faithful functors

0G07 Our goal is to prove fully faithful functors between derived categories are siblings of Fourier-Mukai functors, following [Orl97] and [Bal08].

0G08 Situation 57.13.1. Here k is a field. We have proper smooth schemes X and Y over k. We have a k-linear, exact, fully faithful functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

Before reading on, it makes sense to read at least some of Derived Categories, Section 13.41.

Recall that X is regular and hence has the resolution property (Varieties, Lemma 33.25.3 and Derived Categories of Schemes, Lemma 36.36.8). Thus on $X \times X$ we may choose a resolution

$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_\Delta \to 0$$

where each $\mathcal{E}_i$ and $\mathcal{G}_i$ is a finite locally free $\mathcal{O}_X$-module, see Lemma 57.9.3. Using the complex

0G09 (57.13.1.1)
$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0$$

in $D_{perf}(\mathcal{O}_{X\times X})$ as in Derived Categories, Example 13.41.2 if for each n we denote

$$M_n = (\mathcal{E}_n \boxtimes \mathcal{G}_n \to \ldots \to \mathcal{E}_0 \boxtimes \mathcal{G}_0)[-n]$$

we obtain an infinite Postnikov system for the complex (57.13.1.1). This means the morphisms $M_0 \to M_1[1] \to M_2[2] \to \ldots$ and $M_n \to \mathcal{E}_n \boxtimes \mathcal{G}_n$ and $\mathcal{E}_n \boxtimes \mathcal{G}_n \to M_{n-1}$ satisfy certain conditions documented in Derived Categories, Definition 13.41.1. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{E}_n \boxtimes \mathcal{G}_n \to \mathcal{E}_{n-1} \boxtimes \mathcal{G}_{n-1})$$

Observe that since $\mathcal{O}_\Delta$ is flat over X via pr_1 the same is true for $\mathcal{F}_n$ for all n (this is a convenient though not essential observation). We have

$$H^q(M_n[n]) = \begin{cases} \mathcal{O}_\Delta & \text{if} \quad q = 0 \\ \mathcal{F}_n & \text{if} \quad q = -n \\ 0 & \text{if} \quad q \neq 0, -n \end{cases}$$

Thus for $n \geq \dim(X \times X)$ we have

$$M_n[n] \cong \mathcal{O}_\Delta \oplus \mathcal{F}_n[n]$$

in $D_{perf}(\mathcal{O}_{X\times X})$ by Lemma 57.9.5.

We are interested in the complex

0G0A (57.13.1.2)
$$\ldots \to \mathcal{E}_2 \boxtimes F(\mathcal{G}_2) \to \mathcal{E}_1 \boxtimes F(\mathcal{G}_1) \to \mathcal{E}_0 \boxtimes F(\mathcal{G}_0)$$

in $D_{perf}(\mathcal{O}_{X\times Y})$ as the "totalization" of this complex should give us the kernel of the Fourier-Mukai functor we are trying to construct. For all $i, j \geq 0$ we have

$$\begin{aligned} \mathrm{Ext}^q_{X\times Y}(\mathcal{E}_i \boxtimes F(\mathcal{G}_i), \mathcal{E}_j \boxtimes F(\mathcal{G}_j)) &= \bigoplus\nolimits_p \mathrm{Ext}^{q+p}_X(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \mathrm{Ext}^{-p}_Y(F(\mathcal{G}_i), F(\mathcal{G}_j)) \\ &= \bigoplus\nolimits_p \mathrm{Ext}^{q+p}_X(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \mathrm{Ext}^{-p}_X(\mathcal{G}_i, \mathcal{G}_j) \end{aligned}$$

The second equality holds because F is fully faithful and the first by Derived Categories of Schemes, Lemma 36.25.1. We find these Ext^q are zero for $q < 0$. Hence by Derived Categories, Lemma 13.41.6 we can build an infinite Postnikov system $K_0, K_1, K_2, \ldots$ in $D_{perf}(\mathcal{O}_{X \times Y})$ for the complex (57.13.1.2). Parallel to what happens with $M_0, M_1, M_2, \ldots$ this means we obtain morphisms $K_0 \to K_1[1] \to K_2[2] \to \ldots$ and $K_n \to \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$ and $\mathcal{E}_n \boxtimes F(\mathcal{G}_n) \to K_{n-1}$ in $D_{perf}(\mathcal{O}_{X \times Y})$ satisfying certain conditions documented in Derived Categories, Definition 13.41.1.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module whose support has a finite number of points, i.e., with $\dim(\text{Supp}(\mathcal{F})) = 0$. Consider the exact functor of triangulated categories

$$D_{perf}(\mathcal{O}_{X \times Y}) \longrightarrow D_{perf}(\mathcal{O}_Y), \quad N \longmapsto R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y}} N)$$

It follows that the objects $R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y}} K_i)$ form a Postnikov system for the complex in $D_{perf}(\mathcal{O}_Y)$ with terms

$$R\text{pr}_{2,*}((\mathcal{F} \otimes \mathcal{E}_i) \boxtimes F(\mathcal{G}_i)) = \Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k F(\mathcal{G}_i) = F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i)$$

Here we have used that $\mathcal{F} \otimes \mathcal{E}_i$ has vanishing higher cohomology as its support has dimension 0. On the other hand, applying the exact functor

$$D_{perf}(\mathcal{O}_{X \times X}) \longrightarrow D_{perf}(\mathcal{O}_Y), \quad N \longmapsto F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} N))$$

we find that the objects $F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} M_n))$ form a second infinite Postnikov system for the complex in $D_{perf}(\mathcal{O}_Y)$ with terms

$$F(R\text{pr}_{2,*}((\mathcal{F} \otimes \mathcal{E}_i) \boxtimes \mathcal{G}_i)) = F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i)$$

This is the same as before! By uniqueness of Postnikov systems (Derived Categories, Lemma 13.41.6) which applies because

$$\text{Ext}^q_Y(F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i), F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_j) \otimes_k \mathcal{G}_j)) = 0, \quad q < 0$$

as F is fully faithful, we find a system of isomorphisms

$$F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} M_n[n])) \cong R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y}} K_n[n])$$

in $D_{perf}(\mathcal{O}_Y)$ compatible with the morphisms in $D_{perf}(\mathcal{O}_Y)$ induced by the morphisms

$$M_{n-1}[n-1] \to M_n[n] \quad \text{and} \quad K_{n-1}[n-1] \to K_n[n]$$
$$M_n \to \mathcal{E}_n \boxtimes \mathcal{G}_n \quad \text{and} \quad K_n \to \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$$
$$\mathcal{E}_n \boxtimes \mathcal{G}_n \to M_{n-1} \quad \text{and} \quad \mathcal{E}_n \boxtimes F(\mathcal{G}_n) \to K_{n-1}$$

which are part of the structure of Postnikov systems. For n sufficiently large we obtain a direct sum decomposition

$$F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} M_n[n])) = F(\mathcal{F}) \oplus F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{F}_n))[n]$$

corresponding to the direct sum decomposition of M_n constructed above (we are using the flatness of $\mathcal{F}_n$ over X via pr_1 to write a usual tensor product in the formula above, but this isn't essential for the argument). By Lemma 57.9.9 we find there exists an integer $m \geq 0$ such that the first summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval $[-m, m]$ and the second summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval $[-m-n, m+\dim(X)-n]$. We conclude the system $K_0 \to K_1[1] \to K_2[2] \to \ldots$ in $D_{perf}(\mathcal{O}_{X \times Y})$ satisfies the assumptions of Lemma 57.9.10 after possibly replacing m by a larger integer. We conclude we can write

$$K_n[n] = K \oplus C_n$$

for $n \gg 0$ compatible with transition maps and with C_n having nonzero cohomology sheaves only in the range $[-m-n, m-n]$. Denote G the Fourier-Mukai functor corresponding to K. Putting everything together we find

$$\begin{gathered} G(\mathcal{F}) \oplus R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} C_n) \cong \\ R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n[n]) \cong \\ F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} M_n[n])) \cong \\ F(\mathcal{F}) \oplus F(R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}_n))[n] \end{gathered}$$

Looking at the degrees that objects live in we conclude that for $n \gg m$ we obtain an isomorphism

$$F(\mathcal{F}) \cong G(\mathcal{F})$$

Moreover, recall that this holds for every coherent $\mathcal{F}$ on X whose support has dimension 0.

0G0B Lemma 57.13.2. Let k be a field. Let X and Y be smooth proper schemes over k. Given a k-linear, exact, fully faithful functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$ which is a sibling to F.

Proof. Apply Lemma 57.12.6 to F and the functor G constructed above. □

The following theorem is also true without assuming X is projective, see [Ola20].

0G0C Theorem 57.13.3 (Orlov). Let k be a field. Let X and Y be smooth proper schemes over k with X projective over k. Any k-linear fully faithful exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is a Fourier-Mukai functor for some kernel in $D_{perf}(\mathcal{O}_{X\times Y})$. [Orl97, Theorem 2.2]; this is shown in [Ola20] without the assumption that X be projective

Proof. Let F' be the Fourier-Mukai functor which is a sibling of F as in Lemma 57.13.2. By Proposition 57.10.6 we have $F \cong F'$ provided we can show that $\text{Coh}(\mathcal{O}_X)$ has enough negative objects. However, if $X = \text{Spec}(k)$ for example, then this isn't true. Thus we first decompose $X = \coprod X_i$ into its connected (and irreducible) components and we argue that it suffices to prove the result for each of the (fully faithful) composition functors

$$F_i : D_{perf}(\mathcal{O}_{X_i}) \to D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$$

Details omitted. Thus we may assume X is irreducible.

The case $\dim(X) = 0$. Here X is the spectrum of a finite (separable) extension k'/k and hence $D_{perf}(\mathcal{O}_X)$ is equivalent to the category of graded k'-vector spaces such that $\mathcal{O}_X$ corresponds to the trivial 1-dimensional vector space in degree 0. It is straightforward to see that any two siblings $F, F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ are isomorphic. Namely, we are given an isomorphism $F(\mathcal{O}_X) \cong F'(\mathcal{O}_X)$ compatible the action of the k-algebra $k' = \text{End}_{D_{perf}(\mathcal{O}_X)}(\mathcal{O}_X)$ which extends canonically to an isomorphism on any graded k'-vector space.

The case $\dim(X) > 0$. Here X is a projective smooth variety of dimension > 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We have to show there exists a coherent module $\mathcal{N}$ such that

(1) there is a surjection $\mathcal{N} \to \mathcal{F}$ and
(2) $\text{Hom}(\mathcal{F}, \mathcal{N}) = 0$.

Choose an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. We claim that $\mathcal{N} = (\mathcal{L}^{\otimes n})^{\oplus r}$ will work for $n \ll 0$ and r large enough. Condition (1) follows from Properties, Proposition 28.26.13. Finally, we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{L}^{\otimes n}) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{L}^{\otimes n})) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{L}^{\otimes n})$$

Since the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is torsion free, this vanishes for $n \ll 0$ by Varieties, Lemma 33.48.1. This finishes the proof. □

0G0D Proposition 57.13.4. Let k be a field. Let X and Y be smooth proper schemes over k. If $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is a k-linear exact equivalence of triangulated categories then there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$ which is an equivalence and a sibling of F.

Proof. The functor F' of Lemma 57.13.2 is an equivalence by Lemma 57.10.3. □

0G0E Lemma 57.13.5. Let k be a field. Let X be a smooth proper scheme over k. Let $K \in D_{perf}(\mathcal{O}_{X\times X})$. If the Fourier-Mukai functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ is isomorphic to the identity functor, then $K \cong \Delta_*\mathcal{O}_X$ in ${}_{perf}(\mathcal{O}_{X\times X})$.

Proof. Let i be the minimal integer such that the cohomology sheaf $H^i(K)$ is nonzero. Let $\mathcal{E}$ and $\mathcal{G}$ be finite locally free $\mathcal{O}_X$-modules. Then

$$\begin{aligned} H^i(X \times X, K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} (\mathcal{E} \boxtimes \mathcal{G})) &= H^i(X, R\mathrm{pr}_{2,*}(K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} (\mathcal{E} \boxtimes \mathcal{G}))) \\ &= H^i(X, \Phi_K(\mathcal{E}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \\ &\cong H^i(X, \mathcal{E} \otimes \mathcal{G}) \end{aligned}$$

which is zero if $i < 0$. On the other hand, we can choose $\mathcal{E}$ and $\mathcal{G}$ such that there is a surjection $\mathcal{E}^\vee \boxtimes \mathcal{G}^\vee \to H^i(K)$ by Lemma 57.9.1. In this case the left hand side of the equalities is nonzero. Hence we conclude that $H^i(K) = 0$ for $i < 0$.

Let i be the maximal integer such that $H^i(K)$ is nonzero. The same argument with $\mathcal{E}$ and $\mathcal{G}$ support of dimension 0 shows that $i \leq 0$. Hence we conclude that K is given by a single coherent $\mathcal{O}_{X\times X}$-module $\mathcal{K}$ sitting in degree 0.

Since $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})$ is $\mathcal{F}$, by taking $\mathcal{F}$ supported at closed points we see that the support of $\mathcal{K}$ is finite over X via pr_2. Since $R\mathrm{pr}_{2,*}(\mathcal{K}) \cong \mathcal{O}_X$ we conclude by Functors and Morphisms, Lemma 56.7.6 that $\mathcal{K} = s_*\mathcal{O}_X$ for some section $s : X \to X \times X$ of the second projection. Then $\Phi_K(M) = f^*M$ where $f = \mathrm{pr}_1 \circ s$ and this can happen only if s is the diagonal morphism as desired. □

57.14. A category of Fourier-Mukai kernels

0G0F Let S be a scheme. We claim there is a category with

(1) Objects are proper smooth schemes over S.
(2) Morphisms from X to Y are isomorphism classes of objects of $D_{perf}(\mathcal{O}_{X\times_S Y})$.
(3) Composition of the isomorphism class of $K \in D_{perf}(\mathcal{O}_{X\times_S Y})$ and the isomorphism class of K' in $D_{perf}(\mathcal{O}_{Y\times_S Z})$ is the isomorphism class of

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y\times_S Z}} L\mathrm{pr}_{23}^*K')$$

which is in $D_{perf}(\mathcal{O}_{X\times_S Z})$ by Derived Categories of Schemes, Lemma 36.30.4.

(4) The identity morphism from X to X is the isomorphism class of $\Delta_{X/S,*}\mathcal{O}_X$ which is in $D_{perf}(\mathcal{O}_{X \times_S X})$ by More on Morphisms, Lemma 37.61.12 and the fact that $\Delta_{X/S}$ is a perfect morphism by Divisors, Lemma 31.22.11 and More on Morphisms, Lemma 37.61.7.

Let us check that associativity of composition of morphisms holds; we omit verifying that the identity morphisms are indeed identities. To see this suppose we have X, Y, Z, W and $c \in D_{perf}(\mathcal{O}_{X \times_S Y})$, $c' \in D_{perf}(\mathcal{O}_{Y \times_S Z})$, and $c'' \in D_{perf}(\mathcal{O}_{Z \times_S W})$. Then we have

$$\begin{aligned} c'' \circ (c' \circ c) &\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{134,*}_{13}\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{12} c \otimes \mathrm{pr}^{123,*}_{23} c') \otimes \mathrm{pr}^{134,*}_{34} c'') \\ &\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}\mathrm{pr}^{1234,*}_{123}(\mathrm{pr}^{123,*}_{12} c \otimes \mathrm{pr}^{123,*}_{23} c') \otimes \mathrm{pr}^{134,*}_{34} c'') \\ &\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}(\mathrm{pr}^{1234,*}_{12} c \otimes \mathrm{pr}^{1234,*}_{23} c') \otimes \mathrm{pr}^{134,*}_{34} c'') \\ &\cong \mathrm{pr}^{134}_{14,*}\mathrm{pr}^{1234}_{134,*}((\mathrm{pr}^{1234,*}_{12} c \otimes \mathrm{pr}^{1234,*}_{23} c') \otimes \mathrm{pr}^{1234,*}_{34} c'') \\ &\cong \mathrm{pr}^{1234}_{14,*}((\mathrm{pr}^{1234,*}_{12} c \otimes \mathrm{pr}^{1234,*}_{23} c') \otimes \mathrm{pr}^{1234,*}_{34} c'') \end{aligned}$$

Here we use the notation

$$p^{1234}_{134} : X \times_S Y \times_S Z \times_S W \to X \times_S Z \times_S W \quad \text{and} \quad p^{134}_{14} : X \times_S Z \times_S W \to X \times_S W$$

the projections and similarly for other indices. We also write pr_* instead of $R\mathrm{pr}_*$ and pr^* instead of $L\mathrm{pr}^*$ and we drop all super and sub scripts on $\otimes$. The first equality is the definition of the composition. The second equality holds because $\mathrm{pr}^{134,*}_{13}\mathrm{pr}^{123}_{13,*} = \mathrm{pr}^{1234}_{134,*}\mathrm{pr}^{1234,*}_{123}$ by base change (Derived Categories of Schemes, Lemma 36.22.5). The third equality holds because pullbacks compose correctly and pass through tensor products, see Cohomology, Lemmas 20.27.2 and 20.27.3. The fourth equality follows from the "projection formula" for p^{1234}_{134}, see Derived Categories of Schemes, Lemma 36.22.1. The fifth equality is that proper pushforward is compatible with composition, see Cohomology, Lemma 20.28.2. Since tensor product is associative this concludes the proof of associativity of composition.

0G0G Lemma 57.14.1. Let $S' \to S$ be a morphism of schemes. The rule which sends

(1) a smooth proper scheme X over S to $X' = S' \times_S X$, and
(2) the isomorphism class of an object K of $D_{perf}(\mathcal{O}_{X \times_S Y})$ to the isomorphism class of $L(X' \times_{S'} Y' \to X \times_S Y)^*K$ in $D_{perf}(\mathcal{O}_{X' \times_{S'} Y'})$

is a functor from the category defined for S to the category defined for S'.

Proof. To see this suppose we have X, Y, Z and $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$ and $M \in D_{perf}(\mathcal{O}_{Y \times_S Z})$. Denote $K' \in D_{perf}(\mathcal{O}_{X' \times_{S'} Y'})$ and $M' \in D_{perf}(\mathcal{O}_{Y' \times_{S'} Z'})$ their pullbacks as in the statement of the lemma. The diagram

$$\begin{tikzcd} X' \times_{S'} Y' \times_{S'} Z' \arrow[r] \arrow[d, "\mathrm{pr}'_{13}"'] & X \times_S Y \times_S Z \arrow[d, "\mathrm{pr}_{13}"] \\ X' \times_{S'} Z' \arrow[r] & X \times_S Z \end{tikzcd}$$

is cartesian and pr_{13} is proper and smooth. By Derived Categories of Schemes, Lemma 36.30.4 we see that the derived pullback by the lower horizontal arrow of the composition

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}^*_{12}K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S Z}} L\mathrm{pr}^*_{23}M)$$

indeed is (canonically) isomorphic to

$$R\mathrm{pr}'_{13,*}(L(\mathrm{pr}'_{12})^*K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X' \times_{S'} Y' \times_{S'} Z'}} L(\mathrm{pr}'_{23})^*M')$$

as desired. Some details omitted. □

57.15. Relative equivalences

0G0H In this section we prove some lemmas about the following concept.

0G0I Definition 57.15.1. Let S be a scheme. Let $X \to S$ and $Y \to S$ be smooth proper morphisms. An object $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$ is said to be the Fourier-Mukai kernel of a relative equivalence from X to Y over S if there exist an object $K' \in D_{perf}(\mathcal{O}_{X \times_S Y})$ such that

$$\Delta_{X/S,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S X}} L\mathrm{pr}_{23}^*K')$$

in $D(\mathcal{O}_{X \times_S X})$ and

$$\Delta_{Y/S,*}\mathcal{O}_Y \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K' \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times_S X \times_S Y}} L\mathrm{pr}_{23}^*K)$$

in $D(\mathcal{O}_{Y \times_S Y})$. In other words, the isomorphism class of K defines an invertible arrow in the category defined in Section 57.14.

The language is intentionally cumbersome.

0G0J Lemma 57.15.2. With notation as in Definition 57.15.1 let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Then the corresponding Fourier-Mukai functors $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ (Lemma 57.8.2) and $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ (Lemma 57.8.4) are equivalences.

Proof. Immediate from Lemma 57.8.3 and Example 57.8.6. □

0G0K Lemma 57.15.3. With notation as in Definition 57.15.1 let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Let $S_1 \to S$ be a morphism of schemes. Let $X_1 = S_1 \times_S X$ and $Y_1 = S_1 \times_S Y$. Then the pullback $K_1 = L(X_1 \times_{S_1} Y_1 \to X \times_S Y)^*K$ is the Fourier-Mukai kernel of a relative equivalence from X_1 to Y_1 over S_1.

Proof. Let $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ be the object assumed to exist in Definition 57.15.1. Denote K'_1 the pullback of K' by $Y_1 \times_{S_1} X_1 \to Y \times_S X$. Then it suffices to prove that we have

$$\Delta_{X_1/S_1,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K_1 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_1 \times_{S_1} Y_1 \times_{S_1} X_1}} L\mathrm{pr}_{23}^*K'_1)$$

in $D(\mathcal{O}_{X_1 \times_{S_1} X_1})$ and similarly for the other condition. Since

$$\begin{tikzcd} X_1 \times_{S_1} Y_1 \times_{S_1} X_1 \arrow[r] \arrow[d, "\mathrm{pr}_{13}"'] & X \times_S Y \times_S X \arrow[d, "\mathrm{pr}_{13}"] \\ X_1 \times_{S_1} X_1 \arrow[r] & X \times_S X \end{tikzcd}$$

is cartesian it suffices by Derived Categories of Schemes, Lemma 36.30.4 to prove that

$$\Delta_{X_1/S_1,*}\mathcal{O}_{X_1} \cong L(X_1 \times_{S_1} X_1 \to X \times_S X)^*\Delta_{X/S,*}\mathcal{O}_X$$

This in turn will be true if X and $X_1 \times_{S_1} X_1$ are tor independent over $X \times_S X$, see Derived Categories of Schemes, Lemma 36.22.5. This tor independence can be seen directly but also follows from the more general More on Morphisms, Lemma 37.69.1 applied to the square with corners X, X, X, S and its base change by $S_1 \to S$. $\square$

0G0L Lemma 57.15.4. Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. Let $0 \in I$. Let $X_0 \to S_0$ and $Y_0 \to S_0$ be smooth proper morphisms. We set $X_i = S_i \times_{S_0} X_0$ for $i \geq 0$ and $X = S \times_{S_0} X_0$ and similarly for Y_0. If K is the Fourier-Mukai kernel of a relative equivalence from X to Y over S then for some $i \geq 0$ there exists a Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i.

Proof. Let $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ be the object assumed to exist in Definition 57.15.1. Since $X \times_S Y = \lim X_i \times_{S_i} Y_i$ there exists an i and objects K_i and K'_i in $D_{perf}(\mathcal{O}_{Y_i \times_{S_i} X_i})$ whose pullbacks to $Y \times_S X$ give K and K'. See Derived Categories of Schemes, Lemma 36.29.3. By Derived Categories of Schemes, Lemma 36.30.4 the object

$$R\text{pr}_{13,*}(L\text{pr}_{12}^*K_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}} L\text{pr}_{23}^*K'_i)$$

is perfect and its pullback to $X \times_S X$ is equal to

$$R\text{pr}_{13,*}(L\text{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S X}} L\text{pr}_{23}^*K') \cong \Delta_{X/S,*}\mathcal{O}_X$$

See proof of Lemma 57.15.3. On the other hand, since $X_i \to S$ is smooth and separated the object

$$\Delta_{i,*}\mathcal{O}_{X_i}$$

of $D(\mathcal{O}_{X_i \times_{S_i} X_i})$ is also perfect (by More on Morphisms, Lemmas 37.62.18 and 37.61.13) and its pullback to $X \times_S X$ is equal to

$$\Delta_{X/S,*}\mathcal{O}_X$$

See proof of Lemma 57.15.3. Thus by Derived Categories of Schemes, Lemma 36.29.3 after increasing i we may assume that

$$\Delta_{i,*}\mathcal{O}_{X_i} \cong R\text{pr}_{13,*}(L\text{pr}_{12}^*K_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}} L\text{pr}_{23}^*K'_i)$$

as desired. The same works for the roles of K and K' reversed. $\square$

57.16. No deformations

0G0M The title of this section refers to Lemma 57.16.4

0G0N Lemma 57.16.1. Let $(R, \mathfrak{m}, \kappa) \to (A, \mathfrak{n}, \lambda)$ be a flat local ring homorphism of local rings which is essentially of finite presentation. Let $\overline{f}_1, \ldots, \overline{f}_r \in \mathfrak{n}/\mathfrak{m}A \subset A/\mathfrak{m}A$ be a regular sequence. Let $K \in D(A)$. Assume

(1) K is perfect,
(2) $K \otimes^{\mathbf{L}}_A A/\mathfrak{m}A$ is isomorphic in $D(A/\mathfrak{m}A)$ to the Koszul complex on $\overline{f}_1, \ldots, \overline{f}_r$.

Then K is isomorphic in $D(A)$ to a Koszul complex on a regular sequence $f_1, \ldots, f_r \in A$ lifting the given elements $\overline{f}_1, \ldots, \overline{f}_r$. Moreover, $A/(f_1, \ldots, f_r)$ is flat over R.

Proof. Let us use chain complexes in the proof of this lemma. The Koszul complex $K_\bullet(\overline{f}_1, \ldots, \overline{f}_r)$ is defined in More on Algebra, Definition 15.28.2. By More on Algebra, Lemma 15.75.4 we can represent K by a complex

$$K_\bullet : A \to A^{\oplus r} \to \ldots \to A^{\oplus r} \to A$$

whose tensor product with $A/\mathfrak{m}A$ is equal (!) to $K_\bullet(\overline{f}_1, \ldots, \overline{f}_r)$. Denote $f_1, \ldots, f_r \in A$ the components of the arrow $A^{\oplus r} \to A$. These f_i are lifts of the $\overline{f}_i$. By Algebra, Lemma 10.128.6 $f_1, \ldots, f_r$ form a regular sequence in A and $A/(f_1, \ldots, f_r)$ is flat over R. Let $J = (f_1, \ldots, f_r) \subset A$. Consider the diagram

$$\begin{tikzcd} K_\bullet \arrow[rr, dotted, "\varphi_\bullet"] \arrow[dr] & & K_\bullet(f_1, \ldots, f_r) \arrow[dl] \\ & A/J & \end{tikzcd}$$

Since $f_1, \ldots, f_r$ is a regular sequence the south-west arrow is a quasi-isomorphism (see More on Algebra, Lemma 15.30.2). Hence we can find the dotted arrow making the diagram commute for example by Algebra, Lemma 10.71.4. Reducing modulo $\mathfrak{m}$ we obtain a commutative diagram

$$\begin{tikzcd} K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) \arrow[rr, "\overline{\varphi}_\bullet"] \arrow[dr] & & K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) \arrow[dl] \\ & (A/\mathfrak{m}A)/(\overline{f}_1, \ldots, \overline{f}_r) & \end{tikzcd}$$

by our choice of $K_\bullet$. Thus $\overline{\varphi}$ is an isomorphism in the derived category $D(A/\mathfrak{m}A)$. It follows that $\overline{\varphi} \otimes^{\mathbf{L}}_{A/\mathfrak{m}A} \lambda$ is an isomorphism. Since $\overline{f}_i \in \mathfrak{n}/\mathfrak{m}A$ we see that

$$\mathrm{Tor}_i^{A/\mathfrak{m}A}(K_\bullet(\overline{f}_1, \ldots, \overline{f}_r), \lambda) = K_i(\overline{f}_1, \ldots, \overline{f}_r) \otimes_{A/\mathfrak{m}A} \lambda$$

Hence φ_i mod $\mathfrak{n}$ is invertible. Since A is local this means that φ_i is an isomorphism and the proof is complete. $\square$

0G0P Lemma 57.16.2. Let $R \to S$ be a finite type flat ring map of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Let $K \in D(S)$ be perfect. Let $f_1, \ldots, f_r \in \mathfrak{q}S_\mathfrak{q}$ be a regular sequence such that $S_\mathfrak{q}/(f_1, \ldots, f_r)$ is flat over R and such that $K \otimes^{\mathbf{L}}_S S_\mathfrak{q}$ is isomorphic to the Koszul complex on $f_1, \ldots, f_r$. Then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that

(1) $f_1, \ldots, f_r$ are the images of $f'_1, \ldots, f'_r \in S_g$,
(2) $f'_1, \ldots, f'_r$ form a regular sequence in S_g,
(3) $S_g/(f'_1, \ldots, f'_r)$ is flat over R,
(4) $K \otimes^{\mathbf{L}}_S S_g$ is isomorphic to the Koszul complex on $f_1, \ldots, f_r$.

Proof. We can find $g \in S$, $g \notin \mathfrak{q}$ with property (1) by the definition of localizations. After replacing g by gg' for some $g' \in S$, $g' \notin \mathfrak{q}$ we may assume (2) holds, see Algebra, Lemma 10.68.6. By Algebra, Theorem 10.129.4 we find that $S_g/(f'_1, \ldots, f'_r)$ is flat over R in an open neighbourhood of $\mathfrak{q}$. Hence after once more replacing g by gg' for some $g' \in S$, $g' \notin \mathfrak{q}$ we may assume (3) holds as well. Finally, we get (4) for a further replacement by More on Algebra, Lemma 15.74.17. $\square$

For a generalization of the following lemma, please see More on Morphisms of Spaces, Lemma 76.49.6.

0G0Q Lemma 57.16.3. Let S be a Noetherian scheme. Let $s \in S$. Let $p : X \to Y$ be a morphism of schemes over S. Assume

(1) $Y \to S$ and $X \to S$ proper,
(2) X is flat over S,

(3) $X_s \to Y_s$ an isomorphism.

Then there exists an open neighbourhood $U \subset S$ of s such that the base change $X_U \to Y_U$ is an isomorphism.

Proof. The morphism p is proper by Morphisms, Lemma 29.41.6. By Cohomology of Schemes, Lemma 30.21.2 there is an open $Y_s \subset V \subset Y$ such that $p|_{p^{-1}(V)} : p^{-1}(V) \to V$ is finite. By More on Morphisms, Theorem 37.16.1 there is an open $X_s \subset U \subset X$ such that $p|_U : U \to Y$ is flat. After removing the images of $X \setminus U$ and $Y \setminus V$ (which are closed subsets not containing s) we may assume p is flat and finite. Then p is open (Morphisms, Lemma 29.25.10) and $Y_s \subset p(X) \subset Y$ hence after shrinking S we may assume p is surjective. As $p_s : X_s \to Y_s$ is an isomorphism, the map

$$p^\sharp : \mathcal{O}_Y \longrightarrow p_*\mathcal{O}_X$$

of coherent $\mathcal{O}_Y$-modules (p is finite) becomes an isomorphism after pullback by $i : Y_s \to Y$ (by Cohomology of Schemes, Lemma 30.5.1 for example). By Nakayama's lemma, this implies that $\mathcal{O}_{Y,y} \to (p_*\mathcal{O}_X)_y$ is surjective for all $y \in Y_s$. Hence there is an open $Y_s \subset V \subset Y$ such that $p^\sharp|_V$ is surjective (Modules, Lemma 17.9.4). Hence after shrinking S once more we may assume $p^\sharp$ is surjective which means that p is a closed immersion (as p is already finite). Thus now p is a surjective flat closed immersion of Noetherian schemes and hence an isomorphism, see Morphisms, Section 29.26. ☐

0G0R Lemma 57.16.4. Let k be a field. Let S be a finite type scheme over k with k-rational point s. Let $Y \to S$ be a smooth proper morphism. Let $X = Y_s \times S \to S$ be the constant family with fibre Y_s. Let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Assume the restriction

$$L(Y_s \times_S Y_s \to X \times_S Y)^* K \cong \Delta_{Y_s/k,*}\mathcal{O}_{Y_s}$$

in $D(\mathcal{O}_{Y_s \times Y_s})$. Then there is an open neighbourhood $s \in U \subset S$ such that $Y|_U$ is isomorphic to $Y_s \times U$ over U.

Proof. Denote $i : Y_s \times Y_s = X_s \times Y_s \to X \times_S Y$ the natural closed immersion. (We will write Y_s and not X_s for the fibre of X over s from now on.) Let $z \in Y_s \times Y_s = (X \times_S Y)_s \subset X \times_S Y$ be a closed point. As indicated we think of z both as a closed point of $Y_s \times Y_s$ as well as a closed point of $X \times_S Y$.

Case I: $z \notin \Delta_{Y_s/k}(Y_s)$. Denote $\mathcal{O}_z$ the coherent $\mathcal{O}_{Y_s \times Y_s}$-module supported at z whose value is $\kappa(z)$. Then $i_*\mathcal{O}_z$ is the coherent $\mathcal{O}_{X \times_S Y}$-module supported at z whose value is $\kappa(z)$. Our assumption means that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} i_*\mathcal{O}_z = Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{Y_s \times Y_s}} \mathcal{O}_z = 0$$

Hence by Lemma 57.11.3 we find an open neighbourhood $U(z) \subset X \times_S Y$ of z such that $K|_{U(z)} = 0$. In this case we set $Z(z) = \emptyset$ as closed subscheme of $U(z)$.

Case II: $z \in \Delta_{Y_s/k}(Y_s)$. Since Y_s is smooth over k we know that $\Delta_{Y_s/k} : Y_s \to Y_s \times Y_s$ is a regular immersion, see More on Morphisms, Lemma 37.62.18. Choose a regular sequence $\overline{f}_1, \ldots, \overline{f}_r \in \mathcal{O}_{Y_s \times Y_s, z}$ cutting out the ideal sheaf of $\Delta_{Y_s/k}(Y_s)$. Since a regular sequence is Koszul-regular (More on Algebra, Lemma 15.30.2) our assumption means that

$$K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y, z}} \mathcal{O}_{Y_s \times Y_s, z} \in D(\mathcal{O}_{Y_s \times Y_s, z})$$

is represented by the Koszul complex on $\overline{f}_1, \ldots, \overline{f}_r$ over $\mathcal{O}_{Y_s \times Y_s, z}$. By Lemma 57.16.1 applied to $\mathcal{O}_{S,s} \to \mathcal{O}_{X \times_S Y, z}$ we conclude that $K_z \in D(\mathcal{O}_{X \times_S Y, z})$ is represented by the Koszul complex on a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_{X \times_S Y, z}$ lifting the regular sequence $\overline{f}_1, \ldots, \overline{f}_r$ such that moreover $\mathcal{O}_{X \times_S Y}/(f_1, \ldots, f_r)$ is flat over $\mathcal{O}_{S,s}$. By some limit arguments (Lemma 57.16.2) we conclude that there exists an affine open neighbourhood $U(z) \subset X \times_S Y$ of z and a closed subscheme $Z(z) \subset U(z)$ such that

(1) $Z(z) \to U(z)$ is a regular closed immersion,
(2) $K|_{U(z)}$ is quasi-isomorphic to $\mathcal{O}_{Z(z)}$,
(3) $Z(z) \to S$ is flat,
(4) $Z(z)_s = \Delta_{Y_s/k}(Y_s) \cap U(z)_s$ as closed subschemes of $U(z)_s$.

By property (2), for $z, z' \in Y_s \times Y_s$, we find that $Z(z) \cap U(z') = Z(z') \cap U(z)$ as closed subschemes. Hence we obtain an open neighbourhood

$$U = \bigcup\nolimits_{z \in Y_s \times Y_s \text{ closed}} U(z)$$

of $Y_s \times Y_s$ in $X \times_S Y$ and a closed subscheme $Z \subset U$ such that (1) $Z \to U$ is a regular closed immersion, (2) $Z \to S$ is flat, and (3) $Z_s = \Delta_{Y_s/k}(Y_s)$. Since $X \times_S Y \to S$ is proper, after replacing S by an open neighbourhood of s we may assume $U = X \times_S Y$. Since the projections $Z_s \to Y_s$ and $Z_s \to X_s$ are isomorphisms, we conclude that after shrinking S we may assume $Z \to Y$ and $Z \to X$ are isomorphisms, see Lemma 57.16.3. This finishes the proof. □

0G0S Lemma 57.16.5. Let k be an algebraically closed field. Let X be a smooth proper scheme over k. Let $f : Y \to S$ be a smooth proper morphism with S of finite type over k. Let K be the Fourier-Mukai kernel of a relative equivalence from $X \times S$ to Y over S. Then S can be covered by open subschemes U such that there is a U-isomorphism $f^{-1}(U) \cong Y_0 \times U$ for some Y_0 proper and smooth over k.

Proof. Choose a closed point $s \in S$. Since k is algebraically closed this is a k-rational point. Set $Y_0 = Y_s$. The restriction K_0 of K to $X \times Y_0$ is the Fourier-Mukai kernel of a relative equivalence from X to Y_0 over $\text{Spec}(k)$ by Lemma 57.15.3. Let K'_0 in $D_{perf}(\mathcal{O}_{Y_0 \times X})$ be the object assumed to exist in Definition 57.15.1. Then K'_0 is the Fourier-Mukai kernel of a relative equivalence from Y_0 to X over $\text{Spec}(k)$ by the symmetry inherent in Definition 57.15.1. Hence by Lemma 57.15.3 we see that the pullback

$$M = (Y_0 \times X \times S \to Y_0 \times X)^* K'_0$$

on $(Y_0 \times S) \times_S (X \times S) = Y_0 \times X \times S$ is the Fourier-Mukai kernel of a relative equivalence from $Y_0 \times S$ to $X \times S$ over S. Now consider the kernel

$$K_{new} = R\text{pr}_{13,*}(L\text{pr}_{12}^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{(Y_0 \times S) \times_S (X \times S) \times_S Y}} L\text{pr}_{23}^* K)$$

on $(Y_0 \times S) \times_S Y$. This is the Fourier-Mukai kernel of a relative equivalence from $Y_0 \times S$ to Y over S since it is the composition of two invertible arrows in the category constructed in Section 57.14. Moreover, this composition passes through base change (Lemma 57.14.1). Hence we see that the pullback of K_{new} to $((Y_0 \times S) \times_S Y)_s = Y_0 \times Y_0$ is equal to the composition of K_0 and K'_0 and hence equal to the identity in this category. In other words, we have

$$L(Y_0 \times Y_0 \to (Y_0 \times S) \times_S Y)^* K_{new} \cong \Delta_{Y_0/k,*}\mathcal{O}_{Y_0}$$

Thus by Lemma 57.16.4 we conclude that $Y \to S$ is isomorphic to $Y_0 \times S$ in an open neighbourhood of s. This finishes the proof. □

57.17. Countability

0G0T In this section we prove some elementary lemmas about countability of certain sets. Let $\mathcal{C}$ be a category. In this section we will say that $\mathcal{C}$ is countable if

(1) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Mor}_\mathcal{C}(X, Y)$ is countable, and
(2) the set of isomorphism classes of objects of $\mathcal{C}$ is countable.

0G0U **Lemma 57.17.1.** *Let R be a countable Noetherian ring. Then the category of schemes of finite type over R is countable.*

Proof. Omitted. □

0G0V **Lemma 57.17.2.** *Let $\mathcal{A}$ be a countable abelian category. Then $D^b(\mathcal{A})$ is countable.*

Proof. It suffices to prove the statement for $D(\mathcal{A})$ as the others are full subcategories of this one. Since every object in $D(\mathcal{A})$ is a complex of objects of $\mathcal{A}$ it is immediate that the set of isomorphism classes of objects of $D^b(\mathcal{A})$ is countable. Moreover, for bounded complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{A}$ it is clear that $\mathrm{Hom}_{K^b(\mathcal{A})}(A^\bullet, B^\bullet)$ is countable. We have

$$\mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{colim}_{s:(A')^\bullet \to A^\bullet \text{ qis and } (A')^\bullet \text{ bounded}} \mathrm{Hom}_{K^b(\mathcal{A})}((A')^\bullet, B^\bullet)$$

by Derived Categories, Lemma 13.11.6. Thus this is a countable set as a countable colimit of □

0G0W **Lemma 57.17.3.** *Let X be a scheme of finite type over a countable Noetherian ring. Then the categories $D_{perf}(\mathcal{O}_X)$ and $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ are countable.*

Proof. Observe that X is Noetherian by Morphisms, Lemma 29.15.6. Hence $D_{perf}(\mathcal{O}_X)$ is a full subcategory of $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.6. Thus it suffices to prove the result for $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$. Recall that $D^b_{\mathrm{Coh}}(\mathcal{O}_X) = D^b(\mathrm{Coh}(\mathcal{O}_X))$ by Derived Categories of Schemes, Proposition 36.11.2. Hence by Lemma 57.17.2 it suffices to prove that $\mathrm{Coh}(\mathcal{O}_X)$ is countable. This we omit. □

0G0X **Lemma 57.17.4.** *Let K be an algebraically closed field. Let S be a finite type scheme over K. Let $X \to S$ and $Y \to S$ be finite type morphisms. There exists a countable set I and for $i \in I$ a pair $(S_i \to S, h_i)$ with the following properties*

(1) *$S_i \to S$ is a morphism of finite type, set $X_i = X \times_S S_i$ and $Y_i = Y \times_S S_i$,*
(2) *$h_i : X_i \to Y_i$ is an isomorphism over S_i, and*
(3) *for any closed point $s \in S(K)$ if $X_s \cong Y_s$ over $K = \kappa(s)$ then s is in the image of $S_i \to S$ for some i.*

Proof. The field K is the filtered union of its countable subfields. Dually, $\mathrm{Spec}(K)$ is the cofiltered limit of the spectra of the countable subfields of K. Hence Limits, Lemma 32.10.1 guarantees that we can find a countable subfield k and morphisms $X_0 \to S_0$ and $Y_0 \to S_0$ of schemes of finite type over k such that $X \to S$ and $Y \to S$ are the base changes of these.

By Lemma 57.17.1 there is a countable set I and pairs $(S_{0,i} \to S_0, h_{0,i})$ such that

(1) $S_{0,i} \to S_0$ is a morphism of finite type, set $X_{0,i} = X_0 \times_{S_0} S_{0,i}$ and $Y_{0,i} = Y_0 \times_{S_0} S_{0,i}$,
(2) $h_{0,i} : X_{0,i} \to Y_{0,i}$ is an isomorphism over $S_{0,i}$.

such that every pair $(T \to S_0, h_T)$ with $T \to S_0$ of finite type and $h_T : X_0 \times_{S_0} T \to Y_0 \times_{S_0} T$ an isomorphism is isomorphic to one of these. Denote $(S_i \to S, h_i)$ the base change of $(S_{0,i} \to S_0, h_{0,i})$ by $\text{Spec}(K) \to \text{Spec}(k)$. We claim this works.

Let $s \in S(K)$ and let $h_s : X_s \to Y_s$ be an isomorphism over $K = \kappa(s)$. We can write K as the filtered union of its finitely generated k-subalgebras. Hence by Limits, Proposition 32.6.1 and Lemma 32.10.1 we can find such a finitely generated k-subalgebra $K \supset A \supset k$ such that

(1) there is a commutative diagram

$$\begin{tikzcd} \text{Spec}(K) \arrow[r] \arrow[d, "s"'] & \text{Spec}(A) \arrow[d, "s'"] \\ S \arrow[r] & S_0 \end{tikzcd}$$

for some morphism $s' : \text{Spec}(A) \to S_0$ over k,

(2) h_s is the base change of an isomorphism $h_{s'} : X_0 \times_{S_0,s'} \text{Spec}(A) \to X_0 \times_{S_0,s'} \text{Spec}(A)$ over A.

Of course, then $(s' : \text{Spec}(A) \to S_0, h_{s'})$ is isomorphic to the pair $(S_{0,i} \to S_0, h_{0,i})$ for some $i \in I$. This concludes the proof because the commutative diagram in (1) shows that s is in the image of the base change of s' to $\text{Spec}(K)$. □

0G0Y Lemma 57.17.5. Let K be an algebraically closed field. There exists a countable set I and for $i \in I$ a pair $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ with the following properties

(1) S_i is a scheme of finite type over K,
(2) $X_i \to S_i$ and $Y_i \to S_i$ are proper smooth morphisms of schemes,
(3) $M_i \in D_{perf}(\mathcal{O}_{X_i \times_{S_i} Y_i})$ is the Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i, and
(4) for any smooth proper schemes X and Y over K such that there is a K-linear exact equivalence $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ there exists an $i \in I$ and a $s \in S_i(K)$ such that $X \cong (X_i)_s$ and $Y \cong (Y_i)_s$.

Proof. Choose a countable subfield $k \subset K$ for example the prime field. By Lemmas 57.17.1 and 57.17.3 there exists a countable set of isomorphism classes of systems over k satisfying parts (1), (2), (3) of the lemma. Thus we can choose a countable set I and for each $i \in I$ such a system

$$(S_{0,i}/k, X_{0,i} \to S_{0,i}, Y_{0,i} \to S_{0,i}, M_{0,i})$$

over k such that each isomorphism class occurs at least once. Denote $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ the base change of the displayed system to K. This system has properties (1), (2), (3), see Lemma 57.15.3. Let us prove property (4).

Consider smooth proper schemes X and Y over K such that there is a K-linear exact equivalence $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$. By Proposition 57.13.4 we may assume that there exists an object $M \in D_{perf}(\mathcal{O}_{X \times Y})$ such that $F = \Phi_M$ is the corresponding Fourier-Mukai functor. By Lemma 57.8.9 there is an M' in $D_{perf}(\mathcal{O}_{Y \times X})$ such that $\Phi_{M'}$ is the right adjoint to Φ_M. Since Φ_M is an equivalence, this means that $\Phi_{M'}$ is the quasi-inverse to Φ_M. By Lemma 57.8.9 we see that the Fourier-Mukai functors defined by the objects

$$A = R\text{pr}_{13,*}(L\text{pr}_{12}^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y \times X}} L\text{pr}_{23}^*M')$$

in $D_{perf}(\mathcal{O}_{X \times X})$ and

$$B = R\text{pr}_{13,*}(L\text{pr}_{12}^* M' \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times X \times Y}} L\text{pr}_{23}^* M)$$

in $D_{perf}(\mathcal{O}_{Y \times Y})$ are isomorphic to $\text{id} : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ and $\text{id} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_Y)$ Hence $A \cong \Delta_{X/K,*}\mathcal{O}_X$ and $B \cong \Delta_{Y/K,*}\mathcal{O}_Y$ by Lemma 57.13.5. Hence we see that M is the Fourier-Mukai kernel of a relative equivalence from X to Y over K by definition.

We can write K as the filtered colimit of its finite type k-subalgebras $A \subset K$. By Limits, Lemma 32.10.1 we can find X_0, Y_0 of finite type over A whose base changes to K produces X and Y. By Limits, Lemmas 32.13.1 and 32.8.9 after enlarging A we may assume X_0 and Y_0 are smooth and proper over A. By Lemma 57.15.4 after enlarging A we may assume M is the pullback of some $M_0 \in D_{perf}(\mathcal{O}_{X_0 \times_{\text{Spec}(A)} Y_0})$ which is the Fourier-Mukai kernel of a relative equivalence from X_0 to Y_0 over $\text{Spec}(A)$. Thus we see that $(S_0/k, X_0 \to S_0, Y_0 \to S_0, M_0)$ is isomorphic to $(S_{0,i}/k, X_{0,i} \to S_{0,i}, Y_{0,i} \to S_{0,i}, M_{0,i})$ for some $i \in I$. Since $S_i = S_{0,i} \times_{\text{Spec}(k)} \text{Spec}(K)$ we conclude that (4) is true with $s : \text{Spec}(K) \to S_i$ induced by the morphism $\text{Spec}(K) \to \text{Spec}(A) \cong S_{0,i}$ we get from $A \subset K$. $\square$

57.18. Countability of derived equivalent varieties

0G0Z In this section we prove a result of Anel and Toën, see [AT09].

0G10 Definition 57.18.1. Let k be a field. Let X and Y be smooth projective schemes over k. We say X and Y are derived equivalent if there exists a k-linear exact equivalence $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

Here is the result

0G11 Theorem 57.18.2. Let K be an algebraically closed field. Let $\mathbf{X}$ be a smooth proper scheme over K. There are at most countably many isomorphism classes of smooth proper schemes $\mathbf{Y}$ over K which are derived equivalent to $\mathbf{X}$. Slight improvement of [AT09]

Proof. Choose a countable set I and for $i \in I$ systems $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ satisfying properties (1), (2), (3), and (4) of Lemma 57.17.5. Pick $i \in I$ and set $S = S_i$, $X = X_i$, $Y = Y_i$, and $M = M_i$. Clearly it suffice to show that the set of isomorphism classes of fibres Y_s for $s \in S(K)$ such that $X_s \cong \mathbf{X}$ is countable. This we prove in the next paragraph.

Let S be a finite type scheme over K, let $X \to S$ and $Y \to S$ be proper smooth morphisms, and let $M \in D_{perf}(\mathcal{O}_{X \times_S Y})$ be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. We will show the set of isomorphism classes of fibres Y_s for $s \in S(K)$ such that $X_s \cong \mathbf{X}$ is countable. By Lemma 57.17.4 applied to the families $\mathbf{X} \times S \to S$ and $X \to S$ there exists a countable set I and for $i \in I$ a pair $(S_i \to S, h_i)$ with the following properties

(1) $S_i \to S$ is a morphism of finite type, set $X_i = X \times_S S_i$,
(2) $h_i : \mathbf{X} \times S_i \to X_i$ is an isomorphism over S_i, and
(3) for any closed point $s \in S(K)$ if $\mathbf{X} \cong X_s$ over $K = \kappa(s)$ then s is in the image of $S_i \to S$ for some i.

Set $Y_i = Y \times_S S_i$. Denote $M_i \in D_{perf}(\mathcal{O}_{X_i \times_{S_i} Y_i})$ the pullback of M. By Lemma 57.15.3 M_i is the Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i. Since I is countable, by property (3) it suffices to prove that the set of isomorphism classes of fibres $Y_{i,s}$ for $s \in S_i(K)$ is countable. In fact, this number is finite by Lemma 57.16.5 and the proof is complete. □

57.19. Other chapters

CHAPTER 58

Fundamental Groups of Schemes

0BQ6 58.1. Introduction

0BQ7 In this chapter we discuss Grothendieck's fundamental group of a scheme and applications. A foundational reference is [Gro71]. A nice introduction is [Len]. Other references [Mur67] and [GM71].

58.2. Schemes étale over a point

04JI In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

04JJ Definition 58.2.1. Let G be a topological group. A G-set, sometimes called a discrete G-set, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A morphism of G-sets $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted G-Sets.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 7.6.5) provided we endow G with the discrete topology.

Recall that if L/K is an infinite Galois extension then the Galois group $G = \text{Gal}(L/K)$ comes endowed with a canonical topology, see Fields, Section 9.22.

03QR Lemma 58.2.2. Let K be a field. Let K^{sep} be a separable closure of K. Consider the profinite group $G = \text{Gal}(K^{sep}/K)$. The functor

$$\begin{matrix} \text{schemes étale over } K & \longrightarrow & G\text{-Sets} \\ X/K & \longmapsto & \text{Mor}_{\text{Spec}(K)}(\text{Spec}(K^{sep}), X) \end{matrix}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i\in I} \text{Spec}(K_i)$ with each K_i a finite separable extension of K (Morphisms, Lemma 29.36.7). The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i \text{Hom}_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory (Fields, Theorem 9.22.4) the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i \text{Spec}((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. $\square$

03QS Remark 58.2.3. Under the correspondence of Lemma 58.2.2, the coverings in the small étale site $\mathrm{Spec}(K)_{\acute{e}tale}$ of K correspond to surjective families of maps in G-Sets.

58.3. Galois categories

0BMQ In this section we discuss some of the material the reader can find in [Gro71, Exposé V, Sections 4, 5, and 6].

Let $F : \mathcal{C} \to \textit{Sets}$ be a functor. Recall that by our conventions categories have a set of objects and for any pair of objects a set of morphisms. There is a canonical injective map

0BS7 (58.3.0.1)
$$\mathrm{Aut}(F) \longrightarrow \prod\nolimits_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Aut}(F(X))$$

For a set E we endow $\mathrm{Aut}(E)$ with the compact open topology, see Topology, Example 5.30.2. Of course this is the discrete topology when E is finite, which is the case of interest in this section[1]. We endow $\mathrm{Aut}(F)$ with the topology induced from the product topology on the right hand side of (58.3.0.1). In particular, the action maps
$$\mathrm{Aut}(F) \times F(X) \longrightarrow F(X)$$
are continuous when $F(X)$ is given the discrete topology because this is true for the action maps $\mathrm{Aut}(E) \times E \to E$ for any set E. The universal property of our topology on $\mathrm{Aut}(F)$ is the following: suppose that G is a topological group and $G \to \mathrm{Aut}(F)$ is a group homomorphism such that the induced actions $G \times F(X) \to F(X)$ are continuous for all $X \in \mathrm{Ob}(\mathcal{C})$ where $F(X)$ has the discrete topology. Then $G \to \mathrm{Aut}(F)$ is continuous.

The following lemma tells us that the group of automorphisms of a functor to the category of finite sets is automatically a profinite group.

0BMR Lemma 58.3.1. Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor. The map (58.3.0.1) identifies $\mathrm{Aut}(F)$ with a closed subgroup of $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Aut}(F(X))$. In particular, if $F(X)$ is finite for all X, then $\mathrm{Aut}(F)$ is a profinite group.

Proof. Let $\xi = (\gamma_X) \in \prod \mathrm{Aut}(F(X))$ be an element not in $\mathrm{Aut}(F)$. Then there exists a morphism $f : X \to X'$ of $\mathcal{C}$ and an element $x \in F(X)$ such that $F(f)(\gamma_X(x)) \neq \gamma_{X'}(F(f)(x))$. Consider the open neighbourhood $U = \{\gamma \in \mathrm{Aut}(F(X)) \mid \gamma(x) = \gamma_X(x)\}$ of γ_X and the open neighbourhood $U' = \{\gamma' \in \mathrm{Aut}(F(X')) \mid \gamma'(F(f)(x)) = \gamma_{X'}(F(f)(x))\}$. Then $U \times U' \times \prod_{X'' \neq X, X'} \mathrm{Aut}(F(X''))$ is an open neighbourhood of ξ not meeting $\mathrm{Aut}(F)$. The final statement follows from the fact that $\prod \mathrm{Aut}(F(X))$ is a profinite space if each $F(X)$ is finite. $\square$

0BMS Example 58.3.2. Let G be a topological group. An important example will be the forgetful functor

0BMT (58.3.2.1)
$$\textit{Finite-}G\textit{-Sets} \longrightarrow \textit{Sets}$$

where Finite-G-Sets is the full subcategory of G-Sets whose objects are the finite G-sets. The category G-Sets of G-sets is defined in Definition 58.2.1.

[1] When we discuss the pro-étale fundamental group the general case will be of interest.

Let G be a topological group. The profinite completion of G will be the profinite group

$$G^\wedge = \lim_{U \subset G \text{ open, normal, finite index}} G/U$$

with its profinite topology. Observe that the limit is cofiltered as a finite intersection of open, normal subgroups of finite index is another. The universal property of the profinite completion is that any continuous map $G \to H$ to a profinite group H factors canonically as $G \to G^\wedge \to H$.

0BMU Lemma 58.3.3. Let G be a topological group. The automorphism group of the functor (58.3.2.1) endowed with its profinite topology from Lemma 58.3.1 is the profinite completion of G.

Proof. Denote F_G the functor (58.3.2.1). Any morphism $X \to Y$ in Finite-G-Sets commutes with the action of G. Thus any $g \in G$ defines an automorphism of F_G and we obtain a canonical homomorphism $G \to \text{Aut}(F_G)$ of groups. Observe that any finite G-set X is a finite disjoint union of G-sets of the form G/H_i with canonical G-action where $H_i \subset G$ is an open subgroup of finite index. Then $U_i = \bigcap gH_ig^{-1}$ is open, normal, and has finite index. Moreover U_i acts trivially on G/H_i hence $U = \bigcap U_i$ acts trivially on $F_G(X)$. Hence the action $G \times F_G(X) \to F_G(X)$ is continuous. By the universal property of the topology on $\text{Aut}(F_G)$ the map $G \to \text{Aut}(F_G)$ is continuous. By Lemma 58.3.1 and the universal property of profinite completion there is an induced continuous group homomorphism

$$G^\wedge \longrightarrow \text{Aut}(F_G)$$

Moreover, since G/U acts faithfully on G/U this map is injective. If the image is dense, then the map is surjective and hence a homeomorphism by Topology, Lemma 5.17.8.

Let $\gamma \in \text{Aut}(F_G)$ and let $X \in \text{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that γ and g induce the same action on $F_G(X)$. This will finish the proof. As before we see that X is a finite disjoint union of G/H_i. With U_i and U as above, the finite G-set $Y = G/U$ surjects onto G/H_i for all i and hence it suffices to find $g \in G$ such that γ and g induce the same action on $F_G(G/U) = G/U$. Let $e \in G$ be the neutral element and say that $\gamma(eU) = g_0U$ for some $g_0 \in G$. For any $g_1 \in G$ the morphism

$$R_{g_1} : G/U \longrightarrow G/U, \quad gU \longmapsto gg_1U$$

of Finite-G-Sets commutes with the action of γ. Hence

$$\gamma(g_1U) = \gamma(R_{g_1}(eU)) = R_{g_1}(\gamma(eU)) = R_{g_1}(g_0U) = g_0g_1U$$

Thus we see that $g = g_0$ works. □

Recall that an exact functor is one which commutes with all finite limits and finite colimits. In particular such a functor commutes with equalizers, coequalizers, fibred products, pushouts, etc.

0BMV Lemma 58.3.4. Let G be a topological group. Let F : Finite-G-Sets $\to$ Sets be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the functor (58.3.2.1).

Proof. Let X be a nonempty object of Finite-G-Sets. The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

is cocartesian. Hence we conclude that $F(X)$ is nonempty. Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod\nolimits_{gU \in G/U} G/U$$

where the summand corresponding to gU corresponds to the orbit of (eU, gU) on the left hand side. Then we see that

$$F(G/U) \times F(G/U) = F(G/U \times G/U) = \coprod\nolimits_{gU \in G/U} F(G/U)$$

Hence $|F(G/U)| = |G/U|$ as $F(G/U)$ is nonempty. Thus we see that

$$\lim_{U \subset G \text{ open, normal, finite idex}} F(G/U)$$

is nonempty (Categories, Lemma 4.21.7). Pick $\gamma = (\gamma_U)$ an element in this limit. Denote F_G the functor (58.3.2.1). We can identify F_G with the functor

$$X \longmapsto \operatorname{colim}_U \operatorname{Mor}(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$ (details omitted). Hence the element γ determines a well defined map

$$t : F_G \longrightarrow F$$

Namely, given $x \in X$ choose U and $f : G/U \to X$ sending eU to x and then set $t_X(x) = F(f)(\gamma_U)$. We will show that t induces a bijective map $t_{G/U} : F_G(G/U) \to F(G/U)$ for any U. This implies in a straightforward manner that t is an isomorphism (details omitted). Since $|F_G(G/U)| = |F(G/U)|$ it suffices to show that $t_{G/U}$ is surjective. The image contains at least one element, namely $t_{G/U}(eU) = F(\text{id}_{G/U})(\gamma_U) = \gamma_U$. For $g \in G$ denote $R_g : G/U \to G/U$ right multiplication. Then set of fixed points of $F(R_g) : F(G/U) \to F(G/U)$ is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ because F commutes with equalizers. It follows that if $g_1, \ldots, g_{|G/U|}$ is a system of representatives for G/U, then the elements $F(R_{g_i})(\gamma_U)$ are pairwise distinct and hence fill out $F(G/U)$. Then

$$t_{G/U}(g_iU) = F(R_{g_i})(\gamma_U)$$

and the proof is complete. □

0BMW **Example 58.3.5.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor such that $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$. By Lemma 58.3.1 we see that $G = \text{Aut}(F)$ comes endowed with the structure of a profinite topological group in a canonical manner. We obtain a functor

0BMX (58.3.5.1) $$\mathcal{C} \longrightarrow \text{Finite-}G\text{-Sets}, \quad X \longmapsto F(X)$$

where $F(X)$ is endowed with the induced action of G. This action is continuous by our construction of the topology on $\text{Aut}(F)$.

The purpose of defining Galois categories is to single out those pairs $(\mathcal{C}, F)$ for which the functor (58.3.5.1) is an equivalence. Our definition of a Galois category is as follows.

0BMY Definition 58.3.6. Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a Galois category if Different from the definition in [Gro71, Exposé V, Definition 5.1]. Compare with [BS13, Definition 7.2.1].

(1) $\mathcal{C}$ has finite limits and finite colimits,
0BMZ (2) every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
(3) $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$, and
(4) F reflects isomorphisms[2] and is exact[3].

Here we say $X \in \text{Ob}(\mathcal{C})$ is connected if it is not initial and for any monomorphism $Y \to X$ either Y is initial or $Y \to X$ is an isomorphism.

Warning: This definition is not the same (although eventually we'll see it is equivalent) as the definition given in most references. Namely, in [Gro71, Exposé V, Definition 5.1] a Galois category is defined to be a category equivalent to Finite-G-Sets for some profinite group G. Then Grothendieck characterizes Galois categories by a list of axioms (G1) – (G6) which are weaker than our axioms above. The motivation for our choice is to stress the existence of finite limits and finite colimits and exactness of the functor F. The price we'll pay for this later is that we'll have to work a bit harder to apply the results of this section.

0BN0 Lemma 58.3.7. Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y \in \text{Arrows}(\mathcal{C})$. Then

(1) F is faithful,
(2) $X \to Y$ is a monomorphism $\Leftrightarrow F(X) \to F(Y)$ is injective,
(3) $X \to Y$ is an epimorphism $\Leftrightarrow F(X) \to F(Y)$ is surjective,
(4) an object A of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$,
(5) an object Z of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton,
(6) if X and Y are connected, then $X \to Y$ is an epimorphism,
0BN1 (7) if X is connected and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$,
(8) if $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} Y_j$ where X_i, Y_j are connected, then there is map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.

Proof. Proof of (1). Suppose $a, b : X \to Y$ with $F(a) = F(b)$. Let E be the equalizer of a and b. Then $F(E) = F(X)$ and we see that $E = X$ because F reflects isomorphisms.

Proof of (2). This is true because F turns the morphism $X \to X \times_Y X$ into the map $F(X) \to F(X) \times_{F(Y)} F(X)$ and F reflects isomorphisms.

Proof of (3). This is true because F turns the morphism $Y \amalg_X Y \to Y$ into the map $F(Y) \amalg_{F(X)} F(Y) \to F(Y)$ and F reflects isomorphisms.

Proof of (4). There exists an initial object A and certainly $F(A) = \emptyset$. On the other hand, if X is an object with $F(X) = \emptyset$, then the unique map $A \to X$ induces a bijection $F(A) \to F(X)$ and hence $A \to X$ is an isomorphism.

[2]Namely, given a morphism f of $\mathcal{C}$ if $F(f)$ is an isomorphism, then f is an isomorphism.
[3]This means that F commutes with finite limits and colimits, see Categories, Section 4.23.

Proof of (5). There exists a final object Z and certainly $F(Z)$ is a singleton. On the other hand, if X is an object with $F(X)$ a singleton, then the unique map $X \to Z$ induces a bijection $F(X) \to F(Z)$ and hence $X \to Z$ is an isomorphism.

Proof of (6). The equalizer E of the two maps $Y \to Y \amalg_X Y$ is not an initial object of $\mathcal{C}$ because $X \to Y$ factors through E and $F(X) \neq \emptyset$. Hence $E = Y$ and we conclude.

Proof of (7). The equalizer E of a and b comes with a monomorphism $E \to X$ and $F(E) \subset F(X)$ is the set of elements where $F(a)$ and $F(b)$ agree. To finish use that either E is initial or $E = X$.

Proof of (8). For each i, j we see that $E_{ij} = X_i \times_Y Y_j$ is either initial or equal to X_i. Picking $s \in F(X_i)$ we see that $E_{ij} = X_i$ if and only if s maps to an element of $F(Y_j) \subset F(Y)$, hence this happens for a unique $j = \alpha(i)$. □

By the lemma above we see that, given a connected object X of a Galois category $(\mathcal{C}, F)$, the automorphism group $\text{Aut}(X)$ has order at most $|F(X)|$. Namely, given $s \in F(X)$ and $g \in \text{Aut}(X)$ we see that $g(s) = s$ if and only if $g = \text{id}_X$ by (7). We say X is Galois if equality holds. Equivalently, X is Galois if it is connected and $\text{Aut}(X)$ acts transitively on $F(X)$.

0BN2 Lemma 58.3.8. Let $(\mathcal{C}, F)$ be a Galois category. For any connected object X of $\mathcal{C}$ there exists a Galois object Y and a morphism $Y \to X$.

Proof. We will use the results of Lemma 58.3.7 without further mention. Let $n = |F(X)|$. Consider X^n endowed with its natural action of S_n. Let

$$X^n = \coprod\nolimits_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a t such that $F(Z_t)$ contains $(s_1, \ldots, s_n)$ with s_i pairwise distinct. If $(s'_1, \ldots, s'_n) \in F(Z_t)$ is another element, then we claim s'_i are pairwise distinct as well. Namely, if not, say $s'_i = s'_j$, then Z_t is the image of an connected component of X^{n-1} under the diagonal morphism

$$\Delta_{ij} : X^{n-1} \longrightarrow X^n$$

Since morphisms of connected objects are epimorphisms and induce surjections after applying F it would follow that $s_i = s_j$ which is not the case.

Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. Looking at the action of S_n on

$$F(X)^n = F(X^n) = \coprod\nolimits_{t' \in T} F(Z_{t'})$$

we see that $G = \{g \in S_n \mid g(s_1, \ldots, s_n) \in F(Z_t)\}$. Now pick a second element $(s'_1, \ldots, s'_n) \in F(Z_t)$. Above we have seen that s'_i are pairwise distinct. Thus we can find a $g \in S_n$ with $g(s_1, \ldots, s_n) = (s'_1, \ldots, s'_n)$. In other words, the action of G on $F(Z_t)$ is transitive and the proof is complete. □

Here is a key lemma.

0BN3 Lemma 58.3.9. Let $(\mathcal{C}, F)$ be a Galois category. Let $G = \text{Aut}(F)$ be as in Example 58.3.5. For any connected X in $\mathcal{C}$ the action of G on $F(X)$ is transitive. Compare with [BS13, Definition 7.2.4].

Proof. We will use the results of Lemma 58.3.7 without further mention. Let I be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$ let X_i be a representative of the isomorphism class. Choose $\gamma_i \in F(X_i)$ for each $i \in I$. We define a partial ordering on I by setting $i \geq i'$ if and only if there is a morphism $f_{ii'} : X_i \to X_{i'}$. Given such a morphism we can post-compose by an automorphism $X_{i'} \to X_{i'}$ to assure that $F(f_{ii'})(\gamma_i) = \gamma_{i'}$. With this normalization the morphism $f_{ii'}$ is unique. Observe that I is a directed partially ordered set: (Categories, Definition 4.21.1) if $i_1, i_2 \in I$ there exists a Galois object Y and a morphism $Y \to X_{i_1} \times X_{i_2}$ by Lemma 58.3.8 applied to a connected component of $X_{i_1} \times X_{i_2}$. Then $Y \cong X_i$ for some $i \in I$ and $i \geq i_1$, $i \geq I_2$.

We claim that the functor F is isomorphic to the functor F' which sends X to

$$F'(X) = \mathrm{colim}_I \, \mathrm{Mor}_\mathcal{C}(X_i, X)$$

via the transformation of functors $t : F' \to F$ defined as follows: given $f : X_i \to X$ we set $t_X(f) = F(f)(\gamma_i)$. Using (7) we find that t_X is injective. To show surjectivity, let $\gamma \in F(X)$. Then we can immediately reduce to the case where X is connected by the definition of a Galois category. Then we may assume X is Galois by Lemma 58.3.8. In this case X is isomorphic to X_i for some i and we can choose the isomorphism $X_i \to X$ such that γ_i maps to γ (by definition of Galois objects). We conclude that t is an isomorphism.

Set $A_i = \mathrm{Aut}(X_i)$. We claim that for $i \geq i'$ there is a canonical map $h_{ii'} : A_i \to A_{i'}$ such that for all $a \in A_i$ the diagram

$$\begin{tikzcd} X_i \arrow[r, "f_{ii'}"'] \arrow[d, "a"'] & X_{i'} \arrow[d, "h_{ii'}(a)"] \\ X_i \arrow[r, "f_{ii'}"] & X_{i'} \end{tikzcd}$$

commutes. Namely, just let $h_{ii'}(a) = a' : X_{i'} \to X_{i'}$ be the unique automorphism such that $F(a')(\gamma_{i'}) = F(f_{ii'} \circ a)(\gamma_i)$. As before this makes the diagram commute and moreover the choice is unique. It follows that $h_{i'i''} \circ h_{ii'} = h_{ii''}$ if $i \geq i' \geq i''$. Since $F(X_i) \to F(X_{i'})$ is surjective we see that $A_i \to A_{i'}$ is surjective. Taking the inverse limit we obtain a group

$$A = \lim_I A_i$$

This is a profinite group since the automorphism groups are finite. The map $A \to A_i$ is surjective for all i by Categories, Lemma 4.21.7.

Since elements of A act on the inverse system X_i we get an action of A (on the right) on F' by pre-composing. In other words, we get a homomorphism $A^{opp} \to G$. Since $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the lemma is true. □

0BN4 Proposition 58.3.10. Let $(\mathcal{C}, F)$ be a Galois category. Let $G = \mathrm{Aut}(F)$ be as in Example 58.3.5. The functor $F : \mathcal{C} \to \textit{Finite-}G\textit{-Sets}$ (58.3.5.1) an equivalence.

This is a weak version of [Gro71, Exposé V]. The proof is borrowed from [BS13, Theorem 7.2.5].

Proof. We will use the results of Lemma 58.3.7 without further mention. In particular we know the functor is faithful. By Lemma 58.3.9 we know that for any connected X the action of G on $F(X)$ is transitive. Hence F preserves the decomposition into connected components (existence of which is an axiom of a Galois category). Let X and Y be objects and let $s : F(X) \to F(Y)$ be a map. Then the graph $\Gamma_s \subset F(X) \times F(Y)$ of s is a union of connected components. Hence there exists a union of connected components Z of $X \times Y$, which comes equipped with a monomorphism $Z \to X \times Y$, with $F(Z) = \Gamma_s$. Since $F(Z) \to F(X)$ is bijective we see that $Z \to X$ is an isomorphism and we conclude that $s = F(f)$ where $f : X \cong Z \to Y$ is the composition. Hence F is fully faithful.

To finish the proof we show that F is essentially surjective. It suffices to show that G/H is in the essential image for any open subgroup $H \subset G$ of finite index. By definition of the topology on G there exists a finite collection of objects X_i such that

$$\mathrm{Ker}(G \longrightarrow \prod\nolimits_i \mathrm{Aut}(F(X_i)))$$

is contained in H. We may assume X_i is connected for all i. We can choose a Galois object Y mapping to a connected component of $\prod X_i$ using Lemma 58.3.8. Choose an isomorphism $F(Y) = G/U$ in G-sets for some open subgroup $U \subset G$. As Y is Galois, the group $\mathrm{Aut}(Y) = \mathrm{Aut}_{G\text{-Sets}}(G/U)$ acts transitively on $F(Y) = G/U$. This implies that U is normal. Since $F(Y)$ surjects onto $F(X_i)$ for each i we see that $U \subset H$. Let $M \subset \mathrm{Aut}(Y)$ be the finite subgroup corresponding to

$$(H/U)^{opp} \subset (G/U)^{opp} = \mathrm{Aut}_{G\text{-Sets}}(G/U) = \mathrm{Aut}(Y).$$

Set $X = Y/M$, i.e., X is the coequalizer of the arrows $m : Y \to Y$, $m \in M$. Since F is exact we see that $F(X) = G/H$ and the proof is complete. ☐

0BN5 Lemma 58.3.11. Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. There exists an isomorphism $t : F' \circ H \to F$. The choice of t determines a continuous homomorphism $h : G' = \mathrm{Aut}(F') \to \mathrm{Aut}(F) = G$ and a 2-commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"'] \arrow[d] & \mathcal{C}' \arrow[d] \\ \text{Finite-}G\text{-Sets} \arrow[r, "h"] & \text{Finite-}G'\text{-Sets} \end{tikzcd}$$

The map h is independent of t up to an inner automorphism of G. Conversely, given a continuous homomorphism $h : G' \to G$ there is an exact functor $H : \mathcal{C} \to \mathcal{C}'$ and an isomorphism t recovering h as above.

Proof. By Proposition 58.3.10 and Lemma 58.3.3 we may assume $\mathcal{C} = \text{Finite-}G\text{-Sets}$ and F is the forgetful functor and similarly for $\mathcal{C}'$. Thus the existence of t follows from Lemma 58.3.4. The map h comes from transport of structure via t. The commutativity of the diagram is obvious. Uniqueness of h up to inner conjugation by an element of G comes from the fact that the choice of t is unique up to an element of G. The final statement is straightforward. ☐

58.4. Functors and homomorphisms

0BTQ Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$, $(\mathcal{C}'', F'')$ be Galois categories. Set $G = \mathrm{Aut}(F)$, $G' = \mathrm{Aut}(F')$, and $G'' = \mathrm{Aut}(F'')$. Let $H : \mathcal{C} \to \mathcal{C}'$ and $H' : \mathcal{C}' \to \mathcal{C}''$ be exact functors. Let $h : G' \to G$ and $h' : G'' \to G'$ be the corresponding continuous homomorphism as in Lemma 58.3.11. In this section we consider the corresponding 2-commutative diagram

0BTR (58.4.0.1)
$$\begin{tikzcd} \mathcal{C} \arrow[rr, "H"'] \arrow[d] & & \mathcal{C}' \arrow[rr, "H'"'] \arrow[d] & & \mathcal{C}'' \arrow[d] \\ \text{Finite-}G\text{-Sets} \arrow[rr, "h"] & & \text{Finite-}G'\text{-Sets} \arrow[rr, "h'"] & & \text{Finite-}G''\text{-Sets} \end{tikzcd}$$

and we relate exactness properties of the sequence $1 \to G'' \to G' \to G \to 1$ to properties of the functors H and H'.

0BN6 Lemma 58.4.1. In diagram (58.4.0.1) the following are equivalent

(1) $h : G' \to G$ is surjective,
(2) $H : \mathcal{C} \to \mathcal{C}'$ is fully faithful,
(3) if $X \in \mathrm{Ob}(\mathcal{C})$ is connected, then $H(X)$ is connected,
(4) if $X \in \mathrm{Ob}(\mathcal{C})$ is connected and there is a morphism $*' \to H(X)$ in $\mathcal{C}'$, then there is a morphism $* \to X$, and
(5) for any object X of $\mathcal{C}$ the map $\mathrm{Mor}_\mathcal{C}(*, X) \to \mathrm{Mor}_{\mathcal{C}'}(*', H(X))$ is bijective.

Here $*$ and $*'$ are final objects of $\mathcal{C}$ and $\mathcal{C}'$.

Proof. The implications (5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (5) are clear.

Assume (3). Let X be a connected object of $\mathcal{C}$ and let $*' \to H(X)$ be a morphism. Since $H(X)$ is connected by (3) we see that $*' \to H(X)$ is an isomorphism. Hence the G'-set corresponding to $H(X)$ has exactly one element, which means the G-set corresponding to X has one element which means X is isomorphic to the final object of $\mathcal{C}$, in particular there is a map $* \to X$. In this way we see that (3) $\Rightarrow$ (4).

If (1) is true, then the functor Finite-G-Sets $\to$ Finite-G'-Sets is fully faithful: in this case a map of G-sets commutes with the action of G if and only if it commutes with the action of G'. Thus (1) $\Rightarrow$ (2).

If (1) is true, then for a G-set X the G-orbits and G'-orbits agree. Thus (1) $\Rightarrow$ (3).

To finish the proof it suffices to show that (4) implies (1). If (1) is false, i.e., if h is not surjective, then there is an open subgroup $U \subset G$ containing $h(G')$ which is not equal to G. Then the finite G-set $M = G/U$ has a transitive action but G' has a fixed point. The object X of $\mathcal{C}$ corresponding to M would contradict (3). In this way we see that (3) $\Rightarrow$ (1) and the proof is complete. $\square$

0BS8 Lemma 58.4.2. In diagram (58.4.0.1) the following are equivalent

(1) $h \circ h'$ is trivial, and
(2) the image of $H' \circ H$ consists of objects isomorphic to finite coproducts of final objects.

Proof. We may replace H and H' by the canonical functors Finite-G-Sets $\to$ Finite-G'-Sets $\to$ Finite-G''-Sets determined by h and h'. Then we are saying that the action of G'' on every G-set is trivial if and only if the homomorphism $G'' \to G$ is trivial. This is clear. $\square$

0BS9 Lemma 58.4.3. In diagram (58.4.0.1) the following are equivalent

(1) the sequence $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ is exact in the following sense: h is surjective, $h \circ h'$ is trivial, and $\mathrm{Ker}(h)$ is the smallest closed normal subgroup containing $\mathrm{Im}(h')$,

(2) H is fully faithful and an object X' of $\mathcal{C}'$ is in the essential image of H if and only if $H'(X')$ is isomorphic to a finite coproduct of final objects, and
(3) H is fully faithful, $H \circ H'$ sends every object to a finite coproduct of final objects, and for an object X' of $\mathcal{C}'$ such that $H'(X')$ is a finite coproduct of final objects there exists an object X of $\mathcal{C}$ and an epimorphism $H(X) \to X'$.

Proof. By Lemmas 58.4.1 and 58.4.2 we may assume that H is fully faithful, h is surjective, $H' \circ H$ maps objects to disjoint unions of the final object, and $h \circ h'$ is trivial. Let $N \subset G'$ be the smallest closed normal subgroup containing the image of h'. It is clear that $N \subset \text{Ker}(h)$. We may assume the functors H and H' are the canonical functors Finite-G-Sets $\to$ Finite-G'-Sets $\to$ Finite-G''-Sets determined by h and h'.

Suppose that (2) holds. This means that for a finite G'-set X' such that G'' acts trivially, the action of G' factors through G. Apply this to $X' = G'/U'N$ where U' is a small open subgroup of G'. Then we see that $\text{Ker}(h) \subset U'N$ for all U'. Since N is closed this implies $\text{Ker}(h) \subset N$, i.e., (1) holds.

Suppose that (1) holds. This means that $N = \text{Ker}(h)$. Let X' be a finite G'-set such that G'' acts trivially. This means that $\text{Ker}(G' \to \text{Aut}(X'))$ is a closed normal subgroup containing $\text{Im}(h')$. Hence $N = \text{Ker}(h)$ is contained in it and the G'-action on X' factors through G, i.e., (2) holds.

Suppose that (3) holds. This means that for a finite G'-set X' such that G'' acts trivially, there is a surjection of G'-sets $X \to X'$ where X is a G-set. Clearly this means the action of G' on X' factors through G, i.e., (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate. This finishes the proof. $\square$

0BN7 Lemma 58.4.4. In diagram (58.4.0.1) the following are equivalent

(1) h' is injective, and
(2) for every connected object X'' of $\mathcal{C}''$ there exists an object X' of $\mathcal{C}'$ and a diagram
$$X'' \leftarrow Y'' \to H(X')$$
in $\mathcal{C}''$ where $Y'' \to X''$ is an epimorphism and $Y'' \to H(X')$ is a monomorphism.

Proof. We may replace H' by the corresponding functor between the categories of finite G'-sets and finite G''-sets.

Assume $h' : G'' \to G'$ is injective. Let $H'' \subset G''$ be an open subgroup. Since the topology on G'' is the induced topology from G' there exists an open subgroup $H' \subset G'$ such that $(h')^{-1}(H') \subset H''$. Then the desired diagram is
$$G''/H'' \leftarrow G''/(h')^{-1}(H') \to G'/H'$$
Conversely, assume (2) holds for the functor Finite-G'-Sets $\to$ Finite-G''-Sets. Let $g'' \in \text{Ker}(h')$. Pick any open subgroup $H'' \subset G''$. By assumption there exists a finite G'-set X' and a diagram
$$G''/H'' \leftarrow Y'' \to X'$$
of G''-sets with the left arrow surjective and the right arrow injective. Since g'' is in the kernel of h' we see that g'' acts trivially on X'. Hence g'' acts trivially on Y''

and hence trivially on G''/H''. Thus $g'' \in H''$. As this holds for all open subgroups we conclude that g'' is the identity element as desired. $\square$

0BTS Lemma 58.4.5. In diagram (58.4.0.1) the following are equivalent

(1) the image of h' is normal, and
(2) for every connected object X' of $\mathcal{C}'$ such that there is a morphism from the final object of $\mathcal{C}''$ to $H'(X')$ we have that $H'(X')$ is isomorphic to a finite coproduct of final objects.

Proof. This translates into the following statement for the continuous group homomorphism $h' : G'' \to G'$: the image of h' is normal if and only if every open subgroup $U' \subset G'$ which contains $h'(G'')$ also contains every conjugate of $h'(G'')$. The result follows easily from this; some details omitted. $\square$

58.5. Finite étale morphisms

0BL6 In this section we prove enough basic results on finite étale morphisms to be able to construct the étale fundamental group.

Let X be a scheme. We will use the notation FÉt_X to denote the category of schemes finite and étale over X. Thus

(1) an object of FÉt_X is a finite étale morphism $Y \to X$ with target X, and
(2) a morphism in FÉt_X from $Y \to X$ to $Y' \to X$ is a morphism $Y \to Y'$ making the diagram
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$
commute.

We will often call an object of FÉt_X a finite étale cover of X (even if Y is empty). It turns out that there is a stack $p : \text{FÉt} \to Sch$ over the category of schemes whose fibre over X is the category FÉt_X just defined. See Examples of Stacks, Section 95.6.

0BN8 Example 58.5.1. Let k be an algebraically closed field and $X = \text{Spec}(k)$. In this case FÉt_X is equivalent to the category of finite sets. This works more generally when k is separably algebraically closed. The reason is that a scheme étale over k is the disjoint union of spectra of fields finite separable over k, see Morphisms, Lemma 29.36.7.

0BN9 Lemma 58.5.2. Let X be a scheme. The category FÉt_X has finite limits and finite colimits and for any morphism $X' \to X$ the base change functor $\text{FÉt}_X \to \text{FÉt}_{X'}$ is exact.

Proof. Finite limits and left exactness. By Categories, Lemma 4.18.4 it suffices to show that FÉt_X has a final object and fibred products. This is clear because the category of all schemes over X has a final object (namely X) and fibred products. Also, fibred products of schemes finite étale over X are finite étale over X. Moreover, it is clear that base change commutes with these operations and hence base change is left exact (Categories, Lemma 4.23.2).

Finite colimits and right exactness. By Categories, Lemma 4.18.7 it suffices to show that FÉt_X has finite coproducts and coequalizers. Finite coproducts are given by disjoint unions (the empty coproduct is the empty scheme). Let $a, b : Z \to Y$ be two morphisms of FÉt_X. Since $Z \to X$ and $Y \to X$ are finite étale we can write $Z = \underline{\text{Spec}}(\mathcal{C})$ and $Y = \underline{\text{Spec}}(\mathcal{B})$ for some finite locally free $\mathcal{O}_X$-algebras $\mathcal{C}$ and $\mathcal{B}$. The morphisms a, b induce two maps $a^\sharp, b^\sharp : \mathcal{B} \to \mathcal{C}$. Let $\mathcal{A} = \text{Eq}(a^\sharp, b^\sharp)$ be their equalizer. If

$$\underline{\text{Spec}}(\mathcal{A}) \longrightarrow X$$

is finite étale, then it is clear that this is the coequalizer (after all we can write any object of FÉt_X as the relative spectrum of a sheaf of $\mathcal{O}_X$-algebras). This we may do after replacing X by the members of an étale covering (Descent, Lemmas 35.23.23 and 35.23.29). Thus by Étale Morphisms, Lemma 41.18.3 we may assume that $Y = \coprod_{i=1,\ldots,n} X$ and $Z = \coprod_{j=1,\ldots,m} X$. Then

$$\mathcal{C} = \prod\nolimits_{1 \leq j \leq m} \mathcal{O}_X \quad \text{and} \quad \mathcal{B} = \prod\nolimits_{1 \leq i \leq n} \mathcal{O}_X$$

After a further replacement by the members of an open covering we may assume that a, b correspond to maps $a_s, b_s : \{1, \ldots, m\} \to \{1, \ldots, n\}$, i.e., the summand X of Z corresponding to the index j maps into the summand X of Y corresponding to the index $a_s(j)$, resp. $b_s(j)$ under the morphism a, resp. b. Let $\{1, \ldots, n\} \to T$ be the coequalizer of a_s, b_s. Then we see that

$$\mathcal{A} = \prod\nolimits_{t \in T} \mathcal{O}_X$$

whose spectrum is certainly finite étale over X. We omit the verification that this is compatible with base change. Thus base change is a right exact functor. $\square$

0BNA Remark 58.5.3. Let X be a scheme. Consider the natural functors $F_1 : \text{FÉt}_X \to \textit{Sch}$ and $F_2 : \text{FÉt}_X \to \textit{Sch}/X$. Then

1. (1) The functors F_1 and F_2 commute with finite colimits.
2. (2) The functor F_2 commutes with finite limits,
3. (3) The functor F_1 commutes with connected finite limits, i.e., with equalizers and fibre products.

The results on limits are immediate from the discussion in the proof of Lemma 58.5.2 and Categories, Lemma 4.16.2. It is clear that F_1 and F_2 commute with finite coproducts. By the dual of Categories, Lemma 4.23.2 we need to show that F_1 and F_2 commute with coequalizers. In the proof of Lemma 58.5.2 we saw that coequalizers in FÉt_X look étale locally like this

$$\coprod\nolimits_{j \in J} U \underset{b}{\overset{a}{\rightrightarrows}} \coprod\nolimits_{i \in I} U \longrightarrow \coprod\nolimits_{t \in \text{Coeq}(a,b)} U$$

which is certainly a coequalizer in the category of schemes. Hence the statement follows from the fact that being a coequalizer is fpqc local as formulated precisely in Descent, Lemma 35.13.8.

0BL7 Lemma 58.5.4. Let X be a scheme. Given U, V finite étale over X there exists a scheme W finite étale over X such that

$$\text{Mor}_X(X, W) = \text{Mor}_X(U, V)$$

and such that the same remains true after any base change.

Proof. By More on Morphisms, Lemma 37.68.4 there exists a scheme W representing $Mor_X(U, V)$. (Use that an étale morphism is locally quasi-finite by Morphisms, Lemmas 29.36.6 and that a finite morphism is separated.) This scheme clearly satisfies the formula after any base change. To finish the proof we have to show that $W \to X$ is finite étale. This we may do after replacing X by the members of an étale covering (Descent, Lemmas 35.23.23 and 35.23.6). Thus by Étale Morphisms, Lemma 41.18.3 we may assume that $U = \coprod_{i=1,\ldots,n} X$ and $V = \coprod_{j=1,\ldots,m} X$. In this case $W = \coprod_{\alpha:\{1,\ldots,n\}\to\{1,\ldots,m\}} X$ by inspection (details omitted) and the proof is complete. $\square$

Let X be a scheme. A geometric point of X is a morphism $\mathrm{Spec}(k) \to X$ where k is algebraically closed. Such a point is usually denoted $\overline{x}$, i.e., by an overlined small case letter. We often use $\overline{x}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{x})$ to denote k. We say $\overline{x}$ lies over x to indicate that $x \in X$ is the image of $\overline{x}$. We will discuss this further in Étale Cohomology, Section 59.29. Given $\overline{x}$ and an étale morphism $U \to X$ we can consider

$$|U_{\overline{x}}| : \text{the underlying set of points of the scheme } U_{\overline{x}} = U \times_X \overline{x}$$

Since $U_{\overline{x}}$ as a scheme over $\overline{x}$ is a disjoint union of copies of $\overline{x}$ (Morphisms, Lemma 29.36.7) we can also describe this set as

$$|U_{\overline{x}}| = \left\{ \begin{matrix} \text{commutative} \\ \text{diagrams} \end{matrix} \begin{tikzcd} \overline{x} \arrow[r, "\overline{u}"] \arrow[rd, "\overline{x}"'] & U \arrow[d] \\ & X \end{tikzcd} \right\}$$

The assignment $U \mapsto |U_{\overline{x}}|$ is a functor which is often denoted $F_{\overline{x}}$.

0BNB **Lemma 58.5.5.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point. The functor*

$$F_{\overline{x}} : \text{FÉt}_X \longrightarrow \text{Sets}, \quad Y \longmapsto |Y_{\overline{x}}|$$

defines a Galois category (Definition 58.3.6).

Proof. After identifying $\text{FÉt}_{\overline{x}}$ with the category of finite sets (Example 58.5.1) we see that our functor $F_{\overline{x}}$ is nothing but the base change functor for the morphism $\overline{x} \to X$. Thus we see that FÉt_X has finite limits and finite colimits and that $F_{\overline{x}}$ is exact by Lemma 58.5.2. We will also use that finite limits in FÉt_X agree with the corresponding finite limits in the category of schemes over X, see Remark 58.5.3.

If $Y' \to Y$ is a monomorphism in FÉt_X then we see that $Y' \to Y' \times_Y Y'$ is an isomorphism, and hence $Y' \to Y$ is a monomorphism of schemes. It follows that $Y' \to Y$ is an open immersion (Étale Morphisms, Theorem 41.14.1). Since Y' is finite over X and Y separated over X, the morphism $Y' \to Y$ is finite (Morphisms, Lemma 29.44.14), hence closed (Morphisms, Lemma 29.44.11), hence it is the inclusion of an open and closed subscheme of Y. It follows that Y is a connected objects of the category FÉt_X (as in Definition 58.3.6) if and only if Y is connected as a scheme. Then it follows from Topology, Lemma 5.7.7 that Y is a finite coproduct of its connected components both as a scheme and in the sense of Definition 58.3.6.

Let $Y \to Z$ be a morphism in FÉt_X which induces a bijection $F_{\overline{x}}(Y) \to F_{\overline{x}}(Z)$. We have to show that $Y \to Z$ is an isomorphism. By the above we may assume Z is connected. Since $Y \to Z$ is finite étale and hence finite locally free it suffices to show that $Y \to Z$ is finite locally free of degree 1. This is true in a neighbourhood of any point of Z lying over $\overline{x}$ and since Z is connected and the degree is locally constant we conclude. □

58.6. Fundamental groups

0BQ8 In this section we define Grothendieck's algebraic fundamental group. The following definition makes sense thanks to Lemma 58.5.5.

0BNC Definition 58.6.1. Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X. The fundamental group of X with base point $\overline{x}$ is the group

$$\pi_1(X, \overline{x}) = \text{Aut}(F_{\overline{x}})$$

of automorphisms of the fibre functor $F_{\overline{x}} : \text{FÉt}_X \to \text{Sets}$ endowed with its canonical profinite topology from Lemma 58.3.1.

Combining the above with the material from Section 58.3 we obtain the following theorem.

0BND Theorem 58.6.2. Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.

(1) The fibre functor $F_{\overline{x}}$ defines an equivalence of categories

$$\text{FÉt}_X \longrightarrow \text{Finite-}\pi_1(X, \overline{x})\text{-Sets}$$

(2) Given a second geometric point $\overline{x}'$ of X there exists an isomorphism $t : F_{\overline{x}} \to F_{\overline{x}'}$. This gives an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x}')$ compatible with the equivalences in (1). This isomorphism is independent of t up to inner conjugation.
(3) Given a morphism $f : X \to Y$ of connected schemes denote $\overline{y} = f \circ \overline{x}$. There is a canonical continuous homomorphism

$$f_* : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$$

such that the diagram

$$\begin{tikzcd} \text{FÉt}_Y \arrow[r, "\text{base change}"'] \arrow[d, "F_{\overline{y}}"'] & \text{FÉt}_X \arrow[d, "F_{\overline{x}}"] \\ \text{Finite-}\pi_1(Y, \overline{y})\text{-Sets} \arrow[r, "f_*"] & \text{Finite-}\pi_1(X, \overline{x})\text{-Sets} \end{tikzcd}$$

is commutative.

Proof. Part (1) follows from Lemma 58.5.5 and Proposition 58.3.10. Part (2) is a special case of Lemma 58.3.11. For part (3) observe that the diagram

$$\begin{tikzcd} \text{FÉt}_Y \arrow[r] \arrow[d, "F_{\overline{y}}"'] & \text{FÉt}_X \arrow[d, "F_{\overline{x}}"] \\ \text{Sets} \arrow[r, equal] & \text{Sets} \end{tikzcd}$$

is commutative (actually commutative, not just 2-commutative) because $\overline{y} = f \circ \overline{x}$. Hence we can apply Lemma 58.3.11 with the implied transformation of functors to get (3). □

0BNE **Lemma 58.6.3.** Let K be a field and set $X = \mathrm{Spec}(K)$. Let $\overline{K}$ be an algebraic closure and denote $\overline{x} : \mathrm{Spec}(\overline{K}) \to X$ the corresponding geometric point. Let $K^{sep} \subset \overline{K}$ be the separable algebraic closure.

(1) The functor of Lemma 58.2.2 induces an equivalence
$$\textit{FÉt}_X \longrightarrow \textit{Finite-Gal}(K^{sep}/K)\textit{-Sets}.$$
compatible with $F_{\overline{x}}$ and the functor $\textit{Finite-Gal}(K^{sep}/K)\textit{-Sets} \to \textit{Sets}$.

(2) This induces a canonical isomorphism
$$\mathrm{Gal}(K^{sep}/K) \longrightarrow \pi_1(X, \overline{x})$$
of profinite topological groups.

Proof. The functor of Lemma 58.2.2 is the same as the functor $F_{\overline{x}}$ because for any Y étale over X we have
$$\mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y) = \mathrm{Mor}_X(\mathrm{Spec}(K^{sep}), Y)$$
Namely, as seen in the proof of Lemma 58.2.2 we have $Y = \coprod_{i \in I} \mathrm{Spec}(L_i)$ with L_i/K finite separable over K. Hence any K-algebra homomorphism $L_i \to \overline{K}$ factors through K^{sep}. Also, note that $F_{\overline{x}}(Y)$ is finite if and only if I is finite if and only if $Y \to X$ is finite étale. This proves (1).

Part (2) is a formal consequence of (1), Lemma 58.3.11, and Lemma 58.3.3. (Please also see the remark below.) □

0BQ9 **Remark 58.6.4.** In the situation of Lemma 58.6.3 let us give a more explicit construction of the isomorphism $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$. Observe that $\mathrm{Gal}(K^{sep}/K) = \mathrm{Aut}(\overline{K}/K)$ as $\overline{K}$ is the perfection of K^{sep}. Since $F_{\overline{x}}(Y) = \mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y)$ we may consider the map
$$\mathrm{Aut}(\overline{K}/K) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y), \quad (\sigma, \overline{y}) \mapsto \sigma \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma)$$
This is an action because
$$\sigma\tau \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma\tau) = \overline{y} \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma) = \sigma \cdot (\tau \cdot \overline{y})$$
The action is functorial in $Y \in \textit{FÉt}_X$ and we obtain the desired map.

58.7. Galois covers of connected schemes

03SF Let X be a connected scheme with geometric point $\overline{x}$. Since $F_{\overline{x}} : \textit{FÉt}_X \to \textit{Sets}$ is a Galois category (Lemma 58.5.5) the material in Section 58.3 applies. In this section we explicity transfer some of the terminology and results to the setting of schemes and finite étale morphisms.

We will say a finite étale morphism $Y \to X$ is a *Galois cover* if Y defines a Galois object of $\textit{FÉt}_X$. For a finite étale morphism $Y \to X$ with $G = \mathrm{Aut}_X(Y)$ the following are equivalent

(1) Y is a Galois cover of X,
(2) Y is connected and $|G|$ is equal to the degree of $Y \to X$,
(3) Y is connected and G acts transitively on $F_{\overline{x}}(Y)$, and
(4) Y is connected and G acts simply transitively on $F_{\overline{x}}(Y)$.

This follows immediately from the discussion in Section 58.3.

For any finite étale morphism $f : Y \to X$ with Y connected, there is a finite étale Galois cover $Y' \to X$ which dominates Y (Lemma 58.3.8).

The Galois objects of FÉt_X correspond, via the equivalence

$$F_{\overline{x}} : \text{FÉt}_X \to \text{Finite-}\pi_1(X, \overline{x})\text{-Sets}$$

of Theorem 58.6.2, with the finite $\pi_1(X, \overline{x})$-Sets of the form $G = \pi_1(X, \overline{x})/H$ where H is a normal open subgroup. Equivalently, if G is a finite group and $\pi_1(X, \overline{x}) \to G$ is a continuous surjection, then G viewed as a $\pi_1(X, \overline{x})$-set corresponds to a Galois covering.

If $Y_i \to X$, $i = 1, 2$ are finite étale Galois covers with Galois groups G_i, then there exists a finite étale Galois cover $Y \to X$ whose Galois group is a subgroup of $G_1 \times G_2$. Namely, take the corresponding continuous homomorphisms $\pi_1(X, \overline{x}) \to G_i$ and let G be the image of the induced continuous homomorphism $\pi_1(X, \overline{x}) \to G_1 \times G_2$.

58.8. Topological invariance of the fundamental group

0BTT The main result of this section is that a universal homeomorphism of connected schemes induces an isomorphism on fundamental groups. See Proposition 58.8.4.

Instead of directly proving two schemes have the same fundamental group, we often prove that their categories of finite étale coverings are the same. This of course implies that their fundamental groups are equal provided they are connected.

0BQA **Lemma 58.8.1.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes such that the base change functor $\text{FÉt}_Y \to \text{FÉt}_X$ is an equivalence of categories. In this case

(1) f induces a homeomorphism $\pi_0(X) \to \pi_0(Y)$,
(2) if X or equivalently Y is connected, then $\pi_1(X, \overline{x}) = \pi_1(Y, \overline{y})$.

Proof. Let $Y = Y_0 \amalg Y_1$ be a decomposition into nonempty open and closed subschemes. We claim that $f(X)$ meets both Y_i. Namely, if not, say $f(X) \subset Y_1$, then we can consider the finite étale morphism $V = Y_1 \to Y$. This is not an isomorphism but $V \times_Y X \to X$ is an isomorphism, which is a contradiction.

Suppose that $X = X_0 \amalg X_1$ is a decomposition into open and closed subschemes. Consider the finite étale morphism $U = X_1 \to X$. Then $U = X \times_Y V$ for some finite étale morphism $V \to Y$. The degree of the morphism $V \to Y$ is locally constant, hence we obtain a decomposition $Y = \coprod_{d \geq 0} Y_d$ into open and closed subschemes such that $V \to Y$ has degree d over Y_d. Since $f^{-1}(Y_d) = \emptyset$ for $d > 1$ we conclude that $Y_d = \emptyset$ for $d > 1$ by the above. And we conclude that $f^{-1}(Y_i) = X_i$ for $i = 0, 1$.

It follows that f^{-1} induces a bijection between the set of open and closed subsets of Y and the set of open and closed subsets of X. Note that X and Y are spectral spaces, see Properties, Lemma 28.2.4. By Topology, Lemma 5.12.10 the lattice of open and closed subsets of a spectral space determines the set of connected components. Hence $\pi_0(X) \to \pi_0(Y)$ is bijective. Since $\pi_0(X)$ and $\pi_0(Y)$ are profinite spaces (Topology, Lemma 5.22.5) we conclude that $\pi_0(X) \to \pi_0(Y)$ is a homeomorphism by Topology, Lemma 5.17.8. This proves (1). Part (2) is immediate. $\square$

The following lemma tells us that the fundamental group of a henselian pair is the fundamental group of the closed subset.

09ZS **Lemma 58.8.2.** Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. The functor

$$\text{FÉt}_X \longrightarrow \text{FÉt}_Z, \quad U \longmapsto U \times_X Z$$

is an equivalence of categories.

Proof. This is a translation of More on Algebra, Lemma 15.13.2. □

The following lemma tells us that the fundamental group of a thickening is the same as the fundamental group of the original. We will use this in the proof of the strong proposition concerning universal homeomorphisms below.

0BQB **Lemma 58.8.3.** Let $X \subset X'$ be a thickening of schemes. The functor

$$\text{FÉt}_{X'} \longrightarrow \text{FÉt}_X, \quad U' \longmapsto U' \times_{X'} X$$

is an equivalence of categories.

Proof. For a discussion of thickenings see More on Morphisms, Section 37.2. Let $U' \to X'$ be an étale morphism such that $U = U' \times_{X'} X \to X$ is finite étale. Then $U' \to X'$ is finite étale as well. This follows for example from More on Morphisms, Lemma 37.3.4. Now, if $X \subset X'$ is a finite order thickening then this remark combined with Étale Morphisms, Theorem 41.15.2 proves the lemma. Below we will prove the lemma for general thickenings, but we suggest the reader skip the proof.

Let $X' = \bigcup X'_i$ be an affine open covering. Set $X_i = X \times_{X'} X'_i$, $X'_{ij} = X'_i \cap X'_j$, $X_{ij} = X \times_{X'} X'_{ij}$, $X'_{ijk} = X'_i \cap X'_j \cap X'_k$, $X_{ijk} = X \times_{X'} X'_{ijk}$. Suppose that we can prove the theorem for each of the thickenings $X_i \subset X'_i$, $X_{ij} \subset X'_{ij}$, and $X_{ijk} \subset X'_{ijk}$. Then the result follows for $X \subset X'$ by relative glueing of schemes, see Constructions, Section 27.2. Observe that the schemes X'_i, X'_{ij}, X'_{ijk} are each separated as open subschemes of affine schemes. Repeating the argument one more time we reduce to the case where the schemes X'_i, X'_{ij}, X'_{ijk} are affine.

In the affine case we have $X' = \mathrm{Spec}(A')$ and $X = \mathrm{Spec}(A'/I')$ where I' is a locally nilpotent ideal. Then (A', I') is a henselian pair (More on Algebra, Lemma 15.11.2) and the result follows from Lemma 58.8.2 (which is much easier in this case). □

The "correct" way to prove the following proposition would be to deduce it from the invariance of the étale site, see Étale Cohomology, Theorem 59.45.2.

0BQN **Proposition 58.8.4.** Let $f : X \to Y$ be a universal homeomorphism of schemes. Then

$$\text{FÉt}_Y \longrightarrow \text{FÉt}_X, \quad V \longmapsto V \times_Y X$$

is an equivalence. Thus if X and Y are connected, then f induces an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$ of fundamental groups.

Proof. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 29.45.5. In particular, the diagonal $\Delta : X \to X \times_Y X$ is a thickening by Morphisms, Lemma

29.10.2. Thus by Lemma 58.8.3 we see that given a finite étale morphism $U \to X$ there is a unique isomorphism

$$\varphi : U \times_Y X \to X \times_Y U$$

of schemes finite étale over $X \times_Y X$ which pulls back under Δ to $\mathrm{id} : U \to U$ over X. Since $X \to X \times_Y X \times_Y X$ is a thickening as well (it is bijective and a closed immersion) we conclude that (U, φ) is a descent datum relative to X/Y. By Étale Morphisms, Proposition 41.20.6 we conclude that $U = X \times_Y V$ for some $V \to Y$ quasi-compact, separated, and étale. We omit the proof that $V \to Y$ is finite (hints: the morphism $U \to V$ is surjective and $U \to Y$ is integral). We conclude that $\text{FÉt}_Y \to \text{FÉt}_X$ is essentially surjective.

Arguing in the same manner as above we see that given $V_1 \to Y$ and $V_2 \to Y$ in FÉt_Y any morphism $a : X \times_Y V_1 \to X \times_Y V_2$ over X is compatible with the canonical descent data. Thus a descends to a morphism $V_1 \to V_2$ over Y by Étale Morphisms, Lemma 41.20.3. □

58.9. Finite étale covers of proper schemes

0BQC In this section we show that the fundamental group of a connected proper scheme over a henselian local ring is the same as the fundamental group of its special fibre. We also prove a variant of this result for a henselian pair.

We also show that the fundamental group of a connected proper scheme over an algebraically closed field k does not change if we replace k by an algebraically closed extension.

Instead of stating and proving the results in the connected case we prove the results in general and we leave it to the reader to deduce the result for fundamental groups using Lemma 58.8.1.

0A48 **Lemma 58.9.1.** Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Then the functor

$$\text{FÉt}_X \to \text{FÉt}_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof given here is an example of applying algebraization and approximation. We proceed in a number of stages.

Essential surjectivity when A is a complete local Noetherian ring. Let $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^{n+1})$. By Étale Morphisms, Theorem 41.15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 37.3.3. In this case the morphism $U_0 \to \mathrm{Spec}(A/\mathfrak{m})$ is proper as X_0 is proper over $A/\mathfrak{m}$. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 30.28.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 37.12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is a complete local Noetherian ring. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 41.14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when A is a henselian local Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to the maximal ideal. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \text{colim}\, A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 32.10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 32.8.3 and 32.8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Theorem 16.13.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of $V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when A is a henselian local Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let $(A, \mathfrak m)$ be a henselian local ring. Set $S = \text{Spec}(A)$ and denote $s \in S$ the closed point. By Limits, Lemma 32.13.3 we can write $X \to \text{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i of finite type over $\mathbf{Z}$. For each i let $s_i \in S_i$ be the image of s. Since $S = \lim S_i$ and $A = \mathcal{O}_{S,s}$ we have $A = \text{colim}\, \mathcal{O}_{S_i,s_i}$. The ring $A_i = \mathcal{O}_{S_i,s_i}$ is a Noetherian local G-ring (More on Algebra, Proposition 15.50.12). By More on Algebra, Lemma 15.12.5 we see that $A = \text{colim}\, A_i^h$. By More on Algebra, Lemma 15.50.8 the rings A_i^h are G-rings. Thus we see that $A = \text{colim}\, A_i^h$ and

$$X = \lim(X_i \times_{S_i} \text{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \text{Spec}(A_i^h)$ (by Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} s_i)$. Thus we formally deduce the result for $X/\text{Spec}(A)$ from the result for the $(X_i \times_{S_i} \text{Spec}(A_i^h))/\text{Spec}(A_i^h)$ which we dealt with above. $\square$

0GS2 Lemma 58.9.2. Let (A, I) be a henselian pair. Let X be a proper scheme over A. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Then the functor

$$\text{FÉt}_X \to \text{FÉt}_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 58.9.1.

Essential surjectivity when A is Noetherian and I-adically complete. Let $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I^{n+1})$. By Étale Morphisms, Theorem 41.15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 37.3.3. In this case the morphism $U_0 \to \mathrm{Spec}(A/I)$ is proper as X_0 is proper over A/I. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 30.28.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 37.12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is Noetherian and I-adically complete. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 41.14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when (A, I) is a henselian pair and A is a Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to I. Observe that $A^\wedge$ is a Noetherian ring which is $IA^\wedge$-adically complete, see Algebra, Lemmas 10.97.4 and 10.97.6. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \mathrm{colim}\, A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 32.10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 32.8.3 and 32.8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Lemma 16.14.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of

$V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when $(A, I$ is a henselian pair and A is a Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let (A, I) be a henselian pair. Set $S = \mathrm{Spec}(A)$ and denote $S_0 = \mathrm{Spec}(A/I)$. By Limits, Lemma 32.13.3 we can write $X \to \mathrm{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i affine and of finite type over $\mathbf{Z}$. Write $S_i = \mathrm{Spec}(A_i)$ and denote $I_i \subset A_i$ the inverse image of I by the map $A_i \to A$. Set $S_{i,0} = \mathrm{Spec}(A_i/I_i)$. Since $S = \lim S_i$ we have $A = \mathrm{colim}\, A_i$. Thus we also have $I = \mathrm{colim}\, I_i$ and $A/I = \mathrm{colim}\, A_i/I_i$. The ring A_i is a Noetherian G-ring (More on Algebra, Proposition 15.50.12). Denote (A_i^h, I_i^h) the henselization of the pair (A_i, I_i). By More on Algebra, Lemma 15.12.5 we see that $A = \mathrm{colim}\, A_i^h$. By More on Algebra, Lemma 15.50.15 the rings A_i^h are G-rings. Thus we see that $A = \mathrm{colim}\, A_i^h$ and

$$X = \lim(X_i \times_{S_i} \mathrm{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \mathrm{Spec}(A_i^h)$ (by Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} S_{i,0})$. Thus we formally deduce the result for $X/\mathrm{Spec}(A)$ from the result for the $(X_i \times_{S_i} \mathrm{Spec}(A_i^h))/\mathrm{Spec}(A_i^h)$ which we dealt with above. $\square$

0A49 Lemma 58.9.3. Let k'/k be an extension of algebraically closed fields. Let X be a proper scheme over k. Then the functor

$$U \longmapsto U_{k'}$$

is an equivalence of categories between schemes finite étale over X and schemes finite étale over $X_{k'}$.

Proof. Let us prove the functor is essentially surjective. Let $U' \to X_{k'}$ be a finite étale morphism. Write $k' = \mathrm{colim}\, A_i$ as a filtered colimit of finite type k-algebras. By Limits, Lemma 32.10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X_{k'}$ is U'. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 32.8.3 and 32.8.10). Since k is algebraically closed we can find a k-valued point t in $\mathrm{Spec}(A_i)$. Let $U = (U_i)_t$ be the fibre of U_i over t. Let A_i^h be the henselization of $(A_i)_{\mathfrak m}$ where $\mathfrak m$ is the maximal ideal corresponding to the point t. By Lemma 58.9.1 we see that $(U_i)_{A_i^h} = U \times \mathrm{Spec}(A_i^h)$ as schemes over $X_{A_i^h}$. Now since A_i^h is algebraic over A_i (see for example discussion in Smoothing Ring Maps, Example 16.13.3) and since k' is algebraically closed we can find a ring map $A_i^h \to k'$ extending the given inclusion $A_i \subset k'$. Hence we conclude that U' is isomorphic to the base change of U. The proof of fully faithfulness is exactly the same. $\square$

58.10. Local connectedness

0BQD In this section we ask when $\pi_1(U) \to \pi_1(X)$ is surjective for U a dense open of a scheme X. We will see that this is the case (roughly) when $U \cap B$ is connected for any small "ball" B around a point $x \in X \setminus U$.

0BQE Lemma 58.10.1. Let $f : X \to Y$ be a morphism of schemes. If $f(X)$ is dense in Y then the base change functor $\textit{FÉt}_Y \to \textit{FÉt}_X$ is faithful.

Proof. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given W finite étale over Y and a morphism $s : X \to W$ over X there is at most one section $t : Y \to W$ such that $s = t \circ f$. Consider two sections $t_1, t_2 : Y \to W$ such that $s = t_1 \circ f = t_2 \circ f$. Since the equalizer of t_1 and t_2 is closed in Y (Schemes, Lemma 26.21.5) and since $f(X)$ is dense in Y we see that t_1 and t_2 agree on Y_{red}. Then it follows that t_1 and t_2 have the same image which is an open and closed subscheme of W mapping isomorphically to Y (Étale Morphisms, Proposition 41.6.1) hence they are equal. $\square$

The condition in the following lemma that the punctured spectrum of the strict henselization is connected follows for example from the assumption that the local ring is geometrically unibranch, see More on Algebra, Lemma 15.106.5. There is a partial converse in Properties, Lemma 28.15.3.

0BLQ Lemma 58.10.2. Let $(A, \mathfrak m)$ be a local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak m\}$. If the punctured spectrum of the strict henselization of A is connected, then

$$\textit{FÉt}_X \longrightarrow \textit{FÉt}_U, \quad Y \longmapsto Y \times_X U$$

is a fully faithful functor.

Proof. Assume A is strictly henselian. In this case any finite étale cover Y of X is isomorphic to a finite disjoint union of copies of X. Thus it suffices to prove that any morphism $U \to U \amalg \ldots \amalg U$ over U, extends uniquely to a morphism $X \to X \amalg \ldots \amalg X$ over X. If U is connected (in particular nonempty), then this is true.

The general case. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given Y finite étale over X any morphism $s : U \to Y$ over X extends to a morphism $t : X \to Y$ over X. Let A^{sh} be the strict henselization of A and denote $X^{sh} = \mathrm{Spec}(A^{sh})$, $U^{sh} = U \times_X X^{sh}$, $Y^{sh} = Y \times_X X^{sh}$. By the first paragraph and our assumption on A, we can extend the base change $s^{sh} : U^{sh} \to Y^{sh}$ of s to $t^{sh} : X^{sh} \to Y^{sh}$. Set $A' = A^{sh} \otimes_A A^{sh}$. Then the two pullbacks t'_1, t'_2 of t^{sh} to $X' = \mathrm{Spec}(A')$ are extensions of the pullback s' of s to $U' = U \times_X X'$. As $A \to A'$ is flat we see that $U' \subset X'$ is (topologically) dense by going down for $A \to A'$ (Algebra, Lemma 10.39.19). Thus $t'_1 = t'_2$ by Lemma 58.10.1. Hence t^{sh} descends to a morphism $t : X \to Y$ for example by Descent, Lemma 35.13.7. $\square$

In view of Lemma 58.10.2 it is interesting to know when the punctured spectrum of a ring (and of its strict henselization) is connected. There is a famous lemma due to Hartshorne which gives a sufficient condition, see Local Cohomology, Lemma 51.3.1.

0BQF Lemma 58.10.3. Let X be a scheme. Let $U \subset X$ be a dense open. Assume

(1) the underlying topological space of X is Noetherian, and
(2) for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.

Then $\textit{FÉt}_X \to \textit{Fét}_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 58.10.1.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 58.10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 32.20.3 we find an open $U \subset U'$ containing x and an extension $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ of φ. Since the underlying topological space of X is Noetherian this finishes the proof by Noetherian induction on the complement of the open over which φ is defined. $\square$

0BSA Lemma 58.10.4. Let X be a scheme. Let $U \subset X$ be a dense open. Assume

(1) $U \to X$ is quasi-compact,
(2) every point of $X \setminus U$ is closed, and
(3) for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.

Then $\mathrm{F\acute{E}t}_X \to \mathrm{F\acute{e}t}_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 58.10.1.

Let $x \in X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. Since every point of $X \setminus U$ is closed V is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 58.10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 32.20.3 (this uses that U is retrocompact in X) we find an open $U \subset U'_x$ containing x and an extension $\varphi'_x : (Y_1)_{U'_x} \to (Y_2)_{U'_x}$ of φ. Note that given two points $x, x' \in X \setminus U$ the morphisms φ'_x and $\varphi'_{x'}$ agree over $U'_x \cap U'_{x'}$ as U is dense in that open (Lemma 58.10.1). Thus we can extend φ to $\bigcup U'_x = X$ as desired. $\square$

0BQG Lemma 58.10.5. Let X be a scheme. Let $U \subset X$ be a dense open. Assume

(1) every quasi-compact open of X has finitely many irreducible components,
(2) for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.

Then $\mathrm{F\acute{E}t}_X \to \mathrm{F\acute{e}t}_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 58.10.1. We will prove existence by showing that we can enlarge U if $U \neq X$ and using Zorn's lemma to finish the proof.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 58.10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ (uniquely) to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. Choose an affine neighbourhood $W \subset X$ of x. Since $U \cap W$ is dense in W it contains the generic points $\eta_1, \ldots, \eta_n$ of W. Choose an affine open $W' \subset W \cap U$ containing $\eta_1, \ldots, \eta_n$. Set $V' = W' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By Limits, Lemma 32.20.3 applied to $x \in W \supset W'$ we find an open $W' \subset W'' \subset W$ with $x \in W''$ and a morphism $\varphi'' : (Y_1)_{W''} \to (Y_2)_{W''}$ agreeing with φ over W'. Since W' is dense in $W'' \cap U$, we see by Lemma 58.10.1 that φ and φ'' agree over $U \cap W'$. Thus φ and φ'' glue to a morphism φ' over $U' = U \cup W''$ agreeing with φ over U. Observe that $x \in U'$ so that we've extended φ to a strictly larger open.

Consider the set $\mathcal{S}$ of pairs (U', φ') where $U \subset U'$ and φ' is an extension of φ. We endow $\mathcal{S}$ with a partial ordering in the obvious manner. If (U'_i, φ'_i) is a totally ordered subset, then it has a maximum (U', φ'). Just take $U' = \bigcup U'_i$ and let $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ be the morphism agreeing with φ'_i over U'_i. Thus Zorn's lemma applies and $\mathcal{S}$ has a maximal element. By the argument above we see that this maximal element is an extension of φ over all of X. □

0BSB Lemma 58.10.6. Let $(A, \mathfrak{m})$ be a local ring. Set $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Let U^{sh} be the punctured spectrum of the strict henselization A^{sh} of A. Assume U is quasi-compact and U^{sh} is connected. Then the sequence

$$\pi_1(U^{sh}, \overline{u}) \to \pi_1(U, \overline{u}) \to \pi_1(X, \overline{u}) \to 1$$

is exact in the sense of Lemma 58.4.3 part (1).

Proof. The map $\pi_1(U) \to \pi_1(X)$ is surjective by Lemmas 58.10.2 and 58.4.1.

Write $X^{sh} = \text{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X U^{sh}$. It follows that the composition $\pi_1(U^{sh}) \to \pi_1(U) \to \pi_1(X)$ is trivial, see Lemma 58.4.2.

To finish the proof, it suffices according to Lemma 58.4.3 to show the following: Given a finite étale morphism $V \to U$ such that $V \times_U U^{sh}$ is a disjoint union of copies of U^{sh}, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X U$ over U. The assumption implies that there exists a finite étale morphism $Y^{sh} \to X^{sh}$ and an isomorphism $V \times_U U^{sh} \cong Y^{sh} \times_{X^{sh}} U^{sh}$. Consider the following diagram

$$\begin{tikzcd}
U \arrow[d] & U^{sh} \arrow[l] \arrow[d] & U^{sh} \times_U U^{sh} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & U^{sh} \times_U U^{sh} \times_U U^{sh} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
X & X^{sh} \arrow[l] & X^{sh} \times_X X^{sh} \arrow[l, shift left] \arrow[l, shift right] & X^{sh} \times_X X^{sh} \times_X X^{sh} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

Since $U \subset X$ is quasi-compact by assumption, all the downward arrows are quasi-compact open immersions. Let $\xi \in X^{sh} \times_X X^{sh}$ be a point not in $U^{sh} \times_U U^{sh}$. Then ξ lies over the closed point x^{sh} of X^{sh}. Consider the local ring homomorphism

$$A^{sh} = \mathcal{O}_{X^{sh}, x^{sh}} \to \mathcal{O}_{X^{sh} \times_X X^{sh}, \xi}$$

determined by the first projection $X^{sh} \times_X X^{sh}$. This is a filtered colimit of local homomorphisms which are localizations étale ring maps. Since A^{sh} is strictly henselian, we conclude that it is an isomorphism. Since this holds for every ξ in the complement it follows there are no specializations among these points and hence every such ξ is a closed point (you can also prove this directly). As the local ring at ξ is isomorphic to A^{sh}, it is strictly henselian and has connected punctured spectrum. Similarly for points ξ of $X^{sh} \times_X X^{sh} \times_X X^{sh}$ not in $U^{sh} \times_U U^{sh} \times_U U^{sh}$. It follows from Lemma 58.10.4 that pullback along the vertical arrows induce fully faithful functors on the categories of finite étale schemes. Thus the canonical descent datum on $V \times_U U^{sh}$ relative to the fpqc covering $\{U^{sh} \to U\}$ translates into a descent datum for Y^{sh} relative to the fpqc covering $\{X^{sh} \to X\}$. Since $Y^{sh} \to X^{sh}$ is finite hence affine, this descent datum is effective (Descent, Lemma 35.37.1). Thus we get an affine morphism $Y \to X$ and an isomorphism $Y \times_X X^{sh} \to Y^{sh}$ compatible with descent data. By fully faithfulness of descent data (as in Descent, Lemma 35.35.11) we get an isomorphism $V \to U \times_X Y$. Finally, $Y \to X$ is finite étale as $Y^{sh} \to X^{sh}$ is, see Descent, Lemmas 35.23.29 and 35.23.23. □

Let X be an irreducible scheme. Let $\eta \in X$ be the generic point. The canonical morphism $\eta \to X$ induces a canonical map

0BQH (58.10.6.1)
$$\mathrm{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

The identification on the left hand side is Lemma 58.6.3.

0BQI Lemma 58.10.7. Let X be an irreducible, geometrically unibranch scheme. For any nonempty open $U \subset X$ the canonical map

$$\pi_1(U, \overline{u}) \longrightarrow \pi_1(X, \overline{u})$$

is surjective. The map (58.10.6.1) $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective as well.

Proof. By Lemma 58.8.3 we may replace X by its reduction. Thus we may assume that X is an integral scheme. By Lemma 58.4.1 the assertion of the lemma translates into the statement that the functors $\mathrm{FÉt}_X \to \mathrm{FÉt}_U$ and $\mathrm{FÉt}_X \to \mathrm{FÉt}_\eta$ are fully faithful.

The result for $\mathrm{FÉt}_X \to \mathrm{FÉt}_U$ follows from Lemma 58.10.5 and the fact that for a local ring A which is geometrically unibranch its strict henselization has an irreducible spectrum. See More on Algebra, Lemma 15.106.5.

Observe that the residue field $\kappa(\eta) = \mathcal{O}_{X,\eta}$ is the filtered colimit of $\mathcal{O}_X(U)$ over $U \subset X$ nonempty open affine. Hence $\mathrm{FÉt}_\eta$ is the colimit of the categories $\mathrm{FÉt}_U$ over such U, see Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. A formal argument then shows that fully faithfulness for $\mathrm{FÉt}_X \to \mathrm{FÉt}_\eta$ follows from the fully faithfulness of the functors $\mathrm{FÉt}_X \to \mathrm{FÉt}_U$. □

0BSC Lemma 58.10.8. Let X be a scheme. Let $x_1, \ldots, x_n \in X$ be a finite number of closed points such that

(1) $U = X \setminus \{x_1, \ldots, x_n\}$ is connected and is a retrocompact open of X, and
(2) for each i the punctured spectrum U_i^{sh} of the strict henselization of $\mathcal{O}_{X,x_i}$ is connected.

Then the map $\pi_1(U) \to \pi_1(X)$ is surjective and the kernel is the smallest closed normal subgroup of $\pi_1(U)$ containing the image of $\pi_1(U_i^{sh}) \to \pi_1(U)$ for $i = 1, \ldots, n$.

Proof. Surjectivity follows from Lemmas 58.10.4 and 58.4.1. We can consider the sequence of maps

$$\pi_1(U) \to \ldots \to \pi_1(X \setminus \{x_1, x_2\}) \to \pi_1(X \setminus \{x_1\}) \to \pi_1(X)$$

A group theory argument then shows it suffices to prove the statement on the kernel in the case $n = 1$ (details omitted). Write $x = x_1$, $U^{sh} = U_1^{sh}$, set $A = \mathcal{O}_{X,x}$, and let A^{sh} be the strict henselization. Consider the diagram

$$\begin{tikzcd} U \arrow[d] & \mathrm{Spec}(A) \setminus \{\mathfrak m\} \arrow[l] \arrow[d] & U^{sh} \arrow[l] \arrow[d] \\ X & \mathrm{Spec}(A) \arrow[l] & \mathrm{Spec}(A^{sh}) \arrow[l] \end{tikzcd}$$

By Lemma 58.4.3 we have to show finite étale morphisms $V \to U$ which pull back to trivial coverings of U^{sh} extend to finite étale schemes over X. By Lemma 58.10.6 we know the corresponding statement for finite étale schemes over the punctured spectrum of A. However, by Limits, Lemma 32.20.1 schemes of finite presentation over X are the same thing as schemes of finite presentation over U and A glued over the punctured spectrum of A. This finishes the proof. □

58.11. Fundamental groups of normal schemes

0BQJ Let X be an integral, geometrically unibranch scheme. In the previous section we have seen that the fundamental group of X is a quotient of the Galois group of the function field K of X. Since the map is continuous the kernel is a normal closed subgroup of the Galois group. Hence this kernel corresponds to a Galois extension M/K by Galois theory (Fields, Theorem 9.22.4). In this section we will determine M when X is a normal integral scheme.

Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Consider the normalization $Y \to X$ of X in the morphism $\mathrm{Spec}(L) \to X$ as defined in Morphisms, Section 29.53. We will say (in this setting) that X is unramified in L if $Y \to X$ is an unramified morphism of schemes. In Lemma 58.13.4 we will elucidate this condition. Observe that the scheme theoretic fibre of $Y \to X$ over $\mathrm{Spec}(K)$ is $\mathrm{Spec}(L)$. Hence the field extension L/K is separable if X is unramified in L, see Morphisms, Lemmas 29.35.11.

0BQK **Lemma 58.11.1.** *In the situation above the following are equivalent*

1. *X is unramified in L,*
2. *$Y \to X$ is étale, and*
3. *$Y \to X$ is finite étale.*

Proof. Observe that $Y \to X$ is an integral morphism. In each case the morphism $Y \to X$ is locally of finite type by definition. Hence we find that in each case $Y \to X$ is finite by Morphisms, Lemma 29.44.4. In particular we see that (2) is equivalent to (3). An étale morphism is unramified, hence (2) implies (1).

Conversely, assume $Y \to X$ is unramified. Since a normal scheme is geometrically unibranch (Properties, Lemma 28.15.2), we see that the morphism $Y \to X$ is étale by More on Morphisms, Lemma 37.37.2. We also give a direct proof in the next paragraph.

Let $x \in X$. We can choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$Y \times_X U = \coprod V_j \longrightarrow U$$

is a disjoint union of closed immersions, see Étale Morphisms, Lemma 41.17.3. Shrinking we may assume U is quasi-compact. Then U has finitely many irreducible components (Descent, Lemma 35.16.3). Since U is normal (Descent, Lemma

35.18.2) the irreducible components of U are open and closed (Properties, Lemma 28.7.5) and we may assume U is irreducible. Then U is an integral scheme whose generic point ξ maps to the generic point of X. On the other hand, we know that $Y \times_X U$ is the normalization of U in $\mathrm{Spec}(L) \times_X U$ by More on Morphisms, Lemma 37.19.2. Every point of $\mathrm{Spec}(L) \times_X U$ maps to ξ. Thus every V_j contains a point mapping to ξ by Morphisms, Lemma 29.53.9. Thus $V_j \to U$ is an isomorphism as $U = \overline{\{\xi\}}$. Thus $Y \times_X U \to U$ is étale. By Descent, Lemma 35.23.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x). $\square$

0BQL Lemma 58.11.2. Let X be a normal integral scheme with function field K. Let $Y \to X$ be a finite étale morphism. If Y is connected, then Y is an integral normal scheme and Y is the normalization of X in the function field of Y.

Proof. The scheme Y is normal by Descent, Lemma 35.18.2. Since $Y \to X$ is flat every generic point of Y maps to the generic point of X by Morphisms, Lemma 29.25.9. Since $Y \to X$ is finite we see that Y has a finite number of irreducible components. Thus Y is the disjoint union of a finite number of integral normal schemes by Properties, Lemma 28.7.5. Thus if Y is connected, then Y is an integral normal scheme.

Let L be the function field of Y and let $Y' \to X$ be the normalization of X in L. By Morphisms, Lemma 29.53.4 we obtain a factorization $Y' \to Y \to X$ and $Y' \to Y$ is the normalization of Y in L. Since Y is normal it is clear that $Y' = Y$ (this can also be deduced from Morphisms, Lemma 29.54.8). $\square$

0BQM Proposition 58.11.3. Let X be a normal integral scheme with function field K. Then the canonical map (58.10.6.1)

$$\mathrm{Gal}(K^{sep}/K) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

is identified with the quotient map $\mathrm{Gal}(K^{sep}/K) \to \mathrm{Gal}(M/K)$ where $M \subset K^{sep}$ is the union of the finite subextensions L such that X is unramified in L.

Proof. The normal scheme X is geometrically unibranch (Properties, Lemma 28.15.2). Hence Lemma 58.10.7 applies to X. Thus $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective and top horizontal arrow of the commutative diagram

$$\begin{tikzcd} \text{FÉt}_X \arrow[r] \arrow[d] \arrow[dr, "c"] & \text{FÉt}_\eta \arrow[d] \\ \text{Finite-}\pi_1(X, \overline{\eta})\text{-sets} \arrow[r] & \text{Finite-Gal}(K^{sep}/K)\text{-sets} \end{tikzcd}$$

is fully faithful. The left vertical arrow is the equivalence of Theorem 58.6.2 and the right vertical arrow is the equivalence of Lemma 58.6.3. The lower horizontal arrow is induced by the map of the proposition. By Lemmas 58.11.1 and 58.11.2 we see that the essential image of c consists of $\mathrm{Gal}(K^{sep}/K)$-Sets isomorphic to sets of the form

$$S = \mathrm{Hom}_K(\prod\nolimits_{i=1,\ldots,n} L_i, K^{sep}) = \coprod\nolimits_{i=1,\ldots,n} \mathrm{Hom}_K(L_i, K^{sep})$$

with L_i/K finite separable such that X is unramified in L_i. Thus if $M \subset K^{sep}$ is as in the statement of the lemma, then $\mathrm{Gal}(K^{sep}/M)$ is exactly the subgroup of $\mathrm{Gal}(K^{sep}/K)$ acting trivially on every object in the essential image of c. On the other hand, the essential image of c is exactly the category of S such that the $\text{Gal}(K^{sep}/K)$-action factors through the surjection $\text{Gal}(K^{sep}/K) \to \pi_1(X, \overline{\eta})$. We conclude that $\text{Gal}(K^{sep}/M)$ is the kernel. Hence $\text{Gal}(K^{sep}/M)$ is a normal subgroup, M/K is Galois, and we have a short exact sequence

$$1 \to \text{Gal}(K^{sep}/M) \to \text{Gal}(K^{sep}/K) \to \text{Gal}(M/K) \to 1$$

by Galois theory (Fields, Theorem 9.22.4 and Lemma 9.22.5). The proof is done. $\square$

0BSM Lemma 58.11.4. Let $(A, \mathfrak{m})$ be a normal local ring. Set $X = \text{Spec}(A)$. Let A^{sh} be the strict henselization of A. Let K and K^{sh} be the fraction fields of A and A^{sh}. Then the sequence

$$\pi_1(\text{Spec}(K^{sh})) \to \pi_1(\text{Spec}(K)) \to \pi_1(X) \to 1$$

is exact in the sense of Lemma 58.4.3 part (1).

Proof. Note that A^{sh} is a normal domain, see More on Algebra, Lemma 15.45.6. The map $\pi_1(\text{Spec}(K)) \to \pi_1(X)$ is surjective by Proposition 58.11.3.

Write $X^{sh} = \text{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X \text{Spec}(K^{sh})$. It follows that the composition $\pi_1(\text{Spec}(K^{sh})) \to \pi_1(X)$ is trivial, see Lemma 58.4.2.

To finish the proof, it suffices according to Lemma 58.4.3 to show the following: Given a finite étale morphism $V \to \text{Spec}(K)$ such that $V \times_{\text{Spec}(K)} \text{Spec}(K^{sh})$ is a disjoint union of copies of $\text{Spec}(K^{sh})$, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X \text{Spec}(K)$ over $\text{Spec}(K)$. Write $V = \text{Spec}(L)$, so L is a finite product of finite separable extensions of K. Let $B \subset L$ be the integral closure of A in L. If $A \to B$ is étale, then we can take $Y = \text{Spec}(B)$ and the proof is complete. By Algebra, Lemma 10.147.4 (and a limit argument we omit) we see that $B \otimes_A A^{sh}$ is the integral closure of A^{sh} in $L^{sh} = L \otimes_K K^{sh}$. Our assumption is that L^{sh} is a product of copies of K^{sh} and hence B^{sh} is a product of copies of A^{sh}. Thus $A^{sh} \to B^{sh}$ is étale. As $A \to A^{sh}$ is faithfully flat it follows that $A \to B$ is étale (Descent, Lemma 35.23.29) as desired. $\square$

58.12. Group actions and integral closure

0BSN In this section we continue the discussion of More on Algebra, Section 15.110. Recall that a normal local ring is a domain by definition.

0BSP Lemma 58.12.1. Let A be a normal domain whose fraction field K is separably algebraically closed. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal. Then the residue field $\kappa(\mathfrak{p})$ is algebraically closed.

Proof. Assume the lemma is not true to get a contradiction. Then there exists a monic irreducible polynomial $P(T) \in \kappa(\mathfrak{p})[T]$ of degree $d > 1$. After replacing P by $a^d P(a^{-1}T)$ for suitable $a \in A$ (to clear denominators) we may assume that P is the image of a monic polynomial Q in $A[T]$. Observe that Q is irreducible in $K[T]$. Namely a factorization over K leads to a factorization over A by Algebra, Lemma 10.38.5 which we could reduce modulo $\mathfrak{p}$ to get a factorization of P. As K is separably closed, Q is not a separable polynomial (Fields, Definition 9.12.2). Then the characteristic of K is $p > 0$ and Q has vanishing linear term (Fields, Definition

9.12.2). However, then we can replace Q by $Q + aT$ where $a \in \mathfrak{p}$ is nonzero to get a contradiction. $\square$

0BSQ **Lemma 58.12.2.** A normal local ring with separably closed fraction field is strictly henselian.

Proof. Let $(A, \mathfrak{m}, \kappa)$ be normal local with separably closed fraction field K. If $A = K$, then we are done. If not, then the residue field κ is algebraically closed by Lemma 58.12.1 and it suffices to check that A is henselian. Let $f \in A[T]$ be monic and let $a_0 \in \kappa$ be a root of multiplicity 1 of the reduction $\overline{f} \in \kappa[T]$. Let $f = \prod f_i$ be the factorization in $K[T]$. By Algebra, Lemma 10.38.5 we have $f_i \in A[T]$. Thus a_0 is a root of f_i for some i. After replacing f by f_i we may assume f is irreducible. Then, since the derivative f' cannot be zero in $A[T]$ as a_0 is a single root, we conclude that f is linear due to the fact that K is separably algebraically closed. Thus A is henselian, see Algebra, Definition 10.153.1. $\square$

0BSS **Lemma 58.12.3.** Let G be a finite group acting on a ring R. Let $R^G \to A$ be a ring map. Let $\mathfrak{q}' \subset A \otimes_{R^G} R$ be a prime lying over the prime $\mathfrak{q} \subset R$. Then

$$I_\mathfrak{q} = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = \mathrm{id}_{\kappa(\mathfrak{q})}\}$$

is equal to

$$I_{\mathfrak{q}'} = \{\sigma \in G \mid \sigma(\mathfrak{q}') = \mathfrak{q}' \text{ and } \sigma \bmod \mathfrak{q}' = \mathrm{id}_{\kappa(\mathfrak{q}')}\}$$

Proof. Since $\mathfrak{q}$ is the inverse image of $\mathfrak{q}'$ and since $\kappa(\mathfrak{q}) \subset \kappa(\mathfrak{q}')$, we get $I_{\mathfrak{q}'} \subset I_\mathfrak{q}$. Conversely, if $\sigma \in I_\mathfrak{q}$, the σ acts trivially on the fibre ring $A \otimes_{R^G} \kappa(\mathfrak{q})$. Thus σ fixes all the primes lying over $\mathfrak{q}$ and induces the identity on their residue fields. $\square$

0BST **Lemma 58.12.4.** Let G be a finite group acting on a ring R. Let $\mathfrak{q} \subset R$ be a prime. Set

$$I = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = \mathrm{id}_\mathfrak{q}\}$$

Then $R^G \to R^I$ is étale at $R^I \cap \mathfrak{q}$.

Proof. The strategy of the proof is to use étale localization to reduce to the case where $R \to R^I$ is a local isomorphism at $R^I \cap \mathfrak{p}$. Let $R^G \to A$ be an étale ring map. We claim that if the result holds for the action of G on $A \otimes_{R^G} R$ and some prime $\mathfrak{q}'$ of $A \otimes_{R^G} R$ lying over $\mathfrak{q}$, then the result is true.

To check this, note that since $R^G \to A$ is flat we have $A = (A \otimes_{R^G} R)^G$, see More on Algebra, Lemma 15.110.7. By Lemma 58.12.3 the group I does not change. Then a second application of More on Algebra, Lemma 15.110.7 shows that $A \otimes_{R^G} R^I = (A \otimes_{R^G} R)^I$ (because $R^I \to A \otimes_{R^G} R^I$ is flat). Thus

$$\begin{tikzcd}
\mathrm{Spec}((A \otimes_{R^G} R)^I) \arrow[r] \arrow[d] & \mathrm{Spec}(R^I) \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(R^G)
\end{tikzcd}$$

is cartesian and the horizontal arrows are étale. Thus if the left vertical arrow is étale in some open neighbourhood W of $(A \otimes_{R^G} R)^I \cap \mathfrak{q}'$, then the right vertical arrow is étale at the points of the (open) image of W in $\mathrm{Spec}(R^I)$, see Descent, Lemma 35.14.5. In particular the morphism $\mathrm{Spec}(R^I) \to \mathrm{Spec}(R^G)$ is étale at $R^I \cap \mathfrak{q}$.

Let $\mathfrak p = R^G \cap \mathfrak q$. By More on Algebra, Lemma 15.110.8 the fibre of $\text{Spec}(R) \to \text{Spec}(R^G)$ over $\mathfrak p$ is finite. Moreover the residue field extensions at these points are algebraic, normal, with finite automorphism groups by More on Algebra, Lemma 15.110.9. Thus we may apply More on Morphisms, Lemma 37.42.1 to the integral ring map $R^G \to R$ and the prime $\mathfrak p$. Combined with the claim above we reduce to the case where $R = A_1 \times \ldots \times A_n$ with each A_i having a single prime $\mathfrak q_i$ lying over $\mathfrak p$ such that the residue field extensions $\kappa(\mathfrak q_i)/\kappa(\mathfrak p)$ are purely inseparable. Of course $\mathfrak q$ is one of these primes, say $\mathfrak q = \mathfrak q_1$.

It may not be the case that G permutes the factors A_i (this would be true if the spectrum of A_i were connected, for example if R^G was local). This we can fix as follows; we suggest the reader think this through for themselves, perhaps using idempotents instead of topology. Recall that the product decomposition gives a corresponding disjoint union decomposition of $\text{Spec}(R)$ by open and closed subsets U_i. Since G is finite, we can refine this covering by a finite disjoint union decomposition $\text{Spec}(R) = \coprod_{j \in J} W_j$ by open and closed subsets W_j, such that for all $j \in J$ there exists a $j' \in J$ with $\sigma(W_j) = W_{j'}$. The union of the W_j not meeting $\{\mathfrak q_1, \ldots, \mathfrak q_n\}$ is a closed subset not meeting the fibre over $\mathfrak p$ hence maps to a closed subset of $\text{Spec}(R^G)$ not meeting $\mathfrak p$ as $\text{Spec}(R) \to \text{Spec}(R^G)$ is closed. Hence after replacing R^G by a principal localization (permissible by the claim) we may assume each W_j meets one of the points $\mathfrak q_i$. Then we set $U_i = W_j$ if $\mathfrak q_i \in W_j$. The corresponding product decomposition $R = A_1 \times \ldots \times A_n$ is one where G permutes the factors A_i.

Thus we may assume we have a product decomposition $R = A_1 \times \ldots \times A_n$ compatible with G-action, where each A_i has a single prime $\mathfrak q_i$ lying over $\mathfrak p$ and the field extensions $\kappa(\mathfrak q_i)/\kappa(\mathfrak p)$ are purely inseparable. Write $A' = A_2 \times \ldots \times A_n$ so that

$$R = A_1 \times A'$$

Since $\mathfrak q = \mathfrak q_1$ we find that every $\sigma \in I$ preserves the product decomposition above. Hence

$$R^I = (A_1)^I \times (A')^I$$

Observe that $I = D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\}$ because $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is purely inseparable. Since the action of G on primes over $\mathfrak p$ is transitive (More on Algebra, Lemma 15.110.8) we conclude that, the index of I in G is n and we can write $G = eI \amalg \sigma_2 I \amalg \ldots \amalg \sigma_n I$ so that $A_i = \sigma_i(A_1)$ for $i = 2, \ldots, n$. It follows that

$$R^G = (A_1)^I.$$

Thus the map $R^G \to R^I$ is étale at $R^I \cap \mathfrak q$ and the proof is complete. $\square$

The following lemma generalizes More on Algebra, Lemma 15.112.8.

0BSU Lemma 58.12.5. Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = \text{Gal}(L/K)$ and let B be the integral closure of A in L. Let $\mathfrak q \subset B$. Set

$$I = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q \text{ and } \sigma \bmod \mathfrak q = \text{id}_{\kappa(\mathfrak q)}\}$$

Then $(B^I)_{B^I \cap \mathfrak q}$ is a filtered colimit of étale A-algebras.

Proof. We can write L as the filtered colimit of finite Galois extensions of K. Hence it suffices to prove this lemma in case L/K is a finite Galois extension, see Algebra, Lemma 10.154.3. Since $A = B^G$ as A is integrally closed in $K = L^G$ the result follows from Lemma 58.12.4. $\square$

58.13. Ramification theory

0BSD In this section we continue the discussion of More on Algebra, Section 15.112 and we relate it to our discussion of the fundamental groups of schemes.

Let $(A, \mathfrak{m}, \kappa)$ be a normal local ring with fraction field K. Choose a separable algebraic closure K^{sep}. Let A^{sep} be the integral closure of A in K^{sep}. Choose maximal ideal $\mathfrak{m}^{sep} \subset A^{sep}$. Let $A \subset A^h \subset A^{sh}$ be the henselization and strict henselization. Observe that A^h and A^{sh} are normal rings as well (More on Algebra, Lemma 15.45.6). Denote K^h and K^{sh} their fraction fields. Since $(A^{sep})_{\mathfrak{m}^{sep}}$ is strictly henselian by Lemma 58.12.2 we can choose an A-algebra map $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$. Namely, first choose a κ-embedding[4] $\kappa(\mathfrak{m}^{sh}) \to \kappa(\mathfrak{m}^{sep})$ and then extend (uniquely) to an A-algebra homomorphism by Algebra, Lemma 10.155.10. We get the following diagram

$$
\begin{tikzcd}
K^{sep} & K^{sh} \arrow[l] & K^h \arrow[l] & K \arrow[l] \\
(A^{sep})_{\mathfrak{m}^{sep}} \arrow[u] & A^{sh} \arrow[l] \arrow[u] & A^h \arrow[l] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}
$$

We can take the fundamental groups of the spectra of these rings. Of course, since K^{sep}, $(A^{sep})_{\mathfrak{m}^{sep}}$, and A^{sh} are strictly henselian, for them we obtain trivial groups. Thus the interesting part is the following

0BSV (58.13.0.1)
$$
\begin{tikzcd}
\pi_1(U^{sh}) \arrow[r] \arrow[rd, "1"'] & \pi_1(U^h) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\
& \pi_1(X^h) \arrow[r] & \pi_1(X)
\end{tikzcd}
$$

Here X^h and X are the spectra of A^h and A and U^{sh}, U^h, U are the spectra of K^{sh}, K^h, and K. The label 1 means that the map is trivial; this follows as it factors through the trivial group $\pi_1(X^{sh})$. On the other hand, the profinite group $G = \mathrm{Gal}(K^{sep}/K)$ acts on A^{sep} and we can make the following definitions

$$D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\} \supset I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = \mathrm{id}_{\kappa(\mathfrak{m}^{sep})}\}$$

These groups are sometimes called the decomposition group and the inertia group especially when A is a discrete valuation ring.

0BSW **Lemma 58.13.1.** *In the situation described above, via the isomorphism $\pi_1(U) = \mathrm{Gal}(K^{sep}/K)$ the diagram (58.13.0.1) translates into the diagram*

$$
\begin{tikzcd}
I \arrow[r] \arrow[rd, "1"'] & D \arrow[r] \arrow[d] & \mathrm{Gal}(K^{sep}/K) \arrow[d] \\
& \mathrm{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa) \arrow[r] & \mathrm{Gal}(M/K)
\end{tikzcd}
$$

where $K^{sep}/M/K$ is the maximal subextension unramified with respect to A. Moreover, the vertical arrows are surjective, the kernel of the left vertical arrow is I and

[4] This is possible because $\kappa(\mathfrak{m}^{sh})$ is a separable algebraic closure of κ and $\kappa(\mathfrak{m}^{sep})$ is an algebraic closure of κ by Lemma 58.12.1.

the kernel of the right vertical arrow is the smallest closed normal subgroup of $\text{Gal}(K^{sep}/K)$ containing I.

Proof. By construction the group D acts on $(A^{sep})_{\mathfrak{m}^{sep}}$ over A. By the uniqueness of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ given the map on residue fields (Algebra, Lemma 10.155.10) we see that the image of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ is contained in $((A^{sep})_{\mathfrak{m}^{sep}})^I$. On the other hand, Lemma 58.12.5 shows that $((A^{sep})_{\mathfrak{m}^{sep}})^I$ is a filtered colimit of étale extensions of A. Since A^{sh} is the maximal such extension, we conclude that $A^{sh} = ((A^{sep})_{\mathfrak{m}^{sep}})^I$. Hence $K^{sh} = (K^{sep})^I$.

Recall that I is the kernel of a surjective map $D \to \text{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa)$, see More on Algebra, Lemma 15.110.10. We have $\text{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$ as we have seen above that these fields are the algebraic and separable algebraic closures of κ. On the other hand, any automorphism of A^{sh} over A is an automorphism of A^{sh} over A^h by the uniqueness in Algebra, Lemma 10.155.6. Furthermore, A^{sh} is the colimit of finite étale extensions $A^h \subset A'$ which correspond 1-to-1 with finite separable extension κ'/κ, see Algebra, Remark 10.155.4. Thus

$$\text{Aut}(A^{sh}/A) = \text{Aut}(A^{sh}/A^h) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$$

Let κ'/κ be a finite Galois extension with Galois group G. Let $A^h \subset A'$ be the finite étale extension corresponding to $\kappa \subset \kappa'$ by Algebra, Lemma 10.153.7. Then it follows that $(A')^G = A^h$ by looking at fraction fields and degrees (small detail omitted). Taking the colimit we conclude that $(A^{sh})^{\text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)} = A^h$. Combining all of the above, we find $A^h = ((A^{sep})_{\mathfrak{m}^{sep}})^D$. Hence $K^h = (K^{sep})^D$.

Since U, U^h, U^{sh} are the spectra of the fields K, K^h, K^{sh} we see that the top lines of the diagrams correspond via Lemma 58.6.3. By Lemma 58.8.2 we have $\pi_1(X^h) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$. The exactness of the sequence $1 \to I \to D \to \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa) \to 1$ was pointed out above. By Proposition 58.11.3 we see that $\pi_1(X) = \text{Gal}(M/K)$. Finally, the statement on the kernel of $\text{Gal}(K^{sep}/K) \to \text{Gal}(M/K) = \pi_1(X)$ follows from Lemma 58.11.4. This finishes the proof. $\square$

Let X be a normal integral scheme with function field K. Let K^{sep} be a separable algebraic closure of K. Let $X^{sep} \to X$ be the normalization of X in K^{sep}. Since $G = \text{Gal}(K^{sep}/K)$ acts on K^{sep} we obtain a right action of G on X^{sep}. For $y \in X^{sep}$ define

$$D_y = \{\sigma \in G \mid \sigma(y) = y\} \supset I_y = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}_y = \text{id}_{\kappa(y)}\}$$

similarly to the above. On the other hand, for $x \in X$ let $\mathcal{O}_{X,x}^{sh}$ be a strict henselization, let K_x^{sh} be the fraction field of $\mathcal{O}_{X,x}^{sh}$ and choose a K-embedding $K_x^{sh} \to K^{sep}$.

0BTD Lemma 58.13.2. Let X be a normal integral scheme with function field K. With notation as above, the following three subgroups of $\text{Gal}(K^{sep}/K) = \pi_1(\text{Spec}(K))$ are equal

(1) the kernel of the surjection $\text{Gal}(K^{sep}/K) \longrightarrow \pi_1(X)$,
(2) the smallest normal closed subgroup containing I_y for all $y \in X^{sep}$, and
(3) the smallest normal closed subgroup containing $\text{Gal}(K^{sep}/K_x^{sh})$ for all $x \in X$.

Proof. The equivalence of (2) and (3) follows from Lemma 58.13.1 which tells us that I_y is conjugate to $\text{Gal}(K^{sep}/K_x^{sh})$ if y lies over x. By Lemma 58.11.4 we see that $\text{Gal}(K^{sep}/K_x^{sh})$ maps trivially to $\pi_1(\text{Spec}(\mathcal{O}_{X,x}))$ and therefore the subgroup $N \subset G = \text{Gal}(K^{sep}/K)$ of (2) and (3) is contained in the kernel of $G \longrightarrow \pi_1(X)$.

To prove the other inclusion, since N is normal, it suffices to prove: given $N \subset U \subset G$ with U open normal, the quotient map $G \to G/U$ factors through $\pi_1(X)$. In other words, if L/K is the Galois extension corresponding to U, then we have to show that X is unramified in L (Section 58.11, especially Proposition 58.11.3). It suffices to do this when X is affine (we do this so we can refer to algebra results in the rest of the proof). Let $Y \to X$ be the normalization of X in L. The inclusion $L \subset K^{sep}$ induces a morphism $\pi : X^{sep} \to Y$. For $y \in X^{sep}$ the inertia group of $\pi(y)$ in $\text{Gal}(L/K)$ is the image of I_y in $\text{Gal}(L/K)$; this follows from More on Algebra, Lemma 15.110.11. Since $N \subset U$ all these inertia groups are trivial. We conclude that $Y \to X$ is étale by applying Lemma 58.12.4. (Alternative: you can use Lemma 58.11.4 to see that the pullback of Y to $\text{Spec}(\mathcal{O}_{X,x})$ is étale for all $x \in X$ and then conclude from there with a bit more work.) □

0BTE Example 58.13.3. Let X be a normal integral Noetherian scheme with function field K. Purity of branch locus (see below) tells us that if X is regular, then it suffices in Lemma 58.13.2 to consider the inertia groups $I = \pi_1(\text{Spec}(K_x^{sh}))$ for points x of codimension 1 in X. In general this is not enough however. Namely, let $Y = \mathbf{A}^n_k = \text{Spec}(k[t_1, \ldots, t_n])$ where k is a field not of characteristic 2. Let $G = \{\pm 1\}$ be the group of order 2 acting on Y by multiplication on the coordinates. Set

$$X = \text{Spec}(k[t_i t_j, i, j \in \{1, \ldots, n\}])$$

The embedding $k[t_i t_j] \subset k[t_1, \ldots, t_n]$ defines a degree 2 morphism $Y \to X$ which is unramified everywhere except over the maximal ideal $\mathfrak{m} = (t_i t_j)$ which is a point of codimension n in X.

0BTF Lemma 58.13.4. Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Let $Y \to X$ be the normalization of X in L. The following are equivalent

(1) X is unramified in L as defined in Section 58.11,
(2) $Y \to X$ is an unramified morphism of schemes,
(3) $Y \to X$ is an étale morphism of schemes,
(4) $Y \to X$ is a finite étale morphism of schemes,
(5) for $x \in X$ the projection $Y \times_X \text{Spec}(\mathcal{O}_{X,x}) \to \text{Spec}(\mathcal{O}_{X,x})$ is unramified,
(6) same as in (5) but with $\mathcal{O}^h_{X,x}$,
(7) same as in (5) but with $\mathcal{O}^{sh}_{X,x}$,
(8) for $x \in X$ the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$.

If L/K is Galois with Galois group G, then these are also equivalent to

(9) for $y \in Y$ the group $I_y = \{g \in G \mid g(y) = y \text{ and } g \bmod \mathfrak{m}_y = \text{id}_{\kappa(y)}\}$ is trivial.

Proof. The equivalence of (1) and (2) is the definition of (1). The equivalence of (2), (3), and (4) is Lemma 58.11.1. It is straightforward to prove that (4) ⇒ (5), (5) ⇒ (6), (6) ⇒ (7).

Assume (7). Observe that $\mathcal{O}^{sh}_{X,x}$ is a normal local domain (More on Algebra, Lemma 15.45.6). Let $L^{sh} = L \otimes_K K_x^{sh}$ where K_x^{sh} is the fraction field of $\mathcal{O}^{sh}_{X,x}$. Then $L^{sh} = \prod_{i=1,\ldots,n} L_i$ with L_i/K_x^{sh} finite separable. By Algebra, Lemma 10.147.4

(and a limit argument we omit) we see that $Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ is the integral closure of $\mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ in L^{sh}. Hence by Lemma 58.11.1 (applied to the factors L_i of L^{sh}) we see that $Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^{sh}) \to \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ is finite étale. Looking at the generic point we see that the degree is equal to $[L : K]$ and hence we see that (8) is true.

Assume (8). Assume that $x \in X$ and that the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$. Observe that this means that Y has $\geq [L : K]$ geometric points lying over x. We will show that $Y \to X$ is finite étale over a neighbourhood of x. This will prove (1) holds. To prove this we may assume $X = \mathrm{Spec}(R)$, the point x corresponds to the prime $\mathfrak{p} \subset R$, and $Y = \mathrm{Spec}(S)$. We apply More on Morphisms, Lemma 37.42.1 and we find an étale neighbourhood $(U, u) \to (X, x)$ such that $Y \times_X U = V_1 \amalg \ldots \amalg V_m$ such that V_i has a unique point v_i lying over u with $\kappa(v_i)/\kappa(u)$ purely inseparable. Shrinking U if necessary we may assume U is a normal integral scheme with generic point ξ (use Descent, Lemmas 35.16.3 and 35.18.2 and Properties, Lemma 28.7.5). By our remark on geometric points we see that $m \geq [L : K]$. On the other hand, by More on Morphisms, Lemma 37.19.2 we see that $\coprod V_i \to U$ is the normalization of U in $\mathrm{Spec}(L) \times_X U$. As $K \subset \kappa(\xi)$ is finite separable, we can write $\mathrm{Spec}(L) \times_X U = \mathrm{Spec}(\prod_{i=1,\ldots,n} L_i)$ with $L_i/\kappa(\xi)$ finite and $[L : K] = \sum [L_i : \kappa(\xi)]$. Since V_j is nonempty for each j and $m \geq [L : K]$ we conclude that $m = n$ and $[L_i : \kappa(\xi)] = 1$ for all i. Then $V_j \to U$ is an isomorphism in particular étale, hence $Y \times_X U \to U$ is étale. By Descent, Lemma 35.23.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x).

Assume L/K is Galois and (9) holds. Then $Y \to X$ is étale by Lemma 58.12.5. We omit the proof that (1) implies (9). □

In the case of infinite Galois extensions of discrete valuation rings we can say a tiny bit more. To do so we introduce the following notation. A subset $S \subset \mathbf{N}$ of integers is multiplicativity directed if $1 \in S$ and for $n, m \in S$ there exists $k \in S$ with $n|k$ and $m|k$. Define a partial ordering on S by the rule $n \geq_S m$ if and only if $m|n$. Given a field κ we obtain an inverse system of finite groups $\{\mu_n(\kappa)\}_{n \in S}$ with transition maps

$$\mu_n(\kappa) \longrightarrow \mu_m(\kappa), \quad \zeta \longmapsto \zeta^{n/m}$$

for $n \geq_S m$. Then we can form the profinite group

$$\lim_{n \in S} \mu_n(\kappa)$$

Observe that the limit is cofiltered (as S is directed). The construction is functorial in κ. In particular $\mathrm{Aut}(\kappa)$ acts on this profinite group. For example, if $S = \{1, n\}$, then this gives $\mu_n(\kappa)$. If $S = \{1, \ell, \ell^2, \ell^3, \ldots\}$ for some prime ℓ different from the characteristic of κ this produces $\lim_n \mu_{\ell^n}(\kappa)$ which is sometimes called the ℓ-adic Tate module of the multiplicative group of κ (compare with More on Algebra, Example 15.93.5).

0BUA Lemma 58.13.5. Let A be a discrete valuation ring with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let B be the integral closure of A in L. Let $\mathfrak{m}$ be a maximal ideal of B. Let $G = \mathrm{Gal}(L/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m} = \mathrm{id}_{\kappa(\mathfrak{m})}\}$. The decomposition group D fits into a canonical exact sequence

$$1 \to I \to D \to \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A) \to 1$$

The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

(1) P is a normal subgroup of D,
(2) P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,
(3) there is a multiplicatively directed $S \subset \mathbf{N}$ such that $\kappa(\mathfrak{m})$ contains a primitive nth root of unity for each $n \in S$ (elements of S are prime to p),
(4) there exists a canonical surjective map

$$\theta_{can} : I \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$.

Proof. This is mostly a reformulation of the results on finite Galois extensions proved in More on Algebra, Section 15.112. The surjectivity of the map $D \to \text{Aut}(\kappa(\mathfrak{m})/\kappa)$ is More on Algebra, Lemma 15.110.10. This gives the first exact sequence.

To construct the second short exact sequence let Λ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$. Set $G_\lambda = \text{Gal}(L_\lambda/K)$. Recall that G_λ is an inverse system of finite groups with surjective transition maps and that $G = \lim_{\lambda \in \Lambda} G_\lambda$, see Fields, Lemma 9.22.3. We let B_λ be the integral closure of A in L_λ. Then we set $\mathfrak{m}_\lambda = \mathfrak{m} \cap B_\lambda$ and we denote $P_\lambda, I_\lambda, D_\lambda$ the wild inertia, inertia, and decomposition group of $\mathfrak{m}_\lambda$, see More on Algebra, Lemma 15.112.5. For $\lambda \geq \lambda'$ the restriction defines a commutative diagram

$$\begin{tikzcd}
P_\lambda \arrow[r] \arrow[d] & I_\lambda \arrow[r] \arrow[d] & D_\lambda \arrow[r] \arrow[d] & G_\lambda \arrow[d] \\
P_{\lambda'} \arrow[r] & I_{\lambda'} \arrow[r] & D_{\lambda'} \arrow[r] & G_{\lambda'}
\end{tikzcd}$$

with surjective vertical maps, see More on Algebra, Lemma 15.112.10.

From the definitions it follows immediately that $I = \lim I_\lambda$ and $D = \lim D_\lambda$ under the isomorphism $G = \lim G_\lambda$ above. Since $L = \text{colim}\, L_\lambda$ we have $B = \text{colim}\, B_\lambda$ and $\kappa(\mathfrak{m}) = \text{colim}\, \kappa(\mathfrak{m}_\lambda)$. Since the transition maps of the system D_λ are compatible with the maps $D_\lambda \to \text{Aut}(\kappa(\mathfrak{m}_\lambda)/\kappa)$ (see More on Algebra, Lemma 15.112.10) we see that the map $D \to \text{Aut}(\kappa(\mathfrak{m})/\kappa)$ is the limit of the maps $D_\lambda \to \text{Aut}(\kappa(\mathfrak{m}_\lambda)/\kappa)$.

There exist canonical maps

$$\theta_{\lambda,can} : I_\lambda \longrightarrow \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda))$$

where $n_\lambda = |I_\lambda|/|P_\lambda|$, where $\mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda))$ has order n_λ, such that $\theta_{\lambda,can}(\tau\sigma\tau^{-1}) = \tau(\theta_{\lambda,can}(\sigma))$ for $\tau \in D_\lambda$ and $\sigma \in I_\lambda$, and such that we get commutative diagrams

$$\begin{tikzcd}
I_\lambda \arrow[r, "\theta_{\lambda,can}"'] \arrow[d] & \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda)) \arrow[d, "(-)^{n_\lambda/n_{\lambda'}}"] \\
I_{\lambda'} \arrow[r, "\theta_{\lambda',can}"] & \mu_{n_{\lambda'}}(\kappa(\mathfrak{m}_{\lambda'}))
\end{tikzcd}$$

see More on Algebra, Remark 15.112.11.

Let $S \subset \mathbf{N}$ be the collection of integers n_λ. Since Λ is directed, we see that S is multiplicatively directed. By the displayed commutative diagrams above we can take the limits of the maps $\theta_{\lambda,can}$ to obtain

$$\theta_{can} : I \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m})).$$

This map is continuous (small detail omitted). Since the transition maps of the system of I_λ are surjective and Λ is directed, the projections $I \to I_\lambda$ are surjective. For every λ the diagram

$$\begin{tikzcd} I \arrow[r, "\theta_{can}"'] \arrow[d] & \lim_{n \in S} \mu_n(\kappa(\mathfrak{m})) \arrow[d] \\ I_\lambda \arrow[r, "\theta_{\lambda,can}"] & \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda)) \end{tikzcd}$$

commutes. Hence the image of θ_{can} surjects onto the finite group $\mu_{n_\lambda}(\kappa(\mathfrak{m})) = \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda))$ of order n_λ (see above). It follows that the image of θ_{can} is dense. On the other hand θ_{can} is continuous and the source is a profinite group. Hence θ_{can} is surjective by a topological argument.

The property $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$ follows from the corresponding properties of the maps $\theta_{\lambda,can}$ and the compatibility of the map $D \to \text{Aut}(\kappa(\mathfrak{m}))$ with the maps $D_\lambda \to \text{Aut}(\kappa(\mathfrak{m}_\lambda))$. Setting $P = \text{Ker}(\theta_{can})$ this implies that P is a normal subgroup of D. Setting $I_t = I/P$ we obtain the isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$ from the surjectivity of θ_{can}.

To finish the proof we show that $P = \lim P_\lambda$ which proves that P is a pro-p-group. Recall that the tame inertia group $I_{\lambda,t} = I_\lambda/P_\lambda$ has order n_λ. Since the transition maps $P_\lambda \to P_{\lambda'}$ are surjective and Λ is directed, we obtain a short exact sequence

$$1 \to \lim P_\lambda \to I \to \lim I_{\lambda,t} \to 1$$

(details omitted). Since for each λ the map $\theta_{\lambda,can}$ induces an isomorphism $I_{\lambda,t} \cong \mu_{n_\lambda}(\kappa(\mathfrak{m}))$ the desired result follows. $\square$

0BUB Lemma 58.13.6. Let A be a discrete valuation ring with fraction field K. Let K^{sep} be a separable closure of K. Let A^{sep} be the integral closure of A in K^{sep}. Let $\mathfrak{m}^{sep}$ be a maximal ideal of A^{sep}. Let $\mathfrak{m} = \mathfrak{m}^{sep} \cap A$, let $\kappa = A/\mathfrak{m}$, and let $\overline{\kappa} = A^{sep}/\mathfrak{m}^{sep}$. Then $\overline{\kappa}$ is an algebraic closure of κ. Let $G = \text{Gal}(K^{sep}/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = \text{id}_{\kappa(\mathfrak{m}^{sep})}\}$. The decomposition group D fits into a canonical exact sequence

$$1 \to I \to D \to \text{Gal}(\kappa^{sep}/\kappa) \to 1$$

where $\kappa^{sep} \subset \overline{\kappa}$ is the separable closure of κ. The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

(1) P is a normal subgroup of D,
(2) P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,

(3) there exists a canonical surjective map

$$\theta_{can} : I \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$.

Proof. The field $\overline{\kappa}$ is the algebraic closure of κ by Lemma 58.12.1. Most of the statements immediately follow from the corresponding parts of Lemma 58.13.5. For example because $\text{Aut}(\overline{\kappa}/\kappa) = \text{Gal}(\kappa^{sep}/\kappa)$ we obtain the first sequence. Then the only other assertion that needs a proof is the fact that with S as in Lemma 58.13.5 the limit $\lim_{n \in S} \mu_n(\overline{\kappa})$ is equal to $\lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$. To see this it suffices to show that every integer n prime to p divides an element of S. Let $\pi \in A$ be a uniformizer and consider the splitting field L of the polynomial $X^n - \pi$. Since the polynomial is separable we see that L is a finite Galois extension of K. Choose an embedding $L \to K^{sep}$. Observe that if B is the integral closure of A in L, then the ramification index of $A \to B_{\mathfrak{m}^{sep} \cap B}$ is divisible by n (because π has an nth root in B; in fact the ramification index equals n but we do not need this). Then it follows from the construction of the S in the proof of Lemma 58.13.5 that n divides an element of S. □

58.14. Geometric and arithmetic fundamental groups

0BTU In this section we work out what happens when comparing the fundamental group of a scheme X over a field k with the fundamental group of $X_{\overline{k}}$ where $\overline{k}$ is the algebraic closure of k.

0BTV **Lemma 58.14.1.** Let I be a directed set. Let X_i be an inverse system of quasi-compact and quasi-separated schemes over I with affine transition morphisms. Let $X = \lim X_i$ as in Limits, Section 32.2. Then there is an equivalence of categories

$$\text{colim}\, \text{FÉt}_{X_i} = \text{FÉt}_X$$

If X_i is connected for all sufficiently large i and $\overline{x}$ is a geometric point of X, then

$$\pi_1(X, \overline{x}) = \lim \pi_1(X_i, \overline{x})$$

Proof. The equivalence of categories follows from Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. The second statement is formal given the statement on categories. □

0BTW **Lemma 58.14.2.** Let k be a field with perfection k^{perf}. Let X be a connected scheme over k. Then $X_{k^{perf}}$ is connected and $\pi_1(X_{k^{perf}}) \to \pi_1(X)$ is an isomorphism.

Proof. Special case of topological invariance of the fundamental group. See Proposition 58.8.4. To see that $\text{Spec}(k^{perf}) \to \text{Spec}(k)$ is a universal homeomorphism you can use Algebra, Lemma 10.46.10. □

0BTX **Lemma 58.14.3.** Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-compact and quasi-separated scheme over k. If the base change $X_{\overline{k}}$ is connected, then there is a short exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\text{Spec}(k)) \to 1$$

of profinite topological groups.

Proof. Connected objects of $\text{FÉt}_{\text{Spec}(k)}$ are of the form $\text{Spec}(k') \to \text{Spec}(k)$ with k'/k a finite separable extension. Then $X_{\text{Spec}\, k'}$ is connected, as the morphism $X_{\overline{k}} \to X_{\text{Spec}(k')}$ is surjective and $X_{\overline{k}}$ is connected by assumption. Thus $\pi_1(X) \to \pi_1(\text{Spec}(k))$ is surjective by Lemma 58.4.1.

Before we go on, note that we may assume that k is a perfect field. Namely, we have $\pi_1(X_{k^{perf}}) = \pi_1(X)$ and $\pi_1(\text{Spec}(k^{perf})) = \pi_1(\text{Spec}(k))$ by Lemma 58.14.2.

It is clear that the composition of the functors $\text{FÉt}_{\text{Spec}(k)} \to \text{FÉt}_X \to \text{FÉt}_{X_{\overline{k}}}$ sends objects to disjoint unions of copies of $X_{\text{Spec}(\overline{k})}$. Therefore the composition $\pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\text{Spec}(k))$ is the trivial homomorphism by Lemma 58.4.2.

Let $U \to X$ be a finite étale morphism with U connected. Observe that $U \times_X X_{\overline{k}} = U_{\overline{k}}$. Suppose that $U_{\overline{k}} \to X_{\overline{k}}$ has a section $s : X_{\overline{k}} \to U_{\overline{k}}$. Then $s(X_{\overline{k}})$ is an open connected component of $U_{\overline{k}}$. For $\sigma \in \text{Gal}(\overline{k}/k)$ denote s^σ the base change of s by $\text{Spec}(\sigma)$. Since $U_{\overline{k}} \to X_{\overline{k}}$ is finite étale it has only a finite number of sections. Thus

$$\overline{T} = \bigcup s^\sigma(X_{\overline{k}})$$

is a finite union and we see that $\overline{T}$ is a $\text{Gal}(\overline{k}/k)$-stable open and closed subset. By Varieties, Lemma 33.7.10 we see that $\overline{T}$ is the inverse image of a closed subset $T \subset U$. Since $U_{\overline{k}} \to U$ is open (Morphisms, Lemma 29.23.4) we conclude that T is open as well. As U is connected we see that $T = U$. Hence $U_{\overline{k}}$ is a (finite) disjoint union of copies of $X_{\overline{k}}$. By Lemma 58.4.5 we conclude that the image of $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is normal.

Let $V \to X_{\overline{k}}$ be a finite étale cover. Recall that $\overline{k}$ is the union of finite separable extensions of k. By Lemma 58.14.1 we find a finite separable extension k'/k and a finite étale morphism $U \to X_{k'}$ such that $V = X_{\overline{k}} \times_{X_{k'}} U = U \times_{\text{Spec}(k')} \text{Spec}(\overline{k})$. Then the composition $U \to X_{k'} \to X$ is finite étale and $U \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$ contains $V = U \times_{\text{Spec}(k')} \text{Spec}(\overline{k})$ as an open and closed subscheme. (Because $\text{Spec}(\overline{k})$ is an open and closed subscheme of $\text{Spec}(k') \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$ via the multiplication map $k' \otimes_k \overline{k} \to \overline{k}$.) By Lemma 58.4.4 we conclude that $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is injective.

Finally, we have to show that for any finite étale morphism $U \to X$ such that $U_{\overline{k}}$ is a disjoint union of copies of $X_{\overline{k}}$ there is a finite étale morphism $V \to \text{Spec}(k)$ and a surjection $V \times_{\text{Spec}(k)} X \to U$. See Lemma 58.4.3. Arguing as above using Lemma 58.14.1 we find a finite separable extension k'/k such that there is an isomorphism $U_{k'} \cong \coprod_{i=1,\ldots,n} X_{k'}$. Thus setting $V = \coprod_{i=1,\ldots,n} \text{Spec}(k')$ we conclude. □

58.15. Homotopy exact sequence

0BUM In this section we discuss the following result. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups. See Proposition 58.15.2.

0BUN Lemma 58.15.1. Let $f : X \to S$ be a proper morphism of schemes. Let $X \to S' \to S$ be the Stein factorization of f, see More on Morphisms, Theorem 37.53.5. If f is of finite presentation, flat, with geometrically reduced fibres, then $S' \to S$ is finite étale. [Gro71, Expose X, Proposition 1.2, p. 262].

Proof. This follows from Derived Categories of Schemes, Lemma 36.32.8 and the information contained in More on Morphisms, Theorem 37.53.5. ☐

0C0J Proposition 58.15.2. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Proof. Let $Y \to X$ be a finite étale morphism. Consider the Stein factorization

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

of $Y \to S$. By Lemma 58.15.1 the morphism $T \to S$ is finite étale. In this way we obtain a functor $\text{FÉt}_X \to \text{FÉt}_S$. For any finite étale morphism $U \to S$ a morphism $Y \to U \times_S X$ over X is the same thing as a morphism $Y \to U$ over S and such a morphism factors uniquely through the Stein factorization, i.e., corresponds to a unique morphism $T \to U$ (by the construction of the Stein factorization as a relative normalization in More on Morphisms, Lemma 37.53.1 and factorization by Morphisms, Lemma 29.53.4). Thus we see that the functors $\text{FÉt}_X \to \text{FÉt}_S$ and $\text{FÉt}_S \to \text{FÉt}_X$ are adjoints. Note that the Stein factorization of $U \times_S X \to S$ is U, because the fibres of $U \times_S X \to U$ are geometrically connected.

By the discussion above and Categories, Lemma 4.24.4 we conclude that $\text{FÉt}_S \to \text{FÉt}_X$ is fully faithful, i.e., $\pi_1(X) \to \pi_1(S)$ is surjective (Lemma 58.4.1).

It is immediate that the composition $\text{FÉt}_S \to \text{FÉt}_X \to \text{FÉt}_{X_{\overline{s}}}$ sends any U to a disjoint union of copies of $X_{\overline{s}}$. Hence $\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S)$ is trivial by Lemma 58.4.2.

Let $Y \to X$ be a finite étale morphism with Y connected such that $Y \times_X X_{\overline{s}}$ contains a connected component Z isomorphic to $X_{\overline{s}}$. Consider the Stein factorization T as above. Let $\overline{t} \in T_{\overline{s}}$ be the point corresponding to the fibre Z. Observe that T is connected (as the image of a connected scheme) and by the surjectivity above $T \times_S X$ is connected. Now consider the factorization

$$\pi : Y \longrightarrow T \times_S X$$

Let $\overline{x} \in X_{\overline{s}}$ be any closed point. Note that $\kappa(\overline{t}) = \kappa(\overline{s}) = \kappa(\overline{x})$ is an algebraically closed field. Then the fibre of π over $(\overline{t}, \overline{x})$ consists of a unique point, namely the unique point $\overline{z} \in Z$ corresponding to $\overline{x} \in X_{\overline{s}}$ via the isomorphism $Z \to X_{\overline{s}}$. We conclude that the finite étale morphism π has degree 1 in a neighbourhood of $(\overline{t}, \overline{x})$. Since $T \times_S X$ is connected it has degree 1 everywhere and we find that $Y \cong T \times_S X$. Thus $Y \times_X X_{\overline{s}}$ splits completely. Combining all of the above we see that Lemmas 58.4.3 and 58.4.5 both apply and the proof is complete. ☐

58.16. Specialization maps

0BUP In this section we construct specialization maps. Let $f : X \to S$ be a proper morphism of schemes with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points in S. Let $\overline{s}$ and $\overline{s}'$ be geometric points lying over s and s'. Then there is a specialization map

$$sp : \pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_{\overline{s}})$$

The construction of this map is as follows. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$, see Algebra, Definition 10.155.3. Since $s' \rightsquigarrow s$ the point s' corresponds to a point of $\mathrm{Spec}(\mathcal{O}_{S,s})$ and hence there is at least one point (and potentially many points) of $\mathrm{Spec}(A)$ over s' whose residue field is a separable algebraic extension of $\kappa(s')$. Since $\kappa(\overline{s}')$ is algebraically closed we can choose a morphism $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ giving rise to a commutative diagram

$$\begin{tikzcd} \overline{s}' \arrow[r, "\varphi"'] \arrow[dr] & \mathrm{Spec}(A) \arrow[d] & \overline{s} \arrow[l] \arrow[dl] \\ & S & \end{tikzcd}$$

The specialization map is the composition

$$\pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_A) = \pi_1(X_{\kappa(s)^{sep}}) = \pi_1(X_{\overline{s}})$$

where the first equality is Lemma 58.9.1 and the second follows from Lemmas 58.14.2 and 58.9.3. By construction the specialization map fits into a commutative diagram

$$\begin{tikzcd} \pi_1(X_{\overline{s}'}) \arrow[rr, "sp"'] \arrow[dr] & & \pi_1(X_{\overline{s}}) \arrow[dl] \\ & \pi_1(X) & \end{tikzcd}$$

provided that X is connected. The specialization map depends on the choice of $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ above and we will write sp_φ if we want to indicate this.

0C0K **Lemma 58.16.1.** *Consider a commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

of schemes where f and g are proper with geometrically connected fibres. Let $t' \rightsquigarrow t$ be a specialization of points in T and consider a specialization map $sp : \pi_1(Y_{\overline{t}'}) \to \pi_1(Y_{\overline{t}})$ as above. Then there is a commutative diagram

$$\begin{tikzcd} \pi_1(Y_{\overline{t}'}) \arrow[r, "sp"'] \arrow[d] & \pi_1(Y_{\overline{t}}) \arrow[d] \\ \pi_1(X_{\overline{s}'}) \arrow[r, "sp"] & \pi_1(X_{\overline{s}}) \end{tikzcd}$$

of specialization maps where $\overline{s}$ and $\overline{s}'$ are the images of $\overline{t}$ and $\overline{t}'$.

Proof. Let B be the strict henselization of $\mathcal{O}_{T,t}$ with respect to $\kappa(t) \subset \kappa(t)^{sep} \subset \kappa(\overline{t})$. Pick $\psi : \overline{t}' \to \mathrm{Spec}(B)$ lifting $\overline{t}' \to T$ as in the construction of the specialization map. Let s and s' denote the images of t and t' in S. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$. Since $\kappa(\overline{s}) = \kappa(\overline{t})$, by the functoriality of strict henselization (Algebra, Lemma 10.155.10) we obtain a ring map $A \to B$ fitting into the commutative diagram

$$\begin{tikzcd} \overline{t}' \arrow[r, "\psi"'] \arrow[d] & \mathrm{Spec}(B) \arrow[r] \arrow[d] & T \arrow[d] \\ \overline{s}' \arrow[r, "\varphi"] & \mathrm{Spec}(A) \arrow[r] & S \end{tikzcd}$$

Here the morphism $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ is simply taken to be the composition $\overline{t}' \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Applying base change we obtain a commutative diagram

$$\begin{tikzcd} Y_{\overline{t}'} \arrow[r] \arrow[d] & Y_B \arrow[d] \\ X_{\overline{s}'} \arrow[r] & X_A \end{tikzcd}$$

and from the construction of the specialization map the commutativity of this diagram implies the commutativity of the diagram of the lemma. ☐

0C0L Lemma 58.16.2. Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s'' \rightsquigarrow s' \rightsquigarrow s$ be specializations of points of S. A composition of specialization maps $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is a specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$.

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct the first specialization map. Let $\mathcal{O}_{S,s'} \to A'$ be the strict henselization constructed using $\kappa(s') \subset \kappa(\overline{s}')$. By functoriality of strict henselization, there is a map $A \to A'$ such that the composition with $A' \to \kappa(\overline{s}')$ is the given map (Algebra, Lemma 10.154.6). Next, let $A' \to \kappa(\overline{s}'')$ be the map used to construct the second specialization map. Then it is clear that the composition of the first and second specialization maps is the specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$ constructed using $A \to A' \to \kappa(\overline{s}'')$. ☐

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian valuation ring with algebraically closed fraction field and let $\mathrm{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \mathrm{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_\eta) \to \pi_1(X_s)$$

for the base change $X_R/\mathrm{Spec}(R)$. Note that this makes sense as both η and s have algebraically closed residue fields.

0C0M Lemma 58.16.3. Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. Then there exists a strictly henselian valuation ring R over S with algebraically closed fraction field such that sp is isomorphic to sp_R defined above.

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a valuation ring with fraction field $\kappa(\overline{s}')$ dominating the image of A. See Algebra, Lemma 10.50.2. Observe that R is strictly henselian for example by Lemma 58.12.2 and Algebra, Lemma 10.50.3. Then the lemma is clear. □

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian discrete valuation ring and let $\mathrm{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \mathrm{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_{\overline{\eta}}) \to \pi_1(X_s)$$

for the base change $X_R/\mathrm{Spec}(R)$. Note that this makes sense as s has algebraically closed residue field.

0C0N **Lemma 58.16.4.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \leadsto s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. If S is Noetherian, then there exists a strictly henselian discrete valuation ring R over S such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a discrete valuation ring dominating the image of A, see Algebra, Lemma 10.119.13. Choose a diagram of fields

$$\begin{tikzcd} \kappa(\overline{s}) \arrow[r] & k \\ A/\mathfrak{m}_A \arrow[u] \arrow[r] & R/\mathfrak{m}_R \arrow[u] \end{tikzcd}$$

with k algebraically closed. Let R^{sh} be the strict henselization of R constructed using $R \to k$. Then R^{sh} is a discrete valuation ring by More on Algebra, Lemma 15.45.11. Denote η, o the generic and closed point of $\mathrm{Spec}(R^{sh})$. Since the diagram of schemes

$$\begin{tikzcd} \overline{\eta} \arrow[r] \arrow[d] & \mathrm{Spec}(R^{sh}) \arrow[d] & \mathrm{Spec}(k) \arrow[l] \arrow[d] \\ \overline{s}' \arrow[r] & \mathrm{Spec}(A) & \overline{s} \arrow[l] \end{tikzcd}$$

commutes, we obtain a commutative diagram

$$\begin{tikzcd} \pi_1(X_{\overline{\eta}}) \arrow[r, "sp_{R^{sh}}"'] \arrow[d] & \pi_1(X_o) \arrow[d] \\ \pi_1(X_{\overline{s}'}) \arrow[r, "sp"] & X_{\overline{s}} \end{tikzcd}$$

of specialization maps by the construction of these maps. Since the vertical arrows are isomorphisms (Lemma 58.9.3), this proves the lemma. □

58.17. Restriction to a closed subscheme

0EJW In this section we prove some results about the restriction functor

$$\text{FÉt}_X \longrightarrow \text{FÉt}_Y, \quad U \longmapsto V = U \times_X Y$$

where X is a scheme and Y is a closed subscheme. Using the topological invariance of the fundamental group, we can relate the study of this functor to the completion functor on finite locally free modules.

In the following lemmas we use the concept of coherent formal modules defined in Cohomology of Schemes, Section 30.23. Given a Noetherian scheme and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we will say an object $(\mathcal{F}_n)$ of $\text{Coh}(X, \mathcal{I})$ is finite locally free if each $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_X/\mathcal{I}^n$-module.

0EL8 **Lemma 58.17.1.** Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Assume the completion functor

$$\text{Coh}(\mathcal{O}_X) \longrightarrow \text{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $\text{FÉt}_X \to \text{FÉt}_Y$ is fully faithful.

Proof. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given U finite étale over X and a morphism $t : Y \to U$ over X there exists a unique section $s : X \to U$ such that $t = s|_Y$. Picture

$$\begin{tikzcd} & U \arrow[d, "f"] \\ Y \arrow[r] \arrow[ur] & X \arrow[u, dotted, bend left] \end{tikzcd}$$

Finding the dotted arrow s is the same thing as finding an $\mathcal{O}_X$-algebra map

$$s^\sharp : f_*\mathcal{O}_U \longrightarrow \mathcal{O}_X$$

which reduces modulo the ideal sheaf of Y to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 58.8.3 we can lift t uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \longrightarrow \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on X. Since $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_X$-module, we conclude that we get a unique $\mathcal{O}_X$-module map $\sigma : f_*\mathcal{O}_U \to \mathcal{O}_X$ whose completion is $\lim t_n^\sharp$. To see that σ is an algebra homomorphism, we need to check that the diagram

$$\begin{tikzcd} f_*\mathcal{O}_U \otimes_{\mathcal{O}_X} f_*\mathcal{O}_U \arrow[r] \arrow[d, "\sigma\otimes\sigma"'] & f_*\mathcal{O}_U \arrow[d, "\sigma"] \\ \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we conclude. $\square$

0EL9 **Lemma 58.17.2.** Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Assume the completion functor

$$\textit{Coh}(\mathcal{O}_X) \longrightarrow \textit{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is an equivalence on full subcategories of finite locally free objects (see above). Then the restriction functor $\textit{FÉt}_X \to \textit{FÉt}_Y$ is an equivalence.

Proof. The restriction functor is fully faithful by Lemma 58.17.1.

Let $U_1 \to Y$ be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For $n \geq 1$ let Y_n be the nth infinitesimal neighbourhood of Y. By Lemma 58.8.3 there is a unique finite étale morphism $\pi_n : U_n \to Y_n$ whose base change to $Y = Y_1$ recovers $U_1 \to Y_1$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$. We may and do view $\mathcal{F}_n$ as an $\mathcal{O}_X$-module on X wich is locally isomorphic to $(\mathcal{O}_X/f^{n+1}\mathcal{O}_X)^{\oplus r}$. This $(\mathcal{F}_n)$ is a finite locally free object of $\textit{Coh}(X, \mathcal{I})$. By assumption there exists a finite locally free $\mathcal{O}_X$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_X$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_X} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F})^\wedge \longrightarrow \mathcal{F}^\wedge$$

in the category $\textit{Coh}(X, \mathcal{I})$. Hence by assumption we may assume there is a map $\mu : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ whose restriction to Y_n gives the multiplication maps above. By faithfulness of the functor in the statement of the lemma, we conclude that μ defines a commutative $\mathcal{O}_X$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$U = \underline{\mathrm{Spec}}_X((\mathcal{F}, \mu))$$

we obtain a finite locally free scheme $\pi : U \to X$ whose restriction to Y is isomorphic to U_1. The the discriminant of π is the zero set of the section

$$\det(Q_\pi) : \mathcal{O}_X \longrightarrow \wedge^{top}(\pi_*\mathcal{O}_U)^{\otimes -2}$$

constructed in Discriminants, Section 49.3. Since the restriction of this to Y_n is an isomorphism for all n by Discriminants, Lemma 49.3.1 we conclude that it is an isomorphism. Thus π is étale by Discriminants, Lemma 49.3.1. $\square$

0ELA **Lemma 58.17.3.** Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume the completion functor

$$\mathrm{colim}_\mathcal{V} \textit{Coh}(\mathcal{O}_V) \longrightarrow \textit{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

defines is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $\mathrm{colim}_\mathcal{V} \textit{FÉt}_V \to \textit{FÉt}_Y$ is fully faithful.

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 4.19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 58.17.1; we urge the reader to skip it.

Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given U finite étale over $V \in \mathcal{V}$ and a morphism $t : Y \to U$ over V there exists a $V' \geq V$ and a morphism $s : V' \to U$ over V such that $t = s|_Y$. Picture

$$\begin{tikzcd} & & U \arrow[d, "f"] \\ Y \arrow[r] \arrow[rru] & V' \arrow[r] \arrow[ru, dotted] & V \end{tikzcd}$$

Finding the dotted arrow s is the same thing as finding an $\mathcal{O}_{V'}$-algebra map

$$s^\sharp : f_*\mathcal{O}_U|_{V'} \longrightarrow \mathcal{O}_{V'}$$

which reduces modulo the ideal sheaf of Y to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 58.8.3 we can lift t uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \longrightarrow \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on V. Observe that $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_V$-module. Hence we get a $V' \geq V$ a map $\sigma : f_*\mathcal{O}_U|_{V'} \to \mathcal{O}_{V'}$ whose completion is $\lim t_n^\sharp$. To see that σ is an algebra homomorphism, we need to check that the diagram

$$\begin{tikzcd} (f_*\mathcal{O}_U \otimes_{\mathcal{O}_V} f_*\mathcal{O}_U)|_{V'} \arrow[r] \arrow[d, "\sigma \otimes \sigma"'] & f_*\mathcal{O}_U|_{V'} \arrow[d, "\sigma"] \\ \mathcal{O}_{V'} \otimes_{\mathcal{O}_{V'}} \mathcal{O}_{V'} \arrow[r] & \mathcal{O}_{V'} \end{tikzcd}$$

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we deduce that there exists a $V'' \geq V'$ such that $\sigma|_{V''}$ is an algebra homomorphism as desired. □

0EK1 Lemma 58.17.4. Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume the completion functor

$$\mathrm{colim}_{\mathcal{V}} \, \mathrm{Coh}(\mathcal{O}_V) \longrightarrow \mathrm{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

defines an equivalence of the full subcategories of finite locally free objects (see explanation above). Then the restriction functor

$$\mathrm{colim}_{\mathcal{V}} \, \mathrm{FÉt}_V \to \mathrm{FÉt}_Y$$

is an equivalence.

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 4.19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 58.17.2; we urge the reader to skip it.

The restriction functor is fully faithful by Lemma 58.17.3.

Let $U_1 \to Y$ be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For $n \geq 1$ let Y_n be the nth infinitesimal neighbourhood of Y. By Lemma 58.8.3 there is a unique finite étale morphism $\pi_n : U_n \to Y_n$ whose base change to $Y = Y_1$ recovers $U_1 \to Y_1$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$. We may and do view $\mathcal{F}_n$

as an $\mathcal{O}_X$-module on X wich is locally isomorphic to $(\mathcal{O}_X/f^{n+1}\mathcal{O}_X)^{\oplus r}$. This $(\mathcal{F}_n)$ is a finite locally free object of $\textit{Coh}(X, \mathcal{I})$. By assumption there exists a $V \in \mathcal{V}$ and a finite locally free $\mathcal{O}_V$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_V$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_V} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{F})^\wedge \longrightarrow \mathcal{F}^\wedge$$

in the category $\textit{Coh}(X, \mathcal{I})$. Hence by assumption after shrinking V we may assume there is a map $\mu : \mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{F} \to \mathcal{F}$ whose restriction to Y_n gives the multiplication maps above. After possibly shrinking further we may assume μ defines a commutative $\mathcal{O}_V$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$U = \underline{\mathrm{Spec}}_V((\mathcal{F}, \mu))$$

we obtain a finite locally free scheme over V whose restriction to Y is isomorphic to U_1. It follows that $U \to V$ is étale at all points lying over Y, see More on Morphisms, Lemma 37.12.3. Thus after shrinking V once more we may assume $U \to V$ is finite étale. This finishes the proof. $\square$

0EJX Lemma 58.17.5. Let X be a scheme and let $Y \subset X$ be a closed subscheme. If every connected component of X meets Y, then the restriction functor $\textit{FÉt}_X \to \textit{FÉt}_Y$ is faithful.

Proof. Let $a, b : U \to U'$ be two morphisms of schemes finite étale over X whose restriction to Y are the same. The image of a connected component of U is an connected component of X; this follows from Topology, Lemma 5.7.7 applied to the restriction of $U \to X$ to a connected component of X. Hence the image of every connected component of U meets Y by assumption. We conclude that $a = b$ after restriction to each connected component of U by Étale Morphisms, Proposition 41.6.3. Since the equalizer of a and b is an open subscheme of U (as the diagonal of U' over X is open) we conclude. $\square$

0EJZ Lemma 58.17.6. Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Assume one of the following holds

1. X is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism, or
2. X has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^{\otimes m}) \to \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$ is an isomorphism for all $m \gg 0$, or
3. for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism.

Then the restriction functor $\textit{FÉt}_X \to \textit{FÉt}_Y$ is fully faithful.

Proof. This lemma follows formally from Lemma 58.17.1 and Algebraic and Formal Geometry, Lemma 52.15.1. $\square$

0EK0 Lemma 58.17.7. Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Let $\mathcal{V}$

be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume one of the following holds

(1) X is quasi-affine and
$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$$
is an isomorphism, or
(2) X has an ample invertible module $\mathcal{L}$ and
$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$
is an isomorphism for all $m \gg 0$, or
(3) for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map
$$\mathrm{colim}_{V' \geq V} \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism.

Then the functor
$$\mathrm{colim}_{\mathcal{V}} \mathrm{FÉt}_V \to \mathrm{FÉt}_Y$$
is fully faithful.

Proof. This lemma follows formally from Lemma 58.17.3 and Algebraic and Formal Geometry, Lemma 52.15.2. □

58.18. Pushouts and fundamental groups

0EK3 Here is the main result.

0EK4 Lemma 58.18.1. In More on Morphisms, Situation 37.67.1, for example if $Z \to Y$ and $Z \to X$ are closed immersions of schemes, there is an equivalence of categories
$$\mathrm{FÉt}_{Y \amalg_Z X} \longrightarrow \mathrm{FÉt}_Y \times_{\mathrm{FÉt}_Z} \mathrm{FÉt}_X$$

Proof. The pushout exists by More on Morphisms, Proposition 37.67.3. The functor is given by sending a scheme U finite étale over the pushout to the base changes $Y' = U \times_{Y \amalg_Z X} Y$ and $X' = U \times_{Y \amalg_Z X} X$ and the natural isomorphism $Y' \times_Y Z \to X' \times_X Z$ over Z. To prove this functor is an equivalence we use More on Morphisms, Lemma 37.67.7 to construct a quasi-inverse functor. The only thing left to prove is to show that given a morphism $U \to Y \amalg_Z X$ which is separated, quasi-finite and étale such that $X' \to X$ and $Y' \to Y$ are finite, then $U \to Y \amalg_Z X$ is finite. This can either be deduced from the corresponding algebra fact (More on Algebra, Lemma 15.6.7) or it can be seen because
$$X' \amalg Y' \to U$$
is surjective and X' and Y' are proper over $Y \amalg_Z X$ (this uses the description of the pushout in More on Morphisms, Proposition 37.67.3) and then we can apply Morphisms, Lemma 29.41.10 to conclude that U is proper over $Y \amalg_Z X$. Since a quasi-finite and proper morphism is finite (More on Morphisms, Lemma 37.44.1) we win. □

58.19. Finite étale covers of punctured spectra, I

0BLE We first prove some results á la Lefschetz.

0BLF **Situation 58.19.1.** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA.

Recall that for a scheme X the category of schemes finite étale over X is denoted FÉt_X, see Section 58.5. In Situation 58.19.1 we will study the base change functors

$$\begin{tikzcd} \text{FÉt}_X \arrow[r] \arrow[d] & \text{FÉt}_U \arrow[d] \\ \text{FÉt}_{X_0} \arrow[r] & \text{FÉt}_{U_0} \end{tikzcd}$$

In many case the right vertical arrow is faithful.

0BLG **Lemma 58.19.2.** In Situation 58.19.1. Assume one of the following holds

(1) $\dim(A/\mathfrak{p}) \geq 2$ for every minimal prime $\mathfrak{p} \subset A$ with $f \notin \mathfrak{p}$, or
(2) every connected component of U meets U_0.

Then

$$\text{FÉt}_U \longrightarrow \text{FÉt}_{U_0}, \quad V \longmapsto V_0 = V \times_U U_0$$

is a faithful functor.

Proof. Case (2) is immediate from Lemma 58.17.5. Assumption (1) implies every irreducible component of U meets U_0, see Algebra, Lemma 10.60.13. Hence (1) follows from (2). $\square$

Before we prove something more interesting, we need a couple of lemmas.

0BLH **Lemma 58.19.3.** In Situation 58.19.1. Let $V \to U$ be a finite morphism. Let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A, let $X' = \mathrm{Spec}(A^\wedge)$ and let U' and V' be the base changes of U and V to X'. If $Y' \to X'$ is a finite morphism such that $V' = Y' \times_{X'} U'$, then there exists a finite morphism $Y \to X$ such that $V = Y \times_X U$ and $Y' = Y \times_X X'$.

Proof. This is a straightforward application of More on Algebra, Proposition 15.89.15. Namely, choose generators $f_1, \ldots, f_t$ of $\mathfrak{m}$. For each i write $V \times_U D(f_i) = \mathrm{Spec}(B_i)$. For $1 \leq i, j \leq n$ we obtain an isomorphism $\alpha_{ij} : (B_i)_{f_j} \to (B_j)_{f_i}$ of $A_{f_i f_j}$-algebras because the spectrum of both represent $V \times_U D(f_i f_j)$. Write $Y' = \mathrm{Spec}(B')$. Since $V \times_U U' = Y \times_{X'} U'$ we get isomorphisms $\alpha_i : B'_{f_i} \to B_i \otimes_A A^\wedge$. A straightforward argument shows that $(B', B_i, \alpha_i, \alpha_{ij})$ is an object of $\mathrm{Glue}(A \to A^\wedge, f_1, \ldots, f_t)$, see More on Algebra, Remark 15.89.10. Applying the proposition cited above (and using More on Algebra, Remark 15.89.19 to obtain the algebra structure) we find an A-algebra B such that $\mathrm{Can}(B)$ is isomorphic to $(B', B_i, \alpha_i, \alpha_{ij})$. Setting $Y = \mathrm{Spec}(B)$ we see that $Y \to X$ is a morphism which comes equipped with compatible isomorphisms $V \cong Y \times_X U$ and $Y' = Y \times_X X'$ as desired. $\square$

0BLI **Lemma 58.19.4.** In Situation 58.19.1 assume A is henselian or more generally that $(A, (f))$ is a henselian pair. Let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A, let $X' = \mathrm{Spec}(A^\wedge)$ and let U' and U'_0 be the base changes of U and U_0 to X'. If $\text{FÉt}_{U'} \to \text{FÉt}_{U'_0}$ is fully faithful, then $\text{FÉt}_U \to \text{FÉt}_{U_0}$ is fully faithful.

Proof. Assume $\text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U'_0}$ is a fully faithful. Since $X' \to X$ is faithfully flat, it is immediate that the functor $V \to V_0 = V \times_U U_0$ is faithful. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given V finite étale over U we have

$$\text{Mor}_U(U, V) = \text{Mor}_{U_0}(U_0, V_0)$$

The we assume we have a morphism $s_0 : U_0 \to V_0$ over U_0 and we will produce a morphism $s : U \to V$ over U.

By our assumption there does exist a morphism $s' : U' \to V'$ whose restriction to V'_0 is the base change s'_0 of s_0. Since $V' \to U'$ is finite étale this means that $V' = s'(U') \amalg W'$ for some $W' \to U'$ finite and étale. Choose a finite morphism $Z' \to X'$ such that $W' = Z' \times_{X'} U'$. This is possible by Zariski's main theorem in the form stated in More on Morphisms, Lemma 37.43.3 (small detail omitted). Then

$$V' = s'(U') \amalg W' \longrightarrow X' \amalg Z' = Y'$$

is an open immersion such that $V' = Y' \times_{X'} U'$. By Lemma 58.19.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. Write $Y = \text{Spec}(B)$ so that $Y' = \text{Spec}(B \otimes_A A^\wedge)$. Then $B \otimes_A A^\wedge$ has an idempotent e' corresponding to the open and closed subscheme X' of $Y' = X' \amalg Z'$.

The case A is henselian (slightly easier). The image $\overline{e}$ of e' in $B \otimes_A \kappa(\mathfrak m) = B/\mathfrak m B$ lifts to an idempotent e of B as A is henselian (because B is a product of local rings by Algebra, Lemma 10.153.3). Then we see that e maps to e' by uniqueness of lifts of idempotents (using that $B \otimes_A A^\wedge$ is a product of local rings). Let $Y_1 \subset Y$ be the open and closed subscheme corresponding to e. Then $Y_1 \times_X X' = s'(X')$ which implies that $Y_1 \to X$ is an isomorphism (by faithfully flat descent) and gives the desired section.

The case where $(A, (f))$ is a henselian pair. Here we use that s' is a lift of s'_0. Namely, let $Y_{0,1} \subset Y_0 = Y \times_X X_0$ be the closure of $s_0(U_0) \subset V_0 = Y_0 \times_{X_0} U_0$. As $X' \to X$ is flat, the base change $Y'_{0,1} \subset Y'_0$ is the closure of $s'_0(U'_0)$ which is equal to $X'_0 \subset Y'_0$ (see Morphisms, Lemma 29.25.16). Since $Y'_0 \to Y_0$ is submersive (Morphisms, Lemma 29.25.12) we conclude that $Y_{0,1}$ is open and closed in Y_0. Let $e_0 \in B/fB$ be the corresponding idempotent. By More on Algebra, Lemma 15.11.6 we can lift e_0 to an idempotent $e \in B$. Then we conclude as before. □

In Situation 58.19.1 fully faithfulness of the restriction functor $\text{FÉt}_U \longrightarrow \text{FÉt}_{U_0}$ holds under fairly mild assumptions. In particular, the assumptions often do not imply U is a connected scheme, but the conclusion guarantees that U and U_0 have the same number of connected components.

0EK5 **Lemma 58.19.5.** In Situation 58.19.1. Assume

(a) A has a dualizing complex,
(b) the pair $(A, (f))$ is henselian,
(c) one of the following is true
   (i) A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 3, or
   (ii) for every prime $\mathfrak p \subset A$, $f \notin \mathfrak p$ we have $\text{depth}(A_\mathfrak p) + \dim(A/\mathfrak p) > 2$.

Then the restriction functor $\text{FÉt}_U \longrightarrow \text{FÉt}_{U_0}$ is fully faithful.

Proof. Let A' be the $\mathfrak{m}$-adic completion of A. We will show that the hypotheses remain true for A'. This is clear for conditions (a) and (b). Condition (c)(ii) is preserved by Local Cohomology, Lemma 51.11.3. Next, assume (c)(i) holds. Since A is universally catenary (Dualizing Complexes, Lemma 47.17.4) we see that every irreducible component of $\text{Spec}(A')$ not contained in $V(f)$ has dimension ≥ 3, see More on Algebra, Proposition 15.109.5. Since $A \to A'$ is flat with Gorenstein fibres, the condition that A_f is (S_2) implies that A'_f is (S_2). References used: Dualizing Complexes, Section 47.23, More on Algebra, Section 15.51, and Algebra, Lemma 10.163.4. Thus by Lemma 58.19.4 we may assume that A is a Noetherian complete local ring.

Assume A is a complete local ring in addition to the other assumptions. By Lemma 58.17.1 the result follows from Algebraic and Formal Geometry, Lemma 52.15.6. ☐

0BM6 Lemma 58.19.6. In Situation 58.19.1. Assume [BdJ14, Corollary 1.11]

(1) $H^1_\mathfrak{m}(A)$ and $H^2_\mathfrak{m}(A)$ are annihilated by a power of f, and
(2) A is henselian or more generally $(A, (f))$ is a henselian pair.

Then the restriction functor $\text{FÉt}_U \longrightarrow \text{FÉt}_{U_0}$ is fully faithful.

Proof. By Lemma 58.19.4 we may assume that A is a Noetherian complete local ring. (The assumptions carry over; use Dualizing Complexes, Lemma 47.9.3.) By Lemma 58.17.1 the result follows from Algebraic and Formal Geometry, Lemma 52.15.5. ☐

0BLJ Lemma 58.19.7. In Situation 58.19.1 assume A has depth ≥ 3 and A is henselian or more generally $(A, (f))$ is a henselian pair. Then the restriction functor $\text{FÉt}_U \to \text{FÉt}_{U_0}$ is fully faithful.

Proof. The assumption of depth forces $H^1_\mathfrak{m}(A) = H^2_\mathfrak{m}(A) = 0$, see Dualizing Complexes, Lemma 47.11.1. Hence Lemma 58.19.6 applies. ☐

58.20. Purity in local case, I

0BM7 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$ be the punctured spectrum. We say purity holds for $(A, \mathfrak{m})$ if the restriction functor

$$\text{FÉt}_X \longrightarrow \text{FÉt}_U$$

is essentially surjective. In this section we try to understand how the question changes when one passes from X to a hypersurface X_0 in X, in other words, we study a kind of local Lefschetz property for the fundamental groups of punctured spectra. These results will be useful to proceed by induction on dimension in the proofs of our main results on local purity, namely, Lemma 58.21.3, Proposition 58.25.3, and Proposition 58.26.4.

0BM8 Lemma 58.20.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let $\pi : Y \to X$ be a finite morphism such that $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for all closed points $y \in Y$. Then Y is the spectrum of $B = \mathcal{O}_Y(\pi^{-1}(U))$.

Proof. Set $V = \pi^{-1}(U)$ and denote $\pi' : V \to U$ the restriction of π. Consider the $\mathcal{O}_X$-module map

$$\pi_*\mathcal{O}_Y \longrightarrow j_*\pi'_*\mathcal{O}_V$$

where $j : U \to X$ is the inclusion morphism. We claim Divisors, Lemma 31.5.11 applies to this map. If so, then $B = \Gamma(Y, \mathcal{O}_Y)$ and we see that the lemma holds. Let $x \in X$ be the closed point. It suffices to show that $\text{depth}((\pi_*\mathcal{O}_Y)_x) \geq 2$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to x. By Algebra, Lemma 10.72.11 it suffices to show that $\text{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. Since this is the assumption of the lemma the proof is complete. □

0BLK Lemma 58.20.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Assume A has depth ≥ 2. The following are equivalent

(1) $V = Y \times_X U$ for some $Y \to X$ finite étale,
(2) $B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.

Proof. Denote $\pi : V \to U$ the given finite étale morphism. Assume Y as in (1) exists. Let $x \in X$ be the point corresponding to $\mathfrak{m}$. Let $y \in Y$ be a point mapping to x. We claim that $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$. This is true because $Y \to X$ is étale and hence $A = \mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ have the same depth (Algebra, Lemma 10.163.2). Hence Lemma 58.20.1 applies and $Y = \text{Spec}(B)$.

The implication (2) ⇒ (1) is easier and the details are omitted. □

0BM9 Lemma 58.20.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Assume A is normal of dimension ≥ 2. The functor

$$\text{FÉt}_U \longrightarrow \left\{ \begin{matrix} \text{finite normal } A\text{-algebras } B \text{ such} \\ \text{that } \text{Spec}(B) \to X \text{ is étale over } U \end{matrix} \right\}, \quad V \longmapsto \Gamma(V, \mathcal{O}_V)$$

is an equivalence. Moreover, $V = Y \times_X U$ for some $Y \to X$ finite étale if and only if $B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.

Proof. Observe that $\text{depth}(A) \geq 2$ because A is normal (Serre's criterion for normality, Algebra, Lemma 10.157.4). Thus the final statement follows from Lemma 58.20.2. Given $\pi : V \to U$ finite étale, set $B = \Gamma(V, \mathcal{O}_V)$. If we can show that B is normal and finite over A, then we obtain the displayed functor. Since there is an obvious quasi-inverse functor, this is also all that we have to show.

Since A is normal, the scheme V is normal (Descent, Lemma 35.18.2). Hence V is a finite disjoint union of integral schemes (Properties, Lemma 28.7.6). Thus we may assume V is integral. In this case the function field L of V (Morphisms, Section 29.49) is a finite separable extension of the fraction field of A (because we get it by looking at the generic fibre of $V \to U$ and using Morphisms, Lemma 29.36.7). By Algebra, Lemma 10.161.8 the integral closure $B' \subset L$ of A in L is finite over A. By More on Algebra, Lemma 15.23.20 we see that B' is a reflexive A-module, which in turn implies that $\text{depth}_A(B') \geq 2$ by More on Algebra, Lemma 15.23.18.

Let $f \in \mathfrak{m}$. Then $B_f = \Gamma(V \times_U D(f), \mathcal{O}_V)$ (Properties, Lemma 28.17.1). Hence $B'_f = B_f$ because B_f is normal (see above), finite over A_f with fraction field L. It follows that $V = \text{Spec}(B') \times_X U$. Then we conclude that $B = B'$ from Lemma 58.20.1 applied to $\text{Spec}(B') \to X$. This lemma applies because the localizations $B'_{\mathfrak{m}'}$ of B' at maximal ideals $\mathfrak{m}' \subset B'$ lying over $\mathfrak{m}$ have depth ≥ 2 by Algebra, Lemma 10.72.11 and the remark on depth in the preceding paragraph. □

0BLL Lemma 58.20.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A, let $X' = \mathrm{Spec}(A^\wedge)$ and let U' and V' be the base changes of U and V to X'. The following are equivalent

(1) $V = Y \times_X U$ for some $Y \to X$ finite étale, and
(2) $V' = Y' \times_{X'} U'$ for some $Y' \to X'$ finite étale.

Proof. The implication (1) $\Rightarrow$ (2) follows from taking the base change of a solution $Y \to X$. Let $Y' \to X'$ be as in (2). By Lemma 58.19.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. By descent we see that $Y \to X$ is finite étale (Algebra, Lemmas 10.83.2 and 10.143.3). This finishes the proof. $\square$

The point of the following two lemmas is that the assumptions do not force A to have depth ≥ 3. For example if A is a complete normal local domain of dimension ≥ 3 and $f \in \mathfrak{m}$ is nonzero, then the assumptions are satisfied.

0EK6 Lemma 58.20.5. In Situation 58.19.1. Let V be finite étale over U. Assume

(a) A has a dualizing complex,
(b) the pair $(A, (f))$ is henselian,
(c) one of the following is true
  (i) A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 3, or
  (ii) for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ we have $\mathrm{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 2$.
(d) $V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 58.20.4. (The assumptions carry over; see proof of Lemma 58.19.5.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 15.13.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 15.11.4. Then we can use Lemma 58.19.5 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. $\square$

0BLS Lemma 58.20.6. In Situation 58.19.1. Let V be finite étale over U. Assume

(1) $H^1_\mathfrak{m}(A)$ and $H^2_\mathfrak{m}(A)$ are annihilated by a power of f,
(2) $V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 58.20.4. (The assumptions carry over; use Dualizing Complexes, Lemma 47.9.3.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 15.13.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 15.11.4. Then we can use Lemma 58.19.6 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. $\square$

0BLM Lemma 58.20.7. In Situation 58.19.1. Let V be finite étale over U. Assume

(1) A has depth ≥ 3,
(2) $V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. The assumption of depth forces $H^1_\mathfrak{m}(A) = H^2_\mathfrak{m}(A) = 0$, see Dualizing Complexes, Lemma 47.11.1. Hence Lemma 58.20.6 applies. $\square$

58.21. Purity of branch locus

0BJE We will use the discriminant of a finite locally free morphism. See Discriminants, Section 49.3.

0BJG **Lemma 58.21.1.** Let $(A, \mathfrak m)$ be a Noetherian local ring with $\dim(A) \geq 1$. Let $f \in \mathfrak m$. Then there exist a $\mathfrak p \in V(f)$ with $\dim(A_\mathfrak p) = 1$.

Proof. By induction on $\dim(A)$. If $\dim(A) = 1$, then $\mathfrak p = \mathfrak m$ works. If $\dim(A) > 1$, then let $Z \subset \mathrm{Spec}(A)$ be an irreducible component of dimension > 1. Then $V(f) \cap Z$ has dimension > 0 (Algebra, Lemma 10.60.13). Pick a prime $\mathfrak q \in V(f) \cap Z$, $\mathfrak q \neq \mathfrak m$ corresponding to a closed point of the punctured spectrum of A; this is possible by Properties, Lemma 28.6.4. Then $\mathfrak q$ is not the generic point of Z. Hence $0 < \dim(A_\mathfrak q) < \dim(A)$ and $f \in \mathfrak q A_\mathfrak q$. By induction on the dimension we can find $f \in \mathfrak p \subset A_\mathfrak q$ with $\dim((A_\mathfrak q)_\mathfrak p) = 1$. Then $\mathfrak p \cap A$ works. $\square$

0BJH **Lemma 58.21.2.** Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$. Assume

(1) f is flat,
(2) f is quasi-finite at x,
(3) x is not a generic point of an irreducible component of X,
(4) for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal O_{X,x'}) = 1$ our f is unramified at x'.

Then f is étale at x.

Proof. Observe that the set of points where f is unramified is the same as the set of points where f is étale and that this set is open. See Morphisms, Definitions 29.35.1 and 29.36.1 and Lemma 29.36.16. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 35.23.29 and 35.31.1). Thus we can apply More on Morphisms, Lemma 37.41.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over $y = f(x)$. Then it follows that f is finite locally free (Morphisms, Lemma 29.48.2).

Assume f is finite locally free and that x is the unique point of X lying over $y = f(x)$. By Discriminants, Lemma 49.3.1 we find a locally principal closed subscheme $D_\pi \subset Y$ such that $y' \in D_\pi$ if and only if there exists an $x' \in X$ with $f(x') = y'$ and f ramified at x'. Thus we have to prove that $y \notin D_\pi$. Assume $y \in D_\pi$ to get a contradiction.

By condition (3) we have $\dim(\mathcal O_{X,x}) \geq 1$. We have $\dim(\mathcal O_{X,x}) = \dim(\mathcal O_{Y,y})$ by Algebra, Lemma 10.112.7. By Lemma 58.21.1 we can find $y' \in D_\pi$ specializing to y with $\dim(\mathcal O_{Y,y'}) = 1$. Choose $x' \in X$ with $f(x') = y'$ where f is ramified. Since f is finite it is closed, and hence $x' \rightsquigarrow x$. We have $\dim(\mathcal O_{X,x'}) = \dim(\mathcal O_{Y,y'}) = 1$ as before. This contradicts property (4). $\square$

0BMA **Lemma 58.21.3.** Let $(A, \mathfrak m)$ be a regular local ring of dimension $d \geq 2$. Set $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak m\}$. Then

(1) the functor $\text{FÉt}_X \to \text{FÉt}_U$ is essentially surjective, i.e., purity holds for A,
(2) any finite $A \to B$ with B normal which induces a finite étale morphism on punctured spectra is étale.

Proof. Recall that a regular local ring is normal by Algebra, Lemma 10.157.5. Hence (1) and (2) are equivalent by Lemma 58.20.3. We prove the lemma by induction on d.

The case $d = 2$. In this case $A \to B$ is flat. Namely, we have going down for $A \to B$ by Algebra, Proposition 10.38.7. Then $\dim(B_{\mathfrak m'}) = 2$ for all maximal ideals $\mathfrak m' \subset B$ by Algebra, Lemma 10.112.7. Then $B_{\mathfrak m'}$ is Cohen-Macaulay by Algebra, Lemma 10.157.4. Hence and this is the important step Algebra, Lemma 10.128.1 applies to show $A \to B_{\mathfrak m'}$ is flat. Then Algebra, Lemma 10.39.18 shows $A \to B$ is flat. Thus we can apply Lemma 58.21.2 (or you can directly argue using the easier Discriminants, Lemma 49.3.1) to see that $A \to B$ is étale.

The case $d \geq 3$. Let $V \to U$ be finite étale. Let $f \in \mathfrak m_A$, $f \not\in \mathfrak m_A^2$. Then A/fA is a regular local ring of dimension $d - 1 \geq 2$, see Algebra, Lemma 10.106.3. Let U_0 be the punctured spectrum of A/fA and let $V_0 = V \times_U U_0$. By Lemma 58.20.7 it suffices to show that V_0 is in the essential image of $\textit{FÉt}_{\mathrm{Spec}(A/fA)} \to \textit{FÉt}_{U_0}$. This follows from the induction hypothesis. $\square$

0BMB **Lemma 58.21.4** (Purity of branch locus). Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume [Nag59] and [Gro71, Exp. X, Thm. 3.1]

1. $\mathcal{O}_{X,x}$ is normal,
2. $\mathcal{O}_{Y,y}$ is regular,
3. f is quasi-finite at x,
4. $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) \geq 1$
5. for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.

Then f is étale at x.

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.

An uninteresting case is when $d = 1$. In that case we are assuming that f is unramified at x and that $\mathcal{O}_{Y,y}$ is a discrete valuation ring (Algebra, Lemma 10.119.7). Then $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ (otherwise the map would not be quasi-finite at x) and we see that f is flat at x. Since flat + unramified is étale we conclude (some details omitted).

The case $d \geq 2$. We will use induction on d to reduce to the case discussed in Lemma 58.21.3. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 35.23.29 and 35.31.1). Thus we can apply More on Morphisms, Lemma 37.41.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over y. Here we use that étale extensions of local rings do not change dimension, normality, and regularity, see More on Algebra, Section 15.44 and Étale Morphisms, Section 41.19.

Next, we can base change by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume that Y is the spectrum of a regular local ring. It follows that $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ as every point of X necessarily specializes to x.

The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite and necessarily injective (by equality of dimensions). We conclude we have going down for $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ by Algebra, Proposition 10.38.7 (and the fact that a regular ring is a normal ring by Algebra, Lemma 10.157.5). Pick $x' \in X$, $x' \neq x$ with image $y' = f(x')$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of a normal domain. Similarly, $\mathcal{O}_{Y,y'}$ is regular (see Algebra, Lemma 10.110.6). We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'})$ by Algebra, Lemma 10.112.7 (we checked going down above). Of course these dimensions are strictly less than d as $x' \neq x$ and by induction on d we conclude that f is étale at x'.

Thus we arrive at the following situation: We have a finite local homomorphism $A \to B$ of Noetherian local rings of dimension $d \geq 2$, with A regular, B normal, which induces a finite étale morphism $V \to U$ on punctured spectra. Our goal is to show that $A \to B$ is étale. This follows from Lemma 58.21.3 and the proof is complete. $\square$

The following lemma is sometimes useful to find the maximal open subset over which a finite étale morphism extends.

0EY6 Lemma 58.21.5. Let $j : U \to X$ be an open immersion of locally Noetherian schemes such that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$. Let $\pi : V \to U$ be finite étale. Then

(1) $\mathcal{B} = j_*\pi_*\mathcal{O}_V$ is a reflexive coherent $\mathcal{O}_X$-algebra, set $Y = \underline{\text{Spec}}_X(\mathcal{B})$,
(2) $Y \to X$ is the unique finite morphism such that $V = Y \times_X U$ and $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$,
(3) $Y \to X$ is étale at y if and only if $Y \to X$ is flat at y, and
(4) $Y \to X$ is étale if and only if $\mathcal{B}$ is finite locally free as an $\mathcal{O}_X$-module.

Moreover, (a) the construction of $\mathcal{B}$ and $Y \to X$ commutes with base change by flat morphisms $X' \to X$ of locally Noetherian schemes, and (b) if $V' \to U'$ is a finite étale morphism with $U \subset U' \subset X$ open which restricts to $V \to U$ over U, then there is a unique isomorphism $Y' \times_X U' = V'$ over U'.

Proof. Observe that $\pi_*\mathcal{O}_V$ is a finite locally free $\mathcal{O}_U$-module, in particular reflexive. By Divisors, Lemma 31.12.12 the module $j_*\pi_*\mathcal{O}_V$ is the unique reflexive coherent module on X restricting to $\pi_*\mathcal{O}_V$ over U. This proves (1).

By construction $Y \times_X U = V$. Since $\mathcal{B}$ is coherent, we see that $Y \to X$ is finite. We have $\text{depth}(\mathcal{B}_x) \geq 2$ for $x \in X \setminus U$ by Divisors, Lemma 31.12.11. Hence $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$ by Algebra, Lemma 10.72.11. Conversely, suppose that $\pi' : Y' \to X$ is a finite morphism such that $V = Y' \times_X U$ and $\text{depth}(\mathcal{O}_{Y',y'}) \geq 2$ for $y' \in Y' \setminus V$. Then $\pi'_*\mathcal{O}_{Y'}$ restricts to $\pi_*\mathcal{O}_V$ over U and satisfies $\text{depth}((\pi'_*\mathcal{O}_{Y'})_x) \geq 2$ for $x \in X \setminus U$ by Algebra, Lemma 10.72.11. Then $\pi'_*\mathcal{O}_{Y'}$ is canonically isomorphic to $j_*\pi_*\mathcal{O}_V$ for example by Divisors, Lemma 31.5.11. This proves (2).

If $Y \to X$ is étale at y, then $Y \to X$ is flat at y. Conversely, suppose that $Y \to X$ is flat at y. If $y \in V$, then $Y \to X$ is étale at y. If $y \notin V$, then we check (1), (2), (3), and (4) of Lemma 58.21.2 hold to see that $Y \to X$ is étale at y. Parts (1) and (2) are clear and so is (3) since $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$. If $y' \rightsquigarrow y$ is a specialization and $\dim(\mathcal{O}_{Y,y'}) = 1$, then $y' \in V$ since otherwise the depth of this local ring would be 2 a contradiction by Algebra, Lemma 10.72.3. Hence $Y \to X$ is étale at y' and we conclude (4) of Lemma 58.21.2 holds too. This finishes the proof of (3).

Part (4) follows from (3) and the fact that $((Y \to X)_*\mathcal{O}_Y)_x$ is a flat $\mathcal{O}_{X,x}$-module if and only if $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-module for all $y \in Y$ mapping to x, see Algebra, Lemma 10.39.18. Here we also use that a finite flat module over a Noetherian ring is finite locally free, see Algebra, Lemma 10.78.2 (and Algebra, Lemma 10.31.4).

As to the final assertions of the lemma, part (a) follows from flat base change, see Cohomology of Schemes, Lemma 30.5.2 and part (b) follows from the uniqueness in (2) applied to the restriction $Y \times_X U'$. $\square$

0EY7 Lemma 58.21.6. Let $j : U \to X$ be an open immersion of Noetherian schemes such that purity holds for $\mathcal{O}_{X,x}$ for all $x \notin U$. Then

$$\text{FÉt}_X \longrightarrow \text{FÉt}_U$$

is essentially surjective.

Proof. Let $V \to U$ be a finite étale morphism. By Noetherian induction it suffices to extend $V \to U$ to a finite étale morphism to a strictly larger open subset of X. Let $x \in X \setminus U$ be the generic point of an irreducible component of $X \setminus U$. Then the inverse image U_x of U in $\text{Spec}(\mathcal{O}_{X,x})$ is the punctured spectrum of $\mathcal{O}_{X,x}$. By assumption $V_x = V \times_U U_x$ is the restriction of a finite étale morphism $Y_x \to \text{Spec}(\mathcal{O}_{X,x})$ to U_x. By Limits, Lemma 32.20.3 we find an open subscheme $U \subset U' \subset X$ containing x and a morphism $V' \to U'$ of finite presentation whose restriction to U recovers $V \to U$ and whose restriction to $\text{Spec}(\mathcal{O}_{X,x})$ recovering Y_x. Finally, the morphism $V' \to U'$ is finite étale after possible shrinking U' to a smaller open by Limits, Lemma 32.20.4. $\square$

58.22. Finite étale covers of punctured spectra, II

0BLU In this section we prove some variants of the material discussed in Section 58.19. Suppose we have a Noetherian local ring $(A, \mathfrak{m})$ and $f \in \mathfrak{m}$. We set $X = \text{Spec}(A)$ and $X_0 = \text{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA. All of this is exactly as in Situation 58.19.1. The difference is that we will consider the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

In other words, we will not try to lift finite étale coverings of U_0 to all of U, but just to some open neighbourhood U' of U_0 in U.

0BLN Lemma 58.22.1. In Situation 58.19.1. Let $U' \subset U$ be open and contain U_0. Assume for $\mathfrak{p} \subset A$ minimal with $\mathfrak{p} \in U'$, $\mathfrak{p} \notin U_0$ we have $\dim(A/\mathfrak{p}) \geq 2$. Then

$$\text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}, \quad V' \longmapsto V_0 = V' \times_{U'} U_0$$

is a faithful functor. Moreover, there exists a U' satisfying the assumption and any smaller open $U'' \subset U'$ containing U_0 also satisfies this assumption. In particular, the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

is faithful.

Proof. By Algebra, Lemma 10.60.13 we see that $V(\mathfrak{p})$ meets U_0 for every prime $\mathfrak{p}$ of A with $\dim(A/\mathfrak{p}) \geq 2$. Thus the displayed functor is faithful for a U as in the statement by Lemma 58.17.5. To see the existence of such a U' note that for $\mathfrak{p} \subset A$ with $\mathfrak{p} \in U$, $\mathfrak{p} \notin U_0$ with $\dim(A/\mathfrak{p}) = 1$ then $\mathfrak{p}$ corresponds to a closed point of U and hence $V(\mathfrak{p}) \cap U_0 = \emptyset$. Thus we can take U' to be the complement of the irreducible components of X which do not meet U_0 and have dimension 1. $\square$

0DXX Lemma 58.22.2. In Situation 58.19.1 assume

(1) A has a dualizing complex and is f-adically complete,
(2) every irreducible component of X not contained in X_0 has dimension ≥ 3.

Then the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

is fully faithful.

Proof. To prove this we may replace A by its reduction by the topological invariance of the fundamental group, see Lemma 58.8.3. Then the result follows from Lemma 58.17.3 and Algebraic and Formal Geometry, Lemma 52.15.7. ☐

0BLP Lemma 58.22.3. In Situation 58.19.1 assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor.
(3) $H^1_{\mathfrak m}(A/fA)$ is a finite A-module.

Then the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

is fully faithful.

Proof. Follows from Lemma 58.17.3 and Algebraic and Formal Geometry, Lemma 52.15.8. ☐

58.23. Finite étale covers of punctured spectra, III

0EK7 In this section we study when in Situation 58.19.1. the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

is an equivalence of categories.

0DXY Lemma 58.23.1. In Situation 58.19.1 assume

(1) A has a dualizing complex and is f-adically complete,
(2) one of the following is true
    (a) A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or
    (b) if $\mathfrak p \notin V(f)$ and $V(\mathfrak p) \cap V(f) \neq \{\mathfrak m\}$, then $\text{depth}(A_{\mathfrak p}) + \dim(A/\mathfrak p) > 3$.

Then the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

is an equivalence.

Proof. This follows from Lemma 58.17.4 and Algebraic and Formal Geometry, Lemma 52.24.1. ☐

0BLV Lemma 58.23.2. In Situation 58.19.1 assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor,
(3) $H^1_{\mathfrak m}(A/fA)$ and $H^2_{\mathfrak m}(A/fA)$ are finite A-modules.

Then the restriction functor

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \longrightarrow \text{FÉt}_{U_0}$$

is an equivalence.

Proof. This follows from Lemma 58.17.4 and Algebraic and Formal Geometry, Lemma 52.24.2. ☐

0BLW Remark 58.23.3. Let $(A, \mathfrak{m})$ be a complete local Noetherian ring and $f \in \mathfrak{m}$ nonzero. Suppose that A_f is (S_2) and every irreducible component of $\text{Spec}(A)$ has dimension ≥ 4. Then Lemma 58.23.1 tells us that the category

$$\text{colim}_{U' \subset U \text{ open}, U_0 \subset U} \text{ category of schemes finite étale over } U'$$

is equivalent to the category of schemes finite étale over U_0. For example this holds if A is a normal domain of dimension ≥ 4!

58.24. Finite étale covers of punctured spectra, IV

0EK8 Let X, X_0, U, U_0 be as in Situation 58.19.1. In this section we ask when the restriction functor

$$\text{FÉt}_U \longrightarrow \text{FÉt}_{U_0}$$

is essentially surjective. We will do this by taking results from Section 58.23 and then filling in the gaps using purity. Recall that we say purity holds for a Noetherian local ring $(A, \mathfrak{m})$ if the restriction functor $\text{FÉt}_X \to \text{FÉt}_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

0EK9 Lemma 58.24.1. In Situation 58.19.1 assume

(1) A has a dualizing complex and is f-adically complete,
(2) one of the following is true
  (a) A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or
  (b) if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.
(3) for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$.

Then the restriction functor $\text{FÉt}_U \to \text{FÉt}_{U_0}$ is essentially surjective.

Proof. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 58.23.1 there exists an open $U' \subset U$ containing U_0 and a finite étale morphism $V' \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ as in (3). By assumption we can find finite étale morphisms $V'_i \to \text{Spec}(A_{\mathfrak{p}_i})$ agreeing with $V' \to U'$ over $U' \times_U \text{Spec}(A_{\mathfrak{p}_i})$. By Limits, Lemma 32.20.1 applied n times we see that $V' \to U'$ extends to a finite étale morphism $V \to U$. $\square$

0EKA Lemma 58.24.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor,
(3) $H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules,
(4) for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$.

Then the restriction functor $\text{FÉt}_U \to \text{FÉt}_{U_0}$ is essentially surjective.

Proof. The proof is identical to the proof of Lemma 58.24.1 using Lemma 58.23.2 in stead of Lemma 58.23.1. $\square$

58.25. Purity in local case, II

0BPB This section is the continuation of Section 58.20. Recall that we say purity holds for a Noetherian local ring $(A, \mathfrak{m})$ if the restriction functor $\text{FÉt}_X \to \text{FÉt}_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

0DXZ Lemma 58.25.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume

(1) A has a dualizing complex and is f-adically complete,
(2) one of the following is true
  (a) A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or
  (b) if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.
(3) for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$, and
(4) purity holds for A.

Then purity holds for A/fA.

Proof. Denote $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$ the punctured spectrum. Similarly we have $X_0 = \text{Spec}(A/fA)$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 58.24.1 we find a finite étale morphism $V \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. By assumption (5) we find that $V \to U$ extends to a finite étale morphism $Y \to X$. Then the restriction of Y to X_0 is the desired extension of $V_0 \to U_0$. $\square$

0BPC Lemma 58.25.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume

(1) A is f-adically complete,
(2) f is a nonzerodivisor,
(3) $H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules,
(4) for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$,
(5) purity holds for A.

Then purity holds for A/fA.

Proof. The proof is identical to the proof of Lemma 58.25.1 using Lemma 58.24.2 in stead of Lemma 58.24.1. $\square$

Now we can bootstrap the earlier results to prove that purity holds for complete intersections of dimension ≥ 3. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 23.8.

0BPD Proposition 58.25.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is a complete intersection of dimension ≥ 3, then purity holds for A in the sense that any finite étale cover of the punctured spectrum extends.

Proof. By Lemma 58.20.4 we may assume that A is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where B is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on r. The base case is $r = 0$ which follows from Lemma 58.21.3 which applies to regular rings of dimension ≥ 2.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 58.25.2 to $f_r \in A'$. This is permissible: condition (1) holds as $f_1, \ldots, f_r$ is a regular sequence, condition (2) holds as B and hence A' is complete, condition (3) holds as $A = A'/f_rA'$ is Cohen-Macaulay of dimension $\dim(A) \geq 3$, see Dualizing Complexes, Lemma 47.11.1, condition (4) holds by induction hypothesis as $\dim((A'_{f_r})_\mathfrak{p}) \geq 3$ for a maximal prime $\mathfrak{p}$ of A'_{f_r} and as $(A'_{f_r})_\mathfrak{p} = B_\mathfrak{q}/(f_1, \ldots, f_{r-1})$ for some $\mathfrak{q} \subset B$, condition (5) holds by induction hypothesis. $\square$

58.26. Purity in local case, III

0EY8 In this section is a continuation of the discussion in Sections 58.20 and 58.25.

0EY9 **Lemma 58.26.1.** Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. Set $Y = \mathrm{Spec}(B)$ and $Y_0 = V(x_1, \ldots, x_d)$. For any open subscheme $V \subset Y$ with $V_0 = V \cap Y_0$ equal to $Y_0 \setminus \{\mathfrak{m}_B\}$ the restriction functor

$$\text{FÉt}_V \longrightarrow \text{FÉt}_{V_0}$$

is fully faithful.

Proof. Set $I = (x_1, \ldots, x_d)$. Set $X = \mathrm{Spec}(A)$. If we use the map $Y \to X$ to identify Y_0 with X, then V_0 is identified with the punctured spectrum U of A. Pushing forward modules by this affine morphism we get

$$\begin{aligned} \lim_n \Gamma(V_0, \mathcal{O}_V/I^n\mathcal{O}_V) &= \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n\mathcal{O}_Y) \\ &= \lim_n \Gamma(U, \mathcal{O}_U[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n) \\ &= \lim_n A[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n \\ &= B \end{aligned}$$

Namely, as the depth of A is ≥ 2 we have $\Gamma(U, \mathcal{O}_U) = A$, see Local Cohomology, Lemma 51.8.2. Thus for any $V \subset Y$ open as in the lemma we get

$$B = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(V, \mathcal{O}_V) \to \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n\mathcal{O}_Y) = B$$

which implies both arrows are isomorphisms (small detail omitted). By Algebraic and Formal Geometry, Lemma 52.15.1 we conclude that $\mathrm{Coh}(\mathcal{O}_V) \to \mathrm{Coh}(V, I\mathcal{O}_V)$ is fully faithful on the full subcategory of finite locally free objects. Thus we conclude by Lemma 58.17.1. $\square$

0EYA **Lemma 58.26.2.** Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. For any open $V \subset Y = \mathrm{Spec}(B)$ which contains

(1) any prime $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A \neq \mathfrak{m}$,
(2) the prime $\mathfrak{m}B$

the functor $\text{FÉt}_Y \to \text{FÉt}_V$ is an equivalence. In particular purity holds for B.

Proof. A prime $\mathfrak{q} \subset B$ which is not contained in V lies over $\mathfrak{m}$. In this case $A \to B_\mathfrak{q}$ is a flat local homomorphism and hence $\mathrm{depth}(B_\mathfrak{q}) \geq 2$ (Algebra, Lemma 10.163.2). Thus the functor is fully faithful by Lemma 58.10.3 combined with Local Cohomology, Lemma 51.3.1.

Let $W \to V$ be a finite étale morphism. Let $B \to C$ be the unique finite ring map such that $\mathrm{Spec}(C) \to Y$ is the finite morphism extending $W \to V$ constructed in Lemma 58.21.5. Observe that $C = \Gamma(W, \mathcal{O}_W)$.

Set $Y_0 = V(x_1, \ldots, x_d)$ and $V_0 = V \cap Y_0$. Set $X = \mathrm{Spec}(A)$. If we use the map $Y \to X$ to identify Y_0 with X, then V_0 is identified with the punctured spectrum U of A. Thus we may view $W_0 = W \times_Y Y_0$ as a finite étale scheme over U. Then

$$W_0 \times_U (U \times_X Y) \quad \text{and} \quad W \times_V (U \times_X Y)$$

are schemes finite étale over $U \times_X Y$ which restrict to isomorphic finite étale schemes over V_0. By Lemma 58.26.1 applied to the open $U \times_X Y$ we obtain an isomorphism

$$W_0 \times_U (U \times_X Y) \longrightarrow W \times_V (U \times_X Y)$$

over $U \times_X Y$.

Observe that $C_0 = \Gamma(W_0, \mathcal{O}_{W_0})$ is a finite A-algebra by Lemma 58.21.5 applied to $W_0 \to U \subset X$ (exactly as we did for $B \to C$ above). Since the construction in Lemma 58.21.5 is compatible with flat base change and with change of opens, the isomorphism above induces an isomorphism

$$\Psi : C \longrightarrow C_0 \otimes_A B$$

of finite B-algebras. However, we know that $\mathrm{Spec}(C) \to Y$ is étale at all points above at least one point of Y lying over $\mathfrak{m} \in X$. Since Ψ is an isomorphism, we conclude that $\mathrm{Spec}(C_0) \to X$ is étale above $\mathfrak{m}$ (small detail omitted). Of course this means that $A \to C_0$ is finite étale and hence $B \to C$ is finite étale. $\square$

0EYB Lemma 58.26.3. Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ be an open subscheme. Assume

(1) f is smooth,
(2) S is Noetherian,
(3) for $s \in S$ with $\mathrm{depth}(\mathcal{O}_{S,s}) \leq 1$ we have $X_s = U_s$,
(4) $U_s \subset X_s$ is dense for all $s \in S$.

Then $\mathrm{FÉt}_X \to \mathrm{FÉt}_U$ is an equivalence.

Proof. The functor is fully faithful by Lemma 58.10.3 combined with Local Cohomology, Lemma 51.3.1 (plus an application of Algebra, Lemma 10.163.2 to check the depth condition).

Let $\pi : V \to U$ be a finite étale morphism. Let $Y \to X$ be the finite morphism constructed in Lemma 58.21.5. We have to show that $Y \to X$ is finite étale. To show that this is true for all points $x \in X$ mapping to a given point $s \in S$ we may perform a base change by a flat morphism $S' \to S$ of Noetherian schemes such that s is in the image. This follows from the compatibility of the construction in Lemma 58.21.5 with flat base change.

After enlarging U we may assume $U \subset X$ is the maximal open over which $Y \to X$ is finite étale. Let $Z \subset X$ be the complement of U. To get a contradiction, assume $Z \neq \emptyset$. Let $s \in S$ be a point in the image of $Z \to S$ such that no strict generalization of s is in the image. Then after base change to $\mathrm{Spec}(\mathcal{O}_{S,s})$ we see that $S = \mathrm{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring of depth ≥ 2 and Z contained in the closed fibre X_s and nowhere dense in X_s. Choose a closed point $z \in Z$. Then $\kappa(z)/\kappa$ is finite (by the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1). Choose a finite flat morphism $(S', s') \to (S, s)$ of local schemes realizing the residue field extension $\kappa(z)/\kappa$, see Algebra, Lemma 10.159.3. After doing a base change by $S' \to S$ we reduce to the case where $\kappa(z) = \kappa$.

By More on Morphisms, Lemma 37.38.5 there exists a locally closed subscheme $S' \subset X$ passing through z such that $S' \to S$ is étale at z. After performing the base change by $S' \to S$, we may assume there is a section $\sigma : S \to X$ such that $\sigma(s) = z$. Choose an affine neighbourhood $\mathrm{Spec}(B) \subset X$ of s. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \mathrm{Ker}(\sigma)$ and denote $B^\wedge$ the I-adic completion of B. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 10.139.4. Observe that $d > 0$ since otherwise we see that $X \to S$ is étale at z which would imply that z is a generic point of X_s and hence $z \in U$ by assumption (4). Similarly, if $d > 0$, then $\mathfrak{m}B^\wedge$ maps into U via the morphism $\text{Spec}(B^\wedge) \to X$. It suffices prove $Y \to X$ is finite étale after base change to $\text{Spec}(B^\wedge)$. Since $B \to B^\wedge$ is flat (Algebra, Lemma 10.97.2) this follows from Lemma 58.26.2 and the uniqueness in the construction of $Y \to X$. □

0EYC Proposition 58.26.4. Let $A \to B$ be a local homomorphism of local Noetherian rings. Assume A has depth ≥ 2, $A \to B$ is formally smooth for the $\mathfrak{m}_B$-adic topology, and $\dim(B) > \dim(A)$. For any open $V \subset Y = \text{Spec}(B)$ which contains

(1) any prime $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A \neq \mathfrak{m}_A$,
(2) the prime $\mathfrak{m}_A B$

the functor $\text{FÉt}_Y \to \text{FÉt}_V$ is an equivalence. In particular purity holds for B.

Proof. A prime $\mathfrak{q} \subset B$ which is not contained in V lies over $\mathfrak{m}_A$. In this case $A \to B_\mathfrak{q}$ is a flat local homomorphism and hence $\text{depth}(B_\mathfrak{q}) \geq 2$ (Algebra, Lemma 10.163.2). Thus the functor is fully faithful by Lemma 58.10.3 combined with Local Cohomology, Lemma 51.3.1.

Denote $A^\wedge$ and $B^\wedge$ the completions of A and B with respect to their maximal ideals. Observe that the assumptions of the proposition hold for $A^\wedge \to B^\wedge$, see More on Algebra, Lemmas 15.43.1, 15.43.2, and 15.37.4. By the uniqueness and compatibility with flat base change of the construction of Lemma 58.21.5 it suffices to prove the essential surjectivity for $A^\wedge \to B^\wedge$ and the inverse image of V (details omitted; compare with Lemma 58.20.4 for the case where V is the punctured spectrum). By More on Algebra, Proposition 15.49.2 this means we may assume $A \to B$ is regular.

Let $W \to V$ be a finite étale morphism. By Popescu's theorem (Smoothing Ring Maps, Theorem 16.12.1) we can write $B = \text{colim}\, B_i$ as a filtered colimit of smooth A-algebras. We can pick an i and an open $V_i \subset \text{Spec}(B_i)$ whose inverse image is V (Limits, Lemma 32.4.11). After increasing i we may assume there is a finite étale morphism $W_i \to V_i$ whose base change to V is $W \to V$, see Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. We may assume the complement of V_i is contained in the closed fibre of $\text{Spec}(B_i) \to \text{Spec}(A)$ as this is true for V (either choose V_i this way or use the lemma above to show this is true for i large enough). Let η be the generic point of the closed fibre of $\text{Spec}(B) \to \text{Spec}(A)$. Since $\eta \in V$, the image of η is in V_i. Hence after replacing V_i by an affine open neighbourhood of the image of the closed point of $\text{Spec}(B)$, we may assume that the closed fibre of $\text{Spec}(B_i) \to \text{Spec}(A)$ is irreducible and that its generic point is contained in V_i (details omitted; use that a scheme smooth over a field is a disjoint union of irreducible schemes). At this point we may apply Lemma 58.26.3 to see that $W_i \to V_i$ extends to a finite étale morphism $\text{Spec}(C_i) \to \text{Spec}(B_i)$ and pulling back to $\text{Spec}(B)$ we conclude that W is in the essential image of the functor $\text{FÉt}_Y \to \text{FÉt}_V$ as desired. □

58.27. Lefschetz for the fundamental group

0ELB Of course we have already proven a bunch of results of this type in the local case. In this section we discuss the projective case.

0ELC Proposition 58.27.1. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Assume that for all $x \in X \setminus Y$ we have

$$\text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 1$$

Then the restriction functor $\text{FÉt}_X \to \text{FÉt}_Y$ is fully faithful. In fact, for any open subscheme $V \subset X$ containing Y the restriction functor $\text{FÉt}_V \to \text{FÉt}_Y$ is fully faithful.

Proof. The first statement is a formal consequence of Lemma 58.17.6 and Algebraic and Formal Geometry, Proposition 52.28.1. The second statement follows from Lemma 58.17.6 and Algebraic and Formal Geometry, Lemma 52.28.2. $\square$

0ELD Proposition 58.27.2. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Let $\mathcal{V}$ be the set of open subschemes of X containing Y ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have

$$\text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

Then the restriction functor

$$\text{colim}_{\mathcal{V}} \text{FÉt}_V \to \text{FÉt}_Y$$

is an equivalence.

Proof. This is a formal consequence of Lemma 58.17.4 and Algebraic and Formal Geometry, Proposition 52.28.7. $\square$

0ELE Proposition 58.27.3. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Assume that for all $x \in X \setminus Y$ we have

$$\text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

and that for $x \in X \setminus Y$ closed purity holds for $\mathcal{O}_{X,x}$. Then the restriction functor $\text{FÉt}_X \to \text{FÉt}_Y$ is an equivalence. If X or equivalently Y is connected, then

$$\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{y})$$

is an isomorphism for any geometric point $\overline{y}$ of Y.

Proof. Fully faithfulness holds by Proposition 58.27.1. By Proposition 58.27.2 any object of FÉt_Y is isomorphic to the fibre product $U \times_V Y$ for some finite étale morphism $U \to V$ where $V \subset X$ is an open subscheme containing Y. The complement $T = X \setminus V$ is[5] a finite set of closed points of $X \setminus Y$. Say $T = \{x_1, \ldots, x_n\}$. By assumption we can find finite étale morphisms $V_i' \to \text{Spec}(\mathcal{O}_{X,x_i})$ agreeing with $U \to V$ over $V \times_X \text{Spec}(\mathcal{O}_{X,x_i})$. By Limits, Lemma 32.20.1 applied n times we see that $U \to V$ extends to a finite étale morphism $U' \to X$ as desired. See Lemma 58.8.1 for the final statement. $\square$

58.28. Purity of ramification locus

0EA1 In this section we discuss the analogue of purity of branch locus for generically finite morphisms. Apparently, this result is due to Gabber. A special case is van der Waerden's purity theorem for the locus where a birational morphism from a normal variety to a smooth variety is not an isomorphism.

[5] Namely, T is proper over k (being closed in X) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 29.43.18) and hence finite over k (Morphisms, Lemma 29.44.11). Thus T is a finite set of closed points.

0EA2 Lemma 58.28.1. Let A be a Noetherian normal local domain of dimension 2. Assume A is Nagata, has a dualizing module ω_A, and has a resolution of singularities $f : X \to \mathrm{Spec}(A)$. Let ω_X be as in Resolution of Surfaces, Remark 54.7.7. If $\omega_X \cong \mathcal{O}_X(E)$ for some effective Cartier divisor $E \subset X$ supported on the exceptional fibre, then A defines a rational singularity. If f is a minimal resolution, then $E = 0$.

Proof. There is a trace map $Rf_*\omega_X \to \omega_A$, see Duality for Schemes, Section 48.7. By Grauert-Riemenschneider (Resolution of Surfaces, Proposition 54.7.8) we have $R^1f_*\omega_X = 0$. Thus the trace map is a map $f_*\omega_X \to \omega_A$. Then we can consider

$$\mathcal{O}_{\mathrm{Spec}(A)} = f_*\mathcal{O}_X \to f_*\omega_X \to \omega_A$$

where the first map comes from the map $\mathcal{O}_X \to \mathcal{O}_X(E) = \omega_X$ which is assumed to exist in the statement of the lemma. The composition is an isomorphism by Divisors, Lemma 31.2.11 as it is an isomorphism over the punctured spectrum of A (by the assumption in the lemma and the fact that f is an isomorphism over the punctured spectrum) and A and ω_A are A-modules of depth 2 (by Algebra, Lemma 10.157.4 and Dualizing Complexes, Lemma 47.17.5). Hence $f_*\omega_X \to \omega_A$ is surjective whence an isomorphism. Thus $Rf_*\omega_X = \omega_A$ which by duality implies $Rf_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Spec}(A)}$. Whence $H^1(X, \mathcal{O}_X) = 0$ which implies that A defines a rational singularity (see discussion in Resolution of Surfaces, Section 54.8 in particular Lemmas 54.8.7 and 54.8.1). If f is minimal, then $E = 0$ because the map $f^*\omega_A \to \omega_X$ is surjective by a repeated application of Resolution of Surfaces, Lemma 54.9.7 and $\omega_A \cong A$ as we've seen above. ☐

0EA3 Lemma 58.28.2. Let $f : X \to \mathrm{Spec}(A)$ be a finite type morphism. Let $x \in X$ be a point. Assume

(1) A is an excellent regular local ring,
(2) $\mathcal{O}_{X,x}$ is normal of dimension 2,
(3) f is étale outside of $\overline{\{x\}}$.

Then f is étale at x.

Proof. We first replace X by an affine open neighbourhood of x. Observe that $\mathcal{O}_{X,x}$ is an excellent local ring (More on Algebra, Lemma 15.52.2). Thus we can choose a minimal resolution of singularities $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$, see Resolution of Surfaces, Theorem 54.14.5. After possibly replacing X by an affine open neighbourhood of x we can find a proper morphism $b : X' \to X$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) = W$, see Limits, Lemma 32.20.1. After shrinking X further, we may assume X' is regular. Namely, we know W is regular and X' is excellent and the regular locus of the spectrum of an excellent ring is open. Since $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is projective (as a sequence of normalized blowing ups), we may assume after shrinking X that b is projective (details omitted). Let $U = X \setminus \overline{\{x\}}$. Since $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is an isomorphism over the punctured spectrum, we may assume $b : X' \to X$ is an isomorphism over U. Thus we may and will think of U as an open subscheme of X' as well. Set $f' = f \circ b : X' \to \mathrm{Spec}(A)$.

Since A is regular we see that $\mathcal{O}_Y$ is a dualizing complex for Y. Hence $f^!\mathcal{O}_Y$ is a dualzing complex on X (Duality for Schemes, Lemma 48.17.7). The Cohen-Macaulay locus of X is open by Duality for Schemes, Lemma 48.23.1 (this can also be proven using excellency). Since $\mathcal{O}_{X,x}$ is Cohen-Macaulay, after shrinking X we may assume X is Cohen-Macaulay. Observe that an étale morphism is a local complete intersection. Thus Duality for Schemes, Lemma 48.29.3 applies with $r = 0$ and we get a map

$$\mathcal{O}_X \longrightarrow \omega_{X/Y} = H^0(f^!\mathcal{O}_Y)$$

which is an isomorphism over $X \setminus \overline{\{x\}}$. Since $\omega_{X/Y}$ is (S_2) by Duality for Schemes, Lemma 48.21.5 we find this map is an isomorphism by Divisors, Lemma 31.2.11. This already shows that X and in particular $\mathcal{O}_{X,x}$ is Gorenstein.

Set $\omega_{X'/Y} = H^0((f')^!\mathcal{O}_Y)$. Arguing in exactly the same manner as above we find that $(f')^!\mathcal{O}_Y = \omega_{X'/Y}[0]$ is a dualizing complex for X'. Since X' is regular the morphism $X' \to Y$ is a local complete intersection morphism, see More on Morphisms, Lemma 37.62.11. By Duality for Schemes, Lemma 48.29.2 there exists a map

$$\mathcal{O}_{X'} \longrightarrow \omega_{X'/Y}$$

which is an isomorphism over U. We conclude $\omega_{X'/Y} = \mathcal{O}_{X'}(E)$ for some effective Cartier divisor $E \subset X'$ disjoint from U.

Since $\omega_{X/Y} = \mathcal{O}_Y$ we see that $\omega_{X'/Y} = b^!f^!\mathcal{O}_Y = b^!\mathcal{O}_X$. Returning to $W \to \text{Spec}(\mathcal{O}_{X,x})$ we see that $\omega_W = \mathcal{O}_W(E|_W)$. By Lemma 58.28.1 we find $E|_W = 0$. This means that $f' : X' \to Y$ is étale by (the already used) Duality for Schemes, Lemma 48.29.2. This immediately finishes the proof, as étaleness of f' forces b to be an isomorphism. $\square$

0EA4 Lemma 58.28.3 (Purity of ramification locus). Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume

(1) $\mathcal{O}_{X,x}$ is normal of dimension ≥ 1,
(2) $\mathcal{O}_{Y,y}$ is regular,
(3) f is locally of finite type, and
(4) for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is étale at x'.

Then f is étale at x.

This result for complex spaces can be found on page 170 of [Fis76]. In general this is [Zon14, Theorem 2.4] attributed to Gabber.

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x})$.

An uninteresting case is $d = 1$ since in that case the morphism f is étale at x by assumption. Assume $d \geq 2$.

We can base change by $\text{Spec}(\mathcal{O}_{Y,y}) \to Y$ without affecting the conclusion of the lemma, see Morphisms, Lemma 29.36.17. Thus we may assume $Y = \text{Spec}(A)$ where A is a regular local ring and y corresponds to the maximal ideal $\mathfrak{m}$ of A.

Let $x' \rightsquigarrow x$ be a specialization with $x' \neq x$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of $\mathcal{O}_{X,x}$. If x' is not a generic point of X, then $1 \leq \dim(\mathcal{O}_{X,x'}) < d$ and we conclude that f is étale at x' by induction hypothesis. Thus we may assume that f is étale at all points specializing to x. Since the set of points where f is étale is open in X (by definition) we may after replacing X by an open neighbourhood of x assume that f is étale away from $\overline{\{x\}}$. In particular, we see that f is étale except at points lying over the closed point $y \in Y = \text{Spec}(A)$.

Let $X' = X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge)$. Let $x' \in X'$ be the unique point lying over x. By the above we see that X' is étale over $\text{Spec}(A^\wedge)$ away from the closed fibre and hence X' is normal away from the closed fibre. Since X is normal we conclude that X' is normal by Resolution of Surfaces, Lemma 54.11.6. Then if we can show $X' \to \text{Spec}(A^\wedge)$ is étale at x', then f is étale at x (by the aforementioned

Morphisms, Lemma 29.36.17). Thus we may and do assume A is a regular complete local ring.

The case $d = 2$ now follows from Lemma 58.28.2.

Assume $d > 2$. Let $t \in \mathfrak{m}$, $t \not\in \mathfrak{m}^2$. Set $Y_0 = \mathrm{Spec}(A/tA)$ and $X_0 = X \times_Y Y_0$. Then $X_0 \to Y_0$ is étale away from the fibre over the closed point. Since $d > 2$ we have $\dim(\mathcal{O}_{X_0,x}) = d-1$ is ≥ 2. The normalization $X_0' \to X_0$ is surjective and finite (as we're working over a complete local ring and such rings are Nagata). Let $x' \in X_0'$ be a point mapping to x. By induction hypothesis the morphism $X_0' \to Y$ is étale at x'. From the inclusions $\kappa(y) \subset \kappa(x) \subset \kappa(x')$ we conclude that $\kappa(x)$ is finite over $\kappa(y)$. Hence x is a closed point of the fibre of $X \to Y$ over y. But since x is also a generic point of this fibre, we conclude that f is quasi-finite at x and we reduce to the case of purity of branch locus, see Lemma 58.21.4. □

58.29. Affineness of complement of ramification locus

0ECA Let $f : X \to Y$ be a finite type morphism of Noetherian schemes with X normal and Y regular. Let $V \subset X$ be the maximal open subscheme where f is étale. The discussion in [DG67, Chapter IV, Section 21.12] suggests that $V \to X$ might be an affine morphism. Observe that if $V \to X$ is affine, then we deduce purity of ramification locus (Lemma 58.28.3) by using Divisors, Lemma 31.16.4. Thus affineness of $V \to X$ is a "strong" form of purity for the ramification locus. In this section we prove $V \to X$ is affine when X and Y are equicharacteristic and excellent, see Theorem 58.29.3. It seems reasonable to guess the result remains true for X and Y of mixed characteristic (but still excellent).

0ECB **Lemma 58.29.1.** *Let $(A, \mathfrak{m})$ be a regular local ring which contains a field. Let $f : V \to \mathrm{Spec}(A)$ be étale and quasi-compact. Assume that $\mathfrak{m} \not\in f(V)$ and assume that $g : V \to \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is affine. Then $H^i(V, \mathcal{O}_V)$, $i > 0$ is isomorphic to a direct sum of copies of the injective hull of the residue field of A.*

Proof. Denote $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ the punctured spectrum. Thus $g : V \to U$ is affine. We have $H^i(V, \mathcal{O}_V) = H^i(U, g_*\mathcal{O}_V)$ by Cohomology of Schemes, Lemma 30.2.4. The $\mathcal{O}_U$-module $g_*\mathcal{O}_V$ is quasi-coherent by Schemes, Lemma 26.24.1. For any quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$ the cohomology $H^i(U, \mathcal{F})$, $i > 0$ is $\mathfrak{m}$-power torsion, see for example Local Cohomology, Lemma 51.2.2. In particular, the A-modules $H^i(V, \mathcal{O}_V)$, $i > 0$ are $\mathfrak{m}$-power torsion. For any flat ring map $A \to A'$ we have $H^i(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'})$ where $V' = V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ by flat base change Cohomology of Schemes, Lemma 30.5.2. If we take A' to be the completion of A (flat by More on Algebra, Section 15.43), then we see that

$$H^i(V, \mathcal{O}_V) = H^i(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'}), \quad \text{for } i > 0$$

The first equality by the torsion property we just proved and More on Algebra, Lemma 15.89.3. Moreover, the injective hull of the residue field k is the same for A and A', see Dualizing Complexes, Lemma 47.7.4. In this way we reduce to the case $A = k[[x_1, \ldots, x_d]]$, see Algebra, Section 10.160.

Assume the characteristic of k is $p > 0$. Since $F : A \to A$, $a \mapsto a^p$ is flat (Local Cohomology, Lemma 51.17.6) and since $V \times_{\mathrm{Spec}(A), \mathrm{Spec}(F)} \mathrm{Spec}(A) \cong V$ as schemes over $\mathrm{Spec}(A)$ by Étale Morphisms, Lemma 41.14.3 the above gives $H^i(V, \mathcal{O}_V) \otimes_{A,F} A \cong H^i(V, \mathcal{O}_V)$. Thus we get the result by Local Cohomology, Lemma 51.18.2.

Assume the characteristic of k is 0. By Local Cohomology, Lemma 51.19.3 there are additive operators D_j, $j = 1, \ldots, d$ on $H^i(V, \mathcal{O}_V)$ satisfying the Leibniz rule with respect to $\partial_j = \partial/\partial x_j$. Thus we get the result by Local Cohomology, Lemma 51.18.1. $\square$

0ECC Lemma 58.29.2. In the situation of Lemma 58.29.1 assume that $H^i(V, \mathcal{O}_V) = 0$ for $i \geq \dim(A) - 1$. Then V is affine.

Proof. Let $k = A/\mathfrak{m}$. Since $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = \emptyset$, by cohomology and base change we have

$$R\Gamma(V, \mathcal{O}_V) \otimes_A^{\mathbf{L}} k = 0$$

See Derived Categories of Schemes, Lemma 36.22.5. Thus there is a spectral sequence (More on Algebra, Example 15.62.4)

$$E_2^{p,q} = \mathrm{Tor}_{-p}(k, H^q(V, \mathcal{O}_V)), \quad d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

and $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ converging to zero. By Lemma 58.29.1, Dualizing Complexes, Lemma 47.21.9, and our assumption $H^i(V, \mathcal{O}_V) = 0$ for $i \geq \dim(A) - 1$ we conclude that there is no nonzero differential entering or leaving the $(p, q) = (0, 0)$ spot. Thus $H^0(V, \mathcal{O}_V) \otimes_A k = 0$. This means that if $\mathfrak{m} = (x_1, \ldots, x_d)$ then we have an open covering $V = \bigcup V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{x_i})$ by affine open subschemes $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{x_i})$ (because V is affine over the punctured spectrum of A) such that $x_1, \ldots, x_d$ generate the unit ideal in $\Gamma(V, \mathcal{O}_V)$. This implies V is affine by Properties, Lemma 28.27.3. $\square$

0ECD Theorem 58.29.3. Let Y be an excellent regular scheme over a field. Let $f : X \to Y$ be a finite type morphism of schemes with X normal. Let $V \subset X$ be the maximal open subscheme where f is étale. Then the inclusion morphism $V \to X$ is affine.

Proof. Let $x \in X$ with image $y \in Y$. It suffices to prove that $V \cap W$ is affine for some affine open neighbourhood W of x. Since $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the limit of the schemes W, this holds if and only if

$$V_x = V \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$$

is affine (Limits, Lemma 32.4.13). Thus, if the theorem holds for the morphism $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$, then the theorem holds. In particular, we may assume Y is regular of finite dimension, which allows us to do induction on the dimension $d = \dim(Y)$. Combining this with the same argument again, we may assume that Y is local with closed point y and that $V \cap (X \setminus f^{-1}(\{y\}) \to X \setminus f^{-1}(\{y\})$ is affine.

Let $x \in X$ be a point lying over y. If $x \in V$, then there is nothing to prove. Observe that $f^{-1}(\{y\}) \cap V$ is a finite set of closed points (the fibres of an étale morphism are discrete). Thus after replacing X by an affine open neighbourhood of x we may assume $y \notin f(V)$. We have to prove that V is affine.

Let $e(V)$ be the maximum i with $H^i(V, \mathcal{O}_V) \neq 0$. As X is affine the integer $e(V)$ is the maximum of the numbers $e(V_x)$ where $x \in X \setminus V$, see Local Cohomology, Lemma 51.4.6 and the characterization of cohomological dimension in Local Cohomology, Lemma 51.4.1. We have $e(V_x) \leq \dim(\mathcal{O}_{X,x}) - 1$ by Local Cohomology, Lemma 51.4.7. If $\dim(\mathcal{O}_{X,x}) \geq 2$ then purity of ramification locus (Lemma 58.28.3) shows that V_x is strictly smaller than the punctured spectrum of $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is normal and excellent, this implies $e(V_x) \leq \dim(\mathcal{O}_{X,x}) - 2$ by Hartshorne-Lichtenbaum vanishing (Local Cohomology, Lemma 51.16.7). On the other hand, since $X \to Y$ is of finite type and $V \subset X$ is dense (after possibly replacing X by the closure of V), we see that $\dim(\mathcal{O}_{X,x}) \leq d$ by the dimension formula (Morphisms, Lemma 29.52.1). Whence $e(V) \leq \max(0, d-2)$. Thus V is affine by Lemma 58.29.2 if $d \geq 2$. If $d = 1$ or $d = 0$, then the punctured spectrum of $\mathcal{O}_{Y,y}$ is affine and hence V is affine. □

58.30. Specialization maps in the smooth proper case

0BUQ In this section we discuss the following result. Let $f : X \to S$ be a proper smooth morphism of schemes. Let $s \rightsquigarrow s'$ be a specialization of points in S. Then the specialization map

$$sp : \pi_1(X_{\overline{s}}) \longrightarrow \pi_1(X_{\overline{s}'})$$

of Section 58.16 is surjective and

(1) if the characteristic of $\kappa(s')$ is zero, then it is an isomorphism, or
(2) if the characteristic of $\kappa(s')$ is $p > 0$, then it induces an isomorphism on maximal prime-to-p quotients.

0C0P **Lemma 58.30.1.** Let $f : X \to S$ be a flat proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If X_s is geometrically reduced, then the specialization map $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is surjective.

Proof. Since X_s is geometrically reduced, we may assume all fibres are geometrically reduced after possibly shrinking S, see More on Morphisms, Lemma 37.26.7. Let $\mathcal{O}_{S,s} \to A \to \kappa(\overline{s}')$ be as in the construction of the specialization map, see Section 58.16. Thus it suffices to show that

$$\pi_1(X_{\overline{s}'}) \to \pi_1(X_A)$$

is surjective. This follows from Proposition 58.15.2 and $\pi_1(\text{Spec}(A)) = \{1\}$. □

0C0Q **Proposition 58.30.2.** Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If the characteristic to $\kappa(s)$ is zero, then the specialization map

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is an isomorphism.

Proof. The map is surjective by Lemma 58.30.1. Thus we have to show it is injective.

We may assume S is affine. Then S is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to S$ is the base change of $X_0 \to S_0$ where S_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to S_0$ is smooth and proper. See Limits, Lemma 32.10.1, 32.8.9, and 32.13.1. By Lemma 58.16.1 we reduce to the case where the base is Noetherian.

Applying Lemma 58.16.4 we reduce to the case where the base S is the spectrum of a strictly henselian discrete valuation ring A and we are looking at the specialization map over A. Let K be the fraction field of A. Choose an algebraic closure $\overline{K}$ which corresponds to a geometric generic point $\overline{\eta}$ of $\text{Spec}(A)$. For $\overline{K}/L/K$ finite separable, let $B \subset L$ be the integral closure of A in L. This is a discrete valuation ring by More on Algebra, Remark 15.111.6.

Let $X \to \mathrm{Spec}(A)$ be as in the previous paragraph. To show injectivity of the specialization map it suffices to prove that every finite étale cover V of $X_{\overline{\eta}}$ is the base change of a finite étale cover $Y \to X$. Namely, then $\pi_1(X_{\overline{\eta}}) \to \pi_1(X) = \pi_1(X_s)$ is injective by Lemma 58.4.4.

Given V we can first descend V to $V' \to X_{K^{sep}}$ by Lemma 58.14.2 and then to $V'' \to X_L$ by Lemma 58.14.1. Let $Z \to X_B$ be the normalization of X_B in V''. Observe that Z is normal and that $Z_L = V''$ as schemes over X_L. Hence $Z \to X_B$ is finite étale over the generic fibre. The problem is that we do not know that $Z \to X_B$ is everywhere étale. Since $X \to \mathrm{Spec}(A)$ has geometrically connected smooth fibres, we see that the special fibre X_s is geometrically irreducible. Hence the special fibre of $X_B \to \mathrm{Spec}(B)$ is irreducible; let ξ_B be its generic point. Let $\xi_1, \ldots, \xi_r$ be the points of Z mapping to ξ_B. Our first (and it will turn out only) problem is now that the extensions

$$\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$$

of discrete valuation rings may be ramified. Let e_i be the ramification index of this extension. Note that since the characteristic of $\kappa(s)$ is zero, the ramification is tame!

To get rid of the ramification we are going to choose a further finite separable extension $K^{sep}/L'/L/K$ such that the ramification index e of the induced extensions B'/B is divisible by e_i. Consider the normalized base change Z' of Z with respect to $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$, see discussion in More on Morphisms, Section 37.65. Let $\xi_{i,j}$ be the points of Z' mapping to $\xi_{B'}$ and to ξ_i in Z. Then the local rings

$$\mathcal{O}_{Z',\xi_{i,j}}$$

are localizations of the integral closure of $\mathcal{O}_{Z,\xi_i}$ in $L' \otimes_L F_i$ where F_i is the fraction field of $\mathcal{O}_{Z,\xi_i}$; details omitted. Hence Abhyankar's lemma (More on Algebra, Lemma 15.114.4) tells us that

$$\mathcal{O}_{X_{B'},\xi_{B'}} \subset \mathcal{O}_{Z',\xi_{i,j}}$$

is unramified. We conclude that the morphism $Z' \to X_{B'}$ is étale away from codimension 1. Hence by purity of branch locus (Lemma 58.21.4) we see that $Z' \to X_{B'}$ is finite étale!

However, since the residue field extension induced by $A \to B'$ is trivial (as the residue field of A is algebraically closed being separably closed of characteristic zero) we conclude that Z' is the base change of a finite étale cover $Y \to X$ by applying Lemma 58.9.1 twice (first to get Y over A, then to prove that the pullback to B is isomorphic to Z'). This finishes the proof. □

Let G be a profinite group. Let p be a prime number. The maximal prime-to-p quotient is by definition

$$G' = \lim_{U \subset G \text{ open, normal, index prime to } p} G/U$$

If X is a connected scheme and p is given, then the maximal prime-to-p quotient of $\pi_1(X)$ is denoted $\pi'_1(X)$.

0C0R Theorem 58.30.3. Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \leadsto s$ be a specialization. If the characteristic of $\kappa(s)$ is p, then the specialization map

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is surjective and induces an isomorphism

$$\pi'_1(X_{\overline{s}'}) \cong \pi'_1(X_{\overline{s}})$$

of the maximal prime-to-p quotients

Proof. This is proved in exactly the same manner as Proposition 58.30.2 with the following differences

(1) Given X/A we no longer show that the functor $\text{FÉt}_X \to \text{FÉt}_{X_{\overline{\eta}}}$ is essentially surjective. We show only that Galois objects whose Galois group has order prime to p are in the essential image. This will be enough to conclude the injectivity of $\pi'_1(X_{\overline{s}'}) \to \pi'_1(X_{\overline{s}})$ by exactly the same argument.
(2) The extensions $\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$ are tamely ramified as the associated extension of fraction fields is Galois with group of order prime to p. See More on Algebra, Lemma 15.112.2.
(3) The extension κ_B/κ_A is no longer necessarily trivial, but it is purely inseparable. Hence the morphism $X_{\kappa_B} \to X_{\kappa_A}$ is a universal homeomorphism and induces an isomorphism of fundamental groups by Proposition 58.8.4.

□

58.31. Tame ramification

0BSE Let $X \to Y$ be a finite étale morphism of schemes of finite type over $\mathbf{Z}$. There are many ways to define what it means for f to be tamely ramified at ∞. The article [KS10] discusses to what extent these notions agree.

In this section we discuss a different more elementary question which precedes the notion of tameness at infinity. Please compare with the (slightly different) discussion in [GM71]. Assume we are given

(1) a locally Noetherian scheme X,
(2) a dense open $U \subset X$,
(3) a finite étale morphism $f : Y \to U$

such that for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring. Setting K_ξ equal to the fraction field of $\mathcal{O}_{X,\xi}$ we obtain a cartesian square

$$\begin{tikzcd} \mathrm{Spec}(K_\xi) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{X,\xi}) \arrow[r] & X \end{tikzcd}$$

of schemes. In particular, we see that $Y \times_U \mathrm{Spec}(K_\xi)$ is the spectrum of a finite separable algebra L_ξ/K_ξ. Then we say Y is unramified over X in codimension 1, resp. Y is tamely ramified over X in codimension 1 if L_ξ/K_ξ is unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$ for every (Z, ξ) as above, see More on Algebra, Definition 15.111.7. More precisely, we decompose L_ξ into a product of finite separable field extensions of K_ξ and we require each of these to be unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$.

0EYD Lemma 58.31.1. Let $X' \to X$ be a morphism of locally Noetherian schemes. Let $U \subset X$ be a dense open. Assume

(1) $U' = f^{-1}(U)$ is dense open in X',
(2) for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring,
(3) for every prime divisor $Z' \subset X'$ with $Z' \cap U' = \emptyset$ the local ring $\mathcal{O}_{X',\xi'}$ of X' at the generic point ξ' of Z' is a discrete valuation ring,
(4) if $\xi' \in X'$ is as in (3), then $\xi = f(\xi')$ is as in (2).

Then if $f : Y \to U$ is finite étale and Y is unramified, resp. tamely ramified over X in codimension 1, then $Y' = Y \times_X X' \to U'$ is finite étale and Y' is unramified, resp. tamely ramified over X' in codimension 1.

Proof. The only interesting fact in this lemma is the commutative algebra result given in More on Algebra, Lemma 15.114.9. □

Using the terminology introduced above, we can reformulate our purity results obtained earlier in the following pleasing manner.

0H2W Lemma 58.31.2. Let X be a locally Noetherian scheme. Let $U \subset X$ be open and dense. Let $Y \to U$ be a finite étale morphism. Assume

(1) Y is unramified over X in codimension 1, and
(2) $\mathcal{O}_{X,x}$ is regular for all $x \in X \setminus U$.

Then there exists a finite étale morphism $Y' \to X$ whose restriction to $X \setminus D$ is Y.

Proof. Let $\xi \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$ of codimension 1. Then $\mathcal{O}_{X,\xi}$ is a discrete valuation ring. As in the discussion above, write $Y \times_U \mathrm{Spec}(K_\xi) = \mathrm{Spec}(L_\xi)$. Denote B_ξ the integral closure of $\mathcal{O}_{X,\xi}$ in L_ξ. Our assumption that Y is unramified over X in codimension 1 signifies that $\mathcal{O}_{X,\xi} \to B_\xi$ is finite étale. Thus we get $Y_\xi \to \mathrm{Spec}(\mathcal{O}_{X,\xi})$ finite étale and an isomorphism

$$Y \times_U \mathrm{Spec}(K_\xi) \cong Y_\xi \times_{\mathrm{Spec}(\mathcal{O}_{X,\xi})} \mathrm{Spec}(K_\xi)$$

over $\mathrm{Spec}(K_\xi)$. By Limits, Lemma 32.20.3 we find an open subscheme $U \subset U' \subset X$ containing ξ and a morphism $Y' \to U'$ of finite presentation whose restriction to U recovers Y and whose restriction to $\mathrm{Spec}(\mathcal{O}_{X,\xi})$ recovers Y_ξ. Finally, the morphism $Y' \to U'$ is finite étale after possible shrinking U' to a smaller open by Limits, Lemma 32.20.4. Repeating the argument with the other generic points of $X \setminus U$ of codimension 1 we may assume that we have a finite étale morphism $Y' \to U'$ extending $Y \to U$ to an open subscheme containing $U' \subset X$ containing U and all codimension 1 points of $X \setminus U$. We finish by applying Lemma 58.21.6 to $Y' \to U'$. Namely, all local rings $\mathcal{O}_{X,x}$ for $x \in X \setminus U'$ are regular and have $\dim(\mathcal{O}_{X,x}) \geq 2$. Hence we have purity for $\mathcal{O}_{X,x}$ by Lemma 58.21.3. □

0EYE Lemma 58.31.3. Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor such that D is a regular scheme. Let $Y \to X \setminus D$ be a finite étale morphism. If Y is unramified over X in codimension 1, then there exists a finite étale morphism $Y' \to X$ whose restriction to $X \setminus D$ is Y.

Proof. This is a special case of Lemma 58.31.2. First, D is nowhere dense in X (see discussion in Divisors, Section 31.13) and hence $X \setminus D$ is dense in X. Second, the ring $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$ by Algebra, Lemma 10.106.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. □

0EYF Example 58.31.4 (Standard tamely ramified morphism). Let A be a Noetherian ring. Let $f \in A$ be a nonzerodivisor such that A/fA is reduced. This implies that $A_\mathfrak{p}$ is a discrete valuation ring with uniformizer f for any minimal prime $\mathfrak{p}$ over f. Let $e \geq 1$ be an integer which is invertible in A. Set

$$C = A[x]/(x^e - f)$$

Then $\text{Spec}(C) \to \text{Spec}(A)$ is a finite locally free morphism which is étale over the spectrum of A_f. The finite étale morphism

$$\text{Spec}(C_f) \longrightarrow \text{Spec}(A_f)$$

is tamely ramified over $\text{Spec}(A)$ in codimension 1. The tameness follows immediately from the characterization of tamely ramified extensions in More on Algebra, Lemma 15.114.7.

Here is a version of Abhyankar's lemma for regular divisors.

0EYG Lemma 58.31.5 (Abhyankar's lemma for regular divisor). Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor such that D is a regular scheme. Let $Y \to X \setminus D$ be a finite étale morphism. If Y is tamely ramified over X in codimension 1, then étale locally on X the morphism $Y \to X$ is as given as a finite disjoint union of standard tamely ramified morphisms as described in Example 58.31.4.

Proof. Before we start we note that $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$. This follows from Algebra, Lemma 10.106.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. Below we will also use that regular rings are normal, see Algebra, Lemma 10.157.5.

To prove the lemma we may work locally on X. Thus we may assume $X = \text{Spec}(A)$ and $D \subset X$ is given by a nonzerodivisor $f \in A$. Then $Y = \text{Spec}(B)$ as a finite étale scheme over A_f. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of A over f. Then $A_i = A_{\mathfrak{p}_i}$ is a discrete valuation ring; denote its fraction field K_i. By assumption

$$K_i \otimes_{A_f} B = \prod L_{ij}$$

is a finite product of fields each tamely ramified with respect to A_i. Choose $e \geq 1$ sufficiently divisible (namely, divisible by all ramification indices for L_{ij} over A_i as in More on Algebra, Remark 15.111.6). Warning: at this point we do not know that e is invertible on A.

Consider the finite free A-algebra

$$A' = A[x]/(x^e - f)$$

Observe that $f' = x$ is a nonzerodivisor in A' and that $A'/f'A' \cong A/fA$ is a regular ring. Set $B' = B \otimes_A A' = B \otimes_{A_f} A'_{f'}$. By Abhyankar's lemma (More on Algebra, Lemma 15.114.4) we see that $\text{Spec}(B')$ is unramified over $\text{Spec}(A')$ in codimension 1. Namely, by Lemma 58.31.1 we see that $\text{Spec}(B')$ is still at least tamely ramified over $\text{Spec}(A')$ in codimension 1. But Abhyankar's lemma tells us that the ramification indices have all become equal to 1. By Lemma 58.31.3 we conclude that $\mathrm{Spec}(B') \to \mathrm{Spec}(A'_{f'})$ extends to a finite étale morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A')$.

For a point $x \in D$ corresponding to $\mathfrak{p} \in V(f)$ denote A^{sh} a strict henselization of $A_\mathfrak{p} = \mathcal{O}_{X,x}$. Observe that A^{sh} and $A^{sh}/fA^{sh} = (A/fA)^{sh}$ (Algebra, Lemma 10.156.4) are regular local rings, see More on Algebra, Lemma 15.45.10. Observe that A' has a unique prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ with identical residue field. Thus

$$(A')^{sh} = A^{sh} \otimes_A A' = A^{sh}[x]/(x^e - f)$$

is a strictly henselian local ring finite over A^{sh} (Algebra, Lemma 10.156.3). Since f' is a nonzerodivisor in $(A')^{sh}$ and since $(A')^{sh}/f'(A')^{sh} = A^{sh}/fA^{sh}$ is regular, we conclude that $(A')^{sh}$ is a regular local ring (see above). Observe that the induced extension

$$Q(A^{sh}) \subset Q((A')^{sh}) = Q(A^{sh})[x]/(x^e - f)$$

of fraction fields has degree e (and not less). Since $A' \to C$ is finite étale we see that $A^{sh} \otimes_A C$ is a finite product of copies of $(A')^{sh}$ (Algebra, Lemma 10.153.6). We have the inclusions

$$A^{sh}_f \subset A^{sh} \otimes_A B \subset A^{sh} \otimes_A B' = A^{sh} \otimes_A C_{f'}$$

and each of these rings is Noetherian and normal; this follows from Algebra, Lemma 10.163.9 for the ring in the middle. Taking total quotient rings, using the product decomposition of $A^{sh} \otimes_A C$ and using Fields, Lemma 9.24.3 we conclude that there is an isomorphism

$$Q(A^{sh}) \otimes_A B \cong \prod\nolimits_{i \in I} F_i, \quad F_i \cong Q(A^{sh})[x]/(x^{e_i} - f)$$

of $Q(A^{sh})$-algebras for some finite set I and integers $e_i|e$. Since $A^{sh} \otimes_A B$ is a normal ring, it must be the integral closure of A^{sh} in its total quotient ring. We conclude that we have an isomorphism

$$A^{sh} \otimes_A B \cong \prod A^{sh}_f[x]/(x^{e_i} - f)$$

over A^{sh}_f because the algebras $A^{sh}[x]/(x^{e_i} - f)$ are regular and hence normal. The discriminant of $A^{sh}[x]/(x^{e_i} - f)$ over A^{sh} is $e_i^{e_i} f^{e_i - 1}$ (up to sign; calculation omitted). Since $A_f \to B$ is finite étale we see that e_i must be invertible in A^{sh}_f. On the other hand, since $A_f \to B$ is tamely ramified over $\mathrm{Spec}(A)$ in codimension 1, by Lemma 58.31.1 the ring map $A^{sh}_f \to A^{sh} \otimes_A B$ is tamely ramified over $\mathrm{Spec}(A^{sh})$ in codimension 1. This implies e_i is nonzero in A^{sh}/fA^{sh} (as it must map to an invertible element of the fraction field of this domain by definition of tamely ramified extensions). We conclude that $V(e_i) \subset \mathrm{Spec}(A^{sh})$ has codimension ≥ 2 which is absurd unless it is empty. In other words, e_i is an invertible element of A^{sh}. We conclude that the pullback of Y to $\mathrm{Spec}(A^{sh})$ is indeed a finite disjoint union of standard tamely ramified morphisms.

To finish the proof, we write $A^{sh} = \mathrm{colim}\, A_\lambda$ as a filtered colimit of étale A-algebras A_λ. The isomorphism

$$A^{sh} \otimes_A B \cong \prod\nolimits_{i \in I} A^{sh}_f[x]/(x^{e_i} - f)$$

descends to an isomorphism

$$A_\lambda \otimes_A B \cong \prod\nolimits_{i \in I} (A_\lambda)_f[x]/(x^{e_i} - f)$$

for suitably large λ. After increasing λ a bit more we may assume e_i is invertible in A_λ. Then $\mathrm{Spec}(A_\lambda) \to \mathrm{Spec}(A)$ is the desired étale neighbourhood of x and the proof is complete. $\square$

0EYH Lemma 58.31.6. In the situation of Lemma 58.31.5 the normalization of X in Y is a finite locally free morphism $\pi : Y' \to X$ such that

(1) the restriction of Y' to $X \setminus D$ is isomorphic to Y,
(2) $D' = \pi^{-1}(D)_{red}$ is an effective Cartier divisor on Y', and
(3) D' is a regular scheme.

Moreover, étale locally on X the morphism $Y' \to X$ is a finite disjoint union of morphisms

$$\mathrm{Spec}(A[x]/(x^e - f)) \to \mathrm{Spec}(A)$$

where A is a Noetherian ring, $f \in A$ is a nonzerodivisor with A/fA regular, and $e \geq 1$ is invertible in A.

Proof. This is just an addendum to Lemma 58.31.5 and in fact the truth of this lemma follows almost immediately if you've read the proof of that lemma. But we can also deduce the lemma from the result of Lemma 58.31.5. Namely, taking the normalization of X in Y commutes with étale base change, see More on Morphisms, Lemma 37.19.2. Hence we see that we may prove the statements on the local structure of $Y' \to X$ étale locally on X. Thus, by Lemma 58.31.5 we may assume that $X = \mathrm{Spec}(A)$ where A is a Noetherian ring, that we have a nonzerodivisor $f \in A$ such that A/fA is regular, and that Y is a finite disjoint union of spectra of rings $A_f[x]/(x^e - f)$ where e is invertible in A. We omit the verification that the integral closure of A in $A_f[x]/(x^e - f)$ is equal to $A' = A[x]/(x^e - f)$. (To see this argue that the localizations of A' at primes lying over (f) are regular.) We omit the details. $\square$

0EYI Lemma 58.31.7. In the situation of Lemma 58.31.5 let $Y' \to X$ be as in Lemma 58.31.6. Let R be a discrete valuation ring with fraction field K. Let

$$t : \mathrm{Spec}(R) \to X$$

be a morphism such that the scheme theoretic inverse image $t^{-1}D$ is the reduced closed point of $\mathrm{Spec}(R)$.

(1) If $t|_{\mathrm{Spec}(K)}$ lifts to a point of Y, then we get a lift $t' : \mathrm{Spec}(R) \to Y'$ such that $Y' \to X$ is étale along $t'(\mathrm{Spec}(R))$.
(2) If $\mathrm{Spec}(K) \times_X Y$ is isomorphic to a disjoint union of copies of $\mathrm{Spec}(K)$, then $Y' \to X$ is finite étale over an open neighbourhood of $t(\mathrm{Spec}(R))$.

Proof. By the valuative criterion of properness applied to the finite morphism $Y' \to X$ we see that $\mathrm{Spec}(K)$-valued points of Y matching $t|_{\mathrm{Spec}(K)}$ as maps into X lift uniquely to morphisms $t' : \mathrm{Spec}(R) \to Y'$. Thus statement (1) make sense.

Choose an étale neighbourhood $(U, u) \to (X, t(\mathfrak{m}_R))$ such that $U = \mathrm{Spec}(A)$ and such that $Y' \times_X U \to U$ has a description as in Lemma 58.31.6 for some $f \in A$. Then $\mathrm{Spec}(R) \times_X U \to \mathrm{Spec}(R)$ is étale and surjective. If R' denotes the local ring of $\mathrm{Spec}(R) \times_X U$ lying over the closed point of $\mathrm{Spec}(R)$, then R' is a discrete valuation ring and $R \subset R'$ is an unramified extension of discrete valuation rings (More on Algebra, Lemma 15.44.4). The assumption on t signifies that the map $A \to R'$ corresponding to

$$\mathrm{Spec}(R') \to \mathrm{Spec}(R) \times_X U \to U$$

maps f to a uniformizer $\pi \in R'$. Now suppose that

$$Y' \times_X U = \coprod\nolimits_{i \in I} \mathrm{Spec}(A[x]/(x^{e_i} - f))$$

for some $e_i \geq 1$. Then we see that

$$\mathrm{Spec}(R') \times_U (Y' \times_X U) = \coprod\nolimits_{i \in I} \mathrm{Spec}(R'[x]/(x^{e_i} - \pi))$$

The rings $R'[x]/(x^{e_i} - f)$ are discrete valuation rings (More on Algebra, Lemma 15.114.2) and hence have no map into the fraction field of R' unless $e_i = 1$.

Proof of (1). In this case the map $t' : \mathrm{Spec}(R) \to Y'$ base changes to determine a corresponding map $t'' : \mathrm{Spec}(R') \to Y' \times_X U$ which must map into a summand corresponding to $i \in I$ with $e_i = 1$ by the discussion above. Thus clearly we see that $Y' \times_X U \to U$ is étale along the image of t''. Since being étale is a property one can check after étale base chamge, this proves (1).

Proof of (2). In this case the assumption implies that $e_i = 1$ for all $i \in I$. Thus $Y' \times_X U \to U$ is finite étale and we conclude as before. ☐

0EYJ Lemma 58.31.8. Let S be an integral normal Noetherian scheme with generic point η. Let $f : X \to S$ be a smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of f. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 58.7) with group G of order invertible on S such that Z has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group G whose restriction to X_η is Z.

Proof. First assume $S = \mathrm{Spec}(R)$ is the spectrum of a discrete valuation ring R with closed point $s \in S$. Then X_s is an effective Cartier divisor in X and X_s is regular as a scheme smooth over a field. Moreover the generic fibre X_η is the open subscheme $X \setminus X_s$. It follows from More on Algebra, Lemma 15.112.2 and the assumption on G that Z is tamely ramified over X in codimension 1. Let $Z' \to X$ be as in Lemma 58.31.6. Observe that the action of G on Z extends to an action of G on Z'. By Lemma 58.31.7 we see that $Z' \to X$ is finite étale over an open neighbourhood of $\sigma(y)$. Since X_s is irreducible, this implies $Z \to X_\eta$ is unramified over X in codimension 1. Then we get a finite étale morphism $Y \to X$ whose restriction to X_η is Z by Lemma 58.31.3. Of course $Y \cong Z'$ (details omitted; hint: compute étale locally) and hence Y is a Galois cover with group G.

General case. Let $U \subset S$ be a maximal open subscheme such that there exists a finite étale Galois cover $Y \to X \times_S U$ with group G whose restriction to X_η is isomorphic to Z. Assume $U \neq S$ to get a contradiction. Let $s \in S \setminus U$ be a generic point of an irreducible component of $S \setminus U$. Then the inverse image U_s of U in $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the punctured spectrum of $\mathcal{O}_{S,s}$. We claim $Y \times_S U_s \to X \times_S U_s$ is the restriction of a finite étale Galois cover $Y'_s \to X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ with group G.

Let us first prove the claim produces the desired contradiction. By Limits, Lemma 32.20.3 we find an open subscheme $U \subset U' \subset S$ containing s and a morphism $Y'' \to U'$ of finite presentation whose restriction to U recovers $Y' \to U$ and whose restriction to $\mathrm{Spec}(\mathcal{O}_{S,s})$ recovers Y'_s. Moreover, by the equivalence of categories given in the lemma, we may assume after shrinking U' there is a morphism $Y'' \to U' \times_S X$ and there is an action of G on Y'' over $U' \times_S X$ compatible with the given morphisms and actions after base change to U and $\mathrm{Spec}(\mathcal{O}_{S,s})$. After shrinking U' further if necessary, we may assume $Y'' \to U \times_S X$ is finite étale, see Limits, Lemma 32.20.4. This means we have found a strictly larger open of S over which Y extends to a finite étale Galois cover with group G which gives the contradiction we were looking for.

Proof of the claim. We may and do replace S by $\text{Spec}(\mathcal{O}_{S,s})$. Then $S = \text{Spec}(A)$ where $(A, \mathfrak{m})$ is a local normal domain. Also $U \subset S$ is the punctured spectrum and we have a finite étale Galois cover $Y \to X \times_S U$ with group G. If $\dim(A) = 1$, then we can construct the extension of Y to a Galois covering of X by the first paragraph of the proof. Thus we may assume $\dim(A) \geq 2$ and hence $\text{depth}(A) \geq 2$ as S is normal, see Algebra, Lemma 10.157.4. Since $X \to S$ is flat, we conclude that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for every point $x \in X$ mapping to s, see Algebra, Lemma 10.163.2. Let

$$Y' \longrightarrow X$$

be the finite morphism constructed in Lemma 58.21.5 using $Y \to X \times_S U$. Observe that we obtain a canonical G-action on Y. Thus all that remains is to show that Y' is étale over X. In fact, by Lemma 58.26.3 (for example) it even suffices to show that $Y' \to X$ is étale over the (unique) generic point of the fibre X_s. This we do by a local calculation in a (formal) neighbourhood of $\sigma(s)$.

Choose an affine open $\text{Spec}(B) \subset X$ containing $\sigma(s)$. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \text{Ker}(\sigma)$ and denote $B^\wedge$ the I-adic completion of B. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 10.139.4. Of course $B \to B^\wedge$ is flat (Algebra, Lemma 10.97.2) and the image of $\text{Spec}(B^\wedge) \to X$ contains the generic point of X_s. Let $V \subset \text{Spec}(B^\wedge)$ be the inverse image of U. Consider the finite étale morphism

$$W = Y \times_{(X \times_S U)} V \longrightarrow V$$

By the compatibility of the construction of Y' with flat base change in Lemma 58.21.5 we find that the base chang $Y' \times_X \text{Spec}(B^\wedge) \to \text{Spec}(B^\wedge)$ is constructed from $W \to V$ over $\text{Spec}(B^\wedge)$ by the procedure in Lemma 58.21.5. Set $V_0 = V \cap V(x_1, \ldots, x_d) \subset V$ and $W_0 = W \times_V V_0$. This is a normal integral scheme which maps into $\sigma(S)$ by the morphism $\text{Spec}(B^\wedge) \to X$ and in fact is identified with $\sigma(U)$. Hence we know that $W_0 \to V_0 = U$ completely decomposes as this is true for its generic fibre by our assumption on $Z \to X_\eta$ having a $\kappa(\eta)$-rational point lying over $\sigma(\eta)$ (and of course the G-action then implies the whole fibre $Z_{\sigma(\eta)}$ is a disjoint union of copies of the scheme $\eta = \text{Spec}(\kappa(\eta))$). Finally, by Lemma 58.26.1 we have

$$W_0 \times_U V \cong W$$

This shows that W is a disjoint union of copies of V and hence $Y' \times_X \text{Spec}(B^\wedge)$ is a disjoint union of copies of $\text{Spec}(B^\wedge)$ and the proof is complete. □

0EZJ Lemma 58.31.9. Let S be a quasi-compact and quasi-separated integral normal scheme with generic point η. Let $f : X \to S$ be a quasi-compact and quasi-separated smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of f. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 58.7) with group G of order invertible on S such that Z has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group G whose restriction to X_η is Z.

Proof. If S is Noetherian, then this is the result of Lemma 58.31.8. The general case follows from this by a standard limit argument. We strongly urge the reader to skip the proof.

We can write $S = \lim S_i$ as a directed limit of a system of schemes with affine transition morphisms and with S_i of finite type over $\mathbf{Z}$, see Limits, Proposition 32.5.4. For each i let $S \to S_i' \to S_i$ be the normalization of S_i in S, see Morphisms, Section 29.53. Combining Algebra, Proposition 10.162.16 Morphisms, Lemmas 29.53.15 and 29.53.13 we conclude that S_i' is of finite type over $\mathbf{Z}$, finite over S_i, and that S_i' is an integral normal scheme such that $S \to S_i'$ is dominant. By Morphisms, Lemma 29.53.5 we obtain transition morphisms $S_{i'}' \to S_i'$ compatible with the transition morphisms $S_{i'} \to S_i$ and with the morphisms with source S. We claim that $S = \lim S_i'$. Proof of claim omitted (hint: look on affine opens over a chosen affine open in S_i for some i to translate this into a straightforward algebra problem). We conclude that we may write $S = \lim S_i$ as a directed limit of a system of normal integral schemes S_i with affine transition morphisms and with S_i of finite type over $\mathbf{Z}$.

For some i we can find a smooth morphism $X_i \to S_i$ of finite presentation whose base change to S is $X \to S$. See Limits, Lemmas 32.10.1 and 32.8.9. After increasing i we may assume the section σ lifts to a section $\sigma_i : S_i \to X_i$ (by the equivalence of categories in Limits, Lemma 32.10.1). We may replace X_i by the open subscheme X_i^0 of it studied in More on Morphisms, Section 37.29 since the image of $X \to X_i$ clearly maps into it (openness by More on Morphisms, Lemma 37.29.6). Thus we may assume the fibres of $X_i \to S_i$ are geometrically connected. After increasing i we may assume $|G|$ is invertible on S_i. Let $\eta_i \in S_i$ be the generic point. Since X_η is the limit of the schemes X_{i,η_i} we can use the exact same arguments to descent $Z \to X_\eta$ to some finite étale Galois cover $Z_i \to X_{i,\eta_i}$ after possibly increasing i. See Lemma 58.14.1. After possibly increasing i once more we may assume Z_i has a $\kappa(\eta_i)$-rational point mapping to $\sigma_i(\eta_i)$. Then we apply the lemma in the Noetherian case and we pullback to X to conclude. □

58.32. Tricks in positive characteristic

0G1E In Piotr Achinger's paper [Ach17] it is shown that an affine scheme in positive characteristic is always a $K(\pi, 1)$. In this section we explain the more elementary parts of [Ach17]. Namely, we show that for a field k of positive characteristic an affine scheme étale over $\mathbf{A}^n_k$ is actually finite étale over $\mathbf{A}^n_k$ (by a different morphism). We also show that a closed immersion of connected affine schemes in positive characteristic induces an injective map on étale fundamental groups.

Let k be a field of characteristic $p > 0$. Let

$$k[x_1, \ldots, x_n] \longrightarrow A$$

be a surjection of finite type k-algebras whose source is the polynomial algebra on $x_1, \ldots, x_n$. Denote $I \subset k[x_1, \ldots, x_n]$ the kernel so that we have $A = k[x_1, \ldots, x_n]/I$. We do not assume A is nonzero (in other words, we allow the case where A is the zero ring and $I = k[x_1, \ldots, x_n]$). Finally, we assume given a finite étale ring map $\pi : A \to B$.

Suppose given $k, n, k[x_1, \ldots, x_n] \to A, I, \pi : A \to B$. Let C be a k-algebra. Consider commutative diagrams

$$\begin{tikzcd} & B \\ C \arrow[r] & C/\varphi(I)C \arrow[u, "\tau"] \\ k[x_1, \ldots, x_n] \arrow[u, "\varphi"] \arrow[r] & A \arrow[u] \arrow[uu, bend right, "\pi"'] \end{tikzcd}$$

where φ is an étale k-algebra map and τ is a surjective k-algebra map. Let C, φ, τ be given. For any $r \geq 0$ and $y_1, \ldots, y_r \in C$ which generate C as an algebra over $\mathrm{Im}(\varphi)$ let $s = s(r, y_1, \ldots, y_r) \in \{0, \ldots, r\}$ be the maximal element such that y_i is integral over $\mathrm{Im}(\varphi)$ for $1 \leq i \leq s$. We define $NF(C, \varphi, \tau)$ to be the minimum value of $r - s = r - s(r, y_1, \ldots, y_r)$ for all choices of r and $y_1, \ldots, y_r$ as above. Observe that $NF(C, \varphi, \tau)$ is 0 if and only if φ is finite.

0G1F Lemma 58.32.1. In the situation above, if $NF(C, \varphi, \tau) > 0$, then there exist an étale k-algebra map φ' and a surjective k-algebra map τ' fitting into the commutative diagram

$$\begin{tikzcd} & B \\ C \arrow[r] & C/\varphi'(I)C \arrow[u, "\tau'"'] \\ k[x_1, \ldots, x_n] \arrow[u, "\varphi'"] \arrow[r] & A \arrow[u] \arrow[uu, bend right, "\pi"'] \end{tikzcd}$$

with $NF(C, \varphi', \tau') < NF(C, \varphi, \tau)$.

Proof. Choose $r \geq 0$ and $y_1, \ldots, y_r \in C$ which generate C over $\mathrm{Im}(\varphi)$ and let $0 \leq s \leq r$ be such that $y_1, \ldots, y_s$ are integral over $\mathrm{Im}(\varphi)$ such that $r - s = NF(C, \varphi, \tau) > 0$. Since B is finite over A, the image of y_{s+1} in B satisfies a monic polynomial over A. Hence we can find $d \geq 1$ and $f_1, \ldots, f_d \in k[x_1, \ldots, x_n]$ such that

$$z = y_{s+1}^d + \varphi(f_1) y_{s+1}^{d-1} + \ldots + \varphi(f_d) \in J = \mathrm{Ker}(C \to C/\varphi(I)C \xrightarrow{\tau} B)$$

Since $\varphi : k[x_1, \ldots, x_n] \to C$ is étale, we can find a nonzero and nonconstant polynomial $g \in k[T_1, \ldots, T_{n+1}]$ such that

$$g(\varphi(x_1), \ldots, \varphi(x_n), z) = 0 \quad \text{in} \quad C$$

To see this you can use for example that $C \otimes_{\varphi, k[x_1, \ldots, x_n]} k(x_1, \ldots, x_n)$ is a finite product of finite separable field extensions of $k(x_1, \ldots, x_n)$ (see Algebra, Lemmas 10.143.4) and hence z satisfies a monic polynomial over $k(x_1, \ldots, x_n)$. Clearing denominators we obtain g.

The existence of g and Algebra, Lemma 10.115.2 produce integers $e_1, e_2, \ldots, e_n \geq 1$ such that z is integral over the subring C' of C generated by $t_1 = \varphi(x_1) + z^{pe_1}, \ldots, t_n = \varphi(x_n) + z^{pe_n}$. Of course, the elements $\varphi(x_1), \ldots, \varphi(x_n)$ are also integral over C' as are the elements $y_1, \ldots, y_s$. Finally, by our choice of z the element y_{s+1} is integral over C' too.

Consider the ring map

$$\varphi' : k[x_1, \ldots, x_n] \longrightarrow C, \quad x_i \longmapsto t_i$$

with image C'. Since $\mathrm{d}(\varphi(x_i)) = \mathrm{d}(t_i) = \mathrm{d}(\varphi'(x_i))$ in $\Omega_{C/k}$ (and this is where we use the characteristic of k is $p > 0$) we conclude that φ' is étale because φ is étale, see Algebra, Lemma 10.151.9. Observe that $\varphi'(x_i) - \varphi(x_i) = t_i - \varphi(x_i) = z^{pe_i}$ is in the kernel J of the map $C \to C/\varphi(I)C \to B$ by our choice of z as an element of J. Hence for $f \in I$ the element

$$\varphi'(f) = f(t_1, \ldots, t_n) = f(\varphi(x_1) + z^{pe_1}, \ldots, \varphi(x_n) + z^{pe_n}) = \varphi(f) + \text{element of } (z)$$

is in J as well. In other words, $\varphi'(I)C \subset J$ and we obtain a surjection

$$\tau' : C/\varphi'(I)C \longrightarrow C/J \cong B$$

of algebras étale over A. Finally, the algebra C is generated by the elements $\varphi(x_1), \ldots, \varphi(x_n), y_1, \ldots, y_r$ over $C' = \mathrm{Im}(\varphi')$ with $\varphi(x_1), \ldots, \varphi(x_n), y_1, \ldots, y_{s+1}$ integral over $C' = \mathrm{Im}(\varphi')$. Hence $NF(C, \varphi', \tau') < r - s = NF(C, \varphi, \tau)$. This finishes the proof. □

0G1G Lemma 58.32.2. Let k be a field of characteristic $p > 0$. Let $X \to \mathbf{A}^n_k$ be an étale morphism with X affine. Then there exists a finite étale morphism $X \to \mathbf{A}^n_k$.

Proof. Write $X = \mathrm{Spec}(C)$. Set $A = 0$ and denote $I = k[x_1, \ldots, x_n]$. By assumption there exists some étale k-algebra map $\varphi : k[x_1, \ldots, x_n] \to C$. Denote $\tau : C/\varphi(I)C \to 0$ the unique surjection. We may choose φ and τ such that $N(C, \varphi, \tau)$ is minimal. By Lemma 58.32.1 we get $N(C, \varphi, \tau) = 0$. Hence φ is finite étale. □

0G1H Lemma 58.32.3. Let k be a field of characteristic $p > 0$. Let $Z \subset \mathbf{A}^n_k$ be a closed subscheme. Let $Y \to Z$ be finite étale. There exists a finite étale morphism $f : U \to \mathbf{A}^n_k$ such that there is an open and closed immersion $Y \to f^{-1}(Z)$ over Z.

Proof. Let us turn the problem into algebra. Write $\mathbf{A}^n_k = \mathrm{Spec}(k[x_1, \ldots, x_n])$. Then $Z = \mathrm{Spec}(A)$ where $A = k[x_1, \ldots, x_n]/I$ for some ideal $I \subset k[x_1, \ldots, x_n]$. Write $Y = \mathrm{Spec}(B)$ so that $Y \to Z$ corresponds to the finite étale k-algebra map $A \to B$.

By Algebra, Lemma 10.143.10 there exists an étale ring map

$$\varphi : k[x_1, \ldots, x_n] \to C$$

and a surjective A-algebra map $\tau : C/\varphi(I)C \to B$. (We can even choose C, φ, τ such that τ is an isomorphism, but we won't use this). We may choose φ and τ such that $N(C, \varphi, \tau)$ is minimal. By Lemma 58.32.1 we get $N(C, \varphi, \tau) = 0$. Hence φ is finite étale.

Let $f : U = \mathrm{Spec}(C) \to \mathbf{A}^n_k$ be the finite étale morphism corresponding to φ. The morphism $Y \to f^{-1}(Z) = \mathrm{Spec}(C/\varphi(I)C)$ induced by τ is a closed immersion as τ is surjective and open as it is an étale morphism by Morphisms, Lemma 29.36.18. This finishes the proof. □

Here is the main result.

0G1I Proposition 58.32.4. Let p be a prime number. Let $i : Z \to X$ be a closed immersion of connected affine schemes over $\mathbf{F}_p$. For any geometric point $\overline{z}$ of Z the map

$$\pi_1(Z, \overline{z}) \to \pi_1(X, \overline{z})$$

is injective.

Proof. Let $Y \to Z$ be a finite étale morphism. It suffices to construct a finite étale morphism $f : U \to X$ such that Y is isomorphic to an open and closed subscheme of $f^{-1}(Z)$, see Lemma 58.4.4. Write $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$ so the closed immersion $Y \to X$ is given by a surjection $R \to A$. We may write $A = \mathrm{colim}\, A_i$ as the filtered colimit of its $\mathbf{F}_p$-subalgebras of finite type. By Lemma 58.14.1 we can find an i and a finite étale morphism $Y_i \to Z_i = \mathrm{Spec}(A_i)$ such that $Y = Z \times_{Z_i} Y_i$. Choose a surjection $\mathbf{F}_p[x_1, \ldots, x_n] \to A_i$. This determines a closed immersion

$$Z_i = \mathrm{Spec}(A_i) \longrightarrow X_i = \mathbf{A}^n_{\mathbf{F}_p} = \mathrm{Spec}(\mathbf{F}_p[x_1, \ldots, x_n])$$

By the universal property of polynomial algebras and since $R \to A$ is surjective, we can find a commutative diagram

$$\begin{tikzcd} \mathbf{F}_p[x_1, \ldots, x_n] \arrow[r] \arrow[d] & A_i \arrow[d] \\ R \arrow[r] & A \end{tikzcd}$$

of $\mathbf{F}_p$-algebras. Thus we have a commutative diagram

$$\begin{tikzcd} Y_i \arrow[r] & Z_i \arrow[r] & X_i \\ Y \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & X \arrow[u] \end{tikzcd}$$

whose right square is cartesian. Clearly, if we can find $f_i : U_i \to X_i$ finite étale such that Y_i is isomorphic to an open and closed subscheme of $f_i^{-1}(Z_i)$, then the base change $f : U \to X$ of f_i by $X \to X_i$ is a solution to our problem. Thus we conclude by applying Lemma 58.32.3 to $Y_i \to Z_i \to X_i = \mathbf{A}^n_{\mathbf{F}_p}$. $\square$

58.33. Other chapters

CHAPTER 59

Étale Cohomology

03N1 ## 59.1. Introduction

03N2 This chapter is the first in a series of chapter on the étale cohomology of schemes. In this chapter we discuss the very basics of the étale topology and cohomology of abelian sheaves in this topology. Many of the topics discussed may be safely skipped on a first reading; please see the advice in the next section as to how to decide what to skip.

The initial version of this chapter was formed by the notes of the first part of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. The second part of the course can be found in the chapter on the trace formula, see The Trace Formula, Section 64.1.

59.2. Which sections to skip on a first reading?

04JG We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word "topos", or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:

(1) Comparing big and small topoi, Section 59.99.
(2) Recovering morphisms, Section 59.40.
(3) Push and pull, Section 59.41.
(4) Property (A), Section 59.42.
(5) Property (B), Section 59.43.
(6) Property (C), Section 59.44.
(7) Topological invariance of the small étale site, Section 59.45.
(8) Integral universally injective morphisms, Section 59.47.
(9) Big sites and pushforward, Section 59.48.
(10) Exactness of big lower shriek, Section 59.49.

Besides these sections there are some sporadic results that may be skipped that the reader can recognize by the keywords given above.

59.3. Prologue

03N3 These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that it fails to give the results that we would want is that if X is a complex variety and $\mathcal{F}$ is a constant sheaf then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of a constant sheaf are surjective. We say that the sheaf is flasque. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are "not enough" open sets in the Zariski topology to detect this higher cohomology.

On the other hand, if X is a smooth projective complex variety, then

$$H_{Betti}^{2\dim X}(X(\mathbf{C}), \Lambda) = \Lambda \quad \text{for } \Lambda = \mathbf{Z},\ \mathbf{Z}/n\mathbf{Z},$$

where $X(\mathbf{C})$ means the set of complex points of X. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

59.4. The étale topology

03N4 It is very hard to simply "add" extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If X is a smooth projective variety over $\mathbf{C}$, then this means

(1) U is a disjoint union of smooth varieties, and
(2) φ is (analytically) locally an isomorphism.

The word "analytically" refers to the usual (transcendental) topology over $\mathbf{C}$. So the second condition means that the derivative of φ has full rank everywhere (and in particular all the components of U have the same dimension as X).

A double cover – loosely defined as a finite degree 2 map between varieties – for example

$$\mathrm{Spec}(\mathbf{C}[t]) \longrightarrow \mathrm{Spec}(\mathbf{C}[t]), \quad t \longmapsto t^2$$

will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when $t = 0$. For a finite map between varieties over $\mathbf{C}$ to be étale all the fibers should have the same number of points. Removing the point $t = 0$ from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely open subsets of X, even though their images always are.

03N5 **Definition 59.4.1.** A family of morphisms $\{\varphi_i : U_i \to X\}_{i\in I}$ is called an *étale covering* if each φ_i is an étale morphism and their images cover X, i.e., $X = \bigcup_{i\in I} \varphi_i(U_i)$.

This "defines" the étale topology. In other words, we can now say what the sheaves are. An étale sheaf $\mathcal{F}$ of sets (resp. abelian groups, vector spaces, etc) on X is the data:

(1) for each étale morphism $\varphi : U \to X$ a set (resp. abelian group, vector space, etc) $\mathcal{F}(U)$,

(2) for each pair U, U' of étale schemes over X, and each morphism $U \to U'$ over X (which is automatically étale) a restriction map $\rho_U^{U'} : \mathcal{F}(U') \to \mathcal{F}(U)$

These data have to satisfy the condition that $\rho_U^U = \text{id}$ in case of the identity morphism $U \to U$ and that $\rho_U^{U'} \circ \rho_{U'}^{U''} = \rho_U^{U''}$ when we have morphisms $U \to U' \to U''$ of schemes étale over X as well as the following sheaf axiom:

(*) for every étale covering $\{\varphi_i : U_i \to U\}_{i \in I}$, the diagram

$$\emptyset \longrightarrow \mathcal{F}(U) \longrightarrow \Pi_{i \in I} \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is exact in the category of sets (resp. abelian groups, vector spaces, etc).

03N6 Remark 59.4.2. In the last statement, it is essential not to forget the case where $i = j$ which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

59.5. Feats of the étale topology

03N7 For a natural number $n \in \mathbf{N} = \{1, 2, 3, 4, \ldots\}$ it is true that

$$H^2_{\acute{e}tale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}.$$

More generally, if X is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of $X(\mathbf{C})$, i.e.,

$$\dim_{\mathbf{F}_q} H^{2i}_{\acute{e}tale}(X, \mathbf{F}_q) = \dim_{\mathbf{F}_q} H^{2i}_{Betti}(X(\mathbf{C}), \mathbf{F}_q).$$

This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has

$$H^2_{\acute{e}tale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}) = 0.$$

By contrast $H^2_{Betti}(\mathbf{P}^1(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}$ as the topological space $\mathbf{P}^1(\mathbf{C})$ is homeomorphic to a 2-sphere. There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.

59.6. A computation

03N8 How do we compute the cohomology of $\mathbf{P}^1_{\mathbf{C}}$ with coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$? We use Čech cohomology. A covering of $\mathbf{P}^1_{\mathbf{C}}$ is given by the two standard opens U_0, U_1, which are both isomorphic to $\mathbf{A}^1_{\mathbf{C}}$, and whose intersection is isomorphic to $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\} = \mathbf{G}_{m,\mathbf{C}}$. It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence

$$H^{i-1}_{\acute{e}tale}(U_0 \cap U_1, \Lambda) \to H^i_{\acute{e}tale}(\mathbf{P}^1_C, \Lambda) \to H^i_{\acute{e}tale}(U_0, \Lambda) \oplus H^i_{\acute{e}tale}(U_1, \Lambda) \to H^i_{\acute{e}tale}(U_0 \cap U_1, \Lambda).$$

To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,

$$H^q_{\acute{e}tale}(\mathbf{A}^1_{\mathbf{C}}, \Lambda) = 0 \quad \text{for } q \geq 1,$$

and

$$H^q_{étale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \text{ and} \\ 0 & \text{for } q \geq 2. \end{cases}$$

These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

Higher cohomology. This is taken care of by the following general fact: if X is an affine curve over $\mathbf{C}$, then

$$H^q_{étale}(X, \mathbf{Z}/n\mathbf{Z}) = 0 \quad \text{for } q \geq 2.$$

This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.

Cohomology in degree 1. We use the following identifications:

$$\begin{aligned} H^1_{étale}(X, \mathbf{Z}/n\mathbf{Z}) &= \left\{ \begin{matrix} \text{sheaves of sets } \mathcal{F} \text{ on the étale site } X_{étale} \text{ endowed with an} \\ \text{action } \mathbf{Z}/n\mathbf{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbf{Z}/n\mathbf{Z}\text{-torsor.} \end{matrix} \right\} \Big/ \cong \\ &= \left\{ \begin{matrix} \text{morphisms } Y \to X \text{ which are finite étale together} \\ \text{with a free } \mathbf{Z}/n\mathbf{Z} \text{ action such that } X = Y/(\mathbf{Z}/n\mathbf{Z}). \end{matrix} \right\} \Big/ \cong . \end{aligned}$$

The first identification is very general (it is true for any cohomology theory on a site) and has nothing to do with the étale topology. The second identification is a consequence of descent theory. The last set describes a collection of geometric objects on which we can get our hands.

The curve $\mathbf{A}^1_{\mathbf{C}}$ has no nontrivial finite étale covering and hence $H^1_{étale}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = 0$. This can be seen either topologically or by using the argument in the next paragraph.

Let us describe the finite étale coverings $\varphi : Y \to \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$. It suffices to consider the case where Y is connected, which we assume. We are going to find out what Y can be by applying the Riemann-Hurwitz formula (of course this is a bit silly, and you can go ahead and skip the next section if you like). Say that this morphism is n to 1, and consider a projective compactification

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\varphi"] & \bar{Y} \arrow[d, "\bar{\varphi}"] \\ \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} \arrow[r, hook] & \mathbf{P}^1_{\mathbf{C}} \end{tikzcd}$$

Even though φ is étale and does not ramify, $\bar{\varphi}$ may ramify at 0 and ∞. Say that the preimages of 0 are the points $y_1, \ldots, y_r$ with indices of ramification $e_1, \ldots e_r$, and that the preimages of ∞ are the points $y'_1, \ldots, y'_s$ with indices of ramification $d_1, \ldots d_s$. In particular, $\sum e_i = n = \sum d_j$. Applying the Riemann-Hurwitz formula, we get

$$2g_Y - 2 = -2n + \sum (e_i - 1) + \sum (d_j - 1)$$

and therefore $g_Y = 0$, $r = s = 1$ and $e_1 = d_1 = n$. Hence $Y \cong \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, and it is easy to see that $\varphi(z) = \lambda z^n$ for some $\lambda \in \mathbf{C}^*$. After reparametrizing Y we may assume $\lambda = 1$. Thus our covering is given by taking the nth root of the coordinate on $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$.

Remember that we need to classify the coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ together with free $\mathbf{Z}/n\mathbf{Z}$-actions on them. In our case any such action corresponds to an automorphism of Y sending z to $\zeta_n z$, where ζ_n is a primitive nth root of unity. There are $\phi(n)$ such actions (here $\phi(n)$ means the Euler function). Thus there are exactly $\phi(n)$ connected finite étale coverings with a given free $\mathbf{Z}/n\mathbf{Z}$-action, each corresponding to a primitive nth root of unity. We leave it to the reader to see that the disconnected finite étale degree n coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ with a given free $\mathbf{Z}/n\mathbf{Z}$-action correspond one-to-one with nth roots of 1 which are not primitive. In other words, this computation shows that

$$H^1_{\acute{e}tale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}(\mu_n(\mathbf{C}), \mathbf{Z}/n\mathbf{Z}) \cong \mathbf{Z}/n\mathbf{Z}.$$

The first identification is canonical, the second isn't, see Remark 59.69.5. Since the proof of Riemann-Hurwitz does not use the computation of cohomology, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).

59.7. Nontorsion coefficients

03N9 To study nontorsion coefficients, one makes the following definition:

$$H^i_{\acute{e}tale}(X, \mathbf{Q}_\ell) := \left(\lim_n H^i_{\acute{e}tale}(X, \mathbf{Z}/\ell^n\mathbf{Z})\right) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

The symbol $\lim_n$ denote the limit of the system of cohomology groups $H^i_{\acute{e}tale}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ indexed by n, see Categories, Section 4.21. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

59.8. Sheaf theory

03NA At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding too much material here, just enough so the material later in this chapter makes sense.

59.9. Presheaves

03NB A reference for this section is Sites, Section 7.2.

03NC Definition 59.9.1. Let $\mathcal{C}$ be a category. A presheaf of sets (respectively, an abelian presheaf) on $\mathcal{C}$ is a functor $\mathcal{C}^{opp} \to \textit{Sets}$ (resp. $\textit{Ab}$).

Terminology. If $U \in \mathrm{Ob}(\mathcal{C})$, then elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U. For $\varphi : V \to U$ in $\mathcal{C}$, the map $\mathcal{F}(\varphi) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the restriction map and is often denoted $s \mapsto s|_V$ or sometimes $s \mapsto \varphi^* s$. The notation $s|_V$ is ambiguous since the restriction map depends on φ, but it is a standard abuse of notation. We also use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

Saying that $\mathcal{F}$ is a functor means that if $W \to V \to U$ are morphisms in $\mathcal{C}$ and $s \in \Gamma(U, \mathcal{F})$ then $(s|_V)|_W = s|_W$, with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms $\mathrm{id}_U : U \to U$ are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on $\mathcal{C}$ is denoted $\textit{PSh}(\mathcal{C})$ (resp. $\textit{PAb}(\mathcal{C})$). It is the category of functors from $\mathcal{C}^{opp}$ to $\textit{Sets}$ (resp. $\textit{Ab}$), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories $\textit{PSh}(\mathcal{C})$ and $\textit{PAb}(\mathcal{C})$ when the category

$\mathcal{C}$ is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn't small it should be listed in Categories, Remark 4.2.2.)

03ND Example 59.9.2. Given an object $X \in \mathrm{Ob}(\mathcal{C})$, we consider the functor

$$\begin{array}{rccc} h_X : & \mathcal{C}^{opp} & \longrightarrow & \text{Sets} \\ & U & \longmapsto & h_X(U) = \mathrm{Mor}_\mathcal{C}(U, X) \\ & V \xrightarrow{\varphi} U & \longmapsto & \varphi \circ - : h_X(U) \to h_X(V). \end{array}$$

It is a presheaf, called the representable presheaf associated to X. It is not true that representable presheaves are sheaves in every topology on every site.

03NE Lemma 59.9.3 (Yoneda). Let $\mathcal{C}$ be a category, and $X, Y \in \mathrm{Ob}(\mathcal{C})$. There is a natural bijection

$$\begin{array}{ccc} \mathrm{Mor}_\mathcal{C}(X, Y) & \longrightarrow & \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_X, h_Y) \\ \psi & \longmapsto & h_\psi = \psi \circ - : h_X \to h_Y. \end{array}$$

Proof. See Categories, Lemma 4.3.5. □

59.10. Sites

03NF

03NG Definition 59.10.1. Let $\mathcal{C}$ be a category. A family of morphisms with fixed target $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ is the data of

(1) an object $U \in \mathcal{C}$,
(2) a set I (possibly empty), and
(3) for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $\mathcal{C}$ with target U.

There is a notion of a morphism of families of morphisms with fixed target. A special case of that is the notion of a refinement. A reference for this material is Sites, Section 7.8.

03NH Definition 59.10.2. A site[1] consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target called coverings, such that

(1) (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,
(2) (locality) if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$, then
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j) \in \prod_{i \in I} \{i\} \times I_i}$$
is also a covering, and
(3) (base change) if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then
    (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$, and
    (b) $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

For us the category underlying a site is always "small", i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 7.6.3.

[1]What we call a site is a called a category endowed with a pretopology in [AGV71, Exposé II, Définition 1.3]. In [Art62] it is called a category with a Grothendieck topology.

03NI Example 59.10.3. If X is a topological space, then it has an associated site X_{Zar} defined as follows: the objects of X_{Zar} are the open subsets of X, the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fiber products. All the verifications are trivial and everything works as expected.

59.11. Sheaves

03NJ

03NK Definition 59.11.1. A presheaf $\mathcal{F}$ of sets (resp. abelian presheaf) on a site $\mathcal{C}$ is said to be a separated presheaf if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$ the map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i)$$

is injective. Here the map is $s \mapsto (s|_{U_i})_{i\in I}$. The presheaf $\mathcal{F}$ is a sheaf if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$, the diagram

03NL (59.11.1.1) $$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{i,j\in I} \mathcal{F}(U_i \times_U U_j),$$

where the first map is $s \mapsto (s|_{U_i})_{i\in I}$ and the two maps on the right are $(s_i)_{i\in I} \mapsto (s_i|_{U_i\times_U U_j})$ and $(s_i)_{i\in I} \mapsto (s_j|_{U_i\times_U U_j})$, is an equalizer diagram in the category of sets (resp. abelian groups).

03NM Remark 59.11.2. For the empty covering (where $I = \emptyset$), this implies that $\mathcal{F}(\emptyset)$ is an empty product, which is a final object in the corresponding category (a singleton, for both Sets and Ab).

03NN Example 59.11.3. Working this out for the site X_{Zar} associated to a topological space, see Example 59.10.3, gives the usual notion of sheaves.

03NO Definition 59.11.4. We denote $Sh(\mathcal{C})$ (resp. $\text{Ab}(\mathcal{C})$) the full subcategory of $\text{PSh}(\mathcal{C})$ (resp. $\text{PAb}(\mathcal{C})$) whose objects are sheaves. This is the category of sheaves of sets (resp. abelian sheaves) on $\mathcal{C}$.

59.12. The example of G-sets

03NP Let G be a group and define a site $\mathcal{T}_G$ as follows: the underlying category is the category of G-sets, i.e., its objects are sets endowed with a left G-action and the morphisms are equivariant maps; and the coverings of $\mathcal{T}_G$ are the families $\{\varphi_i : U_i \to U\}_{i\in I}$ satisfying $U = \bigcup_{i\in I} \varphi_i(U_i)$.

There is a special object in the site $\mathcal{T}_G$, namely the G-set G endowed with its natural action by left translations. We denote it ${}_GG$. Observe that there is a natural group isomorphism

$$\begin{matrix} \rho : & G^{opp} & \longrightarrow & \text{Aut}_{G\text{-Sets}}({}_GG) \\ & g & \longmapsto & (h \mapsto hg). \end{matrix}$$

In particular, for any presheaf $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ inherits a G-action via ρ. (Note that by contravariance of $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ is again a left G-set.) In fact, the functor

$$\begin{matrix} Sh(\mathcal{T}_G) & \longrightarrow & G\text{-Sets} \\ \mathcal{F} & \longmapsto & \mathcal{F}({}_GG) \end{matrix}$$

is an equivalence of categories. Its quasi-inverse is the functor $X \mapsto h_X$. Without giving the complete proof (which can be found in Sites, Section 7.9) let us try to explain why this is true.

(1) If S is a G-set, we can decompose it into orbits $S = \coprod_{i \in I} O_i$. The sheaf axiom for the covering $\{O_i \to S\}_{i \in I}$ says that

$$\mathcal{F}(S) \longrightarrow \prod_{i \in I} \mathcal{F}(O_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j)$$

is an equalizer. Observing that fibered products in G-Sets are induced from fibered products in Sets, and using the fact that $\mathcal{F}(\emptyset)$ is a G-singleton, we get that

$$\prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j) = \prod_{i \in I} \mathcal{F}(O_i)$$

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that $\mathcal{F}(S) = \prod_{i \in I} \mathcal{F}(O_i)$.

(2) If S is the G-set $S = G/H$ and $\mathcal{F}$ is a sheaf on $\mathcal{T}_G$, then we claim that

$$\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$$

and in particular $\mathcal{F}(\{*\}) = \mathcal{F}({}_GG)^G$. To see this, let's use the sheaf axiom for the covering $\{{}_GG \to G/H\}$ of S. We have

$$\begin{array}{rcl} {}_GG \times_{G/H} {}_GG & \cong & G \times H \\ (g_1, g_2) & \longmapsto & (g_1, g_1^{-1} g_2) \end{array}$$

is a disjoint union of copies of ${}_GG$ (as a G-set). Hence the sheaf axiom reads

$$\mathcal{F}(G/H) \longrightarrow \mathcal{F}({}_GG) \rightrightarrows \prod_{h \in H} \mathcal{F}({}_GG)$$

where the two maps on the right are $s \mapsto (s)_{h \in H}$ and $s \mapsto (hs)_{h \in H}$. Therefore $\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$ as claimed.

This doesn't quite prove the claimed equivalence of categories, but it shows at least that a sheaf $\mathcal{F}$ is entirely determined by its sections over ${}_GG$. Details (and set theoretical remarks) can be found in Sites, Section 7.9.

03NQ

59.13. Sheafification

03NR **Definition 59.13.1.** Let $\mathcal{F}$ be a presheaf on the site $\mathcal{C}$ and $\mathcal{U} = \{U_i \to U\} \in \text{Cov}(\mathcal{C})$. We define the zeroth Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ by

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod\nolimits_{i \in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$

There is a canonical map $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$, $s \mapsto (s|_{U_i})_{i \in I}$. We say that a morphism of coverings from a covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ to $\mathcal{U}$ is a triple (χ, α, χ_j), where $\chi : V \to U$ is a morphism, $\alpha : J \to I$ is a map of sets, and for all $j \in J$ the morphism χ_j fits into a commutative diagram

$$\begin{tikzcd} V_j \arrow[r, "\chi_j"'] \arrow[d] & U_{\alpha(j)} \arrow[d] \\ V \arrow[r, "\chi"] & U. \end{tikzcd}$$

Given the data $\chi, \alpha, \{\chi_j\}_{j \in J}$ we define

$$\begin{array}{rcl} \check{H}^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{V}, \mathcal{F}) \\ (s_i)_{i \in I} & \longmapsto & \left(\chi_j^* \left(s_{\alpha(j)}\right)\right)_{j \in J}. \end{array}$$

We then claim that

(1) the map is well-defined, and
(2) depends only on χ and is independent of the choice of $\alpha, \{\chi_j\}_{j \in J}$.

We omit the proof of the first fact. To see part (2), consider another triple (ψ, β, ψ_j) with $\chi = \psi$. Then we have the commutative diagram

$$\begin{tikzcd} V_j \arrow[rr, "{(\chi_j, \psi_j)}"'] \arrow[dd] & & U_{\alpha(j)} \times_U U_{\beta(j)} \arrow[dl] \arrow[dr] & \\ & U_{\alpha(j)} \arrow[dr] & & U_{\beta(j)} \arrow[dl] \\ V \arrow[rr, "\chi = \psi"] & & U. & \end{tikzcd}$$

Given a section $s \in \mathcal{F}(\mathcal{U})$, its image in $\mathcal{F}(V_j)$ under the map given by $(\chi, \alpha, \{\chi_j\}_{j \in J})$ is $\chi_j^* s_{\alpha(j)}$, and its image under the map given by $(\psi, \beta, \{\psi_j\}_{j \in J})$ is $\psi_j^* s_{\beta(j)}$. These two are equal since by assumption $s \in \check{H}^0(\mathcal{U}, \mathcal{F})$ and hence both are equal to the pullback of the common value

$$s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}} = s_{\beta(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}}$$

pulled back by the map (χ_j, ψ_j) in the diagram.

03NS Theorem 59.13.2. Let $\mathcal{C}$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$.

(1) The rule

$$U \mapsto \mathcal{F}^+(U) := \operatorname{colim}_{\mathcal{U} \text{ covering of } U} \check{H}^0(\mathcal{U}, \mathcal{F})$$

is a presheaf. And the colimit is a directed one.
(2) There is a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.
(3) If $\mathcal{F}$ is a separated presheaf then $\mathcal{F}^+$ is a sheaf and the map in (2) is injective.
(4) $\mathcal{F}^+$ is a separated presheaf.
(5) $\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf, and the canonical map induces a functorial isomorphism

$$\operatorname{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G})$$

for any $\mathcal{G} \in Sh(\mathcal{C})$.

Proof. See Sites, Theorem 7.10.10. $\square$

In other words, this means that the natural map $\mathcal{F} \to \mathcal{F}^\#$ is a left adjoint to the forgetful functor $Sh(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$.

59.14. Cohomology

03NT The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

03NU **Theorem 59.14.1.** The category of abelian sheaves on a site is an abelian category which has enough injectives.

Proof. See Modules on Sites, Lemma 18.3.1 and Injectives, Theorem 19.7.4. □

So we can define cohomology as the right-derived functors of the sections functor: if $U \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{F} \in \mathrm{Ab}(\mathcal{C})$,

$$H^p(U, \mathcal{F}) := R^p\Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U, -)$ is left exact. This is true and is part of why the category $\mathrm{Ab}(\mathcal{C})$ is abelian, see Modules on Sites, Lemma 18.3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 21.2.

59.15. The fpqc topology

03NV Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

03NW **Definition 59.15.1.** Let T be a scheme. An *fpqc covering* of T is a family $\{\varphi_i : T_i \to T\}_{i \in I}$ such that

1. each φ_i is a flat morphism and $\bigcup_{i \in I} \varphi_i(T_i) = T$, and
2. for each affine open $U \subset T$ there exists a finite set K, a map $\mathbf{i} : K \to I$ and affine opens $U_{\mathbf{i}(k)} \subset T_{\mathbf{i}(k)}$ such that $U = \bigcup_{k \in K} \varphi_{\mathbf{i}(k)}(U_{\mathbf{i}(k)})$.

03NX **Remark 59.15.2.** The first condition corresponds to fp, which stands for fidèlement plat, faithfully flat in french, and the second to qc, quasi-compact. The second part of the first condition is unnecessary when the second condition holds.

03NY **Example 59.15.3.** Examples of fpqc coverings.

1. Any Zariski open covering of T is an fpqc covering.
2. A family $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is a faithfully flat ring map.
3. If $f : X \to Y$ is flat, surjective and quasi-compact, then $\{f : X \to Y\}$ is an fpqc covering.
4. The morphism $\varphi : \coprod_{x \in \mathbf{A}^1_k} \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_k, x}) \to \mathbf{A}^1_k$, where k is a field, is flat and surjective. It is not quasi-compact, and in fact the family $\{\varphi\}$ is not an fpqc covering.
5. Write $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$. Denote $i_x : D(x) \to \mathbf{A}^2_k$ and $i_y : D(y) \to \mathbf{A}^2_k$ the standard opens. Then the families $\{i_x, i_y, \mathrm{Spec}(k[[x, y]]) \to \mathbf{A}^2_k\}$ and $\{i_x, i_y, \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^2_k, 0}) \to \mathbf{A}^2_k\}$ are fpqc coverings.

03NZ **Lemma 59.15.4.** The collection of fpqc coverings on the category of schemes satisfies the axioms of site.

Proof. See Topologies, Lemma 34.9.7. □

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 34.9. It comes from our requirement that sites are "small", but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Although this does not strictly speaking prevent us from defining "partial" fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

03X6 Definition 59.15.5. Let S be a scheme. The category of schemes over S is denoted Sch/S. Consider a functor $\mathcal{F} : (Sch/S)^{opp} \to \text{Sets}$, in other words a presheaf of sets. We say $\mathcal{F}$ satisfies the sheaf property for the fpqc topology if for every fpqc covering $\{U_i \to U\}_{i\in I}$ of schemes over S the diagram (59.11.1.1) is an equalizer diagram.

We similarly say that $\mathcal{F}$ satisfies the sheaf property for the Zariski topology if for every open covering $U = \bigcup_{i\in I} U_i$ the diagram (59.11.1.1) is an equalizer diagram. See Schemes, Definition 26.15.3. Clearly, this is equivalent to saying that for every scheme T over S the restriction of $\mathcal{F}$ to the opens of T is a (usual) sheaf.

03O1 Lemma 59.15.6. Let $\mathcal{F}$ be a presheaf on Sch/S. Then $\mathcal{F}$ satisfies the sheaf property for the fpqc topology if and only if

(1) $\mathcal{F}$ satisfies the sheaf property with respect to the Zariski topology, and
(2) for every faithfully flat morphism $\text{Spec}(B) \to \text{Spec}(A)$ of affine schemes over S, the sheaf axiom holds for the covering $\{\text{Spec}(B) \to \text{Spec}(A)\}$. Namely, this means that

$$\mathcal{F}(\text{Spec}(A)) \longrightarrow \mathcal{F}(\text{Spec}(B)) \rightrightarrows \mathcal{F}(\text{Spec}(B \otimes_A B))$$

is an equalizer diagram.

Proof. See Topologies, Lemma 34.9.13. $\square$

An alternative way to think of a presheaf $\mathcal{F}$ on Sch/S which satisfies the sheaf condition for the fpqc topology is as the following data:

(1) for each T/S, a usual (i.e., Zariski) sheaf $\mathcal{F}_T$ on T_{Zar},
(2) for every map $f : T' \to T$ over S, a restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$

such that

(a) the restriction mappings are functorial,
(b) if $f : T' \to T$ is an open immersion then the restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism, and
(c) for every faithfully flat morphism $\text{Spec}(B) \to \text{Spec}(A)$ over S, the diagram

$$\mathcal{F}_{\text{Spec}(A)}(\text{Spec}(A)) \longrightarrow \mathcal{F}_{\text{Spec}(B)}(\text{Spec}(B)) \rightrightarrows \mathcal{F}_{\text{Spec}(B\otimes_A B)}(\text{Spec}(B \otimes_A B))$$

is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on Sch/S satisfying the sheaf condition for the Zariski topology. By Lemma 59.15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

03O2 Example 59.15.7. Consider the presheaf

$$\begin{array}{cccc} \mathcal{F} : & (Sch/S)^{opp} & \longrightarrow & \text{Ab} \\ & T/S & \longmapsto & \Gamma(T, \Omega_{T/S}). \end{array}$$

The compatibility of differentials with localization implies that $\mathcal{F}$ is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case $S = \mathrm{Spec}(\mathbf{F}_p)$ and the morphism

$$\varphi : V = \mathrm{Spec}(\mathbf{F}_p[v]) \to U = \mathrm{Spec}(\mathbf{F}_p[u])$$

given by mapping u to v^p. The family $\{\varphi\}$ is an fpqc covering, yet the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ sends the generator $\mathrm{d}u$ to $\mathrm{d}(v^p) = 0$, so it is the zero map, and the diagram

$$\mathcal{F}(U) \xrightarrow{0} \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is not an equalizer. We will see later that $\mathcal{F}$ does in fact give rise to a sheaf on the étale and smooth sites.

03O3 Lemma 59.15.8. Any representable presheaf on Sch/S satisfies the sheaf condition for the fpqc topology.

Proof. See Descent, Lemma 35.13.7. □

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that the fpqc topology is weaker than the canonical topology, or that the fpqc topology is subcanonical. In the setting of sites this is discussed in Sites, Section 7.12.

03O4 Remark 59.15.9. The fpqc is finer than the Zariski, étale, smooth, syntomic, and fppf topologies. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf on the Zariski, étale, smooth, syntomic, and fppf sites. In particular representable presheaves will be sheaves on the étale site of a scheme for example.

03O5 Example 59.15.10. Let S be a scheme. Consider the additive group scheme $\mathbf{G}_{a,S} = \mathbf{A}^1_S$ over S, see Groupoids, Example 39.5.3. The associated representable presheaf is given by

$$h_{\mathbf{G}_{a,S}}(T) = \mathrm{Mor}_S(T, \mathbf{G}_{a,S}) = \Gamma(T, \mathcal{O}_T).$$

By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor

$$\begin{array}{rccc} \mathcal{O} : & (Sch/S)^{opp} & \longrightarrow & \text{Rings} \\ & T/S & \longmapsto & \Gamma(T, \mathcal{O}_T) \end{array}$$

which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of $\mathcal{O}$-module, and so on and so forth.

59.16. Faithfully flat descent

03O6 In this section we discuss faithfully flat descent for quasi-coherent modules. More precisely, we will prove quasi-coherent modules satisfy effective descent with respect to fpqc coverings.

03O7 Definition 59.16.1. Let $\mathcal{U} = \{t_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. A descent datum for quasi-coherent sheaves with respect to $\mathcal{U}$ is a collection $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ where

(1) $\mathcal{F}_i$ is a quasi-coherent sheaf on T_i, and

(2) $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ is an isomorphism of modules on $T_i \times_T T_j$,

such that the cocycle condition holds: the diagrams

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{01}^*\varphi_{ij}"] \arrow[dr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] & & \mathrm{pr}_1^*\mathcal{F}_j \arrow[dl, "\mathrm{pr}_{12}^*\varphi_{jk}"] \\ & \mathrm{pr}_2^*\mathcal{F}_k & \end{tikzcd}$$

commute on $T_i \times_T T_j \times_T T_k$. This descent datum is called effective if there exist a quasi-coherent sheaf $\mathcal{F}$ over T and $\mathcal{O}_{T_i}$-module isomorphisms $\varphi_i : t_i^*\mathcal{F} \cong \mathcal{F}_i$ compatible with the maps φ_{ij}, namely

$$\varphi_{ij} = \mathrm{pr}_1^*(\varphi_j) \circ \mathrm{pr}_0^*(\varphi_i)^{-1}.$$

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.

03O8 Theorem 59.16.2. If $\mathcal{V} = \{T_i \to T\}_{i\in I}$ is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to $\mathcal{V}$ are effective.

Proof. See Descent, Proposition 35.5.2. □

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 6.33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is a faithfully flat ring map. This is a lemma in algebra, which we now present.

Descent of modules. If $A \to B$ is a ring map, we consider the complex

$$(B/A)_\bullet : B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

where B is in degree 0, $B \otimes_A B$ in degree 1, etc, and the maps are given by

$$\begin{array}{rcl} b & \mapsto & 1 \otimes b - b \otimes 1, \\ b_0 \otimes b_1 & \mapsto & 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1, \\ & \text{etc.} & \end{array}$$

03O9 Lemma 59.16.3. If $A \to B$ is faithfully flat, then the complex $(B/A)_\bullet$ is exact in positive degrees, and $H^0((B/A)_\bullet) = A$.

Proof. See Descent, Lemma 35.3.6. □

Grothendieck proves this in three steps. Firstly, he assumes that the map $A \to B$ has a section, and constructs an explicit homotopy to the complex where A is the only nonzero term, in degree 0. Secondly, he observes that to prove the result, it suffices to do so after a faithfully flat base change $A \to A'$, replacing B with $B' = B \otimes_A A'$. Thirdly, he applies the faithfully flat base change $A \to A' = B$ and remark that the map $A' = B \to B' = B \otimes_A B$ has a natural section.

The same strategy proves the following lemma.

03OA Lemma 59.16.4. If $A \to B$ is faithfully flat and M is an A-module, then the complex $(B/A)_\bullet \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_\bullet \otimes_A M) = M$.

Proof. See Descent, Lemma 35.3.6. □

03OB Definition 59.16.5. Let $A \to B$ be a ring map and N a B-module. A descent datum for N with respect to $A \to B$ is an isomorphism $\varphi : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-modules

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\varphi_{01}"] \arrow[dr, "\varphi_{02}"'] & & B \otimes_A N \otimes_A B \arrow[dl, "\varphi_{12}"] \\ & B \otimes_A B \otimes_A N & \end{tikzcd}$$

commutes where $\varphi_{01} = \varphi \otimes \mathrm{id}_B$ and similarly for φ_{12} and φ_{02}.

If $N' = B \otimes_A M$ for some A-module M, then it has a canonical descent datum given by the map

$$\begin{array}{rccc} \varphi_{\mathrm{can}} : & N' \otimes_A B & \to & B \otimes_A N' \\ & b_0 \otimes m \otimes b_1 & \mapsto & b_0 \otimes b_1 \otimes m. \end{array}$$

03OC Definition 59.16.6. A descent datum (N, φ) is called effective if there exists an A-module M such that $(N, \varphi) \cong (B \otimes_A M, \varphi_{\mathrm{can}})$, with the obvious notion of isomorphism of descent data.

Theorem 59.16.2 is a consequence the following result.

03OD Theorem 59.16.7. If $A \to B$ is faithfully flat then descent data with respect to $A \to B$ are effective.

Proof. See Descent, Proposition 35.3.9. See also Descent, Remark 35.3.11 for an alternative view of the proof. □

03OE Remarks 59.16.8. The results on descent of modules have several applications:

(1) The exactness of the Čech complex in positive degrees for the covering $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is faithfully flat. This will give some vanishing of cohomology.
(2) If (N, φ) is a descent datum with respect to a faithfully flat map $A \to B$, then the corresponding A-module is given by

$$M = \mathrm{Ker} \begin{pmatrix} N & \longrightarrow & B \otimes_A N \\ n & \longmapsto & 1 \otimes n - \varphi(n \otimes 1) \end{pmatrix}.$$

See Descent, Proposition 35.3.9.

59.17. Quasi-coherent sheaves

03OF We can apply the descent of modules to study quasi-coherent sheaves.

03OG Proposition 59.17.1. For any quasi-coherent sheaf $\mathcal{F}$ on S the presheaf

$$\begin{array}{rccc} \mathcal{F}^a : & Sch/S & \to & \mathrm{Ab} \\ & (f : T \to S) & \mapsto & \Gamma(T, f^*\mathcal{F}) \end{array}$$

is an $\mathcal{O}$-module which satisfies the sheaf condition for the fpqc topology.

Proof. This is proved in Descent, Lemma 35.8.1. We indicate the proof here. As established in Lemma 59.15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for Zariski coverings is standard scheme theory, since $\Gamma(U, i^*\mathcal{F}) = \mathcal{F}(U)$ when $i : U \hookrightarrow S$ is an open immersion.

For $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ with $A \to B$ faithfully flat and $\mathcal{F}|_{\mathrm{Spec}(A)} = \widetilde{M}$ this corresponds to the fact that $M = H^0\left((B/A)_\bullet \otimes_A M\right)$, i.e., that

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M$$

is exact by Lemma 59.16.4. □

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 18.23. Let $\mathcal{C}$ be a category, and let U be an object of $\mathcal{C}$. Then $\mathcal{C}/U$ indicates the category of objects over U, see Categories, Example 4.2.13. If $\mathcal{C}$ is a site, then $\mathcal{C}/U$ is a site as well, namely the coverings of V/U are families $\{V_i/U \to V/U\}$ of morphisms of $\mathcal{C}/U$ with fixed target such that $\{V_i \to V\}$ is a covering of $\mathcal{C}$. Moreover, given any sheaf $\mathcal{F}$ on $\mathcal{C}$ the restriction $\mathcal{F}|_{\mathcal{C}/U}$ (defined in the obvious manner) is a sheaf as well. See Sites, Section 7.25 for details.

03OH Definition 59.17.2. Let $\mathcal{C}$ be a ringed site, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called quasi-coherent if for all $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$\bigoplus\nolimits_{k\in K} \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}|_{\mathcal{C}/U_i}.$$

The direct sum over K is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k\in K} \mathcal{O}(V)$ and similarly for the other.

Although it is useful to be able to give a general definition as above this notion is not well behaved in general.

03OI Remark 59.17.3. In the case where $\mathcal{C}$ has a final object, e.g. S, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 18.23.3.

03OJ Theorem 59.17.4 (Meta theorem on quasi-coherent sheaves). Let S be a scheme. Let $\mathcal{C}$ be a site. Assume that

(1) the underlying category $\mathcal{C}$ is a full subcategory of Sch/S,
(2) any Zariski covering of $T \in \mathrm{Ob}(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,
(3) S/S is an object of $\mathcal{C}$,
(4) every covering of $\mathcal{C}$ is an fpqc covering of schemes.

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. After some formal arguments this is exactly Theorem 59.16.2. Details omitted. In Descent, Proposition 35.8.9 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of S, as well as the small Zariski and étale sites of S. □

In other words, there is no difference between quasi-coherent modules on the scheme S and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(Sch/S)_{fppf}$, $S_{\acute{e}tale}$, etc can be found in Descent, Sections 35.8, 35.9, and 35.10. In this chapter we will sometimes refer to a “site as in Theorem 59.17.4” in order to conveniently state results which hold in any of those situations.

59.18. Čech cohomology

03OK Our next goal is to use descent theory to show that $H^i(\mathcal{C}, \mathcal{F}^a) = H^i_{Zar}(S, \mathcal{F})$ for all quasi-coherent sheaves $\mathcal{F}$ on S, and any site $\mathcal{C}$ as in Theorem 59.17.4. To this end, we introduce Čech cohomology on sites. See [Art62] and Cohomology on Sites, Sections 21.8, 21.9 and 21.10 for more details.

03OL **Definition 59.18.1.** Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in \text{PAb}(\mathcal{C})$ an abelian presheaf. We define the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \ldots$$

where the first term is in degree 0, and the maps are the usual ones. Again, it is essential to allow the case $i_0 = i_1$ etc. The Čech cohomology groups are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

03OM **Lemma 59.18.2.** *The functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, -)$ is exact on the category $\text{PAb}(\mathcal{C})$.*

In other words, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0$$

is a short exact sequence of complexes.

Proof. This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in Ab, it is automatically an abelian category: a sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact in PAb if and only if for all $U \in \text{Ob}(\mathcal{C})$, the sequence $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact in Ab. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 21.9.1. $\square$

This shows that $\check{H}^\bullet(\mathcal{U}, -)$ is a δ-functor. We now proceed to show that it is a universal δ-functor. We thus need to show that it is an effaceable functor. We start by recalling the Yoneda lemma.

03ON **Lemma 59.18.3** (Yoneda Lemma)**.** *For any presheaf $\mathcal{F}$ on a category $\mathcal{C}$ there is a functorial isomorphism*

$$\text{Hom}_{\text{PSh}(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

Proof. See Categories, Lemma 4.3.5. $\square$

Given a set E we denote (in this section) $\mathbf{Z}[E]$ the free abelian group on E. In a formula $\mathbf{Z}[E] = \bigoplus_{e \in E} \mathbf{Z}$, i.e., $\mathbf{Z}[E]$ is a free $\mathbf{Z}$-module having a basis consisting of the elements of E. Using this notation we introduce the free abelian presheaf on a presheaf of sets.

03OO **Definition 59.18.4.** Let $\mathcal{C}$ be a category. Given a presheaf of sets $\mathcal{G}$, we define the free abelian presheaf on $\mathcal{G}$, denoted $\mathbf{Z}_\mathcal{G}$, by the rule

$$\mathbf{Z}_\mathcal{G}(U) = \mathbf{Z}[\mathcal{G}(U)]$$

for $U \in \text{Ob}(\mathcal{C})$ with restriction maps induced by the restriction maps of $\mathcal{G}$. In the special case $\mathcal{G} = h_U$ we write simply $\mathbf{Z}_U = \mathbf{Z}_{h_U}$.

The functor $\mathcal{G} \mapsto \mathbf{Z}_\mathcal{G}$ is left adjoint to the forgetful functor $\text{PAb}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$. Thus, for any presheaf $\mathcal{F}$, there is a canonical isomorphism

$$\text{Hom}_{\text{PAb}(\mathcal{C})}(\mathbf{Z}_U, \mathcal{F}) = \text{Hom}_{\text{PSh}(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

03OP Lemma 59.18.5. The Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ can be described explicitly as follows

$$\begin{aligned} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) &= \left(\prod_{i_0 \in I} \text{Hom}_{\text{PAb}(\mathcal{C})}(\mathbf{Z}_{U_{i_0}}, \mathcal{F}) \to \prod_{i_0, i_1 \in I} \text{Hom}_{\text{PAb}(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U U_{i_1}}, \mathcal{F}) \to \ldots \right) \\ &= \text{Hom}_{\text{PAb}(\mathcal{C})}\left(\left(\bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \ldots \right), \mathcal{F} \right) \end{aligned}$$

Proof. This follows from the formula above. See Cohomology on Sites, Lemma 21.9.3. $\square$

This reduces us to studying only the complex in the first argument of the last Hom.

03OQ Lemma 59.18.6. The complex of abelian presheaves

$$\mathbf{Z}_\mathcal{U}^\bullet \quad : \quad \bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 in $\text{PAb}(\mathcal{C})$.

Proof. For any $V \in \text{Ob}(\mathcal{C})$ the complex of abelian groups $\mathbf{Z}_\mathcal{U}^\bullet(V)$ is

$$\begin{gathered} \mathbf{Z}\left[\coprod\nolimits_{i_0 \in I} \text{Mor}_\mathcal{C}(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod\nolimits_{i_0, i_1 \in I} \text{Mor}_\mathcal{C}(V, U_{i_0} \times_U U_{i_1})\right] \leftarrow \ldots = \\ \bigoplus\nolimits_{\varphi : V \to U} \left(\mathbf{Z}\left[\coprod\nolimits_{i_0 \in I} \text{Mor}_\varphi(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod\nolimits_{i_0, i_1 \in I} \text{Mor}_\varphi(V, U_{i_0}) \times \text{Mor}_\varphi(V, U_{i_1})\right] \leftarrow \ldots \right) \end{gathered}$$

where

$$\text{Mor}_\varphi(V, U_i) = \{V \to U_i \text{ such that } V \to U_i \to U \text{ equals } \varphi\}.$$

Set $S_\varphi = \coprod_{i \in I} \text{Mor}_\varphi(V, U_i)$, so that

$$\mathbf{Z}_\mathcal{U}^\bullet(V) = \bigoplus_{\varphi : V \to U} \left(\mathbf{Z}[S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \leftarrow \ldots \right).$$

Thus it suffices to show that for each $S = S_\varphi$, the complex

$$\mathbf{Z}[S] \leftarrow \mathbf{Z}[S \times S] \leftarrow \mathbf{Z}[S \times S \times S] \leftarrow \ldots$$

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix $s \in S$ and define $K : n_{(s_0, \ldots, s_p)} \mapsto n_{(s, s_0, \ldots, s_p)}$. One easily checks that K is a nullhomotopy for the operator

$$\delta : \eta_{(s_0, \ldots, s_p)} \mapsto \sum\nolimits_{i=0}^p (-1)^p \eta_{(s_0, \ldots, \hat{s}_i, \ldots, s_p)}.$$

See Cohomology on Sites, Lemma 21.9.4 for more details. $\square$

03OR Lemma 59.18.7. Let $\mathcal{C}$ be a category. If $\mathcal{I}$ is an injective object of $\text{PAb}(\mathcal{C})$ and $\mathcal{U}$ is a family of morphisms with fixed target in $\mathcal{C}$, then $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for all $p > 0$.

Proof. The Čech complex is the result of applying the functor $\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(-, \mathcal{I})$ to the complex $\mathbf{Z}_{\mathcal{U}}^\bullet$, i.e.,

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = H^p(\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{\mathcal{U}}^\bullet, \mathcal{I})).$$

But we have just seen that $\mathbf{Z}_{\mathcal{U}}^\bullet$ is exact in negative degrees, and the functor $\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(-, \mathcal{I})$ is exact, hence $\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{\mathcal{U}}^\bullet, \mathcal{I})$ is exact in positive degrees. □

03OS **Theorem 59.18.8.** On $\mathrm{PAb}(\mathcal{C})$ the functors $\check{H}^p(\mathcal{U}, -)$ are the right derived functors of $\check{H}^0(\mathcal{U}, -)$.

Proof. By the Lemma 59.18.7, the functors $\check{H}^p(\mathcal{U}, -)$ are universal δ-functors since they are effaceable. So are the right derived functors of $\check{H}^0(\mathcal{U}, -)$. Since they agree in degree 0, they agree by the universal property of universal δ-functors. For more details see Cohomology on Sites, Lemma 21.9.6. □

03OT **Remark 59.18.9.** Observe that all of the preceding statements are about presheaves so we haven't made use of the topology yet.

59.19. The Čech-to-cohomology spectral sequence

03OU This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

03OV **Lemma 59.19.1.** The forgetful functor $\mathrm{Ab}(\mathcal{C}) \to \mathrm{PAb}(\mathcal{C})$ transforms injectives into injectives.

Proof. This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 21.10.1. □

03OW **Theorem 59.19.2.** Let $\mathcal{C}$ be a site. For any covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of $U \in \mathrm{Ob}(\mathcal{C})$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $V \mapsto H^q(V, \mathcal{F})$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $\mathrm{Ab}(\mathcal{C})$, and consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ and the maps

$$\begin{tikzcd} \Gamma(U, I^\bullet) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet) \\ & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[u] \end{tikzcd}$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet)$ to the left column, and the vertical map is induced by $\mathcal{F} \to \mathcal{I}^0$ and lands in the bottom row. By assumption, $\mathcal{I}^\bullet$ is a complex of injectives in $\mathrm{Ab}(\mathcal{C})$, hence by Lemma 59.19.1, it is a complex of injectives in $\mathrm{PAb}(\mathcal{C})$. Thus, the rows of the double complex are exact in positive degrees (Lemma 59.18.7), and the kernel of $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{I}^\bullet)$ is equal to $\Gamma(U, \mathcal{I}^\bullet)$, since $\mathcal{I}^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, \mathcal{F})$.

For the vertical direction, the qth cohomology group of the pth column is

$$\prod_{i_0,\ldots,i_p} H^q(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = \prod_{i_0,\ldots,i_p} \underline{H}^q(\mathcal{F})(U_{i_0} \times_U \ldots \times_U U_{i_p})$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the E_2-page is as prescribed. For more details see Cohomology on Sites, Lemma 21.10.6. □

03OX Remark 59.19.3. This is a Grothendieck spectral sequence for the composition of functors

$$\mathrm{Ab}(\mathcal{C}) \longrightarrow \mathrm{PAb}(\mathcal{C}) \xrightarrow{\check{H}^0} \mathrm{Ab}.$$

59.20. Big and small sites of schemes

03X7 Let S be a scheme. Let τ be one of the topologies we will be discussing. Thus $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Of course if you are only interested in the étale topology, then you can simply assume $\tau = étale$ throughout. Moreover, we will discuss étale morphisms, étale coverings, and étale sites in more detail starting in Section 59.25. In order to proceed with the discussion of cohomology of quasi-coherent sheaves it is convenient to introduce the big τ-site and in case $\tau \in \{étale, Zariski\}$, the small τ-site of S. In order to do this we first introduce the notion of a τ-covering.

03X8 Definition 59.20.1. (See Topologies, Definitions 34.7.1, 34.6.1, 34.5.1, 34.4.1, and 34.3.1.) Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. A family of morphisms of schemes $\{f_i : T_i \to T\}_{i\in I}$ with fixed target is called a τ-covering if and only if each f_i is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have $\bigcup f_i(T_i) = T$.

The class of all τ-coverings satisfies the axioms (1), (2) and (3) of Definition 59.10.2 (our definition of a site), see Topologies, Lemmas 34.7.3, 34.6.3, 34.5.3, 34.4.3, and 34.3.2.

Let us introduce the sites we will be working with. Contrary to what happens in [AGV71], we do not want to choose a universe. Instead we pick a "partial universe" (which is a suitably large set as in Sets, Section 3.5), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme S is contained in the partial universe. Having picked the underlying category we pick a suitably large set of τ-coverings which turns this into a site. The details are in the chapter on topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of S (just as in [AGV71] the actual choice of universe doesn't matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

03XB Definition 59.20.2. Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$.

(1) A big τ-site of S is any of the sites $(Sch/S)_\tau$ constructed as explained above and in more detail in Topologies, Definitions 34.7.8, 34.6.8, 34.5.8, 34.4.8, and 34.3.7.

(2) If $\tau \in \{\acute{e}tale, Zariski\}$, then the small τ-site of S is the full subcategory S_τ of $(Sch/S)_\tau$ whose objects are schemes T over S whose structure morphism $T \to S$ is étale, resp. an open immersion. A covering in S_τ is a covering $\{U_i \to U\}$ in $(Sch/S)_\tau$ such that U is an object of S_τ.

The underlying category of the site $(Sch/S)_\tau$ has reasonable "closure" properties, i.e., given a scheme T in it any locally closed subscheme of T is isomorphic to an object of $(Sch/S)_\tau$. Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given scheme, given an affine scheme $\mathrm{Spec}(R)$ one can complete, localize, or take the quotient of R by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in $(Sch/S)_\tau$ will take you outside of it. Also note that, given an object T of $(Sch/S)_\tau$ there will exist τ-coverings $\{T_i \to T\}_{i\in I}$ (as in Definition 59.20.1) which are not coverings in $(Sch/S)_\tau$ for example because the schemes T_i are not objects of the category $(Sch/S)_\tau$. But our choice of the sites $(Sch/S)_\tau$ is such that there always does exist a covering $\{U_j \to T\}_{j\in J}$ of $(Sch/S)_\tau$ which refines the covering $\{T_i \to T\}_{i\in I}$, see Topologies, Lemmas 34.7.7, 34.6.7, 34.5.7, 34.4.7, and 34.3.6. We will mostly ignore these issues in this chapter.

If $\mathcal{F}$ is a sheaf on $(Sch/S)_\tau$ or S_τ, then we denote

$$H^p_\tau(U, \mathcal{F}), \text{ in particular } H^p_\tau(S, \mathcal{F})$$

the cohomology groups of $\mathcal{F}$ over the object U of the site, see Section 59.14. Thus we have $H^p_{fppf}(S, \mathcal{F})$, $H^p_{syntomic}(S, \mathcal{F})$, $H^p_{smooth}(S, \mathcal{F})$, $H^p_{\acute{e}tale}(S, \mathcal{F})$, and $H^p_{Zar}(S, \mathcal{F})$. The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

03YX Lemma 59.20.3. Let $\tau \in \{\acute{e}tale, Zariski\}$. If $\mathcal{F}$ is an abelian sheaf defined on $(Sch/S)_\tau$, then the cohomology groups of $\mathcal{F}$ over S agree with the cohomology groups of $\mathcal{F}|_{S_\tau}$ over S.

Proof. By Topologies, Lemmas 34.3.14 and 34.4.14 the functors $S_\tau \to (Sch/S)_\tau$ satisfy the hypotheses of Sites, Lemma 7.21.8. Hence our lemma follows from Cohomology on Sites, Lemma 21.7.2. □

The category of sheaves on the big or small étale site of S depends only on the full subcategory of $(Sch/S)_{\acute{e}tale}$ or $S_{\acute{e}tale}$ consisting of affines and one only needs to consider the standard étale coverings between them (as defined below). This gives rise to sites $(\mathrm{Aff}/S)_{\acute{e}tale}$ and $S_{affine,\acute{e}tale}$, see Topologies, Definition 34.4.8. The comparison results are proven in Topologies, Lemmas 34.4.11 and 34.4.12. Here is our definition of standard coverings in some of the topologies we will consider in this chapter.

03X9 Definition 59.20.4. (See Topologies, Definitions 34.7.5, 34.6.5, 34.5.5, 34.4.5, and 34.3.4.) Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let T be an affine scheme. A standard τ-covering of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, and each f_j flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in T and $T = \bigcup f_j(U_j)$.

03XA Lemma 59.20.5. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Any τ-covering of an affine scheme can be refined by a standard τ-covering.

Proof. See Topologies, Lemmas 34.7.4, 34.6.4, 34.5.4, 34.4.4, and 34.3.3. □

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. We will prove invariance of the small étale topos in Lemma 59.21.2 below. For notation and terminology used in this lemma we refer to Topologies, Section 34.12.

03YY Lemma 59.20.6. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $(Sch/S)_\tau$ and $(Sch'/S)_\tau$ be two big τ-sites of S, and assume that the first is contained in the second. In this case

(1) for any abelian sheaf $\mathcal{F}'$ defined on $(Sch'/S)_\tau$ and any object U of $(Sch/S)_\tau$ we have
$$H^p_\tau(U, \mathcal{F}'|_{(Sch/S)_\tau}) = H^p_\tau(U, \mathcal{F}')$$
In words: the cohomology of $\mathcal{F}'$ over U computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over U.

(2) for any abelian sheaf $\mathcal{F}$ on $(Sch/S)_\tau$ there is an abelian sheaf $\mathcal{F}'$ on $(Sch/S)'_\tau$ whose restriction to $(Sch/S)_\tau$ is isomorphic to $\mathcal{F}$.

Proof. By Topologies, Lemma 34.12.2 the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 7.21.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 21.7.2. □

59.21. The étale topos

04HP A topos is the category of sheaves of sets on a site, see Sites, Definition 7.15.1. Hence it is customary to refer to the use the phrase "étale topos of a scheme" to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.

04HQ Definition 59.21.1. Let S be a scheme.

(1) The étale topos, or the small étale topos of S is the category $Sh(S_{étale})$ of sheaves of sets on the small étale site of S.
(2) The Zariski topos, or the small Zariski topos of S is the category $Sh(S_{Zar})$ of sheaves of sets on the small Zariski site of S.
(3) For $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ a big τ-topos is the category of sheaves of set on a big τ-topos of S.

Note that the small Zariski topos of S is simply the category of sheaves of sets on the underlying topological space of S, see Topologies, Lemma 34.3.12. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topoi do depend on those choices.

It turns out that the big or small étale topos only depends on the full subcategory of $(Sch/S)_{étale}$ or $S_{étale}$ consisting of affines, see Topologies, Lemmas 34.4.11 and 34.4.12. We will use this for example in the proof of the following lemma.

0958 Lemma 59.21.2. Let S be a scheme. The étale topos of S is independent (up to canonical equivalence) of the construction of the small étale site in Definition 59.20.2.

Proof. We have to show, given two big étale sites $Sch_{étale}$ and $Sch'_{étale}$ containing S, then $Sh(S_{étale}) \cong Sh(S'_{étale})$ with obvious notation. By Topologies, Lemma 34.12.1 we may assume $Sch_{étale} \subset Sch'_{étale}$. By Sets, Lemma 3.9.9 any affine scheme étale over S is isomorphic to an object of both $Sch_{étale}$ and $Sch'_{étale}$. Thus the induced functor $S_{affine,étale} \to S'_{affine,étale}$ is an equivalence. Moreover, it is clear that both this functor and a quasi-inverse map transform standard étale coverings into standard étale coverings. Hence the result follows from Topologies, Lemma 34.4.12. □

59.22. Cohomology of quasi-coherent sheaves

03OY We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ with $A \to B$ faithfully flat.

03OZ Lemma 59.22.1. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $(Sch/S)_\tau$, or on S_τ in case $\tau = étale$, and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a standard τ-covering of this site. Let $V = \coprod_{i \in I} U_i$. Then

(1) V is an affine scheme,
(2) $\mathcal{V} = \{V \to U\}$ is an fpqc covering and also a τ-covering unless $\tau = Zariski$,
(3) the Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ agree.

Proof. The defintion of a standard τ-covering is given in Topologies, Definition 34.3.4, 34.4.5, 34.5.5, 34.6.5, and 34.7.5. By definition each of the schemes U_i is affine and I is a finite set. Hence V is an affine scheme. It is clear that $V \to U$ is flat and surjective, hence $\mathcal{V}$ is an fpqc covering, see Example 59.15.3. Excepting the Zariski case, the covering $\mathcal{V}$ is also a τ-covering, see Topologies, Definition 34.4.1, 34.5.1, 34.6.1, and 34.7.1.

Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and hence there is a map of Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$, see Cohomology on Sites, Equation (21.8.2.1). Next, we observe that if $T = \coprod_{j \in J} T_j$ is a disjoint union of schemes in the site on which $\mathcal{F}$ is defined then the family of morphisms with fixed target $\{T_j \to T\}_{j \in J}$ is a Zariski covering, and so

03XC (59.22.1.1)
$$\mathcal{F}(T) = \mathcal{F}(\coprod\nolimits_{j \in J} T_j) = \prod\nolimits_{j \in J} \mathcal{F}(T_j)$$

by the sheaf condition of $\mathcal{F}$. This implies the map of Čech complexes above is an isomorphism in each degree because

$$V \times_U \ldots \times_U V = \coprod\nolimits_{i_0, \ldots i_p} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

as schemes. □

Note that Equality (59.22.1.1) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

03P0 Remark 59.22.2. In the statement of Lemma 59.22.1 the covering $\mathcal{U}$ is a refinement of $\mathcal{V}$ but not the other way around. Coverings of the form $\{V \to U\}$ do not form an initial subcategory of the category of all coverings of U. Yet it is still true that we can compute Čech cohomology $\check{H}^n(U, \mathcal{F})$ (which is defined as the colimit over the opposite of the category of coverings $\mathcal{U}$ of U of the Čech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$) in terms of the coverings $\{V \to U\}$. We will formulate a precise lemma (it only works for sheaves) and add it here if we ever need it.

03P1 Lemma 59.22.3 (Locality of cohomology). Let $\mathcal{C}$ be a site, $\mathcal{F}$ an abelian sheaf on $\mathcal{C}$, U an object of $\mathcal{C}$, $p > 0$ an integer and $\xi \in H^p(U, \mathcal{F})$. Then there exists a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of U in $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then ξ is represented by a cocycle $\tilde{\xi} \in \mathcal{I}^p(U)$ with $d^p(\tilde{\xi}) = 0$. By assumption, the sequence $\mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1}$ in exact in $\text{Ab}(\mathcal{C})$, which means that there exists a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\tilde{\xi}|_{U_i} = d^{p-1}(\xi_i)$ for some $\xi_i \in \mathcal{I}^{p-1}(U_i)$. Since the cohomology class $\xi|_{U_i}$ is represented by the cocycle $\tilde{\xi}|_{U_i}$ which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 21.7.3. □

03P2 Theorem 59.22.4. Let S be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(Sch/S)_\tau$ for $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ or $S_{étale}$. Then

$$H^p(S, \mathcal{F}) = H^p_\tau(S, \mathcal{F}^a)$$

for all $p \geq 0$ where

(1) the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme S, and
(2) the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition 59.17.1) on the site $\mathcal{C}$.

Proof. We are going to show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that U is affine. Then we want to prove that $H^p_\tau(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on p.

p = 1 Pick $\xi \in H^1_\tau(U, \mathcal{F}^a)$. By Lemma 59.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. Applying the spectral sequence of Theorem 59.19.2, we see that ξ comes from a cohomology class $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F}^a)$. Consider the covering $\mathcal{V} = \{\coprod_{i \in I} U_i \to U\}$. By Lemma 59.22.1, $\check{H}^\bullet(\mathcal{U}, \mathcal{F}^a) = \check{H}^\bullet(\mathcal{V}, \mathcal{F}^a)$. On the other hand, since $\mathcal{V}$ is a covering of the form $\{\text{Spec}(B) \to \text{Spec}(A)\}$ and $f^*\mathcal{F} = \widetilde{M}$ for some A-module M, we see the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ is none other than the complex $(B/A)_\bullet \otimes_A M$. Now by Lemma 59.16.4, $H^p((B/A)_\bullet \otimes_A M) = 0$ for $p > 0$, hence $\check{\xi} = 0$ and so $\xi = 0$.

p > 1 Pick $\xi \in H^p_\tau(U, \mathcal{F}^a)$. By Lemma 59.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. We apply the spectral sequence of Theorem 59.19.2. Observe that the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a))$$

vanish for all $0 < q < p$. We see that ξ must come from a $\check{\xi} \in \check{H}^p(\mathcal{U}, \mathcal{F}^a)$. Replacing $\mathcal{U}$ with the covering $\mathcal{V}$ containing only one morphism and using Lemma 59.16.4 again, we see that the Čech cohomology class $\check{\xi}$ must be zero, hence $\xi = 0$.

Next, assume that U is separated. Choose an affine open covering $U = \bigcup_{i \in I} U_i$ of U. The family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine since U is separated. So all rows of the spectral sequence of Theorem 59.19.2 are zero, except the zeroth row. Therefore

$$H^p_\tau(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$$

where the last equality results from standard scheme theory, see Cohomology of Schemes, Lemma 30.2.6.

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won't treat it. It follows from Descent, Proposition 35.9.3 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 21.7.1. ☐

03P3 Remark 59.22.5. Comment on Theorem 59.22.4. Since S is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof shows that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

59.23. Examples of sheaves

03YZ Let S and τ be as in Section 59.20. We have already seen that any representable presheaf is a sheaf on $(Sch/S)_\tau$ or S_τ, see Lemma 59.15.8 and Remark 59.15.9. Here are some special cases.

03P4 Definition 59.23.1. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 59.20.

(1) The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.
(2) Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ is denoted $\mathcal{O}_S^*$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.
(3) The constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbf{Z}/n\mathbf{Z}$.

The first is a sheaf by Theorem 59.17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{m,S}$, see Groupoids, Section 39.4. The third by the finite étale group scheme $\mathbf{Z}/n\mathbf{Z}_S$ sometimes denoted $(\mathbf{Z}/n\mathbf{Z})_S$ which is just n copies of S endowed with the obvious group scheme structure over S, see Groupoids, Example 39.5.6 and the following remark.

03P5 Remark 59.23.2. Let G be an abstract group. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 59.20 the sheafification $\underline{G}$ of the constant presheaf associated to G in the Zariski topology of the site already gives

$$\Gamma(U, \underline{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme G_S according to Groupoids, Example 39.5.6. By Lemma 59.15.8 any representable presheaf satisfies the sheaf condition for the τ-topology as well, and hence we conclude that the Zariski sheafification $\underline{G}$ above is also the τ-sheafification.

04HS Definition 59.23.3. Let S be a scheme. The structure sheaf of S is the sheaf of rings $\mathcal{O}_S$ on any of the sites S_{Zar}, $S_{étale}$, or $(Sch/S)_\tau$ discussed above.

If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{étale}}$ to stress the fact that we are working on the small étale site of S.

03P6 Remark 59.23.4. In the terminology introduced above a special case of Theorem 59.22.4 is

$$H^p_{fppf}(X, \mathbf{G}_a) = H^p_{étale}(X, \mathbf{G}_a) = H^p_{Zar}(X, \mathbf{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{fppf}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of X.

59.24. Picard groups

03P7 The following theorem is sometimes called "Hilbert 90".

03P8 Theorem 59.24.1. For any scheme X we have canonical identifications

$$\begin{aligned} H^1_{fppf}(X, \mathbf{G}_m) &= H^1_{syntomic}(X, \mathbf{G}_m) \\ &= H^1_{smooth}(X, \mathbf{G}_m) \\ &= H^1_{étale}(X, \mathbf{G}_m) \\ &= H^1_{Zar}(X, \mathbf{G}_m) \\ &= \mathrm{Pic}(X) \\ &= H^1(X, \mathcal{O}^*_X) \end{aligned}$$

Proof. Let τ be one of the topologies considered in Section 59.20. By Cohomology on Sites, Lemma 21.6.1 we see that $H^1_\tau(X, \mathbf{G}_m) = H^1_\tau(X, \mathcal{O}^*_\tau) = \mathrm{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(Sch/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 59.17.4 or the more precise Descent, Proposition 35.8.9 we see that $\mathrm{Pic}(\mathcal{O}_\tau) = \mathrm{Pic}(X)$. The last equality is proved in the same way. $\square$

59.25. The étale site

03P9 At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.

59.26. Étale morphisms

03PA For more details, see Morphisms, Section 29.36 for the formal definition and Étale Morphisms, Sections 41.11, 41.12, 41.13, 41.14, 41.16, and 41.19 for a survey of interesting properties of étale morphisms.

Recall that an algebra A over an algebraically closed field k is smooth if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme X over k is smooth over k if it is locally of finite type and each affine open is the spectrum of a smooth k-algebra. If k is not algebraically closed then a k-algebra A is a smooth k-algebra if $A \otimes_k \overline{k}$ is a smooth $\overline{k}$-algebra. A ring map $A \to B$ is smooth if it is flat, finitely presented, and for all primes $\mathfrak{p} \subset A$ the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is smooth over the residue field $\kappa(\mathfrak{p})$. More generally, a morphism of schemes is smooth if it is flat, locally of finite presentation, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 29.34. Using this we may define an étale morphism as follows.

03PB Definition 59.26.1. A morphism of schemes is étale if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \to S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

03PC Proposition 59.26.2. Facts on étale morphisms.

(1) Let k be a field. A morphism of schemes $U \to \mathrm{Spec}(k)$ is étale if and only if $U \cong \coprod_{i \in I} \mathrm{Spec}(k_i)$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.
(2) Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:
   (a) φ is étale,
   (b) φ is locally finitely presented, flat, and all its fibres are étale,
   (c) φ is flat, unramified and locally of finite presentation.
(3) A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.
(4) The base change of an étale morphism is étale.
(5) Compositions of étale morphisms are étale.
(6) Fibre products and products of étale morphisms are étale.
(7) An étale morphism has relative dimension 0.
(8) Let $Y \to X$ be an étale morphism. If X is reduced (respectively regular) then so is Y.
(9) Étale morphisms are open.
(10) If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.

Proof. We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 29.36.7. (2) Morphisms, Lemmas 29.36.8 and 29.36.16. (3) Algebra, Lemma 10.143.2. (4) Morphisms, Lemma 29.36.4. (5) Morphisms, Lemma 29.36.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 29.36.6 and 29.29.5. (8) See Algebra, Lemmas 10.163.7 and 10.163.5, see also more results of this kind in Étale Morphisms, Section 41.19. (9) See Morphisms, Lemma 29.25.10 and 29.36.12. (10) See Morphisms, Lemma 29.36.18. ☐

03PD Definition 59.26.3. A ring map $A \to B$ is called standard étale if $B \cong \left(A[t]/(f)\right)_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B.

It is true that a standard étale ring map is étale. Namely, suppose that $B = \left(A[t]/(f)\right)_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B. Then $A[t]/(f)$ is a finite free A-module of rank equal to the degree of the monic polynomial f. Hence B, as a localization of this free algebra is finitely presented and flat over A. To finish the proof that B is étale it suffices to show that the fibre rings

$$\kappa(\mathfrak p) \otimes_A B \cong \kappa(\mathfrak p) \otimes_A (A[t]/(f))_g \cong \kappa(\mathfrak p)[t, 1/\overline{g}]/(\overline{f})$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(\mathfrak p)[t]$ are the images of f and g. Let

$$\overline{f} = \overline{f}_1 \ldots \overline{f}_a \overline{f}_{a+1}^{e_1} \ldots \overline{f}_{a+b}^{e_b}$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $e_1, \ldots, e_b > 0$. By assumption $\mathrm{d}\overline{f}/\mathrm{d}t$ is invertible in $\kappa(\mathfrak p)[t, 1/\overline{g}]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$\kappa(\mathfrak p)[t, 1/\overline{g}]/(\overline{f}) \cong \prod\nolimits_{i \in I} \kappa(\mathfrak p)[t]/(\overline{f}_i)$$

where $I \subset \{1, \ldots, a\}$ is the subset of indices i such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $\mathrm{d}\overline{f}/\mathrm{d}t$ in the factor $\kappa(\mathfrak p)[t]/(\overline{f}_i)$ is clearly equal to a unit times $\mathrm{d}\overline{f}_i/\mathrm{d}t$. Hence we conclude that $\kappa_i = \kappa(\mathfrak p)[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(\mathfrak p)$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa_i/\kappa(\mathfrak p)$ is finite separable as desired.

It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is étale at a prime $\mathfrak q$ of B if there exists $h \in B$, $h \notin \mathfrak q$ such that $A \to B_h$ is étale. Here is the result.

03PE Theorem 59.26.4. A ring map $A \to B$ is étale at a prime $\mathfrak q$ if and only if there exists $g \in B$, $g \notin \mathfrak q$ such that B_g is standard étale over A.

Proof. See Algebra, Proposition 10.144.4. □

59.27. Étale coverings

03PF We recall the definition.

03PG Definition 59.27.1. An étale covering of a scheme U is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}_{i \in I}$ such that

(1) each φ_i is an étale morphism,
(2) the U_i cover U, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

03PH Lemma 59.27.2. Any étale covering is an fpqc covering.

Proof. (See also Topologies, Lemma 34.9.6.) Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since φ_i is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of V. Further, since V is quasi-compact, this covering has a finite refinement. □

So any statement which is true for fpqc coverings remains true a fortiori for étale coverings. For instance, the étale site is subcanonical.

03PI Definition 59.27.3. (For more details see Section 59.20, or Topologies, Section 34.4.) Let S be a scheme. The big étale site over S is the site $(Sch/S)_{étale}$, see Definition 59.20.2. The small étale site over S is the site $S_{étale}$, see Definition 59.20.2. We define similarly the big and small Zariski sites on S, denoted $(Sch/S)_{Zar}$ and S_{Zar}.

Loosely speaking the big étale site of S is made up out of schemes over S and coverings the étale coverings. The small étale site of S is made up out of schemes étale over S with coverings the étale coverings. Actually any morphism between objects of $S_{étale}$ is étale, in virtue of Proposition 59.26.2, hence to check that $\{U_i \to U\}_{i \in I}$ in $S_{étale}$ is a covering it suffices to check that $\coprod U_i \to U$ is surjective.

The small étale site has fewer objects than the big étale site, it contains only the "opens" of the étale topology on S. It is a full subcategory of the big étale site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

03PJ Proposition 59.27.4. Let S be a scheme and $\mathcal{F}$ an abelian sheaf on $(Sch/S)_{étale}$. Then $\mathcal{F}|_{S_{étale}}$ is a sheaf on $S_{étale}$ and

$$H^p_{étale}(S, \mathcal{F}|_{S_{étale}}) = H^p_{étale}(S, \mathcal{F})$$

for all $p \geq 0$.

Proof. This is a special case of Lemma 59.20.3. $\square$

In accordance with the general notation introduced in Section 59.20 we write $H^p_{étale}(S, \mathcal{F})$ for the above cohomology group.

59.28. Kummer theory

03PK Let $n \in \mathbf{N}$ and consider the functor μ_n defined by

$$\begin{array}{ccc} Sch^{opp} & \longrightarrow & \mathrm{Ab} \\ S & \longmapsto & \mu_n(S) = \{t \in \Gamma(S, \mathcal{O}_S^*) \mid t^n = 1\}. \end{array}$$

By Groupoids, Example 39.5.2 this is a representable functor, and the scheme representing it is denoted μ_n also. By Lemma 59.15.8 this functor satisfies the sheaf condition for the fpqc topology (in particular, it also satisfies the sheaf condition for the étale, Zariski, etc topology).

03PL Lemma 59.28.1. If $n \in \mathcal{O}_S^*$ then

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on both the small and big étale site of S.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find an étale cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 27.3 or 27.4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, n and T^{n-1} are invertible in B, so $nT^{n-1} \in B^*$ and the ring map $A \to B$ is standard étale, in particular étale. Since this holds for every affine open of U we conclude that π is étale. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is an étale covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. $\square$

03PM Remark 59.28.2. Lemma 59.28.1 is false when "étale" is replaced with "Zariski". Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 34.5.2 it follows that the sequence is also exact in the smooth topology.

By Theorem 59.24.1 and Lemma 59.28.1 and general properties of cohomology we obtain the long exact cohomology sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^0_{étale}(S, \mu_{n,S}) \arrow[r] & \Gamma(S, \mathcal{O}_S^*) \arrow[r, "(\cdot)^n"] & \Gamma(S, \mathcal{O}_S^*) \arrow[dll, bend left] \\
& H^1_{étale}(S, \mu_{n,S}) \arrow[r] & \mathrm{Pic}(S) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(S) \arrow[dll, bend left] \\
& H^2_{étale}(S, \mu_{n,S}) \arrow[r] & \ldots
\end{tikzcd}
$$

at least if n is invertible on S. When n is not invertible on S we can apply the following lemma.

040N Lemma 59.28.3. For any $n \in \mathbf{N}$ the sequence

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on the site $(Sch/S)_{fppf}$ and $(Sch/S)_{syntomic}$.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Since the syntomic topology is weaker than the fppf topology, see Topologies, Lemma 34.7.2, it suffices to prove this for the syntomic topology. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find a syntomic cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 27.3 or 27.4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, B is a global relative complete intersection over A, so the ring map $A \to B$ is standard syntomic, in particular syntomic. Since this holds for every affine open of U we conclude that π is syntomic. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. $\square$

040O Remark 59.28.4. Lemma 59.28.3 is false for the smooth, étale, or Zariski topology.

By Theorem 59.24.1 and Lemma 59.28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0_{fppf}(S, \mu_{n,S}) \arrow[r] & \Gamma(S, \mathcal{O}_S^*) \arrow[r, "(\cdot)^n"] & \Gamma(S, \mathcal{O}_S^*) \arrow[dll, bend left] \\ & H^1_{fppf}(S, \mu_{n,S}) \arrow[r] & \mathrm{Pic}(S) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(S) \arrow[dll, bend left] \\ & H^2_{fppf}(S, \mu_{n,S}) \arrow[r] & \ldots \end{tikzcd}$$

for any scheme S and any integer n. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbf{N}$ and let S be any scheme. There is another more direct way to describe the first cohomology group with values in μ_n. Consider pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on S and $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization of the nth tensor power of $\mathcal{L}$. Let $(\mathcal{L}', \alpha')$ be a second such pair. An isomorphism $\varphi : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible sheaves such that the diagram

$$\begin{tikzcd} \mathcal{L}^{\otimes n} \arrow[r, "\alpha"'] \arrow[d, "\varphi^{\otimes n}"'] & \mathcal{O}_S \arrow[d, "1"] \\ (\mathcal{L}')^{\otimes n} \arrow[r, "\alpha'"] & \mathcal{O}_S \end{tikzcd}$$

commutes. Thus we have
(59.28.4.1)

040P
$$Isom_S((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')) = \begin{cases} \emptyset & \text{if they are not isomorphic} \\ H^0(S, \mu_{n,S}) \cdot \varphi & \text{if } \varphi \text{ isomorphism of pairs} \end{cases}$$

Moreover, given two pairs $(\mathcal{L}, \alpha)$, $(\mathcal{L}', \alpha')$ the tensor product

$$(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') = (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')$$

is another pair. The pair $(\mathcal{O}_S, 1)$ is an identity for this tensor product operation, and an inverse is given by

$$(\mathcal{L}, \alpha)^{-1} = (\mathcal{L}^{\otimes -1}, \alpha^{\otimes -1}).$$

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

$$(\mathcal{L}, \alpha)^{\otimes n} = (\mathcal{L}^{\otimes n}, \alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S, 1)$$

is an isomorphism hence every element of this group has order dividing n. We warn the reader that this group is in general not the n-torsion in $\mathrm{Pic}(S)$.

040Q Lemma 59.28.5. Let S be a scheme. There is a canonical identification

$$H^1_{étale}(S, \mu_n) = \text{group of pairs } (\mathcal{L}, \alpha) \text{ up to isomorphism as above}$$

if n is invertible on S. In general we have

$$H^1_{fppf}(S, \mu_n) = \text{group of pairs } (\mathcal{L}, \alpha) \text{ up to isomorphism as above.}$$

The same result holds with fppf replaced by syntomic.

Proof. We first prove the second isomorphism. Let $(\mathcal{L}, \alpha)$ be a pair as above. Choose an affine open covering $S = \bigcup U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Say $s_i \in \mathcal{L}(U_i)$ is a generator. Then $\alpha(s_i^{\otimes n}) = f_i \in \mathcal{O}_S^*(U_i)$. Writing $U_i = \mathrm{Spec}(A_i)$ we see there exists a global relative complete intersection $A_i \to B_i = A_i[T]/(T^n - f_i)$ such that f_i maps to an nth power in B_i. In other words, setting $V_i = \mathrm{Spec}(B_i)$ we obtain a syntomic covering $\mathcal{V} = \{V_i \to S\}_{i \in I}$ and trivializations $\varphi_i : (\mathcal{L}, \alpha)|_{V_i} \to (\mathcal{O}_{V_i}, 1)$.

We will use this result (the existence of the covering $\mathcal{V}$) to associate to this pair a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$. We give two (equivalent) constructions.

First construction: using Čech cohomology. Over the double overlaps $V_i \times_S V_j$ we have the isomorphism

$$(\mathcal{O}_{V_i \times_S V_j}, 1) \xrightarrow{\mathrm{pr}_0^* \varphi_i^{-1}} (\mathcal{L}|_{V_i \times_S V_j}, \alpha|_{V_i \times_S V_j}) \xrightarrow{\mathrm{pr}_1^* \varphi_j} (\mathcal{O}_{V_i \times_S V_j}, 1)$$

of pairs. By (59.28.4.1) this is given by an element $\zeta_{ij} \in \mu_n(V_i \times_S V_j)$. We omit the verification that these ζ_{ij}'s give a 1-cocycle, i.e., give an element $(\zeta_{i_0 i_1}) \in \check{C}(\mathcal{V}, \mu_n)$ with $d(\zeta_{i_0 i_1}) = 0$. Thus its class is an element in $\check{H}^1(\mathcal{V}, \mu_n)$ and by Theorem 59.19.2 it maps to a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$.

Second construction: Using torsors. Consider the presheaf

$$\mu_n(\mathcal{L}, \alpha) : U \longmapsto \mathit{Isom}_U((\mathcal{O}_U, 1), (\mathcal{L}, \alpha)|_U)$$

on $(Sch/S)_{syntomic}$. We may view this as a subpresheaf of $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, \mathcal{L})$ (internal hom sheaf, see Modules on Sites, Section 18.27). Since the conditions defining this subpresheaf are local, we see that it is a sheaf. By (59.28.4.1) this sheaf has a free action of the sheaf $\mu_{n,S}$. Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover $\mathcal{V}$. Hence $\mu_n(\mathcal{L}, \alpha)$ is a $\mu_{n,S}$-torsor and by Cohomology on Sites, Lemma 21.4.3 we obtain a corresponding element of $H^1_{syntomic}(S, \mu_{n,S})$.

Ok, now we have to still show the following

(1) The two constructions give the same cohomology class.
(2) Isomorphic pairs give rise to the same cohomology class.
(3) The cohomology class of $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha')$ is the sum of the cohomology classes of $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$.
(4) If the cohomology class is trivial, then the pair is trivial.
(5) Any element of $H^1_{syntomic}(S, \mu_{n,S})$ is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Čech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 21.4.2.

Part (5) can be seen as follows (although a direct proof would be preferable). Suppose $\xi \in H^1_{syntomic}(S, \mu_{n,S})$. Then ξ maps to an element $\overline{\xi} \in H^1_{syntomic}(S, \mathbf{G}_{m,S})$ with $n\overline{\xi} = 0$. By Theorem 59.24.1 we see that $\overline{\xi}$ corresponds to an invertible sheaf $\mathcal{L}$ whose nth tensor power is isomorphic to $\mathcal{O}_S$. Hence there exists a pair $(\mathcal{L}, \alpha')$ whose cohomology class ξ' has the same image $\overline{\xi'}$ in $H^1_{syntomic}(S, \mathbf{G}_{m,S})$. Thus it suffices to show that $\xi - \xi'$ is the class of a pair. By construction, and the long exact cohomology sequence above, we see that $\xi - \xi' = \partial(f)$ for some $f \in H^0(S, \mathcal{O}_S^*)$. Consider the pair $(\mathcal{O}_S, f)$. We omit the verification that the cohomology class of this pair is $\partial(f)$, which finishes the proof of the first identification (with fppf replaced with syntomic).

To see the first, note that if n is invertible on S, then the covering $\mathcal{V}$ constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma 59.28.1). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma 59.28.1. $\square$

59.29. Neighborhoods, stalks and points

03PN We can associate to any geometric point of S a stalk functor which is exact. A map of sheaves on $S_{étale}$ is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Altogether this means that the small étale site of a scheme S has enough points. It also turns out that any point of the small étale topos of S (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of S can be understood in terms of geometric points and neighbourhoods.

03PO **Definition 59.29.1.** Let S be a scheme.

(1) A geometric point of S is a morphism $\mathrm{Spec}(k) \to S$ where k is algebraically closed. Such a point is usually denoted $\overline{s}$, i.e., by an overlined small case letter. We often use $\overline{s}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{s})$ to denote k.
(2) We say $\overline{s}$ lies over s to indicate that $s \in S$ is the image of $\overline{s}$.
(3) An étale neighborhood of a geometric point $\overline{s}$ of S is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \overline{s} \arrow[ur, "\bar{u}"] \arrow[r, "\overline{s}"] & S \end{tikzcd}$$

where φ is an étale morphism of schemes. We write $(U, \overline{u}) \to (S, \overline{s})$.
(4) A morphism of étale neighborhoods $(U, \overline{u}) \to (U', \overline{u}')$ is an S-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

03PP **Remark 59.29.2.** Since U and U' are étale over S, any S-morphism between them is also étale, see Proposition 59.26.2. In particular all morphisms of étale neighborhoods are étale.

04HT **Remark 59.29.3.** Let S be a scheme and $s \in S$ a point. In More on Morphisms, Definition 37.35.1 we defined the notion of an étale neighbourhood $(U, u) \to (S, s)$ of (S, s). If $\overline{s}$ is a geometric point of S lying over s, then any étale neighbourhood $(U, \overline{u}) \to (S, \overline{s})$ gives rise to an étale neighbourhood (U, u) of (S, s) by taking $u \in U$ to be the unique point of U such that $\overline{u}$ lies over u. Conversely, given an étale neighbourhood (U, u) of (S, s) the residue field extension $\kappa(u)/\kappa(s)$ is finite separable (see Proposition 59.26.2) and hence we can find an embedding $\kappa(u) \subset \kappa(\overline{s})$ over $\kappa(s)$. In other words, we can find a geometric point $\overline{u}$ of U lying over u such that $(U, \overline{u})$ is an étale neighbourhood of $(S, \overline{s})$. We will use these observations to go between the two types of étale neighbourhoods.

03PQ Lemma 59.29.4. Let S be a scheme, and let $\overline{s}$ be a geometric point of S. The category of étale neighborhoods is cofiltered. More precisely:

(1) Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{s}$ in S. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.
(2) Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Proposition 59.26.2. The map $\overline{s} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2. For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_S U'. \end{tikzcd}$$

Since $\overline{u}$ and $\overline{u}'$ agree over S with $\overline{s}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Proposition 59.26.2). Hence the vertical arrow (h_1, h_2) is étale by Remark 59.29.2 above. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem. □

03PR Lemma 59.29.5. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $(U, \overline{u})$ be an étale neighborhood of $\overline{s}$. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{u}_i : \overline{s} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.

Proof. As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is not empty for some i. Then look at the cartesian diagram

$$\begin{tikzcd} \overline{s} \times_{\overline{u}, U, \varphi_i} U_i \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & U_i \arrow[d, "\varphi_i"] \\ \mathrm{Spec}(k) = \overline{s} \arrow[r, "\overline{u}"] \arrow[u, bend left, "\sigma"] & U \end{tikzcd}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Proposition 59.26.2. Therefore, $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is a disjoint union of finite separable extensions of k, by Proposition 59.26.2. Here $\overline{s} = \mathrm{Spec}(k)$. But k is algebraically closed, so all these extensions are trivial, and there exists a section σ of pr_1. The composition $\mathrm{pr}_2 \circ \sigma$ gives a map compatible with $\overline{u}$. □

040R Definition 59.29.6. Let S be a scheme. Let $\mathcal{F}$ be a presheaf on $S_{\acute{e}tale}$. Let $\overline{s}$ be a geometric point of S. The stalk of $\mathcal{F}$ at $\overline{s}$ is

$$\mathcal{F}_{\overline{s}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{s}$ in S.

By Lemma 59.29.4, this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{s}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.19.

04FM Lemma 59.29.7. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Consider the functor

$$\begin{aligned} u : S_{\acute{e}tale} &\longrightarrow Sets, \\ U &\longmapsto |U_{\overline{s}}| = \{\overline{u} \text{ such that } (U, \overline{u}) \text{ is an étale neighbourhood of } \overline{s}\}. \end{aligned}$$

Here $|U_{\overline{s}}|$ denotes the underlying set of the geometric fibre. Then u defines a point p of the site $S_{\acute{e}tale}$ (Sites, Definition 7.32.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 7.32.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defined above.

Proof. In the proof of Lemma 59.29.5 we have seen that the scheme $U_{\overline{s}}$ is a disjoint union of schemes isomorphic to $\overline{s}$. Thus we can also think of $|U_{\overline{s}}|$ as the set of geometric points of U lying over $\overline{s}$, i.e., as the collection of morphisms $\overline{u} : \overline{s} \to U$ fitting into the diagram of Definition 59.29.1. From this it follows that $u(S)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{\acute{e}tale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $S_{\acute{e}tale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 59.29.5. Hence Sites, Proposition 7.33.3 applies, so p is a point of the site $S_{\acute{e}tale}$. Finally, our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 7.32.1.1 which proves the final assertion. □

04FN Remark 59.29.8. Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ and $\overline{s}' : \mathrm{Spec}(k') \to S$ be two geometric points of S. A morphism $a : \overline{s} \to \overline{s}'$ of geometric points is simply a morphism $a : \mathrm{Spec}(k) \to \mathrm{Spec}(k')$ such that $\overline{s}' \circ a = \overline{s}$. Given such a morphism we obtain a functor from the category of étale neighbourhoods of $\overline{s}'$ to the category of étale neighbourhoods of $\overline{s}$ by the rule $(U, \overline{u}') \mapsto (U, \overline{u}' \circ a)$. Hence we obtain a canonical map

$$\mathcal{F}_{\overline{s}'} = \mathrm{colim}_{(U, \overline{u}')} \mathcal{F}(U) \longrightarrow \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U) = \mathcal{F}_{\overline{s}}$$

from Categories, Lemma 4.14.8. Using the description of elements of stalks as triples this maps the element of $\mathcal{F}_{\overline{s}'}$ represented by the triple $(U, \overline{u}', \sigma)$ to the element of $\mathcal{F}_{\overline{s}}$ represented by the triple $(U, \overline{u}' \circ a, \sigma)$. Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 7.37.2, that a defines a morphism $a : p \to p'$ between the points p, p' of the site $S_{\acute{e}tale}$ associated to $\overline{s}, \overline{s}'$ by Lemma 59.29.7. There are more general morphisms of points (corresponding to specializations of points of S) which we will describe later, and which will not be isomorphisms, see Section 59.75.

03PT Lemma 59.29.9. Let S be a scheme. Let $\overline{s}$ be a geometric point of S.

(1) The stalk functor $\mathrm{PAb}(S_{\acute{e}tale}) \to \mathrm{Ab}$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.

(2) We have $(\mathcal{F}^\#)_{\overline{s}} = \mathcal{F}_{\overline{s}}$ for any presheaf of sets $\mathcal{F}$ on $S_{étale}$.
(3) The functor $\text{Ab}(S_{étale}) \to \text{Ab}$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.
(4) Similarly the functors $\text{PSh}(S_{étale}) \to \text{Sets}$ and $Sh(S_{étale}) \to \text{Sets}$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.23.1) and commute with arbitrary colimits.

Proof. Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 18.36. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ comes from a point of the small étale site of S, see Lemma 59.29.7. We will only give a direct proof of (1), (2) and (3), and omit a direct proof of (4).

Exactness as a functor on $\text{PAb}(S_{étale})$ is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

$$\kappa : \mathcal{F}_{\overline{s}} \longrightarrow (\mathcal{F}^\#)_{\overline{s}}$$

induced by the natural morphism $\mathcal{F} \to \mathcal{F}^\#$, see Theorem 59.13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let $\sigma \in \mathcal{F}_{\overline{s}}$. There exists an étale neighborhood $(U, \overline{u}) \to (S, \overline{s})$ such that σ is the image of some section $s \in \mathcal{F}(U)$. If $\kappa(\sigma) = 0$ in $(\mathcal{F}^\#)_{\overline{s}}$ then there exists a morphism of étale neighborhoods $(U', \overline{u}') \to (U, \overline{u})$ such that $s|_{U'}$ is zero in $\mathcal{F}^\#(U')$. It follows there exists an étale covering $\{U'_i \to U'\}_{i \in I}$ such that $s|_{U'_i} = 0$ in $\mathcal{F}(U'_i)$ for all i. By Lemma 59.29.5 there exist $i \in I$ and a morphism $\overline{u}'_i : \overline{s} \to U'_i$ such that $(U'_i, \overline{u}'_i) \to (U', \overline{u}') \to (U, \overline{u})$ are morphisms of étale neighborhoods. Hence $\sigma = 0$ since $(U'_i, \overline{u}'_i) \to (U, \overline{u})$ is a morphism of étale neighbourhoods such that we have $s|_{U'_i} = 0$. This proves κ is injective.

To show that the functor $\text{Ab}(S_{étale}) \to \text{Ab}$ is exact, consider any short exact sequence in $\text{Ab}(S_{étale})$: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. This gives us the exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{H}/^p\mathcal{G} \to 0,$$

where $/^p$ denotes the quotient in $\text{PAb}(S_{étale})$. Taking stalks at $\overline{s}$, we see that $(\mathcal{H}/^p\mathcal{G})_{\overline{s}} = (\mathcal{H}/\mathcal{G})_{\overline{s}} = 0$, since the sheafification of $\mathcal{H}/^p\mathcal{G}$ is 0. Therefore,

$$0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0 = (\mathcal{H}/^p\mathcal{G})_{\overline{s}}$$

is exact, since taking stalks is exact as a functor from presheaves. □

03PU Theorem 59.29.10. Let S be a scheme. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{s}} : \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is injective (resp. surjective) for all geometric points of S. A sequence of abelian sheaves on $S_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.

Proof. The necessity of exactness on stalks follows from Lemma 59.29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of sheaves such that $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is surjective for all geometric points. Fix $U \in \text{Ob}(S_{étale})$ and $s \in \mathcal{G}(U)$. For every $u \in U$ choose some $\overline{u} \to U$ lying over u and an étale neighborhood $(V_u, \overline{v}_u) \to (U, \overline{u})$ such that $s|_{V_u} = \alpha(s_{V_u})$

for some $s_{V_u} \in \mathcal{F}(V_u)$. This is possible since α is surjective on stalks. Then $\{V_u \to U\}_{u \in U}$ is an étale covering on which the restrictions of s are in the image of the map α. Thus, α is surjective, see Sites, Section 7.11. $\square$

040S Remarks 59.29.11. On points of the geometric sites.

(1) Theorem 59.29.10 says that the family of points of $S_{étale}$ given by the geometric points of S (Lemma 59.29.7) is conservative, see Sites, Definition 7.38.1. In particular $S_{étale}$ has enough points.

04FP (2) Suppose $\mathcal{F}$ is a sheaf on the big étale site of S. Let $T \to S$ be an object of the big étale site of S, and let $\bar{t}$ be a geometric point of T. Then we define $\mathcal{F}_{\overline{t}}$ as the stalk of the restriction $\mathcal{F}|_{T_{étale}}$ of $\mathcal{F}$ to the small étale site of T. In other words, we can define the stalk of $\mathcal{F}$ at any geometric point of any scheme $T/S \in \text{Ob}((Sch/S)_{étale})$.

(3) The big étale site of S also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

04HU Lemma 59.29.12. Let S be a scheme.

(1) Let p be a point of the small étale site $S_{étale}$ of S given by a functor $u : S_{étale} \to \text{Sets}$. Then there exists a geometric point $\overline{s}$ of S such that p is isomorphic to the point of $S_{étale}$ associated to $\overline{s}$ in Lemma 59.29.7.

(2) Let $p : Sh(pt) \to Sh(S_{étale})$ be a point of the small étale topos of S. Then p comes from a geometric point of S, i.e., the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to a stalk functor as defined in Definition 59.29.6.

Proof. By Sites, Lemma 7.32.7 there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition 7.33.3 the functor u has the following properties: (a) $u(S) = \{*\}$, (b) $u(U \times_V W) = u(U) \times_{u(V)} u(W)$, and (c) if $\{U_i \to U\}$ is an étale covering, then $\coprod u(U_i) \to u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma 7.32.7 we can write $u(U) = p^{-1}(h_U^\#)$, in other words $u(U)$ is the stalk of the representable sheaf h_U. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\#$ and we get $u(U) = u(V) \amalg u(W)$ since p^{-1} is exact.

Consider the restriction of u to S_{Zar}. By Sites, Examples 7.33.5 and 7.33.6 there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{*\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \notin S'$. Note that if $\varphi : U \to S$ is an object of $S_{étale}$ then $\varphi(U) \subset S$ is open (see Proposition 59.26.2) and $\{U \to \varphi(U)\}$ is an étale covering. Hence we conclude that $u(U) = \emptyset \Leftrightarrow s \in \varphi(U)$.

Pick a geometric point $\overline{s} : \overline{s} \to S$ lying over s, see Definition 59.29.1 for customary abuse of notation. Suppose that $\varphi : U \to S$ is an object of $S_{étale}$ with U affine. Note that φ is separated, and that the fibre U_s of φ over s is an affine scheme over $\text{Spec}(\kappa(s))$ which is the spectrum of a finite product of finite separable extensions k_i of $\kappa(s)$. Hence we may apply Étale Morphisms, Lemma 41.18.2 to get an étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \to V$ an isomorphism and W having no point lying over $\overline{v}$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking V a bit we can assume that V has exactly one point lying over s, and hence W has no point lying over s. By the above this then gives $u(W) = \emptyset$. Hence we obtain

$$u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod\nolimits_{i=1,\ldots,n} u(V)$$

Note that $u(V) \neq \emptyset$ as s is in the image of $V \to S$. In particular, we see that in this situation $u(U)$ is a finite set with n elements.

Consider the limit

$$\lim\nolimits_{(V,\overline{v})} u(V)$$

over the category of étale neighbourhoods $(V, \overline{v})$ of $\overline{s}$. It is clear that we get the same value when taking the limit over the subcategory of $(V, \overline{v})$ with V affine. By the previous paragraph (applied with the roles of V and U switched) we see that in this case $u(V)$ is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma 59.29.4. Hence by Categories, Section 4.20 the limit is nonempty. Pick an element x from this limit. This means we obtain a $x_{V,\overline{v}} \in u(V)$ for every étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that for every morphism of étale neighbourhoods $\varphi : (V', \overline{v}') \to (V, \overline{v})$ we have $u(\varphi)(x_{V',\overline{v}'}) = x_{V,\overline{v}}$.

We will use the choice of x to construct a functorial bijective map

$$c : |U_{\overline{s}}| \longrightarrow u(U)$$

for $U \in \mathrm{Ob}(S_{étale})$ which will conclude the proof. See Lemma 59.29.7 and its proof for a description of $|U_{\overline{s}}|$. First we claim that it suffices to construct the map for U affine. We omit the proof of this claim. Assume $U \to S$ in $S_{étale}$ with U affine, and let $\overline{u} : \overline{s} \to U$ be an element of $|U_{\overline{s}}|$. Choose a $(V, \overline{v})$ such that $U \times_S V$ decomposes as in the third paragraph of the proof. Then the pair $(\overline{u}, \overline{v})$ gives a geometric point of $U \times_S V$ lying over $\overline{v}$ and determines one of the components U_i of $U \times_S V$. More precisely, there exists a section $\sigma : V \to U \times_S V$ of the projection pr_U such that $(\overline{u}, \overline{v}) = \sigma \circ \overline{v}$. Set $c(\overline{u}) = u(\mathrm{pr}_U)(u(\sigma)(x_{V,\overline{v}})) \in u(U)$. We have to check this is independent of the choice of $(V, \overline{v})$. By Lemma 59.29.4 the category of étale neighbourhoods is cofiltered. Hence it suffice to show that given a morphism of étale neighbourhood $\varphi : (V', \overline{v}') \to (V, \overline{v})$ and a choice of a section $\sigma' : V' \to U \times_S V'$ of the projection such that $(\overline{u}, \overline{v'}) = \sigma' \circ \overline{v}'$ we have $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$. Consider the diagram

$$\begin{tikzcd} V' \arrow[r, "\varphi"'] \arrow[d, "\sigma'"] & V \arrow[d, "\sigma"] \\ U \times_S V' \arrow[r, "1 \times \varphi"] & U \times_S V \end{tikzcd}$$

Now, it may not be the case that this diagram commutes. The reason is that the schemes V' and V may not be connected, and hence the decompositions used to construct σ' and σ above may not be unique. But we do know that $\sigma \circ \varphi \circ \overline{v}' = (1 \times \varphi) \circ \sigma' \circ \overline{v}'$ by construction. Hence, since $U \times_S V$ is étale over S, there exists an open neighbourhood $V'' \subset V'$ of $\overline{v'}$ such that the diagram does commute when restricted to V'', see Morphisms, Lemma 29.35.17. This means we may extend the diagram above to

$$
\begin{tikzcd}
V'' \arrow[r] \arrow[d, "\sigma'|_{V''}"] & V' \arrow[r, "\varphi"'] \arrow[d, "\sigma'"] & V \arrow[d, "\sigma"] \\
U \times_S V'' \arrow[r] & U \times_S V' \arrow[r, "1\times\varphi"] & U \times_S V
\end{tikzcd}
$$

such that the left square and the outer rectangle commute. Since u is a functor this implies that $x_{V'',\overline{v}'}$ maps to the same element in $u(U \times_S V)$ no matter which route we take through the diagram. On the other hand, it maps to the elements $x_{V',\overline{v}'}$ and $x_{V,\overline{v}}$ in $u(V')$ and $u(V)$. This implies the desired equality $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$.

In a similar manner one proves that the construction $c : |U_{\overline{s}}| \to u(U)$ is functorial in U; details omitted. And finally, by the results of the third paragraph it is clear that the map c is bijective which ends the proof of the lemma. □

59.30. Points in other topologies

06VW In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 7.38 and Topologies, Section 34.2 ff for the terminology used in this section. All of the geometric sites have enough points.

06VX **Lemma 59.30.1.** Let S be a scheme. All of the following sites have enough points $S_{affine,Zar}$, S_{Zar}, $S_{affine,étale}$, $S_{étale}$, $(Sch/S)_{Zar}$, $(\textit{Aff}/S)_{Zar}$, $(Sch/S)_{étale}$, $(\textit{Aff}/S)_{étale}$, $(Sch/S)_{smooth}$, $(\textit{Aff}/S)_{smooth}$, $(Sch/S)_{syntomic}$, $(\textit{Aff}/S)_{syntomic}$, $(Sch/S)_{fppf}$, and $(\textit{Aff}/S)_{fppf}$.

Proof. For each of the big sites the associated topos is equivalent to the topos defined by the site $(\textit{Aff}/S)_\tau$, see Topologies, Lemmas 34.3.10, 34.4.11, 34.5.9, 34.6.9, and 34.7.11. The result for the sites $(\textit{Aff}/S)_\tau$ follows immediately from Deligne's result Sites, Lemma 7.39.4.

The result for S_{Zar} is clear. The result for $S_{affine,Zar}$ follows from Deligne's result. The result for $S_{étale}$ either follows from (the proof of) Theorem 59.29.10 or from Topologies, Lemma 34.4.12 and Deligne's result applied to $S_{affine,étale}$. □

The lemma above guarantees the existence of points, but it doesn't tell us what these points look like. We can explicitly construct some points as follows. Suppose $\overline{s} : \mathrm{Spec}(k) \to S$ is a geometric point with k algebraically closed. Consider the functor

$$u : (Sch/S)_{fppf} \longrightarrow \textit{Sets}, \quad u(U) = U(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), U).$$

Note that $U \mapsto U(k)$ commutes with finite limits as $S(k) = \{\overline{s}\}$ and $(U_1 \times_U U_2)(k) = U_1(k) \times_{U(k)} U_2(k)$. Moreover, if $\{U_i \to U\}$ is an fppf covering, then $\coprod U_i(k) \to U(k)$ is surjective. By Sites, Proposition 7.33.3 we see that u defines a point p of $(Sch/S)_{fppf}$ with stalks

$$\mathcal{F}_p = \mathrm{colim}_{(U,x)} \mathcal{F}(U)$$

where the colimit is over pairs $U \to S$, $x \in U(k)$ as usual. But... this category has an initial object, namely $(\mathrm{Spec}(k), \mathrm{id})$, hence we see that

$$\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(k))$$

which isn't terribly interesting! In fact, in general these points won't form a conservative family of points. A more interesting type of point is described in the following remark.

06VY Remark 59.30.2. Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let (p, u) be a point of the site $(\mathit{Aff}/S)_{fppf}$, see Sites, Sections 7.32 and 7.33. Let $B = \mathcal{O}_p$ be the stalk of the structure sheaf at the point p. Recall that This is discussed in [Sch14].

$$B = \mathrm{colim}_{(U,x)} \mathcal{O}(U) = \mathrm{colim}_{(\mathrm{Spec}(C), x_C)} C$$

where $x_C \in u(\mathrm{Spec}(C))$. It can happen that $\mathrm{Spec}(B)$ is an object of $(\mathit{Aff}/S)_{fppf}$ and that there is an element $x_B \in u(\mathrm{Spec}(B))$ mapping to the compatible system x_C. In this case the system of neighbourhoods has an initial object and it follows that $\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(B))$ for any sheaf $\mathcal{F}$ on $(\mathit{Aff}/S)_{fppf}$. It is straightforward to see that if $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(B))$ defines a point of $\mathit{Sh}((\mathit{Aff}/S)_{fppf})$, then B has to be a local A-algebra such that for every faithfully flat, finitely presented ring map $B \to B'$ there is a section $B' \to B$. Conversely, for any such A-algebra B the functor $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(B))$ is the stalk functor of a point. Details omitted. It is not clear what a general point of the site $(\mathit{Aff}/S)_{fppf}$ looks like.

59.31. Supports of abelian sheaves

04FQ First we talk about supports of local sections.

04HV Lemma 59.31.1. Let S be a scheme. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of S (see Sites, Example 7.10.2). Then there exists a unique open $W \subset S$ such that $\mathcal{F} = h_W$.

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to S$ in $\mathrm{Ob}(S_{\acute{e}tale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\{\varphi : U \to \varphi(U)\}$ is a covering. Hence we can take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$. □

04FR Lemma 59.31.2. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{\acute{e}tale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subset $W \subset U$ such that

(1) $W \subset U$ is the largest Zariski open subset of U such that $\sigma|_W = 0$,
(2) for every $\varphi : V \to U$ in $S_{\acute{e}tale}$ we have
$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$
(3) for every geometric point $\overline{u}$ of U we have
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 17.5.2. Let $\varphi : V \to U$ be an arrow of $S_{\acute{e}tale}$. Note that $\varphi(V) \subset U$ is an open subset and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let $s \in S$. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. By Remark 59.29.8 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over s is well defined.

04FS Definition 59.31.3. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$.

(1) The support of $\mathcal{F}$ is the set of points $s \in S$ such that $\mathcal{F}_{\overline{s}} \neq 0$ for any (some) geometric point $\overline{s}$ lying over s.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The support of σ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 59.31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = \mathrm{Spec}(\mathbf{A}^1_{\mathbf{C}})$. Let $i_t : \mathrm{Spec}(\mathbf{C}) \to S$ be the inclusion of the point $t \in \mathbf{C}$. We will see later that $\mathcal{F}_t = i_{t,*}(\mathbf{Z}/2\mathbf{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 59.46. Then

$$\bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{F}_n$$

is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subset S$. This is true because taking stalks commutes with colimits, see Lemma 59.29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

04FT Lemma 59.31.4. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $U \in \mathrm{Ob}(S_{étale})$ and $\sigma \in \mathcal{F}(U)$.

(1) The support of σ is closed in U.
(2) The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.
(3) If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $S_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.
(4) The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.
(5) If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.
(6) If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 17.5.2. Part (4) is a direct consequence of Lemma 59.31.2 part (3). Parts (5) and (6) follow from the other parts. □

04FU Lemma 59.31.5. The support of a sheaf of rings on $S_{étale}$ is closed.

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

59.32. Henselian rings

03QD We begin by stating a theorem which has already been used many times in the Stacks project. There are many versions of this result; here we just state the algebraic version.

03QE Theorem 59.32.1. Let $A \to B$ be finite type ring map and $\mathfrak{p} \subset A$ a prime ideal. Then there exist an étale ring map $A \to A'$ and a prime $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that

(1) $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,
(2) $B \otimes_A A' = B_1 \times \ldots \times B_r \times C$,
(3) $A' \to B_i$ is finite and there exists a unique prime $q_i \subset B_i$ lying over $\mathfrak{p}'$, and
(4) all irreducible components of the fibre $\mathrm{Spec}(C \otimes_{A'} \kappa(\mathfrak{p}'))$ of C over $\mathfrak{p}'$ have dimension at least 1.

Proof. See Algebra, Lemma 10.145.3, or see [GD67, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 37.41. $\square$

Recall Hensel's lemma. There are many versions of this lemma. Here are two:

(f) if $f \in \mathbf{Z}_p[T]$ monic and $f \bmod p = g_0 h_0$ with $gcd(g_0, h_0) = 1$ then f factors as $f = gh$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,
(r) if $f \in \mathbf{Z}_p[T]$, monic $a_0 \in \mathbf{F}_p$, $\bar{f}(a_0) = 0$ but $\bar{f}'(a_0) \neq 0$ then there exists $a \in \mathbf{Z}_p$ with $f(a) = 0$ and $\bar{a} = a_0$.

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

03QF Definition 59.32.2. (See Algebra, Definition 10.153.1.) A local ring $(R, \mathfrak{m}, \kappa)$ is called henselian if for all $f \in R[T]$ monic, for all $a_0 \in \kappa$ such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$, there exists an $a \in R$ such that $f(a) = 0$ and $a \bmod \mathfrak{m} = a_0$.

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition 10.160.1) that a complete local ring is a local ring $(R, \mathfrak{m})$ such that $R \cong \lim_n R/\mathfrak{m}^n$, i.e., it is complete and separated for the $\mathfrak{m}$-adic topology.

03QG Theorem 59.32.3. Complete local rings are henselian.

Proof. Newton's method. See Algebra, Lemma 10.153.9. $\square$

03QH Theorem 59.32.4. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent:

(1) R is henselian,
(2) for any $f \in R[T]$ and any factorization $\bar{f} = g_0 h_0$ in $\kappa[T]$ with $\gcd(g_0, h_0) = 1$, there exists a factorization $f = gh$ in $R[T]$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,
(3) any finite R-algebra S is isomorphic to a finite product of local rings finite over R,
(4) any finite type R-algebra A is isomorphic to a product $A \cong A' \times C$ where $A' \cong A_1 \times \ldots \times A_r$ is a product of finite local R-algebras and all the irreducible components of $C \otimes_R \kappa$ have dimension at least 1,
(5) if A is an étale R-algebra and $\mathfrak{n}$ is a maximal ideal of A lying over $\mathfrak{m}$ such that $\kappa \cong A/\mathfrak{n}$, then there exists an isomorphism $\varphi : A \cong R \times A'$ such that $\varphi(\mathfrak{n}) = \mathfrak{m} \times A' \subset R \times A'$.

Proof. This is just a subset of the results from Algebra, Lemma 10.153.3. Note that part (5) above corresponds to part (8) of Algebra, Lemma 10.153.3 but is formulated slightly differently. □

03QJ Lemma 59.32.5. If R is henselian and A is a finite R-algebra, then A is a finite product of henselian local rings.

Proof. See Algebra, Lemma 10.153.4. □

03QK Definition 59.32.6. A local ring R is called strictly henselian if it is henselian and its residue field is separably closed.

03QI Example 59.32.7. In the case $R = \mathbf{C}[[t]]$, the étale R-algebras are finite products of the trivial extension $R \to R$ and the extensions $R \to R[X, X^{-1}]/(X^n - t)$. The latter ones factor through the open $D(t) \subset \text{Spec}(R)$, so any étale covering can be refined by the covering $\{\text{id} : \text{Spec}(R) \to \text{Spec}(R)\}$. We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

03QL Theorem 59.32.8. Let $(R, \mathfrak{m}, \kappa)$ be a local ring and $\kappa \subset \kappa^{sep}$ a separable algebraic closure. There exist canonical flat local ring maps $R \to R^h \to R^{sh}$ where

(1) R^h, R^{sh} are filtered colimits of étale R-algebras,
(2) R^h is henselian, R^{sh} is strictly henselian,
(3) $\mathfrak{m}R^h$ (resp. $\mathfrak{m}R^{sh}$) is the maximal ideal of R^h (resp. R^{sh}), and
(4) $\kappa = R^h/\mathfrak{m}R^h$, and $\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$ as extensions of κ.

Proof. The structure of R^h and R^{sh} is described in Algebra, Lemmas 10.155.1 and 10.155.2. □

The rings constructed in Theorem 59.32.8 are called respectively the henselization and the strict henselization of the local ring R, see Algebra, Definition 10.155.3. Many of the properties of R are reflected in its (strict) henselization, see More on Algebra, Section 15.45.

59.33. Stalks of the structure sheaf

04HW In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding "usual" point.

04HX Lemma 59.33.1. Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of κ in $\kappa(\overline{s})$. Then there is a canonical identification

$$(\mathcal{O}_{S,s})^{sh} \cong (\mathcal{O}_S)_{\overline{s}}$$

where the left hand side is the strict henselization of the local ring $\mathcal{O}_{S,s}$ as described in Theorem 59.32.8 and right hand side is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{\acute{e}tale}$ at the geometric point $\overline{s}$.

Proof. Let $\text{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. Thus we have $\kappa(\mathfrak{p}) \subset \kappa^{sep} \subset \kappa(\overline{s})$. Recall that

$$(\mathcal{O}_S)_{\overline{s}} = \text{colim}_{(U, \overline{u})} \mathcal{O}(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \overline{u})$ such that U is affine and $U \to S$ factors through $\mathrm{Spec}(A)$. In other words, we see that

$$(\mathcal{O}_S)_{\overline{s}} = \mathrm{colim}_{(B, \mathfrak{q}, \phi)} B$$

where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ and a $\kappa(\mathfrak{p})$-algebra map $\phi : \kappa(\mathfrak{q}) \to \kappa(\overline{s})$. Note that since $\kappa(\mathfrak{q})$ is finite separable over $\kappa(\mathfrak{p})$ the image of ϕ is contained in κ^{sep}. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 10.155.11. □

03PS Definition 59.33.2. Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over the point $s \in S$.

(1) The étale local ring of S at $\overline{s}$ is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{\acute{e}tale}$ at $\overline{s}$. We sometimes call this the strict henselization of $\mathcal{O}_{S,s}$ relative to the geometric point $\overline{s}$. Notation used: $\mathcal{O}^{sh}_{S,\overline{s}}$.
(2) The henselization of $\mathcal{O}_{S,s}$ is the henselization of the local ring of S at s. See Algebra, Definition 10.155.3, and Theorem 59.32.8. Notation: $\mathcal{O}^h_{S,s}$.
(3) The strict henselization of S at $\overline{s}$ is the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$.
(4) The henselization of S at s is the scheme $\mathrm{Spec}(\mathcal{O}^h_{S,s})$.

Let $f : T \to S$ be a morphism of schemes. Let $\overline{t}$ be a geometric point of T with image $\overline{s}$ in S. Let $t \in T$ and $s \in S$ be their images. Then we obtain a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{O}^h_{T,t}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}^{sh}_{T,\overline{t}}) \arrow[r] \arrow[d] & T \arrow[d, "f"] \\
\mathrm{Spec}(\mathcal{O}^h_{S,s}) \arrow[r] & \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \arrow[r] & S
\end{tikzcd}$$

of henselizations and strict henselizations of T and S. You can prove this by choosing affine neighbourhoods of t and s and using the functoriality of (strict) henselizations given by Algebra, Lemmas 10.155.8 and 10.155.12.

04HY Lemma 59.33.3. Let S be a scheme. Let $s \in S$. Then we have

$$\mathcal{O}^h_{S,s} = \mathrm{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category of étale neighbourhoods (U, u) of (S, s) such that $\kappa(s) = \kappa(u)$.

Proof. This lemma is a copy of More on Morphisms, Lemma 37.35.5. □

03QM Remark 59.33.4. Let S be a scheme. Let $s \in S$. If S is locally Noetherian then $\mathcal{O}^h_{S,s}$ is also Noetherian and it has the same completion:

$$\widehat{\mathcal{O}_{S,s}} \cong \widehat{\mathcal{O}^h_{S,s}}.$$

In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}^h_{S,s} \subset \widehat{\mathcal{O}_{S,s}}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}^h_{S,s}$, but this is not true for the completion in general. Insert future references here.

04HZ Lemma 59.33.5. Let S be a scheme. The small étale site $S_{étale}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 18.40.4.

Proof. This follows because the stalks $(\mathcal{O}_S)_{\overline{s}} = \mathcal{O}^{sh}_{S,\overline{s}}$ are local, and because $S_{étale}$ has enough points, see Lemma 59.33.1, Theorem 59.29.10, and Remarks 59.29.11. See Modules on Sites, Lemmas 18.40.2 and 18.40.3 for the fact that this implies the small étale site is locally ringed. $\square$

59.34. Functoriality of small étale topos

04I0 So far we haven't yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 34.4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \to Y$ be a morphism of schemes. We obtain a functor

04I1 (59.34.0.1) $$u : Y_{étale} \longrightarrow X_{étale}, \quad V/Y \longmapsto X \times_Y V/X.$$

This functor has the following important properties

(1) u(final object) = final object,
(2) u preserves fibre products,
(3) if $\{V_j \to V\}$ is a covering in $Y_{étale}$, then $\{u(V_j) \to u(V)\}$ is a covering in $X_{étale}$.

Each of these is easy to check (omitted). As a consequence we obtain what is called a morphism of sites

$$f_{small} : X_{étale} \longrightarrow Y_{étale},$$

see Sites, Definition 7.14.1 and Sites, Proposition 7.14.7. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{small,*}$ (pushforward) and f^{-1}_{small} (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

59.35. Direct images

03PV Let us define the pushforward of a presheaf.

03PW Definition 59.35.1. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{étale}$. The direct image, or pushforward of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow \text{Sets}, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_*\mathcal{F}$ is the composition of functors $\mathcal{F} \circ u$ where u is as in Equation (59.34.0.1). This makes it clear that the construction is functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* = f_{small,*} : \text{PSh}(X_{étale}) \longrightarrow \text{PSh}(Y_{étale})$$

Note that if $\mathcal{F}$ is a presheaf of abelian groups, then $f_*\mathcal{F}$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : \text{PAb}(X_{\acute{e}tale}) \longrightarrow \text{PAb}(Y_{\acute{e}tale})$$

as before (i.e., defined by exactly the same rule).

03PX Remark 59.35.2. We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_j \to V\}$ is an étale covering in $Y_{\acute{e}tale}$ then $\{X \times_Y V_j \to X \times_Y V\}$ is an étale covering in $X_{\acute{e}tale}$. Hence the sheaf condition for $\mathcal{F}$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_*\mathcal{F}$ with respect to $\{V_i \to V\}$. Thus if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$.

03PY Definition 59.35.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a sheaf of sets on $X_{\acute{e}tale}$. The direct image, or pushforward of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{\acute{e}tale}^{opp} \longrightarrow \text{Sets}, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X)$$

which is a sheaf by Remark 59.35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : \mathit{Sh}(X_{\acute{e}tale}) \longrightarrow \mathit{Sh}(Y_{\acute{e}tale})$$

and

$$f_* = f_{small,*} : \text{Ab}(X_{\acute{e}tale}) \longrightarrow \text{Ab}(Y_{\acute{e}tale})$$

called direct image again.

The functor f_* on abelian sheaves is left exact. (See Homology, Section 12.7 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact on $X_{\acute{e}tale}$, then for every $U/X \in \text{Ob}(X_{\acute{e}tale})$ the sequence of abelian groups $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact. Hence for every $V/Y \in \text{Ob}(Y_{\acute{e}tale})$ the sequence of abelian groups $0 \to f_*\mathcal{F}_1(V) \to f_*\mathcal{F}_2(V) \to f_*\mathcal{F}_3(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

04I2 Definition 59.35.4. Let $f : X \to Y$ be a morphism of schemes. The right derived functors $\{R^pf_*\}_{p\geq 1}$ of $f_* : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(Y_{\acute{e}tale})$ are called higher direct images.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

59.36. Inverse image

03PZ In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 34.4.

03Q0 Definition 59.36.1. Let $f : X \to Y$ be a morphism of schemes. The inverse image, or pullback[2] functors are the functors

$$f^{-1} = f^{-1}_{small} : \mathit{Sh}(Y_{\acute{e}tale}) \longrightarrow \mathit{Sh}(X_{\acute{e}tale})$$

and

$$f^{-1} = f^{-1}_{small} : \text{Ab}(Y_{\acute{e}tale}) \longrightarrow \text{Ab}(X_{\acute{e}tale})$$

[2]We use the notation f^{-1} for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve f^* for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.

which are left adjoint to $f_* = f_{small,*}$. Thus f^{-1} is characterized by the fact that

$$\mathrm{Hom}_{Sh(X_{\acute{e}tale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Sh(Y_{\acute{e}tale})}(\mathcal{G}, f_*\mathcal{F})$$

functorially, for any $\mathcal{F} \in Sh(X_{\acute{e}tale})$ and $\mathcal{G} \in Sh(Y_{\acute{e}tale})$. We similarly have

$$\mathrm{Hom}_{\mathrm{Ab}(X_{\acute{e}tale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Ab}(Y_{\acute{e}tale})}(\mathcal{G}, f_*\mathcal{F})$$

for $\mathcal{F} \in \mathrm{Ab}(X_{\acute{e}tale})$ and $\mathcal{G} \in \mathrm{Ab}(Y_{\acute{e}tale})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 59.36.3 below. The general machinery shows that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

04I3 (59.36.1.1) $$U/X \longmapsto \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V/Y)$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X \times_Y V/X)$. To see this apply Sites, Proposition 7.14.7 to the functor u of Equation (59.34.0.1) and use the description of $u_s = (u_p\ )^\#$ in Sites, Sections 7.13 and 7.5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

03Q1 **Lemma 59.36.2.** Let $f : X \to Y$ be a morphism of schemes.

(1) The functor $f^{-1} : \mathrm{Ab}(Y_{\acute{e}tale}) \to \mathrm{Ab}(X_{\acute{e}tale})$ is exact.
(2) The functor $f^{-1} : Sh(Y_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.23.1.
(3) Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{\acute{e}tale}$. Then there is a canonical identification
$$(f^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$
where $\overline{y} = f \circ \overline{x}$.
(4) For any $V \to Y$ étale we have $f^{-1}h_V = h_{X \times_Y V}$.

Proof. The exactness of f^{-1} on sheaves of sets is a consequence of Sites, Proposition 7.14.7 applied to our functor u of Equation (59.34.0.1). In fact the exactness of pullback is part of the definition of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf $\mathcal{G}$ on $Y_{\acute{e}tale}$ the underlying sheaf of sets of $f^{-1}\mathcal{F}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{F}$, see Sites, Section 7.44. See also Modules on Sites, Lemma 18.31.2. In the literature (1) and (2) are sometimes deduced from (3) via Theorem 59.29.10.

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma 7.34.2. We will also prove (3) directly as follows. Note that by Lemma 59.29.9 taking stalks commutes with sheafification. Now recall that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$U \longrightarrow \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V),$$

see Equation (59.36.1.1). Thus we have

$$\begin{aligned} (f^{-1}\mathcal{G})_{\overline{x}} &= \mathrm{colim}_{(U,\overline{u})} f^{-1}\mathcal{G}(U) \\ &= \mathrm{colim}_{(U,\overline{u})} \mathrm{colim}_{a : U \to X \times_Y V} \mathcal{G}(V) \\ &= \mathrm{colim}_{(V,\overline{v})} \mathcal{G}(V) \\ &= \mathcal{G}_{\overline{y}} \end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$.

Part (4) can be proved in a similar manner by identifying the colimits which define $f^{-1}h_V$. Or you can use Yoneda's lemma (Categories, Lemma 4.3.5) and the functorial equalities

$$\mathrm{Mor}_{Sh(X_{\acute{e}tale})}(f^{-1}h_V, \mathcal{F}) = \mathrm{Mor}_{Sh(Y_{\acute{e}tale})}(h_V, f_*\mathcal{F}) = f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)$$

combined with the fact that representable presheaves are sheaves. See also Sites, Lemma 7.13.5 for a completely general result. □

The pair of functors (f_*, f^{-1}) define a morphism of small étale topoi

$$f_{small} : Sh(X_{\acute{e}tale}) \longrightarrow Sh(Y_{\acute{e}tale})$$

Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

03Q2 Remark 59.36.3. More generally, let $\mathcal{C}_1, \mathcal{C}_2$ be sites, and assume they have final objects and fibre products. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ be a functor satisfying:

(1) if $\{V_i \to V\}$ is a covering of $\mathcal{C}_2$, then $\{u(V_i) \to u(V)\}$ is a covering of $\mathcal{C}_1$ (we say that u is continuous), and
(2) u commutes with finite limits (i.e., u is left exact, i.e., u preserves fibre products and final objects).

Then one can define $f_* : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ by $f_*\mathcal{F}(V) = \mathcal{F}(u(V))$. Moreover, there exists an exact functor f^{-1} which is left adjoint to f_*, see Sites, Definition 7.14.1 and Proposition 7.14.7. Warning: It is not enough to require simply that u is continuous and commutes with fibre products in order to get a morphism of topoi.

59.37. Functoriality of big topoi

04DI Given a morphism of schemes $f : X \to Y$ there are a whole host of morphisms of topoi associated to f, see Topologies, Section 34.11 for a list. Perhaps the most used ones are the morphisms of topoi

$$f_{big} = f_{big,\tau} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

where $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. These each correspond to a continuous functor

$$(Sch/Y)_\tau \longrightarrow (Sch/X)_\tau, \quad V/Y \longmapsto X \times_Y V/X$$

which preserves final objects, fibre products and covering, and hence defines a morphism of sites

$$f_{big} : (Sch/X)_\tau \longrightarrow (Sch/Y)_\tau.$$

See Topologies, Sections 34.3, 34.4, 34.5, 34.6, and 34.7. In particular, pushforward along f_{big} is given by the rule

$$(f_{big,*}\mathcal{F})(V/Y) = \mathcal{F}(X \times_Y V/X)$$

It turns out that these morphisms of topoi have an inverse image functor f_{big}^{-1} which is very easy to describe. Namely, we have

$$(f_{big}^{-1}\mathcal{G})(U/X) = \mathcal{G}(U/Y)$$

where the structure morphism of U/Y is the composition of the structure morphism $U \to X$ with f, see Topologies, Lemmas 34.3.16, 34.4.16, 34.5.10, 34.6.10, and 34.7.12.

59.38. Functoriality and sheaves of modules

04I4 In this section we are going to reformulate some of the material explained in Descent, Sections 35.8, 35.9, and 35.10 in the setting of étale topologies. Let $f : X \to Y$ be a morphism of schemes. We have seen above, see Sections 59.34, 59.35, and 59.36 that this induces a morphism f_{small} of small étale sites. In Descent, Remark 35.8.4 we have seen that f also induces a natural map

$$f_{small}^\sharp : \mathcal{O}_{Y_{\acute{e}tale}} \longrightarrow f_{small,*}\mathcal{O}_{X_{\acute{e}tale}}$$

of sheaves of rings on $Y_{\acute{e}tale}$ such that $(f_{small}, f_{small}^\sharp)$ is a morphism of ringed sites. See Modules on Sites, Definition 18.6.1 for the definition of a morphism of ringed sites. Let us just recall here that $f_{small}^\sharp$ is defined by the compatible system of maps

$$\text{pr}_V^\sharp : \mathcal{O}(V) \longrightarrow \mathcal{O}(X \times_Y V)$$

for V varying over the objects of $Y_{\acute{e}tale}$.

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, then we have

$$(g_{small}, g_{small}^\sharp) \circ (f_{small}, f_{small}^\sharp) = ((g \circ f)_{small}, (g \circ f)_{small}^\sharp)$$

as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 18.13.1 we see that given a morphism $f : X \to Y$ of schemes we get well defined pullback and direct image functors

$$f_{small}^* : \text{Mod}(\mathcal{O}_{Y_{\acute{e}tale}}) \longrightarrow \text{Mod}(\mathcal{O}_{X_{\acute{e}tale}}),$$
$$f_{small,*} : \text{Mod}(\mathcal{O}_{X_{\acute{e}tale}}) \longrightarrow \text{Mod}(\mathcal{O}_{Y_{\acute{e}tale}})$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of schemes, then we have $(g \circ f)_{small}^* = f_{small}^* \circ g_{small}^*$ and $(g \circ f)_{small,*} = g_{small,*} \circ f_{small,*}$ because of what we said about compositions.

There is quite a bit of difference between the category of all $\mathcal{O}_X$ modules on X and the category between all $\mathcal{O}_{X_{\acute{e}tale}}$-modules on $X_{\acute{e}tale}$. But the results of Descent, Sections 35.8, 35.9, and 35.10 tell us that there is not much difference between considering quasi-coherent modules on S and quasi-coherent modules on $S_{\acute{e}tale}$. (We have already seen this in Theorem 59.17.4 for example.) In particular, if $f : X \to Y$ is any morphism of schemes, then the pullback functors f_{small}^* and f^* match for quasi-coherent sheaves, see Descent, Proposition 35.9.4. Moreover, the same is true for pushforward provided f is quasi-compact and quasi-separated, see Descent, Lemma 35.9.5.

A few words about functoriality of the structure sheaf on big sites. Let $f : X \to Y$ be a morphism of schemes. Choose any of the topologies $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Then the morphism $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ becomes a morphism of ringed sites by a map

$$f_{big}^\sharp : \mathcal{O}_Y \longrightarrow f_{big,*}\mathcal{O}_X$$

see Descent, Remark 35.8.4. In fact it is given by the same construction as in the case of small sites explained above.

59.39. Comparing topologies

09XL In this section we start studying what happens when you compare sheaves with respect to different topologies.

09XM Lemma 59.39.1. Let S be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $S_{étale}$. Let $s, t \in \mathcal{F}(S)$. Then there exists an open $W \subset S$ characterized by the following property: A morphism $f : T \to S$ factors through W if and only if $s|_T = t|_T$ (restriction is pullback by f_{small}).

Proof. Consider the presheaf which assigns to $U \in \mathrm{Ob}(S_{étale})$ the empty set if $s|_U \neq t|_U$ and a singleton else. It is clear that this is a subsheaf of the final object of $Sh(S_{étale})$. By Lemma 59.31.1 we find an open $W \subset S$ representing this presheaf. For a geometric point $\overline{x}$ of S we see that $\overline{x} \in W$ if and only if the stalks of s and t at $\overline{x}$ agree. By the description of stalks of pullbacks in Lemma 59.36.2 we see that W has the desired property. $\square$

09XN Lemma 59.39.2. Let S be a scheme. Let $\tau \in \{Zariski, étale\}$. Consider the morphism

$$\pi_S : (Sch/S)_\tau \longrightarrow S_\tau$$

of Topologies, Lemma 34.3.14 or 34.4.14. Let $\mathcal{F}$ be a sheaf on S_τ. Then $\pi_S^{-1}\mathcal{F}$ is given by the rule

$$(\pi_S^{-1}\mathcal{F})(T) = \Gamma(T_\tau, f_{small}^{-1}\mathcal{F})$$

where $f : T \to S$. Moreover, $\pi_S^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to fpqc coverings.

Proof. Observe that we have a morphism $i_f : Sh(T_\tau) \to Sh(Sch/S)_\tau)$ such that $\pi_S \circ i_f = f_{small}$ as morphisms $T_\tau \to S_\tau$, see Topologies, Lemmas 34.3.13, 34.3.17, 34.4.13, and 34.4.17. Since pullback is transitive we see that $i_f^{-1}\pi_S^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$ as desired.

Let $\{g_i : T_i \to T\}_{i\in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on T_τ and given sections $s_i \in \Gamma(T_i, g_{i,small}^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section s of $\mathcal{G}$ over T whose pullback to T_i agrees with s_i.

Let $V \to T$ be an object of T_τ and let $t \in \mathcal{G}(V)$. For every i there is a largest open $W_i \subset T_i \times_T V$ such that the pullbacks of s_i and t agree as sections of the pullback of $\mathcal{G}$ to $W_i \subset T_i \times_T V$, see Lemma 59.39.1. Because s_i and s_j agree over $T_i \times_T T_j$ we find that W_i and W_j pullback to the same open over $T_i \times_T T_j \times_T V$. By Descent, Lemma 35.13.6 we find an open $W \subset V$ whose inverse image to $T_i \times_T V$ recovers W_i.

By construction of $g_{i,small}^{-1}\mathcal{G}$ there exists a τ-covering $\{T_{ij} \to T_i\}_{j\in J_i}$, for each j an open immersion or étale morphism $V_{ij} \to T$, a section $t_{ij} \in \mathcal{G}(V_{ij})$, and commutative diagrams

$$\begin{tikzcd} T_{ij} \arrow[r] \arrow[d] & V_{ij} \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$

such that $s_i|_{T_{ij}}$ is the pullback of t_{ij}. In other words, after replacing the covering $\{T_i \to T\}$ by $\{T_{ij} \to T\}$ we may assume there are factorizations $T_i \to V_i \to T$ with

$V_i \in \text{Ob}(T_\tau)$ and sections $t_i \in \mathcal{G}(V_i)$ pulling back to s_i over T_i. By the result of the previous paragraph we find opens $W_i \subset V_i$ such that $t_i|_{W_i}$ "agrees with" every s_j over $T_j \times_T W_i$. Note that $T_i \to V_i$ factors through W_i. Hence $\{W_i \to T\}$ is a τ-covering and the lemma is proven. $\square$

0A3H Lemma 59.39.3. Let S be a scheme. Let $f : T \to S$ be a morphism such that

(1) f is flat and quasi-compact, and
(2) the geometric fibres of f are connected.

Let $\mathcal{F}$ be a sheaf on $S_{\acute{e}tale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(T, f_{small}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(T, f_{small}^{-1}\mathcal{F})$. Since f is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\alpha \in \Gamma(T, f_{small}^{-1}\mathcal{F})$. Since $\{T \to S\}$ is an fpqc covering we can use Lemma 59.39.2 to see that suffices to prove that α pulls back to the same section over $T \times_S T$ by the two projections. Let $\overline{s} \to S$ be a geometric point. It suffices to show the agreement holds over $(T \times_S T)_{\overline{s}}$ as every geometric point of $T \times_S T$ is contained in one of these geometric fibres. In other words, we are trying to show that $\alpha|_{T_{\overline{s}}}$ pulls back to the same section over

$$(T \times_S T)_{\overline{s}} = T_{\overline{s}} \times_{\overline{s}} T_{\overline{s}}$$

by the two projections to $T_{\overline{s}}$. However, since $\mathcal{F}|_{T_{\overline{s}}}$ is the pullback of $\mathcal{F}|_{\overline{s}}$ it is a constant sheaf with value $\mathcal{F}_{\overline{s}}$. Since $T_{\overline{s}}$ is connected by assumption, any section of a constant sheaf is constant. Hence $\alpha|_{T_{\overline{s}}}$ corresponds to an element of $\mathcal{F}_{\overline{s}}$. Thus the two pullbacks to $(T \times_S T)_{\overline{s}}$ both correspond to this same element and we conclude. $\square$

Here is a version of Lemma 59.39.3 where we do not assume that the morphism is flat.

0EZK Lemma 59.39.4. Let S be a scheme. Let $f : X \to S$ be a morphism such that

(1) f is submersive, and
(2) the geometric fibres of f are connected.

Let $\mathcal{F}$ be a sheaf on $S_{\acute{e}tale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(X, f_{small}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(X, f_{small}^{-1}\mathcal{F})$. Since f is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\tau \in \Gamma(X, f_{small}^{-1}\mathcal{F})$. It suffices to find an étale covering $\{U_i \to S\}$ and sections $\sigma_i \in \mathcal{F}(U_i)$ such that σ_i pulls back to $\tau|_{X \times_S U_i}$. Namely, the injectivity shown above guarantees that σ_i and σ_j restrict to the same section of $\mathcal{F}$ over $U_i \times_S U_j$. Thus we obtain a unique section $\sigma \in \mathcal{F}(S)$ which restricts to σ_i over U_i. Then the pullback of σ to X is τ because this is true locally.

Let $\overline{x}$ be a geometric point of X with image $\overline{s}$ in S. Consider the image of τ in the stalk

$$(f_{small}^{-1}\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{s}}$$

See Lemma 59.36.2. We can find an étale neighbourhood $U \to S$ of $\overline{s}$ and a section $\sigma \in \mathcal{F}(U)$ mapping to this image in the stalk. Thus after replacing S by U and X by $X \times_S U$ we may assume there exits a section σ of $\mathcal{F}$ over S whose image in $(f_{small}^{-1}\mathcal{F})_{\overline{x}}$ is the same as τ.

By Lemma 59.39.1 there exists a maximal open $W \subset X$ such that $f_{small}^{-1}\sigma$ and τ agree over W and the formation of W commutes with further pullback. Observe that the pullback of $\mathcal{F}$ to the geometric fibre $X_{\overline{s}}$ is the pullback of $\mathcal{F}_{\overline{s}}$ viewed as a sheaf on $\overline{s}$ by $X_{\overline{s}} \to \overline{s}$. Hence we see that τ and σ give sections of the constant sheaf with value $\mathcal{F}_{\overline{s}}$ on $X_{\overline{s}}$ which agree in one point. Since $X_{\overline{s}}$ is connected by assumption, we conclude that W contains X_s. The same argument for different geometric fibres shows that W contains every fibre it meets. Since f is submersive, we conclude that W is the inverse image of an open neighbourhood of s in S. This finishes the proof. □

0A3I Lemma 59.39.5. Let K/k be an extension of fields with k separably algebraically closed. Let S be a scheme over k. Denote $p : S_K = S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to S$ the projection. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(S_K, p_{small}^{-1}\mathcal{F})$.

Proof. Follows from Lemma 59.39.3. Namely, it is clear that p is flat and quasi-compact as the base change of $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. On the other hand, if $\overline{s} : \mathrm{Spec}(L) \to S$ is a geometric point, then the fibre of p over $\overline{s}$ is the spectrum of $K \otimes_k L$ which is irreducible hence connected by Algebra, Lemma 10.47.2. □

59.40. Recovering morphisms

04JH In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 59.40.5.

04I5 Lemma 59.40.1. Let $f : X \to Y$ be a morphism of schemes. The morphism of ringed sites $(f_{small}, f_{small}^\sharp)$ associated to f is a morphism of locally ringed sites, see Modules on Sites, Definition 18.40.9.

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 59.33.5. Moreover, we know that $X_{étale}$ has enough points, see Theorem 59.29.10 and Remarks 59.29.11. Hence it suffices to prove that $(f_{small}, f_{small}^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 18.40.8. To see this take a point p of $X_{étale}$. By Lemma 59.29.12 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 59.36.2 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of stalks

$$(\mathcal{O}_Y)_{\overline{y}} \longrightarrow (\mathcal{O}_X)_{\overline{x}}$$

is a local ring map. Suppose that $a \in (\mathcal{O}_Y)_{\overline{y}}$ is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that a is the equivalence class of a triple $(V, \overline{v}, a)$ with $V \to Y$ étale, $\overline{v} : \overline{x} \to V$ over Y, and $a \in \mathcal{O}(V)$. It maps to the equivalence class of $(X \times_Y V, \overline{x} \times \overline{v}, \mathrm{pr}_V^\sharp(a))$ in the local ring $(\mathcal{O}_X)_{\overline{x}}$. But it is clear that being in the maximal ideal means that pulling back $\mathrm{pr}_V^\sharp(a)$ to an element of $\kappa(\overline{x})$ gives zero. Hence also pulling back a to $\kappa(\overline{x})$ is zero. Which means that a lies in the maximal ideal of $(\mathcal{O}_Y)_{\overline{y}}$. □

04IJ Lemma 59.40.2. Let X, Y be schemes. Let $f : X \to Y$ be a morphism of schemes. Let t be a 2-morphism from $(f_{small}, f_{small}^\sharp)$ to itself, see Modules on Sites, Definition 18.8.1. Then $t = \mathrm{id}$.

Proof. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{tikzcd} f_{small}^{-1}\mathcal{O}_Y \arrow[dr, "f_{small}^\sharp"'] & & f_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "f_{small}^\sharp"] \\ & \mathcal{O}_X & \end{tikzcd}$$

is commutative. Suppose $V \to Y$ is étale with V affine. By Morphisms, Lemma 29.39.2 we may choose an immersion $i : V \to \mathbf{A}^n_Y$ over Y. In terms of sheaves this means that i induces an injection $h_i : h_V \to \prod_{j=1,\ldots,n} \mathcal{O}_Y$ of sheaves. The base change i' of i to X is an immersion (Schemes, Lemma 26.18.2). Hence $i' : X \times_Y V \to \mathbf{A}^n_X$ is an immersion, which in turn means that $h_{i'} : h_{X \times_Y V} \to \prod_{j=1,\ldots,n} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1}h_V = h_{X \times_Y V}$ of Lemma 59.36.2 the map $h_{i'}$ is equal to

$$f_{small}^{-1}h_V \xrightarrow{f^{-1}h_i} \prod\nolimits_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod\nolimits_{j=1,\ldots,n} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1}h_V \to f_{small}^{-1}h_V$ fits into the commutative diagram

$$\begin{tikzcd} f_{small}^{-1}h_V \arrow[r, "f^{-1}h_i"] \arrow[d, "t"] & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] \arrow[d, "\prod t"] & \prod_{j=1,\ldots,n} \mathcal{O}_X \arrow[d, "\text{id}"] \\ f_{small}^{-1}h_V \arrow[r, "f^{-1}h_i"] & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Topologies, Lemma 34.4.12 with Sites, Lemma 7.12.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. □

04LW Lemma 59.40.3. Let X, Y be schemes. Any two morphisms $a, b : X \to Y$ of schemes for which there exists a 2-isomorphism $(a_{small}, a_{small}^\sharp) \cong (b_{small}, b_{small}^\sharp)$ in the 2-category of ringed topoi are equal.

Proof. Let us argue this carefuly since it is a bit confusing. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. Consider any open $V \subset Y$. Note that h_V is a subsheaf of the final sheaf $*$. Thus both $a_{small}^{-1}h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1}h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1}h_V = h_{a^{-1}(V)} \to b_{small}^{-1}h_V = h_{b^{-1}(V)}$$

has to be the identity, and $a^{-1}(V) = b^{-1}(V)$. It follows that a and b are equal on underlying topological spaces. Next, take a section $f \in \mathcal{O}_Y(V)$. This determines and is determined by a map of sheaves of sets $f : h_V \to \mathcal{O}_Y$. Pull this back and apply t to get a commutative diagram

$$\begin{tikzcd}
h_{b^{-1}(V)} \arrow[r, equal] & b_{small}^{-1}h_V \arrow[d, "b_{small}^{-1}(f)"] & & a_{small}^{-1}h_V \arrow[ll, "t"'] \arrow[d, "a_{small}^{-1}(f)"] \arrow[r, equal] & h_{a^{-1}(V)} \\
 & b_{small}^{-1}\mathcal{O}_Y \arrow[dr, "b^\sharp"'] & & a_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "a^\sharp"] & \\
 & & \mathcal{O}_X & &
\end{tikzcd}$$

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section 18.8. Above we have seen that the composition of the top horizontal arrows comes from the identity $a^{-1}(V) = b^{-1}(V)$. Thus the commutativity of the diagram tells us that $a_{small}^\sharp(f) = b_{small}^\sharp(f)$ in $\mathcal{O}_X(a^{-1}(V)) = \mathcal{O}_X(b^{-1}(V))$. Since this holds for every open V and every $f \in \mathcal{O}_Y(V)$ we conclude that $a = b$ as morphisms of schemes. $\square$

04I6 Lemma 59.40.4. Let X, Y be affine schemes. Let

$$(g, g^\#) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is 2-isomorphic to $(f_{small}, f_{small}^\sharp)$, see Modules on Sites, Definition 18.8.1.

Proof. In this proof we write $\mathcal{O}_X$ for the structure sheaf of the small étale site $X_{\acute{e}tale}$, and similarly for $\mathcal{O}_Y$. Say $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. Since $B = \Gamma(Y_{\acute{e}tale}, \mathcal{O}_Y)$, $A = \Gamma(X_{\acute{e}tale}, \mathcal{O}_X)$ we see that $g^\sharp$ induces a ring map $\varphi : B \to A$. Let $f = \mathrm{Spec}(\varphi) : X \to Y$ be the corresponding morphism of affine schemes. We will show this f does the job.

Let $V \to Y$ be an affine scheme étale over Y. Thus we may write $V = \mathrm{Spec}(C)$ with C an étale B-algebra. We can write

$$C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$$

with P_i polynomials such that $\Delta = \det(\partial P_i/\partial x_j)$ is invertible in C, see for example Algebra, Lemma 10.143.2. If T is a scheme over Y, then a T-valued point of V is given by n sections of $\Gamma(T, \mathcal{O}_T)$ which satisfy the polynomial equations $P_1 = 0, \ldots, P_n = 0$. In other words, the sheaf h_V on $Y_{\acute{e}tale}$ is the equalizer of the two maps

$$\begin{tikzcd}
\prod_{i=1,\ldots,n} \mathcal{O}_Y \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \prod_{j=1,\ldots,n} \mathcal{O}_Y
\end{tikzcd}$$

where $b(h_1, \ldots, h_n) = 0$ and $a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n))$. Since g^{-1} is exact we conclude that the top row of the following solid commutative diagram is an equalizer diagram as well:

$$\begin{tikzcd}
g^{-1}h_V \arrow[r] \arrow[d, dotted] & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[r, shift left, "g^{-1}a"] \arrow[r, shift right, "g^{-1}b"'] \arrow[d, "\prod g^\sharp"] & \prod_{j=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[d, "\prod g^\sharp"] \\
h_{X \times_Y V} \arrow[r] & \prod_{i=1,\ldots,n} \mathcal{O}_X \arrow[r, shift left, "a'"] \arrow[r, shift right, "b'"'] & \prod_{j=1,\ldots,n} \mathcal{O}_X
\end{tikzcd}$$

Here b' is the zero map and a' is the map defined by the images $P'_i = \varphi(P_i) \in A[x_1, \ldots, x_n]$ via the same rule $a'(h_1, \ldots, h_n) = (P'_1(h_1, \ldots, h_n), \ldots, P'_n(h_1, \ldots, h_n))$. that a was defined by. The commutativity of the diagram follows from the fact that $\varphi = g^\sharp$ on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since $X \times_Y V$ is the affine scheme $\mathrm{Spec}(A \otimes_B C)$ and $A \otimes_B C = A[x_1, \ldots, x_n]/(P'_1, \ldots, P'_n)$. Thus we obtain a unique dotted arrow $g^{-1}h_V \to h_{X \times_Y V}$ fitting into the diagram

We claim that the map of sheaves $g^{-1}h_V \to h_{X \times_Y V}$ is an isomorphism. Since the small étale site of X has enough points (Theorem 59.29.10) it suffices to prove this on stalks. Hence let $\overline{x}$ be a geometric point of X, and denote p the associate point of the small étale topos of X. Set $q = g \circ p$. This is a point of the small étale topos of Y. By Lemma 59.29.12 we see that q corresponds to a geometric point $\overline{y}$ of Y. Consider the map of stalks

$$(g^\sharp)_p : (\mathcal{O}_Y)_{\overline{y}} = \mathcal{O}_{Y,q} = (g^{-1}\mathcal{O}_Y)_p \longrightarrow \mathcal{O}_{X,p} = (\mathcal{O}_X)_{\overline{x}}$$

Since $(g, g^\sharp)$ is a morphism of locally ringed topoi $(g^\sharp)_p$ is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 10.153.12 we conclude that $(g^{-1}h_V)_p \to (h_{X \times_Y V})_p$ is an isomorphism as desired.

We claim that the isomorphisms $g^{-1}h_V \to h_{X \times_Y V}$ are functorial. Namely, suppose that $V_1 \to V_2$ is a morphism of affine schemes étale over Y. Write $V_i = \mathrm{Spec}(C_i)$ with

$$C_i = B[x_{i,1}, \ldots, x_{i,n_i}]/(P_{i,1}, \ldots, P_{i,n_i})$$

The morphism $V_1 \to V_2$ is given by a B-algebra map $C_2 \to C_1$ which in turn is given by some polynomials $Q_j \in B[x_{1,1}, \ldots, x_{1,n_1}]$ for $j = 1, \ldots, n_2$. Then it is an easy matter to show that the diagram of sheaves

$$\begin{tikzcd}
h_{V_1} \arrow[r] \arrow[d] & \prod_{i=1,\ldots,n_1} \mathcal{O}_Y \arrow[d, "{Q_1,\ldots,Q_{n_2}}"] \\
h_{V_2} \arrow[r] & \prod_{i=1,\ldots,n_2} \mathcal{O}_Y
\end{tikzcd}$$

is commutative, and pulling back to $X_{étale}$ we obtain the solid commutative diagram

$$\begin{tikzcd}
g^{-1}h_{V_1} \arrow[r] \arrow[drr] \arrow[dd, dotted] & \prod_{i=1,\ldots,n_1} g^{-1}\mathcal{O}_Y \arrow[drr, "{Q_1,\ldots,Q_{n_2}}"] \arrow[dd, "g^\sharp"] & & \\
& & g^{-1}h_{V_2} \arrow[r] \arrow[dd, dotted] & \prod_{i=1,\ldots,n_2} g^{-1}\mathcal{O}_Y \arrow[dd, "g^\sharp"] \\
h_{X \times_Y V_1} \arrow[r] \arrow[drr] & \prod_{i=1,\ldots,n_1} \mathcal{O}_X \arrow[drr, "{Q'_1,\ldots,Q'_{n_2}}"] & & \\
& & h_{X \times_Y V_2} \arrow[r] & \prod_{i=1,\ldots,n_2} \mathcal{O}_X
\end{tikzcd}$$

where $Q'_j \in A[x_{1,1}, \ldots, x_{1,n_1}]$ is the image of Q_j via φ. Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we see that the whole diagram commutes. This proves functoriality (and also that the construction of $g^{-1}h_V \to h_{X \times_Y V}$ is independent of the choice of the presentation, although we strictly speaking do not need to show this).

At this point we are able to show that $f_{small,*} \cong g_*$. Namely, let $\mathcal{F}$ be a sheaf on $X_{\acute{e}tale}$. For every $V \in \text{Ob}(X_{\acute{e}tale})$ affine we have

$$\begin{aligned}(g_*\mathcal{F})(V) &= \text{Mor}_{Sh(Y_{\acute{e}tale})}(h_V, g_*\mathcal{F}) \\ &= \text{Mor}_{Sh(X_{\acute{e}tale})}(g^{-1}h_V, \mathcal{F}) \\ &= \text{Mor}_{Sh(X_{\acute{e}tale})}(h_{X \times_Y V}, \mathcal{F}) \\ &= \mathcal{F}(X \times_Y V) \\ &= f_{small,*}\mathcal{F}(V)\end{aligned}$$

where in the third equality we use the isomorphism $g^{-1}h_V \cong h_{X \times_Y V}$ constructed above. These isomorphisms are clearly functorial in $\mathcal{F}$ and functorial in V as the isomorphisms $g^{-1}h_V \cong h_{X \times_Y V}$ are functorial. Now any sheaf on $Y_{\acute{e}tale}$ is determined by the restriction to the subcategory of affine schemes (Topologies, Lemma 34.4.12), and hence we obtain an isomorphism of functors $f_{small,*} \cong g_*$ as desired.

Finally, we have to check that, via the isomorphism $f_{small,*} \cong g_*$ above, the maps $f_{small}^\sharp$ and $g^\sharp$ agree. By construction this is already the case for the global sections of $\mathcal{O}_Y$, i.e., for the elements of B. We only need to check the result on sections over an affine V étale over Y (by Topologies, Lemma 34.4.12 again). Writing $V = \text{Spec}(C)$, $C = B[x_i]/(P_j)$ as before it suffices to check that the coordinate functions x_i are mapped to the same sections of $\mathcal{O}_X$ over $X \times_Y V$. And this is exactly what it means that the diagram

$$\begin{tikzcd} g^{-1}h_V \arrow[r] \arrow[d, dotted] & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[d, "\prod g^\sharp"] \\ h_{X \times_Y V} \arrow[r] & \prod_{i=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

commutes. Thus the lemma is proved. □

Here is a version for general schemes.

04I7 Theorem 59.40.5. Let X, Y be schemes. Let

$$(g, g^\#) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is isomorphic to $(f_{small}, f_{small}^\sharp)$. In other words, the construction

$$Sch \longrightarrow \text{Locally ringed topoi}, \quad X \longrightarrow (X_{\acute{e}tale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. You can prove this theorem by carefully adjusting the arguments of the proof of Lemma 59.40.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 59.40.2. Unfortunately, this is a bit messy.

Let us prove existence when Y is affine. In this case choose an affine open covering $X = \bigcup U_i$. For each i the inclusion morphism $j_i : U_i \to X$ induces a morphism of locally ringed topoi $(j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(X_{étale}), \mathcal{O}_X)$ by Lemma 59.40.1. We can compose this with $(g, g^\sharp)$ to obtain a morphism of locally ringed topoi

$$(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(Y_{étale}), \mathcal{O}_Y)$$

see Modules on Sites, Lemma 18.40.10. By Lemma 59.40.4 there exists a unique morphism of schemes $f_i : U_i \to Y$ and a 2-isomorphism

$$t_i : (f_{i,small}, f_{i,small}^\sharp) \longrightarrow (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp).$$

Set $U_{i,i'} = U_i \cap U_{i'}$, and denote $j_{i,i'} : U_{i,i'} \to U_i$ the inclusion morphism. Since we have $j_i \circ j_{i,i'} = j_{i'} \circ j_{i',i}$ we see that

$$\begin{aligned} (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) \circ (j_{i,i',small}, j_{i,i',small}^\sharp) = \\ (g, g^\sharp) \circ (j_{i',small}, j_{i',small}^\sharp) \circ (j_{i',i,small}, j_{i',i,small}^\sharp) \end{aligned}$$

Hence by uniqueness (see Lemma 59.40.3) we conclude that $f_i \circ j_{i,i'} = f_{i'} \circ j_{i',i}$, in other words the morphisms of schemes $f_i = f \circ j_i$ are the restrictions of a global morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $^\sharp$ to ease the notation)

$$\begin{tikzcd} g \circ j_{i,small} \circ j_{i,i',small} \arrow[r, "t_i \star \mathrm{id}_{j_{i,i',small}}"] \arrow[d, equal] & f_{small} \circ j_{i,small} \circ j_{i,i',small} \arrow[d, equal] \\ g \circ j_{i',small} \circ j_{i',i,small} \arrow[r, "t_{i'} \star \mathrm{id}_{j_{i',i,small}}"] & f_{small} \circ j_{i',small} \circ j_{i',i,small} \end{tikzcd}$$

The notation $\star$ indicates horizontal composition, see Categories, Definition 4.29.1 in general and Sites, Section 7.36 for our particular case. By the result of Lemma 59.40.2 this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{étale}$ the isomorphisms $t_i : f_{small}^{-1}\mathcal{G}|_{U_i} \to g^{-1}\mathcal{G}|_{U_i}$ agree over $U_{i,i'}$ and we obtain a global isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f_{small}^\sharp$ and $g^\sharp$ (because it is compatible with these maps locally). This proves the theorem in case Y is affine.

In the general case, let $V \subset Y$ be an affine open. Then h_V is a subsheaf of the final sheaf $*$ on $Y_{étale}$. As g is exact we see that $g^{-1}h_V$ is a subsheaf of the final sheaf on $X_{étale}$. Hence by Lemma 59.31.1 there exists an open subscheme $W \subset X$ such that $g^{-1}h_V = h_W$. By Modules on Sites, Lemma 18.40.12 there exists a commutative diagram of morphisms of locally ringed topoi

$$\begin{tikzcd} (Sh(W_{étale}), \mathcal{O}_W) \arrow[r] \arrow[d, "g'"'] & (Sh(X_{étale}), \mathcal{O}_X) \arrow[d, "g"] \\ (Sh(V_{étale}), \mathcal{O}_V) \arrow[r] & (Sh(Y_{étale}), \mathcal{O}_Y) \end{tikzcd}$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \to Y$ and $W \to X$) and where g' is induced from g. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \to V$ and a 2-isomorphism $t : (f'_{small}, (f'_{small})^\sharp) \to (g', (g')^\sharp)$. Exactly as before these morphisms f' (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness, so we get a morphism $f : X \to Y$. Moreover, the 2-isomorphisms t are compatible on overlaps by Lemma 59.40.2 again and we obtain a global 2-isomorphism $(f_{small}, (f_{small})^\sharp) \to (g, (g)^\sharp)$. as desired. Some details omitted. □

59.41. Push and pull

04C6 Let $f : X \to Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \to X$ and $u \in U$ there exist an étale morphism $V \to Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \to U$ over X with u in the image of h.
(B) For every $V \to Y$ étale, and every étale covering $\{U_i \to X \times_Y V\}$ there exists an étale covering $\{V_j \to V\}$ such that for each j we have $X \times_Y V_j = \coprod W_{ij}$ where $W_{ij} \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i.
(C) For every $U \to X$ étale, there exists a $V \to Y$ étale and a surjective morphism $X \times_Y V \to U$ over X.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{small,*}$. We will work this out in the next few sections.

59.42. Property (A)

04DJ Please see Section 59.41 for the definition of property (A).

04DK Lemma 59.42.1. Let $f : X \to Y$ be a morphism of schemes. Assume (A).

(1) $f_{small,*} : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(Y_{\acute{e}tale})$ reflects injections and surjections,
(2) $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,
(3) $f_{small,*} : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(Y_{\acute{e}tale})$ is faithful.

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let U be an object of $X_{\acute{e}tale}$. By assumption we can find a covering $\{W_i \to U\}$ in $X_{\acute{e}tale}$ such that each W_i is an open and closed subscheme of $X \times_Y V_i$ for some object V_i of $Y_{\acute{e}tale}$. The sheaf condition shows that

$$\mathcal{F}(U) \subset \prod \mathcal{F}(W_i)$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i) = f_{small,*}\mathcal{F}(V_i)$. Hence it is clear that $f_{small,*}$ reflects injections.

Next, suppose that $a : \mathcal{G} \to \mathcal{F}$ is a map of abelian sheaves such that $f_{small,*}a$ is surjective. Let $s \in \mathcal{F}(U)$ with U as above. With W_i, V_i as above we see that it suffices to show that $s|_{W_i}$ is étale locally the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i)$ it suffices to show that for any $V \in \text{Ob}(Y_{\acute{e}tale})$ any element $s \in \mathcal{F}(X \times_Y V)$ is étale locally on $X \times_Y V$ the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(X \times_Y V) = f_{small,*}\mathcal{F}(V)$ we see by assumption that there exists a covering $\{V_j \to V\}$ such that s is the image of $s_j \in f_{small,*}\mathcal{G}(V_j) = \mathcal{G}(X \times_Y V_j)$. This proves $f_{small,*}$ reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma 18.15.1. □

04DL Lemma 59.42.2. Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.

Proof. Let $U \to X$ be an étale morphism and $u \in U$. The geometric statement (A) reduces directly to the case where U and Y are affine schemes. Denote $x \in X$ and $y \in Y$ the images of u. Since $X \to Y$ is locally quasi-finite, and $U \to X$ is locally quasi-finite (see Morphisms, Lemma 29.36.6) we see that $U \to Y$ is locally quasi-finite (see Morphisms, Lemma 29.20.12). Moreover both $X \to Y$ and $U \to Y$ are separated. Thus More on Morphisms, Lemma 37.41.5 applies to both morphisms. This means we may pick an étale neighbourhood $(V, v) \to (Y, y)$ such that

$$X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'$$

and points $w \in W$, $w' \in W'$ such that

(1) W, R are open and closed in $X \times_Y V$,
(2) W', R' are open and closed in $U \times_Y V$,
(3) $W \to V$ and $W' \to V$ are finite,
(4) w, w' map to v,
(5) $\kappa(v) \subset \kappa(w)$ and $\kappa(v) \subset \kappa(w')$ are purely inseparable, and
(6) no other point of W or W' maps to v.

Here is a commutative diagram

$$\begin{tikzcd} U \arrow[d] & U \times_Y V \arrow[l] \arrow[d] & W' \amalg R' \arrow[l] \arrow[d] \\ X \arrow[d] & X \times_Y V \arrow[l] \arrow[d] & W \amalg R \arrow[l] \\ Y & V \arrow[l] & \end{tikzcd}$$

After shrinking V we may assume that W' maps into W: just remove the image the inverse image of R in W'; this is a closed set (as $W' \to V$ is finite) not containing v. Then $W' \to W$ is finite because both $W \to V$ and $W' \to V$ are finite. Hence $W' \to W$ is finite étale, and there is exactly one point in the fibre over w with $\kappa(w) = \kappa(w')$. Hence $W' \to W$ is an isomorphism in an open neighbourhood W° of w, see Étale Morphisms, Lemma 41.14.2. Since $W \to V$ is finite the image of $W \setminus W^\circ$ is a closed subset T of V not containing v. Thus after replacing V by $V \setminus T$ we may assume that $W' \to W$ is an isomorphism. Now the decomposition $X \times_Y V = W \amalg R$ and the morphism $W \to U$ are as desired and we win. □

04DM Lemma 59.42.3. Let $f : X \to Y$ be an integral morphism of schemes. Then property (A) holds.

Proof. Let $U \to X$ be étale, and let $u \in U$ be a point. We have to find $V \to Y$ étale, a disjoint union decomposition $X \times_Y V = W \amalg W'$ and an X-morphism $W \to U$ with u in the image. We may shrink U and Y and assume U and Y are affine. In this case also X is affine, since an integral morphism is affine by definition. Write $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $U = \mathrm{Spec}(C)$. Then $A \to B$ is an integral ring map, and $B \to C$ is an étale ring map. By Algebra, Lemma 10.143.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \to C'$ such that $C = B \otimes_{B'} C'$. Thus the question reduces to the étale morphism $U' = \mathrm{Spec}(C') \to X' = \mathrm{Spec}(B')$ over the finite morphism $X' \to Y$. In this case the result follows from Lemma 59.42.2. □

04C9 **Lemma 59.42.4.** Let $f : X \to Y$ be a morphism of schemes. Denote $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ the associated morphism of small étale topoi. Assume at least one of the following

(1) f is integral, or
(2) f is separated and locally quasi-finite.

Then the functor $f_{small,*} : \mathrm{Ab}(X_{étale}) \to \mathrm{Ab}(Y_{étale})$ has the following properties

(1) the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is always surjective,
(2) $f_{small,*}$ is faithful, and
(3) $f_{small,*}$ reflects injections and surjections.

Proof. Combine Lemmas 59.42.2, 59.42.3, and 59.42.1. □

59.43. Property (B)

04DN Please see Section 59.41 for the definition of property (B).

04DO **Lemma 59.43.1.** Let $f : X \to Y$ be a morphism of schemes. Assume (B) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ transforms surjections into surjections.

Proof. This follows from Sites, Lemma 7.41.2. □

04DP **Lemma 59.43.2.** Let $f : X \to Y$ be a morphism of schemes. Suppose

(1) $V \to Y$ is an étale morphism of schemes,
(2) $\{U_i \to X \times_Y V\}$ is an étale covering, and
(3) $v \in V$ is a point.

Assume that for any such data there exists an étale neighbourhood $(V', v') \to (V, v)$, a disjoint union decomposition $X \times_Y V' = \coprod W'_i$, and morphisms $W'_i \to U_i$ over $X \times_Y V$. Then property (B) holds.

Proof. Omitted. □

04DQ **Lemma 59.43.3.** Let $f : X \to Y$ be a finite morphism of schemes. Then property (B) holds.

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 59.43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is finite over Y, this also implies that X is affine. During the proof we may (finitely often) replace (V, v) by an étale neighbourhood (V', v') and correspondingly the covering $\{U_i \to X \times_Y V\}$ by $\{V' \times_V U_i \to X \times_Y V'\}$.

Since $X \times_Y V \to V$ is finite there exist finitely many (pairwise distinct) points $x_1, \ldots, x_n \in X \times_Y V$ mapping to v. We may apply More on Morphisms, Lemma 37.41.5 to $X \times_Y V \to V$ and the points $x_1, \ldots, x_n$ lying over v and find an étale neighbourhood $(V', v') \to (V, v)$ such that

$$X \times_Y V' = R \amalg \coprod T_a$$

with $T_a \to V'$ finite with exactly one point p_a lying over v' and moreover $\kappa(v') \subset \kappa(p_a)$ purely inseparable, and such that $R \to V'$ has empty fibre over v'. Because $X \to Y$ is finite, also $R \to V'$ is finite. Hence after shrinking V' we may assume that $R = \emptyset$. Thus we may assume that $X \times_Y V = X_1 \amalg \ldots \amalg X_n$ with exactly one point $x_l \in X_l$ lying over v with moreover $\kappa(v) \subset \kappa(x_l)$ purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood (V, v).

For each l choose an i_l and a point $u_l \in U_{i_l}$ mapping to x_l. Now we apply property (A) for the finite morphism $X \times_Y V \to V$ and the étale morphisms $U_{i_l} \to X \times_Y V$ and the points u_l. This is permissible by Lemma 59.42.3 This gives produces an étale neighbourhood $(V', v') \to (V, v)$ and decompositions

$$X \times_Y V' = W_l \amalg R_l$$

and X-morphisms $a_l : W_l \to U_{i_l}$ whose image contains u_{i_l}. Here is a picture:

$$\begin{tikzcd}
 & & & U_{i_l} \arrow[d] & \\
W_l \arrow[r] \arrow[urrr] & W_l \amalg R_l \arrow[r, equal] & X \times_Y V' \arrow[r] \arrow[d] & X \times_Y V \arrow[r] \arrow[d] & X \arrow[d] \\
 & & V' \arrow[r] & V \arrow[r] & Y
\end{tikzcd}$$

After replacing (V, v) by (V', v') we conclude that each x_l is contained in an open and closed neighbourhood W_l such that the inclusion morphism $W_l \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i. Replacing W_l by $W_l \cap X_l$ we see that these open and closed sets are disjoint and moreover that $\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n$. Since $X \times_Y V \to V$ is finite we may shrink V and assume that $X \times_Y V = W_1 \amalg \ldots \amalg W_n$ as desired. □

04DR Lemma 59.43.4. Let $f : X \to Y$ be an integral morphism of schemes. Then property (B) holds.

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 59.43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is integral over Y, this also implies that X and $X \times_Y V$ are affine. We may refine the covering $\{U_i \to X \times_Y V\}$, and hence we may assume that $\{U_i \to X \times_Y V\}_{i=1,\ldots,n}$ is a standard étale covering. Write $Y = \text{Spec}(A)$, $X = \text{Spec}(B)$, $V = \text{Spec}(C)$, and $U_i = \text{Spec}(B_i)$. Then $A \to B$ is an integral ring map, and $B \otimes_A C \to B_i$ are étale ring maps. By Algebra, Lemma 10.143.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \otimes_A C \to B'_i$ for $i = 1, \ldots, n$ such that $B_i = B \otimes_{B'} B'_i$. Thus the question reduces to the étale covering $\{\text{Spec}(B'_i) \to X' \times_Y V\}_{i=1,\ldots,n}$ with $X' = \text{Spec}(B')$ finite over Y. In this case the result follows from Lemma 59.43.3. □

04C2 Lemma 59.43.5. Let $f : X \to Y$ be a morphism of schemes. Assume f is integral (for example finite). Then

(1) $f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),
(2) $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,
(3) $f_{small,*} : \text{Ab}(X_{étale}) \to \text{Ab}(Y_{étale})$ is faithful and reflects injections and surjections, and
(4) $f_{small,*} : \text{Ab}(X_{étale}) \to \text{Ab}(Y_{étale})$ is exact.

Proof. Parts (2), (3) we have seen in Lemma 59.42.4. Part (1) follows from Lemmas 59.43.4 and 59.43.1. Part (4) is a consequence of part (1), see Modules on Sites, Lemma 18.15.2. □

59.44. Property (C)

04DS Please see Section 59.41 for the definition of property (C).

04DT Lemma 59.44.1. Let $f : X \to Y$ be a morphism of schemes. Assume (C) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections.

Proof. Follows from Sites, Lemma 7.41.4. We omit the verification that property (C) implies that the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ satisfies the assumption of Sites, Lemma 7.41.4. □

04DU Remark 59.44.2. Property (C) holds if $f : X \to Y$ is an open immersion. Namely, if $U \in \text{Ob}(X_{étale})$, then we can view U also as an object of $Y_{étale}$ and $U \times_Y X = U$. Hence property (C) does not imply that $f_{small,*}$ is exact as this is not the case for open immersions (in general).

04DV Lemma 59.44.3. Let $f : X \to Y$ be a morphism of schemes. Assume that for any $V \to Y$ étale we have that

(1) $X \times_Y V \to V$ has property (C), and
(2) $X \times_Y V \to V$ is closed.

Then the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous, see Sites, Definition 7.42.3.

Proof. Let $V \to Y$ be an object of $Y_{étale}$ and let $\{U_i \to X \times_Y V\}_{i \in I}$ be a covering of $X_{étale}$. By assumption (1) for each i we can find an étale morphism $h_i : V_i \to V$ and a surjective morphism $X \times_Y V_i \to U_i$ over $X \times_Y V$. Note that $\bigcup h_i(V_i) \subset V$ is an open set containing the closed set $Z = \text{Im}(X \times_Y V \to V)$. Let $h_0 : V_0 = V \setminus Z \to V$ be the open immersion. It is clear that $\{V_i \to V\}_{i \in I \cup \{0\}}$ is an étale covering such that for each $i \in I \cup \{0\}$ we have either $V_i \times_Y X = \emptyset$ (namely if $i = 0$), or $V_i \times_Y X \to V \times_Y X$ factors through $U_i \to X \times_Y V$ (if $i \neq 0$). Hence the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. □

04DW Lemma 59.44.4. Let $f : X \to Y$ be an integral morphism of schemes which defines a homeomorphism of X with a closed subset of Y. Then property (C) holds.

Proof. Let $g : U \to X$ be an étale morphism. We need to find an object $V \to Y$ of $Y_{étale}$ and a surjective morphism $X \times_Y V \to U$ over X. Suppose that for every $u \in U$ we can find an object $V_u \to Y$ of $Y_{étale}$ and a morphism $h_u : X \times_Y V_u \to U$ over X with $u \in \text{Im}(h_u)$. Then we can take $V = \coprod V_u$ and $h = \coprod h_u$ and we win. Hence given a point $u \in U$ we find a pair (V_u, h_u) as above. To do this we may shrink U and assume that U is affine. In this case $g : U \to X$ is locally quasi-finite. Let $g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\}$. Since there are no specializations $u_i \rightsquigarrow u$ we may replace U by an affine neighbourhood so that $g^{-1}(g(\{u\})) = \{u\}$.

The image $g(U) \subset X$ is open, hence $f(g(U))$ is locally closed in Y. Choose an open $V \subset Y$ such that $f(g(U)) = f(X) \cap V$. It follows that g factors through $X \times_Y V$ and that the resulting $\{U \to X \times_Y V\}$ is an étale covering. Since f has property (B) , see Lemma 59.43.4, we see that there exists an étale covering $\{V_j \to V\}$ such that $X \times_Y V_j \to X \times_Y V$ factor through U. This implies that $V' = \coprod V_j$ is étale over Y and that there is a morphism $h : X \times_Y V' \to U$ whose image surjects onto $g(U)$. Since u is the only point in its fibre it must be in the image of h and we win. □

We urge the reader to think of the following lemma as a way station[3] on the journey towards the ultimate truth regarding $f_{small,*}$ for integral universally injective morphisms.

04DX Lemma 59.44.5. Let $f : X \to Y$ be a morphism of schemes. Assume that f is universally injective and integral (for example a closed immersion). Then

(1) $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections,
(2) $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ commutes with pushouts and coequalizers (and more generally finite connected colimits),
(3) $f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),
(4) the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any sheaf (of sets or of abelian groups) $\mathcal{F}$ on $X_{étale}$,
(5) the functor $f_{small,*}$ is faithful (on sheaves of sets and on abelian sheaves),
(6) $f_{small,*} : \text{Ab}(X_{étale}) \to \text{Ab}(Y_{étale})$ is exact, and
(7) the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous.

Proof. By Lemmas 59.42.3, 59.43.4 and 59.44.4 we know that the morphism f has properties (A), (B), and (C). Moreover, by Lemma 59.44.3 we know that the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. Now we have

(1) property (C) implies (1) by Lemma 59.44.1,
(2) almost continuous implies (2) by Sites, Lemma 7.42.6,
(3) property (B) implies (3) by Lemma 59.43.1.

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 7.41.1 and Modules on Sites, Lemma 18.15.2. Property (7) we saw above. □

59.45. Topological invariance of the small étale site

04DY In the following theorem we show that the small étale site is a topological invariant in the following sense: If $f : X \to Y$ is a morphism of schemes which is a universal homeomorphism, then $X_{étale} \cong Y_{étale}$ as sites. This improves the result of Étale Morphisms, Theorem 41.15.2. We first prove the result for morphisms and then we state the result for categories.

0BTY Theorem 59.45.1. Let X and Y be two schemes over a base scheme S. Let $S' \to S$ be a universal homeomorphism. Denote X' (resp. Y') the base change to S'. If X is étale over S, then the map

$$\text{Mor}_S(Y, X) \longrightarrow \text{Mor}_{S'}(Y', X')$$

is bijective.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. Thus we may and do assume both X and Y are étale over S. In other words, the theorem states that the base change functor is a fully faithful functor from the category of schemes étale over S to the category of schemes étale over S'.

[3] A way station is a place where people stop to eat and rest when they are on a long journey.

Consider the forgetful functor

0BTZ (59.45.1.1)
$$\begin{matrix}\text{descent data } (X', \varphi') \text{ relative to } S'/S \\ \text{with } X' \text{ étale over } S'\end{matrix} \longrightarrow \text{schemes } X' \text{ étale over } S'$$

We claim this functor is an equivalence. On the other hand, the functor

0BU0 (59.45.1.2)
$$\text{schemes } X \text{ étale over } S \longrightarrow \begin{matrix}\text{descent data } (X', \varphi') \text{ relative to } S'/S \\ \text{with } X' \text{ étale over } S'\end{matrix}$$

is fully faithful by Étale Morphisms, Lemma 41.20.3. Thus the claim implies the theorem.

Proof of the claim. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 29.45.5. In particular, the diagonal $\Delta : S' \to S' \times_S S'$ is a thickening by Morphisms, Lemma 29.10.2. Thus by Étale Morphisms, Theorem 41.15.1 we see that given $X' \to S'$ étale there is a unique isomorphism

$$\varphi' : X' \times_S S' \to S' \times_S X'$$

of schemes étale over $S' \times_S S'$ which pulls back under Δ to $\mathrm{id} : X' \to X'$ over S'. Since $S' \to S' \times_S S' \times_S S'$ is a thickening as well (it is bijective and a closed immersion) we conclude that (X', φ') is a descent datum relative to S'/S. The canonical nature of the construction of φ' shows that it is compatible with morphisms between schemes étale over S'. In other words, we obtain a quasi-inverse $X' \mapsto (X', \varphi')$ of the functor (59.45.1.1). This proves the claim and finishes the proof of the theorem. □

04DZ Theorem 59.45.2. Let $f : X \to Y$ be a morphism of schemes. Assume f is integral, universally injective and surjective (i.e., f is a universal homeomorphism, see Morphisms, Lemma 29.45.5). The functor [DG67, IV Theorem 18.1.2]

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{\text{schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{schemes } U \text{ étale over } X\}$$

We give two proofs. The first uses effectivity of descent for quasi-compact, separated, étale morphisms relative to surjective integral morphisms. The second uses the material on properties (A), (B), and (C) discussed earlier in the chapter.

First proof. By Theorem 59.45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if U and Y are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each U_i maps into an affine open of Y. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over X. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i}$$

as schemes over X we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i',i}$ of schemes over Y. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 26.14. Applying Schemes, Lemma

26.14.2 we obtain a scheme $V \to Y$ by glueing the schemes V_i along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to show the lemma in case U and Y are affine. Recall that in the proof of Theorem 59.45.1 we showed that U comes with a unique descent datum (U, φ) relative to X/Y. By Étale Morphisms, Proposition 41.20.6 (which applies because $U \to X$ is quasi-compact and separated as well as étale by our reduction to the affine case) there exists an étale morphism $V \to Y$ such that $X \times_Y V \cong U$ and the proof is complete. $\square$

Second proof. By Theorem 59.45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if U and Y are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each U_i maps into an affine open of Y. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over X. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i}$$

as schemes over X we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i',i}$ of schemes over Y. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 26.14. Applying Schemes, Lemma 26.14.2 we obtain a scheme $V \to Y$ by glueing the schemes V_i along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to prove that the functor

04E0 (59.45.2.1) $$\{\text{affine schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{affine schemes } U \text{ étale over } X\}$$

is essentially surjective when X and Y are affine.

Let $U \to X$ be an affine scheme étale over X. We have to find $V \to Y$ étale (and affine) such that $X \times_Y V$ is isomorphic to U over X. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemmas 29.36.6 and 29.57.9. Hence we can do induction on the integer n bounding the degree of the fibres of $U \to X$. See Morphisms, Lemma 29.57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $U \to X$ is an open immersion (see Étale Morphisms, Theorem 41.14.1), and the result is clear. Assume $n > 1$.

By Lemma 59.44.4 there exists an étale morphism of schemes $W \to Y$ and a surjective morphism $W_X \to U$ over X. As U is quasi-compact we may replace W by a disjoint union of finitely many affine opens of W, hence we may assume that W is affine as well. Here is a diagram

$$\begin{tikzcd} U & U \times_Y W \arrow[l] \arrow[d] \arrow[r, equal] & W_X \amalg R \\ X \arrow[from=u] & W_X \arrow[l] \arrow[d] & \\ Y \arrow[from=u] & W \arrow[l] & \end{tikzcd}$$

The disjoint union decomposition arises because by construction the étale morphism of affine schemes $U \times_Y W \to W_X$ has a section. OK, and now we see that the morphism $R \to X \times_Y W$ is an étale morphism of affine schemes whose fibres have degree universally bounded by $n - 1$. Hence by induction assumption there exists a scheme $V' \to W$ étale such that $R \cong W_X \times_W V'$. Taking $V'' = W \amalg V'$ we find a scheme V'' étale over W whose base change to W_X is isomorphic to $U \times_Y W$ over $X \times_Y W$.

At this point we can use descent to find V over Y whose base change to X is isomorphic to U over X. Namely, by the fully faithfulness of the functor (59.45.2.1) corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular φ satisfies the cocycle condition. Hence by Descent, Lemma 35.37.1 we see that φ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with φ. Note that $V \to Y$ is étale, by Descent, Lemma 35.23.29. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y V \times_Y W = (X \times_Y W) \times_W (W \times_Y V) \cong W_X \times_W V'' \cong U \times_Y W$$

which we have by construction. Some details omitted. $\square$

05YX Remark 59.45.3. In the situation of Theorem 59.45.2 it is also true that $V \mapsto V_X$ induces an equivalence between those étale morphisms $V \to Y$ with V affine and those étale morphisms $U \to X$ with U affine. This follows for example from Limits, Proposition 32.11.2.

03SI Proposition 59.45.4 (Topological invariance of étale cohomology). Let $X_0 \to X$ be a universal homeomorphism of schemes (for example the closed immersion defined by a nilpotent sheaf of ideals). Then

(1) the étale sites $X_{étale}$ and $(X_0)_{étale}$ are isomorphic,
(2) the étale topoi $Sh(X_{étale})$ and $Sh((X_0)_{étale})$ are equivalent, and
(3) $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_0, \mathcal{F}|_{X_0})$ for all q and for any abelian sheaf $\mathcal{F}$ on $X_{étale}$.

Proof. The equivalence of categories $X_{étale} \to (X_0)_{étale}$ is given by Theorem 59.45.2. We omit the proof that under this equivalence the étale coverings correspond. Hence (1) holds. Parts (2) and (3) follow formally from (1). $\square$

59.46. Closed immersions and pushforward

04E1 Before stating and proving Proposition 59.46.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section $i : Z \to X$ is a closed immersion. The functor

$$Sch/X \longrightarrow Sch/Z, \quad U \longmapsto U_Z = Z \times_X U$$

will be denoted $U \mapsto U_Z$ as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme U_Z is a closed subscheme of U.

04FV **Lemma 59.46.1.** Let $i : Z \to X$ be a closed immersion of schemes. Let U, U' be schemes étale over X. Let $h : U_Z \to U'_Z$ be a morphism over Z. Then there exists a diagram

$$U \xleftarrow{a} W \xrightarrow{b} U'$$

such that $a_Z : W_Z \to U_Z$ is an isomorphism and $h = b_Z \circ (a_Z)^{-1}$.

Proof. Consider the scheme $M = U \times_X U'$. The graph $\Gamma_h \subset M_Z$ of h is open. This is true for example as Γ_h is the image of a section of the étale morphism $\text{pr}_{1,Z} : M_Z \to U_Z$, see Étale Morphisms, Proposition 41.6.1. Hence there exists an open subscheme $W \subset M$ whose intersection with the closed subset M_Z is Γ_h. Set $a = \text{pr}_1|_W$ and $b = \text{pr}_2|_W$. $\square$

04FW **Lemma 59.46.2.** Let $i : Z \to X$ be a closed immersion of schemes. Let $V \to Z$ be an étale morphism of schemes. There exist étale morphisms $U_i \to X$ and morphisms $U_{i,Z} \to V$ such that $\{U_{i,Z} \to V\}$ is a Zariski covering of V.

Proof. Since we only have to find a Zariski covering of V consisting of schemes of the form U_Z with U étale over X, we may Zariski localize on X and V. Hence we may assume X and V affine. In the affine case this is Algebra, Lemma 10.143.10. $\square$

If $\overline{x} : \text{Spec}(k) \to X$ is a geometric point of X, then either $\overline{x}$ factors (uniquely) through the closed subscheme Z, or $Z_{\overline{x}} = \emptyset$. If $\overline{x}$ factors through Z we say that $\overline{x}$ is a geometric point of Z (because it is) and we use the notation "$\overline{x} \in Z$" to indicate this.

04FX **Lemma 59.46.3.** Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be a sheaf of sets on $Z_{étale}$. Let $\overline{x}$ be a geometric point of X. Then

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \begin{cases} * & \text{if} \quad \overline{x} \notin Z \\ \mathcal{G}_{\overline{x}} & \text{if} \quad \overline{x} \in Z \end{cases}$$

where $*$ denotes a singleton set.

Proof. Note that $i_{small,*}\mathcal{G}|_{U_{étale}} = *$ is the final object in the category of étale sheaves on U, i.e., the sheaf which associates a singleton set to each scheme étale over U. This explains the value of $(i_{small,*}\mathcal{G})_{\overline{x}}$ if $\overline{x} \notin Z$.

Next, suppose that $\overline{x} \in Z$. Note that

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \text{colim}_{(U,\overline{u})} \mathcal{G}(U_Z)$$

and on the other hand

$$\mathcal{G}_{\overline{x}} = \text{colim}_{(V,\overline{v})} \mathcal{G}(V).$$

Let $\mathcal{C}_1 = \{(U, \overline{u})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in X, and let $\mathcal{C}_2 = \{(V, \overline{v})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in Z. The canonical map

$$\mathcal{G}_{\overline{x}} \longrightarrow (i_{small,*}\mathcal{G})_{\overline{x}}$$

corresponds to the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, $F(U, \overline{u}) = (U_Z, \overline{x})$. Now Lemmas 59.46.2 and 59.46.1 imply that $\mathcal{C}_1$ is cofinal in $\mathcal{C}_2$, see Categories, Definition 4.17.1. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma 4.17.2. $\square$

04CA **Proposition 59.46.4.** Let $i : Z \to X$ be a closed immersion of schemes.

(1) The functor

$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) the functor

$$i_{small,*} : \text{Ab}(Z_{étale}) \longrightarrow \text{Ab}(X_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let's discuss the case of sheaves of sets. For any sheaf $\mathcal{G}$ on Z the morphism $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism by Lemma 59.46.3 (and Theorem 59.29.10). This implies formally that $i_{small,*}$ is fully faithful, see Sites, Lemma 7.41.1. It is clear that $i_{small,*}\mathcal{G}|_{U_{étale}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $\mathcal{F}$ is a sheaf of sets on X such that $\mathcal{F}|_{U_{étale}} \cong *$. Consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1}\mathcal{F}$$

Combining Lemmas 59.46.3 and 59.36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

59.47. Integral universally injective morphisms

04FY Here is the general version of Proposition 59.46.4.

04FZ Proposition 59.47.1. Let $f : X \to Y$ be a morphism of schemes which is integral and universally injective.

(1) The functor

$$f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $Y_{étale}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

(2) the functor

$$f_{small,*} : \text{Ab}(X_{étale}) \longrightarrow \text{Ab}(Y_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(X)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. We may factor f as

$$X \xrightarrow{h} Z \xrightarrow{i} Y$$

where h is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 59.46.4 to i and apply Theorem 59.45.2 to h. □

59.48. Big sites and pushforward

04E2 In this section we prove some technical results on $f_{big,*}$ for certain types of morphisms of schemes.

04C7 **Lemma 59.48.1.** Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{big}^{-1} f_{big,*}\mathcal{F} \to \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ on $(Sch/X)_\tau$.

Proof. In this case the functor $(Sch/X)_\tau \to (Sch/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 7.21.7. $\square$

04C8 **Remark 59.48.2.** In the situation of Lemma 59.48.1 it is true that the canonical map $\mathcal{F} \to f_{big}^{-1} f_{big!}\mathcal{F}$ is an isomorphism for any sheaf of sets $\mathcal{F}$ on $(Sch/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{big!}$ for sheaves of abelian groups is defined in Modules on Sites, Section 18.16 and is in general different from $f_{big!}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 18.16.4.

04E3 **Lemma 59.48.3.** Let $f : X \to Y$ be a closed immersion of schemes. Let $U \to X$ be a syntomic (resp. smooth, resp. étale) morphism. Then there exist syntomic (resp. smooth, resp. étale) morphisms $V_i \to Y$ and morphisms $V_i \times_Y X \to U$ such that $\{V_i \times_Y X \to U\}$ is a Zariski covering of U.

Proof. Let us prove the lemma when $\tau = syntomic$. The question is local on U. Thus we may assume that U is an affine scheme mapping into an affine of Y. Hence we reduce to proving the following case: $Y = \text{Spec}(A)$, $X = \text{Spec}(A/I)$, and $U = \text{Spec}(\overline{B})$, where $A/I \to \overline{B}$ be a syntomic ring map. By Algebra, Lemma 10.136.18 we can find elements $\overline{g}_i \in \overline{B}$ such that $\overline{B}_{\overline{g}_i} = A_i/IA_i$ for certain syntomic ring maps $A \to A_i$. This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 10.137.20. In the étale case use Algebra, Lemma 10.143.10. $\square$

04E4 **Lemma 59.48.4.** Let $f : X \to Y$ be a closed immersion of schemes. Let $\{U_i \to X\}$ be a syntomic (resp. smooth, resp. étale) covering. There exists a syntomic (resp. smooth, resp. étale) covering $\{V_j \to Y\}$ such that for each j, either $V_j \times_Y X = \emptyset$, or the morphism $V_j \times_Y X \to X$ factors through U_i for some i.

Proof. For each i we can choose syntomic (resp. smooth, resp. étale) morphisms $g_{ij} : V_{ij} \to Y$ and morphisms $V_{ij} \times_Y X \to U_i$ over X, such that $\{V_{ij} \times_Y X \to U_i\}$ are Zariski coverings, see Lemma 59.48.3. This in particular implies that $\bigcup_{ij} g_{ij}(V_{ij})$ contains the closed subset $f(X)$. Hence the family of syntomic (resp. smooth, resp. étale) maps g_{ij} together with the open immersion $Y \setminus f(X) \to Y$ forms the desired syntomic (resp. smooth, resp. étale) covering of Y. $\square$

04C3 **Lemma 59.48.5.** Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$. The functor $V \mapsto X \times_Y V$ defines an almost cocontinuous functor (see Sites, Definition 7.42.3) $(Sch/Y)_\tau \to (Sch/X)_\tau$ between big τ sites.

Proof. We have to show the following: given a morphism $V \to Y$ and any syntomic (resp. smooth, resp. étale) covering $\{U_i \to X \times_Y V\}$, there exists a smooth (resp. smooth, resp. étale) covering $\{V_j \to V\}$ such that for each j, either $X \times_Y V_j$ is empty, or $X \times_Y V_j \to Z \times_Y V$ factors through one of the U_i. This follows on applying Lemma 59.48.4 above to the closed immersion $X \times_Y V \to V$. $\square$

04C4 Lemma 59.48.6. Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$.

(1) The pushforward $f_{big,*} : Sh((Sch/X)_\tau) \to Sh((Sch/Y)_\tau)$ commutes with coequalizers and pushouts.
(2) The pushforward $f_{big,*} : \text{Ab}((Sch/X)_\tau) \to \text{Ab}((Sch/Y)_\tau)$ is exact.

Proof. This follows from Sites, Lemma 7.42.6, Modules on Sites, Lemma 18.15.3, and Lemma 59.48.5 above. $\square$

04C5 Remark 59.48.7. In Lemma 59.48.6 the case $\tau = fppf$ is missing. The reason is that given a ring A, an ideal I and a faithfully flat, finitely presented ring map $A/I \to \overline{B}$, there is no reason to think that one can find any flat finitely presented ring map $A \to B$ with $B/IB \neq 0$ such that $A/I \to B/IB$ factors through $\overline{B}$. Hence the proof of Lemma 59.48.5 does not work for the fppf topology. In fact it is likely false that $f_{big,*} : \text{Ab}((Sch/X)_{fppf}) \to \text{Ab}((Sch/Y)_{fppf})$ is exact when f is a closed immersion. If you know an example, please email stacks.project@gmail.com.

59.49. Exactness of big lower shriek

04CB This is just the following technical result. Note that the functor $f_{big!}$ has nothing whatsoever to do with cohomology with compact support in general.

04CC Lemma 59.49.1. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a morphism of schemes. Let

$$f_{big} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

be the corresponding morphism of topoi as in Topologies, Lemma 34.3.16, 34.4.16, 34.5.10, 34.6.10, or 34.7.12.

(1) The functor $f_{big}^{-1} : \text{Ab}((Sch/Y)_\tau) \to \text{Ab}((Sch/X)_\tau)$ has a left adjoint
$$f_{big!} : \text{Ab}((Sch/X)_\tau) \to \text{Ab}((Sch/Y)_\tau)$$
which is exact.
(2) The functor $f_{big}^* : \text{Mod}((Sch/Y)_\tau, \mathcal{O}) \to \text{Mod}((Sch/X)_\tau, \mathcal{O})$ has a left adjoint
$$f_{big!} : \text{Mod}((Sch/X)_\tau, \mathcal{O}) \to \text{Mod}((Sch/Y)_\tau, \mathcal{O})$$
which is exact.

Moreover, the two functors $f_{big!}$ agree on underlying sheaves of abelian groups.

Proof. Recall that f_{big} is the morphism of topoi associated to the continuous and cocontinuous functor $u : (Sch/X)_\tau \to (Sch/Y)_\tau$, $U/X \mapsto U/Y$. Moreover, we have $f_{big}^{-1}\mathcal{O} = \mathcal{O}$. Hence the existence of $f_{big!}$ follows from Modules on Sites, Lemma 18.16.2, respectively Modules on Sites, Lemma 18.41.1. Note that if U is an object of $(Sch/X)_\tau$ then the functor u induces an equivalence of categories

$$u' : (Sch/X)_\tau/U \longrightarrow (Sch/Y)_\tau/U$$

because both sides of the arrow are equal to $(Sch/U)_\tau$. Hence the agreement of $f_{big!}$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark 18.41.2. The exactness of $f_{big!}$ follows from Modules on Sites, Lemma

18.16.3 as the functor u above which commutes with fibre products and equalizers. $\square$

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

07AJ Lemma 59.49.2. Let X be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (Sch/X)_\tau$ be full subcategories with the following properties:

(1) For an object U/X of $\mathcal{C}_t$,
    (a) if $\{U_i \to U\}$ is a covering of $(Sch/X)_\tau$, then U_i/X is an object of $\mathcal{C}_t$,
    (b) $U \times \mathbf{A}^1/X$ is an object of $\mathcal{C}_t$.
(2) X/X is an object of $\mathcal{C}_t$.

We endow $\mathcal{C}_t$ with the structure of a site whose coverings are exactly those coverings $\{U_i \to U\}$ of $(Sch/X)_\tau$ with $U \in \mathrm{Ob}(\mathcal{C}_t)$. Then

(a) The functor $\mathcal{C}_1 \to \mathcal{C}_2$ is fully faithful, continuous, and cocontinuous.

Denote $g : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ the corresponding morphism of topoi. Denote $\mathcal{O}_t$ the restriction of $\mathcal{O}$ to $\mathcal{C}_t$. Denote $g_!$ the functor of Modules on Sites, Definition 18.16.1.

(b) The canonical map $g_!\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism.

Proof. Assertion (a) is immediate from the definitions. In this proof all schemes are schemes over X and all morphisms of schemes are morphisms of schemes over X. Note that g^{-1} is given by restriction, so that for an object U of $\mathcal{C}_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}_1$ which associates to V in $\mathcal{C}_2$ the group

$$\mathrm{colim}_{V \to U}\, \mathcal{O}(U)$$

where U runs over the objects of $\mathcal{C}_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if

$$V \to U \to U'$$

are morphisms with $U, U' \in \mathrm{Ob}(\mathcal{C}_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Surjectivity. Let V be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbf{A}^1$ induced by h and the structure morphism $V \to X$. Writing $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[x])$ we see the element $x \in \mathcal{O}(X \times \mathbf{A}^1)$ pulls back to h. Since $X \times \mathbf{A}^1$ is an object of $\mathcal{C}_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

Injectivity. Let V be a scheme. Let $s = \sum_{i=1,\ldots,n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any i we can use the morphism $f_i : U_i \to X \times \mathbf{A}^1$ to see that $(V \to U_i, f_i)$ defines the same element of the colimit as $(f_i : V \to X \times \mathbf{A}^1, x)$. Then we can consider

$$f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n$$

and we see that s is equivalent in the colimit to

$$\sum\nolimits_{i=1,\ldots,n}(f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_i) = (f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_1 + \ldots + x_n)$$

Now, if $x_1 + \ldots + x_n$ restricts to zero on V, then we see that $f_1 \times \ldots \times f_n$ factors through $X \times \mathbf{A}^{n-1} = V(x_1 + \ldots + x_n)$. Hence we see that s is equivalent to zero in the colimit. $\square$

59.50. Étale cohomology

03Q3 In the following sections we prove some basic results on étale cohomology. Here is an example of something we know for cohomology of topological spaces which also holds for étale cohomology.

0A50 **Lemma 59.50.1** (Mayer-Vietoris for étale cohomology). *Let X be a scheme. Suppose that $X = U \cup V$ is a union of two opens. For any abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ there exists a long exact cohomology sequence*

$$\begin{matrix} 0 \to H^0_{\acute{e}tale}(X, \mathcal{F}) \to H^0_{\acute{e}tale}(U, \mathcal{F}) \oplus H^0_{\acute{e}tale}(V, \mathcal{F}) \to H^0_{\acute{e}tale}(U \cap V, \mathcal{F}) \\ \to H^1_{\acute{e}tale}(X, \mathcal{F}) \to H^1_{\acute{e}tale}(U, \mathcal{F}) \oplus H^1_{\acute{e}tale}(V, \mathcal{F}) \to H^1_{\acute{e}tale}(U \cap V, \mathcal{F}) \to \ldots \end{matrix}$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Observe that if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V) \to 0$$

is exact. This is true in the first and middle spots as $\mathcal{I}$ is a sheaf. It is true on the right, because $\mathcal{I}(U) \to \mathcal{I}(U \cap V)$ is surjective by Cohomology on Sites, Lemma 21.12.6. Another way to prove it would be to show that the cokernel of the map $\mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is the first Čech cohomology group of $\mathcal{I}$ with respect to the covering $X = U \cup V$ which vanishes by Lemmas 59.18.7 and 59.19.1. Thus, if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution, then

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0$$

is a short exact sequence of complexes and the associated long exact cohomology sequence is the sequence of the statement of the lemma. $\square$

0EYK **Lemma 59.50.2** (Relative Mayer-Vietoris). *Let $f : X \to Y$ be a morphism of schemes. Suppose that $X = U \cup V$ is a union of two open subschemes. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

on $Y_{\acute{e}tale}$. This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ on $X_{\acute{e}tale}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any W in $Y_{\acute{e}tale}$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(W \times_Y X) \to \mathcal{I}^n(W \times_Y U) \oplus \mathcal{I}^n(W \times_Y V) \to \mathcal{I}^n(W \times_Y (U \cap V)) \to 0$$

was shown to be short exact in the proof of Lemma 59.50.1. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$. $\square$

59.51. Colimits

03Q4 We recall that if $(\mathcal{F}_i, \varphi_{ii'})$ is a diagram of sheaves on a site $\mathcal{C}$ its colimit (in the category of sheaves) is the sheafification of the presheaf $U \mapsto \text{colim}_i \mathcal{F}_i(U)$. See Sites, Lemma 7.10.13. If the system is directed, U is a quasi-compact object of $\mathcal{C}$ which has a cofinal system of coverings by quasi-compact objects, then $\mathcal{F}(U) = \text{colim}\, \mathcal{F}_i(U)$, see Sites, Lemma 7.17.7. See Cohomology on Sites, Lemma 21.16.1 for a result dealing with higher cohomology groups of colimits of abelian sheaves.

In Cohomology on Sites, Lemma 21.16.5 we generalize this result to a system of sheaves on an inverse system of sites. Here is the corresponding notion in the case of a system of étale sheaves living on an inverse system of schemes.

0EZL **Definition 59.51.1.** Let I be a preordered set. Let $(X_i, f_{i'i})$ be an inverse system of schemes over I. A system $(\mathcal{F}_i, \varphi_{i'i})$ of sheaves on $(X_i, f_{i'i})$ is given by

(1) a sheaf $\mathcal{F}_i$ on $(X_i)_{étale}$ for all $i \in I$,
(2) for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i \to \mathcal{F}_{i'}$ of sheaves on $(X_{i'})_{étale}$

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$.

In the situation of Definition 59.51.1, assume I is a directed set and the transition morphisms $f_{i'i}$ affine. Let $X = \lim X_i$ be the limit in the category of schemes, see Limits, Section 32.2. Denote $f_i : X \to X_i$ the projection morphisms and consider the maps

$$f_i^{-1}\mathcal{F}_i = f_{i'}^{-1}f_{i'i}^{-1}\mathcal{F}_i \xrightarrow{f_{i'}^{-1}\varphi_{i'i}} f_{i'}^{-1}\mathcal{F}_{i'}$$

This turns $f_i^{-1}\mathcal{F}_i$ into a system of sheaves on $X_{étale}$ over I (it is a good exercise to check this). We often want to know whether there is an isomorphism

$$H^q_{étale}(X, \text{colim}\, f_i^{-1}\mathcal{F}_i) = \text{colim}\, H^q_{étale}(X_i, \mathcal{F}_i)$$

It will turn out this is true if X_i is quasi-compact and quasi-separated for all i, see Theorem 59.51.3.

0EYL **Lemma 59.51.2.** Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse system of schemes over I with affine transition morphisms. Let $X = \lim_{i \in I} X_i$. With notation as in Topologies, Lemma 34.4.12 we have

$$X_{affine,étale} = \text{colim}(X_i)_{affine,étale}$$

as sites in the sense of Sites, Lemma 7.18.2.

Proof. Let us first prove this when X and X_i are quasi-compact and quasi-separated for all i (as this is true in all cases of interest). In this case any object of $X_{affine,étale}$, resp. $(X_i)_{affine,étale}$ is of finite presentation over X. Moreover, the category of schemes of finite presentation over X is the colimit of the categories of schemes of finite presentation over X_i, see Limits, Lemma 32.10.1. The same holds for the subcategories of affine objects étale over X by Limits, Lemmas 32.4.13 and 32.8.10. Finally, if $\{U^j \to U\}$ is a covering of $X_{affine,étale}$ and if $U_i^j \to U_i$ is morphism of affine schemes étale over X_i whose base change to X is $U^j \to U$, then we see that the base change of $\{U_i^j \to U_i\}$ to some $X_{i'}$ is a covering for i' large enough, see Limits, Lemma 32.8.15.

In the general case, let U be an object of $X_{affine,étale}$. Then $U \to X$ is étale and separated (as U is separated) but in general not quasi-compact. Still, $U \to X$ is locally of finite presentation and hence by Limits, Lemma 32.10.5 there exists an i, a quasi-compact and quasi-separated scheme U_i, and a morphism $U_i \to X_i$ which is locally of finite presentation whose base change to X is $U \to X$. Then $U = \lim_{i' \geq i} U_{i'}$ where $U_{i'} = U_i \times_{X_i} X_{i'}$. After increasing i we may assume U_i is affine, see Limits, Lemma 32.4.13. To check that $U_i \to X_i$ is étale for i sufficiently large, choose a finite affine open covering $U_i = U_{i,1} \cup \ldots \cup U_{i,m}$ such that $U_{i,j} \to U_i \to X_i$ maps into an affine open $W_{i,j} \subset X_i$. Then we can apply Limits, Lemma 32.8.10 to see that $U_{i,j} \to W_{i,j}$ is étale after possibly increasing i. In this way we see that the functor $\text{colim}(X_i)_{affine,\acute{e}tale} \to X_{affine,\acute{e}tale}$ is essentially surjective. Fully faithfulness follows directly from the already used Limits, Lemma 32.10.5. The statement on coverings is proved in exactly the same manner as done in the first paragraph of the proof. □

Using the above we get the following general result on colimits and cohomology.

09YQ Theorem 59.51.3. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \text{colim}\, f_i^{-1}\mathcal{F}_i$. Then

$$\text{colim}_{i \in I} H^p_{\acute{e}tale}(X_i, \mathcal{F}_i) = H^p_{\acute{e}tale}(X, \mathcal{F}).$$

for all $p \geq 0$.

Proof. By Topologies, Lemma 34.4.12 we can compute the cohomology of $\mathcal{F}$ on $X_{affine,\acute{e}tale}$. Thus the result by a combination of Lemma 59.51.2 and Cohomology on Sites, Lemma 21.16.5. □

The following two results are special cases of the theorem above.

03Q5 Lemma 59.51.4. Let X be a quasi-compact and quasi-separated scheme. Let I be a directed set. Let $(\mathcal{F}_i, \varphi_{ij})$ be a system of abelian sheaves on $X_{\acute{e}tale}$ over I. Then

$$\text{colim}_{i \in I} H^p_{\acute{e}tale}(X, \mathcal{F}_i) = H^p_{\acute{e}tale}(X, \text{colim}_{i \in I} \mathcal{F}_i).$$

Proof. This is a special case of Theorem 59.51.3. We also sketch a direct proof. We prove it for all X at the same time, by induction on p.

(1) For any quasi-compact and quasi-separated scheme X and any étale covering $\mathcal{U}$ of X, show that there exists a refinement $\mathcal{V} = \{V_j \to X\}_{j \in J}$ with J finite and each V_j quasi-compact and quasi-separated such that all $V_{j_0} \times_X \ldots \times_X V_{j_p}$ are also quasi-compact and quasi-separated.
(2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for $p = 0$ and all X.
(3) Using the locality of cohomology (Lemma 59.22.3), the Čech-to-cohomology spectral sequence (Theorem 59.19.2) and the fact that the induction hypothesis applies to all $V_{j_0} \times_X \ldots \times_X V_{j_p}$ in the above situation, prove the induction step $p \to p + 1$.

□

03Q6 Lemma 59.51.5. Let A be a ring, $(I, \leq)$ a directed set and (B_i, φ_{ij}) a system of A-algebras. Set $B = \text{colim}_{i \in I} B_i$. Let $X \to \text{Spec}(A)$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ an abelian sheaf on $X_{\acute{e}tale}$. Denote

$Y_i = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B_i)$, $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$, $\mathcal{G}_i = (Y_i \to X)^{-1}\mathcal{F}$ and $\mathcal{G} = (Y \to X)^{-1}\mathcal{F}$. Then

$$H^p_{\acute{e}tale}(Y, \mathcal{G}) = \mathrm{colim}_{i \in I} H^p_{\acute{e}tale}(Y_i, \mathcal{G}_i).$$

Proof. This is a special case of Theorem 59.51.3. We also outline a direct proof as follows.

(1) Given $V \to Y$ étale with V quasi-compact and quasi-separated, there exist $i \in I$ and $V_i \to Y_i$ such that $V = V_i \times_{Y_i} Y$. If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = \mathrm{colim}\, B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$. This is proved in Algebra, Lemma 10.143.3.
(2) In the situation of (1) show that $\mathcal{G}(V) = \mathrm{colim}_{i' \geq i} \mathcal{G}_{i'}(V_{i'})$ where $V_{i'}$ is the base change of V_i to $Y_{i'}$.
(3) By (1), we see that for every étale covering $\mathcal{V} = \{V_j \to Y\}_{j \in J}$ with J finite and the V_js quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $\mathcal{V}_i = \{V_{ij} \to Y_i\}_{j \in J}$ such that $\mathcal{V} \cong \mathcal{V}_i \times_{Y_i} Y$.
(4) Show that (2) and (3) imply
$$\check{H}^*(\mathcal{V}, \mathcal{G}) = \mathrm{colim}_{i \in I} \check{H}^*(\mathcal{V}_i, \mathcal{G}_i).$$
(5) Cleverly use the Čech-to-cohomology spectral sequence (Theorem 59.19.2).

$\square$

03Q8 Lemma 59.51.6. Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F} \in \mathrm{Ab}(X_{\acute{e}tale})$. Then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$(V \to Y) \longmapsto H^p_{\acute{e}tale}(X \times_Y V, \mathcal{F}|_{X \times_Y V}).$$

More generally, for $K \in D(X_{\acute{e}tale})$ we have that $R^p f_* K$ is the sheaf associated to the presheaf

$$(V \to Y) \longmapsto H^p_{\acute{e}tale}(X \times_Y V, K|_{X \times_Y V}).$$

Proof. This lemma is valid for topological spaces, and the proof in this case is the same. See Cohomology on Sites, Lemma 21.7.4 for the case of a sheaf and see Cohomology on Sites, Lemma 21.20.3 for the case of a complex of abelian sheaves. $\square$

09Z1 Lemma 59.51.7. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume the structure morphisms $g_i : X_i \to S$ and $g : X \to S$ are quasi-compact and quasi-separated. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i$. Then

$$\mathrm{colim}_{i \in I} R^p g_{i,*} \mathcal{F}_i = R^p g_* \mathcal{F}$$

for all $p \geq 0$.

Proof. Recall (Lemma 59.51.6) that $R^p g_{i,*}\mathcal{F}_i$ is the sheaf associated to the presheaf $U \mapsto H^p_{\acute{e}tale}(U \times_S X_i, \mathcal{F}_i)$ and similarly for $R^p g_* \mathcal{F}$. Moreover, the colimit of a system of sheaves is the sheafification of the colimit on the level of presheaves. Note that every object of $S_{\acute{e}tale}$ has a covering by quasi-compact and quasi-separated objects (e.g., affine schemes). Moreover, if U is a quasi-compact and quasi-separated object, then we have

$$\mathrm{colim}\, H^p_{\acute{e}tale}(U \times_S X_i, \mathcal{F}_i) = H^p_{\acute{e}tale}(U \times_S X, \mathcal{F})$$

by Theorem 59.51.3. Thus the lemma follows. □

0EYM Lemma 59.51.8. Let I be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over I. Assume g_i is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $h_{i'i} : S_{i'} \to S_i$ are affine. Let $g : X \to S$ be the limit of the morphisms g_i, see Limits, Section 32.2. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on $(X_i, f_{i'i})$. Set $\mathcal{F} = \text{colim}\, f_i^{-1}\mathcal{F}_i$. Then

$$R^p g_* \mathcal{F} = \text{colim}_{i \in I}\, h_i^{-1} R^p g_{i,*}\mathcal{F}_i$$

for all $p \geq 0$.

Proof. How is the map of the lemma constructed? For $i' \geq i$ we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_{i'}"'] \arrow[d, "g"'] & X_{i'} \arrow[r, "f_{i'i}"'] \arrow[d, "g_{i'}"'] & X_i \arrow[d, "g_i"] \\ S \arrow[r, "h_{i'}"] & S_{i'} \arrow[r, "h_{i'i}"] & S_i \end{tikzcd}$$

If we combine the base change map $h_{i'i}^{-1}Rg_{i,*}\mathcal{F}_i \to Rg_{i',*}f_{i'i}^{-1}\mathcal{F}_i$ (Cohomology on Sites, Lemma 21.15.1 or Remark 21.19.3) with the map $Rg_{i',*}\varphi_{i'i}$, then we obtain $\psi_{i'i} : h_{i'i}^{-1}R^p g_{i,*}\mathcal{F}_i \to R^p g_{i',*}\mathcal{F}_{i'}$. Similarly, using the left square in the diagram we obtain maps $\psi_i : h_i^{-1}R^p g_{i,*}\mathcal{F}_i \to R^p g_*\mathcal{F}$. The maps $h_{i'}^{-1}\psi_{i'i}$ and ψ_i are the maps used in the statement of the lemma. For this to make sense, we have to check that $\psi_{i''i} = \psi_{i''i'} \circ h_{i''i'}^{-1}\psi_{i'i}$ and $\psi_{i'} \circ h_{i'}^{-1}\psi_{i'i} = \psi_i$; this follows from Cohomology on Sites, Remark 21.19.5.

Proof of the equality. First proof using dimension shifting[4]. For any U affine and étale over X by Theorem 59.51.3 we have

$$g_*\mathcal{F}(U) = H^0(U \times_S X, \mathcal{F}) = \text{colim}\, H^0(U_i \times_{S_i} X_i, \mathcal{F}_i) = \text{colim}\, g_{i,*}\mathcal{F}_i(U_i)$$

where the colimit is over i large enough such that there exists an i and U_i affine étale over S_i whose base change is U over S (see Lemma 59.51.2). The right hand side is equal to $(\text{colim}\, h_i^{-1}g_{i,*}\mathcal{F}_i)(U)$ by Sites, Lemma 7.18.4. This proves the lemma for $p = 0$. If $(\mathcal{G}_i, \varphi_{i'i})$ is a system with $\mathcal{G} = \text{colim}\, f_i^{-1}\mathcal{G}_i$ such that $\mathcal{G}_i$ is an injective abelian sheaf on X_i for all i, then for any U affine and étale over X by Theorem 59.51.3 we have

$$H^p(U \times_S X, \mathcal{G}) = \text{colim}\, H^p(U_i \times_{S_i} X_i, \mathcal{G}_i) = 0$$

for $p > 0$ (same colimit as before). Hence $R^p g_*\mathcal{G} = 0$ and we get the result for $p > 0$ for such a system. In general we may choose a short exact sequence of systems

$$0 \to (\mathcal{F}_i, \varphi_{i'i}) \to (\mathcal{G}_i, \varphi_{i'i}) \to (\mathcal{Q}_i, \varphi_{i'i}) \to 0$$

where $(\mathcal{G}_i, \varphi_{i'i})$ is as above, see Cohomology on Sites, Lemma 21.16.4. By induction the lemma holds for $p - 1$ and by the above we have vanishing for p and $(\mathcal{G}_i, \varphi_{i'i})$. Hence the result for p and $(\mathcal{F}_i, \varphi_{i'i})$ by the long exact sequence of cohomology.

[4]You can also use this method to produce the maps in the lemma.

Second proof. Recall that $S_{affine,étale} = \text{colim}(S_i)_{affine,étale}$, see Lemma 59.51.2. Thus if U is an object of $S_{affine,étale}$, then we can write $U = U_i \times_{S_i} S$ for some i and some U_i in $(S_i)_{affine,étale}$ and

$$(\text{colim}_{i \in I} h_i^{-1} R^p g_{i,*}\mathcal{F}_i)(U) = \text{colim}_{i' \geq i}(R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$$

by Sites, Lemma 7.18.4 and the construction of the transition maps in the system described above. Since $R^p g_{i',*}\mathcal{F}_{i'}$ is the sheaf associated to the presheaf $U_{i'} \mapsto H^p(U_{i'} \times_{S_{i'}} X_{i'}, \mathcal{F}_{i'})$ and since $R^p g_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p(U \times_S X, \mathcal{F})$ (Lemma 59.51.6) we obtain a canonical commutative diagram

$$\begin{tikzcd}
\text{colim}_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, \mathcal{F}_{i'}) \arrow[r] \arrow[d] & \text{colim}_{i' \geq i}(R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'}) \arrow[d] \\
H^p(U \times_S X, \mathcal{F}) \arrow[r] & R^p g_*\mathcal{F}(U)
\end{tikzcd}$$

Observe that the left hand vertical arrow is an isomorphism by Theorem 59.51.3. We're trying to show that the right hand vertical arrow is an isomorphism. However, we already know that the source and target of this arrow are sheaves on $S_{affine,étale}$. Hence it suffices to show: (1) an element in the target, locally comes from an element in the source and (2) an element in the source which maps to zero in the target locally vanishes. Part (1) follows immediately from the above and the fact that the lower horizontal arrow comes from a map of presheaves which becomes an isomorphism after sheafification. For part (2), say $\xi \in \text{colim}_{i' \geq i}(R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$ is in the kernel. Choose an $i' \geq i$ and $\xi_{i'} \in (R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$ representing ξ. Choose a standard étale covering $\{U_{i',k} \to U_i \times_{S_i} S_{i'}\}_{k=1,\ldots,m}$ such that $\xi_{i'}|_{U_{i',k}}$ comes from $\xi_{i',k} \in H^p(U_{i',k} \times_{S_{i'}} X_{i'}, \mathcal{F}_{i'})$. Since it is enough to prove that ξ dies locally, we may replace U by the members of the étale covering $\{U_{i',k} \times_{S_{i'}} S \to U = U_i \times_{S_i} S\}$. After this replacement we see that ξ is the image of an element ξ' of the group $\text{colim}_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, \mathcal{F}_{i'})$ in the diagram above. Since ξ' maps to zero in $R^p g_*\mathcal{F}(U)$ we can do another replacement and assume that ξ' maps to zero in $H^p(U \times_S X, \mathcal{F})$. However, since the left vertical arrow is an isomorphism we then conclude $\xi' = 0$ hence $\xi = 0$ as desired. □

0EYN Lemma 59.51.9. Let $X = \lim_{i \in I} X_i$ be a directed limit of schemes with affine transition morphisms $f_{i'i}$ and projection morphisms $f_i : X \to X_i$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then

(1) there are canonical maps $\varphi_{i'i} : f_{i'i}^{-1} f_{i,*}\mathcal{F} \to f_{i',*}\mathcal{F}$ such that $(f_{i,*}\mathcal{F}, \varphi_{i'i})$ is a system of sheaves on $(X_i, f_{i'i})$ as in Definition 59.51.1, and
(2) $\mathcal{F} = \text{colim}\, f_i^{-1} f_{i,*}\mathcal{F}$.

Proof. Via Topologies, Lemma 34.4.12 and Lemma 59.51.2 this is a special case of Sites, Lemma 7.18.5. □

0DV2 Lemma 59.51.10. Let I be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over I. Assume g_i is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $X_{i'} \to X_i$ and $S_{i'} \to S_i$ are affine. Let $g : X \to S$ be the limit of the morphisms g_i, see Limits, Section 32.2. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $\mathcal{F}$ be an abelian sheaf on X. Then we have

$$R^p g_*\mathcal{F} = \text{colim}_{i \in I} h_i^{-1} R^p g_{i,*}(f_{i,*}\mathcal{F})$$

Proof. Formal combination of Lemmas 59.51.8 and 59.51.9. □

59.52. Colimits and complexes

0GIR In this section we discuss taking cohomology of systems of complexes in various settings, continuing the discussion for sheaves started in Section 59.51. We strongly urge the reader not to read this section unless absolutely necessary.

0EZM Lemma 59.52.1. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $\mathcal{F}_i^\bullet$ be a complex of abelian sheaves on $X_{i,\acute{e}tale}$. Let $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i^\bullet \to \mathcal{F}_{i'}^\bullet$ be a map of complexes on $X_{i,\acute{e}tale}$ such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$. Assume there is an integer a such that $\mathcal{F}_i^n = 0$ for $n < a$ and all $i \in I$. Then we have

$$H^p_{\acute{e}tale}(X, \operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet) = \operatorname{colim} H^p_{\acute{e}tale}(X_i, \mathcal{F}_i^\bullet)$$

where $f_i : X \to X_i$ is the projection.

Proof. This is a consequence of Theorem 59.51.3. Set $\mathcal{F}^\bullet = \operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet$. The theorem tells us that

$$\operatorname{colim}_{i \in I} H^p_{\acute{e}tale}(X_i, \mathcal{F}_i^n) = H^p_{\acute{e}tale}(X, \mathcal{F}^n)$$

for all $n, p \in \mathbf{Z}$. Let us use the spectral sequences

$$E_{1,i}^{s,t} = H^t_{\acute{e}tale}(X_i, \mathcal{F}_i^s) \Rightarrow H^{s+t}_{\acute{e}tale}(X_i, \mathcal{F}_i^\bullet)$$

and

$$E_1^{s,t} = H^t_{\acute{e}tale}(X, \mathcal{F}^s) \Rightarrow H^{s+t}_{\acute{e}tale}(X, \mathcal{F}^\bullet)$$

of Derived Categories, Lemma 13.21.3. Since $\mathcal{F}_i^n = 0$ for $n < a$ (with a independent of i) we see that only a fixed finite number of terms $E_{1,i}^{s,t}$ (independent of i) and $E_1^{s,t}$ contribute to $H^q_{\acute{e}tale}(X_i, \mathcal{F}_i^\bullet)$ and $H^q_{\acute{e}tale}(X, \mathcal{F}^\bullet)$ and $E_1^{s,t} = \operatorname{colim} E_{i,i}^{s,t}$. This implies what we want. Some details omitted. (There is an alternative argument using "stupid" truncations of complexes which avoids using spectral sequences.) □

0GIS Lemma 59.52.2. Let X be a quasi-compact and quasi-sepated scheme. Let $K_i \in D(X_{\acute{e}tale})$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $R\Gamma(X, \bigoplus_i K_i) = \bigoplus_i R\Gamma(X, K_i)$.

Proof. We have to show that $H^p(X, \bigoplus_i K_i) = \bigoplus_i H^p(X, K_i)$ for all $p \in \mathbf{Z}$. Choose complexes $\mathcal{F}_i^\bullet$ representing K_i such that $\mathcal{F}_i^n = 0$ for $n < a$. The direct sum of the complexes $\mathcal{F}_i^\bullet$ represents the object $\bigoplus K_i$ by Injectives, Lemma 19.13.4. Since $\bigoplus \mathcal{F}^\bullet$ is the filtered colimit of the finite direct sums, the result follows from Lemma 59.52.1. □

0GIT Lemma 59.52.3. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $K \in D^+(S_{\acute{e}tale})$. Then

$$\operatorname{colim}_{i \in I} H^p_{\acute{e}tale}(X_i, K|_{X_i}) = H^p_{\acute{e}tale}(X, K|_X).$$

for all $p \in \mathbf{Z}$ where $K|_{X_i}$ and $K|_X$ are the pullbacks of K to X_i and X.

Proof. We may represent K by a bounded below complex $\mathcal{G}^\bullet$ of abelian sheaves on $S_{étale}$. Say $\mathcal{G}^n = 0$ for $n < a$. Denote $\mathcal{F}_i^\bullet$ and $\mathcal{F}^\bullet$ the pullbacks of this complex of X_i and X. These complexes represent the objects $K|_{X_i}$ and $K|_X$ and we have $\mathcal{F}^\bullet = \text{colim}\, f_i^{-1}\mathcal{F}_i^\bullet$ termwise. Hence the lemma follows from Lemma 59.52.1. □

0GIU Lemma 59.52.4. Let I, $g_i : X_i \to S_i$, $g : X \to S$, f_i, g_i, h_i be as in Lemma 59.51.8. Let $0 \in I$ and $K_0 \in D^+(X_{0,étale})$. For $i \geq 0$ denote K_i the pullback of K_0 to X_i. Denote K the pullback of K to X. Then

$$R^pg_*K = \text{colim}_{i\geq 0}\, h_i^{-1}R^pg_{i,*}K_i$$

for all $p \in \mathbf{Z}$.

Proof. Fix an integer $p_0 \in \mathbf{Z}$. Let a be an integer such that $H^j(K_0) = 0$ for $j < a$. We will prove the formula holds for all $p \leq p_0$ by descending induction on a. If $a > p_0$, then we see that the left and right hand side of the formula are zero for $p \leq p_0$ by trivial vanishing, see Derived Categories, Lemma 13.16.1. Assume $a \leq p_0$. Consider the distinguished triangle

$$H^a(K_0)[-a] \to K_0 \to \tau_{\geq a+1}K_0$$

Pulling back this distinguished triangle to X_i and X gives compatible distinguished triangles for K_i and K. For $p \leq p_0$ we consider the commutative diagram

$$\begin{tikzcd}
\text{colim}_{i\geq 0}\, h_i^{-1}R^{p-1}g_{i,*}(\tau_{\geq a+1}K_i) \arrow[r, "\alpha"'] \arrow[d] & R^{p-1}g_*(\tau_{\geq a+1}K) \arrow[d] \\
\text{colim}_{i\geq 0}\, h_i^{-1}R^pg_{i,*}(H^a(K_i)[-a]) \arrow[r, "\beta"'] \arrow[d] & R^pg_*(H^a(K)[-a]) \arrow[d] \\
\text{colim}_{i\geq 0}\, h_i^{-1}R^pg_{i,*}K_i \arrow[r, "\gamma"'] \arrow[d] & R^pg_*K \arrow[d] \\
\text{colim}_{i\geq 0}\, R^pg_{i,*}\tau_{\geq a+1}K_i \arrow[r, "\delta"'] \arrow[d] & R^pg_*\tau_{\geq a+1}K \arrow[d] \\
\text{colim}_{i\geq 0}\, R^{p+1}g_{i,*}(H^a(K_i)[-a]) \arrow[r, "\epsilon"] & R^{p+1}g_*(H^a(K)[-a])
\end{tikzcd}$$

with exact columns. The arrows β and ϵ are isomorphisms by Lemma 59.51.8. The arrows α and δ are isomorphisms by induction hypothesis. Hence γ is an isomorphism as desired. □

0GIV Lemma 59.52.5. Let I, $g_i : X_i \to S_i$, $g : X \to S$, $f_{ii'}$, f_i, g_i, h_i be as in Lemma 59.51.8. Let $\mathcal{F}_i^\bullet$ be a complex of abelian sheaves on $X_{i,étale}$. Let $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i^\bullet \to \mathcal{F}_{i'}^\bullet$ be a map of complexes on $X_{i,étale}$ such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$. Assume there is an integer a such that $\mathcal{F}_i^n = 0$ for $n < a$ and all $i \in I$. Then

$$R^pg_*(\text{colim}\, f_i^{-1}\mathcal{F}_i^\bullet) = \text{colim}_{i\geq 0}\, h_i^{-1}R^pg_{i,*}\mathcal{F}_i^\bullet$$

for all $p \in \mathbf{Z}$.

Proof. This is a consequence of Lemma 59.51.8. Set $\mathcal{F}^\bullet = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i^\bullet$. The lemma tells us that

$$\mathrm{colim}_{i \in I}\, h_i^{-1} R^p g_{i,*}\mathcal{F}_i^n = R^p g_*\mathcal{F}^n$$

for all $n, p \in \mathbf{Z}$. Let us use the spectral sequences

$$E_{1,i}^{s,t} = R^t g_{i,*}\mathcal{F}_i^s \Rightarrow R^{s+t} g_{i,*}\mathcal{F}_i^\bullet$$

and

$$E_1^{s,t} = R^t g_*\mathcal{F}^s \Rightarrow R^{s+t} g_*\mathcal{F}^\bullet$$

of Derived Categories, Lemma 13.21.3. Since $\mathcal{F}_i^n = 0$ for $n < a$ (with a independent of i) we see that only a fixed finite number of terms $E_{1,i}^{s,t}$ (independent of i) and $E_1^{s,t}$ contribute and $E_1^{s,t} = \mathrm{colim}\, E_{i,i}^{s,t}$. This implies what we want. Some details omitted. (There is an alternative argument using "stupid" truncations of complexes which avoids using spectral sequences.) ☐

0GIW Lemma 59.52.6. Let $f : X \to Y$ be a quasi-compact and quasi-sepated morphism of schemes. Let $K_i \in D(X_{\acute{e}tale})$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $Rf_*(\bigoplus_i K_i) = \bigoplus_i Rf_*K_i$.

Proof. We have to show that $R^pf_*(\bigoplus_i K_i) = \bigoplus_i R^pf_*K_i$ for all $p \in \mathbf{Z}$. Choose complexes $\mathcal{F}_i^\bullet$ representing K_i such that $\mathcal{F}_i^n = 0$ for $n < a$. The direct sum of the complexes $\mathcal{F}_i^\bullet$ represents the object $\bigoplus K_i$ by Injectives, Lemma 19.13.4. Since $\bigoplus \mathcal{F}^\bullet$ is the filtered colimit of the finite direct sums, the result follows from Lemma 59.52.5. ☐

59.53. Stalks of higher direct images

03Q7 The stalks of higher direct images can often be computed as follows.

03Q9 Theorem 59.53.1. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $\mathcal{F}$ an abelian sheaf on $X_{\acute{e}tale}$, and $\overline{s}$ a geometric point of S lying over $s \in S$. Then

$$(R^nf_*\mathcal{F})_{\overline{s}} = H^n_{\acute{e}tale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{F})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection. For $K \in D^+(X_{\acute{e}tale})$ and $n \in \mathbf{Z}$ we have

$$(R^nf_*K)_{\overline{s}} = H^n_{\acute{e}tale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$$

In fact, we have

$$(Rf_*K)_{\overline{s}} = R\Gamma_{\acute{e}tale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$$

in $D^+(\mathrm{Ab})$.

Proof. Let $\mathcal{I}$ be the category of étale neighborhoods of $\overline{s}$ on S. By Lemma 59.51.6 we have

$$(R^nf_*\mathcal{F})_{\overline{s}} = \mathrm{colim}_{(V,\overline{v}) \in \mathcal{I}^{opp}}\, H^n_{\acute{e}tale}(X \times_S V, \mathcal{F}|_{X \times_S V}).$$

We may replace $\mathcal{I}$ by the initial subcategory consisting of affine étale neighbourhoods of $\overline{s}$. Observe that

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{(V,\overline{v}) \in \mathcal{I}} V$$

by Lemma 59.33.1 and Limits, Lemma 32.2.1. Since fibre products commute with limits we also obtain

$$X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{(V,\overline{v}) \in \mathcal{I}} X \times_S V$$

We conclude by Lemma 59.51.5. For the second variant, use the same argument using Lemma 59.52.3 instead of Lemma 59.51.5.

To see that the last statement is true, it suffices to produce a map $(Rf_*K)_{\overline{s}} \to R\Gamma_{étale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$ in $D^+(\mathrm{Ab})$ which realizes the ismorphisms on cohomology groups in degree n above for all n. To do this, choose a bounded below complex $\mathcal{J}^\bullet$ of injective abelian sheaves on $X_{étale}$ representing K. The complex $f_*\mathcal{J}^\bullet$ represents Rf_*K. Thus the complex

$$(f_*\mathcal{J}^\bullet)_{\overline{s}} = \mathrm{colim}_{(V,\overline{v}) \in \mathcal{I}^{opp}}(f_*\mathcal{J}^\bullet)(V)$$

represents $(Rf_*K)_{\overline{s}}$. For each V we have maps

$$(f_*\mathcal{J}^\bullet)(V) = \Gamma(X \times_S V, \mathcal{J}^\bullet) \longrightarrow \Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{J}^\bullet)$$

and the target complex represents $R\Gamma_{étale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$ in $D^+(\mathrm{Ab})$. Taking the colimit of these maps we obtain the result. $\square$

0GIX Remark 59.53.2. Let $f : X \to S$ be a morphism of schemes. Let $K \in D(X_{étale})$. Let $\overline{s}$ be a geometric point of S. There are always canonical maps

$$(Rf_*K)_{\overline{s}} \longrightarrow R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K) \longrightarrow R\Gamma(X_{\overline{s}}, K|_{X_{\overline{s}}})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection. Namely, consider the commutative diagram

$$\begin{tikzcd}
X_{\overline{s}} \arrow[r] \arrow[d, "f_{\overline{s}}"] & X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \arrow[r, "p"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\overline{s} \arrow[r, "i"] & \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \arrow[r, "j"] & S
\end{tikzcd}$$

We have the base change maps

$$i^{-1}Rf'_*(p^{-1}K) \to Rf_{\overline{s},*}(K|_{X_{\overline{s}}}) \quad \text{and} \quad j^{-1}Rf_*K \to Rf'_*(p^{-1}K)$$

(Cohomology on Sites, Remark 21.19.3) for the two squares in this diagram. Taking global sections we obtain the desired maps. By Cohomology on Sites, Remark 21.19.5 the composition of these two maps is the usual (base change) map $(Rf_*K)_{\overline{s}} \to R\Gamma(X_{\overline{s}}, K|_{X_{\overline{s}}})$.

59.54. The Leray spectral sequence

03QA

03QB Lemma 59.54.1. Let $f : X \to Y$ be a morphism and $\mathcal{I}$ an injective object of $\mathrm{Ab}(X_{étale})$. Let $V \in \mathrm{Ob}(Y_{étale})$. Then

(1) for any covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ we have $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$,
(2) $f_*\mathcal{I}$ is acyclic for the functor $\Gamma(V, -)$, and
(3) if $g : Y \to Z$, then $f_*\mathcal{I}$ is acyclic for g_*.

Proof. Observe that $\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{V} \times_Y X, \mathcal{I})$ which has vanishing higher cohomology groups by Lemma 59.18.7. This proves (1). The second statement follows as a sheaf which has vanishing higher Čech cohomology groups for any covering has vanishing higher cohomology groups. This a wonderful exercise in using the Čech-to-cohomology spectral sequence, but see Cohomology on Sites, Lemma 21.10.9 for details and a more precise and general statement. Part (3) is a consequence of (2) and the description of R^pg_* in Lemma 59.51.6. $\square$

Using the formalism of Grothendieck spectral sequences, this gives the following.

03QC Proposition 59.54.2 (Leray spectral sequence). Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an étale sheaf on X. Then there is a spectral sequence

$$E_2^{p,q} = H^p_{\acute{e}tale}(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}_{\acute{e}tale}(X, \mathcal{F}).$$

Proof. See Lemma 59.54.1 and see Derived Categories, Section 13.22. □

59.55. Vanishing of finite higher direct images

03QN The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

03QO Lemma 59.55.1. Let R be a strictly henselian local ring. Set $S = \mathrm{Spec}(R)$ and let $\overline{s}$ be its closed point. Then the global sections functor $\Gamma(S, -) : \mathrm{Ab}(S_{\acute{e}tale}) \to \mathrm{Ab}$ is exact. In fact we have $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ for any sheaf of sets $\mathcal{F}$. In particular

$$\forall p \geq 1, \quad H^p_{\acute{e}tale}(S, \mathcal{F}) = 0$$

for all $\mathcal{F} \in \mathrm{Ab}(S_{\acute{e}tale})$.

Proof. If we show that $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ then $\Gamma(S, -)$ is exact as the stalk functor is exact. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{s}$. Pick an affine open neighborhood $\mathrm{Spec}(A)$ of $\overline{u}$ in U. Then $R \to A$ is étale and $\kappa(\overline{s}) = \kappa(\overline{u})$. By Theorem 59.32.4 we see that $A \cong R \times A'$ as an R-algebra compatible with maps to $\kappa(\overline{s}) = \kappa(\overline{u})$. Hence we get a section

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] & U \arrow[d] \\ & S \arrow[ul] \end{tikzcd}$$

It follows that in the system of étale neighbourhoods of $\overline{s}$ the identity map $(S, \overline{s}) \to (S, \overline{s})$ is cofinal. Hence $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$. The final statement of the lemma follows as the higher derived functors of an exact functor are zero, see Derived Categories, Lemma 13.16.9. □

03QP Proposition 59.55.2. Let $f : X \to Y$ be a finite morphism of schemes.

(1) For any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have
$$(f_*\mathcal{F})_{\overline{y}} = \prod\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$
for $\mathcal{F}$ in $Sh(X_{\acute{e}tale})$ and
$$(f_*\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$
for $\mathcal{F}$ in $\mathrm{Ab}(X_{\acute{e}tale})$.

(2) For any $q \geq 1$ we have $R^q f_*\mathcal{F} = 0$ for $\mathcal{F}$ in $\mathrm{Ab}(X_{\acute{e}tale})$.

Proof. Let $X^{sh}_{\overline{y}}$ denote the fiber product $X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$. By Theorem 59.53.1 the stalk of $R^q f_*\mathcal{F}$ at $\overline{y}$ is computed by $H^q_{\acute{e}tale}(X^{sh}_{\overline{y}}, \mathcal{F})$. Since f is finite, $X^{sh}_{\overline{y}}$ is finite over $\mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$, thus $X^{sh}_{\overline{y}} = \mathrm{Spec}(A)$ for some ring A finite over $\mathcal{O}^{sh}_{Y,\overline{y}}$. Since the latter is strictly henselian, Lemma 59.32.5 implies that A is a finite product of henselian local rings $A = A_1 \times \ldots \times A_r$. Since the residue field of $\mathcal{O}^{sh}_{Y,\overline{y}}$ is separably closed the same is true for each A_i. Hence A_i is strictly henselian. This implies that $X^{sh}_{\overline{y}} = \coprod_{i=1}^r \mathrm{Spec}(A_i)$. The vanishing of Lemma 59.55.1 implies that

$(R^q f_*\mathcal{F})_{\overline{y}} = 0$ for $q > 0$ which implies (2) by Theorem 59.29.10. Part (1) follows from the corresponding statement of Lemma 59.55.1. □

0959 Lemma 59.55.3. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f a finite morphism. For any sheaf of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 7.45. It suffices to check on stalks (Theorem 59.29.10). Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point. We have

$$\begin{aligned} (f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} & = \prod\nolimits_{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'} ((g')^{-1}\mathcal{F})_{\overline{x}'} \\ & = \prod\nolimits_{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'} \mathcal{F}_{g' \circ \overline{x}'} \\ & = \prod\nolimits_{\overline{x} : \mathrm{Spec}(k) \to X,\ f \circ \overline{x} = g \circ \overline{y}'} \mathcal{F}_{\overline{x}} \\ & = (f_*\mathcal{F})_{g \circ \overline{y}'} \\ & = (g^{-1}f_*\mathcal{F})_{\overline{y}'} \end{aligned}$$

The first equality by Proposition 59.55.2. The second equality by Lemma 59.36.2. The third equality holds because the diagram is a cartesian square and hence the map

$$\{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'\} \longrightarrow \{\overline{x} : \mathrm{Spec}(k) \to X,\ f \circ \overline{x} = g \circ \overline{y}'\}$$

sending $\overline{x}'$ to $g' \circ \overline{x}'$ is a bijection. The fourth equality by Proposition 59.55.2. The fifth equality by Lemma 59.36.2. □

0EYP Lemma 59.55.4. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f an integral morphism. For any sheaf of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. The question is local on Y and hence we may assume Y is affine. Then we can write $X = \lim X_i$ with $f_i : X_i \to Y$ finite (this is easy in the affine case, but see Limits, Lemma 32.7.3 for a reference). Denote $p_{i'i} : X_{i'} \to X_i$ the transition morphisms and $p_i : X \to X_i$ the projections. Setting $\mathcal{F}_i = p_{i,*}\mathcal{F}$ we obtain from Lemma 59.51.9 a system $(\mathcal{F}_i, \varphi_{i'i})$ with $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$. We get $f_*\mathcal{F} = \mathrm{colim}\, f_{i,*}\mathcal{F}_i$ from Lemma 59.51.7. Set $X'_i = Y' \times_Y X_i$ with projections f'_i and g'_i. Then $X' = \lim X'_i$

as limits commute with limits. Denote $p'_i : X' \to X'_i$ the projections. We have

$$\begin{aligned} g^{-1}f_*\mathcal{F} &= g^{-1}\operatorname{colim} f_{i,*}\mathcal{F}_i \\ &= \operatorname{colim} g^{-1}f_{i,*}\mathcal{F}_i \\ &= \operatorname{colim} f'_{i,*}(g'_i)^{-1}\mathcal{F}_i \\ &= f'_*(\operatorname{colim}(p'_i)^{-1}(g'_i)^{-1}\mathcal{F}_i) \\ &= f'_*(\operatorname{colim}(g')^{-1}p_i^{-1}\mathcal{F}_i) \\ &= f'_*(g')^{-1}\operatorname{colim} p_i^{-1}\mathcal{F}_i \\ &= f'_*(g')^{-1}\mathcal{F} \end{aligned}$$

as desired. For the first equality see above. For the second use that pullback commutes with colimits. For the third use the finite case, see Lemma 59.55.3. For the fourth use Lemma 59.51.7. For the fifth use that $g'_i \circ p'_i = p_i \circ g'$. For the sixth use that pullback commutes with colimits. For the seventh use $\mathcal{F} = \operatorname{colim} p_i^{-1}\mathcal{F}_i$. □

The following lemma is a case of cohomological descent dealing with étale sheaves and finite surjective morphisms. We will significantly generalize this result once we prove the proper base change theorem.

09Z2 Lemma 59.55.5. Let $f : X \to Y$ be a surjective finite morphism of schemes. Set $f_n : X_n \to Y$ equal to the $(n+1)$-fold fibre product of X over Y. For $\mathcal{F} \in \textit{Ab}(Y_{\acute{e}tale})$ set $\mathcal{F}_n = f_{n,*}f_n^{-1}\mathcal{F}$. There is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots$$

on $X_{\acute{e}tale}$. Moreover, there is a spectral sequence

$$E_1^{p,q} = H^q_{\acute{e}tale}(X_p, f_p^{-1}\mathcal{F})$$

converging to $H^{p+q}(Y_{\acute{e}tale}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. If we prove the first statement of the lemma, then we obtain a spectral sequence with $E_1^{p,q} = H^q_{\acute{e}tale}(Y, \mathcal{F})$ converging to $H^{p+q}(Y_{\acute{e}tale}, \mathcal{F})$, see Derived Categories, Lemma 13.21.3. On the other hand, since $R^if_{p,*}f_p^{-1}\mathcal{F} = 0$ for $i > 0$ (Proposition 59.55.2) we get

$$H^q_{\acute{e}tale}(X_p, f_p^{-1}\mathcal{F}) = H^q_{\acute{e}tale}(Y, f_{p,*}f_p^{-1}\mathcal{F}) = H^q_{\acute{e}tale}(Y, \mathcal{F}_p)$$

by Proposition 59.54.2 and we get the spectral sequence of the lemma.

To prove the first statement of the lemma, observe that X_n forms a simplicial scheme over Y, see Simplicial, Example 14.3.5. Observe moreover, that for each of the projections $d_j : X_{n+1} \to X_n$ there is a map $d_j^{-1}f_n^{-1}\mathcal{F} \to f_{n+1}^{-1}\mathcal{F}$. These maps induce maps

$$\delta_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$$

for $j = 0, \ldots, n+1$. We use the alternating sum of these maps to define the differentials $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Similarly, there is a canonical augmentation $\mathcal{F} \to \mathcal{F}_0$, namely this is just the canonical map $\mathcal{F} \to f_*f^{-1}\mathcal{F}$. To check that this sequence of sheaves is an exact complex it suffices to check on stalks at geometric points (Theorem 59.29.10). Thus we let $\overline{y} : \operatorname{Spec}(k) \to Y$ be a geometric point. Let $E = \{\overline{x} : \operatorname{Spec}(k) \to X \mid f(\overline{x}) = \overline{y}\}$. Then E is a finite nonempty set and we see that

$$(\mathcal{F}_n)_{\overline{y}} = \bigoplus\nolimits_{e \in E^{n+1}} \mathcal{F}_{\overline{y}}$$

by Proposition 59.55.2 and Lemma 59.36.2. Thus we have to see that given an abelian group M the sequence

$$0 \to M \to \bigoplus\nolimits_{e \in E} M \to \bigoplus\nolimits_{e \in E^2} M \to \ldots$$

is exact. Here the first map is the diagonal map and the map $\bigoplus_{e \in E^{n+1}} M \to \bigoplus_{e \in E^{n+2}} M$ is the alternating sum of the maps induced by the $(n+2)$ projections $E^{n+2} \to E^{n+1}$. This can be shown directly or deduced by applying Simplicial, Lemma 14.26.9 to the map $E \to \{*\}$. □

09Z3 Remark 59.55.6. In the situation of Lemma 59.55.5 if $\mathcal{G}$ is a sheaf of sets on $Y_{étale}$, then we have

$$\Gamma(Y, \mathcal{G}) = \text{Equalizer}(\ \Gamma(X_0, f_0^{-1}\mathcal{G}) \rightrightarrows \Gamma(X_1, f_1^{-1}\mathcal{G})\ )$$

This is proved in exactly the same way, by showing that the sheaf $\mathcal{G}$ is the equalizer of the two maps $f_{0,*}f_0^{-1}\mathcal{G} \to f_{1,*}f_1^{-1}\mathcal{G}$.

59.56. Galois action on stalks

03QW In this section we define an action of the absolute Galois group of a residue field of a point s of S on the stalk functor at any geometric point lying over s.

Galois action on stalks. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s))$. Define an action of σ on the stalk $\mathcal{F}_{\overline{s}}$ of a sheaf $\mathcal{F}$ as follows

04JK (59.56.0.1)
$$\begin{matrix} \mathcal{F}_{\overline{s}} & \longrightarrow & \mathcal{F}_{\overline{s}} \\ (U, \overline{u}, t) & \longmapsto & (U, \overline{u} \circ \text{Spec}(\sigma), t). \end{matrix}$$

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 59.29.6. This is a left action, since if $\sigma_i \in \text{Aut}(\kappa(\overline{s})/\kappa(s))$ then

$$\begin{aligned} \sigma_1 \cdot (\sigma_2 \cdot (U, \overline{u}, t)) &= \sigma_1 \cdot (U, \overline{u} \circ \text{Spec}(\sigma_2), t) \\ &= (U, \overline{u} \circ \text{Spec}(\sigma_2) \circ \text{Spec}(\sigma_1), t) \\ &= (U, \overline{u} \circ \text{Spec}(\sigma_1 \circ \sigma_2), t) \\ &= (\sigma_1 \circ \sigma_2) \cdot (U, \overline{u}, t) \end{aligned}$$

It is clear that this action is functorial in the sheaf $\mathcal{F}$. We note that we could have defined this action by referring directly to Remark 59.29.8.

03QX Definition 59.56.1. Let S be a scheme. Let $\overline{s}$ be a geometric point lying over the point s of S. Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa(s)$ in the algebraically closed field $\kappa(\overline{s})$.

(1) In this situation the absolute Galois group of $\kappa(s)$ is $\text{Gal}(\kappa(s)^{sep}/\kappa(s))$. It is sometimes denoted $\text{Gal}_{\kappa(s)}$.
(2) The geometric point $\overline{s}$ is called algebraic if $\kappa(s) \subset \kappa(\overline{s})$ is an algebraic closure of $\kappa(s)$.

03QY Example 59.56.2. The geometric point $\text{Spec}(\mathbf{C}) \to \text{Spec}(\mathbf{Q})$ is not algebraic.

Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ be as in the definition. Note that as $\kappa(\overline{s})$ is algebraically closed the map

$$\text{Aut}(\kappa(\overline{s})/\kappa(s)) \longrightarrow \text{Gal}(\kappa(s)^{sep}/\kappa(s)) = \text{Gal}_{\kappa(s)}$$

is surjective. Suppose $(U, \overline{u})$ is an étale neighbourhood of $\overline{s}$, and say $\overline{u}$ lies over the point u of U. Since $U \to S$ is étale, the residue field extension $\kappa(u)/\kappa(s)$ is finite separable. This implies the following

(1) If $\sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s)^{sep})$ then σ acts trivially on $\mathcal{F}_{\overline{s}}$.
(2) More precisely, the action of $\text{Aut}(\kappa(\overline{s})/\kappa(s))$ determines and is determined by an action of the absolute Galois group $\text{Gal}_{\kappa(s)}$ on $\mathcal{F}_{\overline{s}}$.
(3) Given $(U, \overline{u}, t)$ representing an element ξ of $\mathcal{F}_{\overline{s}}$ any element of $\text{Gal}(\kappa(s)^{sep}/K)$ acts trivially, where $\kappa(s) \subset K \subset \kappa(s)^{sep}$ is the image of $\overline{u}^\sharp : \kappa(u) \to \kappa(\overline{s})$.

Altogether we see that $\mathcal{F}_{\overline{s}}$ becomes a $\text{Gal}_{\kappa(s)}$-set (see Fundamental Groups, Definition 58.2.1). Hence we may think of the stalk functor as a functor

$$Sh(S_{\acute{e}tale}) \longrightarrow \text{Gal}_{\kappa(s)}\text{-Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

and from now on we usually do think about the stalk functor in this way.

03QT **Theorem 59.56.3.** Let $S = \text{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = \text{Gal}_{\kappa(s)}$ denote the absolute Galois group. Taking stalks induces an equivalence of categories

$$Sh(S_{\acute{e}tale}) \longrightarrow G\text{-Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. Let us construct the inverse to this functor. In Fundamental Groups, Lemma 58.2.2 we have seen that given a G-set M there exists an étale morphism $X \to \text{Spec}(K)$ such that $\text{Mor}_K(\text{Spec}(K^{sep}), X)$ is isomorphic to M as a G-set. Consider the sheaf $\mathcal{F}$ on $\text{Spec}(K)_{\acute{e}tale}$ defined by the rule $U \mapsto \text{Mor}_K(U, X)$. This is a sheaf as the étale topology is subcanonical. Then we see that $\mathcal{F}_{\overline{s}} = \text{Mor}_K(\text{Spec}(K^{sep}), X) = M$ as G-sets (details omitted). This gives the inverse of the functor and we win. $\square$

04JL **Remark 59.56.4.** Another way to state the conclusion of Theorem 59.56.3 and Fundamental Groups, Lemma 58.2.2 is to say that every sheaf on $\text{Spec}(K)_{\acute{e}tale}$ is representable by a scheme X étale over $\text{Spec}(K)$. This does not mean that every sheaf is representable in the sense of Sites, Definition 7.12.3. The reason is that in our construction of $\text{Spec}(K)_{\acute{e}tale}$ we chose a sufficiently large set of schemes étale over $\text{Spec}(K)$, whereas sheaves on $\text{Spec}(K)_{\acute{e}tale}$ form a proper class.

04JM **Lemma 59.56.5.** Assumptions and notations as in Theorem 59.56.3. There is a functorial bijection

$$\Gamma(S, \mathcal{F}) = (\mathcal{F}_{\overline{s}})^G$$

Proof. We can prove this using formal arguments and the result of Theorem 59.56.3 as follows. Given a sheaf $\mathcal{F}$ corresponding to the G-set $M = \mathcal{F}_{\overline{s}}$ we have

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \text{Mor}_{Sh(S_{\acute{e}tale})}(h_{\text{Spec}(K)}, \mathcal{F}) \\ &= \text{Mor}_{G\text{-Sets}}(\{*\}, M) \\ &= M^G \end{aligned}$$

Here the first identification is explained in Sites, Sections 7.2 and 7.12, the second results from Theorem 59.56.3 and the third is clear. We will also give a direct proof[5].

Suppose that $t \in \Gamma(S, \mathcal{F})$ is a global section. Then the triple $(S, \overline{s}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is clearly invariant under the action of G. Conversely, suppose

[5]For the doubting Thomases out there.

that $(U, \overline{u}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is invariant. Then we may shrink U and assume $U = \text{Spec}(L)$ for some finite separable field extension of K, see Proposition 59.26.2. In this case the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$ is injective, because for any morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective since $U' \to U$ is a covering of $S_{\acute{e}tale}$. After enlarging L a bit we may assume $K \subset L$ is a finite Galois extension. At this point we use that

$$\text{Spec}(L) \times_{\text{Spec}(K)} \text{Spec}(L) = \coprod\nolimits_{\sigma \in \text{Gal}(L/K)} \text{Spec}(L)$$

where the maps $\text{Spec}(L) \to \text{Spec}(L \otimes_K L)$ come from the ring maps $a \otimes b \mapsto a\sigma(b)$. Hence we see that the condition that $(U, \overline{u}, t)$ is invariant under all of G implies that $t \in \mathcal{F}(\text{Spec}(L))$ maps to the same element of $\mathcal{F}(\text{Spec}(L) \times_{\text{Spec}(K)} \text{Spec}(L))$ via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of $\mathcal{F}$ for the étale covering $\{\text{Spec}(L) \to \text{Spec}(K)\}$ kicks in and we conclude that t comes from a unique section of $\mathcal{F}$ over $\text{Spec}(K)$. $\square$

04JN Remark 59.56.6. Let S be a scheme and let $\overline{s} : \text{Spec}(k) \to S$ be a geometric point of S. By definition this means that k is algebraically closed. In particular the absolute Galois group of k is trivial. Hence by Theorem 59.56.3 the category of sheaves on $\text{Spec}(k)_{\acute{e}tale}$ is equivalent to the category of sets. The equivalence is given by taking sections over $\text{Spec}(k)$. This finally provides us with an alternative definition of the stalk functor. Namely, the functor

$$Sh(S_{\acute{e}tale}) \longrightarrow \text{Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

is isomorphic to the functor

$$Sh(S_{\acute{e}tale}) \longrightarrow Sh(\text{Spec}(k)_{\acute{e}tale}) = \text{Sets}, \quad \mathcal{F} \longmapsto \overline{s}^*\mathcal{F}$$

To prove this rigorously one can use Lemma 59.36.2 part (3) with $f = \overline{s}$. Moreover, having said this the general case of Lemma 59.36.2 part (3) follows from functoriality of pullbacks.

59.57. Group cohomology

0A2H In the following, if we write $H^i(G, M)$ we will mean that G is a topological group and M a discrete G-module with continuous G-action and $H^i(G, -)$ is the ith right derived functor on the category Mod_G of such G-modules, see Definitions 59.57.1 and 59.57.2. This includes the case of an abstract group G, which simply means that G is viewed as a topological group with the discrete topology.

When the module has a nondiscrete topology, we will use the notation $H^i_{cont}(G, M)$ to indicate the continuous cohomology groups introduced in [Tat76], see Section 59.58.

04JP Definition 59.57.1. Let G be a topological group.

(1) A G-module, sometimes called a discrete G-module, is an abelian group M endowed with a left action $a : G \times M \to M$ by group homomorphisms such that a is continuous when M is given the discrete topology.
(2) A morphism of G-modules $f : M \to N$ is a G-equivariant homomorphism from M to N.
(3) The category of G-modules is denoted Mod_G.

Let R be a ring.

(1) An R-G-module is an R-module M endowed with a left action $a : G \times M \to M$ by R-linear maps such that a is continuous when M is given the discrete topology.
(2) A morphism of R-G-modules $f : M \to N$ is a G-equivariant R-module map from M to N.
(3) The category of R-G-modules is denoted $\text{Mod}_{R,G}$.

The condition that $a : G \times M \to M$ is continuous is equivalent with the condition that the stabilizer of any $x \in M$ is open in G. If G is an abstract group then this corresponds to the notion of an abelian group endowed with a G-action provided we endow G with the discrete topology. Observe that $\text{Mod}_{\mathbf{Z},G} = \text{Mod}_G$.

The category Mod_G has enough injectives, see Injectives, Lemma 19.3.1. Consider the left exact functor

$$\text{Mod}_G \longrightarrow \text{Ab}, \quad M \longmapsto M^G = \{x \in M \mid g \cdot x = x \ \forall g \in G\}$$

We sometimes denote $M^G = H^0(G, M)$ and sometimes we write $M^G = \Gamma_G(M)$. This functor has a total right derived functor $R\Gamma_G(M)$ and ith right derived functor $R^i\Gamma_G(M) = H^i(G, M)$ for any $i \geq 0$.

The same construction works for $H^0(G, -) : \text{Mod}_{R,G} \to \text{Mod}_R$. We will see in Lemma 59.57.3 that this agrees with the cohomology of the underlying G-module.

04JR Definition 59.57.2. Let G be a topological group. Let M be a discrete G-module with continuous G-action. In other words, M is an object of the category Mod_G introduced in Definition 59.57.1.

(1) The right derived functors $H^i(G, M)$ of $H^0(G, M)$ on the category Mod_G are called the continuous group cohomology groups of M.
(2) If G is an abstract group endowed with the discrete topology then the $H^i(G, M)$ are called the group cohomology groups of M.
(3) If G is a Galois group, then the groups $H^i(G, M)$ are called the Galois cohomology groups of M.
(4) If G is the absolute Galois group of a field K, then the groups $H^i(G, M)$ are sometimes called the Galois cohomology groups of K with coefficients in M. In this case we sometimes write $H^i(K, M)$ instead of $H^i(G, M)$.

0DVD Lemma 59.57.3. Let G be a topological group. Let R be a ring. For every $i \geq 0$ the diagram

$$\begin{tikzcd} \text{Mod}_{R,G} \arrow[r, "H^i(G,-)"'] \arrow[d] & \text{Mod}_R \arrow[d] \\ \text{Mod}_G \arrow[r, "H^i(G,-)"] & \text{Ab} \end{tikzcd}$$

whose vertical arrows are the forgetful functors is commutative.

Proof. Let us denote the forgetful functor $F : \text{Mod}_{R,G} \to \text{Mod}_G$. Then F has a left adjoint $H : \text{Mod}_G \to \text{Mod}_{R,G}$ given by $H(M) = M \otimes_{\mathbf{Z}} R$. Observe that every object of Mod_G is a quotient of a direct sum of modules of the form $\mathbf{Z}[G/U]$ where $U \subset G$ is an open subgroup. Here $\mathbf{Z}[G/U]$ denotes the G-modules of finite $\mathbf{Z}$-linear combinations of right U congruence classes in G endowed with left G-action. Thus every bounded above complex in Mod_G is quasi-isomorphic to a bounded above complex in Mod_G whose underlying terms are flat $\mathbf{Z}$-modules (Derived Categories, Lemma 13.15.4). Thus it is clear that LH exists on $D^-(\text{Mod}_G)$ and is computed by evaluating H on any complex whose terms are flat $\mathbf{Z}$-modules; this follows from Derived Categories, Lemma 13.15.7 and Proposition 13.16.8. We conclude from Derived Categories, Lemma 13.30.2 that

$$\text{Ext}^i(\mathbf{Z}, F(M)) = \text{Ext}^i(R, M)$$

for M in $\text{Mod}_{R,G}$. Observe that $H^0(G, -) = \text{Hom}(\mathbf{Z}, -)$ on Mod_G where $\mathbf{Z}$ denotes the G-module with trivial action. Hence $H^i(G, -) = \text{Ext}^i(\mathbf{Z}, -)$ on Mod_G. Similarly we have $H^i(G, -) = \text{Ext}^i(R, -)$ on $\text{Mod}_{R,G}$. Combining everything we see that the lemma is true. □

0DVE Lemma 59.57.4. Let G be a topological group. Let R be a ring. Let M, N be R-G-modules. If M is finite projective as an R-module, then $\text{Ext}^i(M, N) = H^i(G, M^\vee \otimes_R N)$ (for notation see proof).

Proof. The module $M^\vee = \text{Hom}_R(M, R)$ endowed with the contragredient action of G. Namely $(g \cdot \lambda)(m) = \lambda(g^{-1} \cdot m)$ for $g \in G$, $\lambda \in M^\vee$, $m \in M$. The action of G on $M^\vee \otimes_R N$ is the diagonal one, i.e., given by $g \cdot (\lambda \otimes n) = g \cdot \lambda \otimes g \cdot n$. Note that for a third R-G-module E we have $\text{Hom}(E, M^\vee \otimes_R N) = \text{Hom}(M \otimes_R E, N)$. Namely, this is true on the level of R-modules by Algebra, Lemmas 10.12.8 and 10.78.9 and the definitions of G-actions are chosen such that it remains true for R-G-modules. It follows that $M^\vee \otimes_R N$ is an injective R-G-module if N is an injective R-G-module. Hence if $N \to N^\bullet$ is an injective resolution, then $M^\vee \otimes_R N \to M^\vee \otimes_R N^\bullet$ is an injective resolution. Then

$$\text{Hom}(M, N^\bullet) = \text{Hom}(R, M^\vee \otimes_R N^\bullet) = (M^\vee \otimes_R N^\bullet)^G$$

Since the left hand side computes $\text{Ext}^i(M, N)$ and the right hand side computes $H^i(G, M^\vee \otimes_R N)$ the proof is complete. □

0DVF Lemma 59.57.5. Let G be a topological group. Let k be a field. Let V be a k-G-module. If G is topologically finitely generated and $\dim_k(V) < \infty$, then $\dim_k H^1(G, V) < \infty$.

Proof. Let $g_1, \ldots, g_r \in G$ be elements which topologically generate G, i.e., this means that the subgroup generated by $g_1, \ldots, g_r$ is dense. By Lemma 59.57.4 we see that $H^1(G, V)$ is the k-vector space of extensions

$$0 \to V \to E \to k \to 0$$

of k-G-modules. Choose $e \in E$ mapping to $1 \in k$. Write

$$g_i \cdot e = v_i + e$$

for some $v_i \in V$. This is possible because $g_i \cdot 1 = 1$. We claim that the list of elements $v_1, \ldots, v_r \in V$ determine the isomorphism class of the extension E. Once we prove this the lemma follows as this means that our Ext vector space is isomorphic to a subquotient of the k-vector space $V^{\oplus r}$; some details omitted. Since E is an object of the category defined in Definition 59.57.1 we know there is an open subgroup U such that $u \cdot e = e$ for all $u \in U$. Now pick any $g \in G$. Then gU contains a word w in the elements $g_1, \ldots, g_r$. Say $gu = w$. Since the element $w \cdot e$ is determined by $v_1, \ldots, v_r$, we see that $g \cdot e = (gu) \cdot e = w \cdot e$ is too. □

0DV3 Lemma 59.57.6. Let G be a profinite topological group. Then

(1) $H^i(G, M)$ is torsion for $i > 0$ and any G-module M, and

(2) $H^i(G, M) = 0$ if M is a $\mathbf{Q}$-vector space.

Proof. Proof of (1). By dimension shifting we see that it suffices to show that $H^1(G, M)$ is torsion for every G-module M. Choose an exact sequence $0 \to M \to I \to N \to 0$ with I an injective object of the category of G-modules. Then any element of $H^1(G, M)$ is the image of an element $y \in N^G$. Choose $x \in I$ mapping to y. The stabilizer $U \subset G$ of x is open, hence has finite index r. Let $g_1, \ldots, g_r \in G$ be a system of representatives for G/U. Then $\sum g_i(x)$ is an invariant element of I which maps to ry. Thus r kills the element of $H^1(G, M)$ we started with. Part (2) follows as then $H^i(G, M)$ is both a $\mathbf{Q}$-vector space and torsion. □

59.58. Tate's continuous cohomology

0DVG Tate's continuous cohomology ([Tat76]) is defined by the complex of continuous inhomogeneous cochains. We can define this when M is an arbitrary topological abelian group endowed with a continuous G-action. Namely, we consider the complex

$$C^\bullet_{cont}(G, M) : M \to \text{Maps}_{cont}(G, M) \to \text{Maps}_{cont}(G \times G, M) \to \ldots$$

where the boundary map is defined for $n \geq 1$ by the rule

$$\begin{aligned} \text{d}(f)(g_1, \ldots, g_{n+1}) = g_1(f(g_2, \ldots, g_{n+1})) \\ + \sum\nolimits_{j=1,\ldots,n} (-1)^j f(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n) \end{aligned}$$

and for $n = 0$ sends $m \in M$ to the map $g \mapsto g(m) - m$. We define

$$H^i_{cont}(G, M) = H^i(C^\bullet_{cont}(G, M))$$

Since the terms of the complex involve continuous maps from G and self products of G into the topological module M, it is not clear that this turns a short exact sequence of topological modules into a long exact cohomology sequence. Another difficulty is that the category of topological abelian groups isn't an abelian category!

However, a short exact sequence of discrete G-modules does give rise to a short exact sequence of complexes of continuous cochains and hence a long exact cohomology sequence of continuous cohomology groups $H^i_{cont}(G, -)$. Therefore, on the category Mod_G of Definition 59.57.1 the functors $H^i_{cont}(G, M)$ form a cohomological δ-functor as defined in Homology, Section 12.12. Since the cohomology $H^i(G, M)$ of Definition 59.57.2 is a universal δ-functor (Derived Categories, Lemma 13.16.6) we obtain canonical maps

$$H^i(G, M) \longrightarrow H^i_{cont}(G, M)$$

for $M \in \text{Mod}_G$. It is known that these maps are isomorphisms when G is an abstract group (i.e., G has the discrete topology) or when G is a profinite group (insert future reference here). If you know an example showing this map is not an isomorphism for a topological group G and $M \in \text{Ob}(\text{Mod}_G)$ please email stacks.project@gmail.com.

59.59. Cohomology of a point

03QQ As a consequence of the discussion in the preceding sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

04JQ **Lemma 59.59.1.** *Let $S = \text{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = \text{Gal}_{\kappa(s)}$ denote the absolute Galois group. The stalk functor induces an equivalence of categories*

$$\text{Ab}(S_{\acute{e}tale}) \longrightarrow \text{Mod}_G, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. In Theorem 59.56.3 we have seen the equivalence between sheaves of sets and G-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a G-module is just a G-set endowed with a commutative group law. □

03QU **Lemma 59.59.2.** *Notation and assumptions as in Lemma 59.59.1. Let $\mathcal{F}$ be an abelian sheaf on $\text{Spec}(K)_{\acute{e}tale}$ which corresponds to the G-module M. Then*

(1) *in $D(\text{Ab})$ we have a canonical isomorphism $R\Gamma(S, \mathcal{F}) = R\Gamma_G(M)$,*
(2) *$H^0_{\acute{e}tale}(S, \mathcal{F}) = M^G$, and*
(3) *$H^q_{\acute{e}tale}(S, \mathcal{F}) = H^q(G, M)$.*

Proof. Combine Lemma 59.59.1 with Lemma 59.56.5. □

03QV **Example 59.59.3.** Sheaves on $\text{Spec}(K)_{\acute{e}tale}$. Let $G = \text{Gal}(K^{sep}/K)$ be the absolute Galois group of K.

(1) The constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ corresponds to the module $\mathbf{Z}/n\mathbf{Z}$ with trivial G-action,
(2) the sheaf $\mathbf{G}_m|_{\text{Spec}(K)_{\acute{e}tale}}$ corresponds to $(K^{sep})^*$ with its G-action,
(3) the sheaf $\mathbf{G}_a|_{\text{Spec}(K^{sep})}$ corresponds to $(K^{sep}, +)$ with its G-action, and
(4) the sheaf $\mu_n|_{\text{Spec}(K^{sep})}$ corresponds to $\mu_n(K^{sep})$ with its G-action.

By Remark 59.23.4 and Theorem 59.24.1 we have the following identifications for cohomology groups:

$$\begin{aligned} H^0_{\acute{e}tale}(S_{\acute{e}tale}, \mathbf{G}_m) &= \Gamma(S, \mathcal{O}_S^*) \\ H^1_{\acute{e}tale}(S_{\acute{e}tale}, \mathbf{G}_m) &= H^1_{Zar}(S, \mathcal{O}_S^*) = \text{Pic}(S) \\ H^i_{\acute{e}tale}(S_{\acute{e}tale}, \mathbf{G}_a) &= H^i_{Zar}(S, \mathcal{O}_S) \end{aligned}$$

Also, for any quasi-coherent sheaf $\mathcal{F}$ on $S_{\acute{e}tale}$ we have

$$H^i(S_{\acute{e}tale}, \mathcal{F}) = H^i_{Zar}(S, \mathcal{F}),$$

see Theorem 59.22.4. In particular, this gives the following sequence of equalities

$$0 = \text{Pic}(\text{Spec}(K)) = H^1_{\acute{e}tale}(\text{Spec}(K)_{\acute{e}tale}, \mathbf{G}_m) = H^1(G, (K^{sep})^*)$$

which is none other than Hilbert's 90 theorem. Similarly, for $i \geq 1$,

$$0 = H^i(\text{Spec}(K), \mathcal{O}) = H^i_{\acute{e}tale}(\text{Spec}(K)_{\acute{e}tale}, \mathbf{G}_a) = H^i(G, K^{sep})$$

where the K^{sep} indicates K^{sep} as a Galois module with addition as group law. In this way we may consider the work we have done so far as a complicated way of computing Galois cohomology groups.

The following result is a curiosity and should be skipped on a first reading.

0D1W Lemma 59.59.4. Let R be a local ring of dimension 0. Let $S = \mathrm{Spec}(R)$. Then every $\mathcal{O}_S$-module on $S_{\acute{e}tale}$ is quasi-coherent.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module on $S_{\acute{e}tale}$. We have to show that $\mathcal{F}$ is determined by the R-module $M = \Gamma(S, \mathcal{F})$. More precisely, if $\pi : X \to S$ is étale we have to show that $\Gamma(X, \mathcal{F}) = \Gamma(X, \pi^*\widetilde{M})$.

Let $\mathfrak{m} \subset R$ be the maximal ideal and let κ be the residue field. By Algebra, Lemma 10.153.10 the local ring R is henselian. If $X \to S$ is étale, then the underlying topological space of X is discrete by Morphisms, Lemma 29.36.7 and hence X is a disjoint union of affine schemes each having one point. Moreover, if $X = \mathrm{Spec}(A)$ is affine and has one point, then $R \to A$ is finite étale by Algebra, Lemma 10.153.5. We have to show that $\Gamma(X, \mathcal{F}) = M \otimes_R A$ in this case.

The functor $A \mapsto A/\mathfrak{m}A$ defines an equivalence of the category of finite étale R-algebras with the category of finite separable κ-algebras by Algebra, Lemma 10.153.7. Let us first consider the case where $A/\mathfrak{m}A$ is a Galois extension of κ with Galois group G. For each $\sigma \in G$ let $\sigma : A \to A$ denote the corresponding automorphism of A over R. Let $N = \Gamma(X, \mathcal{F})$. Then $\mathrm{Spec}(\sigma) : X \to X$ is an automorphism over S and hence pullback by this defines a map $\sigma : N \to N$ which is a σ-linear map: $\sigma(an) = \sigma(a)\sigma(n)$ for $a \in A$ and $n \in N$. We will apply Galois descent to the quasi-coherent module $\widetilde{N}$ on X endowed with the isomorphisms coming from the action on σ on N. See Descent, Lemma 35.6.2. This lemma tells us there is an isomorphism $N = N^G \otimes_R A$. On the other hand, it is clear that $N^G = M$ by the sheaf property for $\mathcal{F}$. Thus the required isomorphism holds.

The general case (with A local and finite étale over R) is deduced from the Galois case as follows. Choose $A \to B$ finite étale such that B is local with residue field Galois over κ. Let $G = \mathrm{Aut}(B/R) = \mathrm{Gal}(\kappa_B/\kappa)$. Let $H \subset G$ be the Galois group corresponding to the Galois extension κ_B/κ_A. Then as above one shows that $\Gamma(X, \mathcal{F}) = \Gamma(\mathrm{Spec}(B), \mathcal{F})^H$. By the result for Galois extensions (used twice) we get

$$\Gamma(X, \mathcal{F}) = (M \otimes_R B)^H = M \otimes_R A$$

as desired. $\square$

59.60. Cohomology of curves

03R0 The next task at hand is to compute the étale cohomology of a smooth curve over an algebraically closed field with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology.

59.61. Brauer groups

03R1 Brauer groups of fields are defined using finite central simple algebras. In this section we review the relevant facts about Brauer groups, most of which are discussed in the chapter Brauer Groups, Section 11.1. For other references, see [Ser62], [Ser97] or [Wei48].

03R2 Theorem 59.61.1. Let K be a field. For a unital, associative (not necessarily commutative) K-algebra A the following are equivalent

(1) A is finite central simple K-algebra,

(2) A is a finite dimensional K-vector space, K is the center of A, and A has no nontrivial two-sided ideal,
(3) there exists $d \geq 1$ such that $A \otimes_K \bar{K} \cong \mathrm{Mat}(d \times d, \bar{K})$,
(4) there exists $d \geq 1$ such that $A \otimes_K K^{sep} \cong \mathrm{Mat}(d \times d, K^{sep})$,
(5) there exist $d \geq 1$ and a finite Galois extension K'/K such that $A \otimes_K K' \cong \mathrm{Mat}(d \times d, K')$,
(6) there exist $n \geq 1$ and a finite central skew field D over K such that $A \cong \mathrm{Mat}(n \times n, D)$.

The integer d is called the degree of A.

Proof. This is a copy of Brauer Groups, Lemma 11.8.6. □

03R4 Lemma 59.61.2. Let A be a finite central simple algebra over K. Then

$$\begin{array}{ccc} A \otimes_K A^{opp} & \longrightarrow & \mathrm{End}_K(A) \\ a \otimes a' & \longmapsto & (x \mapsto axa') \end{array}$$

is an isomorphism of algebras over K.

Proof. See Brauer Groups, Lemma 11.4.10. □

03R3 Definition 59.61.3. Two finite central simple algebras A_1 and A_2 over K are called similar, or equivalent if there exist $m, n \geq 1$ such that $\mathrm{Mat}(n \times n, A_1) \cong \mathrm{Mat}(m \times m, A_2)$. We write $A_1 \sim A_2$.

By Brauer Groups, Lemma 11.5.1 this is an equivalence relation.

03R5 Definition 59.61.4. Let K be a field. The Brauer group of K is the set $\mathrm{Br}(K)$ of similarity classes of finite central simple algebras over K, endowed with the group law induced by tensor product (over K). The class of A in $\mathrm{Br}(K)$ is denoted by $[A]$. The neutral element is $[K] = [\mathrm{Mat}(d \times d, K)]$ for any $d \geq 1$.

The previous lemma implies that inverses exist and that $-[A] = [A^{opp}]$. The Brauer group of a field is always torsion. In fact, we will see that $[A]$ has order dividing $\deg(A)$ for any finite central simple algebra A (see Lemma 59.62.2). In general the Brauer group is not finitely generated, for example the Brauer group of a non-Archimedean local field is $\mathbf{Q}/\mathbf{Z}$. The Brauer group of $\mathbf{C}(x, y)$ is uncountable.

03R6 Lemma 59.61.5. Let K be a field and let K^{sep} be a separable algebraic closure. Then the set of isomorphism classes of central simple algebras of degree d over K is in bijection with the non-abelian cohomology $H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep}))$.

Sketch of proof. The Skolem-Noether theorem (see Brauer Groups, Theorem 11.6.1) implies that for any field L the group $\mathrm{Aut}_{L\text{-Algebras}}(\mathrm{Mat}_d(L))$ equals $\mathrm{PGL}_d(L)$. By Theorem 59.61.1, we see that central simple algebras of degree d correspond to forms of the K-algebra $\mathrm{Mat}_d(K)$. Combined we see that isomorphism classes of degree d central simple algebras correspond to elements of $H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep}))$. For more details on twisting, see for example [Sil86]. □

If A is a finite central simple algebra of degree d over a field K, we denote ξ_A the corresponding cohomology class in $H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep}))$. Consider the short exact sequence

$$1 \to (K^{sep})^* \to \mathrm{GL}_d(K^{sep}) \to \mathrm{PGL}_d(K^{sep}) \to 1,$$

which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map

$$\delta_d : H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep})) \longrightarrow H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*).$$

Explicitly, this is given as follows: if ξ is a cohomology class represented by the 1-cocycle (g_σ), then $\delta_d(\xi)$ is the class of the 2-cocycle

0A2I (59.61.5.1)
$$(\sigma, \tau) \longmapsto \tilde{g}_\sigma^{-1}\tilde{g}_{\sigma\tau}\sigma(\tilde{g}_\tau^{-1}) \in (K^{sep})^*$$

where $\tilde{g}_\sigma \in \mathrm{GL}_d(K^{sep})$ is a lift of g_σ. Using this we can make explicit the map

$$\delta : \mathrm{Br}(K) \longrightarrow H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*), \quad [A] \longmapsto \delta_{\deg A}(\xi_A)$$

as follows. Assume A has degree d over K. Choose an isomorphism $\varphi : \mathrm{Mat}_d(K^{sep}) \to A \otimes_K K^{sep}$. For $\sigma \in \mathrm{Gal}(K^{sep}/K)$ choose an element $\tilde{g}_\sigma \in \mathrm{GL}_d(K^{sep})$ such that $\varphi^{-1} \circ \sigma(\varphi)$ is equal to the map $x \mapsto \tilde{g}_\sigma x \tilde{g}_\sigma^{-1}$. The class in H^2 is defined by the two cocycle (59.61.5.1).

03R7 Theorem 59.61.6. Let K be a field with separable algebraic closure K^{sep}. The map $\delta : \mathrm{Br}(K) \to H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$ defined above is a group isomorphism.

Sketch of proof. To prove that δ defines a group homomorphism, i.e., that $\delta(A \otimes_K B) = \delta(A) + \delta(B)$, one computes directly with cocycles.

Injectivity of δ. In the abelian case ($d = 1$), one has the identification

$$H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{GL}_d(K^{sep})) = H^1_{\acute{e}tale}(\mathrm{Spec}(K), \mathrm{GL}_d(\mathcal{O}))$$

the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of δ. (See [Del77].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a K-vector space V with a left A-module structure and such that $\dim_K V = \deg A$. In the case where V exists, one has $A \cong \mathrm{End}_K(V)$.

For surjectivity, pick a cohomology class $\xi \in H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$, then there exists a finite Galois extension $K^{sep}/K'/K$ such that ξ is the image of some $\xi' \in H^2(\mathrm{Gal}(K'|K), (K')^*)$. Then write down an explicit central simple algebra over K using the data K', ξ'. □

59.62. The Brauer group of a scheme

0A2J Let S be a scheme. An $\mathcal{O}_S$-algebra $\mathcal{A}$ is called Azumaya if it is étale locally a matrix algebra, i.e., if there exists an étale covering $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ such that $\varphi_i^*\mathcal{A} \cong \mathrm{Mat}_{d_i}(\mathcal{O}_{U_i})$ for some $d_i \geq 1$. Two such $\mathcal{A}$ and $\mathcal{B}$ are called equivalent if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{G}$ which have positive rank at every $s \in S$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$$

as $\mathcal{O}_S$-algebras. The Brauer group of S is the set $\mathrm{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

0A2K Lemma 59.62.1. Let S be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be finite locally free sheaves of $\mathcal{O}_S$-modules of positive rank. If there exists an isomorphism $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$ of $\mathcal{O}_S$-algebras, then there exists an invertible sheaf $\mathcal{L}$ on S such that $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \cong \mathcal{G}$ and such that this isomorphism induces the given isomorphism of endomorphism algebras.

Proof. Fix an isomorphism $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$. Consider the sheaf $\mathcal{L} \subset \mathcal{H}om(\mathcal{F}, \mathcal{G})$ generated as an $\mathcal{O}_S$-module by the local isomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$ such that conjugation by φ is the given isomorphism of endomorphism algebras. A local calculation (reducing to the case that $\mathcal{F}$ and $\mathcal{G}$ are finite free and S is affine) shows that $\mathcal{L}$ is invertible. Another local calculation shows that the evaluation map

$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \longrightarrow \mathcal{G}$$

is an isomorphism. □

The argument given in the proof of the following lemma can be found in [Sal81].

0A2L Lemma 59.62.2. Let S be a scheme. Let $\mathcal{A}$ be an Azumaya algebra which is locally free of rank d^2 over S. Then the class of $\mathcal{A}$ in the Brauer group of S is annihilated by d.

Argument taken from [Sal81].

Proof. Choose an étale covering $\{U_i \to S\}$ and choose isomorphisms $\mathcal{A}|_{U_i} \to \mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)$ for some locally free $\mathcal{O}_{U_i}$-modules $\mathcal{F}_i$ of rank d. (We may assume $\mathcal{F}_i$ is free.) Consider the composition

$$p_i : \mathcal{F}_i^{\otimes d} \to \wedge^d(\mathcal{F}_i) \to \mathcal{F}_i^{\otimes d}$$

The first arrow is the usual projection and the second arrow is the isomorphism of the top exterior power of $\mathcal{F}_i$ with the submodule of sections of $\mathcal{F}_i^{\otimes d}$ which transform according to the sign character under the action of the symmetric group on d letters. Then $p_i^2 = d!p_i$ and the rank of p_i is 1. Using the given isomorphism $\mathcal{A}|_{U_i} \to \mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)$ and the canonical isomorphism

$$\mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)^{\otimes d} = \mathcal{H}om(\mathcal{F}_i^{\otimes d}, \mathcal{F}_i^{\otimes d})$$

we may think of p_i as a section of $\mathcal{A}^{\otimes d}$ over U_i. We claim that $p_i|_{U_i \times_S U_j} = p_j|_{U_i \times_S U_j}$ as sections of $\mathcal{A}^{\otimes d}$. Namely, applying Lemma 59.62.1 we obtain an invertible sheaf $\mathcal{L}_{ij}$ and a canonical isomorphism

$$\mathcal{F}_i|_{U_i \times_S U_j} \otimes \mathcal{L}_{ij} \longrightarrow \mathcal{F}_j|_{U_i \times_S U_j}.$$

Using this isomorphism we see that p_i maps to p_j. Since $\mathcal{A}^{\otimes d}$ is a sheaf on $S_{\acute{e}tale}$ (Proposition 59.17.1) we find a canonical global section $p \in \Gamma(S, \mathcal{A}^{\otimes d})$. A local calculation shows that

$$\mathcal{H} = \mathrm{Im}(\mathcal{A}^{\otimes d} \to \mathcal{A}^{\otimes d}, f \mapsto fp)$$

is a locally free module of rank d^d and that (left) multiplication by $\mathcal{A}^{\otimes d}$ induces an isomorphism $\mathcal{A}^{\otimes d} \to \mathcal{H}om(\mathcal{H}, \mathcal{H})$. In other words, $\mathcal{A}^{\otimes d}$ is the trivial element of the Brauer group of S as desired. □

In this setting, the analogue of the isomorphism δ of Theorem 59.61.6 is a map

$$\delta_S : \mathrm{Br}(S) \to H^2_{\acute{e}tale}(S, \mathbf{G}_m).$$

It is true that δ_S is injective. If S is quasi-compact or connected, then $\mathrm{Br}(S)$ is a torsion group, so in this case the image of δ_S is contained in the cohomological Brauer group of S

$$\mathrm{Br}'(S) := H^2_{\acute{e}tale}(S, \mathbf{G}_m)_{\text{torsion}}.$$

So if S is quasi-compact or connected, there is an inclusion $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$. This is not always an equality: there exists a nonseparated singular surface S for which $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$ is a strict inclusion. If S is quasi-projective, then $\mathrm{Br}(S) = \mathrm{Br}'(S)$.

However, it is not known whether this holds for a smooth proper variety over $\mathbf{C}$, say.

59.63. The Artin-Schreier sequence

0A3J Let p be a prime number. Let S be a scheme in characteristic p. The Artin-Schreier sequence is the short exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{F-1} \mathbf{G}_{a,S} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$.

0A3K **Lemma 59.63.1.** Let p be a prime. Let S be a scheme of characteristic p.

(1) If S is affine, then $H^q_{\acute{e}tale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2$.
(2) If S is a quasi-compact and quasi-separated scheme of dimension d, then $H^q_{\acute{e}tale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2 + d$.

Proof. Recall that the étale cohomology of the structure sheaf is equal to its cohomology on the underlying topological space (Theorem 59.22.4). The first statement follows from the Artin-Schreier exact sequence and the vanishing of cohomology of the structure sheaf on an affine scheme (Cohomology of Schemes, Lemma 30.2.2). The second statement follows by the same argument from the vanishing of Cohomology, Proposition 20.22.4 and the fact that S is a spectral space (Properties, Lemma 28.2.4). ☐

0A3L **Lemma 59.63.2.** Let k be an algebraically closed field of characteristic $p > 0$. Let V be a finite dimensional k-vector space. Let $F : V \to V$ be a frobenius linear map, i.e., an additive map such that $F(\lambda v) = \lambda^p F(v)$ for all $\lambda \in k$ and $v \in V$. Then $F - 1 : V \to V$ is surjective with kernel a finite dimensional $\mathbf{F}_p$-vector space of dimension $\leq \dim_k(V)$.

Proof. If $F = 0$, then the statement holds. If we have a filtration of V by F-stable subvector spaces such that the statement holds for each graded piece, then it holds for (V, F). Combining these two remarks we may assume the kernel of F is zero.

Choose a basis $v_1, \ldots, v_n$ of V and write $F(v_i) = \sum a_{ij} v_j$. Observe that $v = \sum \lambda_i v_i$ is in the kernel if and only if $\sum \lambda_i^p a_{ij} v_j = 0$. Since k is algebraically closed this implies the matrix (a_{ij}) is invertible. Let (b_{ij}) be its inverse. Then to see that $F - 1$ is surjective we pick $w = \sum \mu_i v_i \in V$ and we try to solve

$$(F-1)(\sum \lambda_i v_i) = \sum \lambda_i^p a_{ij} v_j - \sum \lambda_j v_j = \sum \mu_j v_j$$

This is equivalent to

$$\sum \lambda_j^p v_j - \sum b_{ij} \lambda_i v_j = \sum b_{ij} \mu_i v_j$$

in other words

$$\lambda_j^p - \sum b_{ij} \lambda_i = \sum b_{ij} \mu_i, \quad j = 1, \ldots, \dim(V).$$

The algebra

$$A = k[x_1, \ldots, x_n]/(x_j^p - \sum b_{ij} x_i - \sum b_{ij} \mu_i)$$

is standard smooth over k (Algebra, Definition 10.137.6) because the matrix (b_{ij}) is invertible and the partial derivatives of x_j^p are zero. A basis of A over k is the set of monomials $x_1^{e_1} \ldots x_n^{e_n}$ with $e_i < p$, hence $\dim_k(A) = p^n$. Since k is algebraically closed we see that $\mathrm{Spec}(A)$ has exactly p^n points. It follows that $F-1$ is surjective and every fibre has p^n points, i.e., the kernel of $F-1$ is a group with p^n elements. □

0A3M Lemma 59.63.3. Let X be a separated scheme of finite type over a field k. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then $\dim_k H^d(X, \mathcal{F}) < \infty$ where $d = \dim(X)$.

Proof. We will prove this by induction on d. The case $d = 0$ holds because in that case X is the spectrum of a finite dimensional k-algebra A (Varieties, Lemma 33.20.2) and every coherent sheaf $\mathcal{F}$ corresponds to a finite A-module $M = H^0(X, \mathcal{F})$ which has $\dim_k M < \infty$.

Assume $d > 0$ and the result has been shown for separated schemes of finite type of dimension $< d$. The scheme X is Noetherian. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow \dim_k H^d(X, \mathcal{F}) < \infty$$

We are going to use the result of Cohomology of Schemes, Lemma 30.12.4 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of cohomology

$$H^d(X, \mathcal{F}_1) \to H^d(X, \mathcal{F}) \to H^d(X, \mathcal{F}_2)$$

Thus if $\mathcal{P}$ holds for $\mathcal{F}_1$ and $\mathcal{F}_2$, then it holds for $\mathcal{F}$.

Let $Z \subset X$ be an integral closed subscheme. Let $\mathcal{I}$ be a coherent sheaf of ideals on Z. To finish the proof we have to show that $H^d(X, i_*\mathcal{I}) = H^d(Z, \mathcal{I})$ is finite dimensional. If $\dim(Z) < d$, then the result holds because the cohomology group will be zero (Cohomology, Proposition 20.20.7). In this way we reduce to the situation discussed in the following paragraph.

Assume X is a variety of dimension d and $\mathcal{F} = \mathcal{I}$ is a coherent ideal sheaf. In this case we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

where $i : Z \to X$ is the closed subscheme defined by $\mathcal{I}$. By induction hypothesis we see that $H^{d-1}(Z, \mathcal{O}_Z) = H^{d-1}(X, i_*\mathcal{O}_Z)$ is finite dimensional. Thus we see that it suffices to prove the result for the structure sheaf.

We can apply Chow's lemma (Cohomology of Schemes, Lemma 30.18.1) to the morphism $X \to \mathrm{Spec}(k)$. Thus we get a diagram

$$\begin{tikzcd} X \arrow[dr, "g"'] & X' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^n_k \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

as in the statement of Chow's lemma. Also, let $U \subset X$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. We may assume X' is a variety as well, see Cohomology of Schemes, Remark 30.18.2. The morphism $i' = (i, \pi) : X' \to \mathbf{P}^n_X$ is a closed immersion (loc. cit.). Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_k}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^n_X}(1)$$

is π-relatively ample (for example by Morphisms, Lemma 29.39.7). Hence by Cohomology of Schemes, Lemma 30.16.2 there exists an $n \geq 0$ such that $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Choose any nonzero global section s of $\mathcal{L}^{\otimes n}$. Since $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$, the section s corresponds to section of $\mathcal{G}$, i.e., a map $\mathcal{O}_X \to \mathcal{G}$. Since $s|_U \neq 0$ as X' is a variety and $\mathcal{L}$ invertible, we see that $\mathcal{O}_X|_U \to \mathcal{G}|_U$ is nonzero. As $\mathcal{G}|_U = \mathcal{L}^{\otimes n}|_{\pi^{-1}(U)}$ is invertible we conclude that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{G} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is coherent and supported on a proper closed subscheme of X. Arguing as before using our induction hypothesis, we see that it suffices to prove $\dim H^d(X, \mathcal{G}) < \infty$.

By the Leray spectral sequence (Cohomology, Lemma 20.13.6) we see that $H^d(X, \mathcal{G}) = H^d(X', \mathcal{L}^{\otimes n})$. Let $\overline{X}' \subset \mathbf{P}^n_k$ be the closure of X'. Then $\overline{X}'$ is a projective variety of dimension d over k and $X' \subset \overline{X}'$ is a dense open. The invertible sheaf $\mathcal{L}$ is the restriction of $\mathcal{O}_{\overline{X}'}(n)$ to X. By Cohomology, Proposition 20.22.4 the map

$$H^d(\overline{X}', \mathcal{O}_{\overline{X}'}(n)) \longrightarrow H^d(X', \mathcal{L}^{\otimes n})$$

is surjective. Since the cohomology group on the left has finite dimension by Cohomology of Schemes, Lemma 30.14.1 the proof is complete. □

0A3N Lemma 59.63.4. Let X be separated of finite type over an algebraically closed field k of characteristic $p > 0$. Then $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for $q \geq dim(X) + 1$.

Proof. Let $d = \dim(X)$. By the vanishing established in Lemma 59.63.1 it suffices to show that $H^{d+1}_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$. By Lemma 59.63.3 we see that $H^d(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. Hence the long exact cohomology sequence associated to the Artin-Schreier sequence ends with

$$H^d(X, \mathcal{O}_X) \xrightarrow{F-1} H^d(X, \mathcal{O}_X) \to H^{d+1}_{étale}(X, \mathbf{Z}/p\mathbf{Z}) \to 0$$

By Lemma 59.63.2 the map $F - 1$ in this sequence is surjective. This proves the lemma. □

0A3P Lemma 59.63.5. Let X be a proper scheme over an algebraically closed field k of characteristic $p > 0$. Then

(1) $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is a finite $\mathbf{Z}/p\mathbf{Z}$-module for all q, and
(2) $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) \to H^q_{étale}(X_{k'}, \underline{\mathbf{Z}/p\mathbf{Z}}))$ is an isomorphism if k'/k is an extension of algebraically closed fields.

Proof. By Cohomology of Schemes, Lemma 30.19.2) and the comparison of cohomology of Theorem 59.22.4 the cohomology groups $H^q_{étale}(X, \mathbf{G}_a) = H^q(X, \mathcal{O}_X)$ are finite dimensional k-vector spaces. Hence by Lemma 59.63.2 the long exact cohomology sequence associated to the Artin-Schreier sequence, splits into short exact sequences

$$0 \to H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) \to H^q(X, \mathcal{O}_X) \xrightarrow{F-1} H^q(X, \mathcal{O}_X) \to 0$$

and moreover the $\mathbf{F}_p$-dimension of the cohomology groups $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is equal to the k-dimension of the vector space $H^q(X, \mathcal{O}_X)$. This proves the first statement. The second statement follows as $H^q(X, \mathcal{O}_X) \otimes_k k' \to H^q(X_{k'}, \mathcal{O}_{X_{k'}})$ is an isomorphism by flat base change (Cohomology of Schemes, Lemma 30.5.2). □

59.64. Locally constant sheaves

09Y8 This section is the analogue of Modules on Sites, Section 18.43 for the étale site.

03RU **Definition 59.64.1.** Let X be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$.

(1) Let E be a set. We say $\mathcal{F}$ is the *constant sheaf with value E* if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto E$. Notation: $\underline{E}_X$ or $\underline{E}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the values are finite sets.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{étale}$.

(1) Let A be an abelian group. We say $\mathcal{F}$ is the *constant sheaf with value A* if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto A$. Notation: $\underline{A}_X$ or $\underline{A}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as an abelian sheaf to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the values are finite abelian groups.

Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{étale}$.

(1) Let M be a Λ-module. We say $\mathcal{F}$ is the *constant sheaf with value M* if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto M$. Notation: $\underline{M}_X$ or $\underline{M}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as a sheaf of Λ-modules to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

095A **Lemma 59.64.2.** Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally constant sheaf of sets, abelian groups, or Λ-modules on $Y_{étale}$, the same is true for $f^{-1}\mathcal{G}$ on $X_{étale}$.

Proof. Holds for any morphism of topoi, see Modules on Sites, Lemma 18.43.2. $\square$

095B **Lemma 59.64.3.** Let $f : X \to Y$ be a finite étale morphism of schemes. If $\mathcal{F}$ is a (finite) locally constant sheaf of sets, (finite) locally constant sheaf of abelian groups, or (finite type) locally constant sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.

Proof. The construction of f_* commutes with étale localization. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. Thus the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are (finite) locally constant sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. $\square$

03RV **Lemma 59.64.4.** Let X be a scheme and $\mathcal{F}$ a sheaf of sets on $X_{étale}$. Then the following are equivalent

(1) $\mathcal{F}$ is finite locally constant, and
(2) $\mathcal{F} = h_U$ for some finite étale morphism $U \to X$.

Proof. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. Thus (2) implies (1). Conversely, if $\mathcal{F}$ is finite locally constant, then there exists an étale covering $\{X_i \to X\}$ such that $\mathcal{F}|_{X_i}$ is representable by $U_i \to X_i$ finite étale. Arguing exactly as in the proof of Descent, Lemma 35.39.1 we obtain a descent datum for schemes (U_i, φ_{ij}) relative to $\{X_i \to X\}$ (details omitted). This descent datum is effective for example by Descent, Lemma 35.37.1 and the resulting morphism of schemes $U \to X$ is finite étale by Descent, Lemmas 35.23.23 and 35.23.29. □

095C Lemma 59.64.5. Let X be a scheme.

(1) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.
(2) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.
(3) Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $X_{étale}$. If $\mathcal{F}$ is of finite type, then there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.

Proof. This holds on any site, see Modules on Sites, Lemma 18.43.3. □

03RX Lemma 59.64.6. Let X be a scheme.

(1) The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.
(2) The category of finite locally constant abelian sheaves is a weak Serre subcategory of $\textit{Ab}(X_{étale})$.
(3) Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $\textit{Mod}(X_{étale}, \Lambda)$.

Proof. This holds on any site, see Modules on Sites, Lemma 18.43.5. □

095D Lemma 59.64.7. Let X be a scheme. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $X_{étale}$ is a locally constant sheaf of Λ-modules.

Proof. This holds on any site, see Modules on Sites, Lemma 18.43.6. □

09BF Lemma 59.64.8. Let X be a connected scheme. Let Λ be a ring and let $\mathcal{F}$ be a locally constant sheaf of Λ-modules. Then there exists a Λ-module M and an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i} \cong \underline{M}|_{U_i}$.

Proof. Choose an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As étale morphisms are open we see that $U_M = \bigcup_{i \in I_M} \text{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. As X is connected only one U_M is nonempty and the lemma follows. □

59.65. Locally constant sheaves and the fundamental group

0DV4 We can relate locally constant sheaves to the fundamental group of a scheme in some cases.

0DV5 **Lemma 59.65.1.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

1. *There is an equivalence of categories*
$$\left\{\begin{matrix}\text{finite locally constant}\\ \text{sheaves of sets on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\text{finite } \pi_1(X, \overline{x})\text{-sets}\right\}$$
2. *There is an equivalence of categories*
$$\left\{\begin{matrix}\text{finite locally constant}\\ \text{sheaves of abelian groups on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\text{finite } \pi_1(X, \overline{x})\text{-modules}\right\}$$
3. *Let Λ be a finite ring. There is an equivalence of categories*
$$\left\{\begin{matrix}\text{finite type, locally constant}\\ \text{sheaves of } \Lambda\text{-modules on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite } \pi_1(X, \overline{x})\text{-modules endowed}\\ \text{with commuting } \Lambda\text{-module structure}\end{matrix}\right\}$$

Proof. We observe that $\pi_1(X, \overline{x})$ is a profinite topological group, see Fundamental Groups, Definition 58.6.1. The left hand categories are defined in Section 59.64. The notation used in the right hand categories is taken from Fundamental Groups, Definition 58.2.1 for sets and Definition 59.57.1 for abelian groups. This explains the notation.

Assertion (1) follows from Lemma 59.64.4 and Fundamental Groups, Theorem 58.6.2. Parts (2) and (3) follow immediately from this by endowing the underlying (sheaves of) sets with additional structure. For example, a finite locally constant sheaf of abelian groups on $X_{\acute{e}tale}$ is the same thing as a finite locally constant sheaf of sets $\mathcal{F}$ together with a map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ satisfying the usual axioms. The equivalence in (1) sends products to products and hence sends $+$ to an addition on the corresponding finite $\pi_1(X, \overline{x})$-set. Since $\pi_1(X, \overline{x})$-modules are the same thing as $\pi_1(X, \overline{x})$-sets with a compatible abelian group structure we obtain (2). Part (3) is proved in exactly the same way. $\square$

0GIY **Lemma 59.65.2.** *Let X be an irreducible, geometrically unibranch scheme. Let $\overline{x}$ be a geometric point of X. Let Λ be a ring. There is an equivalence of categories*
$$\left\{\begin{matrix}\text{finite type, locally constant}\\ \text{sheaves of } \Lambda\text{-modules on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite } \Lambda\text{-modules } M \text{ endowed}\\ \text{with a continuous } \pi_1(X, \overline{x})\text{-action}\end{matrix}\right\}$$

Proof. The proof given in Lemma 59.65.1 does not work as a finite Λ-module M may not have a finite underlying set.

Let $\nu : X^\nu \to X$ be the normalization morphism. By Morphisms, Lemma 29.54.11 this is a universal homeomorphism. By Fundamental Groups, Proposition 58.8.4 this induces an isomorphism $\pi_1(X^\nu, \overline{x}) \to \pi_1(X, \overline{x})$ and by Theorem 59.45.2 we get an equivalence of category between finite type, locally constant Λ-modules on $X_{\acute{e}tale}$ and on $X^\nu_{\acute{e}tale}$. This reduces us to the case where X is an integral normal scheme.

Assume X is an integral normal scheme. Let $\eta \in X$ be the generic point. Let $\overline{\eta}$ be a geometric point lying over η. By Fundamental Groups, Proposition 58.11.3 have a continuous surjection
$$\text{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

whose kernel is described in Fundamental Groups, Lemma 58.13.2. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $X_{étale}$. Let $M = \mathcal{F}_{\overline{\eta}}$ be the stalk of $\mathcal{F}$ at $\overline{\eta}$. We obtain a continuous action of $\text{Gal}(\kappa(\eta)^{sep}/\kappa(\eta))$ on M by Section 59.56. Our goal is to show that this action factors through the displayed surjection. Since $\mathcal{F}$ is of finite type, M is a finite Λ-module. Since $\mathcal{F}$ is locally constant, for every $x \in X$ the restriction of $\mathcal{F}$ to $\text{Spec}(\mathcal{O}_{X,x}^{sh})$ is constant. Hence the action of $\text{Gal}(K^{sep}/K_x^{sh})$ (with notation as in Fundamental Groups, Lemma 58.13.2) on M is trivial. We conclude we have the factorization as desired.

On the other hand, suppose we have a finite Λ-module M with a continuous action of $\pi_1(X, \overline{\eta})$. We are going to construct an $\mathcal{F}$ such that $M \cong \mathcal{F}_{\overline{\eta}}$ as $\Lambda[\pi_1(X, \overline{\eta})]$-modules. Choose generators $m_1, \ldots, m_r \in M$. Since the action of $\pi_1(X, \overline{\eta})$ on M is continuous, for each i there exists an open subgroup N_i of the profinite group $\pi_1(X, \overline{\eta})$ such that every $\gamma \in H_i$ fixes m_i. We conclude that every element of the open subgroup $H = \bigcap_{i=1,\ldots,r} H_i$ fixes every element of M. After shrinking H we may assume H is an open normal subgroup of $\pi_1(X, \overline{\eta})$. Set $G = \pi_1(X, \overline{\eta})/H$. Let $f : Y \to X$ be the corresponding Galois finite étale G-cover. We can view $f_*\underline{\mathbf{Z}}$ as a sheaf of $\mathbf{Z}[G]$-modules on $X_{étale}$. Then we just take

$$\mathcal{F} = f_*\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}[G]}} \underline{M}$$

We leave it to the reader to compute $\mathcal{F}_{\overline{\eta}}$. We also omit the verification that this construction is the inverse to the construction in the previous paragraph. $\square$

0DV6 Remark 59.65.3. The equivalences of Lemmas 59.65.1 and 59.65.2 are compatible with pullbacks. For example, suppose $f : Y \to X$ is a morphism of connected schemes. Let $\overline{y}$ be geometric point of Y and set $\overline{x} = f(\overline{y})$. Then the diagram

$$\begin{tikzcd} \text{finite locally constant sheaves of sets on } Y_{étale} \arrow[r] & \text{finite } \pi_1(Y, \overline{y})\text{-sets} \\ \text{finite locally constant sheaves of sets on } X_{étale} \arrow[r] \arrow[u, "f^{-1}"] & \text{finite } \pi_1(X, \overline{x})\text{-sets} \arrow[u] \end{tikzcd}$$

is commutative, where the vertical arrow on the right comes from the continuous homomorphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$ induced by f. This follows immediately from the commutative diagram in Fundamental Groups, Theorem 58.6.2. A similar result holds for the other cases.

59.66. Méthode de la trace

03SH A reference for this section is [AGV71, Exposé IX, §5]. The material here will be used in the proof of Lemma 59.83.9 below.

Let $f : Y \to X$ be an étale morphism of schemes. There is a sequence

$$f_!, f^{-1}, f_*$$

of adjoint functors between $\text{Ab}(X_{étale})$ and $\text{Ab}(Y_{étale})$. The functor $f_!$ is discussed in Section 59.70. The adjunction map $\text{id} \to f_*f^{-1}$ is called restriction. The adjunction map $f_!f^{-1} \to \text{id}$ is often called the trace map. If f is finite étale, then $f_* = f_!$ (Lemma 59.70.7) and we can view this as a map $f_*f^{-1} \to \text{id}$.

03SE Definition 59.66.1. Let $f : Y \to X$ be a finite étale morphism of schemes. The map $f_*f^{-1} \to \text{id}$ described above and explicitly below is called the trace.

Let $f : Y \to X$ be a finite étale morphism of schemes. The trace map is characterized by the following two properties:

(1) it commutes with étale localization on X and
(2) if $Y = \coprod_{i=1}^d X$ then the trace map is the sum map $f_*f^{-1}\mathcal{F} = \mathcal{F}^{\oplus d} \to \mathcal{F}$.

By Étale Morphisms, Lemma 41.18.3 every finite étale morphism $f : Y \to X$ is étale locally on X of the form given in (2) for some integer $d \geq 0$. Hence we can define the trace map using the characterization given; in particular we do not need to know about the existence of $f_!$ and the agreement of $f_!$ with f_* in order to construct the trace map. This description shows that if f has constant degree d, then the composition

$$\mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$$

is multiplication by d. The "méthode de la trace" is the following observation: if $\mathcal{F}$ is an abelian sheaf on $X_{étale}$ such that multiplication by d on $\mathcal{F}$ is an isomorphism, then the map

$$H^n_{étale}(X, \mathcal{F}) \longrightarrow H^n_{étale}(Y, f^{-1}\mathcal{F})$$

is injective. Namely, we have

$$H^n_{étale}(Y, f^{-1}\mathcal{F}) = H^n_{étale}(X, f_*f^{-1}\mathcal{F})$$

by the vanishing of the higher direct images (Proposition 59.55.2) and the Leray spectral sequence (Proposition 59.54.2). Thus we can consider the maps

$$H^n_{étale}(X, \mathcal{F}) \to H^n_{étale}(Y, f^{-1}\mathcal{F}) = H^n_{étale}(X, f_*f^{-1}\mathcal{F}) \xrightarrow{trace} H^n_{étale}(X, \mathcal{F})$$

and the composition is an isomorphism (under our assumption on $\mathcal{F}$ and f). In particular, if $H^q_{étale}(Y, f^{-1}\mathcal{F}) = 0$ then $H^q_{étale}(X, \mathcal{F}) = 0$ as well. Indeed, multiplication by d induces an isomorphism on $H^q_{étale}(X, \mathcal{F})$ which factors through $H^q_{étale}(Y, f^{-1}\mathcal{F}) = 0$.

This is often combined with the following.

0A3R Lemma 59.66.2. Let S be a connected scheme. Let ℓ be a prime number. Let $\mathcal{F}$ be a finite type, locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on $S_{étale}$. Then there exists a finite étale morphism $f : T \to S$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are $\underline{\mathbf{Z}/\ell\mathbf{Z}}_T$.

Proof. Choose a geometric point $\overline{s}$ of S. Via the equivalence of Lemma 59.65.1 the sheaf $\mathcal{F}$ corresponds to a finite dimensional $\mathbf{F}_\ell$-vector space V with a continuous $\pi_1(S, \overline{s})$-action. Let $G \subset \text{Aut}(V)$ be the image of the homomorphism $\rho : \pi_1(S, \overline{s}) \to \text{Aut}(V)$ giving the action. Observe that G is finite. The surjective continuous homomorphism $\overline{\rho} : \pi_1(S, \overline{s}) \to G$ corresponds to a Galois object $Y \to S$ of FÉt_S with automorphism group $G = \text{Aut}(Y/S)$, see Fundamental Groups, Section 58.7. Let $H \subset G$ be an ℓ-Sylow subgroup. We claim that $T = Y/H \to S$ works. Namely, let $\overline{t} \in T$ be a geometric point over $\overline{s}$. The image of $\pi_1(T, \overline{t}) \to \pi_1(S, \overline{s})$ is $(\overline{\rho})^{-1}(H)$ as follows from the functorial nature of fundamental groups. Hence the action of $\pi_1(T, \overline{t})$ on V corresponding to $f^{-1}\mathcal{F}$ is through the map $\pi_1(T, \overline{t}) \to H$, see Remark 59.65.3. As H is a finite ℓ-group, the irreducible constituents of the representation $\rho|_{\pi_1(T,\overline{t})}$ are each trivial of rank 1 (this is a simple lemma on representation theory of finite groups; insert future reference here). Via the equivalence of Lemma 59.65.1 this means $f^{-1}\mathcal{F}$ is a successive extension of constant sheaves with value $\underline{\mathbf{Z}/\ell\mathbf{Z}}_T$.

Moreover the degree of $T = Y/H \to S$ is prime to ℓ as it is equal to the index of H in G. $\square$

0GIZ Lemma 59.66.3. Let Λ be a Noetherian ring. Let ℓ be a prime number and $n \geq 1$. Let H be a finite ℓ-group. Let M be a finite $\Lambda[H]$-module annihilated by ℓ^n. Then there is a finite filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$ by $\Lambda[H]$-submodules such that H acts trivially on M_{i+1}/M_i for all $i = 0, \ldots, t-1$.

Proof. Omitted. Hint: Show that the augmentation ideal $\mathfrak{m}$ of the noncommutative ring $\mathbf{Z}/\ell^n\mathbf{Z}[H]$ is nilpotent. $\square$

0GJ0 Lemma 59.66.4. Let S be an irreducible, geometrically unibranch scheme. Let ℓ be a prime number and $n \geq 1$. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $S_{\acute{e}tale}$ which is annihilated by ℓ^n. Then there exists a finite étale morphism $f : T \to S$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are of the form $\underline{M}_T$ for some finite Λ-modules M.

Proof. Choose a geometric point $\overline{s}$ of S. Via the equivalence of Lemma 59.65.2 the sheaf $\mathcal{F}$ corresponds to a finite Λ-module M with a continuous $\pi_1(S, \overline{s})$-action. Let $G \subset \text{Aut}(V)$ be the image of the homomorphism $\rho : \pi_1(S, \overline{s}) \to \text{Aut}(M)$ giving the action. Observe that G is finite as M is a finite Λ-module (see proof of Lemma 59.65.2). The surjective continuous homomorphism $\overline{\rho} : \pi_1(S, \overline{s}) \to G$ corresponds to a Galois object $Y \to S$ of FÉt_S with automorphism group $G = \text{Aut}(Y/S)$, see Fundamental Groups, Section 58.7. Let $H \subset G$ be an ℓ-Sylow subgroup. We claim that $T = Y/H \to S$ works. Namely, let $\overline{t} \in T$ be a geometric point over $\overline{s}$. The image of $\pi_1(T, \overline{t}) \to \pi_1(S, \overline{s})$ is $(\overline{\rho})^{-1}(H)$ as follows from the functorial nature of fundamental groups. Hence the action of $\pi_1(T, \overline{t})$ on M corresponding to $f^{-1}\mathcal{F}$ is through the map $\pi_1(T, \overline{t}) \to H$, see Remark 59.65.3. Let $0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$ be as in Lemma 59.66.3. This induces a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t = f^{-1}\mathcal{F}$ such that the successive quotients are constant with value M_{i+1}/M_i. Finally, the degree of $T = Y/H \to S$ is prime to ℓ as it is equal to the index of H in G. $\square$

59.67. Galois cohomology

0A2M In this section we prove a result on Galois cohomology (Proposition 59.67.4) using étale cohomology and the trick from Section 59.66. This will allow us to prove vanishing of higher étale cohomology groups over the spectrum of a field.

0DV7 Lemma 59.67.1. Let ℓ be a prime number and n an integer > 0. Let S be a quasi-compact and quasi-separated scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed system of S-schemes each $X_i \to S$ being finite étale of constant degree relatively prime to ℓ. The following are equivalent:

(1) there exists an ℓ-power torsion sheaf $\mathcal{G}$ on S such that $H^n_{\acute{e}tale}(S, \mathcal{G}) \neq 0$ and
(2) there exists an ℓ-power torsion sheaf $\mathcal{F}$ on X such that $H^n_{\acute{e}tale}(X, \mathcal{F}) \neq 0$.

In fact, given $\mathcal{G}$ we can take $\mathcal{F} = g^{-1}\mathcal{F}$ and given $\mathcal{F}$ we can take $\mathcal{G} = g_*\mathcal{F}$.

Proof. Let $g : X \to S$ and $g_i : X_i \to S$ denote the structure morphisms. Fix an ℓ-power torsion sheaf $\mathcal{G}$ on S with $H^n_{\acute{e}tale}(S, \mathcal{G}) \neq 0$. The system given by $\mathcal{G}_i = g_i^{-1}\mathcal{G}$

satisify the conditions of Theorem 59.51.3 with colimit sheaf given by $g^{-1}\mathcal{G}$. This tells us that:

$$\mathrm{colim}_{i \in I} H^n_{\acute{e}tale}(X_i, g_i^{-1}\mathcal{G}) = H^n_{\acute{e}tale}(X, \mathcal{G})$$

By virtue of the g_i being finite étale morphism of degree prime to ℓ we can apply "la méthode de la trace" and we find the maps

$$H^n_{\acute{e}tale}(S, \mathcal{G}) \to H^n_{\acute{e}tale}(X_i, g_i^{-1}\mathcal{G})$$

are all injective (and compatible with the transition maps). See Section 59.66. Thus, the colimit is non-zero, i.e., $H^n(X, g^{-1}\mathcal{G}) \neq 0$, giving us the desired result with $\mathcal{F} = g^{-1}\mathcal{G}$.

Conversely, suppose given an ℓ-power torsion sheaf $\mathcal{F}$ on X with $H^n_{\acute{e}tale}(X, \mathcal{F}) \neq 0$. We note that since the g_i are finite morphisms the higher direct images vanish (Proposition 59.55.2). Then, by applying Lemma 59.51.7 we may also conclude the same for g. The vanishing of the higher direct images tells us that $H^n_{\acute{e}tale}(X, \mathcal{F}) = H^n(S, g_*\mathcal{F}) \neq 0$ by Leray (Proposition 59.54.2) giving us what we want with $\mathcal{G} = g_*\mathcal{F}$. □

0DV8 Lemma 59.67.2. Let ℓ be a prime number and n an integer > 0. Let K be a field with $G = Gal(K^{sep}/K)$ and let $H \subset G$ be a maximal pro-ℓ subgroup with L/K being the corresponding field extension. Then $H^n_{\acute{e}tale}(\mathrm{Spec}(K), \mathcal{F}) = 0$ for all ℓ-power torsion $\mathcal{F}$ if and only if $H^n_{\acute{e}tale}(\mathrm{Spec}(L), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.

Proof. Write $L = \bigcup L_i$ as the union of its finite subextensions over K. Our choice of H implies that $[L_i : K]$ is prime to ℓ. Thus $\mathrm{Spec}(L) = \lim_{i \in I} \mathrm{Spec}(L_i)$ as in Lemma 59.67.1. Thus we may replace K by L and assume that the absolute Galois group G of K is a profinite pro-ℓ group.

Assume $H^n(\mathrm{Spec}(K), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$. Let $\mathcal{F}$ be an ℓ-power torsion sheaf on $\mathrm{Spec}(K)_{\acute{e}tale}$. We will show that $H^n_{\acute{e}tale}(\mathrm{Spec}(K), \mathcal{F}) = 0$. By the correspondence specified in Lemma 59.59.1 our sheaf $\mathcal{F}$ corresponds to an ℓ-power torsion G-module M. Any finite set of elements $x_1, \ldots, x_m \in M$ must be fixed by an open subgroup U by continuity. Let M' be the module spanned by the orbits of $x_1, \ldots, x_m$. This is a finite abelian ℓ-group as each x_i is killed by a power of ℓ and the orbits are finite. Since M is the filtered colimit of these submodules M', we see that $\mathcal{F}$ is the filtered colimit of the corresponding subsheaves $\mathcal{F}' \subset \mathcal{F}$. Applying Theorem 59.51.3 to this colimit, we reduce to the case where $\mathcal{F}$ is a finite locally constant sheaf.

Let M be a finite abelian ℓ-group with a continuous action of the profinite pro-ℓ group G. Then there is a G-invariant filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that $M_{i+1}/M_i \cong \mathbf{Z}/\ell\mathbf{Z}$ with trivial G-action (this is a simple lemma on representation theory of finite groups; insert future reference here). Thus the corresponding sheaf $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

with successive quotients isomorphic to $\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Thus by induction and the long exact cohomology sequence we conclude. □

0DV9 Lemma 59.67.3. Let ℓ be a prime number and n an integer > 0. Let K be a field with $G = Gal(K^{sep}/K)$ and let $H \subset G$ be a maximal pro-ℓ subgroup with L/K being the corresponding field extension. Then $H^q_{\acute{e}tale}(\text{Spec}(K), \mathcal{F}) = 0$ for $q \geq n$ and all ℓ-torsion sheaves $\mathcal{F}$ if and only if $H^n_{\acute{e}tale}(\text{Spec}(L), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.

Proof. The forward direction is trivial, so we need only prove the reverse direction. We proceed by induction on q. The case of $q = n$ is Lemma 59.67.2. Now let $\mathcal{F}$ be an ℓ-power torsion sheaf on $\text{Spec}(K)$. Let $f : \text{Spec}(K^{sep}) \to \text{Spec}(K)$ be the inclusion of a geometric point. Then consider the exact sequence:

$$0 \to \mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \to f_*f^{-1}\mathcal{F}/\mathcal{F} \to 0$$

Note that K^{sep} may be written as the filtered colimit of finite separable extensions. Thus f is the limit of a directed system of finite étale morphisms. We may, as was seen in the proof of Lemma 59.67.1, conclude that f has vanishing higher direct images. Thus, we may express the higher cohomology of $f_*f^{-1}\mathcal{F}$ as the higher cohomology on the geometric point which clearly vanishes. Hence, as everything here is still ℓ-torsion, we may use the inductive hypothesis in conjunction with the long-exact cohomology sequence to conclude the result for $q + 1$. □

03R8 Proposition 59.67.4. Let K be a field with separable algebraic closure K^{sep}. Assume that for any finite extension K' of K we have $\text{Br}(K') = 0$. Then [Ser97, Chapter II, Section 3, Proposition 5]

(1) $H^q(\text{Gal}(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$, and
(2) $H^q(\text{Gal}(K^{sep}/K), M) = 0$ for any torsion $\text{Gal}(K^{sep}/K)$-module M and any $q \geq 2$,

Proof. Set $p = \text{char}(K)$. By Lemma 59.59.2, Theorem 59.61.6, and Example 59.59.3 the proposition is equivalent to showing that if $H^2(\text{Spec}(K'), \mathbf{G}_m|_{\text{Spec}(K')_{\acute{e}tale}}) = 0$ for all finite extensions K'/K then:

- $H^q(\text{Spec}(K), \mathbf{G}_m|_{\text{Spec}(K)_{\acute{e}tale}}) = 0$ for all $q \geq 1$, and
- $H^q(\text{Spec}(K), \mathcal{F}) = 0$ for any torsion sheaf $\mathcal{F}$ and any $q \geq 2$.

We prove the second part first. Since $\mathcal{F}$ is a torsion sheaf, we may use the ℓ-primary decomposition as well as the compatibility of cohomology with colimits (i.e, direct sums, see Theorem 59.51.3) to reduce to showing $H^q(\text{Spec}(K), \mathcal{F}) = 0$, $q \geq 2$ for all ℓ-power torsion sheaves for every prime ℓ. This allows us to analyze each prime individually.

Suppose that $\ell \neq p$. For any extension K'/K consider the Kummer sequence (Lemma 59.28.1)

$$0 \to \mu_{\ell,\text{Spec}\,K'} \to \mathbf{G}_{m,\text{Spec}\,K'} \xrightarrow{(\cdot)^\ell} \mathbf{G}_{m,\text{Spec}\,K'} \to 0$$

Since $H^q(\text{Spec}\,K', \mathbf{G}_m|_{\text{Spec}(K')_{\acute{e}tale}}) = 0$ for $q = 2$ by assumption and for $q = 1$ by Theorem 59.24.1 combined with $\text{Pic}(K) = (0)$. Thus, by the long-exact cohomology sequence we may conclude that $H^2(\text{Spec}\,K', \mu_\ell) = 0$ for any separable K'/K. Now let H be a maximal pro-ℓ subgroup of the absolute Galois group of K and let L be the corresponding extension. We can write L as the colimit of finite extensions, applying Theorem 59.51.3 to this colimit we see that $H^2(\text{Spec}(L), \mu_\ell) = 0$. Now μ_ℓ must be the constant sheaf. If it weren't, that would imply there exists a Galois extension of degree relatively prime to ℓ of L which is not true by definition of L (namely, the extension one gets by adjoining the ℓth roots of unity to L). Hence, via Lemma 59.67.3, we conclude the result for $\ell \neq p$.

Now suppose that $\ell = p$. We consider the Artin-Schrier exact sequence (Section 59.63)

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_{\mathrm{Spec}\, K} \longrightarrow \mathbf{G}_{a,\mathrm{Spec}\, K} \xrightarrow{F-1} \mathbf{G}_{a,\mathrm{Spec}\, K} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$. Then note that the higher Cohomology of $\mathbf{G}_{a,\mathrm{Spec}\, K}$ vanishes, by Remark 59.23.4 and the vanishing of the higher cohomology of the structure sheaf of an affine scheme (Cohomology of Schemes, Lemma 30.2.2). Note this can be applied to any field of characteristic p. In particular, we can apply it to the field extension L defined by a maximal pro-p subgroup H. This allows us to conclude $H^n(\mathrm{Spec}\, L, \underline{\mathbf{Z}/p\mathbf{Z}}_{\mathrm{Spec}\, L}) = 0$ for $n \geq 2$, from which the result follows for $\ell = p$, by Lemma 59.67.3.

To finish the proof we still have to show that $H^q(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$. Set $G = \mathrm{Gal}(K^{sep}/K)$ and set $M = (K^{sep})^*$ viewed as a G-module. We have already shown (above) that $H^1(G, M) = 0$ and $H^2(G, M) = 0$. Consider the exact sequence

$$0 \to A \to M \to M \otimes \mathbf{Q} \to B \to 0$$

of G-modules. By the above we have $H^i(G, A) = 0$ and $H^i(G, B) = 0$ for $i > 1$ since A and B are torsion G-modules. By Lemma 59.57.6 we have $H^i(G, M \otimes \mathbf{Q}) = 0$ for $i > 0$. It is a pleasant exercise to see that this implies that $H^i(G, M) = 0$ also for $i \geq 3$. □

03R9 Definition 59.67.5. A field K is called C_r if for every $0 < d^r < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree d, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an α is called a nontrivial solution of f.

03RA Example 59.67.6. An algebraically closed field is C_r.

In fact, we have the following simple lemma.

03RB Lemma 59.67.7. Let k be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.

Proof. This follows from dimension theory, for example in the form of Varieties, Lemma 33.34.2 applied $s - 1$ times. □

The following result computes the Brauer group of C_1 fields.

03RC Theorem 59.67.8. Let K be a C_1 field. Then $\mathrm{Br}(K) = 0$.

Proof. Let D be a finite dimensional division algebra over K with center K. We have seen that

$$D \otimes_K K^{sep} \cong \mathrm{Mat}_d(K^{sep})$$

uniquely up to inner isomorphism. Hence the determinant $\det : \mathrm{Mat}_d(K^{sep}) \to K^{sep}$ is Galois invariant and descends to a homogeneous degree d map

$$\det = N_{\mathrm{red}} : D \longrightarrow K$$

called the reduced norm. Since K is C_1, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\mathrm{red}}(x) = 0$. This clearly implies that x is not invertible, which is a contradiction. Hence $\mathrm{Br}(K) = 0$. □

03RE Definition 59.67.9. Let k be a field. A variety is separated, integral scheme of finite type over k. A curve is a variety of dimension 1.

03RD Theorem 59.67.10 (Tsen's theorem). The function field of a variety of dimension r over an algebraically closed field k is C_r.

Proof. For projective space one can show directly that the field $k(x_1, \ldots, x_r)$ is C_r (exercise).

General case. Without loss of generality, we may assume X to be projective. Let $f \in k(X)[T_1, \ldots, T_n]_d$ with $0 < d^r < n$. Say the coefficients of f are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then $f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\alpha) = 0$. Then by asymptotic Riemann-Roch (Varieties, Proposition 33.45.13) there exists a $c > 0$ such that

- the number of variables is $n \dim_k \Gamma(X, \mathcal{O}_X(eH)) \sim ne^r c$, and
- the number of equations is $\dim_k \Gamma(X, \mathcal{O}_X((de+1)H)) \sim (de+1)^r c$.

Since $n > d^r$, there are more variables than equations. The equations are homogeneous hence there is a solution by Lemma 59.67.7. $\square$

03RF Lemma 59.67.11. Let C be a curve over an algebraically closed field k. Then the Brauer group of the function field of C is zero: $\mathrm{Br}(k(C)) = 0$.

Proof. This is clear from Tsen's theorem, Theorem 59.67.10 and Theorem 59.67.8. $\square$

03RG Lemma 59.67.12. Let k be an algebraically closed field and K/k a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{étale}(\mathrm{Spec}(K), \mathbf{G}_m) = 0$.

Proof. Recall that $H^q_{étale}(\mathrm{Spec}(K), \mathbf{G}_m) = H^q(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$ by Lemma 59.59.2. Thus by Proposition 59.67.4 it suffices to show that if K'/K is a finite field extension, then $\mathrm{Br}(K') = 0$. Now observe that $K' = \mathrm{colim}\, K''$, where K'' runs over the finitely generated subextensions of k contained in K' of transcendence degree 1. Note that $\mathrm{Br}(K') = \mathrm{colim}\, \mathrm{Br}(K'')$ which reduces us to a finitely generated field extension K''/k of transcendence degree 1. Such a field is the function field of a curve over k, hence has trivial Brauer group by Lemma 59.67.11. $\square$

59.68. Higher vanishing for the multiplicative group

03RH In this section, we fix an algebraically closed field k and a smooth curve X over k. We denote $i_x : x \hookrightarrow X$ the inclusion of a closed point of X and $j : \eta \hookrightarrow X$ the inclusion of the generic point. We also denote X_0 the set of closed points of X.

03RI Theorem 59.68.1 (The Fundamental Exact Sequence). There is a short exact sequence of étale sheaves on X

$$0 \longrightarrow \mathbf{G}_{m,X} \longrightarrow j_*\mathbf{G}_{m,\eta} \longrightarrow \bigoplus\nolimits_{x \in X_0} i_{x*}\underline{\mathbf{Z}} \longrightarrow 0.$$

Proof. Let $\varphi : U \to X$ be an étale morphism. Then by properties of étale morphisms (Proposition 59.26.2), $U = \coprod_i U_i$ where each U_i is a smooth curve mapping to X. The above sequence for U is a product of the corresponding sequences for each U_i, so it suffices to treat the case where U is connected, hence irreducible. In this case, there is a well known exact sequence

$$1 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \longrightarrow k(U)^* \longrightarrow \bigoplus\nolimits_{y \in U_0} \mathbf{Z}_y.$$

This amounts to a sequence

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \longrightarrow \Gamma(\eta \times_X U, \mathcal{O}_{\eta \times_X U}^*) \longrightarrow \bigoplus\nolimits_{x \in X_0} \Gamma(x \times_X U, \underline{\mathbf{Z}})$$

which, unfolding definitions, is nothing but a sequence

$$0 \longrightarrow \mathbf{G}_m(U) \longrightarrow j_*\mathbf{G}_{m,\eta}(U) \longrightarrow \left(\bigoplus\nolimits_{x \in X_0} i_{x*}\underline{\mathbf{Z}}\right)(U).$$

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall that if U is a nonsingular curve and D is a divisor on U, then there exists a Zariski open covering $\{U_j \to U\}$ of U such that $D|_{U_j} = \text{div}(f_j)$ for some $f_j \in k(U)^*$. □

03RJ Lemma 59.68.2. For any $q \geq 1$, $R^q j_* \mathbf{G}_{m,\eta} = 0$.

Proof. We need to show that $(R^q j_* \mathbf{G}_{m,\eta})_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of X.

Assume that $\bar{x}$ lies over a closed point x of X. Let $\text{Spec}(A)$ be an affine open neighbourhood of x in X, and K the fraction field of A. Then

$$\text{Spec}(\mathcal{O}_{X,\bar{x}}^{sh}) \times_X \eta = \text{Spec}(\mathcal{O}_{X,\bar{x}}^{sh} \otimes_A K).$$

The ring $\mathcal{O}_{X,\bar{x}}^{sh} \otimes_A K$ is a localization of the discrete valuation ring $\mathcal{O}_{X,\bar{x}}^{sh}$, so it is either $\mathcal{O}_{X,\bar{x}}^{sh}$ again, or its fraction field $K_{\bar{x}}^{sh}$. But since some local uniformizer gets inverted, it must be the latter. Hence

$$(R^q j_* \mathbf{G}_{m,\eta})_{(X,\bar{x})} = H^q_{\acute{e}tale}(\text{Spec}\, K_{\bar{x}}^{sh}, \mathbf{G}_m).$$

Now recall that $\mathcal{O}_{X,\bar{x}}^{sh} = \text{colim}_{(U,\bar{u}) \to \bar{x}} \mathcal{O}(U) = \text{colim}_{A \subset B} B$ where $A \to B$ is étale, hence $K_{\bar{x}}^{sh}$ is an algebraic extension of $K = k(X)$, and we may apply Lemma 59.67.12 to get the vanishing.

Assume that $\bar{x} = \bar{\eta}$ lies over the generic point η of X (in fact, this case is superfluous). Then $\mathcal{O}_{X,\bar{\eta}}^{sh} = \kappa(\eta)^{sep}$ and thus

$$\begin{aligned} (R^q j_* \mathbf{G}_{m,\eta})_{\bar{\eta}} &= H^q_{\acute{e}tale}(\text{Spec}(\kappa(\eta)^{sep}) \times_X \eta, \mathbf{G}_m) \\ &= H^q_{\acute{e}tale}(\text{Spec}(\kappa(\eta)^{sep}), \mathbf{G}_m) \\ &= 0 \quad \text{for } q \geq 1 \end{aligned}$$

since the corresponding Galois group is trivial. □

03RK Lemma 59.68.3. For all $p \geq 1$, $H^p_{\acute{e}tale}(X, j_*\mathbf{G}_{m,\eta}) = 0$.

Proof. The Leray spectral sequence reads

$$E_2^{p,q} = H^p_{\acute{e}tale}(X, R^q j_* \mathbf{G}_{m,\eta}) \Rightarrow H^{p+q}_{\acute{e}tale}(\eta, \mathbf{G}_{m,\eta}),$$

which vanishes for $p + q \geq 1$ by Lemma 59.67.12. Taking $q = 0$, we get the desired vanishing. □

03RL Lemma 59.68.4. For all $q \geq 1$, $H^q_{\acute{e}tale}(X, \bigoplus_{x \in X_0} i_{x*}\underline{\mathbf{Z}}) = 0$.

Proof. For X quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}})$. But then the inclusion i_x of a closed point is finite so $R^p i_{x*}\underline{\mathbf{Z}} = 0$ for all $p \geq 1$ by Proposition 59.55.2. Applying the Leray spectral sequence, we see that $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}}) = H^q_{\acute{e}tale}(x, \underline{\mathbf{Z}})$. Finally, since x is the spectrum of an algebraically closed field, all higher cohomology on x vanishes. □

Concluding this series of lemmata, we get the following result.

03RM Theorem 59.68.5. Let X be a smooth curve over an algebraically closed field. Then

$$H^q_{étale}(X, \mathbf{G}_m) = 0 \quad \text{for all } q \geq 2.$$

Proof. See discussion above. □

We also get the cohomology long exact sequence

$$0 \to H^0_{étale}(X, \mathbf{G}_m) \to H^0_{étale}(X, j_*\mathbf{G}_{m\eta}) \to H^0_{étale}(X, \bigoplus i_{x*}\underline{\mathbf{Z}}) \to H^1_{étale}(X, \mathbf{G}_m) \to 0$$

although this is the familiar

$$0 \to H^0_{Zar}(X, \mathcal{O}_X^*) \to k(X)^* \to \text{Div}(X) \to \text{Pic}(X) \to 0.$$

59.69. Picard groups of curves

03RN Our next step is to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

Let X be a smooth projective curve over an algebraically closed field k. Let $g = \dim_k H^1(X, \mathcal{O}_X)$ be the genus of X. There exists a short exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The abelian group $\text{Pic}^0(X)$ can be identified with $\text{Pic}^0(X) = \underline{\text{Pic}}^0_{X/k}(k)$, i.e., the k-valued points of an abelian variety $\underline{\text{Pic}}^0_{X/k}$ over k of dimension g. Consequently, if $n \in k^*$ then $\text{Pic}^0(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ as abelian groups. See Picard Schemes of Curves, Section 44.6 and Groupoids, Section 39.9. This key fact, namely the description of the torsion in the Picard group of a smooth projective curve over an algebraically closed field does not appear to have an elementary proof.

03RQ Lemma 59.69.1. Let X be a smooth projective curve of genus g over an algebraically closed field k and let $n \geq 1$ be invertible in k. Then there are canonical identifications

$$H^q_{étale}(X, \mu_n) = \begin{cases} \mu_n(k) & \text{if } q = 0, \\ \text{Pic}^0(X)[n] & \text{if } q = 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Since $\mu_n \cong \underline{\mathbf{Z}/n\mathbf{Z}}$, this gives (noncanonical) identifications

$$H^q_{étale}(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & \text{if } q = 0, \\ (\mathbf{Z}/n\mathbf{Z})^{2g} & \text{if } q = 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Proof. Theorems 59.24.1 and 59.68.5 determine the étale cohomology of $\mathbf{G}_m$ on X in terms of the Picard group of X. The Kummer sequence $0 \to \mu_{n,X} \to \mathbf{G}_{m,X} \to$

$\mathbf{G}_{m,X} \to 0$ (Lemma 59.28.1) then gives us the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & \mu_n(k) \arrow[r] & k^* \arrow[r, "(\cdot)^n"] & k^* \arrow[dll, bend left] \\ & H^1_{\acute{e}tale}(X, \mu_n) \arrow[r] & \text{Pic}(X) \arrow[r, "(\cdot)^n"] & \text{Pic}(X) \arrow[dll, bend left] \\ & H^2_{\acute{e}tale}(X, \mu_n) \arrow[r] & 0 \arrow[r] & 0 \ldots \end{tikzcd}$$

The nth power map $k^* \to k^*$ is surjective since k is algebraically closed. So we need to compute the kernel and cokernel of the map $n : \text{Pic}(X) \to \text{Pic}(X)$. Consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \text{Pic}^0(X) \arrow[r] \arrow[d, "(\cdot)^n"] & \text{Pic}(X) \arrow[r, "\deg"'] \arrow[d, "(\cdot)^n"] & \mathbf{Z} \arrow[r] \arrow[d, hook, "n"] & 0 \\ 0 \arrow[r] & \text{Pic}^0(X) \arrow[r] & \text{Pic}(X) \arrow[r, "\deg"] & \mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

The group $\text{Pic}^0(X)$ is the k-points of the group scheme $\underline{\text{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 44.6.7. The same lemma tells us that $\underline{\text{Pic}}^0_{X/k}$ is a g-dimensional abelian variety over k as defined in Groupoids, Definition 39.9.1. Hence the left vertical map is surjective by Groupoids, Proposition 39.9.11. Applying the snake lemma gives canonical identifications as stated in the lemma.

To get the noncanonical identifications of the lemma we need to show the kernel of $n : \text{Pic}^0(X) \to \text{Pic}^0(X)$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$. This is also part of Groupoids, Proposition 39.9.11. □

0AMB Lemma 59.69.2. Let $\pi : X \to Y$ be a nonconstant morphism of smooth projective curves over an algebraically closed field k and let $n \geq 1$ be invertible in k. The map

$$\pi^* : H^2_{\acute{e}tale}(Y, \mu_n) \longrightarrow H^2_{\acute{e}tale}(X, \mu_n)$$

is given by multiplication by the degree of π.

Proof. Observe that the statement makes sense as we have identified both cohomology groups $H^2_{\acute{e}tale}(Y, \mu_n)$ and $H^2_{\acute{e}tale}(X, \mu_n)$ with $\mathbf{Z}/n\mathbf{Z}$ in Lemma 59.69.1. In fact, if $\mathcal{L}$ is a line bundle of degree 1 on Y with class $[\mathcal{L}] \in H^1_{\acute{e}tale}(Y, \mathbf{G}_m)$, then the coboundary of $[\mathcal{L}]$ is the generator of $H^2_{\acute{e}tale}(Y, \mu_n)$. Here the coboundary is the coboundary of the long exact sequence of cohomology associated to the Kummer sequence. Thus the result of the lemma follows from the fact that the degree of the line bundle $\pi^*\mathcal{L}$ on X is $\deg(\pi)$. Some details omitted. □

03RR Lemma 59.69.3. Let X be an affine smooth curve over an algebraically closed field k and $n \in k^*$. Let $X \subset \overline{X}$ be a smooth projective compactification (Varieties, Remark 33.43.9). Let g be the genus of $\overline{X}$ and let r be the number of points of $\overline{X} \setminus X$. Then

(1) $H^0_{\acute{e}tale}(X, \mu_n) = \mu_n(k)$;
(2) $H^1_{\acute{e}tale}(X, \mu_n) \cong (\mathbf{Z}/n\mathbf{Z})^{2g+r-1}$, and
(3) $H^q_{\acute{e}tale}(X, \mu_n) = 0$ for all $q \geq 2$.

Proof. Write $X = \overline{X} - \{x_1, \ldots, x_r\}$. Then $\text{Pic}(X) = \text{Pic}(\overline{X})/R$, where R is the subgroup generated by $\mathcal{O}_{\overline{X}}(x_i)$, $1 \leq i \leq r$. Since $r \geq 1$, we see that $\text{Pic}^0(\overline{X}) \to \text{Pic}(X)$ is surjective, hence $\text{Pic}(X)$ is divisible (see discussion in proof of Lemma 59.69.1). Applying the Kummer sequence, we get (1) and (3). For (2), recall that

$$\begin{aligned} H^1_{\acute{e}tale}(X, \mu_n) &= \{(\mathcal{L}, \alpha) | \mathcal{L} \in \text{Pic}(X), \alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_X\}/\cong \\ &= \{(\bar{\mathcal{L}},\ D,\ \bar{\alpha})\}/\tilde{R} \end{aligned}$$

where $\bar{\mathcal{L}} \in \text{Pic}^0(\overline{X})$, D is a divisor on $\overline{X}$ supported on $\{x_1, \ldots, x_r\}$ and $\bar{\alpha} : \bar{\mathcal{L}}^{\otimes n} \cong \mathcal{O}_{\bar{X}}(D)$ is an isomorphism. Note that D must have degree 0. Further $\tilde{R}$ is the subgroup of triples of the form $(\mathcal{O}_{\overline{X}}(D'), nD', 1^{\otimes n})$ where D' is supported on $\{x_1, \ldots, x_r\}$ and has degree 0. Thus, we get an exact sequence

$$0 \longrightarrow H^1_{\acute{e}tale}(\overline{X}, \mu_n) \longrightarrow H^1_{\acute{e}tale}(X, \mu_n) \longrightarrow \bigoplus_{i=1}^r \mathbf{Z}/n\mathbf{Z} \xrightarrow{\sum} \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

where the middle map sends the class of a triple $(\bar{\mathcal{L}}, D, \bar{\alpha})$ with $D = \sum_{i=1}^r a_i(x_i)$ to the r-tuple $(a_i)_{i=1}^r$. It now suffices to use Lemma 59.69.1 to count ranks. $\square$

03RS Remark 59.69.4. The "natural" way to prove the previous corollary is to excise X from $\bar{X}$. This is possible, we just haven't developed that theory.

0A44 Remark 59.69.5. Let k be an algebraically closed field. Let n be an integer prime to the characteristic of k. Recall that

$$\mathbf{G}_{m,k} = \mathbf{A}^1_k \setminus \{0\} = \mathbf{P}^1_k \setminus \{0, \infty\}$$

We claim there is a canonical isomorphism

$$H^1_{\acute{e}tale}(\mathbf{G}_{m,k}, \mu_n) = \mathbf{Z}/n\mathbf{Z}$$

What does this mean? This means there is an element 1_k in $H^1_{\acute{e}tale}(\mathbf{G}_{m,k}, \mu_n)$ such that for every morphism $\text{Spec}(k') \to \text{Spec}(k)$ the pullback map on étale cohomology for the map $\mathbf{G}_{m,k'} \to \mathbf{G}_{m,k}$ maps 1_k to $1_{k'}$. (In particular this element is fixed under all automorphisms of k.) To see this, consider the $\mu_{n,\mathbf{Z}}$-torsor $\mathbf{G}_{m,\mathbf{Z}} \to \mathbf{G}_{m,\mathbf{Z}}$, $x \mapsto x^n$. By the identification of torsors with first cohomology, this pulls back to give our canonical elements 1_k. Twisting back we see that there are canonical identifications

$$H^1_{\acute{e}tale}(\mathbf{G}_{m,k}, \mathbf{Z}/n\mathbf{Z}) = \text{Hom}(\mu_n(k), \mathbf{Z}/n\mathbf{Z}),$$

i.e., these isomorphisms are compatible with respect to maps of algebraically closed fields, in particular with respect to automorphisms of k.

59.70. Extension by zero

03S2 The general material in Modules on Sites, Section 18.19 allows us to make the following definition.

03S3 Definition 59.70.1. Let $j : U \to X$ be an étale morphism of schemes.

(1) The restriction functor $j^{-1} : Sh(X_{\acute{e}tale}) \to Sh(U_{\acute{e}tale})$ has a left adjoint $j_!^{Sh} : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$.
(2) The restriction functor $j^{-1} : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(U_{\acute{e}tale})$ has a left adjoint which is denoted $j_! : \text{Ab}(U_{\acute{e}tale}) \to \text{Ab}(X_{\acute{e}tale})$ and called extension by zero.

(3) Let Λ be a ring. The restriction functor $j^{-1} : \text{Mod}(X_{\acute{e}tale}, \Lambda) \to \text{Mod}(U_{\acute{e}tale}, \Lambda)$ has a left adjoint which is denoted $j_! : \text{Mod}(U_{\acute{e}tale}, \Lambda) \to \text{Mod}(X_{\acute{e}tale}, \Lambda)$ and called extension by zero.

If $\mathcal{F}$ is an abelian sheaf on $X_{\acute{e}tale}$, then $j_!\mathcal{F} \neq j_!^{Sh}\mathcal{F}$ in general. On the other hand $j_!$ for sheaves of Λ-modules agrees with $j_!$ on underlying abelian sheaves (Modules on Sites, Remark 18.19.6). The functor $j_!$ is characterized by the functorial isomorphism

$$\text{Hom}_X(j_!\mathcal{F}, \mathcal{G}) = \text{Hom}_U(\mathcal{F}, j^{-1}\mathcal{G})$$

for all $\mathcal{F} \in \text{Ab}(U_{\acute{e}tale})$ and $\mathcal{G} \in \text{Ab}(X_{\acute{e}tale})$. Similarly for sheaves of Λ-modules.

To describe the functors in Definition 59.70.1 more explicitly, recall that j^{-1} is just the restriction via the functor $U_{\acute{e}tale} \to X_{\acute{e}tale}$. In other words, $j^{-1}\mathcal{G}(U') = \mathcal{G}(U')$ for U' étale over U. On the other hand, for $\mathcal{F} \in \text{Ab}(U_{\acute{e}tale})$ we consider the presheaf

0F4K (59.70.1.1)
$$j_{p!}\mathcal{F} : X_{\acute{e}tale} \longrightarrow \text{Ab}, \quad V \longmapsto \bigoplus\nolimits_{V \to U} \mathcal{F}(V \to U)$$

Then $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$. This is proven in Modules on Sites, Lemma 18.19.2; more generally see the discussion in Modules on Sites, Sections 18.19 and 18.16.

03S4 Exercise 59.70.2. Prove directly that the functor $j_!$ defined as the sheafification of the functor $j_{p!}$ given in (59.70.1.1) is a left adjoint to j^{-1}.

03S5 Proposition 59.70.3. Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{F}$ in $\text{Ab}(U_{\acute{e}tale})$. If $\overline{x} : \text{Spec}(k) \to X$ is a geometric point of X, then

$$(j_!\mathcal{F})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}:\text{Spec}(k)\to U,\ j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.$$

In particular, $j_!$ is an exact functor.

Proof. Exactness of $j_!$ is very general, see Modules on Sites, Lemma 18.19.3. Of course it does also follow from the description of stalks. The formula for the stalk follows from Modules on Sites, Lemma 18.38.1 and the description of points of the small étale site in terms of geometric points, see Lemma 59.29.12.

For later use we note that the isomorphism

$$\begin{aligned}
(j_!\mathcal{F})_{\overline{x}} &= (j_{p!}\mathcal{F})_{\overline{x}} \\
&= \text{colim}_{(V,\overline{v})}\, j_{p!}\mathcal{F}(V) \\
&= \text{colim}_{(V,\overline{v})} \bigoplus\nolimits_{\varphi:V\to U} \mathcal{F}(V \xrightarrow{\varphi} U) \\
&\to \bigoplus\nolimits_{\overline{u}:\text{Spec}(k)\to U,\ j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.
\end{aligned}$$

constructed in Modules on Sites, Lemma 18.38.1 sends $(V, \overline{v}, \varphi, s)$ to the class of s in the stalk of $\mathcal{F}$ at $\overline{u} = \varphi(\overline{v})$. $\square$

0F70 Lemma 59.70.4. Let $j : U \to X$ be an open immersion of schemes. For any abelian sheaf $\mathcal{F}$ on $U_{\acute{e}tale}$, the adjunction mappings $j^{-1}j_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to j^{-1}j_!\mathcal{F}$ are isomorphisms. In fact, $j_!\mathcal{F}$ is the unique abelian sheaf on $X_{\acute{e}tale}$ whose restriction to U is $\mathcal{F}$ and whose stalks at geometric points of $X \setminus U$ are zero.

Proof. We encourage the reader to prove the first statement by working through the definitions, but here we just use that it is a special case of the very general Modules on Sites, Lemma 18.19.8. For the second statement, observe that if $\mathcal{G}$ is an abelian sheaf on $X_{\acute{e}tale}$ whose restriction to U is $\mathcal{F}$, then we obtain by adjointness a map $j_!\mathcal{F} \to \mathcal{G}$. This map is then an isomorphism at stalks of geometric points of U by Proposition 59.70.3. Thus if $\mathcal{G}$ has vanishing stalks at geometric points of $X \setminus U$, then $j_!\mathcal{F} \to \mathcal{G}$ is an isomorphism by Theorem 59.29.10. □

03S6 **Lemma 59.70.5** (Extension by zero commutes with base change). Let $f : Y \to X$ be a morphism of schemes. Let $j : V \to X$ be an étale morphism. Consider the fibre product

$$\begin{tikzcd} V' = Y \times_X V \arrow[r, "j'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ V \arrow[r, "j"] & X \end{tikzcd}$$

Then we have $j'_! f'^{-1} = f^{-1} j_!$ on abelian sheaves and on sheaves of modules.

Proof. This is true because $j'_! f'^{-1}$ is left adjoint to $f'_*(j')^{-1}$ and $f^{-1}j_!$ is left adjoint to $j^{-1}f_*$. Further $f'_*(j')^{-1} = j^{-1}f_*$ because f_* commutes with étale localization (by construction). In fact, the lemma holds very generally in the setting of a morphism of sites, see Modules on Sites, Lemma 18.20.1. □

0F4L **Lemma 59.70.6.** Let $j : U \to X$ be separated and étale. Then there is a functorial injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ on abelian sheaves and sheaves of Λ-modules.

Proof. We prove this in the case of abelian sheaves. Let us construct a canonical map

$$j_{p!}\mathcal{F} \to j_*\mathcal{F}$$

of abelian presheaves on $X_{\acute{e}tale}$ for any abelian sheaf $\mathcal{F}$ on $U_{\acute{e}tale}$ where $j_{p!}$ is as in (59.70.1.1). Sheafification of this map will be the desired map $j_!\mathcal{F} \to j_*\mathcal{F}$. Evaluating both sides on $V \to X$ étale we obtain

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U) \quad \text{and} \quad j_*\mathcal{F}(V) = \mathcal{F}(V \times_X U)$$

For each φ we have an open and closed immersion

$$\Gamma_\varphi = (1, \varphi) : V \longrightarrow V \times_X U$$

over U. It is open as it is a morphism between schemes étale over U and it is closed as it is a section of a scheme separated over V (Schemes, Lemma 26.21.11). Thus for a section $s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U)$ there exists a unique section s'_φ in $\mathcal{F}(V \times_X U)$ which pulls back to s_φ by Γ_φ and which restricts to zero on the complement of the image of Γ_φ.

To show that our map is injective suppose that $\sum_{i=1,\ldots,n} s_{\varphi_i}$ is an element of $j_{p!}\mathcal{F}(V)$ in the formula above maps to zero in $j_*\mathcal{F}(V)$. Our task is to show that $\sum_{i=1,\ldots,n} s_{\varphi_i}$ restricts to zero on the members of an étale covering of V. Looking at all pairwise equalizers (which are open and closed in V) of the morphisms $\varphi_i : V \to U$ and working locally on V, we may assume the images of the morphisms $\Gamma_{\varphi_1}, \ldots, \Gamma_{\varphi_n}$ are pairwise disjoint. Since our assumption is that $\sum_{i=1,\ldots,n} s'_{\varphi_i} = 0$ we then immediately conclude that $s'_{\varphi_i} = 0$ for each i (by the disjointness of the supports of these sections), whence $s_{\varphi_i} = 0$ for all i as desired. □

03S7 **Lemma 59.70.7.** Let $j : U \to X$ be finite and étale. Then the map $j_! \to j_*$ of Lemma 59.70.6 is an isomorphism on abelian sheaves and sheaves of Λ-modules.

Proof. It suffices to check $j_!\mathcal{F} \to j_*\mathcal{F}$ is an isomorphism étale locally on X. Thus we may assume $U \to X$ is a finite disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. We omit the proof in this case. $\square$

095L **Lemma 59.70.8.** Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. For every abelian sheaf $\mathcal{F}$ on $X_{étale}$ there is a canonical short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{étale}$.

Proof. We obtain the maps by the adjointness properties of the functors involved. For a geometric point $\overline{x}$ in X we have either $\overline{x} \in U$ in which case the map on the left hand side is an isomorphism on stalks and the stalk of $i_*i^{-1}\mathcal{F}$ is zero or $\overline{x} \in Z$ in which case the map on the right hand side is an isomorphism on stalks and the stalk of $j_!j^{-1}\mathcal{F}$ is zero. Here we have used the description of stalks of Lemma 59.46.3 and Proposition 59.70.3. $\square$

0GJ1 **Lemma 59.70.9.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} U \arrow[r, "j'"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "j"] & Y \end{tikzcd}$$

where f is finite, g is étale, and j is an open immersion. Then $f_* \circ j'_! = j_! \circ g_*$ as functors $\textit{Ab}(U_{étale}) \to \textit{Ab}(Y_{étale})$.

Proof. Let $\mathcal{F}$ be an object of $\textit{Ab}(U_{étale})$. Let $\overline{y}$ be a geometric point of Y not contained in the open V. Then

$$(f_*j'_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{\overline{x},\ f(\overline{x})=\overline{y}} (j'_!\mathcal{F})_{\overline{x}} = 0$$

by Proposition 59.55.2 and because the stalk of $j'_!\mathcal{F}$ at $\overline{x} \notin U$ are zero by Lemma 59.70.4. On the other hand, we have

$$j^{-1}f_*j'_!\mathcal{F} = g_*(j')^{-1}j'_!\mathcal{F} = g_*\mathcal{F}$$

by Lemmas 59.55.3 and Lemma 59.70.4. Hence by the characterization of $j_!$ in Lemma 59.70.4 we see that $f_*j'_!\mathcal{F} = j_!g_*\mathcal{F}$. We omit the verification that this identification is functorial in $\mathcal{F}$. $\square$

59.71. Constructible sheaves

05BE Let X be a scheme. A constructible locally closed subscheme of X is a locally closed subscheme $T \subset X$ such that the underlying topological space of T is a constructible subset of X. If $T, T' \subset X$ are locally closed subschemes with the same underlying topological space, then $T_{étale} \cong T'_{étale}$ by the topological invariance of the étale site (Theorem 59.45.2). Thus in the following definition we may assume our locally closed subschemes are reduced.

03RW **Definition 59.71.1.** Let X be a scheme.

(1) A sheaf of sets on $X_{\acute{e}tale}$ is constructible if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(2) A sheaf of abelian groups on $X_{\acute{e}tale}$ is constructible if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(3) Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{\acute{e}tale}$ is constructible if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

It seems that this is the accepted definition. An alternative, which lends itself more readily to generalizations beyond the étale site of a scheme, would have been to define constructible sheaves by starting with h_U, $j_{U!}\mathbf{Z}/n\mathbf{Z}$, and $j_{U!}\underline{\Lambda}$ where U runs over all quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$, and then take the smallest full subcategory of $Sh(X_{\acute{e}tale})$, $\text{Ab}(X_{\acute{e}tale})$, and $\text{Mod}(X_{\acute{e}tale}, \underline{\Lambda})$ containing these and closed under finite limits and colimits. It follows from Lemma 59.71.6 and Lemmas 59.73.5, 59.73.7, and 59.73.6 that this produces the same category if X is quasi-compact and quasi-separated. In general this does not produce the same category however.

A disjoint union decomposition $U = \coprod U_i$ of a scheme by locally closed subschemes will be called a partition of U (compare with Topology, Section 5.28).

095E Lemma 59.71.2. Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. The following are equivalent

(1) $\mathcal{F}$ is constructible,
(2) there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is constructible, and
(3) there exists a partition $X = \bigcup X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.

A similar statement holds for abelian sheaves and sheaves of Λ-modules if Λ is Noetherian.

Proof. It is clear that (1) implies (2).

Assume (2). For every $x \in X$ we can find an i and an affine open neighbourhood $V_x \subset U_i$ of x. Hence we can find a finite affine open covering $X = \bigcup V_j$ such that for each j there exists a finite decomposition $V_j = \coprod V_{j,k}$ by locally closed constructible subsets such that $\mathcal{F}|_{V_{j,k}}$ is finite locally constant. By Topology, Lemma 5.15.5 each $V_{j,k}$ is constructible as a subset of X. By Topology, Lemma 5.28.7 we can find a finite stratification $X = \coprod X_l$ with constructible locally closed strata such that each $V_{j,k}$ is a union of X_l. Thus (3) holds.

Assume (3) holds. Let $U \subset X$ be an affine open. Then $U \cap X_i$ is a constructible locally closed subset of U (for example by Properties, Lemma 28.2.1) and $U = \coprod U \cap X_i$ is a partition of U as in Definition 59.71.1. Thus (1) holds. $\square$

09YR Lemma 59.71.3. Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) on $X_{\acute{e}tale}$. If there exist constructible locally closed subschemes $T_i \subset X$ such that (a) $X = \bigcup T_j$ and (b) $\mathcal{F}|_{T_j}$ is constructible, then $\mathcal{F}$ is constructible.

Proof. First, we can assume the covering is finite as X is quasi-compact in the spectral topology (Topology, Lemma 5.23.2 and Properties, Lemma 28.2.4). Observe that each T_i is a quasi-compact and quasi-separated scheme in its own right (because it is constructible in X; details omitted). Thus we can find a finite partition $T_i = \coprod T_{i,j}$ into locally closed constructible parts of T_i such that $\mathcal{F}|_{T_{i,j}}$ is finite locally constant (Lemma 59.71.2). By Topology, Lemma 5.15.12 we see that $T_{i,j}$ is a constructible locally closed subscheme of X. Then we can apply Topology, Lemma 5.28.7 to $X = \bigcup T_{i,j}$ to find the desired partition of X. □

095F Lemma 59.71.4. Let X be a scheme. Checking constructibility of a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) can be done Zariski locally on X.

Proof. The statement means if $X = \bigcup U_i$ is an open covering such that $\mathcal{F}|_{U_i}$ is constructible, then $\mathcal{F}$ is constructible. If $U \subset X$ is affine open, then $U = \bigcup U \cap U_i$ and $\mathcal{F}|_{U \cap U_i}$ is constructible (it is trivial that the restriction of a constructible sheaf to an open is constructible). It follows from Lemma 59.71.2 that $\mathcal{F}|_U$ is constructible, i.e., a suitable partition of U exists. □

095G Lemma 59.71.5. Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules (with Λ Noetherian) on $Y_{étale}$, the same is true for $f^{-1}\mathcal{F}$ on $X_{étale}$.

Proof. By Lemma 59.71.4 this reduces to the case where X and Y are affine. By Lemma 59.71.2 it suffices to find a finite partition of X by constructible locally closed subschemes such that $f^{-1}\mathcal{F}$ is finite locally constant on each of them. To find it we just pull back the partition of Y adapted to $\mathcal{F}$ and use Lemma 59.64.2. □

03RZ Lemma 59.71.6. Let X be a scheme.

(1) The category of constructible sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.
(2) The category of constructible abelian sheaves is a weak Serre subcategory of $\textit{Ab}(X_{étale})$.
(3) Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $\textit{Mod}(X_{étale}, \Lambda)$.

Proof. We prove (3). We will use the criterion of Homology, Lemma 12.10.3. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of constructible sheaves of Λ-modules. We have to show that $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ are constructible. Similarly, suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is a short exact sequence of sheaves of Λ-modules with $\mathcal{F}$, $\mathcal{G}$ constructible. We have to show that $\mathcal{E}$ is constructible. In both cases we can replace X with the members of an affine open covering. Hence we may assume X is affine. Then we may further replace X by the members of a finite partition of X by constructible locally closed subschemes on which $\mathcal{F}$ and $\mathcal{G}$ are of finite type and locally constant. Thus we may apply Lemma 59.64.6 to conclude.

The proofs of (1) and (2) are very similar and are omitted. □

09YS Lemma 59.71.7. Let X be a quasi-compact and quasi-separated scheme.

(1) Let $\mathcal{F} \to \mathcal{G}$ be a map of constructible sheaves of sets on $X_{étale}$. Then the set of points $x \in X$ where $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is surjective, resp. injective, resp. is isomorphic to a given map of sets, is constructible in X.
(2) Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. The support of $\mathcal{F}$ is constructible.
(3) Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. The support of $\mathcal{F}$ is constructible.

Proof. Proof of (1). Let $X = \coprod X_i$ be a partition of X by locally closed constructible subschemes such that both $\mathcal{F}$ and $\mathcal{G}$ are finite locally constant over the parts (use Lemma 59.71.2 for both $\mathcal{F}$ and $\mathcal{G}$ and choose a common refinement). Then apply Lemma 59.64.5 to the restriction of the map to each part.

The proof of (2) and (3) is omitted. $\square$

The following lemma will turn out to be very useful later on. It roughly says that the category of constructible sheaves has a kind of weak "Noetherian" property.

095P Lemma 59.71.8. Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F} = \operatorname{colim}_{i\in I} \mathcal{F}_i$ be a filtered colimit of sheaves of sets, abelian sheaves, or sheaves of modules.

(1) If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of sets, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.
(2) If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of abelian groups, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.
(3) Let Λ be a Noetherian ring. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of Λ-modules, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.

Proof. Proof of (1). We will use without further mention that finite limits and colimits of constructible sheaves are constructible (Lemma 59.64.6). For each i let $T_i \subset X$ be the set of points $x \in X$ where $\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}}$ is not surjective. Because $\mathcal{F}_i$ and $\mathcal{F}$ are constructible T_i is a constructible subset of X (Lemma 59.71.7). Since the stalks of $\mathcal{F}$ are finite and since $\mathcal{F} = \operatorname{colim}_{i\in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 28.2.4 the constructible topology on X is quasi-compact by Topology, Lemma 5.23.2. Thus $T_i = \emptyset$ for i large enough. Thus $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $S_{i'} \subset X$ the set of points x such that the number of elements in $\operatorname{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}})$ is equal to the number of elements in $\operatorname{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{i',\overline{x}})$. Because $\mathcal{F}_i$, $\mathcal{F}_{i'}$ and $\mathcal{F}$ are constructible $S_{i'}$ is a constructible subset of X (details omitted; hint: use Lemma 59.71.7). Since the stalks of $\mathcal{F}_i$ and $\mathcal{F}$ are finite and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired.

Proof of (2). Observe that a constructible abelian sheaf is a constructible sheaf of sets. Thus case (2) follows from (1).

Proof of (3). We will use without further mention that the category of constructible sheaves of Λ-modules is abelian (Lemma 59.64.6). For each i let $\mathcal{Q}_i$ be the cokernel of the map $\mathcal{F}_i \to \mathcal{F}$. The support T_i of $\mathcal{Q}_i$ is a constructible subset of X as $\mathcal{Q}_i$ is constructible (Lemma 59.71.7). Since the stalks of $\mathcal{F}$ are finite Λ-modules and since $\mathcal{F} = \operatorname{colim}_{i\in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since

X is a spectral space by Properties, Lemma 28.2.4 the constructible topology on X is quasi-compact by Topology, Lemma 5.23.2. Thus $T_i = \emptyset$ for i large enough. This proves the first assertion. For the second, assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $\mathcal{K}_{i'}$ the image of $\text{Ker}(\mathcal{F}_i \to \mathcal{F})$ in $\mathcal{F}_{i'}$. The support $S_{i'}$ of $\mathcal{K}_{i'}$ is a constructible subset of X as $\mathcal{K}_{i'}$ is constructible. Since the stalks of $\text{Ker}(\mathcal{F}_i \to \mathcal{F})$ are finite Λ-modules and since $\mathcal{F} = \text{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired. □

095I Lemma 59.71.9. Let X be a scheme. Let Λ be a Noetherian ring. The tensor product of two constructible sheaves of Λ-modules on $X_{\acute{e}tale}$ is a constructible sheaf of Λ-modules.

Proof. The question immediately reduces to the case where X is affine. Since any two partitions of X with constructible locally closed strata have a common refinement of the same type and since pullbacks commute with tensor product we reduce to Lemma 59.64.7. □

0GKB Lemma 59.71.10. Let $\Lambda \to \Lambda'$ be a homomorphism of Noetherian rings. Let X be a scheme. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$. Then $\mathcal{F} \otimes_{\underline{\Lambda}} \underline{\Lambda'}$ is a constructible sheaf of Λ'-modules.

Proof. Omitted. Hint: affine locally you can use the same stratification. □

59.72. Auxiliary lemmas on morphisms

095J Some lemmas that are useful for proving functoriality properties of constructible sheaves.

03S0 Lemma 59.72.1. Let $U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes (for example an étale morphism of Noetherian schemes). Then there exists a partition $X = \coprod_i X_i$ by constructible locally closed subschemes such that $X_i \times_X U \to X_i$ is finite étale for all i.

Proof. If $U \to X$ is separated, then this is More on Morphisms, Lemma 37.45.4. In general, we may assume X is affine. Choose a finite affine open covering $U = \bigcup U_j$. Apply the previous case to all the morphisms $U_j \to X$ and $U_j \cap U_{j'} \to X$ and choose a common refinement $X = \coprod X_i$ of the resulting partitions. After refining the partition further we may assume X_i affine as well. Fix i and set $V = U \times_X X_i$. The morphisms $V_j = U_j \times_X X_i \to X_i$ and $V_{jj'} = (U_j \cap U_{j'}) \times_X X_i \to X_i$ are finite étale. Hence V_j and $V_{jj'}$ are affine schemes and $V_{jj'} \subset V_j$ is closed as well as open (since $V_{jj'} \to X_i$ is proper, so Morphisms, Lemma 29.41.7 applies). Then $V = \bigcup V_j$ is separated because $\mathcal{O}(V_j) \to \mathcal{O}(V_{jj'})$ is surjective, see Schemes, Lemma 26.21.7. Thus the previous case applies to $V \to X_i$ and we can further refine the partition if needed (it actually isn't but we don't need this). □

In the Noetherian case one can prove the preceding lemma by Noetherian induction and the following amusing lemma.

03S1 Lemma 59.72.2. Let $f : X \to Y$ be a morphism of schemes which is quasi-compact, quasi-separated, and locally of finite type. If η is a generic point of an irreducible component of Y such that $f^{-1}(\eta)$ is finite, then there exists an open $V \subset Y$ containing η such that $f^{-1}(V) \to V$ is finite.

Proof. This is Morphisms, Lemma 29.51.1. □

The statement of the following lemma can be strengthened a bit.

095K Lemma 59.72.3. Let $f : Y \to X$ be a quasi-finite and finitely presented morphism of affine schemes.

(1) There exists a surjective morphism of affine schemes $X' \to X$ and a closed subscheme $Z' \subset Y' = X' \times_X Y$ such that
  (a) $Z' \subset Y'$ is a thickening, and
  (b) $Z' \to X'$ is a finite étale morphism.
(2) There exists a finite partition $X = \coprod X_i$ by locally closed, constructible, affine strata, and surjective finite locally free morphisms $X'_i \to X_i$ such that the reduction of $Y'_i = X'_i \times_X Y \to X'_i$ is isomorphic to $\coprod_{j=1}^{n_i} (X'_i)_{red} \to (X'_i)_{red}$ for some n_i.

Proof. Setting $X' = \coprod X'_i$ we see that (2) implies (1). Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Write A as a filtered colimit of finite type $\mathbf{Z}$-algebras A_i. Since B is an A-algebra of finite presentation, we see that there exists $0 \in I$ and a finite type ring map $A_0 \to B_0$ such that $B = \text{colim}\, B_i$ with $B_i = A_i \otimes_{A_0} B_0$, see Algebra, Lemma 10.127.8. For i sufficiently large we see that $A_i \to B_i$ is quasi-finite, see Limits, Lemma 32.18.2. Thus we reduce to the case of finite type algebras over $\mathbf{Z}$, in particular we reduce to the Noetherian case. (Details omitted.)

Assume X and Y Noetherian. In this case any locally closed subset of X is constructible. By Lemma 59.72.2 and Noetherian induction we see that there is a finite partition $X = \coprod X_i$ of X by locally closed strata such that $Y \times_X X_i \to X_i$ is finite. We can refine this partition to get affine strata. Thus after replacing X by $X' = \coprod X_i$ we may assume $Y \to X$ is finite.

Assume X and Y Noetherian and $Y \to X$ finite. Suppose that we can prove (2) after base change by a surjective, flat, quasi-finite morphism $U \to X$. Thus we have a partition $U = \coprod U_i$ and finite locally free morphisms $U'_i \to U_i$ such that $U'_i \times_X Y \to U'_i$ is isomorphic to $\coprod_{j=1}^{n_i} (U'_i)_{red} \to (U'_i)_{red}$ for some n_i. Then, by the argument in the previous paragraph, we can find a partition $X = \coprod X_j$ with locally closed affine strata such that $X_j \times_X U_i \to X_j$ is finite for all i, j. By Morphisms, Lemma 29.48.2 each $X_j \times_X U_i \to X_j$ is finite locally free. Hence $X_j \times_X U'_i \to X_j$ is finite locally free (Morphisms, Lemma 29.48.3). It follows that $X = \coprod X_j$ and $X'_j = \coprod_i X_j \times_X U'_i$ is a solution for $Y \to X$. Thus it suffices to prove the result (in the Noetherian case) after a surjective flat quasi-finite base change.

Applying Morphisms, Lemma 29.48.6 we see we may assume that Y is a closed subscheme of an affine scheme Z which is (set theoretically) a finite union $Z = \bigcup_{i \in I} Z_i$ of closed subschemes mapping isomorphically to X. In this case we will find a finite partition of $X = \coprod X_j$ with affine locally closed strata that works (in other words $X'_j = X_j$). Set $T_i = Y \cap Z_i$. This is a closed subscheme of X. As X is Noetherian we can find a finite partition of $X = \coprod X_j$ by affine locally closed subschemes, such that each $X_j \times_X T_i$ is (set theoretically) a union of strata $X_j \times_X Z_i$. Replacing X by X_j we see that we may assume $I = I_1 \amalg I_2$ with $Z_i \subset Y$ for $i \in I_1$ and $Z_i \cap Y = \emptyset$ for $i \in I_2$. Replacing Z by $\bigcup_{i \in I_1} Z_i$ we see that we may assume $Y = Z$. Finally, we can replace X again by the members of a partition as above such that for every $i, i' \subset I$ the intersection $Z_i \cap Z_{i'}$ is either empty or (set theoretically) equal to Z_i and $Z_{i'}$. This clearly means that Y is (set theoretically) equal to a disjoint union of the Z_i which is what we wanted to show. □

59.73. More on constructible sheaves

095M Let Λ be a Noetherian ring. Let X be a scheme. We often consider $X_{\acute{e}tale}$ as a ringed site with sheaf of rings $\underline{\Lambda}$. In case of abelian sheaves we often take $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for a suitable integer n.

03S8 Lemma 59.73.1. Let $j : U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes.

(1) The sheaf h_U is a constructible sheaf of sets.
(2) The sheaf $j_!\underline{M}$ is a constructible abelian sheaf for a finite abelian group M.
(3) If Λ is a Noetherian ring and M is a finite Λ-module, then $j_!\underline{M}$ is a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$.

Proof. By Lemma 59.72.1 there is a partition $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale. The restriction of h_U to X_i is $h_{j^{-1}(X_i)}$ which is finite locally constant by Lemma 59.64.4. For cases (2) and (3) we note that

$$j_!(\underline{M})|_{X_i} = \pi_{i!}(\underline{M}) = \pi_{i*}(\underline{M})$$

by Lemmas 59.70.5 and 59.70.7. Thus it suffices to show the lemma for $\pi : Y \to X$ finite étale. This is Lemma 59.64.3. □

03SA Lemma 59.73.2. Let X be a quasi-compact and quasi-separated scheme.

(1) Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of sets.
(2) Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible abelian sheaves.
(3) Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of Λ-modules.

Proof. Let $\mathcal{B}$ be the collection of quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Modules on Sites, Lemma 18.30.7 any sheaf of sets is a filtered colimit of sheaves of the form

$$\text{Coequalizer}\left(\coprod\nolimits_{j=1,\ldots,m} h_{V_j} \rightrightarrows \coprod\nolimits_{i=1,\ldots,n} h_{U_i} \right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Lemmas 59.73.1 and 59.71.6 these coequalizers are constructible. This proves (1).

Let Λ be a Noetherian ring. By Modules on Sites, Lemma 18.30.7 Λ-modules $\mathcal{F}$ is a filtered colimit of modules of the form

$$\text{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Lemmas 59.73.1 and 59.71.6 these cokernels are constructible. This proves (3).

Proof of (2). First write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n]$ is the n-torsion subsheaf. Then we can view $\mathcal{F}[n]$ as a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and apply (3). □

095Q Lemma 59.73.3. Let $f : X \to Y$ be a surjective morphism of quasi-compact and quasi-separated schemes.

(1) Let $\mathcal{F}$ be a sheaf of sets on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.
(2) Let $\mathcal{F}$ be an abelian sheaf on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.
(3) Let Λ be a Noetherian ring. Let $\mathcal{F}$ be sheaf of Λ-modules on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.

Proof. One implication follows from Lemma 59.71.5. For the converse, assume $f^{-1}\mathcal{F}$ is constructible. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of constructible sheaves (of sets, abelian groups, or modules) using Lemma 59.73.2. Since f^{-1} is a left adjoint it commutes with colimits (Categories, Lemma 4.24.5) and we see that $f^{-1}\mathcal{F} = \operatorname{colim} f^{-1}\mathcal{F}_i$. By Lemma 59.71.8 we see that $f^{-1}\mathcal{F}_i \to f^{-1}\mathcal{F}$ is surjective for all i large enough. Since f is surjective we conclude (by looking at stalks using Lemma 59.36.2 and Theorem 59.29.10) that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i large enough. Thus $\mathcal{F}$ is the quotient of a constructible sheaf $\mathcal{G}$. Applying the argument once more to $\mathcal{G} \times_{\mathcal{F}} \mathcal{G}$ or the kernel of $\mathcal{G} \to \mathcal{F}$ we conclude using that f^{-1} is exact and that the category of constructible sheaves (of sets, abelian groups, or modules) is preserved under finite (co)limits or (co)kernels inside $Sh(Y_{étale})$, $Sh(X_{étale})$, $\text{Ab}(Y_{étale})$, $\text{Ab}(X_{étale})$, $\text{Mod}(Y_{étale}, \Lambda)$, and $\text{Mod}(X_{étale}, \Lambda)$, see Lemma 59.71.6. $\square$

095H Lemma 59.73.4. Let $f : X \to Y$ be a finite étale morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, constructible sheaf of abelian groups, or constructible sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.

Proof. By Lemma 59.71.4 it suffices to check this Zariski locally on Y and by Lemma 59.73.3 we may replace Y by an étale cover (the construction of f_* commutes with étale localization). A finite étale morphism is étale locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. Thus, in the case of sheaves of sets, the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are constructible sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. $\square$

09Y9 Lemma 59.73.5. Let X be a quasi-compact and quasi-separated scheme. The category of constructible sheaves of sets is the full subcategory of $Sh(X_{étale})$ consisting of sheaves $\mathcal{F}$ which are coequalizers

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

such that $\mathcal{F}_i$, $i = 0, 1$ is a finite coproduct of sheaves of the form h_U with U a quasi-compact and quasi-separated object of $X_{étale}$.

Proof. In the proof of Lemma 59.73.2 we have seen that sheaves of this form are constructible. For the converse, suppose that for every constructible sheaf of sets $\mathcal{F}$ we can find a surjection $\mathcal{F}_0 \to \mathcal{F}$ with $\mathcal{F}_0$ as in the lemma. Then we find our surjection $\mathcal{F}_1 \to \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ because the latter is constructible by Lemma 59.71.6.

By Topology, Lemma 5.28.7 we may choose a finite stratification $X = \coprod_{i \in I} X_i$ such that $\mathcal{F}$ is finite locally constant on each stratum. We will prove the result by induction on the cardinality of I. Let $i \in I$ be a minimal element in the partial ordering of I. Then $X_i \subset X$ is closed. By induction, there exist finitely many quasi-compact and quasi-separated objects U_α of $(X \setminus X_i)_{\acute{e}tale}$ and a surjective map $\coprod h_{U_\alpha} \to \mathcal{F}|_{X \setminus X_i}$. These determine a map

$$\coprod h_{U_\alpha} \to \mathcal{F}$$

which is surjective after restricting to $X \setminus X_i$. By Lemma 59.64.4 we see that $\mathcal{F}|_{X_i} = h_V$ for some scheme V finite étale over X_i. Let $\overline{v}$ be a geometric point of V lying over $\overline{x} \in X_i$. We may think of $\overline{v}$ as an element of the stalk $\mathcal{F}_{\overline{x}} = V_{\overline{x}}$. Thus we can find an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ and a section $s \in \mathcal{F}(U)$ whose stalk at $\overline{x}$ gives $\overline{v}$. Thinking of s as a map $s : h_U \to \mathcal{F}$, restricting to X_i we obtain a morphism $s|_{X_i} : U \times_X X_i \to V$ over X_i which maps $\overline{u}$ to $\overline{v}$. Since V is quasi-compact (finite over the closed subscheme X_i of the quasi-compact scheme X) a finite number $s^{(1)}, \ldots, s^{(m)}$ of these sections of $\mathcal{F}$ over $U^{(1)}, \ldots, U^{(m)}$ will determine a jointly surjective map

$$\coprod s^{(j)}|_{X_i} : \coprod U^{(j)} \times_X X_i \longrightarrow V$$

Then we obtain the surjection

$$\coprod h_{U_\alpha} \amalg \coprod h_{U^{(j)}} \to \mathcal{F}$$

as desired. □

095N Lemma 59.73.6. Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules is exactly the category of modules of the form

$$\text{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. In fact, we can even assume U_i and V_j affine.

Proof. In the proof of Lemma 59.73.2 we have seen modules of this form are constructible. Since the category of constructible modules is abelian (Lemma 59.71.6) it suffices to prove that given a constructible module $\mathcal{F}$ there is a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

for some affine objects U_i in $X_{\acute{e}tale}$. By Modules on Sites, Lemma 18.30.7 there is a surjection

$$\Psi : \bigoplus\nolimits_{i \in I} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

with U_i affine and the direct sum over a possibly infinite index set I. For every finite subset $I' \subset I$ set

$$T_{I'} = \text{Supp}(\text{Coker}(\bigoplus\nolimits_{i \in I'} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}))$$

By the very definition of constructible sheaves, the set $T_{I'}$ is a constructible subset of X. We want to show that $T_{I'} = \emptyset$ for some I'. Since every stalk $\mathcal{F}_{\overline{x}}$ is a finite type Λ-module and since Ψ is surjective, for every $x \in X$ there is an I' such that $x \notin T_{I'}$. In other words we have $\emptyset = \bigcap_{I' \subset I \text{ finite}} T_{I'}$. Since X is a spectral space by Properties, Lemma 28.2.4 the constructible topology on X is quasi-compact by Topology, Lemma 5.23.2. Thus $T_{I'} = \emptyset$ for some $I' \subset I$ finite as desired. □

09YT Lemma 59.73.7. Let X be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form

$$\text{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\mathbf{Z}/m_j\mathbf{Z}}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\mathbf{Z}/n_i\mathbf{Z}}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$ and m_j, n_i positive integers. In fact, we can even assume U_i and V_j affine.

Proof. This follows from Lemma 59.73.6 applied with $\Lambda = \mathbf{Z}/n\mathbf{Z}$ and the fact that, since X is quasi-compact, every constructible abelian sheaf is annihilated by some positive integer n (details omitted). $\square$

09Z4 Lemma 59.73.8. Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{\acute{e}tale}$. Let $\mathcal{G} = \text{colim}\, \mathcal{G}_i$ be a filtered colimit of sheaves of sets, abelian groups, or Λ-modules. Then

$$\text{Mor}(\mathcal{F}, \mathcal{G}) = \text{colim}\, \text{Mor}(\mathcal{F}, \mathcal{G}_i)$$

in the category of sheaves of sets, abelian groups, or Λ-modules on $X_{\acute{e}tale}$.

Proof. The case of sheaves of sets. By Lemma 59.73.5 it suffices to prove the lemma for h_U where U is a quasi-compact and quasi-separated object of $X_{\acute{e}tale}$. Recall that $\text{Mor}(h_U, \mathcal{G}) = \mathcal{G}(U)$. Hence the result follows from Sites, Lemma 7.17.7.

In the case of abelian sheaves or sheaves of modules, the result follows in the same way using Lemmas 59.73.7 and 59.73.6. For the case of abelian sheaves, we add that $\text{Mor}(j_{U!}\underline{\mathbf{Z}/n\mathbf{Z}}, \mathcal{G})$ is equal to the n-torsion elements of $\mathcal{G}(U)$. $\square$

095R Lemma 59.73.9. Let $f : X \to Y$ be a finite and finitely presented morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{\acute{e}tale}$, then $f_*\mathcal{F}$ is too.

Proof. It suffices to prove this when X and Y are affine by Lemma 59.71.4. By Lemmas 59.55.3 and 59.73.3 we may base change to any affine scheme surjective over X. By Lemma 59.72.3 this reduces us to the case of a finite étale morphism (because a thickening leads to an equivalence of étale topoi and even small étale sites, see Theorem 59.45.2). The finite étale case is Lemma 59.73.4. $\square$

09YU Lemma 59.73.10. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$.

(1) The category of constructible sheaves of sets on $X_{\acute{e}tale}$ is the colimit of the categories of constructible sheaves of sets on $(X_i)_{\acute{e}tale}$.
(2) The category of constructible abelian sheaves on $X_{\acute{e}tale}$ is the colimit of the categories of constructible abelian sheaves on $(X_i)_{\acute{e}tale}$.
(3) Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{\acute{e}tale}$ is the colimit of the categories of constructible sheaves of Λ-modules on $(X_i)_{\acute{e}tale}$.

Proof. Proof of (1). Denote $f_i : X \to X_i$ the projection maps. There are 3 parts to the proof corresponding to "faithful", "fully faithful", and "essentially surjective".

Faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a, b : \mathcal{F}_0 \to \mathcal{G}_0$ are maps such that $f_0^{-1}a = f_0^{-1}b$. Let $E \subset X_0$ be the set of points $x \in X_0$ such that $a_{\overline{x}} = b_{\overline{x}}$. By Lemma 59.71.7 the subset $E \subset X_0$ is constructible. By assumption $X \to X_0$ maps into E. By Limits, Lemma 32.4.10 we find an $i \geq 0$ such that $X_i \to X_0$ maps into E. Hence $f_{i0}^{-1}a = f_{i0}^{-1}b$.

Fully faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a : f_0^{-1}\mathcal{F}_0 \to f_0^{-1}\mathcal{G}_0$ is a map. We claim there is an i and a map $a_i : f_{i0}^{-1}\mathcal{F}_0 \to f_{i0}^{-1}\mathcal{G}_0$ which pulls back to a on X. By Lemma 59.73.5 we can replace $\mathcal{F}_0$ by a finite coproduct of sheaves represented by quasi-compact and quasi-separated objects of $(X_0)_{\acute{e}tale}$. Thus we have to show: If $U_0 \to X_0$ is such an object of $(X_0)_{\acute{e}tale}$, then

$$f_0^{-1}\mathcal{G}(U) = \text{colim}_{i \geq 0} f_{i0}^{-1}\mathcal{G}(U_i)$$

where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. This is a special case of Theorem 59.51.3.

Essentially surjective. We have to show every constructible $\mathcal{F}$ on X is isomorphic to $f_i^{-1}\mathcal{F}$ for some constructible $\mathcal{F}_i$ on X_i. Applying Lemma 59.73.5 and using the results of the previous two paragraphs, we see that it suffices to prove this for h_U for some quasi-compact and quasi-separated object U of $X_{\acute{e}tale}$. In this case we have to show that U is the base change of a quasi-compact and quasi-separated scheme étale over X_i for some i. This follows from Limits, Lemmas 32.10.1 and 32.8.10.

Proof of (3). The argument is very similar to the argument for sheaves of sets, but using Lemma 59.73.6 instead of Lemma 59.73.5. Details omitted. Part (2) follows from part (3) because every constructible abelian sheaf over a quasi-compact scheme is a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n. □

0GL2 **Lemma 59.73.11.** Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$.

(1) The category of finite locally constant sheaves on $X_{\acute{e}tale}$ is the colimit of the categories of finite locally constant sheaves on $(X_i)_{\acute{e}tale}$.
(2) The category of finite locally constant abelian sheaves on $X_{\acute{e}tale}$ is the colimit of the categories of finite locally constant abelian sheaves on $(X_i)_{\acute{e}tale}$.
(3) Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $X_{\acute{e}tale}$ is the colimit of the categories of finite type, locally constant sheaves of Λ-modules on $(X_i)_{\acute{e}tale}$.

Proof. By Lemma 59.73.10 the functor in each case is fully faithful. By the same lemma, all we have to show to finish the proof in case (1) is the following: given a constructible sheaf $\mathcal{F}_i$ on X_i whose pullback $\mathcal{F}$ to X is finite locally constant, there exists an $i' \geq i$ such that the pullback $\mathcal{F}_{i'}$ of $\mathcal{F}_i$ to $X_{i'}$ is finite locally constant. By assumption there exists an étale covering $\mathcal{U} = \{U_j \to X\}_{j \in J}$ such that $\mathcal{F}|_{U_j} \cong \underline{S_j}$ for some finite set S_j. We may assume U_j is affine for all $j \in J$. Since X is quasi-compact, we may assume J finite. By Lemma 59.51.2 we can find an $i' \geq i$ and an étale covering $\mathcal{U}_{i'} = \{U_{i',j} \to X_{i'}\}_{j \in J}$ whose base change to X is $\mathcal{U}$. Then $\mathcal{F}_{i'}|_{U_{i',j}}$ and $\underline{S_j}$ are constructible sheaves on $(U_{i',j})_{\acute{e}tale}$ whose pullbacks to U_j are isomorphic. Hence after increasing i' we get that $\mathcal{F}_{i'}|_{U_{i',j}}$ and $\underline{S_j}$ are isomorphic.

Thus $\mathcal{F}_{i'}$ is finite locally constant. The proof in cases (2) and (3) is exactly the same. $\square$

09BG Lemma 59.73.12. Let X be an irreducible scheme with generic point η.

(1) Let $S' \subset S$ be an inclusion of sets. If we have $\underline{S'} \subset \mathcal{G} \subset \underline{S}$ in $Sh(X_{\acute{e}tale})$ and $S' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{S'}$.
(2) Let $A' \subset A$ be an inclusion of abelian groups. If we have $\underline{A'} \subset \mathcal{G} \subset \underline{A}$ in $\textit{Ab}(X_{\acute{e}tale})$ and $A' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{A'}$.
(3) Let $M' \subset M$ be an inclusion of modules over a ring Λ. If we have $\underline{M'} \subset \mathcal{G} \subset \underline{M}$ in $\textit{Mod}(X_{\acute{e}tale}, \underline{\Lambda})$ and $M' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{M'}$.

Proof. This is true because for every étale morphism $U \to X$ with $U \neq \emptyset$ the point η is in the image. $\square$

09Z5 Lemma 59.73.13. Let X be an integral normal scheme with function field K. Let E be a set.

(1) Let $g : \mathrm{Spec}(K) \to X$ be the inclusion of the generic point. Then $g_*\underline{E} = \underline{E}$.
(2) Let $j : U \to X$ be the inclusion of a nonempty open. Then $j_*\underline{E} = \underline{E}$.

Proof. Proof of (1). Let $x \in X$ be a point. Let $\mathcal{O}^{sh}_{X,\overline{x}}$ be a strict henselization of $\mathcal{O}_{X,x}$. By More on Algebra, Lemma 15.45.6 we see that $\mathcal{O}^{sh}_{X,\overline{x}}$ is a normal domain. Hence $\mathrm{Spec}(K) \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ is irreducible. It follows that the stalk $(g_*\underline{E}_{\overline{x}}$ is equal to E, see Theorem 59.53.1.

Proof of (2). Since g factors through j there is a map $j_*\underline{E} \to g_*\underline{E}$. This map is injective because for every scheme V étale over X the set $\mathrm{Spec}(K) \times_X V$ is dense in $U \times_X V$. On the other hand, we have a map $\underline{E} \to j_*\underline{E}$ and we conclude. $\square$

0F0M Lemma 59.73.14. Let X be a quasi-compact and quasi-separated scheme. Let $\eta \in X$ be a generic point of an irreducible component of X.

(1) Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\acute{e}tale}$ whose stalk $\mathcal{F}_{\overline{\eta}}$ is zero. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing η.
(2) Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{\acute{e}tale}$ whose stalk $\mathcal{F}_{\overline{\eta}}$ is zero. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible sheaves of Λ-modules $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing η.

Proof. Proof of (1). We can write $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 59.73.2. Choose $i \in I$. Since $\mathcal{F}|_\eta$ is zero by assumption, we see that there exists an $i'(i) \geq i$ such that $\mathcal{F}_i|_\eta \to \mathcal{F}_{i'(i)}|_\eta$ is zero, see Lemma 59.71.8. Then $\mathcal{G}_i = \mathrm{Im}(\mathcal{F}_i \to \mathcal{F}_{i'(i)})$ is a constructible abelian sheaf (Lemma 59.71.6) whose stalk at η is zero. Hence the support E_i of $\mathcal{G}_i$ is a constructible subset of X not containing η. Since η is a generic point of an irreducible component of X, we see that $\eta \notin Z_i = \overline{E_i}$ by Topology, Lemma 5.15.15. Define a new directed set I' by using the set I with ordering defined by the rule i_1 is bigger or equal to i_2 if and only if $i_1 \geq i'(i_2)$. Then the sheaves $\mathcal{G}_i$ form a system over I' with colimit $\mathcal{F}$ and the proof is complete.

The proof in case (2) is exactly the same and we omit it. $\square$

59.74. Constructible sheaves on Noetherian schemes

03RY If X is a Noetherian scheme then any locally closed subset is a constructible locally closed subset (Topology, Lemma 5.16.1). Hence an abelian sheaf $\mathcal{F}$ on $X_{étale}$ is constructible if and only if there exists a finite partition $X = \coprod X_i$ such that $\mathcal{F}|_{X_i}$ is finite locally constant. (By convention a partition of a topological space has locally closed parts, see Topology, Section 5.28.) In other words, we can omit the adjective "constructible" in Definition 59.71.1. Actually, the category of constructible sheaves on Noetherian schemes has some additional properties which we will catalogue in this section.

09BH **Proposition 59.74.1.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring.*

(1) *Any sub or quotient sheaf of a constructible sheaf of sets is constructible.*
(2) *The category of constructible abelian sheaves on $X_{étale}$ is a (strong) Serre subcategory of $\text{Ab}(X_{étale})$. In particular, every sub and quotient sheaf of a constructible abelian sheaf on $X_{étale}$ is constructible.*
(3) *The category of constructible sheaves of Λ-modules on $X_{étale}$ is a (strong) Serre subcategory of $\text{Mod}(X_{étale}, \Lambda)$. In particular, every submodule and quotient module of a constructible sheaf of Λ-modules on $X_{étale}$ is constructible.*

Proof. Proof of (1). Let $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{S}$ for some finite set S. Then $S' = \mathcal{G}_{\overline{\eta}} \subset S$ say $S' = \{s_1, \ldots, s_t\}$. Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset S$. Since σ_i maps to an element $s_i \in S' \subset S = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\text{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{S'} \subset \mathcal{G} \subset \underline{S}$. Then we see that $\underline{S'} = \mathcal{G}$ by Lemma 59.73.12.

Let $\mathcal{F} \to \mathcal{Q}$ be a surjection with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Then set $\mathcal{G} = \mathcal{F} \times_{\mathcal{Q}} \mathcal{F}$. By the first part of the proof we see that $\mathcal{G}$ is constructible as a subsheaf of $\mathcal{F} \times \mathcal{F}$. This in turn implies that $\mathcal{Q}$ is constructible, see Lemma 59.71.6.

Proof of (3). we already know that constructible sheaves of modules form a weak Serre subcategory, see Lemma 59.71.6. Thus it suffices to show the statement on submodules.

Let $\mathcal{G} \subset \mathcal{F}$ be a submodule of a constructible sheaf of Λ-modules on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{M}$ for some finite Λ-module M. Then $M' = \mathcal{G}_{\overline{\eta}} \subset M$. Pick finitely many elements $s_1, \ldots, s_t$ generating M' as a Λ-module. (This is possible as Λ is Noetherian and M is finite.) Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset M$. Since σ_i maps to an element

$s_i \in M' \subset M = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\mathrm{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{M'} \subset \mathcal{G} \subset \underline{M}$. Then we see that $\underline{M'} = \mathcal{G}$ by Lemma 59.73.12.

Proof of (2). This follows in the usual manner from (3). Details omitted. □

The following lemma tells us that every object of the abelian category of constructible sheaves on X is "Noetherian", i.e., satisfies a.c.c. for subobjects.

09YV Lemma 59.74.2. Let X be a Noetherian scheme. Let Λ be a Noetherian ring. Consider inclusions

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots \subset \mathcal{F}$$

in the category of sheaves of sets, abelian groups, or Λ-modules. If $\mathcal{F}$ is constructible, then for some n we have $\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}_{n+2} = \ldots$.

Proof. By Proposition 59.74.1 we see that $\mathcal{F}_i$ and $\mathrm{colim}\, \mathcal{F}_i$ are constructible. Then the lemma follows from Lemma 59.71.8. □

09Z6 Lemma 59.74.3. Let X be a Noetherian scheme.

(1) Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{\acute{e}tale}$. There exist an injective map of sheaves
$$\mathcal{F} \longrightarrow \prod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite morphism and E_i is a finite set.

(2) Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{\acute{e}tale}$. There exist an injective map of abelian sheaves
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite abelian group.

(3) Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$. There exist an injective map of sheaves of modules
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite Λ-module.

Moreover, we may assume each Y_i is irreducible, reduced, maps onto an irreducible and reduced closed subscheme $Z_i \subset X$ such that $Y_i \to Z_i$ is finite étale over a nonempty open of Z_i.

Proof. Proof of (1). Because we have the ascending chain condition for subsheaves of $\mathcal{F}$ (Lemma 59.74.2), it suffices to show that for every point $x \in X$ we can find a map $\varphi : \mathcal{F} \to f_*\underline{E}$ where $f : Y \to X$ is finite and E is a finite set such that $\varphi_{\overline{x}} : \mathcal{F}_{\overline{x}} \to (f_*S)_{\overline{x}}$ is injective. (This argument can be avoided by picking a partition of X as in Lemma 59.71.2 and constructing a $Y_i \to X$ for each irreducible component of each part.) Let $Z \subset X$ be the induced reduced scheme structure (Schemes, Definition 26.12.5) on $\overline{\{x\}}$. Since $\mathcal{F}$ is constructible, there is a finite separable extension $K/\kappa(x)$ such that $\mathcal{F}|_{\mathrm{Spec}(K)}$ is the constant sheaf with value E for some finite set E. Let $Y \to Z$ be the normalization of Z in $\mathrm{Spec}(K)$. By Morphisms, Lemma 29.53.13 we see that Y is a normal integral scheme. As $K/\kappa(x)$ is a finite extension, it is clear that K is the function field of Y. Denote $g : \mathrm{Spec}(K) \to Y$ the inclusion. The map $\mathcal{F}|_{\mathrm{Spec}(K)} \to \underline{E}$ is adjoint to a map

$\mathcal{F}|_Y \to g_*\underline{E} = \underline{E}$ (Lemma 59.73.13). This in turn is adjoint to a map $\varphi : \mathcal{F} \to f_*\underline{E}$. Observe that the stalk of φ at a geometric point $\overline{x}$ is injective: we may take a lift $\overline{y} \in Y$ of $\overline{x}$ and the commutative diagram

$$\begin{tikzcd} \mathcal{F}_{\overline{x}} \arrow[r, equal] \arrow[d] & (\mathcal{F}|_Y)_{\overline{y}} \arrow[d, equal] \\ (f_*\underline{E})_{\overline{x}} \arrow[r] & \underline{E}_{\overline{y}} \end{tikzcd}$$

proves the injectivity. We are not yet done, however, as the morphism $f : Y \to Z$ is integral but in general not finite[6].

To fix the problem stated in the last sentence of the previous paragraph, we write $Y = \lim_{i \in I} Y_i$ with Y_i irreducible, integral, and finite over Z. Namely, apply Properties, Lemma 28.22.13 to $f_*\mathcal{O}_Y$ viewed as a sheaf of $\mathcal{O}_Z$-algebras and apply the functor $\underline{\mathrm{Spec}}_Z$. Then $f_*\underline{E} = \mathrm{colim}\, f_{i,*}\underline{E}$ by Lemma 59.51.7. By Lemma 59.73.8 the map $\mathcal{F} \to f_*\underline{E}$ factors through $f_{i,*}\underline{E}$ for some i. Since $Y_i \to Z$ is a finite morphism of integral schemes and since the function field extension induced by this morphism is finite separable, we see that the morphism is finite étale over a nonempty open of Z (use Algebra, Lemma 10.140.9; details omitted). This finishes the proof of (1).

The proofs of (2) and (3) are identical to the proof of (1). $\square$

In the following lemma we use a standard trick to reduce a very general statement to the Noetherian case.

09Z7 **Lemma 59.74.4.** Let X be a quasi-compact and quasi-separated scheme. [AGV71, Exposee IX, Proposition 2.14]

(1) Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{étale}$. There exist an injective map of sheaves
$$\mathcal{F} \longrightarrow \prod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and E_i is a finite set.

(2) Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. There exist an injective map of abelian sheaves
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite abelian group.

(3) Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. There exist an injective map of sheaves of modules
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite Λ-module.

Proof. We will reduce this lemma to the Noetherian case by absolute Noetherian approximation. Namely, by Limits, Proposition 32.5.4 we can write $X = \lim_{t \in T} X_t$ with each X_t of finite type over $\mathrm{Spec}(\mathbf{Z})$ and with affine transition morphisms. By Lemma 59.73.10 the category of constructible sheaves (of sets, abelian groups, or

[6]If X is a Nagata scheme, for example of finite type over a field, then $Y \to Z$ is finite.

Λ-modules) on $X_{étale}$ is the colimit of the corresponding categories for X_t. Thus our constructible sheaf $\mathcal{F}$ is the pullback of a similar constructible sheaf $\mathcal{F}_t$ over X_t for some t. Then we apply the Noetherian case (Lemma 59.74.3) to find an injection

$$\mathcal{F}_t \longrightarrow \prod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i} \quad \text{or} \quad \mathcal{F}_t \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$

over X_t for some finite morphisms $f_i : Y_i \to X_t$. Since X_t is Noetherian the morphisms f_i are of finite presentation. Since pullback is exact and since formation of $f_{i,*}$ commutes with base change (Lemma 59.55.3), we conclude. □

0F0N Lemma 59.74.5. Let X be a Noetherian scheme. Let $E \subset X$ be a subset closed under specialization.

(1) Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$ whose support is contained in E. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subset contained in E.
(2) Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$ whose support is contained in E. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ is a filtered colimit of constructible sheaves of Λ-modules $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subset contained in E.

Proof. Proof of (1). We can write $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 59.73.2. By Proposition 59.74.1 the image $\mathcal{F}_i' \subset \mathcal{F}$ of the map $\mathcal{F}_i \to \mathcal{F}$ is constructible. Thus $\mathcal{F} = \operatorname{colim} \mathcal{F}_i'$ and the support of $\mathcal{F}_i'$ is contained in E. Since the support of $\mathcal{F}_i'$ is constructible (by our definition of constructible sheaves), we see that its closure is also contained in E, see for example Topology, Lemma 5.23.6.

The proof in case (2) is exactly the same and we omit it. □

59.75. Specializations and étale sheaves

0GJ2 Topological picture: Let X be a topological space and let $x' \rightsquigarrow x$ be a specialization of points in X. Then every open neighbourhood of x contains x'. Hence for any sheaf $\mathcal{F}$ on X there is a specialization map

$$sp : \mathcal{F}_x \longrightarrow \mathcal{F}_{x'}$$

of stalks sending the equivalence class of the pair (U, s) in $\mathcal{F}_x$ to the equivalence class of the pair (U, s) in $\mathcal{F}_{x'}$; see Sheaves, Section 6.11 for the description of stalks in terms of equivalence classes of pairs. Of course this map is functorial in $\mathcal{F}$, i.e., sp is a transformation of functors.

For sheaves in the étale topology we can mimick this construction, see [AGV71, Exposee VII, 7.7, page 397]. To do this suppose we have a scheme S, a geometric point $\overline{s}$ of S, and a geometric point $\overline{t}$ of $\operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. For any sheaf $\mathcal{F}$ on $S_{étale}$ we will construct the specialization map

$$sp : \mathcal{F}_{\overline{s}} \longrightarrow \mathcal{F}_{\overline{t}}$$

Here we have abused language: instead of writing $\mathcal{F}_{\overline{t}}$ we should write $\mathcal{F}_{p(\overline{t})}$ where $p : \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to S$ is the canonical morphism. Recall that

$$\mathcal{F}_{\overline{s}} = \operatorname{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where the colimit is over all étale neighbourhoods $(U, \overline{u})$ of $(S, \overline{s})$, see Section 59.29. Since $\mathcal{O}^{sh}_{S,\overline{s}}$ is the stalk of the structure sheaf, we find for every étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ a canonical map $\mathcal{O}_{U,u} \to \mathcal{O}^{sh}_{S,\overline{s}}$. Hence we get a unique factorization

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to U \to S$$

If $\overline{v}$ denotes the image of $\overline{t}$ in U, then we see that $(U, \overline{v})$ is an étale neighbourhood of $(S, \overline{t})$. This construction defines a functor from the category of étale neighbourhoods of $(S, \overline{s})$ to the category of étale neighbourhoods of $(S, \overline{t})$. Thus we may define the map $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ by sending the equivalence class of $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$ to the equivalence class of $(U, \overline{v}, \sigma)$.

Let $K \in D(S_{étale})$. With $\overline{s}$ and $\overline{t}$ as above we have the specialization map

$$sp : K_{\overline{s}} \longrightarrow K_{\overline{t}} \quad \text{in} \quad D(\mathrm{Ab})$$

Namely, if K is represented by the complex $\mathcal{F}^\bullet$ of abelian sheaves, then we simply that the map

$$K_{\overline{s}} = \mathcal{F}^\bullet_{\overline{s}} \longrightarrow \mathcal{F}^\bullet_{\overline{t}} = K_{\overline{t}}$$

which is termwise given by the specialization maps for sheaves constructed above. This is independent of the choice of complex representing K by the exactness of the stalk functors (i.e., taking stalks of complexes is well defined on the derived category).

Clearly the construction is functorial in the sheaf $\mathcal{F}$ on $S_{étale}$. If we think of the stalk functors as morphisms of topoi $\overline{s}, \overline{t} : \mathrm{Sets} \to Sh(S_{étale})$, then we may think of sp as a 2-morphism

$$\begin{tikzcd} \mathrm{Sets} \arrow[r, bend left, "\overline{t}"] \arrow[r, bend right, "\overline{s}"'] \arrow[r, phantom, "\Downarrow sp"] & Sh(S_{étale}) \end{tikzcd}$$

of topoi.

0GJ3 Remark 59.75.1 (Alternative description of sp). Let S, $\overline{s}$, and $\overline{t}$ be as above. Another way to describe the specialization map is to use that

$$\mathcal{F}_{\overline{s}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{F}) \quad \text{and} \quad \mathcal{F}_{\overline{t}} = \Gamma(\overline{t}, \overline{t}^{-1}p^{-1}\mathcal{F})$$

The first equality follows from Theorem 59.53.1 applied to $\mathrm{id}_S : S \to S$ and the second equality follows from Lemma 59.36.2. Then we can think of sp as the map

$$sp : \mathcal{F}_{\overline{s}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{F}) \xrightarrow{\text{pullback by } \overline{t}} \Gamma(\overline{t}, \overline{t}^{-1}p^{-1}\mathcal{F}) = \mathcal{F}_{\overline{t}}$$

0GJ4 Remark 59.75.2 (Yet another description of sp). Let S, $\overline{s}$, and $\overline{t}$ be as above. Another alternative is to use the unique morphism

$$c : \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}) \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$$

over S which is compatible with the given morphism $\overline{t} \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ and the morphism $\overline{t} \to \mathrm{Spec}(\mathcal{O}^{sh}_{t,\overline{t}})$. The uniqueness and existence of the displayed arrow follows from Algebra, Lemma 10.154.6 applied to $\mathcal{O}_{S,s}$, $\mathcal{O}^{sh}_{S,\overline{t}}$, and $\mathcal{O}^{sh}_{S,\overline{s}} \to \kappa(\overline{t})$. We obtain

$$sp : \mathcal{F}_{\overline{s}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), \mathcal{F}) \xrightarrow{\text{pullback by } c} \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), \mathcal{F}) = \mathcal{F}_{\overline{t}}$$

(with obvious notational conventions). In fact this procedure also works for objects K in $D(S_{\acute{e}tale})$: the specialization map for K is equal to the map

$$sp : K_{\overline{s}} = R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), K) \xrightarrow{\text{pullback by } c} R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), K) = K_{\overline{t}}$$

The equality signs are valid as taking global sections over the striclty henselian schemes $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ and $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}})$ is exact (and the same as taking stalks at $\overline{s}$ and $\overline{t}$) and hence no subtleties related to the fact that K may be unbounded arise.

0GJ5 Remark 59.75.3 (Lifting specializations). Let S be a scheme and let $t \leadsto s$ be a specialization of point on S. Choose geometric points $\overline{t}$ and $\overline{s}$ lying over t and s. Since t corresponds to a point of $\mathrm{Spec}(\mathcal{O}_{S,s})$ by Schemes, Lemma 26.13.2 and since $\mathcal{O}_{S,s} \to \mathcal{O}^{sh}_{S,\overline{s}}$ is faithfully flat, we can find a point $t' \in \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ mapping to t. As $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is a limit of schemes étale over S we see that $\kappa(t')/\kappa(t)$ is a separable algebraic extension (usually not finite of course). Since $\kappa(\overline{t})$ is algebraically closed, we can choose an embedding $\kappa(t') \to \kappa(\overline{t})$ as extensions of $\kappa(t)$. This choice gives us a commutative diagram

$$\begin{tikzcd}
\overline{t} \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \arrow[d] & \overline{s} \arrow[l] \arrow[d] \\
t \arrow[r] & S & s \arrow[l]
\end{tikzcd}$$

of points and geometric points. Thus if $t \leadsto s$ we can always "lift" $\overline{t}$ to a geometric point of the strict henselization of S at $\overline{s}$ and get specialization maps as above.

0GJ6 Lemma 59.75.4. Let $g : S' \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf on $S_{\acute{e}tale}$. Let $\overline{s}'$ be a geometric point of S', and let $\overline{t}'$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S',\overline{s}'})$. Denote $\overline{s} = g(\overline{s}')$ and $\overline{t} = h(\overline{t}')$ where $h : \mathrm{Spec}(\mathcal{O}^{sh}_{S',\overline{s}'}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the canonical morphism. For any sheaf $\mathcal{F}$ on $S_{\acute{e}tale}$ the specialization map

$$sp : (f^{-1}\mathcal{F})_{\overline{s}'} \longrightarrow (f^{-1}\mathcal{F})_{\overline{t}'}$$

is equal to the specialization map $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ via the identifications $(f^{-1}\mathcal{F})_{\overline{s}'} = \mathcal{F}_{\overline{s}}$ and $(f^{-1}\mathcal{F})_{\overline{t}'} = \mathcal{F}_{\overline{t}}$ of Lemma 59.36.2.

Proof. Omitted. □

0GJ7 Lemma 59.75.5. Let S be a scheme such that every quasi-compact open of S has finite number of irreducible components (for example if S has a Noetherian underlying topological space, or if S is locally Noetherian). Let $\mathcal{F}$ be a sheaf of sets on $S_{\acute{e}tale}$. The following are equivalent

(1) $\mathcal{F}$ is finite locally constant, and
(2) all stalks of $\mathcal{F}$ are finite sets and all specialization maps $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ are bijective.

Proof. Assume (2). Let $\overline{s}$ be a geometric point of S lying over $s \in S$. In order to prove (1) we have to find an étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ such that $\mathcal{F}|_U$ is constant. We may and do assume S is affine.

Since $\mathcal{F}_{\overline{s}}$ is finite, we can choose $(U, \overline{u})$, $n \geq 0$, and pairwise distinct elements $\sigma_1, \ldots, \sigma_n \in \mathcal{F}(U)$ such that $\{\sigma_1, \ldots, \sigma_n\} \subset \mathcal{F}(U)$ maps bijectively to $\mathcal{F}_{\overline{s}}$ via the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$. Consider the map

$$\varphi : \underline{\{1, \ldots, n\}} \longrightarrow \mathcal{F}|_U$$

on $U_{étale}$ defined by $\sigma_1, \ldots, \sigma_n$. This map is a bijection on stalks at $\overline{u}$ by construction. Let us consider the subset

$$E = \{u' \in U \mid \varphi_{\overline{u}'} \text{ is bijective}\} \subset U$$

Here $\overline{u}'$ is any geometric point of U lying over u' (the condition is independent of the choice by Remark 59.29.8). The image $u \in U$ of $\overline{u}$ is in E. By our assumption on the specialization maps for $\mathcal{F}$, by Remark 59.75.3, and by Lemma 59.75.4 we see that E is closed under specializations and generalizations in the topological space U.

After shrinking U we may assume U is affine too. By Descent, Lemma 35.16.3 we see that U has a finite number of irreducible components. After removing the irreducible components which do not pass through u, we may assume every irreducible component of U passes through u. Since U is a sober topological space it follows that $E = U$ and we conclude that φ is an isomorphism by Theorem 59.29.10. Thus (1) follows.

We omit the proof that (1) implies (2). □

0GKC **Lemma 59.75.6.** Let S be a scheme such that every quasi-compact open of S has finite number of irreducible components (for example if S has a Noetherian underlying topological space, or if S is locally Noetherian). Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $S_{étale}$. The following are equivalent

(1) $\mathcal{F}$ is a finite type, locally constant sheaf of Λ-modules, and
(2) all stalks of $\mathcal{F}$ are finite Λ-modules and all specialization maps $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ are bijective.

Proof. The proof of this lemma is the same as the proof of Lemma 59.75.5. Assume (2). Let $\overline{s}$ be a geometric point of S lying over $s \in S$. In order to prove (1) we have to find an étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ such that $\mathcal{F}|_U$ is constant. We may and do assume S is affine.

Since $M = \mathcal{F}_{\overline{s}}$ is a finite Λ-module and Λ is Noetherian, we can choose a presentation

$$\Lambda^{\oplus m} \xrightarrow{A} \Lambda^{\oplus n} \to M \to 0$$

for some matrix $A = (a_{ji})$ with coefficients in Λ. We can choose $(U, \overline{u})$ and elements $\sigma_1, \ldots, \sigma_n \in \mathcal{F}(U)$ such that $\sum a_{ji}\sigma_i = 0$ in $\mathcal{F}(U)$ and such that the images of σ_i in $\mathcal{F}_{\overline{s}} = M$ are the images of the standard basis element of Λ^n in the presentation of M given above. Consider the map

$$\varphi : \underline{M} \longrightarrow \mathcal{F}|_U$$

on $U_{étale}$ defined by $\sigma_1, \ldots, \sigma_n$. This map is a bijection on stalks at $\overline{u}$ by construction. Let us consider the subset

$$E = \{u' \in U \mid \varphi_{\overline{u}'} \text{ is bijective}\} \subset U$$

Here $\overline{u}'$ is any geometric point of U lying over u' (the condition is independent of the choice by Remark 59.29.8). The image $u \in U$ of $\overline{u}$ is in E. By our assumption on the specialization maps for $\mathcal{F}$, by Remark 59.75.3, and by Lemma 59.75.4 we see that E is closed under specializations and generalizations in the topological space U.

After shrinking U we may assume U is affine too. By Descent, Lemma 35.16.3 we see that U has a finite number of irreducible components. After removing the irreducible components which do not pass through u, we may assume every irreducible component of U passes through u. Since U is a sober topological space it follows that $E = U$ and we conclude that φ is an isomorphism by Theorem 59.29.10. Thus (1) follows.

We omit the proof that (1) implies (2). $\square$

0GJ8 Lemma 59.75.7. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $K \in D^+(X_{étale})$. Let $\overline{s}$ be a geometric point of S and let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. We have a commutative diagram

$$\begin{tikzcd} (Rf_*K)_{\overline{s}} \arrow[rr, "sp"'] \arrow[d, equal] & & (Rf_*K)_{\overline{t}} \arrow[d, equal] \\ R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), K) \arrow[rr] & & R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), K) \end{tikzcd}$$

where the bottom horizontal arrow arises as pullback by the morphism $\mathrm{id}_X \times c$ where $c : \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the morphism introduced in Remark 59.75.2. The vertical arrows are given by Theorem 59.53.1.

Proof. This follows immediately from the description of sp in Remark 59.75.2. $\square$

0GJ9 Remark 59.75.8. Let $f : X \to S$ be a morphism of schemes. Let $K \in D(X_{étale})$. Let $\overline{s}$ be a geometric point of S and let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Let c be as in Remark 59.75.2. We can always make a commutative diagram

$$\begin{tikzcd} (Rf_*K)_{\overline{s}} \arrow[r] \arrow[d, "sp"'] & R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), K) \arrow[r] \arrow[d, "(\mathrm{id}_X \times c)^{-1}"'] & R\Gamma(X_{\overline{s}}, K) \\ (Rf_*K)_{\overline{t}} \arrow[r] & R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), K) \arrow[r] & R\Gamma(X_{\overline{t}}, K) \end{tikzcd}$$

where the horizontal arrows are those of Remark 59.53.2. In general there won't be a vertical map on the right between the cohomologies of K on the fibres fitting into this diagram, even in the case of Lemma 59.75.7.

59.76. Complexes with constructible cohomology

095V Let Λ be a ring. Denote $D(X_{étale}, \Lambda)$ the derived category of sheaves of Λ-modules on $X_{étale}$. We denote by $D^b(X_{étale}, \Lambda)$ (respectively D^+, D^-) the full subcategory of bounded (resp. above, below) complexes in $D(X_{étale}, \Lambda)$.

095W Definition 59.76.1. Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_c(X_{étale}, \Lambda)$ the full subcategory of $D(X_{étale}, \Lambda)$ of complexes whose cohomology sheaves are constructible sheaves of Λ-modules.

This definition makes sense by Lemma 59.71.6 and Derived Categories, Section 13.17. Thus we see that $D_c(X_{étale}, \Lambda)$ is a strictly full, saturated triangulated subcategory of $D(X_{étale}, \Lambda)$.

095X Lemma 59.76.2. Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then

(1) $K|_U \in D_c(U_{étale}, \Lambda)$ if $K \in D_c(X_{étale}, \Lambda)$, and

(2) $j_!M \in D_c(X_{étale}, \Lambda)$ if $M \in D_c(U_{étale}, \Lambda)$ and the morphism j is quasi-compact and quasi-separated.

Proof. The first assertion is clear. The second follows from the fact that $j_!$ is exact and Lemma 59.73.1. □

095Y Lemma 59.76.3. Let Λ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_c(Y_{étale}, \Lambda)$ then $Lf^*K \in D_c(X_{étale}, \Lambda)$.

Proof. This follows as $f^{-1} = f^*$ is exact and Lemma 59.71.5. □

095Z Lemma 59.76.4. Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $K \in D(X_{étale}, \Lambda)$ and $b \in \mathbf{Z}$ such that $H^b(K)$ is constructible. Then there exist a sheaf $\mathcal{F}$ which is a finite direct sum of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}[-b] \to K$ in $D(X_{étale}, \Lambda)$ inducing a surjection $\mathcal{F} \to H^b(K)$.

Proof. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Modules on Sites, Lemma 18.30.6 we may choose a surjection $\bigoplus_{i \in I} j_{U_i!}\underline{\Lambda} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$ with U_i affine. For $I' \subset I$ finite, denote $\mathcal{H}_{I'} \subset H^b(K)$ the image of $\bigoplus_{i \in I'} j_{U_i!}\underline{\Lambda}$. By Lemma 59.71.8 we see that $\mathcal{H}_{I'} = H^b(K)$ for some $I' \subset I$ finite. The lemma follows taking $\mathcal{F} = \bigoplus_{i \in I'} j_{U_i!}\underline{\Lambda}$. □

0960 Lemma 59.76.5. Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $K \in D^-(X_{étale}, \Lambda)$. Then the following are equivalent

(1) K is in $D_c(X_{étale}, \Lambda)$,
(2) K can be represented by a bounded above complex whose terms are finite direct sums of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine,
(3) K can be represented by a bounded above complex of flat constructible sheaves of Λ-modules.

Proof. It is clear that (2) implies (3) and that (3) implies (1). Assume K is in $D_c^-(X_{étale}, \Lambda)$. Say $H^i(K) = 0$ for $i > b$. By induction on a we will construct a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ such that each $\mathcal{F}^i$ is a finite direct sum of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}^\bullet \to K$ which induces an isomorphism $H^i(\mathcal{F}^\bullet) \to H^i(K)$ for $i > a$ and a surjection $H^a(\mathcal{F}^\bullet) \to H^a(K)$. For $a = b$ this can be done by Lemma 59.76.4. Given such a datum choose a distinguished triangle

$$\mathcal{F}^\bullet \to K \to L \to \mathcal{F}^\bullet[1]$$

Then we see that $H^i(L) = 0$ for $i \geq a$. Choose $\mathcal{F}^{a-1}[-a+1] \to L$ as in Lemma 59.76.4. The composition $\mathcal{F}^{a-1}[-a+1] \to L \to \mathcal{F}^\bullet$ corresponds to a map $\mathcal{F}^{a-1} \to \mathcal{F}^a$ such that the composition with $\mathcal{F}^a \to \mathcal{F}^{a+1}$ is zero. By TR4 we obtain a map

$$(\mathcal{F}^{a-1} \to \ldots \to \mathcal{F}^b) \to K$$

in $D(X_{étale}, \Lambda)$. This finishes the induction step and the proof of the lemma. □

0961 Lemma 59.76.6. Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X_{étale}, \Lambda)$. Then $K \otimes_\Lambda^{\mathbf{L}} L$ is in $D_c^-(X_{étale}, \Lambda)$.

Proof. This follows from Lemmas 59.76.5 and 59.71.9. □

59.77. Tor finite with constructible cohomology

0F4M Let X be a scheme and let Λ be a Noetherian ring. An often used subcategory of the derived category $D_c(X_{étale}, \Lambda)$ defined in Section 59.76 is the full subcategory consisting of objects having (locally) finite tor dimension. Here is the formal definition.

03TQ Definition 59.77.1. Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_{ctf}(X_{étale}, \Lambda)$ the full subcategory of $D_c(X_{étale}, \Lambda)$ consisting of objects having locally finite tor dimension.

This is a strictly full, saturated triangulated subcategory of $D_c(X_{étale}, \Lambda)$ and $D(X_{étale}, \Lambda)$. By our conventions, see Cohomology on Sites, Definition 21.46.1, we see that

$$D_{ctf}(X_{étale}, \Lambda) \subset D^b_c(X_{étale}, \Lambda) \subset D^b(X_{étale}, \Lambda)$$

if X is quasi-compact. A good way to think about objects of $D_{ctf}(X_{étale}, \Lambda)$ is given in Lemma 59.77.3.

03TS Remark 59.77.2. Objects in the derived category $D_{ctf}(X_{étale}, \Lambda)$ in some sense have better global properties than the perfect objects in $D(\mathcal{O}_X)$. Namely, it can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. The following lemma shows this does not happen for D_{ctf} on a Noetherian scheme.

03TT Lemma 59.77.3. Let Λ be a Noetherian ring. Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(X_{étale}, \Lambda)$. The following are equivalent

(1) $K \in D_{ctf}(X_{étale}, \Lambda)$, and
(2) K can be represented by a finite complex of constructible flat sheaves of Λ-modules.

In fact, if K has tor amplitude in $[a, b]$ then we can represent K by a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ with $\mathcal{F}^p$ a constructible flat sheaf of Λ-modules.

Proof. It is clear that a finite complex of constructible flat sheaves of Λ-modules has finite tor dimension. It is also clear that it is an object of $D_c(X_{étale}, \Lambda)$. Thus we see that (2) implies (1).

Assume (1). Choose $a, b \in \mathbf{Z}$ such that $H^i(K \otimes^{\mathbf{L}}_\Lambda \mathcal{G}) = 0$ if $i \notin [a, b]$ for all sheaves of Λ-modules $\mathcal{G}$. We will prove the final assertion holds by induction on $b - a$. If $a = b$, then $K = H^a(K)[-a]$ is a flat constructible sheaf and the result holds. Next, assume $b > a$. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Lemma 59.73.6 we can find finitely many affine schemes U_i étale over X and a surjection $\bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to H^b(K)$. After replacing U_i by standard étale coverings $\{U_{ij} \to U_i\}$ we may assume this surjection lifts to a map $\mathcal{F} = \bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$. This map determines a distinguished triangle

$$\mathcal{F}[-b] \to K \to L \to \mathcal{F}[-b+1]$$

in $D(X_{étale}, \Lambda)$. Since $D_{ctf}(X_{étale}, \Lambda)$ is a triangulated subcategory we see that L is in it too. In fact L has tor amplitude in $[a, b-1]$ as $\mathcal{F}$ surjects onto $H^b(K)$ (details omitted). By induction hypothesis we can find a finite complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}$

of flat constructible sheaves of Λ-modules representing L. The map $L \to \mathcal{F}[-b+1]$ corresponds to a map $\mathcal{F}^b \to \mathcal{F}$ annihilating the image of $\mathcal{F}^{b-1} \to \mathcal{F}^b$. Then it follows from axiom TR3 that K is represented by the complex

$$\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1} \to \mathcal{F}^b$$

which finishes the proof. $\square$

03TR Remark 59.77.4. Let Λ be a Noetherian ring. Let X be a scheme. For a bounded complex $K^\bullet$ of constructible flat Λ-modules on $X_{\acute{e}tale}$ each stalk $K^p_{\bar{x}}$ is a finite projective Λ-module. Hence the stalks of the complex are perfect complexes of Λ-modules.

0962 Lemma 59.77.5. Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then

(1) $K|_U \in D_{ctf}(U_{\acute{e}tale}, \Lambda)$ if $K \in D_{ctf}(X_{\acute{e}tale}, \Lambda)$, and
(2) $j_!M \in D_{ctf}(X_{\acute{e}tale}, \Lambda)$ if $M \in D_{ctf}(U_{\acute{e}tale}, \Lambda)$ and the morphism j is quasi-compact and quasi-separated.

Proof. Perhaps the easiest way to prove this lemma is to reduce to the case where X is affine and then apply Lemma 59.77.3 to translate it into a statement about finite complexes of flat constructible sheaves of Λ-modules where the result follows from Lemma 59.73.1. $\square$

0963 Lemma 59.77.6. Let Λ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_{ctf}(Y_{\acute{e}tale}, \Lambda)$ then $Lf^*K \in D_{ctf}(X_{\acute{e}tale}, \Lambda)$.

Proof. Apply Lemma 59.77.3 to reduce this to a question about finite complexes of flat constructible sheaves of Λ-modules. Then the statement follows as $f^{-1} = f^*$ is exact and Lemma 59.71.5. $\square$

09BI Lemma 59.77.7. Let X be a connected scheme. Let Λ be a Noetherian ring. Let $K \in D_{ctf}(X_{\acute{e}tale}, \Lambda)$ have locally constant cohomology sheaves. Then there exists a finite complex of finite projective Λ-modules $M^\bullet$ and an étale covering $\{U_i \to X\}$ such that $K|_{U_i} \cong \underline{M^\bullet}|_{U_i}$ in $D(U_{i,\acute{e}tale}, \Lambda)$.

Proof. Choose an étale covering $\{U_i \to X\}$ such that $K|_{U_i}$ is constant, say $K|_{U_i} \cong \underline{M_i^\bullet}_{U_i}$ for some finite complex of finite Λ-modules $M_i^\bullet$. See Cohomology on Sites, Lemma 21.53.1. Observe that $U_i \times_X U_j$ is empty if $M_i^\bullet$ is not isomorphic to $M_j^\bullet$ in $D(\Lambda)$. For each complex of Λ-modules $M^\bullet$ let $I_{M^\bullet} = \{i \in I \mid M_i^\bullet \cong M^\bullet \text{ in } D(\Lambda)\}$. As étale morphisms are open we see that $U_{M^\bullet} = \bigcup_{i \in I_{M^\bullet}} \text{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_{M^\bullet}$ is a disjoint open covering of X. As X is connected only one $U_{M^\bullet}$ is nonempty. As K is in $D_{ctf}(X_{\acute{e}tale}, \Lambda)$ we see that $M^\bullet$ is a perfect complex of Λ-modules, see More on Algebra, Lemma 15.74.2. Hence we may assume $M^\bullet$ is a finite complex of finite projective Λ-modules. $\square$

59.78. Torsion sheaves

0DDB A brief section on torsion abelian sheaves and their étale cohomology. Let $\mathcal{C}$ be a site. We have shown in Cohomology on Sites, Lemma 21.19.8 that any object in $D(\mathcal{C})$ whose cohomology sheaves are torsion sheaves, can be represented by a complex all of whose terms are torsion.

0DDC Lemma 59.78.1. Let X be a quasi-compact and quasi-separated scheme.

(1) If $\mathcal{F}$ is a torsion abelian sheaf on $X_{\acute{e}tale}$, then $H^n_{\acute{e}tale}(X, \mathcal{F})$ is a torsion abelian group for all n.
(2) If K in $D^+(X_{\acute{e}tale})$ has torsion cohomology sheaves, then $H^n_{\acute{e}tale}(X, K)$ is a torsion abelian group for all n.

Proof. To prove (1) we write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[d]$ is the d-torsion subsheaf. By Lemma 59.51.4 we have $H^n_{\acute{e}tale}(X, \mathcal{F}) = \text{colim}\, H^n_{\acute{e}tale}(X, \mathcal{F}[d])$. This proves (1) as $H^n_{\acute{e}tale}(X, \mathcal{F}[d])$ is annihilated by d.

To prove (2) we can use the spectral sequence $E_2^{p,q} = H^p_{\acute{e}tale}(X, H^q(K))$ converging to $H^n_{\acute{e}tale}(X, K)$ (Derived Categories, Lemma 13.21.3) and the result for sheaves. ☐

0DDD Lemma 59.78.2. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes.

(1) If $\mathcal{F}$ is a torsion abelian sheaf on $X_{\acute{e}tale}$, then $R^n f_*\mathcal{F}$ is a torsion abelian sheaf on $Y_{\acute{e}tale}$ for all n.
(2) If K in $D^+(X_{\acute{e}tale})$ has torsion cohomology sheaves, then Rf_*K is an object of $D^+(Y_{\acute{e}tale})$ whose cohomology sheaves are torsion abelian sheaves.

Proof. Proof of (1). Recall that $R^n f_*\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^n_{\acute{e}tale}(X \times_Y V, \mathcal{F})$ on $Y_{\acute{e}tale}$. See Cohomology on Sites, Lemma 21.7.4. If we choose V affine, then $X \times_Y V$ is quasi-compact and quasi-separated because f is, hence we can apply Lemma 59.78.1 to see that $H^n_{\acute{e}tale}(X \times_Y V, \mathcal{F})$ is torsion.

Proof of (2). Recall that $R^n f_*K$ is the sheaf associated to the presheaf $V \mapsto H^n_{\acute{e}tale}(X \times_Y V, K)$ on $Y_{\acute{e}tale}$. See Cohomology on Sites, Lemma 21.20.6. If we choose V affine, then $X \times_Y V$ is quasi-compact and quasi-separated because f is, hence we can apply Lemma 59.78.1 to see that $H^n_{\acute{e}tale}(X \times_Y V, K)$ is torsion. ☐

59.79. Cohomology with support in a closed subscheme

09XP Let X be a scheme and let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z\}$$

be the sections with support in Z (Definition 59.31.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{\acute{e}tale}) \longrightarrow D(\text{Ab})$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\acute{e}tale}$. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 21.12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{\acute{e}tale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(\text{Ab})$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{\acute{e}tale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we can consider the subsheaf of sections with support in Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Definition 59.31.3. Using the equivalence of Proposition 59.46.4 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$\text{Ab}(X_{étale}) \longrightarrow \text{Ab}(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

09XQ Lemma 59.79.1. Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\text{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \text{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Section 59.46) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

09XR Lemma 59.79.2. Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.

Proof. This is true because the functor i_* is exact and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 21.14.2). □

0A45 Lemma 59.79.3. Let $i : Z \to X$ be a closed immersion of schemes. Let $j : U \to X$ be the inclusion of the complement of Z. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. There is a distinguished triangle

$$i_*R\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to Rj_*(\mathcal{F}|_U) \to i_*R\mathcal{H}_Z(\mathcal{F})[1]$$

in $D(X_{étale})$. This produces an exact sequence

$$0 \to i_*\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to i_*\mathcal{H}^1_Z(\mathcal{F}) \to 0$$

and isomorphisms $R^pj_*(\mathcal{F}|_U) \cong i_*\mathcal{H}^{p+1}_Z(\mathcal{F})$ for $p \geq 1$.

Proof. To get the distinguished triangle, choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then we obtain a short exact sequence of complexes

$$0 \to i_*\mathcal{H}_Z(\mathcal{I}^\bullet) \to \mathcal{I}^\bullet \to j_*(\mathcal{I}^\bullet|_U) \to 0$$

by the discussion above. Thus the distinguished triangle by Derived Categories, Section 13.12. □

Let X be a scheme and let $Z \subset X$ be a closed subscheme. We denote $D_Z(X_{étale})$ the strictly full saturated triangulated subcategory of $D(X_{étale})$ consisting of complexes whose cohomology sheaves are supported on Z. Note that $D_Z(X_{étale})$ only depends on the underlying closed subset of X.

0AEG Lemma 59.79.4. Let $i : Z \to X$ be a closed immersion of schemes. The map $Ri_{small,*} = i_{small,*} : D(Z_{étale}) \to D(X_{étale})$ induces an equivalence $D(Z_{étale}) \to D_Z(X_{étale})$ with quasi-inverse

$$i_{small}^{-1}|_{D_Z(X_{étale})} = R\mathcal{H}_Z|_{D_Z(X_{étale})}$$

Proof. Recall that i_{small}^{-1} and $i_{small,*}$ is an adjoint pair of exact functors such that $i_{small}^{-1}i_{small,*}$ is isomorphic to the identify functor on abelian sheaves. See Proposition 59.46.4 and Lemma 59.36.2. Thus $i_{small,*} : D(Z_{étale}) \to D_Z(X_{étale})$ is fully faithful and i_{small}^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X_{étale})$ and consider the adjunction map $K \to i_{small,*}i_{small}^{-1}K$. Using exactness of $i_{small,*}$ and i_{small}^{-1} this induces the adjunction maps $H^n(K) \to i_{small,*}i_{small}^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z_{étale}) \to D_Z(X_{étale})$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i_{small}^{-1}K$ if K is an object of $D_Z(X_{étale})$. To do this we can use that $K = i_{small,*}i_{small}^{-1}K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i_{small}^{-1}K$. Since $i_{small,*}$ is the right adjoint to the exact functor i_{small}^{-1}, the complex $i_{small,*}\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.9). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_{small,*}\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. $\square$

0A46 Lemma 59.79.5. Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and denote $\mathcal{F}^a$ the associated quasi-coherent sheaf on the small étale site of X (Proposition 59.17.1). Then

(1) $H^q_Z(X, \mathcal{F})$ agrees with $H^q_Z(X_{étale}, \mathcal{F}^a)$,
(2) if the complement of Z is retrocompact in X, then $i_*\mathcal{H}^q_Z(\mathcal{F}^a)$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules equal to $(i_*\mathcal{H}^q_Z(\mathcal{F}))^a$.

Proof. Let $j : U \to X$ be the inclusion of the complement of Z. The statement (1) on cohomology groups follows from the long exact sequences for cohomology with supports and the agreements $H^q(X_{étale}, \mathcal{F}^a) = H^q(X, \mathcal{F})$ and $H^q(U_{étale}, \mathcal{F}^a) = H^q(U, \mathcal{F})$, see Theorem 59.22.4. If $j : U \to X$ is a quasi-compact morphism, i.e., if $U \subset X$ is retrocompact, then R^qj_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Cohomology of Schemes, Lemma 30.4.5) and commutes with taking associated sheaf on étale sites (Descent, Lemma 35.9.5). We conclude by applying Lemma 59.79.3. $\square$

59.80. Schemes with strictly henselian local rings

0EZN In this section we collect some results about the étale cohomology of schemes whose local rings are strictly henselian. For example, here is a fun generalization of Lemma 59.55.1.

09AX Lemma 59.80.1. Let S be a scheme all of whose local rings are strictly henselian. Then for any abelian sheaf $\mathcal{F}$ on $S_{étale}$ we have $H^i(S_{étale}, \mathcal{F}) = H^i(S_{Zar}, \mathcal{F})$.

Proof. Let $\epsilon : S_{étale} \to S_{Zar}$ be the morphism of sites given by the inclusion functor. The Zariski sheaf $R^p\epsilon_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p_{étale}(U, \mathcal{F})$. Thus the stalk at $x \in X$ is $\text{colim}\, H^p_{étale}(U, \mathcal{F}) = H^p_{étale}(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{G}_x)$ where $\mathcal{G}_x$ denotes the pullback of $\mathcal{F}$ to $\text{Spec}(\mathcal{O}_{X,x})$, see Lemma 59.51.5. Thus the higher direct images of $R^p\epsilon_*\mathcal{F}$ are zero by Lemma 59.55.1 and we conclude by the Leray spectral sequence. $\square$

0GY0 Lemma 59.80.2. Let R be a ring all of whose local rings are strictly henselian. Let $\mathcal{F}$ be a sheaf on $\mathrm{Spec}(R)_{\acute{e}tale}$. Assume that for all $f, g \in R$ the kernel of

$$H^1_{\acute{e}tale}(D(f+g), \mathcal{F}) \longrightarrow H^1_{\acute{e}tale}(D(f(f+g)), \mathcal{F}) \oplus H^1_{\acute{e}tale}(D(g(f+g)), \mathcal{F})$$

is zero. Then $H^q_{\acute{e}tale}(\mathrm{Spec}(R), \mathcal{F}) = 0$ for $q > 0$.

Proof. By Lemma 59.80.1 we see that étale cohomology of $\mathcal{F}$ agrees with Zariski cohomology on any open of $\mathrm{Spec}(R)$. We will prove by induction on i the statement: for $h \in R$ we have $H^q_{\acute{e}tale}(D(h), \mathcal{F}) = 0$ for $1 \leq q \leq i$. The base case $i = 0$ is trivial. Assume $i \geq 1$.

Let $\xi \in H^q_{\acute{e}tale}(D(h), \mathcal{F})$ for some $1 \leq q \leq i$ and $h \in R$. If $q < i$ then we are done by induction, so we assume $q = i$. After replacing R by R_h we may assume $\xi \in H^i_{\acute{e}tale}(\mathrm{Spec}(R), \mathcal{F})$; some details omitted. Let $I \subset R$ be the set of elements $f \in R$ such that $\xi|_{D(f)} = 0$. Since ξ is Zariski locally trivial, it follows that for every prime $\mathfrak{p}$ of R there exists an $f \in I$ with $f \not\in \mathfrak{p}$. Thus if we can show that I is an ideal, then $1 \in I$ and we're done. It is clear that $f \in I$, $r \in R$ implies $rf \in I$. Thus we assume that $f, g \in I$ and we show that $f + g \in I$. If $q = i = 1$, then this is exactly the assumption of the lemma! Whence the result for $i = 1$. For $q = i > 1$, note that

$$D(f+g) = D(f(f+g)) \cup D(g(f+g))$$

By Mayer-Vietoris (Cohomology, Lemma 20.8.2 which applies as étale cohomology on open subschemes of $\mathrm{Spec}(R)$ equals Zariski cohomology) we have an exact sequence

$$\begin{tikzcd} H^{i-1}_{\acute{e}tale}(D(fg(f+g)), \mathcal{F}) \arrow[d] \\ H^i_{\acute{e}tale}(D(f+g), \mathcal{F}) \arrow[d] \\ H^i_{\acute{e}tale}(D(f(f+g)), \mathcal{F}) \oplus H^i_{\acute{e}tale}(D(g(f+g)), \mathcal{F}) \end{tikzcd}$$

and the result follows as the first group is zero by induction. $\square$

09AY Lemma 59.80.3. Let S be an affine scheme such that (1) all points are closed, and (2) all residue fields are separably algebraically closed. Then for any abelian sheaf $\mathcal{F}$ on $S_{\acute{e}tale}$ we have $H^i(S_{\acute{e}tale}, \mathcal{F}) = 0$ for $i > 0$.

Proof. Condition (1) implies that the underlying topological space of S is profinite, see Algebra, Lemma 10.26.5. Thus the higher cohomology groups of an abelian sheaf on the topological space S (i.e., Zariski cohomology) is trivial, see Cohomology, Lemma 20.22.3. The local rings are strictly henselian by Algebra, Lemma 10.153.10. Thus étale cohomology of S is computed by Zariski cohomology by Lemma 59.80.1 and the proof is done. $\square$

The spectrum of an absolutely integrally closed ring is an example of a scheme all of whose local rings are strictly henselian, see More on Algebra, Lemma 15.14.7. It turns out that normal domains with separably closed fraction fields have an even stronger property as explained in the following lemma.

09Z9 **Lemma 59.80.4.** Let X be an integral normal scheme with separably closed function field.

(1) A separated étale morphism $U \to X$ is a disjoint union of open immersions.
(2) All local rings of X are strictly henselian.

Proof. Let R be a normal domain whose fraction field is separably algebraically closed. Let $R \to A$ be an étale ring map. Then $A \otimes_R K$ is as a K-algebra a finite product $\prod_{i=1,\ldots,n} K$ of copies of K. Let e_i, $i = 1, \ldots, n$ be the corresponding idempotents of $A \otimes_R K$. Since A is normal (Algebra, Lemma 10.163.9) the idempotents e_i are in A (Algebra, Lemma 10.37.12). Hence $A = \prod Ae_i$ and we may assume $A \otimes_R K = K$. Since $A \subset A \otimes_R K = K$ (by flatness of $R \to A$ and since $R \subset K$) we conclude that A is a domain. By the same argument we conclude that $A \otimes_R A \subset (A \otimes_R A) \otimes_R K = K$. It follows that the map $A \otimes_R A \to A$ is injective as well as surjective. Thus $R \to A$ defines an open immersion by Morphisms, Lemma 29.10.2 and Étale Morphisms, Theorem 41.14.1.

Let $f : U \to X$ be a separated étale morphism. Let $\eta \in X$ be the generic point and let $f^{-1}(\{\eta\}) = \{\xi_i\}_{i \in I}$. The result of the previous paragraph shows the following: For any affine open $U' \subset U$ whose image in X is contained in an affine we have $U' = \coprod_{i \in I} U'_i$ where U'_i is the set of point of U' which are specializations of ξ_i. Moreover, the morphism $U'_i \to X$ is an open immersion. It follows that $U_i = \overline{\{\xi_i\}}$ is an open and closed subscheme of U and that $U_i \to X$ is locally on the source an isomorphism. By Morphisms, Lemma 29.49.7 the fact that $U_i \to X$ is separated, implies that $U_i \to X$ is injective and we conclude that $U_i \to X$ is an open immersion, i.e., (1) holds.

Part (2) follows from part (1) and the description of the strict henselization of $\mathcal{O}_{X,x}$ as the local ring at $\overline{x}$ on the étale site of X (Lemma 59.33.1). It can also be proved directly, see Fundamental Groups, Lemma 58.12.2. $\square$

0EZP **Lemma 59.80.5.** Let $f : X \to Y$ be a morphism of schemes where X is an integral normal scheme with separably closed function field. Then $R^q f_*\underline{M} = 0$ for $q > 0$ and any abelian group M.

Proof. Recall that $R^q f_*\underline{M}$ is the sheaf associated to the presheaf $V \mapsto H^q_{étale}(V \times_Y X, M)$ on $Y_{étale}$, see Lemma 59.51.6. If V is affine, then $V \times_Y X \to X$ is separated and étale. Hence $V \times_Y X = \coprod U_i$ is a disjoint union of open subschemes U_i of X, see Lemma 59.80.4. By Lemma 59.80.1 we see that $H^q_{étale}(U_i, M)$ is equal to $H^q_{Zar}(U_i, M)$. This vanishes by Cohomology, Lemma 20.20.2. $\square$

09ZA **Lemma 59.80.6.** Let X be an affine integral normal scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. Let $V \to Z$ be an étale morphism with V affine. Then V is a finite disjoint union of open subschemes of Z. If $V \to Z$ is surjective and finite étale, then $V \to Z$ has a section.

Proof. By Algebra, Lemma 10.143.10 we can lift V to an affine scheme U étale over X. Apply Lemma 59.80.4 to $U \to X$ to get the first statement.

The final statement is a consequence of the first. Let $V = \coprod_{i=1,\ldots,n} V_i$ be a finite decomposition into open and closed subschemes with $V_i \to Z$ an open immersion. As $V \to Z$ is finite we see that $V_i \to Z$ is also closed. Let $U_i \subset Z$ be the image. Then we have a decomposition into open and closed subschemes

$$Z = \coprod_{(A,B)} \bigcap_{i \in A} U_i \cap \bigcap_{i \in B} U_i^c$$

where the disjoint union is over $\{1, \ldots, n\} = A \amalg B$ where A has at least one element. Each of the strata is contained in a single U_i and we find our section. $\square$

09ZB Lemma 59.80.7. Let X be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^1_{étale}(Z, \underline{M}) = 0$.

Proof. By Cohomology on Sites, Lemma 21.4.3 an element of $H^1_{étale}(Z, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $Z_{étale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over Z, Lemma 59.64.4. Of course $V \to Z$ is surjective as a torsor is locally trivial. Since $V \to Z$ has a section by Lemma 59.80.6 we are done. $\square$

09ZC Lemma 59.80.8. Let X be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^q_{étale}(Z, \underline{M}) = 0$ for $q \geq 1$.

Proof. Write $X = \text{Spec}(R)$ and $Z = \text{Spec}(R')$ so that we have a surjection of rings $R \to R'$. All local rings of R' are strictly henselian by Lemma 59.80.4 and Algebra, Lemma 10.156.4. Furthermore, we see that for any $f' \in R'$ there is a surjection $R_f \to R'_{f'}$ where $f \in R$ is a lift of f'. Since R_f is a normal domain with separably closed fraction field we see that $H^1_{étale}(D(f'), \underline{M}) = 0$ by Lemma 59.80.7. Thus we may apply Lemma 59.80.2 to $Z = \text{Spec}(R')$ to conclude. $\square$

09ZD Lemma 59.80.9. Let X be an affine scheme.

(1) There exists an integral surjective morphism $X' \to X$ such that for every closed subscheme $Z' \subset X'$, every finite abelian group M, and every $q \geq 1$ we have $H^q_{étale}(Z', \underline{M}) = 0$.
(2) For any closed subscheme $Z \subset X$, finite abelian group M, $q \geq 1$, and $\xi \in H^q_{étale}(Z, \underline{M})$ there exists a finite surjective morphism $X' \to X$ of finite presentation such that ξ pulls back to zero in $H^q_{étale}(X' \times_X Z, \underline{M})$.

Proof. Write $X = \text{Spec}(A)$. Write $A = \mathbf{Z}[x_i]/J$ for some ideal J. Let R be the integral closure of $\mathbf{Z}[x_i]$ in an algebraic closure of the fraction field of $\mathbf{Z}[x_i]$. Let $A' = R/JR$ and set $X' = \text{Spec}(A')$. This gives an example as in (1) by Lemma 59.80.8.

Proof of (2). Let $X' \to X$ be the integral surjective morphism we found above. Certainly, ξ maps to zero in $H^q_{étale}(X' \times_X Z, \underline{M})$. We may write X' as a limit $X' = \lim X'_i$ of schemes finite and of finite presentation over X; this is easy to do in our current affine case, but it is a special case of the more general Limits, Lemma 32.7.3. By Lemma 59.51.5 we see that ξ maps to zero in $H^q_{étale}(X'_i \times_X Z, \underline{M})$ for some i large enough. $\square$

59.81. Absolutely integrally closed vanishing

0GY1 Recall that we say a ring R is absolutely integrally closed if every monic polynomial over R has a root in R (More on Algebra, Definition 15.14.1). In this section we prove that the étale cohomology of $\mathrm{Spec}(R)$ with coefficients in a finite torsion group vanishes in positive degrees (Proposition 59.81.5) thereby slightly improving the earlier Lemma 59.80.8. We suggest the reader skip this section.

0GY2 **Lemma 59.81.1.** Let A be a ring. Let $a, b \in A$ such that $aA+bA = A$ and $a \bmod bA$ is a root of unity. Then there exists a monogenic extension $A \subset B$ and an element $y \in B$ such that $u = a - by$ is a unit.

Proof. Say $a^n \equiv 1 \bmod bA$. In particular a^i is a unit modulo $b^m A$ for all $i, m \geq 1$. We claim there exist $a_1, \ldots, a_n \in A$ such that

$$1 = a^n + a_1 a^{n-1} b + a_2 a^{n-2} b^2 + \ldots + a_n b^n$$

Namely, since $1-a^n \in bA$ we can find an element $a_1 \in A$ such that $1-a^n-a_1a^{n-1}b \in b^2A$ using the unit property of a^{n-1} modulo bA. Next, we can find an element $a_2 \in A$ such that $1-a^n-a_1a^{n-1}b-a_2a^{n-2}b^2 \in b^3A$. And so on. Eventually we find $a_1, \ldots, a_{n-1} \in A$ such that $1-(a^n+a_1a^{n-1}b+a_2a^{n-2}b^2+\ldots+a_{n-1}ab^{n-1}) \in b^nA$. This allows us to find $a_n \in A$ such that the displayed equality holds.

With $a_1, \ldots, a_n$ as above we claim that setting

$$B = A[y]/(y^n + a_1 y^{n-1} + a_2 y^{n-2} + \ldots + a_n)$$

works. Namely, suppose that $\mathfrak q \subset B$ is a prime ideal lying over $\mathfrak p \subset A$. To get a contradiction assume $u = a - by$ is in $\mathfrak q$. If $b \in \mathfrak p$ then $a \not\in \mathfrak p$ as $aA + bA = A$ and hence u is not in $\mathfrak q$. Thus we may assume $b \not\in \mathfrak p$, i.e., $b \not\in \mathfrak q$. This implies that $y \bmod \mathfrak q$ is equal to $a/b \bmod \mathfrak q$. However, then we obtain

$$0 = y^n+a_1y^{n-1}+a_2y^{n-2}+\ldots+a_n = b^{-n}(a^n+a_1a^{n-1}b+a_2a^{n-2}b^2+\ldots+a_nb^n) = b^{-n}$$

a contradiction. This finishes the proof. $\square$

In order to explain the proof we need to introduce some group schemes. Fix a prime number ℓ. Let

$$A = \mathbf{Z}[\zeta] = \mathbf{Z}[x]/(x^{\ell-1} + x^{\ell-2} + \ldots + 1)$$

In other words A is the monogenic extension of $\mathbf{Z}$ generated by a primitive ℓth root of unity ζ. We set

$$\pi = \zeta - 1$$

A calculation (omitted) shows that ℓ is divisible by $\pi^{\ell-1}$ in A. Our first group scheme over A is

$$G = \mathrm{Spec}(A[s, \frac{1}{\pi s + 1}])$$

with group law given by the comultiplication

$$\mu : A[s, \frac{1}{\pi s+1}] \longrightarrow A[s, \frac{1}{\pi s+1}] \otimes_A A[s, \frac{1}{\pi s+1}], \quad s \longmapsto \pi s \otimes s + s \otimes 1 + 1 \otimes s$$

With this choice we have

$$\mu(\pi s + 1) = (\pi s + 1) \otimes (\pi s + 1)$$

and hence we indeed have an A-algebra map as indicated. We omit the verification that this indeed defines a group law. Our second group scheme over A is

$$H = \text{Spec}(A[t, \frac{1}{\pi^\ell t + 1}])$$

with group law given by the comultiplication

$$\mu : A[t, \frac{1}{\pi^\ell t + 1}] \longrightarrow A[t, \frac{1}{\pi^\ell t + 1}] \otimes_A A[t, \frac{1}{\pi^\ell t + 1}], \quad t \longmapsto \pi^\ell t \otimes t + t \otimes 1 + 1 \otimes t$$

The same verification as before shows that this defines a group law. Next, we observe that the polynomial

$$\Phi(s) = \frac{(\pi s + 1)^\ell - 1}{\pi^\ell}$$

is in $A[s]$ and of degree ℓ and monic in s. Namely, the coefficicient of s^i for $0 < i < \ell$ is equal to $\binom{\ell}{i}\pi^{i-\ell}$ and since $\pi^{\ell-1}$ divides ℓ in A this is an element of A. We obtain a ring map

$$A[t, \frac{1}{\pi^\ell t + 1}] \longrightarrow A[s, \frac{1}{\pi s + 1}], \quad t \longmapsto \Phi(s)$$

which the reader easily verifies is compatible with the comultiplications. Thus we get a morphism of group schemes

$$f : G \to H$$

The following lemma in particular shows that this morphism is faithfully flat (in fact we will see that it is finite étale surjective).

0GY3 **Lemma 59.81.2.** We have

$$A[s, \frac{1}{\pi s + 1}] = \left(A[t, \frac{1}{\pi^\ell t + 1}]\right)[s]/(\Phi(s) - t)$$

In particular, the Hopf algebra of G is a monogenic extension of the Hopf algebra of H.

Proof. Follows from the discussion above and the shape of $\Phi(s)$. In particular, note that using $\Phi(s) = t$ the element $\frac{1}{\pi^\ell t+1}$ becomes the element $\frac{1}{(\pi s+1)^\ell}$. $\square$

Next, let us compute the kernel of f. Since the origin of H is given by $t = 0$ in H we see that the kernel of f is given by $\Phi(s) = 0$. Now observe that the A-valued points $\sigma_0, \ldots, \sigma_{\ell-1}$ of G given by

$$\sigma_i : s = \frac{\zeta^i - 1}{\pi} = \frac{\zeta^i - 1}{\zeta - 1} = \zeta^{i-1} + \zeta^{i-2} + \ldots + 1, \quad i = 0, 1, \ldots, \ell - 1$$

are certainly contained in $\text{Ker}(f)$. Moreover, these are all pairwise distinct in all fibres of $G \to \text{Spec}(A)$. Also, the reader computes that $\sigma_i +_G \sigma_j = \sigma_{i+j \bmod \ell}$. Hence we find a closed immersion of group schemes

$$\underline{\mathbf{Z}/\ell\mathbf{Z}}_A \longrightarrow \text{Ker}(f)$$

sending i to σ_i. However, by construction $\text{Ker}(f)$ is finite flat over $\text{Spec}(A)$ of degree ℓ. Hence we conclude that this map is an isomorphism. All in all we conclude that we have a short exact sequence

0GY4 (59.81.2.1)
$$0 \to \underline{\mathbf{Z}/\ell\mathbf{Z}}_A \to G \to H \to 0$$

of group schemes over A.

0GY5 **Lemma 59.81.3.** Let R be an A-algebra which is absolutely integrally closed. Then $G(R) \to H(R)$ is surjective.

Proof. Let $h \in H(R)$ correspond to the A-algebra map $A[t, \frac{1}{\pi^\ell t+1}] \to R$ sending t to $a \in A$. Since $\Phi(s)$ is monic we can find $b \in A$ with $\Phi(b) = a$. By Lemma 59.81.2 sending s to b we obtain a unique A-algebra map $A[s, \frac{1}{\pi s+1}] \to R$ compatible with the map $A[t, \frac{1}{\pi^\ell t+1}] \to R$ above. This in turn corresponds to an element $g \in G(R)$ mapping to $h \in H(R)$. $\square$

0GY6 **Lemma 59.81.4.** Let R be an A-algebra which is absolutely integrally closed. Let $I, J \subset R$ be ideals with $I + J = R$. There exists a $g \in G(R)$ such that $g \bmod I = \sigma_0$ and $g \bmod J = \sigma_1$.

Proof. Choose $x \in I$ such that $x \equiv 1 \bmod J$. We may and do replace I by xR and J by $(x-1)R$. Then we are looking for an $s \in R$ such that

(1) $1 + \pi s$ is a unit,
(2) $s \equiv 0 \bmod xR$, and
(3) $s \equiv 1 \bmod (x-1)R$.

The last two conditions say that $s = x + x(x-1)y$ for some $y \in R$. The first condition says that $1 + \pi s = 1 + \pi x + \pi x(x-1)y$ needs to be a unit of R. However, note that $1 + \pi x$ and $\pi x(x-1)$ generate the unit ideal of R and that $1 + \pi x$ is an ℓth root of 1 modulo $\pi x(x-1)$[7]. Thus we win by Lemma 59.81.1 and the fact that R is absolutely integrally closed. $\square$

0GY7 **Proposition 59.81.5.** Let R be an absolutely integrally closed ring. Let M be a finite abelian group. Then $H^i_{\acute{e}tale}(\text{Spec}(R), \underline{M}) = 0$ for $i > 0$.

Proof. Since any finite abelian group has a finite filtration whose subquotients are cyclic of prime order, we may assume $M = \mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number.

Observe that all local rings of R are strictly henselian, see More on Algebra, Lemma 15.14.7. Furthermore, any localization of R is also absolutely integrally closed by More on Algebra, Lemma 15.14.3. Thus Lemma 59.80.2 tells us it suffices to show that the kernel of

$$H^1_{\acute{e}tale}(D(f+g), \mathbf{Z}/\ell\mathbf{Z}) \longrightarrow H^1_{\acute{e}tale}(D(f(f+g)), \mathbf{Z}/\ell\mathbf{Z}) \oplus H^1_{\acute{e}tale}(D(g(f+g)), \mathbf{Z}/\ell\mathbf{Z})$$

is zero for any $f, g \in R$. After replacing R by R_{f+g} we reduce to the following claim: given $\xi \in H^1_{\acute{e}tale}(\text{Spec}(R), \mathbf{Z}/\ell\mathbf{Z})$ and an affine open covering $\text{Spec}(R) = U \cup V$ such that $\xi|_U$ and $\xi|_V$ are trivial, then $\xi = 0$.

Let $A = \mathbf{Z}[\zeta]$ as above. Since $\mathbf{Z} \subset A$ is monogenic, we can find a ring map $A \to R$. From now on we think of R as an A-algebra and we think of $\text{Spec}(R)$ as a scheme over $\text{Spec}(A)$. If we base change the short exact sequence (59.81.2.1) to $\text{Spec}(R)$ and take étale cohomology we obtain

$$G(R) \to H(R) \to H^1_{\acute{e}tale}(\text{Spec}(R), \mathbf{Z}/\ell\mathbf{Z}) \to H^1_{\acute{e}tale}(\text{Spec}(R), G)$$

Please keep this in mind during the rest of the proof.

Let $\tau \in \Gamma(U \cap V, \mathbf{Z}/\ell\mathbf{Z})$ be a section whose boundary in the Mayer-Vietoris sequence (Lemma 59.50.1) gives ξ. For $i = 0, 1, \ldots, \ell - 1$ let $A_i \subset U \cap V$ be the open and

[7] Because $1 + \pi x$ is congruent to 1 modulo π, congruent to 1 modulo x, and congruent to $1 + \pi = \zeta$ modulo $x - 1$ and because we have $(\pi) \cap (x) \cap (x-1) = (\pi x(x-1))$ in $A[x]$.

closed subset where τ has the value $i \bmod \ell$. Thus we have a finite disjoint union decomposition

$$U \cap V = A_0 \amalg \ldots \amalg A_{\ell-1}$$

such that τ is constant on each A_i. For $i = 0, 1, \ldots, \ell - 1$ denote $\tau_i \in H^0(U \cap V, \mathbf{Z}/\ell\mathbf{Z})$ the element which is equal to 1 on A_i and equal to 0 on A_j for $j \neq i$. Then τ is a sum of multiples of the τ_i[8]. Hence it suffices to show that the cohomology class corresponding to τ_i is trivial. This reduces us to the case where τ takes only two distinct values, namely 1 and 0.

Assume τ takes only the values 1 and 0. Write

$$U \cap V = A \amalg B$$

where A is the locus where $\tau = 0$ and B is the locus where $\tau = 1$. Then A and B are disjoint closed subsets. Denote $\overline{A}$ and $\overline{B}$ the closures of A and B in $\text{Spec}(R)$. Then we have a "banana": namely we have

$$\overline{A} \cap \overline{B} = Z_1 \amalg Z_2$$

with $Z_1 \subset U$ and $Z_2 \subset V$ disjoint closed subsets. Set $T_1 = \text{Spec}(R) \setminus V$ and $T_2 = \text{Spec}(R) \setminus U$. Observe that $Z_1 \subset T_1 \subset U$, $Z_2 \subset T_2 \subset V$, and $T_1 \cap T_2 = \emptyset$. Topologically we can write

$$\text{Spec}(R) = \overline{A} \cup \overline{B} \cup T_1 \cup T_2$$

We suggest drawing a picture to visualize this. In order to prove that ξ is zero, we may and do replace R by its reduction (Proposition 59.45.4). Below, we think of A, $\overline{A}$, B, $\overline{B}$, T_1, T_2 as reduced closed subschemes of $\text{Spec}(R)$. Next, as scheme structures on Z_1 and Z_2 we use

$$Z_1 = \overline{A} \cap (\overline{B} \cup T_1) \quad \text{and} \quad Z_2 = \overline{A} \cap (\overline{B} \cup T_2)$$

(scheme theoretic unions and intersections as in Morphisms, Definition 29.4.4).

Denote X the G-torsor over $\text{Spec}(R)$ corresponding to the image of ξ in $H^1(\text{Spec}(R), G)$. If X is trivial, then ξ comes from an element $h \in H(R)$ (see exact sequence of cohomology above). However, then by Lemma 59.81.3 the element h lifts to an element of $G(R)$ and we conclude $\xi = 0$ as desired. Thus our goal is to prove that X is trivial.

Recall that the embedding $\mathbf{Z}/\ell\mathbf{Z} \to G(R)$ sends $i \bmod \ell$ to $\sigma_i \in G(R)$. Observe that $\overline{A}$ is the spectrum of an absolutely integrally closed ring (namely a qotient of R). By Lemma 59.81.4 we can find $g \in G(\overline{A})$ with $g|_{\overline{A} \cap Z_1} = \sigma_0$ and $g|_{\overline{A} \cap Z_2} = \sigma_1$ (scheme theoretically). Then we can define

(1) $g_1 \in G(U)$ which is g on $\overline{A} \cap U$, which is σ_0 on $\overline{B} \cap U$, and σ_0 on T_1, and
(2) $g_2 \in G(V)$ which is g on $\overline{A} \cap V$, which is σ_1 on $\overline{B} \cap V$, and σ_1 on T_2.

Namely, to find g_1 as in (1) we glue the section σ_0 on $\Omega = (\overline{B} \cup T_1) \cap U$ to the restriction of the section g on $\Omega' = \overline{A} \cap U$. Note that $U = \Omega \cup \Omega'$ (scheme theoretically) because U is reduced and $\Omega \cap \Omega' = Z_1$ (scheme theoretically) by our choice of Z_1. Hence by Morphisms, Lemma 29.4.6 we have that U is the pushout of Ω and Ω' along Z_1. Thus we can find g_1. Similarly for the existence of g_2 in (2). Then we have

$$\tau = g_2|_{A \cup B} - g_1|_{A \cup B} \quad \text{(addition in group law)}$$

[8]Modulo calculation errors we have $\tau = \sum i\tau_i$.

and we see that X is trivial thereby finishing the proof. □

59.82. Affine analog of proper base change

09Z8 In this section we discuss a result by Ofer Gabber, see [Gab94]. This was also proved by Roland Huber, see [Hub93b]. We have already done some of the work needed for Gabber's proof in Section 59.80.

09ZE Lemma 59.82.1. Let X be an affine scheme. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Let $Z \subset X$ be a closed subscheme. Let $\xi \in H^q_{étale}(Z, \mathcal{F}|_Z)$ for some $q > 0$. Then there exists an injective map $\mathcal{F} \to \mathcal{F}'$ of torsion abelian sheaves on $X_{étale}$ such that the image of ξ in $H^q_{étale}(Z, \mathcal{F}'|_Z)$ is zero.

Proof. By Lemmas 59.73.2 and 59.51.4 we can find a map $\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ a constructible abelian sheaf and ξ coming from an element ζ of $H^q_{étale}(Z, \mathcal{G}|_Z)$. Suppose we can find an injective map $\mathcal{G} \to \mathcal{G}'$ of torsion abelian sheaves on $X_{étale}$ such that the image of ζ in $H^q_{étale}(Z, \mathcal{G}'|_Z)$ is zero. Then we can take $\mathcal{F}'$ to be the pushout

$$\mathcal{F}' = \mathcal{G}' \amalg_\mathcal{G} \mathcal{F}$$

and we conclude the result of the lemma holds. (Observe that restriction to Z is exact, so commutes with finite limits and colimits and moreover it commutes with arbitrary colimits as a left adjoint to pushforward.) Thus we may assume $\mathcal{F}$ is constructible.

Assume $\mathcal{F}$ is constructible. By Lemma 59.74.4 it suffices to prove the result when $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite abelian group and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 59.73.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 59.55.3) we see that the restriction of $f_*\underline{M}$ to Z is equal to the pushforward of $\underline{M}$ via $Y \times_X Z \to Z$. By the Leray spectral sequence (Proposition 59.54.2) and vanishing of higher direct images (Proposition 59.55.2), we find

$$H^q_{étale}(Z, f_*\underline{M}|_Z) = H^q_{étale}(Y \times_X Z, \underline{M}).$$

By Lemma 59.80.9 we can find a finite surjective morphism $Y' \to Y$ of finite presentation such that ξ maps to zero in $H^q(Y' \times_X Z, \underline{M})$. Denoting $f' : Y' \to X$ the composition $Y' \to Y \to X$ we claim the map

$$f_*\underline{M} \longrightarrow f'_*\underline{M}$$

is injective which finishes the proof by what was said above. To see the desired injectivity we can look at stalks. Namely, if $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point, then

$$(f_*\underline{M})_{\overline{x}} = \bigoplus\nolimits_{f(\overline{y}) = \overline{x}} M$$

by Proposition 59.55.2 and similarly for the other sheaf. Since $Y' \to Y$ is surjective and finite we see that the induced map on geometric points lifting $\overline{x}$ is surjective too and we conclude. □

The lemma above will take care of higher cohomology groups in Gabber's result. The following lemma will be used to deal with global sections.

09ZF Lemma 59.82.2. Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a closed immersion. Assume that

(1) for any sheaf $\mathcal{F}$ on X_{Zar} the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}\mathcal{F})$ is bijective, and

(2) for any finite morphism $X' \to X$ assumption (1) holds for $Z \times_X X' \to X'$.

Then for any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i_{small}^{-1}\mathcal{F})$.

Proof. Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}tale}$. There is a canonical (base change) map

$$i^{-1}(\mathcal{F}|_{X_{Zar}}) \longrightarrow (i_{small}^{-1}\mathcal{F})|_{Z_{Zar}}$$

of sheaves on Z_{Zar}. We will show this map is injective by looking at stalks. The stalk on the left hand side at $z \in Z$ is the stalk of $\mathcal{F}|_{X_{Zar}}$ at z. The stalk on the right hand side is the colimit over all elementary étale neighbourhoods $(U, u) \to (X, z)$ such that $U \times_X Z \to Z$ has a section over a neighbourhood of z. As étale morphisms are open, the image of $U \to X$ is an open neighbourhood U_0 of z in X. The map $\mathcal{F}(U_0) \to \mathcal{F}(U)$ is injective by the sheaf condition for $\mathcal{F}$ with respect to the étale covering $U \to U_0$. Taking the colimit over all U and U_0 we obtain injectivity on stalks.

It follows from this and assumption (1) that the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i_{small}^{-1}\mathcal{F})$ is injective. By (2) the same thing is true on all X' finite over X.

Let $s \in \Gamma(Z, i_{small}^{-1}\mathcal{F})$. By construction of $i_{small}^{-1}\mathcal{F}$ there exists an étale covering $\{V_j \to Z\}$, étale morphisms $U_j \to X$, sections $s_j \in \mathcal{F}(U_j)$ and morphisms $V_j \to U_j$ over X such that $s|_{V_j}$ is the pullback of s_j. Observe that every nonempty closed subscheme $T \subset X$ meets Z by assumption (1) applied to the sheaf $(T \to X)_*\underline{\mathbf{Z}}$ for example. Thus we see that $\coprod U_j \to X$ is surjective. By More on Morphisms, Lemma 37.45.7 we can find a finite surjective morphism $X' \to X$ such that $X' \to X$ Zariski locally factors through $\coprod U_j \to X$. It follows that $s|_{Z'}$ Zariski locally comes from a section of $\mathcal{F}|_{X'}$. In other words, $s|_{Z'}$ comes from $t' \in \Gamma(X', \mathcal{F}|_{X'})$ by assumption (2). By injectivity we conclude that the two pullbacks of t' to $X' \times_X X'$ are the same (after all this is true for the pullbacks of s to $Z' \times_Z Z'$). Hence we conclude t' comes from a section of $\mathcal{F}$ over X by Remark 59.55.6. $\square$

0CAM Lemma 59.82.3. Let $Z \subset X$ be a closed subset of a topological space X. Assume

(1) X is a spectral space (Topology, Definition 5.23.1), and
(2) for $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty).

If $Z = Z_1 \amalg Z_2$ with Z_i closed in Z, then there exists a decomposition $X = X_1 \amalg X_2$ with X_i closed in X and $Z_i = Z \cap X_i$.

Proof. Observe that Z_i is quasi-compact. Hence the set of points W_i specializing to Z_i is closed in the constructible topology by Topology, Lemma 5.24.7. Assumption (2) implies that $X = W_1 \amalg W_2$. Let $x \in \overline{W_1}$. By Topology, Lemma 5.23.6 part (1) there exists a specialization $x_1 \rightsquigarrow x$ with $x_1 \in W_1$. Thus $\overline{\{x\}} \subset \overline{\{x_1\}}$ and we see that $x \in W_1$. In other words, setting $X_i = W_i$ does the job. $\square$

09ZG Lemma 59.82.4. Let $Z \subset X$ be a closed subset of a topological space X. Assume

(1) X is a spectral space (Topology, Definition 5.23.1), and
(2) for $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty).

Then for any sheaf $\mathcal{F}$ on X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

Proof. If $x \rightsquigarrow x'$ is a specialization of points, then there is a canonical map $\mathcal{F}_{x'} \to \mathcal{F}_x$ compatible with sections over opens and functorial in $\mathcal{F}$. Since every point of X specializes to a point of Z it follows that $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ is injective. The difficult part is to show that it is surjective.

Denote $\mathcal{B}$ be the set of all quasi-compact opens of X. Write $\mathcal{F}$ as a filtered colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ where each $\mathcal{F}_i$ is as in Modules, Equation (17.19.2.1). See Modules, Lemma 17.19.2. Then $\mathcal{F}|_Z = \operatorname{colim} \mathcal{F}_i|_Z$ as restriction to Z is a left adjoint (Categories, Lemma 4.24.5 and Sheaves, Lemma 6.21.8). By Sheaves, Lemma 6.29.1 the functors $\Gamma(X, -)$ and $\Gamma(Z, -)$ commute with filtered colimits. Hence we may assume our sheaf $\mathcal{F}$ is as in Modules, Equation (17.19.2.1).

Suppose that we have an embedding $\mathcal{F} \subset \mathcal{G}$. Then we have

$$\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \cap \Gamma(X, \mathcal{G})$$

where the intersection takes place in $\Gamma(Z, \mathcal{G}|_Z)$. This follows from the first remark of the proof because we can check whether a global section of $\mathcal{G}$ is in $\mathcal{F}$ by looking at the stalks and because every point of X specializes to a point of Z.

By Modules, Lemma 17.19.4 there is an injection $\mathcal{F} \to \prod (Z_i \to X)_* \underline{S_i}$ where the product is finite, $Z_i \subset X$ is closed, and S_i is finite. Thus it suffices to prove surjectivity for the sheaves $(Z_i \to X)_* \underline{S_i}$. Observe that

$$\Gamma(X, (Z_i \to X)_* \underline{S_i}) = \Gamma(Z_i, \underline{S_i}) \quad \text{and} \quad \Gamma(X, (Z_i \to X)_* \underline{S_i}|_Z) = \Gamma(Z \cap Z_i, \underline{S_i})$$

Moreover, conditions (1) and (2) are inherited by Z_i; this is clear for (2) and follows from Topology, Lemma 5.23.5 for (1). Thus it suffices to prove the lemma in the case of a (finite) constant sheaf. This case is a restatement of Lemma 59.82.3 which finishes the proof. □

0CAF Example 59.82.5. Lemma 59.82.4 is false if X is not spectral. Here is an example: Let Y be a T_1 topological space, and $y \in Y$ a non-open point. Let $X = Y \amalg \{x\}$, endowed with the topology whose closed sets are $\emptyset$, $\{y\}$, and all $F \amalg \{x\}$, where F is a closed subset of Y. Then $Z = \{x, y\}$ is a closed subset of X, which satisfies assumption (2) of Lemma 59.82.4. But X is connected, while Z is not. The conclusion of the lemma thus fails for the constant sheaf with value $\{0, 1\}$ on X.

09ZH Lemma 59.82.6. Let (A, I) be a henselian pair. Set $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

Proof. Recall that the spectrum of any ring is a spectral space, see Algebra, Lemma 10.26.2. By More on Algebra, Lemma 15.11.16 we see that $\overline{\{x\}} \cap Z$ is connected for every $x \in X$. By Lemma 59.82.4 we see that the statement is true for sheaves on X_{Zar}. For any finite morphism $X' \to X$ we have $X' = \operatorname{Spec}(A')$ and $Z \times_X X' = \operatorname{Spec}(A'/IA')$ with (A', IA') a henselian pair, see More on Algebra, Lemma 15.11.8 and we get the same statement for sheaves on $(X')_{Zar}$. Thus we can apply Lemma 59.82.2 to conclude. □

Finally, we can state and prove Gabber's theorem.

09ZI Theorem 59.82.7 (Gabber). Let (A, I) be a henselian pair. Set $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$. For any torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $H^q_{\acute{e}tale}(X, \mathcal{F}) = H^q_{\acute{e}tale}(Z, \mathcal{F}|_Z)$.

Proof. The result holds for $q = 0$ by Lemma 59.82.6. Let $q \geq 1$. Suppose the result has been shown in all degrees $< q$. Let $\mathcal{F}$ be a torsion abelian sheaf. Let $\mathcal{F} \to \mathcal{F}'$ be an injective map of torsion abelian sheaves (to be chosen later) with cokernel $\mathcal{Q}$ so that we have the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{Q} \to 0$$

of torsion abelian sheaves on $X_{étale}$. This gives a map of long exact cohomology sequences over X and Z part of which looks like

$$\begin{tikzcd} H^{q-1}_{étale}(X, \mathcal{F}') \arrow[r] \arrow[d] & H^{q-1}_{étale}(X, \mathcal{Q}) \arrow[r] \arrow[d] & H^q_{étale}(X, \mathcal{F}) \arrow[r] \arrow[d] & H^q_{étale}(X, \mathcal{F}') \arrow[d] \\ H^{q-1}_{étale}(Z, \mathcal{F}'|_Z) \arrow[r] & H^{q-1}_{étale}(Z, \mathcal{Q}|_Z) \arrow[r] & H^q_{étale}(Z, \mathcal{F}|_Z) \arrow[r] & H^q_{étale}(Z, \mathcal{F}'|_Z) \end{tikzcd}$$

Using this commutative diagram of abelian groups with exact rows we will finish the proof.

Injectivity for $\mathcal{F}$. Let ξ be a nonzero element of $H^q_{étale}(X, \mathcal{F})$. By Lemma 59.82.1 applied with $Z = X$ (!) we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(X, \mathcal{Q})$ and bijectivity for $q - 1$ implies ξ does not map to zero in $H^q_{étale}(Z, \mathcal{F}|_Z)$.

Surjectivity for $\mathcal{F}$. Let ξ be an element of $H^q_{étale}(Z, \mathcal{F}|_Z)$. By Lemma 59.82.1 applied with $Z = Z$ we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(Z, \mathcal{Q}|_Z)$ and bijectivity for $q-1$ implies ξ is in the image of the vertical map. $\square$

0A51 Lemma 59.82.8. Let X be a scheme with affine diagonal which can be covered by $n + 1$ affine opens. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{A}$ be a torsion sheaf of rings on $X_{étale}$ and let $\mathcal{I}$ be an injective sheaf of $\mathcal{A}$-modules on $X_{étale}$. Then $H^q_{étale}(Z, \mathcal{I}|_Z) = 0$ for $q > n$.

Proof. We will prove this by induction on n. If $n = 0$, then X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Let A^h be the filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism. Then (A^h, IA^h) is a henselian pair and $A/I = A^h/IA^h$, see More on Algebra, Lemma 15.12.1 and its proof. Set $X^h = \mathrm{Spec}(A^h)$. By Theorem 59.82.7 we see that

$$H^q_{étale}(Z, \mathcal{I}|_Z) = H^q_{étale}(X^h, \mathcal{I}|_{X^h})$$

By Theorem 59.51.3 we have

$$H^q_{étale}(X^h, \mathcal{I}|_{X^h}) = \mathrm{colim}_{A \to B}\, H^q_{étale}(\mathrm{Spec}(B), \mathcal{I}|_{\mathrm{Spec}(B)})$$

where the colimit is over the A-algebras B as above. Since the morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are étale, the restriction $\mathcal{I}|_{\mathrm{Spec}(B)}$ is an injective sheaf of $\mathcal{A}|_{\mathrm{Spec}(B)}$-modules (Cohomology on Sites, Lemma 21.7.1). Thus the cohomology groups on the right are zero and we get the result in this case.

Induction step. We can use Mayer-Vietoris to do the induction step. Namely, suppose that $X = U \cup V$ where U is a union of n affine opens and V is affine. Then, using that the diagonal of X is affine, we see that $U \cap V$ is the union of n affine opens. Mayer-Vietoris gives an exact sequence

$$H^{q-1}_{étale}(U \cap V \cap Z, \mathcal{I}|_Z) \to H^q_{étale}(Z, \mathcal{I}|_Z) \to H^q_{étale}(U \cap Z, \mathcal{I}|_Z) \oplus H^q_{étale}(V \cap Z, \mathcal{I}|_Z)$$

and by our induction hypothesis we obtain vanishing for $q > n$ as desired. $\square$

59.83. Cohomology of torsion sheaves on curves

03SB The goal of this section is to prove the basic finiteness and vanishing results for cohomology of torsion sheaves on curves, see Theorem 59.83.10. In Section 59.84 we will discuss constructible sheaves of torsion modules over a Noetherian ring.

0A52 Situation 59.83.1. Here k is an algebraically closed field, X is a separated, finite type scheme of dimension ≤ 1 over k, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{\acute{e}tale}$.

In Situation 59.83.1 we want to prove the following statements

0A53 (1) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 2$,
0A54 (2) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 1$ if X is affine,
0A55 (3) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \text{char}(k) > 0$ and $\mathcal{F}$ is p-power torsion,
0A56 (4) $H^q_{\acute{e}tale}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\text{char}(k)$,
0A57 (5) $H^q_{\acute{e}tale}(X, \mathcal{F})$ is finite if X is proper and $\mathcal{F}$ constructible,
0A58 (6) $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension k'/k of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\text{char}(k)$,
0A59 (7) $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension k'/k of algebraically closed fields if X is proper,
0A5A (8) $H^2_{\acute{e}tale}(X, \mathcal{F}) \to H^2_{\acute{e}tale}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Given any Situation 59.83.1 we will say that "statements (1) – (8) hold" if those statements that apply to the given situation are true. We start the proof with the following consequence of our computation of cohomology with constant coefficients.

0A5B Lemma 59.83.2. In Situation 59.83.1 assume X is smooth and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$ for some prime number ℓ. Then statements (1) – (8) hold for $\mathcal{F}$.

Proof. Since X is smooth, we see that X is a finite disjoint union of smooth curves. Hence we may assume X is a smooth curve.

Case I: ℓ different from the characteristic of k. This case follows from Lemma 59.69.1 (projective case) and Lemma 59.69.3 (affine case). Statement (6) on cohomology and extension of algebraically closed ground field follows from the fact that the genus g and the number of "punctures" r do not change when passing from k to k'. Statement (8) follows as $H^2_{\acute{e}tale}(U, \mathcal{F})$ is zero as soon as $U \neq X$, because then U is affine (Varieties, Lemmas 33.43.2 and 33.43.10).

Case II: ℓ is equal to the characteristic of k. Vanishing by Lemma 59.63.4. Statements (5) and (7) follow from Lemma 59.63.5. $\square$

0A47 Remark 59.83.3 (Invariance under extension of algebraically closed ground field). Let k be an algebraically closed field of characteristic $p > 0$. In Section 59.63 we have seen that there is an exact sequence

$$k[x] \to k[x] \to H^1_{\acute{e}tale}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to 0$$

where the first arrow maps $f(x)$ to $f^p - f$. A set of representatives for the cokernel is formed by the polynomials

$$\sum\nolimits_{p \not| n} \lambda_n x^n$$

with $\lambda_n \in k$. (If k is not algebraically closed you have to add some constants to this as well.) In particular when k'/k is an algebraically closed extension, then the map

$$H^1_{\acute{e}tale}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to H^1_{\acute{e}tale}(\mathbf{A}^1_{k'}, \mathbf{Z}/p\mathbf{Z})$$

is not an isomorphism in general. In particular, the map $\pi_1(\mathbf{A}^1_{k'}) \to \pi_1(\mathbf{A}^1_k)$ between étale fundamental groups (insert future reference here) is not an isomorphism either. Thus the étale homotopy type of the affine line depends on the algebraically closed ground field. From Lemma 59.83.2 above we see that this is a phenomenon which only happens in characteristic p with p-power torsion coefficients.

0A5C Lemma 59.83.4. Let k be an algebraically closed field. Let X be a separated finite type scheme over k of dimension ≤ 1. Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of torsion abelian sheaves on X. If statements (1) – (8) hold for $\mathcal{F}_1$ and $\mathcal{F}_2$, then they hold for $\mathcal{F}$.

Proof. This is mostly immediate from the definitions and the long exact sequence of cohomology. Also observe that $\mathcal{F}$ is constructible (resp. of torsion prime to the characteristic of k) if and only if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are constructible (resp. of torsion prime to the characteristic of k). See Proposition 59.74.1. Some details omitted. $\square$

0A5D Lemma 59.83.5. Let k be an algebraically closed field. Let $f : X \to Y$ be a finite morphism of separated finite type schemes over k of dimension ≤ 1. Let $\mathcal{F}$ be a torsion abelian sheaf on X. If statements (1) – (8) hold for $\mathcal{F}$, then they hold for $f_*\mathcal{F}$.

Proof. Namely, we have $H^q_{\acute{e}tale}(X, \mathcal{F}) = H^q_{\acute{e}tale}(Y, f_*\mathcal{F})$ by the vanishing of $R^q f_*$ for $q > 0$ (Proposition 59.55.2) and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). For (8) use that formation of f_* commutes with arbitrary base change (Lemma 59.55.3). $\square$

0GJA Lemma 59.83.6. In Situation 59.83.1 assume $\mathcal{F}$ constructible. Let $j : X' \to X$ be the inclusion of a dense open subscheme. Then statements (1) – (8) hold for $\mathcal{F}$ if and only if they hold for $j_!j^{-1}\mathcal{F}$.

Proof. Since X' is dense, we see that $Z = X \setminus X'$ has dimension 0 and hence is a finite set $Z = \{x_1, \ldots, x_n\}$ of k-rational points. Consider the short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of Lemma 59.70.8. Observe that $H^q_{\acute{e}tale}(X, i_*i^{-1}\mathcal{F}) = H^q_{\acute{e}tale}(Z, i^*\mathcal{F})$. Namely, $i : Z \to X$ is a closed immersion, hence finite, hence we have the vanishing of $R^q i_*$ for $q > 0$ by Proposition 59.55.2, and hence the equality follows from the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). Since Z is a disjoint union of spectra of algebraically closed fields, we conclude that $H^q_{\acute{e}tale}(Z, i^*\mathcal{F}) = 0$ for $q > 0$ and

$$H^0_{\acute{e}tale}(Z, i^{-1}\mathcal{F}) = \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{x_i}$$

which is finite as $\mathcal{F}_{x_i}$ is finite due to the assumption that $\mathcal{F}$ is constructible. The long exact cohomology sequence gives an exact sequence

$$0 \to H^0_{\acute{e}tale}(X, j_!j^{-1}\mathcal{F}) \to H^0_{\acute{e}tale}(X, \mathcal{F}) \to H^0_{\acute{e}tale}(Z, i^{-1}\mathcal{F}) \to H^1_{\acute{e}tale}(X, j_!j^{-1}\mathcal{F}) \to H^1_{\acute{e}tale}(X, \mathcal{F}) \to 0$$

and isomorphisms $H^q_{\acute{e}tale}(X, j_!j^{-1}\mathcal{F}) \to H^q_{\acute{e}tale}(X, \mathcal{F})$ for $q > 1$.

At this point it is easy to deduce each of (1) – (8) holds for $\mathcal{F}$ if and only if it holds for $j_!j^{-1}\mathcal{F}$. We make a few small remarks to help the reader: (a) if $\mathcal{F}$ is torsion prime to the characteristic of k, then so is $j_!j^{-1}\mathcal{F}$, (b) the sheaf $j_!j^{-1}\mathcal{F}$ is constructible, (c) we have $H^0_{\acute{e}tale}(Z, i^{-1}\mathcal{F}) = H^0_{\acute{e}tale}(Z_{k'}, i^{-1}\mathcal{F}|_{Z_{k'}})$, and (d) if $U \subset X$ is an open, then $U' = U \cap X'$ is dense in U. $\square$

03SG **Lemma 59.83.7.** In Situation 59.83.1 assume X is smooth. Let $j : U \to X$ an open immersion. Let ℓ be a prime number. Let $\mathcal{F} = j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Then statements (1) – (8) hold for $\mathcal{F}$.

Proof. Since X is smooth, it is a disjoint union of smooth curves and hence we may assume X is a curve (i.e., irreducible). Then either $U = \emptyset$ and there is nothing to prove or $U \subset X$ is dense. In this case the lemma follows from Lemmas 59.83.2 and 59.83.6. $\square$

0A3Q **Lemma 59.83.8.** In Situation 59.83.1 assume X reduced. Let $j : U \to X$ an open immersion. Let ℓ be a prime number and $\mathcal{F} = j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Then statements (1) – (8) hold for $\mathcal{F}$.

Proof. The difference with Lemma 59.83.7 is that here we do not assume X is smooth. Let $\nu : X^\nu \to X$ be the normalization morphism. Then ν is finite (Varieties, Lemma 33.27.1) and X^ν is smooth (Varieties, Lemma 33.43.8). Let $j^\nu : U^\nu \to X^\nu$ be the inverse image of U. By Lemma 59.83.7 the result holds for $j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. By Lemma 59.83.5 the result holds for $\nu_*j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. In general it won't be true that $\nu_*j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ is equal to $j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ but we can work around this as follows. As X is reduced the morphism $\nu : X^\nu \to X$ is an isomorphism over a dense open $j' : X' \to X$ (Varieties, Lemma 33.27.1). Over this open we have agreement

$$(j')^{-1}(\nu_*j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}) = (j')^{-1}(j_!\underline{\mathbf{Z}/\ell\mathbf{Z}})$$

Using Lemma 59.83.6 twice for $j' : X' \to X$ and the sheaves above we conclude. $\square$

03SD **Lemma 59.83.9.** In Situation 59.83.1 assume X reduced. Let $j : U \to X$ an open immersion with U connected. Let ℓ be a prime number. Let $\mathcal{G}$ a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U. Let $\mathcal{F} = j_!\mathcal{G}$. Then statements (1) – (8) hold for $\mathcal{F}$.

Proof. Let $f : V \to U$ be a finite étale morphism of degree prime to ℓ as in Lemma 59.66.2. The discussion in Section 59.66 gives maps

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \to \mathcal{G}$$

whose composition is an isomorphism. Hence it suffices to prove the lemma with $\mathcal{F} = j_!f_*f^{-1}\mathcal{G}$. By Zariski's Main theorem (More on Morphisms, Lemma 37.43.3) we can choose a diagram

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f"] & Y \arrow[d, "\overline{f}"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

with $\overline{f} : Y \to X$ finite and j' an open immersion with dense image. We may replace Y by its reduction (this does not change V as V is reduced being étale over U). Since f is finite and V dense in Y we have $V = U \times_X Y$. By Lemma 59.70.9 we have

$$j_!f_*f^{-1}\mathcal{G} = \overline{f}_*j'_!f^{-1}\mathcal{G}$$

By Lemma 59.83.5 it suffices to consider $j'_!f^{-1}\mathcal{G}$. The existence of the filtration given by Lemma 59.66.2, the fact that $j'_!$ is exact, and Lemma 59.83.4 reduces us to the case $\mathcal{F} = j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ which is Lemma 59.83.8. $\square$

03SC Theorem 59.83.10. If k is an algebraically closed field, X is a separated, finite type scheme of dimension ≤ 1 over k, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{\acute{e}tale}$, then

(1) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 2$,
(2) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 1$ if X is affine,
(3) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \text{char}(k) > 0$ and $\mathcal{F}$ is p-power torsion,
(4) $H^q_{\acute{e}tale}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\text{char}(k)$,
(5) $H^q_{\acute{e}tale}(X, \mathcal{F})$ is finite if X is proper and $\mathcal{F}$ constructible,
(6) $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension k'/k of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\text{char}(k)$,
(7) $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension k'/k of algebraically closed fields if X is proper,
(8) $H^2_{\acute{e}tale}(X, \mathcal{F}) \to H^2_{\acute{e}tale}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Proof. The theorem says that in Situation 59.83.1 statements (1) – (8) hold. Our first step is to replace X by its reduction, which is permissible by Proposition 59.45.4. By Lemma 59.73.2 we can write $\mathcal{F}$ as a filtered colimit of constructible abelian sheaves. Taking cohomology commutes with colimits, see Lemma 59.51.4. Moreover, pullback via $X_{k'} \to X$ commutes with colimits as a left adjoint. Thus it suffices to prove the statements for a constructible sheaf.

In this paragraph we use Lemma 59.83.4 without further mention. Writing $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is ℓ_i-primary for some prime ℓ_i, we may assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ. Now consider the exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.$$

Thus we see that it suffices to assume that $\mathcal{F}$ is ℓ-torsion. This means that $\mathcal{F}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces for some prime number ℓ.

By definition this means there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces. Since $\dim(X) \leq 1$ we may assume, after shrinking U, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of ≥ 2 components of U). By Lemma 59.83.6 we reduce to the case $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{G}$ is a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U.

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with U_i irreducible we have

$$j_!\mathcal{G} = j_{1!}(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_{n!}(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_{i!}(\mathcal{G}|_{U_i})$ is handled in Lemma 59.83.9. $\square$

03RT Theorem 59.83.11. Let X be a finite type, dimension 1 scheme over an algebraically closed field k. Let $\mathcal{F}$ be a torsion sheaf on $X_{\acute{e}tale}$. Then

$$H^q_{\acute{e}tale}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$

If X affine then also $H^2_{\acute{e}tale}(X, \mathcal{F}) = 0$.

Proof. If X is separated, this follows immediately from the more precise Theorem 59.83.10. If X is nonseparated, choose an affine open covering $X = X_1 \cup \ldots \cup X_n$. By induction on n we may assume the vanishing holds over $U = X_1 \cup \ldots \cup X_{n-1}$. Then Mayer-Vietoris (Lemma 59.50.1) gives

$$H^2_{\acute{e}tale}(U, \mathcal{F}) \oplus H^2_{\acute{e}tale}(X_n, \mathcal{F}) \to H^2_{\acute{e}tale}(U \cap X_n, \mathcal{F}) \to H^3_{\acute{e}tale}(X, \mathcal{F}) \to 0$$

However, since $U \cap X_n$ is an open of an affine scheme and hence affine by our dimension assumption, the group $H^2_{\acute{e}tale}(U \cap X_n, \mathcal{F})$ vanishes by Theorem 59.83.10. $\square$

0A5E Lemma 59.83.12. Let k'/k be an extension of separably closed fields. Let X be a proper scheme over k of dimension ≤ 1. Let $\mathcal{F}$ be a torsion abelian sheaf on X. Then the map $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.

Proof. We have seen this for algebraically closed fields in Theorem 59.83.10. Given $k \subset k'$ as in the statement of the lemma we can choose a diagram

$$\begin{tikzcd} k' \arrow[r] & \overline{k}' \\ k \arrow[u] \arrow[r] & \overline{k} \arrow[u] \end{tikzcd}$$

where $k \subset \overline{k}$ and $k' \subset \overline{k}'$ are the algebraic closures. Since k and k' are separably closed the field extensions $\overline{k}/k$ and $\overline{k}'/k'$ are algebraic and purely inseparable. In this case the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{k}'} \to X_{k'}$ are universal homeomorphisms. Thus the cohomology of $\mathcal{F}$ may be computed on $X_{\overline{k}}$ and the cohomology of $\mathcal{F}|_{X_{k'}}$ may be computed on $X_{\overline{k}'}$, see Proposition 59.45.4. Hence we deduce the general case from the case of algebraically closed fields. $\square$

59.84. Cohomology of torsion modules on curves

0GJB In this section we repeat the arguments of Section 59.83 for constructible sheaves of modules over a Noetherian ring which are torsion. We start with the most interesting step.

0GJC Lemma 59.84.1. Let Λ be a Noetherian ring, let M be a finite Λ-module which is annihilated by an integer $n > 0$, let k be an algebraically closed field, and let X be a separated, finite type scheme of dimension ≤ 1 over k. Then

(1) $H^q_{\acute{e}tale}(X, \underline{M})$ is a finite Λ-module if n is prime to $\text{char}(k)$,
(2) $H^q_{\acute{e}tale}(X, \underline{M})$ is a finite Λ-module if X is proper.

Proof. If $n = \ell n'$ for some prime number ℓ, then we get a short exact sequence $0 \to M[\ell] \to M \to M' \to 0$ of finite Λ-modules and M' is annihilated by n'. This produces a corresponding short exact sequence of constant sheaves, which in turn gives rise to an exact sequence of cohomology modules

$$H^q_{\acute{e}tale}(X, \underline{M[n]}) \to H^q_{\acute{e}tale}(X, \underline{M}) \to H^q_{\acute{e}tale}(X, \underline{M'})$$

Thus, if we can show the result in case M is annihilated by a prime number, then by induction on n we win.

Let ℓ be a prime number such that ℓ annihilates M. Then we can replace Λ by the $\mathbf{F}_\ell$-algebra $\Lambda/\ell\Lambda$. Namely, the cohomology of $\mathcal{F}$ as a sheaf of Λ-modules is the same as the cohomology of $\mathcal{F}$ as a sheaf of $\Lambda/\ell\Lambda$-modules, for example by Cohomology on Sites, Lemma 21.12.4.

Assume ℓ be a prime number such that ℓ annihilates M and Λ. Let us reduce to the case where M is a finite free Λ-module. Namely, choose a short exact sequence

$$0 \to N \to \Lambda^{\oplus m} \to M \to 0$$

This determines an exact sequence

$$H^q_{\acute{e}tale}(X, \underline{\Lambda^{\oplus m}}) \to H^q_{\acute{e}tale}(X, \underline{M}) \to H^{q+1}_{\acute{e}tale}(X, \underline{N})$$

By descending induction on q we get the result for M if we know the result for $\Lambda^{\oplus m}$. Here we use that we know that our cohomology groups vanish in degrees > 2 by Theorem 59.83.10.

Let ℓ be a prime number and assume that ℓ annihilates Λ. It remains to show that the cohomology groups $H^q_{\acute{e}tale}(X, \underline{\Lambda})$ are finite Λ-modules. We will use a trick to show this; the "correct" argument uses a coefficient theorem which we will show later. Choose a basis $\Lambda = \bigoplus_{i \in I} \mathbf{F}_\ell e_i$ such that $e_0 = 1$ for some $0 \in I$. The choice of this basis determines an isomorphism

$$\underline{\Lambda} = \bigoplus \underline{\mathbf{F}_\ell} e_i$$

of sheaves on $X_{\acute{e}tale}$. Thus we see that

$$H^q_{\acute{e}tale}(X, \underline{\Lambda}) = H^q_{\acute{e}tale}(X, \bigoplus \underline{\mathbf{F}_\ell} e_i) = \bigoplus H^q_{\acute{e}tale}(X, \underline{\mathbf{F}_\ell}) e_i$$

since taking cohomology over X commutes with direct sums by Theorem 59.51.3 (or Lemma 59.51.4 or Lemma 59.52.2). Since we already know that $H^q_{\acute{e}tale}(X, \underline{\mathbf{F}_\ell})$ is a finite dimensional $\mathbf{F}_\ell$-vector space (by Theorem 59.83.10), we see that $H^q_{\acute{e}tale}(X, \underline{\Lambda})$ is free over Λ of the same rank. Namely, given a basis $\xi_1, \ldots, \xi_m$ of $H^q_{\acute{e}tale}(X, \underline{\mathbf{F}_\ell})$ we see that $\xi_1 e_0, \ldots, \xi_m e_0$ form a Λ-basis for $H^q_{\acute{e}tale}(X, \underline{\Lambda})$. $\square$

0GJD Lemma 59.84.2. Let Λ be a Noetherian ring, let k be an algebraically closed field, let $f : X \to Y$ be a finite morphism of separated finite type schemes over k of dimension ≤ 1, and let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{\acute{e}tale}$. If $H^q_{\acute{e}tale}(X, \mathcal{F})$ is a finite Λ-module, then so is $H^q_{\acute{e}tale}(Y, f_*\mathcal{F})$.

Proof. Namely, we have $H^q_{\acute{e}tale}(X, \mathcal{F}) = H^q_{\acute{e}tale}(Y, f_*\mathcal{F})$ by the vanishing of $R^q f_*$ for $q > 0$ (Proposition 59.55.2) and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). $\square$

0GJE Lemma 59.84.3. Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated finite type scheme over k of dimension ≤ 1, let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$, and let $j : X' \to X$ be the inclusion of a dense open subscheme. Then $H^q_{\acute{e}tale}(X, \mathcal{F})$ is a finite Λ-module if and only if $H^q_{\acute{e}tale}(X, j_!j^{-1}\mathcal{F})$ is a finite Λ-module.

Proof. Since X' is dense, we see that $Z = X \setminus X'$ has dimension 0 and hence is a finite set $Z = \{x_1, \ldots, x_n\}$ of k-rational points. Consider the short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of Lemma 59.70.8. Observe that $H^q_{\acute{e}tale}(X, i_*i^{-1}\mathcal{F}) = H^q_{\acute{e}tale}(Z, i^*\mathcal{F})$. Namely, $i : Z \to X$ is a closed immersion, hence finite, hence we have the vanishing of $R^q i_*$ for $q > 0$ by Proposition 59.55.2, and hence the equality follows from the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). Since Z is a disjoint union of spectra of algebraically closed fields, we conclude that $H^q_{\acute{e}tale}(Z, i^*\mathcal{F}) = 0$ for $q > 0$ and

$$H^0_{\acute{e}tale}(Z, i^{-1}\mathcal{F}) = \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{x_i}$$

which is a finite Λ-module $\mathcal{F}_{x_i}$ is finite due to the assumption that $\mathcal{F}$ is a constructible sheaf of Λ-modules. The long exact cohomology sequence gives an exact sequence

$$0 \to H^0_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^0_{étale}(X, \mathcal{F}) \to H^0_{étale}(Z, i^{-1}\mathcal{F}) \to H^1_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^1_{étale}(X, \mathcal{F}) \to 0$$

and isomorphisms $H^0_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^0_{étale}(X, \mathcal{F})$ for $q > 1$. The lemma follows easily from this. ☐

0GJF Lemma 59.84.4. Let Λ be a Noetherian ring, let M be a finite Λ-module which is annihilated by an integer $n > 0$, let k be an algebraically closed field, let X be a separated, finite type scheme of dimension ≤ 1 over k, and let $j : U \to X$ be an open immersion. Then

(1) $H^q_{étale}(X, j_!\underline{M})$ is a finite Λ-module if n is prime to char(k),
(2) $H^q_{étale}(X, j_!\underline{M})$ is a finite Λ-module if X is proper.

Proof. Since $\dim(X) \leq 1$ there is an open $V \subset X$ which is disjoint from U such that $X' = U \cup V$ is dense open in X (details omitted). If $j' : X' \to X$ denotes the inclusion morphism, then we see that $j_!\underline{M}$ is a direct summand of $j'_!\underline{M}$. Hence it suffices to prove the lemma in case U is open and dense in X. This case follows from Lemmas 59.84.3 and 59.84.1. ☐

0GJG Lemma 59.84.5. Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated finite type scheme over k of dimension ≤ 1, and let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of sheaves of Λ-modules on $X_{étale}$. If $H^q_{étale}(X, \mathcal{F}_i)$, $i = 1, 2$ are finite Λ-modules then $H^q_{étale}(X, \mathcal{F})$ is a finite Λ-module.

Proof. Immediate from the long exact sequence of cohomology. ☐

0GJH Lemma 59.84.6. Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated, finite type scheme of dimension ≤ 1 over k, let $j : U \to X$ be an open immersion with U connected, let ℓ be a prime number, let $n > 0$, and let $\mathcal{G}$ be a finite type, locally constant sheaf of Λ-modules on $U_{étale}$ annihilated by ℓ^n. Then

(1) $H^q_{étale}(X, j_!\mathcal{G})$ is a finite Λ-module if ℓ is prime to char(k),
(2) $H^q_{étale}(X, j_!\mathcal{G})$ is a finite Λ-module if X is proper.

Proof. Let $f : V \to U$ be a finite étale morphism of degree prime to ℓ as in Lemma 59.66.4. The discussion in Section 59.66 gives maps

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \to \mathcal{G}$$

whose composition is an isomorphism. Hence it suffices to prove the finiteness of $H^q_{étale}(X, j_!f_*f^{-1}\mathcal{G})$. By Zariski's Main theorem (More on Morphisms, Lemma 37.43.3) we can choose a diagram

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f"] & Y \arrow[d, "\overline{f}"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

with $\overline{f} : Y \to X$ finite and j' an open immersion with dense image. Since f is finite and V dense in Y we have $V = U \times_X Y$. By Lemma 59.70.9 we have

$$j_!f_*f^{-1}\mathcal{G} = \overline{f}_*j'_!f^{-1}\mathcal{G}$$

By Lemma 59.84.2 it suffices to consider $j'_! f^{-1}\mathcal{G}$. The existence of the filtration given by Lemma 59.66.4, the fact that $j'_!$ is exact, and Lemma 59.84.5 reduces us to the case $\mathcal{F} = j'_! \underline{M}$ for a finite Λ-module M which is Lemma 59.84.4. □

0GJI Theorem 59.84.7. Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated, finite type scheme of dimension ≤ 1 over k, and let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$ which is torsion. Then

0GJJ (1) $H^q_{étale}(X, \mathcal{F})$ is a finite Λ-module if $\mathcal{F}$ is torsion prime to $\text{char}(k)$,
0GJK (2) $H^q_{étale}(X, \mathcal{F})$ is a finite Λ-module if X is proper.

Proof. without further mention. Write $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is annihilated by $\ell_i^{n_i}$ for some prime ℓ_i and integer $n_i > 0$. By Lemma 59.84.5 it suffices to prove the theorem for $\mathcal{F}_i$. Thus we may and do assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ and integer $n > 0$.

Since $\mathcal{F}$ is constructible as a sheaf of Λ-modules, there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is a finite type, locally constant sheaf of Λ-modules. Since $\dim(X) \leq 1$ we may assume, after shrinking U, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of ≥ 2 components of U). By Lemma 59.84.3 we reduce to the case $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{G}$ is a finite type, locally constant sheaf of Λ-modules on U (and annihilated by ℓ^n).

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with U_i irreducible we have

$$j_!\mathcal{G} = j_{1!}(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_{n!}(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_{i!}(\mathcal{G}|_{U_i})$ is handled in Lemma 59.84.6. □

59.85. First cohomology of proper schemes

0A5F In Fundamental Groups, Section 58.9 we have seen, in some sense, that taking $R^1 f_* \underline{G}$ commutes with base change if $f : X \to Y$ is a proper morphism and G is a finite group (not necessarily commutative). In this section we deduce a useful consequence of these results.

0A5G Lemma 59.85.1. Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Let M be a finite abelian group. Then $H^1_{étale}(X, \underline{M}) = H^1_{étale}(X_0, \underline{M})$.

Proof. By Cohomology on Sites, Lemma 21.4.3 an element of $H^1_{étale}(X, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $X_{étale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over X, Lemma 59.64.4. Conversely, a scheme V finite étale over X with an M-action which turns it into an M-torsor over X gives rise to a cohomology class. The same translation between cohomology classes over X_0 and torsors finite étale over X_0 holds. Thus the lemma is a consequence of the equivalence of categories of Fundamental Groups, Lemma 58.9.1. □

The following technical lemma is a key ingredient in the proof of the proper base change theorem. The argument works word for word for any proper scheme over A whose special fibre has dimension ≤ 1, but in fact the conclusion will be a consequence of the proper base change theorem and we only need this particular version in its proof.

0A5H **Lemma 59.85.2.** Let A be a henselian local ring. Let $X = \mathbf{P}^1_A$. Let $X_0 \subset X$ be the closed fibre. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{\acute{e}tale}$. Then $H^q_{\acute{e}tale}(X_0, \mathcal{I}|_{X_0}) = 0$ for $q > 0$.

Proof. Observe that X is a separated scheme which can be covered by 2 affine opens. Hence for $q > 1$ this follows from Gabber's affine variant of the proper base change theorem, see Lemma 59.82.8. Thus we may assume $q = 1$. Let $\xi \in H^1_{\acute{e}tale}(X_0, \mathcal{I}|_{X_0})$. Goal: show that ξ is 0. By Lemmas 59.73.2 and 59.51.4 we can find a map $\mathcal{F} \to \mathcal{I}$ with $\mathcal{F}$ a constructible sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules and ξ coming from an element ζ of $H^1_{\acute{e}tale}(X_0, \mathcal{F}|_{X_0})$. Suppose we have an injective map $\mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{\acute{e}tale}$. Since $\mathcal{I}$ is injective we can extend the given map $\mathcal{F} \to \mathcal{I}$ to a map $\mathcal{F}' \to \mathcal{I}$. In this situation we may replace $\mathcal{F}$ by $\mathcal{F}'$ and ζ by the image of ζ in $H^1_{\acute{e}tale}(X_0, \mathcal{F}'|_{X_0})$. Also, if $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ is a direct sum, then we may replace $\mathcal{F}$ by $\mathcal{F}_i$ and ζ by the image of ζ in $H^1_{\acute{e}tale}(X_0, \mathcal{F}_i|_{X_0})$.

By Lemma 59.74.4 and the remarks above we may assume $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite $\mathbf{Z}/\ell\mathbf{Z}$-module and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 59.73.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 59.55.3) we see that the restriction of $f_*\underline{M}$ to X_0 is equal to the pushforward of $\underline{M}$ via the induced morphism $Y_0 \to X_0$ of special fibres. By the Leray spectral sequence (Proposition 59.54.2) and vanishing of higher direct images (Proposition 59.55.2), we find

$$H^1_{\acute{e}tale}(X_0, f_*\underline{M}|_{X_0}) = H^1_{\acute{e}tale}(Y_0, \underline{M}).$$

Since $Y \to \mathrm{Spec}(A)$ is proper we can use Lemma 59.85.1 to see that the $H^1_{\acute{e}tale}(Y_0, \underline{M})$ is equal to $H^1_{\acute{e}tale}(Y, \underline{M})$. Thus we see that our cohomology class ζ lifts to a cohomology class

$$\tilde{\zeta} \in H^1_{\acute{e}tale}(Y, \underline{M}) = H^1_{\acute{e}tale}(X, f_*\underline{M})$$

However, $\tilde{\zeta}$ maps to zero in $H^1_{\acute{e}tale}(X, \mathcal{I})$ as $\mathcal{I}$ is injective and by commutativity of

$$\begin{tikzcd} H^1_{\acute{e}tale}(X, f_*\underline{M}) \arrow[r] \arrow[d] & H^1_{\acute{e}tale}(X, \mathcal{I}) \arrow[d] \\ H^1_{\acute{e}tale}(X_0, (f_*\underline{M})|_{X_0}) \arrow[r] & H^1_{\acute{e}tale}(X_0, \mathcal{I}|_{X_0}) \end{tikzcd}$$

we conclude that the image ξ of ζ is zero as well. $\square$

59.86. Preliminaries on base change

0EZQ If you are interested in either the smooth base change theorem or the proper base change theorem, you should skip directly to the corresponding sections. In this section and the next few sections we consider commutative diagrams

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

of schemes; we usually assume this diagram is cartesian, i.e., $Y = X \times_S T$. A commutative diagram as above gives rise to a commutative diagram

$$\begin{tikzcd} X_{\acute{e}tale} \arrow[d, "f_{small}"'] & Y_{\acute{e}tale} \arrow[l, "h_{small}"] \arrow[d, "e_{small}"] \\ S_{\acute{e}tale} & T_{\acute{e}tale} \arrow[l, "g_{small}"'] \end{tikzcd}$$

of small étale sites. Let us use the notation

$$f^{-1} = f_{small}^{-1}, \quad g_* = g_{small,*}, \quad e^{-1} = e_{small}^{-1}, \text{ and } \quad h_* = h_{small,*}.$$

By Sites, Section 7.45 we get a base change or pullback map

$$f^{-1}g_*\mathcal{F} \longrightarrow h_*e^{-1}\mathcal{F}$$

for a sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$. If $\mathcal{F}$ is an abelian sheaf on $T_{\acute{e}tale}$, then we get a derived base change map

$$f^{-1}Rg_*\mathcal{F} \longrightarrow Rh_*e^{-1}\mathcal{F}$$

see Cohomology on Sites, Lemma 21.15.1. Finally, if K is an arbitrary object of $D(T_{\acute{e}tale})$ there is a base change map

$$f^{-1}Rg_*K \longrightarrow Rh_*e^{-1}K$$

see Cohomology on Sites, Remark 21.19.3.

0EZR **Lemma 59.86.1.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

Let $\{U_i \to X\}$ be an étale covering such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and consider the cartesian diagrams

$$\begin{tikzcd} U_i \arrow[d, "f_i"'] & U_i \times_X Y \arrow[l, "h_i"] \arrow[d, "e_i"] \\ V_i & V_i \times_S T \arrow[l, "g_i"'] \end{tikzcd}$$

Let $\mathcal{F}$ be a sheaf on $T_{\acute{e}tale}$. Let K in $D(T_{\acute{e}tale})$. Set $K_i = K|_{V_i \times_S T}$ and $\mathcal{F}_i = \mathcal{F}|_{V_i \times_S T}$.

(1) If $f_i^{-1}g_{i,*}\mathcal{F}_i = h_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.
(2) If $f_i^{-1}Rg_{i,*}K_i = Rh_{i,*}e_i^{-1}K_i$ for all i, then $f^{-1}Rg_*K = Rh_*e^{-1}K$.
(3) If $\mathcal{F}$ is an abelian sheaf and $f_i^{-1}R^qg_{i,*}\mathcal{F}_i = R^qh_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$.

Proof. Proof of (1). First we observe that

$$(f^{-1}g_*\mathcal{F})|_{U_i} = f_i^{-1}(g_*\mathcal{F}|_{V_i}) = f_i^{-1}g_{i,*}\mathcal{F}_i$$

The first equality because $U_i \to X \to S$ is equal to $U_i \to V_i \to S$ and the second equality because $g_*\mathcal{F}|_{V_i} = g_{i,*}\mathcal{F}_i$ by Sites, Lemma 7.28.2. Similarly we have

$$(h_*e^{-1}\mathcal{F})|_{U_i} = h_{i,*}(e^{-1}\mathcal{F}|_{U_i \times_X Y}) = h_{i,*}e_i^{-1}\mathcal{F}_i$$

Thus if the base change maps $f_i^{-1}g_{i,*}\mathcal{F}_i \to h_{i,*}e_i^{-1}\mathcal{F}_i$ are isomorphisms for all i, then the base change map $f^{-1}g_*\mathcal{F} \to h_*e^{-1}\mathcal{F}$ restricts to an isomorphism over U_i for all i and we conclude it is an isomorphism as $\{U_i \to X\}$ is an étale covering.

For the other two statements we replace the appeal to Sites, Lemma 7.28.2 by an appeal to Cohomology on Sites, Lemma 21.20.4. $\square$

0EZS Lemma 59.86.2. Consider a tower of cartesian diagrams of schemes

$$\begin{tikzcd} W \arrow[d, "i"'] & Z \arrow[l, "j"] \arrow[d, "k"] \\ X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

Let K in $D(T_{\acute{e}tale})$. If

$$f^{-1}Rg_*K \to Rh_*e^{-1}K \quad \text{and} \quad i^{-1}Rh_*e^{-1}K \to Rj_*k^{-1}e^{-1}K$$

are isomorphisms, then $(f \circ i)^{-1}Rg_*K \to Rj_*(e \circ k)^{-1}K$ is an isomorphism. Similarly, if $\mathcal{F}$ is an abelian sheaf on $T_{\acute{e}tale}$ and if

$$f^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F} \quad \text{and} \quad i^{-1}R^qh_*e^{-1}\mathcal{F} \to R^qj_*k^{-1}e^{-1}\mathcal{F}$$

are isomorphisms, then $(f \circ i)^{-1}R^qg_*\mathcal{F} \to R^qj_*(e \circ k)^{-1}\mathcal{F}$ is an isomorphism.

Proof. This is formal, provided one checks that the composition of these base change maps is the base change maps for the outer rectangle, see Cohomology on Sites, Remark 21.19.5. $\square$

0EZT Lemma 59.86.3. Let I be a directed set. Consider an inverse system of cartesian diagrams of schemes

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] & Y_i \arrow[l, "h_i"] \arrow[d, "e_i"] \\ S_i & T_i \arrow[l, "g_i"] \end{tikzcd}$$

with affine transition morphisms and with g_i quasi-compact and quasi-separated. Set $X = \lim X_i$, $S = \lim S_i$, $T = \lim T_i$ and $Y = \lim Y_i$ to obtain the cartesian diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on (T_i) as in Definition 59.51.1. Set $\mathcal{F} = \operatorname{colim} p_i^{-1}\mathcal{F}_i$ on T where $p_i : T \to T_i$ is the projection. Then we have the following

(1) If $f_i^{-1}g_{i,*}\mathcal{F}_i = h_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.
(2) If $\mathcal{F}_i$ is an abelian sheaf for all i and $f_i^{-1}R^qg_{i,*}\mathcal{F}_i = R^qh_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$.

Proof. We prove (2) and we omit the proof of (1). We will use without further mention that pullback of sheaves commutes with colimits as it is a left adjoint. Observe that h_i is quasi-compact and quasi-separated as a base change of g_i. Denoting $q_i : Y \to Y_i$ the projections, observe that $e^{-1}\mathcal{F} = \text{colim}\, e^{-1}p_i^{-1}\mathcal{F}_i = \text{colim}\, q_i^{-1}e_i^{-1}\mathcal{F}_i$. By Lemma 59.51.8 this gives

$$R^q h_* e^{-1}\mathcal{F} = \text{colim}\, r_i^{-1} R^q h_{i,*} e_i^{-1}\mathcal{F}_i$$

where $r_i : X \to X_i$ is the projection. Similarly, we have

$$f^{-1}Rg_*\mathcal{F} = f^{-1}\,\text{colim}\, s_i^{-1}R^q g_{i,*}\mathcal{F}_i = \text{colim}\, r_i^{-1}f_i^{-1}R^q g_{i,*}\mathcal{F}_i$$

where $s_i : S \to S_i$ is the projection. The lemma follows. □

0GJL **Lemma 59.86.4.** Let I, X_i, Y_i, S_i, T_i, f_i, h_i, e_i, g_i, X, Y, S, T, f, h, e, g be as in the statement of Lemma 59.86.3. Let $0 \in I$ and let $K_0 \in D^+(T_{0,\acute{e}tale})$. For $i \in I$, $i \geq 0$ denote K_i the pullback of K_0 to T_i. Denote K the pullback of K_0 to T. If $f_i^{-1}Rg_{i,*}K_i = Rh_{i,*}e_i^{-1}K_i$ for all $i \geq 0$, then $f^{-1}Rg_*K = Rh_*e^{-1}K$.

Proof. It suffices to show that the base change map $f^{-1}Rg_*K \to Rh_*e^{-1}K$ induces an isomorphism on cohomology sheaves. In other words, we have to show that $f^{-1}R^pg_*K \to R^ph_*e^{-1}K$ is an isomorphism for all $p \in \mathbf{Z}$ if we are given that $f_i^{-1}R^pg_{i,*}K_i \to R^ph_{i,*}e_i^{-1}K_i$ is an isomorphism for all $i \geq 0$ and $p \in \mathbf{Z}$. At this point we can argue exactly as in the proof of Lemma 59.86.3 replacing reference to Lemma 59.51.8 by a reference to Lemma 59.52.4. □

0EZU **Lemma 59.86.5.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

where $g : T \to S$ is quasi-compact and quasi-separated. Let $\mathcal{F}$ be an abelian sheaf on $T_{\acute{e}tale}$. Let $q \geq 0$. The following are equivalent

(1) For every geometric point $\overline{x}$ of X with image $\overline{s} = f(\overline{x})$ we have

$$H^q(\text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S T, \mathcal{F}) = H^q(\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S T, \mathcal{F})$$

(2) $f^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Since $Y = X \times_S T$ we have $\text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y = \text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S T$. Thus the map in (1) is the map of stalks at $\overline{x}$ for the map in (2) by Theorem 59.53.1 (and Lemma 59.36.2). Thus the result by Theorem 59.29.10. □

0EZV **Lemma 59.86.6.** Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s}$ in S. Let $\text{Spec}(K) \to \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ be a morphism with K a separably closed field. Let $\mathcal{F}$ be an abelian sheaf on $\text{Spec}(K)_{\acute{e}tale}$. Let $q \geq 0$. The following are equivalent

(1) $H^q(\text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \text{Spec}(K), \mathcal{F}) = H^q(\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \text{Spec}(K), \mathcal{F})$
(2) $H^q(\text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \text{Spec}(K), \mathcal{F}) = H^q(\text{Spec}(K), \mathcal{F})$

Proof. Observe that $\mathrm{Spec}(K) \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the spectrum of a filtered colimit of étale algebras over K. Since K is separably closed, each étale K-algebra is a finite product of copies of K. Thus we can write

$$\mathrm{Spec}(K) \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{i \in I} \coprod\nolimits_{a \in A_i} \mathrm{Spec}(K)$$

as a cofiltered limit where each term is a disjoint union of copies of $\mathrm{Spec}(K)$ over a finite set A_i. Note that A_i is nonempty as we are given $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. It follows that

$$\begin{aligned} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \mathrm{Spec}(K) &= \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \left(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \mathrm{Spec}(K)\right) \\ &= \lim_{i \in I} \coprod\nolimits_{a \in A_i} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \mathrm{Spec}(K) \end{aligned}$$

Since taking cohomology in our setting commutes with limits of schemes (Theorem 59.51.3) we conclude. $\square$

59.87. Base change for pushforward

0EZW This section is preliminary and should be skipped on a first reading. In this section we discuss for what morphisms $f : X \to S$ we have $f^{-1}g_* = h_*e^{-1}$ on all sheaves (of sets) for every cartesian diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

with g quasi-compact and quasi-separated.

0EZX **Lemma 59.87.1.** Consider the cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

Assume that f is flat and every object U of $X_{étale}$ has a covering $\{U_i \to U\}$ such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and $U_i \to V_i$ quasi-compact with geometrically connected fibres. Then for any sheaf $\mathcal{F}$ of sets on $T_{étale}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Let $U \to X$ be an étale morphism such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale and $U \to V$ quasi-compact with geometrically connected fibres. Observe that $U \to V$ is flat (More on Flatness, Lemma 38.2.3). We claim that

$$\begin{aligned} f^{-1}g_*\mathcal{F}(U) &= g_*\mathcal{F}(V) \\ &= \mathcal{F}(V \times_S T) \\ &= e^{-1}\mathcal{F}(U \times_X Y) \\ &= h_*e^{-1}\mathcal{F}(U) \end{aligned}$$

Namely, thinking of U as an object of $X_{étale}$ and V as an object of $S_{étale}$ we see that the first equality follows from Lemma 59.39.3[9]. Thinking of $V \times_S T$ as an

[9]Strictly speaking, we are also using that the restriction of $f^{-1}g_*\mathcal{F}$ to $U_{étale}$ is the pullback via $U \to V$ of the restriction of $g_*\mathcal{F}$ to $V_{étale}$. See Sites, Lemma 7.28.2.

object of $T_{\acute{e}tale}$ the second equality follows from the definition of g_*. Observe that $U \times_X Y = U \times_S T$ (because $Y = X \times_S T$) and hence $U \times_X Y \to V \times_S T$ has geometrically connected fibres as a base change of $U \to V$. Thinking of $U \times_X Y$ as an object of $Y_{\acute{e}tale}$, we see that the third equality follows from Lemma 59.39.3 as before. Finally, the fourth equality follows from the definition of h_*.

Since by assumption every object of $X_{\acute{e}tale}$ has an étale covering to which the argument of the previous paragraph applies we see that the lemma is true. $\square$

0EYS Lemma 59.87.2. Consider a cartesian diagram of schemes

$$\begin{tikzcd} X & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S \arrow[from=u, "f"'] & T \arrow[l, "g"'] \end{tikzcd}$$

where f is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$.

Proof. Combine Lemma 59.87.1 with More on Morphisms, Lemma 37.46.3. $\square$

0EZY Lemma 59.87.3. Consider the cartesian diagrams of schemes

$$\begin{tikzcd} X & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S \arrow[from=u, "f"'] & T \arrow[l, "g"'] \end{tikzcd}$$

Assume that S is the spectrum of a separably closed field. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$.

Proof. We may work locally on X. Hence we may assume X is affine. Then we can write X as a cofiltered limit of affine schemes of finite type over S. By Lemma 59.86.3 we may assume that X is of finite type over S. Then Lemma 59.87.1 applies because any scheme of finite type over a separably closed field is a finite disjoint union of connected and geometrically connected schemes (see Varieties, Lemma 33.7.6). $\square$

0EZZ Lemma 59.87.4. Consider a cartesian diagram of schemes

$$\begin{tikzcd} X & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S \arrow[from=u, "f"'] & T \arrow[l, "g"'] \end{tikzcd}$$

Assume that

(1) f is flat and open,
(2) the residue fields of S are separably algebraically closed,
(3) given an étale morphism $U \to X$ with U affine we can write U as a finite disjoint union of open subschemes of X (for example if X is a normal integral scheme with separably closed function field),
(4) any nonempty open of a fibre X_s of f is connected (for example if X_s is irreducible or empty).

Then for any sheaf $\mathcal{F}$ of sets on $T_{\acute{e}tale}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Omitted. Hint: the assumptions almost trivially imply the condition of Lemma 59.87.1. The for example in part (3) follows from Lemma 59.80.4. ☐

The following lemma doesn't really belong here but there does not seem to be a good place for it anywhere.

0EYR Lemma 59.87.5. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1} : Sh(S_{étale}) \to Sh(X_{étale})$ commutes with products.

Proof. Let I be a set and let $\mathcal{G}_i$ be a sheaf on $S_{étale}$ for $i \in I$. Let $U \to X$ be an étale morphism such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale and $U \to V$ flat of finite presentation with geometrically connected fibres. Then we have

$$\begin{aligned} f^{-1}(\prod \mathcal{G}_i)(U) &= (\prod \mathcal{G}_i)(V) \\ &= \prod \mathcal{G}_i(V) \\ &= \prod f^{-1}\mathcal{G}_i(U) \\ &= (\prod f^{-1}\mathcal{G}_i)(U) \end{aligned}$$

where we have used Lemma 59.39.3 in the first and third equality (we are also using that the restriction of $f^{-1}\mathcal{G}$ to $U_{étale}$ is the pullback via $U \to V$ of the restriction of $\mathcal{G}$ to $V_{étale}$, see Sites, Lemma 7.28.2). By More on Morphisms, Lemma 37.46.3 every object U of $X_{étale}$ has an étale covering $\{U_i \to U\}$ such that the discussion in the previous paragraph applies to U_i. The lemma follows. ☐

0F00 Lemma 59.87.6. Let $f : X \to S$ be a flat morphism of schemes such that for every geometric point $\overline{x}$ of X the map

$$\mathcal{O}^{sh}_{S,f(\overline{x})} \longrightarrow \mathcal{O}^{sh}_{X,\overline{x}}$$

has geometrically connected fibres. Then for every cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

with g quasi-compact and quasi-separated we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ of sets on $T_{étale}$.

Proof. It suffices to check equality on stalks, see Theorem 59.29.10. By Theorem 59.53.1 we have

$$(h_*e^{-1}\mathcal{F})_{\overline{x}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y, e^{-1}\mathcal{F})$$

and we have similarly

$$(f^{-1}g_*^{-1}\mathcal{F})_{\overline{x}} = (g_*^{-1}\mathcal{F})_{f(\overline{x})} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T, \mathcal{F})$$

These sets are equal by an application of Lemma 59.39.3 to the morphism

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T$$

which is a base change of $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$ because $Y = X \times_S T$. ☐

59.88. Base change for higher direct images

0F01 This section is the analogue of Section 59.87 for higher direct images. This section is preliminary and should be skipped on a first reading.

0F02 Remark 59.88.1. Let $f : X \to S$ be a morphism of schemes. Let n be an integer. We will say $BC(f, n, q_0)$ is true if for every commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & S' \arrow[l] & T \arrow[l, "g"] \end{tikzcd}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$ annihilated by n the base change map

$$(f')^{-1}R^q g_*\mathcal{F} \longrightarrow R^q h_* e^{-1}\mathcal{F}$$

is an isomorphism for $q \leq q_0$.

0F03 Lemma 59.88.2. With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Then for every commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & S' \arrow[l] & T \arrow[l, "g"] \end{tikzcd}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$ annihilated by n

(1) the base change map $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is injective,
(2) if $\mathcal{F} \subset \mathcal{G}$ where $\mathcal{G}$ on $T_{\acute{e}tale}$ is annihilated by n, then

$$\mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}\right) \subset \mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{G} \to R^q h_* e^{-1}\mathcal{G}\right)$$

(3) if in (2) the sheaf $\mathcal{G}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules, then

$$\mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}\right) \subset R^q h_* e^{-1}\mathcal{G}$$

Proof. Choose a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ where $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Consider the induced diagram

$$\begin{tikzcd} (f')^{-1}R^{q-1}g_*\mathcal{I} \arrow[r] \arrow[d, "\cong"'] & (f')^{-1}R^{q-1}g_*\mathcal{Q} \arrow[r] \arrow[d, "\cong"'] & (f')^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[d] \\ R^{q-1}h_*e^{-1}\mathcal{I} \arrow[r] & R^{q-1}h_*e^{-1}\mathcal{Q} \arrow[r] & R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q h_* e^{-1}\mathcal{I} \end{tikzcd}$$

with exact rows. We have the zero in the right upper corner as $\mathcal{I}$ is injective. The left two vertical arrows are isomorphisms by $BC(f, n, q-1)$. We conclude that part (1) holds. The above also shows that

$$\mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}\right) \subset R^q h_* e^{-1}\mathcal{I}$$

hence part (3) holds. To prove (2) choose $\mathcal{F} \subset \mathcal{G} \subset \mathcal{I}$. □

0F04 Lemma 59.88.3. With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Consider commutative diagrams

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l] \arrow[d, "f'"] & Y \arrow[l, "h"] \arrow[d, "e"] & Y' \arrow[l, "\pi'"] \arrow[d, "e'"] \\ S & S' \arrow[l] & T \arrow[l, "g"] & T' \arrow[l, "\pi"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[d, "f'"] & & Y' \arrow[ll, "h'=h\circ\pi'"] \arrow[d, "e'"] \\ S' & & T' \arrow[ll, "g'=g\circ\pi"] \end{tikzcd}$$

where all squares are cartesian, g quasi-compact and quasi-separated, and π is integral surjective. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$ annihilated by n and set $\mathcal{F}' = \pi^{-1}\mathcal{F}$. If the base change map

$$(f')^{-1}R^q g'_*\mathcal{F}' \longrightarrow R^q h'_*(e')^{-1}\mathcal{F}'$$

is an isomorphism, then the base change map $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Observe that $\mathcal{F} \to \pi_*\pi^{-1}\mathcal{F}'$ is injective as π is surjective (check on stalks). Thus by Lemma 59.88.2 we see that it suffices to show that the base change map

$$(f')^{-1}R^q g_*\pi_*\mathcal{F}' \longrightarrow R^q h_* e^{-1}\pi_*\mathcal{F}'$$

is an isomorphism. This follows from the assumption because we have $R^q g_*\pi_*\mathcal{F}' = R^q g'_*\mathcal{F}'$, we have $e^{-1}\pi_*\mathcal{F}' = \pi'_*(e')^{-1}\mathcal{F}'$, and we have $R^q h_*\pi'_*(e')^{-1}\mathcal{F}' = R^q h'_*(e')^{-1}\mathcal{F}'$. This follows from Lemmas 59.55.4 and 59.43.5 and the relative leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). $\square$

0F05 Lemma 59.88.4. With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Consider commutative diagrams

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l] \arrow[d, "f'"] & X'' \arrow[l, "\pi'"] \arrow[d, "f''"] & Y \arrow[l, "h'"] \arrow[d, "e"] \\ S & S' \arrow[l] & S'' \arrow[l, "\pi"] & T \arrow[l, "g'"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[d, "f'"] & & Y \arrow[ll, "h=h'\circ\pi'"] \arrow[d, "e"] \\ S' & & T \arrow[ll, "g=g'\circ\pi"] \end{tikzcd}$$

where all squares are cartesian, g' quasi-compact and quasi-separated, and π is integral. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$ annihilated by n. If the base change map

$$(f')^{-1}R^q g_*\mathcal{F} \longrightarrow R^q h_* e^{-1}\mathcal{F}$$

is an isomorphism, then the base change map $(f'')^{-1}R^q g'_*\mathcal{F} \to R^q h'_* e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Since π and π' are integral we have $R\pi_* = \pi_*$ and $R\pi'_* = \pi'_*$, see Lemma 59.43.5. We also have $(f')^{-1}\pi_* = \pi'_*(f'')^{-1}$. Thus we see that $\pi'_*(f'')^{-1}R^q g'_*\mathcal{F} = (f')^{-1}R^q g_*\mathcal{F}$ and $\pi'_* R^q h'_* e^{-1}\mathcal{F} = R^q h_* e^{-1}\mathcal{F}$. Thus the assumption means that our map becomes an isomorphism after applying the functor π'_*. Hence we see that it is an isomorphism by Lemma 59.43.5. $\square$

0F06 Lemma 59.88.5. Let T be a quasi-compact and quasi-separated scheme. Let P be a property for quasi-compact and quasi-separated schemes over T. Assume

(1) If $T'' \to T'$ is a thickening of quasi-compact and quasi-separated schemes over T, then $P(T'')$ if and only if $P(T')$.
(2) If $T' = \lim T_i$ is a limit of an inverse system of quasi-compact and quasi-separated schemes over T with affine transition morphisms and $P(T_i)$ holds for all i, then $P(T')$ holds.

(3) If $Z \subset T'$ is a closed subscheme with quasi-compact complement $V \subset T'$ and $P(T')$ holds, then either $P(V)$ or $P(Z)$ holds.

Then $P(T)$ implies $P(\mathrm{Spec}(K))$ for some morphism $\mathrm{Spec}(K) \to T$ where K is a field.

Proof. Consider the set $\mathfrak{T}$ of closed subschemes $T' \subset T$ such that $P(T')$. By assumption (2) this set has a minimal element, say T'. By assumption (1) we see that T' is reduced. Let $\eta \in T'$ be the generic point of an irreducible component of T'. Then $\eta = \mathrm{Spec}(K)$ for some field K and $\eta = \lim V$ where the limit is over the affine open subschemes $V \subset T'$ containing η. By assumption (3) and the minimality of T' we see that $P(V)$ holds for all these V. Hence $P(\eta)$ by (2) and the proof is complete. $\square$

0F07 Lemma 59.88.6. With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram

$$\begin{tikzcd} X & X' \arrow[l] & Y \arrow[l, "h"'] \\ S \arrow[u, leftarrow, "f"] & S' \arrow[l] \arrow[u, leftarrow, "f'"] & \mathrm{Spec}(K) \arrow[l, "g"] \arrow[u, leftarrow, "e"'] \end{tikzcd}$$

where $X' = X \times_S S'$, $Y = X' \times_{S'} \mathrm{Spec}(K)$, K is a field, and $\mathcal{F}$ is an abelian sheaf on $\mathrm{Spec}(K)$ annihilated by n such that $(f')^{-1}R^q g_* \mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is not an isomorphism.

Proof. Choose a commutative diagram

$$\begin{tikzcd} X & X' \arrow[l] & Y \arrow[l, "h"'] \\ S \arrow[u, leftarrow, "f"] & S' \arrow[l] \arrow[u, leftarrow, "f'"] & T \arrow[l, "g"] \arrow[u, leftarrow, "e"'] \end{tikzcd}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and an abelian sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$ annihilated by n such that the base change map $(f')^{-1}R^q g_* \mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is not an isomorphism. Of course we may and do replace S' by an affine open of S'; this implies that T is quasi-compact and quasi-separated. By Lemma 59.88.2 we see $(f')^{-1}R^q g_* \mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is injective. Pick a geometric point $\overline{x}$ of X' and an element ξ of $(R^q h_* q^{-1}\mathcal{F})_{\overline{x}}$ which is not in the image of the map $((f')^{-1}R^q g_* \mathcal{F})_{\overline{x}} \to (R^q h_* e^{-1}\mathcal{F})_{\overline{x}}$.

Consider a morphism $\pi : T' \to T$ with T' quasi-compact and quasi-separated and denote $\mathcal{F}' = \pi^{-1}\mathcal{F}$. Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of π and $e' : Y' \to T'$ the base change of e. Picture

$$\begin{tikzcd} X' & Y \arrow[l, "h"'] & Y' \arrow[l, "\pi'"'] \\ S' \arrow[u, leftarrow, "f'"] & T \arrow[l, "g"] \arrow[u, leftarrow, "e"'] & T' \arrow[l, "\pi"] \arrow[u, leftarrow, "e'"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' & & Y' \arrow[ll, "h' = h \circ \pi'"'] \\ S' \arrow[u, leftarrow, "f'"] & & T' \arrow[ll, "g' = g \circ \pi"] \arrow[u, leftarrow, "e'"'] \end{tikzcd}$$

Using pullback maps we obtain a canonical commutative diagram

$$\begin{tikzcd} (f')^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & (f')^{-1}R^q g'_*\mathcal{F}' \arrow[d] \\ R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q h'_*(e')^{-1}\mathcal{F}' \end{tikzcd}$$

of abelian sheaves on X'. Let $P(T')$ be the property

- The image ξ' of ξ in $(Rh'_*(e')^{-1}\mathcal{F}')_{\overline{x}}$ is not in the image of the map $(f^{-1}R^q g'_*\mathcal{F}')_{\overline{x}} \to (R^q h'_*(e')^{-1}\mathcal{F}')_{\overline{x}}$.

We claim that hypotheses (1), (2), and (3) of Lemma 59.88.5 hold for P which proves our lemma.

Condition (1) of Lemma 59.88.5 holds for P because the étale topology of a scheme and a thickening of the scheme is the same. See Proposition 59.45.4.

Suppose that I is a directed set and that T_i is an inverse system over I of quasi-compact and quasi-separated schemes over T with affine transition morphisms. Set $T' = \lim T_i$. Denote $\mathcal{F}'$ and $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to T', resp. T_i. Consider the diagrams

$$\begin{tikzcd} X & Y \arrow[l, "h"'] \arrow[d, "e"] & Y_i \arrow[l, "\pi'_i"'] \arrow[d, "e_i"] \\ S \arrow[u, leftarrow, "f'"'] & T \arrow[l, "g"'] & T_i \arrow[l, "\pi_i"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X & Y_i \arrow[l, "h_i = h\circ\pi'_i"'] \arrow[d, "e_i"] \\ S \arrow[u, leftarrow, "f'"'] & T_i \arrow[l, "g_i = g\circ\pi_i"'] \end{tikzcd}$$

as in the previous paragraph. It is clear that $\mathcal{F}'$ on T' is the colimit of the pullbacks of $\mathcal{F}_i$ to T' and that $(e')^{-1}\mathcal{F}'$ is the colimit of the pullbacks of $e_i^{-1}\mathcal{F}_i$ to Y'. By Lemma 59.51.8 we have

$$R^q h'_*(e')^{-1}\mathcal{F}' = \operatorname{colim} R^q h_{i,*}e_i^{-1}\mathcal{F}_i \quad \text{and} \quad (f')^{-1}R^q g'_*\mathcal{F}' = \operatorname{colim}(f')^{-1}R^q g_{i,*}\mathcal{F}_i$$

It follows that if $P(T_i)$ is true for all i, then $P(T')$ holds. Thus condition (2) of Lemma 59.88.5 holds for P.

The most interesting is condition (3) of Lemma 59.88.5. Assume T' is a quasi-compact and quasi-separated scheme over T such that $P(T')$ is true. Let $Z \subset T'$ be a closed subscheme with complement $V \subset T'$ quasi-compact. Consider the diagram

$$\begin{tikzcd} Y' \times_{T'} Z \arrow[r, "i'"'] \arrow[d, "e_Z"'] & Y' \arrow[d, "e'"'] & Y' \times_{T'} V \arrow[l, "j'"] \arrow[d, "e_V"] \\ Z \arrow[r, "i"] & T' & V \arrow[l, "j"'] \end{tikzcd}$$

Choose an injective map $j^{-1}\mathcal{F}' \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on V. Looking at stalks we see that the map

$$\mathcal{F}' \to \mathcal{G} = j_*\mathcal{J} \oplus i_*i^{-1}\mathcal{F}'$$

is injective. Thus ξ' maps to a nonzero element of

$$\begin{aligned} & \operatorname{Coker}\left(((f')^{-1}R^q g'_*\mathcal{G})_{\overline{x}} \to (R^q h'_*(e')^{-1}\mathcal{G})_{\overline{x}}\right) = \\ & \operatorname{Coker}\left(((f')^{-1}R^q g'_*j_*\mathcal{J})_{\overline{x}} \to (R^q h'_*(e')^{-1}j_*\mathcal{J})_{\overline{x}}\right) \oplus \\ & \operatorname{Coker}\left(((f')^{-1}R^q g'_*i_*i^{-1}\mathcal{F}')_{\overline{x}} \to (R^q h'_*(e')^{-1}i_*i^{-1}\mathcal{F}')_{\overline{x}}\right) \end{aligned}$$

by part (2) of Lemma 59.88.2. If ξ' does not map to zero in the second summand, then we use

$$(f')^{-1}R^q g'_* i_* i^{-1}\mathcal{F}' = (f')^{-1}R^q(g' \circ i)_* i^{-1}\mathcal{F}'$$

(because $Ri_* = i_*$ by Proposition 59.55.2) and

$$R^q h'_*(e')^{-1} i_* i^{-1}\mathcal{F} = R^q h'_* i'_* e_Z^{-1} i^{-1}\mathcal{F} = R^q(h' \circ i')_* e_Z^{-1} i^{-1}\mathcal{F}'$$

(first equality by Lemma 59.55.3 and the second because $Ri'_* = i'_*$ by Proposition 59.55.2) to we see that we have $P(Z)$. Finally, suppose ξ' does not map to zero in the first summand. We have

$$(e')^{-1} j_* \mathcal{J} = j'_* e_V^{-1}\mathcal{J} \quad \text{and} \quad R^a j'_* e_V^{-1}\mathcal{J} = 0, \quad a = 1, \ldots, q-1$$

by $BC(f, n, q-1)$ applied to the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Y' \arrow[l] \arrow[d, "e'"'] & Y \arrow[l, "j'"] \arrow[d, "e_V"] \\ S & T' \arrow[l] & V \arrow[l, "j"'] \end{tikzcd}$$

and the fact that $\mathcal{J}$ is injective. By the relative Leray spectral sequence for $h' \circ j'$ (Cohomology on Sites, Lemma 21.14.7) we deduce that

$$R^q h'_*(e')^{-1} j_* \mathcal{J} = R^q h'_* j'_* e_V^{-1}\mathcal{J} \longrightarrow R^q(h' \circ j')_* e_V^{-1}\mathcal{J}$$

is injective. Thus ξ maps to a nonzero element of $(R^q(h' \circ j')_* e_V^{-1}\mathcal{J})_{\overline{x}}$. Applying part (3) of Lemma 59.88.2 to the injection $j^{-1}\mathcal{F}' \to \mathcal{J}$ we conclude that $P(V)$ holds. $\square$

0F08 Lemma 59.88.7. With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\ S & S' \arrow[l] & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

with both squares cartesian, where

(1) S' is affine, integral, and normal with algebraically closed function field,
(2) K is algebraically closed and $\mathrm{Spec}(K) \to S'$ is dominant (in other words K is an extension of the function field of S')

and there exists an integer $d|n$ such that $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ is nonzero.

Conversely, nonvanishing of $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ in the lemma implies $BC(f, n, q)$ isn't true as Lemma 59.80.5 shows that $R^q(\mathrm{Spec}(K) \to S')_*\mathbf{Z}/d\mathbf{Z} = 0$.

Proof. First choose a diagram and $\mathcal{F}$ as in Lemma 59.88.6. We may and do assume S' is affine (this is obvious, but see proof of the lemma in case of doubt). By Lemma 59.88.3 we may assume K is algebraically closed. Then $\mathcal{F}$ corresponds to a $\mathbf{Z}/n\mathbf{Z}$-module. Such a modules is a direct sum of copies of $\mathbf{Z}/d\mathbf{Z}$ for varying $d|n$ hence we may assume $\mathcal{F}$ is constant with value $\mathbf{Z}/d\mathbf{Z}$. By Lemma 59.88.4 we may replace S' by the normalization of S' in $\mathrm{Spec}(K)$ which finishes the proof. $\square$

59.89. Smooth base change

0EYQ In this section we prove the smooth base change theorem.

0EYT **Lemma 59.89.1.** Let K/k be an extension of fields. Let X be a smooth affine curve over k with a rational point $x \in X(k)$. Let $\mathcal{F}$ be an abelian sheaf on $\mathrm{Spec}(K)$ annihilated by an integer n invertible in k. Let $q > 0$ and

$$\xi \in H^q(X_K, (X_K \to \mathrm{Spec}(K))^{-1}\mathcal{F})$$

There exist

(1) finite extensions K'/K and k'/k with $k' \subset K'$,
(2) a finite étale Galois cover $Z \to X_{k'}$ with group G

such that the order of G divides a power of n, such that $Z \to X_{k'}$ is split over $x_{k'}$, and such that ξ dies in $H^q(Z_{K'}, (Z_{K'} \to \mathrm{Spec}(K))^{-1}\mathcal{F})$.

Proof. For $q > 1$ we know that ξ dies in $H^q(X_{\overline{K}}, (X_{\overline{K}} \to \mathrm{Spec}(K))^{-1}\mathcal{F})$ (Theorem 59.83.10). By Lemma 59.51.5 we see that this means there is a finite extension K'/K such that ξ dies in $H^q(X_{K'}, (X_{K'} \to \mathrm{Spec}(K))^{-1}\mathcal{F})$. Thus we can take $k' = k$ and $Z = X$ in this case.

Assume $q = 1$. Recall that $\mathcal{F}$ corresponds to a discrete module M with continuous Gal_K-action, see Lemma 59.59.1. Since M is n-torsion, it is the uninon of finite Gal_K-stable subgroups. Thus we reduce to the case where M is a finite abelian group annihilated by n, see Lemma 59.51.4. After replacing K by a finite extension we may assume that the action of Gal_K on M is trivial. Thus we may assume $\mathcal{F} = \underline{M}$ is the constant sheaf with value a finite abelian group M annihilated by n.

We can write M as a direct sum of cyclic groups. Any two finite étale Galois coverings whose Galois groups have order invertible in k, can be dominated by a third one whose Galois group has order invertible in k (Fundamental Groups, Section 58.7). Thus it suffices to prove the lemma when $M = \mathbf{Z}/d\mathbf{Z}$ where $d|n$.

Assume $M = \mathbf{Z}/d\mathbf{Z}$ where $d|n$. In this case $\overline{\xi} = \xi|_{X_{\overline{K}}}$ is an element of

$$H^1(X_{\overline{k}}, \mathbf{Z}/d\mathbf{Z}) = H^1(X_{\overline{K}}, \mathbf{Z}/d\mathbf{Z})$$

See Theorem 59.83.10. This group classifies $\mathbf{Z}/d\mathbf{Z}$-torsors, see Cohomology on Sites, Lemma 21.4.3. The torsor corresponding to $\overline{\xi}$ (viewed as a sheaf on $X_{\overline{k},\acute{e}tale}$) in turn gives rise to a finite étale morphism $T \to X_{\overline{k}}$ endowed an action of $\mathbf{Z}/d\mathbf{Z}$ transitive on the fibre of T over $x_{\overline{k}}$, see Lemma 59.64.4. Choose a connected component $T' \subset T$ (if $\overline{\xi}$ has order d, then T is already connected). Then $T' \to X_{\overline{k}}$ is a finite étale Galois cover whose Galois group is a subgroup $G \subset \mathbf{Z}/d\mathbf{Z}$ (small detail omitted). Moreover the element $\overline{\xi}$ maps to zero under the map $H^1(X_{\overline{k}}, \mathbf{Z}/d\mathbf{Z}) \to H^1(T', \mathbf{Z}/d\mathbf{Z})$ as this is one of the defining properties of T.

Next, we use a limit argument to choose a finite extension k'/k contained in $\overline{k}$ such that $T' \to X_{\overline{k}}$ descends to a finite étale Galois cover $Z \to X_{k'}$ with group G. See Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. After increasing k' we may assume that Z splits over $x_{k'}$. The image of ξ in $H^1(Z_{\overline{K}}, \mathbf{Z}/d\mathbf{Z})$ is zero by construction. Thus by Lemma 59.51.5 we can find a finite subextension $\overline{K}/K'/K$ containing k' such that ξ dies in $H^1(Z_{K'}, \mathbf{Z}/d\mathbf{Z})$ and this finishes the proof. $\square$

0EYU **Theorem 59.89.2** (Smooth base change). Consider a cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

where f is smooth and g quasi-compact and quasi-separated. Then

$$f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$$

for any q and any abelian sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$ all of whose stalks at geometric points are torsion of orders invertible on S.

First proof of smooth base change. This proof is very long but more direct (using less general theory) than the second proof given below.

The theorem is local on $X_{\acute{e}tale}$. More precisely, suppose we have $U \to X$ étale such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale. Then we can consider the cartesian square

$$\begin{tikzcd} U \arrow[d, "f'"'] & U \times_X Y \arrow[l, "h'"'] \arrow[d, "e'"] \\ V & V \times_S T \arrow[l, "g'"] \end{tikzcd}$$

and setting $\mathcal{F}' = \mathcal{F}|_{V\times_S T}$ we have $f^{-1}R^qg_*\mathcal{F}|_U = (f')^{-1}R^qg'_*\mathcal{F}'$ and $R^qh_*e^{-1}\mathcal{F}|_U = R^qh'_*(e')^{-1}\mathcal{F}'$ (as follows from the compatibility of localization with morphisms of sites, see Sites, Lemma 7.28.2 and and Cohomology on Sites, Lemma 21.20.4). Thus it suffices to produce an étale covering of X by $U \to X$ and factorizations $U \to V \to S$ as above such that the theorem holds for the diagram with f', h', g', e'.

By the local structure of smooth morphisms, see Morphisms, Lemma 29.36.20, we may assume X and S are affine and $X \to S$ factors through an étale morphism $X \to \mathbf{A}^d_S$. If we have a tower of cartesian diagrams

$$\begin{tikzcd} W \arrow[d, "i"'] & Z \arrow[l, "j"'] \arrow[d, "k"] \\ X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

and the theorem holds for the bottom and top squares, then the theorem holds for the outer rectangle; this is formal. Writing $X \to S$ as the composition

$$X \to \mathbf{A}^{d-1}_S \to \mathbf{A}^{d-2}_S \to \ldots \to \mathbf{A}^1_S \to S$$

we conclude that it suffices to prove the theorem when X and S are affine and $X \to S$ has relative dimension 1.

For every $n \geq 1$ invertible on S, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}[n]$ by our assumption on the stalks of $\mathcal{F}$. The functors e^{-1} and f^{-1} commute with colimits as they are left adjoints. The functors R^qh_* and R^qg_* commute with filtered colimits by Lemma 59.51.7. Thus it suffices to prove the theorem for $\mathcal{F}[n]$. From now on we fix an integer n, we work with sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules and we assume S is a scheme over $\mathrm{Spec}(\mathbf{Z}[1/n])$.

Next, we reduce to the case where T is affine. Since g is quasi-compact and quasi-separate and S is affine, the scheme T is quasi-compact and quasi-separated. Thus we can use the induction principle of Cohomology of Schemes, Lemma 30.4.1. Hence it suffices to show that if $T = W \cup W'$ is an open covering and the theorem holds for the squares

$$\begin{tikzcd} X \arrow[d] & e^{-1}(W) \arrow[l, "i"'] \arrow[d] \\ S & W \arrow[l, "a"] \end{tikzcd} \quad \begin{tikzcd} X \arrow[d] & e^{-1}(W') \arrow[l, "j"'] \arrow[d] \\ S & W' \arrow[l, "b"] \end{tikzcd} \quad \begin{tikzcd} X \arrow[d] & e^{-1}(W \cap W') \arrow[l, "k"'] \arrow[d] \\ S & W \cap W' \arrow[l, "c"] \end{tikzcd}$$

then the theorem holds for the original diagram. To see this we consider the diagram

$$\begin{tikzcd} f^{-1}R^{q-1}c_*\mathcal{F}|_{W\cap W'} \arrow[r] \arrow[d, "\cong"] & f^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & f^{-1}R^q a_*\mathcal{F}|_W \oplus f^{-1}R^q b_*\mathcal{F}|_{W'} \arrow[d, "\cong"'] \\ R^q k_* e^{-1}\mathcal{F}|_{e^{-1}(W\cap W')} \arrow[r] & R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q i_* e^{-1}\mathcal{F}|_{e^{-1}(W)} \oplus R^q j_* e^{-1}\mathcal{F}|_{e^{-1}(W')} \end{tikzcd}$$

whose rows are the long exact sequences of Lemma 59.50.2. Thus the 5-lemma gives the desired conclusion.

Summarizing, we may assume S, X, T, and Y affine, $\mathcal{F}$ is n torsion, $X \to S$ is smooth of relative dimension 1, and S is a scheme over $\mathbf{Z}[1/n]$. We will prove the theorem by induction on q. The base case $q = 0$ is handled by Lemma 59.87.2. Assume $q > 0$ and the theorem holds for all smaller degrees. Choose a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ where $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Consider the induced diagram

$$\begin{tikzcd} f^{-1}R^{q-1}g_*\mathcal{I} \arrow[r] \arrow[d, "\cong"'] & f^{-1}R^{q-1}g_*\mathcal{Q} \arrow[r] \arrow[d, "\cong"'] & f^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[d] \\ R^{q-1}h_*e^{-1}\mathcal{I} \arrow[r] & R^{q-1}h_*e^{-1}\mathcal{Q} \arrow[r] & R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q h_* e^{-1}\mathcal{I} \end{tikzcd}$$

with exact rows. We have the zero in the right upper corner as $\mathcal{I}$ is injective. The left two vertical arrows are isomorphisms by induction hypothesis. Thus it suffices to prove that $R^q h_* e^{-1}\mathcal{I} = 0$.

Write $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ and say the morphism $T \to S$ is given by the ring map $A \to B$. We can write $A \to B = \mathrm{colim}_{i \in I}(A_i \to B_i)$ as a filtered colimit of maps of rings of finite type over $\mathbf{Z}[1/n]$ (see Algebra, Lemma 10.127.14). For $i \in I$ we set $S_i = \mathrm{Spec}(A_i)$ and $T_i = \mathrm{Spec}(B_i)$. For i large enough we can find a smooth morphism $X_i \to S_i$ of relative dimension 1 such that $X = X_i \times_{S_i} S$, see Limits, Lemmas 32.10.1, 32.8.9, and 32.18.4. Set $Y_i = X_i \times_{S_i} T_i$ to get squares

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] & Y_i \arrow[l, "h_i"'] \arrow[d, "e_i"] \\ S_i & T_i \arrow[l, "g_i"] \end{tikzcd}$$

Observe that $\mathcal{I}_i = (T \to T_i)_*\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on T_i, see Cohomology on Sites, Lemma 21.14.2. We have $\mathcal{I} = \text{colim}(T \to T_i)^{-1}\mathcal{I}_i$ by Lemma 59.51.9. Pulling back by e we get $e^{-1}\mathcal{I} = \text{colim}(Y \to Y_i)^{-1}e_i^{-1}\mathcal{I}_i$. By Lemma 59.51.8 applied to the system of morphisms $Y_i \to X_i$ with limit $Y \to X$ we have

$$R^q h_* e^{-1}\mathcal{I} = \text{colim}(X \to X_i)^{-1} R^q h_{i,*} e_i^{-1}\mathcal{I}_i$$

This reduces us to the case where T and S are affine of finite type over $\mathbf{Z}[1/n]$.

Summarizing, we have an integer $q \geq 1$ such that the theorem holds in degrees $< q$, the schemes S and T affine of finite type type over $\mathbf{Z}[1/n]$, we have $X \to S$ smooth of relative dimension 1 with X affine, and $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and we have to show that $R^q h_* e^{-1}\mathcal{I} = 0$. We will do this by induction on $\dim(T)$.

The base case is $T = \emptyset$, i.e., $\dim(T) < 0$. If you don't like this, you can take as your base case the case $\dim(T) = 0$. In this case $T \to S$ is finite (in fact even $T \to \text{Spec}(\mathbf{Z}[1/n])$ is finite as the target is Jacobson; details omitted), so h is finite too and hence has vanishing higher direct images (see references below).

Assume $\dim(T) = d \geq 0$ and we know the result for all situations where T has lower dimension. Pick U affine and étale over X and a section ξ of $R^q h_* q^{-1}\mathcal{I}$ over U. We have to show that ξ is zero. Of course, we may replace X by U (and correspondingly Y by $U \times_X Y$) and assume $\xi \in H^0(X, R^q h_* e^{-1}\mathcal{I})$. Moreover, since $R^q h_* e^{-1}\mathcal{I}$ is a sheaf, it suffices to prove that ξ is zero locally on X. Hence we may replace X by the members of an étale covering. In particular, using Lemma 59.51.6 we may assume that ξ is the image of an element $\tilde{\xi} \in H^q(Y, e^{-1}\mathcal{I})$. In terms of $\tilde{\xi}$ our task is to show that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ for some étale covering $\{U_i \to X\}$.

By More on Morphisms, Lemma 37.38.8 we may assume that $X \to S$ factors as $X \to V \to S$ where $V \to S$ is étale and $X \to V$ is a smooth morphism of affine schemes of relative dimension 1, has a section, and has geometrically connected fibres. Observe that $\dim(V \times_S T) \leq \dim(T) = d$ for example by More on Algebra, Lemma 15.44.2. Hence we may then replace S by V and T by $V \times_S T$ (exactly as in the discussion in the first paragraph of the proof). Thus we may assume $X \to S$ is smooth of relative dimension 1, geometrically connected fibres, and has a section $\sigma : S \to X$.

Let $\pi : T' \to T$ be a finite surjective morphism. We will use below that $\dim(T') \leq \dim(T) = d$, see Algebra, Lemma 10.112.3. Choose an injective map $\pi^{-1}\mathcal{I} \to \mathcal{I}'$ into an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Then $\mathcal{I} \to \pi_*\mathcal{I}'$ is injective and hence has a splitting (as $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules). Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of π and $e' : Y' \to T'$ the base change of e. Picture

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] & Y' \arrow[l, "\pi'"'] \arrow[d, "e'"] \\ S & T \arrow[l, "g"'] & T' \arrow[l, "\pi"'] \end{tikzcd}$$

By Proposition 59.55.2 and Lemma 59.55.3 we have $R\pi'_*(e')^{-1}\mathcal{I}' = e^{-1}\pi_*\mathcal{I}'$. Thus by the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.5) we have

$$H^q(Y', (e')^{-1}\mathcal{I}') = H^q(Y, e^{-1}\pi_*\mathcal{I}') \supset H^q(Y, e^{-1}\mathcal{I})$$

and this remains true after base change by any $U \to X$ étale. Thus we may replace T by T', $\mathcal{I}$ by $\mathcal{I}'$ and $\tilde{\xi}$ by its image in $H^q(Y', (e')^{-1}\mathcal{I}')$.

Suppose we have a factorization $T \to S' \to S$ where $\pi : S' \to S$ is finite. Setting $X' = S' \times_S X$ we can consider the induced diagram

$$\begin{tikzcd} X & X' \arrow[l, "\pi'"'] \arrow[d, "f'"] & Y \arrow[l, "h'"'] \arrow[d, "e"] \\ S \arrow[from=u, "f"'] & S' \arrow[l, "\pi"'] & T \arrow[l, "g"'] \end{tikzcd}$$

Since π' has vanishing higher direct images we see that $R^q h_* e^{-1}\mathcal{I} = \pi'_* R^q h'_* e^{-1}\mathcal{I}$ by the Leray spectral sequence. Hence $H^0(X, R^q h_* e^{-1}\mathcal{I}) = H^0(X', R^q h'_* e^{-1}\mathcal{I})$. Thus ξ is zero if and only if the corresponding section of $R^q h'_* e^{-1}\mathcal{I}$ is zero[10]. Thus we may replace S by S' and X by X'. Observe that $\sigma : S \to X$ base changes to $\sigma' : S' \to X'$ and hence after this replacement it is still true that $X \to S$ has a section σ and geometrically connected fibres.

We will use that S and T are Nagata schemes, see Algebra, Proposition 10.162.16 which will guarantee that various normalizations are finite, see Morphisms, Lemmas 29.53.15 and 29.54.10. In particular, we may first replace T by its normalization and then replace S by the normalization of S in T. Then $T \to S$ is a disjoint union of dominant morphisms of integral normal schemes, see Morphisms, Lemma 29.53.13. Clearly we may argue one connnected component at a time, hence we may assume $T \to S$ is a dominant morphism of integral normal schemes.

Let $s \in S$ and $t \in T$ be the generic points. By Lemma 59.89.1 there exist finite field extensions $K/\kappa(t)$ and $k/\kappa(s)$ such that k is contained in K and a finite étale Galois covering $Z \to X_k$ with Galois group G of order dividing a power of n split over $\sigma(\mathrm{Spec}(k))$ such that $\tilde{\xi}$ maps to zero in $H^q(Z_K, e^{-1}\mathcal{I}|_{Z_K})$. Let $T' \to T$ be the normalization of T in $\mathrm{Spec}(K)$ and let $S' \to S$ be the normalization of S in $\mathrm{Spec}(k)$. Then we obtain a commutative diagram

$$\begin{tikzcd} S' \arrow[d] & T' \arrow[l] \arrow[d] \\ S & T \arrow[l] \end{tikzcd}$$

whose vertical arrows are finite. By the arguments given above we may and do replace S and T by S' and T' (and correspondingly X by $X \times_S S'$ and Y by $Y \times_T T'$). After this replacement we conclude we have a finite étale Galois covering $Z \to X_s$ of the generic fibre of $X \to S$ with Galois group G of order dividing a power of n split over $\sigma(s)$ such that $\tilde{\xi}$ maps to zero in $H^q(Z_t, (Z_t \to Y)^{-1}e^{-1}\mathcal{I})$. Here $Z_t = Z \times_S t = Z \times_s t = Z \times_{X_s} Y_t$. Since n is invertible on S, by Fundamental Groups, Lemma 58.31.8 we can find a finite étale morphism $U \to X$ whose restriction to X_s is Z.

At this point we replace X by U and Y by $U \times_X Y$. After this replacement it may no longer be the case that the fibres of $X \to S$ are geometrically connected (there still is a section but we won't use this), but what we gain is that after this

[10]This step can also be seen another way. Namely, we have to show that there is an étale covering $\{U_i \to X\}$ such that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$. However, if we prove there is an étale covering $\{U'_j \to X'\}$ such that $\tilde{\xi}$ dies in $H^q(U'_i \times_{X'} Y, e^{-1}\mathcal{I})$, then by property (B) for $X' \to X$ (Lemma 59.43.3) there exists an étale covering $\{U_i \to X\}$ such that $U_i \times_X X'$ is a disjoint union of schemes over X' each of which factors through U'_j for some j. Thus we see that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ as desired.

replacement $\tilde{\xi}$ maps to zero in $H^q(Y_t, e^{-1}\mathcal{I})$, i.e., $\tilde{\xi}$ restricts to zero on the generic fibre of $Y \to T$.

Recall that t is the spectrum of the function field of T, i.e., as a scheme t is the limit of the nonempty affine open subschemes of T. By Lemma 59.51.5 we conclude there exists a nonempty open subscheme $V \subset T$ such that $\tilde{\xi}$ maps to zero in $H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$.

Denote $Z = T \setminus V$. Consider the diagram

$$\begin{tikzcd} Y \times_T Z \arrow[r, "i'"'] \arrow[d, "e_Z"'] & Y \arrow[d, "e"'] & Y \times_T V \arrow[l, "j'"] \arrow[d, "e_V"] \\ Z \arrow[r, "i"] & T & V \arrow[l, "j"'] \end{tikzcd}$$

Choose an injection $i^{-1}\mathcal{I} \to \mathcal{I}'$ into an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on Z. Looking at stalks we see that the map

$$\mathcal{I} \to j_*\mathcal{I}|_V \oplus i_*\mathcal{I}'$$

is injective and hence splits as $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Thus it suffices to show that $\tilde{\xi}$ maps to zero in

$$H^q(Y, e^{-1}j_*\mathcal{I}|_V) \oplus H^q(Y, e^{-1}i_*\mathcal{I}')$$

at least after replacing X by the members of an étale covering. Observe that

$$e^{-1}j_*\mathcal{I}|_V = j'_*e_V^{-1}\mathcal{I}|_V, \quad e^{-1}i_*\mathcal{I}' = i'_*e_Z^{-1}\mathcal{I}'$$

By induction hypothesis on q we see that

$$R^a j'_* e_V^{-1}\mathcal{I}|_V = 0, \quad a = 1, \ldots, q-1$$

By the Leray spectral sequence for j' and the vanishing above it follows that

$$H^q(Y, j'_*(e_V^{-1}\mathcal{I}|_V)) \longrightarrow H^q(Y \times_T V, e_V^{-1}\mathcal{I}_V) = H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$$

is injective. Thus the vanishing of the image of $\tilde{\xi}$ in the first summand above because we know $\tilde{\xi}$ vanishes in $H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$. Since $\dim(Z) < \dim(T) = d$ by induction the image of $\tilde{\xi}$ in the second summand

$$H^q(Y, e^{-1}i_*\mathcal{I}') = H^q(Y, i'_*e_Z^{-1}\mathcal{I}') = H^q(Y \times_T Z, e_Z^{-1}\mathcal{I}')$$

dies after replacing X by the members of a suitable étale covering. This finishes the proof of the smooth base change theorem. □

Second proof of smooth base change. This proof is the same as the longer first proof; it is shorter only in that we have split out the arguments used in a number of lemmas.

The case of $q = 0$ is Lemma 59.87.2. Thus we may assume $q > 0$ and the result is true for all smaller degrees.

For every $n \geq 1$ invertible on S, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$ by our assumption on the stalks of $\mathcal{F}$. The functors e^{-1} and f^{-1} commute with colimits as they are left adjoints. The functors R^qh_* and R^qg_* commute with filtered colimits by Lemma 59.51.7. Thus it suffices to prove the theorem for $\mathcal{F}[n]$. From now on we fix an integer n invertible on S and we work with sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules.

By Lemma 59.86.1 the question is étale local on X and S. By the local structure of smooth morphisms, see Morphisms, Lemma 29.36.20, we may assume X and S are affine and $X \to S$ factors through an étale morphism $X \to \mathbf{A}^d_S$. Writing $X \to S$ as the composition

$$X \to \mathbf{A}^{d-1}_S \to \mathbf{A}^{d-2}_S \to \ldots \to \mathbf{A}^1_S \to S$$

we conclude from Lemma 59.86.2 that it suffices to prove the theorem when X and S are affine and $X \to S$ has relative dimension 1.

By Lemma 59.88.7 it suffices to show that $R^qh_*\mathbf{Z}/d\mathbf{Z} = 0$ for $d|n$ whenever we have a cartesian diagram

$$\begin{tikzcd} X \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\ S & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

where $X \to S$ is affine and smooth of relative dimension 1, S is the spectrum of a normal domain A with algebraically closed fraction field L, and K/L is an extension of algebraically closed fields.

Recall that $R^qh_*\mathbf{Z}/d\mathbf{Z}$ is the sheaf associated to the presheaf

$$U \longmapsto H^q(U \times_X Y, \mathbf{Z}/d\mathbf{Z}) = H^q(U \times_S \mathrm{Spec}(K), \mathbf{Z}/d\mathbf{Z})$$

on $X_{étale}$ (Lemma 59.51.6). Thus it suffices to show: given U and $\xi \in H^q(U \times_S \mathrm{Spec}(K), \mathbf{Z}/d\mathbf{Z})$ there exists an étale covering $\{U_i \to U\}$ such that ξ dies in $H^q(U_i \times_S \mathrm{Spec}(K), \mathbf{Z}/d\mathbf{Z})$.

Of course we may take U affine. Then $U \times_S \mathrm{Spec}(K)$ is a (smooth) affine curve over K and hence we have vanishing for $q > 1$ by Theorem 59.83.10.

Final case: $q = 1$. We may replace U by the members of an étale covering as in More on Morphisms, Lemma 37.38.8. Then $U \to S$ factors as $U \to V \to S$ where $U \to V$ has geometrically connected fibres, U, V are affine, $V \to S$ is étale, and there is a section $\sigma : V \to U$. By Lemma 59.80.4 we see that V is isomorphic to a (finite) disjoint union of (affine) open subschemes of S. Clearly we may replace S by one of these and X by the corresponding component of U. Thus we may assume $X \to S$ has geometrically connected fibres, has a section σ, and $\xi \in H^1(Y, \mathbf{Z}/d\mathbf{Z})$. Since K and L are algebraically closed we have

$$H^1(X_L, \mathbf{Z}/d\mathbf{Z}) = H^1(Y, \mathbf{Z}/d\mathbf{Z})$$

See Lemma 59.83.12. Thus there is a finite étale Galois covering $Z \to X_L$ with Galois group $G \subset \mathbf{Z}/d\mathbf{Z}$ which annihilates ξ. You can either see this by looking at the statement or proof of Lemma 59.89.1 or by using directly that ξ corresponds to a $\mathbf{Z}/d\mathbf{Z}$-torsor over X_L. Finally, by Fundamental Groups, Lemma 58.31.9 we find a (necessarily surjective) finite étale morphism $X' \to X$ whose restriction to X_L is $Z \to X_L$. Since ξ dies in X'_K this finishes the proof. □

The following immediate consquence of the smooth base change theorem is what is often used in practice.

0F09 **Lemma 59.89.3.** Let S be a scheme. Let $S' = \lim S_i$ be a directed inverse limit of schemes S_i smooth over S with affine transition morphisms. Let $f : X \to S$ be quasi-compact and quasi-separated and form the fibre square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then

$$g^{-1}Rf_*E = R(f')_*(g')^{-1}E$$

for any $E \in D^+(X_{\acute{e}tale})$ whose cohomology sheaves $H^q(E)$ have stalks which are torsion of orders invertible on S.

Proof. Consider the spectral sequences

$$E_2^{p,q} = R^pf_*H^q(E) \quad \text{and} \quad {E'}_2^{p,q} = R^pf'_*H^q((g')^{-1}E) = R^pf'_*(g')^{-1}H^q(E)$$

converging to R^nf_*E and $R^nf'_*(g')^{-1}E$. These spectral sequences are constructed in Derived Categories, Lemma 13.21.3. Combining the smooth base change theorem (Theorem 59.89.2) with Lemma 59.86.3 we see that

$$g^{-1}R^pf_*H^q(E) = R^p(f')_*(g')^{-1}H^q(E)$$

Combining all of the above we get the lemma. □

59.90. Applications of smooth base change

0F0A In this section we discuss some more or less immediate consequences of the smooth base change theorem.

0F1C **Lemma 59.90.1.** Let L/K be an extension of fields. Let $g : T \to S$ be a quasi-compact and quasi-separated morphism of schemes over K. Denote $g_L : T_L \to S_L$ the base change of g to $\text{Spec}(L)$. Let $E \in D^+(T_{\acute{e}tale})$ have cohomology sheaves whose stalks are torsion of orders invertible in K. Let E_L be the pullback of E to $(T_L)_{\acute{e}tale}$. Then $Rg_{L,*}E_L$ is the pullback of Rg_*E to S_L.

Proof. If L/K is separable, then L is a filtered colimit of smooth K-algebras, see Algebra, Lemma 10.158.11. Thus the lemma in this case follows immediately from Lemma 59.89.3. In the general case, let K' and L' be the perfect closures (Algebra, Definition 10.45.5) of K and L. Then $\text{Spec}(K') \to \text{Spec}(K)$ and $\text{Spec}(L') \to \text{Spec}(L)$ are universal homeomorphisms as K'/K and L'/L are purely inseparable (see Algebra, Lemma 10.46.7). Thus we have $(T_{K'})_{\acute{e}tale} = T_{\acute{e}tale}$, $(S_{K'})_{\acute{e}tale} = S_{\acute{e}tale}$, $(T_{L'})_{\acute{e}tale} = (T_L)\acute{e}tale$, and $(S_{L'})_{\acute{e}tale} = (S_L)_{\acute{e}tale}$ by the topological invariance of étale cohomology, see Proposition 59.45.4. This reduces the lemma to the case of the field extension L'/K' which is separable (by definition of perfect fields, see Algebra, Definition 10.45.1). □

0F0B **Lemma 59.90.2.** Let K/k be an extension of separably closed fields. Let X be a quasi-compact and quasi-separated scheme over k. Let $E \in D^+(X_{\acute{e}tale})$ have cohomology sheaves whose stalks are torsion of orders invertible in k. Then

(1) the maps $H^q_{\acute{e}tale}(X, E) \to H^q_{\acute{e}tale}(X_K, E|_{X_K})$ are isomorphisms, and
(2) $E \to R(X_K \to X)_*E|_{X_K}$ is an isomorphism.

Proof. Proof of (1). First let $\overline{k}$ and $\overline{K}$ be the algebraic closures of k and K. The morphisms $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ and $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(K)$ are universal homeomorphisms as $\overline{k}/k$ and $\overline{K}/K$ are purely inseparable (see Algebra, Lemma 10.46.7). Thus $H^q_{\acute{e}tale}(X, \mathcal{F}) = H^q_{\acute{e}tale}(X_{\overline{k}}, \mathcal{F}_{X_{\overline{k}}})$ by the topological invariance of étale cohomology, see Proposition 59.45.4. Similarly for X_K and $X_{\overline{K}}$. Thus we may assume k and K are algebraically closed. In this case K is a limit of smooth k-algebras, see Algebra, Lemma 10.158.11. We conclude our lemma is a special case of Theorem 59.89.2 as reformulated in Lemma 59.89.3.

Proof of (2). For any quasi-compact and quasi-separated U in $X_{\acute{e}tale}$ the above shows that the restriction of the map $E \to R(X_K \to X)_*E|_{X_K}$ determines an isomorphism on cohomology. Since every object of $X_{\acute{e}tale}$ has an étale covering by such U this proves the desired statement. $\square$

0F1D Lemma 59.90.3. With $f : X \to S$ and n as in Remark 59.88.1 assume n is invertible on S and that for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\ S & S' \arrow[l] & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

with both squares cartesian, where S' is affine, integral, and normal with algebraically closed function field K and there exists an integer $d|n$ such that $R^qh_*(\mathbf{Z}/d\mathbf{Z})$ is nonzero.

Proof. First choose a diagram and $\mathcal{F}$ as in Lemma 59.88.7. We may and do assume S' is affine (this is obvious, but see proof of the lemma in case of doubt). Let K' be the function field of S' and let $Y' = X' \times_{S'} \mathrm{Spec}(K')$ to get the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y' \arrow[l, "h'"'] \arrow[d] & Y \arrow[l] \arrow[d] \\ S & S' \arrow[l] & \mathrm{Spec}(K') \arrow[l] & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

By Lemma 59.90.2 the total direct image $R(Y \to Y')_*\mathbf{Z}/d\mathbf{Z}$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ in $D(Y'_{\acute{e}tale})$; here we use that n is invertible on S. Thus $Rh'_*\mathbf{Z}/d\mathbf{Z} = Rh_*\mathbf{Z}/d\mathbf{Z}$ by the relative Leray spectral sequence. This finishes the proof. $\square$

59.91. The proper base change theorem

095S The proper base change theorem is stated and proved in this section. Our approach follows roughly the proof in [AGV71, XII, Theorem 5.1] using Gabber's ideas (from the affine case) to slightly simplify the arguments.

0A0B Lemma 59.91.1. Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $Z = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on the topological space associated to X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

Proof. We will use Lemma 59.82.4 to prove this. First observe that the underlying topological space of X is spectral by Properties, Lemma 28.2.4. Let $Y \subset X$ be an irreducible closed subscheme. To finish the proof we show that $Y \cap Z = Y \times_{\mathrm{Spec}(A)}$

$\mathrm{Spec}(A/I)$ is connected. Replacing X by Y we may assume that X is irreducible and we have to show that Z is connected. Let $X \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the Stein factorization of f (More on Morphisms, Theorem 37.53.5). Then $A \to B$ is integral and (B, IB) is a henselian pair (More on Algebra, Lemma 15.11.8). Thus we may assume the fibres of $X \to \mathrm{Spec}(A)$ are geometrically connected. On the other hand, the image $T \subset \mathrm{Spec}(A)$ of f is irreducible and closed as X is proper over A. Hence $T \cap V(I)$ is connected by More on Algebra, Lemma 15.11.16. Now $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) \to T \cap V(I)$ is a surjective closed map with connected fibres. The result now follows from Topology, Lemma 5.7.5. □

0A0C Lemma 59.91.2. Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $i : Z \to X$ be the closed immersion of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$ into X. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{small}\mathcal{F})$.

Proof. This follows from Lemma 59.82.2 and 59.91.1 and the fact that any scheme finite over X is proper over $\mathrm{Spec}(A)$. □

0A3S Lemma 59.91.3. Let A be a henselian local ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $X_0 \subset X$ be the fibre of f over the closed point. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})$.

Proof. This is a special case of Lemma 59.91.2. □

Let $f : X \to S$ be a morphism of schemes. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. The fibre of f at $\overline{s}$ is the scheme $X_{\overline{s}} = \mathrm{Spec}(k) \times_{\overline{s}, S} X$ viewed as a scheme over $\mathrm{Spec}(k)$. If $\mathcal{F}$ is a sheaf on $X_{étale}$, then denote $\mathcal{F}_{\overline{s}} = p^{-1}_{small}\mathcal{F}$ the pullback of $\mathcal{F}$ to $(X_{\overline{s}})_{étale}$. In the following we will consider the set

$$\Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

Let $s \in S$ be the image point of $\overline{s}$. Let $\kappa(s)^{sep}$ be the separable algebraic closure of $\kappa(s)$ in k as in Definition 59.56.1. By Lemma 59.39.5 pullback defines a bijection

$$\Gamma(X_{\kappa(s)^{sep}}, p^{-1}_{sep}\mathcal{F}) \longrightarrow \Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

where $p_{sep} : X_{\kappa(s)^{sep}} = \mathrm{Spec}(\kappa(s)^{sep}) \times_S X \to X$ is the projection.

0A3T Lemma 59.91.4. Let $f : X \to S$ be a proper morphism of schemes. Let $\overline{s} \to S$ be a geometric point. For any sheaf $\mathcal{F}$ on $X_{étale}$ the canonical map

$$(f_*\mathcal{F})_{\overline{s}} \longrightarrow \Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

is bijective.

Proof. By Theorem 59.53.1 (for sheaves of sets) we have

$$(f_*\mathcal{F})_{\overline{s}} = \Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}_{small}\mathcal{F})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection. Since the residue field of the strictly henselian local ring $\mathcal{O}^{sh}_{S,\overline{s}}$ is $\kappa(s)^{sep}$ we conclude from the discussion above the lemma and Lemma 59.91.3. □

0A3U Lemma 59.91.5. Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ with projections $f' : X' \to Y'$ and $g' : X' \to X$. Let $\mathcal{F}$ be any sheaf on $X_{étale}$. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$.

Proof. There is a canonical map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$. Namely, it is adjoint to the map

$$f_*\mathcal{F} \longrightarrow g_*f'_*(g')^{-1}\mathcal{F} = f_*g'_*(g')^{-1}\mathcal{F}$$

which is f_* applied to the canonical map $\mathcal{F} \to g'_*(g')^{-1}\mathcal{F}$. To check this map is an isomorphism we can compute what happens on stalks. Let $y' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image y in Y. By Lemma 59.91.4 the stalks are $\Gamma(X'_{y'}, \mathcal{F}_{y'})$ and $\Gamma(X_y, \mathcal{F}_y)$ respectively. Here the sheaves $\mathcal{F}_y$ and $\mathcal{F}_{y'}$ are the pullbacks of $\mathcal{F}$ by the projections $X_y \to X$ and $X'_{y'} \to X$. Thus we see that the groups agree by Lemma 59.39.5. We omit the verification that this isomorphism is compatible with our map. ☐

At this point we start discussing the proper base change theorem. To do so we introduce some notation. consider a commutative diagram

0A29 (59.91.5.1)
$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of morphisms of schemes. Then we obtain a commutative diagram of sites

$$\begin{tikzcd} X'_{\acute{e}tale} \arrow[r, "g'_{small}"'] \arrow[d, "f'_{small}"'] & X_{\acute{e}tale} \arrow[d, "f_{small}"] \\ Y'_{\acute{e}tale} \arrow[r, "g_{small}"] & Y_{\acute{e}tale} \end{tikzcd}$$

For any object E of $D(X_{\acute{e}tale})$ we obtain a canonical base change map

0A2A (59.91.5.2)
$$g_{small}^{-1}Rf_{small,*}E \longrightarrow Rf'_{small,*}(g'_{small})^{-1}E$$

in $D(Y'_{\acute{e}tale})$. See Cohomology on Sites, Remark 21.19.3 where we use the constant sheaf $\mathbf{Z}$ as our sheaf of rings. We will usually omit the subscripts $_{small}$ in this formula. For example, if $E = \mathcal{F}[0]$ where $\mathcal{F}$ is an abelian sheaf on $X_{\acute{e}tale}$, the base change map is a map

0A4A (59.91.5.3)
$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

in $D(Y'_{\acute{e}tale})$.

The map (59.91.5.2) has no chance of being an isomorphism in the generality given above. The goal is to show it is an isomorphism if the diagram (59.91.5.1) is cartesian, $f : X \to Y$ proper, the cohomology sheaves of E are torsion, and E is bounded below. To study this question we introduce the following terminology. Let us say that cohomology commutes with base change for $f : X \to Y$ if (59.91.5.3) is an isomorphism for every diagram (59.91.5.1) where $X' = Y' \times_Y X$ and every torsion abelian sheaf $\mathcal{F}$.

0A4B Lemma 59.91.6. Let $f : X \to Y$ be a proper morphism of schemes. The following are equivalent

(1) cohomology commutes with base change for f (see above),
(2) for every prime number ℓ and every injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules $\mathcal{I}$ on $X_{\acute{e}tale}$ and every diagram (59.91.5.1) where $X' = Y' \times_Y X$ the sheaves $R^qf'_*(g')^{-1}\mathcal{I}$ are zero for $q > 0$.

Proof. It is clear that (1) implies (2). Conversely, assume (2) and let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Let $Y' \to Y$ be a morphism of schemes and let $X' = Y' \times_Y X$ with projections $g' : X' \to X$ and $f' : X' \to Y'$ as in diagram (59.91.5.1). We want to show the maps of sheaves

$$g^{-1}R^q f_*\mathcal{F} \longrightarrow R^q f'_*(g')^{-1}\mathcal{F}$$

are isomorphisms for all $q \geq 0$.

For every $n \geq 1$, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$. The functors g^{-1} and $(g')^{-1}$ commute with arbitrary colimits (as left adjoints). Taking higher direct images along f or f' commutes with filtered colimits by Lemma 59.51.7. Hence we see that

$$g^{-1}R^q f_*\mathcal{F} = \operatorname{colim} g^{-1}R^q f_*\mathcal{F}[n] \quad \text{and} \quad R^q f'_*(g')^{-1}\mathcal{F} = \operatorname{colim} R^q f'_*(g')^{-1}\mathcal{F}[n]$$

Thus it suffices to prove the result in case $\mathcal{F}$ is annihilated by a positive integer n.

If $n = \ell n'$ for some prime number ℓ, then we obtain a short exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$$

Observe that $\mathcal{F}/\mathcal{F}[\ell]$ is annihilated by n'. Moreover, if the result holds for both $\mathcal{F}[\ell]$ and $\mathcal{F}/\mathcal{F}[\ell]$, then the result holds by the long exact sequence of higher direct images (and the 5 lemma). In this way we reduce to the case that $\mathcal{F}$ is annihilated by a prime number ℓ.

Assume $\mathcal{F}$ is annihilated by a prime number ℓ. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $D(X_{étale}, \mathbf{Z}/\ell\mathbf{Z})$. Applying assumption (2) and Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we see that

$$f'_*(g')^{-1}\mathcal{I}^\bullet$$

computes $Rf'_*(g')^{-1}\mathcal{F}$. We conclude by applying Lemma 59.91.5. $\square$

0A4C Lemma 59.91.7. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes. Assume

(1) cohomology commutes with base change for f,
(2) cohomology commutes with base change for $g \circ f$, and
(3) f is surjective.

Then cohomology commutes with base change for g.

Proof. We will use the equivalence of Lemma 59.91.6 without further mention. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Y_{étale}$. Choose an injective map of sheaves $f^{-1}\mathcal{I} \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Z_{étale}$. Since f is surjective the map $\mathcal{I} \to f_*\mathcal{J}$ is injective (look at stalks in geometric points). Since $\mathcal{I}$ is injective we see that $\mathcal{I}$ is a direct summand of $f_*\mathcal{J}$. Thus it suffices to prove the desired vanishing for $f_*\mathcal{J}$.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 59.91.5 we have $b^{-1}f_*\mathcal{J} = f'_*a^{-1}\mathcal{J}$. On the other hand, we know that $R^q f'_*a^{-1}\mathcal{J}$ and $R^q(g' \circ f')_*a^{-1}\mathcal{J}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 21.14.7)

$$R^p g'_* R^q f'_* a^{-1}\mathcal{J} \Rightarrow R^{p+q}(g' \circ f')_* a^{-1}\mathcal{J}$$

we conclude that $R^p g'_*(b^{-1}f_*\mathcal{J}) = R^p g'_*(f'_*a^{-1}\mathcal{J}) = 0$ for $p > 0$ as desired. $\square$

0A4D Lemma 59.91.8. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes. Assume

(1) cohomology commutes with base change for f, and
(2) cohomology commutes with base change for g.

Then cohomology commutes with base change for $g \circ f$.

Proof. We will use the equivalence of Lemma 59.91.6 without further mention. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Then $f_*\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Y_{étale}$ (Cohomology on Sites, Lemma 21.14.2). The result follows formally from this, but we will also spell it out.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 59.91.5 we have $b^{-1}f_*\mathcal{I} = f'_*a^{-1}\mathcal{I}$. On the other hand, we know that $R^qf'_*a^{-1}\mathcal{I}$ and $R^q(g')_*b^{-1}f_*\mathcal{I}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 21.14.7)

$$R^pg'_*R^qf'_*a^{-1}\mathcal{I} \Rightarrow R^{p+q}(g' \circ f')_*a^{-1}\mathcal{I}$$

we conclude that $R^p(g' \circ f')_*a^{-1}\mathcal{I} = 0$ for $p > 0$ as desired. □

0A4E Lemma 59.91.9. Let $f : X \to Y$ be a finite morphism of schemes. Then cohomology commutes with base change for f.

Proof. Observe that a finite morphism is proper, see Morphisms, Lemma 29.44.11. Moreover, the base change of a finite morphism is finite, see Morphisms, Lemma 29.44.6. Thus the result follows from Lemma 59.91.6 combined with Proposition 59.55.2. □

0A4F Lemma 59.91.10. To prove that cohomology commutes with base change for every proper morphism of schemes it suffices to prove it holds for the morphism $\mathbf{P}^1_S \to S$ for every scheme S.

Proof. Let $f : X \to Y$ be a proper morphism of schemes. Let $Y = \bigcup Y_i$ be an affine open covering and set $X_i = f^{-1}(Y_i)$. If we can prove cohomology commutes with base change for $X_i \to Y_i$, then cohomology commutes with base change for f. Namely, the formation of the higher direct images commutes with Zariski (and even étale) localization on the base, see Lemma 59.51.6. Thus we may assume Y is affine.

Let Y be an affine scheme and let $X \to Y$ be a proper morphism. By Chow's lemma there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X' \to X$ is proper and surjective, see Limits, Lemma 32.12.1. Since $X \to Y$ is proper, we find that $X' \to Y$ is proper (Morphisms, Lemma 29.41.4). Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 29.41.7). It follows that $X' \to X \times_Y \mathbf{P}^n_Y = \mathbf{P}^n_X$ is a closed immersion (as an immersion with closed image).

By Lemma 59.91.7 it suffices to prove cohomology commutes with base change for π and $X' \to Y$. These morphisms both factor as a closed immersion followed by a projection $\mathbf{P}^n_S \to S$ (for some S). By Lemma 59.91.9 the result holds for closed immersions (as closed immersions are finite). By Lemma 59.91.8 it suffices to prove the result for projections $\mathbf{P}^n_S \to S$.

For every $n \geq 1$ there is a finite surjective morphism

$$\mathbf{P}^1_S \times_S \ldots \times_S \mathbf{P}^1_S \longrightarrow \mathbf{P}^n_S$$

given on coordinates by

$$((x_1 : y_1), (x_2 : y_2), \ldots, (x_n : y_n)) \longmapsto (F_0 : \ldots : F_n)$$

where $F_0, \ldots, F_n$ in $x_1, \ldots, y_n$ are the polynomials with integer coefficients such that

$$\prod (x_i t + y_i) = F_0 t^n + F_1 t^{n-1} + \ldots + F_n$$

Applying Lemmas 59.91.7, 59.91.9, and 59.91.8 one more time we conclude that the lemma is true. $\square$

095T Theorem 59.91.11. Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and consider the cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$. Then the base change map

$$g^{-1} R f_* \mathcal{F} \longrightarrow R f'_* (g')^{-1} \mathcal{F}$$

is an isomorphism.

Proof. In the terminology introduced above, this means that cohomology commutes with base change for every proper morphism of schemes. By Lemma 59.91.10 it suffices to prove that cohomology commutes with base change for the morphism $\mathbf{P}^1_S \to S$ for every scheme S.

Let S be the spectrum of a strictly henselian local ring with closed point s. Set $X = \mathbf{P}^1_S$ and $X_0 = X_s = \mathbf{P}^1_s$. Let $\mathcal{F}$ be a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{\acute{e}tale}$. The key to our proof is that

$$H^q_{\acute{e}tale}(X, \mathcal{F}) = H^q_{\acute{e}tale}(X_0, \mathcal{F}|_{X_0}).$$

Namely, choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by injective sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules. Then $\mathcal{I}^\bullet|_{X_0}$ is a resolution of $\mathcal{F}|_{X_0}$ by right $H^0_{\acute{e}tale}(X_0, -)$-acyclic objects, see Lemma 59.85.2. Leray's acyclicity lemma tells us the right hand side is computed by the complex $H^0_{\acute{e}tale}(X_0, \mathcal{I}^\bullet|_{X_0})$ which is equal to $H^0_{\acute{e}tale}(X, \mathcal{I}^\bullet)$ by Lemma 59.91.3. This complex computes the left hand side.

Assume S is general and $\mathcal{F}$ is a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{\acute{e}tale}$. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point of S lying over $s \in S$. We have

$$(R^q f_* \mathcal{F})_{\overline{s}} = H^q_{\acute{e}tale}(\mathbf{P}^1_{\mathcal{O}^{sh}_{S,\overline{s}}}, \mathcal{F}|_{\mathbf{P}^1_{\mathcal{O}^{sh}_{S,\overline{s}}}}) = H^q_{\acute{e}tale}(\mathbf{P}^1_{\kappa(s)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(s)^{sep}}})$$

where $\kappa(s)^{sep}$ is the residue field of $\mathcal{O}^{sh}_{S,\overline{s}}$, i.e., the separable algebraic closure of $\kappa(s)$ in k. The first equality by Theorem 59.53.1 and the second equality by the displayed formula in the previous paragraph.

Finally, consider any morphism of schemes $g : T \to S$ where S and $\mathcal{F}$ are as above. Set $f' : \mathbf{P}^1_T \to T$ the projection and let $g' : \mathbf{P}^1_T \to \mathbf{P}^1_S$ the morphism induced by g. Consider the base change map

$$g^{-1}R^qf_*\mathcal{F} \longrightarrow R^qf'_*(g')^{-1}\mathcal{F}$$

Let $\overline{t}$ be a geometric point of T with image $\overline{s} = g(\overline{t})$. By our discussion above the map on stalks at $\overline{t}$ is the map

$$H^q_{\acute{e}tale}(\mathbf{P}^1_{\kappa(s)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(s)^{sep}}}) \longrightarrow H^q_{\acute{e}tale}(\mathbf{P}^1_{\kappa(t)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(t)^{sep}}})$$

Since $\kappa(s)^{sep} \subset \kappa(t)^{sep}$ this map is an isomorphism by Lemma 59.83.12.

This proves cohomology commutes with base change for $\mathbf{P}^1_S \to S$ and sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules. In particular, for an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules the higher direct images of any base change are zero. In other words, condition (2) of Lemma 59.91.6 holds and the proof is complete. $\square$

0DDE Lemma 59.91.12. Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $E \in D^+(X_{\acute{e}tale})$ have torsion cohomology sheaves. Then the base change map (59.91.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.

Proof. This is a simple consequence of the proper base change theorem (Theorem 59.91.11) using the spectral sequences

$$E_2^{p,q} = R^pf_*H^q(E) \quad \text{and} \quad E'^{p,q}_2 = R^pf'_*(g')^{-1}H^q(E)$$

converging to R^nf_*E and $R^nf'_*(g')^{-1}E$. The spectral sequences are constructed in Derived Categories, Lemma 13.21.3. Some details omitted. $\square$

0DDF Lemma 59.91.13. Let $f : X \to Y$ be a proper morphism of schemes. Let $\overline{y} \to Y$ be a geometric point.

(1) For a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $(R^nf_*\mathcal{F})_{\overline{y}} = H^n_{\acute{e}tale}(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.
(2) For $E \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves we have $(R^nf_*E)_{\overline{y}} = H^n_{\acute{e}tale}(X_{\overline{y}}, E|_{X_{\overline{y}}})$.

Proof. In the statement, $\mathcal{F}_{\overline{y}}$ denotes the pullback of $\mathcal{F}$ to the scheme theoretic fibre $X_{\overline{y}} = \overline{y} \times_Y X$. Since pulling back by $\overline{y} \to Y$ produces the stalk of $\mathcal{F}$, the first statement of the lemma is a special case of Theorem 59.91.11. The second one is a special case of Lemma 59.91.12. $\square$

59.92. Applications of proper base change

0A5I In this section we discuss some more or less immediate consequences of the proper base change theorem.

0DDG Lemma 59.92.1. Let K/k be an extension of separably closed fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\acute{e}tale}$. Then the map $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_K, \mathcal{F}|_{X_K})$ is an isomorphism for $q \geq 0$.

Proof. Looking at stalks we see that this is a special case of Theorem 59.91.11. $\square$

095U Lemma 59.92.2. Let $f : X \to Y$ be a proper morphism of schemes all of whose fibres have dimension $\leq n$. Then for any abelian torsion sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $R^q f_* \mathcal{F} = 0$ for $q > 2n$.

Proof. We will prove this by induction on n for all proper morphisms.

If $n = 0$, then f is a finite morphism (More on Morphisms, Lemma 37.44.1) and the result is true by Proposition 59.55.2.

If $n > 0$, then using Lemma 59.91.13 we see that it suffices to prove $H^i_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $i > 2n$ and X a proper scheme, $\dim(X) \leq n$ over an algebraically closed field k and $\mathcal{F}$ is a torsion abelian sheaf on X.

If $n = 1$ this follows from Theorem 59.83.11. Assume $n > 1$. By Proposition 59.45.4 we may replace X by its reduction. Let $\nu : X^\nu \to X$ be the normalization. This is a surjective birational finite morphism (see Varieties, Lemma 33.27.1) and hence an isomorphism over a dense open $U \subset X$ (Morphisms, Lemma 29.50.5). Then we see that $c : \mathcal{F} \to \nu_*\nu^{-1}\mathcal{F}$ is injective (as ν is surjective) and an isomorphism over U. Denote $i : Z \to X$ the inclusion of the complement of U. Since U is dense in X we have $\dim(Z) < \dim(X) = n$. By Proposition 59.46.4 have $\text{Coker}(c) = i_*\mathcal{G}$ for some abelian torsion sheaf $\mathcal{G}$ on $Z_{\acute{e}tale}$. Then $H^q_{\acute{e}tale}(X, \text{Coker}(c)) = H^q_{\acute{e}tale}(Z, \mathcal{F})$ (by Proposition 59.55.2 and the Leray spectral sequence) and by induction hypothesis we conclude that the cokernel of c has cohomology in degrees $\leq 2(n-1)$. Thus it suffices to prove the result for $\nu_*\nu^{-1}\mathcal{F}$. As ν is finite this reduces us to showing that $H^i_{\acute{e}tale}(X^\nu, \nu^{-1}\mathcal{F})$ is zero for $i > 2n$. This case is treated in the next paragraph.

Assume X is integral normal proper scheme over k of dimension n. Choose a nonconstant rational function f on X. The graph $X' \subset X \times \mathbf{P}^1_k$ of f sits into a diagram

$$X \xleftarrow{b} X' \xrightarrow{f} \mathbf{P}^1_k$$

Observe that b is an isomorphism over an open subscheme $U \subset X$ whose complement is a closed subscheme $Z \subset X$ of codimension ≥ 2. Namely, U is the domain of definition of f which contains all codimension 1 points of X, see Morphisms, Lemmas 29.49.9 and 29.42.5 (combined with Serre's criterion for normality, see Properties, Lemma 28.12.5). Moreover the fibres of b have dimension ≤ 1 (as closed subschemes of $\mathbf{P}^1$). Hence $R^i b_* b^{-1}\mathcal{F}$ is nonzero only if $i \in \{0, 1, 2\}$ by induction. Choose a distinguished triangle

$$\mathcal{F} \to Rb_*b^{-1}\mathcal{F} \to Q \to \mathcal{F}[1]$$

Using that $\mathcal{F} \to b_*b^{-1}\mathcal{F}$ is injective as before and using what we just said, we see that Q has nonzero cohomology sheaves only in degrees $0, 1, 2$ sitting on Z. Moreover, these cohomology sheaves are torsion by Lemma 59.78.2. By induction we see that $H^i(X, Q)$ is zero for $i > 2 + 2\dim(Z) \leq 2 + 2(n-2) = 2n - 2$. Thus it suffices to prove that $H^i(X', b^{-1}\mathcal{F}) = 0$ for $i > 2n$. At this point we use the morphism

$$f : X' \to \mathbf{P}^1_k$$

whose fibres have dimension $< n$. Hence by induction we see that $R^i f_* b^{-1}\mathcal{F} = 0$ for $i > 2(n-1)$. We conclude by the Leray spectral seqence

$$H^i(\mathbf{P}^1_k, R^j f_* b^{-1}\mathcal{F}) \Rightarrow H^{i+j}(X', b^{-1}\mathcal{F})$$

and the fact that $\dim(\mathbf{P}^1_k) = 1$. □

When working with mod n coefficients we can do proper base change for unbounded complexes.

0F0C Lemma 59.92.3. Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $n \geq 1$ be an integer. Let $E \in D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$. Then the base change map (59.91.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.

Proof. It is enough to prove this when Y and Y' are quasi-compact. By Morphisms, Lemma 29.28.5 we see that the dimension of the fibres of $f : X \to Y$ and $f' : X' \to Y'$ are bounded. Thus Lemma 59.92.2 implies that

$$f_* : \mathrm{Mod}(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathrm{Mod}(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$$

and

$$f'_* : \mathrm{Mod}(X'_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathrm{Mod}(Y'_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$$

have finite cohomological dimension in the sense of Derived Categories, Lemma 13.32.2. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathbf{Z}/n\mathbf{Z}$-modules each of whose terms $\mathcal{I}^n$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules representing E. See Injectives, Theorem 19.12.6. By the usual proper base change theorem we find that $R^qf'_*(g')^{-1}\mathcal{I}^n = 0$ for $q > 0$, see Theorem 59.91.11. Hence we conclude by Derived Categories, Lemma 13.32.2 that we may compute $Rf'_*(g')^{-1}E$ by the complex $f'_*(g')^{-1}\mathcal{I}^\bullet$. Another application of the usual proper base change theorem shows that this is equal to $g^{-1}f_*\mathcal{I}^\bullet$ as desired. $\square$

0F0E Lemma 59.92.4. Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^+(X_{\acute{e}tale})$ and $K \in D^+(\mathbf{Z})$. Then

$$R\Gamma(X, E \otimes_{\mathbf{Z}}^{\mathbf{L}} \underline{K}) = R\Gamma(X, E) \otimes_{\mathbf{Z}}^{\mathbf{L}} K$$

Proof. Say $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. We may represent K by a bounded below complex $K^\bullet$ of torsion free $\mathbf{Z}$-modules. (Choose a K-flat complex $L^\bullet$ representing K and then take $K^\bullet = \tau_{\geq b-1}L^\bullet$. This works because $\mathbf{Z}$ has global dimension 1. See More on Algebra, Lemma 15.66.2.) We may represent E by a bounded below complex $\mathcal{E}^\bullet$. Then $E \otimes_{\mathbf{Z}}^{\mathbf{L}} \underline{K}$ is represented by

$$\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathbf{Z}} \underline{K}^\bullet)$$

Using distinguished triangles

$$\sigma_{\geq -b+n+1}K^\bullet \to K^\bullet \to \sigma_{\leq -b+n}K^\bullet$$

and the trivial vanishing

$$H^n(X, \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathbf{Z}} \sigma_{\geq -a+n+1}\underline{K}^\bullet) = 0$$

and

$$H^n(R\Gamma(X, E) \otimes_{\mathbf{Z}}^{\mathbf{L}} \sigma_{\geq -a+n+1}K^\bullet) = 0$$

we reduce to the case where $K^\bullet$ is a bounded complex of flat $\mathbf{Z}$-modules. Repeating the argument we reduce to the case where $K^\bullet$ is equal to a single flat $\mathbf{Z}$-module sitting in some degree. Next, using the stupid trunctions for $\mathcal{E}^\bullet$ we reduce in exactly the same manner to the case where $\mathcal{E}^\bullet$ is a single abelian sheaf sitting in some degree. Thus it suffices to show that

$$H^n(X, \mathcal{E} \otimes_{\mathbf{Z}} \underline{M}) = H^n(X, \mathcal{E}) \otimes_{\mathbf{Z}} M$$

when M is a flat $\mathbf{Z}$-module and $\mathcal{E}$ is an abelian sheaf on X. In this case we write M is a filtered colimit of finite free $\mathbf{Z}$-modules (Lazard's theorem, see Algebra, Theorem 10.81.4). By Theorem 59.51.3 this reduces us to the case of finite free $\mathbf{Z}$-module M in which case the result is trivially true. $\square$

0F0F Lemma 59.92.5. Let $f : X \to Y$ be a proper morphism of schemes. Let $E \in D^+(X_{étale})$ have torsion cohomology sheaves. Let $K \in D^+(Y_{étale})$. Then

$$Rf_*E \otimes_{\mathbf{Z}}^{\mathbf{L}} K = Rf_*(E \otimes_{\mathbf{Z}}^{\mathbf{L}} f^{-1}K)$$

in $D^+(Y_{étale})$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 21.50. We will check the equality on stalks. Recall that computing derived tensor products commutes with pullbacks. See Cohomology on Sites, Lemma 21.18.4. Thus we have

$$(E \otimes_{\mathbf{Z}}^{\mathbf{L}} f^{-1}K)_{\overline{x}} = E_{\overline{x}} \otimes_{\mathbf{Z}}^{\mathbf{L}} K_{\overline{y}}$$

where $\overline{y}$ is the image of $\overline{x}$ in Y. Since $\mathbf{Z}$ has global dimension 1 we see that this complex has vanishing cohomology in degree $< -1 + a + b$ if $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. Moreover, since $H^i(E)$ is a torsion abelian sheaf for each i, the same is true for the cohomology sheaves of the complex $E \otimes_{\mathbf{Z}}^{\mathbf{L}} K$. Namely, we have

$$(E \otimes_{\mathbf{Z}}^{\mathbf{L}} f^{-1}K) \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q} = (E \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q}) \otimes_{\mathbf{Q}}^{\mathbf{L}} (f^{-1}K \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q})$$

which is zero in the derived category. In this way we see that Lemma 59.91.13 applies to both sides to see that it suffices to show

$$R\Gamma(X_{\overline{y}}, E|_{X_{\overline{y}}} \otimes_{\mathbf{Z}}^{\mathbf{L}} (X_{\overline{y}} \to \overline{y})^{-1}K_{\overline{y}}) = R\Gamma(X_{\overline{y}}, E|_{X_{\overline{y}}}) \otimes_{\mathbf{Z}}^{\mathbf{L}} K_{\overline{y}}$$

This is shown in Lemma 59.92.4. $\square$

59.93. Local acyclicity

0GJM In this section we deduce local acyclicity of smooth morphisms from the smooth base change theorem. In SGA 4 or SGA 4.5 the authors first prove a version of local acyclicity for smooth morphisms and then deduce the smooth base change theorem.

We will use the formulation of local acyclicity given by Deligne [Del77, Definition 2.12, page 242]. Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$ in S. Let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh})$. We obtain a commutative diagram

$$\begin{tikzcd} F_{\overline{x},\overline{t}} = \overline{t} \times_{\mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh})} \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}^{sh}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}^{sh}) \arrow[r] \arrow[d] & X \arrow[d] \\ \overline{t} \arrow[r] & \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh}) \arrow[r] & S \end{tikzcd}$$

The scheme $F_{\overline{x},\overline{t}}$ is called a variety of vanishing cycles of f at $\overline{x}$. Let K be an object of $D(X_{étale})$. For any morphism of schemes $g : Y \to X$ we write $R\Gamma(Y, K)$ instead of $R\Gamma(Y_{étale}, g_{small}^{-1}K)$. Since $\mathcal{O}_{X,\overline{x}}^{sh}$ is strictly henselian we have $K_{\overline{x}} = R\Gamma(\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}^{sh}), K)$. Thus we obtain a canonical map

0GJN (59.93.0.1) $$\alpha_{K,\overline{x},\overline{t}} : K_{\overline{x}} \longrightarrow R\Gamma(F_{\overline{x},\overline{t}}, K)$$

by pulling back cohomology along $F_{\overline{x},\overline{t}} \to \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$.

0GJP Definition 59.93.1. Let $f : X \to S$ be a morphism of schemes. Let K be an object of $D(X_{\acute{e}tale})$. [Del77, Definition 2.12, page 242] and [Del77, Definition (1.3), page 54]

(1) Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$. We say f is locally acyclic at $\overline{x}$ relative to K if for every geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ the map (59.93.0.1) is an isomorphism[11].
(2) We say f is locally acyclic relative to K if f is locally acyclic at $\overline{x}$ relative to K for every geometric point $\overline{x}$ of X.
(3) We say f is universally locally acyclic relative to K if for any morphism $S' \to S$ of schemes the base change $f' : X' \to S'$ is locally acyclic relative to the pullback of K to X'.
(4) We say f is locally acyclic if for all geometric points $\overline{x}$ of X and any integer n prime to the characteristic of $\kappa(\overline{x})$, the morphism f is locally acyclic at $\overline{x}$ relative to the constant sheaf with value $\mathbf{Z}/n\mathbf{Z}$.
(5) We say f is universally locally acyclic if for any morphism $S' \to S$ of schemes the base change $f' : X' \to S'$ is locally acyclic.

Let M be an abelian group. Then local acyclicity of $f : X \to S$ with respect to the constant sheaf $\underline{M}$ boils down to the requirement that

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \begin{cases} M & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0 \end{cases}$$

for any geometric point $\overline{x}$ of X and any geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$. In this way we see that being locally acyclic corresponds to the vanishing of the higher cohomology groups of the geometric fibres $F_{\overline{x},\overline{t}}$ of the maps between the strict henselizations at $\overline{x}$ and $\overline{s}$.

0GJQ Proposition 59.93.2. Let $f : X \to S$ be a smooth morphism of schemes. Then f is universally locally acyclic.

Proof. Since the base change of a smooth morphism is smooth, it suffices to show that smooth morphisms are locally acyclic. Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$. Let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$. Since we are trying to prove a property of the ring map $\mathcal{O}^{sh}_{S,\overline{s}} \to \mathcal{O}^{sh}_{X,\overline{x}}$ (see discussion following Definition 59.93.1) we may and do replace $f : X \to S$ by the base change $X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Thus we may and do assume that S is the spectrum of a strictly henselian local ring and that $\overline{s}$ lies over the closed point of S.

We will apply Lemma 59.86.5 to the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X_{\overline{t}} \arrow[l, "h"'] \arrow[d, "e"] \\ S & \overline{t} \arrow[l, "g"] \end{tikzcd}$$

and the sheaf $\mathcal{F} = \underline{M}$ where $M = \mathbf{Z}/n\mathbf{Z}$ for some integer n prime to the characteristic of the residue field of $\overline{x}$. We know that the map $f^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}$ is an isomorphism by smooth base change, see Theorem 59.89.2 (the assumption on

[11] We do not assume $\overline{t}$ is an algebraic geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Often using Lemma 59.90.2 one may reduce to this case.

torsion holds by our choice of n). Thus Lemma 59.86.5 gives us the middle equality in

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \overline{t}, \underline{M}) = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \overline{t}, \underline{M}) = H^q(\overline{t}, \underline{M})$$

For the outer two equalities we use that $S = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Since $\overline{t}$ is the spectrum of a separably closed field we conclude that

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \begin{cases} M & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

which is what we had to show (see discussion following Definition 59.93.1). □

0GJR **Lemma 59.93.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a locally constant abelian sheaf on $X_{\acute{e}tale}$ such that for every geometric point $\overline{x}$ of X the abelian group $\mathcal{F}_{\overline{x}}$ is a torsion group all of whose elements have order prime to the characteristic of the residue field of $\overline{x}$. If f is locally acyclic, then f is locally acyclic relative to $\mathcal{F}$.*

Proof. Namely, let $\overline{x}$ be a geometric point of X. Since $\mathcal{F}$ is locally constant we see that the restriction of $\mathcal{F}$ to $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ is isomorphic to the constant sheaf $\underline{M}$ with $M = \mathcal{F}_{\overline{x}}$. By assumption we can write $M = \mathrm{colim}\, M_i$ as a filtered colimit of finite abelian groups M_i of order prime to the characteristic of the residue field of $\overline{x}$. Consider a geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$. Since $F_{\overline{x},\overline{t}}$ is affine, we have

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \mathrm{colim}\, H^q(F_{\overline{x},\overline{t}}, \underline{M_i})$$

by Lemma 59.51.4. For each i we can write $M_i = \bigoplus \mathbf{Z}/n_{i,j}\mathbf{Z}$ as a finite direct sum for some integers $n_{i,j}$ prime to the characteristic of the residue field of $\overline{x}$. Since f is locally acyclic we see that

$$H^q(F_{\overline{x},\overline{t}}, \underline{\mathbf{Z}/n_{i,j}\mathbf{Z}}) = \begin{cases} \mathbf{Z}/n_{i,j}\mathbf{Z} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

See discussion following Definition 59.93.1. Taking the direct sums and the colimit we conclude that

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \begin{cases} M & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

and we win. □

0GJS **Lemma 59.93.4.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let K be an object of $D(X_{\acute{e}tale})$. Let $\overline{x}'$ be a geometric point of X' with image $\overline{x}$ in X. If

(1) *f is locally acyclic at $\overline{x}$ relative to K and*
(2) *g is locally quasi-finite, or $S' = \lim S_i$ is a directed inverse limit of schemes locally quasi-finite over S with affine transition morphisms, or $g : S' \to S$ is integral,*

then f' locally acyclic at $\overline{x}'$ relative to $(g')^{-1}K$.

Proof. Denote $\overline{s}'$ and $\overline{s}$ the images of $\overline{x}'$ and $\overline{x}$ in S' and S. Let $\overline{t}'$ be a geometric point of the spectrum of $\text{Spec}(\mathcal{O}^{sh}_{S',\overline{s}'})$ and denote $\overline{t}$ the image in $\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. By Algebra, Lemma 10.156.6 and our assumptions on g we have

$$\mathcal{O}^{sh}_{X,\overline{x}} \otimes_{\mathcal{O}^{sh}_{S,\overline{s}}} \mathcal{O}^{sh}_{S',\overline{s}'} \longrightarrow \mathcal{O}^{sh}_{X',\overline{x}'}$$

is an isomorphism. Since by our conventions $\kappa(\overline{t}) = \kappa(\overline{t}')$ we conclude that

$$F_{\overline{x}',\overline{t}'} = \text{Spec}\left(\mathcal{O}^{sh}_{X',\overline{x}'} \otimes_{\mathcal{O}^{sh}_{S',\overline{s}'}} \kappa(\overline{t}')\right) = \text{Spec}\left(\mathcal{O}^{sh}_{X,\overline{x}} \otimes_{\mathcal{O}^{sh}_{S,\overline{s}}} \kappa(\overline{t})\right) = F_{\overline{x},\overline{t}}$$

In other words, the varieties of vanishing cycles of f' at $\overline{x}'$ are examples of varieties of vanishing cycles of f at $\overline{x}$. The lemma follows immediately from this and the definitions. $\square$

59.94. The cospecialization map

0GJT Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$ in S. Let $\overline{t}$ be a geometric point of $\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Let $K \in D(X_{\acute{e}tale})$. For any morphism $g : Y \to X$ of schemes we write $K|_Y$ instead of $g^{-1}_{small}K$ and $R\Gamma(Y, K)$ instead of $R\Gamma(Y_{\acute{e}tale}, g^{-1}_{small}K)$. We claim that if

(1) K is bounded below, i.e., $K \in D^+(X_{\acute{e}tale})$,
(2) f is locally acyclic relative to K

then there is a cospecialization map

$$cosp : R\Gamma(X_{\overline{t}}, K) \longrightarrow R\Gamma(X_{\overline{s}}, K)$$

which will be closely related to the specialization map considered in Section 59.75 and especially Remark 59.75.8.

To construct the map we consider the morphisms

$$X_{\overline{t}} \xrightarrow{h} X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \xleftarrow{i} X_{\overline{s}}$$

The unit of the adjunction between h^{-1} and Rh_* gives a map

$$\beta_{K,\overline{s},\overline{t}} : K|_{X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \longrightarrow Rh_*(K|_{X_{\overline{t}}})$$

in $D((X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}))_{\acute{e}tale})$. Lemma 59.94.1 below shows that the pullback $i^{-1}\beta_{K,\overline{s},\overline{t}}$ is an isomorphism under the assumptions above. Thus we can define the cospecialization map as the composition

$$\begin{aligned} R\Gamma(X_{\overline{t}}, K) &= R\Gamma(X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), Rh_*(K|_{X_{\overline{t}}})) \\ &\xrightarrow{i^{-1}} R\Gamma(X_{\overline{s}}, i^{-1}Rh_*(K|_{X_{\overline{t}}})) \\ &\xrightarrow{(i^{-1}\beta_{K,\overline{s},\overline{t}})^{-1}} R\Gamma(X_{\overline{s}}, i^{-1}(K|_{X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})})) \\ &= R\Gamma(X_{\overline{s}}, K) \end{aligned}$$

0GJU **Lemma 59.94.1.** *The map $i^{-1}\beta_{K,\overline{s},\overline{t}}$ is an isomorphism.*

Proof. The construction of the maps h, i, $\beta_{K,\overline{s},\overline{t}}$ only depends on the base change of X and K to $\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Thus we may and do assume that S is a strictly henselian scheme with closed point $\overline{s}$. Observe that the local acyclicity of f relative to K is preserved by this base change (for example by Lemma 59.93.4 or just directly by comparing strictly henselian rings in this very special case).

Let $\overline{x}$ be a geometric point of $X_{\overline{s}}$. Or equivalently, let $\overline{x}$ be a geometric point whose image by f is $\overline{s}$. Let us compute the stalk of $i^{-1}\beta_{K,\overline{s},\overline{t}}$ at $\overline{x}$. First, we have

$$(i^{-1}\beta_{K,\overline{s},\overline{t}})_{\overline{x}} = (\beta_{K,\overline{s},\overline{t}})_{\overline{x}}$$

since pullback preserves stalks, see Lemma 59.36.2. Since we are in the situation $S = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ we see that $h : X_{\overline{t}} \to X$ has the property that $X_{\overline{t}} \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) = F_{\overline{x},\overline{t}}$. Thus we see that

$$(\beta_{K,\overline{s},\overline{t}})_{\overline{x}} : K_{\overline{x}} \longrightarrow Rh_*(K|_{X_{\overline{t}}})_{\overline{x}} = R\Gamma(F_{\overline{x},\overline{t}}, K)$$

where the equal sign is Theorem 59.53.1. It follows that the map $(\beta_{K,\overline{s},\overline{t}})_{\overline{x}}$ is none other than the map $\alpha_{K,\overline{x},\overline{t}}$ used in Definition 59.93.1. The result follows as we may check whether a map is an isomorphism in stalks by Theorem 59.29.10. $\square$

The cospecialization map when it exists is trying to be the inverse of the specialization map.

0GJV Lemma 59.94.2. In the situation above, if in addition f is quasi-compact and quasi-separated, then the diagram

$$\begin{tikzcd} (Rf_*K)_{\overline{s}} \arrow[r] \arrow[d, "sp"'] & R\Gamma(X_{\overline{s}}, K) \\ (Rf_*K)_{\overline{t}} \arrow[r] & R\Gamma(X_{\overline{t}}, K) \arrow[u, "cosp"'] \end{tikzcd}$$

is commutative.

Proof. As in the proof of Lemma 59.94.1 we may replace S by $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Then our maps simplify to $h : X_{\overline{t}} \to X$, $i : X_{\overline{s}} \to X$, and $\beta_{K,\overline{s},\overline{t}} : K \to Rh_*(K|_{X_{\overline{t}}})$. Using that $(Rf_*K)_{\overline{s}} = R\Gamma(X, K)$ by Theorem 59.53.1 the composition of sp with the base change map $(Rf_*K)_{\overline{t}} \to R\Gamma(X_{\overline{t}}, K)$ is just pullback of cohomology along h. This is the same as the map

$$R\Gamma(X, K) \xrightarrow{\beta_{K,\overline{s},\overline{t}}} R\Gamma(X, Rh_*(K|_{X_{\overline{t}}})) = R\Gamma(X_{\overline{t}}, K)$$

Now the map $cosp$ first inverts the $=$ sign in this displayed formula, then pulls back along i, and finally applies the inverse of $i^{-1}\beta_{K,\overline{s},\overline{t}}$. Hence we get the desired commutativity. $\square$

0GJW Lemma 59.94.3. Let $f : X \to S$ be a morphism of schemes. Let $K \in D(X_{\acute{e}tale})$. Assume

(1) K is bounded below, i.e., $K \in D^+(X_{\acute{e}tale})$,
(2) f is locally acyclic relative to K,
(3) f is proper, and
(4) K has torsion cohomology sheaves.

Then for every geometric point $\overline{s}$ of S and every geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ both the specialization map $sp : (Rf_*K)_{\overline{s}} \to (Rf_*K)_{\overline{t}}$ and the cospecialization map $cosp : R\Gamma(X_{\overline{t}}, K) \to R\Gamma(X_{\overline{s}}, K)$ are isomorphisms.

Proof. By the proper base change theorem (in the form of Lemma 59.91.13) we have $(Rf_*K)_{\overline{s}} = R\Gamma(X_{\overline{s}}, K)$ and similarly for $\overline{t}$. The "correct" proof would be to show that the argument in Lemma 59.94.2 shows that sp and $cosp$ are inverse isomorphisms in this case. Instead we will show directly that *cosp* is an isomorphism. From the discussion above we see that *cosp* is an isomorphism if and only if pullback by i

$$R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh}), Rh_*(K|_{X_{\overline{t}}})) \longrightarrow R\Gamma(X_{\overline{s}}, i^{-1}Rh_*(K|_{X_{\overline{t}}}))$$

is an isomorphism in $D^+(\mathrm{Ab})$. This is true by the proper base change theorem for the proper morphism $f' : X \times_S \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh}) \to \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh})$ by the morphism $\overline{s} \to \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh})$ and the complex $K' = Rh_*(K|_{X_{\overline{t}}})$. The complex K' is bounded below and has torsion cohomology sheaves by Lemma 59.78.2. Since $\mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh})$ is strictly henselian with $\overline{s}$ lying over the closed point, we see that the source of the displayed arrow equals $(Rf'_*K')_{\overline{s}}$ and the target equals $R\Gamma(X_{\overline{s}}, K')$ and the displayed map is an isomorphism by the already used Lemma 59.91.13. Thus we see that three out of the four arrows in the diagram of Lemma 59.94.2 are isomorphisms and we conclude. □

0GKD Lemma 59.94.4. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Assume

(1) f is smooth and proper
(2) $\mathcal{F}$ is locally constant, and
(3) $\mathcal{F}_{\overline{x}}$ is a torsion group all of whose elements have order prime to the residue characteristic of $\overline{x}$ for every geometric point $\overline{x}$ of X.

Then for every geometric point $\overline{s}$ of S and every geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh})$ the specialization map $sp : (Rf_*\mathcal{F})_{\overline{s}} \to (Rf_*\mathcal{F})_{\overline{t}}$ is an isomorphism.

Proof. This follows from Lemmas 59.94.3 and 59.93.3 and Proposition 59.93.2. □

59.95. Cohomological dimension

0F0P We can deduce some bounds on the cohomological dimension of schemes and on the cohomological dimension of fields using the results in Section 59.83 and one, seemingly innocuous, application of the proper base change theorem (in the proof of Proposition 59.95.6).

0F0Q Definition 59.95.1. Let X be a quasi-compact and quasi-separated scheme. The cohomological dimension of X is the smallest element

$$\mathrm{cd}(X) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $H^i_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $i > \mathrm{cd}(X)$. If $X = \mathrm{Spec}(A)$ we sometimes call this the cohomological dimension of A.

If the scheme is in characteristic p, then we often can obtain sharper bounds for the vanishing of cohomology of p-power torsion sheaves. We will address this elsewhere (insert future reference here).

0F0R Lemma 59.95.2. Let $X = \lim X_i$ be a directed limit of a system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then $\mathrm{cd}(X) \leq \max \mathrm{cd}(X_i)$.

Proof. Denote $f_i : X \to X_i$ the projections. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$. Then we have $\mathcal{F} = \lim f_i^{-1}f_{i,*}\mathcal{F}$ by Lemma 59.51.9. Thus $H^q_{\acute{e}tale}(X, \mathcal{F}) = \mathrm{colim}\, H^q_{\acute{e}tale}(X_i, f_{i,*}\mathcal{F})$ by Theorem 59.51.3. The lemma follows. □

0F0S **Lemma 59.95.3.** Let K be a field. Let X be a 1-dimensional affine scheme of finite type over K. Then $\text{cd}(X) \leq 1 + \text{cd}(K)$.

Proof. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$. Consider the Leray spectral sequence for the morphism $f : X \to \text{Spec}(K)$. We obtain

$$E_2^{p,q} = H^p(\text{Spec}(K), R^q f_* \mathcal{F})$$

converging to $H^{p+q}_{\acute{e}tale}(X, \mathcal{F})$. The stalk of $R^q f_* \mathcal{F}$ at a geometric point $\text{Spec}(\overline{K}) \to \text{Spec}(K)$ is the cohomology of the pullback of $\mathcal{F}$ to $X_{\overline{K}}$. Hence it vanishes in degrees ≥ 2 by Theorem 59.83.10. $\square$

0F0T **Lemma 59.95.4.** Let L/K be a field extension. Then we have $\text{cd}(L) \leq \text{cd}(K) + \text{trdeg}_K(L)$.

Proof. If $\text{trdeg}_K(L) = \infty$, then this is clear. If not then we can find a sequence of extensions $L = L_r/L_{r-1}/\ldots/L_1/L_0 = K$ such that $\text{trdeg}_{L_i}(L_{i+1}) = 1$ and $r = \text{trdeg}_K(L)$. Hence it suffices to prove the lemma in the case that $r = 1$. In this case we can write $L = \text{colim}\, A_i$ as a filtered colimit of its finite type K-subalgebras. By Lemma 59.95.2 it suffices to prove that $\text{cd}(A_i) \leq 1 + \text{cd}(K)$. This follows from Lemma 59.95.3. $\square$

0F0U **Lemma 59.95.5.** Let K be a field. Let X be a scheme of finite type over K. Let $x \in X$. Set $a = \text{trdeg}_K(\kappa(x))$ and $d = \dim_x(X)$. Then there is a map

$$K(t_1, \ldots, t_a)^{sep} \longrightarrow \mathcal{O}_{X,x}^{sh}$$

such that

(1) the residue field of $\mathcal{O}_{X,x}^{sh}$ is a purely inseparable extension of $K(t_1, \ldots, t_a)^{sep}$,
(2) $\mathcal{O}_{X,x}^{sh}$ is a filtered colimit of finite type $K(t_1, \ldots, t_a)^{sep}$-algebras of dimension $\leq d - a$.

Proof. We may assume X is affine. By Noether normalization, after possibly shrinking X again, we can choose a finite morphism $\pi : X \to \mathbf{A}^d_K$, see Algebra, Lemma 10.115.5. Since $\kappa(x)$ is a finite extension of the residue field of $\pi(x)$, this residue field has transcendence degree a over K as well. Thus we can find a finite morphism $\pi' : \mathbf{A}^d_K \to \mathbf{A}^d_K$ such that $\pi'(\pi(x))$ corresponds to the generic point of the linear subspace $\mathbf{A}^a_K \subset \mathbf{A}^d_K$ given by setting the last $d-a$ coordinates equal to zero. Hence the composition

$$X \xrightarrow{\pi' \circ \pi} \mathbf{A}^d_K \xrightarrow{p} \mathbf{A}^a_K$$

of $\pi' \circ \pi$ and the projection p onto the first a coordinates maps x to the generic point $\eta \in \mathbf{A}^a_K$. The induced map

$$K(t_1, \ldots, t_a)^{sep} = \mathcal{O}^{sh}_{\mathbf{A}^a_k, \eta} \longrightarrow \mathcal{O}^{sh}_{X,x}$$

on étale local rings satisfies (1) since it is clear that the residue field of $\mathcal{O}^{sh}_{X,x}$ is an algebraic extension of the separably closed field $K(t_1, \ldots, t_a)^{sep}$. On the other hand, if $X = \text{Spec}(B)$, then $\mathcal{O}^{sh}_{X,x} = \text{colim}\, B_j$ is a filtered colimit of étale B-algebras B_j. Observe that B_j is quasi-finite over $K[t_1, \ldots, t_d]$ as B is finite over $K[t_1, \ldots, t_d]$. We may similarly write $K(t_1, \ldots, t_a)^{sep} = \text{colim}\, A_i$ as a filtered colimit of étale $K[t_1, \ldots, t_a]$-algebras. For every i we can find an j such that

$A_i \to K(t_1, \ldots, t_a)^{sep} \to \mathcal{O}_{X,x}^{sh}$ factors through a map $\psi_{i,j} : A_i \to B_j$. Then B_j is quasi-finite over $A_i[t_{a+1}, \ldots, t_d]$. Hence

$$B_{i,j} = B_j \otimes_{\psi_{i,j}, A_i} K(t_1, \ldots, t_a)^{sep}$$

has dimension $\leq d - a$ as it is quasi-finite over $K(t_1, \ldots, t_a)^{sep}[t_{a+1}, \ldots, t_d]$. The proof of (2) is now finished as $\mathcal{O}_{X,x}^{sh}$ is a filtered colimit[12] of the algebras $B_{i,j}$. Some details omitted. $\square$

0F0V Proposition 59.95.6. Let K be a field. Let X be an affine scheme of finite type over K. Then we have $\text{cd}(X) \leq \dim(X) + \text{cd}(K)$.

Proof. We will prove this by induction on $\dim(X)$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$.

The case $\dim(X) = 0$. In this case the structure morphism $f : X \to \text{Spec}(K)$ is finite. Hence we see that $R^i f_* \mathcal{F} = 0$ for $i > 0$, see Proposition 59.55.2. Thus $H^i_{étale}(X, \mathcal{F}) = H^i_{étale}(\text{Spec}(K), f_*\mathcal{F})$ by the Leray spectral sequence for f (Cohomology on Sites, Lemma 21.14.5) and the result is clear.

The case $\dim(X) = 1$. This is Lemma 59.95.3.

Assume $d = \dim(X) > 1$ and the proposition holds for finite type affine schemes of dimension $< d$ over fields. By Noether normalization, see for example Varieties, Lemma 33.18.2, there exists a finite morphism $f : X \to \mathbf{A}^d_K$. Recall that $R^i f_* \mathcal{F} = 0$ for $i > 0$ by Proposition 59.55.2. By the Leray spectral sequence for f (Cohomology on Sites, Lemma 21.14.5) we conclude that it suffices to prove the result for $\pi_* \mathcal{F}$ on $\mathbf{A}^d_K$.

Interlude I. Let $j : X \to Y$ be an open immersion of smooth d-dimensional varieties over K (not necessarily affine) whose complement is the support of an effective Cartier divisor D. The sheaves $R^q j_* \mathcal{F}$ for $q > 0$ are supported on D. We claim that $(R^q j_* \mathcal{F})_{\overline{y}} = 0$ for $a = \text{trdeg}_K(\kappa(y)) > d - q$. Namely, by Theorem 59.53.1 we have

$$(R^q j_* \mathcal{F})_{\overline{y}} = H^q(\text{Spec}(\mathcal{O}^{sh}_{Y,y}) \times_Y X, \mathcal{F})$$

Choose a local equation $f \in \mathfrak{m}_y = \mathcal{O}_{Y,y}$ for D. Then we have

$$\text{Spec}(\mathcal{O}^{sh}_{Y,y}) \times_Y X = \text{Spec}(\mathcal{O}^{sh}_{Y,y}[1/f])$$

Using Lemma 59.95.5 we get an embedding

$$K(t_1, \ldots, t_a)^{sep}(x) = K(t_1, \ldots, t_a)^{sep}[x]_{(x)}[1/x] \longrightarrow \mathcal{O}^{sh}_{Y,y}[1/f]$$

Since the transcendence degree over K of the fraction field of $\mathcal{O}^{sh}_{Y,y}$ is d, we see that $\mathcal{O}^{sh}_{Y,y}[1/f]$ is a filtered colimit of $(d - a - 1)$-dimensional finite type algebras over the field $K(t_1, \ldots, t_a)^{sep}(x)$ which itself has cohomological dimension 1 by Lemma 59.95.4. Thus by induction hypothesis and Lemma 59.95.2 we obtain the desired vanishing.

Interlude II. Let Z be a smooth variety over K of dimension $d - 1$. Let $E_a \subset Z$ be the set of points $z \in Z$ with $\text{trdeg}_K(\kappa(z)) \leq a$. Observe that E_a is closed under

[12] Let R be a ring. Let $A = \text{colim}_{i \in I} A_i$ be a filtered colimit of finitely presented R-algebras. Let $B = \text{colim}_{j \in J} B_j$ be a filtered colimit of R-algebras. Let $A \to B$ be an R-algebra map. Assume that for all $i \in I$ there is a $j \in J$ and an R-algebra map $\psi_{i,j} : A_i \to B_j$. Say $(i', j', \psi_{i',j'}) \geq (i, j, \psi_{i,j})$ if $i' \geq i$, $j' \geq j$, and $\psi_{i,j}$ and $\psi_{i',j'}$ are compatible. Then the collection of triples forms a directed set and $B = \text{colim}\, B_j \otimes_{\psi_{i,j} A_i} A$.

specialization, see Varieties, Lemma 33.20.3. Suppose that $\mathcal{G}$ is a torsion abelian sheaf on Z whose support is contained in E_a. Then we claim that $H^b_{étale}(Z, \mathcal{G}) = 0$ for $b > a + \text{cd}(K)$. Namely, we can write $\mathcal{G} = \text{colim}\, \mathcal{G}_i$ with $\mathcal{G}_i$ a torsion abelian sheaf supported on a closed subscheme Z_i contained in E_a, see Lemma 59.74.5. Then the induction hypothesis kicks in to imply the desired vanishing for $\mathcal{G}_i$[13]. Finally, we conclude by Theorem 59.51.3.

Consider the commutative diagram

$$\begin{tikzcd} \mathbf{A}^d_K \arrow[rr, "j"] \arrow[dr, "f"'] & & \mathbf{P}^1_K \times_K \mathbf{A}^{d-1}_K \arrow[dl, "g"] \\ & \mathbf{A}^{d-1}_K & \end{tikzcd}$$

Observe that j is an open immersion of smooth d-dimensional varieties whose complement is an effective Cartier divisor D. Thus we may use the results obtained in interlude I. We are going to study the relative Leray spectral sequence

$$E_2^{p,q} = R^p g_* R^q j_* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

Since $R^q j_* \mathcal{F}$ for $q > 0$ is supported on D and since $g|_D : D \to \mathbf{A}^{d-1}_K$ is an isomorphism, we find $R^p g_* R^q j_* \mathcal{F} = 0$ for $p > 0$ and $q > 0$. Moreover, we have $R^q j_* \mathcal{F} = 0$ for $q > d$. On the other hand, g is a proper morphism of relative dimension 1. Hence by Lemma 59.92.2 we see that $R^p g_* j_* \mathcal{F} = 0$ for $p > 2$. Thus the E_2-page of the spectral sequence looks like this

$$\begin{matrix} g_* R^d j_* \mathcal{F} & 0 & 0 \\ \ldots & \ldots & \ldots \\ g_* R^2 j_* \mathcal{F} & 0 & 0 \\ g_* R^1 j_* \mathcal{F} & 0 & 0 \\ g_* j_* \mathcal{F} & R^1 g_* j_* \mathcal{F} & R^2 g_* j_* \mathcal{F} \end{matrix}$$

We conclude that $R^q f_* \mathcal{F} = g_* R^q j_* \mathcal{F}$ for $q > 2$. By interlude I we see that the support of $R^q f_* \mathcal{F}$ for $q > 2$ is contained in the set of points of $\mathbf{A}^{d-1}_K$ whose residue field has transcendence degree $\leq d - q$. By interlude II

$$H^p(\mathbf{A}^{d-1}_K, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - q + \text{cd}(K) \text{ and } q > 2$$

On the other hand, by Theorem 59.53.1 we have $R^2 f_* \mathcal{F}_{\overline{\eta}} = H^2(\mathbf{A}^1_{\overline{\eta}}, \mathcal{F}) = 0$ (vanishing by the case of dimension 1) where η is the generic point of $\mathbf{A}^{d-1}_K$. Hence by interlude II again we see

$$H^p(\mathbf{A}^{d-1}_K, R^2 f_* \mathcal{F}) = 0 \text{ for } p > d - 2 + \text{cd}(K)$$

Finally, we have

$$H^p(\mathbf{A}^{d-1}_K, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - 1 + \text{cd}(K) \text{ and } q = 0, 1$$

by induction hypothesis. Combining everything we just said with the Leray spectral sequence $H^p(\mathbf{A}^{d-1}_K, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(\mathbf{A}^d_K, \mathcal{F})$ we conclude. $\square$

[13] Here we first use Proposition 59.46.4 to write $\mathcal{G}_i$ as the pushforward of a sheaf on Z_i, the induction hypothesis gives the vanishing for this sheaf on Z_i, and the Leray spectral sequence for $Z_i \to Z$ gives the vanishing for $\mathcal{G}_i$.

0F0W Lemma 59.95.7. Let K be a field. Let X be an affine scheme of finite type over K. Let $E_a \subset X$ be the set of points $x \in X$ with $\text{trdeg}_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$ whose support is contained in E_a. Then $H^b_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $b > a + \text{cd}(K)$.

Proof. We can write $\mathcal{F} = \text{colim}\, \mathcal{F}_i$ with $\mathcal{F}_i$ a torsion abelian sheaf supported on a closed subscheme Z_i contained in E_a, see Lemma 59.74.5. Then Proposition 59.95.6 gives the desired vanishing for $\mathcal{F}_i$. Details omitted; hints: first use Proposition 59.46.4 to write $\mathcal{F}_i$ as the pushforward of a sheaf on Z_i, use the vanishing for this sheaf on Z_i, and use the Leray spectral sequence for $Z_i \to Z$ to get the vanishing for $\mathcal{F}_i$. Finally, we conclude by Theorem 59.51.3. □

0F0X Lemma 59.95.8. Let $f : X \to Y$ be an affine morphism of schemes of finite type over a field K. Let $E_a(X)$ be the set of points $x \in X$ with $\text{trdeg}_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$ whose support is contained in E_a. Then $R^q f_* \mathcal{F}$ has support contained in $E_{a-q}(Y)$.

Proof. The question is local on Y hence we can assume Y is affine. Then X is affine too and we can choose a diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & \mathbf{A}^{n+m}_K \arrow[d, "\text{pr}"] \\ Y \arrow[r, "j"] & \mathbf{A}^n_K \end{tikzcd}$$

where the horizontal arrows are closed immersions and the vertical arrow on the right is the projection (details omitted). Then $j_* R^q f_* \mathcal{F} = R^q \text{pr}_* i_* \mathcal{F}$ by the vanishing of the higher direct images of i and j, see Proposition 59.55.2. Moreover, the description of the stalks of j_* in the proposition shows that it suffices to prove the vanishing for $j_* R^q f_* \mathcal{F}$. Thus we may assume f is the projection morphism $\text{pr} : \mathbf{A}^{n+m}_K \to \mathbf{A}^n_K$ and an abelian torsion sheaf $\mathcal{F}$ on $\mathbf{A}^{n+m}_K$ satisfying the assumption in the statement of the lemma.

Let y be a point in $\mathbf{A}^n_K$. By Theorem 59.53.1 we have

$$(R^q \text{pr}_* \mathcal{F})_{\overline{y}} = H^q(\mathbf{A}^{n+m}_K \times_{A^n_K} \text{Spec}(\mathcal{O}^{sh}_{Y,y}), \mathcal{F}) = H^q(\mathbf{A}^m_{\mathcal{O}^{sh}_{Y,y}}, \mathcal{F})$$

Say $b = \text{trdeg}_K(\kappa(y))$. From Lemma 59.95.5 we get an embedding

$$L = K(t_1, \ldots, t_b)^{sep} \longrightarrow \mathcal{O}^{sh}_{Y,y}$$

Write $\mathcal{O}^{sh}_{Y,y} = \text{colim}\, B_i$ as the filtered colimit of finite type L-subalgebras $B_i \subset \mathcal{O}^{sh}_{Y,y}$ containing the ring $K[T_1, \ldots, T_n]$ of regular functions on $\mathbf{A}^n_K$. Then we get

$$\mathbf{A}^m_{\mathcal{O}^{sh}_{Y,y}} = \lim \mathbf{A}^m_{B_i}$$

If $z \in \mathbf{A}^m_{B_i}$ is a point in the support of $\mathcal{F}$, then the image x of z in $\mathbf{A}^{m+n}_K$ satisfies $\text{trdeg}_K(\kappa(x)) \leq a$ by our assumption on $\mathcal{F}$ in the lemma. Since $\mathcal{O}^{sh}_{Y,y}$ is a filtered colimit of étale algebras over $K[T_1, \ldots, T_n]$ and since $B_i \subset \mathcal{O}^{sh}_{Y,y}$ we see that $\kappa(z)/\kappa(x)$ is algebraic (some details omitted). Then $\text{trdeg}_K(\kappa(z)) \leq a$ and hence $\text{trdeg}_L(\kappa(z)) \leq a - b$. By Lemma 59.95.7 we see that

$$H^q(\mathbf{A}^m_{B_i}, \mathcal{F}) = 0 \text{ for } q > a - b$$

Thus by Theorem 59.51.3 we get $(Rf_* \mathcal{F})_{\overline{y}} = 0$ for $q > a - b$ as desired. □

59.96. Finite cohomological dimension

0F0Y We continue the discussion started in Section 59.95.

0F0Z Definition 59.96.1. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. The cohomological dimension of f is the smallest element

$$\mathrm{cd}(f) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $R^i f_* \mathcal{F} = 0$ for $i > \mathrm{cd}(f)$.

0F10 Lemma 59.96.2. Let K be a field.

(1) If $f : X \to Y$ is a morphism of finite type schemes over K, then $\mathrm{cd}(f) < \infty$.
(2) If $\mathrm{cd}(K) < \infty$, then $\mathrm{cd}(X) < \infty$ for any finite type scheme X over K.

Proof. Proof of (1). We may assume Y is affine. We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove this. If X is affine too, then the result holds by Lemma 59.95.8. Thus it suffices to show that if $X = U \cup V$ and the result is true for $U \to Y$, $V \to Y$, and $U \cap V \to Y$, then it is true for f. This follows from the relative Mayer-Vietoris sequence, see Lemma 59.50.2.

Proof of (2). We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove this. If X is affine, then the result holds by Proposition 59.95.6. Thus it suffices to show that if $X = U \cup V$ and the result is true for U, V, and $U \cap V$, then it is true for X. This follows from the Mayer-Vietoris sequence, see Lemma 59.50.1. □

0F11 Lemma 59.96.3. Cohomology and direct sums. Let $n \geq 1$ be an integer.

(1) Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes with $\mathrm{cd}(f) < \infty$. Then the functor
$$Rf_* : D(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.
(2) Let X be a quasi-compact and quasi-separated scheme with $\mathrm{cd}(X) < \infty$. Then the functor
$$R\Gamma(X, -) : D(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(\mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.

Proof. Proof of (1). Since $\mathrm{cd}(f) < \infty$ we see that

$$f_* : \mathrm{Mod}(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathrm{Mod}(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$

has finite cohomological dimension in the sense of Derived Categories, Lemma 13.32.2. Let I be a set and for $i \in I$ let E_i be an object of $D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$. Choose a K-injective complex $\mathcal{I}_i^\bullet$ of $\mathbf{Z}/n\mathbf{Z}$-modules each of whose terms $\mathcal{I}_i^n$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules representing E_i. See Injectives, Theorem 19.12.6. Then $\bigoplus E_i$ is represented by the complex $\bigoplus \mathcal{I}_i^\bullet$ (termwise direct sum), see Injectives, Lemma 19.13.4. By Lemma 59.51.7 we have

$$R^q f_*(\bigoplus \mathcal{I}_i^n) = \bigoplus R^q f_*(\mathcal{I}_i^n) = 0$$

for $q > 0$ and any n. Hence we conclude by Derived Categories, Lemma 13.32.2 that we may compute $Rf_*(\bigoplus E_i)$ by the complex

$$f_*(\bigoplus \mathcal{I}_i^\bullet) = \bigoplus f_*(\mathcal{I}_i^\bullet)$$

(equality again by Lemma 59.51.7) which represents $\bigoplus Rf_*E_i$ by the already used Injectives, Lemma 19.13.4.

Proof of (2). This is identical to the proof of (1) and we omit it. $\square$

0F0D Lemma 59.96.4. Let $f : X \to Y$ be a proper morphism of schemes. Let $n \geq 1$ be an integer. Then the functor

$$Rf_* : D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$$

commutes with direct sums.

Proof. It is enough to prove this when Y is quasi-compact. By Morphisms, Lemma 29.28.5 we see that the dimension of the fibres of $f : X \to Y$ is bounded. Thus Lemma 59.92.2 implies that $\text{cd}(f) < \infty$. Hence the result by Lemma 59.96.3. $\square$

0F12 Lemma 59.96.5. Let X be a quasi-compact and quasi-separated scheme such that $\text{cd}(X) < \infty$. Let Λ be a torsion ring. Let $E \in D(X_{\acute{e}tale}, \Lambda)$ and $K \in D(\Lambda)$. Then

$$R\Gamma(X, E \otimes_\Lambda^{\mathbf{L}} \underline{K}) = R\Gamma(X, E) \otimes_\Lambda^{\mathbf{L}} K$$

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 21.50. Let $T(K)$ be the property that the statement of the lemma holds for $K \in D(\Lambda)$. We will check conditions (1), (2), and (3) of More on Algebra, Remark 15.59.11 hold for T to conclude. Property (1) holds because both sides of the equality commute with direct sums, see Lemma 59.96.3. Property (2) holds because we are comparing exact functors between triangulated categories and we can use Derived Categories, Lemma 13.4.3. Property (3) says the lemma holds when $K = \Lambda[k]$ for any shift $k \in \mathbf{Z}$ and this is obvious. $\square$

0F0G Lemma 59.96.6. Let $f : X \to Y$ be a proper morphism of schemes. Let Λ be a torsion ring. Let $E \in D(X_{\acute{e}tale}, \Lambda)$ and $K \in D(Y_{\acute{e}tale}, \Lambda)$. Then

$$Rf_*E \otimes_\Lambda^{\mathbf{L}} K = Rf_*(E \otimes_\Lambda^{\mathbf{L}} f^{-1}K)$$

in $D(Y_{\acute{e}tale}, \Lambda)$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 21.50. We will check the equality on stalks at $\overline{y}$. By the proper base change (in the form of Lemma 59.92.3 where $Y' = \overline{y}$) this reduces to the case where Y is the spectrum of an algebraically closed field. This is shown in Lemma 59.96.5 where we use that $\text{cd}(X) < \infty$ by Lemma 59.92.2. $\square$

59.97. Künneth in étale cohomology

0F13 We first prove a Künneth formula in case one of the factors is proper. Then we use this formula to prove a base change property for open immersions. This then gives a "base change by morphisms towards spectra of fields" (akin to smooth base change). Finally we use this to get a more general Künneth formula.

0F1E Remark 59.97.1. Consider a cartesian diagram in the category of schemes:

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"] \arrow[rd, "c"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

Let Λ be a ring and let $E \in D(X_{étale}, \Lambda)$ and $K \in D(Y_{étale}, \Lambda)$. Then there is a canonical map

$$Rf_*E \otimes_\Lambda^{\mathbf{L}} Rg_*K \longrightarrow Rc_*(p^{-1}E \otimes_\Lambda^{\mathbf{L}} q^{-1}K)$$

For example we can define this using the canonical maps $Rf_*E \to Rc_*p^{-1}E$ and $Rg_*K \to Rc_*q^{-1}K$ and the relative cup product defined in Cohomology on Sites, Remark 21.19.7. Or you can use the adjoint to the map

$$c^{-1}(Rf_*E \otimes_\Lambda^{\mathbf{L}} Rg_*K) = p^{-1}f^{-1}Rf_*E \otimes_\Lambda^{\mathbf{L}} q^{-1}g^{-1}Rg_*K \to p^{-1}E \otimes_\Lambda^{\mathbf{L}} q^{-1}K$$

which uses the adjunction maps $f^{-1}Rf_*E \to E$ and $g^{-1}Rg_*K \to K$.

0F14 Lemma 59.97.2. Let k be a separably closed field. Let X be a proper scheme over k. Let Y be a quasi-compact and quasi-separated scheme over k.

(1) If $E \in D^+(X_{étale})$ has torsion cohomology sheaves and $K \in D^+(Y_{étale})$, then

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, \mathrm{pr}_1^{-1}E \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathrm{pr}_2^{-1}K) = R\Gamma(X, E) \otimes_{\mathbf{Z}}^{\mathbf{L}} R\Gamma(Y, K)$$

(2) If $n \geq 1$ is an integer, Y is of finite type over k, $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$, and $K \in D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$, then

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, \mathrm{pr}_1^{-1}E \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} \mathrm{pr}_2^{-1}K) = R\Gamma(X, E) \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} R\Gamma(Y, K)$$

Proof. Proof of (1). By Lemma 59.92.5 we have

$$R\mathrm{pr}_{2,*}(\mathrm{pr}_1^{-1}E \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathrm{pr}_2^{-1}K) = R\mathrm{pr}_{2,*}(\mathrm{pr}_1^{-1}E) \otimes_{\mathbf{Z}}^{\mathbf{L}} K$$

By proper base change (in the form of Lemma 59.91.12) this is equal to the object

$$\underline{R\Gamma(X, E)} \otimes_{\mathbf{Z}}^{\mathbf{L}} K$$

of $D(Y_{étale})$. Taking $R\Gamma(Y, -)$ on this object reproduces the left hand side of the equality in (1) by the Leray spectral sequence for pr_2. Thus we conclude by Lemma 59.92.4.

Proof of (2). This is exactly the same as the proof of (1) except that we use Lemmas 59.96.6, 59.92.3, and 59.96.5 as well as $\mathrm{cd}(Y) < \infty$ by Lemma 59.96.2. □

0F1F Lemma 59.97.3. Let K be a separably closed field. Let X be a scheme of finite type over K. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$ whose support is contained in the set of closed points of X. Then $H^q(X, \mathcal{F}) = 0$ for $q > 0$ and $\mathcal{F}$ is globally generated.

Proof. (If $\mathcal{F}$ is torsion, then the vanishing follows immediately from Lemma 59.95.7.) By Lemma 59.74.5 we can write $\mathcal{F}$ as a filtered colimit of constructible sheaves $\mathcal{F}_i$ of $\mathbf{Z}$-modules whose supports $Z_i \subset X$ are finite sets of closed points. By Proposition 59.46.4 such a sheaf is of the form $(Z_i \to X)_*\mathcal{G}_i$ where $\mathcal{G}_i$ is a sheaf on Z_i. As K is separably closed, the scheme Z_i is a finite disjoint union of spectra of separably closed fields. Recall that $H^q(Z_i, \mathcal{G}_i) = H^q(X, \mathcal{F}_i)$ by the Leray spectral sequence for $Z_i \to X$ and vanising of higher direct images for this morphism (Proposition 59.55.2). By Lemmas 59.59.1 and 59.59.2 we see that $H^q(Z_i, \mathcal{G}_i)$ is zero for $q > 0$ and that $H^0(Z_i, \mathcal{G}_i)$ generates $\mathcal{G}_i$. We conclude the vanishing of $H^q(X, \mathcal{F}_i)$ for $q > 0$ and that $\mathcal{F}_i$ is generated by global sections. By Theorem 59.51.3 we see that $H^q(X, \mathcal{F}) = 0$ for $q > 0$. The proof is now done because a filtered colimit of globally generated sheaves of abelian groups is globally generated (details omitted). □

0F1G Lemma 59.97.4. Let K be a separably closed field. Let X be a scheme of finite type over K. Let $Q \in D(X_{étale})$. Assume that $Q_{\overline{x}}$ is nonzero only if x is a closed point of X. Then

$$Q = 0 \Leftrightarrow H^i(X, Q) = 0 \text{ for all } i$$

Proof. The implication from left to right is trivial. Thus we need to prove the reverse implication.

Assume Q is bounded below; this cases suffices for almost all applications. If Q is not zero, then we can look at the smallest i such that the cohomology sheaf $H^i(Q)$ is nonzero. By Lemma 59.97.3 we have $H^i(X, Q) = H^0(X, H^i(Q)) \neq 0$ and we conclude.

General case. Let $\mathcal{B} \subset \mathrm{Ob}(X_{étale})$ be the quasi-compact objects. By Lemma 59.97.3 the assumptions of Cohomology on Sites, Lemma 21.23.11 are satisfied. We conclude that $H^q(U, Q) = H^0(U, H^q(Q))$ for all $U \in \mathcal{B}$. In particular, this holds for $U = X$. Thus the conclusion by Lemma 59.97.3 as Q is zero in $D(X_{étale})$ if and only if $H^q(Q)$ is zero for all q. $\square$

0F1H Lemma 59.97.5. Let K be a field. Let $j : U \to X$ be an open immersion of schemes of finite type over K. Let Y be a scheme of finite type over K. Consider the diagram

$$\begin{tikzcd} Y \times_{\mathrm{Spec}(K)} X \arrow[d, "q"'] & Y \times_{\mathrm{Spec}(K)} U \arrow[l, "h"'] \arrow[d, "p"] \\ X & U \arrow[l, "j"'] \end{tikzcd}$$

Then the base change map $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism for $\mathcal{F}$ an abelian sheaf on $U_{étale}$ whose stalks are torsion of orders invertible in K.

Proof. Write $\mathcal{F} = \mathrm{colim}\, \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in K. Since cohomology commutes with filtered colimits in our situation (for a precise reference see Lemma 59.86.3), it suffices to prove the lemma for $\mathcal{F}[n]$. Thus we may assume $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n invertible in K (we will use this at the very end of the proof). In the proof we use the short hand $X \times_K Y$ for the fibre product over $\mathrm{Spec}(K)$. We will prove the lemma by induction on $\dim(X) + \dim(Y)$. The lemma is trivial if $\dim(X) \leq 0$, since in this case U is an open and closed subscheme of X. Choose a point $z \in X \times_K Y$. We will show the stalk at $\overline{z}$ is an isomorphism.

Suppose that $z \mapsto x \in X$ and assume $\mathrm{trdeg}_K(\kappa(x)) > 0$. Set $X' = \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ and denote $U' \subset X'$ the inverse image of U. Consider the base change

$$\begin{tikzcd} Y \times_K X' \arrow[d, "q'"'] & Y \times_K U' \arrow[l, "h'"'] \arrow[d, "p'"] \\ X' & U' \arrow[l, "j'"'] \end{tikzcd}$$

of our diagram by $X' \to X$. Observe that $X' \to X$ is a filtered colimit of étale morphisms. By smooth base change in the form of Lemma 59.89.3 the pullback of $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ to X' to $Y \times_K X'$ is the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ to U'. (In this step it would suffice to use étale base change which is an essentially trivial result.) So it suffices to show that

$(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\overline{z}$. By Lemma 59.95.5 there is a separably closed field L/K such that $X' = \lim X_i$ with X_i affine of finite type over L and $\dim(X_i) < \dim(X)$. For i large enough there exists an open $U_i \subset X_i$ restricting to U' in X'. We may apply the induction hypothesis to the diagram

$$\begin{tikzcd} Y \times_K X_i \arrow[d, "q_i"'] & Y \times_K U_i \arrow[l, "h_i"'] \arrow[d, "p_i"] \\ X_i & U_i \arrow[l, "j_i"'] \end{tikzcd} \quad \text{equal to} \quad \begin{tikzcd} Y_L \times_L X_i \arrow[d, "q_i"'] & Y_L \times_L U_i \arrow[l, "h_i"'] \arrow[d, "p_i"] \\ X_i & U_i \arrow[l, "j_i"'] \end{tikzcd}$$

over the field L and the pullback of $\mathcal{F}$ to these diagrams. By Lemma 59.86.3 we conclude that the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}$ is an isomorphism.

Suppose that $z \mapsto y \in Y$ and assume $\text{trdeg}_K(\kappa(y)) > 0$. Let $Y' = \text{Spec}(\mathcal{O}^{sh}_{X,x})$. By Lemma 59.95.5 there is a separably closed field L/K such that $Y' = \lim Y_i$ with Y_i affine of finite type over L and $\dim(Y_i) < \dim(Y)$. In particular Y' is a scheme over L. Denote with a subscript L the base change from schemes over K to schemes over L. Consider the commutative diagrams

$$\begin{tikzcd} Y' \times_K X \arrow[d, "f"'] & Y' \times_K U \arrow[l, "h'"'] \arrow[d, "f'"] \\ Y \times_K X \arrow[d, "q"'] & Y \times_K U \arrow[l, "h"'] \arrow[d, "p"] \\ X & U \arrow[l, "j"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y' \times_L X_L \arrow[d, "q'"'] & Y' \times_L U_L \arrow[l, "h'"'] \arrow[d, "p'"] \\ X_L \arrow[d] & U_L \arrow[l, "j_L"'] \arrow[d] \\ X & U \arrow[l, "j"'] \end{tikzcd}$$

and observe the top and bottom rows are the same on the left and the right. By smooth base change we see that $f^{-1}Rh_*p^{-1}\mathcal{F} = Rh'_*(f')^{-1}p^{-1}\mathcal{F}$ (similarly to the previous paragraph). By smooth base change for $\text{Spec}(L) \to \text{Spec}(K)$ (Lemma 59.90.1) we see that $Rj_{L,*}\mathcal{F}_L$ is the pullback of $Rj_*\mathcal{F}$ to X_L. Combining these two observations, we conclude that it suffices to prove the base change map for the upper square in the diagram on the right is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\overline{z}$[14]. Then using that $Y' = \lim Y_i$ and argueing exactly as in the previous paragraph we see that the induction hypothesis forces our map over $Y' \times_K X$ to be an isomorphism.

Thus any counter example with $\dim(X) + \dim(Y)$ minimal would only have nonisomorphisms $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ on stalks at closed points of $X \times_K Y$ (because a point z of $X \times_K Y$ is a closed point if and only if both the image of z in X and in Y are closed). Since it is enough to prove the isomorphism locally, we may assume X and Y are affine. However, then we can choose an open dense immersion $Y \to Y'$ with Y' projective. (Choose a closed immersion $Y \to \mathbf{A}^n_K$ and let Y' be the scheme theoretic closure of Y in $\mathbf{P}^n_K$.) Then $\dim(Y') = \dim(Y)$ and hence we get a "minimal" counter example with Y projective over K. In the next paragraph we show that this can't happen.

Consider a diagram as in the statement of the lemma such that $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism at all non-closed points of $X \times_K Y$ and such that

[14]Here we use that a "vertical composition" of base change maps is a base change map as explained in Cohomology on Sites, Remark 21.19.4.

Y is projective. The restriction of the map to $(X \times_K Y)_{K^{sep}}$ is the corresponding map for the diagram of the lemma base changed to K^{sep}. Thus we may and do assume K is separably algebraically closed. Choose a distinguished triangle

$$q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F} \to Q \to (q^{-1}Rj_*\mathcal{F})[1]$$

in $D((X \times_K Y)_{étale})$. Since Q is supported in closed points we see that it suffices to prove $H^i(X \times_K Y, Q) = 0$ for all i, see Lemma 59.97.4. Thus it suffices to prove that $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ induces an isomorphism on cohomology. Recall that $\mathcal{F}$ is annihilated by n invertible in K. By the Künneth formula of Lemma 59.97.2 we have

$$\begin{aligned} R\Gamma(X \times_K Y, q^{-1}Rj_*\mathcal{F}) &= R\Gamma(X, Rj_*\mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z}) \\ &= R\Gamma(U, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z}) \end{aligned}$$

and

$$R\Gamma(X \times_K Y, Rh_*p^{-1}\mathcal{F}) = R\Gamma(U \times_K Y, p^{-1}\mathcal{F}) = R\Gamma(U, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z})$$

This finishes the proof. □

0F1I Lemma 59.97.6. Let K be a field. For any commutative diagram

$$\begin{tikzcd} X & X' \arrow[l] & Y \arrow[l, "h"'] \\ \mathrm{Spec}(K) \arrow[u, leftarrow] & S' \arrow[l] \arrow[u, leftarrow, "f'"] & T \arrow[l, "g"'] \arrow[u, leftarrow, "e"'] \end{tikzcd}$$

of schemes over K with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ whose stalks are torsion of orders invertible in K the base change map

$$(f')^{-1}Rg_*\mathcal{F} \longrightarrow Rh_*e^{-1}\mathcal{F}$$

is an isomorphism.

Proof. The question is local on X, hence we may assume X is affine. By Limits, Lemma 32.7.2 we can write $X = \lim X_i$ as a cofiltered limit with affine transition morphisms of schemes X_i of finite type over K. Denote $X'_i = X_i \times_{\mathrm{Spec}(K)} S'$ and $Y_i = X'_i \times_{S'} T$. By Lemma 59.86.3 it suffices to prove the statement for the squares with corners X_i, Y_i, S_i, T_i. Thus we may assume X is of finite type over K. Similarly, we may write $\mathcal{F} = \mathrm{colim}\, \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in K. The same lemma used above reduces us to the case where $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n invertible in K.

We may replace K by its algebraic closure $\overline{K}$. Namely, formation of direct image commutes with base change to $\overline{K}$ according to Lemma 59.90.1 (works for both g and h). And it suffices to prove the agreement after restriction to $X'_{\overline{K}}$. Next, we may replace X by its reduction as we have the topological invariance of étale cohomology, see Proposition 59.45.4. After this replacement the morphism $X \to \mathrm{Spec}(K)$ is flat, finite presentation, with geometrically reduced fibres and the same is true for any base change, in particular for $X' \to S'$. Hence $(f')^{-1}g_*\mathcal{F} \to Rh_*e^{-1}\mathcal{F}$ is an isomorphism by Lemma 59.87.2.

At this point we may apply Lemma 59.90.3 to see that it suffices to prove: given a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y \arrow[l, "h"] \arrow[d] \\ \mathrm{Spec}(K) & S' \arrow[l] & \mathrm{Spec}(L) \arrow[l] \end{tikzcd}$$

with both squares cartesian, where S' is affine, integral, and normal with algebraically closed function field K, then $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ is zero for $q > 0$ and $d|n$. Observe that this vanishing is equivalent to the statement that

$$(f')^{-1}R^q(\mathrm{Spec}(L) \to S')_*\mathbf{Z}/d\mathbf{Z} \longrightarrow R^q h_*\mathbf{Z}/d\mathbf{Z}$$

is an isomorphism, because the left hand side is zero for example by Lemma 59.80.5.

Write $S' = \mathrm{Spec}(B)$ so that L is the fraction field of B. Write $B = \bigcup_{i \in I} B_i$ as the union of its finite type K-subalgebras B_i. Let J be the set of pairs (i, g) where $i \in I$ and $g \in B_i$ nonzero with ordering $(i', g') \geq (i, g)$ if and only if $i' \geq i$ and g maps to an invertible element of $(B_{i'})_{g'}$. Then $L = \mathrm{colim}_{(i,g) \in J}(B_i)_g$. For $j = (i, g) \in J$ set $S_j = \mathrm{Spec}(B_i)$ and $U_j = \mathrm{Spec}((B_i)_g)$. Then

$$\begin{tikzcd} X' \arrow[d] & Y \arrow[l, "h"] \arrow[d] \\ S' & \mathrm{Spec}(L) \arrow[l] \end{tikzcd} \quad \text{is the colimit of} \quad \begin{tikzcd} X \times_K S_j \arrow[d] & X \times_K U_j \arrow[l, "h_j"] \arrow[d] \\ S_j & U_j \arrow[l] \end{tikzcd}$$

Thus we may apply Lemma 59.86.3 to see that it suffices to prove base change holds in the diagrams on the right which is what we proved in Lemma 59.97.5. ☐

0F1J Lemma 59.97.7. Let K be a field. Let $n \geq 1$ be invertible in K. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "p"] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ \mathrm{Spec}(K) & S' \arrow[l] & T \arrow[l, "g"'] \end{tikzcd}$$

of schemes with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated. The canonical map

$$p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \longrightarrow Rh_*(h^{-1}p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

is an isomorphism if E in $D^+(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$ has tor amplitude in $[a, \infty]$ for some $a \in \mathbf{Z}$ and F in $D^+(T_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. This lemma is a generalization of Lemma 59.97.6 to objects of the derived category; the assertion of our lemma is true because in Lemma 59.97.6 the scheme X over K is arbitrary. We strongly urge the reader to skip the laborious proof (alternative: read only the last paragraph).

We may represent E by a bounded below K-flat complex $\mathcal{E}^\bullet$ consisting of flat $\mathbf{Z}/n\mathbf{Z}$-modules. See Cohomology on Sites, Lemma 21.46.4. Choose an integer b such that $H^i(F) = 0$ for $i < b$. Choose a large integer N and consider the short exact sequence

$$0 \to \sigma_{\geq N+1}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq N}\mathcal{E}^\bullet \to 0$$

of stupid truncations. This produces a distinguished triangle $E'' \to E \to E' \to E''[1]$ in $D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$. For fixed F both sides of the arrow in the statement of the lemma are exact functors in E. Observe that

$$p^{-1}E'' \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} (f')^{-1}Rg_*F \quad \text{and} \quad Rh_*(h^{-1}p^{-1}E'' \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} e^{-1}F)$$

are sitting in degrees $\geq N + b$. Hence, if we can prove the lemma for the object E', then we see that the lemma holds in degrees $\leq N + b$ and we will conclude. Some details omitted. Thus we may assume E is represented by a bounded complex of flat $\mathbf{Z}/n\mathbf{Z}$-modules. Doing another argument of the same nature, we may assume E is given by a single flat $\mathbf{Z}/n\mathbf{Z}$-module $\mathcal{E}$.

Next, we use the same arguments for the variable F to reduce to the case where F is given by a single sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules $\mathcal{F}$. Say $\mathcal{F}$ is annihilated by an integer $m|n$. If ℓ is a prime number dividing m and $m > \ell$, then we can look at the short exact sequence $0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$ and reduce to smaller m. This finally reduces us to the case where $\mathcal{F}$ is annihilated by a prime number ℓ dividing n. In this case observe that

$$p^{-1}\mathcal{E} \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} (f')^{-1}Rg_*\mathcal{F} = p^{-1}(\mathcal{E}/\ell\mathcal{E}) \otimes_{\mathbf{F}_\ell}^{\mathbf{L}} (f')^{-1}Rg_*\mathcal{F}$$

by the flatness of $\mathcal{E}$. Similarly for the other term. This reduces us to the case where we are working with sheaves of $\mathbf{F}_\ell$-vector spaces which is discussed

Assume ℓ is a prime number invertible in K. Assume $\mathcal{E}$, $\mathcal{F}$ are sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$ and $T_{étale}$. We want to show that

$$p^{-1}\mathcal{E} \otimes_{\mathbf{F}_\ell} (f')^{-1}R^qg_*\mathcal{F} \longrightarrow R^qh_*(h^{-1}p^{-1}\mathcal{E} \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F})$$

is an isomorphism for every $q \geq 0$. This question is local on X hence we may assume X is affine. We can write $\mathcal{E}$ as a filtered colimit of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$, see Lemma 59.73.2. Since tensor products commute with filtered colimits and since higher direct images do too (Lemma 59.51.7) we may assume $\mathcal{E}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$. Then we can choose an integer m and finite and finitely presented morphisms $\pi_i : X_i \to X$, $i = 1, \ldots, m$ such that there is an injective map

$$\mathcal{E} \to \bigoplus\nolimits_{i=1,\ldots,m} \pi_{i,*}\mathbf{F}_\ell$$

See Lemma 59.74.4. Observe that the direct sum is a constructible sheaf as well (Lemma 59.73.9). Thus the cokernel is constructible too (Lemma 59.71.6). By dimension shifting, i.e., induction on q, on the category of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$, it suffices to prove the result for the sheaves $\pi_{i,*}\mathbf{F}_\ell$ (details omitted; hint: start with proving injectivity for $q = 0$ for all constructible $\mathcal{E}$). To prove this case we extend the diagram of the lemma to

$$\begin{tikzcd}
X_i \arrow[d, "\pi_i"] & X'_i \arrow[l, "p_i"'] \arrow[d, "\pi'_i"] & Y_i \arrow[l, "h_i"'] \arrow[d, "\rho_i"] \\
X \arrow[d] & X' \arrow[l, "p"'] \arrow[d, "f'"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\
\mathrm{Spec}(K) & S' \arrow[l] & T \arrow[l, "g"']
\end{tikzcd}$$

with all squares cartesian. In the equations below we are going to use that $R\pi_{i,*} = \pi_{i,*}$ and similarly for π'_i, ρ_i, we are going to use the Leray spectral sequence, we are going to use Lemma 59.55.3, and we are going to use Lemma 59.96.6 (although this lemma is almost trivial for finite morphisms) for π_i, π'_i, ρ_i. Doing so we see that

$$\begin{aligned} p^{-1}\pi_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} (f')^{-1}R^q g_*\mathcal{F} &= \pi'_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} (f')^{-1}R^q g_*\mathcal{F} \\ &= \pi'_{i,*}((\pi'_i)^{-1}(f')^{-1}R^q g_*\mathcal{F}) \end{aligned}$$

Similarly, we have

$$\begin{aligned} R^q h_*(h^{-1}p^{-1}\pi_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F}) &= R^q h_*(\rho_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F}) \\ &= R^q h_*(\rho_i^{-1}e^{-1}\mathcal{F}) \\ &= \pi'_{i,*}R^q h_{i,*}\rho_i^{-1}e^{-1}\mathcal{F}) \end{aligned}$$

Simce $R^q h_{i,*}\rho_i^{-1}e^{-1}\mathcal{F} = (\pi'_i)^{-1}(f')^{-1}R^q g_*\mathcal{F}$ by Lemma 59.97.6 we conclude. □

0F1N Lemma 59.97.8. Let K be a field. Let $n \geq 1$ be invertible in K. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "p"'] \arrow[d, "f'"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ \mathrm{Spec}(K) & S' \arrow[l] & T \arrow[l, "g"'] \end{tikzcd}$$

of schemes of finite type over K with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$. The canonical map

$$p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \longrightarrow Rh_*(h^{-1}p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

is an isomorphism for E in $D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$ and F in $D(T_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. We will reduce this to Lemma 59.97.7 using that our functors commute with direct sums. We suggest the reader skip the proof. Recall that derived tensor product commutes with direct sums. Recall that (derived) pullback commutes with direct sums. Recall that Rh_* and Rg_* commute with direct sums, see Lemmas 59.96.2 and 59.96.3 (this is where we use our schemes are of finite type over K).

To finish the proof we can argue as follows. First we write $E = \text{hocolim}\tau_{\leq N}E$. Since our functors commute with direct sums, they commute with homotopy colimits. Hence it suffices to prove the lemma for E bounded above. Similarly for F we may assume F is bounded above. Then we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Then

$$\mathcal{E}^\bullet = \operatorname{colim} \sigma_{\geq -N}\mathcal{E}^\bullet$$

(stupid truncations). Thus we may assume $\mathcal{E}^\bullet$ is a bounded complex of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. For F we choose a bounded above complex of flat(!) sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Then we reduce to the case where F is represented by a bounded complex of flat sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. At this point Lemma 59.97.7 kicks in and we conclude. □

0F1P Lemma 59.97.9. Let k be a separably closed field. Let X and Y be finite type schemes over k. Let $n \geq 1$ be an integer invertible in k. Then for $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$ and $K \in D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$ we have

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, \mathrm{pr}_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \mathrm{pr}_2^{-1}K) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, K)$$

Proof. By Lemma 59.97.8 we have

$$R\mathrm{pr}_{1,*}(\mathrm{pr}_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \mathrm{pr}_2^{-1}K) = E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \underline{R\Gamma(Y, K)}$$

We conclude by Lemma 59.96.5 which we may use because $\mathrm{cd}(X) < \infty$ by Lemma 59.96.2. $\square$

59.98. Comparing chaotic and Zariski topologies

0F1K When constructing the structure sheaf of an affine scheme, we first construct the values on affine opens, and then we extend to all opens. A similar construction is often useful for constructing complexes of abelian groups on a scheme X. Recall that $X_{affine,Zar}$ denotes the category of affine opens of X with topology given by standard Zariski coverings, see Topologies, Definition 34.3.7. We remind the reader that the topos of $X_{affine,Zar}$ is the small Zariski topos of X, see Topologies, Lemma 34.3.11. In this section we denote X_{affine} the same underlying category with the chaotic topology, i.e., such that sheaves agree with presheaves. We obtain a morphisms of sites

$$\epsilon : X_{affine,Zar} \longrightarrow X_{affine}$$

as in Cohomology on Sites, Section 21.27.

0F1L Lemma 59.98.1. In the situation above let K be an object of $D^+(X_{affine})$. Then K is in the essential image of the (fully faithful) functor $R\epsilon_*; D(X_{affine,Zar}) \to D(X_{affine})$ if and only if the following two conditions hold

(1) $R\Gamma(\emptyset, K)$ is zero in $D(\mathrm{Ab})$, and
(2) if $U = V \cup W$ with $U, V, W \subset X$ affine open and $V, W \subset U$ standard open (Algebra, Definition 10.17.3), then the map $c^K_{U,V,W,V\cap W}$ of Cohomology on Sites, Lemma 21.26.1 is a quasi-isomorphism.

Proof. (The functor $R\epsilon_*$ is fully faithful by the discussion in Cohomology on Sites, Section 21.27.) Except for a snafu having to do with the empty set, this follows from the very general Cohomology on Sites, Lemma 21.29.2 whose hypotheses hold by Schemes, Lemma 26.11.7 and Cohomology on Sites, Lemma 21.29.3.

To get around the snafu, denote $X_{affine,almost-chaotic}$ the site where the empty object $\emptyset$ has two coverings, namely, $\{\emptyset \to \emptyset\}$ and the empty covering (see Sites, Example 7.6.4 for a discussion). Then we have morphisms of sites

$$X_{affine,Zar} \to X_{affine,almost-chaotic} \to X_{affine}$$

The argument above works for the first arrow. Then we leave it to the reader to see that an object K of $D^+(X_{affine})$ is in the essential image of the (fully faithful) functor $D(X_{affine,almost-chaotic}) \to D(X_{affine})$ if and only if $R\Gamma(\emptyset, K)$ is zero in $D(\mathrm{Ab})$. $\square$

59.99. Comparing big and small topoi

0757 Let S be a scheme. In Topologies, Lemma 34.4.14 we have introduced comparison morphisms $\pi_S : (Sch/S)_{étale} \to S_{étale}$ and $i_S : Sh(S_{étale}) \to Sh((Sch/S)_{étale})$ with $\pi_S \circ i_S = \text{id}$ and $\pi_{S,*} = i_S^{-1}$. More generally, if $f : T \to S$ is an object of $(Sch/S)_{étale}$, then there is a morphism $i_f : Sh(T_{étale}) \to Sh((Sch/S)_{étale})$ such that $f_{small} = \pi_S \circ i_f$, see Topologies, Lemmas 34.4.13 and 34.4.17. In Descent, Remark 35.8.4 we have extended these to a morphism of ringed sites

$$\pi_S : ((Sch/S)_{étale}, \mathcal{O}) \to (S_{étale}, \mathcal{O}_S)$$

and morphisms of ringed topoi

$$i_S : (Sh(S_{étale}), \mathcal{O}_S) \to (Sh((Sch/S)_{étale}), \mathcal{O})$$

and

$$i_f : (Sh(T_{étale}), \mathcal{O}_T) \to (Sh((Sch/S)_{étale}, \mathcal{O}))$$

Note that the restriction $i_S^{-1} = \pi_{S,*}$ (see Topologies, Definition 34.4.15) transforms $\mathcal{O}$ into $\mathcal{O}_S$. Similarly, i_f^{-1} transforms $\mathcal{O}$ into $\mathcal{O}_T$. See Descent, Remark 35.8.4. Hence $i_S^*\mathcal{F} = i_S^{-1}\mathcal{F}$ and $i_f^*\mathcal{F} = i_f^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_{étale}$. In particular i_S^* and i_f^* are exact functors. The functor i_S^* is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{S_{étale}}$ (and this does not conflict with the notation in Topologies, Definition 34.4.15).

0758 **Lemma 59.99.1.** Let S be a scheme. Let T be an object of $(Sch/S)_{étale}$.

(1) If $\mathcal{I}$ is injective in $\text{Ab}((Sch/S)_{étale})$, then
  (a) $i_f^{-1}\mathcal{I}$ is injective in $\text{Ab}(T_{étale})$,
  (b) $\mathcal{I}|_{S_{étale}}$ is injective in $\text{Ab}(S_{étale})$,
(2) If $\mathcal{I}^\bullet$ is a K-injective complex in $\text{Ab}((Sch/S)_{étale})$, then
  (a) $i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $\text{Ab}(T_{étale})$,
  (b) $\mathcal{I}^\bullet|_{S_{étale}}$ is a K-injective complex in $\text{Ab}(S_{étale})$,

The corresponding statements for modules do not hold.

Proof. Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor $\pi_{S,*} = i_S^{-1}$ is a right adjoint of the exact functor π_S^{-1}, see Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that i_f^{-1} is a right adjoint of the exact functor $i_{f,!}$. This functor is constructed in Topologies, Lemma 34.4.13 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 18.16.2. It is shown in Modules on Sites, Lemma 18.16.3 that it is exact. Second proof. We can use that $i_f = i_T \circ f_{big}$ as is shown in Topologies, Lemma 34.4.17. Since f_{big} is a localization, we see that pullback by it preserves injectives and K-injectives, see Cohomology on Sites, Lemmas 21.7.1 and 21.20.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor i_T^{-1} to conclude.

Let $S = \text{Spec}(\mathbf{Z})$ and consider the map $2 : \mathcal{O}_S \to \mathcal{O}_S$. This is an injective map of $\mathcal{O}_S$-modules on $S_{étale}$. However, the pullback $\pi_S^*(2) : \mathcal{O} \to \mathcal{O}$ is not injective as we see by evaluating on $\text{Spec}(\mathbf{F}_2)$. Now choose an injection $\alpha : \mathcal{O} \to \mathcal{I}$ into an injective $\mathcal{O}$-module $\mathcal{I}$ on $(Sch/S)_{étale}$. Then consider the diagram

$$\begin{tikzcd} \mathcal{O}_S \arrow[r] \arrow[d, "2"'] & \mathcal{I}|_{S_{étale}} \\ \mathcal{O}_S \arrow[ur, dotted, "\alpha|_{S_{étale}}"] & \end{tikzcd}$$

Then the dotted arrow cannot exist in the category of $\mathcal{O}_S$-modules because it would mean (by adjunction) that the injective map α factors through the noninjective map $\pi_S^*(2)$ which cannot be the case. Thus $\mathcal{I}|_{S_{étale}}$ is not an injective $\mathcal{O}_S$-module. $\square$

Let $f : T \to S$ be a morphism of schemes. The commutative diagram of Topologies, Lemma 34.4.17 (3) leads to a commutative diagram of ringed sites

$$\begin{tikzcd} (T_{étale}, \mathcal{O}_T) \arrow[d, "f_{small}"'] & ((Sch/T)_{étale}, \mathcal{O}) \arrow[l, "\pi_T"] \arrow[d, "f_{big}"] \\ (S_{étale}, \mathcal{O}_S) & ((Sch/S)_{étale}, \mathcal{O}) \arrow[l, "\pi_S"'] \end{tikzcd}$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_S^\sharp$, and $\pi_T^\sharp$. In particular this means that

0759 (59.99.1.1) $$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}})$$

for any sheaf $\mathcal{F}$ on $(Sch/T)_{étale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (59.99.1.1) is an isomorphism of $\mathcal{O}_S$-modules on $S_{étale}$.

075A **Lemma 59.99.2.** Let $f : T \to S$ be a morphism of schemes.

(1) For K in $D((Sch/T)_{étale})$ we have $(Rf_{big,*}K)|_{S_{étale}} = Rf_{small,*}(K|_{T_{étale}})$ in $D(S_{étale})$.
(2) For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $(Rf_{big,*}K)|_{S_{étale}} = Rf_{small,*}(K|_{T_{étale}})$ in $D(\text{Mod}(S_{étale}, \mathcal{O}_S))$.

More generally, let $g : S' \to S$ be an object of $(Sch/S)_{étale}$. Consider the fibre product

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then

(3) For K in $D((Sch/T)_{étale})$ we have $i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$ in $D(S'_{étale})$.
(4) For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $i_g^*(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^*K)$ in $D(\text{Mod}(S'_{étale}, \mathcal{O}_{S'}))$.
(5) For K in $D((Sch/T)_{étale})$ we have $g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$ in $D((Sch/S')_{étale})$.
(6) For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $g_{big}^*(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^*K)$ in $D(\text{Mod}(S'_{étale}, \mathcal{O}_{S'}))$.

Proof. Part (1) follows from Lemma 59.99.1 and (59.99.1.1) on choosing a K-injective complex of abelian sheaves representing K.

Part (3) follows from Lemma 59.99.1 and Topologies, Lemma 34.4.19 on choosing a K-injective complex of abelian sheaves representing K.

Part (5) is Cohomology on Sites, Lemma 21.21.1.

Part (6) is Cohomology on Sites, Lemma 21.21.2.

Part (2) can be proved as follows. Above we have seen that $\pi_S \circ f_{big} = f_{small} \circ \pi_T$ as morphisms of ringed sites. Hence we obtain $R\pi_{S,*} \circ Rf_{big,*} = Rf_{small,*} \circ R\pi_{T,*}$ by Cohomology on Sites, Lemma 21.19.2. Since the restriction functors $\pi_{S,*}$ and $\pi_{T,*}$ are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to $f' : T' \to S'$. $\square$

Let S be a scheme and let $\mathcal{H}$ be an abelian sheaf on $(Sch/S)_{étale}$. Recall that $H^n_{étale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Sch/S)_{étale}$.

0DDH Lemma 59.99.3. Let $f : T \to S$ be a morphism of schemes. Then

(1) For K in $D(S_{étale})$ we have $H^n_{étale}(S, \pi_S^{-1}K) = H^n(S_{étale}, K)$.
(2) For K in $D(S_{étale}, \mathcal{O}_S)$ we have $H^n_{étale}(S, L\pi_S^*K) = H^n(S_{étale}, K)$.
(3) For K in $D(S_{étale})$ we have $H^n_{étale}(T, \pi_S^{-1}K) = H^n(T_{étale}, f_{small}^{-1}K)$.
(4) For K in $D(S_{étale}, \mathcal{O}_S)$ we have $H^n_{étale}(T, L\pi_S^*K) = H^n(T_{étale}, Lf_{small}^*K)$.
(5) For M in $D((Sch/S)_{étale})$ we have $H^n_{étale}(T, M) = H^n(T_{étale}, i_f^{-1}M)$.
(6) For M in $D((Sch/S)_{étale}, \mathcal{O})$ we have $H^n_{étale}(T, M) = H^n(T_{étale}, i_f^*M)$.

Proof. To prove (5) represent M by a K-injective complex of abelian sheaves and apply Lemma 59.99.1 and work out the definitions. Part (3) follows from this as $i_f^{-1}\pi_S^{-1} = f_{small}^{-1}$. Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 21.37.5. Then part (4) follows because $Lf_{small}^* = i_f^* \circ L\pi_S^*$. Part (2) is a special case of (4). $\square$

0DDI Lemma 59.99.4. Let S be a scheme. For $K \in D(S_{étale})$ the map

$$K \longrightarrow R\pi_{S,*}\pi_S^{-1}K$$

is an isomorphism.

Proof. This is true because both π_S^{-1} and $\pi_{S,*} = i_S^{-1}$ are exact functors and the composition $\pi_{S,*} \circ \pi_S^{-1}$ is the identity functor. $\square$

0DDJ Lemma 59.99.5. Let $f : T \to S$ be a proper morphism of schemes. Then we have

(1) $\pi_S^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_T^{-1}$ as functors $Sh(T_{étale}) \to Sh((Sch/S)_{étale})$,
(2) $\pi_S^{-1}Rf_{small,*}K = Rf_{big,*}\pi_T^{-1}K$ for K in $D^+(T_{étale})$ whose cohomology sheaves are torsion,
(3) $\pi_S^{-1}Rf_{small,*}K = Rf_{big,*}\pi_T^{-1}K$ for K in $D(T_{étale}, \mathbf{Z}/n\mathbf{Z})$, and
(4) $\pi_S^{-1}Rf_{small,*}K = Rf_{big,*}\pi_T^{-1}K$ for all K in $D(T_{étale})$ if f is finite.

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf on $T_{étale}$. Let $g : S' \to S$ be an object of $(Sch/S)_{étale}$. Consider the fibre product

$$\begin{tikzcd} T' \arrow[r, "f'"] \arrow[d, "g'"'] & S' \arrow[d, "g"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

Then we have

$$(f_{big,*}\pi_T^{-1}\mathcal{F})(S') = (\pi_T^{-1}\mathcal{F})(T') = ((g'_{small})^{-1}\mathcal{F})(T') = (f'_{small,*}(g'_{small})^{-1}\mathcal{F})(S')$$

the second equality by Lemma 59.39.2. On the other hand

$$(\pi_S^{-1}f_{small,*}\mathcal{F})(S') = (g_{small}^{-1}f_{small,*}\mathcal{F})(S')$$

again by Lemma 59.39.2. Hence by proper base change for sheaves of sets (Lemma 59.91.5) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_S^{-1}f_{small,*}\mathcal{F} = f_{big,*}\pi_T^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_S^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_T^{-1}$.

Proof of (2). There is a canonical base change map $\pi_S^{-1}Rf_{small,*}K \to Rf_{big,*}\pi_T^{-1}K$ for any K in $D(T_{étale})$, see Cohomology on Sites, Remark 21.19.3. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by $i_g : Sh(S'_{étale}) \to Sh((Sch/S)_{étale})$ is an isomorphism for any object $g : S' \to S$ of $(Sch/S)_{étale}$. Let T', g', f' be as in the previous paragraph. The pullback of the base change map is

$$\begin{aligned} g_{small}^{-1}Rf_{small,*}K &= i_g^{-1}\pi_S^{-1}Rf_{small,*}K \\ &\to i_g^{-1}Rf_{big,*}\pi_T^{-1}K \\ &= Rf'_{small,*}(i_{g'}^{-1}\pi_T^{-1}K) \\ &= Rf'_{small,*}((g'_{small})^{-1}K) \end{aligned}$$

where we have used $\pi_S \circ i_g = g_{small}$, $\pi_T \circ i_{g'} = g'_{small}$, and Lemma 59.99.2. This map is an isomorphism by the proper base change theorem (Lemma 59.91.12) provided K is bounded below and the cohomology sheaves of K are torsion.

The proof of part (3) is the same as the proof of part (2), except we use Lemma 59.92.3 instead of Lemma 59.91.12.

Proof of (4). If f is finite, then the functors $f_{small,*}$ and $f_{big,*}$ are exact. This follows from Proposition 59.55.2 for f_{small}. Since any base change f' of f is finite too, we conclude from Lemma 59.99.2 part (3) that $f_{big,*}$ is exact too (as the higher derived functors are zero). Thus this case follows from part (1). □

59.100. Comparing fppf and étale topologies

0DDK A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 21.31. We first review some material from Topologies, Sections 34.11 and 34.4.

Let S be a scheme and let $(Sch/S)_{fppf}$ be an fppf site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_{fppf}$ is continuous and defines a morphism of sites

$$\epsilon_S : (Sch/S)_{fppf} \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 21.27. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the fppf sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies.

0DDL Lemma 59.100.1. With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. The rule

$$(Sch/S)_{fppf} \longrightarrow \text{Sets}, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{étale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{fppf}$.

Proof. The statement about the étale topology is the content of Lemma 59.39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the fppf topology. This is shown in Lemma 59.39.2 as well. ☐

In the situation of Lemma 59.100.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_{fppf} \longrightarrow S_{étale}$$

0DDM Lemma 59.100.2. With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_{fppf}$. Then there are commutative diagrams of topoi

$$\begin{tikzcd} Sh((Sch/X)_{fppf}) \arrow[r, "f_{big,fppf}"'] \arrow[d, "\epsilon_X"'] & Sh((Sch/Y)_{fppf}) \arrow[d, "\epsilon_Y"] \\ Sh((Sch/X)_{étale}) \arrow[r, "f_{big,étale}"] & Sh((Sch/Y)_{étale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((Sch/X)_{fppf}) \arrow[r, "f_{big,fppf}"'] \arrow[d, "a_X"'] & Sh((Sch/Y)_{fppf}) \arrow[d, "a_Y"] \\ Sh(X_{étale}) \arrow[r, "f_{small}"] & Sh(Y_{étale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 34.11. ☐

0DDN Lemma 59.100.3. In Lemma 59.100.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,fppf,*} \circ a_X^{-1}$.

Proof. You can prove this by repeating the proof of Lemma 59.99.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 59.100.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1}f_{small,*}\mathcal{F} \to f_{big,fppf,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1}f_{small,*}\mathcal{F} &= \epsilon_{Y,*}a_Y^{-1}f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*}f_{big,fppf,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 59.99.5. ☐

0DEU Lemma 59.100.4. In Lemma 59.100.2 assume f is flat, locally of finite presentation, and surjective. Then the functor

$$Sh(Y_{\acute{e}tale}) \longrightarrow \left\{(\mathcal{G}, \mathcal{H}, \alpha)\middle| \begin{array}{l}\mathcal{G} \in Sh(X_{\acute{e}tale}),\ \mathcal{H} \in Sh((Sch/Y)_{fppf}),\\ \alpha : a_X^{-1}\mathcal{G} \to f_{big,fppf}^{-1}\mathcal{H} \text{ an isomorphism}\end{array}\right\}$$

sending $\mathcal{F}$ to $(f_{small}^{-1}\mathcal{F}, a_Y^{-1}\mathcal{F}, can)$ is an equivalence.

Proof. The functor a_X^{-1} is fully faithful (as $a_{X,*}a_X^{-1} = \text{id}$ by Lemma 59.100.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor a_Y^{-1} is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of f. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is true for f_i and f_{ij} for all i, j, then the lemma is true for f. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{\acute{e}tale}, X_{\acute{e}tale}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 7.26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on Y.

Note that $\{X \to Y\}$ is an fppf covering. Working étale locally on Y, we may assume there exists a morphism $s : X' \to X$ such that the composition $f' = f \circ s : X' \to Y$ is surjective finite locally free, see More on Morphisms, Lemma 37.48.1. Claim: if the lemma is true for f', then it is true for f. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for f, we can pullback by s to get a triple $(s_{small}^{-1}\mathcal{G}, \mathcal{H}, s_{big,fppf}^{-1}\alpha)$ for f'. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{\acute{e}tale}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case described in the next paragraph.

Assume f is surjective finite locally free. Let $(\mathcal{G}, \mathcal{H}, \alpha)$ be a triple. In this case consider the triple

$$(\mathcal{G}_1, \mathcal{H}_1, \alpha_1) = (f_{small}^{-1}f_{small,*}\mathcal{G}, f_{big,fppf,*}f_{big,fppf}^{-1}\mathcal{H}, \alpha_1)$$

where α_1 comes from the identifications

$$\begin{aligned} a_X^{-1}f_{small}^{-1}f_{small,*}\mathcal{G} &= f_{big,fppf}^{-1}a_Y^{-1}f_{small,*}\mathcal{G} \\ &= f_{big,fppf}^{-1}f_{big,fppf,*}a_X^{-1}\mathcal{G} \\ &\to f_{big,fppf}^{-1}f_{big,fppf,*}f_{big,fppf}^{-1}\mathcal{H} \end{aligned}$$

where the third equality is Lemma 59.100.3 and the arrow is given by α. This triple is in the image of our functor because $\mathcal{F}_1 = f_{small,*}\mathcal{F}$ is a solution (to see this use Lemma 59.100.3 again; details omitted). There is a canonical map of triples

$$(\mathcal{G}, \mathcal{H}, \alpha) \to (\mathcal{G}_1, \mathcal{H}_1, \alpha_1)$$

which uses the unit $\text{id} \to f_{big,fppf,*}f_{big,fppf}^{-1}$ on the second entry (it is enough to prescribe morphisms on the second entry by the first paragraph of the proof). Since $\{f : X \to Y\}$ is an fppf covering the map $\mathcal{H} \to \mathcal{H}_1$ is injective (details omitted). Set

$$\mathcal{G}_2 = \mathcal{G}_1 \amalg_{\mathcal{G}} \mathcal{G}_1 \quad \mathcal{H}_2 = \mathcal{H}_1 \amalg_{\mathcal{H}} \mathcal{H}_1$$

and let α_2 be the induced isomorphism (pullback functors are exact, so this makes sense). Then $\mathcal{H}$ is the equalizer of the two maps $\mathcal{H}_1 \to \mathcal{H}_2$. Repeating the arguments above for the triple $(\mathcal{G}_2, \mathcal{H}_2, \alpha_2)$ we find an injective morphism of triples

$$(\mathcal{G}_2, \mathcal{H}_2, \alpha_2) \to (\mathcal{G}_3, \mathcal{H}_3, \alpha_3)$$

such that this last triple is in the image of our functor. Say it corresponds to $\mathcal{F}_3$ in $Sh(Y_{\acute{e}tale})$. By fully faithfulness we obtain two maps $\mathcal{F}_1 \to \mathcal{F}_3$ and we can let $\mathcal{F}$ be the equalizer of these two maps. By exactness of the pullback functors involved we find that $a_Y^{-1}\mathcal{F} = \mathcal{H}$ as desired. □

0F0H Lemma 59.100.5. Consider the comparison morphism $\epsilon : (Sch/S)_{fppf} \to (Sch/S)_{\acute{e}tale}$. Let $\mathcal{P}$ denote the class of finite morphisms of schemes. For X in $(Sch/S)_{\acute{e}tale}$ denote $\mathcal{A}'_X \subset \text{Ab}((Sch/X)_{\acute{e}tale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ with $\mathcal{F}$ in $\text{Ab}(X_{\acute{e}tale})$. Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 21.30.1 hold.

Proof. We first show that $\mathcal{A}'_X \subset \text{Ab}((Sch/X)_{\acute{e}tale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 59.99.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $\text{Ab}((Sch/X)_{\acute{e}tale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 59.99.4. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a finite morphism of schemes is a finite morphism of schemes, see Morphisms, Lemma 29.44.6.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 34.4.17.

Cohomology on Sites, Property (3) is Lemma 59.100.1.

Cohomology on Sites, Property (4) holds by Lemma 59.99.5 part (4).

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 37.48.1. □

0DDS Lemma 59.100.6. With notation as above.

(1) For $X \in \text{Ob}((Sch/S)_{fppf})$ and an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.
(2) For a finite morphism $f : X \to Y$ in $(Sch/S)_{fppf}$ and abelian sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,fppf,*}(a_X^{-1}\mathcal{F})$ for all i.
(3) For a scheme X and K in $D^+(X_{\acute{e}tale})$ the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.
(4) For a finite morphism $f : X \to Y$ of schemes and K in $D^+(X_{\acute{e}tale})$ we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$.
(5) For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$.

Proof. By Lemma 59.100.5 the lemmas in Cohomology on Sites, Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 21.30.2 is the essential image of $a_X^{-1} : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}((Sch/X)_{fppf})$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 21.30.3.

Part (3) follows from Cohomology on Sites, Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 21.30.8 part (2) for the same reason.

Part (5). We use that

$$\begin{aligned} R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K &= Rf_{big,étale,*}R\epsilon_{X,*}a_X^{-1}K \\ &= Rf_{big,étale,*}\pi_X^{-1}K \\ &= \pi_Y^{-1}Rf_{small,*}K \\ &= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \end{aligned}$$

The first equality by the commutative diagram in Lemma 59.100.2 and Cohomology on Sites, Lemma 21.19.2. The second equality is (3). The third is Lemma 59.99.5 part (2). The fourth is (3) again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,fppf,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{étale})$ and M in $D^+((Sch/Y)_{fppf})$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by (1) because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

0DDT Lemma 59.100.7. Let X be a scheme. For $K \in D^+(X_{étale})$ the map

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_{fppf}) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^qa_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 59.100.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^qa_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 59.100.6 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 59.99.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DDU Lemma 59.100.8. For a scheme X and $a_X : Sh((Sch/X)_{fppf}) \to Sh(X_{étale})$ as above:

(1) $H^q(X_{étale}, \mathcal{F}) = H^q_{fppf}(X, a_X^{-1}\mathcal{F})$ for an abelian sheaf $\mathcal{F}$ on $X_{étale}$,
(2) $H^q(X_{étale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$.

Example: if A is an abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{fppf}(X, \underline{A})$.

Proof. This follows from Lemma 59.100.7 by Cohomology on Sites, Remark 21.14.4. □

59.101. Comparing fppf and étale topologies: modules

0DEV We continue the discussion in Section 59.100 but in this section we briefly discuss what happens for sheaves of modules.

Let S be a scheme. The morphisms of sites ϵ_S, π_S, and their composition a_S introduced in Section 59.100 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_S : ((Sch/S)_{fppf}, \mathcal{O}) \longrightarrow ((Sch/S)_{étale}, \mathcal{O})$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_S : ((Sch/S)_{étale}, \mathcal{O}) \longrightarrow (S_{étale}, \mathcal{O}_S)$$

The third is the morphism

$$a_S : ((Sch/S)_{fppf}, \mathcal{O}) \longrightarrow (S_{étale}, \mathcal{O}_S)$$

We already know that the category of quasi-coherent modules on the scheme S is the same as the category of quasi-coherent modules on $(S_{étale}, \mathcal{O}_S)$, see Descent, Proposition 35.8.9. Since we are interested in stating a comparison between étale and fppf cohomology, we will in the rest of this section think of quasi-coherent sheaves in terms of the small étale site. Let us review what we already know about quasi-coherent modules on these sites.

0DEW Lemma 59.101.1. Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on $S_{étale}$.

(1) The rule
$$\mathcal{F}^a : (Sch/S)_{étale} \longrightarrow \text{Ab}, \quad (f : T \to S) \longmapsto \Gamma(T, f^*_{small}\mathcal{F})$$
satisfies the sheaf condition for fppf and a fortiori étale coverings,
(2) $\mathcal{F}^a = \pi_S^*\mathcal{F}$ on $(Sch/S)_{étale}$,
(3) $\mathcal{F}^a = a_S^*\mathcal{F}$ on $(Sch/S)_{fppf}$,
(4) the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_S$-modules and quasi-coherent modules on $((Sch/S)_{étale}, \mathcal{O})$,
(5) the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_S$-modules and quasi-coherent modules on $((Sch/S)_{fppf}, \mathcal{O})$,
(6) we have $\epsilon_{S,*}a_S^*\mathcal{F} = \pi_S^*\mathcal{F}$ and $a_{S,*}a_S^*\mathcal{F} = \mathcal{F}$,
(7) we have $R^i\epsilon_{S,*}(a_S^*\mathcal{F}) = 0$ and $R^ia_{S,*}(a_S^*\mathcal{F}) = 0$ for $i > 0$.

Proof. We urge the reader to find their own proof of these results based on the material in Descent, Sections 35.8, 35.9, and 35.10.

We first explain why the notation in this lemma is consistent with our earlier use of the notation $\mathcal{F}^a$ in Sections 59.17 and 59.22 and in Descent, Section 35.8. Namely, we know by Descent, Proposition 35.8.9 that there exists a quasi-coherent module $\mathcal{F}_0$ on the scheme S (in other words on the small Zariski site) such that $\mathcal{F}$ is the restriction of the rule

$$\mathcal{F}_0^a : (Sch/S)_{étale} \longrightarrow \text{Ab}, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$

to the subcategory $S_{étale} \subset (Sch/S)_{étale}$ where here f^* denotes usual pullback of sheaves of modules on schemes. Since $\mathcal{F}_0^a$ is pullback by the morphism of ringed sites

$$((Sch/S)_{étale}, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_{S_{Zar}})$$

by Descent, Remark 35.8.6 it follows immediately (from composition of pullbacks) that $\mathcal{F}^a = \mathcal{F}_0^a$. This proves the sheaf property even for fpqc coverings by Descent, Lemma 35.8.1 (see also Proposition 59.17.1). Then (2) and (3) follow again by Descent, Remark 35.8.6 and (4) and (5) follow from Descent, Proposition 35.8.9 (see also the meta result Theorem 59.17.4).

Part (6) is immediate from the description of the sheaf $\mathcal{F}^a = \pi_S^*\mathcal{F} = a_S^*\mathcal{F}$.

For any abelian $\mathcal{H}$ on $(Sch/S)_{fppf}$ the higher direct image $R^p\epsilon_{S,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, \mathcal{H})$ on $(Sch/S)_{étale}$. See Cohomology on Sites, Lemma 21.7.4. Hence to prove $R^p\epsilon_{S,*}a_S^*\mathcal{F} = R^p\epsilon_{S,*}\mathcal{F}^a = 0$ for $p > 0$ it suffices to show that any scheme U over S has an étale covering $\{U_i \to U\}_{i \in I}$ such that $H^p_{fppf}(U_i, \mathcal{F}^a) = 0$ for $p > 0$. If we take an open covering by affines, then the required vanishing follows from comparison with usual cohomology (Descent, Proposition 35.9.3 or Theorem 59.22.4) and the vanishing of cohomology of quasi-coherent sheaves on affine schemes afforded by Cohomology of Schemes, Lemma 30.2.2.

To show that $R^p a_{S,*}a_S^{-1}\mathcal{F} = R^p a_{S,*}\mathcal{F}^a = 0$ for $p > 0$ we argue in exactly the same manner. This finishes the proof. $\square$

0DEX Lemma 59.101.2. Let S be a scheme. For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module on $S_{étale}$ the maps

$$\pi_S^*\mathcal{F} \longrightarrow R\epsilon_{S,*}(a_S^*\mathcal{F}) \quad \text{and} \quad \mathcal{F} \longrightarrow Ra_{S,*}(a_S^*\mathcal{F})$$

are isomorphisms with $a_S : Sh((Sch/S)_{fppf}) \to Sh(S_{étale})$ as above.

Proof. This is an immediate consequence of parts (6) and (7) of Lemma 59.101.1. $\square$

0H0U Lemma 59.101.3. Let $S = \text{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules. Consider the complex $\mathcal{F}^\bullet$ of presheaves of $\mathcal{O}$-modules on $(\text{Aff}/S)_{fppf}$ given by the rule

$$(U/S) = (\text{Spec}(B)/\text{Spec}(A)) \longmapsto M^\bullet \otimes_A B$$

Then this is a complex of modules and the canonical map

$$M^\bullet \longrightarrow R\Gamma((\text{Aff}/S)_{fppf}, \mathcal{F}^\bullet)$$

is a quasi-isomorphism.

Proof. Each $\mathcal{F}^n$ is a sheaf of modules as it agrees with the restriction of the module $\mathcal{G}^n = (\widetilde{M^n})^a$ of Lemma 59.101.1 to $(\text{Aff}/S)_{fppf} \subset (Sch/S)_{fppf}$. Since this inclusion defines an equivalence of ringed topoi (Topologies, Lemma 34.7.11), we have

$$R\Gamma((\text{Aff}/S)_{fppf}, \mathcal{F}^\bullet) = R\Gamma((Sch/S)_{fppf}, \mathcal{G}^\bullet)$$

We observe that $M^\bullet = R\Gamma(S, \widetilde{M^\bullet})$ for example by Derived Categories of Schemes, Lemma 36.3.5. Hence we are trying to show the comparison map

$$R\Gamma(S, \widetilde{M^\bullet}) \longrightarrow R\Gamma((Sch/S)_{fppf}, (\widetilde{M^\bullet})^a)$$

is an isomorphism. If $M^\bullet$ is bounded below, then this holds by Descent, Proposition 35.9.3 and the first spectral sequence of Derived Categories, Lemma 13.21.3. For the general case, let us write $M^\bullet = \lim M_n^\bullet$ with $M_n^\bullet = \tau_{\geq -n} M^\bullet$. Whence the system M_n^p is eventually constant with value M^p. We claim that

$$(\widetilde{M^\bullet})^a = R\lim (\widetilde{M_n^\bullet})^a$$

Namely, it suffices to show that the natural map from left to right induces an isomorphism on cohomology over any affine object $U = \mathrm{Spec}(B)$ of $(Sch/S)_{fppf}$. For $i \in \mathbf{Z}$ and $n > |i|$ we have

$$H^i(U, (\widetilde{M_n^\bullet})^a) = H^i(\tau_{\geq -n} M^\bullet \otimes_A B) = H^i(M^\bullet \otimes_A B)$$

The first equality holds by the bounded below case treated above. Thus we see from Cohomology on Sites, Lemma 21.23.2 that the claim holds. Then we finally get

$$\begin{aligned} R\Gamma((Sch/S)_{fppf}, (\widetilde{M^\bullet})^a) &= R\Gamma((Sch/S)_{fppf}, R\lim (\widetilde{M_n^\bullet})^a) \\ &= R\lim R\Gamma((Sch/S)_{fppf}, (\widetilde{M_n^\bullet})^a) \\ &= R\lim M_n^\bullet \\ &= M^\bullet \end{aligned}$$

as desired. The second equality holds because $R\lim$ commutes with $R\Gamma$, see Cohomology on Sites, Lemma 21.23.2. $\square$

59.102. Comparing ph and étale topologies

0DDV A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 21.31. We first review some material from Topologies, Sections 34.11 and 34.4.

Let S be a scheme and let $(Sch/S)_{ph}$ be a ph site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_{ph}$ is continuous (by More on Morphisms, Lemma 37.48.7 and Topologies, Lemma 34.7.2) and defines a morphism of sites

$$\epsilon_S : (Sch/S)_{ph} \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 21.27. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the ph sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies.

0DDW Lemma 59.102.1. With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. The rule

$$(Sch/S)_{ph} \longrightarrow \text{Sets}, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{étale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{ph}$.

Proof. The statement about the étale topology is the content of Lemma 59.39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the ph topology. By Topologies, Lemma 34.8.15 it suffices to show that given a proper surjective morphism $V \to U$ of schemes over S we have an equalizer diagram

$$(\pi_S^{-1}\mathcal{F})(U) \longrightarrow (\pi_S^{-1}\mathcal{F})(V) \rightrightarrows (\pi_S^{-1}\mathcal{F})(V \times_U V)$$

Set $\mathcal{G} = \pi_S^{-1}\mathcal{F}|_{U_{étale}}$. Consider the commutative diagram

$$\begin{tikzcd} V \times_U V \arrow[r] \arrow[d] \arrow[dr, "g"] & V \arrow[d, "f"] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

We have

$$(\pi_S^{-1}\mathcal{F})(V) = \Gamma(V, f^{-1}\mathcal{G}) = \Gamma(U, f_*f^{-1}\mathcal{G})$$

where we use f_* and f^{-1} to denote functorialities between small étale sites. Second, we have

$$(\pi_S^{-1}\mathcal{F})(V \times_U V) = \Gamma(V \times_U V, g^{-1}\mathcal{G}) = \Gamma(U, g_*g^{-1}\mathcal{G})$$

The two maps in the equalizer diagram come from the two maps

$$f_*f^{-1}\mathcal{G} \longrightarrow g_*g^{-1}\mathcal{G}$$

Thus it suffices to prove $\mathcal{G}$ is the equalizer of these two maps of sheaves. Let $\overline{u}$ be a geometric point of U. Set $\Omega = \mathcal{G}_{\overline{u}}$. Taking stalks at $\overline{u}$ by Lemma 59.91.4 we obtain the two maps

$$H^0(V_{\overline{u}}, \underline{\Omega}) \longrightarrow H^0((V \times_U V)_{\overline{u}}, \underline{\Omega}) = H^0(V_{\overline{u}} \times_{\overline{u}} V_{\overline{u}}, \underline{\Omega})$$

where $\underline{\Omega}$ indicates the constant sheaf with value Ω. Of course these maps are the pullback by the projection maps. Then it is clear that the sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection, and sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection. The sections in the intersection of the images of these pullback maps are constant on all of $V_{\overline{u}} \times_{\overline{u}} V_{\overline{u}}$, i.e., these come from elements of Ω as desired. □

In the situation of Lemma 59.102.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_{ph} \longrightarrow S_{étale}$$

0DDX Lemma 59.102.2. With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_{ph}$. Then there are commutative diagrams of topoi

$$\begin{tikzcd} Sh((Sch/X)_{ph}) \arrow[r, "f_{big,ph}"'] \arrow[d, "\epsilon_X"'] & Sh((Sch/Y)_{ph}) \arrow[d, "\epsilon_Y"] \\ Sh((Sch/X)_{étale}) \arrow[r, "f_{big,étale}"] & Sh((Sch/Y)_{étale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((Sch/X)_{ph}) \arrow[r, "f_{big,ph}"'] \arrow[d, "a_X"'] & Sh((Sch/Y)_{ph}) \arrow[d, "a_Y"] \\ Sh(X_{étale}) \arrow[r, "f_{small}"] & Sh(Y_{étale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 34.11. □

0DDY Lemma 59.102.3. In Lemma 59.102.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,ph,*} \circ a_X^{-1}$.

Proof. You can prove this by repeating the proof of Lemma 59.99.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 59.102.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,ph,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,ph,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 59.99.5. □

0F0I Lemma 59.102.4. Consider the comparison morphism $\epsilon : (Sch/S)_{ph} \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of proper morphisms of schemes. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset \text{Ab}((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$ Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 21.30.1 hold.

Proof. We first show that $\mathcal{A}'_X \subset \text{Ab}((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 59.99.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $\text{Ab}((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 59.99.4. In particular we see that $\pi_{X,*}\mathcal{G}$ is an abelian torsion sheaf on $X_{étale}$. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 29.41.5.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 34.4.17.

Cohomology on Sites, Property (3) is Lemma 59.102.1.

Cohomology on Sites, Property (4) holds by Lemma 59.99.5 part (2) and the fact that $R^i f_{small}\mathcal{F}$ is torsion if $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$, see Lemma 59.78.2.

Cohomology on Sites, Property (5) follows from More on Morphisms, Lemma 37.48.1 combined with the fact that a finite morphism is proper and a surjective proper morphism defines a ph covering, see Topologies, Lemma 34.8.6. □

0DE4 Lemma 59.102.5. With notation as above.

(1) For $X \in \mathrm{Ob}((Sch/S)_{ph})$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.
(2) For a proper morphism $f : X \to Y$ in $(Sch/S)_{ph}$ and abelian torsion sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,ph,*}(a_X^{-1}\mathcal{F})$ for all i.
(3) For a scheme X and K in $D^+(X_{étale})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.
(4) For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$.

Proof. By Lemma 59.102.4 the lemmas in Cohomology on Sites, Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 21.30.2 is the full subcategory of $\mathrm{Ab}((Sch/X)_{ph})$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 21.30.3.

Part (3) follows from Cohomology on Sites, Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 21.30.8 part (2) for the same reason. □

0DE5 Lemma 59.102.6. Let X be a scheme. For $K \in D^+(X_{étale})$ with torsion cohomology sheaves the map

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_{ph}) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 21.19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 59.102.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 59.102.5 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 59.99.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DE6 Lemma 59.102.7. For a scheme X and $a_X : Sh((Sch/X)_{ph}) \to Sh(X_{étale})$ as above:

(1) $H^q(X_{étale}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$,

(2) $H^q(X_{étale}, K) = H^q_{ph}(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$ with torsion cohomology sheaves.

Example: if A is a torsion abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{ph}(X, \underline{A})$.

Proof. This follows from Lemma 59.102.6 by Cohomology on Sites, Remark 21.14.4. □

59.103. Comparing h and étale topologies

0EW7 A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 21.31. Moreover, this section is almost word for word the same as the section comparing the ph and étale topologies. We first review some material from Topologies, Sections 34.11 and 34.4 and More on Flatness, Section 38.34.

Let S be a scheme and let $(Sch/S)_h$ be an h site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_h$ is continuous (by More on Flatness, Lemma 38.34.6 and Topologies, Lemma 34.7.2) and defines a morphism of sites

$$\epsilon_S : (Sch/S)_h \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 21.27. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the h sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies.

0EW8 Lemma 59.103.1. With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. The rule

$$(Sch/S)_h \longrightarrow \text{Sets}, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{étale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_h$.

Proof. The statement about the étale topology is the content of Lemma 59.39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the h topology. However, in Lemma 59.102.1 we have shown that $\pi_S^{-1}\mathcal{F}$ is a sheaf even in the stronger ph topology. □

In the situation of Lemma 59.103.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_h \longrightarrow S_{étale}$$

0EW9 Lemma 59.103.2. With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_h$. Then there are commutative diagrams of topoi

$$\begin{tikzcd} Sh((Sch/X)_h) \arrow[r, "f_{big,h}"'] \arrow[d, "\epsilon_X"'] & Sh((Sch/Y)_h) \arrow[d, "\epsilon_Y"] \\ Sh((Sch/X)_{étale}) \arrow[r, "f_{big,étale}"] & Sh((Sch/Y)_{étale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((Sch/X)_h) \arrow[r, "f_{big,h}"'] \arrow[d, "a_X"'] & Sh((Sch/Y)_h) \arrow[d, "a_Y"] \\ Sh(X_{étale}) \arrow[r, "f_{small}"] & Sh(Y_{étale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows similarly to what was said in Topologies, Section 34.11. □

0EWA Lemma 59.103.3. In Lemma 59.103.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,h,*} \circ a_X^{-1}$.

Proof. You can prove this by repeating the proof of Lemma 59.99.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 59.103.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,h,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,h,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 59.99.5. □

0F0J Lemma 59.103.4. Consider the comparison morphism $\epsilon : (Sch/S)_h \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of proper morphisms. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset \text{Ab}((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$ Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 21.30.1 hold.

Proof. We first show that $\mathcal{A}'_X \subset \text{Ab}((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 59.99.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $\text{Ab}((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 59.99.4. In particular we see that $\pi_{X,*}\mathcal{G}$ is an abelian torsion sheaf on $X_{étale}$. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes, the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 29.41.5, and the fact that the base change of a morphism of finite presentation is a morphism of finite presentation, see Morphisms, Lemma 29.21.4.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 34.4.17.

Cohomology on Sites, Property (3) is Lemma 59.103.1.

Cohomology on Sites, Property (4) holds by Lemma 59.99.5 part (2) and the fact that $R^i f_{small}\mathcal{F}$ is torsion if $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$, see Lemma 59.78.2.

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 37.48.1 combined with the fact that a surjective finite locally free morphism is surjective, proper, and of finite presentation and hence defines a h-covering by More on Flatness, Lemma 38.34.7. ☐

0EWF Lemma 59.103.5. With notation as above.

(1) For $X \in \mathrm{Ob}((Sch/S)_h)$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.
(2) For a proper morphism $f : X \to Y$ in $(Sch/S)_h$ and abelian torsion sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,h,*}(a_X^{-1}\mathcal{F})$ for all i.
(3) For a scheme X and K in $D^+(X_{étale})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.
(4) For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,h,*}(a_X^{-1}K)$.

Proof. By Lemma 59.103.4 the lemmas in Cohomology on Sites, Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 21.30.2 is the full subcategory of $\mathrm{Ab}((Sch/X)_h)$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 21.30.3.

Part (3) follows from Cohomology on Sites, Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 21.30.8 part (2) for the same reason. ☐

0EWG Lemma 59.103.6. Let X be a scheme. For $K \in D^+(X_{étale})$ with torsion cohomology sheaves the map

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_h) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 21.19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 59.103.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 59.103.5 we have

$R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 59.99.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0EWH Lemma 59.103.7. For a scheme X and $a_X : Sh((Sch/X)_h) \to Sh(X_{\acute{e}tale})$ as above:

(1) $H^q(X_{\acute{e}tale}, \mathcal{F}) = H^q_h(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,
(2) $H^q(X_{\acute{e}tale}, K) = H^q_h(X, a_X^{-1}K)$ for $K \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves.

Example: if A is a torsion abelian group, then $H^q_{\acute{e}tale}(X, \underline{A}) = H^q_h(X, \underline{A})$.

Proof. This follows from Lemma 59.103.6 by Cohomology on Sites, Remark 21.14.4. □

59.104. Descending étale sheaves

0GEX We prove that étale sheaves "glue" in the fppf and h topology and related results. We have already shown the following related results

(1) Lemma 59.39.2 tells us that a sheaf on the small étale site of a scheme S determines a sheaf on the big étale site of S satisfying the sheaf condition for fpqc coverings (and a fortiori for Zariski, étale, smooth, syntomic, and fppf coverings),
(2) Lemma 59.100.1 is a restatement of the previous point for the fppf topology,
(3) Lemma 59.102.1 proves the same for the ph topology,
(4) Lemma 59.103.1 proves the same for the h topology,
(5) Lemma 59.100.4 is a version of fppf descent for étale sheaves, and
(6) Remark 59.55.6 tells us that we have descent of étale sheaves for finite surjective morphisms (we will clarify and strengthen this below).

In the chapter on simplicial spaces we will prove some additional results on this, see for example Simplicial Spaces, Sections 85.33 and 85.36.

In order to conveniently express our results we need some notation. Let $\mathcal{U} = \{f_i : X_i \to X\}$ be a family of morphisms of schemes with fixed target. A descent datum for étale sheaves with respect to $\mathcal{U}$ is a family $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ where

(1) $\mathcal{F}_i$ is in $Sh(X_{i,\acute{e}tale})$, and
(2) $\varphi_{ij} : \text{pr}_{0,small}^{-1}\mathcal{F}_i \longrightarrow \text{pr}_{1,small}^{-1}\mathcal{F}_j$ is an isomorphism in $Sh((X_i \times_X X_j)_{\acute{e}tale})$

such that the cocycle condition holds: the diagrams

$$\begin{tikzcd} \text{pr}_{0,small}^{-1}\mathcal{F}_i \arrow[rr, "\text{pr}_{01,small}^{-1}\varphi_{ij}"] \arrow[dr, "\text{pr}_{02,small}^{-1}\varphi_{ik}"'] & & \text{pr}_{1,small}^{-1}\mathcal{F}_j \arrow[dl, "\text{pr}_{12,small}^{-1}\varphi_{jk}"] \\ & \text{pr}_{2,small}^{-1}\mathcal{F}_k & \end{tikzcd}$$

commute in $Sh((X_i \times_X X_j \times_X X_k)_{\acute{e}tale})$. There is an obvious notion of morphisms of descent data and we obtain a category of descent data. A descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is called effective if there exist a $\mathcal{F}$ in $Sh(X_{\acute{e}tale})$ and isomorphisms $\varphi_i : f_{i,small}^{-1}\mathcal{F} \to \mathcal{F}_i$ in $Sh(X_{i,\acute{e}tale})$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \text{pr}_{1,small}^{-1}(\varphi_j) \circ \text{pr}_{0,small}^{-1}(\varphi_i^{-1})$$

Another way to say this is the following. Given an object $\mathcal{F}$ of $Sh(X_{étale})$ we obtain the canonical descent datum $(f_{i,small}^{-1}\mathcal{F}_i, c_{ij})$ where c_{ij} is the canonical isomorphism

$$c_{ij} : \mathrm{pr}_{0,small}^{-1} f_{i,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1} f_{j,small}^{-1}\mathcal{F}$$

The descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is effective if and only if it is isomorphic to the canonical descent datum associated to some $\mathcal{F}$ in $Sh(X_{étale})$.

If the family consists of a single morphism $\{X \to Y\}$, then we think of a descent datum as a pair $(\mathcal{F}, \varphi)$ where $\mathcal{F}$ is an object of $Sh(X_{étale})$ and φ is an isomorphism

$$\mathrm{pr}_{0,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}$$

in $Sh((X \times_Y X)_{étale})$ such that the cocycle condition holds:

$$\begin{tikzcd}
\mathrm{pr}_{0,small}^{-1}\mathcal{F} \arrow[rr, "\mathrm{pr}_{01,small}^{-1}\varphi"] \arrow[dr, "\mathrm{pr}_{02,small}^{-1}\varphi"'] & & \mathrm{pr}_{1,small}^{-1}\mathcal{F} \arrow[dl, "\mathrm{pr}_{12,small}^{-1}\varphi"] \\
& \mathrm{pr}_{2,small}^{-1}\mathcal{F} &
\end{tikzcd}$$

commutes in $Sh((X \times_Y X \times_Y X)_{étale})$. There is a notion of morphisms of descent data and effectivity exactly as before.

We first prove effective descent for surjective integral morphisms.

0GEY Lemma 59.104.1. Let $f : X \to Y$ be a morphism of schemes which has a section. Then the functor

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

sending $\mathcal{G}$ in $Sh(Y_{étale})$ to the canonical descent datum is an equivalence of categories.

Proof. This is formal and depends only on functoriality of the pullback functors. We omit the details. Hint: If $s : Y \to X$ is a section, then a quasi-inverse is the functor sending $(\mathcal{F}, \varphi)$ to $s_{small}^{-1}\mathcal{F}$. $\square$

0GEZ Lemma 59.104.2. Let $f : X \to Y$ be a surjective integral morphism of schemes. The functor

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. In this proof we drop the subscript $_{small}$ from our pullback and pushforward functors. Denote $X_1 = X \times_Y X$ and denote $f_1 : X_1 \to Y$ the morphism $f \circ \mathrm{pr}_0 = f \circ \mathrm{pr}_1$. Let $(\mathcal{F}, \varphi)$ be a descent datum for $\{X \to Y\}$. Let us set $\mathcal{F}_1 = \mathrm{pr}_0^{-1}\mathcal{F}$. We may think of φ as defining an isomorphism $\mathcal{F}_1 \to \mathrm{pr}_1^{-1}\mathcal{F}$. We claim that the rule which sends a descent datum $(\mathcal{F}, \varphi)$ to the sheaf

$$\mathcal{G} = \text{Equalizer}\left(f_*\mathcal{F} \rightrightarrows f_{1,*}\mathcal{F}_1 \right)$$

is a quasi-inverse to the functor in the statement of the lemma. The first of the two arrows comes from the map

$$f_*\mathcal{F} \to f_*\mathrm{pr}_{0,*}\mathrm{pr}_0^{-1}\mathcal{F} = f_{1,*}\mathcal{F}_1$$

and the second arrow comes from the map

$$f_*\mathcal{F} \to f_*\mathrm{pr}_{1,*}\mathrm{pr}_1^{-1}\mathcal{F} \xleftarrow{\varphi} f_*\mathrm{pr}_{0,*}\mathrm{pr}_0^{-1}\mathcal{F} = f_{1,*}\mathcal{F}_1$$

where the arrow pointing left is invertible. To prove this works we have to show that the canonical map $f^{-1}\mathcal{G} \to \mathcal{F}$ is an isomorphism; details omitted. In order to prove this it suffices to check after pulling back by any collection of morphisms $\mathrm{Spec}(k) \to Y$ where k is an algebraically closed field. Namely, the corresponing base changes $X_k \to X$ are jointly surjective and we can check whether a map of sheaves on $X_{étale}$ is an isomorphism by looking at stalks on geometric points, see Theorem 59.29.10. By Lemma 59.55.4 the construction of $\mathcal{G}$ from the descent datum $(\mathcal{F}, \varphi)$ commutes with any base change. Thus we may assume Y is the spectrum of an algebraically closed point (note that base change preserves the properties of the morphism f, see Morphisms, Lemma 29.9.4 and 29.44.6). In this case the morphism $X \to Y$ has a section, so we know that the functor is an equivalence by Lemma 59.104.1. However, the reader may show that the functor is an equivalence if and only if the construction above is a quasi-inverse; details omitted. This finishes the proof. □

0GF0 Lemma 59.104.3. Let $f : X \to Y$ be a surjective proper morphism of schemes. The functor

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. The exact same proof as given in Lemma 59.104.2 works, except the appeal to Lemma 59.55.4 should be replaced by an appeal to Lemma 59.91.5. □

0GF1 Lemma 59.104.4. Let $f : X \to Y$ be a morphism of schemes. Let $Z \to Y$ be a surjective integral morphism of schemes or a surjective proper morphism of schemes. If the functors

$$Sh(Z_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Z \to Z\}$$

and

$$Sh((Z \times_Y Z)_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y (Z \times_Y Z) \to Z \times_Y Z\}$$

are equivalences of categories, then

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence.

Proof. Formal consequence of the definitions and Lemmas 59.104.2 and 59.104.3. Details omitted. □

0GF2 Lemma 59.104.5. Let $f : X \to Y$ be a morphism of schemes which is surjective, flat, locally of finite presentation. The functor

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Exactly as in the proof of Lemma 59.104.2 we claim a quasi-inverse is given by the functor sending $(\mathcal{F}, \varphi)$ to

$$\mathcal{G} = \mathrm{Equalizer}\left(f_*\mathcal{F} \rightrightarrows f_{1,*}\mathcal{F}_1 \right)$$

and in order to prove this it suffices to show that $f^{-1}\mathcal{G} \to \mathcal{F}$ is an isomorphism. This we may check locally, hence we may and do assume Y is affine. Then we can find a finite surjective morphism $Z \to Y$ such that there exists an open covering $Z = \bigcup W_i$ such that $W_i \to Y$ factors through X. See More on Morphisms, Lemma 37.48.6. Applying Lemma 59.104.4 we see that it suffices to prove the lemma after replacing Y by Z and $Z \times_Y Z$ and f by its base change. Thus we may assume f has sections Zariski locally. Of course, using that the problem is local on Y we reduce to the case where we have a section which is Lemma 59.104.1. ☐

0GF3 Lemma 59.104.6. Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. The functor

$$Sh(X_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. We have Lemma 59.104.5 for the morphism $f : \coprod X_i \to X$. Then a formal argument shows that descent data for f are the same thing as descent data for the covering, compare with Descent, Lemma 35.34.5. Details omitted. ☐

0GF4 Lemma 59.104.7. Let $f : X' \to X$ be a proper morphism of schemes. Let $i : Z \to X$ be a closed immersion. Set $E = Z \times_X X'$. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

If f is an isomorphism over $X \setminus Z$, then the functor

$$Sh(X_{\acute{e}tale}) \longrightarrow Sh(X'_{\acute{e}tale}) \times_{Sh(E_{\acute{e}tale})} Sh(Z_{\acute{e}tale})$$

is an equivalence of categories.

Proof. We will work with the 2-fibre product category as constructed in Categories, Example 4.31.3. The functor sends $\mathcal{F}$ to the triple $(f^{-1}\mathcal{F}, i^{-1}\mathcal{F}, c)$ where $c : j^{-1}f^{-1}\mathcal{F} \to g^{-1}i^{-1}\mathcal{F}$ is the canonical isomorphism. We will construct a quasi-inverse functor. Let $(\mathcal{F}', \mathcal{G}, \alpha)$ be an object of the right hand side of the arrow. We obtain an isomorphism

$$i^{-1}f_*\mathcal{F}' = g_*j^{-1}\mathcal{F}' \xrightarrow{g_*\alpha} g_*g^{-1}\mathcal{G}$$

The first equality is Lemma 59.91.5. Using this we obtain maps $i_*\mathcal{G} \to i_*g_*g^{-1}\mathcal{G}$ and $f'_*\mathcal{F}' \to i_*g_*g^{-1}\mathcal{G}$. We set

$$\mathcal{F} = f_*\mathcal{F}' \times_{i_*g_*g^{-1}\mathcal{G}} i_*\mathcal{G}$$

and we claim that $\mathcal{F}$ is an object of the left hand side of the arrow whose image in the right hand side is isomorphic to the triple we started out with. Let us compute the stalk of $\mathcal{F}$ at a geometric point $\overline{x}$ of X.

If $\overline{x}$ is not in Z, then on the one hand $\overline{x}$ comes from a unique geometric point $\overline{x}'$ of X' and $\mathcal{F}'_{\overline{x}'} = (f_*\mathcal{F}')_{\overline{x}}$ and on the other hand we have $(i_*\mathcal{G})_{\overline{x}}$ and $(i_*g_*g^{-1}\mathcal{G})_{\overline{x}}$ are singletons. Hence we see that $\mathcal{F}_{\overline{x}}$ equals $\mathcal{F}'_{\overline{x}'}$.

If $\overline{x}$ is in Z, i.e., $\overline{x}$ is the image of a geometric point $\overline{z}$ of Z, then we obtain $(i_*\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{z}}$ and

$$(i_*g_*g^{-1}\mathcal{G})_{\overline{x}} = (g_*g^{-1}\mathcal{G})_{\overline{z}} = \Gamma(E_{\overline{z}}, g^{-1}\mathcal{G}|_{E_{\overline{z}}})$$

(by the proper base change for pushforward used above) and similarly

$$(f_*\mathcal{F}')_{\overline{x}} = \Gamma(X'_{\overline{x}}, \mathcal{F}'|_{X'_{\overline{x}}})$$

Since we have the identification $E_{\overline{z}} = X'_{\overline{x}}$ and since α defines an isomorphism between the sheaves $\mathcal{F}'|_{X'_{\overline{x}}}$ and $g^{-1}\mathcal{G}|_{E_{\overline{z}}}$ we conclude that we get

$$\mathcal{F}_{\overline{x}} = \mathcal{G}_{\overline{z}}$$

in this case.

To finish the proof, we observe that there are canonical maps $i^{-1}\mathcal{F} \to \mathcal{G}$ and $f^{-1}\mathcal{F} \to \mathcal{F}'$ compatible with α which on stalks produce the isomorphisms we saw above. We omit the careful construction of these maps. □

0GF5 Lemma 59.104.8. Let S be a scheme. Then the category fibred in groupoids

$$p : \mathcal{S} \longrightarrow (Sch/S)_h$$

whose fibre category over U is the category $Sh(U_{étale})$ of sheaves on the small étale site of U is a stack in groupoids.

Proof. To prove the lemma we will check conditions (1), (2), and (3) of More on Flatness, Lemma 38.37.13.

Condition (1) holds because we have glueing for sheaves (and Zariski coverings are étale coverings). See Sites, Lemma 7.26.4.

To see condition (2), suppose that $f : X \to Y$ is a surjective, flat, proper morphism of finite presentation over S with Y affine. Then we have descent for $\{X \to Y\}$ by either Lemma 59.104.5 or Lemma 59.104.3.

Condition (3) follows immediately from the more general Lemma 59.104.7. □

59.105. Blow up squares and étale cohomology

0EW4 Blow up squares are introduced in More on Flatness, Section 38.36. Using the proper base change theorem we can see that we have a Mayer-Vietoris type result for blow up squares.

0EW5 Lemma 59.105.1. Let X be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

For $K \in D^+(X_{étale})$ with torsion cohomology sheaves we have a distinguished triangle

$$K \to Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \to Rc_*(K|_E) \to K[1]$$

in $D(X_{étale})$ where $c = i \circ \pi = b \circ j$.

Proof. The notation $K|_{X'}$ stands for $b_{small}^{-1}K$. Choose a bounded below complex $\mathcal{F}^\bullet$ of abelian sheaves representing K. Observe that $i_*(\mathcal{F}^\bullet|_Z)$ represents $Ri_*(K|_Z)$ because i_* is exact (Proposition 59.55.2). Choose a quasi-isomorphism $b_{small}^{-1}\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injective abelian sheaves on $X'_{étale}$. This map is adjoint to a map $\mathcal{F}^\bullet \to b_*(\mathcal{I}^\bullet)$ and $b_*(\mathcal{I}^\bullet)$ represents $Rb_*(K|_{X'})$. We have $\pi_*(\mathcal{I}^\bullet|_E) = (b_*\mathcal{I}^\bullet)|_Z$ by Lemma 59.91.5 and by Lemma 59.91.12 this complex represents $R\pi_*(K|_E)$. Hence the map

$$Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \to Rc_*(K|_E)$$

is represented by the surjective map of bounded below complexes

$$i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet) \to i_*\left(b_*(\mathcal{I}^\bullet)|_Z\right)$$

To get our distinguished triangle it suffices to show that the canonical map $\mathcal{F}^\bullet \to i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet)$ maps quasi-isomorphically onto the kernel of the map of complexes displayed above (namely a short exact sequence of complexes determines a distinguished triangle in the derived category, see Derived Categories, Section 13.12). We may check this on stalks at a geometric point $\overline{x}$ of X. If $\overline{x}$ is not in Z, then $X' \to X$ is an isomorphism over an open neighbourhood of $\overline{x}$. Thus, if $\overline{x}'$ denotes the corresponding geometric point of X' in this case, then we have to show that

$$\mathcal{F}^\bullet_{\overline{x}} \to \mathcal{I}^\bullet_{\overline{x}'}$$

is a quasi-isomorphism. This is true by our choice of $\mathcal{I}^\bullet$. If $\overline{x}$ is in Z, then $b_(\mathcal{I}^\bullet)_{\overline{x}} \to i_*\left(b_*(\mathcal{I}^\bullet)|_Z\right)_{\overline{x}}$ is an isomorphism of complexes of abelian groups. Hence the kernel is equal to $i_*(\mathcal{F}^\bullet|_Z)_{\overline{x}} = \mathcal{F}^\bullet_{\overline{x}}$ as desired. □

0EW3 Lemma 59.105.2. Let X be a scheme and let $K \in D^+(X_{étale})$ have torsion cohomology sheaves. Let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d, "b"] \\ Z \arrow[r] & X \end{tikzcd}$$

Then there is a canonical long exact sequence

$$H^p_{étale}(X, K) \to H^p_{étale}(X', K|_{X'}) \oplus H^p_{étale}(Z, K|_Z) \to H^p_{étale}(E, K|_E) \to H^{p+1}_{étale}(X, K)$$

First proof. This follows immediately from Lemma 59.105.1 and the fact that

$$R\Gamma(X, Rb_*(K|_{X'})) = R\Gamma(X', K|_{X'})$$

(see Cohomology on Sites, Section 21.14) and similarly for the others. □

Second proof. By Lemma 59.102.7 these cohomology groups are the cohomology of X, X', E, Z with values in some complex of abelian sheaves on the site $(Sch/X)_{ph}$. (Namely, the object $a_X^{-1}K$ of the derived category, see Lemma 59.102.1 above and recall that $K|_{X'} = b_{small}^{-1}K$.) By More on Flatness, Lemma 38.36.1 the ph sheafification of the diagram of representable presheaves is cocartesian. Thus the lemma follows from the very general Cohomology on Sites, Lemma 21.26.3 applied to the site $(Sch/X)_{ph}$ and the commutative diagram of the lemma. □

0EW6 Lemma 59.105.3. Let X be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square

$$\begin{tikzcd} E \arrow[r, "j"'] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Suppose given

(1) an object K' of $D^+(X'_{étale})$ with torsion cohomology sheaves,
(2) an object L of $D^+(Z_{étale})$ with torsion cohomology sheaves, and
(3) an isomorphism $\gamma : K'|_E \to L|_E$.

Then there exists an object K of $D^+(X_{étale})$ and isomorphisms $f : K|_{X'} \to K'$, $g : K|_Z \to L$ such that $\gamma = g|_E \circ f^{-1}|_E$. Moreover, given

(1) an object M of $D^+(X_{étale})$ with torsion cohomology sheaves,
(2) a morphism $\alpha : K' \to M|_{X'}$ of $D(X'_{étale})$,
(3) a morphism $\beta : L \to M|_Z$ of $D(Z_{étale})$,

such that

$$\alpha|_E = \beta|_E \circ \gamma.$$

Then there exists a morphism $M \to K$ in $D(X_{étale})$ whose restriction to X' is $a \circ f$ and whose restriction to Z is $b \circ g$.

Proof. If K exists, then Lemma 59.105.1 tells us a distinguished triangle that it fits in. Thus we simply choose a distinguished triangle

$$K \to Ri_*(L) \oplus Rb_*(K') \to Rc_*(L|_E) \to K[1]$$

where $c = i \circ \pi = b \circ j$. Here the map $Ri_*(L) \to Rc_*(L|_E)$ is Ri_* applied to the adjunction mapping $E \to R\pi_*(L|_E)$. The map $Rb_*(K') \to Rc_*(L|_E)$ is the composition of the canonical map $Rb_*(K') \to Rc_*(K'|_E)) = R$ and $Rc_*(\gamma)$. The maps g and f of the statement of the lemma are the adjoints of these maps. If we restrict this distinguished triangle to Z then the map $Rb_*(K) \to Rc_*(L|_E)$ becomes an isomorphism by the base change theorem (Lemma 59.91.12) and hence the map $g : K|_Z \to L$ is an isomorphism. Looking at the distinguished triangle we see that $f : K|_{X'} \to K'$ is an isomorphism over $X' \setminus E = X \setminus Z$. Moreover, we have $\gamma \circ f|_E = g|_E$ by construction. Then since γ and g are isomorphisms we conclude that f induces isomorphisms on stalks at geometric points of E as well. Thus f is an isomorphism.

For the final statement, we may replace K' by $K|_{X'}$, L by $K|_Z$, and γ by the canonical identification. Observe that α and β induce a commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d, dotted] & Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \arrow[r] \arrow[d, "\beta\oplus\alpha"'] & Rc_*(K|_E) \arrow[r] \arrow[d, "\alpha|_E"'] & K[1] \arrow[d, dotted] \\ M \arrow[r] & Ri_*(M|_Z) \oplus Rb_*(M|_{X'}) \arrow[r] & Rc_*(M|_E) \arrow[r] & M[1] \end{tikzcd}$$

Thus by the axioms of a derived category we get a dotted arrow producing a morphism of distinguished triangles. ☐

59.106. Almost blow up squares and the h topology

0EWL In this section we continue the discussion in More on Flatness, Section 38.37. For the convenience of the reader we recall that an almost blow up square is a commutative diagram

0EWM (59.106.0.1)
$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d, "b"] \\ Z \arrow[r] & X \end{tikzcd}$$

of schemes satisfying the following conditions:

(1) $Z \to X$ is a closed immersion of finite presentation,
(2) $E = b^{-1}(Z)$ is a locally principal closed subscheme of X',
(3) b is proper and of finite presentation,
(4) the closed subscheme $X'' \subset X'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$ supported on E (Properties, Lemma 28.24.5) is the blow up of X in Z.

It follows that the morphism b induces an isomorphism $X' \setminus E \to X \setminus Z$.

We are going to give a criterion for "h sheafiness" for objects in the derived category of the big fppf site $(Sch/S)_{fppf}$ of a scheme S. On the same underlying category we have a second topology, namely the h topology (More on Flatness, Section 38.34). Recall that fppf coverings are h coverings (More on Flatness, Lemma 38.34.6). Hence we may consider the morphism

$$\epsilon : (Sch/S)_h \longrightarrow (Sch/S)_{fppf}$$

See Cohomology on Sites, Section 21.27. In particular, we have a fully faithful functor

$$R\epsilon_* : D((Sch/S)_h) \longrightarrow D((Sch/S)_{fppf})$$

and we can ask: what is the essential image of this functor?

0EWN Lemma 59.106.1. With notation as above, if K is in the essential image of $R\epsilon_*$, then the maps $c^K_{X,Z,X',E}$ of Cohomology on Sites, Lemma 21.26.1 are quasi-isomorphisms.

Proof. Denote $^\#$ sheafification in the h topology. We have seen in More on Flatness, Lemma 38.37.7 that $h_X^\# = h_Z^\# \amalg_{h_E^\#} h_{X'}^\#$. On the other hand, the map $h_E^\# \to h_{X'}^\#$ is injective as $E \to X'$ is a monomorphism. Thus this lemma is a special case of Cohomology on Sites, Lemma 21.29.3 (which itself is a formal consequence of Cohomology on Sites, Lemma 21.26.3). $\square$

0EWQ Proposition 59.106.2. Let K be an object of $D^+((Sch/S)_{fppf})$. Then K is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$ if and only if $c^K_{X,X',Z,E}$ is a quasi-isomorphism for every almost blow up square (59.106.0.1) in $(Sch/S)_h$ with X affine.

Proof. We prove this by applying Cohomology on Sites, Lemma 21.29.2 whose hypotheses hold by Lemma 59.106.1 and More on Flatness, Proposition 38.37.9. $\square$

0EWR Lemma 59.106.3. Let K be an object of $D^+((Sch/S)_{fppf})$. Then K is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$ if and only if $c^K_{X,X',Z,E}$ is a quasi-isomorphism for every almost blow up square as in More on Flatness, Examples 38.37.10 and 38.37.11.

Proof. We prove this by applying Cohomology on Sites, Lemma 21.29.2 whose hypotheses hold by Lemma 59.106.1 and More on Flatness, Lemma 38.37.12 $\square$

59.107. Cohomology of the structure sheaf in the h topology

0EWS Let p be a prime number. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site with $p\mathcal{O} = 0$. Then we set $\operatorname{colim}_F \mathcal{O}$ equal to the colimit in the category of sheaves of rings of the system

$$\mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \ldots$$

where $F : \mathcal{O} \to \mathcal{O}$, $f \mapsto f^p$ is the Frobenius endomorphism.

0EWT Lemma 59.107.1. Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Consider the sheaf $\mathcal{O}^{perf} = \text{colim}_F \mathcal{O}$ on $(Sch/S)_{fppf}$. Then $\mathcal{O}^{perf}$ is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$.

Proof. We prove this using the criterion of Lemma 59.106.3. Before check the conditions, we note that for a quasi-compact and quasi-separated object X of $(Sch/S)_{fppf}$ we have

$$H^i_{fppf}(X, \mathcal{O}^{perf}) = \text{colim}_F H^i_{fppf}(X, \mathcal{O})$$

See Cohomology on Sites, Lemma 21.16.1. We will also use that $H^i_{fppf}(X, \mathcal{O}) = H^i(X, \mathcal{O})$, see Descent, Proposition 35.9.3.

Let A, f, J be as in More on Flatness, Example 38.37.10 and consider the associated almost blow up square. Since X, X', Z, E are affine, we have no higher cohomology of $\mathcal{O}$. Hence we only have to check that

$$0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 38.38.2.

Let X, X', Z, E be as in More on Flatness, Example 38.37.11. Since X and Z are affine we have $H^p(X, \mathcal{O}_X) = H^p(Z, \mathcal{O}_X) = 0$ for $p > 0$. By More on Flatness, Lemma 38.38.1 we have $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$. Since $E = \mathbf{P}^1_Z$ and Z is affine we also have $H^p(E, \mathcal{O}_E) = 0$ for $p > 0$. As in the previous paragraph we reduce to checking that

$$0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 38.38.2. □

0EWU Proposition 59.107.2. Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. Then

$$H^i((Sch/S)_h, \mathcal{O}^h) = \text{colim}_F H^i(S, \mathcal{O})$$

Here on the left hand side by $\mathcal{O}^h$ we mean the h sheafification of the structure sheaf.

Proof. This is just a reformulation of Lemma 59.107.1. Recall that $\mathcal{O}^h = \mathcal{O}^{perf} = \text{colim}_F \mathcal{O}$, see More on Flatness, Lemma 38.38.7. By Lemma 59.107.1 we see that $\mathcal{O}^{perf}$ viewed as an object of $D((Sch/S)_{fppf})$ is of the form $R\epsilon_* K$ for some $K \in D((Sch/S)_h)$. Then $K = \epsilon^{-1}\mathcal{O}^{perf}$ which is actually equal to $\mathcal{O}^{perf}$ because $\mathcal{O}^{perf}$ is an h sheaf. See Cohomology on Sites, Section 21.27. Hence $R\epsilon_*\mathcal{O}^{perf} = \mathcal{O}^{perf}$ (with apologies for the confusing notation). Thus the lemma now follows from Leray

$$R\Gamma_h(S, \mathcal{O}^{perf}) = R\Gamma_{fppf}(S, R\epsilon_*\mathcal{O}^{perf}) = R\Gamma_{fppf}(S, \mathcal{O}^{perf})$$

and the fact that

$$H^i_{fppf}(S, \mathcal{O}^{perf}) = H^i_{fppf}(S, \text{colim}_F \mathcal{O}) = \text{colim}_F H^i_{fppf}(S, \mathcal{O})$$

as S is quasi-compact and quasi-separated (see proof of Lemma 59.107.1). □

59.108. Other chapters

CHAPTER 60

Crystalline Cohomology

07GI
60.1. Introduction

07GJ This chapter is based on a lecture series given by Johan de Jong held in 2012 at Columbia University. The goals of this chapter are to give a quick introduction to crystalline cohomology. A reference is the book [Ber74].

We have moved the more elementary purely algebraic discussion of divided power rings to a preliminary chapter as it is also useful in discussing Tate resolutions in commutative algebra. Please see Divided Power Algebra, Section 23.1.

60.2. Divided power envelope

07H7 The construction of the following lemma will be dubbed the divided power envelope. It will play an important role later.

07H8 Lemma 60.2.1. Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. There exists a homomorphism of divided power rings

$$(A, I, \gamma) \longrightarrow (D, \bar{J}, \bar{\gamma})$$

such that

$$\mathrm{Hom}_{(A,I,\gamma)}((D, \bar{J}, \bar{\gamma}), (C, K, \delta)) = \mathrm{Hom}_{(A,I)}((B, J), (C, K))$$

functorially in the divided power algebra (C, K, δ) over (A, I, γ). Here the LHS is morphisms of divided power rings over (A, I, γ) and the RHS is morphisms of (ring, ideal) pairs over (A, I).

Proof. Denote $\mathcal{C}$ the category of divided power rings (C, K, δ). Consider the functor $F : \mathcal{C} \longrightarrow \textit{Sets}$ defined by

$$F(C, K, \delta) = \left\{ (\varphi, \psi) \middle| \begin{array}{c} \varphi : (A, I, \gamma) \to (C, K, \delta) \text{ homomorphism of divided power rings} \\ \psi : (B, J) \to (C, K) \text{ an } A\text{-algebra homomorphism with } \psi(J) \subset K \end{array} \right\}$$

We will show that Divided Power Algebra, Lemma 23.3.3 applies to this functor which will prove the lemma. Suppose that $(\varphi, \psi) \in F(C, K, \delta)$. Let $C' \subset C$ be the subring generated by $\varphi(A)$, $\psi(B)$, and $\delta_n(\psi(f))$ for all $f \in J$. Let $K' \subset K \cap C'$ be the ideal of C' generated by $\varphi(I)$ and $\delta_n(\psi(f))$ for $f \in J$. Then $(C', K', \delta|_{K'})$ is a divided power ring and C' has cardinality bounded by the cardinal $\kappa = |A| \otimes |B|^{\aleph_0}$. Moreover, φ factors as $A \to C' \to C$ and ψ factors as $B \to C' \to C$. This proves assumption (1) of Divided Power Algebra, Lemma 23.3.3 holds. Assumption (2) is clear as limits in the category of divided power rings commute with the forgetful functor $(C, K, \delta) \mapsto (C, K)$, see Divided Power Algebra, Lemma 23.3.2 and its proof. $\square$

07H9 **Definition 60.2.2.** Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. The divided power algebra $(D, \bar{J}, \bar{\gamma})$ constructed in Lemma 60.2.1 is called the *divided power envelope of J in B relative to (A, I, γ)* and is denoted $D_B(J)$ or $D_{B,\gamma}(J)$.

Let $(A, I, \gamma) \to (C, K, \delta)$ be a homomorphism of divided power rings. The universal property of $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ is

$$\begin{matrix} \text{ring maps } B \to C \\ \text{which map } J \text{ into } K \end{matrix} \longleftrightarrow \begin{matrix} \text{divided power homomorphisms} \\ (D, \bar{J}, \bar{\gamma}) \to (C, K, \delta) \end{matrix}$$

and the correspondence is given by precomposing with the map $B \to D$ which corresponds to id_D. Here are some properties of $(D, \bar{J}, \bar{\gamma})$ which follow directly from the universal property. There are A-algebra maps

07HA (60.2.2.1)
$$B \longrightarrow D \longrightarrow B/J$$

The first arrow maps J into $\bar{J}$ and $\bar{J}$ is the kernel of the second arrow. The elements $\bar{\gamma}_n(x)$ where $n > 0$ and x is an element in the image of $J \to D$ generate $\bar{J}$ as an ideal in D and generate D as a B-algebra.

07HB **Lemma 60.2.3.** *Let (A, I, γ) be a divided power ring. Let $\varphi : B' \to B$ be a surjection of A-algebras with kernel K. Let $IB \subset J \subset B$ be an ideal. Let $J' \subset B'$ be the inverse image of J. Write $D_{B',\gamma}(J') = (D', \bar{J}', \bar{\gamma})$. Then $D_{B,\gamma}(J) = (D'/K', \bar{J}'/K', \bar{\gamma})$ where K' is the ideal generated by the elements $\bar{\gamma}_n(k)$ for $n \geq 1$ and $k \in K$.*

Proof. Write $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. The universal property of D' gives us a homomorphism $D' \to D$ of divided power algebras. As $B' \to B$ and $J' \to J$ are surjective, we see that $D' \to D$ is surjective (see remarks above). It is clear that $\bar{\gamma}_n(k)$ is in the kernel for $n \geq 1$ and $k \in K$, i.e., we obtain a homomorphism $D'/K' \to D$. Conversely, there exists a divided power structure on $\bar{J}'/K' \subset D'/K'$, see Divided Power Algebra, Lemma 23.4.3. Hence the universal property of D gives an inverse $D \to D'/K'$ and we win. $\square$

In the situation of Definition 60.2.2 we can choose a surjection $P \to B$ where P is a polynomial algebra over A and let $J' \subset P$ be the inverse image of J. The previous lemma describes $D_{B,\gamma}(J)$ in terms of $D_{P,\gamma}(J')$. Note that γ extends to a divided power structure γ' on IP by Divided Power Algebra, Lemma 23.4.2. Hence $D_{P,\gamma}(J') = D_{P,\gamma'}(J')$ is an example of a special case of divided power envelopes we describe in the following lemma.

07HC **Lemma 60.2.4.** *Let (B, I, γ) be a divided power algebra. Let $I \subset J \subset B$ be an ideal. Let $(D, \bar{J}, \bar{\gamma})$ be the divided power envelope of J relative to γ. Choose elements $f_t \in J$, $t \in T$ such that $J = I + (f_t)$. Then there exists a surjection*

$$\Psi : B\langle x_t \rangle \longrightarrow D$$

of divided power rings mapping x_t to the image of f_t in D. The kernel of Ψ is generated by the elements $x_t - f_t$ and all

$$\delta_n \left(\sum r_t x_t - r_0 \right)$$

whenever $\sum r_t f_t = r_0$ in B for some $r_t \in B$, $r_0 \in I$.

Proof. In the statement of the lemma we think of $B\langle x_t\rangle$ as a divided power ring with ideal $J' = IB\langle x_t\rangle + B\langle x_t\rangle_+$, see Divided Power Algebra, Remark 23.5.2. The existence of Ψ follows from the universal property of divided power polynomial rings. Surjectivity of Ψ follows from the fact that its image is a divided power subring of D, hence equal to D by the universal property of D. It is clear that $x_t - f_t$ is in the kernel. Set

$$\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus\nolimits_{t \in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$$

If $(r_0, r_t) \in \mathcal{R}$ then it is clear that $\sum r_t x_t - r_0$ is in the kernel. As Ψ is a homomorphism of divided power rings and $\sum r_t x_t - r_0 \in J'$ it follows that $\delta_n(\sum r_t x_t - r_0)$ is in the kernel as well. Let $K \subset B\langle x_t\rangle$ be the ideal generated by $x_t - f_t$ and the elements $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. To show that $K = \mathrm{Ker}(\Psi)$ it suffices to show that δ extends to $B\langle x_t\rangle/K$. Namely, if so the universal property of D gives a map $D \to B\langle x_t\rangle/K$ inverse to Ψ. Hence we have to show that $K \cap J'$ is preserved by δ_n, see Divided Power Algebra, Lemma 23.4.3. Let $K' \subset B\langle x_t\rangle$ be the ideal generated by the elements

(1) $\delta_m(\sum r_t x_t - r_0)$ where $m > 0$ and $(r_0, r_t) \in \mathcal{R}$,
(2) $x_{t'}^{[m]}(x_t - f_t)$ where $m > 0$ and $t', t \in I$.

We claim that $K' = K \cap J'$. The claim proves that $K \cap J'$ is preserved by δ_n, $n > 0$ by the criterion of Divided Power Algebra, Lemma 23.4.3 (2)(c) and a computation of δ_n of the elements listed which we leave to the reader. To prove the claim note that $K' \subset K \cap J'$. Conversely, if $h \in K \cap J'$ then, modulo K' we can write

$$h = \sum r_t(x_t - f_t)$$

for some $r_t \in B$. As $h \in K \cap J' \subset J'$ we see that $r_0 = \sum r_t f_t \in I$. Hence $(r_0, r_t) \in \mathcal{R}$ and we see that

$$h = \sum r_t x_t - r_0$$

is in K' as desired. □

07KE Lemma 60.2.5. Let (A, I, γ) be a divided power ring. Let B be an A-algebra and $IB \subset J \subset B$ an ideal. Let x_i be a set of variables. Then

$$D_{B[x_i],\gamma}(JB[x_i] + (x_i)) = D_{B,\gamma}(J)\langle x_i\rangle$$

Proof. One possible proof is to deduce this from Lemma 60.2.4 as any relation between x_i in $B[x_i]$ is trivial. On the other hand, the lemma follows from the universal property of the divided power polynomial algebra and the universal property of divided power envelopes. □

Conditions (1) and (2) of the following lemma hold if $B \to B'$ is flat at all primes of $V(IB') \subset \mathrm{Spec}(B')$ and is very closely related to that condition, see Algebra, Lemma 10.99.8. It in particular says that taking the divided power envelope commutes with localization.

07HD Lemma 60.2.6. Let (A, I, γ) be a divided power ring. Let $B \to B'$ be a homomorphism of A-algebras. Assume that

(1) $B/IB \to B'/IB'$ is flat, and
(2) $\mathrm{Tor}_1^B(B', B/IB) = 0$.

Then for any ideal $IB \subset J \subset B$ the canonical map

$$D_B(J) \otimes_B B' \longrightarrow D_{B'}(JB')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a B-algebra map $D \to D_{B'}(JB')$, whence a map as in the lemma. It suffices to show that the divided powers $\bar{\gamma}$ extend to $D \otimes_B B'$ since then the universal property of $D_{B'}(JB')$ will produce a map $D_{B'}(JB') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose a surjection $P \to B'$ where P is a polynomial algebra over B. In particular $B \to P$ is flat, hence $D \to D \otimes_B P$ is flat by Algebra, Lemma 10.39.7. Then $\bar{\gamma}$ extends to $D \otimes_B P$ by Divided Power Algebra, Lemma 23.4.2; we will denote this extension $\bar{\gamma}$ also. Set $\mathfrak{a} = \text{Ker}(P \to B')$ so that we have the short exact sequence

$$0 \to \mathfrak{a} \to P \to B' \to 0$$

Thus $\text{Tor}_1^B(B', B/IB) = 0$ implies that $\mathfrak{a} \cap IP = I\mathfrak{a}$. Now we have the following commutative diagram

$$\begin{tikzcd}
B/J \otimes_B \mathfrak{a} \arrow[r, "\beta"'] & B/J \otimes_B P \arrow[r] & B/J \otimes_B B' \\
D \otimes_B \mathfrak{a} \arrow[r, "\alpha"] \arrow[u] & D \otimes_B P \arrow[r] \arrow[u] & D \otimes_B B' \arrow[u] \\
\bar{J} \otimes_B \mathfrak{a} \arrow[r] \arrow[u] & \bar{J} \otimes_B P \arrow[r] \arrow[u] & \bar{J} \otimes_B B' \arrow[u]
\end{tikzcd}$$

This diagram is exact even with 0's added at the top and the right. We have to show the divided powers on the ideal $\bar{J} \otimes_B P$ preserve the ideal $\text{Im}(\alpha) \cap \bar{J} \otimes_B P$, see Divided Power Algebra, Lemma 23.4.3. Consider the exact sequence

$$0 \to \mathfrak{a}/I\mathfrak{a} \to P/IP \to B'/IB' \to 0$$

(which uses that $\mathfrak{a} \cap IP = I\mathfrak{a}$ as seen above). As B'/IB' is flat over B/IB this sequence remains exact after applying $B/J \otimes_{B/IB} -$, see Algebra, Lemma 10.39.12. Hence

$$\text{Ker}(B/J \otimes_{B/IB} \mathfrak{a}/I\mathfrak{a} \to B/J \otimes_{B/IB} P/IP) = \text{Ker}(\mathfrak{a}/J\mathfrak{a} \to P/JP)$$

is zero. Thus β is injective. It follows that $\text{Im}(\alpha) \cap \bar{J} \otimes_B P$ is the image of $\bar{J} \otimes \mathfrak{a}$. Now if $f \in \bar{J}$ and $a \in \mathfrak{a}$, then $\bar{\gamma}_n(f \otimes a) = \bar{\gamma}_n(f) \otimes a^n$ hence the result is clear. $\square$

The following lemma is a special case of [dJ95, Proposition 2.1.7] which in turn is a generalization of [Ber74, Proposition 2.8.2].

07HE Lemma 60.2.7. Let $(B, I, \gamma) \to (B', I', \gamma')$ be a homomorphism of divided power rings. Let $I \subset J \subset B$ and $I' \subset J' \subset B'$ be ideals. Assume

(1) $B/I \to B'/I'$ is flat, and
(2) $J' = JB' + I'$.

Then the canonical map

$$D_{B,\gamma}(J) \otimes_B B' \longrightarrow D_{B',\gamma'}(J')$$

is an isomorphism.

Proof. Set $D = D_{B,\gamma}(J)$. Choose elements $f_t \in J$ which generate J/I. Set $\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus_{t\in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$ as in the proof of Lemma 60.2.4. This lemma shows that

$$D = B\langle x_t\rangle/K$$

where K is generated by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. Thus we see that

07HF (60.2.7.1)
$$D \otimes_B B' = B'\langle x_t\rangle/K'$$

where K' is generated by the images in $B'\langle x_t\rangle$ of the generators of K listed above. Let $f'_t \in B'$ be the image of f_t. By assumption (1) we see that the elements $f'_t \in J'$ generate J'/I' and we see that $x_t - f'_t \in K'$. Set

$$\mathcal{R}' = \{(r'_0, r'_t) \in I' \oplus \bigoplus\nolimits_{t\in T} B' \mid \sum r'_t f'_t = r'_0 \text{ in } B'\}$$

To finish the proof we have to show that $\delta'_n(\sum r'_t x_t - r'_0) \in K'$ for $(r'_0, r'_t) \in \mathcal{R}'$, because then the presentation (60.2.7.1) of $D \otimes_B B'$ is identical to the presentation of $D_{B',\gamma'}(J')$ obtain in Lemma 60.2.4 from the generators f'_t. Suppose that $(r'_0, r'_t) \in \mathcal{R}'$. Then $\sum r'_t f'_t = 0$ in B'/I'. As $B/I \to B'/I'$ is flat by assumption (1) we can apply the equational criterion of flatness (Algebra, Lemma 10.39.11) to see that there exist an $m > 0$ and $r_{jt} \in B$ and $c_j \in B'$, $j = 1, \ldots, m$ such that

$$r_{j0} = \sum\nolimits_t r_{jt} f_t \in I \text{ for } j = 1, \ldots, m$$

and

$$i'_t = r'_t - \sum\nolimits_j c_j r_{jt} \in I' \text{ for all } t$$

Note that this also implies that $r'_0 = \sum_t i'_t f_t + \sum_j c_j r_{j0}$. Then we have

$$\begin{aligned}\delta'_n(\sum\nolimits_t r'_t x_t - r'_0) &= \delta'_n(\sum\nolimits_t i'_t x_t + \sum\nolimits_{t,j} c_j r_{jt} x_t - \sum\nolimits_t i'_t f_t - \sum\nolimits_j c_j r_{j0}) \\ &= \delta'_n(\sum\nolimits_t i'_t(x_t - f_t) + \sum\nolimits_j c_j(\sum\nolimits_t r_{jt} x_t - r_{j0}))\end{aligned}$$

Since $\delta_n(a+b) = \sum_{m=0,\ldots,n} \delta_m(a)\delta_{n-m}(b)$ and since $\delta_m(\sum i'_t(x_t - f_t))$ is in the ideal generated by $x_t - f_t \in K'$ for $m > 0$, it suffices to prove that $\delta_n(\sum c_j(\sum r_{jt} x_t - r_{j0}))$ is in K'. For this we use

$$\delta_n(\sum\nolimits_j c_j(\sum\nolimits_t r_{jt} x_t - r_{j0})) = \sum c_1^{n_1} \ldots c_m^{n_m} \delta_{n_1}(\sum r_{1t} x_t - r_{10}) \ldots \delta_{n_m}(\sum r_{mt} x_t - r_{m0})$$

where the sum is over $n_1 + \ldots + n_m = n$. This proves what we want. $\square$

60.3. Some explicit divided power thickenings

07HG The constructions in this section will help us to define the connection on a crystal in modules on the crystalline site.

07HH Lemma 60.3.1. Let (A, I, γ) be a divided power ring. Let M be an A-module. Let $B = A \oplus M$ as an A-algebra where M is an ideal of square zero and set $J = I \oplus M$. Set

$$\delta_n(x + z) = \gamma_n(x) + \gamma_{n-1}(x)z$$

for $x \in I$ and $z \in M$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. We have to check conditions (1) – (5) of Divided Power Algebra, Definition 23.2.1. We will prove this directly for this case, but please see the proof of the next lemma for a method which avoids calculations. Conditions (1) and (3) are clear. Condition (2) follows from

$$\begin{aligned}\delta_n(x+z)\delta_m(x+z) &= (\gamma_n(x)+\gamma_{n-1}(x)z)(\gamma_m(x)+\gamma_{m-1}(x)z)\\ &= \gamma_n(x)\gamma_m(x)+\gamma_n(x)\gamma_{m-1}(x)z+\gamma_{n-1}(x)\gamma_m(x)z\\ &= \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)+\left(\frac{(n+m-1)!}{n!(m-1)!}+\frac{(n+m-1)!}{(n-1)!m!}\right)\gamma_{n+m-1}(x)z\\ &= \frac{(n+m)!}{n!m!}\delta_{n+m}(x+z)\end{aligned}$$

Condition (5) follows from

$$\begin{aligned}\delta_n(\delta_m(x+z)) &= \delta_n(\gamma_m(x)+\gamma_{m-1}(x)z)\\ &= \gamma_n(\gamma_m(x))+\gamma_{n-1}(\gamma_m(x))\gamma_{m-1}(x)z\\ &= \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)+\frac{((n-1)m)!}{(n-1)!(m!)^{n-1}}\gamma_{(n-1)m}(x)\gamma_{m-1}(x)z\\ &= \frac{(nm)!}{n!(m!)^n}(\gamma_{nm}(x)+\gamma_{nm-1}(x)z)\end{aligned}$$

by elementary number theory. To prove (4) we have to see that

$$\delta_n(x+x'+z+z') = \gamma_n(x+x')+\gamma_{n-1}(x+x')(z+z')$$

is equal to

$$\sum\nolimits_{i=0}^{n}(\gamma_i(x)+\gamma_{i-1}(x)z)(\gamma_{n-i}(x')+\gamma_{n-i-1}(x')z')$$

This follows easily on collecting the coefficients of 1, z, and z' and using condition (4) for γ. $\square$

07HI Lemma 60.3.2. Let (A, I, γ) be a divided power ring. Let M, N be A-modules. Let $q : M \times M \to N$ be an A-bilinear map. Let $B = A \oplus M \oplus N$ as an A-algebra with multiplication

$$(x, z, w) \cdot (x', z', w') = (xx', xz' + x'z, xw' + x'w + q(z, z') + q(z', z))$$

and set $J = I \oplus M \oplus N$. Set

$$\delta_n(x, z, w) = (\gamma_n(x), \gamma_{n-1}(x)z, \gamma_{n-1}(x)w + \gamma_{n-2}(x)q(z, z))$$

for $(x, z, w) \in J$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. Suppose we want to prove that property (4) of Divided Power Algebra, Definition 23.2.1 is satisfied. Pick (x, z, w) and (x', z', w') in J. Pick a map

$$A_0 = \mathbf{Z}\langle s, s'\rangle \longrightarrow A, \quad s \longmapsto x, s' \longmapsto x'$$

which is possible by the universal property of divided power polynomial rings. Set $M_0 = A_0 \oplus A_0$ and $N_0 = A_0 \oplus A_0 \oplus M_0 \otimes_{A_0} M_0$. Let $q_0 : M_0 \times M_0 \to N_0$ be the obvious map. Define $M_0 \to M$ as the A_0-linear map which sends the basis vectors of M_0 to z and z'. Define $N_0 \to N$ as the A_0 linear map which sends the first two basis vectors of N_0 to w and w' and uses $M_0 \otimes_{A_0} M_0 \to M \otimes_A M \xrightarrow{q} N$ on the last summand. Then we see that it suffices to prove the identity (4) for the situation (A_0, M_0, N_0, q_0). Similarly for the other identities. This reduces us to the case of a

$\mathbf{Z}$-torsion free ring A and A-torsion free modules. In this case all we have to do is show that

$$n!\delta_n(x, z, w) = (x, z, w)^n$$

in the ring A, see Divided Power Algebra, Lemma 23.2.2. To see this note that

$$(x, z, w)^2 = (x^2, 2xz, 2xw + 2q(z, z))$$

and by induction

$$(x, z, w)^n = (x^n, nx^{n-1}z, nx^{n-1}w + n(n-1)x^{n-2}q(z, z))$$

On the other hand,

$$n!\delta_n(x, z, w) = (n!\gamma_n(x), n!\gamma_{n-1}(x)z, n!\gamma_{n-1}(x)w + n!\gamma_{n-2}(x)q(z, z))$$

which matches. This finishes the proof. □

60.4. Compatibility

07HJ This section isn't required reading; it explains how our discussion fits with that of [Ber74]. Consider the following technical notion.

07HK Definition 60.4.1. Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring map. We say δ is compatible with γ if there exists a divided power structure $\bar{\gamma}$ on $J + IB$ such that

$$(A, I, \gamma) \to (B, J + IB, \bar{\gamma}) \quad \text{and} \quad (B, J, \delta) \to (B, J + IB, \bar{\gamma})$$

are homomorphisms of divided power rings.

Let p be a prime number. Let (A, I, γ) be a divided power ring. Let $A \to C$ be a ring map with p nilpotent in C. Assume that γ extends to IC (see Divided Power Algebra, Lemma 23.4.2). In this situation, the (big affine) crystalline site of $\mathrm{Spec}(C)$ over $\mathrm{Spec}(A)$ as defined in [Ber74] is the opposite of the category of systems

$$(B, J, \delta, A \to B, C \to B/J)$$

where

(1) (B, J, δ) is a divided power ring with p nilpotent in B,
(2) δ is compatible with γ, and
(3) the diagram

$$\begin{tikzcd} B \arrow[r] & B/J \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

is commutative.

The conditions "γ extends to C and δ compatible with γ" are used in [Ber74] to ensure that the crystalline cohomology of $\mathrm{Spec}(C)$ is the same as the crystalline cohomology of $\mathrm{Spec}(C/IC)$. We will avoid this issue by working exclusively with C such that $IC = 0$[1]. In this case, for a system $(B, J, \delta, A \to B, C \to B/J)$ as above, the commutativity of the displayed diagram above implies $IB \subset J$ and compatibility is equivalent to the condition that $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings.

[1] Of course there will be a price to pay.

60.5. Affine crystalline site

07HL In this section we discuss the algebraic variant of the crystalline site. Our basic situation in which we discuss this material will be as follows.

07MD **Situation 60.5.1.** Here p is a prime number, (A, I, γ) is a divided power ring such that A is a $\mathbf{Z}_{(p)}$-algebra, and $A \to C$ is a ring map such that $IC = 0$ and such that p is nilpotent in C.

Usually the prime number p will be contained in the divided power ideal I.

07HM **Definition 60.5.2.** In Situation 60.5.1.

(1) A divided power thickening of C over (A, I, γ) is a homomorphism of divided power algebras $(A, I, \gamma) \to (B, J, \delta)$ such that p is nilpotent in B and a ring map $C \to B/J$ such that

$$\begin{tikzcd} B \arrow[r] & B/J \\ & C \arrow[u] \\ A \arrow[uu] \arrow[r] & A/I \arrow[u] \end{tikzcd}$$

is commutative.

(2) A homomorphism of divided power thickenings

$$(B, J, \delta, C \to B/J) \longrightarrow (B', J', \delta', C \to B'/J')$$

is a homomorphism $\varphi : B \to B'$ of divided power A-algebras such that $C \to B/J \to B'/J'$ is the given map $C \to B'/J'$.

(3) We denote $\mathrm{CRIS}(C/A, I, \gamma)$ or simply $\mathrm{CRIS}(C/A)$ the category of divided power thickenings of C over (A, I, γ).

(4) We denote $\mathrm{Cris}(C/A, I, \gamma)$ or simply $\mathrm{Cris}(C/A)$ the full subcategory consisting of $(B, J, \delta, C \to B/J)$ such that $C \to B/J$ is an isomorphism. We often denote such an object $(B \to C, \delta)$ with $J = \mathrm{Ker}(B \to C)$ being understood.

Note that for a divided power thickening (B, J, δ) as above the ideal J is locally nilpotent, see Divided Power Algebra, Lemma 23.2.6. There is a canonical functor

07KF (60.5.2.1) $$\mathrm{CRIS}(C/A) \longrightarrow C\text{-algebras}, \quad (B, J, \delta) \longmapsto B/J$$

This category does not have equalizers or fibre products in general. It also doesn't have an initial object (= empty colimit) in general.

07HN **Lemma 60.5.3.** In Situation 60.5.1.

(1) $\mathrm{CRIS}(C/A)$ has finite products (but not infinite ones),
(2) $\mathrm{CRIS}(C/A)$ has all finite nonempty colimits and (60.5.2.1) commutes with these, and
(3) $\mathrm{Cris}(C/A)$ has all finite nonempty colimits and $\mathrm{Cris}(C/A) \to \mathrm{CRIS}(C/A)$ commutes with them.

Proof. The empty product, i.e., the final object in the category of divided power thickenings of C over (A, I, γ), is the zero ring viewed as an A-algebra endowed with the zero ideal and the unique divided powers on the zero ideal and finally endowed with the unique homomorphism of C to the zero ring. If $(B_t, J_t, \delta_t)_{t \in T}$ is a family of objects of $\text{CRIS}(C/A)$ then we can form the product $(\prod_t B_t, \prod_t J_t, \prod_t \delta_t)$ as in Divided Power Algebra, Lemma 23.3.2. The map $C \to \prod B_t / \prod J_t = \prod B_t/J_t$ is clear. However, we are only guaranteed that p is nilpotent in $\prod_t B_t$ if T is finite.

Given two objects (B, J, γ) and (B', J', γ') of $\text{CRIS}(C/A)$ we can form a cocartesian diagram

$$\begin{tikzcd} (B, J, \delta) \arrow[r] & (B'', J'', \delta'') \\ (A, I, \gamma) \arrow[u] \arrow[r] & (B', J', \delta') \arrow[u] \end{tikzcd}$$

in the category of divided power rings. Then we see that we have

$$B''/J'' = B/J \otimes_{A/I} B'/J' \longleftarrow C \otimes_{A/I} C$$

see Divided Power Algebra, Remark 23.3.5. Denote $J'' \subset K \subset B''$ the ideal such that

$$\begin{tikzcd} B''/J'' \arrow[r] & B''/K \\ C \otimes_{A/I} C \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

is a pushout, i.e., $B''/K \cong B/J \otimes_C B'/J'$. Let $D_{B''}(K) = (D, \bar{K}, \bar{\delta})$ be the divided power envelope of K in B'' relative to (B'', J'', δ''). Then it is easily verified that $(D, \bar{K}, \bar{\delta})$ is a coproduct of (B, J, δ) and (B', J', δ') in $\text{CRIS}(C/A)$.

Next, we come to coequalizers. Let $\alpha, \beta : (B, J, \delta) \to (B', J', \delta')$ be morphisms of $\text{CRIS}(C/A)$. Consider $B'' = B'/(\alpha(b) - \beta(b))$. Let $J'' \subset B''$ be the image of J'. Let $D_{B''}(J'') = (D, \bar{J}, \bar{\delta})$ be the divided power envelope of J'' in B'' relative to (B', J', δ'). Then it is easily verified that $(D, \bar{J}, \bar{\delta})$ is the coequalizer of (B, J, δ) and (B', J', δ') in $\text{CRIS}(C/A)$.

By Categories, Lemma 4.18.6 we have all finite nonempty colimits in $\text{CRIS}(C/A)$. The constructions above shows that (60.5.2.1) commutes with them. This formally implies part (3) as $\text{Cris}(C/A)$ is the fibre category of (60.5.2.1) over C. □

07KH Remark 60.5.4. In Situation 60.5.1 we denote $\text{Cris}^\wedge(C/A)$ the category whose objects are pairs $(B \to C, \delta)$ such that

(1) B is a p-adically complete A-algebra,
(2) $B \to C$ is a surjection of A-algebras,
(3) δ is a divided power structure on $\text{Ker}(B \to C)$,
(4) $A \to B$ is a homomorphism of divided power rings.

Morphisms are defined as in Definition 60.5.2. Then $\text{Cris}(C/A) \subset \text{Cris}^\wedge(C/A)$ is the full subcategory consisting of those B such that p is nilpotent in B. Conversely, any object $(B \to C, \delta)$ of $\text{Cris}^\wedge(C/A)$ is equal to the limit

$$(B \to C, \delta) = \lim_e (B/p^e B \to C, \delta)$$

where for $e \gg 0$ the object $(B/p^eB \to C, \delta)$ lies in $\text{Cris}(C/A)$, see Divided Power Algebra, Lemma 23.4.5. In particular, we see that $\text{Cris}^\wedge(C/A)$ is a full subcategory of the category of pro-objects of $\text{Cris}(C/A)$, see Categories, Remark 4.22.5.

07KG Lemma 60.5.5. In Situation 60.5.1. Let $P \to C$ be a surjection of A-algebras with kernel J. Write $D_{P,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. Let $(D^\wedge, J^\wedge, \bar{\gamma}^\wedge)$ be the p-adic completion of D, see Divided Power Algebra, Lemma 23.4.5. For every $e \geq 1$ set $P_e = P/p^eP$ and $J_e \subset P_e$ the image of J and write $D_{P_e,\gamma}(J_e) = (D_e, \bar{J}_e, \bar{\gamma})$. Then for all e large enough we have

(1) $p^eD \subset \bar{J}$ and $p^eD^\wedge \subset \bar{J}^\wedge$ are preserved by divided powers,
(2) $D^\wedge/p^eD^\wedge = D/p^eD = D_e$ as divided power rings,
(3) $(D_e, \bar{J}_e, \bar{\gamma})$ is an object of $\text{Cris}(C/A)$,
(4) $(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is equal to $\lim_e(D_e, \bar{J}_e, \bar{\gamma})$, and
(5) $(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is an object of $\text{Cris}^\wedge(C/A)$.

Proof. Part (1) follows from Divided Power Algebra, Lemma 23.4.5. It is a general property of p-adic completion that $D/p^eD = D^\wedge/p^eD^\wedge$. Since D/p^eD is a divided power ring and since $P \to D/p^eD$ factors through P_e, the universal property of D_e produces a map $D_e \to D/p^eD$. Conversely, the universal property of D produces a map $D \to D_e$ which factors through D/p^eD. We omit the verification that these maps are mutually inverse. This proves (2). If e is large enough, then $p^eC = 0$, hence we see (3) holds. Part (4) follows from Divided Power Algebra, Lemma 23.4.5. Part (5) is clear from the definitions. □

07HP Lemma 60.5.6. In Situation 60.5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. With $(D_e, \bar{J}_e, \bar{\gamma})$ as in Lemma 60.5.5: for every object (B, J_B, δ) of $\text{CRIS}(C/A)$ there exists an e and a morphism $D_e \to B$ of $\text{CRIS}(C/A)$.

Proof. We can find an A-algebra homomorphism $P \to B$ lifting the map $C \to B/J_B$. By our definition of $\text{CRIS}(C/A)$ we see that $p^eB = 0$ for some e hence $P \to B$ factors as $P \to P_e \to B$. By the universal property of the divided power envelope we conclude that $P_e \to B$ factors through D_e. □

07KI Lemma 60.5.7. In Situation 60.5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. Let $(D, \bar{J}, \bar{\gamma})$ be the p-adic completion of $D_{P,\gamma}(J)$. For every object $(B \to C, \delta)$ of $\text{Cris}^\wedge(C/A)$ there exists a morphism $D \to B$ of $\text{Cris}^\wedge(C/A)$.

Proof. We can find an A-algebra homomorphism $P \to B$ compatible with maps to C. By our definition of $\text{Cris}(C/A)$ we see that $P \to B$ factors as $P \to D_{P,\gamma}(J) \to B$. As B is p-adically complete we can factor this map through D. □

60.6. Module of differentials

07HQ In this section we develop a theory of modules of differentials for divided power rings.

07HR Definition 60.6.1. Let A be a ring. Let (B, J, δ) be a divided power ring. Let $A \to B$ be a ring map. Let M be an B-module. A divided power A-derivation into

M is a map $\theta : B \to M$ which is additive, annihilates the elements of A, satisfies the Leibniz rule $\theta(bb') = b\theta(b') + b'\theta(b)$ and satisfies

$$\theta(\delta_n(x)) = \delta_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and all $x \in J$.

In the situation of the definition, just as in the case of usual derivations, there exists a universal divided power A-derivation

$$\mathrm{d}_{B/A,\delta} : B \to \Omega_{B/A,\delta}$$

such that any divided power A-derivation $\theta : B \to M$ is equal to $\theta = \xi \circ d_{B/A,\delta}$ for some unique B-linear map $\xi : \Omega_{B/A,\delta} \to M$. If $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then we can forget the divided powers on A and consider the divided power derivations of B over A. Here are some basic properties of the universal module of (divided power) differentials.

07HS Lemma 60.6.2. Let A be a ring. Let (B, J, δ) be a divided power ring and $A \to B$ a ring map.

(1) Consider $B[x]$ with divided power ideal $(JB[x], \delta')$ where δ' is the extension of δ to $B[x]$. Then
$$\Omega_{B[x]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[x] \oplus B[x]\mathrm{d}x.$$
(2) Consider $B\langle x\rangle$ with divided power ideal $(JB\langle x\rangle + B\langle x\rangle_+, \delta')$. Then
$$\Omega_{B\langle x\rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle x\rangle \oplus B\langle x\rangle\mathrm{d}x.$$
(3) Let $K \subset J$ be an ideal preserved by δ_n for all $n > 0$. Set $B' = B/K$ and denote δ' the induced divided power on J/K. Then $\Omega_{B'/A,\delta'}$ is the quotient of $\Omega_{B/A,\delta} \otimes_B B'$ by the B'-submodule generated by $\mathrm{d}k$ for $k \in K$.

Proof. These are proved directly from the construction of $\Omega_{B/A,\delta}$ as the free B-module on the elements $\mathrm{d}b$ modulo the relations

(1) $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$, $b, b' \in B$,
(2) $\mathrm{d}a = 0$, $a \in A$,
(3) $\mathrm{d}(bb') = b\mathrm{d}b' + b'\mathrm{d}b$, $b, b' \in B$,
(4) $\mathrm{d}\delta_n(f) = \delta_{n-1}(f)\mathrm{d}f$, $f \in J$, $n > 1$.

Note that the last relation explains why we get "the same" answer for the divided power polynomial algebra and the usual polynomial algebra: in the first case x is an element of the divided power ideal and hence $\mathrm{d}x^{[n]} = x^{[n-1]}\mathrm{d}x$. □

Let (A, I, γ) be a divided power ring. In this setting the correct version of the powers of I is given by the divided powers

$$I^{[n]} = \text{ideal generated by } \gamma_{e_1}(x_1) \ldots \gamma_{e_t}(x_t) \text{ with } \sum e_j \geq n \text{ and } x_j \in I.$$

Of course we have $I^n \subset I^{[n]}$. Note that $I^{[1]} = I$. Sometimes we also set $I^{[0]} = A$.

07HT Lemma 60.6.3. Let $(A, I, \gamma) \to (B, J, \delta)$ be a homomorphism of divided power rings. Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself over (A, I, γ), i.e., such that

$$\begin{tikzcd}
(B, J, \delta) \arrow[r] & (B(1), J(1), \delta(1)) \\
(A, I, \gamma) \arrow[u] \arrow[r] & (B, J, \delta) \arrow[u]
\end{tikzcd}$$

is cocartesian. Denote $K = \mathrm{Ker}(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. The fact that $K \cap J(1) \subset J(1)$ is preserved by the divided power structure follows from the fact that $B(1) \to B$ is a homomorphism of divided power rings.

Recall that K/K^2 has a canonical B-module structure. Denote $s_0, s_1 : B \to B(1)$ the two coprojections and consider the map $\mathrm{d} : B \to K/K^2 + (K \cap J(1))^{[2]}$ given by $b \mapsto s_1(b) - s_0(b)$. It is clear that d is additive, annihilates A, and satisfies the Leibniz rule. We claim that d is a divided power A-derivation. Let $x \in J$. Set $y = s_1(x)$ and $z = s_0(x)$. Denote δ the divided power structure on $J(1)$. We have to show that $\delta_n(y) - \delta_n(z) = \delta_{n-1}(y)(y - z)$ modulo $K^2 + (K \cap J(1))^{[2]}$ for $n \geq 1$. The equality holds for $n = 1$. Assume $n > 1$. Note that $\delta_i(y-z)$ lies in $(K \cap J(1))^{[2]}$ for $i > 1$. Calculating modulo $K^2 + (K \cap J(1))^{[2]}$ we have

$$\delta_n(z) = \delta_n(z - y + y) = \sum\nolimits_{i=0}^{n} \delta_i(z - y)\delta_{n-i}(y) = \delta_{n-1}(y)\delta_1(z - y) + \delta_n(y)$$

This proves the desired equality.

Let M be a B-module. Let $\theta : B \to M$ be a divided power A-derivation. Set $D = B \oplus M$ where M is an ideal of square zero. Define a divided power structure on $J \oplus M \subset D$ by setting $\delta_n(x + m) = \delta_n(x) + \delta_{n-1}(x)m$ for $n > 1$, see Lemma 60.3.1. There are two divided power algebra homomorphisms $B \to D$: the first is given by the inclusion and the second by the map $b \mapsto b + \theta(b)$. Hence we get a canonical homomorphism $B(1) \to D$ of divided power algebras over (A, I, γ). This induces a map $K \to M$ which annihilates K^2 (as M is an ideal of square zero) and $(K \cap J(1))^{[2]}$ as $M^{[2]} = 0$. The composition $B \to K/K^2 + (K \cap J(1))^{[2]} \to M$ equals θ by construction. It follows that d is a universal divided power A-derivation and we win. □

07HU Remark 60.6.4. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. The universal module $\Omega_{B/A,\delta}$ comes with a little bit of extra structure, namely the B-submodule N of $\Omega_{B/A,\delta}$ generated by $\mathrm{d}_{B/A,\delta}(J)$. In terms of the isomorphism given in Lemma 60.6.3 this corresponds to the image of $K \cap J(1)$ in $\Omega_{B/A,\delta}$. Consider the A-algebra $D = B \oplus \Omega^1_{B/A,\delta}$ with ideal $\bar{J} = J \oplus N$ and divided powers $\bar{\delta}$ as in the proof of the lemma. Then $(D, \bar{J}, \bar{\delta})$ is a divided power ring and the two maps $B \to D$ given by $b \mapsto b$ and $b \mapsto b + \mathrm{d}_{B/A,\delta}(b)$ are homomorphisms of divided power rings over A. Moreover, N is the smallest submodule of $\Omega_{B/A,\delta}$ such that this is true.

07HV Lemma 60.6.5. In Situation 60.5.1. Let (B, J, δ) be an object of $\mathrm{CRIS}(C/A)$. Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself in $\mathrm{CRIS}(C/A)$. Denote $K = \mathrm{Ker}(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. Word for word the same as the proof of Lemma 60.6.3. The only point that has to be checked is that the divided power ring $D = B \oplus M$ is an object of $\text{CRIS}(C/A)$ and that the two maps $B \to C$ are morphisms of $\text{CRIS}(C/A)$. Since $D/(J \oplus M) = B/J$ we can use $C \to B/J$ to view D as an object of $\text{CRIS}(C/A)$ and the statement on morphisms is clear from the construction. $\square$

07HW Lemma 60.6.6. Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map and let $IB \subset J \subset B$ be an ideal. Let $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ be the divided power envelope. Then we have

$$\Omega_{D/A,\bar{\gamma}} = \Omega_{B/A} \otimes_B D$$

First proof. Let M be a D-module. We claim that an A-derivation $\vartheta : B \to M$ is the same thing as a divided power A-derivation $\theta : D \to M$. The claim implies the statement by the Yoneda lemma.

Consider the square zero thickening $D \oplus M$ of D. There is a divided power structure δ on $\bar{J} \oplus M$ if we set the higher divided power operations zero on M. In other words, we set $\delta_n(x+m) = \bar{\gamma}_n(x) + \bar{\gamma}_{n-1}(x)m$ for any $x \in \bar{J}$ and $m \in M$, see Lemma 60.3.1. Consider the A-algebra map $B \to D \oplus M$ whose first component is given by the map $B \to D$ and whose second component is ϑ. By the universal property we get a corresponding homomorphism $D \to D \oplus M$ of divided power algebras whose second component is the divided power A-derivation θ corresponding to ϑ. $\square$

Second proof. We will prove this first when B is flat over A. In this case γ extends to a divided power structure γ' on IB, see Divided Power Algebra, Lemma 23.4.2. Hence $D = D_{B,\gamma'}(J)$ is equal to a quotient of the divided power ring (D', J', δ) where $D' = B\langle x_t \rangle$ and $J' = IB\langle x_t \rangle + B\langle x_t \rangle_+$ by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$, see Lemma 60.2.4 for notation and explanation. Write $\mathrm{d} : D' \to \Omega_{D'/A,\delta}$ for the universal derivation. Note that

$$\Omega_{D'/A,\delta} = \Omega_{B/A} \otimes_B D' \oplus \bigoplus D' \mathrm{d}x_t,$$

see Lemma 60.6.2. We conclude that $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{D'/A,\delta} \otimes_{D'} D$ by the submodule generated by d applied to the generators of the kernel of $D' \to D$ listed above, see Lemma 60.6.2. Since $\mathrm{d}(x_t - f_t) = -\mathrm{d}f_t + \mathrm{d}x_t$ we see that we have $\mathrm{d}x_t = \mathrm{d}f_t$ in the quotient. In particular we see that $\Omega_{B/A} \otimes_B D \to \Omega_{D/A,\gamma}$ is surjective with kernel given by the images of d applied to the elements $\delta_n(\sum r_t x_t - r_0)$. However, given a relation $\sum r_t f_t - r_0 = 0$ in B with $r_t \in B$ and $r_0 \in IB$ we see that

$$\begin{aligned} \mathrm{d}\delta_n(\sum r_t x_t - r_0) &= \delta_{n-1}(\sum r_t x_t - r_0)\mathrm{d}(\sum r_t x_t - r_0) \\ &= \delta_{n-1}(\sum r_t x_t - r_0)\left(\sum r_t \mathrm{d}(x_t - f_t) + \sum (x_t - f_t)\mathrm{d}r_t\right) \end{aligned}$$

because $\sum r_t f_t - r_0 = 0$ in B. Hence this is already zero in $\Omega_{B/A} \otimes_A D$ and we win in the case that B is flat over A.

In the general case we write B as a quotient of a polynomial ring $P \to B$ and let $J' \subset P$ be the inverse image of J. Then $D = D'/K'$ with notation as in Lemma 60.2.3. By the case handled in the first paragraph of the proof we have $\Omega_{D'/A,\bar{\gamma}'} = \Omega_{P/A} \otimes_P D'$. Then $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{P/A} \otimes_P D$ by the submodule generated by $\mathrm{d}\bar{\gamma}'_n(k)$ where k is an element of the kernel of $P \to B$, see Lemma 60.6.2 and the description of K' from Lemma 60.2.3. Since $\mathrm{d}\bar{\gamma}'_n(k) = \bar{\gamma}'_{n-1}(k)\mathrm{d}k$ we see again that it suffices to divided by the submodule generated by $\mathrm{d}k$ with $k \in \mathrm{Ker}(P \to B)$

and since $\Omega_{B/A}$ is the quotient of $\Omega_{P/A} \otimes_A B$ by these elements (Algebra, Lemma 10.131.9) we win. □

07HZ Remark 60.6.7. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Set $\Omega^i_{B/A,\delta} = \wedge^i_B \Omega_{B/A,\delta}$ where $\Omega_{B/A,\delta}$ is the target of the universal divided power A-derivation $\mathrm{d} = \mathrm{d}_{B/A} : B \to \Omega_{B/A,\delta}$. Note that $\Omega_{B/A,\delta}$ is the quotient of $\Omega_{B/A}$ by the B-submodule generated by the elements $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ for $x \in J$. We claim Algebra, Lemma 10.132.1 applies. To see this it suffices to verify the elements $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ of Ω_B are mapped to zero in $\Omega^2_{B/A,\delta}$. We observe that

$$\mathrm{d}(\delta_{n-1}(x)) \wedge \mathrm{d}x = \delta_{n-2}(x)\mathrm{d}x \wedge \mathrm{d}x = 0$$

in $\Omega^2_{B/A,\delta}$ as desired. Hence we obtain a divided power de Rham complex

$$\Omega^0_{B/A,\delta} \to \Omega^1_{B/A,\delta} \to \Omega^2_{B/A,\delta} \to \ldots$$

which will play an important role in the sequel.

07I0 Remark 60.6.8. Let $A \to B$ be a ring map. Let $\Omega_{B/A} \to \Omega$ be a quotient satisfying the assumptions of Algebra, Lemma 10.132.1. Let M be a B-module. A connection is an additive map

$$\nabla : M \longrightarrow M \otimes_B \Omega$$

such that $\nabla(bm) = b\nabla(m) + m \otimes \mathrm{d}b$ for $b \in B$ and $m \in M$. In this situation we can define maps

$$\nabla : M \otimes_B \Omega^i \longrightarrow M \otimes_B \Omega^{i+1}$$

by the rule $\nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes \mathrm{d}\omega$. This works because if $b \in B$, then

$$\begin{aligned}\nabla(bm \otimes \omega) - \nabla(m \otimes b\omega) &= \nabla(bm) \wedge \omega + bm \otimes \mathrm{d}\omega - \nabla(m) \wedge b\omega - m \otimes \mathrm{d}(b\omega)\\ &= b\nabla(m) \wedge \omega + m \otimes \mathrm{d}b \wedge \omega + bm \otimes \mathrm{d}\omega\\ &\quad - b\nabla(m) \wedge \omega - bm \otimes \mathrm{d}(\omega) - m \otimes \mathrm{d}b \wedge \omega = 0\end{aligned}$$

As is customary we say the connection is integrable if and only if the composition

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2$$

is zero. In this case we obtain a complex

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2 \xrightarrow{\nabla} M \otimes_B \Omega^3 \xrightarrow{\nabla} M \otimes_B \Omega^4 \to \ldots$$

which is called the de Rham complex of the connection.

07KJ Remark 60.6.9. Consider a commutative diagram of rings

$$\begin{tikzcd} B \arrow[r, "\varphi"'] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

Let $\Omega_{B/A} \to \Omega$ and $\Omega_{B'/A'} \to \Omega'$ be quotients satisfying the assumptions of Algebra, Lemma 10.132.1. Assume there is a map $\varphi : \Omega \to \Omega'$ which fits into a commutative diagram

$$\begin{tikzcd} \Omega_{B/A} \arrow[r] \arrow[d] & \Omega_{B'/A'} \arrow[d] \\ \Omega \arrow[r, "\varphi"] & \Omega' \end{tikzcd}$$

where the top horizontal arrow is the canonical map $\Omega_{B/A} \to \Omega_{B'/A'}$ induced by $\varphi : B \to B'$. In this situation, given any pair (M, ∇) where M is a B-module and $\nabla : M \to M \otimes_B \Omega$ is a connection we obtain a base change $(M \otimes_B B', \nabla')$ where

$$\nabla' : M \otimes_B B' \longrightarrow (M \otimes_B B') \otimes_{B'} \Omega' = M \otimes_B \Omega'$$

is defined by the rule

$$\nabla'(m \otimes b') = \sum m_i \otimes b' \mathrm{d}\varphi(b_i) + m \otimes \mathrm{d}b'$$

if $\nabla(m) = \sum m_i \otimes \mathrm{d}b_i$. If ∇ is integrable, then so is ∇', and in this case there is a canonical map of de Rham complexes (Remark 60.6.8)

07PY (60.6.9.1) $$M \otimes_B \Omega^\bullet \longrightarrow (M \otimes_B B') \otimes_{B'} (\Omega')^\bullet = M \otimes_B (\Omega')^\bullet$$

which maps $m \otimes \eta$ to $m \otimes \varphi(\eta)$.

07KK Lemma 60.6.10. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Let p be a prime number. Assume that A is a $\mathbf{Z}_{(p)}$-algebra and that p is nilpotent in B/J. Then we have

$$\lim_e \Omega_{B_e/A,\bar\delta} = \lim_e \Omega_{B/A,\delta}/p^e\Omega_{B/A,\delta} = \lim_e \Omega_{B^\wedge/A,\delta^\wedge}/p^e\Omega_{B^\wedge/A,\delta^\wedge}$$

see proof for notation and explanation.

Proof. By Divided Power Algebra, Lemma 23.4.5 we see that δ extends to $B_e = B/p^eB$ for all sufficiently large e. Hence the first limit make sense. The lemma also produces a divided power structure $\delta^\wedge$ on the completion $B^\wedge = \lim_e B_e$, hence the last limit makes sense. By Lemma 60.6.2 and the fact that $\mathrm{d}p^e = 0$ (always) we see that the surjection $\Omega_{B/A,\delta} \to \Omega_{B_e/A,\bar\delta}$ has kernel $p^e\Omega_{B/A,\delta}$. Similarly for the kernel of $\Omega_{B^\wedge/A,\delta^\wedge} \to \Omega_{B_e/A,\bar\delta}$. Hence the lemma is clear. □

60.7. Divided power schemes

07I1 Some remarks on how to globalize the previous notions.

07I2 Definition 60.7.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. A divided power structure γ on $\mathcal{I}$ is a sequence of maps $\gamma_n : \mathcal{I} \to \mathcal{I}$, $n \geq 1$ such that for any object U of $\mathcal{C}$ the triple

$$(\mathcal{O}(U), \mathcal{I}(U), \gamma)$$

is a divided power ring.

To be sure this applies in particular to sheaves of rings on topological spaces. But it's good to be a little bit more general as the structure sheaf of the crystalline site lives on a... site! A triple $(\mathcal{C}, \mathcal{I}, \gamma)$ as in the definition above is sometimes called a divided power topos in this chapter. Given a second $(\mathcal{C}', \mathcal{I}', \gamma')$ and given a morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ we say that $(f, f^\sharp)$ induces a morphism of divided power topoi if $f^\sharp(f^{-1}\mathcal{I}') \subset \mathcal{I}$ and the diagrams

$$\begin{tikzcd} f^{-1}\mathcal{I}' \arrow[r, "f^\sharp"'] \arrow[d, "f^{-1}\gamma'_n"'] & \mathcal{I} \arrow[d, "\gamma_n"] \\ f^{-1}\mathcal{I}' \arrow[r, "f^\sharp"] & \mathcal{I} \end{tikzcd}$$

are commutative for all $n \geq 1$. If f comes from a morphism of sites induced by a functor $u : \mathcal{C}' \to \mathcal{C}$ then this just means that

$$(\mathcal{O}'(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}(u(U')), \mathcal{I}(u(U')), \gamma)$$

is a homomorphism of divided power rings for all $U' \in \mathrm{Ob}(\mathcal{C}')$.

In the case of schemes we require the divided power ideal to be quasi-coherent. But apart from this the definition is exactly the same as in the case of topoi. Here it is.

07I3 Definition 60.7.2. A divided power scheme is a triple $(S, \mathcal{I}, \gamma)$ where S is a scheme, $\mathcal{I}$ is a quasi-coherent sheaf of ideals, and γ is a divided power structure on $\mathcal{I}$. A morphism of divided power schemes $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \to S'$ such that $f^{-1}\mathcal{I}'\mathcal{O}_S \subset \mathcal{I}$ and such that

$$(\mathcal{O}_{S'}(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}_S(f^{-1}U'), \mathcal{I}(f^{-1}U'), \gamma)$$

is a homomorphism of divided power rings for all $U' \subset S'$ open.

Recall that there is a 1-to-1 correspondence between quasi-coherent sheaves of ideals and closed immersions, see Morphisms, Section 29.2. Thus given a divided power scheme $(T, \mathcal{J}, \gamma)$ we get a canonical closed immersion $U \to T$ defined by $\mathcal{J}$. Conversely, given a closed immersion $U \to T$ and a divided power structure γ on the sheaf of ideals $\mathcal{J}$ associated to $U \to T$ we obtain a divided power scheme $(T, \mathcal{J}, \gamma)$. In many situations we only want to consider such triples (U, T, γ) when the morphism $U \to T$ is a thickening, see More on Morphisms, Definition 37.2.1.

07I4 Definition 60.7.3. A triple (U, T, γ) as above is called a divided power thickening if $U \to T$ is a thickening.

Fibre products of divided power schemes exist when one of the three is a divided power thickening. Here is a formal statement.

07ME Lemma 60.7.4. Let $(U', T', \delta') \to (S'_0, S', \gamma')$ and $(S_0, S, \gamma) \to (S'_0, S', \gamma')$ be morphisms of divided power schemes. If (U', T', δ') is a divided power thickening, then there exists a divided power scheme (T_0, T, δ) and

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

which is a cartesian diagram in the category of divided power schemes.

Proof. Omitted. Hints: If T exists, then $T_0 = S_0 \times_{S'_0} U'$ (argue as in Divided Power Algebra, Remark 23.3.5). Since T' is a divided power thickening, we see that T (if it exists) will be a divided power thickening too. Hence we can define T as the scheme with underlying topological space the underlying topological space of $T_0 = S_0 \times_{S'_0} U'$ and as structure sheaf on affine pieces the ring given by Lemma 60.5.3. $\square$

We make the following observation. Suppose that (U, T, γ) is triple as above. Assume that T is a scheme over $\mathbf{Z}_{(p)}$ and that p is locally nilpotent on U. Then

(1) p locally nilpotent on $T \Leftrightarrow U \to T$ is a thickening (see Divided Power Algebra, Lemma 23.2.6), and

(2) $p^e\mathcal{O}_T$ is locally on T preserved by γ for $e \gg 0$ (see Divided Power Algebra, Lemma 23.4.5).

This suggest that good results on divided power thickenings will be available under the following hypotheses.

07MF Situation 60.7.5. Here p is a prime number and $(S, \mathcal{I}, \gamma)$ is a divided power scheme over $\mathbf{Z}_{(p)}$. We set $S_0 = V(\mathcal{I}) \subset S$. Finally, $X \to S_0$ is a morphism of schemes such that p is locally nilpotent on X.

It is in this situation that we will define the big and small crystalline sites.

60.8. The big crystalline site

07I5 We first define the big site. Given a divided power scheme $(S, \mathcal{I}, \gamma)$ we say $(T, \mathcal{J}, \delta)$ is a divided power scheme over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes. Similarly, we say a divided power thickening (U, T, δ) is a divided power thickening over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes.

07I6 Definition 60.8.1. In Situation 60.7.5.

(1) A divided power thickening of X relative to $(S, \mathcal{I}, \gamma)$ is given by a divided power thickening (U, T, δ) over $(S, \mathcal{I}, \gamma)$ and an S-morphism $U \to X$.
(2) A morphism of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$ is defined in the obvious manner.

The category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$ is denoted $\text{CRIS}(X/S, \mathcal{I}, \gamma)$ or simply $\text{CRIS}(X/S)$.

For any (U, T, δ) in $\text{CRIS}(X/S)$ we have that p is locally nilpotent on T, see discussion preceding Situation 60.7.5. A good way to visualize all the data associated to (U, T, δ) is the commutative diagram

$$\begin{tikzcd} T \arrow[dd] & U \arrow[l] \arrow[d] \\ & X \arrow[d] \\ S & S_0 \arrow[l] \end{tikzcd}$$

where $S_0 = V(\mathcal{I}) \subset S$. Morphisms of $\text{CRIS}(X/S)$ can be similarly visualized as huge commutative diagrams. In particular, there is a canonical forgetful functor

07I7 (60.8.1.1) $$\text{CRIS}(X/S) \longrightarrow Sch/X, \quad (U, T, \delta) \longmapsto U$$

as well as its one sided inverse (and left adjoint)

07I8 (60.8.1.2) $$Sch/X \longrightarrow \text{CRIS}(X/S), \quad U \longmapsto (U, U, \emptyset)$$

which is sometimes useful.

07I9 Lemma 60.8.2. In Situation 60.7.5. The category $\text{CRIS}(X/S)$ has all finite nonempty limits, in particular products of pairs and fibre products. The functor (60.8.1.1) commutes with limits.

Proof. Omitted. Hint: See Lemma 60.5.3 for the affine case. See also Divided Power Algebra, Remark 23.3.5. $\square$

07IA Lemma 60.8.3. In Situation 60.7.5. Let

$$\begin{tikzcd} (U_3, T_3, \delta_3) \arrow[r] \arrow[d] & (U_2, T_2, \delta_2) \arrow[d] \\ (U_1, T_1, \delta_1) \arrow[r] & (U, T, \delta) \end{tikzcd}$$

be a fibre square in the category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$. If $T_2 \to T$ is flat and $U_2 = T_2 \times_T U$, then $T_3 = T_1 \times_T T_2$ (as schemes).

Proof. This is true because a divided power structure extends uniquely along a flat ring map. See Divided Power Algebra, Lemma 23.4.2. □

The lemma above means that the base change of a flat morphism of divided power thickenings is another flat morphism, and in fact is the "usual" base change of the morphism. This implies that the following definition makes sense.

07IB Definition 60.8.4. In Situation 60.7.5.

(1) A family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ of divided power thickenings of X/S is a Zariski, étale, smooth, syntomic, or fppf covering if and only if
   (a) $U_i = U \times_T T_i$ for all i and
   (b) $\{T_i \to T\}$ is a Zariski, étale, smooth, syntomic, or fppf covering.
(2) The big crystalline site of X over $(S, \mathcal{I}, \gamma)$, is the category $\text{CRIS}(X/S)$ endowed with the Zariski topology.
(3) The topos of sheaves on $\text{CRIS}(X/S)$ is denoted $(X/S)_{\text{CRIS}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\text{CRIS}}$[2].

There are some obvious functorialities concerning these topoi.

07IC Remark 60.8.5 (Functoriality). Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Set $S_0 = V(\mathcal{I})$ and $S'_0 = V(\mathcal{I}')$. Let

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ S_0 \arrow[r] & S'_0 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. Then we get a continuous and cocontinuous functor

$$\text{CRIS}(X/S) \longrightarrow \text{CRIS}(Y/S')$$

by letting (U, T, δ) correspond to (U, T, δ) with $U \to X \to Y$ as the S'-morphism from U to Y. Hence we get a morphism of topoi

$$f_{\text{CRIS}} : (X/S)_{\text{CRIS}} \longrightarrow (Y/S')_{\text{CRIS}}$$

see Sites, Section 7.21.

[2]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

07ID Remark 60.8.6 (Comparison with Zariski site). In Situation 60.7.5. The functor (60.8.1.1) is cocontinuous (details omitted) and commutes with products and fibred products (Lemma 60.8.2). Hence we obtain a morphism of topoi

$$U_{X/S} : (X/S)_{\text{CRIS}} \longrightarrow Sh((Sch/X)_{Zar})$$

from the big crystalline topos of X/S to the big Zariski topos of X. See Sites, Section 7.21.

07IE Remark 60.8.7 (Structure morphism). In Situation 60.7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 60.8.1 with S_0 instead of X. Hence we get a site $\text{CRIS}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of morphisms of ringed topoi

$$\begin{tikzcd} (X/S)_{\text{CRIS}} \arrow[r, "f_{\text{CRIS}}"'] \arrow[d, "U_{X/S}"'] & (S_0/S)_{\text{CRIS}} \arrow[d, "U_{S_0/S}"] & \\ Sh((Sch/X)_{Zar}) \arrow[r, "f_{big}"] & Sh((Sch/S_0)_{Zar}) \arrow[rd] & \\ & & Sh((Sch/S)_{Zar}) \end{tikzcd}$$

by Remark 60.8.5. We think of the composition $(X/S)_{\text{CRIS}} \to Sh((Sch/S)_{Zar})$ as the structure morphism of the big crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$(X/S)_{\text{CRIS}} \longrightarrow Sh((Sch/S)_{Zar})$$

is defined as we can take the lower route through the diagram above. Thus it is the morphism of topoi corresponding to the cocontinuous functor $\text{CRIS}(X/S) \to (Sch/S)_{Zar}$ given by the rule $(U, T, \delta)/S \mapsto U/S$, see Sites, Section 7.21.

07MG Remark 60.8.8 (Compatibilities). The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 60.8.5 we obtain a commutative diagram of ringed topoi

$$\begin{tikzcd} (X/S)_{\text{CRIS}} \arrow[r] \arrow[d] & (Y/S')_{\text{CRIS}} \arrow[d] \\ Sh((Sch/S)_{Zar}) \arrow[r] & Sh((Sch/S')_{Zar}) \end{tikzcd}$$

where the vertical arrows are the structure morphisms.

60.9. The crystalline site

07IF Since (60.8.1.1) commutes with products and fibre products, we see that looking at those (U, T, δ) such that $U \to X$ is an open immersion defines a full subcategory preserved under fibre products (and more generally finite nonempty limits). Hence the following definition makes sense.

07IG Definition 60.9.1. In Situation 60.7.5.

(1) The (small) crystalline site of X over $(S, \mathcal{I}, \gamma)$, denoted $\text{Cris}(X/S, \mathcal{I}, \gamma)$ or simply $\text{Cris}(X/S)$ is the full subcategory of $\text{CRIS}(X/S)$ consisting of those (U, T, δ) in $\text{CRIS}(X/S)$ such that $U \to X$ is an open immersion. It comes endowed with the Zariski topology.
(2) The topos of sheaves on $\text{Cris}(X/S)$ is denoted $(X/S)_{\text{cris}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\text{cris}}$[3].

For any (U, T, δ) in $\text{Cris}(X/S)$ the morphism $U \to X$ defines an object of the small Zariski site X_{Zar} of X. Hence a canonical forgetful functor

07IH (60.9.1.1) $$\text{Cris}(X/S) \longrightarrow X_{Zar}, \quad (U, T, \delta) \longmapsto U$$

and a left adjoint

07II (60.9.1.2) $$X_{Zar} \longrightarrow \text{Cris}(X/S), \quad U \longmapsto (U, U, \emptyset)$$

which is sometimes useful.

We can compare the small and big crystalline sites, just like we can compare the small and big Zariski sites of a scheme, see Topologies, Lemma 34.3.14.

07IJ **Lemma 60.9.2.** Assumptions as in Definition 60.8.1. The inclusion functor

$$\text{Cris}(X/S) \to \text{CRIS}(X/S)$$

commutes with finite nonempty limits, is fully faithful, continuous, and cocontinuous. There are morphisms of topoi

$$(X/S)_{\text{cris}} \xrightarrow{i} (X/S)_{\text{CRIS}} \xrightarrow{\pi} (X/S)_{\text{cris}}$$

whose composition is the identity and of which the first is induced by the inclusion functor. Moreover, $\pi_* = i^{-1}$.

Proof. For the first assertion see Lemma 60.8.2. This gives us a morphism of topoi $i : (X/S)_{\text{cris}} \to (X/S)_{\text{CRIS}}$ and a left adjoint $i_!$ such that $i^{-1}i_! = i^{-1}i_* = \text{id}$, see Sites, Lemmas 7.21.5, 7.21.6, and 7.21.7. We claim that $i_!$ is exact. If this is true, then we can define π by the rules $\pi^{-1} = i_!$ and $\pi_* = i^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_!$ is right exact and preserves fibre products (see references given). Hence it suffices to show that $i_! * = *$ where $*$ indicates the final object in the category of sheaves of sets. To see this it suffices to produce a set of objects (U_i, T_i, δ_i), $i \in I$ of $\text{Cris}(X/S)$ such that

$$\coprod\nolimits_{i \in I} h_{(U_i, T_i, \delta_i)} \to *$$

is surjective in $(X/S)_{\text{CRIS}}$ (details omitted; hint: use that $\text{Cris}(X/S)$ has products and that the functor $\text{Cris}(X/S) \to \text{CRIS}(X/S)$ commutes with them). In the affine case this follows from Lemma 60.5.6. We omit the proof in general. $\square$

07IK **Remark 60.9.3** (Functoriality). Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ S_0 \arrow[r] & S'_0 \end{tikzcd}$$

[3] This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. By analogy with Topologies, Lemma 34.3.17 we define

$$f_{\text{cris}} : (X/S)_{\text{cris}} \longrightarrow (Y/S')_{\text{cris}}$$

by the formula $f_{\text{cris}} = \pi_Y \circ f_{\text{CRIS}} \circ i_X$ where i_X and π_Y are as in Lemma 60.9.2 for X and Y and where f_{CRIS} is as in Remark 60.8.5.

07IL Remark 60.9.4 (Comparison with Zariski site). In Situation 60.7.5. The functor (60.9.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$u_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(X_{Zar})$$

relating the small crystalline topos of X/S with the small Zariski topos of X. See Sites, Section 7.21.

07KL Lemma 60.9.5. In Situation 60.7.5. Let $X' \subset X$ and $S' \subset S$ be open subschemes such that X' maps into S'. Then there is a fully faithful functor $\text{Cris}(X'/S') \to \text{Cris}(X/S)$ which gives rise to a morphism of topoi fitting into the commutative diagram

$$\begin{tikzcd}
(X'/S')_{\text{cris}} \arrow[r] \arrow[d, "u_{X'/S'}"'] & (X/S)_{\text{cris}} \arrow[d, "u_{X/S}"] \\
Sh(X'_{Zar}) \arrow[r] & Sh(X_{Zar})
\end{tikzcd}$$

Moreover, this diagram is an example of localization of morphisms of topoi as in Sites, Lemma 7.31.1.

Proof. The fully faithful functor comes from thinking of objects of $\text{Cris}(X'/S')$ as divided power thickenings (U, T, δ) of X where $U \to X$ factors through $X' \subset X$ (since then automatically $T \to S$ will factor through S'). This functor is clearly cocontinuous hence we obtain a morphism of topoi as indicated. Let $h_{X'} \in Sh(X_{Zar})$ be the representable sheaf associated to X' viewed as an object of X_{Zar}. It is clear that $Sh(X'_{Zar})$ is the localization $Sh(X_{Zar})/h_{X'}$. On the other hand, the category $\text{Cris}(X/S)/u_{X/S}^{-1}h_{X'}$ (see Sites, Lemma 7.30.3) is canonically identified with $\text{Cris}(X'/S')$ by the functor above. This finishes the proof. $\square$

07IM Remark 60.9.6 (Structure morphism). In Situation 60.7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 60.8.1 with S_0 instead of X. Hence we get a site $\text{Cris}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of ringed topoi

$$\begin{tikzcd}
(X/S)_{\text{cris}} \arrow[r, "f_{\text{cris}}"'] \arrow[d, "u_{X/S}"'] & (S_0/S)_{\text{cris}} \arrow[d, "u_{S_0/S}"] & \\
Sh(X_{Zar}) \arrow[r, "f_{small}"] & Sh(S_{0,Zar}) \arrow[dr] & \\
& & Sh(S_{Zar})
\end{tikzcd}$$

see Remark 60.9.3. We think of the composition $(X/S)_{\text{cris}} \to Sh(S_{Zar})$ as the structure morphism of the crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$\tau_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(S_{Zar})$$

is defined as we can take the lower route through the diagram above.

07MH Remark 60.9.7 (Compatibilities). The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 60.9.3 we obtain a commutative diagram of ringed topoi

$$\begin{tikzcd} (X/S)_{\text{cris}} \arrow[r] \arrow[d] & (Y/S')_{\text{cris}} \arrow[d] \\ Sh((Sch/S)_{Zar}) \arrow[r] & Sh((Sch/S')_{Zar}) \end{tikzcd}$$

where the vertical arrows are the structure morphisms.

60.10. Sheaves on the crystalline site

07IN Notation and assumptions as in Situation 60.7.5. In order to discuss the small and big crystalline sites of X/S simultaneously in this section we let

$$\mathcal{C} = \text{CRIS}(X/S) \quad \text{or} \quad \mathcal{C} = \text{Cris}(X/S).$$

A sheaf $\mathcal{F}$ on $\mathcal{C}$ gives rise to a restriction $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$. Namely, $\mathcal{F}_T$ is the Zariski sheaf on the scheme T defined by the rule

$$\mathcal{F}_T(W) = \mathcal{F}(U \cap W, W, \delta|_W)$$

for $W \subset T$ is open. Moreover, if $f : T \to T'$ is a morphism between objects (U, T, δ) and (U', T', δ') of $\mathcal{C}$, then there is a canonical comparison map

07IP (60.10.0.1) $$c_f : f^{-1}\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T.$$

Namely, if $W' \subset T'$ is open then f induces a morphism

$$f|_{f^{-1}W'} : (U \cap f^{-1}(W'), f^{-1}W', \delta|_{f^{-1}W'}) \longrightarrow (U' \cap W', W', \delta|_{W'})$$

of $\mathcal{C}$, hence we can use the restriction mapping $(f|_{f^{-1}W'})^*$ of $\mathcal{F}$ to define a map $\mathcal{F}_{T'}(W') \to \mathcal{F}_T(f^{-1}W')$. These maps are clearly compatible with further restriction, hence define an f-map from $\mathcal{F}_{T'}$ to $\mathcal{F}_T$ (see Sheaves, Section 6.21 and especially Sheaves, Definition 6.21.7). Thus a map c_f as in (60.10.0.1). Note that if f is an open immersion, then c_f is an isomorphism, because in that case $\mathcal{F}_T$ is just the restriction of $\mathcal{F}_{T'}$ to T.

Conversely, given Zariski sheaves $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$ and comparison maps c_f as above which (a) are isomorphisms for open immersions, and (b) satisfy a suitable cocycle condition, we obtain a sheaf on $\mathcal{C}$. This is proved exactly as in Topologies, Lemma 34.3.20.

The structure sheaf on $\mathcal{C}$ is the sheaf $\mathcal{O}_{X/S}$ defined by the rule

$$\mathcal{O}_{X/S} : (U, T, \delta) \longmapsto \Gamma(T, \mathcal{O}_T)$$

This is a sheaf by the definition of coverings in $\mathcal{C}$. Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules. In this case the comparison mappings (60.10.0.1) define a comparison map

07IQ (60.10.0.2) $$c_f : f^*\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T$$

of $\mathcal{O}_T$-modules.

Another type of example comes by starting with a sheaf $\mathcal{G}$ on $(Sch/X)_{Zar}$ or X_{Zar} (depending on whether $\mathcal{C} = \text{CRIS}(X/S)$ or $\mathcal{C} = \text{Cris}(X/S)$). Then $\underline{\mathcal{G}}$ defined by the rule

$$\underline{\mathcal{G}} : (U, T, \delta) \longmapsto \mathcal{G}(U)$$

is a sheaf on $\mathcal{C}$. In particular, if we take $\mathcal{G} = \mathbf{G}_a = \mathcal{O}_X$, then we obtain

$$\underline{\mathbf{G}_a} : (U, T, \delta) \longmapsto \Gamma(U, \mathcal{O}_U)$$

There is a surjective map of sheaves $\mathcal{O}_{X/S} \to \underline{\mathbf{G}_a}$ defined by the canonical maps $\Gamma(T, \mathcal{O}_T) \to \Gamma(U, \mathcal{O}_U)$ for objects (U, T, δ). The kernel of this map is denoted $\mathcal{J}_{X/S}$, hence a short exact sequence

$$0 \to \mathcal{J}_{X/S} \to \mathcal{O}_{X/S} \to \underline{\mathbf{G}_a} \to 0$$

Note that $\mathcal{J}_{X/S}$ comes equipped with a canonical divided power structure. After all, for each object (U, T, δ) the third component δ is a divided power structure on the kernel of $\mathcal{O}_T \to \mathcal{O}_U$. Hence the (big) crystalline topos is a divided power topos.

60.11. Crystals in modules

07IR It turns out that a crystal is a very general gadget. However, the definition may be a bit hard to parse, so we first give the definition in the case of modules on the crystalline sites.

07IS **Definition 60.11.1.** In Situation 60.7.5. Let $\mathcal{C} = \text{CRIS}(X/S)$ or $\mathcal{C} = \text{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathcal{C}$.

(1) We say $\mathcal{F}$ is *locally quasi-coherent* if for every object (U, T, δ) of $\mathcal{C}$ the restriction $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module.
(2) We say $\mathcal{F}$ is *quasi-coherent* if it is quasi-coherent in the sense of Modules on Sites, Definition 18.23.1.
(3) We say $\mathcal{F}$ is a *crystal in $\mathcal{O}_{X/S}$-modules* if all the comparison maps (60.10.0.2) are isomorphisms.

It turns out that we can relate these notions as follows.

07IT **Lemma 60.11.2.** With notation $X/S, \mathcal{I}, \gamma, \mathcal{C}, \mathcal{F}$ as in Definition 60.11.1. The following are equivalent

(1) $\mathcal{F}$ is quasi-coherent, and
(2) $\mathcal{F}$ is locally quasi-coherent and a crystal in $\mathcal{O}_{X/S}$-modules.

Proof. Assume (1). Let $f : (U', T', \delta') \to (U, T, \delta)$ be an object of $\mathcal{C}$. We have to prove (a) $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module and (b) $c_f : f^*\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism. The assumption means that we can find a covering $\{(T_i, U_i, \delta_i) \to (T, U, \delta)\}$ and for each i the restriction of $\mathcal{F}$ to $\mathcal{C}/(T_i, U_i, \delta_i)$ has a global presentation. Since it suffices to prove (a) and (b) Zariski locally, we may replace

$f : (T', U', \delta') \to (T, U, \delta)$ by the base change to (T_i, U_i, δ_i) and assume that $\mathcal{F}$ restricted to $\mathcal{C}/(T, U, \delta)$ has a global presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \mathcal{F}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow 0$$

It is clear that this gives a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

and hence (a) holds. Moreover, the presentation restricts to T' to give a similar presentation of $\mathcal{F}_{T'}$, whence (b) holds.

Assume (2). Let (U, T, δ) be an object of $\mathcal{C}$. We have to find a covering of (U, T, δ) such that $\mathcal{F}$ has a global presentation when we restrict to the localization of $\mathcal{C}$ at the members of the covering. Thus we may assume that T is affine. In this case we can choose a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

as $\mathcal{F}_T$ is assumed to be a quasi-coherent $\mathcal{O}_T$-module. Then by the crystal property of $\mathcal{F}$ we see that this pulls back to a presentation of $\mathcal{F}_{T'}$ for any morphism $f : (U', T', \delta') \to (U, T, \delta)$ of $\mathcal{C}$. Thus the desired presentation of $\mathcal{F}|_{\mathcal{C}/(U,T,\delta)}$. □

07IU Definition 60.11.3. If $\mathcal{F}$ satisfies the equivalent conditions of Lemma 60.11.2, then we say that $\mathcal{F}$ is a crystal in quasi-coherent modules. We say that $\mathcal{F}$ is a crystal in finite locally free modules if, in addition, $\mathcal{F}$ is finite locally free.

Of course, as Lemma 60.11.2 shows, this notation is somewhat heavy since a quasi-coherent module is always a crystal. But it is standard terminology in the literature.

07IV Remark 60.11.4. To formulate the general notion of a crystal we use the language of stacks and strongly cartesian morphisms, see Stacks, Definition 8.4.1 and Categories, Definition 4.33.1. In Situation 60.7.5 let $p : \mathcal{C} \to \text{Cris}(X/S)$ be a stack. A crystal in objects of $\mathcal{C}$ on X relative to S is a cartesian section $\sigma : \text{Cris}(X/S) \to \mathcal{C}$, i.e., a functor σ such that $p \circ \sigma = \text{id}$ and such that $\sigma(f)$ is strongly cartesian for all morphisms f of $\text{Cris}(X/S)$. Similarly for the big crystalline site.

60.12. Sheaf of differentials

07IW In this section we will stick with the (small) crystalline site as it seems more natural. We globalize Definition 60.6.1 as follows.

07IX Definition 60.12.1. In Situation 60.7.5 let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\text{Cris}(X/S)$. An S-derivation $D : \mathcal{O}_{X/S} \to \mathcal{F}$ is a map of sheaves such that for every object (U, T, δ) of $\text{Cris}(X/S)$ the map

$$D : \Gamma(T, \mathcal{O}_T) \longrightarrow \Gamma(T, \mathcal{F})$$

is a divided power $\Gamma(V, \mathcal{O}_V)$-derivation where $V \subset S$ is any open such that $T \to S$ factors through V.

This means that D is additive, satisfies the Leibniz rule, annihilates functions coming from S, and satisfies $D(f^{[n]}) = f^{[n-1]}D(f)$ for a local section f of the divided power ideal $\mathcal{J}_{X/S}$. This is a special case of a very general notion which we now describe.

Please compare the following discussion with Modules on Sites, Section 18.33. Let $\mathcal{C}$ be a site, let $\mathcal{A} \to \mathcal{B}$ be a map of sheaves of rings on $\mathcal{C}$, let $\mathcal{J} \subset \mathcal{B}$ be a sheaf of ideals, let δ be a divided power structure on $\mathcal{J}$, and let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Then there is a notion of a divided power $\mathcal{A}$-derivation $D : \mathcal{B} \to \mathcal{F}$. This means that D is $\mathcal{A}$-linear, satisfies the Leibniz rule, and satisfies $D(\delta_n(x)) = \delta_{n-1}(x)D(x)$ for local sections x of $\mathcal{J}$. In this situation there exists a universal divided power $\mathcal{A}$-derivation

$$\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta} : \mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

Moreover, $\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta}$ is the composition

$$\mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

where the first map is the universal derivation constructed in the proof of Modules on Sites, Lemma 18.33.2 and the second arrow is the quotient by the submodule generated by the local sections $\mathrm{d}_{\mathcal{B}/\mathcal{A}}(\delta_n(x)) - \delta_{n-1}(x)\mathrm{d}_{\mathcal{B}/\mathcal{A}}(x)$.

We translate this into a relative notion as follows. Suppose $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a morphism of ringed topoi, $\mathcal{J} \subset \mathcal{O}$ a sheaf of ideals, δ a divided power structure on $\mathcal{J}$, and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. In this situation we say $D : \mathcal{O} \to \mathcal{F}$ is a divided power $\mathcal{O}'$-derivation if D is a divided power $f^{-1}\mathcal{O}'$-derivation as defined above. Moreover, we write

$$\Omega_{\mathcal{O}/\mathcal{O}',\delta} = \Omega_{\mathcal{O}/f^{-1}\mathcal{O}',\delta}$$

which is the receptacle of the universal divided power $\mathcal{O}'$-derivation.

Applying this to the structure morphism

$$(X/S)_{\mathrm{Cris}} \longrightarrow Sh(S_{Zar})$$

(see Remark 60.9.6) we recover the notion of Definition 60.12.1 above. In particular, there is a universal divided power derivation

$$d_{X/S} : \mathcal{O}_{X/S} \to \Omega_{X/S}$$

Note that we omit from the notation the decoration indicating the module of differentials is compatible with divided powers (it seems unlikely anybody would ever consider the usual module of differentials of the structure sheaf on the crystalline site).

07IY Lemma 60.12.2. Let $(T, \mathcal{J}, \delta)$ be a divided power scheme. Let $T \to S$ be a morphism of schemes. The quotient $\Omega_{T/S} \to \Omega_{T/S,\delta}$ described above is a quasi-coherent $\mathcal{O}_T$-module. For $W \subset T$ affine open mapping into $V \subset S$ affine open we have

$$\Gamma(W, \Omega_{T/S,\delta}) = \Omega_{\Gamma(W,\mathcal{O}_W)/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 60.6.

Proof. Omitted. $\square$

07IZ Lemma 60.12.3. In Situation 60.7.5. For (U, T, δ) in $\mathrm{Cris}(X/S)$ the restriction $(\Omega_{X/S})_T$ to T is $\Omega_{T/S,\delta}$ and the restriction $\mathrm{d}_{X/S}|_T$ is equal to $\mathrm{d}_{T/S,\delta}$.

Proof. Omitted. $\square$

07J0 Lemma 60.12.4. In Situation 60.7.5. For any affine object (U, T, δ) of $\text{Cris}(X/S)$ mapping into an affine open $V \subset S$ we have

$$\Gamma((U, T, \delta), \Omega_{X/S}) = \Omega_{\Gamma(T, \mathcal{O}_T)/\Gamma(V, \mathcal{O}_V), \delta}$$

where the right hand side is as constructed in Section 60.6.

Proof. Combine Lemmas 60.12.2 and 60.12.3. □

07J1 Lemma 60.12.5. In Situation 60.7.5. Let (U, T, δ) be an object of $\text{Cris}(X/S)$. Let

$$(U(1), T(1), \delta(1)) = (U, T, \delta) \times (U, T, \delta)$$

in $\text{Cris}(X/S)$. Let $\mathcal{K} \subset \mathcal{O}_{T(1)}$ be the quasi-coherent sheaf of ideals corresponding to the closed immersion $\Delta : T \to T(1)$. Then $\mathcal{K} \subset \mathcal{J}_{T(1)}$ is preserved by the divided structure on $\mathcal{J}_{T(1)}$ and we have

$$(\Omega_{X/S})_T = \mathcal{K}/\mathcal{K}^{[2]}$$

Proof. Note that $U = U(1)$ as $U \to X$ is an open immersion and as (60.9.1.1) commutes with products. Hence we see that $\mathcal{K} \subset \mathcal{J}_{T(1)}$. Given this fact the lemma follows by working affine locally on T and using Lemmas 60.12.4 and 60.6.5. □

It turns out that $\Omega_{X/S}$ is not a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. But it does satisfy two closely related properties (compare with Lemma 60.11.2).

07KM Lemma 60.12.6. In Situation 60.7.5. The sheaf of differentials $\Omega_{X/S}$ has the following two properties:

(1) $\Omega_{X/S}$ is locally quasi-coherent, and
(2) for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $\text{Cris}(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*(\Omega_{X/S})_{T'} \to (\Omega_{X/S})_T$ is surjective.

Proof. Part (1) follows from a combination of Lemmas 60.12.2 and 60.12.3. Part (2) follows from the fact that $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ is a quotient of $\Omega_{T/S}$ and that $f^*\Omega_{T'/S} \to \Omega_{T/S}$ is surjective. □

60.13. Two universal thickenings

07KN The constructions in this section will help us define a connection on a crystal in modules on the crystalline site. In some sense the constructions here are the "sheafified, universal" versions of the constructions in Section 60.3.

07J2 Remark 60.13.1. In Situation 60.7.5. Let (U, T, δ) be an object of $\text{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 60.12.3. We explicitly describe a first order thickening T' of T. Namely, set

$$\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$$

with algebra structure such that $\Omega_{T/S,\delta}$ is an ideal of square zero. Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Set $\mathcal{J}' = \mathcal{J} \oplus \Omega_{T/S,\delta}$. Define a divided power structure on $\mathcal{J}'$ by setting

$$\delta'_n(f, \omega) = (\delta_n(f), \delta_{n-1}(f)\omega),$$

see Lemma 60.3.1. There are two ring maps

$$p_0, p_1 : \mathcal{O}_T \to \mathcal{O}_{T'}$$

The first is given by $f \mapsto (f, 0)$ and the second by $f \mapsto (f, \text{d}_{T/S,\delta} f)$. Note that both are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}'$ and so is the quotient map $\mathcal{O}_{T'} \to \mathcal{O}_T$. Thus we get an object (U, T', δ') of $\text{Cris}(X/S)$ and a commutative diagram

$$\begin{tikzcd} & T \arrow[ld, "\text{id}"'] \arrow[d, "i"] \arrow[rd, "\text{id}"] & \\ T & T' \arrow[l, "p_0"'] \arrow[r, "p_1"] & T \end{tikzcd}$$

of $\text{Cris}(X/S)$ such that i is a first order thickening whose ideal sheaf is identified with $\Omega_{T/S,\delta}$ and such that $p_1^* - p_0^* : \mathcal{O}_T \to \mathcal{O}_{T'}$ is identified with the universal derivation $\text{d}_{T/S,\delta}$ composed with the inclusion $\Omega_{T/S,\delta} \to \mathcal{O}_{T'}$.

07J3 Remark 60.13.2. In Situation 60.7.5. Let (U, T, δ) be an object of $\text{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 60.12.3. We also write $\Omega^2_{T/S,\delta}$ for its second exterior power. We explicitly describe a second order thickening T'' of T. Namely, set

$$\mathcal{O}_{T''} = \mathcal{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta}$$

with algebra structure defined in the following way

$$(f, \omega_1, \omega_2, \eta)\cdot(f', \omega_1', \omega_2', \eta') = (ff', f\omega_1' + f'\omega_1, f\omega_2' + f'\omega_2, f\eta' + f'\eta + \omega_1 \wedge \omega_2' + \omega_1' \wedge \omega_2).$$

Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Let $\mathcal{J}''$ be the inverse image of $\mathcal{J}$ under the projection $\mathcal{O}_{T''} \to \mathcal{O}_T$. Define a divided power structure on $\mathcal{J}''$ by setting

$$\delta_n''(f, \omega_1, \omega_2, \eta) = (\delta_n(f), \delta_{n-1}(f)\omega_1, \delta_{n-1}(f)\omega_2, \delta_{n-1}(f)\eta + \delta_{n-2}(f)\omega_1 \wedge \omega_2)$$

see Lemma 60.3.2. There are three ring maps $q_0, q_1, q_2 : \mathcal{O}_T \to \mathcal{O}_{T''}$ given by

$$\begin{aligned} q_0(f) &= (f, 0, 0, 0), \\ q_1(f) &= (f, \text{d}f, 0, 0), \\ q_2(f) &= (f, \text{d}f, \text{d}f, 0) \end{aligned}$$

where $\text{d} = \text{d}_{T/S,\delta}$. Note that all three are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}''$. There are three ring maps $q_{01}, q_{12}, q_{02} : \mathcal{O}_{T'} \to \mathcal{O}_{T''}$ where $\mathcal{O}_{T'}$ is as in Remark 60.13.1. Namely, set

$$\begin{aligned} q_{01}(f, \omega) &= (f, \omega, 0, 0), \\ q_{12}(f, \omega) &= (f, \text{d}f, \omega, \text{d}\omega), \\ q_{02}(f, \omega) &= (f, \omega, \omega, 0) \end{aligned}$$

These are also compatible with the given divided power structures. Let's do the verifications for q_{12}: Note that q_{12} is a ring homomorphism as

$$\begin{aligned} q_{12}(f, \omega)q_{12}(g, \eta) &= (f, \text{d}f, \omega, \text{d}\omega)(g, \text{d}g, \eta, \text{d}\eta) \\ &= (fg, f\text{d}g + g\text{d}f, f\eta + g\omega, f\text{d}\eta + g\text{d}\omega + \text{d}f \wedge \eta + \text{d}g \wedge \omega) \\ &= q_{12}(fg, f\eta + g\omega) = q_{12}((f, \omega)(g, \eta)) \end{aligned}$$

Note that q_{12} is compatible with divided powers because

$$\begin{aligned} \delta_n''(q_{12}(f, \omega)) &= \delta_n''((f, \text{d}f, \omega, \text{d}\omega)) \\ &= (\delta_n(f), \delta_{n-1}(f)\text{d}f, \delta_{n-1}(f)\omega, \delta_{n-1}(f)\text{d}\omega + \delta_{n-2}(f)\text{d}(f) \wedge \omega) \\ &= q_{12}((\delta_n(f), \delta_{n-1}(f)\omega)) = q_{12}(\delta_n'(f, \omega)) \end{aligned}$$

The verifications for q_{01} and q_{02} are easier. Note that $q_0 = q_{01} \circ p_0$, $q_1 = q_{01} \circ p_1$, $q_1 = q_{12} \circ p_0$, $q_2 = q_{12} \circ p_1$, $q_0 = q_{02} \circ p_0$, and $q_2 = q_{02} \circ p_1$. Thus (U, T'', δ'') is an object of $\text{Cris}(X/S)$ and we get morphisms

$$T'' \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} T' \substack{\longrightarrow \\ \longrightarrow} T$$

of $\text{Cris}(X/S)$ satisfying the relations described above. In applications we will use $q_i : T'' \to T$ and $q_{ij} : T'' \to T'$ to denote the morphisms associated to the ring maps described above.

60.14. The de Rham complex

07J4 In Situation 60.7.5. Working on the (small) crystalline site, we define $\Omega^i_{X/S} = \wedge^i_{\mathcal{O}_{X/S}} \Omega_{X/S}$ for $i \geq 0$. The universal S-derivation $\text{d}_{X/S}$ gives rise to the de Rham complex

$$\mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

on $\text{Cris}(X/S)$, see Lemma 60.12.4 and Remark 60.6.7.

60.15. Connections

07J5 In Situation 60.7.5. Given an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ on $\text{Cris}(X/S)$ a connection is a map of abelian sheaves

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S}$$

such that $\nabla(fs) = f\nabla(s) + s \otimes \text{d}f$ for local sections s, f of $\mathcal{F}$ and $\mathcal{O}_{X/S}$. Given a connection there are canonical maps $\nabla : \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{i+1}_{X/S}$ defined by the rule $\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes \text{d}\omega$ as in Remark 60.6.8. We say the connection is integrable if $\nabla \circ \nabla = 0$. If ∇ is integrable we obtain the de Rham complex

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots$$

on $\text{Cris}(X/S)$. It turns out that any crystal in $\mathcal{O}_{X/S}$-modules comes equipped with a canonical integrable connection.

07J6 **Lemma 60.15.1.** *In Situation 60.7.5. Let $\mathcal{F}$ be a crystal in $\mathcal{O}_{X/S}$-modules on $\text{Cris}(X/S)$. Then $\mathcal{F}$ comes equipped with a canonical integrable connection.*

Proof. Say (U, T, δ) is an object of $\text{Cris}(X/S)$. Let (U, T', δ') be the infinitesimal thickening of T by $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ constructed in Remark 60.13.1. It comes with projections $p_0, p_1 : T' \to T$ and a diagonal $i : T \to T'$. By assumption we get isomorphisms

$$p_0^*\mathcal{F}_T \xrightarrow{c_0} \mathcal{F}_{T'} \xleftarrow{c_1} p_1^*\mathcal{F}_T$$

of $\mathcal{O}_{T'}$-modules. Pulling $c = c_1^{-1} \circ c_0$ back to T by i we obtain the identity map of $\mathcal{F}_T$. Hence if $s \in \Gamma(T, \mathcal{F}_T)$ then $\nabla(s) = p_1^*s - c(p_0^*s)$ is a section of $p_1^*\mathcal{F}_T$ which vanishes on pulling back by i. Hence $\nabla(s)$ is a section of

$$\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta}$$

because this is the kernel of $p_1^*\mathcal{F}_T \to \mathcal{F}_T$ as $\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$ by construction. It is easily verified that $\nabla(fs) = f\nabla(s) + s \otimes \text{d}(f)$ using the description of d in Remark 60.13.1.

The collection of maps

$$\nabla : \Gamma(T, \mathcal{F}_T) \to \Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta})$$

so obtained is functorial in T because the construction of T' is functorial in T. Hence we obtain a connection.

To show that the connection is integrable we consider the object (U, T'', δ'') constructed in Remark 60.13.2. Because $\mathcal{F}$ is a sheaf we see that

$$\begin{tikzcd} q_0^*\mathcal{F}_T \arrow[rr, "q_{01}^*c"] \arrow[dr, "q_{02}^*c"'] & & q_1^*\mathcal{F}_T \arrow[dl, "q_{12}^*c"] \\ & q_2^*\mathcal{F}_T & \end{tikzcd}$$

is a commutative diagram of $\mathcal{O}_{T''}$-modules. For $s \in \Gamma(T, \mathcal{F}_T)$ we have $c(p_0^*s) = p_1^*s - \nabla(s)$. Write $\nabla(s) = \sum p_1^*s_i \cdot \omega_i$ where s_i is a local section of $\mathcal{F}_T$ and ω_i is a local section of $\Omega_{T/S,\delta}$. We think of ω_i as a local section of the structure sheaf of $\mathcal{O}_{T'}$ and hence we write product instead of tensor product. On the one hand

$$\begin{aligned} q_{12}^*c \circ q_{01}^*c(q_0^*s) &= q_{12}^*c(q_1^*s - \sum q_1^*s_i \cdot q_{01}^*\omega_i) \\ &= q_2^*s - \sum q_2^*s_i \cdot q_{12}^*\omega_i - \sum q_2^*s_i \cdot q_{01}^*\omega_i + \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i \end{aligned}$$

and on the other hand

$$q_{02}^*c(q_0^*s) = q_2^*s - \sum q_2^*s_i \cdot q_{02}^*\omega_i.$$

From the formulae of Remark 60.13.2 we see that $q_{01}^*\omega_i + q_{12}^*\omega_i - q_{02}^*\omega_i = \mathrm{d}\omega_i$. Hence the difference of the two expressions above is

$$\sum q_2^*s_i \cdot \mathrm{d}\omega_i - \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i$$

Note that $q_{12}^*\omega \cdot q_{01}^*\omega' = \omega' \wedge \omega = -\omega \wedge \omega'$ by the definition of the multiplication on $\mathcal{O}_{T''}$. Thus the expression above is $\nabla^2(s)$ viewed as a section of the subsheaf $\mathcal{F}_T \otimes \Omega^2_{T/S,\delta}$ of $q_2^*\mathcal{F}$. Hence we get the integrability condition. $\square$

60.16. Cosimplicial algebra

07KP This section should be moved somewhere else. A cosimplicial ring is a cosimplicial object in the category of rings. Given a ring R, a cosimplicial R-algebra is a cosimplicial object in the category of R-algebras. A cosimplicial ideal in a cosimplicial ring A_* is given by an ideal $I_n \subset A_n$ for all n such that $A(f)(I_n) \subset I_m$ for all $f : [n] \to [m]$ in Δ.

Let A_* be a cosimplicial ring. Let $\mathcal{C}$ be the category of pairs (A, M) where A is a ring and M is a module over A. A morphism $(A, M) \to (A', M')$ consists of a ring map $A \to A'$ and an A-module map $M \to M'$ where M' is viewed as an A-module via $A \to A'$ and the A'-module structure on M'. Having said this we can define a cosimplicial module M_* over A_* as a cosimplicial object (A_*, M_*) of $\mathcal{C}$ whose first entry is equal to A_*. A homomorphism $\varphi_* : M_* \to N_*$ of cosimplicial modules over A_* is a morphism $(A_*, M_*) \to (A_*, N_*)$ of cosimplicial objects in $\mathcal{C}$ whose first component is 1_{A_*}.

A homotopy between homomorphisms $\varphi_*, \psi_* : M_* \to N_*$ of cosimplicial modules over A_* is a homotopy between the associated maps $(A_*, M_*) \to (A_*, N_*)$ whose first component is the trivial homotopy (dual to Simplicial, Example 14.26.3). We spell out what this means. Such a homotopy is a homotopy

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$$

between φ_* and ψ_* as homomorphisms of cosimplicial abelian groups such that for each n the map $h_n : M_n \to \prod_{\alpha \in \Delta[1]_n} N_n$ is A_n-linear. The following lemma is a version of Simplicial, Lemma 14.28.4 for cosimplicial modules.

07KQ **Lemma 60.16.1.** Let A_* be a cosimplicial ring. Let $\varphi_*, \psi_* : K_* \to M_*$ be homomorphisms of cosimplicial A_*-modules.

07KR (1) If φ_* and ψ_* are homotopic, then
$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} L_* \longrightarrow M_* \otimes_{A_*} L_*$$
are homotopic for any cosimplicial A_*-module L_*.

07KS (2) If φ_* and ψ_* are homotopic, then
$$\wedge^i(\varphi_*), \wedge^i(\psi_*) : \wedge^i(K_*) \longrightarrow \wedge^i(M_*)$$
are homotopic.

07KT (3) If φ_* and ψ_* are homotopic, and $A_* \to B_*$ is a homomorphism of cosimplicial rings, then
$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} B_* \longrightarrow M_* \otimes_{A_*} B_*$$
are homotopic as homomorphisms of cosimplicial B_*-modules.

07KU (4) If $I_* \subset A_*$ is a cosimplicial ideal, then the induced maps
$$\varphi_*^\wedge, \psi_*^\wedge : K_*^\wedge \longrightarrow M_*^\wedge$$
between completions are homotopic.

(5) Add more here as needed, for example symmetric powers.

Proof. Let $h : M_* \longrightarrow \text{Hom}(\Delta[1], N_*)$ be the given homotopy. In degree n we have
$$h_n = (h_{n,\alpha}) : K_n \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} K_n$$
see Simplicial, Section 14.28. In order for a collection of $h_{n,\alpha}$ to form a homotopy, it is necessary and sufficient if for every $f : [n] \to [m]$ we have
$$h_{m,\alpha} \circ M_*(f) = N_*(f) \circ h_{n,\alpha \circ f}$$
see Simplicial, Equation (14.28.1.1). We also should have that $\psi_n = h_{n,0:[n]\to[1]}$ and $\varphi_n = h_{n,1:[n]\to[1]}$.

In each of the cases of the lemma we can produce the corresponding maps. Case (1). We can use the homotopy $h \otimes 1$ defined in degree n by setting
$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1_{L_n} : K_n \otimes_{A_n} L_n \longrightarrow M_n \otimes_{A_n} L_n.$$
Case (2). We can use the homotopy $\wedge^i h$ defined in degree n by setting
$$\wedge^i(h)_{n,\alpha} = \wedge^i(h_{n,\alpha}) : \wedge_{A_n}(K_n) \longrightarrow \wedge^i_{A_n}(M_n).$$
Case (3). We can use the homotopy $h \otimes 1$ defined in degree n by setting
$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1 : K_n \otimes_{A_n} B_n \longrightarrow M_n \otimes_{A_n} B_n.$$
Case (4). We can use the homotopy $h^\wedge$ defined in degree n by setting
$$(h^\wedge)_{n,\alpha} = h^\wedge_{n,\alpha} : K_n^\wedge \longrightarrow M_n^\wedge.$$
This works because each $h_{n,\alpha}$ is A_n-linear. $\square$

60.17. Crystals in quasi-coherent modules

07J7 In Situation 60.5.1. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We are going to classify crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Before we do so we fix some notation.

Choose a polynomial ring $P = A[x_i]$ over A and a surjection $P \to C$ of A-algebras with kernel $J = \mathrm{Ker}(P \to C)$. Set

07J8 (60.17.0.1) $$D = \lim_e D_{P,\gamma}(J)/p^e D_{P,\gamma}(J)$$

for the p-adically completed divided power envelope. This ring comes with a divided power ideal $\bar{J}$ and divided power structure $\bar{\gamma}$, see Lemma 60.5.5. Set $D_e = D/p^e D$ and denote $\bar{J}_e$ the image of $\bar{J}$ in D_e. We will use the short hand

07J9 (60.17.0.2) $$\Omega_D = \lim_e \Omega_{D_e/A,\bar{\gamma}} = \lim_e \Omega_{D/A,\bar{\gamma}}/p^e\Omega_{D/A,\bar{\gamma}}$$

for the p-adic completion of the module of divided power differentials, see Lemma 60.6.10. It is also the p-adic completion of $\Omega_{D_{P,\gamma}(J)/A,\bar{\gamma}}$ which is free on $\mathrm{d}x_i$, see Lemma 60.6.6. Hence any element of Ω_D can be written uniquely as a sum $\sum f_i \mathrm{d}x_i$ with for all e only finitely many f_i not in $p^e D$. Moreover, the maps $\mathrm{d}_{D_e/A,\bar{\gamma}} : D_e \to \Omega_{D_e/A,\bar{\gamma}}$ fit together to define a divided power A-derivation

07JA (60.17.0.3) $$\mathrm{d} : D \longrightarrow \Omega_D$$

on p-adic completions.

We will also need the "products $\mathrm{Spec}(D(n))$ of $\mathrm{Spec}(D)$", see Proposition 60.21.1 and its proof for an explanation. Formally these are defined as follows. For $n \geq 0$ let $J(n) = \mathrm{Ker}(P \otimes_A \ldots \otimes_A P \to C)$ where the tensor product has $n+1$ factors. We set

07JF (60.17.0.4) $$D(n) = \lim_e D_{P\otimes_A \ldots \otimes_A P,\gamma}(J(n))/p^e D_{P\otimes_A \ldots \otimes_A P,\gamma}(J(n))$$

equal to the p-adic completion of the divided power envelope. We denote $\bar{J}(n)$ its divided power ideal and $\bar{\gamma}(n)$ its divided powers. We also introduce $D(n)_e = D(n)/p^e D(n)$ as well as the p-adically completed module of differentials

07L0 (60.17.0.5) $$\Omega_{D(n)} = \lim_e \Omega_{D(n)_e/A,\bar{\gamma}} = \lim_e \Omega_{D(n)/A,\bar{\gamma}}/p^e\Omega_{D(n)/A,\bar{\gamma}}$$

and derivation

07L1 (60.17.0.6) $$\mathrm{d} : D(n) \longrightarrow \Omega_{D(n)}$$

Of course we have $D = D(0)$. Note that the rings $D(0), D(1), D(2), \ldots$ form a cosimplicial object in the category of divided power rings.

07L2 **Lemma 60.17.1.** Let D and $D(n)$ be as in (60.17.0.1) and (60.17.0.4). The coprojection $P \to P \otimes_A \ldots \otimes_A P$, $f \mapsto f \otimes 1 \otimes \ldots \otimes 1$ induces an isomorphism

07L3 (60.17.1.1) $$D(n) = \lim_e D\langle \xi_i(j)\rangle/p^e D\langle \xi_i(j)\rangle$$

of algebras over D with

$$\xi_i(j) = x_i \otimes 1 \otimes \ldots \otimes 1 - 1 \otimes \ldots \otimes 1 \otimes x_i \otimes 1 \otimes \ldots \otimes 1$$

for $j = 1, \ldots, n$ where the second x_i is placed in the $j+1$st slot; recall that $D(n)$ is constructed starting with the $n+1$-fold tensor product of P over A.

Proof. We have

$$P \otimes_A \ldots \otimes_A P = P[\xi_i(j)]$$

and $J(n)$ is generated by J and the elements $\xi_i(j)$. Hence the lemma follows from Lemma 60.2.5. □

07L4 Lemma 60.17.2. Let D and $D(n)$ be as in (60.17.0.1) and (60.17.0.4). Then $(D, \bar{J}, \bar{\gamma})$ and $(D(n), \bar{J}(n), \bar{\gamma}(n))$ are objects of $\text{Cris}^\wedge(C/A)$, see Remark 60.5.4, and

$$D(n) = \coprod\nolimits_{j=0,\ldots,n} D$$

in $\text{Cris}^\wedge(C/A)$.

Proof. The first assertion is clear. For the second, if $(B \to C, \delta)$ is an object of $\text{Cris}^\wedge(C/A)$, then we have

$$\text{Mor}_{\text{Cris}^\wedge(C/A)}(D, B) = \text{Hom}_A((P, J), (B, \text{Ker}(B \to C)))$$

and similarly for $D(n)$ replacing (P, J) by $(P \otimes_A \ldots \otimes_A P, J(n))$. The property on coproducts follows as $P \otimes_A \ldots \otimes_A P$ is a coproduct. □

In the lemma below we will consider pairs (M, ∇) satisfying the following conditions

07JB (1) M is a p-adically complete D-module,

07JC (2) $\nabla : M \to M \otimes^\wedge_D \Omega_D$ is a connection, i.e., $\nabla(fm) = m \otimes \text{d}f + f\nabla(m)$,

07JD (3) ∇ is integrable (see Remark 60.6.8), and

07JE (4) ∇ is topologically quasi-nilpotent: If we write $\nabla(m) = \sum \theta_i(m)\text{d}x_i$ for some operators $\theta_i : M \to M$, then for any $m \in M$ there are only finitely many pairs (i, k) such that $\theta_i^k(m) \notin pM$.

The operators θ_i are sometimes denoted $\nabla_{\partial/\partial x_i}$ in the literature. In the following lemma we construct a functor from crystals in quasi-coherent modules on $\text{Cris}(X/S)$ to the category of such pairs. We will show this functor is an equivalence in Proposition 60.17.4.

07JG Lemma 60.17.3. In the situation above there is a functor

$$\begin{matrix} \text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } \text{Cris}(X/S) \end{matrix} \longrightarrow \begin{matrix} \text{pairs } (M, \nabla) \text{ satisfying} \\ (1), (2), (3), \text{ and } (4) \end{matrix}$$

Proof. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on X/S. Set $T_e = \text{Spec}(D_e)$ so that $(X, T_e, \bar{\gamma})$ is an object of $\text{Cris}(X/S)$ for $e \gg 0$. We have morphisms

$$(X, T_e, \bar{\gamma}) \to (X, T_{e+1}, \bar{\gamma}) \to \ldots$$

which are closed immersions. We set

$$M = \lim_e \Gamma((X, T_e, \bar{\gamma}), \mathcal{F}) = \lim_e \Gamma(T_e, \mathcal{F}_{T_e}) = \lim_e M_e$$

Note that since $\mathcal{F}$ is locally quasi-coherent we have $\mathcal{F}_{T_e} = \widetilde{M_e}$. Since $\mathcal{F}$ is a crystal we have $M_e = M_{e+1}/p^e M_{e+1}$. Hence we see that $M_e = M/p^e M$ and that M is p-adically complete, see Algebra, Lemma 10.98.2.

By Lemma 60.15.1 we know that $\mathcal{F}$ comes endowed with a canonical integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/S}$. If we evaluate this connection on the objects T_e constructed above we obtain a canonical integrable connection

$$\nabla : M \longrightarrow M \otimes^\wedge_D \Omega_D$$

To see that this is topologically nilpotent we work out what this means.

Now we can do the same procedure for the rings $D(n)$. This produces a p-adically complete $D(n)$-module $M(n)$. Again using the crystal property of $\mathcal{F}$ we obtain isomorphisms

$$M \otimes^\wedge_{D,p_0} D(1) \to M(1) \leftarrow M \otimes^\wedge_{D,p_1} D(1)$$

compare with the proof of Lemma 60.15.1. Denote c the composition from left to right. Pick $m \in M$. Write $\xi_i = x_i \otimes 1 - 1 \otimes x_i$. Using (60.17.1.1) we can write uniquely

$$c(m \otimes 1) = \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

for some $\theta_K(m) \in M$ where the sum is over multi-indices $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$. Set $\theta_i = \theta_K$ where K has a 1 in the ith spot and zeros elsewhere. We have

$$\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i.$$

as can be seen by comparing with the definition of ∇. Namely, the defining equation is $p_1^*m = \nabla(m) - c(p_0^*m)$ in Lemma 60.15.1 but the sign works out because in the Stacks project we consistently use $\mathrm{d}f = p_1(f) - p_0(f)$ modulo the ideal of the diagonal squared, and hence $\xi_i = x_i \otimes 1 - 1 \otimes x_i$ maps to $-\mathrm{d}x_i$ modulo the ideal of the diagonal squared.

Denote $q_i : D \to D(2)$ and $q_{ij} : D(1) \to D(2)$ the coprojections corresponding to the indices i, j. As in the last paragraph of the proof of Lemma 60.15.1 we see that

$$q_{02}^*c = q_{12}^*c \circ q_{01}^*c.$$

This means that

$$\sum\nolimits_{K''} \theta_{K''}(m) \otimes \prod \zeta_i''^{[k_i'']} = \sum\nolimits_{K', K} \theta_{K'}(\theta_K(m)) \otimes \prod \zeta_i'^{[k_i']} \prod \zeta_i^{[k_i]}$$

in $M \otimes^\wedge_{D,q_2} D(2)$ where

$$\begin{aligned} \zeta_i &= x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1, \\ \zeta_i' &= 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i, \\ \zeta_i'' &= x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i. \end{aligned}$$

In particular $\zeta_i'' = \zeta_i + \zeta_i'$ and we have that $D(2)$ is the p-adic completion of the divided power polynomial ring in ζ_i, ζ_i' over $q_2(D)$, see Lemma 60.17.1. Comparing coefficients in the expression above it follows immediately that $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ (this provides an alternative proof of the integrability of ∇) and that

$$\theta_K(m) = (\prod \theta_i^{k_i})(m).$$

In particular, as the sum expressing $c(m \otimes 1)$ above has to converge p-adically we conclude that for each i and each $m \in M$ only a finite number of $\theta_i^k(m)$ are allowed to be nonzero modulo p. □

07JH **Proposition 60.17.4.** The functor

$$\begin{matrix} \text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } \mathrm{Cris}(X/S) \end{matrix} \longrightarrow \begin{matrix} \text{pairs } (M, \nabla) \text{ satisfying} \\ (1), (2), (3), \text{ and } (4) \end{matrix}$$

of Lemma 60.17.3 is an equivalence of categories.

Proof. Let (M, ∇) be given. We are going to construct a crystal in quasi-coherent modules $\mathcal{F}$. Write $\nabla(m) = \sum \theta_i(m) \text{d}x_i$. Then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ and we can set $\theta_K(m) = (\prod \theta_i^{k_i})(m)$ for any multi-index $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$.

Let (U, T, δ) be any object of $\text{Cris}(X/S)$ with T affine. Say $T = \text{Spec}(B)$ and the ideal of $U \to T$ is $J_B \subset B$. By Lemma 60.5.6 there exists an integer e and a morphism

$$f : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$$

where $T_e = \text{Spec}(D_e)$ as in the proof of Lemma 60.17.3. Choose such an e and f; denote $f : D \to B$ also the corresponding divided power A-algebra map. We will set $\mathcal{F}_T$ equal to the quasi-coherent sheaf of $\mathcal{O}_T$-modules associated to the B-module

$$M \otimes_{D,f} B.$$

However, we have to show that this is independent of the choice of f. Suppose that $g : D \to B$ is a second such morphism. Since f and g are morphisms in $\text{Cris}(X/S)$ we see that the image of $f - g : D \to B$ is contained in the divided power ideal J_B. Write $\xi_i = f(x_i) - g(x_i) \in J_B$. By analogy with the proof of Lemma 60.17.3 we define an isomorphism

$$c_{f,g} : M \otimes_{D,f} B \longrightarrow M \otimes_{D,g} B$$

by the formula

$$m \otimes 1 \longmapsto \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

which makes sense by our remarks above and the fact that ∇ is topologically quasi-nilpotent (so the sum is finite!). A computation shows that

$$c_{g,h} \circ c_{f,g} = c_{f,h}$$

if given a third morphism $h : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$. It is also true that $c_{f,f} = 1$. Hence these maps are all isomorphisms and we see that the module $\mathcal{F}_T$ is independent of the choice of f.

If $a : (U', T', \delta') \to (U, T, \delta)$ is a morphism of affine objects of $\text{Cris}(X/S)$, then choosing $f' = f \circ a$ it is clear that there exists a canonical isomorphism $a^*\mathcal{F}_T \to \mathcal{F}_{T'}$. We omit the verification that this map is independent of the choice of f. Using these maps as the restriction maps it is clear that we obtain a crystal in quasi-coherent modules on the full subcategory of $\text{Cris}(X/S)$ consisting of affine objects. We omit the proof that this extends to a crystal on all of $\text{Cris}(X/S)$. We also omit the proof that this procedure is a functor and that it is quasi-inverse to the functor constructed in Lemma 60.17.3. □

07L5 Lemma 60.17.5. In Situation 60.5.1. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. There are homomorphisms of divided power A-algebras

$$a : D \longrightarrow D', \quad b : D' \longrightarrow D$$

compatible with the maps $D \to C$ and $D' \to C$ such that $a \circ b = \text{id}_{D'}$. These maps induce an equivalence of categories of pairs (M, ∇) satisfying (1), (2), (3), and (4) over D and pairs (M', ∇') satisfying (1), (2), (3), and (4)[4] over D'. In particular, the equivalence of categories of Proposition 60.17.4 also holds for the corresponding functor towards pairs over D'.

[4]This condition is tricky to formulate for (M', ∇') over D'. See proof.

Proof. First, suppose that $P' = A[y_1, \ldots, y_m]$ is a polynomial algebra over A. In this case, we can find ring maps $P \to P'$ and $P' \to P$ compatible with the maps to C which induce maps $a : D \to D'$ and $b : D' \to D$ as in the lemma. Using completed base change along a and b we obtain functors between the categories of modules with connection satisfying properties (1), (2), (3), and (4) simply because these these categories are equivalent to the category of quasi-coherent crystals by Proposition 60.17.4 (and this equivalence is compatible with the base change operation as shown in the proof of the proposition).

Proof for general smooth P'. By the first paragraph of the proof we may assume $P = A[y_1, \ldots, y_m]$ which gives us a surjection $P \to P'$ compatible with the map to C. Hence we obtain a surjective map $a : D \to D'$ by functoriality of divided power envelopes and completion. Pick e large enough so that D_e is a divided power thickening of C over A. Then $D_e \to C$ is a surjection whose kernel is locally nilpotent, see Divided Power Algebra, Lemma 23.2.6. Setting $D'_e = D'/p^eD'$ we see that the kernel of $D_e \to D'_e$ is locally nilpotent. Hence by Algebra, Lemma 10.138.17 we can find a lift $\beta_e : P' \to D_e$ of the map $P' \to D'_e$. Note that $D_{e+i+1} \to D_{e+i} \times_{D'_{e+i}} D'_{e+i+1}$ is surjective with square zero kernel for any $i \geq 0$ because $p^{e+i}D \to p^{e+i}D'$ is surjective. Applying the usual lifting property (Algebra, Proposition 10.138.13) successively to the diagrams

$$\begin{tikzcd} P' \arrow[r] & D_{e+i} \times_{D'_{e+i}} D'_{e+i+1} \\ A \arrow[u] \arrow[r] & D_{e+i+1} \arrow[u] \end{tikzcd}$$

we see that we can find an A-algebra map $\beta : P' \to D$ whose composition with a is the given map $P' \to D'$. By the universal property of the divided power envelope we obtain a map $D_{P',\gamma}(J') \to D$. As D is p-adically complete we obtain $b : D' \to D$ such that $a \circ b = \mathrm{id}_{D'}$.

Consider the base change functors

$$F : (M, \nabla) \longmapsto (M \otimes^\wedge_{D,a} D', \nabla') \quad \text{and} \quad G : (M', \nabla') \longmapsto (M' \otimes^\wedge_{D',b} D, \nabla)$$

on modules with connections satisfying (1), (2), and (3). See Remark 60.6.9. Since $a \circ b = \mathrm{id}_{D'}$ we see that $F \circ G$ is the identity functor. Let us say that (M', ∇') has property (4) if this is true for $G(M', \nabla')$. A formal argument now shows that to finish the proof it suffices to show that $G(F(M, \nabla))$ is isomorphic to (M, ∇) in the case that (M, ∇) satisfies all four conditions (1), (2), (3), and (4). For this we use the functorial isomorphism

$$c_{\mathrm{id}_D, b\circ a} : M \otimes_{D,\mathrm{id}_D} D \longrightarrow M \otimes_{D,b\circ a} D$$

of the proof of Proposition 60.17.4 (which requires the topological quasi-nilpotency of ∇ which we have assumed). It remains to prove that this map is horizontal, i.e., compatible with connections, which we omit.

The last statement of the proof now follows. □

07L6 Remark 60.17.6. The equivalence of Proposition 60.17.4 holds if we start with a surjection $P \to C$ where P/A satisfies the strong lifting property of Algebra, Lemma 10.138.17. To prove this we can argue as in the proof of Lemma 60.17.5. (Details will be added here if we ever need this.) Presumably there is also a direct proof of this result, but the advantage of using polynomial rings is that the rings $D(n)$ are p-adic completions of divided power polynomial rings and the algebra is simplified.

60.18. General remarks on cohomology

07JI In this section we do a bit of work to translate the cohomology of modules on the cristalline site of an affine scheme into an algebraic question.

07JJ Lemma 60.18.1. In Situation 60.7.5. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_{X/S}$-module on $\text{Cris}(X/S)$. Then we have

$$H^p((U, T, \delta), \mathcal{F}) = 0$$

for all $p > 0$ and all (U, T, δ) with T or U affine.

Proof. As $U \to T$ is a thickening we see that U is affine if and only if T is affine, see Limits, Lemma 32.11.1. Having said this, let us apply Cohomology on Sites, Lemma 21.10.9 to the collection $\mathcal{B}$ of affine objects (U, T, δ) and the collection Cov of affine open coverings $\mathcal{U} = \{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$. The Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F})$ for such a covering is simply the Čech complex of the quasi-coherent $\mathcal{O}_T$-module $\mathcal{F}_T$ (here we are using the assumption that $\mathcal{F}$ is locally quasi-coherent) with respect to the affine open covering $\{T_i \to T\}$ of the affine scheme T. Hence the Čech cohomology is zero by Cohomology of Schemes, Lemma 30.2.6 and 30.2.2. Thus the hypothesis of Cohomology on Sites, Lemma 21.10.9 are satisfied and we win. □

07JK Lemma 60.18.2. In Situation 60.7.5. Assume moreover X and S are affine schemes. Consider the full subcategory $\mathcal{C} \subset \text{Cris}(X/S)$ consisting of divided power thickenings (X, T, δ) endowed with the chaotic topology (see Sites, Example 7.6.6). For any locally quasi-coherent $\mathcal{O}_{X/S}$-module $\mathcal{F}$ we have

$$R\Gamma(\mathcal{C}, \mathcal{F}|_{\mathcal{C}}) = R\Gamma(\text{Cris}(X/S), \mathcal{F})$$

Proof. Denote $\text{AffineCris}(X/S)$ the fully subcategory of $\text{Cris}(X/S)$ consisting of those objects (U, T, δ) with U and T affine. We turn this into a site by saying a family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}_{i \in I}$ of $\text{AffineCris}(X/S)$ is a covering if and only if it is a covering of $\text{Cris}(X/S)$. With this definition the inclusion functor

$$\text{AffineCris}(X/S) \longrightarrow \text{Cris}(X/S)$$

is a special cocontinuous functor as defined in Sites, Definition 7.29.2. The proof of this is exactly the same as the proof of Topologies, Lemma 34.3.10. Thus we see that the topos of sheaves on $\text{Cris}(X/S)$ is the same as the topos of sheaves on $\text{AffineCris}(X/S)$ via restriction by the displayed inclusion functor. Therefore we have to prove the corresponding statement for the inclusion $\mathcal{C} \subset \text{AffineCris}(X/S)$.

We will use without further mention that $\mathcal{C}$ and $\text{AffineCris}(X/S)$ have products and fibre products (details omitted, see Lemma 60.8.2). The inclusion functor $u : \mathcal{C} \to \text{AffineCris}(X/S)$ is fully faithful, continuous, and commutes with products and fibre products. We claim it defines a morphism of ringed sites

$$f : (\text{AffineCris}(X/S), \mathcal{O}_{X/S}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{X/S}|_{\mathcal{C}})$$

To see this we will use Sites, Lemma 7.14.6. Note that $\mathcal{C}$ has fibre products and u commutes with them so the categories $\mathcal{I}^u_{(U,T,\delta)}$ are disjoint unions of directed categories (by Sites, Lemma 7.5.1 and Categories, Lemma 4.19.8). Hence it suffices to show that $\mathcal{I}^u_{(U,T,\delta)}$ is connected. Nonempty follows from Lemma 60.5.6: since U and T are affine that lemma says there is at least one object (X, T', δ') of $\mathcal{C}$ and a morphism $(U, T, \delta) \to (X, T', \delta')$ of divided power thickenings. Connectedness follows from the fact that $\mathcal{C}$ has products and that u commutes with them (compare with the proof of Sites, Lemma 7.5.2).

Note that $f_*\mathcal{F} = \mathcal{F}|_\mathcal{C}$. Hence the lemma follows if $R^pf_*\mathcal{F} = 0$ for $p > 0$, see Cohomology on Sites, Lemma 21.14.6. By Cohomology on Sites, Lemma 21.7.4 it suffices to show that $H^p(\text{AffineCris}(X/S)/(X, T, \delta), \mathcal{F}) = 0$ for all (X, T, δ). This follows from Lemma 60.18.1 because the topos of the site $\text{AffineCris}(X/S)/(X, T, \delta)$ is equivalent to the topos of the site $\text{Cris}(X/S)/(X, T, \delta)$ used in the lemma. □

07JL Lemma 60.18.3. In Situation 60.5.1. Set $\mathcal{C} = (\text{Cris}(C/A))^{opp}$ and $\mathcal{C}^\wedge = (\text{Cris}^\wedge(C/A))^{opp}$ endowed with the chaotic topology, see Remark 60.5.4 for notation. There is a morphism of topoi

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}^\wedge)$$

such that if $\mathcal{F}$ is a sheaf of abelian groups on $\mathcal{C}$, then

$$R^pg_*\mathcal{F}(B \to C, \delta) = \begin{cases} \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 0 \\ R^1\lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 1 \\ 0 & \text{else} \end{cases}$$

where $B_e = B/p^eB$ for $e \gg 0$.

Proof. Any functor between categories defines a morphism between chaotic topoi in the same direction, for example because such a functor can be considered as a cocontinuous functor between sites, see Sites, Section 7.21. Proof of the description of $g_*\mathcal{F}$ is omitted. Note that in the statement we take $(B_e \to C, \delta)$ is an object of $\text{Cris}(C/A)$ only for e large enough. Let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the transition maps

$$\mathcal{I}(B_e \to C, \delta) \leftarrow \mathcal{I}(B_{e+1} \to C, \delta)$$

are surjective as the morphisms

$$(B_e \to C, \delta) \longrightarrow (B_{e+1} \to C, \delta)$$

are monomorphisms in the category $\mathcal{C}$. Hence for an injective abelian sheaf both sides of the displayed formula of the lemma agree. Taking an injective resolution of $\mathcal{F}$ one easily obtains the result (sheaves are presheaves, so exactness is measured on the level of groups of sections over objects). □

07JM Lemma 60.18.4. Let $\mathcal{C}$ be a category endowed with the chaotic topology. Let X be an object of $\mathcal{C}$ such that every object of $\mathcal{C}$ has a morphism towards X. Assume that $\mathcal{C}$ has products of pairs. Then for every abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ the total cohomology $R\Gamma(\mathcal{C}, \mathcal{F})$ is represented by the complex

$$\mathcal{F}(X) \to \mathcal{F}(X \times X) \to \mathcal{F}(X \times X \times X) \to \ldots$$

associated to the cosimplicial abelian group $[n] \mapsto \mathcal{F}(X^n)$.

Proof. Note that $H^q(X^p, \mathcal{F}) = 0$ for all $q > 0$ as any presheaf is a sheaf on $\mathcal{C}$. The assumption on X is that $h_X \to *$ is surjective. Using that $H^q(X, \mathcal{F}) = H^q(h_X, \mathcal{F})$ and $H^q(\mathcal{C}, \mathcal{F}) = H^q(*, \mathcal{F})$ we see that our statement is a special case of Cohomology on Sites, Lemma 21.13.2. □

60.19. Cosimplicial preparations

07JP In this section we compare crystalline cohomology with de Rham cohomology. We follow [BdJ11].

07L7 Example 60.19.1. Suppose that A_* is any cosimplicial ring. Consider the cosimplicial module M_* defined by the rule

$$M_n = \bigoplus\nolimits_{i=0,\ldots,n} A_n e_i$$

For a map $f : [n] \to [m]$ define $M_*(f) : M_n \to M_m$ to be the unique $A_*(f)$-linear map which maps e_i to $e_{f(i)}$. We claim the identity on M_* is homotopic to 0. Namely, a homotopy is given by a map of cosimplicial modules

$$h : M_* \longrightarrow \text{Hom}(\Delta[1], M_*)$$

see Section 60.16. For $j \in \{0, \ldots, n+1\}$ we let $\alpha^n_j : [n] \to [1]$ be the map defined by $\alpha^n_j(i) = 0 \Leftrightarrow i < j$. Then $\Delta[1]_n = \{\alpha^n_0, \ldots, \alpha^n_{n+1}\}$ and correspondingly $\text{Hom}(\Delta[1], M_*)_n = \prod_{j=0,\ldots,n+1} M_n$, see Simplicial, Sections 14.26 and 14.28. Instead of using this product representation, we think of an element in $\text{Hom}(\Delta[1], M_*)_n$ as a function $\Delta[1]_n \to M_n$. Using this notation, we define h in degree n by the rule

$$h_n(e_i)(\alpha^n_j) = \begin{cases} e_i & \text{if} \quad i < j \\ 0 & \text{else} \end{cases}$$

We first check h is a morphism of cosimplicial modules. Namely, for $f : [n] \to [m]$ we will show that

07L8 (60.19.1.1)
$$h_m \circ M_*(f) = \text{Hom}(\Delta[1], M_*)(f) \circ h_n$$

The left hand side of (60.19.1.1) evaluated at e_i and then in turn evaluated at α^m_j is

$$h_m(e_{f(i)})(\alpha^m_j) = \begin{cases} e_{f(i)} & \text{if} \quad f(i) < j \\ 0 & \text{else} \end{cases}$$

Note that $\alpha^m_j \circ f = \alpha^n_{j'}$ where $0 \leq j' \leq n+1$ is the unique index such that $f(i) < j$ if and only if $i < j'$. Thus the right hand side of (60.19.1.1) evaluated at e_i and then in turn evaluated at α^m_j is

$$M_*(f)(h_n(e_i)(\alpha^m_j \circ f) = M_*(f)(h_n(e_i)(\alpha^n_{j'})) = \begin{cases} e_{f(i)} & \text{if} \quad i < j' \\ 0 & \text{else} \end{cases}$$

It follows from our description of j' that the two answers are equal. Hence h is a map of cosimplicial modules. Let $0 : \Delta[0] \to \Delta[1]$ and $1 : \Delta[0] \to \Delta[1]$ be the obvious maps, and denote $ev_0, ev_1 : \text{Hom}(\Delta[1], M_*) \to M_*$ the corresponding evaluation maps. The reader verifies readily that the compositions

$$ev_0 \circ h, ev_1 \circ h : M_* \longrightarrow M_*$$

are 0 and 1 respectively, whence h is the desired homotopy between 0 and 1.

07L9 Lemma 60.19.2. With notation as in (60.17.0.5) the complex

$$\Omega_{D(0)} \to \Omega_{D(1)} \to \Omega_{D(2)} \to \ldots$$

is homotopic to zero as a $D(*)$-cosimplicial module.

Proof. We are going to use the principle of Simplicial, Lemma 14.28.4 and more specifically Lemma 60.16.1 which tells us that homotopic maps between (co)simplicial objects are transformed by any functor into homotopic maps. The complex of the lemma is equal to the p-adic completion of the base change of the cosimplicial module

$$M_* = \left(\Omega_{P/A} \to \Omega_{P\otimes_A P/A} \to \Omega_{P\otimes_A P\otimes_A P/A} \to \ldots\right)$$

via the cosimplicial ring map $P \otimes_A \ldots \otimes_A P \to D(n)$. This follows from Lemma 60.6.6, see comments following (60.17.0.2). Hence it suffices to show that the cosimplicial module M_* is homotopic to zero (uses base change and p-adic completion). We can even assume $A = \mathbf{Z}$ and $P = \mathbf{Z}[\{x_i\}_{i\in I}]$ as we can use base change with $\mathbf{Z} \to A$. In this case $P^{\otimes n+1}$ is the polynomial algebra on the elements

$$x_i(e) = 1 \otimes \ldots \otimes x_i \otimes \ldots \otimes 1$$

with x_i in the eth slot. The modules of the complex are free on the generators $\mathrm{d}x_i(e)$. Note that if $f : [n] \to [m]$ is a map then we see that

$$M_*(f)(\mathrm{d}x_i(e)) = \mathrm{d}x_i(f(e))$$

Hence we see that M_* is a direct sum over I of copies of the module studied in Example 60.19.1 and we win. □

07LA Lemma 60.19.3. With notation as in (60.17.0.4) and (60.17.0.5), given any cosimplicial module M_* over $D(*)$ and $i > 0$ the cosimplicial module

$$M_0 \otimes^\wedge_{D(0)} \Omega^i_{D(0)} \to M_1 \otimes^\wedge_{D(1)} \Omega^i_{D(1)} \to M_2 \otimes^\wedge_{D(2)} \Omega^i_{D(2)} \to \ldots$$

is homotopic to zero, where $\Omega^i_{D(n)}$ is the p-adic completion of the ith exterior power of $\Omega_{D(n)}$.

Proof. By Lemma 60.19.2 the endomorphisms 0 and 1 of $\Omega_{D(*)}$ are homotopic. If we apply the functor $\wedge^i$ we see that the same is true for the cosimplicial module $\wedge^i\Omega_{D(*)}$, see Lemma 60.16.1. Another application of the same lemma shows the p-adic completion $\Omega^i_{D(*)}$ is homotopy equivalent to zero. Tensoring with M_* we see that $M_* \otimes_{D(*)} \Omega^i_{D(*)}$ is homotopic to zero, see Lemma 60.16.1 again. A final application of the p-adic completion functor finishes the proof. □

60.20. Divided power Poincaré lemma

07LB Just the simplest possible version.

07LC Lemma 60.20.1. Let A be a ring. Let $P = A\langle x_i\rangle$ be a divided power polynomial ring over A. For any A-module M the complex

$$0 \to M \to M \otimes_A P \to M \otimes_A \Omega^1_{P/A,\delta} \to M \otimes_A \Omega^2_{P/A,\delta} \to \ldots$$

is exact. Let D be the p-adic completion of P. Let Ω^i_D be the p-adic completion of the ith exterior power of $\Omega_{D/A,\delta}$. For any p-adically complete A-module M the complex

$$0 \to M \to M \otimes^\wedge_A D \to M \otimes^\wedge_A \Omega^1_D \to M \otimes^\wedge_A \Omega^2_D \to \ldots$$

is exact.

Proof. It suffices to show that the complex

$$E : (0 \to A \to P \to \Omega^1_{P/A,\delta} \to \Omega^2_{P/A,\delta} \to \ldots)$$

is homotopy equivalent to zero as a complex of A-modules. For every multi-index $K = (k_i)$ we can consider the subcomplex $E(K)$ which in degree j consists of

$$\bigoplus\nolimits_{I=\{i_1,\ldots,i_j\}\subset \text{Supp}(K)} A \prod\nolimits_{i \not\in I} x_i^{[k_i]} \prod\nolimits_{i \in I} x_i^{[k_i-1]} \text{d}x_{i_1} \wedge \ldots \wedge \text{d}x_{i_j}$$

Since $E = \bigoplus E(K)$ we see that it suffices to prove each of the complexes $E(K)$ is homotopic to zero. If $K = 0$, then $E(K) : (A \to A)$ is homotopic to zero. If K has nonempty (finite) support S, then the complex $E(K)$ is isomorphic to the complex

$$0 \to A \to \bigoplus\nolimits_{s \in S} A \to \wedge^2(\bigoplus\nolimits_{s \in S} A) \to \ldots \to \wedge^{\#S}(\bigoplus\nolimits_{s \in S} A) \to 0$$

which is homotopic to zero, for example by More on Algebra, Lemma 15.28.5. $\square$

An alternative (more direct) approach to the following lemma is explained in Example 60.25.2.

07LD **Lemma 60.20.2.** Let A be a ring. Let (B, I, δ) be a divided power ring. Let $P = B\langle x_i \rangle$ be a divided power polynomial ring over B with divided power ideal $J = IP + B\langle x_i \rangle_+$ as usual. Let M be a B-module endowed with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_{B/A,\delta}$. Then the map of de Rham complexes

$$M \otimes_B \Omega^*_{B/A,\delta} \longrightarrow M \otimes_P \Omega^*_{P/A,\delta}$$

is a quasi-isomorphism. Let D, resp. D' be the p-adic completion of B, resp. P and let Ω^i_D, resp. $\Omega^i_{D'}$ be the p-adic completion of $\Omega^i_{B/A,\delta}$, resp. $\Omega^i_{P/A,\delta}$. Let M be a p-adically complete D-module endowed with an integral connection $\nabla : M \to M \otimes^\wedge_D \Omega^1_D$. Then the map of de Rham complexes

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_D \Omega^*_{D'}$$

is a quasi-isomorphism.

Proof. Consider the decreasing filtration F^* on $\Omega^*_{B/A,\delta}$ given by the subcomplexes $F^i(\Omega^*_{B/A,\delta}) = \sigma_{\geq i}\Omega^*_{B/A,\delta}$. See Homology, Section 12.15. This induces a decreasing filtration F^* on $\Omega^*_{P/A,\delta}$ by setting

$$F^i(\Omega^*_{P/A,\delta}) = F^i(\Omega^*_{B/A,\delta}) \wedge \Omega^*_{P/A,\delta}.$$

We have a split short exact sequence

$$0 \to \Omega^1_{B/A,\delta} \otimes_B P \to \Omega^1_{P/A,\delta} \to \Omega^1_{P/B,\delta} \to 0$$

and the last module is free on $\text{d}x_i$. It follows from this that $F^i(\Omega^*_{P/A,\delta}) \to \Omega^*_{P/A,\delta}$ is a termwise split injection and that

$$\text{gr}^i_F(\Omega^*_{P/A,\delta}) = \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. Thus we can define a filtration F^* on $M \otimes_B \Omega^*_{B/A,\delta}$ by setting

$$F^i(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B F^i(\Omega^*_{P/A,\delta})$$

and we have

$$\text{gr}^i_F(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. By Lemma 60.20.1 each of these complexes is quasi-isomorphic to $M \otimes_B \Omega^i_{B/A,\delta}$ placed in degree 0. Hence we see that the first displayed map of the lemma is a morphism of filtered complexes which induces a quasi-isomorphism on graded pieces. This implies that it is a quasi-isomorphism, for example by the spectral sequence associated to a filtered complex, see Homology, Section 12.24.

The proof of the second quasi-isomorphism is exactly the same. □

60.21. Cohomology in the affine case

07LE Let's go back to the situation studied in Section 60.17. We start with (A, I, γ) and $A/I \to C$ and set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. Then we choose a polynomial ring P over A and a surjection $P \to C$ with kernel J. We obtain D and $D(n)$ see (60.17.0.1) and (60.17.0.4). Set $T(n)_e = \mathrm{Spec}(D(n)/p^eD(n))$ so that $(X, T(n)_e, \delta(n))$ is an object of $\mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules and set

$$M(n) = \lim\nolimits_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$$

for $n = 0, 1, 2, 3, \ldots$. This forms a cosimplicial module over the cosimplicial ring $D(0), D(1), D(2), \ldots$.

07JN **Proposition 60.21.1.** With notations as above assume that

(1) $\mathcal{F}$ is locally quasi-coherent, and
(2) for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $\mathrm{Cris}(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is surjective.

Then the complex

$$M(0) \to M(1) \to M(2) \to \ldots$$

computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

Proof. Using assumption (1) and Lemma 60.18.2 we see that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is isomorphic to $R\Gamma(\mathcal{C}, \mathcal{F})$. Note that the categories $\mathcal{C}$ used in Lemmas 60.18.2 and 60.18.3 agree. Let $f : T \to T'$ be a closed immersion as in (2). Surjectivity of $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is equivalent to surjectivity of $\mathcal{F}_{T'} \to f_*\mathcal{F}_T$. Hence, if $\mathcal{F}$ satisfies (1) and (2), then we obtain a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_{T'} \to f_*\mathcal{F}_T \to 0$$

of quasi-coherent $\mathcal{O}_{T'}$-modules on T', see Schemes, Section 26.24 and in particular Lemma 26.24.1. Thus, if T' is affine, then we conclude that the restriction map $\mathcal{F}(U', T', \delta') \to \mathcal{F}(U, T, \delta)$ is surjective by the vanishing of $H^1(T', \mathcal{K})$, see Cohomology of Schemes, Lemma 30.2.2. Hence the transition maps of the inverse systems in Lemma 60.18.3 are surjective. We conclude that $R^pg_*(\mathcal{F}|_\mathcal{C}) = 0$ for all $p \geq 1$ where g is as in Lemma 60.18.3. The object D of the category $\mathcal{C}^\wedge$ satisfies the assumption of Lemma 60.18.4 by Lemma 60.5.7 with

$$D \times \ldots \times D = D(n)$$

in $\mathcal{C}$ because $D(n)$ is the $n+1$-fold coproduct of D in $\mathrm{Cris}^\wedge(C/A)$, see Lemma 60.17.2. Thus we win. □

07LF **Lemma 60.21.2.** Assumptions and notation as in Proposition 60.21.1. Then

$$H^j(\mathrm{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for all $i > 0$ and all $j \geq 0$.

Proof. Using Lemma 60.12.6 it follows that $\mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}$ also satisfies assumptions (1) and (2) of Proposition 60.21.1. Write $M(n)_e = \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$ so that $M(n) = \lim_e M(n)_e$. Then

$$\begin{aligned} \lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{H}) &= \lim_e M(n)_e \otimes_{D(n)_e} \Omega_{D(n)}/p^e\Omega_{D(n)} \\ &= \lim_e M(n)_e \otimes_{D(n)} \Omega_{D(n)} \end{aligned}$$

By Lemma 60.19.3 the cosimplicial modules

$$M(0)_e \otimes_{D(0)} \Omega^i_{D(0)} \to M(1)_e \otimes_{D(1)} \Omega^i_{D(1)} \to M(2)_e \otimes_{D(2)} \Omega^i_{D(2)} \to \ldots$$

are homotopic to zero. Because the transition maps $M(n)_{e+1} \to M(n)_e$ are surjective, we see that the inverse limit of the associated complexes are acyclic[5]. Hence the vanishing of cohomology of $\mathcal{H}$ by Proposition 60.21.1. □

07LG Proposition 60.21.3. Assumptions as in Proposition 60.21.1 but now assume that $\mathcal{F}$ is a crystal in quasi-coherent modules. Let (M, ∇) be the corresponding module with connection over D, see Proposition 60.17.4. Then the complex

$$M \otimes^\wedge_D \Omega^*_D$$

computes $R\Gamma(\text{Cris}(X/S), \mathcal{F})$.

Proof. We will prove this using the two spectral sequences associated to the double complex $K^{*,*}$ with terms

$$K^{a,b} = M \otimes^\wedge_D \Omega^a_{D(b)}$$

What do we know so far? Well, Lemma 60.19.3 tells us that each column $K^{a,*}$, $a > 0$ is acyclic. Proposition 60.21.1 tells us that the first column $K^{0,*}$ is quasi-isomorphic to $R\Gamma(\text{Cris}(X/S), \mathcal{F})$. Hence the first spectral sequence associated to the double complex shows that there is a canonical quasi-isomorphism of $R\Gamma(\text{Cris}(X/S), \mathcal{F})$ with $\text{Tot}(K^{*,*})$.

Next, let's consider the rows $K^{*,b}$. By Lemma 60.17.1 each of the $b + 1$ maps $D \to D(b)$ presents $D(b)$ as the p-adic completion of a divided power polynomial algebra over D. Hence Lemma 60.20.2 shows that the map

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_{D(b)} \Omega^*_{D(b)} = K^{*,b}$$

is a quasi-isomorphism. Note that each of these maps defines the same map on cohomology (and even the same map in the derived category) as the inverse is given by the co-diagonal map $D(b) \to D$ (corresponding to the multiplication map $P \otimes_A \ldots \otimes_A P \to P$). Hence if we look at the E_1 page of the second spectral sequence we obtain

$$E_1^{a,b} = H^a(M \otimes^\wedge_D \Omega^*_D)$$

with differentials

$$E_1^{a,0} \xrightarrow{0} E_1^{a,1} \xrightarrow{1} E_1^{a,2} \xrightarrow{0} E_1^{a,3} \xrightarrow{1} \ldots$$

as each of these is the alternation sum of the given identifications $H^a(M \otimes^\wedge_D \Omega^*_D) = E_1^{a,0} = E_1^{a,1} = \ldots$. Thus we see that the E_2 page is equal $H^a(M \otimes^\wedge_D \Omega^*_D)$ on the first row and zero elsewhere. It follows that the identification of $M \otimes^\wedge_D \Omega^*_D$ with the first row induces a quasi-isomorphism of $M \otimes^\wedge_D \Omega^*_D$ with $\text{Tot}(K^{*,*})$. □

[5] Actually, they are even homotopic to zero as the homotopies fit together, but we don't need this. The reason for this roundabout argument is that the limit $\lim_e M(n)_e \otimes_{D(n)} \Omega^i_{D(n)}$ isn't the p-adic completion of $M(n) \otimes_{D(n)} \Omega^i_{D(n)}$ as with the assumptions of the lemma we don't know that $M(n)_e = M(n)_{e+1}/p^e M(n)_{e+1}$. If $\mathcal{F}$ is a crystal then this does hold.

07LH Lemma 60.21.4. Assumptions as in Proposition 60.21.3. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. Let (M', ∇') be the pair over D' corresponding to $\mathcal{F}$, see Lemma 60.17.5. Then the complex

$$M' \otimes^\wedge_{D'} \Omega^*_{D'}$$

computes $R\Gamma(\text{Cris}(X/S), \mathcal{F})$.

Proof. Choose $a : D \to D'$ and $b : D' \to D$ as in Lemma 60.17.5. Note that the base change $M = M' \otimes_{D',b} D$ with its connection ∇ corresponds to $\mathcal{F}$. Hence we know that $M \otimes^\wedge_D \Omega^*_D$ computes the crystalline cohomology of $\mathcal{F}$, see Proposition 60.21.3. Hence it suffices to show that the base change maps (induced by a and b)

$$M' \otimes^\wedge_{D'} \Omega^*_{D'} \longrightarrow M \otimes^\wedge_D \Omega^*_D \quad \text{and} \quad M \otimes^\wedge_D \Omega^*_D \longrightarrow M' \otimes^\wedge_{D'} \Omega^*_{D'}$$

are quasi-isomorphisms. Since $a \circ b = \text{id}_{D'}$ we see that the composition one way around is the identity on the complex $M' \otimes^\wedge_{D'} \Omega^*_{D'}$. Hence it suffices to show that the map

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_D \Omega^*_D$$

induced by $b \circ a : D \to D$ is a quasi-isomorphism. (Note that we have the same complex on both sides as $M = M' \otimes^\wedge_{D',b} D$, hence $M \otimes^\wedge_{D,b\circ a} D = M' \otimes^\wedge_{D',b\circ a\circ b} D = M' \otimes^\wedge_{D',b} D = M$.) In fact, we claim that for any divided power A-algebra homomorphism $\rho : D \to D$ compatible with the augmentation to C the induced map $M \otimes^\wedge_D \Omega^*_D \to M \otimes^\wedge_{D,\rho} \Omega^*_D$ is a quasi-isomorphism.

Write $\rho(x_i) = x_i + z_i$. The elements z_i are in the divided power ideal of D because ρ is compatible with the augmentation to C. Hence we can factor the map ρ as a composition

$$D \xrightarrow{\sigma} D\langle \xi_i \rangle^\wedge \xrightarrow{\tau} D$$

where the first map is given by $x_i \mapsto x_i + \xi_i$ and the second map is the divided power D-algebra map which maps ξ_i to z_i. (This uses the universal properties of polynomial algebra, divided power polynomial algebras, divided power envelopes, and p-adic completion.) Note that there exists an automorphism α of $D\langle \xi_i \rangle^\wedge$ with $\alpha(x_i) = x_i - \xi_i$ and $\alpha(\xi_i) = \xi_i$. Applying Lemma 60.20.2 to $\alpha \circ \sigma$ (which maps x_i to x_i) and using that α is an isomorphism we conclude that σ induces a quasi-isomorphism of $M \otimes^\wedge_D \Omega^*_D$ with $M \otimes^\wedge_{D,\sigma} \Omega^*_{D\langle x_i \rangle^\wedge}$. On the other hand the map τ has as a left inverse the map $D \to D\langle x_i \rangle^\wedge$, $x_i \mapsto x_i$ and we conclude (using Lemma 60.20.2 once more) that τ induces a quasi-isomorphism of $M \otimes^\wedge_{D,\sigma} \Omega^*_{D\langle x_i \rangle^\wedge}$ with $M \otimes^\wedge_{D,\tau\circ\sigma} \Omega^*_D$. Composing these two quasi-isomorphisms we obtain that ρ induces a quasi-isomorphism $M \otimes^\wedge_D \Omega^*_D \to M \otimes^\wedge_{D,\rho} \Omega^*_D$ as desired. □

60.22. Two counter examples

07LI Before we turn to some of the successes of crystalline cohomology, let us give two examples which explain why crystalline cohomology does not work very well if the schemes in question are either not proper over the base, or singular. The first example can be found in [BO83].

07LJ Example 60.22.1. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x, y]/(x^2, xy, y^2)$. We choose the presentation

$$C = P/J = \mathbf{Z}_p[x, y]/(x^2, xy, y^2, p)$$

Let $D = D_{P,\gamma}(J)^\wedge$ with divided power ideal $(\bar{J}, \bar{\gamma})$ as in Section 60.17. We will denote x, y also the images of x and y in D. Consider the element

$$\tau = \bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) - \bar{\gamma}_p(xy)^2 \in D$$

We note that $p\tau = 0$ as

$$p!\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) = x^{2p}\bar{\gamma}_p(y^2) = \bar{\gamma}_p(x^2y^2) = x^py^p\bar{\gamma}_p(xy) = p!\bar{\gamma}_p(xy)^2$$

in D. We also note that $\mathrm{d}\tau = 0$ in Ω_D as

$$\begin{aligned}
\mathrm{d}(\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2)) &= \bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x^2 + \bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y^2 \\
&= 2x\bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x + 2y\bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y \\
&= 2/(p-1)!(x^{2p-1}\bar{\gamma}_p(y^2)\mathrm{d}x + y^{2p-1}\bar{\gamma}_p(x^2)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}\bar{\gamma}_p(xy^2)\mathrm{d}x + y^{p-1}\bar{\gamma}_p(x^2y)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}y^p\bar{\gamma}_p(xy)\mathrm{d}x + x^py^{p-1}\bar{\gamma}_p(xy)\mathrm{d}y) \\
&= 2\bar{\gamma}_{p-1}(xy)\bar{\gamma}_p(xy)(y\mathrm{d}x + x\mathrm{d}y) \\
&= \mathrm{d}(\bar{\gamma}_p(xy)^2)
\end{aligned}$$

Finally, we claim that $\tau \neq 0$ in D. To see this it suffices to produce an object $(B \to \mathbf{F}_p[x,y]/(x^2, xy, y^2), \delta)$ of $\mathrm{Cris}(C/S)$ such that τ does not map to zero in B. To do this take

$$B = \mathbf{F}_p[x, y, u, v]/(x^3, x^2y, xy^2, y^3, xu, yu, xv, yv, u^2, v^2)$$

with the obvious surjection to C. Let $K = \mathrm{Ker}(B \to C)$ and consider the map

$$\delta_p : K \longrightarrow K, \quad ax^2 + bxy + cy^2 + du + ev + fuv \longmapsto a^pu + c^pv$$

One checks this satisfies the assumptions (1), (2), (3) of Divided Power Algebra, Lemma 23.5.3 and hence defines a divided power structure. Moreover, we see that τ maps to uv which is not zero in B. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We draw the following conclusions

(1) $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ has p-torsion, and
(2) pulling back by Frobenius $F^* : H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) \to H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not injective.

Namely, τ defines a nonzero torsion element of $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ by Proposition 60.21.3. Similarly, $F^*(\tau) = \sigma(\tau)$ where $\sigma : D \to D$ is the map induced by any lift of Frobenius on P. If we choose $\sigma(x) = x^p$ and $\sigma(y) = y^p$, then an easy computation shows that $F^*(\tau) = 0$.

The next example shows that even for affine n-space crystalline cohomology does not give the correct thing.

07LK Example 60.22.2. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x_1, \ldots, x_r]$. We choose the presentation

$$C = P/J = P/pP \quad \text{with} \quad P = \mathbf{Z}_p[x_1, \ldots, x_r]$$

Note that pP has divided powers by Divided Power Algebra, Lemma 23.4.2. Hence setting $D = P^\wedge$ with divided power ideal (p) we obtain a situation as in Section 60.17. We conclude that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is represented by the complex

$$D \to \Omega^1_D \to \Omega^2_D \to \ldots \to \Omega^r_D$$

see Proposition 60.21.3. Assuming $r > 0$ we conclude the following

(1) The cristalline cohomology of the cristalline structure sheaf of $X = \mathbf{A}^r_{\mathbf{F}_p}$ over $S = \mathrm{Spec}(\mathbf{Z}_p)$ is zero except in degrees $0, \ldots, r$.
(2) We have $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) = \mathbf{Z}_p$.
(3) The cohomology group $H^r(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is infinite and is not a torsion abelian group.
(4) The cohomology group $H^r(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not separated for the p-adic topology.

While the first two statements are reasonable, parts (3) and (4) are disconcerting! The truth of these statements follows immediately from working out what the complex displayed above looks like. Let's just do this in case $r = 1$. Then we are just looking at the two term complex of p-adically complete modules

$$\mathrm{d} : D = \left(\bigoplus\nolimits_{n \geq 0} \mathbf{Z}_p x^n\right)^\wedge \longrightarrow \Omega^1_D = \left(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p x^{n-1}\mathrm{d}x\right)^\wedge$$

The map is given by $\mathrm{diag}(0, 1, 2, 3, 4, \ldots)$ except that the first summand is missing on the right hand side. Now it is clear that $\bigoplus_{n>0} \mathbf{Z}_p/n\mathbf{Z}_p$ is a subgroup of the cokernel, hence the cokernel is infinite. In fact, the element

$$\omega = \sum\nolimits_{e>0} p^e x^{p^{2e}-1}\mathrm{d}x$$

is clearly not a torsion element of the cokernel. But it gets worse. Namely, consider the element

$$\eta = \sum\nolimits_{e>0} p^e x^{p^e-1}\mathrm{d}x$$

For every $t > 0$ the element η is congruent to $\sum_{e>t} p^e x^{p^e-1}\mathrm{d}x$ modulo the image of d which is divisible by p^t. But η is not in the image of d because it would have to be the image of $a + \sum_{e>0} x^{p^e}$ for some $a \in \mathbf{Z}_p$ which is not an element of the left hand side. In fact, $p^N\eta$ is similarly not in the image of d for any integer N. This implies that η "generates" a copy of $\mathbf{Q}_p$ inside of $H^1_{\mathrm{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/\mathrm{Spec}(\mathbf{Z}_p))$.

60.23. Applications

07LL In this section we collect some applications of the material in the previous sections.

07LM Proposition 60.23.1. In Situation 60.7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $\mathrm{Cris}(X/S)$. The truncation map of complexes

$$(\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots) \longrightarrow \mathcal{F}[0],$$

while not a quasi-isomorphism, becomes a quasi-isomorphism after applying $Ru_{X/S,*}$. In fact, for any $i > 0$, we have

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0.$$

Proof. By Lemma 60.15.1 we get a de Rham complex as indicated in the lemma. We abbreviate $\mathcal{H} = \mathcal{F} \otimes \Omega^i_{X/S}$. Let $X' \subset X$ be an affine open subscheme which maps into an affine open subscheme $S' \subset S$. Then

$$(Ru_{X/S,*}\mathcal{H})|_{X'_{Zar}} = Ru_{X'/S',*}(\mathcal{H}|_{\mathrm{Cris}(X'/S')}),$$

see Lemma 60.9.5. Thus Lemma 60.21.2 shows that $Ru_{X/S,*}\mathcal{H}$ is a complex of sheaves on X_{Zar} whose cohomology on any affine open is trivial. As X has a basis for its topology consisting of affine opens this implies that $Ru_{X/S,*}\mathcal{H}$ is quasi-isomorphic to zero. $\square$

07LN Remark 60.23.2. The proof of Proposition 60.23.1 shows that the conclusion

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for $i > 0$ is true for any $\mathcal{O}_{X/S}$-module $\mathcal{F}$ which satisfies conditions (1) and (2) of Proposition 60.21.1. This applies to the following non-crystals: $\Omega^i_{X/S}$ for all i, and any sheaf of the form $\underline{\mathcal{F}}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, it applies to the sheaf $\underline{\mathcal{O}_X} = \underline{\mathbf{G}_a}$. But note that we need something like Lemma 60.15.1 to produce a de Rham complex which requires $\mathcal{F}$ to be a crystal. Hence (currently) the collection of sheaves of modules for which the full statement of Proposition 60.23.1 holds is exactly the category of crystals in quasi-coherent modules.

In Situation 60.7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $\text{Cris}(X/S)$. Let (U, T, δ) be an object of $\text{Cris}(X/S)$. Proposition 60.23.1 allows us to construct a canonical map

07LP (60.23.2.1)
$$R\Gamma(\text{Cris}(X/S), \mathcal{F}) \longrightarrow R\Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T/S,\delta})$$

Namely, we have $R\Gamma(\text{Cris}(X/S), \mathcal{F}) = R\Gamma(\text{Cris}(X/S), \mathcal{F} \otimes \Omega^*_{X/S})$, we can restrict global cohomology classes to T, and $\Omega_{X/S}$ restricts to $\Omega_{T/S,\delta}$ by Lemma 60.12.3.

60.24. Some further results

07MI In this section we mention some results whose proof is missing. We will formulate these as a series of remarks and we will convert them into actual lemmas and propositions only when we add detailed proofs.

07MJ Remark 60.24.1 (Higher direct images). Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X' \arrow[d] \\ S_0 \arrow[r] & S'_0 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and X'. Let $\mathcal{F}$ be an $\mathcal{O}_{X/S}$-module on $\text{Cris}(X/S)$. Then $Rf_{\text{cris},*}\mathcal{F}$ can be computed as follows.

Given an object (U', T', δ') of $\text{Cris}(X'/S')$ set $U = X \times_{X'} U' = f^{-1}(U')$ (an open subscheme of X). Denote (T_0, T, δ) the divided power scheme over S such that

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

is cartesian in the category of divided power schemes, see Lemma 60.7.4. There is an induced morphism $U \to T_0$ and we obtain a morphism $(U/T)_{\text{cris}} \to (X/S)_{\text{cris}}$, see Remark 60.9.3. Let $\mathcal{F}_U$ be the pullback of $\mathcal{F}$. Let $\tau_{U/T} : (U/T)_{\text{cris}} \to T_{Zar}$ be the structure morphism. Then we have

07MK (60.24.1.1)
$$\left(Rf_{\text{cris},*}\mathcal{F}\right)_{T'} = R(T \to T')_* \left(R\tau_{U/T,*}\mathcal{F}_U\right)$$

where the left hand side is the restriction (see Section 60.10).

Hints: First, show that $\text{Cris}(U/T)$ is the localization (in the sense of Sites, Lemma 7.30.3) of $\text{Cris}(X/S)$ at the sheaf of sets $f_{\text{cris}}^{-1}h_{(U',T',\delta')}$. Next, reduce the statement to the case where $\mathcal{F}$ is an injective module and pushforward of modules using that the pullback of an injective $\mathcal{O}_{X/S}$-module is an injective $\mathcal{O}_{U/T}$-module on $\text{Cris}(U/T)$. Finally, check the result holds for plain pushforward.

07ML Remark 60.24.2 (Mayer-Vietoris). In the situation of Remark 60.24.1 suppose we have an open covering $X = X' \cup X''$. Denote $X''' = X' \cap X''$. Let f', f'', and f'' be the restriction of f to X', X'', and X'''. Moreover, let $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{F}'''$ be the restriction of $\mathcal{F}$ to the crystalline sites of X', X'', and X'''. Then there exists a distinguished triangle

$$Rf_{\text{cris},*}\mathcal{F} \longrightarrow Rf'_{\text{cris},*}\mathcal{F}' \oplus Rf''_{\text{cris},*}\mathcal{F}'' \longrightarrow Rf'''_{\text{cris},*}\mathcal{F}''' \longrightarrow Rf_{\text{cris},*}\mathcal{F}[1]$$

in $D(\mathcal{O}_{X'/S'})$.

Hints: This is a formal consequence of the fact that the subcategories $\text{Cris}(X'/S)$, $\text{Cris}(X''/S)$, $\text{Cris}(X'''/S)$ correspond to open subobjects of the final sheaf on $\text{Cris}(X/S)$ and that the last is the intersection of the first two.

07MM Remark 60.24.3 (Čech complex). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \text{Spec}(A)$ and $S_0 = \text{Spec}(A/I)$. Let X be a separated[6] scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose an affine open covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ of X. Write $U_\lambda = \text{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct a Čech complex which computes $R\Gamma(\text{Cris}(X/S), \mathcal{F})$.

Given $n \geq 0$ and $\lambda_0, \ldots, \lambda_n \in \Lambda$ write $U_{\lambda_0\ldots\lambda_n} = U_{\lambda_0} \cap \ldots \cap U_{\lambda_n}$. This is an affine scheme by assumption. Write $U_{\lambda_0\ldots\lambda_n} = \text{Spec}(C_{\lambda_0\ldots\lambda_n})$. Set

$$P_{\lambda_0\ldots\lambda_n} = P_{\lambda_0} \otimes_A \ldots \otimes_A P_{\lambda_n}$$

which comes with a canonical surjection onto $C_{\lambda_0\ldots\lambda_n}$. Denote the kernel $J_{\lambda_0\ldots\lambda_n}$ and set $D_{\lambda_0\ldots\lambda_n}$ the p-adically completed divided power envelope of $J_{\lambda_0\ldots\lambda_n}$ in $P_{\lambda_0\ldots\lambda_n}$ relative to γ. Let $M_{\lambda_0\ldots\lambda_n}$ be the $P_{\lambda_0\ldots\lambda_n}$-module corresponding to the restriction of $\mathcal{F}$ to $\text{Cris}(U_{\lambda_0\ldots\lambda_n}/S)$ via Proposition 60.17.4. By construction we obtain a cosimplicial divided power ring $D(*)$ having in degree n the ring

$$D(n) = \prod\nolimits_{\lambda_0\ldots\lambda_n} D_{\lambda_0\ldots\lambda_n}$$

(use that divided power envelopes are functorial and the trivial cosimplicial structure on the ring $P(*)$ defined similarly). Since $M_{\lambda_0\ldots\lambda_n}$ is the "value" of $\mathcal{F}$ on the objects $\text{Spec}(D_{\lambda_0\ldots\lambda_n})$ we see that $M(*)$ defined by the rule

$$M(n) = \prod\nolimits_{\lambda_0\ldots\lambda_n} M_{\lambda_0\ldots\lambda_n}$$

forms a cosimplicial $D(*)$-module. Now we claim that we have

$$R\Gamma(\text{Cris}(X/S), \mathcal{F}) = s(M(*))$$

Here $s(-)$ denotes the cochain complex associated to a cosimplicial module (see Simplicial, Section 14.25).

[6]This assumption is not strictly necessary, as using hypercoverings the construction of the remark can be extended to the general case.

Hints: The proof of this is similar to the proof of Proposition 60.21.1 (in particular the result holds for any module satisfying the assumptions of that proposition).

07MN Remark 60.24.4 (Alternating Čech complex). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated quasi-compact scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose a finite affine open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X and a total ordering on Λ. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct an alternating Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

We are going to use the notation introduced in Remark 60.24.3. Denote $\Omega_{\lambda_0 \ldots \lambda_n}$ the p-adically completed module of differentials of $D_{\lambda_0 \ldots \lambda_n}$ over A compatible with the divided power structure. Let ∇ be the integrable connection on $M_{\lambda_0 \ldots \lambda_n}$ coming from Proposition 60.17.4. Consider the double complex $M^{\bullet, \bullet}$ with terms

$$M^{n,m} = \bigoplus\nolimits_{\lambda_0 < \ldots < \lambda_n} M_{\lambda_0 \ldots \lambda_n} \otimes^\wedge_{D_{\lambda_0 \ldots \lambda_n}} \Omega^m_{D_{\lambda_0 \ldots \lambda_n}}.$$

For the differential d_1 (increasing n) we use the usual Čech differential and for the differential d_2 we use the connection, i.e., the differential of the de Rham complex. We claim that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = \mathrm{Tot}(M^{\bullet, \bullet})$$

Here $\mathrm{Tot}(-)$ denotes the total complex associated to a double complex, see Homology, Definition 12.18.3.

Hints: We have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S})$$

by Proposition 60.23.1. The right hand side of the formula is simply the alternating Čech complex for the covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ (which induces an open covering of the final sheaf of $\mathrm{Cris}(X/S)$) and the complex $\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S}$, see Proposition 60.21.3. Now the result follows from a general result in cohomology on sites, namely that the alternating Čech complex computes the cohomology provided it gives the correct answer on all the pieces (insert future reference here).

07MP Remark 60.24.5 (Quasi-coherence). In the situation of Remark 60.24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is quasi-compact and quasi-separated. Then for a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ the sheaves $R^i f_{\mathrm{cris},*}\mathcal{F}$ are locally quasi-coherent.

Hints: We have to show that the restrictions to T' are quasi-coherent $\mathcal{O}_{T'}$-modules, where (U', T', δ') is any object of $\mathrm{Cris}(X'/S')$. It suffices to do this when T' is affine. We use the formula (60.24.1.1), the fact that $T \to T'$ is quasi-compact and quasi-separated (as T is affine over the base change of T' by $S \to S'$), and Cohomology of Schemes, Lemma 30.4.5 to see that it suffices to show that the sheaves $R^i\tau_{U/T,*}\mathcal{F}_U$ are quasi-coherent. Note that $U \to T_0$ is also quasi-compact and quasi-separated, see Schemes, Lemmas 26.21.14 and 26.21.14.

This reduces us to proving that $R^i\tau_{X/S,*}\mathcal{F}$ is quasi-coherent on S in the case that p locally nilpotent on S. Here $\tau_{X/S}$ is the structure morphism, see Remark 60.9.6.

We may work locally on S, hence we may assume S affine (see Lemma 60.9.5). Induction on the number of affines covering X and Mayer-Vietoris (Remark 60.24.2) reduces the question to the case where X is also affine (as in the proof of Cohomology of Schemes, Lemma 30.4.5). Say $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$ so that (A, I, γ) and $A \to C$ are as in Situation 60.5.1. Choose a polynomial algebra P over A and a surjection $P \to C$ as in Section 60.17. Let (M, ∇) be the module corresponding to $\mathcal{F}$, see Proposition 60.17.4. Applying Proposition 60.21.3 we see that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is represented by $M \otimes_D \Omega^*_D$. Note that completion isn't necessary as p is nilpotent in A! We have to show that this is compatible with taking principal opens in $S = \mathrm{Spec}(A)$. Suppose that $g \in A$. Then we conclude that similarly $R\Gamma(\mathrm{Cris}(X_g/S_g), \mathcal{F})$ is computed by $M_g \otimes_{D_g} \Omega^*_{D_g}$ (again this uses that p-adic completion isn't necessary). Hence we conclude because localization is an exact functor on A-modules.

07MQ Remark 60.24.6 (Boundedness). In the situation of Remark 60.24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is of finite type and quasi-separated. Then there exists an integer i_0 such that for any crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ we have $R^i f_{\mathrm{cris},*}\mathcal{F} = 0$ for all $i > i_0$.

Hints: Arguing as in Remark 60.24.5 (using Cohomology of Schemes, Lemma 30.4.5) we reduce to proving that $H^i(\mathrm{Cris}(X/S), \mathcal{F}) = 0$ for $i \gg 0$ in the situation of Proposition 60.21.3 when C is a finite type algebra over A. This is clear as we can choose a finite polynomial algebra and we see that $\Omega^i_D = 0$ for $i \gg 0$.

07MR Remark 60.24.7 (Specific boundedness). In Situation 60.7.5 let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Assume that S_0 has a unique point and that $X \to S_0$ is of finite presentation.

(1) If $\dim X = d$ and X/S_0 has embedding dimension e, then $H^i(\mathrm{Cris}(X/S), \mathcal{F}) = 0$ for $i > d + e$.
(2) If X is separated and can be covered by q affines, and X/S_0 has embedding dimension e, then $H^i(\mathrm{Cris}(X/S), \mathcal{F}) = 0$ for $i > q + e$.

Hints: In case (1) we can use that

$$H^i(\mathrm{Cris}(X/S), \mathcal{F}) = H^i(X_{Zar}, Ru_{X/S,*}\mathcal{F})$$

and that $Ru_{X/S,*}\mathcal{F}$ is locally calculated by a de Rham complex constructed using an embedding of X into a smooth scheme of dimension e over S (see Lemma 60.21.4). These de Rham complexes are zero in all degrees $> e$. Hence (1) follows from Cohomology, Proposition 20.20.7. In case (2) we use the alternating Čech complex (see Remark 60.24.4) to reduce to the case X affine. In the affine case we prove the result using the de Rham complex associated to an embedding of X into a smooth scheme of dimension e over S (it takes some work to construct such a thing).

07MS Remark 60.24.8 (Base change map). In the situation of Remark 60.24.1 assume $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ are affine. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'/S'}$-module. Let $\mathcal{F}$ be the pullback of $\mathcal{F}'$. Then there is a canonical base change map

$$L(S' \to S)^* R\tau_{X'/S',*}\mathcal{F}' \longrightarrow R\tau_{X/S,*}\mathcal{F}$$

where $\tau_{X/S}$ and $\tau_{X'/S'}$ are the structure morphisms, see Remark 60.9.6. On global sections this gives a base change map

07MT (60.24.8.1) $$R\Gamma(\mathrm{Cris}(X'/S'), \mathcal{F}') \otimes^{\mathbf{L}}_{A'} A \longrightarrow R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$$

in $D(A)$.

Hint: Compose the very general base change map of Cohomology on Sites, Remark 21.19.3 with the canonical map $Lf_{\text{cris}}^*\mathcal{F}' \to f_{\text{cris}}^*\mathcal{F}' = \mathcal{F}$.

07MU Remark 60.24.9 (Base change isomorphism). The map (60.24.8.1) is an isomorphism provided all of the following conditions are satisfied:

(1) p is nilpotent in A',
(2) $\mathcal{F}'$ is a crystal in quasi-coherent $\mathcal{O}_{X'/S'}$-modules,
(3) $X' \to S'_0$ is a quasi-compact, quasi-separated morphism,
(4) $X = X' \times_{S'_0} S_0$,
(5) $\mathcal{F}'$ is a flat $\mathcal{O}_{X'/S'}$-module,
(6) $X' \to S'_0$ is a local complete intersection morphism (see More on Morphisms, Definition 37.62.2; this holds for example if $X' \to S'_0$ is syntomic or smooth),
(7) X' and S_0 are Tor independent over S'_0 (see More on Algebra, Definition 15.61.1; this holds for example if either $S_0 \to S'_0$ or $X' \to S'_0$ is flat).

Hints: Condition (1) means that in the arguments below p-adic completion does nothing and can be ignored. Using condition (3) and Mayer Vietoris (see Remark 60.24.2) this reduces to the case where X' is affine. In fact by condition (6), after shrinking further, we can assume that $X' = \text{Spec}(C')$ and we are given a presentation $C' = A'/I'[x_1, \ldots, x_n]/(\bar{f}'_1, \ldots, \bar{f}'_c)$ where $\bar{f}'_1, \ldots, \bar{f}'_c$ is a Koszul-regular sequence in A'/I'. (This means that smooth locally $\bar{f}'_1, \ldots, \bar{f}'_c$ forms a regular sequence, see More on Algebra, Lemma 15.30.17.) We choose a lift of $\bar{f}'_i$ to an element $f'_i \in A'[x_1, \ldots, x_n]$. By (4) we see that $X = \text{Spec}(C)$ with $C = A/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ where $f_i \in A[x_1, \ldots, x_n]$ is the image of f'_i. By property (7) we see that $\bar{f}_1, \ldots, \bar{f}_c$ is a Koszul-regular sequence in $A/I[x_1, \ldots, x_n]$. The divided power envelope of $I'A'[x_1, \ldots, x_n] + (f'_1, \ldots, f'_c)$ in $A'[x_1, \ldots, x_n]$ relative to γ' is

$$D' = A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle/(\xi_i - f'_i)$$

see Lemma 60.2.4. Then you check that $\xi_1 - f'_1, \ldots, \xi_n - f'_n$ is a Koszul-regular sequence in the ring $A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle$. Similarly the divided power envelope of $IA[x_1, \ldots, x_n] + (f_1, \ldots, f_c)$ in $A[x_1, \ldots, x_n]$ relative to γ is

$$D = A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle/(\xi_i - f_i)$$

and $\xi_1 - f_1, \ldots, \xi_n - f_n$ is a Koszul-regular sequence in the ring $A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle$. It follows that $D' \otimes_{A'}^{\mathbf{L}} A = D$. Condition (2) implies $\mathcal{F}'$ corresponds to a pair (M', ∇) consisting of a D'-module with connection, see Proposition 60.17.4. Then $M = M' \otimes_{D'} D$ corresponds to the pullback $\mathcal{F}$. By assumption (5) we see that M' is a flat D'-module, hence

$$M = M' \otimes_{D'} D = M' \otimes_{D'} D' \otimes_{A'}^{\mathbf{L}} A = M' \otimes_{A'}^{\mathbf{L}} A$$

Since the modules of differentials $\Omega_{D'}$ and Ω_D (as defined in Section 60.17) are free D'-modules on the same generators we see that

$$M \otimes_D \Omega_D^\bullet = M' \otimes_{D'} \Omega_{D'}^\bullet \otimes_{D'} D = M' \otimes_{D'} \Omega_{D'}^\bullet \otimes_{A'}^{\mathbf{L}} A$$

which proves what we want by Proposition 60.21.3.

07MV Remark 60.24.10 (Rlim). Let p be a prime number. Let (A, I, γ) be a divided power ring with A an algebra over $\mathbf{Z}_{(p)}$ with p nilpotent in A/I. Set $S = \text{Spec}(A)$

and $S_0 = \mathrm{Spec}(A/I)$. Let X be a scheme over S_0 with p locally nilpotent on X. Let $\mathcal{F}$ be any $\mathcal{O}_{X/S}$-module. For $e \gg 0$ we have $(p^e) \subset I$ is preserved by γ, see Divided Power Algebra, Lemma 23.4.5. Set $S_e = \mathrm{Spec}(A/p^eA)$ for $e \gg 0$. Then $\mathrm{Cris}(X/S_e)$ is a full subcategory of $\mathrm{Cris}(X/S)$ and we denote $\mathcal{F}_e$ the restriction of $\mathcal{F}$ to $\mathrm{Cris}(X/S_e)$. Then

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\lim_e R\Gamma(\mathrm{Cris}(X/S_e), \mathcal{F}_e)$$

Hints: Suffices to prove this for $\mathcal{F}$ injective. In this case the sheaves $\mathcal{F}_e$ are injective modules too, the transition maps $\Gamma(\mathcal{F}_{e+1}) \to \Gamma(\mathcal{F}_e)$ are surjective, and we have $\Gamma(\mathcal{F}) = \lim_e \Gamma(\mathcal{F}_e)$ because any object of $\mathrm{Cris}(X/S)$ is locally an object of one of the categories $\mathrm{Cris}(X/S_e)$ by definition of $\mathrm{Cris}(X/S)$.

07MW Remark 60.24.11 (Comparison). Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) γ extends to a divided power structure on the ideal of X in Y so that (X, Y, γ) is an object of $\mathrm{Cris}(X/S)$,
(2) the restriction $\mathcal{F}_Y$ (see Section 60.10) comes endowed with a canonical integrable connection $\nabla : \mathcal{F}_Y \to \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$, and
(3) we have
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$
in $D(A)$.

Hints: See Divided Power Algebra, Lemma 23.4.2 for (1). See Lemma 60.15.1 for (2). For Part (3) note that there is a map, see (60.23.2.1). This map is an isomorphism when X is affine, see Lemma 60.21.4. This shows that $Ru_{X/S,*}\mathcal{F}$ and $\mathcal{F}_Y \otimes \Omega^\bullet_{Y/S}$ are quasi-isomorphic as complexes on $Y_{Zar} = X_{Zar}$. Since $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(X_{Zar}, Ru_{X/S,*}\mathcal{F})$ the result follows.

07MX Remark 60.24.12 (Perfectness). Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: By Remark 60.24.9 we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes_A^{\mathbf{L}} A/I \cong R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)})$$

By Remark 60.24.11 we have

$$R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)}) = R\Gamma(X, \mathcal{F}_X \otimes \Omega^\bullet_{X/S_0})$$

Using the stupid filtration on the de Rham complex we see that the last displayed complex is perfect in $D(A/I)$ as soon as the complexes

$$R\Gamma(X, \mathcal{F}_X \otimes \Omega^q_{X/S_0})$$

are perfect complexes in $D(A/I)$, see More on Algebra, Lemma 15.74.4. This is true by standard arguments in coherent cohomology using that $\mathcal{F}_X \otimes \Omega^q_{X/S_0}$ is a finite locally free sheaf and $X \to S_0$ is proper and flat (insert future reference here). Applying More on Algebra, Lemma 15.78.4 we see that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes_A^{\mathbf{L}} A/I^n$$

is a perfect object of $D(A/I^n)$ for all n. This isn't quite enough unless A is Noetherian. Namely, even though I is locally nilpotent by our assumption that p is nilpotent, see Divided Power Algebra, Lemma 23.2.6, we cannot conclude that $I^n = 0$ for some n. A counter example is $\mathbf{F}_p\langle x\rangle$. To prove it in general when $\mathcal{F} = \mathcal{O}_{X/S}$ the argument of `https://math.columbia.edu/~dejong/wordpress/?p=2227` works. When the coefficients $\mathcal{F}$ are non-trivial the argument of [Fal99] seems to be as follows. Reduce to the case $pA = 0$ by More on Algebra, Lemma 15.78.4. In this case the Frobenius map $A \to A$, $a \mapsto a^p$ factors as $A \to A/I \xrightarrow{\varphi} A$ (as $x^p = 0$ for $x \in I$). Set $X^{(1)} = X \otimes_{A/I,\varphi} A$. The absolute Frobenius morphism of X factors through a morphism $F_X : X \to X^{(1)}$ (a kind of relative Frobenius). Affine locally if $X = \text{Spec}(C)$ then $X^{(1)} = \text{Spec}(C \otimes_{A/I,\varphi} A)$ and F_X corresponds to $C \otimes_{A/I,\varphi} A \to C$, $c \otimes a \mapsto c^p a$. This defines morphisms of ringed topoi

$$(X/S)_{\text{cris}} \xrightarrow{(F_X)_{\text{cris}}} (X^{(1)}/S)_{\text{cris}} \xrightarrow{u_{X^{(1)}/S}} Sh(X^{(1)}_{Zar})$$

whose composition is denoted Frob_X. One then shows that $R\text{Frob}_{X,*}\mathcal{F}$ is representable by a perfect complex of $\mathcal{O}_{X^{(1)}}$-modules(!) by a local calculation.

07MY Remark 60.24.13 (Complete perfectness). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a p-adically complete ring and p nilpotent in A/I. Set $S = \text{Spec}(A)$ and $S_0 = \text{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\text{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: We know that $K = R\Gamma(\text{Cris}(X/S), \mathcal{F})$ is the derived limit $K = R\lim K_e$ of the cohomologies over A/p^eA, see Remark 60.24.10. Each K_e is a perfect complex of $D(A/p^eA)$ by Remark 60.24.12. Since A is p-adically complete the result follows from More on Algebra, Lemma 15.97.4.

07MZ Remark 60.24.14 (Complete comparison). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a Noetherian p-adically complete ring and p nilpotent in A/I. Set $S = \text{Spec}(A)$ and $S_0 = \text{Spec}(A/I)$. Let Y be a proper smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a finite type crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) there exists a coherent $\mathcal{O}_Y$-module $\mathcal{F}_Y$ endowed with integrable connection

$$\nabla : \mathcal{F}_Y \longrightarrow \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$$

such that $\mathcal{F}_Y/p^e\mathcal{F}_Y$ is the module with connection over A/p^eA found in Remark 60.24.11, and

(2) we have

$$R\Gamma(\text{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$

in $D(A)$.

Hints: The existence of $\mathcal{F}_Y$ is Grothendieck's existence theorem (insert future reference here). The isomorphism of cohomologies follows as both sides are computed as $R\lim$ of the versions modulo p^e (see Remark 60.24.10 for the left hand side; use the theorem on formal functions, see Cohomology of Schemes, Theorem 30.20.5 for the right hand side). Each of the versions modulo p^e are isomorphic by Remark 60.24.11.

60.25. Pulling back along purely inseparable maps

07PZ By an α_p-cover we mean a morphism of the form

$$X' = \mathrm{Spec}(C[z]/(z^p - c)) \longrightarrow \mathrm{Spec}(C) = X$$

where C is an $\mathbf{F}_p$-algebra and $c \in C$. Equivalently, X' is an α_p-torsor over X. An iterated α_p-cover[7] is a morphism of schemes in characteristic p which is locally on the target a composition of finitely many α_p-covers. In this section we prove that pullback along such a morphism induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. In fact, we prove a precise version of this result. We begin with a preliminary lemma whose formulation needs some notation.

Assume we have a ring map $B \to B'$ and quotients $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ satisfying the assumptions of Remark 60.6.9. Thus (60.6.9.1) provides a canonical map of complexes

$$c^\bullet_M : M \otimes_B \Omega^\bullet \longrightarrow M \otimes_B (\Omega')^\bullet$$

for all B-modules M endowed with integrable connection $\nabla : M \to M \otimes_B \Omega_B$.

Suppose we have $a \in B$, $z \in B'$, and a map $\theta : B' \to B'$ satisfying the following assumptions

07Q0 (1) $\mathrm{d}(a) = 0$,

07Q1 (2) $\Omega' = B' \otimes_B \Omega \oplus B'\mathrm{d}z$; we write $\mathrm{d}(f) = \mathrm{d}_1(f) + \partial_z(f)\mathrm{d}z$ with $\mathrm{d}_1(f) \in B' \otimes \Omega$ and $\partial_z(f) \in B'$ for all $f \in B'$,

07Q2 (3) $\theta : B' \to B'$ is B-linear,

07Q3 (4) $\partial_z \circ \theta = a$,

07Q4 (5) $B \to B'$ is universally injective (and hence $\Omega \to \Omega'$ is injective),

07Q5 (6) $af - \theta(\partial_z(f)) \in B$ for all $f \in B'$,

07Q6 (7) $(\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f)) \in \Omega$ for all $f \in B'$ where $\theta \otimes 1 : B' \otimes \Omega \to B' \otimes \Omega$

These conditions are not logically independent. For example, assumption (4) implies that $\partial_z(af - \theta(\partial_z(f))) = 0$. Hence if the image of $B \to B'$ is the collection of elements annihilated by ∂_z, then (6) follows. A similar argument can be made for condition (7).

07Q7 **Lemma 60.25.1.** In the situation above there exists a map of complexes

$$e^\bullet_M : M \otimes_B (\Omega')^\bullet \longrightarrow M \otimes_B \Omega^\bullet$$

such that $c^\bullet_M \circ e^\bullet_M$ and $e^\bullet_M \circ c^\bullet_M$ are homotopic to multiplication by a.

Proof. In this proof all tensor products are over B. Assumption (2) implies that

$$M \otimes (\Omega')^i = (B' \otimes M \otimes \Omega^i) \oplus (B'\mathrm{d}z \otimes M \otimes \Omega^{i-1})$$

for all $i \geq 0$. A collection of additive generators for $M \otimes (\Omega')^i$ is formed by elements of the form $f\omega$ and elements of the form $f\mathrm{d}z \wedge \eta$ where $f \in B'$, $\omega \in M \otimes \Omega^i$, and $\eta \in M \otimes \Omega^{i-1}$.

For $f \in B'$ we write

$$\epsilon(f) = af - \theta(\partial_z(f)) \quad \text{and} \quad \epsilon'(f) = (\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f))$$

so that $\epsilon(f) \in B$ and $\epsilon'(f) \in \Omega$ by assumptions (6) and (7). We define $e^\bullet_M$ by the rules $e^i_M(f\omega) = \epsilon(f)\omega$ and $e^i_M(f\mathrm{d}z \wedge \eta) = \epsilon'(f) \wedge \eta$. We will see below that the collection of maps e^i_M is a map of complexes.

[7] This is nonstandard notation.

We define

$$h^i : M \otimes_B (\Omega')^i \longrightarrow M \otimes_B (\Omega')^{i-1}$$

by the rules $h^i(f\omega) = 0$ and $h^i(f\mathrm{d}z \wedge \eta) = \theta(f)\eta$ for elements as above. We claim that

$$\mathrm{d} \circ h + h \circ \mathrm{d} = a - c_M^\bullet \circ e_M^\bullet$$

Note that multiplication by a is a map of complexes by (1). Hence, since $c_M^\bullet$ is an injective map of complexes by assumption (5), we conclude that $e_M^\bullet$ is a map of complexes. To prove the claim we compute

$$\begin{aligned}(\mathrm{d} \circ h + h \circ \mathrm{d})(f\omega) &= h\left(\mathrm{d}(f) \wedge \omega + f\nabla(\omega)\right) \\ &= \theta(\partial_z(f))\omega \\ &= af\omega - \epsilon(f)\omega \\ &= af\omega - c_M^i(e_M^i(f\omega))\end{aligned}$$

The second equality because $\mathrm{d}z$ does not occur in $\nabla(\omega)$ and the third equality by assumption (6). Similarly, we have

$$\begin{aligned}(\mathrm{d} \circ h + h \circ \mathrm{d})(f\mathrm{d}z \wedge \eta) &= \mathrm{d}(\theta(f)\eta) + h\left(\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta - f\mathrm{d}z \wedge \nabla(\eta)\right) \\ &= \mathrm{d}(\theta(f)) \wedge \eta + \theta(f)\nabla(\eta) - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta - \theta(f)\nabla(\eta) \\ &= \mathrm{d}_1(\theta(f)) \wedge \eta + \partial_z(\theta(f))\mathrm{d}z \wedge \eta - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta \\ &= af\mathrm{d}z \wedge \eta - \epsilon'(f) \wedge \eta \\ &= af\mathrm{d}z \wedge \eta - c_M^i(e_M^i(f\mathrm{d}z \wedge \eta))\end{aligned}$$

The second equality because $\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta = -\mathrm{d}z \wedge \mathrm{d}_1(f) \wedge \eta$. The fourth equality by assumption (4). On the other hand it is immediate from the definitions that $e_M^i(c_M^i(\omega)) = \epsilon(1)\omega = a\omega$. This proves the lemma. □

07Q8 Example 60.25.2. A standard example of the situation above occurs when $B' = B\langle z\rangle$ is the divided power polynomial ring over a divided power ring (B, J, δ) with divided powers δ' on $J' = B'_+ + JB' \subset B'$. Namely, we take $\Omega = \Omega_{B,\delta}$ and $\Omega' = \Omega_{B',\delta'}$. In this case we can take $a = 1$ and

$$\theta(\sum b_m z^{[m]}) = \sum b_m z^{[m+1]}$$

Note that

$$f - \theta(\partial_z(f)) = f(0)$$

equals the constant term. It follows that in this case Lemma 60.25.1 recovers the crystalline Poincaré lemma (Lemma 60.20.2).

07N1 Lemma 60.25.3. In Situation 60.5.1. Assume D and Ω_D are as in (60.17.0.1) and (60.17.0.2). Let $\lambda \in D$. Let D' be the p-adic completion of

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$$

and let $\Omega_{D'}$ be the p-adic completion of the module of divided power differentials of D' over A. For any pair (M, ∇) over D satisfying (1), (2), (3), and (4) the canonical map of complexes (60.6.9.1)

$$c_M^\bullet : M \otimes_D^\wedge \Omega_D^\bullet \longrightarrow M \otimes_D^\wedge \Omega_{D'}^\bullet$$

has the following property: There exists a map $e_M^\bullet$ in the opposite direction such that both $c_M^\bullet \circ e_M^\bullet$ and $e_M^\bullet \circ c_M^\bullet$ are homotopic to multiplication by p.

Proof. We will prove this using Lemma 60.25.1 with $a = p$. Thus we have to find $\theta : D' \to D'$ and prove (1), (2), (3), (4), (5), (6), (7). We first collect some information about the rings D and D' and the modules Ω_D and $\Omega_{D'}$.

Writing

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda)) = D\langle\xi\rangle[z]/(z^p - \xi - \lambda)$$

we see that D' is the p-adic completion of the free D-module

$$\bigoplus\nolimits_{i=0,\ldots,p-1} \bigoplus\nolimits_{n \geq 0} z^i \xi^{[n]} D$$

where $\xi^{[0]} = 1$. It follows that $D \to D'$ has a continuous D-linear section, in particular $D \to D'$ is universally injective, i.e., (5) holds. We think of D' as a divided power algebra over A with divided power ideal $\overline{J}' = \overline{J}D' + (\xi)$. Then D' is also the p-adic completion of the divided power envelope of the ideal generated by $z^p - \lambda$ in D, see Lemma 60.2.4. Hence

$$\Omega_{D'} = \Omega_D \otimes^\wedge_D D' \oplus D'\mathrm{d}z$$

by Lemma 60.6.6. This proves (2). Note that (1) is obvious.

At this point we construct θ. (We wrote a PARI/gp script theta.gp verifying some of the formulas in this proof which can be found in the scripts subdirectory of the Stacks project.) Before we do so we compute the derivative of the elements $z^i\xi^{[n]}$. We have $\mathrm{d}z^i = iz^{i-1}\mathrm{d}z$. For $n \geq 1$ we have

$$\mathrm{d}\xi^{[n]} = \xi^{[n-1]}\mathrm{d}\xi = -\xi^{[n-1]}\mathrm{d}\lambda + pz^{p-1}\xi^{[n-1]}\mathrm{d}z$$

because $\xi = z^p - \lambda$. For $0 < i < p$ and $n \geq 1$ we have

$$\begin{aligned}
\mathrm{d}(z^i\xi^{[n]}) &= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}\xi \\
&= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}(z^p - \lambda) \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i+p-1}\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i-1}(\xi + \lambda)\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + ((i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]})\mathrm{d}z
\end{aligned}$$

the last equality because $\xi\xi^{[n-1]} = n\xi^{[n]}$. Thus we see that

$$\begin{aligned}
\partial_z(z^i) &= iz^{i-1} \\
\partial_z(\xi^{[n]}) &= pz^{p-1}\xi^{[n-1]} \\
\partial_z(z^i\xi^{[n]}) &= (i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]}
\end{aligned}$$

Motivated by these formulas we define θ by the rules

$$\begin{array}{rcll}
\theta(z^j) & = & p\frac{z^{j+1}}{j+1} & j = 0, \ldots p-1, \\
\theta(z^{p-1}\xi^{[m]}) & = & \xi^{[m+1]} & m \geq 1, \\
\theta(z^j\xi^{[m]}) & = & \frac{pz^{j+1}\xi^{[m]} - \theta(p\lambda z^j\xi^{[m-1]})}{(j+1+pm)} & 0 \leq j < p-1, m \geq 1
\end{array}$$

where in the last line we use induction on m to define our choice of θ. Working this out we get (for $0 \leq j < p - 1$ and $1 \leq m$)

$$\theta(z^j\xi^{[m]}) = \frac{pz^{j+1}\xi^{[m]}}{(j+1+pm)} - \frac{p^2\lambda z^{j+1}\xi^{[m-1]}}{(j+1+pm)(j+1+p(m-1))} + \ldots + \frac{(-1)^m p^{m+1}\lambda^m z^{j+1}}{(j+1+pm)\ldots(j+1)}$$

although we will not use this expression below. It is clear that θ extends uniquely to a p-adically continuous D-linear map on D'. By construction we have (3) and (4). It remains to prove (6) and (7).

Proof of (6) and (7). As θ is D-linear and continuous it suffices to prove that $p - \theta \circ \partial_z$, resp. $(\theta \otimes 1) \circ \text{d}_1 - \text{d}_1 \circ \theta$ gives an element of D, resp. Ω_D when evaluated on the elements $z^i\xi^{[n]}$[8]. Set $D_0 = \mathbf{Z}_{(p)}[\lambda]$ and $D_0' = \mathbf{Z}_{(p)}[z, \lambda]\langle\xi\rangle/(\xi - z^p + \lambda)$. Observe that each of the expressions above is an element of D_0' or $\Omega_{D_0'}$. Hence it suffices to prove the result in the case of $D_0 \to D_0'$. Note that D_0 and D_0' are torsion free rings and that $D_0 \otimes \mathbf{Q} = \mathbf{Q}[\lambda]$ and $D_0' \otimes \mathbf{Q} = \mathbf{Q}[z, \lambda]$. Hence $D_0 \subset D_0'$ is the subring of elements annihilated by ∂_z and (6) follows from (4), see the discussion directly preceding Lemma 60.25.1. Similarly, we have $\text{d}_1(f) = \partial_\lambda(f)\text{d}\lambda$ hence

$$((\theta \otimes 1) \circ \text{d}_1 - \text{d}_1 \circ \theta)(f) = (\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f)))\,\text{d}\lambda$$

Applying ∂_z to the coefficient we obtain

$$\begin{aligned} \partial_z\left(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))\right) &= p\partial_\lambda(f) - \partial_z(\partial_\lambda(\theta(f))) \\ &= p\partial_\lambda(f) - \partial_\lambda(\partial_z(\theta(f))) \\ &= p\partial_\lambda(f) - \partial_\lambda(pf) = 0 \end{aligned}$$

whence the coefficient does not depend on z as desired. This finishes the proof of the lemma. □

Note that an iterated α_p-cover $X' \to X$ (as defined in the introduction to this section) is finite locally free. Hence if X is connected the degree of $X' \to X$ is constant and is a power of p.

07Q9 Lemma 60.25.4. Let p be a prime number. Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme over $\mathbf{Z}_{(p)}$ with $p \in \mathcal{I}$. We set $S_0 = V(\mathcal{I}) \subset S$. Let $f : X' \to X$ be an iterated α_p-cover of schemes over S_0 with constant degree q. Let $\mathcal{F}$ be any crystal in quasi-coherent sheaves on X and set $\mathcal{F}' = f_{\text{cris}}^*\mathcal{F}$. In the distinguished triangle

$$Ru_{X/S,*}\mathcal{F} \longrightarrow f_*Ru_{X'/S,*}\mathcal{F}' \longrightarrow E \longrightarrow Ru_{X/S,*}\mathcal{F}[1]$$

the object E has cohomology sheaves annihilated by q.

Proof. Note that $X' \to X$ is a homeomorphism hence we can identify the underlying topological spaces of X and X'. The question is clearly local on X, hence we may assume X, X', and S affine and $X' \to X$ given as a composition

$$X' = X_n \to X_{n-1} \to X_{n-2} \to \ldots \to X_0 = X$$

where each morphism $X_{i+1} \to X_i$ is an α_p-cover. Denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. It suffices to prove that each of the maps

$$R\Gamma(\text{Cris}(X_i/S), \mathcal{F}_i) \longrightarrow R\Gamma(\text{Cris}(X_{i+1}/S), \mathcal{F}_{i+1})$$

fits into a triangle whose third member has cohomology groups annihilated by p. (This uses axiom TR4 for the triangulated category $D(X)$. Details omitted.)

Hence we may assume that $S = \text{Spec}(A)$, $X = \text{Spec}(C)$, $X' = \text{Spec}(C')$ and $C' = C[z]/(z^p - c)$ for some $c \in C$. Choose a polynomial algebra P over A and a surjection $P \to C$. Let D be the p-adically completed divided power envelop

[8]This can be done by direct computation: It turns out that $p - \theta \circ \partial_z$ evaluated on $z^i\xi^{[n]}$ gives zero except for 1 which is mapped to p and ξ which is mapped to $-p\lambda$. It turns out that $(\theta \otimes 1) \circ \text{d}_1 - \text{d}_1 \circ \theta$ evaluated on $z^i\xi^{[n]}$ gives zero except for $z^{p-1}\xi$ which is mapped to $-\lambda$.

of $\text{Ker}(P \to C)$ in P as in (60.17.0.1). Set $P' = P[z]$ with surjection $P' \to C'$ mapping z to the class of z in C'. Choose a lift $\lambda \in D$ of $c \in C$. Then we see that the p-adically completed divided power envelope D' of $\text{Ker}(P' \to C')$ in P' is isomorphic to the p-adic completion of $D[z]\langle \xi \rangle / (\xi - (z^p - \lambda))$, see Lemma 60.25.3 and its proof. Thus we see that the result follows from this lemma by the computation of cohomology of crystals in quasi-coherent modules in Proposition 60.21.3. $\square$

The bound in the following lemma is probably not optimal.

07QA Lemma 60.25.5. With notations and assumptions as in Lemma 60.25.4 the map

$$f^* : H^i(\text{Cris}(X/S), \mathcal{F}) \longrightarrow H^i(\text{Cris}(X'/S), \mathcal{F}')$$

has kernel and cokernel annihilated by q^{i+1}.

Proof. This follows from the fact that E has nonzero cohomology sheaves in degrees -1 and up, so that the spectral sequence $H^a(\mathcal{H}^b(E)) \Rightarrow H^{a+b}(E)$ converges. This combined with the long exact cohomology sequence associated to a distinguished triangle gives the bound. $\square$

In Situation 60.7.5 assume that $p \in \mathcal{I}$. Set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0.$$

Denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism.

07QB Lemma 60.25.6. In the situation above, assume that $X \to S_0$ is smooth of relative dimension d. Then F_{X/S_0} is an iterated α_p-cover of degree p^d. Hence Lemmas 60.25.4 and 60.25.5 apply to this situation. In particular, for any crystal in quasi-coherent modules $\mathcal{G}$ on $\text{Cris}(X^{(1)}/S)$ the map

$$F_{X/S_0}^* : H^i(\text{Cris}(X^{(1)}/S), \mathcal{G}) \longrightarrow H^i(\text{Cris}(X/S), F_{X/S_0, \text{cris}}^* \mathcal{G})$$

has kernel and cokernel annihilated by $p^{d(i+1)}$.

Proof. It suffices to prove the first statement. To see this we may assume that X is étale over $\mathbf{A}^d_{S_0}$, see Morphisms, Lemma 29.36.20. Denote $\varphi : X \to \mathbf{A}^d_{S_0}$ this étale morphism. In this case the relative Frobenius of X/S_0 fits into a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^{(1)} \arrow[d] \\ \mathbf{A}^d_{S_0} \arrow[r] & \mathbf{A}^d_{S_0} \end{tikzcd}$$

where the lower horizontal arrow is the relative frobenius morphism of $\mathbf{A}^d_{S_0}$ over S_0. This is the morphism which raises all the coordinates to the pth power, hence it is an iterated α_p-cover. The proof is finished by observing that the diagram is a fibre square, see Étale Morphisms, Lemma 41.14.3. $\square$

60.26. Frobenius action on crystalline cohomology

07N0 In this section we prove that Frobenius pullback induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. But in order to even formulate this we need to work in a special situation.

07N2 Situation 60.26.1. In Situation 60.7.5 assume the following

(1) $S = \mathrm{Spec}(A)$ for some divided power ring (A, I, γ) with $p \in I$,
(2) there is given a homomorphism of divided power rings $\sigma : A \to A$ such that $\sigma(x) = x^p \bmod pA$ for all $x \in A$.

In Situation 60.26.1 the morphism $\mathrm{Spec}(\sigma) : S \to S$ is a lift of the absolute Frobenius $F_{S_0} : S_0 \to S_0$ and since the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"'] \arrow[d] & X \arrow[d] \\ S_0 \arrow[r, "F_{S_0}"] & S_0 \end{tikzcd}$$

is commutative where $F_X : X \to X$ is the absolute Frobenius morphism of X. Thus we obtain a morphism of crystalline topoi

$$(F_X)_{\mathrm{cris}} : (X/S)_{\mathrm{cris}} \longrightarrow (X/S)_{\mathrm{cris}}$$

see Remark 60.9.3. Here is the terminology concerning F-crystals following the notation of Saavedra, see [SR72].

07N3 Definition 60.26.2. In Situation 60.26.1 an F-crystal on X/S (relative to σ) is a pair $(\mathcal{E}, F_\mathcal{E})$ given by a crystal in finite locally free $\mathcal{O}_{X/S}$-modules $\mathcal{E}$ together with a map

$$F_\mathcal{E} : (F_X)_{\mathrm{cris}}^*\mathcal{E} \longrightarrow \mathcal{E}$$

An F-crystal is called nondegenerate if there exists an integer $i \geq 0$ a map $V : \mathcal{E} \to (F_X)_{\mathrm{cris}}^*\mathcal{E}$ such that $V \circ F_\mathcal{E} = p^i\mathrm{id}$.

07N4 Remark 60.26.3. Let $(\mathcal{E}, F)$ be an F-crystal as in Definition 60.26.2. In the literature the nondegeneracy condition is often part of the definition of an F-crystal. Moreover, often it is also assumed that $F \circ V = p^n\mathrm{id}$. What is needed for the result below is that there exists an integer $j \geq 0$ such that $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ are killed by p^j. If the rank of $\mathcal{E}$ is bounded (for example if X is quasi-compact), then both of these conditions follow from the nondegeneracy condition as formulated in the definition. Namely, suppose R is a ring, $r \geq 1$ is an integer and $K, L \in \mathrm{Mat}(r\times r, R)$ are matrices with $KL = p^i 1_{r\times r}$. Then $\det(K)\det(L) = p^{ri}$. Let L' be the adjugate matrix of L, i.e., $L'L = LL' = \det(L)$. Set $K' = p^{ri}K$ and $j = ri + i$. Then we have $K'L = p^j 1_{r\times r}$ as $KL = p^i$ and

$$LK' = LK\det(L)\det(M) = LKLL'\det(M) = Lp^iL'\det(M) = p^j1_{r\times r}$$

It follows that if V is as in Definition 60.26.2 then setting $V' = p^NV$ where $N > i \cdot \mathrm{rank}(\mathcal{E})$ we get $V' \circ F = p^{N+i}$ and $F \circ V' = p^{N+i}$.

07N5 Theorem 60.26.4. In Situation 60.26.1 let $(\mathcal{E}, F_\mathcal{E})$ be a nondegenerate F-crystal. Assume A is a p-adically complete Noetherian ring and that $X \to S_0$ is proper smooth. Then the canonical map

$$F_\mathcal{E} \circ (F_X)_{\mathrm{cris}}^* : R\Gamma(\mathrm{Cris}(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(\mathrm{Cris}(X/S), \mathcal{E})$$

becomes an isomorphism after inverting p.

Proof. We first write the arrow as a composition of three arrows. Namely, set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0$$

and denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism. Denote $\mathcal{E}^{(1)}$ the base change of $\mathcal{E}$ by $\text{Spec}(\sigma)$, in other words the pullback of $\mathcal{E}$ to $\text{Cris}(X^{(1)}/S)$ by the morphism of crystalline topoi associated to the commutative diagram

$$\begin{tikzcd} X^{(1)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "\text{Spec}(\sigma)"] & S \end{tikzcd}$$

Then we have the base change map

07QC (60.26.4.1)
$$R\Gamma(\text{Cris}(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(\text{Cris}(X^{(1)}/S), \mathcal{E}^{(1)})$$

see Remark 60.24.8. Note that the composition of $F_{X/S_0} : X \to X^{(1)}$ with the projection $X^{(1)} \to X$ is the absolute Frobenius morphism F_X. Hence we see that $F^*_{X/S_0}\mathcal{E}^{(1)} = (F_X)^*_{\text{cris}}\mathcal{E}$. Thus pullback by F_{X/S_0} is a map

07N6 (60.26.4.2)
$$F^*_{X/S_0} : R\Gamma(\text{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \longrightarrow R\Gamma(\text{Cris}(X/S), (F_X)^*_{\text{cris}}\mathcal{E})$$

Finally we can use $F_{\mathcal{E}}$ to get a map

07QD (60.26.4.3)
$$R\Gamma(\text{Cris}(X/S), (F_X)^*_{\text{cris}}\mathcal{E}) \longrightarrow R\Gamma(\text{Cris}(X/S), \mathcal{E})$$

The map of the theorem is the composition of the three maps (60.26.4.1), (60.26.4.2), and (60.26.4.3) above. The first is a quasi-isomorphism modulo all powers of p by Remark 60.24.9. Hence it is a quasi-isomorphism since the complexes involved are perfect in $D(A)$ see Remark 60.24.13. The third map is a quasi-isomorphism after inverting p simply because $F_{\mathcal{E}}$ has an inverse up to a power of p, see Remark 60.26.3. Finally, the second is an isomorphism after inverting p by Lemma 60.25.6. □

60.27. Other chapters

CHAPTER 61

Pro-étale Cohomology

0965 61.1. Introduction

0966 The material in this chapter and more can be found in the preprint [BS13].

The goal of this chapter is to introduce the pro-étale topology and to develop the basic theory of cohomology of abelian sheaves in this topology. A secondary goal is to show how using the pro-étale topology simplifies the introduction of ℓ-adic cohomology in algebraic geometry.

Here is a brief overview of the history of ℓ-adic étale cohomology as we have understood it. In [Gro77, Exposés V and VI] Grothendieck et al developed a theory for dealing with ℓ-adic sheaves as inverse systems of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. In his second paper on the Weil conjectures ([Del80]) Deligne introduced a derived category of ℓ-adic sheaves as a certain 2-limit of categories of complexes of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules on the étale site of a scheme X. This approach is used in the paper by Beilinson, Bernstein, and Deligne ([BBD82]) as the basis for their beautiful theory of perverse sheaves. In a paper entitled "Continuous Étale Cohomology" ([Jan88]) Uwe Jannsen discusses an important variant of the cohomology of a ℓ-adic sheaf on a variety over a field. His paper is followed up by a paper of Torsten Ekedahl ([Eke90]) who discusses the adic formalism needed to work comfortably with derived categories defined as limits.

It turns out that, working with the pro-étale site of a scheme, one can avoid some of the technicalities these authors encountered. This comes at the expense of having to work with non-Noetherian schemes, even when one is only interested in working with ℓ-adic sheaves and cohomology of such on varieties over an algebraically closed field.

A very important and remarkable feature of the (small) pro-étale site of a scheme is that it has enough quasi-compact w-contractible objects. The existence of these objects implies a number of useful and (perhaps) unusual consequences for the derived category of abelian sheaves and for inverse systems of sheaves. This is exactly the feature that will allow us to handle the intricacies of working with ℓ-adic sheaves, but as we will see it has a number of other benefits as well.

61.2. Some topology

0967 Some preliminaries. We have defined spectral spaces and spectral maps of spectral spaces in Topology, Section 5.23. The spectrum of a ring is a spectral space, see Algebra, Lemma 10.26.2.

0968 **Lemma 61.2.1.** Let X be a spectral space. Let $X_0 \subset X$ be the set of closed points. The following are equivalent

(1) Every open covering of X can be refined by a finite disjoint union decomposition $X = \coprod U_i$ with U_i open and closed in X.
(2) The composition $X_0 \to X \to \pi_0(X)$ is bijective.

Moreover, if X_0 is closed in X and every point of X specializes to a unique point of X_0, then these conditions are satisfied.

Proof. We will use without further mention that X_0 is quasi-compact (Topology, Lemma 5.12.9) and $\pi_0(X)$ is profinite (Topology, Lemma 5.23.9). Picture

$$\begin{tikzcd} X_0 \arrow[r] \arrow[rd, "f"'] & X \arrow[d, "\pi"] \\ & \pi_0(X) \end{tikzcd}$$

If (2) holds, the continuous bijective map $f : X_0 \to \pi_0(X)$ is a homeomorphism by Topology, Lemma 5.17.8. Given an open covering $X = \bigcup U_i$, we get an open covering $\pi_0(X) = \bigcup f(X_0 \cap U_i)$. By Topology, Lemma 5.22.4 we can find a finite open covering of the form $\pi_0(X) = \coprod V_j$ which refines this covering. Since $X_0 \to \pi_0(X)$ is bijective each connected component of X has a unique closed point, whence is equal to the set of points specializing to this closed point. Hence $\pi^{-1}(V_j)$ is the set of points specializing to the points of $f^{-1}(V_j)$. Now, if $f^{-1}(V_j) \subset X_0 \cap U_i \subset U_i$, then it follows that $\pi^{-1}(V_j) \subset U_i$ (because the open set U_i is closed under generalizations). In this way we see that the open covering $X = \coprod \pi^{-1}(V_j)$ refines the covering we started out with. In this way we see that (2) implies (1).

Assume (1). Let $x, y \in X$ be closed points. Then we have the open covering $X = (X \setminus \{x\}) \cup (X \setminus \{y\})$. It follows from (1) that there exists a disjoint union decomposition $X = U \amalg V$ with U and V open (and closed) and $x \in U$ and $y \in V$. In particular we see that every connected component of X has at most one closed point. By Topology, Lemma 5.12.8 every connected component (being closed) also does have a closed point. Thus $X_0 \to \pi_0(X)$ is bijective. In this way we see that (1) implies (2).

Assume X_0 is closed in X and every point specializes to a unique point of X_0. Then X_0 is a spectral space (Topology, Lemma 5.23.5) consisting of closed points, hence profinite (Topology, Lemma 5.23.8). Let $x, y \in X_0$ be distinct. By Topology, Lemma 5.22.4 we can find a disjoint union decomposition $X_0 = U_0 \amalg V_0$ with U_0 and V_0 open and closed and $x \in U_0$ and $y \in V_0$. Let $U \subset X$, resp. $V \subset X$ be the set of points specializing to U_0, resp. V_0. Observe that $X = U \amalg V$. By Topology, Lemma 5.24.7 we see that U is an intersection of quasi-compact open subsets. Hence U is closed in the constructible topology. Since U is closed under specialization, we see that U is closed by Topology, Lemma 5.23.6. By symmetry V is closed and hence U and V are both open and closed. This proves that x, y are not in the same connected component of X. In other words, $X_0 \to \pi_0(X)$ is injective. The map is also surjective by Topology, Lemma 5.12.8 and the fact that connected components are closed. In this way we see that the final condition implies (2). □

0969 Example 61.2.2. Let T be a profinite space. Let $t \in T$ be a point and assume that $T \setminus \{t\}$ is not quasi-compact. Let $X = T \times \{0, 1\}$. Consider the topology on X with a subbase given by the sets $U \times \{0, 1\}$ for $U \subset T$ open, $X \setminus \{(t, 0)\}$, and $U \times \{1\}$ for $U \subset T$ open with $t \notin U$. The set of closed points of X is $X_0 = T \times \{0\}$ and

$(t, 1)$ is in the closure of X_0. Moreover, $X_0 \to \pi_0(X)$ is a bijection. This example shows that conditions (1) and (2) of Lemma 61.2.1 do no imply the set of closed points is closed.

It turns out it is more convenient to work with spectral spaces which have the slightly stronger property mentioned in the final statement of Lemma 61.2.1. We give this property a name.

096A Definition 61.2.3. A spectral space X is w-local if the set of closed points X_0 is closed and every point of X specializes to a unique closed point. A continuous map $f : X \to Y$ of w-local spaces is w-local if it is spectral and maps any closed point of X to a closed point of Y.

We have seen in the proof of Lemma 61.2.1 that in this case $X_0 \to \pi_0(X)$ is a homeomorphism and that $X_0 \cong \pi_0(X)$ is a profinite space. Moreover, a connected component of X is exactly the set of points specializing to a given $x \in X_0$.

096B Lemma 61.2.4. Let X be a w-local spectral space. If $Y \subset X$ is closed, then Y is w-local.

Proof. The subset $Y_0 \subset Y$ of closed points is closed because $Y_0 = X_0 \cap Y$. Since X is w-local, every $y \in Y$ specializes to a unique point of X_0. This specialization is in Y, and hence also in Y_0, because $\overline{\{y\}} \subset Y$. In conclusion, Y is w-local. $\square$

096C Lemma 61.2.5. Let X be a spectral space. Let

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

be a cartesian diagram in the category of topological spaces with T profinite. Then Y is spectral and $T = \pi_0(Y)$. If moreover X is w-local, then Y is w-local, $Y \to X$ is w-local, and the set of closed points of Y is the inverse image of the set of closed points of X.

Proof. Note that Y is a closed subspace of $X \times T$ as $\pi_0(X)$ is a profinite space hence Hausdorff (use Topology, Lemmas 5.23.9 and 5.3.4). Since $X \times T$ is spectral (Topology, Lemma 5.23.10) it follows that Y is spectral (Topology, Lemma 5.23.5). Let $Y \to \pi_0(Y) \to T$ be the canonical factorization (Topology, Lemma 5.7.9). It is clear that $\pi_0(Y) \to T$ is surjective. The fibres of $Y \to T$ are homeomorphic to the fibres of $X \to \pi_0(X)$. Hence these fibres are connected. It follows that $\pi_0(Y) \to T$ is injective. We conclude that $\pi_0(Y) \to T$ is a homeomorphism by Topology, Lemma 5.17.8.

Next, assume that X is w-local and let $X_0 \subset X$ be the set of closed points. The inverse image $Y_0 \subset Y$ of X_0 in Y maps bijectively onto T as $X_0 \to \pi_0(X)$ is a bijection by Lemma 61.2.1. Moreover, Y_0 is quasi-compact as a closed subset of the spectral space Y. Hence $Y_0 \to \pi_0(Y) = T$ is a homeomorphism by Topology, Lemma 5.17.8. It follows that all points of Y_0 are closed in Y. Conversely, if $y \in Y$ is a closed point, then it is closed in the fibre of $Y \to \pi_0(Y) = T$ and hence its image x in X is closed in the (homeomorphic) fibre of $X \to \pi_0(X)$. This implies $x \in X_0$ and hence $y \in Y_0$. Thus Y_0 is the collection of closed points of Y and for each $y \in Y_0$ the set of generalizations of y is the fibre of $Y \to \pi_0(Y)$. The lemma follows. □

61.3. Local isomorphisms

096D We start with a definition.

096E **Definition 61.3.1.** Let $\varphi : A \to B$ be a ring map.

(1) We say $A \to B$ is a *local isomorphism* if for every prime $\mathfrak q \subset B$ there exists a $g \in B$, $g \notin \mathfrak q$ such that $A \to B_g$ induces an open immersion $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$.
(2) We say $A \to B$ *identifies local rings* if for every prime $\mathfrak q \subset B$ the canonical map $A_{\varphi^{-1}(\mathfrak q)} \to B_{\mathfrak q}$ is an isomorphism.

We list some elementary properties.

096F **Lemma 61.3.2.** Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.

(1) If $A \to B$ is a local isomorphism, then $A' \to B'$ is a local isomorphism.
(2) If $A \to B$ identifies local rings, then $A' \to B'$ identifies local rings.

Proof. Omitted. □

096G **Lemma 61.3.3.** Let $A \to B$ and $B \to C$ be ring maps.

(1) If $A \to B$ and $B \to C$ are local isomorphisms, then $A \to C$ is a local isomorphism.
(2) If $A \to B$ and $B \to C$ identify local rings, then $A \to C$ identifies local rings.

Proof. Omitted. □

096H **Lemma 61.3.4.** Let A be a ring. Let $B \to C$ be an A-algebra homomorphism.

(1) If $A \to B$ and $A \to C$ are local isomorphisms, then $B \to C$ is a local isomorphism.
(2) If $A \to B$ and $A \to C$ identify local rings, then $B \to C$ identifies local rings.

Proof. Omitted. □

096I **Lemma 61.3.5.** Let $A \to B$ be a local isomorphism. Then

(1) $A \to B$ is étale,
(2) $A \to B$ identifies local rings,
(3) $A \to B$ is quasi-finite.

Proof. Omitted. □

096J **Lemma 61.3.6.** Let $A \to B$ be a local isomorphism. Then there exist $n \geq 0$, $g_1, \ldots, g_n \in B$, $f_1, \ldots, f_n \in A$ such that $(g_1, \ldots, g_n) = B$ and $A_{f_i} \cong B_{g_i}$.

Proof. Omitted. □

096K **Lemma 61.3.7.** Let $p : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $q : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be morphisms of locally ringed spaces. If $\mathcal{O}_Y = p^{-1}\mathcal{O}_X$, then

$$\mathrm{Mor}_{LRS/(X,\mathcal{O}_X)}((Z, \mathcal{O}_Z), (Y, \mathcal{O}_Y)) \longrightarrow \mathrm{Mor}_{Top/X}(Z, Y), \quad (f, f^\sharp) \longmapsto f$$

is bijective. Here $LRS/(X, \mathcal{O}_X)$ is the category of locally ringed spaces over X and Top/X is the category of topological spaces over X.

Proof. This is immediate from the definitions. ☐

096L Lemma 61.3.8. Let A be a ring. Set $X = \text{Spec}(A)$. The functor

$$B \longmapsto \text{Spec}(B)$$

from the category of A-algebras B such that $A \to B$ identifies local rings to the category of topological spaces over X is fully faithful.

Proof. This follows from Lemma 61.3.7 and the fact that if $A \to B$ identifies local rings, then the pullback of the structure sheaf of $\text{Spec}(A)$ via $p : \text{Spec}(B) \to \text{Spec}(A)$ is equal to the structure sheaf of $\text{Spec}(B)$. ☐

61.4. Ind-Zariski algebra

096M We start with a definition; please see Remark 61.6.9 for a comparison with the corresponding definition of the article [BS13].

096N Definition 61.4.1. A ring map $A \to B$ is said to be ind-Zariski if B can be written as a filtered colimit $B = \text{colim}\, B_i$ with each $A \to B_i$ a local isomorphism.

An example of an Ind-Zariski map is a localization $A \to S^{-1}A$, see Algebra, Lemma 10.9.9. The category of ind-Zariski algebras is closed under several natural operations.

096P Lemma 61.4.2. Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-Zariski, then $A' \to B'$ is ind-Zariski.

Proof. Omitted. ☐

096Q Lemma 61.4.3. Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-Zariski, then $A \to C$ is ind-Zariski.

Proof. Omitted. ☐

096R Lemma 61.4.4. Let A be a ring. Let $B \to C$ be an A-algebra homomorphism. If $A \to B$ and $A \to C$ are ind-Zariski, then $B \to C$ is ind-Zariski.

Proof. Omitted. ☐

096S Lemma 61.4.5. A filtered colimit of ind-Zariski A-algebras is ind-Zariski over A.

Proof. Omitted. ☐

096T Lemma 61.4.6. Let $A \to B$ be ind-Zariski. Then $A \to B$ identifies local rings,

Proof. Omitted. ☐

61.5. Constructing w-local affine schemes

096U An affine scheme X is called w-local if its underlying topological space is w-local (Definition 61.2.3). It turns out given any ring A there is a canonical faithfully flat ind-Zariski ring map $A \to A_w$ such that $\text{Spec}(A_w)$ is w-local. The key to constructing A_w is the following simple lemma.

096V Lemma 61.5.1. Let A be a ring. Set $X = \text{Spec}(A)$. Let $Z \subset X$ be a locally closed subscheme which is of the form $D(f) \cap V(I)$ for some $f \in A$ and ideal $I \subset A$. Then

(1) there exists a multiplicative subset $S \subset A$ such that $\mathrm{Spec}(S^{-1}A)$ maps by a homeomorphism to the set of points of X specializing to Z,
(2) the A-algebra $A_Z^\sim = S^{-1}A$ depends only on the underlying locally closed subset $Z \subset X$,
(3) Z is a closed subscheme of $\mathrm{Spec}(A_Z^\sim)$,

If $A \to A'$ is a ring map and $Z' \subset X' = \mathrm{Spec}(A')$ is a locally closed subscheme of the same form which maps into Z, then there is a unique A-algebra map $A_Z^\sim \to (A')_{Z'}^\sim$.

Proof. Let $S \subset A$ be the multiplicative set of elements which map to invertible elements of $\Gamma(Z, \mathcal{O}_Z) = (A/I)_f$. If $\mathfrak p$ is a prime of A which does not specialize to Z, then $\mathfrak p$ generates the unit ideal in $(A/I)_f$. Hence we can write $f^n = g + h$ for some $n \geq 0$, $g \in \mathfrak p$, $h \in I$. Then $g \in S$ and we see that $\mathfrak p$ is not in the spectrum of $S^{-1}A$. Conversely, if $\mathfrak p$ does specialize to Z, say $\mathfrak p \subset \mathfrak q \supset I$ with $f \not\in \mathfrak q$, then we see that $S^{-1}A$ maps to $A_{\mathfrak q}$ and hence $\mathfrak p$ is in the spectrum of $S^{-1}A$. This proves (1).

The isomorphism class of the localization $S^{-1}A$ depends only on the corresponding subset $\mathrm{Spec}(S^{-1}A) \subset \mathrm{Spec}(A)$, whence (2) holds. By construction $S^{-1}A$ maps surjectively onto $(A/I)_f$, hence (3). The final statement follows as the multiplicative subset $S' \subset A'$ corresponding to Z' contains the image of the multiplicative subset S. ☐

Let A be a ring. Let $E \subset A$ be a finite subset. We get a stratification of $X = \mathrm{Spec}(A)$ into locally closed subschemes by looking at the vanishing behaviour of the elements of E. More precisely, given a disjoint union decomposition $E = E' \amalg E''$ we set

(61.5.1.1)

096W $$Z(E', E'') = \bigcap\nolimits_{f \in E'} D(f) \cap \bigcap\nolimits_{f \in E''} V(f) = D(\prod\nolimits_{f \in E'} f) \cap V(\sum\nolimits_{f \in E''} fA)$$

The points of $Z(E', E'')$ are exactly those $x \in X$ such that $f \in E'$ maps to a nonzero element in $\kappa(x)$ and $f \in E''$ maps to zero in $\kappa(x)$. Thus it is clear that

096X (61.5.1.2) $$X = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'')$$

set theoretically. Observe that each stratum is constructible.

096Y Lemma 61.5.2. Let $X = \mathrm{Spec}(A)$ as above. Given any finite stratification $X = \coprod T_i$ by constructible subsets, there exists a finite subset $E \subset A$ such that the stratification (61.5.1.2) refines $X = \coprod T_i$.

Proof. We may write $T_i = \bigcup_j U_{i,j} \cap V_{i,j}^c$ as a finite union for some $U_{i,j}$ and $V_{i,j}$ quasi-compact open in X. Then we may write $U_{i,j} = \bigcup D(f_{i,j,k})$ and $V_{i,j} = \bigcup D(g_{i,j,l})$. Then we set $E = \{f_{i,j,k}\} \cup \{g_{i,j,l}\}$. This does the job, because the stratification (61.5.1.2) is the one whose strata are labeled by the vanishing pattern of the elements of E which clearly refines the given stratification. ☐

We continue the discussion. Given a finite subset $E \subset A$ we set

096Z (61.5.2.1) $$A_E = \prod\nolimits_{E = E' \amalg E''} A^\sim_{Z(E', E'')}$$

with notation as in Lemma 61.5.1. This makes sense because (61.5.1.1) shows that each $Z(E', E'')$ has the correct shape. We take the spectrum of this ring and denote it

0970 (61.5.2.2) $$X_E = \mathrm{Spec}(A_E) = \coprod\nolimits_{E = E' \amalg E''} X_{E', E''}$$

with $X_{E',E''} = \mathrm{Spec}(A^\sim_{Z(E',E'')})$. Note that

0971 (61.5.2.3)
$$Z_E = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'') \longrightarrow X_E$$

is a closed subscheme. By construction the closed subscheme Z_E contains all the closed points of the affine scheme X_E as every point of $X_{E',E''}$ specializes to a point of $Z(E', E'')$.

Let $I(A)$ be the partially ordered set of all finite subsets of A. This is a directed partially ordered set. For $E_1 \subset E_2$ there is a canonical transition map $A_{E_1} \to A_{E_2}$ of A-algebras. Namely, given a decomposition $E_2 = E'_2 \amalg E''_2$ we set $E'_1 = E_1 \cap E'_2$ and $E''_1 = E_1 \cap E''_2$. Then observe that $Z(E'_1, E''_1) \subset Z(E'_2, E''_2)$ hence a unique A-algebra map $A^\sim_{Z(E'_1,E''_1)} \to A^\sim_{Z(E'_2,E''_2)}$ by Lemma 61.5.1. Using these maps collectively we obtain the desired ring map $A_{E_1} \to A_{E_2}$. Observe that the corresponding map of affine schemes

0972 (61.5.2.4)
$$X_{E_2} \longrightarrow X_{E_1}$$

maps Z_{E_2} into Z_{E_1}. By uniqueness we obtain a system of A-algebras over $I(A)$ and we set

0973 (61.5.2.5)
$$A_w = \mathrm{colim}_{E \in I(A)} A_E$$

This A-algebra is ind-Zariski and faithfully flat over A. Finally, we set $X_w = \mathrm{Spec}(A_w)$ and endow it with the closed subscheme $Z = \lim_{E \in I(A)} Z_E$. In a formula

0974 (61.5.2.6)
$$X_w = \lim_{E \in I(A)} X_E \supset Z = \lim_{E \in I(A)} Z_E$$

0975 **Lemma 61.5.3.** Let $X = \mathrm{Spec}(A)$ be an affine scheme. With $A \to A_w$, $X_w = \mathrm{Spec}(A_w)$, and $Z \subset X_w$ as above.

(1) $A \to A_w$ is ind-Zariski and faithfully flat,
(2) $X_w \to X$ induces a bijection $Z \to X$,
(3) Z is the set of closed points of X_w,
(4) Z is a reduced scheme, and
(5) every point of X_w specializes to a unique point of Z.

In particular, X_w is w-local (Definition 61.2.3).

Proof. The map $A \to A_w$ is ind-Zariski by construction. For every E the morphism $Z_E \to X$ is a bijection, hence (2). As $Z \subset X_w$ we conclude $X_w \to X$ is surjective and $A \to A_w$ is faithfully flat by Algebra, Lemma 10.39.16. This proves (1).

Suppose that $y \in X_w$, $y \notin Z$. Then there exists an E such that the image of y in X_E is not contained in Z_E. Then for all $E \subset E'$ also y maps to an element of $X_{E'}$ not contained in $Z_{E'}$. Let $T_{E'} \subset X_{E'}$ be the reduced closed subscheme which is the closure of the image of y. It is clear that $T = \lim_{E \subset E'} T_{E'}$ is the closure of y in X_w. For every $E \subset E'$ the scheme $T_{E'} \cap Z_{E'}$ is nonempty by construction of $X_{E'}$. Hence $\lim T_{E'} \cap Z_{E'}$ is nonempty and we conclude that $T \cap Z$ is nonempty. Thus y is not a closed point. It follows that every closed point of X_w is in Z.

Suppose that $y \in X_w$ specializes to $z, z' \in Z$. We will show that $z = z'$ which will finish the proof of (3) and will imply (5). Let $x, x' \in X$ be the images of z and z'. Since $Z \to X$ is bijective it suffices to show that $x = x'$. If $x \neq x'$, then there exists an $f \in A$ such that $x \in D(f)$ and $x' \in V(f)$ (or vice versa). Set $E = \{f\}$ so that

$$X_E = \mathrm{Spec}(A_f) \amalg \mathrm{Spec}(A^\sim_{V(f)})$$

Then we see that z and z' map x_E and x'_E which are in different parts of the given decomposition of X_E above. But then it impossible for x_E and x'_E to be specializations of a common point. This is the desired contradiction.

Recall that given a finite subset $E \subset A$ we have Z_E is a disjoint union of the locally closed subschemes $Z(E', E'')$ each isomorphic to the spectrum of $(A/I)_f$ where I is the ideal generated by E'' and f the product of the elements of E'. Any nilpotent element b of $(A/I)_f$ is the class of g/f^n for some $g \in A$. Then setting $E' = E \cup \{g\}$ the reader verifies that b is pulls back to zero under the transition map $Z_{E'} \to Z_E$ of the system. This proves (4). □

0976 Remark 61.5.4. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of A_w (Lemma 61.5.3) is at most κ. Namely, each A_E has cardinality at most κ and the set of finite subsets of A has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6.

0977 Lemma 61.5.5 (Universal property of the construction). Let A be a ring. Let $A \to A_w$ be the ring map constructed in Lemma 61.5.3. For any ring map $A \to B$ such that $\text{Spec}(B)$ is w-local, there is a unique factorization $A \to A_w \to B$ such that $\text{Spec}(B) \to \text{Spec}(A_w)$ is w-local.

Proof. Denote $Y = \text{Spec}(B)$ and $Y_0 \subset Y$ the set of closed points. Denote $f : Y \to X$ the given morphism. Recall that Y_0 is profinite, in particular every constructible subset of Y_0 is open and closed. Let $E \subset A$ be a finite subset. Recall that $A_w = \text{colim}\, A_E$ and that the set of closed points of $\text{Spec}(A_w)$ is the limit of the closed subsets $Z_E \subset X_E = \text{Spec}(A_E)$. Thus it suffices to show there is a unique factorization $A \to A_E \to B$ such that $Y \to X_E$ maps Y_0 into Z_E. Since $Z_E \to X = \text{Spec}(A)$ is bijective, and since the strata $Z(E', E'')$ are constructible we see that

$$Y_0 = \coprod f^{-1}(Z(E', E'')) \cap Y_0$$

is a disjoint union decomposition into open and closed subsets. As $Y_0 = \pi_0(Y)$ we obtain a corresponding decomposition of Y into open and closed pieces. Thus it suffices to construct the factorization in case $f(Y_0) \subset Z(E', E'')$ for some decomposition $E = E' \amalg E''$. In this case $f(Y)$ is contained in the set of points of X specializing to $Z(E', E'')$ which is homeomorphic to $X_{E',E''}$. Thus we obtain a unique continuous map $Y \to X_{E',E''}$ over X. By Lemma 61.3.7 this corresponds to a unique morphism of schemes $Y \to X_{E',E''}$ over X. This finishes the proof. □

Recall that the spectrum of a ring is profinite if and only if every point is closed. There are in fact a whole slew of equivalent conditions that imply this. See Algebra, Lemma 10.26.5 or Topology, Lemma 5.23.8.

0978 Lemma 61.5.6. Let A be a ring such that $\text{Spec}(A)$ is profinite. Let $A \to B$ be a ring map. Then $\text{Spec}(B)$ is profinite in each of the following cases:

(1) if $\mathfrak{q}, \mathfrak{q}' \subset B$ lie over the same prime of A, then neither $\mathfrak{q} \subset \mathfrak{q}'$, nor $\mathfrak{q}' \subset \mathfrak{q}$,
(2) $A \to B$ induces algebraic extensions of residue fields,
(3) $A \to B$ is a local isomorphism,
(4) $A \to B$ identifies local rings,
(5) $A \to B$ is weakly étale,
(6) $A \to B$ is quasi-finite,

(7) $A \to B$ is unramified,
(8) $A \to B$ is étale,
(9) B is a filtered colimit of A-algebras as in (1) – (8),
(10) etc.

Proof. By the references mentioned above (Algebra, Lemma 10.26.5 or Topology, Lemma 5.23.8) there are no specializations between distinct points of $\text{Spec}(A)$ and $\text{Spec}(B)$ is profinite if and only if there are no specializations between distinct points of $\text{Spec}(B)$. These specializations can only happen in the fibres of $\text{Spec}(B) \to \text{Spec}(A)$. In this way we see that (1) is true.

The assumption in (2) implies all primes of B are maximal by Algebra, Lemma 10.35.9. Thus (2) holds. If $A \to B$ is a local isomorphism or identifies local rings, then the residue field extensions are trivial, so (3) and (4) follow from (2). If $A \to B$ is weakly étale, then More on Algebra, Lemma 15.104.17 tells us it induces separable algebraic residue field extensions, so (5) follows from (2). If $A \to B$ is quasi-finite, then the fibres are finite discrete topological spaces. Hence (6) follows from (1). Hence (3) follows from (1). Cases (7) and (8) follow from this as unramified and étale ring map are quasi-finite (Algebra, Lemmas 10.151.6 and 10.143.6). If $B = \text{colim}\, B_i$ is a filtered colimit of A-algebras, then $\text{Spec}(B) = \lim \text{Spec}(B_i)$ in the category of topological spaces by Limits, Lemma 32.4.2. Hence if each $\text{Spec}(B_i)$ is profinite, so is $\text{Spec}(B)$ by Topology, Lemma 5.22.3. This proves (9). $\square$

0979 Lemma 61.5.7. Let A be a ring. Let $V(I) \subset \text{Spec}(A)$ be a closed subset which is a profinite topological space. Then there exists an ind-Zariski ring map $A \to B$ such that $\text{Spec}(B)$ is w-local, the set of closed points is $V(IB)$, and $A/I \cong B/IB$.

Proof. Let $A \to A_w$ and $Z \subset Y = \text{Spec}(A_w)$ as in Lemma 61.5.3. Let $T \subset Z$ be the inverse image of $V(I)$. Then $T \to V(I)$ is a homeomorphism by Topology, Lemma 5.17.8. Let $B = (A_w)^\sim_T$, see Lemma 61.5.1. It is clear that B is w-local with closed points $V(IB)$. The ring map $A/I \to B/IB$ is ind-Zariski and induces a homeomorphism on underlying topological spaces. Hence it is an isomorphism by Lemma 61.3.8. $\square$

097A Lemma 61.5.8. Let A be a ring such that $X = \text{Spec}(A)$ is w-local. Let $I \subset A$ be the radical ideal cutting out the set X_0 of closed points in X. Let $A \to B$ be a ring map inducing algebraic extensions on residue fields at primes. Then

(1) every point of $Z = V(IB)$ is a closed point of $\text{Spec}(B)$,
(2) there exists an ind-Zariski ring map $B \to C$ such that
    (a) $B/IB \to C/IC$ is an isomorphism,
    (b) the space $Y = \text{Spec}(C)$ is w-local,
    (c) the induced map $p : Y \to X$ is w-local, and
    (d) $p^{-1}(X_0)$ is the set of closed points of Y.

Proof. By Lemma 61.5.6 applied to $A/I \to B/IB$ all points of $Z = V(IB) = \text{Spec}(B/IB)$ are closed, in fact $\text{Spec}(B/IB)$ is a profinite space. To finish the proof we apply Lemma 61.5.7 to $IB \subset B$. $\square$

61.6. Identifying local rings versus ind-Zariski

097B An ind-Zariski ring map $A \to B$ identifies local rings (Lemma 61.4.6). The converse does not hold (Examples, Section 110.45). However, it turns out that there is a kind of structure theorem for ring maps which identify local rings in terms of ind-Zariski ring maps, see Proposition 61.6.6.

Let A be a ring. Let $X = \mathrm{Spec}(A)$. The space of connected components $\pi_0(X)$ is a profinite space by Topology, Lemma 5.23.9 (and Algebra, Lemma 10.26.2).

097C Lemma 61.6.1. Let A be a ring. Let $X = \mathrm{Spec}(A)$. Let $T \subset \pi_0(X)$ be a closed subset. There exists a surjective ind-Zariski ring map $A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induces a homeomorphism of $\mathrm{Spec}(B)$ with the inverse image of T in X.

Proof. Let $Z \subset X$ be the inverse image of T. Then Z is the intersection $Z = \bigcap Z_\alpha$ of the open and closed subsets of X containing Z, see Topology, Lemma 5.12.12. For each α we have $Z_\alpha = \mathrm{Spec}(A_\alpha)$ where $A \to A_\alpha$ is a local isomorphism (a localization at an idempotent). Setting $B = \mathrm{colim}\, A_\alpha$ proves the lemma. $\square$

097D Lemma 61.6.2. Let A be a ring and let $X = \mathrm{Spec}(A)$. Let T be a profinite space and let $T \to \pi_0(X)$ be a continuous map. There exists an ind-Zariski ring map $A \to B$ such that with $Y = \mathrm{Spec}(B)$ the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \pi_0(Y) \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

is cartesian in the category of topological spaces and such that $\pi_0(Y) = T$ as spaces over $\pi_0(X)$.

Proof. Namely, write $T = \lim T_i$ as the limit of an inverse system finite discrete spaces over a directed set (see Topology, Lemma 5.22.2). For each i let $Z_i = \mathrm{Im}(T \to \pi_0(X) \times T_i)$. This is a closed subset. Observe that $X \times T_i$ is the spectrum of $A_i = \prod_{t \in T_i} A$ and that $A \to A_i$ is a local isomorphism. By Lemma 61.6.1 we see that $Z_i \subset \pi_0(X \times T_i) = \pi_0(X) \times T_i$ corresponds to a surjection $A_i \to B_i$ which is ind-Zariski such that $\mathrm{Spec}(B_i) = X \times_{\pi_0(X)} Z_i$ as subsets of $X \times T_i$. The transition maps $T_i \to T_{i'}$ induce maps $Z_i \to Z_{i'}$ and $X \times_{\pi_0(X)} Z_i \to X \times_{\pi_0(X)} Z_{i'}$. Hence ring maps $B_{i'} \to B_i$ (Lemmas 61.3.8 and 61.4.6). Set $B = \mathrm{colim}\, B_i$. Because $T = \lim Z_i$ we have $X \times_{\pi_0(X)} T = \lim X \times_{\pi_0(X)} Z_i$ and hence $Y = \mathrm{Spec}(B) = \lim \mathrm{Spec}(B_i)$ fits into the cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

of topological spaces. By Lemma 61.2.5 we conclude that $T = \pi_0(Y)$. $\square$

09BJ Example 61.6.3. Let k be a field. Let T be a profinite topological space. There exists an ind-Zariski ring map $k \to A$ such that $\mathrm{Spec}(A)$ is homeomorphic to T. Namely, just apply Lemma 61.6.2 to $T \to \pi_0(\mathrm{Spec}(k)) = \{*\}$. In fact, in this case we have

$$A = \mathrm{colim}\, \mathrm{Map}(T_i, k)$$

whenever we write $T = \lim T_i$ as a filtered limit with each T_i finite.

097E Lemma 61.6.4. Let $A \to B$ be ring map such that

(1) $A \to B$ identifies local rings,

(2) the topological spaces $\text{Spec}(B)$, $\text{Spec}(A)$ are w-local,
(3) $\text{Spec}(B) \to \text{Spec}(A)$ is w-local, and
(4) $\pi_0(\text{Spec}(B)) \to \pi_0(\text{Spec}(A))$ is bijective.

Then $A \to B$ is an isomorphism

Proof. Let $X_0 \subset X = \text{Spec}(A)$ and $Y_0 \subset Y = \text{Spec}(B)$ be the sets of closed points. By assumption Y_0 maps into X_0 and the induced map $Y_0 \to X_0$ is a bijection. As a space $\text{Spec}(A)$ is the disjoint union of the spectra of the local rings of A at closed points. Similarly for B. Hence $X \to Y$ is a bijection. Since $A \to B$ is flat we have going down (Algebra, Lemma 10.39.19). Thus Algebra, Lemma 10.41.11 shows for any prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ we have $B_\mathfrak{q} = B_\mathfrak{p}$. Since $B_\mathfrak{q} = A_\mathfrak{p}$ by assumption, we see that $A_\mathfrak{p} = B_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A. Thus $A = B$ by Algebra, Lemma 10.23.1. □

097F **Lemma 61.6.5.** Let $A \to B$ be ring map such that

(1) $A \to B$ identifies local rings,
(2) the topological spaces $\text{Spec}(B)$, $\text{Spec}(A)$ are w-local, and
(3) $\text{Spec}(B) \to \text{Spec}(A)$ is w-local.

Then $A \to B$ is ind-Zariski.

Proof. Set $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Let $X_0 \subset X$ and $Y_0 \subset Y$ be the set of closed points. Let $A \to A'$ be the ind-Zariski morphism of affine schemes such that with $X' = \text{Spec}(A')$ the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \pi_0(X') \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

is cartesian in the category of topological spaces and such that $\pi_0(X') = \pi_0(Y)$ as spaces over $\pi_0(X)$, see Lemma 61.6.2. By Lemma 61.2.5 we see that X' is w-local and the set of closed points $X'_0 \subset X'$ is the inverse image of X_0.

We obtain a continuous map $Y \to X'$ of underlying topological spaces over X identifying $\pi_0(Y)$ with $\pi_0(X')$. By Lemma 61.3.8 (and Lemma 61.4.6) this corresponds to a morphism of affine schemes $Y \to X'$ over X. Since $Y \to X$ maps Y_0 into X_0 we see that $Y \to X'$ maps Y_0 into X'_0, i.e., $Y \to X'$ is w-local. By Lemma 61.6.4 we see that $Y \cong X'$ and we win. □

The following proposition is a warm up for the type of result we will prove later.

097G **Proposition 61.6.6.** Let $A \to B$ be a ring map which identifies local rings. Then there exists a faithfully flat, ind-Zariski ring map $B \to B'$ such that $A \to B'$ is ind-Zariski.

Proof. Let $A \to A_w$, resp. $B \to B_w$ be the faithfully flat, ind-Zariski ring map constructed in Lemma 61.5.3 for A, resp. B. Since $\text{Spec}(B_w)$ is w-local, there exists a unique factorization $A \to A_w \to B_w$ such that $\text{Spec}(B_w) \to \text{Spec}(A_w)$ is w-local by Lemma 61.5.5. Note that $A_w \to B_w$ identifies local rings, see Lemma 61.3.4. By Lemma 61.6.5 this means $A_w \to B_w$ is ind-Zariski. Since $B \to B_w$ is faithfully flat, ind-Zariski (Lemma 61.5.3) and the composition $A \to B \to B_w$ is ind-Zariski (Lemma 61.4.3) the proposition is proved. □

The proposition above allows us to characterize the affine, weakly contractible objects in the pro-Zariski site of an affine scheme.

09AZ Lemma 61.6.7. Let A be a ring. The following are equivalent

(1) every faithfully flat ring map $A \to B$ identifying local rings has a retraction,
(2) every faithfully flat ind-Zariski ring map $A \to B$ has a retraction, and
(3) A satisfies
   (a) $\mathrm{Spec}(A)$ is w-local, and
   (b) $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.

Proof. The equivalence of (1) and (2) follows immediately from Proposition 61.6.6.

Assume (3)(a) and (3)(b). Let $A \to B$ be faithfully flat and ind-Zariski. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. We will show that $A \to B$ has a retraction.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 61.5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$.

Assume $\mathrm{Spec}(B)$ is w-local and the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. Choose a continuous section to the surjective continuous map $V(IB) \to V(I)$. This is possible as $V(I) \cong \pi_0(\mathrm{Spec}(A))$ is extremally disconnected, see Topology, Proposition 5.26.6. The image is a closed subspace $T \subset \pi_0(\mathrm{Spec}(B)) \cong V(IB)$ mapping homeomorphically onto $\pi_0(A)$. Replacing B by the ind-Zariski quotient ring constructed in Lemma 61.6.1 we see that we may assume $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ is bijective. At this point $A \to B$ is an isomorphism by Lemma 61.6.4.

Assume (1) or equivalently (2). Let $A \to A_w$ be the ring map constructed in Lemma 61.5.3. By (1) there is a retraction $A_w \to A$. Thus $\mathrm{Spec}(A)$ is homeomorphic to a closed subset of $\mathrm{Spec}(A_w)$. By Lemma 61.2.4 we see (3)(a) holds. Finally, let $T \to \pi_0(A)$ be a surjective map with T an extremally disconnected, quasi-compact, Hausdorff topological space (Topology, Lemma 5.26.9). Choose $A \to B$ as in Lemma 61.6.2 adapted to $T \to \pi_0(\mathrm{Spec}(A))$. By (1) there is a retraction $B \to A$. Thus we see that $T = \pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ has a section. A formal categorical argument, using Topology, Proposition 5.26.6, implies that $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected. $\square$

09B0 Lemma 61.6.8. Let A be a ring. There exists a faithfully flat, ind-Zariski ring map $A \to B$ such that B satisfies the equivalent conditions of Lemma 61.6.7.

Proof. We first apply Lemma 61.5.3 to see that we may assume that $\mathrm{Spec}(A)$ is w-local. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(A))$, see Topology, Lemma 5.26.9. Note that T is profinite. Apply Lemma 61.6.2 to find an ind-Zariski ring map $A \to B$ such that $\pi_0(\mathrm{Spec}(B)) \to$

$\pi_0(\mathrm{Spec}(A))$ realizes $T \to \pi_0(\mathrm{Spec}(A))$ and such that

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & \pi_0(\mathrm{Spec}(B)) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \pi_0(\mathrm{Spec}(A)) \end{tikzcd}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(B)$ is w-local, that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is w-local, and that the set of closed points of $\mathrm{Spec}(B)$ is the inverse image of the set of closed points of $\mathrm{Spec}(A)$, see Lemma 61.2.5. Thus condition (3) of Lemma 61.6.7 holds for B. □

0A0D Remark 61.6.9. In each of Lemmas 61.6.1, 61.6.2, Proposition 61.6.6, and Lemma 61.6.8 we find an ind-Zariski ring map with some properties. In the paper [BS13] the authors use the notion of an ind-(Zariski localization) which is a filtered colimit of finite products of principal localizations. It is possible to replace ind-Zariski by ind-(Zariski localization) in each of the results listed above. However, we do not need this and the notion of an ind-Zariski homomorphism of rings as defined here has slightly better formal properties. Moreover, the notion of an ind-Zariski ring map is the natural analogue of the notion of an ind-étale ring map defined in the next section.

61.7. Ind-étale algebra

097H We start with a definition.

097I Definition 61.7.1. A ring map $A \to B$ is said to be ind-étale if B can be written as a filtered colimit of étale A-algebras.

The category of ind-étale algebras is closed under a number of natural operations.

097J Lemma 61.7.2. Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-étale, then $A' \to B'$ is ind-étale.

Proof. This is Algebra, Lemma 10.154.1. □

097K Lemma 61.7.3. Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-étale, then $A \to C$ is ind-étale.

Proof. This is Algebra, Lemma 10.154.2. □

097L Lemma 61.7.4. A filtered colimit of ind-étale A-algebras is ind-étale over A.

Proof. This is Algebra, Lemma 10.154.3. □

097M Lemma 61.7.5. Let A be a ring. Let $B \to C$ be an A-algebra map of ind-étale A-algebras. Then C is an ind-étale B-algebra.

Proof. This is Algebra, Lemma 10.154.5. □

097N Lemma 61.7.6. Let $A \to B$ be ind-étale. Then $A \to B$ is weakly étale (More on Algebra, Definition 15.104.1).

Proof. This follows from More on Algebra, Lemma 15.104.14. □

097P **Lemma 61.7.7.** Let A be a ring and let $I \subset A$ be an ideal. The base change functor

$$\text{ind-étale } A\text{-algebras} \longrightarrow \text{ind-étale } A/I\text{-algebras}, \quad C \longmapsto C/IC$$

has a fully faithful right adjoint v. In particular, given an ind-étale A/I-algebra $\overline{C}$ there exists an ind-étale A-algebra $C = v(\overline{C})$ such that $\overline{C} = C/IC$.

Proof. Let $\overline{C}$ be an ind-étale A/I-algebra. Consider the category $\mathcal{C}$ of factorizations $A \to B \to \overline{C}$ where $A \to B$ is étale. (We ignore some set theoretical issues in this proof.) We will show that this category is directed and that $C = \mathrm{colim}_\mathcal{C} B$ is an ind-étale A-algebra such that $\overline{C} = C/IC$.

We first prove that $\mathcal{C}$ is directed (Categories, Definition 4.19.1). The category is nonempty as $A \to A \to \overline{C}$ is an object. Suppose that $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ are two objects of $\mathcal{C}$. Then $A \to B \otimes_A B' \to \overline{C}$ is another (use Algebra, Lemma 10.143.3). Suppose that $f, g : B \to B'$ are two maps between objects $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ of $\mathcal{C}$. Then a coequalizer is $A \to B' \otimes_{f,B,g} B' \to \overline{C}$. This is an object of $\mathcal{C}$ by Algebra, Lemmas 10.143.3 and 10.143.8. Thus the category $\mathcal{C}$ is directed.

Write $\overline{C} = \mathrm{colim}\, \overline{B_i}$ as a filtered colimit with $\overline{B_i}$ étale over A/I. For every i there exists $A \to B_i$ étale with $\overline{B_i} = B_i/IB_i$, see Algebra, Lemma 10.143.10. Thus $C \to \overline{C}$ is surjective. Since $C/IC \to \overline{C}$ is ind-étale (Lemma 61.7.5) we see that it is flat. Hence $\overline{C}$ is a localization of C/IC at some multiplicative subset $S \subset C/IC$ (Algebra, Lemma 10.108.2). Take an $f \in C$ mapping to an element of $S \subset C/IC$. Choose $A \to B \to \overline{C}$ in $\mathcal{C}$ and $g \in B$ mapping to f in the colimit. Then we see that $A \to B_g \to \overline{C}$ is an object of $\mathcal{C}$ as well. Thus f is an invertible element of C. It follows that $C/IC = \overline{C}$.

Next, we claim that for an ind-étale algebra D over A we have

$$\mathrm{Mor}_A(D, C) = \mathrm{Mor}_{A/I}(D/ID, \overline{C})$$

Namely, let $D/ID \to \overline{C}$ be an A/I-algebra map. Write $D = \mathrm{colim}_{i \in I} D_i$ as a colimit over a directed set I with D_i étale over A. By choice of $\mathcal{C}$ we obtain a transformation $I \to \mathcal{C}$ and hence a map $D \to C$ compatible with maps to $\overline{C}$. Whence the claim.

It follows that the functor v defined by the rule

$$\overline{C} \longmapsto v(\overline{C}) = \mathrm{colim}_{A \to B \to \overline{C}} B$$

is a right adjoint to the base change functor u as required by the lemma. The functor v is fully faithful because $u \circ v = \mathrm{id}$ by construction, see Categories, Lemma 4.24.4. $\square$

61.8. Constructing ind-étale algebras

097Q Let A be a ring. Recall that any étale ring map $A \to B$ is isomorphic to a standard smooth ring map of relative dimension 0. Such a ring map is of the form

$$A \longrightarrow A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

where the determinant of the $n \times n$-matrix with entries $\partial f_i/\partial x_j$ is invertible in the quotient ring. See Algebra, Lemma 10.143.2.

Let $S(A)$ be the set of all faithfully flat[1] standard smooth A-algebras of relative dimension 0. Let $I(A)$ be the partially ordered (by inclusion) set of finite subsets E of $S(A)$. Note that $I(A)$ is a directed partially ordered set. For $E = \{A \to B_1, \ldots, A \to B_n\}$ set

$$B_E = B_1 \otimes_A \ldots \otimes_A B_n$$

Observe that B_E is a faithfully flat étale A-algebra. For $E \subset E'$, there is a canonical transition map $B_E \to B_{E'}$ of étale A-algebras. Namely, say $E = \{A \to B_1, \ldots, A \to B_n\}$ and $E' = \{A \to B_1, \ldots, A \to B_{n+m}\}$ then $B_E \to B_{E'}$ sends $b_1 \otimes \ldots \otimes b_n$ to the element $b_1 \otimes \ldots \otimes b_n \otimes 1 \otimes \ldots \otimes 1$ of $B_{E'}$. This construction defines a system of faithfully flat étale A-algebras over $I(A)$ and we set

$$T(A) = \mathrm{colim}_{E \in I(A)} B_E$$

Observe that $T(A)$ is a faithfully flat ind-étale A-algebra (Algebra, Lemma 10.39.20). By construction given any faithfully flat étale A-algebra B there is a (non-unique) A-algebra map $B \to T(A)$. Namely, pick some $(A \to B_0) \in S(A)$ and an isomorphism $B \cong B_0$. Then the canonical coprojection

$$B \to B_0 \to T(A) = \mathrm{colim}_{E \in I(A)} B_E$$

is the desired map.

097R Lemma 61.8.1. Given a ring A there exists a faithfully flat ind-étale A-algebra C such that every faithfully flat étale ring map $C \to B$ has a retraction.

Proof. Set $T^1(A) = T(A)$ and $T^{n+1}(A) = T(T^n(A))$. Let

$$C = \mathrm{colim}\, T^n(A)$$

This algebra is faithfully flat over each $T^n(A)$ and in particular over A, see Algebra, Lemma 10.39.20. Moreover, C is ind-étale over A by Lemma 61.7.4. If $C \to B$ is étale, then there exists an n and an étale ring map $T^n(A) \to B'$ such that $B = C \otimes_{T^n(A)} B'$, see Algebra, Lemma 10.143.3. If $C \to B$ is faithfully flat, then $\mathrm{Spec}(B) \to \mathrm{Spec}(C) \to \mathrm{Spec}(T^n(A))$ is surjective, hence $\mathrm{Spec}(B') \to \mathrm{Spec}(T^n(A))$ is surjective. In other words, $T^n(A) \to B'$ is faithfully flat. By our construction, there is a $T^n(A)$-algebra map $B' \to T^{n+1}(A)$. This induces a C-algebra map $B \to C$ which finishes the proof. □

097S Remark 61.8.2. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of $T(A)$ is at most κ. Namely, each B_E has cardinality at most κ and the index set $I(A)$ has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. It follows that the ring constructed in the proof of Lemma 61.8.1 has cardinality at most κ as well.

097T Remark 61.8.3. The construction $A \mapsto T(A)$ is functorial in the following sense: If $A \to A'$ is a ring map, then we can construct a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T(A) \arrow[d] \\ A' \arrow[r] & T(A') \end{tikzcd}$$

[1]In the presence of flatness, e.g., for smooth or étale ring maps, this just means that the induced map on spectra is surjective. See Algebra, Lemma 10.39.16.

Namely, given $(A \to A[x_1, \ldots, x_n]/(f_1, \ldots, f_n))$ in $S(A)$ we can use the ring map $\varphi : A \to A'$ to obtain a corresponding element $(A' \to A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi))$ of $S(A')$ where f^φ means the polynomial obtained by applying φ to the coefficients of the polynomial f. Moreover, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \arrow[d] \\ A' \arrow[r] & A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi) \end{tikzcd}$$

which is a in the category of rings. For $E \subset S(A)$ finite, set $E' = \varphi(E)$ and define $B_E \to B_{E'}$ in the obvious manner. Taking the colimit gives the desired map $T(A) \to T(A')$, see Categories, Lemma 4.14.8.

097U Lemma 61.8.4. Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a retraction. Then the same is true for every quotient ring A/I.

Proof. Let $A/I \to \overline{B}$ be faithfully flat étale. By Algebra, Lemma 10.143.10 we can write $\overline{B} = B/IB$ for some étale ring map $A \to B'$. The image U of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open and contains $V(I)$. Hence the complement $Z = \mathrm{Spec}(A) \setminus U$ is quasi-compact and disjoint from $V(I)$. Hence $Z \subset D(f_1) \cup \ldots \cup D(f_r)$ for some $r \geq 0$ and $f_i \in I$. Then $A \to B' = B \times \prod A_{f_i}$ is faithfully flat étale and $\overline{B} = B'/IB'$. Hence the retraction $B' \to A$ to $A \to B'$, induces a retraction to $A/I \to \overline{B}$. $\square$

097V Lemma 61.8.5. Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a retraction. Then every local ring of A at a maximal ideal is strictly henselian.

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Let $A \to B$ be an étale ring map and let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. By the description of the strict henselization $A_{\mathfrak{m}}^{sh}$ in Algebra, Lemma 10.155.11 it suffices to show that $A_{\mathfrak{m}} = B_{\mathfrak{q}}$. Note that there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ lying over $\mathfrak{m}$ and there are no specializations between them as an étale ring map is quasi-finite, see Algebra, Lemma 10.143.6. Thus $\mathfrak{q}_i$ is a maximal ideal and we can find $g \in \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$, $g \notin \mathfrak{q}$ (Algebra, Lemma 10.15.2). After replacing B by B_g we see that $\mathfrak{q}$ is the only prime of B lying over $\mathfrak{m}$. The image $U \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open (Algebra, Proposition 10.41.8). Thus the complement $\mathrm{Spec}(A) \setminus U$ is closed and we can find $f \in A$, $f \notin \mathfrak{p}$ such that $\mathrm{Spec}(A) = U \cup D(f)$. The ring map $A \to B \times A_f$ is faithfully flat and étale, hence has a retraction $\sigma : B \times A_f \to A$ by assumption on A. Observe that σ is étale, hence flat as a map between étale A-algebras (Algebra, Lemma 10.143.8). Since $\mathfrak{q}$ is the only prime of $B \times A_f$ lying over A we find that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ has a retraction which is also flat. Thus $A_{\mathfrak{p}} \to B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ are flat local ring maps whose composition is the identity. Since a flat local homomorphism of local rings is injective we conclude these maps are isomorphisms as desired. $\square$

097W Lemma 61.8.6. Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a retraction. Let $Z \subset \mathrm{Spec}(A)$ be a closed subscheme. Let $A \to A_Z^\sim$ be as constructed in Lemma 61.5.1. Then every faithfully flat étale ring map $A_Z^\sim \to C$ has a retraction.

Proof. There exists an étale ring map $A \to B'$ such that $C = B' \otimes_A A_Z^\sim$ as $A_Z^\sim$-algebras. The image $U' \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is open and contains $V(I)$, hence we can find $f \in I$ such that $\mathrm{Spec}(A) = U' \cup D(f)$. Then $A \to B' \times A_f$

is étale and faithfully flat. By assumption there is a retraction $B' \times A_f \to A$. Localizing we obtain the desired retraction $C \to A_Z^\sim$. ☐

097X Lemma 61.8.7. Let $A \to B$ be a ring map inducing algebraic extensions on residue fields. There exists a commutative diagram

$$\begin{tikzcd} B \arrow[r] & D \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

with the following properties:

(1) $A \to C$ is faithfully flat and ind-étale,
(2) $B \to D$ is faithfully flat and ind-étale,
(3) $\text{Spec}(C)$ is w-local,
(4) $\text{Spec}(D)$ is w-local,
(5) $\text{Spec}(D) \to \text{Spec}(C)$ is w-local,
(6) the set of closed points of $\text{Spec}(D)$ is the inverse image of the set of closed points of $\text{Spec}(C)$,
(7) the set of closed points of $\text{Spec}(C)$ surjects onto $\text{Spec}(A)$,
(8) the set of closed points of $\text{Spec}(D)$ surjects onto $\text{Spec}(B)$,
(9) for $\mathfrak{m} \subset C$ maximal the local ring $C_\mathfrak{m}$ is strictly henselian.

Proof. There is a faithfully flat, ind-Zariski ring map $A \to A'$ such that $\text{Spec}(A')$ is w-local and such that the set of closed points of $\text{Spec}(A')$ maps onto $\text{Spec}(A)$, see Lemma 61.5.3. Let $I \subset A'$ be the ideal such that $V(I)$ is the set of closed points of $\text{Spec}(A')$. Choose $A' \to C'$ as in Lemma 61.8.1. Note that the local rings $C'_{\mathfrak{m}'}$ at maximal ideals $\mathfrak{m}' \subset C'$ are strictly henselian by Lemma 61.8.5. We apply Lemma 61.5.8 to $A' \to C'$ and $I \subset A'$ to get $C' \to C$ with $C'/IC' \cong C/IC$. Note that since $A' \to C'$ is faithfully flat, $\text{Spec}(C'/IC')$ surjects onto the set of closed points of A' and in particular onto $\text{Spec}(A)$. Moreover, as $V(IC) \subset \text{Spec}(C)$ is the set of closed points of C and $C' \to C$ is ind-Zariski (and identifies local rings) we obtain properties (1), (3), (7), and (9).

Denote $J \subset C$ the ideal such that $V(J)$ is the set of closed points of $\text{Spec}(C)$. Set $D' = B \otimes_A C$. The ring map $C \to D'$ induces algebraic residue field extensions. Keep in mind that since $V(J) \to \text{Spec}(A)$ is surjective the map $T = V(JD) \to \text{Spec}(B)$ is surjective too. Apply Lemma 61.5.8 to $C \to D'$ and $J \subset C$ to get $D' \to D$ with $D'/JD' \cong D/JD$. All of the remaining properties given in the lemma are immediate from the results of Lemma 61.5.8. ☐

61.9. Weakly étale versus pro-étale

097Y Recall that a ring homomorphism $A \to B$ is weakly étale if $A \to B$ is flat and $B \otimes_A B \to B$ is flat. We have proved some properties of such ring maps in More on Algebra, Section 15.104. In particular, if $A \to B$ is a local homomorphism, and A is a strictly henselian local rings, then $A = B$, see More on Algebra, Theorem 15.104.24. Using this theorem and the work we've done above we obtain the following structure theorem for weakly étale ring maps.

097Z Proposition 61.9.1. Let $A \to B$ be a weakly étale ring map. Then there exists a faithfully flat, ind-étale ring map $B \to B'$ such that $A \to B'$ is ind-étale.

Proof. The ring map $A \to B$ induces (separable) algebraic extensions of residue fields, see More on Algebra, Lemma 15.104.17. Thus we may apply Lemma 61.8.7 and choose a diagram

$$\begin{tikzcd} B \arrow[r] & D \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

with the properties as listed in the lemma. Note that $C \to D$ is weakly étale by More on Algebra, Lemma 15.104.11. Pick a maximal ideal $\mathfrak{m} \subset D$. By construction this lies over a maximal ideal $\mathfrak{m}' \subset C$. By More on Algebra, Theorem 15.104.24 the ring map $C_{\mathfrak{m}'} \to D_{\mathfrak{m}}$ is an isomorphism. As every point of $\text{Spec}(C)$ specializes to a closed point we conclude that $C \to D$ identifies local rings. Thus Proposition 61.6.6 applies to the ring map $C \to D$. Pick $D \to D'$ faithfully flat and ind-Zariski such that $C \to D'$ is ind-Zariski. Then $B \to D'$ is a solution to the problem posed in the proposition. □

61.10. The V topology and the pro-h topology

0EVM The V topology was introduced in Topologies, Section 34.10. The h topology was introduced in More on Flatness, Section 38.34. A kind of intermediate topology, namely the ph topology, was introduced in Topologies, Section 34.8.

Given a topology τ on a suitable category $\mathcal{C}$ of schemes, we can introduce a "pro-τ topology" on $\mathcal{C}$ as follows. Recall that for X in $\mathcal{C}$ we use h_X to denote the representable presheaf associated to X. Let us temporarily say a morphism $X \to Y$ of $\mathcal{C}$ is a τ-cover[2] if the τ-sheafification of $h_X \to h_Y$ is surjective. Then we can define the pro-τ topology as the coarsest topology such that

(1) the pro-τ topology is finer than the τ topology, and
(2) $X \to Y$ is a pro-τ-cover if Y is affine and $X = \lim X_\lambda$ is a directed limit of affine schemes X_λ over Y such that $h_{X_\lambda} \to h_Y$ is a τ-cover for all λ.

We use this pedantic formulation because we do not want to specify a choice of pro-τ coverings: for different τ different choices of collections of coverings are suitable. For example, in Section 61.12 we will see that in order to define the pro-étale topology looking at families of weakly étale morphisms with some finiteness property works well. More generally, the proposed construction given in this paragraph is meant mainly to motivate the results in this section and we will never implicitly define a pro-τ topology using this method.

The following lemma tells us that the pro-V topology is equal to the V topology.

0EVN Lemma 61.10.1. Let Y be an affine scheme. Let $X = \lim X_i$ be a directed limit of affine schemes over Y. The following are equivalent

(1) $\{X \to Y\}$ is a standard V covering (Topologies, Definition 34.10.1), and
(2) $\{X_i \to Y\}$ is a standard V covering for all i.

[2]This should not be confused with the notion of a covering. For example if $\tau = \textit{étale}$, any morphism $X \to Y$ which has a section is a τ-covering. But our definition of étale coverings $\{V_i \to Y\}_{i \in I}$ forces each $V_i \to Y$ to be étale.

Proof. A singleton $\{X \to Y\}$ is a standard V covering if and only if given a morphism $g : \mathrm{Spec}(V) \to Y$ there is an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(V) \arrow[r, "g"] & Y \end{tikzcd}$$

Thus (1) $\Rightarrow$ (2) is immediate from the definition. Conversely, assume (2) and let $g : \mathrm{Spec}(V) \to Y$ as above be given. Write $\mathrm{Spec}(V) \times_Y X_i = \mathrm{Spec}(A_i)$. Since $\{X_i \to Y\}$ is a standard V covering, we may choose a valuation ring W_i and a ring map $A_i \to W_i$ such that the composition $V \to A_i \to W_i$ is an extension of valuation rings. In particular, the quotient A_i' of A_i by its V-torsion is a faithfully flat V-algebra. Flatness by More on Algebra, Lemma 15.22.10 and surjectivity on spectra because $A_i \to W_i$ factors through A_i'. Thus

$$A = \mathrm{colim}\, A_i'$$

is a faithfully flat V-algebra (Algebra, Lemma 10.39.20). Since $\{\mathrm{Spec}(A) \to \mathrm{Spec}(V)\}$ is a standard fpqc cover, it is a standard V cover (Topologies, Lemma 34.10.2) and hence we can choose $\mathrm{Spec}(W) \to \mathrm{Spec}(A)$ such that $V \to W$ is an extension of valuation rings. Since we can compose with the morphism $\mathrm{Spec}(A) \to X = \mathrm{Spec}(\mathrm{colim}\, A_i)$ the proof is complete. □

The following lemma tells us that the pro-h topology is equal to the pro-ph topology is equal to the V topology.

0EVP **Lemma 61.10.2.** Let $X \to Y$ be a morphism of affine schemes. The following are equivalent

(1) $\{X \to Y\}$ is a standard V covering (Topologies, Definition 34.10.1),
(2) $X = \lim X_i$ is a directed limit of affine schemes over Y such that $\{X_i \to Y\}$ is a ph covering for each i, and
(3) $X = \lim X_i$ is a directed limit of affine schemes over Y such that $\{X_i \to Y\}$ is an h covering for each i.

Proof. Proof of (2) $\Rightarrow$ (1). Recall that a V covering given by a single arrow between affines is a standard V covering, see Topologies, Definition 34.10.7 and Lemma 34.10.6. Recall that any ph covering is a V covering, see Topologies, Lemma 34.10.10. Hence if $X = \lim X_i$ as in (2), then $\{X_i \to Y\}$ is a standard V covering for each i. Thus by Lemma 61.10.1 we see that (1) is true.

Proof of (3) $\Rightarrow$ (2). This is clear because an h covering is always a ph covering, see More on Flatness, Definition 38.34.2.

Proof of (1) $\Rightarrow$ (3). This is the interesting direction, but the interesting content in this proof is hidden in More on Flatness, Lemma 38.34.1. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$. We can write $A = \mathrm{colim}\, A_i$ with A_i of finite presentation over R, see Algebra, Lemma 10.127.2. Set $X_i = \mathrm{Spec}(A_i)$. Then $\{X_i \to Y\}$ is a standard V covering for all i by (1) and Topologies, Lemma 34.10.6. Hence $\{X_i \to Y\}$ is an h covering by More on Flatness, Definition 38.34.2. This finishes the proof. □

The following lemma tells us, roughly speaking, that an h sheaf which is limit preserving satisfies the sheaf condition for V coverings. Please also compare with Remark 61.10.4.

0EVQ **Lemma 61.10.3.** Let S be a scheme. Let F be a contravariant functor defined on the category of all schemes over S. If

(1) F satisfies the sheaf property for the h topology, and
(2) F is limit preserving (Limits, Remark 32.6.2),

then F satisfies the sheaf property for the V topology.

Proof. We will prove this by verifying (1) and (2') of Topologies, Lemma 34.10.12. The sheaf property for Zariski coverings follows from the fact that F has the sheaf property for all h coverings. Finally, suppose that $X \to Y$ is a morphism of affine schemes over S such that $\{X \to Y\}$ is a V covering. By Lemma 61.10.2 we can write $X = \lim X_i$ as a directed limit of affine schemes over Y such that $\{X_i \to Y\}$ is an h covering for each i. We obtain

$$\begin{aligned} & \text{Equalizer}(\ F(X) \rightrightarrows F(X \times_Y X)\ ) \\ & = \text{Equalizer}(\ \text{colim}\, F(X_i) \rightrightarrows \text{colim}\, F(X_i \times_Y X_i)\ ) \\ & = \text{colim}\, \text{Equalizer}(\ F(X_i) \rightrightarrows F(X_i \times_Y X_i)\ ) \\ & = \text{colim}\, F(Y) = F(Y) \end{aligned}$$

which is what we wanted to show. The first equality because F is limit preserving and $X = \lim X_i$ and $X \times_Y X = \lim X_i \times_Y X_i$. The second equality because filtered colimits are exact. The third equality because F satisfies the sheaf property for h coverings. □

0EVR **Remark 61.10.4.** Let S be a scheme contained in a big site Sch_h. Let F be a sheaf of sets on $(Sch/S)_h$ such that $F(T) = \text{colim}\, F(T_i)$ whenever $T = \lim T_i$ is a directed limit of affine schemes in $(Sch/S)_h$. In this situation F extends uniquely to a contravariant functor F' on the category of all schemes over S such that (a) F' satisfies the sheaf property for the h topology and (b) F' is limit preserving. See More on Flatness, Lemma 38.35.4. In this situation Lemma 61.10.3 tells us that F' satisfies the sheaf property for the V topology.

61.11. Constructing w-contractible covers

0980 In this section we construct w-contractible covers of affine schemes.

0981 **Definition 61.11.1.** Let A be a ring. We say A is *w-contractible* if every faithfully flat weakly étale ring map $A \to B$ has a retraction.

We remark that by Proposition 61.9.1 an equivalent definition would be to ask that every faithfully flat, ind-étale ring map $A \to B$ has a retraction. Here is a key observation that will allow us to construct w-contractible rings.

0982 **Lemma 61.11.2.** Let A be a ring. The following are equivalent

(1) A is w-contractible,
(2) every faithfully flat, ind-étale ring map $A \to B$ has a retraction, and
(3) A satisfies
    (a) $\text{Spec}(A)$ is w-local,

(b) $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected, and
(c) for every maximal ideal $\mathfrak{m} \subset A$ the local ring $A_\mathfrak{m}$ is strictly henselian.

Proof. The equivalence of (1) and (2) follows immediately from Proposition 61.9.1.

Assume (3)(a), (3)(b), and (3)(c). Let $A \to B$ be faithfully flat and ind-étale. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ We will show that $A \to B$ has a retraction.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 61.5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. In this case $A \to B$ identifies local rings by condition (3)(c) as it suffices to check this at maximal ideals of B which lie over maximal ideals of A. Thus $A \to B$ has a retraction by Lemma 61.6.7.

Assume (1) or equivalently (2). We have (3)(c) by Lemma 61.8.5. Properties (3)(a) and (3)(b) follow from Lemma 61.6.7. □

0983 Proposition 61.11.3. For every ring A there exists a faithfully flat, ind-étale ring map $A \to D$ such that D is w-contractible.

Proof. Applying Lemma 61.8.7 to $\mathrm{id}_A : A \to A$ we find a faithfully flat, ind-étale ring map $A \to C$ such that C is w-local and such that every local ring at a maximal ideal of C is strictly henselian. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(C))$, see Topology, Lemma 5.26.9. Note that T is profinite. Apply Lemma 61.6.2 to find an ind-Zariski ring map $C \to D$ such that $\pi_0(\mathrm{Spec}(D)) \to \pi_0(\mathrm{Spec}(C))$ realizes $T \to \pi_0(\mathrm{Spec}(C))$ and such that

$$\begin{tikzcd} \mathrm{Spec}(D) \arrow[r] \arrow[d] & \pi_0(\mathrm{Spec}(D)) \arrow[d] \\ \mathrm{Spec}(C) \arrow[r] & \pi_0(\mathrm{Spec}(C)) \end{tikzcd}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(D)$ is w-local, that $\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ is w-local, and that the set of closed points of $\mathrm{Spec}(D)$ is the inverse image of the set of closed points of $\mathrm{Spec}(C)$, see Lemma 61.2.5. Thus it is still true that the local rings of D at its maximal ideals are strictly henselian (as they are isomorphic to the local rings at the corresponding maximal ideals of C). It follows from Lemma 61.11.2 that D is w-contractible. □

0984 Remark 61.11.4. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of the ring D constructed in Proposition 61.11.3 is at most

$$\kappa^{2^{2^{2^\kappa}}}.$$

Namely, the ring map $A \to D$ is constructed as a composition

$$A \to A_w = A' \to C' \to C \to D.$$

Here the first three steps of the construction are carried out in the first paragraph of the proof of Lemma 61.8.7. For the first step we have $|A_w| \leq \kappa$ by Remark 61.5.4. We have $|C'| \leq \kappa$ by Remark 61.8.2. Then $|C| \leq \kappa$ because C is a localization of $(C')_w$ (it is constructed from C' by an application of Lemma 61.5.7 in the proof of Lemma 61.5.8). Thus C has at most 2^κ maximal ideals. Finally, the ring map $C \to D$ identifies local rings and the cardinality of the set of maximal ideals of D is at most $2^{2^{2^\kappa}}$ by Topology, Remark 5.26.10. Since $D \subset \prod_{\mathfrak m \subset D} D_{\mathfrak m}$ we see that D has at most the size displayed above.

0985 Lemma 61.11.5. Let $A \to B$ be a quasi-finite and finitely presented ring map. If the residue fields of A are separably algebraically closed and $\text{Spec}(A)$ is Hausdorff and extremally disconnected, then $\text{Spec}(B)$ is extremally disconnected.

Proof. Set $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Choose a finite partition $X = \coprod X_i$ and $X'_i \to X_i$ as in Étale Cohomology, Lemma 59.72.3. The map of topological spaces $\coprod X_i \to X$ (where the source is the disjoint union in the category of topological spaces) has a section by Topology, Proposition 5.26.6. Hence we see that X is topologically the disjoint union of the strata X_i. Thus we may replace X by the X_i and assume there exists a surjective finite locally free morphism $X' \to X$ such that $(X' \times_X Y)_{red}$ is isomorphic to a finite disjoint union of copies of X'_{red}. Picture

$$\begin{tikzcd} \coprod_{i=1,\ldots,r} X' \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

The assumption on the residue fields of A implies that this diagram is a fibre product diagram on underlying sets of points (details omitted). Since X is extremally disconnected and X' is Hausdorff (Lemma 61.5.6), the continuous map $X' \to X$ has a continuous section σ. Then $\coprod_{i=1,\ldots,r} \sigma(X) \to Y$ is a bijective continuous map. By Topology, Lemma 5.17.8 we see that it is a homeomorphism and the proof is done. $\square$

0986 Lemma 61.11.6. Let $A \to B$ be a finite and finitely presented ring map. If A is w-contractible, so is B.

Proof. We will use the criterion of Lemma 61.11.2. Set $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ and denote $f : Y \to X$ the induced morphism. As $f : Y \to X$ is a finite morphism, we see that the set of closed points Y_0 of Y is the inverse image of the set of closed points X_0 of X. Let $y \in Y$ with image $x \in X$. Then x specializes to a unique closed point $x_0 \in X$. Say $f^{-1}(\{x_0\}) = \{y_1, \ldots, y_n\}$ with y_i closed in Y. Since $R = \mathcal{O}_{X,x_0}$ is strictly henselian and since f is finite, we see that $Y \times_{f,X} \text{Spec}(R)$ is equal to $\coprod_{i=1,\ldots,n} \text{Spec}(R_i)$ where each R_i is a local ring finite over R whose maximal ideal corresponds to y_i, see Algebra, Lemma 10.153.3 part (10). Then y is a point of exactly one of these $\text{Spec}(R_i)$ and we see that y specializes to exactly one of the y_i. In other words, every point of Y specializes to a unique point of Y_0. Thus Y is w-local. For every $y \in Y_0$ with image $x \in X_0$ we see that $\mathcal{O}_{Y,y}$ is strictly henselian by Algebra, Lemma 10.153.4 applied to $\mathcal{O}_{X,x} \to B \otimes_A \mathcal{O}_{X,x}$. It remains to show that Y_0 is extremally disconnected. To do this we look at $X_0 \times_X Y \to X_0$ where $X_0 \subset X$ is the reduced induced scheme structure. Note that the underlying topological space of $X_0 \times_X Y$ agrees with Y_0. Now the desired result follows from Lemma 61.11.5. $\square$

0987 Lemma 61.11.7. Let A be a ring. Let $Z \subset \text{Spec}(A)$ be a closed subset of the form $Z = V(f_1, \ldots, f_r)$. Set $B = A_Z^\sim$, see Lemma 61.5.1. If A is w-contractible, so is B.

Proof. Let $A_Z^\sim \to B$ be a weakly étale faithfully flat ring map. Consider the ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times B$$

this is faithful flat and weakly étale. If A is w-contractible, then there is a retraction σ. Consider the morphism

$$\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(A) \xrightarrow{\mathrm{Spec}(\sigma)} \coprod \mathrm{Spec}(A_{f_i}) \amalg \mathrm{Spec}(B)$$

Every point of $Z \subset \mathrm{Spec}(A_Z^\sim)$ maps into the component $\mathrm{Spec}(B)$. Since every point of $\mathrm{Spec}(A_Z^\sim)$ specializes to a point of Z we find a morphism $\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(B)$ as desired. □

61.12. The pro-étale site

0988 In this section we only discuss the actual definition and construction of the various pro-étale sites and the morphisms between them. The existence of weakly contractible objects will be done in Section 61.13.

The pro-étale topology is a bit like the fpqc topology (see Topologies, Section 34.9) in that the topos of sheaves on the small pro-étale site of a scheme depends on the choice of the underlying category of schemes. Thus we cannot speak of the pro-étale topos of a scheme. However, it will be true that the cohomology groups of a sheaf are unchanged if we enlarge our underlying category of schemes, see Section 61.31.

We will define pro-étale coverings using weakly étale morphisms of schemes, see More on Morphisms, Section 37.64. The reason is that, on the one hand, it is somewhat awkward to define the notion of a pro-étale morphism of schemes, and on the other, Proposition 61.9.1 assures us that we obtain the same sheaves[3] with the definition that follows.

0989 Definition 61.12.1. Let T be a scheme. A pro-étale covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is weakly-étale and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. Here is a lemma that will allow us to recognize pro-étale coverings. It will also allow us to reduce many lemmas about pro-étale coverings to the corresponding results for fpqc coverings.

098A Lemma 61.12.2. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent

(1) $\{f_i : T_i \to T\}_{i \in I}$ is a pro-étale covering,
(2) each f_i is weakly étale and $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,
(3) each f_i is weakly étale and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,
(4) each f_i is weakly étale and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.

If T is quasi-separated, these are also equivalent to

[3] To be precise the pro-étale topology we obtain using our choice of coverings is the same as the one gotten from the general procedure explained in Section 61.10 starting with $\tau = étale$.

(5) each f_i is weakly étale, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.

Proof. The equivalence of (1) and (2) is immediate from the definitions. Hence the lemma follows from Topologies, Lemma 34.9.2. ☐

098B Lemma 61.12.3. Any étale covering and any Zariski covering is a pro-étale covering.

Proof. This follows from the corresponding result for fpqc coverings (Topologies, Lemma 34.9.6), Lemma 61.12.2, and the fact that an étale morphism is a weakly étale morphism, see More on Morphisms, Lemma 37.64.9. ☐

098C Lemma 61.12.4. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a pro-étale covering of T.
(2) If $\{T_i \to T\}_{i\in I}$ is a pro-étale covering and for each i we have a pro-étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a pro-étale covering.
(3) If $\{T_i \to T\}_{i\in I}$ is a pro-étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a pro-étale covering.

Proof. This follows from the fact that composition and base changes of weakly étale morphisms are weakly étale (More on Morphisms, Lemmas 37.64.5 and 37.64.6), Lemma 61.12.2, and the corresponding results for fpqc coverings, see Topologies, Lemma 34.9.7. ☐

098D Lemma 61.12.5. Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a pro-étale covering of T. Then there exists a pro-étale covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. This follows directly from the definition. ☐

Thus we define the corresponding standard coverings of affines as follows.

098E Definition 61.12.6. Let T be an affine scheme. A standard pro-étale covering of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ where each T_j is affine, each f_i is weakly étale, and $T = \bigcup f_i(T_i)$.

We follow the general outline given in Topologies, Section 34.2 for constructing the big pro-étale site we will be working with. However, because we need a bit larger rings to accommodate for the size of certain constructions we modify the constructions slightly.

098G Definition 61.12.7. A big pro-étale site is any site $Sch_{\textit{pro-étale}}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of pro-étale coverings Cov_0 among these schemes.
(2) Change the function *Bound* of Sets, Equation (3.9.1.1) into

$$Bound(\kappa) = \max\{\kappa^{2^{2^{2^\kappa}}}, \kappa^{\aleph_0}, \kappa^+\}.$$

(3) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0 and the function *Bound*.

(4) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of pro-étale coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

It will turn out, see Lemma 61.31.1, that the topology on a big pro-étale site $Sch_{\textit{pro-étale}}$ is in some sense induced from the pro-étale topology on the category of all schemes.

098K Definition 61.12.8. Let S be a scheme. Let $Sch_{\textit{pro-étale}}$ be a big pro-étale site containing S.

(1) The big pro-étale site of S, denoted $(Sch/S)_{\textit{pro-étale}}$, is the site $Sch_{\textit{pro-étale}}/S$ introduced in Sites, Section 7.25.
(2) The small pro-étale site of S, which we denote $S_{\textit{pro-étale}}$, is the full subcategory of $(Sch/S)_{\textit{pro-étale}}$ whose objects are those U/S such that $U \to S$ is weakly étale. A covering of $S_{\textit{pro-étale}}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{\textit{pro-étale}}$ with $U \in \text{Ob}(S_{\textit{pro-étale}})$.
(3) The big affine pro-étale site of S, denoted $(\text{Aff}/S)_{\textit{pro-étale}}$, is the full subcategory of $(Sch/S)_{\textit{pro-étale}}$ whose objects are affine U/S. A covering of $(\text{Aff}/S)_{\textit{pro-étale}}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{\textit{pro-étale}}$ which is a standard pro-étale covering.

It is not completely clear that the small pro-étale site and the big affine pro-étale site are sites. We check this now.

098L Lemma 61.12.9. Let S be a scheme. Let $Sch_{\textit{pro-étale}}$ be a big pro-étale site containing S. Both $S_{\textit{pro-étale}}$ and $(\text{Aff}/S)_{\textit{pro-étale}}$ are sites.

Proof. Let us show that $S_{\textit{pro-étale}}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{\textit{pro-étale}}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{\textit{pro-étale}}$ with $U \in \text{Ob}(S_{\textit{pro-étale}})$ we also have $U_i \in \text{Ob}(S_{\textit{pro-étale}})$. This follows from the definitions as the composition of weakly étale morphisms is weakly étale.

To show that $(\text{Aff}/S)_{\textit{pro-étale}}$ is a site, reasoning as above, it suffices to show that the collection of standard pro-étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This follows from Lemma 61.12.2 and the corresponding result for standard fpqc coverings (Topologies, Lemma 34.9.10). □

098M Lemma 61.12.10. Let S be a scheme. Let $Sch_{\textit{pro-étale}}$ be a big pro-étale site containing S. Let Sch be the category of all schemes.

(1) The categories $Sch_{\textit{pro-étale}}$, $(Sch/S)_{\textit{pro-étale}}$, $S_{\textit{pro-étale}}$, and $(\text{Aff}/S)_{\textit{pro-étale}}$ have fibre products agreeing with fibre products in Sch.
(2) The categories $Sch_{\textit{pro-étale}}$, $(Sch/S)_{\textit{pro-étale}}$, $S_{\textit{pro-étale}}$ have equalizers agreeing with equalizers in Sch.
(3) The categories $(Sch/S)_{\textit{pro-étale}}$, and $S_{\textit{pro-étale}}$ both have a final object, namely S/S.
(4) The category $Sch_{\textit{pro-étale}}$ has a final object agreeing with the final object of Sch, namely $\text{Spec}(\mathbf{Z})$.

Proof. The category $Sch_{pro\text{-}étale}$ contains $\mathrm{Spec}(\mathbf{Z})$ and is closed under products and fibre products by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{pro\text{-}étale})$. The fibre product $V \times_U W$ in $Sch_{pro\text{-}étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{pro\text{-}étale}$. This proves the result for $(Sch/S)_{pro\text{-}étale}$. If $U \to S$, $V \to U$ and $W \to U$ are weakly étale then so is $V \times_U W \to S$ (see More on Morphisms, Section 37.64) and hence we get fibre products for $S_{pro\text{-}étale}$. If U, V, W are affine, so is $V \times_U W$ and hence we get fibre products for $(Aff/S)_{pro\text{-}étale}$.

Let $a, b : U \to V$ be two morphisms in $Sch_{pro\text{-}étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/\mathrm{Spec}(\mathbf{Z})}, V \times_{\mathrm{Spec}(\mathbf{Z})} V, (a,b)} (U \times_{\mathrm{Spec}(\mathbf{Z})} U)$$

which is an object of $Sch_{pro\text{-}étale}$ by what we saw above. Thus $Sch_{pro\text{-}étale}$ has equalizers. If a and b are morphisms over S, then the equalizer (in the category of schemes) is also given by

$$V \times_{\Delta_{V/S}, V \times_S V, (a,b)} (U \times_S U)$$

hence we see that $(Sch/S)_{pro\text{-}étale}$ has equalizers. Moreover, if U and V are weakly-étale over S, then so is the equalizer above as a fibre product of schemes weakly étale over S. Thus $S_{pro\text{-}étale}$ has equalizers. The statements on final objects is clear. □

Next, we check that the big affine pro-étale site defines the same topos as the big pro-étale site.

098N **Lemma 61.12.11.** Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. The functor $(Aff/S)_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{pro\text{-}étale})$ to $Sh((Sch/S)_{pro\text{-}étale})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$. Being cocontinuous just means that any pro-étale covering of T/S, T affine, can be refined by a standard pro-étale covering of T. This is the content of Lemma 61.12.5. Hence (1) holds. We see u is continuous simply because a standard pro-étale covering is a pro-étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

098P **Lemma 61.12.12.** Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$. The functor $T_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ is cocontinuous and induces a morphism of topoi

$$i_f : Sh(T_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{pro\text{-}étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$. In other words, given a weakly étale morphism $j : U \to T$ corresponding to an object of $T_{pro\text{-}\acute{e}tale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 61.12.10. Moreover, $T_{pro\text{-}\acute{e}tale}$ has equalizers and u commutes with them by Lemma 61.12.10. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

098Q **Lemma 61.12.13.** Let S be a scheme. Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site containing S. The inclusion functor $S_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites

$$\pi_S : (Sch/S)_{pro\text{-}\acute{e}tale} \longrightarrow S_{pro\text{-}\acute{e}tale}$$

and a morphism of topoi

$$i_S : Sh(S_{pro\text{-}\acute{e}tale}) \longrightarrow Sh((Sch/S)_{pro\text{-}\acute{e}tale})$$

such that $\pi_S \circ i_S = \text{id}$. Moreover, $i_S = i_{\text{id}_S}$ with i_{id_S} as in Lemma 61.12.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$, in addition to the properties seen in the proof of Lemma 61.12.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.21.8. □

098R **Definition 61.12.14.** In the situation of Lemma 61.12.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small pro-étale site*, and for a sheaf $\mathcal{F}$ on the big pro-étale site we denote $\mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned} \text{Mor}_{Sh(S_{pro\text{-}\acute{e}tale})}(\mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}, \mathcal{G}) &= \text{Mor}_{Sh((Sch/S)_{pro\text{-}\acute{e}tale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \text{Mor}_{Sh(S_{pro\text{-}\acute{e}tale})}(\mathcal{G}, \mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}) &= \text{Mor}_{Sh((Sch/S)_{pro\text{-}\acute{e}tale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F}) \end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{pro\text{-}\acute{e}tale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{pro\text{-}\acute{e}tale}} = \mathcal{G}$.

098S **Lemma 61.12.15.** Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}\acute{e}tale}$. The functor

$$u : (Sch/T)_{pro\text{-}\acute{e}tale} \longrightarrow (Sch/S)_{pro\text{-}\acute{e}tale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{pro\text{-}\acute{e}tale} \longrightarrow (Sch/T)_{pro\text{-}\acute{e}tale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{pro\text{-}\acute{e}tale}) \longrightarrow Sh((Sch/S)_{pro\text{-}\acute{e}tale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 61.12.12). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given

U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

098T Lemma 61.12.16. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$.

(1) We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 61.12.12 and i_T as in Lemma 61.12.13.
(2) The functor $S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi
$$f_{small} : Sh(T_{pro\text{-}étale}) \longrightarrow Sh(S_{pro\text{-}étale}).$$
We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
(3) We have a commutative diagram of morphisms of sites
$$\begin{tikzcd} T_{pro\text{-}étale} \arrow[d, "f_{small}"'] & (Sch/T)_{pro\text{-}étale} \arrow[l, "\pi_T"'] \arrow[d, "f_{big}"] \\ S_{pro\text{-}étale} & (Sch/S)_{pro\text{-}étale} \arrow[l, "\pi_S"] \end{tikzcd}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemmas 61.12.4 and 61.12.10. Moreover, both $S_{pro\text{-}étale}$, $T_{pro\text{-}étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 7.14.7 the functor u corresponds to a morphism of sites $T_{pro\text{-}étale} \to S_{pro\text{-}étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 7.15.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 61.12.14 we have: for $\mathcal{F}$ a sheaf on the big pro-étale site of T

0F60 (61.12.16.1)
$$(f_{big,*}\mathcal{F})|_{S_{pro\text{-}étale}} = f_{small,*}(\mathcal{F}|_{T_{pro\text{-}étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small pro-étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

098U Lemma 61.12.17. Given schemes X, Y, Y in $Sch_{pro\text{-}étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 61.12.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 61.12.16. □

0F61 **Lemma 61.12.18.** Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Consider a cartesian diagram

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in $Sch_{pro\text{-}étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{pro\text{-}étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{pro\text{-}étale}$ (use Lemmas 61.12.12 and 61.12.15). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. $\square$

We can think about a sheaf on the big pro-étale site of S as a collection of sheaves on the small pro-étale site on schemes over S.

098V **Lemma 61.12.19.** Let S be a scheme contained in a big pro-étale site $Sch_{pro\text{-}étale}$. A sheaf $\mathcal{F}$ on the big pro-étale site $(Sch/S)_{pro\text{-}étale}$ is given by the following data:

(1) for every $T/S \in \mathrm{Ob}((Sch/S)_{pro\text{-}étale})$ a sheaf $\mathcal{F}_T$ on $T_{pro\text{-}étale}$,
(2) for every $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.

These data are subject to the following conditions:

(a) given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{pro\text{-}étale}$ the composition $c_g \circ g_{small}^{-1} c_f$ is equal to $c_{f \circ g}$, and
(b) if $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ is weakly étale then c_f is an isomorphism.

Proof. Identical to the proof of Topologies, Lemma 34.4.20. $\square$

098W **Lemma 61.12.20.** Let S be a scheme. Let $S_{affine,pro\text{-}étale}$ denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects. A covering of $S_{affine,pro\text{-}étale}$ will be a standard pro-étale covering, see Definition 61.12.6. Then restriction

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{affine,pro\text{-}étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 7.29.1 by checking that the inclusion functor $S_{affine,pro\text{-}étale} \to S_{pro\text{-}étale}$ is a special cocontinuous functor (see Sites, Definition 7.29.2). $\square$

098X **Lemma 61.12.21.** Let S be an affine scheme. Let S_{app} denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects U such that $\mathcal{O}(S) \to \mathcal{O}(U)$ is ind-étale. A covering of S_{app} will be a standard pro-étale covering, see Definition 61.12.6. Then restriction

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{app}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{app})$.

Proof. By Lemma 61.12.20 we may replace $S_{pro\text{-}étale}$ by $S_{affine,pro\text{-}étale}$. The lemma follows from Sites, Lemma 7.29.1 by checking that the inclusion functor $S_{app} \to S_{affine,pro\text{-}étale}$ is a special cocontinuous functor, see Sites, Definition 7.29.2. The conditions of Sites, Lemma 7.29.1 follow immediately from the definition and the facts (a) any object U of $S_{affine,pro\text{-}étale}$ has a covering $\{V \to U\}$

with V ind-étale over X (Proposition 61.9.1) and (b) the functor u is fully faithful. $\square$

098Z Lemma 61.12.22. Let S be a scheme. The topology on each of the pro-étale sites $Sch_{pro\text{-}\acute{e}tale}$, $S_{pro\text{-}\acute{e}tale}$, $(Sch/S)_{pro\text{-}\acute{e}tale}$, $S_{affine,pro\text{-}\acute{e}tale}$, and $(\mathrm{Aff}/S)_{pro\text{-}\acute{e}tale}$ is subcanonical.

Proof. Combine Lemma 61.12.2 and Descent, Lemma 35.13.7. $\square$

61.13. Weakly contractible objects

0F4N In this section we prove the key fact that our pro-étale sites contain many weakly contractible objects. In fact, the proof of Lemma 61.13.3 is the reason for the shape of the function *Bound* in Definition 61.12.7 (although for readers who are ignoring set theoretical questions, this information is without content).

We first express the notion of w-contractible rings in terms of pro-étale coverings.

098F Lemma 61.13.1. Let $T = \mathrm{Spec}(A)$ be an affine scheme. The following are equivalent

(1) A is w-contractible, and
(2) every pro-étale covering of T can be refined by a Zariski covering of the form $T = \coprod_{i=1,\ldots,n} U_i$.

Proof. Assume A is w-contractible. By Lemma 61.12.5 it suffices to prove we can refine every standard pro-étale covering $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ by a Zariski covering of T. The morphism $\coprod T_i \to T$ is a surjective weakly étale morphism of affine schemes. Hence by Definition 61.11.1 there exists a morphism $\sigma : T \to \coprod T_i$ over T. Then the Zariski covering $T = \coprod \sigma^{-1}(T_i)$ refines $\{f_i : T_i \to T\}$.

Conversely, assume (2). If $A \to B$ is faithfully flat and weakly étale, then $\{\mathrm{Spec}(B) \to T\}$ is a pro-étale covering. Hence there exists a Zariski covering $T = \coprod U_i$ and morphisms $U_i \to \mathrm{Spec}(B)$ over T. Since $T = \coprod U_i$ we obtain $T \to \mathrm{Spec}(B)$, i.e., an A-algebra map $B \to A$. This means A is w-contractible. $\square$

098H Lemma 61.13.2. Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site as in Definition 61.12.7. Let $T = \mathrm{Spec}(A)$ be an affine object of $Sch_{pro\text{-}\acute{e}tale}$. The following are equivalent

(1) A is w-contractible,
(2) T is a weakly contractible (Sites, Definition 7.40.2) object of $Sch_{pro\text{-}\acute{e}tale}$, and
(3) every pro-étale covering of T can be refined by a Zariski covering of the form $T = \coprod_{i=1,\ldots,n} U_i$.

Proof. We have seen the equivalence of (1) and (3) in Lemma 61.13.1.

Assume (3) and let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves on $Sch_{pro\text{-}\acute{e}tale}$. Let $s \in \mathcal{G}(T)$. To prove (2) we will show that s is in the image of $\mathcal{F}(T) \to \mathcal{G}(T)$. We can find a covering $\{T_i \to T\}$ of $Sch_{pro\text{-}\acute{e}tale}$ such that s lifts to a section of $\mathcal{F}$ over T_i (Sites, Definition 7.11.1). By (3) we may assume we have a finite covering $T = \coprod_{j=1,\ldots,m} U_j$ by open and closed subsets and we have $t_j \in \mathcal{F}(U_j)$ mapping to $s|_{U_j}$. Since Zariski coverings are coverings in $Sch_{pro\text{-}\acute{e}tale}$ (Lemma 61.12.3) we conclude that $\mathcal{F}(T) = \prod \mathcal{F}(U_j)$. Thus $t = (t_1, \ldots, t_m) \in \mathcal{F}(T)$ is a section mapping to s.

Assume (2). Let $A \to D$ be as in Proposition 61.11.3. Then $\{V \to T\}$ is a covering of $Sch_{pro\text{-}\acute{e}tale}$. (Note that $V = \mathrm{Spec}(D)$ is an object of $Sch_{pro\text{-}\acute{e}tale}$ by Remark 61.11.4 combined with our choice of the function *Bound* in Definition 61.12.7 and the computation of the size of affine schemes in Sets, Lemma 3.9.5.) Since the topology on $Sch_{pro\text{-}étale}$ is subcanonical (Lemma 61.12.22) we see that $h_V \to h_T$ is a surjective map of sheaves (Sites, Lemma 7.12.4). Since T is assumed weakly contractible, we see that there is an element $f \in h_V(T) = \mathrm{Mor}(T, V)$ whose image in $h_T(T)$ is id_T. Thus $A \to D$ has a retraction $\sigma : D \to A$. Now if $A \to B$ is faithfully flat and weakly étale, then $D \to D \otimes_A B$ has the same properties, hence there is a retraction $D \otimes_A B \to D$ and combined with σ we get a retraction $B \to D \otimes_A B \to D \to A$ of $A \to B$. Thus A is w-contractible and (1) holds. $\square$

098I Lemma 61.13.3. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 61.12.7. For every object T of $Sch_{pro\text{-}étale}$ there exists a covering $\{T_i \to T\}$ in $Sch_{pro\text{-}étale}$ with each T_i affine and the spectrum of a w-contractible ring. In particular, T_i is weakly contractible in $Sch_{pro\text{-}étale}$.

Proof. For those readers who do not care about set-theoretical issues this lemma is a trivial consequence of Lemma 61.13.2 and Proposition 61.11.3. Here are the details. Choose an affine open covering $T = \bigcup U_i$. Write $U_i = \mathrm{Spec}(A_i)$. Choose faithfully flat, ind-étale ring maps $A_i \to D_i$ such that D_i is w-contractible as in Proposition 61.11.3. The family of morphisms $\{\mathrm{Spec}(D_i) \to T\}$ is a pro-étale covering. If we can show that $\mathrm{Spec}(D_i)$ is isomorphic to an object, say T_i, of $Sch_{pro\text{-}étale}$, then $\{T_i \to T\}$ will be combinatorially equivalent to a covering of $Sch_{pro\text{-}étale}$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 61.12.7 and more precisely the application of Sets, Lemma 3.11.1 in the last step. To prove $\mathrm{Spec}(D_i)$ is isomorphic to an object of $Sch_{pro\text{-}étale}$, it suffices to prove that $|D_i| \leq Bound(\mathrm{size}(T))$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 61.12.7 and more precisely the application of Sets, Lemma 3.9.2 in step (3). Since $|A_i| \leq \mathrm{size}(U_i) \leq \mathrm{size}(T)$ by Sets, Lemmas 3.9.4 and 3.9.7 we get $|D_i| \leq \kappa^{2^{2^{2^\kappa}}}$ where $\kappa = \mathrm{size}(T)$ by Remark 61.11.4. Thus by our choice of the function $Bound$ in Definition 61.12.7 we win. $\square$

0990 Lemma 61.13.4. Let S be a scheme. The pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(\mathrm{Aff}/S)_{pro\text{-}étale}$ and if S is affine S_{app} have enough (affine) quasi-compact, weakly contractible objects, see Sites, Definition 7.40.2.

Proof. Follows immediately from Lemma 61.13.3. $\square$

0F4P Lemma 61.13.5. Let S be a scheme. The pro-étale sites $Sch_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$ have the following property: for any object U there exists a covering $\{V \to U\}$ with V a weakly contractible object. If U is quasi-compact, then we may choose V affine and weakly contractible.

Proof. Suppose that $V = \coprod_{j \in J} V_j$ is an object of $(Sch/S)_{pro\text{-}étale}$ which is the disjoint union of weakly contractible objects V_j. Since a disjoint union decomposition is a pro-étale covering we see that $\mathcal{F}(V) = \prod_{j \in J} \mathcal{F}(V_j)$ for any pro-étale sheaf $\mathcal{F}$. Let $\mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves of sets. Since V_j is weakly contractible, the map $\mathcal{F}(V_j) \to \mathcal{G}(V_j)$ is surjective, see Sites, Definition 7.40.2. Thus $\mathcal{F}(V) \to \mathcal{G}(V)$ is surjective as a product of surjective maps of sets and we conclude that V is weakly contractible.

Choose a covering $\{U_i \to U\}_{i \in I}$ with U_i affine and weakly contractible as in Lemma 61.13.3. Take $V = \coprod_{i \in I} U_i$ (there is a set theoretic issue here which we will address below). Then $\{V \to U\}$ is the desired pro-étale covering by a weakly contractible object (to check it is a covering use Lemma 61.12.2). If U is quasi-compact, then it follows immediately from Lemma 61.12.2 that we can choose a finite subset $I' \subset I$ such that $\{U_i \to U\}_{i \in I'}$ is still a covering and then $\{\coprod_{i \in I'} U_i \to U\}$ is the desired covering by an affine and weakly contractible object.

In this paragraph, which we urge the reader to skip, we address set theoretic problems. In order to know that the disjoint union lies in our partial universe, we need to bound the cardinality of the index set I. It is seen immediately from the construction of the covering $\{U_i \to U\}_{i \in I}$ in the proof of Lemma 61.13.3 that $|I| \leq \text{size}(U)$ where the size of a scheme is as defined in Sets, Section 3.9. Moreover, for each i we have $\text{size}(U_i) \leq Bound(\text{size}(U))$; this follows for the bound of the cardinality of $\Gamma(U_i, \mathcal{O}_{U_i})$ in the proof of Lemma 61.13.3 and Sets, Lemma 3.9.4. Thus $\text{size}(\coprod_{i \in I} U_i)) \leq Bound(\text{size}(U))$ by Sets, Lemma 3.9.5. Hence by construction of the big pro-étale site through Sets, Lemma 3.9.2 we see that $\coprod_{i \in I} U_i$ is isomorphic to an object of our site and the proof is complete. $\square$

61.14. Weakly contractible hypercoverings

09A0 The results of Section 61.13 leads to the existence of hypercoverings made up out weakly contractible objects.

09A1 **Lemma 61.14.1.** Let X be a scheme.

(1) For every object U of $X_{\textit{pro-étale}}$ there exists a hypercovering K of U in $X_{\textit{pro-étale}}$ such that each term K_n consists of a single weakly contractible object of $X_{\textit{pro-étale}}$ covering U.
(2) For every quasi-compact and quasi-separated object U of $X_{\textit{pro-étale}}$ there exists a hypercovering K of U in $X_{\textit{pro-étale}}$ such that each term K_n consists of a single affine and weakly contractible object of $X_{\textit{pro-étale}}$ covering U.

Proof. Let $\mathcal{B} \subset \text{Ob}(X_{\textit{pro-étale}})$ be the set of weakly contractible objects of $X_{\textit{pro-étale}}$. Every object T of $X_{\textit{pro-étale}}$ has a covering $\{T_i \to T\}_{i \in I}$ with I finite and $T_i \in \mathcal{B}$ by Lemma 61.13.5. By Hypercoverings, Lemma 25.12.6 we get a hypercovering K of U such that $K_n = \{U_{n,i}\}_{i \in I_n}$ with I_n finite and $U_{n,i}$ weakly contractible. Then we can replace K by the hypercovering of U given by $\{U_n\}$ in degree n where $U_n = \coprod_{i \in I_n} U_{n,i}$ This is allowed by Hypercoverings, Remark 25.12.9.

Let $X_{\textit{qcqs,pro-étale}} \subset X_{\textit{pro-étale}}$ be the full subcategory consisting of quasi-compact and quasi-separated objects. A covering of $X_{\textit{qcqs,pro-étale}}$ will be a finite pro-étale covering. Then $X_{\textit{qcqs,pro-étale}}$ is a site, has fibre products, and the inclusion functor $X_{\textit{qcqs,pro-étale}} \to X_{\textit{pro-étale}}$ is continuous and commutes with fibre products. In particular, if K is a hypercovering of an object U in $X_{\textit{qcqs,pro-étale}}$ then K is a hypercovering of U in $X_{\textit{pro-étale}}$ by Hypercoverings, Lemma 25.12.5. Let $\mathcal{B} \subset \text{Ob}(X_{\textit{qcqs,pro-étale}})$ be the set of affine and weakly contractible objects. By Lemma 61.13.3 and the fact that finite unions of affines are affine, for every object U of $X_{\textit{qcqs,pro-étale}}$ there exists a covering $\{V \to U\}$ of $X_{\textit{qcqs,pro-étale}}$ with $V \in \mathcal{B}$. By Hypercoverings, Lemma 25.12.6 we get a hypercovering K of U such that $K_n = \{U_{n,i}\}_{i \in I_n}$ with I_n finite and $U_{n,i}$ affine and weakly contractible. Then we can replace K by the hypercovering of U given by $\{U_n\}$ in degree n where $U_n = \coprod_{i \in I_n} U_{n,i}$. This is allowed by Hypercoverings, Remark 25.12.9. $\square$

In the following lemma we use the Čech complex $s(\mathcal{F}(K))$ associated to a hypercovering K in a site. See Hypercoverings, Section 25.5. If K is a hypercovering of

U and $K_n = \{U_n \to U\}$, then the Čech complex looks like this:

$$s(\mathcal{F}(K)) = (\mathcal{F}(U_0) \to \mathcal{F}(U_1) \to \mathcal{F}(U_2) \to \ldots)$$

where $s(\mathcal{F}(U_n))$ is placed in cohomological degree n.

09A2 Lemma 61.14.2. Let X be a scheme. Let $E \in D^+(X_{pro\text{-}étale})$ be represented by a bounded below complex $\mathcal{E}^\bullet$ of abelian sheaves. Let K be a hypercovering of $U \in \mathrm{Ob}(X_{pro\text{-}étale})$ with $K_n = \{U_n \to U\}$ where U_n is a weakly contractible object of $X_{pro\text{-}étale}$. Then

$$R\Gamma(U, E) = \mathrm{Tot}(s(\mathcal{E}^\bullet(K)))$$

in $D(\mathrm{Ab})$.

Proof. If $\mathcal{E}$ is an abelian sheaf on $X_{pro\text{-}étale}$, then the spectral sequence of Hypercoverings, Lemma 25.5.3 implies that

$$R\Gamma(X_{pro\text{-}étale}, \mathcal{E}) = s(\mathcal{E}(K))$$

because the higher cohomology groups of any sheaf over U_n vanish, see Cohomology on Sites, Lemma 21.51.1.

If $\mathcal{E}^\bullet$ is bounded below, then we can choose an injective resolution $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ and consider the map of complexes

$$\mathrm{Tot}(s(\mathcal{E}^\bullet(K))) \longrightarrow \mathrm{Tot}(s(\mathcal{I}^\bullet(K)))$$

For every n the map $\mathcal{E}^\bullet(U_n) \to \mathcal{I}^\bullet(U_n)$ is a quasi-isomorphism because taking sections over U_n is exact. Hence the displayed map is a quasi-isomorphism by one of the spectral sequences of Homology, Lemma 12.25.3. Using the result of the first paragraph we see that for every p the complex $s(\mathcal{I}^p(K))$ is acyclic in degrees $n > 0$ and computes $\mathcal{I}^p(U)$ in degree 0. Thus the other spectral sequence of Homology, Lemma 12.25.3 shows $\mathrm{Tot}(s(\mathcal{I}^\bullet(K)))$ computes $R\Gamma(U, E) = \mathcal{I}^\bullet(U)$. □

09A3 Lemma 61.14.3. Let X be a quasi-compact and quasi-separated scheme. The functor $R\Gamma(X, -) : D^+(X_{pro\text{-}étale}) \to D(\mathrm{Ab})$ commutes with direct sums and homotopy colimits.

Proof. The statement means the following: Suppose we have a family of objects E_i of $D^+(X_{pro\text{-}étale})$ such that $\bigoplus E_i$ is an object of $D^+(X_{pro\text{-}étale})$. Then $R\Gamma(X, \bigoplus E_i) = \bigoplus R\Gamma(X, E_i)$. To see this choose a hypercovering K of X with $K_n = \{U_n \to X\}$ where U_n is an affine and weakly contractible scheme, see Lemma 61.14.1. Let N be an integer such that $H^p(E_i) = 0$ for $p < N$. Choose a complex of abelian sheaves $\mathcal{E}_i^\bullet$ representing E_i with $\mathcal{E}_i^p = 0$ for $p < N$. The termwise direct sum $\bigoplus \mathcal{E}_i^\bullet$ represents $\bigoplus E_i$ in $D(X_{pro\text{-}étale})$, see Injectives, Lemma 19.13.4. By Lemma 61.14.2 we have

$$R\Gamma(X, \bigoplus E_i) = \mathrm{Tot}(s((\bigoplus \mathcal{E}_i^\bullet)(K)))$$

and

$$R\Gamma(X, E_i) = \mathrm{Tot}(s(\mathcal{E}_i^\bullet(K)))$$

Since each U_n is quasi-compact we see that

$$\mathrm{Tot}(s((\bigoplus \mathcal{E}_i^\bullet)(K))) = \bigoplus \mathrm{Tot}(s(\mathcal{E}_i^\bullet(K)))$$

by Modules on Sites, Lemma 18.30.3. The statement on homotopy colimits is a formal consequence of the fact that $R\Gamma$ is an exact functor of triangulated categories and the fact (just proved) that it commutes with direct sums. □

09A4 Remark 61.14.4. Let X be a scheme. Because $X_{pro\text{-}\acute{e}tale}$ has enough weakly contractible objects for all K in $D(X_{pro\text{-}\acute{e}tale})$ we have $K = R\lim \tau_{\geq -n}K$ by Cohomology on Sites, Proposition 21.51.2. Since $R\Gamma$ commutes with $R\lim$ by Injectives, Lemma 19.13.6 we see that

$$R\Gamma(X, K) = R\lim R\Gamma(X, \tau_{\geq -n}K)$$

in $D(\text{Ab})$. This will sometimes allow us to extend results from bounded below complexes to all complexes.

61.15. Compact generation

0994 In this section we prove that various derived categories associated to our pro-étale sites are compactly generated as defined in Derived Categories, Definition 13.37.5.

0F4Q Lemma 61.15.1. Let S be a scheme. Let Λ be a ring.

(1) $D(S_{pro\text{-}\acute{e}tale})$ is compactly generated,
(2) $D(S_{pro\text{-}\acute{e}tale}, \Lambda)$ is compactly generated,
(3) $D(S_{pro\text{-}\acute{e}tale}, \mathcal{A})$ is compactly generated for any sheaf of rings $\mathcal{A}$ on $S_{pro\text{-}\acute{e}tale}$,
(4) $D((Sch/S)_{pro\text{-}\acute{e}tale})$ is compactly generated,
(5) $D((Sch/S)_{pro\text{-}\acute{e}tale}, \Lambda)$ is compactly generated, and
(6) $D((Sch/S)_{pro\text{-}\acute{e}tale}, \mathcal{A})$ is compactly generated for any sheaf of rings $\mathcal{A}$ on $(Sch/S)_{pro\text{-}\acute{e}tale}$,

Proof. Proof of (3). Let U be an affine object of $S_{pro\text{-}\acute{e}tale}$ which is weakly contractible. Then $j_{U!}\mathcal{A}_U$ is a compact object of the derived category $D(S_{pro\text{-}\acute{e}tale}, \mathcal{A})$, see Cohomology on Sites, Lemma 21.52.6. Choose a set I and for each $i \in I$ an affine weakly contractible object U_i of $S_{pro\text{-}\acute{e}tale}$ such that every affine weakly contractible object of $S_{pro\text{-}\acute{e}tale}$ is isomorphic to one of the U_i. This is possible because $\text{Ob}(S_{pro\text{-}\acute{e}tale})$ is a set. To finish the proof of (3) it suffices to show that $\bigoplus j_{U_i,!}\mathcal{A}_{U_i}$ is a generator of $D(S_{pro\text{-}\acute{e}tale}, \mathcal{A})$, see Derived Categories, Definition 13.36.3. To see this, let K be a nonzero object of $D(S_{pro\text{-}\acute{e}tale}, \mathcal{A})$. Then there exists an object T of our site $S_{pro\text{-}\acute{e}tale}$ and a nonzero element ξ of $H^n(K)(T)$. In other words, ξ is a nonzero section of the nth cohomology sheaf of K. We may assume K is represented by a complex $\mathcal{K}^\bullet$ of sheaves of $\mathcal{A}$-modules and ξ is the class of a section $s \in \mathcal{K}^n(T)$ with $\text{d}(s) = 0$. Namely, ξ is locally represented as the class of a section (so you get the result after replacing T by a member of a covering of T). Next, we choose a covering $\{T_j \to T\}_{j \in J}$ as in Lemma 61.13.3. Since $H^n(K)$ is a sheaf, we see that for some j the restriction $\xi|_{T_j}$ remains nonzero. Thus $s|_{T_j}$ defines a nonzero map $j_{T_j,!}\mathcal{A}_{T_j} \to K$ in $D(S_{pro\text{-}\acute{e}tale}, \mathcal{A})$. Since $T_j \cong U_i$ for some $i \in I$ we conclude.

The exact same argument works for the big pro-étale site of S. □

61.16. Comparing topologies

0F62 This section is the analogue of Étale Cohomology, Section 59.39.

0F63 Lemma 61.16.1. Let X be a scheme. Let $\mathcal{F}$ be a presheaf of sets on $X_{pro\text{-}\acute{e}tale}$ which sends finite disjoint unions to products. Then $\mathcal{F}^\#(W) = \mathcal{F}(W)$ if W is an affine weakly contractible object of $X_{pro\text{-}\acute{e}tale}$.

Proof. Recall that $\mathcal{F}^\#$ is equal to $(\mathcal{F}^+)^+$, see Sites, Theorem 7.10.10, where $\mathcal{F}^+$ is the presheaf which sends an object U of $X_{pro\text{-}étale}$ to $\operatorname{colim} H^0(\mathcal{U}, \mathcal{F})$ where the colimit is over all pro-étale coverings $\mathcal{U}$ of U. Thus it suffices to prove that (a) $\mathcal{F}^+$ sends finite disjoint unions to products and (b) sends W to $\mathcal{F}(W)$. If $U = U_1 \amalg U_2$, then given a pro-étale covering $\mathcal{U} = \{f_j : V_j \to U\}$ of U we obtain pro-étale coverings $\mathcal{U}_i = \{f_j^{-1}(U_i) \to U_i\}$ and we clearly have

$$H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{U}_1, \mathcal{F}) \times H^0(\mathcal{U}_2, \mathcal{F})$$

because $\mathcal{F}$ sends finite disjoint unions to products (this includes the condition that $\mathcal{F}$ sends the empty scheme to the singleton). This proves (a). Finally, any pro-étale covering of W can be refined by a finite disjoint union decomposition $W = W_1 \amalg \ldots W_n$ by Lemma 61.13.2. Hence $\mathcal{F}^+(W) = \mathcal{F}(W)$ exactly because the value of $\mathcal{F}$ on W is the product of the values of $\mathcal{F}$ on the W_j. This proves (b). $\square$

0F64 Lemma 61.16.2. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf of sets on $X_{pro\text{-}étale}$. If W is an affine weakly contractible object of $X_{pro\text{-}étale}$, then

$$f_{small}^{-1}\mathcal{F}(W) = \operatorname{colim}_{W \to V} \mathcal{F}(V)$$

where the colimit is over morphisms $W \to V$ over Y with $V \in Y_{pro\text{-}étale}$.

Proof. Recall that $f_{small}^{-1}\mathcal{F}$ is the sheaf associated to the presheaf

$$u_p\mathcal{F} : U \mapsto \operatorname{colim}_{U \to V} \mathcal{F}(V)$$

on $X_{étale}$, see Sites, Sections 7.14 and 7.13; we've surpressed from the notation that the colimit is over the opposite of the category $\{U \to V, V \in Y_{pro\text{-}étale}\}$. By Lemma 61.16.1 it suffices to prove that $u_p\mathcal{F}$ sends finite disjoint unions to products. Suppose that $U = U_1 \amalg U_2$ is a disjoint union of open and closed subschemes. There is a functor

$$\{U_1 \to V_1\} \times \{U_2 \to V_2\} \longrightarrow \{U \to V\}, \quad (U_1 \to V_1, U_2 \to V_2) \longmapsto (U \to V_1 \amalg V_2)$$

which is initial (Categories, Definition 4.17.3). Hence the corresponding functor on opposite categories is cofinal and by Categories, Lemma 4.17.2 we see that $u_p\mathcal{F}$ on U is the colimit of the values $\mathcal{F}(V_1 \amalg V_2)$ over the product category. Since $\mathcal{F}$ is a sheaf it sends disjoint unions to products and we conclude $u_p\mathcal{F}$ does too. $\square$

0F65 Lemma 61.16.3. Let S be a scheme. Consider the morphism

$$\pi_S : (Sch/S)_{pro\text{-}étale} \longrightarrow S_{pro\text{-}étale}$$

of Lemma 61.12.13. Let $\mathcal{F}$ be a sheaf on $S_{pro\text{-}étale}$. Then $\pi_S^{-1}\mathcal{F}$ is given by the rule

$$(\pi_S^{-1}\mathcal{F})(T) = \Gamma(T_{pro\text{-}étale}, f_{small}^{-1}\mathcal{F})$$

where $f : T \to S$. Moreover, $\pi_S^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to fpqc coverings.

Proof. Observe that we have a morphism $i_f : Sh(T_{pro\text{-}étale}) \to Sh(Sch/S)_{pro\text{-}étale})$ such that $\pi_S \circ i_f = f_{small}$ as morphisms $T_{pro\text{-}étale} \to S_{pro\text{-}étale}$, see Lemma 61.12.12. Since pullback is transitive we see that $i_f^{-1}\pi_S^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$ as desired.

Let $\{g_i : T_i \to T\}_{i \in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on $T_{pro\text{-}étale}$ and given sections $s_i \in \Gamma(T_i, g_{i,small}^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section s of $\mathcal{G}$ over T whose pullback to T_i agrees with s_i. We will prove this statement when T is affine and the covering is given by a single surjective flat morphism $T' \to T$ of affines and omit the reduction of the general case to this case.

Let $g : T' \to T$ be a surjective flat morphism of affines and let $s' \in g_{small}^{-1}\mathcal{G}(T')$ be a section with $\text{pr}_0^*s' = \text{pr}_1^*s'$ on $T' \times_T T'$. Choose a surjective weakly étale morphism $W \to T'$ with W affine and weakly contractible, see Lemma 61.13.5. By Lemma 61.16.2 the restriction $s'|_W$ is an element of $\text{colim}_{W \to U} \mathcal{G}(U)$. Choose $\phi : W \to U_0$ and $s_0 \in \mathcal{G}(U_0)$ corresponding to s'. Choose a surjective weakly étale morphism $V \to W \times_T W$ with V affine and weakly contractible. Denote $a, b : V \to W$ the induced morphisms. Since $a^*(s'|_W) = b^*(s'|_W)$ and since the category $\{V \to U, U \in T_{pro\text{-}\acute{e}tale}\}$ is cofiltered (this is clear but see Sites, Lemma 7.14.6 if in doubt), we see that the two morphisms $\phi \circ a, \phi \circ b : V \to U_0$ have to be equal. By the results in Descent, Section 35.13 (especially Descent, Lemma 35.13.7) it follows there is a unique morphism $T \to U_0$ such that ϕ is the composition of this morphism with the structure morphism $W \to T$ (small detail omitted). Then we can let s be the pullback of s_0 by this morphism. We omit the verification that s pulls back to s' on T'. □

61.17. Comparing big and small topoi

0F66 This section is the analogue of Étale Cohomology, Section 59.99. In the following we will often denote $\mathcal{F} \mapsto \mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}$ the pullback functor i_S^{-1} corresponding to the morphism of topoi $i_S : Sh(S_{pro\text{-}\acute{e}tale}) \to Sh((Sch/S)_{pro\text{-}\acute{e}tale})$ of Lemma 61.12.13.

0F67 **Lemma 61.17.1.** Let S be a scheme. Let T be an object of $(Sch/S)_{pro\text{-}\acute{e}tale}$.

(1) If $\mathcal{I}$ is injective in $\text{Ab}((Sch/S)_{pro\text{-}\acute{e}tale})$, then
   (a) $i_f^{-1}\mathcal{I}$ is injective in $\text{Ab}(T_{pro\text{-}\acute{e}tale})$,
   (b) $\mathcal{I}|_{S_{pro\text{-}\acute{e}tale}}$ is injective in $\text{Ab}(S_{pro\text{-}\acute{e}tale})$,
(2) If $\mathcal{I}^\bullet$ is a K-injective complex in $\text{Ab}((Sch/S)_{pro\text{-}\acute{e}tale})$, then
   (a) $i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $\text{Ab}(T_{pro\text{-}\acute{e}tale})$,
   (b) $\mathcal{I}^\bullet|_{S_{pro\text{-}\acute{e}tale}}$ is a K-injective complex in $\text{Ab}(S_{pro\text{-}\acute{e}tale})$,

Proof. Proof of (1)(a) and (2)(a): i_f^{-1} is a right adjoint of an exact functor $i_{f,!}$. Namely, recall that i_f corresponds to a cocontinuous functor $u : T_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$ which is continuous and commutes with fibre products and equalizers, see Lemma 61.12.12 and its proof. Hence we obtain $i_{f,!}$ by Modules on Sites, Lemma 18.16.2. It is shown in Modules on Sites, Lemma 18.16.3 that it is exact. Then we conclude (1)(a) and (2)(a) hold by Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9.

Parts (1)(b) and (2)(b) are special cases of (1)(a) and (2)(a) as $i_S = i_{\text{id}_S}$. □

0F68 **Lemma 61.17.2.** Let $f : T \to S$ be a morphism of schemes. For K in $D((Sch/T)_{pro\text{-}\acute{e}tale})$ we have

$$(Rf_{big,*}K)|_{S_{pro\text{-}\acute{e}tale}} = Rf_{small,*}(K|_{T_{pro\text{-}\acute{e}tale}})$$

in $D(S_{pro\text{-}\acute{e}tale})$. More generally, let $S' \in \text{Ob}((Sch/S)_{pro\text{-}\acute{e}tale})$ with structure morphism $g : S' \to S$. Consider the fibre product

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then for K in $D((Sch/T)_{pro\text{-}\acute{e}tale})$ we have

$$i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$$

in $D(S'_{pro\text{-}\acute{e}tale})$ and

$$g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$$

in $D((Sch/S')_{pro\text{-}\acute{e}tale})$.

Proof. The first equality follows from Lemma 61.17.1 and (61.12.16.1) on choosing a K-injective complex of abelian sheaves representing K. The second equality follows from Lemma 61.17.1 and Lemma 61.12.18 on choosing a K-injective complex of abelian sheaves representing K. The third equality follows similarly from Cohomology on Sites, Lemmas 21.7.1 and 21.20.1 and Lemma 61.12.18 on choosing a K-injective complex of abelian sheaves representing K. □

Let S be a scheme and let $\mathcal{H}$ be an abelian sheaf on $(Sch/S)_{pro\text{-}\acute{e}tale}$. Recall that $H^n_{pro\text{-}\acute{e}tale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Sch/S)_{pro\text{-}\acute{e}tale}$.

0F69 Lemma 61.17.3. Let $f : T \to S$ be a morphism of schemes. For K in $D(S_{pro\text{-}\acute{e}tale})$ we have

$$H^n_{pro\text{-}\acute{e}tale}(S, \pi_S^{-1}K) = H^n(S_{pro\text{-}\acute{e}tale}, K)$$

and

$$H^n_{pro\text{-}\acute{e}tale}(T, \pi_S^{-1}K) = H^n(T_{pro\text{-}\acute{e}tale}, f_{small}^{-1}K).$$

For M in $D((Sch/S)_{pro\text{-}\acute{e}tale})$ we have

$$H^n_{pro\text{-}\acute{e}tale}(T, M) = H^n(T_{pro\text{-}\acute{e}tale}, i_f^{-1}M).$$

Proof. To prove the last equality represent M by a K-injective complex of abelian sheaves and apply Lemma 61.17.1 and work out the definitions. The second equality follows from this as $i_f^{-1} \circ \pi_S^{-1} = f_{small}^{-1}$. The first equality is a special case of the second one. □

0F6A Lemma 61.17.4. Let S be a scheme. For $K \in D(S_{pro\text{-}\acute{e}tale})$ the map

$$K \longrightarrow R\pi_{S,*}\pi_S^{-1}K$$

is an isomorphism.

Proof. This is true because both π_S^{-1} and $\pi_{S,*} = i_S^{-1}$ are exact functors and the composition $\pi_{S,*} \circ \pi_S^{-1}$ is the identity functor. □

61.18. Points of the pro-étale site

0991 We first apply Deligne's criterion to show that there are enough points.

0992 Lemma 61.18.1. Let S be a scheme. The pro-étale sites $Sch_{pro\text{-}\acute{e}tale}$, $S_{pro\text{-}\acute{e}tale}$, $(Sch/S)_{pro\text{-}\acute{e}tale}$, $S_{affine,pro\text{-}\acute{e}tale}$, and $(\text{Aff}/S)_{pro\text{-}\acute{e}tale}$ have enough points.

Proof. The big pro-étale topos of S is equivalent to the topos defined by $(\text{Aff}/S)_{pro\text{-}\acute{e}tale}$, see Lemma 61.12.11. The topos of sheaves on $S_{pro\text{-}\acute{e}tale}$ is equivalent to the topos associated to $S_{affine,pro\text{-}\acute{e}tale}$, see Lemma 61.12.20. The result for the sites $(\text{Aff}/S)_{pro\text{-}\acute{e}tale}$ and $S_{affine,pro\text{-}\acute{e}tale}$ follows immediately from Deligne's result Sites, Lemma 7.39.4. The case $Sch_{pro\text{-}\acute{e}tale}$ is handled because it is equal to $(Sch/\operatorname{Spec}(\mathbf{Z}))_{pro\text{-}\acute{e}tale}$. □

Let S be a scheme. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. We define a pro-étale neighbourhood of $\overline{s}$ to be a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "\overline{u}"'] \arrow[dr, "\overline{s}"'] & U \arrow[d] \\ & S \end{tikzcd}$$

with $U \to S$ weakly étale.

0F6B Lemma 61.18.2. Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. The category of pro-étale neighbourhoods of $\overline{s}$ is cofiltered.

Proof. The proof is identitical to the proof of Étale Cohomology, Lemma 59.29.4 but using the corresponding facts about weakly étale morphisms proven in More on Morphisms, Lemmas 37.64.5, 37.64.6, and 37.64.13. □

0F6C Lemma 61.18.3. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\mathcal{U} = \{\varphi_i : S_i \to S\}_{i \in I}$ be a pro-étale covering. Then there exist $i \in I$ and geometric point $\overline{s}_i$ of S_i mapping to $\overline{s}$.

Proof. Immediate from the fact that $\coprod \varphi_i$ is surjective and that residue field extensions induced by weakly étale morphisms are separable algebraic (see for example More on Morphisms, Lemma 37.64.11. □

Let S be a scheme and let $\overline{s}$ be a geometric point of S. For $\mathcal{F}$ in $Sh(S_{pro\text{-}\acute{e}tale})$ define the stalk of $\mathcal{F}$ at $\overline{s}$ by the formula

$$\mathcal{F}_{\overline{s}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$$

where the colimit is over all pro-étale neighbourhoods $(U, \overline{u})$ of $\overline{s}$ with $U \in \mathrm{Ob}(S_{pro\text{-}\acute{e}tale})$. It follows from the two lemmas above that the functor

$$S_{pro\text{-}\acute{e}tale}\mathrm{Sets}, \quad U \longmapsto \{\overline{u} \text{ geometric point of } U \text{ mapping to } \overline{s}\}$$

defines a point of the site $S_{pro\text{-}\acute{e}tale}$, see Sites, Definition 7.32.2 and Lemma 7.33.1. Hence the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defines a point of the topos $Sh(S_{pro\text{-}\acute{e}tale})$, see Sites, Definition 7.32.1 and Lemma 7.32.7. In particular this functor is exact and commutes with arbitrary colimits. In fact, this functor has another description.

0993 Lemma 61.18.4. In the situation above the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is an object of $X_{pro\text{-}\acute{e}tale}$ and there is a canonical isomorphism

$$\mathcal{F}(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})) = \mathcal{F}_{\overline{s}}$$

functorial in $\mathcal{F}$.

Proof. The first statement is clear from the construction of the strict henselization as a filtered colimit of étale algebras over S, or by the characterization of weakly étale morphisms of More on Morphisms, Lemma 37.64.11. The second statement follows as by Olivier's theorem (More on Algebra, Theorem 15.104.24) the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is an initial object of the category of pro-étale neighbourhoods of $\overline{s}$. □

Contrary to the situation with the étale topos of S it is not true that every point of $Sh(S_{pro\text{-}\acute{e}tale})$ is of this form, and it is not true that the collection of points associated to geometric points is conservative. Namely, suppose that $S = \mathrm{Spec}(k)$

where k is an algebraically closed field. Let A be a nonzero abelian group. Consider the sheaf $\mathcal{F}$ on $S_{pro\text{-}\acute{e}tale}$ defined by the

$$\mathcal{F}(U) = \frac{\{\text{functions } U \to A\}}{\{\text{locally constant functions}\}}$$

for U affine and by sheafification in general, see Example 61.19.12. Then $\mathcal{F}(U) = 0$ if $U = S = \mathrm{Spec}(k)$ but in general $\mathcal{F}$ is not zero. Namely, $S_{pro\text{-}\acute{e}tale}$ contains affine objects with infinitely many points. For example, let $E = \lim E_n$ be an inverse limit of finite sets with surjective transition maps, e.g., $E = \mathbf{Z}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$. The scheme $U = \mathrm{Spec}(\mathrm{colim}\,\mathrm{Map}(E_n, k))$ is an object of $S_{pro\text{-}\acute{e}tale}$ because $\mathrm{colim}\,\mathrm{Map}(E_n, k)$ is weakly étale (even ind-Zariski) over k. Thus $\mathcal{F}(U)$ is nonzero as there exist maps $E \to A$ which aren't locally constant. Thus $\mathcal{F}$ is a nonzero abelian sheaf whose stalk at the unique geometric point of S is zero. Since we know that $S_{pro\text{-}\acute{e}tale}$ has enough points, we conclude there must be a point of the pro-étale site which does not come from the construction explained above.

The replacement for arguments using points, is to use affine weakly contractible objects. First, there are enough affine weakly contractible objects by Lemma 61.13.4. Second, if $W \in \mathrm{Ob}(S_{pro\text{-}\acute{e}tale})$ is affine weakly contractible, then the functor

$$Sh(S_{pro\text{-}\acute{e}tale}) \longrightarrow \mathrm{Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is an exact functor $Sh(S_{pro\text{-}\acute{e}tale}) \to \mathrm{Sets}$ which commutes with all limits. The functor

$$\mathrm{Ab}(S_{pro\text{-}\acute{e}tale}) \longrightarrow \mathrm{Ab}, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is exact and commutes with direct sums (as W is quasi-compact, see Sites, Lemma 7.17.7), hence commutes with all limits and colimits. Moreover, we can check exactness of a complex of abelian sheaves by evaluation at these affine weakly contractible objects of $S_{pro\text{-}\acute{e}tale}$, see Cohomology on Sites, Proposition 21.51.2.

A final remark is that the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ for W affine weakly contractible in general isn't a stalk functor of a point of $S_{pro\text{-}\acute{e}tale}$ because it doesn't preserve coproducts of sheaves of sets if W is disconnected. And in fact, W is disconnected as soon as W has more than 1 closed point, i.e., when W is not the spectrum of a strictly henselian local ring (which is the special case discussed above).

61.19. Comparison with the étale site

099R Let X be a scheme. With suitable choices of sites[4] the functor $u : X_{\acute{e}tale} \to X_{pro\text{-}\acute{e}tale}$ sending U/X to U/X defines a morphism of sites

$$\epsilon : X_{pro\text{-}\acute{e}tale} \longrightarrow X_{\acute{e}tale}$$

This follows from Sites, Proposition 7.14.7.

0GLZ **Lemma 61.19.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}tale}$. The rule*

$$X_{pro\text{-}\acute{e}tale} \longrightarrow \mathrm{Sets}, \quad (f : Y \to X) \longmapsto \Gamma(Y_{\acute{e}tale}, f_{\acute{e}tale}^{-1}\mathcal{F})$$

[4] Choose a big pro-étale site $Sch_{pro\text{-}\acute{e}tale}$ containing X as in Definition 61.12.7. Then let $Sch_{\acute{e}tale}$ be the site having the same underlying category as $Sch_{pro\text{-}\acute{e}tale}$ but whose coverings are exactly those pro-étale coverings which are also étale coverings. With these choices let $X_{\acute{e}tale}$ and $X_{pro\text{-}\acute{e}tale}$ be the subcategories defined in Definition 61.12.8 and Topologies, Definition 34.4.8. Compare with Topologies, Remark 34.11.1.

is a sheaf and is equal to $\epsilon^{-1}\mathcal{F}$. Here $f_{\acute{e}tale} : Y_{\acute{e}tale} \to X_{\acute{e}tale}$ is the morphism of small étale sites constructed in Étale Cohomology, Section 59.34.

Proof. By Lemma 61.12.2 any pro-étale covering is an fpqc covering. Hence the formula defines a sheaf on $X_{pro\text{-}\acute{e}tale}$ by Étale Cohomology, Lemma 59.39.2. Let $a : Sh(X_{\acute{e}tale}) \to Sh(X_{pro\text{-}\acute{e}tale})$ be the functor sending $\mathcal{F}$ to the sheaf given by the formula in the lemma. To show that $a = \epsilon^{-1}$ it suffices to show that a is a left adjoint to ϵ_*.

Let $\mathcal{G}$ be an object of $Sh(X_{pro\text{-}\acute{e}tale})$. Recall that $\epsilon_*\mathcal{G}$ is simply given by the restriction of $\mathcal{G}$ to the full subcategory $X_{\acute{e}tale}$. Let $f : Y \to X$ be an object of $X_{pro\text{-}\acute{e}tale}$. We view $Y_{\acute{e}tale}$ as a subcategory of $X_{pro\text{-}\acute{e}tale}$. The restriction maps of the sheaf $\mathcal{G}$ define a map

$$\epsilon_*\mathcal{G} = \mathcal{G}|_{X_{\acute{e}tale}} \longrightarrow f_{\acute{e}tale,*}(\mathcal{G}|_{Y_{\acute{e}tale}})$$

Namely, for U in $X_{\acute{e}tale}$ the value of $f_{\acute{e}tale,*}(\mathcal{G}|_{Y_{\acute{e}tale}})$ on U is $\mathcal{G}(Y \times_X U)$ and there is a restriction map $\mathcal{G}(U) \to \mathcal{G}(Y \times_X U)$. By adjunction this determines a map

$$f_{\acute{e}tale}^{-1}(\epsilon_*\mathcal{G}) \to \mathcal{G}|_{Y_{\acute{e}tale}}$$

Putting these together for all $f : Y \to X$ in $X_{pro\text{-}\acute{e}tale}$ we obtain a canonical map $a(\epsilon_*\mathcal{G}) \to \mathcal{G}$.

Let $\mathcal{F}$ be an object of $Sh(X_{\acute{e}tale})$. It is immediately clear that $\mathcal{F} = \epsilon_* a(\mathcal{F})$.

We claim the maps $\mathcal{F} \to \epsilon_* a(\mathcal{F})$ and $a(\epsilon_*\mathcal{G}) \to \mathcal{G}$ are the unit and counit of the adjunction (see Categories, Section 4.24). To see this it suffices to show that the corresponding maps

$$\mathrm{Mor}_{Sh(X_{pro\text{-}\acute{e}tale})}(a(\mathcal{F}), \mathcal{G}) \to \mathrm{Mor}_{Sh(X_{\acute{e}tale})}(\mathcal{F}, \epsilon^{-1}\mathcal{G})$$

and

$$\mathrm{Mor}_{Sh(X_{\acute{e}tale})}(\mathcal{F}, \epsilon^{-1}\mathcal{G}) \to \mathrm{Mor}_{Sh(X_{pro\text{-}\acute{e}tale})}(a(\mathcal{F}), \mathcal{G})$$

are mutually inverse. We omit the detailed verification. □

099T Lemma 61.19.2. Let X be a scheme. For every sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ the adjunction map $\mathcal{F} \to \epsilon_*\epsilon^{-1}\mathcal{F}$ is an isomorphism, i.e., $\epsilon^{-1}\mathcal{F}(U) = \mathcal{F}(U)$ for U in $X_{\acute{e}tale}$.

Proof. Follows immediately from the description of ϵ^{-1} in Lemma 61.19.1. □

099S Lemma 61.19.3. Let X be a scheme. Let $Y = \lim Y_i$ be the limit of a directed inverse system of quasi-compact and quasi-separated objects of $X_{pro\text{-}\acute{e}tale}$ with affine transition morphisms. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have

$$\epsilon^{-1}\mathcal{F}(Y) = \mathrm{colim}\, \epsilon^{-1}\mathcal{F}(Y_i)$$

Moreover, if Y_i is in $X_{\acute{e}tale}$ we have $\epsilon^{-1}\mathcal{F}(Y) = \mathrm{colim}\, \mathcal{F}(Y_i)$.

Proof. By the description of $\epsilon^{-1}\mathcal{F}$ in Lemma 61.19.1, the displayed formula is a special case of Étale Cohomology, Theorem 59.51.3. (When X, Y, and the Y_i are all affine, see the easier to parse Étale Cohomology, Lemma 59.51.5.) The final statement follows immediately from this and Lemma 61.19.2. □

099U Lemma 61.19.4. Let X be an affine scheme. For injective abelian sheaf $\mathcal{I}$ on $X_{\acute{e}tale}$ we have $H^p(X_{pro\text{-}\acute{e}tale}, \epsilon^{-1}\mathcal{I}) = 0$ for $p > 0$.

Proof. We are going to use Cohomology on Sites, Lemma 21.10.9 to prove this. Let $\mathcal{B} \subset \text{Ob}(X_{pro\text{-}\acute{e}tale})$ be the set of affine objects U of $X_{pro\text{-}\acute{e}tale}$ such that $\mathcal{O}(X) \to \mathcal{O}(U)$ is ind-étale. Let Cov be the set of pro-étale coverings $\{U_i \to U\}_{i=1,\ldots,n}$ with $U \in \mathcal{B}$ such that $\mathcal{O}(U) \to \mathcal{O}(U_i)$ is ind-étale for $i = 1, \ldots, n$. Properties (1) and (2) of Cohomology on Sites, Lemma 21.10.9 hold for $\mathcal{B}$ and Cov by Lemmas 61.7.3, 61.7.2, and 61.12.5 and Proposition 61.9.1.

To check condition (3) suppose that $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ is an element of Cov. We have to show that the higher Cech cohomology groups of $\epsilon^{-1}\mathcal{I}$ with respect to $\mathcal{U}$ are zero. First we write $U_i = \lim_{a \in A_i} U_{i,a}$ as a directed inverse limit with $U_{i,a} \to U$ étale and $U_{i,a}$ affine. We think of $A_1 \times \ldots \times A_n$ as a direct set with ordering $(a_1, \ldots, a_n) \geq (a'_1, \ldots, a'_n)$ if and only if $a_i \geq a'_i$ for $i = 1, \ldots, n$. Observe that $\mathcal{U}_{(a_1,\ldots,a_n)} = \{U_{i,a_i} \to U\}_{i=1,\ldots,n}$ is an étale covering for all $a_1, \ldots, a_n \in A_1 \times \ldots \times A_n$. Observe that

$$U_{i_0} \times_U U_{i_1} \times_U \ldots \times_U U_{i_p} = \lim_{(a_1,\ldots,a_n) \in A_1 \times \ldots \times A_n} U_{i_0,a_{i_0}} \times_U U_{i_1,a_{i_1}} \times_U \ldots \times_U U_{i_p,a_{i_p}}$$

for all $i_0, \ldots, i_p \in \{1, \ldots, n\}$ because limits commute with fibred products. Hence by Lemma 61.19.3 and exactness of filtered colimits we have

$$\check{H}^p(\mathcal{U}, \epsilon^{-1}\mathcal{I}) = \text{colim}\, \check{H}^p(\mathcal{U}_{(a_1,\ldots,a_n)}, \epsilon^{-1}\mathcal{I})$$

Thus it suffices to prove the vanishing for étale coverings of U!

Let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be an étale covering with U_i affine. Write $U = \lim_{b \in B} U_b$ as a directed inverse limit with U_b affine and $U_b \to X$ étale. By Limits, Lemmas 32.10.1, 32.4.13, and 32.8.10 we can choose a $b_0 \in B$ such that for $i = 1, \ldots, n$ there is an étale morphism $U_{i,b_0} \to U_{b_0}$ of affines such that $U_i = U \times_{U_{b_0}} U_{i,b_0}$. Set $U_{i,b} = U_b \times_{U_{b_0}} U_{i,b_0}$ for $b \geq b_0$. For b large enough the family $\mathcal{U}_b = \{U_{i,b} \to U_b\}_{i=1,\ldots,n}$ is an étale covering, see Limits, Lemma 32.8.15. Exactly as before we find that

$$\check{H}^p(\mathcal{U}, \epsilon^{-1}\mathcal{I}) = \text{colim}\, \check{H}^p(\mathcal{U}_b, \epsilon^{-1}\mathcal{I}) = \text{colim}\, \check{H}^p(\mathcal{U}_b, \mathcal{I})$$

the final equality by Lemma 61.19.2. Since each of the Čech complexes on the right hand side is acyclic in positive degrees (Cohomology on Sites, Lemma 21.10.2) it follows that the one on the left is too. This proves condition (3) of Cohomology on Sites, Lemma 21.10.9. Since $X \in \mathcal{B}$ the lemma follows. $\square$

099V Lemma 61.19.5. Let X be a scheme.

(1) For an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $R\epsilon_*(\epsilon^{-1}\mathcal{F}) = \mathcal{F}$.
(2) For $K \in D^+(X_{\acute{e}tale})$ the map $K \to R\epsilon_*\epsilon^{-1}K$ is an isomorphism.

Proof. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\acute{e}tale}$. Recall that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I})$ is the sheaf associated to $U \mapsto H^q(U_{pro\text{-}\acute{e}tale}, \epsilon^{-1}\mathcal{I})$, see Cohomology on Sites, Lemma 21.7.4. By Lemma 61.19.4 we see that this is zero for $q > 0$ and U affine and étale over X. Since every object of $X_{\acute{e}tale}$ has a covering by affine objects, it follows that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I}) = 0$ for $q > 0$.

Let $K \in D^+(X_{\acute{e}tale})$. Choose a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves on $X_{\acute{e}tale}$ representing K. Then $\epsilon^{-1}K$ is represented by $\epsilon^{-1}\mathcal{I}^\bullet$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we see that $R\epsilon_*\epsilon^{-1}K$ is represented by $\epsilon_*\epsilon^{-1}\mathcal{I}^\bullet$. By Lemma 61.19.2 we conclude that $R\epsilon_*\epsilon^{-1}\mathcal{I}^\bullet = \mathcal{I}^\bullet$ and the proof of (2) is complete. Part (1) is a special case of (2). $\square$

099W Lemma 61.19.6. Let X be a scheme.

(1) For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have
$$H^i(X_{étale}, \mathcal{F}) = H^i(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{F})$$
for all i.
(2) For $K \in D^+(X_{étale})$ we have
$$R\Gamma(X_{étale}, K) = R\Gamma(X_{pro\text{-}étale}, \epsilon^{-1}K)$$

Proof. Immediate consequence of Lemma 61.19.5 and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). $\square$

099X Lemma 61.19.7. Let X be a scheme. Let $\mathcal{G}$ be a sheaf of (possibly noncommutative) groups on $X_{étale}$. We have
$$H^1(X_{étale}, \mathcal{G}) = H^1(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{G})$$
where H^1 is defined as the set of isomorphism classes of torsors (see Cohomology on Sites, Section 21.4).

Proof. Since the functor ϵ^{-1} is fully faithful by Lemma 61.19.2 it is clear that the map $H^1(X_{étale}, \mathcal{G}) \to H^1(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{G})$ is injective. To show surjectivity it suffices to show that any $\epsilon^{-1}\mathcal{G}$-torsor $\mathcal{F}$ is étale locally trivial. To do this we may assume that X is affine. Thus we reduce to proving surjectivity for X affine.

Choose a covering $\{U \to X\}$ with (a) U affine, (b) $\mathcal{O}(X) \to \mathcal{O}(U)$ ind-étale, and (c) $\mathcal{F}(U)$ nonempty. We can do this by Proposition 61.9.1 and the fact that standard pro-étale coverings of X are cofinal among all pro-étale coverings of X (Lemma 61.12.5). Write $U = \lim U_i$ as a limit of affine schemes étale over X. Pick $s \in \mathcal{F}(U)$. Let $g \in \epsilon^{-1}\mathcal{G}(U \times_X U)$ be the unique section such that $g \cdot \text{pr}_1^*s = \text{pr}_2^*s$ in $\mathcal{F}(U \times_X U)$. Then g satisfies the cocycle condition
$$\text{pr}_{12}^*g \cdot \text{pr}_{23}^*g = \text{pr}_{13}^*g$$
in $\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U)$. By Lemma 61.19.3 we have
$$\epsilon^{-1}\mathcal{G}(U \times_X U) = \text{colim}\, \mathcal{G}(U_i \times_X U_i)$$
and
$$\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U) = \text{colim}\, \mathcal{G}(U_i \times_X U_i \times_X U_i)$$
hence we can find an i and an element $g_i \in \mathcal{G}(U_i \times_X U_i)$ mapping to g satisfying the cocycle condition. The cocycle g_i then defines a torsor for $\mathcal{G}$ on $X_{étale}$ whose pullback is isomorphic to $\mathcal{F}$ by construction. Some details omitted (namely, the relationship between torsors and 1-cocycles which should be added to the chapter on cohomology on sites). $\square$

09B1 Lemma 61.19.8. Let X be a scheme. Let Λ be a ring.

(1) The essential image of the fully faithful functor $\epsilon^{-1} : \text{Mod}(X_{étale}, \Lambda) \to \text{Mod}(X_{pro\text{-}étale}, \Lambda)$ is a weak Serre subcategory $\mathcal{C}$.
(2) The functor ϵ^{-1} defines an equivalence of categories of $D^+(X_{étale}, \Lambda)$ with $D^+_{\mathcal{C}}(X_{pro\text{-}étale}, \Lambda)$ with question inverse given by $R\epsilon_*$.

Proof. To prove (1) we will prove conditions (1) – (4) of Homology, Lemma 12.10.3. Since ϵ^{-1} is fully faithful (Lemma 61.19.2) and exact, everything is clear except for condition (4). However, if
$$0 \to \epsilon^{-1}\mathcal{F}_1 \to \mathcal{G} \to \epsilon^{-1}\mathcal{F}_2 \to 0$$

is a short exact sequence of sheaves of Λ-modules on $X_{pro\text{-}étale}$, then we get

$$0 \to \epsilon_*\epsilon^{-1}\mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \epsilon_*\epsilon^{-1}\mathcal{F}_2 \to R^1\epsilon_*\epsilon^{-1}\mathcal{F}_1$$

which by Lemma 61.19.5 is the same as a short exact sequence

$$0 \to \mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \mathcal{F}_2 \to 0$$

Pulling pack we find that $\mathcal{G} = \epsilon^{-1}\epsilon_*\mathcal{G}$. This proves (1).

Part (2) follows from part (1) and Cohomology on Sites, Lemma 21.28.5. $\square$

Let Λ be a ring. In Modules on Sites, Section 18.43 we have defined the notion of a locally constant sheaf of Λ-modules on a site. If M is a Λ-module, then $\underline{M}$ is of finite presentation as a sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module, see Modules on Sites, Lemma 18.42.5.

099Y Lemma 61.19.9. Let X be a scheme. Let Λ be a ring. The functor ϵ^{-1} defines an equivalence of categories

$$\left\{\begin{matrix}\text{locally constant sheaves}\\ \text{of } \Lambda\text{-modules on } X_{étale}\\ \text{of finite presentation}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{locally constant sheaves}\\ \text{of } \Lambda\text{-modules on } X_{pro\text{-}étale}\\ \text{of finite presentation}\end{matrix}\right\}$$

Proof. Let $\mathcal{F}$ be a locally constant sheaf of Λ-modules on $X_{pro\text{-}étale}$ of finite presentation. Choose a pro-étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As pro-étale coverings are fpqc coverings and by Descent, Lemma 35.13.6 we see that $U_M = \bigcup_{i \in I_M} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. We may replace X by U_M for some M and assume that $M_i = M$ for all i.

Consider the sheaf $\mathcal{I} = \mathit{Isom}(\underline{M}, \mathcal{F})$. This sheaf is a torsor for $\mathcal{G} = \mathit{Isom}(\underline{M}, \underline{M})$. By Modules on Sites, Lemma 18.43.4 we have $\mathcal{G} = \underline{G}$ where $G = \mathit{Isom}_\Lambda(M, M)$. Since torsors for the étale topology and the pro-étale topology agree by Lemma 61.19.7 it follows that $\mathcal{I}$ has sections étale locally on X. Thus $\mathcal{F}$ is étale locally a constant sheaf which is what we had to show. $\square$

099Z Lemma 61.19.10. Let X be a scheme. Let Λ be a Noetherian ring. Let $D_{flc}(X_{étale}, \Lambda)$, resp. $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ be the full subcategory of $D(X_{étale}, \Lambda)$, resp. $D(X_{pro\text{-}étale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type. Then

$$\epsilon^{-1} : D^+_{flc}(X_{étale}, \Lambda) \longrightarrow D^+_{flc}(X_{pro\text{-}étale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_{flc}(X_{étale}, \Lambda)$ and $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Modules on Sites, Lemma 18.43.5 and Derived Categories, Section 13.17. The statement of the lemma follows by combining Lemmas 61.19.8 and 61.19.9. $\square$

09B2 Lemma 61.19.11. Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. If K_1 is

(1) in the essential image of $\epsilon^{-1} : D(X_{étale}, \Lambda/I) \to D(X_{pro\text{-}étale}, \Lambda/I)$, and
(2) has tor amplitude in $[a, \infty)$ for some $a \in \mathbf{Z}$,

then (1) and (2) hold for K_n as an object of $D(X_{pro\text{-}étale}, \Lambda/I^n)$.

Proof. Assertion (2) for K_n follows from the more general Cohomology on Sites, Lemma 21.46.9. Assertion (1) for K_n follows by induction on n from the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphism

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = K_1 \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

and the fact proven in Lemma 61.19.8 that the essential image of ϵ^{-1} is a triangulated subcategory of $D^+(X_{pro\text{-}étale}, \Lambda/I^n)$. □

0F6D Example 61.19.12. Let X be a scheme. Let A be an abelian group. Denote $fun(-, A)$ the sheaf on $X_{pro\text{-}étale}$ which maps U to the set of all maps $U \to A$ (of sets of points). Consider the sequence of sheaves

$$0 \to \underline{A} \to fun(-, A) \to \mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$. Since the constant sheaf is the pullback from the final topos we see that $\underline{A} = \epsilon^{-1}\underline{A}$. However, if A has more than one element, then neither $fun(-, A)$ nor $\mathcal{F}$ are pulled back from the étale site of X. To work out the values of $\mathcal{F}$ in some cases, assume that all points of X are closed with separably closed residue fields and U is affine. Then all points of U are closed with separably closed residue fields and we have

$$H^1_{pro\text{-}étale}(U, \underline{A}) = H^1_{étale}(U, \underline{A}) = 0$$

by Lemma 61.19.6 and Étale Cohomology, Lemma 59.80.3. Hence in this case we have

$$\mathcal{F}(U) = fun(U, A)/\underline{A}(U)$$

61.20. Derived completion in the constant Noetherian case

099L We continue the discussion started in Algebraic and Formal Geometry, Section 52.6; we assume the reader has read at least some of that section.

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Recall from Modules on Sites, Lemma 18.42.4 that

$$\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$$

is a flat $\underline{\Lambda}$-algebra and that the map $\underline{\Lambda} \to \underline{\Lambda}^\wedge$ identifies quotients by I. Hence Algebraic and Formal Geometry, Lemma 52.6.17 tells us that

$$D_{comp}(\mathcal{C}, \Lambda) = D_{comp}(\mathcal{C}, \underline{\Lambda}^\wedge)$$

In particular the cohomology sheaves $H^i(K)$ of an object K of $D_{comp}(\mathcal{C}, \Lambda)$ are sheaves of $\underline{\Lambda}^\wedge$-modules. For notational convenience we often work with $D_{comp}(\mathcal{C}, \Lambda)$.

099M Lemma 61.20.1. Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. The left adjoint to the inclusion functor $D_{comp}(\mathcal{C}, \Lambda) \to D(\mathcal{C}, \Lambda)$ of Algebraic and Formal Geometry, Proposition 52.6.12 sends K to

$$K^\wedge = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$$

In particular, K is derived complete if and only if $K = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$.

Proof. Choose generators $f_1, \ldots, f_r$ of I. By Algebraic and Formal Geometry, Lemma 52.6.9 we have

$$K^\wedge = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{K_n})$$

where $K_n = K(\Lambda, f_1^n, \ldots, f_r^n)$. In More on Algebra, Lemma 15.94.1 we have seen that the pro-systems $\{K_n\}$ and $\{\Lambda/I^n\}$ of $D(\Lambda)$ are isomorphic. Thus the lemma follows. □

099N Lemma 61.20.2. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then

1. Rf_* sends $D_{comp}(\mathcal{D}, \Lambda)$ into $D_{comp}(\mathcal{C}, \Lambda)$,
2. the map $Rf_* : D_{comp}(\mathcal{D}, \Lambda) \to D_{comp}(\mathcal{C}, \Lambda)$ has a left adjoint $Lf^*_{comp} : D_{comp}(\mathcal{C}, \Lambda) \to D_{comp}(\mathcal{D}, \Lambda)$ which is Lf^* followed by derived completion,
3. Rf_* commutes with derived completion,
4. for K in $D_{comp}(\mathcal{D}, \Lambda)$ we have $Rf_*K = R\lim Rf_*(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$.
5. for M in $D_{comp}(\mathcal{C}, \Lambda)$ we have $Lf^*_{comp}M = R\lim Lf^*(M \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$.

Proof. We have seen (1) and (2) in Algebraic and Formal Geometry, Lemma 52.6.18. Part (3) follows from Algebraic and Formal Geometry, Lemma 52.6.19. For (4) let K be derived complete. Then

$$Rf_*K = Rf_*(R\lim K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}) = R\lim Rf_*(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$$

the first equality by Lemma 61.20.1 and the second because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.23.3). This proves (4). To prove (5), by Lemma 61.20.1 we have

$$Lf^*_{comp}M = R\lim(Lf^*M \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$$

Since Lf^* commutes with derived tensor product by Cohomology on Sites, Lemma 21.18.4 and since $Lf^*\underline{\Lambda/I^n} = \underline{\Lambda/I^n}$ we get (5). □

61.21. Derived completion and weakly contractible objects

099P We continue the discussion in Section 61.20. In this section we will see how the existence of weakly contractible objects simplifies the study of derived complete modules.

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Although the general theory concerning $D_{comp}(\mathcal{C}, \Lambda)$ is quite satisfactory it is hard to explicitly give examples of derived complete complexes. We know that

1. every object M of $D(\mathcal{C}, \Lambda/I^n)$ restricts to a derived complete object of $D(\mathcal{C}, \Lambda)$, and
2. for every $K \in D(\mathcal{C}, \Lambda)$ the derived completion $K^\wedge = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$ is derived complete.

The first type of objects are trivially complete and perhaps not interesting. The problem with (2) is that derived completion in general is somewhat mysterious, even in case $K = \underline{\Lambda}$. Namely, by definition of homotopy limits there is a distinguished triangle

$$R\lim(\underline{\Lambda/I^n}) \to \prod \underline{\Lambda/I^n} \to \prod \underline{\Lambda/I^n} \to R\lim(\underline{\Lambda/I^n})[1]$$

in $D(\mathcal{C}, \Lambda)$ where the products are in $D(\mathcal{C}, \Lambda)$. These are computed by taking products of injective resolutions (Injectives, Lemma 19.13.4), so we see that the sheaf $H^p(\prod \underline{\Lambda/I^n})$ is the sheafification of the presheaf

$$U \longmapsto \prod H^p(U, \Lambda/I^n).$$

As an explicit example, if $X = \mathrm{Spec}(\mathbf{C}[t, t^{-1}])$, $\mathcal{C} = X_{\acute{e}tale}$, $\Lambda = \mathbf{Z}$, $I = (2)$, and $p = 1$, then we get the sheafification of the presheaf

$$U \mapsto \prod H^1(U_{\acute{e}tale}, \mathbf{Z}/2^n\mathbf{Z})$$

for U étale over X. Note that $H^1(X_{\acute{e}tale}, \mathbf{Z}/m\mathbf{Z})$ is cyclic of order m with generator α_m given by the finite étale $\mathbf{Z}/m\mathbf{Z}$-covering given by the equation $t = s^m$ (see Étale Cohomology, Section 59.6). Then the section

$$\alpha = (\alpha_{2^n}) \in \prod H^1(X_{\acute{e}tale}, \mathbf{Z}/2^n\mathbf{Z})$$

of the presheaf above does not restrict to zero on any nonempty étale scheme over X, whence the sheaf associated to the presheaf is not zero.

However, on the pro-étale site this phenomenon does not occur. The reason is that we have enough (quasi-compact) weakly contractible objects. In the following proposition we collect some results about derived completion in the Noetherian constant case for sites having enough weakly contractible objects (see Sites, Definition 7.40.2).

099Q Proposition 61.21.1. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has enough weakly contractible objects. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal.

(1) The category of derived complete sheaves Λ-modules is a weak Serre subcategory of $\mathrm{Mod}(\mathcal{C}, \Lambda)$.
(2) A sheaf $\mathcal{F}$ of Λ-modules satisfies $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ if and only if $\mathcal{F}$ is derived complete and $\bigcap I^n\mathcal{F} = 0$.
(3) The sheaf $\underline{\Lambda}^\wedge$ is derived complete.
(4) If $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ is an inverse system of derived complete sheaves of Λ-modules, then $\lim \mathcal{F}_n$ is derived complete.
(5) An object $K \in D(\mathcal{C}, \Lambda)$ is derived complete if and only if each cohomology sheaf $H^p(K)$ is derived complete.
(6) An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded above if and only if $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ is bounded above.
(7) An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded if $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has finite tor dimension.

Proof. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. We will use the results of Cohomology on Sites, Lemma 21.51.1 and Proposition 21.51.2 without further mention.

Recall that $R\lim$ commutes with $R\Gamma(U, -)$, see Injectives, Lemma 19.13.6. Let $f \in I$. Recall that $T(K, f)$ is the homotopy limit of the system

$$\ldots \xrightarrow{f} K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{C}, \Lambda)$. Thus

$$R\Gamma(U, T(K, f)) = T(R\Gamma(U, K), f).$$

Since we can test isomorphisms of maps between objects of $D(\mathcal{C}, \Lambda)$ by evaluating at $U \in \mathcal{B}$ we conclude an object K of $D(\mathcal{C}, \Lambda)$ is derived complete if and only if for every $U \in \mathcal{B}$ the object $R\Gamma(U, K)$ is derived complete as an object of $D(\Lambda)$.

The remark above implies that items (1), (5) follow from the corresponding results for modules over rings, see More on Algebra, Lemmas 15.91.1 and 15.91.6. In the same way (2) can be deduced from More on Algebra, Proposition 15.91.5 as $(I^n\mathcal{F})(U) = I^n \cdot \mathcal{F}(U)$ for $U \in \mathcal{B}$ (by exactness of evaluating at U).

Proof of (4). The homotopy limit $R\lim \mathcal{F}_n$ is in $D_{comp}(X, \Lambda)$ (see discussion following Algebraic and Formal Geometry, Definition 52.6.4). By part (5) just proved we conclude that $\lim \mathcal{F}_n = H^0(R\lim \mathcal{F}_n)$ is derived complete. Part (3) is a special case of (4).

Proof of (6) and (7). Follows from Lemma 61.20.1 and Cohomology on Sites, Lemma 21.46.9 and the computation of homotopy limits in Cohomology on Sites, Proposition 21.51.2. $\square$

61.22. Cohomology of a point

09B3 Let Λ be a Noetherian ring complete with respect to an ideal $I \subset \Lambda$. Let k be a field. In this section we "compute"

$$H^i(\mathrm{Spec}(k)_{\textit{pro-étale}}, \underline{\Lambda}^\wedge)$$

where $\underline{\Lambda}^\wedge = \lim_m \underline{\Lambda/I^m}$ as before. Let k^{sep} be a separable algebraic closure of k. Then

$$\mathcal{U} = \{\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)\}$$

is a pro-étale covering of $\mathrm{Spec}(k)$. We will use the Čech to cohomology spectral sequence with respect to this covering. Set $U_0 = \mathrm{Spec}(k^{sep})$ and

$$\begin{aligned} U_n &= \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \\ &= \mathrm{Spec}(k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep}) \end{aligned}$$

($n+1$ factors). Note that the underlying topological space $|U_0|$ of U_0 is a singleton and for $n \geq 1$ we have

$$|U_n| = G \times \ldots \times G \quad (n \text{ factors})$$

as profinite spaces where $G = \mathrm{Gal}(k^{sep}/k)$. Namely, every point of U_n has residue field k^{sep} and we identify $(\sigma_1, \ldots, \sigma_n)$ with the point corresponding to the surjection

$$k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep} \longrightarrow k^{sep}, \quad \lambda_0 \otimes \lambda_1 \otimes \ldots \lambda_n \longmapsto \lambda_0\sigma_1(\lambda_1)\ldots\sigma_n(\lambda_n)$$

Then we compute

$$\begin{aligned} R\Gamma((U_n)_{\textit{pro-étale}}, \underline{\Lambda}^\wedge) &= R\lim_m R\Gamma((U_n)_{\textit{pro-étale}}, \underline{\Lambda/I^m}) \\ &= R\lim_m R\Gamma((U_n)_{\textit{étale}}, \underline{\Lambda/I^m}) \\ &= \lim_m H^0(U_n, \underline{\Lambda/I^m}) \\ &= \mathrm{Maps}_{cont}(G \times \ldots \times G, \Lambda) \end{aligned}$$

The first equality because $R\Gamma$ commutes with derived limits and as $\Lambda^\wedge$ is the derived limit of the sheaves $\underline{\Lambda/I^m}$ by Proposition 61.21.1. The second equality by Lemma 61.19.6. The third equality by Étale Cohomology, Lemma 59.80.3. The fourth equality uses Étale Cohomology, Remark 59.23.2 to identify sections of the constant sheaf $\underline{\Lambda/I^m}$. Then it uses the fact that Λ is complete with respect to I and hence equal to $\lim_m \Lambda/I^m$ as a topological space, to see that $\lim_m \text{Map}_{cont}(G, \Lambda/I^m) = \text{Map}_{cont}(G, \Lambda)$ and similarly for higher powers of G. At this point Cohomology on Sites, Lemmas 21.10.3 and 21.10.7 tell us that

$$\Lambda \to \text{Maps}_{cont}(G, \Lambda) \to \text{Maps}_{cont}(G \times G, \Lambda) \to \ldots$$

computes the pro-étale cohomology. In other words, we see that

$$H^i(\text{Spec}(k)_{pro\text{-}\acute{e}tale}, \underline{\Lambda}^\wedge) = H^i_{cont}(G, \Lambda)$$

where the right hand side is Tate's continuous cohomology, see Étale Cohomology, Section 59.58. Of course, this is as it should be.

09B4 Lemma 61.22.1. Let k be a field. Let $G = \text{Gal}(k^{sep}/k)$ be its absolute Galois group. Further,

(1) let M be a profinite abelian group with a continuous G-action, or
(2) let Λ be a Noetherian ring and $I \subset \Lambda$ an ideal an let M be an I-adically complete Λ-module with continuous G-action.

Then there is a canonical sheaf $\underline{M}^\wedge$ on $\text{Spec}(k)_{pro\text{-}\acute{e}tale}$ associated to M such that

$$H^i(\text{Spec}(k), \underline{M}^\wedge) = H^i_{cont}(G, M)$$

as abelian groups or Λ-modules.

Proof. Proof in case (2). Set $M_n = M/I^nM$. Then $M = \lim M_n$ as M is assumed I-adically complete. Since the action of G is continuous we get continuous actions of G on M_n. By Étale Cohomology, Theorem 59.56.3 this action corresponds to a (locally constant) sheaf $\underline{M_n}$ of Λ/I^n-modules on $\text{Spec}(k)_{\acute{e}tale}$. Pull back to $\text{Spec}(k)_{pro\text{-}\acute{e}tale}$ by the comparison morphism ϵ and take the limit

$$\underline{M}^\wedge = \lim \epsilon^{-1}\underline{M_n}$$

to get the sheaf promised in the lemma. Exactly the same argument as given in the introduction of this section gives the comparison with Tate's continuous Galois cohomology. □

61.23. Functoriality of the pro-étale site

09A5 Let $f : X \to Y$ be a morphism of schemes. The functor $Y_{pro\text{-}\acute{e}tale} \to X_{pro\text{-}\acute{e}tale}$, $V \mapsto X \times_Y V$ induces a morphism of sites $f_{pro\text{-}\acute{e}tale} : X_{pro\text{-}\acute{e}tale} \to Y_{pro\text{-}\acute{e}tale}$, see Sites, Proposition 7.14.7. In fact, we obtain a commutative diagram of morphisms of sites

$$\begin{tikzcd} X_{pro\text{-}\acute{e}tale} \arrow[r, "\epsilon"'] \arrow[d, "f_{pro\text{-}\acute{e}tale}"'] & X_{\acute{e}tale} \arrow[d, "f_{\acute{e}tale}"] \\ Y_{pro\text{-}\acute{e}tale} \arrow[r, "\epsilon"] & Y_{\acute{e}tale} \end{tikzcd}$$

where ϵ is as in Section 61.19. In particular we have $\epsilon^{-1}f_{\acute{e}tale}^{-1} = f_{pro\text{-}\acute{e}tale}^{-1}\epsilon^{-1}$. Here is the corresponding result for pushforward.

09A6 Lemma 61.23.1. Let $f : X \to Y$ be a morphism of schemes.

(1) Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. Then we have $f_{pro\text{-}\acute{e}tale,*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1}f_{\acute{e}tale,*}\mathcal{F}$.
(2) Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Then we have $Rf_{pro\text{-}\acute{e}tale,*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1}Rf_{\acute{e}tale,*}\mathcal{F}$.

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. There is a canonical map $\epsilon^{-1}f_{\acute{e}tale,*}\mathcal{F} \to f_{pro\text{-}\acute{e}tale,*}\epsilon^{-1}\mathcal{F}$, see Sites, Section 7.45. To show it is an isomorphism we may work (Zariski) locally on Y, hence we may assume Y is affine. In this case every object of $Y_{pro\text{-}\acute{e}tale}$ has a covering by objects $V = \lim V_i$ which are limits of affine schemes V_i étale over Y (by Proposition 61.9.1 for example). Evaluating the map $\epsilon^{-1}f_{\acute{e}tale,*}\mathcal{F} \to f_{pro\text{-}\acute{e}tale,*}\epsilon^{-1}\mathcal{F}$ on V we obtain a map

$$\operatorname{colim} \Gamma(X \times_Y V_i, \mathcal{F}) \longrightarrow \Gamma(X \times_Y V, \epsilon^*\mathcal{F}).$$

see Lemma 61.19.3 for the left hand side. By Lemma 61.19.3 we have

$$\Gamma(X \times_Y V, \epsilon^*\mathcal{F}) = \Gamma(X \times_Y V, \mathcal{F})$$

Hence the result holds by Étale Cohomology, Lemma 59.51.5.

Proof of (2). Arguing in exactly the same manner as above we see that it suffices to show that

$$\operatorname{colim} H^i_{\acute{e}tale}(X \times_Y V_i, \mathcal{F}) \longrightarrow H^i_{\acute{e}tale}(X \times_Y V, \mathcal{F})$$

which follows once more from Étale Cohomology, Lemma 59.51.5. □

61.24. Finite morphisms and pro-étale sites

09A7 It is not clear that a finite morphism of schemes determines an exact pushforward on abelian pro-étale sheaves.

09A8 **Lemma 61.24.1.** Let $f : Z \to X$ be a finite morphism of schemes which is locally of finite presentation. Then $f_{pro\text{-}\acute{e}tale,*} : \textit{Ab}(Z_{pro\text{-}\acute{e}tale}) \to \textit{Ab}(X_{pro\text{-}\acute{e}tale})$ is exact.

Proof. The prove this we may work (Zariski) locally on X and assume that X is affine, say $X = \operatorname{Spec}(A)$. Then $Z = \operatorname{Spec}(B)$ for some finite A-algebra B of finite presentation. The construction in the proof of Proposition 61.11.3 produces a faithfully flat, ind-étale ring map $A \to D$ with D w-contractible. We may check exactness of a sequence of sheaves by evaluating on $U = \operatorname{Spec}(D)$ be such an object. Then $f_{pro\text{-}\acute{e}tale,*}\mathcal{F}$ evaluated at U is equal to $\mathcal{F}$ evaluated at $V = \operatorname{Spec}(D \otimes_A B)$. Since $D \otimes_A B$ is w-contractible by Lemma 61.11.6 evaluation at V is exact. □

61.25. Closed immersions and pro-étale sites

09A9 It is not clear (and likely false) that a closed immersion of schemes determines an exact pushforward on abelian pro-étale sheaves.

09BK **Lemma 61.25.1.** Let $i : Z \to X$ be a closed immersion morphism of affine schemes. Denote X_{app} and Z_{app} the sites introduced in Lemma 61.12.21. The base change functor

$$u : X_{app} \to Z_{app}, \quad U \longmapsto u(U) = U \times_X Z$$

is continuous and has a fully faithful left adjoint v. For V in Z_{app} the morphism $V \to v(V)$ is a closed immersion identifying V with $u(v(V)) = v(V) \times_X Z$ and every point of $v(V)$ specializes to a point of V. The functor v is cocontinuous and sends coverings to coverings.

Proof. The existence of the adjoint follows immediately from Lemma 61.7.7 and the definitions. It is clear that u is continuous from the definition of coverings in X_{app}.

Write $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Let $V = \mathrm{Spec}(\overline{C})$ be an object of Z_{app} and let $v(V) = \mathrm{Spec}(C)$. We have seen in the statement of Lemma 61.7.7 that V equals $v(V) \times_X Z = \mathrm{Spec}(C/IC)$. Any $g \in C$ which maps to an invertible element of $C/IC = \overline{C}$ is invertible in C. Namely, we have the A-algebra maps $C \to C_g \to C/IC$ and by adjointness we obtain an C-algebra map $C_g \to C$. Thus every point of $v(V)$ specializes to a point of V.

Suppose that $\{V_i \to V\}$ is a covering in Z_{app}. Then $\{v(V_i) \to v(V)\}$ is a finite family of morphisms of Z_{app} such that every point of $V \subset v(V)$ is in the image of one of the maps $v(V_i) \to v(V)$. As the morphisms $v(V_i) \to v(V)$ are flat (since they are weakly étale) we conclude that $\{v(V_i) \to v(V)\}$ is jointly surjective. This proves that v sends coverings to coverings.

Let V be an object of Z_{app} and let $\{U_i \to v(V)\}$ be a covering in X_{app}. Then we see that $\{u(U_i) \to u(v(V)) = V\}$ is a covering of Z_{app}. By adjointness we obtain morphisms $v(u(U_i)) \to U_i$. Thus the family $\{v(u(U_i)) \to v(V)\}$ refines the given covering and we conclude that v is cocontinuous. $\square$

09BL Lemma 61.25.2. Let $Z \to X$ be a closed immersion morphism of affine schemes. The corresponding morphism of topoi $i = i_{pro\text{-}étale}$ is equal to the morphism of topoi associated to the fully faithful cocontinuous functor $v : Z_{app} \to X_{app}$ of Lemma 61.25.1. It follows that

(1) $i^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto \mathcal{F}(v(V))$,
(2) for a weakly contractible object V of Z_{app} we have $i^{-1}\mathcal{F}(V) = \mathcal{F}(v(V))$,
(3) $i^{-1} : Sh(X_{pro\text{-}étale}) \to Sh(Z_{pro\text{-}étale})$ has a left adjoint $i_!^{Sh}$,
(4) $i^{-1} : \mathrm{Ab}(X_{pro\text{-}étale}) \to \mathrm{Ab}(Z_{pro\text{-}étale})$ has a left adjoint $i_!$,
(5) $\mathrm{id} \to i^{-1}i_!^{Sh}$, $\mathrm{id} \to i^{-1}i_!$, and $i^{-1}i_* \to \mathrm{id}$ are isomorphisms, and
(6) i_*, $i_!^{Sh}$ and $i_!$ are fully faithful.

Proof. By Lemma 61.12.21 we may describe $i_{pro\text{-}étale}$ in terms of the morphism of sites $u : X_{app} \to Z_{app}$, $V \mapsto V \times_X Z$. The first statement of the lemma follows from Sites, Lemma 7.22.2 (but with the roles of u and v reversed).

Proof of (1). By the description of i as the morphism of topoi associated to v this holds by the construction, see Sites, Lemma 7.21.1.

Proof of (2). Since the functor v sends coverings to coverings by Lemma 61.25.1 we see that the presheaf $\mathcal{G} : V \mapsto \mathcal{F}(v(V))$ is a separated presheaf (Sites, Definition 7.10.9). Hence the sheafification of $\mathcal{G}$ is $\mathcal{G}^+$, see Sites, Theorem 7.10.10. Next, let V be a weakly contractible object of Z_{app}. Let $\mathcal{V} = \{V_i \to V\}_{i=1,\ldots,n}$ be any covering in Z_{app}. Set $\mathcal{V}' = \{\coprod V_i \to V\}$. Since v commutes with finite disjoint unions (as a left adjoint or by the construction) and since $\mathcal{F}$ sends finite disjoint unions into products, we see that

$$H^0(\mathcal{V}, \mathcal{G}) = H^0(\mathcal{V}', \mathcal{G})$$

(notation as in Sites, Section 7.10; compare with Étale Cohomology, Lemma 59.22.1). Thus we may assume the covering is given by a single morphism, like so $\{V' \to V\}$. Since V is weakly contractible, this covering can be refined by the trivial covering $\{V \to V\}$. It therefore follows that the value of $\mathcal{G}^+ = i^{-1}\mathcal{F}$ on V is simply $\mathcal{F}(v(V))$ and (2) is proved.

Proof of (3). Every object of Z_{app} has a covering by weakly contractible objects (Lemma 61.13.4). By the above we see that we would have $i_!^{Sh}h_V = h_{v(V)}$ for V

weakly contractible if $i_!^{Sh}$ existed. The existence of $i_!^{Sh}$ then follows from Sites, Lemma 7.24.1.

Proof of (4). Existence of $i_!$ follows in the same way by setting $i_!\mathbf{Z}_V = \mathbf{Z}_{v(V)}$ for V weakly contractible in Z_{app}, using similar for direct sums, and applying Homology, Lemma 12.29.6. Details omitted.

Proof of (5). Let V be a contractible object of Z_{app}. Then $i^{-1}i_!^{Sh}h_V = i^{-1}h_{v(V)} = h_{u(v(V))} = h_V$. (It is a general fact that $i^{-1}h_U = h_{u(U)}$.) Since the sheaves h_V for V contractible generate $Sh(Z_{app})$ (Sites, Lemma 7.12.5) we conclude $\text{id} \to i^{-1}i_!^{Sh}$ is an isomorphism. Similarly for the map $\text{id} \to i^{-1}i_!$. Then $(i^{-1}i_*\mathcal{H})(V) = i_*\mathcal{H}(v(V)) = \mathcal{H}(u(v(V))) = \mathcal{H}(V)$ and we find that $i^{-1}i_* \to \text{id}$ is an isomorphism.

The fully faithfulness statements of (6) now follow from Categories, Lemma 4.24.4. □

09AA Lemma 61.25.3. Let $i : Z \to X$ be a closed immersion of schemes. Then

(1) $i^{-1}_{\textit{pro-étale}}$ commutes with limits,
(2) $i_{\textit{pro-étale},*}$ is fully faithful, and
(3) $i^{-1}_{\textit{pro-étale}} i_{\textit{pro-étale},*} \cong \text{id}_{Sh(Z_{\textit{pro-étale}})}$.

Proof. Assertions (2) and (3) are equivalent by Sites, Lemma 7.41.1. Parts (1) and (3) are (Zariski) local on X, hence we may assume that X is affine. In this case the result follows from Lemma 61.25.2. □

09AB Lemma 61.25.4. Let $i : Z \to X$ be an integral universally injective and surjective morphism of schemes. Then $i_{\textit{pro-étale},*}$ and $i^{-1}_{\textit{pro-étale}}$ are quasi-inverse equivalences of categories of pro-étale topoi.

Proof. There is an immediate reduction to the case that X is affine. Then Z is affine too. Set $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Z)$. Then the categories of étale algebras over A and B are equivalent, see Étale Cohomology, Theorem 59.45.2 and Remark 59.45.3. Thus the categories of ind-étale algebras over A and B are equivalent. In other words the categories X_{app} and Z_{app} of Lemma 61.12.21 are equivalent. We omit the verification that this equivalence sends coverings to coverings and vice versa. Thus the result as Lemma 61.12.21 tells us the pro-étale topos is the topos of sheaves on X_{app}. □

09AC Lemma 61.25.5. Let $i : Z \to X$ be a closed immersion of schemes. Let $U \to X$ be an object of $X_{\textit{pro-étale}}$ such that

(1) U is affine and weakly contractible, and
(2) every point of U specializes to a point of $U \times_X Z$.

Then $i^{-1}_{\textit{pro-étale}}\mathcal{F}(U \times_X Z) = \mathcal{F}(U)$ for all abelian sheaves on $X_{\textit{pro-étale}}$.

Proof. Since pullback commutes with restriction, we may replace X by U. Thus we may assume that X is affine and weakly contractible and that every point of X specializes to a point of Z. By Lemma 61.25.2 part (1) it suffices to show that $v(Z) = X$ in this case. Thus we have to show: If A is a w-contractible ring, $I \subset A$ an ideal contained in the Jacobson radical of A and $A \to B \to A/I$ is a factorization with $A \to B$ ind-étale, then there is a unique retraction $B \to A$ compatible with maps to A/I. Observe that $B/IB = A/I \times R$ as A/I-algebras. After replacing B by a localization we may assume $B/IB = A/I$. Note that $\text{Spec}(B) \to \text{Spec}(A)$

is surjective as the image contains $V(I)$ and hence all closed points and is closed under specialization. Since A is w-contractible there is a retraction $B \to A$. Since $B/IB = A/I$ this retraction is compatible with the map to A/I. We omit the proof of uniqueness (hint: use that A and B have isomorphic local rings at maximal ideals of A). $\square$

09BM Lemma 61.25.6. Let $i : Z \to X$ be a closed immersion of schemes. If $X \setminus i(Z)$ is a retrocompact open of X, then $i_{\textit{pro-étale},*}$ is exact.

Proof. The question is local on X hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. There exist $f_1, \ldots, f_r \in I$ such that $Z = V(f_1, \ldots, f_r)$ set theoretically, see Algebra, Lemma 10.29.1. By Lemma 61.25.4 we may assume that $Z = \mathrm{Spec}(A/(f_1, \ldots, f_r))$. In this case the functor $i_{\textit{pro-étale},*}$ is exact by Lemma 61.24.1. $\square$

61.26. Extension by zero

09AD The general material in Modules on Sites, Section 18.19 allows us to make the following definition.

09AE Definition 61.26.1. Let $j : U \to X$ be a weakly étale morphism of schemes.

(1) The restriction functor $j^{-1} : \mathit{Sh}(X_{\textit{pro-étale}}) \to \mathit{Sh}(U_{\textit{pro-étale}})$ has a left adjoint $j_!^{Sh} : \mathit{Sh}(X_{\textit{pro-étale}}) \to \mathit{Sh}(U_{\textit{pro-étale}})$.
(2) The restriction functor $j^{-1} : \mathrm{Ab}(X_{\textit{pro-étale}}) \to \mathrm{Ab}(U_{\textit{pro-étale}})$ has a left adjoint which is denoted $j_! : \mathrm{Ab}(U_{\textit{pro-étale}}) \to \mathrm{Ab}(X_{\textit{pro-étale}})$ and called extension by zero.
(3) Let Λ be a ring. The functor $j^{-1} : \mathrm{Mod}(X_{\textit{pro-étale}}, \Lambda) \to \mathrm{Mod}(U_{\textit{pro-étale}}, \Lambda)$ has a left adjoint $j_! : \mathrm{Mod}(U_{\textit{pro-étale}}, \Lambda) \to \mathrm{Mod}(X_{\textit{pro-étale}}, \Lambda)$ and called extension by zero.

As usual we compare this to what happens in the étale case.

09AF Lemma 61.26.2. Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{G}$ be an abelian sheaf on $U_{\textit{étale}}$. Then $\epsilon^{-1} j_! \mathcal{G} = j_! \epsilon^{-1} \mathcal{G}$ as sheaves on $X_{\textit{pro-étale}}$.

Proof. This is true because both are left adjoints to $j_{\textit{pro-étale},*}\epsilon^{-1} = \epsilon^{-1} j_{\textit{étale},*}$, see Lemma 61.23.1. $\square$

09AG Lemma 61.26.3. Let $j : U \to X$ be a weakly étale morphism of schemes. Let $i : Z \to X$ be a closed immersion such that $U \times_X Z = \emptyset$. Let $V \to X$ be an affine object of $X_{\textit{pro-étale}}$ such that every point of V specializes to a point of $V_Z = Z \times_X V$. Then $j_! \mathcal{F}(V) = 0$ for all abelian sheaves on $U_{\textit{pro-étale}}$.

Proof. Let $\{V_i \to V\}$ be a pro-étale covering. The lemma follows if we can refine this covering to a covering where the members have no morphisms into U over X (see construction of $j_!$ in Modules on Sites, Section 18.19). First refine the covering to get a finite covering with V_i affine. For each i let $V_i = \mathrm{Spec}(A_i)$ and let $Z_i \subset V_i$ be the inverse image of Z. Set $W_i = \mathrm{Spec}(A_{i,Z_i}^\sim)$ with notation as in Lemma 61.5.1. Then $\coprod W_i \to V$ is weakly étale and the image contains all points of V_Z. Hence the image contains all points of V by our assumption on specializations. Thus $\{W_i \to V\}$ is a pro-étale covering refining the given one. But each point in W_i specializes to a point lying over Z, hence there are no morphisms $W_i \to U$ over X. $\square$

09BN Lemma 61.26.4. Let $j : U \to X$ be an open immersion of schemes. Then $\text{id} \cong j^{-1}j_!$ and $j^{-1}j_* \cong \text{id}$ and the functors $j_!$ and j_* are fully faithful.

Proof. See Modules on Sites, Lemma 18.19.8 (and Sites, Lemma 7.27.4 for the case of sheaves of sets) and Categories, Lemma 4.24.4. □

Here is the relationship between extension by zero and restriction to the complementary closed subscheme.

09AH Lemma 61.26.5. Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Assume that j is a quasi-compact morphism. For every abelian sheaf on $X_{\textit{pro-étale}}$ there is a canonical short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{\textit{pro-étale}}$ where all the functors are for the pro-étale topology.

Proof. We obtain the maps by the adjointness properties of the functors involved. It suffices to show that $X_{\textit{pro-étale}}$ has enough objects (Sites, Definition 7.40.2) on which the sequence evaluates to a short exact sequence. Let $V = \text{Spec}(A)$ be an affine object of $X_{\textit{pro-étale}}$ such that A is w-contractible (there are enough objects of this type). Then $V \times_X Z$ is cut out by an ideal $I \subset A$. The assumption that j is quasi-compact implies there exist $f_1, \ldots, f_r \in I$ such that $V(I) = V(f_1, \ldots, f_r)$. We obtain a faithfully flat, ind-Zariski ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times A_{V(I)}^\sim$$

with $A_{V(I)}^\sim$ as in Lemma 61.5.1. Since $V_i = \text{Spec}(A_{f_i}) \to X$ factors through U we have

$$j_!j^{-1}\mathcal{F}(V_i) = \mathcal{F}(V_i) \quad \text{and} \quad i_*i^{-1}\mathcal{F}(V_i) = 0$$

On the other hand, for the scheme $V^\sim = \text{Spec}(A_{V(I)}^\sim)$ we have

$$j_!j^{-1}\mathcal{F}(V^\sim) = 0 \quad \text{and} \quad \mathcal{F}(V^\sim) = i_*i^{-1}\mathcal{F}(V^\sim)$$

the first equality by Lemma 61.26.3 and the second by Lemmas 61.25.5 and 61.11.7. Thus the sequence evaluates to an exact sequence on $\text{Spec}(A_{f_1} \times \ldots \times A_{f_r} \times A_{V(I)}^\sim)$ and the lemma is proved. □

09BP Lemma 61.26.6. Let $j : U \to X$ be a quasi-compact open immersion morphism of schemes. The functor $j_! : \text{Ab}(U_{\textit{pro-étale}}) \to \text{Ab}(X_{\textit{pro-étale}})$ commutes with limits.

Proof. Since $j_!$ is exact it suffices to show that $j_!$ commutes with products. The question is local on X, hence we may assume X affine. Let $\mathcal{G}$ be an abelian sheaf on $U_{\textit{pro-étale}}$. We have $j^{-1}j_*\mathcal{G} = \mathcal{G}$. Hence applying the exact sequence of Lemma 61.26.5 we get

$$0 \to j_!\mathcal{G} \to j_*\mathcal{G} \to i_*i^{-1}j_*\mathcal{G} \to 0$$

where $i : Z \to X$ is the inclusion of the reduced induced scheme structure on the complement $Z = X \setminus U$. The functors j_* and i_* commute with products as right adjoints. The functor i^{-1} commutes with products by Lemma 61.25.3. Hence $j_!$ does because on the pro-étale site products are exact (Cohomology on Sites, Proposition 21.51.2). □

61.27. Constructible sheaves on the pro-étale site

09AI We stick to constructible sheaves of Λ-modules for a Noetherian ring. In the future we intend to discuss constructible sheaves of sets, groups, etc.

09AJ **Definition 61.27.1.** Let X be a scheme. Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{pro\text{-}étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

Again this does not give anything "new".

09AK **Lemma 61.27.2.** Let X be a scheme. Let Λ be a Noetherian ring. The functor ϵ^{-1} defines an equivalence of categories

$$\left\{\begin{matrix}\text{constructible sheaves of}\\ \Lambda\text{-modules on } X_{étale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{constructible sheaves of}\\ \Lambda\text{-modules on } X_{pro\text{-}étale}\end{matrix}\right\}$$

between constructible sheaves of Λ-modules on $X_{étale}$ and constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$.

Proof. By Lemma 61.19.2 the functor ϵ^{-1} is fully faithful and commutes with pullback (restriction) to the strata. Hence ϵ^{-1} of a constructible étale sheaf is a constructible pro-étale sheaf. To finish the proof let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{pro\text{-}étale}$ as in Definition 61.27.1. There is a canonical map

$$\epsilon^{-1}\epsilon_*\mathcal{F} \longrightarrow \mathcal{F}$$

We will show this map is an isomorphism. This will prove that $\mathcal{F}$ is in the essential image of ϵ^{-1} and finish the proof (details omitted).

To prove this we may assume that X is affine. In this case we have a finite partition $X = \coprod_i X_i$ by constructible locally closed strata such that $\mathcal{F}|_{X_i}$ is locally constant of finite type. Let $U \subset X$ be one of the open strata in the partition and let $Z \subset X$ be the reduced induced structure on the complement. By Lemma 61.26.5 we have a short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$. Functoriality gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \epsilon^{-1}\epsilon_*j_!j^{-1}\mathcal{F} \arrow[r] \arrow[d] & \epsilon^{-1}\epsilon_*\mathcal{F} \arrow[r] \arrow[d] & \epsilon^{-1}\epsilon_*i_*i^{-1}\mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & j_!j^{-1}\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & i_*i^{-1}\mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

By induction on the length of the partition we know that on the one hand $\epsilon^{-1}\epsilon_*i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$ and $\epsilon^{-1}\epsilon_*j^{-1}\mathcal{F} \to j^{-1}\mathcal{F}$ are isomorphisms and on the other that $i^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{A}$ and $j^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{B}$ for some constructible sheaves of Λ-modules $\mathcal{A}$ on $Z_{étale}$ and $\mathcal{B}$ on $U_{étale}$. Then

$$\epsilon^{-1}\epsilon_*j_!j^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_*j_!\epsilon^{-1}\mathcal{B} = \epsilon^{-1}\epsilon_*\epsilon^{-1}j_!\mathcal{B} = \epsilon^{-1}j_!\mathcal{B} = j_!\epsilon^{-1}\mathcal{B} = j_!j^{-1}\mathcal{F}$$

the second equality by Lemma 61.26.2, the third equality by Lemma 61.19.2, and the fourth equality by Lemma 61.26.2 again. Similarly, we have

$$\epsilon^{-1}\epsilon_*i_*i^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_*i_*\epsilon^{-1}\mathcal{A} = \epsilon^{-1}\epsilon_*\epsilon^{-1}i_*\mathcal{A} = \epsilon^{-1}i_*\mathcal{A} = i_*\epsilon^{-1}\mathcal{A} = i_*i^{-1}\mathcal{F}$$

this time using Lemma 61.23.1. By the five lemma we conclude the vertical map in the middle of the big diagram is an isomorphism. □

09B5 Lemma 61.27.3. Let X be a scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$ is a weak Serre subcategory of $\text{Mod}(X_{pro\text{-}étale}, \Lambda)$.

Proof. This is a formal consequence of Lemmas 61.27.2 and 61.19.8 and the result for the étale site (Étale Cohomology, Lemma 59.71.6). □

09AL Lemma 61.27.4. Let X be a scheme. Let Λ be a Noetherian ring. Let $D_c(X_{étale}, \Lambda)$, resp. $D_c(X_{pro\text{-}étale}, \Lambda)$ be the full subcategory of $D(X_{étale}, \Lambda)$, resp. $D(X_{pro\text{-}étale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are constructible sheaves of Λ-modules. Then

$$\epsilon^{-1} : D_c^+(X_{étale}, \Lambda) \longrightarrow D_c^+(X_{pro\text{-}étale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_c(X_{étale}, \Lambda)$ and $D_c(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Étale Cohomology, Lemma 59.71.6 and Lemma 61.27.3 and Derived Categories, Section 13.17. The statement of the lemma follows by combining Lemmas 61.19.8 and 61.27.2. □

09BQ Lemma 61.27.5. Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X_{pro\text{-}étale}, \Lambda)$. Then $K \otimes_\Lambda^{\mathbf{L}} L$ is in $D_c^-(X_{pro\text{-}étale}, \Lambda)$.

Proof. Note that $H^i(K \otimes_\Lambda^{\mathbf{L}} L)$ is the same as $H^i(\tau_{\geq i-1} K \otimes_\Lambda^{\mathbf{L}} \tau_{\geq i-1} L)$. Thus we may assume K and L are bounded. In this case we can apply Lemma 61.27.4 to reduce to the case of the étale site, see Étale Cohomology, Lemma 59.76.6. □

09BR Lemma 61.27.6. Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. If K_1 is in $D_c^-(X_{pro\text{-}étale}, \Lambda/I)$, then K_n is in $D_c^-(X_{pro\text{-}étale}, \Lambda/I^n)$ for all n.

Proof. Consider the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = K_1 \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

By Lemma 61.27.5 we see that this tensor product has constructible cohomology sheaves (and vanishing when K_1 has vanishing cohomology). Hence by induction on n using Lemma 61.27.3 we see that each K_n has constructible cohomology sheaves. □

61.28. Constructible adic sheaves

09BS In this section we define the notion of a constructible Λ-sheaf as well as some variants.

09BT Definition 61.28.1. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$.

(1) We say $\mathcal{F}$ is a constructible Λ-sheaf if $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ and each $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of Λ/I^n-modules.
(2) If $\mathcal{F}$ is a constructible Λ-sheaf, then we say $\mathcal{F}$ is lisse if each $\mathcal{F}/I^n\mathcal{F}$ is locally constant.
(3) We say $\mathcal{F}$ is adic lisse[5] if there exists a I-adically complete Λ-module M with M/IM finite such that $\mathcal{F}$ is locally isomorphic to
$$\underline{M}^\wedge = \lim \underline{M/I^nM}.$$
(4) We say $\mathcal{F}$ is adic constructible[6] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $\mathcal{F}|_{U_i}$ is adic lisse.

The definition of a constructible Λ-sheaf is equivalent to the one in [Gro77, Exposé VI, Definition 1.1.1] when $\Lambda = \mathbf{Z}_\ell$ and $I = (\ell)$. It is clear that we have the implications

$$\begin{tikzcd}
\text{lisse adic} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{adic constructible} \arrow[d, Rightarrow] \\
\text{lisse constructible } \Lambda\text{-sheaf} \arrow[r, Rightarrow] & \text{constructible } \Lambda\text{-sheaf}
\end{tikzcd}$$

The vertical arrows can be inverted in some cases (see Lemmas 61.28.2 and 61.28.5). In general neither the category of adic constructible sheaves nor the category of constructible Λ-sheaves is closed under kernels and cokernels.

Namely, let X be an affine scheme whose underlying topological space $|X|$ is homeomorphic to $\Lambda = \mathbf{Z}_\ell$, see Example 61.6.3. Denote $f : |X| \to \mathbf{Z}_\ell = \Lambda$ a homeomorphism. We can think of f as a section of $\underline{\Lambda}^\wedge$ over X and multiplication by f then defines a two term complex
$$\underline{\Lambda}^\wedge \xrightarrow{f} \underline{\Lambda}^\wedge$$
on $X_{\textit{pro-étale}}$. The sheaf $\underline{\Lambda}^\wedge$ is adic lisse. However, the cokernel of the map above, is not adic constructible, as the isomorphism type of the stalks of this cokernel attains infinitely many values: $\mathbf{Z}/\ell^n\mathbf{Z}$ and $\mathbf{Z}_\ell$. The cokernel is a constructible $\mathbf{Z}_\ell$-sheaf. However, the kernel is not even a constructible $\mathbf{Z}_\ell$-sheaf as it is zero a non-quasi-compact open but not zero.

09BU Lemma 61.28.2. Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{\textit{pro-étale}}$. Then there exists a finite partition $X = \coprod X_i$ by locally closed subschemes such that the restriction $\mathcal{F}|_{X_i}$ is lisse.

Proof. Let $R = \bigoplus I^n/I^{n+1}$. Observe that R is a Noetherian ring. Since each of the sheaves $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of $\Lambda/I^n\Lambda$-modules also $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is a constructible sheaf of Λ/I-modules and hence the pullback of a constructible sheaf $\mathcal{G}_n$ on $X_{\acute{e}tale}$ by Lemma 61.27.2. Set $\mathcal{G} = \bigoplus \mathcal{G}_n$. This is a sheaf of R-modules on $X_{\acute{e}tale}$ and the map
$$\mathcal{G}_0 \otimes_{\Lambda/I} \underline{R} \longrightarrow \mathcal{G}$$
is surjective because the maps
$$\mathcal{F}/I\mathcal{F} \otimes \underline{I^n/I^{n+1}} \to I^n\mathcal{F}/I^{n+1}\mathcal{F}$$

[5]This may be nonstandard notation.
[6]This may be nonstandard notation.

are surjective. Hence $\mathcal{G}$ is a constructible sheaf of R-modules by Étale Cohomology, Proposition 59.74.1. Choose a partition $X = \coprod X_i$ such that $\mathcal{G}|_{X_i}$ is a locally constant sheaf of R-modules of finite type (Étale Cohomology, Lemma 59.71.2). We claim this is a partition as in the lemma. Namely, replacing X by X_i we may assume $\mathcal{G}$ is locally constant. It follows that each of the sheaves $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is locally constant. Using the short exact sequences

$$0 \to I^n\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

induction and Modules on Sites, Lemma 18.43.5 the lemma follows. □

09BV Lemma 61.28.3. Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{\textit{pro-étale}}$ such that

(1) $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$,
(2) $\mathcal{F}/I^n\mathcal{F}$ is a constant sheaf of Λ/I^n-modules,
(3) $\mathcal{F}/I\mathcal{F}$ is of finite type.

Then $\mathcal{F} \cong \underline{M}^\wedge$ where M is a finite $\Lambda^\wedge$-module.

Proof. Pick a Λ/I^n-module M_n such that $\mathcal{F}/I^n\mathcal{F} \cong \underline{M_n}$. Since we have the surjections $\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F}$ we conclude that there exist surjections $M_{n+1} \to M_n$ inducing isomorphisms $M_{n+1}/I^nM_{n+1} \to M_n$. Fix a choice of such surjections and set $M = \lim M_n$. Then M is an I-adically complete Λ-module with $M/I^nM = M_n$, see Algebra, Lemma 10.98.2. Since M_1 is a finite type Λ-module (Modules on Sites, Lemma 18.42.5) we see that M is a finite $\Lambda^\wedge$-module. Consider the sheaves

$$\mathcal{I}_n = \mathit{Isom}(\underline{M_n}, \mathcal{F}/I^n\mathcal{F})$$

on $X_{\textit{pro-étale}}$. Modding out by I^n defines a transition map

$$\mathcal{I}_{n+1} \longrightarrow \mathcal{I}_n$$

By our choice of M_n the sheaf $\mathcal{I}_n$ is a torsor under

$$\mathit{Isom}(\underline{M_n}, \underline{M_n}) = \underline{\mathrm{Isom}_\Lambda(M_n, M_n)}$$

(Modules on Sites, Lemma 18.43.4) since $\mathcal{F}/I^n\mathcal{F}$ is (étale) locally isomorphic to $\underline{M_n}$. It follows from More on Algebra, Lemma 15.100.4 that the system of sheaves $(\mathcal{I}_n)$ is Mittag-Leffler. For each n let $\mathcal{I}'_n \subset \mathcal{I}_n$ be the image of $\mathcal{I}_N \to \mathcal{I}_n$ for all $N \gg n$. Then

$$\ldots \to \mathcal{I}'_3 \to \mathcal{I}'_2 \to \mathcal{I}'_1 \to *$$

is a sequence of sheaves of sets on $X_{\textit{pro-étale}}$ with surjective transition maps. Since $*(X)$ is a singleton (not empty) and since evaluating at X transforms surjective maps of sheaves of sets into surjections of sets, we can pick $s \in \lim \mathcal{I}'_n(X)$. The sections define isomorphisms $\underline{M}^\wedge \to \lim \mathcal{F}/I^n\mathcal{F} = \mathcal{F}$ and the proof is done. □

09BW Lemma 61.28.4. Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If $\mathcal{F}$ is a lisse constructible Λ-sheaf on $X_{\textit{pro-étale}}$, then $\mathcal{F}$ is adic lisse.

Proof. By Lemma 61.19.9 we have $\mathcal{F}/I^n\mathcal{F} = \epsilon^{-1}\mathcal{G}_n$ for some locally constant sheaf $\mathcal{G}_n$ of Λ/I^n-modules. By Étale Cohomology, Lemma 59.64.8 there exists a finite Λ/I^n-module M_n such that $\mathcal{G}_n$ is locally isomorphic to $\underline{M_n}$. Choose a covering $\{W_t \to X\}_{t \in T}$ with each W_t affine and weakly contractible. Then $\mathcal{F}|_{W_t}$ satisfies the assumptions of Lemma 61.28.3 and hence $\mathcal{F}|_{W_t} \cong \underline{N_t}^\wedge$ for some finite $\Lambda^\wedge$-module

N_t. Note that $N_t/I^nN_t \cong M_n$ for all t and n. Hence $N_t \cong N_{t'}$ for all $t, t' \in T$, see More on Algebra, Lemma 15.100.5. This proves that $\mathcal{F}$ is adic lisse. $\square$

09BX Lemma 61.28.5. Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then $\mathcal{F}$ is adic constructible.

Proof. This is a consequence of Lemmas 61.28.2 and 61.28.4, the fact that a Noetherian scheme is locally connected (Topology, Lemma 5.9.6), and the definitions. $\square$

It will be useful to identify the constructible Λ-sheaves inside the category of derived complete sheaves of Λ-modules. It turns out that the naive analogue of More on Algebra, Lemma 15.94.5 is wrong in this setting. However, here is the analogue of More on Algebra, Lemma 15.91.7.

09BY Lemma 61.28.6. Let X be a scheme. Let Λ be a ring and let $I \subset \Lambda$ be a finitely generated ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$. If $\mathcal{F}$ is derived complete and $\mathcal{F}/I\mathcal{F} = 0$, then $\mathcal{F} = 0$.

Proof. Assume that $\mathcal{F}/I\mathcal{F}$ is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $\mathcal{G} = \mathcal{F}/(f_1, \ldots, f_i)\mathcal{F}$ is nonzero. If i does not exist, then $\mathcal{F} = 0$ which is what we want to show. Then $\mathcal{G}$ is derived complete as a cokernel of a map between derived complete modules, see Proposition 61.21.1. By our choice of i we have that $f_{i+1} : \mathcal{G} \to \mathcal{G}$ is surjective. Hence

$$\lim(\ldots \to \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G})$$

is nonzero, contradicting the derived completeness of $\mathcal{G}$. $\square$

09BZ Lemma 61.28.7. Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a derived complete sheaf of Λ-modules on $X_{pro\text{-}étale}$ with $\mathcal{F}/I\mathcal{F}$ a locally constant sheaf of Λ/I-modules of finite type. Then there exists an integer t and a surjective map

$$(\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$$

Proof. Since X is weakly contractible, there exists a finite disjoint open covering $X = \coprod U_i$ such that $\mathcal{F}/I\mathcal{F}|_{U_i}$ is isomorphic to the constant sheaf associated to a finite Λ/I-module M_i. Choose finitely many generators m_{ij} of M_i. We can find sections $s_{ij} \in \mathcal{F}(X)$ restricting to m_{ij} viewed as a section of $\mathcal{F}/I\mathcal{F}$ over U_i. Let t be the total number of s_{ij}. Then we obtain a map

$$\alpha : \underline{\Lambda}^{\oplus t} \longrightarrow \mathcal{F}$$

which is surjective modulo I by construction. By Lemma 61.20.1 the derived completion of $\underline{\Lambda}^{\oplus t}$ is the sheaf $(\underline{\Lambda}^\wedge)^{\oplus t}$. Since $\mathcal{F}$ is derived complete we see that α factors through a map

$$\alpha^\wedge : (\underline{\Lambda}^\wedge)^{\oplus t} \longrightarrow \mathcal{F}$$

Then $\mathcal{Q} = \text{Coker}(\alpha^\wedge)$ is a derived complete sheaf of Λ-modules by Proposition 61.21.1. By construction $\mathcal{Q}/I\mathcal{Q} = 0$. It follows from Lemma 61.28.6 that $\mathcal{Q} = 0$ which is what we wanted to show. $\square$

61.29. A suitable derived category

09C0 Let X be a scheme. It will turn out that for many schemes X a suitable derived category of ℓ-adic sheaves can be gotten by considering the derived complete objects K of $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ with the property that $K \otimes_\Lambda^{\mathbf{L}} \mathbf{F}_\ell$ is bounded with constructible cohomology sheaves. Here is the general definition.

09C1 Definition 61.29.1. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. An object K of $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ is called constructible if

(1) K is derived complete with respect to I,
(2) $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has constructible cohomology sheaves and locally has finite tor dimension.

We denote $D_{cons}(X, \Lambda)$ the full subcategory of constructible K in $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$.

Recall that with our conventions a complex of finite tor dimension is bounded (Cohomology on Sites, Definition 21.46.1). In fact, let's collect everything proved so far in a lemma.

09C2 Lemma 61.29.2. In the situation above suppose K is in $D_{cons}(X, \Lambda)$ and X is quasi-compact. Set $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. There exist a, b such that

(1) $K = R\lim K_n$ and $H^i(K) = 0$ for $i \not\in [a, b]$,
(2) each K_n has tor amplitude in $[a, b]$,
(3) each K_n has constructible cohomology sheaves,
(4) each $K_n = \epsilon^{-1} L_n$ for some $L_n \in D_{ctf}(X_{\acute{e}tale}, \Lambda/I^n)$ (Étale Cohomology, Definition 59.77.1).

Proof. By definition of local having finite tor dimension, we can find a, b such that K_1 has tor amplitude in $[a, b]$. Part (2) follows from Cohomology on Sites, Lemma 21.46.9. Then (1) follows as K is derived complete by the description of limits in Cohomology on Sites, Proposition 21.51.2 and the fact that $H^b(K_{n+1}) \to H^b(K_n)$ is surjective as $K_n = K_{n+1} \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. Part (3) follows from Lemma 61.27.6, Part (4) follows from Lemma 61.27.4 and the fact that L_n has finite tor dimension because K_n does (small argument omitted). $\square$

09C3 Lemma 61.29.3. Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ are locally constant. Then there exists a finite disjoint open covering $X = \coprod U_i$ and for each i a finite collection of finite projective $\Lambda^\wedge$-modules $M_a, \ldots, M_b$ such that $K|_{U_i}$ is represented by a complex

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^b})^\wedge$$

in $D(U_{i,pro\text{-}\acute{e}tale}, \Lambda)$ for some maps of sheaves of Λ-modules $(\underline{M^i})^\wedge \to (\underline{M^{i+1}})^\wedge$.

Proof. We freely use the results of Lemma 61.29.2. Choose a, b as in that lemma. We will prove the lemma by induction on $b - a$. Let $\mathcal{F} = H^b(K)$. Note that $\mathcal{F}$ is a derived complete sheaf of Λ-modules by Proposition 61.21.1. Moreover $\mathcal{F}/I\mathcal{F}$ is a locally constant sheaf of Λ/I-modules of finite type. Apply Lemma 61.28.7 to get a surjection $\rho : (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$.

If $a = b$, then $K = \mathcal{F}[-b]$. In this case we see that

$$\mathcal{F} \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I} = \mathcal{F}/I\mathcal{F}$$

As X is weakly contractible and $\mathcal{F}/I\mathcal{F}$ locally constant, we can find a finite disjoint union decomposition $X = \coprod U_i$ by affine opens U_i and Λ/I-modules $\overline{M}_i$ such that $\mathcal{F}/I\mathcal{F}$ restricts to $\underline{\overline{M}_i}$ on U_i. After refining the covering we may assume the map

$$\rho|_{U_i} \bmod I : \underline{\Lambda/I}^{\oplus t} \longrightarrow \underline{\overline{M}_i}$$

is equal to $\underline{\alpha_i}$ for some surjective module map $\alpha_i : \Lambda/I^{\oplus t} \to \overline{M}_i$, see Modules on Sites, Lemma 18.43.3. Note that each $\overline{M}_i$ is a finite Λ/I-module. Since $\mathcal{F}/I\mathcal{F}$ has tor amplitude in $[0, 0]$ we conclude that $\overline{M}_i$ is a flat Λ/I-module. Hence $\overline{M}_i$ is finite projective (Algebra, Lemma 10.78.2). Hence we can find a projector $\overline{p}_i : (\Lambda/I)^{\oplus t} \to (\Lambda/I)^{\oplus t}$ whose image maps isomorphically to $\overline{M}_i$ under the map α_i. We can lift $\overline{p}_i$ to a projector $p_i : (\Lambda^\wedge)^{\oplus t} \to (\Lambda^\wedge)^{\oplus t}$[7]. Then $M_i = \text{Im}(p_i)$ is a finite I-adically complete $\Lambda^\wedge$-module with $M_i/IM_i = \overline{M}_i$. Over U_i consider the maps

$$\underline{M_i}^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}|_{U_i}$$

By construction the composition induces an isomorphism modulo I. The source and target are derived complete, hence so are the cokernel $\mathcal{Q}$ and the kernel $\mathcal{K}$. We have $\mathcal{Q}/I\mathcal{Q} = 0$ by construction hence $\mathcal{Q}$ is zero by Lemma 61.28.6. Then

$$0 \to \mathcal{K}/I\mathcal{K} \to \underline{\overline{M}_i} \to \mathcal{F}/I\mathcal{F} \to 0$$

is exact by the vanishing of Tor_1 see at the start of this paragraph; also use that $\underline{\Lambda}^\wedge/I\overline{\Lambda}^\wedge$ by Modules on Sites, Lemma 18.42.4 to see that $\underline{M_i}^\wedge/I\underline{M_i}^\wedge = \underline{\overline{M}_i}$. Hence $\mathcal{K}/I\mathcal{K} = 0$ by construction and we conclude that $\mathcal{K} = 0$ as before. This proves the result in case $a = b$.

If $b > a$, then we lift the map ρ to a map

$$\tilde{\rho} : (\underline{\Lambda}^\wedge)^{\oplus t}[-b] \longrightarrow K$$

in $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$. This is possible as we can think of K as a complex of $\underline{\Lambda}^\wedge$-modules by discussion in the introduction to Section 61.20 and because $X_{pro\text{-}\acute{e}tale}$ is weakly contractible hence there is no obstruction to lifting the elements $\rho(e_s) \in H^0(X, \mathcal{F})$ to elements of $H^b(X, K)$. Fitting $\tilde{\rho}$ into a distinguished triangle

$$(\underline{\Lambda}^\wedge)^{\oplus t}[-b] \to K \to L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$$

we see that L is an object of $D_{cons}(X, \Lambda)$ such that $L \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has tor amplitude contained in $[a, b-1]$ (details omitted). By induction we can describe L locally as stated in the lemma, say L is isomorphic to

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^{b-1}})^\wedge$$

The map $L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$ corresponds to a map $(\underline{M^{b-1}})^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t}$ which allows us to extend the complex by one. The corresponding complex is isomorphic to K in the derived category by the properties of triangulated categories. This finishes the proof. □

Motivated by what happens for constructible Λ-sheaves we introduce the following notion.

09C4 Definition 61.29.4. Let X be a scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $K \in D(X_{pro\text{-}\acute{e}tale}, \Lambda)$.

[7]Proof: by Algebra, Lemma 10.32.7 we can lift $\overline{p}_i$ to a compatible system of projectors $p_{i,n} : (\Lambda/I^n)^{\oplus t} \to (\Lambda/I^n)^{\oplus t}$ and then we set $p_i = \lim p_{i,n}$ which works because $\Lambda^\wedge = \lim \Lambda/I^n$.

(1) We say K is adic lisse[8] if there exists a finite complex of finite projective $\Lambda^\wedge$-modules $M^\bullet$ such that K is locally isomorphic to
$$\underline{M^{a\wedge}} \to \ldots \to \underline{M^{b\wedge}}$$
(2) We say K is adic constructible[9] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $K|_{U_i}$ is adic lisse.

The difference between the local structure obtained in Lemma 61.29.3 and the structure of an adic lisse complex is that the maps $\underline{M^{i\wedge}} \to \underline{M^{i+1\wedge}}$ in Lemma 61.29.3 need not be constant, whereas in the definition above they are required to be constant.

09C5 Lemma 61.29.5. Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ is isomorphic in $D(X_{pro\text{-}étale}, \Lambda/I^n)$ to a complex of constant sheaves of Λ/I^n-modules. Then
$$H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^n)$$
has the Mittag-Leffler condition.

Proof. Say $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ is isomorphic to $\underline{E_n}$ for some object E_n of $D(\Lambda/I^n)$. Since $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has finite tor dimension and has finite type cohomology sheaves we see that E_1 is perfect (see More on Algebra, Lemma 15.74.2). The transition maps
$$K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$$
locally come from (possibly many distinct) maps of complexes $E_{n+1} \to E_n$ in $D(\Lambda/I^{n+1})$ see Cohomology on Sites, Lemma 21.53.3. For each n choose one such map and observe that it induces an isomorphism $E_{n+1} \otimes_{\Lambda/I^{n+1}}^{\mathbf{L}} \Lambda/I^n \to E_n$ in $D(\Lambda/I^n)$. By More on Algebra, Lemma 15.97.4 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^nM^\bullet \to E_n$ in $D(\Lambda/I^n)$ compatible with the transition maps.

Now observe that for each finite collection of indices $n > m > k$ the triple of maps
$$H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^n) \to H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^m) \to H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^k)$$
is isomorphic to
$$H^0(X, \underline{M^\bullet/I^nM^\bullet}) \to H^0(X, \underline{M^\bullet/I^mM^\bullet}) \to H^0(X, \underline{M^\bullet/I^kM^\bullet})$$
Namely, choose any isomorphism
$$\underline{M^\bullet/I^nM^\bullet} \to K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^n$$
induces similar isomorphisms module I^m and I^k and we see that the assertion is true. Thus to prove the lemma it suffices to show that the system $H^0(X, \underline{M^\bullet/I^nM^\bullet})$ has Mittag-Leffler. Since taking sections over X is exact, it suffices to prove that the system of Λ-modules
$$H^0(M^\bullet/I^nM^\bullet)$$
has Mittag-Leffler. Set $A = \Lambda^\wedge$ and consider the spectral sequence
$$\text{Tor}_{-p}^A(H^q(M^\bullet), A/I^nA) \Rightarrow H^{p+q}(M^\bullet/I^nM^\bullet)$$

[8]This may be nonstandard notation
[9]This may be nonstandard notation.

By More on Algebra, Lemma 15.27.3 the pro-systems $\{\text{Tor}_{-p}^{A}(H^q(M^\bullet), A/I^nA)\}$ are zero for $p > 0$. Thus the pro-system $\{H^0(M^\bullet/I^nM^\bullet)\}$ is equal to the pro-system $\{H^0(M^\bullet)/I^nH^0(M^\bullet)\}$ and the lemma is proved. $\square$

09C6 **Lemma 61.29.6.** Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If K is in $D_{cons}(X, \Lambda)$ such that $K \otimes_\Lambda \underline{\Lambda/I}$ has locally constant cohomology sheaves, then K is adic lisse (Definition 61.29.4).

Proof. Write $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. We will use the results of Lemma 61.29.2 without further mention. By Cohomology on Sites, Lemma 21.53.5 we see that K_n has locally constant cohomology sheaves for all n. We have $K_n = \epsilon^{-1}L_n$ some L_n in $D_{ctf}(X_{\acute{e}tale}, \Lambda/I^n)$ with locally constant cohomology sheaves. By Étale Cohomology, Lemma 59.77.7 there exist perfect $M_n \in D(\Lambda/I^n)$ such that L_n is étale locally isomorphic to $\underline{M_n}$. The maps $L_{n+1} \to L_n$ corresponding to $K_{n+1} \to K_n$ induces isomorphisms $L_{n+1} \otimes_{\Lambda/I^{n+1}}^{\mathbf{L}} \underline{\Lambda/I^n} \to L_n$. Looking locally on X we conclude that there exist maps $M_{n+1} \to M_n$ in $D(\Lambda/I^{n+1})$ inducing isomorphisms $M_{n+1} \otimes_{\Lambda/I^{n+1}} \Lambda/I^n \to M_n$, see Cohomology on Sites, Lemma 21.53.3. Fix a choice of such maps. By More on Algebra, Lemma 15.97.4 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^nM^\bullet \to M_n$ in $D(\Lambda/I^n)$ compatible with the transition maps. To finish the proof we will show that K is locally isomorphic to

$$\underline{M^\bullet}^\wedge = \lim \underline{M^\bullet/I^nM^\bullet} = R\lim \underline{M^\bullet/I^nM^\bullet}$$

Let $E^\bullet$ be the dual complex to $M^\bullet$, see More on Algebra, Lemma 15.74.15 and its proof. Consider the objects

$$H_n = R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, K_n) = \underline{E^\bullet/I^nE^\bullet} \otimes_{\Lambda/I^n}^{\mathbf{L}} K_n$$

of $D(X_{pro\text{-}\acute{e}tale}, \Lambda/I^n)$. Modding out by I^n defines a transition map $H_{n+1} \to H_n$. Set $H = R\lim H_n$. Then H is an object of $D_{cons}(X, \Lambda)$ (details omitted) with $H \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n} = H_n$. Choose a covering $\{W_t \to X\}_{t \in T}$ with each W_t affine and weakly contractible. By our choice of $M^\bullet$ we see that

$$\begin{aligned} H_n|_{W_t} &\cong R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, \underline{M^\bullet/I^nM^\bullet}) \\ &= \underline{\text{Tot}(E^\bullet/I^nE^\bullet \otimes_{\Lambda/I^n} M^\bullet/I^nM^\bullet)} \end{aligned}$$

Thus we may apply Lemma 61.29.5 to $H = R\lim H_n$. We conclude the system $H^0(W_t, H_n)$ satisfies Mittag-Leffler. Since for all $n \gg 1$ there is an element of $H^0(W_t, H_n)$ which maps to an isomorphism in

$$H^0(W_t, H_1) = \text{Hom}(\underline{M^\bullet/IM^\bullet}, K_1)$$

we find an element $(\varphi_{t,n})$ in the inverse limit which produces an isomorphism mod I. Then

$$R\lim \varphi_{t,n} : \underline{M^\bullet}^\wedge|_{W_t} = R\lim \underline{M^\bullet/I^nM^\bullet}|_{W_t} \longrightarrow R\lim K_n|_{W_t} = K|_{W_t}$$

is an isomorphism. This finishes the proof. $\square$

09C7 **Proposition 61.29.7.** Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$. Then K is adic constructible (Definition 61.29.4).

Proof. This is a consequence of Lemma 61.29.6 and the fact that a Noetherian scheme is locally connected (Topology, Lemma 5.9.6), and the definitions. $\square$

61.30. Proper base change

09C8 In this section we explain how to prove the proper base change theorem for derived complete objects on the pro-étale site using the proper base change theorem for étale cohomology following the general theme that we use the pro-étale topology only to deal with "limit issues" and we use results proved for the étale topology to handle everything else.

09C9 **Theorem 61.30.1.** Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes giving rise to the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal such that Λ/I is torsion. Let K be an object of $D(X_{pro\text{-}\acute{e}tale})$ such that

(1) K is derived complete, and
(2) $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ is bounded below with cohomology sheaves coming from $X_{\acute{e}tale}$,
(3) Λ/I^n is a perfect Λ-module[10].

Then the base change map

$$Lg^*_{comp} Rf_* K \longrightarrow Rf'_* L(g')^*_{comp} K$$

is an isomorphism.

Proof. We omit the construction of the base change map (this uses only formal properties of derived pushforward and completed derived pullback, compare with Cohomology on Sites, Remark 21.19.3). Write $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. By Lemma 61.20.1 we have $K = R\lim K_n$ because K is derived complete. By Lemmas 61.20.2 and 61.20.1 we can unwind the left hand side

$$Lg^*_{comp} Rf_* K = R\lim Lg^*(Rf_* K) \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n} = R\lim Lg^* Rf_* K_n$$

the last equality because Λ/I^n is a perfect module and the projection formula (Cohomology on Sites, Lemma 21.50.1). Using Lemma 61.20.2 we can unwind the right hand side

$$Rf'_* L(g')^*_{comp} K = Rf'_* R\lim L(g')^* K_n = R\lim Rf'_* L(g')^* K_n$$

the last equality because Rf'_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.23.3). Thus it suffices to show the maps

$$Lg^* Rf_* K_n \longrightarrow Rf'_* L(g')^* K_n$$

are isomorphisms. By Lemma 61.19.8 and our second condition we can write $K_n = \epsilon^{-1} L_n$ for some $L_n \in D^+(X_{\acute{e}tale}, \Lambda/I^n)$. By Lemma 61.23.1 and the fact that ϵ^{-1} commutes with pullbacks we obtain

$$Lg^* Rf_* K_n = Lg^* Rf_* \epsilon^* L_n = Lg^* \epsilon^{-1} Rf_* L_n = \epsilon^{-1} Lg^* Rf_* L_n$$

and

$$Rf'_* L(g')^* K_n = Rf'_* L(g')^* \epsilon^{-1} L_n = Rf'_* \epsilon^{-1} L(g')^* L_n = \epsilon^{-1} Rf'_* L(g')^* L_n$$

[10]This assumption can be removed if K is a constructible complex, see [BS13].

(this also uses that L_n is bounded below). Finally, by the proper base change theorem for étale cohomology (Étale Cohomology, Theorem 59.91.11) we have

$$Lg^*Rf_*L_n = Rf'_*L(g')^*L_n$$

(again using that L_n is bounded below) and the theorem is proved. □

61.31. Change of partial universe

0F4R We advise the reader to skip this section: here we show that cohomology of sheaves in the pro-étale topology is independent of the choice of partial universe. Namely, the functor g_* of Lemma 61.31.2 below is an embedding of small pro-étale topoi which does not change cohomology. For big pro-étale sites we have Lemmas 61.31.3 and 61.31.4 saying essentially the same thing.

But first, as promised in Section 61.12 we prove that the topology on a big pro-étale site $Sch_{pro\text{-}\acute{e}tale}$ is in some sense induced from the pro-étale topology on the category of all schemes.

098J **Lemma 61.31.1.** Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site as in Definition 61.12.7. Let $T \in \mathrm{Ob}(Sch_{pro\text{-}\acute{e}tale})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary pro-étale covering of T. There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{pro\text{-}\acute{e}tale}$ which refines $\{T_i \to T\}_{i \in I}$.

Proof. Namely, we first let $\{V_k \to T\}$ be a covering as in Lemma 61.13.3. Then the pro-étale coverings $\{T_i \times_T V_k \to V_k\}$ can be refined by a finite disjoint open covering $V_k = V_{k,1} \amalg \ldots \amalg V_{k,n_k}$, see Lemma 61.13.1. Then $\{V_{k,i} \to T\}$ is a covering of $Sch_{pro\text{-}\acute{e}tale}$ which refines $\{T_i \to T\}_{i \in I}$. □

We first state and prove the comparison for the small pro-étale sites. Note that we are not claiming that the small pro-étale topos of a scheme is independent of the choice of partial universe; this isn't true in contrast with the case of the small étale topos (Étale Cohomology, Lemma 59.21.2).

098Y **Lemma 61.31.2.** Let S be a scheme. Let $S_{pro\text{-}\acute{e}tale} \subset S'_{pro\text{-}\acute{e}tale}$ be two small pro-étale sites of S as constructed in Definition 61.12.8. Then the inclusion functor satisfies the assumptions of Sites, Lemma 7.21.8. Hence there exist morphisms of topoi

$$Sh(S_{pro\text{-}\acute{e}tale}) \xrightarrow{g} Sh(S'_{pro\text{-}\acute{e}tale}) \xrightarrow{f} Sh(S_{pro\text{-}\acute{e}tale})$$

whose composition is isomorphic to the identity and with $f_* = g^{-1}$. Moreover,

(1) for $\mathcal{F}' \in \mathrm{Ab}(S'_{pro\text{-}\acute{e}tale})$ we have $H^p(S'_{pro\text{-}\acute{e}tale}, \mathcal{F}') = H^p(S_{pro\text{-}\acute{e}tale}, g^{-1}\mathcal{F}')$,
(2) for $\mathcal{F} \in \mathrm{Ab}(S_{pro\text{-}\acute{e}tale})$ we have

$$H^p(S_{pro\text{-}\acute{e}tale}, \mathcal{F}) = H^p(S'_{pro\text{-}\acute{e}tale}, g_*\mathcal{F}) = H^p(S'_{pro\text{-}\acute{e}tale}, f^{-1}\mathcal{F}).$$

Proof. The inclusion functor is fully faithful and continuous. We have seen that $S_{pro\text{-}\acute{e}tale}$ and $S'_{pro\text{-}\acute{e}tale}$ have fibre products and final objects and that our functor commutes with these (Lemma 61.12.10). It follows from Lemma 61.31.1 that the inclusion functor is cocontinuous. Hence the existence of f and g follows from Sites, Lemma 7.21.8. The equality in (1) is Cohomology on Sites, Lemma 21.7.2. Part (2) follows from (1) as $\mathcal{F} = g^{-1}g_*\mathcal{F} = g^{-1}f^{-1}\mathcal{F}$. □

Next, we prove a corresponding result for the big pro-étale topoi.

0F4S Lemma 61.31.3. Suppose given big sites $Sch_{pro\text{-}\acute{e}tale}$ and $Sch'_{pro\text{-}\acute{e}tale}$ as in Definition 61.12.7. Assume that $Sch_{pro\text{-}\acute{e}tale}$ is contained in $Sch'_{pro\text{-}\acute{e}tale}$. The inclusion functor $Sch_{pro\text{-}\acute{e}tale} \to Sch'_{pro\text{-}\acute{e}tale}$ satisfies the assumptions of Sites, Lemma 7.21.8. There are morphisms of topoi

$$\begin{aligned} g : Sh(Sch_{pro\text{-}\acute{e}tale}) &\longrightarrow Sh(Sch'_{pro\text{-}\acute{e}tale}) \\ f : Sh(Sch'_{pro\text{-}\acute{e}tale}) &\longrightarrow Sh(Sch_{pro\text{-}\acute{e}tale}) \end{aligned}$$

such that $f \circ g \cong \mathrm{id}$. For any object S of $Sch_{pro\text{-}\acute{e}tale}$ the inclusion functor $(Sch/S)_{pro\text{-}\acute{e}tale} \to (Sch'/S)_{pro\text{-}\acute{e}tale}$ satisfies the assumptions of Sites, Lemma 7.21.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_{pro\text{-}\acute{e}tale}) &\longrightarrow Sh((Sch'/S)_{pro\text{-}\acute{e}tale}) \\ f : Sh((Sch'/S)_{pro\text{-}\acute{e}tale}) &\longrightarrow Sh((Sch/S)_{pro\text{-}\acute{e}tale}) \end{aligned}$$

with $f \circ g \cong \mathrm{id}$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 7.21.8 are immediate for the functors $Sch_{pro\text{-}\acute{e}tale} \to Sch'_{pro\text{-}\acute{e}tale}$ and $(Sch/S)_{pro\text{-}\acute{e}tale} \to (Sch'/S)_{pro\text{-}\acute{e}tale}$. Property (a) holds by Lemma 61.31.1. Property (d) holds because fibre products in the categories $Sch_{pro\text{-}\acute{e}tale}$, $Sch'_{pro\text{-}\acute{e}tale}$ exist and are compatible with fibre products in the category of schemes. ☐

0F4T Lemma 61.31.4. Let S be a scheme. Let $(Sch/S)_{pro\text{-}\acute{e}tale}$ and $(Sch'/S)_{pro\text{-}\acute{e}tale}$ be two big pro-étale sites of S as in Definition 61.12.8. Assume that the first is contained in the second. In this case

(1) for any abelian sheaf $\mathcal{F}'$ defined on $(Sch'/S)_{pro\text{-}\acute{e}tale}$ and any object U of $(Sch/S)_{pro\text{-}\acute{e}tale}$ we have
$$H^p(U, \mathcal{F}'|_{(Sch/S)_{pro\text{-}\acute{e}tale}}) = H^p(U, \mathcal{F}')$$
In words: the cohomology of $\mathcal{F}'$ over U computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over U.

(2) for any abelian sheaf $\mathcal{F}$ on $(Sch/S)_{pro\text{-}\acute{e}tale}$ there is an abelian sheaf $\mathcal{F}'$ on $(Sch/S)'_{pro\text{-}\acute{e}tale}$ whose restriction to $(Sch/S)_{pro\text{-}\acute{e}tale}$ is isomorphic to $\mathcal{F}$.

Proof. By Lemma 61.31.3 the inclusion functor $(Sch/S)_{pro\text{-}\acute{e}tale} \to (Sch'/S)_{pro\text{-}\acute{e}tale}$ satisfies the assumptions of Sites, Lemma 7.21.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 21.7.2. ☐

61.32. Other chapters

Preliminaries

CHAPTER 62

Relative Cycles

0H4B 62.1. Introduction

0H4C A foundational reference is [SV00].

In this chapter we only define what are called the universally integral relative cycles in [SV00]. This choice makes the theory somewhat simpler to develop than in the original, but of course we also lose something.

Fix a morphism $X \to S$ of finite type between Noetherian schemes. A family α of r-cycles on fibres of X/S is simply a collection $\alpha = (\alpha_s)_{s \in S}$ where $\alpha_s \in Z_r(X_s)$. It is immediately clear how to base change $g^*\alpha$ of α along any morphism $g : S' \to S$. Then we say α is a relative r-cycle on X/S if α is compatible with specializations, i.e., for any morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring, we require the generic fibre of $g^*\alpha$ to specialize to the closed fibre of $g^*\alpha$. See Section 62.6.

62.2. Conventions and notation

0H4D Please consult the chapter on Chow Homology and Chern Classes for our conventions and notation regarding cycles on schemes locally of finite type over a fixed Noetherian base, see Chow Homology, Section 42.7 ff.

In particular, if X is locally of finite type over a field k, then $Z_r(X)$ denotes the group of cycles of dimension r, see Chow Homology, Example 42.7.2 and Section 42.8. Given an integral closed subscheme $Z \subset X$ with $\dim(Z) = r$ we have $[Z] \in Z_r(X)$ and if X is quasi-compact, then $Z_r(X)$ is free abelian on these classes.

62.3. Cycles relative to fields

0H4E Let k be a field. Let X be a locally algebraic scheme over k. Let $r \geq 0$ be an integer. In this setting we have the group $Z_r(X)$ of r-cycles on X, see Section 62.2.

Base change. For any field extension k'/k there is a base change map $Z_r(X) \to Z_r(X_{k'})$, see Chow Homology, Section 42.67. Namely, given an integral closed subscheme $Z \subset X$ of dimension r we send $[Z] \in Z_r(X)$ to the r-cycle $[Z_{k'}]_r \in Z_r(X_{k'})$ associated to the closed subscheme $Z_{k'} \subset X_{k'}$ (of course in general $Z_{k'}$ is neither irreducible nor reduced). The base change map $Z_r(X) \to Z_r(X_{k'})$ is always injective.

0H4F **Lemma 62.3.1.** Let K/k be a field extension. Let Z be an integral locally algebraic scheme over k. The multiplicity m_{Z',Z_K} of an irreducible component $Z' \subset Z_K$ is 1 or a power of the characteristic of k.

Proof. If the characteristic of k is zero, then k is perfect and the multiplicity is always 1 since X_K is reduced by Varieties, Lemma 33.6.4. Assume the characteristic of k is $p > 0$. Let L be the function field of Z. Since Z is locally algebraic over k, the field extension L/k is finitely generated. The ring $K \otimes_k L$ is Noetherian (Algebra, Lemma 10.31.8). Translated into algebra, we have to show that the length of the artinian local ring $(K \otimes_k L)_{\mathfrak q}$ is a power of p for every minimal prime ideal $\mathfrak q$.

Let L'/L be a finite purely inseparable extension, say of degree p^n. Then $K \otimes_k L \subset K \otimes_k L'$ is a finite free ring map of degree p^n which induces a homeomorphism on spectra and purely inseparable residue field extensions. Hence for every minimal prime $\mathfrak q$ as above there is a unique minimal prime $\mathfrak q' \subset K \otimes_k L'$ lying over it and

$$p^n \text{length}((K \otimes_k L)_{\mathfrak q}) = [\kappa(\mathfrak q') : \kappa(\mathfrak q)]\text{length}((K \otimes_k L')_{\mathfrak q'})$$

by Algebra, Lemma 10.52.12 applied to $M = (K \otimes_k L')_{\mathfrak q'} \cong (K \otimes_k L)_{\mathfrak q}^{\oplus p^n}$. Since $[\kappa(\mathfrak q') : \kappa(\mathfrak q)]$ is a power of p we conclude that it suffices to prove the statement for L' and $\mathfrak q'$.

By the previous paragraph and Algebra, Lemma 10.45.3 we may assume that we have a subfield $L/k'/k$ such that L/k' is separable and k'/k is finite purely inseparable. Then $K \otimes_k k'$ is an Artinian local ring. The argument of the preceding paragraph (applied to $L = k$ and $L' = k'$) shows that $\text{length}(K \otimes_k k')$ is a power of p. Since L/k' is the localization of a smooth k'-algebra (Algebra, Lemma 10.158.10). Hence $S = (K \otimes_k L)_{\mathfrak q}$ is the localization of a smooth $R = K \otimes_k k'$-algebra at a minimal prime. Thus $R \to S$ is a flat local homomorphism of Artinian local rings and $\mathfrak m_R S = \mathfrak m_S$. It follows from Algebra, Lemma 10.52.13 that $\text{length}(K \otimes_k k') = \text{length}(R) = \text{length}(S) = \text{length}((K \otimes_k L)_{\mathfrak q})$ and the proof is finished. $\square$

0H4G Lemma 62.3.2. Let k be a field of characteristic $p > 0$ with perfect closure k^{perf}. Let X be an algebraic scheme over k. Let $r \geq 0$ be an integer. The cokernel of the injective map $Z_r(X) \to Z_r(X_{k^{perf}})$ is a p-power torsion module (More on Algebra, Definition 15.88.1).

Proof. Since X is quasi-compact, the abelian group $Z_r(X)$ is free with basis given by the integral closed subschemes of dimension r. Similarly for $Z_r(X_{k^{perf}})$. Since $X_{k^{perf}} \to X$ is a homeomorphism, it follows that $Z_r(X) \to Z_r(X_{k^{perf}})$ is injective with torsion cokernel. Every element in the cokernel is p-power torsion by Lemma 62.3.1. $\square$

62.4. Specialization of cycles

0H4H Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. There is a specialization map

$$sp_{X/R} : Z_r(X_K) \longrightarrow Z_r(X_\kappa)$$

defined as follows. For an integral closed subscheme $Z \subset X_K$ of dimension r we denote $\overline{Z}$ the scheme theoretic image of $Z \to X$. Then we let $sp_{X/R}$ be the unique $\mathbf{Z}$-linear map such that

$$sp_{X/R}([Z]) = [\overline{Z}_\kappa]_r$$

We briefly discuss why this is well defined. First, observe that the morphism $X_K \to X$ is quasi-compact and hence the morphism $Z \to X$ is quasi-compact. Thus taking the scheme theoretic image of $Z \to X$ commutes with flat base change by

Morphisms, Lemma 29.25.16. In particular, base changing back to X_K we see that $Z = \overline{Z}_K$. Since Z is integral, of course $\overline{Z}$ is integral too and in fact is equal to the unique integral closed subscheme whose generic point is the (image of the) generic point of Z. It follows from Varieties, Lemma 33.19.2 that Z_κ is equidimensional of dimension r.

0H4I Lemma 62.4.1. Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over R. Assume $\dim(\text{Supp}(\mathcal{F}_K)) \leq r$. Then $\dim(\text{Supp}(\mathcal{F}_\kappa)) \leq r$ and

$$sp_{X/R}([\mathcal{F}_K]_r) = [\mathcal{F}_\kappa]_r$$

Proof. The statement on dimension follows from More on Morphisms, Lemma 37.18.4. Let x be a generic point of an integral closed subscheme $Z \subset X_\kappa$ of dimension r. To finish the proof we wil show that the coefficient of $[Z]$ in the left (L) and right hand side (R) of equality are the same.

Let $A = \mathcal{O}_{X,x}$ and $M = \mathcal{F}_x$. Observe that M is a finite A-module flat over R. Let $\pi \in R$ be a uniformizer so that $A/\pi A = \mathcal{O}_{X_\kappa,x}$. By Chow Homology, Lemma 42.3.2 we have

$$\sum\nolimits_i \text{length}_A(A/(\pi, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) = \text{length}_A(M/\pi M)$$

where the sum is over the minimal primes $\mathfrak{q}_i$ in the support of M. Since π is a nonzerodivisor on M we see that $\pi \notin \mathfrak{q}_i$ and hence these primes correspond to those generic points $y_i \in X_K$ of the support of $\mathcal{F}_K$ which specialize to our chosen $x \in X_\kappa$. Thus the left hand side is the coefficient of $[Z]$ in (L). Of course $\text{length}_A(M/\pi M)$ is the coefficient of $[Z]$ in (R). This finishes the proof. $\square$

0H4J Lemma 62.4.2. Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. Let $W \subset X$ be a closed subscheme flat over R. Assume $\dim(W_K) \leq r$. Then $\dim(W_\kappa) \leq r$ and

$$sp_{X/R}([W_K]_r) = [W_\kappa]_r$$

Proof. Taking $\mathcal{F} = \mathcal{O}_W$ this is a special case of Lemma 62.4.1. See Chow Homology, Lemma 42.10.3. $\square$

0H4K Lemma 62.4.3. Let R'/R be an extension of discrete valuation rings inducing fraction field extension K'/K and residue field extension κ'/κ (More on Algebra, Definition 15.111.1). Let X be locally of finite type over R. Denote $X' = X_{R'}$. Then the diagram

$$\begin{tikzcd}
Z_r(X'_{K'}) \arrow[r, "sp_{X'/R'}"'] & Z_r(X'_{\kappa'}) \\
Z_r(X_K) \arrow[u] \arrow[r, "sp_{X/R}"] & Z_r(X_\kappa) \arrow[u]
\end{tikzcd}$$

commutes where $r \geq 0$ and the vertical arrows are base change maps.

Proof. Observe that $X'_{K'} = X_{K'} = X_K \times_{\text{Spec}(K)} \text{Spec}(K')$ and similarly for closed fibres, so that the vertical arrows indeed make sense (see Section 62.3). Now if $Z \subset X_K$ is an integral closed subscheme with scheme theoretic image $\overline{Z} \subset X$, then we see that $Z_{K'} \subset X_{K'}$ is a closed subscheme with scheme theoretic image $\overline{Z}_{R'} \subset X_{R'}$. The base change of $[Z]$ is $[Z_{K'}]_r = [\overline{Z}_{K'}]_r$ by definition. We have

$$sp_{X/R}([Z]) = [\overline{Z}_\kappa]_r \quad \text{and} \quad sp_{X'/R'}([\overline{Z}_{K'}]_r) = [(\overline{Z}_{R'})_{\kappa'}]_r$$

by Lemma 62.4.1. Since $(\overline{Z}_{R'})_{\kappa'} = (\overline{Z}_\kappa)_{\kappa'}$ we conclude. □

0H4L Lemma 62.4.4. Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $f : X' \to X$ be a morphism which is locally of finite type, flat, and of relative dimension e. Then the diagram

$$\begin{tikzcd} Z_{r+e}(X'_K) \arrow[r, "sp_{X'/R}"'] & Z_{r+e}(X'_\kappa) \\ Z_r(X_K) \arrow[u] \arrow[r, "sp_{X/R}"] & Z_r(X_\kappa) \arrow[u] \end{tikzcd}$$

commutes where $r \geq 0$ and the vertical arrows are given by flat pullback.

Proof. Let $Z \subset X$ be an integral closed subscheme dominating R. By the construction of $sp_{X/R}$ we have $sp_{X/R}([Z_K]) = [Z_\kappa]_r$ and this characterizes the specialization map. Set $Z' = f^{-1}(Z) = X' \times_X Z$. Since R is a valuation ring, Z is flat over R. Hence Z' is flat over R and $sp_{X'/R}([Z'_K]_{r+e}) = [Z'_\kappa]_{r+e}$ by Lemma 62.4.2. Since by Chow Homology, Lemma 42.14.4 we have $f_K^*[Z_K] = [Z'_K]_{r+e}$ and $f_\kappa^*[Z_\kappa]_r = [Z'_\kappa]_{r+e}$ we win. □

0H4M Lemma 62.4.5. Let R be a discrete valuation ring with fraction field K and residue field κ. Let $f : X \to Y$ be a proper morphism of schemes locally of finite type over R. Then the diagram

$$\begin{tikzcd} Z_r(X_K) \arrow[r, "sp_{X/R}"'] \arrow[d] & Z_r(X_\kappa) \arrow[d] \\ Z_r(Y_K) \arrow[r, "sp_{Y/R}"] & Z_r(Y_\kappa) \end{tikzcd}$$

commutes where $r \geq 0$ and the vertical arrows are given by proper pushforward.

Proof. Let $Z \subset X$ be an integral closed subscheme dominating R. By the construction of $sp_{X/R}$ we have $sp_{X/R}([Z_K]) = [Z_\kappa]_r$ and this characterizes the specialization map. Set $Z' = f(Z) \subset Y$. Then Z' is an integral closed subscheme of Y dominating R. Thus $sp_{Y/R}([Z'_K]) = [Z'_\kappa]_r$.

We can think of $[Z]$ as an element of $Z_{r+1}(X)$. By definition we have $f_*[Z] = 0$ if $\dim(Z') < r + 1$ and $f_*[Z] = d[Z']$ if $Z \to Z'$ is generically finite of degree d. Since proper pushforward commutes with flat pullback by $Y_K \to Y$ (Chow Homology, Lemma 42.15.1) we see that correspondingly $f_{K,*}[Z_K] = 0$ or $f_{K,*}[Z_K] = d[Z'_K]$. Let us apply Chow Homology, Lemma 42.29.8 to the commutative diagram

$$\begin{tikzcd} X_\kappa \arrow[r, "i"'] \arrow[d] & X \arrow[d] \\ Y_\kappa \arrow[r, "j"] & Y \end{tikzcd}$$

We obtain that $f_{\kappa,*}[Z_\kappa]_r = 0$ or $f_{\kappa,*}[Z_\kappa] = d[Z'_\kappa]_r$ because clearly $i^*[Z] = [Z_k]_r$ and $j^*[Z'] = [Z'_\kappa]_r$. Putting everything together we conclude. □

62.5. Families of cycles on fibres

0H4N Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. A family α of r-cycles on fibres of X/S is a family

$$\alpha = (\alpha_s)_{s \in S}$$

indexed by the points s of the scheme S where $\alpha_s \in Z_r(X_s)$ is an r cycle on the scheme theoretic fibre X_s of f at s. There are various constructions we can perform on families of r-cycles on fibres.

Base change. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a catesian square of morphisms of schemes with f locally of finite type. Let $r \geq 0$ be an integer. Given a family α of r-cycles on fibres of X/S we define the base change $g^*\alpha$ of α to be the family

$$g^*\alpha = (\alpha'_{s'})_{s' \in S'}$$

where $\alpha'_{s'} \in Z_r(X'_{s'})$ is the base change of the cycle α_s with $s' = g(s)$ as in Section 62.3 via the identitification $X'_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ of scheme theoretic fibres.

Restriction. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $U \subset X$ and $V \subset S$ be open subschemes with $f(U) \subset V$. Given a family α of r-cycles on fibres of X/S we can define the restriction $\alpha|_U$ of α to be the family of r-cycles on fibres of U/V

$$\alpha|_U = (\alpha_s|_{U_s})_{s \in V}$$

of restrictions to scheme theoretic fibres.

Flat pullback. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $r, e \geq 0$ be integers. Let $f : X' \to X$ be a flat morphism, locally of finite type, and of relative dimension e. Given a family α of r-cycles on fibres of X/S we define the flat pullback $f^*\alpha$ of α to be the family of $(r+e)$-cycles on fibres

$$f^*\alpha = (f_s^*\alpha_s)_{s \in S}$$

where $f_s^*\alpha_s \in Z_{r+e}(X'_s)$ is the flat pullback of the cycle α_s in $Z_r(X_s)$ by the flat morphism $f_s : X'_s \to X_s$ of relative dimension e of scheme theoretic fibres.

Proper pushforward. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes with X and Y locally of finite type over S and f proper. Let $r \geq 0$ be an integer. Given a family α of r-cycles on fibres of X/S we define the proper pushforward $f_*\alpha$ of α to be the family of r-cycles on fibres of Y/S by

$$f_*\alpha = (f_{s,*}\alpha_s)_{s \in S}$$

where $f_{s,*}\alpha_s \in Z_r(Y_s)$ is the proper pushforward of the cycle α_s in $Z_r(X_s)$ by the proper morphism $f_s : X_s \to Y_s$ of scheme theoretic fibres.

0H4P Lemma 62.5.1. We have the following compatibilities between the operations above: (1) base change is functorial, (2) restriction is a combination of base change and (a special case of) flat pullback, (3) flat pullback commutes with base change, (4) flat pullback is functorial, (5) proper pushforward commutes with base change, (6) proper pushforward is functorial, and (7) proper pushforward commutes with flat pullback.

Proof. Each of these compatibilities follows directly from the corresponding results proved in the chapter on Chow homology applied to the fibres over S of the schemes in question. We omit the precise statements and the detailed proofs. Here are some references. Part (1): Chow Homology, Lemma 42.67.9. Part (2): Obvious. Part (3): Chow Homology, Lemma 42.67.5. Part (4): Chow Homology, Lemma 42.14.3. Part (5): Chow Homology, Lemma 42.67.6. Part (6): Chow Homology, Lemma 42.12.2. Part (7): Chow Homology, Lemma 42.15.1. □

0H4Q Example 62.5.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For $s \in S$ denote $\mathcal{F}_s$ the pullback of $\mathcal{F}$ to X_s. Assume $\dim(\text{Supp}(\mathcal{F}_s)) \leq r$ for all $s \in S$. Then we can associate to $\mathcal{F}$ the family $[\mathcal{F}/X/S]_r$ of r-cycles on fibres of X/S defined by the formula

$$[\mathcal{F}/X/S]_r = ([\mathcal{F}_s]_r)_{s \in S}$$

where $[\mathcal{F}_s]_r$ is given by Chow Homology, Definition 42.10.2.

0H4R Lemma 62.5.3. The construction in Example 62.5.2 is compatible with base change, restriction, and flat pullback.

Proof. See Chow Homology, Lemmas 42.67.3 and 42.14.4. □

0H4S Example 62.5.4. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $Z \subset X$ be a closed subscheme. For $s \in S$ denote Z_s the inverse image of Z in X_s or equivalently the scheme theoretic fibre of Z at s viewed as a closed subscheme of X_s. Assume $\dim(Z_s) \leq r$ for all $s \in S$. Then we can associate to Z the family $[Z/X/S]_r$ of r-cycles on fibres of X/S defined by the formula

$$[Z/X/S]_r = ([Z_s]_r)_{s \in S}$$

where $[Z_s]_r$ is given by Chow Homology, Definition 42.9.2.

0H4T Lemma 62.5.5. The construction in Example 62.5.4 is compatible with base change, restriction, and flat pullback.

Proof. Taking $\mathcal{F} = (Z \to X)_*\mathcal{O}_Z$ this is a special case of Lemma 62.5.3. See Chow Homology, Lemma 42.10.3. □

0H4U Remark 62.5.6 (Support). Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let α be a family of r-cycles on fibres of X/S. We define the support of α to be

$$\text{Supp}(\alpha) = \bigcup\nolimits_{s \in S} \text{Supp}(\alpha_s) \subset X$$

Here $\text{Supp}(\alpha_s) \subset X_s$ is the support of the cycle α_s, see Chow Homology, Definition 42.8.3. The support $\text{Supp}(\alpha)$ is rarely a closed subset of X.

0H4V **Lemma 62.5.7.** Taking the support as in Remark 62.5.6 is compatible with base change, restriction, and flat pullback.

Proof. Omitted. □

0H4W **Lemma 62.5.8.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $g : S' \to S$ be a surjective morphism of schemes. Set $S'' = S' \times_S S'$ and let $f' : X' \to S'$ and $f'' : X'' \to S''$ be the base changes of f. Let $x \in X$ with $\text{trdeg}_{\kappa(f(x))}(\kappa(x)) = r$.

(1) There exists an $x' \in X'$ mapping to x with $\text{trdeg}_{\kappa(f'(x'))}(\kappa(x')) = r$.
(2) If $x'_1, x'_2 \in X'$ are both as in (1), then there exists an $x'' \in X''$ with $\text{trdeg}_{\kappa(f''(x''))}(\kappa(x'')) = r$ and $\text{pr}_i(x'') = x'_i$.

Proof. Part (1) is Morphisms, Lemma 29.28.3. Let x'_1, x'_2 be as in (2). Then since $X'' = X' \times_X X'$ we see that there exists a $x'' \in X''$ mapping to both x'_1 and x'_2 (see for example Descent, Lemma 35.13.1). Denote $s'' \in S''$, $s'_i \in S'$, and $s \in S$ the images of x'', x'_i, and x. Denote $k = \kappa(s)$ and let $Z \subset X_k$ be the integral closed subscheme whose generic point is x. Then x'_i is a generic point of an irreducible component of $Z_{\kappa(s'_i)}$. Let $Z'' \subset Z_{\kappa(s'')}$ be an irreducible component containing x''. Denote $\xi'' \in Z''$ the generic point. Since $\xi'' \rightsquigarrow x''$ we see that ξ'' must also map to x'_i under the two projections. On the other hand, we see that $\text{trdeg}_{\kappa(s'')}(\kappa(\xi'')) = r$ because it is a generic point of an irreducible component of the base change of Z. □

0H4X **Lemma 62.5.9.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $g : S' \to S$ be a morphism of schemes and $X' = S' \times_S X$. Assume that for every $s \in S$ there exists a point $s' \in S'$ with $g(s') = s$ and such that $\kappa(s')/\kappa(s)$ is a separable extension of fields. Then

(1) For families α_1 and α_2 of r-cycles on fibres of X/S if $g^*\alpha_1 = g^*\alpha_2$, then $\alpha_1 = \alpha_2$.
(2) Given a family α' of r-cycles on fibres of X'/S' if $\text{pr}_1^*\alpha' = \text{pr}_2^*\alpha'$ as families of r-cycles on fibres of $(S' \times_S S') \times_S X/(S' \times_S S')$, then there is a unique family α of r-cycles on fibres of X/S such that $g^*\alpha = \alpha'$.

Proof. Part (1) follows from the injectivity of the base change map discussed in Section 62.3. (This argument works as long as $S' \to S$ is surjective.)

Let α' be as in (2). Denote $\alpha'' = \text{pr}_1^*\alpha' = \text{pr}_2^*\alpha'$ the common value.

Let $(X/S)^{(r)}$ be the set of $x \in X$ with $\text{trdeg}_{\kappa(f(x))}(\kappa(x)) = r$ and similarly define $(X'/S')^{(r)}$ and $(X''/S'')^{(r)}$ Taking coefficients, we may think of α' and α'' as functions $\alpha' : (X'/S')^{(r)} \to \mathbf{Z}$ and $\alpha'' : (X''/S'')^{(r)} \to \mathbf{Z}$. Given a function

$$\varphi : (X/S)^{(r)} \to \mathbf{Z}$$

we define $g^*\varphi : (X'/S')^{(r)} \to \mathbf{Z}$ by analogy with our base change operation. Namely, say $x' \in (X'/S')^{(r)}$ maps to $x \in X$, $s' \in S'$, and $s \in Z$. Denote $Z' \subset X'_{s'}$ and $Z \subset X_s$ the integral closed subschemes with generic points x' and x. Note that $\dim(Z') = r$. If $\dim(Z) < r$, then we set $(g^*\varphi)(x') = 0$. If $\dim(Z) = r$, then Z' is an irreducible component of $Z_{s'}$ and hence has a multiplicity $m_{Z', Z_{s'}}$. Call this $m(x', g)$. Then we define

$$(g^*\varphi)(x') = m(x', g)\varphi(x)$$

Note that the coefficients $m(x', g)$ are always positive integers (see for example Lemma 62.3.1). We similarly have base change maps

$$\mathrm{pr}_1^*, \mathrm{pr}_2^* : \mathrm{Map}((X'/S')^{(r)}, \mathbf{Z}) \longrightarrow \mathrm{Map}((X''/S'')^{(r)}, \mathbf{Z})$$

It follows from the associativity of base change that we have $\mathrm{pr}_1^* \circ g^* = \mathrm{pr}_2^* \circ g^*$ (small detail omitted). To be explicity, in terms of the maps of sets this equality just means that for $x'' \in (X''/S'')^{(r)}$ we have

$$m(x'', \mathrm{pr}_1)m(\mathrm{pr}_1(x''), g) = m(x'', \mathrm{pr}_2)m(\mathrm{pr}_2(x''), g)$$

provided that $\mathrm{pr}_1(x'')$ and $\mathrm{pr}_2(x'')$ are in $(X''/S'')^{(r)}$. By Lemma 62.5.8 and an elementary argument[1] using the previous displayed equation, it follows that there exists a unique map

$$\alpha : (X/S)^{(r)} \to \mathbf{Q}$$

such that $g^*\alpha = \alpha'$. To finish the proof it suffices to show that α has integer values (small detail omitted: one needs to see that α determines a locally finite sum on each fibre which follows from the corresponding fact for α'). Given any $x \in (X/S)^{(r)}$ with image $s \in S$ we can pick a point $s' \in S'$ such that $\kappa(s')/\kappa(s)$ is separable. Then we may choose $x' \in (X'/S')^{(r)}$ mapping to s and x and we see that $m(x', g) = 1$ because $Z_{s'}$ is reduced in this case. Whence $\alpha(x) = \alpha'(x')$ is an integer. □

0H4Y Lemma 62.5.10. Let $g : S' \to S$ be a bijective morphism of schemes which induces isomorphisms of residue fields. Let $f : X \to S$ be locally of finite type. Set $X' = S' \times_S X$. Let $r \geq 0$. Then base change by g determines a bijection between the group of families of r-cycles on fibres of X/S and the group of families of r-cycles on fibres of X'/S'.

Proof. Omitted. □

62.6. Relative cycles

0H4Z Here is the definition we will work with; see Section 62.15 for a comparison with the definitions in [SV00].

0H50 Definition 62.6.1. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. A relative r-cycle on X/S is a family α of r-cycles on fibres of X/S such that for every morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring we have

$$sp_{X'/S'}(\alpha_\eta) = \alpha_0$$

where $sp_{X'/S'}$ is as in Section 62.4 and α_η (resp. α_0) is the value of the base change $g^*\alpha$ of α at the generic (resp. closed) point of S'. The group of all relative r-cycles on X/S is denoted $z(X/S, r)$.

0H51 Lemma 62.6.2. Let α be a relative r-cycle on X/S as in Definition 62.6.1. Then any restriction, base change, flat pullback, or proper pushforward of α is a relative r-cycle.

[1]Given $x \in (X/S)^{(r)}$ pick $x' \in (X'/S')^{(r)}$ mapping to x and set $\alpha(x) = \alpha'(x')/m(x', g)$. This is well defined by the formula and the lemma.

Proof. For flat pullback use Lemma 62.4.4. Restriction is a special case of flat pullback. To see it holds for base change use that base change is transitive. For proper pushforward use Lemma 62.4.5. □

0H52 Lemma 62.6.3. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a family of r-cycles on fibres of X/S. Let $\{g_i : S_i \to S\}$ be a h covering (More on Flatness, Definition 38.34.2). Then α is a relative r-cycle if and only if each base change $g_i^*\alpha$ is a relative r-cycle.

Proof. If α is a relative r-cycle, then each base change $g_i^*\alpha$ is a relative r-cycle by Lemma 62.6.2. Assume each $g_i^*\alpha$ is a relative r-cycle. Let $g : S' \to S$ be a morphism where S' is the spectrum of a discrete valuation ring. After replacing S by S', X by $X' = X \times_S S'$, and α by $\alpha' = g^*\alpha$ and using that the base change of a h covering is a h covering (More on Flatness, Lemma 38.34.9) we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with closed point 0 and generic point η. We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Since a h covering is a V covering (by definition), there is an i and a specialization $s' \rightsquigarrow s$ of points of S_i with $g_i(s') = \eta$ and $g_i(s) = 0$, see Topologies, Lemma 34.10.13. By Properties, Lemma 28.5.10 we can find a morphism $h : S' \to S_i$ from the spectrum S' of a discrete valuation ring which maps the generic point η' to s' and maps the closed point $0'$ to s. Denote $\alpha' = h^*g_i^*\alpha$. By assumption we have $sp_{X'/S'}(\alpha'_{\eta'}) = \alpha'_{0'}$. Since $g = g_i \circ h : S' \to S$ is the morphism of schemes induced by an extension of discrete valuation rings we conclude that $sp_{X/S}$ and $sp_{X'/S'}$ are compatible with base change maps on the fibres, see Lemma 62.4.3. We conclude that $sp_{X/S}(\alpha_\eta) = \alpha_0$ because the base change map $Z_r(X_0) \to Z_r(X'_{0'})$ is injective as discussed in Section 62.3. □

0H53 Lemma 62.6.4. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a family of r-cycles on fibres of X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is a relative r-cycle if and only if each flat pullback $f_i^*\alpha$ is a relative r-cycle.

Proof. If α is a relative r-cycle, then each pull back $f_i^*\alpha$ is a relative r-cycle by Lemma 62.6.2. Assume each $f_i^*\alpha$ is a relative r-cycle. Let $g : S' \to S$ be a morphism where S' is the spectrum of a discrete valuation ring. After replacing S by S', X by $X' = X \times_S S'$, and α by $\alpha' = g^*\alpha$ we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with closed point 0 and generic point η. We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Denote $f_{i,0} : X_{i,0} \to X_0$ the base change of f_i to the closed point of S. Similarly for $f_{i,\eta}$. Observe that

$$f_{i,0}^* sp_{X/S}(\alpha_\eta) = sp_{X_i/S}(f_{i,\eta}^*\alpha_\eta) = f_{i,0}^*\alpha_0$$

Namely, the first equality holds by Lemma 62.4.4 and the second by assumption. Since the family of maps $f_{i,0}^* : Z_r(X_0) \to Z_r(X_{i,0})$ is jointly injective (due to the fact that $f_{i,0}$ is jointly surjective), we conclude what we want. □

0H54 Lemma 62.6.5. Let S be a locally Noetherian scheme. Let $i : X \to Y$ be a closed immersion of schemes locally of finite type over S. Let $r \geq 0$. Let α be a family of

r-cycles on fibres of X/S. Then α is a relative r-cycle on X/S if and only if $i_*\alpha$ is a relative r-cycle on Y/S.

Proof. Since base change commutes with i_* (Lemma 62.5.1) it suffices to prove the following: if S is the spectrum of a discrete valuation ring with generic point η and closed point 0, then $sp_{X/S}(\alpha_\eta) = \alpha_0$ if and only if $sp_{Y/S}(i_{\eta,*}\alpha_\eta) = i_{0,*}\alpha_0$. This is true because $i_{0,*} : Z_r(X_0) \to Z_r(Y_0)$ is injective and because $i_{0,*}sp_{X/S}(\alpha_\eta) = sp_{Y/S}(i_{\eta,*}\alpha_\eta)$ by Lemma 62.4.5. □

The following lemma will be strengthened in Lemma 62.6.12.

0H55 Lemma 62.6.6. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f locally of finite type. Let $r \geq 0$. Let α and β be relative r-cycles on X/S. The following are equivalent

(1) $\alpha = \beta$, and
(2) $\alpha_\eta = \beta_\eta$ for any generic point $\eta \in S$ of an irreducible component of S.

Proof. The implication (1) ⇒ (2) is immediate. Assume (2). For every $s \in S$ we can find an η as in (2) which specializes to s. By Properties, Lemma 28.5.10 we can find a morphism $g : S' \to S$ from the spectrum S' of a discrete valuation ring which maps the generic point η' to η and maps the closed point 0 to s. Then α_s and β_s are elements of $Z_r(X_s)$ which base change to the same element of $Z_r(X_{0'})$, namely $sp_{X_{S'}/S'}(\alpha_{\eta'})$ where $\alpha_{\eta'}$ is the base change of α_η. Since the base change map $Z_r(X_s) \to Z_r(X_{0'})$ is injective as discussed in Section 62.3 we conclude $\alpha_s = \beta_s$. □

0H56 Lemma 62.6.7. In the situation of Example 62.5.2 assume S is locally Noetherian and $\mathcal{F}$ is flat over S in dimensions $\geq r$ (More on Flatness, Definition 38.20.10). Then $[\mathcal{F}/X/S]_r$ is a relative r-cycle on X/S.

Proof. By More on Flatness, Lemma 38.20.9 the hypothesis on $\mathcal{F}$ is preserved by any base change. Also, formation of $[\mathcal{F}/X/S]_r$ is compatible with any base change by Lemma 62.5.3. Since the condition of being compatible with specializations is checked after base change to the spectrum of a discrete valuation ring, this reduces us to the case where S is the spectrum of a valuation ring. In this case the set $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X by More on Flatness, Lemma 38.13.11. Since the complement of U in X has fibres of dimension $< r$ over S by assumption, we see that restriction along the inclusion $U \subset X$ induces an isomorphism on the groups of r-cycles on fibres after any base change, compatible with specialization maps and with formation of the relative cycle associated to $\mathcal{F}$. Thus it suffices to show compability with specializations for $[\mathcal{F}|_U/U/S]_r$. Since $\mathcal{F}|_U$ is flat over S, this follows from Lemma 62.4.1 and the definitions. □

0H57 Lemma 62.6.8. In the situation of Example 62.5.4 assume S is locally Noetherian and Z is flat over S in dimensions $\geq r$. Then $[Z/X/S]_r$ is a relative r-cycle on X/S.

Proof. The assumption means that $\mathcal{O}_Z$ is flat over S in dimensions $\geq r$. Thus applying Lemma 62.6.7 with $\mathcal{F} = (Z \to X)_*\mathcal{O}_Z$ we conclude. □

Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism which is of finite type. Let $r \geq 0$. Denote $Hilb(X/S, r)$ the set of closed subschemes $Z \subset X$ such that $Z \to S$ is flat and of relative dimension $\leq r$. By Lemma 62.6.8 for each

$Z \in Hilb(X/S, r)$ we have an element $[Z/X/S]_r \in z(X/S, r)$. Thus we obtain a group homomorphism

0H58 (62.6.8.1) $$\text{free abelian group on } Hilb(X/S, r) \longrightarrow z(X/S, r)$$

sending $\sum n_i[Z_i]$ to $\sum n_i[Z_i/X/S]_r$. A key feature of relative r-cycles is that they are locally (on X and S in suitable topologies) in the image of this map.

0H59 Lemma 62.6.9. Let $f : X \to S$ be a finite type morphism of schemes with S Noetherian. Let $r \geq 0$. Let α be a relative r-cycle on X/S. Then there is a proper, completely decomposed (More on Morphisms, Definition 37.78.1) morphism $g : S' \to S$ such that $g^*\alpha$ is in the image of (62.6.8.1).

Proof. By Noetherian induction, we may assume the result holds for the pullback of α by any closed immersion $g : S' \to S$ which is not an isomorphism.

Let $S_1 \subset S$ be an irreducible component (viewed as an integral closed subscheme). Let $S_2 \subset S$ be the closure of the complement of S' (viewed as a reduced closed subscheme). If $S_2 \neq \emptyset$, then the result holds for the pullback of α by $S_1 \to S$ and $S_2 \to S$. If $g_1 : S'_1 \to S_1$ and $g_2 : S'_2 \to S_2$ are the corresponding completely decomposed proper morphisms, then $S' = S'_1 \amalg S'_2 \to S$ is a completely decomposed proper morphism and we see the result holds for S[2] . Thus we may assume $S' \to S$ is bijective and we reduce to the case described in the next paragraph.

Assume S is integral. Let $\eta \in S$ be the generic point and let $K = \kappa(\eta)$ be the function field of S. Then α_η is an r-cycle on X_K. Write $\alpha_\eta = \sum n_i[Y_i]$. Taking the closure of Y_i we obtain integral closed subschemes $Z_i \subset X$ whose base change to η is Y_i. By generic flatness (for example Morphisms, Proposition 29.27.1), we see that Z_i is flat over a nonempty open U of S for each i. Applying More on Flatness, Lemma 38.31.1 we can find a U-admissible blowing up $g : S' \to S$ such that the strict transform $Z'_i \subset X_{S'}$ of Z_i is flat over S'. Then $\beta = \sum n_i[Z'_i/X_{S'}/S']_r$ is in the image of (62.6.8.1) and $\beta = g^*\alpha$ by Lemma 62.6.6.

However, this does not finish the proof as $S' \to S$ may not be completely decomposed. This is easily fixed: denoting $T \subset S$ the complement of U (viewed as a closed subscheme), by Noetherian induction we can find a completely decomposed proper morphism $T' \to T$ such that $(T' \to S)^*\alpha$ is in the image of (62.6.8.1). Then $S' \amalg T' \to S$ does the job. $\square$

0H5A Lemma 62.6.10. Let $f : X \to S$ be a finite type morphism of schemes with S the spectrum of a discrete valuation ring. Let $r \geq 0$. Then (62.6.8.1) is surjective.

Proof. This of course follows from Lemma 62.6.9 but we can also see it directly as follows. Say α is a relative r-cycle on X/S. Write $\alpha_\eta = \sum n_i[Z_i]$ (the sum is finite). Denote $\overline{Z}_i \subset X$ the closure of Z_i as in Section 62.4. Then $\alpha = \sum n_i[\overline{Z}_i/X/S]$. $\square$

0H5B Lemma 62.6.11. Let $f : X \to S$ be a morphism of schemes. Let $r \geq 0$. Assume S locally Noetherian and f smooth of relative dimension r. Let $\alpha \in z(X/S, r)$. Then the support of α is open and closed in X (see proof for a more precise result).

---

[2]Namely, any closed subscheme of $S'_1 \times_S X$ flat and of relative dimension $\leq r$ over S'_1 may be viewed as a closed subscheme of $S' \times_S X$ flat and of relative dimension $\leq r$ over S'.

Proof. Let $x \in X$ with image $s \in S$. Since f is smooth, there is a unique irreducible component $Z(x)$ of X_s which contains x. Then $\dim(Z(x)) = r$. Let n_x be the coefficient of $Z(x)$ in the cycle α_s. We will show the function $x \mapsto n_x$ is locally constant on X.

Let $g : S' \to S$ be a morphism of locally Noetherian schemes. Let X' be the base change of X and let $\alpha' = g^*\alpha$ be the base change of α. Let $x' \in X'$ map to $s' \in S'$, $x \in X$, and $s \in S$. We claim $n_{x'} = n_x$. Namely, since $Z(x)$ is smooth over $\kappa(s)$ we see that $Z(x) \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ is reduced. Since $Z(x')$ is an irreducible component of this scheme, we see that the coefficient $n_{x'}$ of $Z(x')$ in $\alpha'_{s'}$ is the same as the coefficient n_x of $Z(x)$ in α_s by the definition of base change in Section 62.3 thereby proving the claim.

Since X is locally Noetherian, to show that $x \mapsto n_x$ is locally constant, it suffices to show: if $x' \leadsto x$ is a specialization in X, then $n_{x'} = n_x$. Choose a morphism $S' \to X$ where S' is the spectrum of a discrete valuation ring mapping the generic point η to x' and the closed point 0 to x. See Properties, Lemma 28.5.10. Then the base change $X' \to S'$ of f by $S' \to S$ has a section $\sigma : S' \to X'$ such that $\sigma(\eta) \leadsto \sigma(0)$ is a specialization of points of X' mapping to $x' \leadsto x$ in X. Thus we reduce to the claim in the next paragraph.

Let S be the spectrum of a discrete valuation ring with generic point η and closed point 0 and we have a section $\sigma : S \to X$. Claim: $n_{\sigma(\eta)} = n_{\sigma(0)}$. By the discussion in More on Morphisms, Section 37.29 and especially More on Morphisms, Lemma 37.29.6 after replacing X by an open subscheme, we may assume the fibres of $X \to S$ are connected. Since these fibres are smooth, they are irreducible. Then we see that $\alpha_\eta = n[X_\eta]$ with $n = n_{\sigma(\eta)}$ and the relation $sp_{X/S}(\alpha_\eta) = \alpha_0$ implies $\alpha_0 = n[X_0]$, i.e., $n_{\sigma(0)} = n$ as desired. □

0H5C Lemma 62.6.12. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ and $\alpha, \beta \in z(X/S, r)$. The set $E = \{s \in S : \alpha_s = \beta_s\}$ is closed in S.

Proof. The question is local on S, thus we may assume S is affine. Let $X = \bigcup U_i$ be an affine open covering. Let $E_i = \{s \in S : \alpha_s|_{U_{i,s}} = \beta_s|_{U_{i,s}}\}$. Then $E = \bigcap E_i$. Hence it suffices to prove the lemma for $U_i \to S$ and the restriction of α and β to U_i. This reduces us to the case discussed in the next paragraph.

Assume X and S are quasi-compact. Set $\gamma = \alpha - \beta$. Then $E = \{s \in S : \gamma_s = 0\}$. By Lemma 62.6.8 there exists a jointly surjective finite family of proper morphisms $\{g_i : S_i \to S\}$ such that $g_i^*\gamma$ is in the image of (62.6.8.1). Observe that $E_i = g_i^{-1}(E)$ is the set of point $t \in S_i$ such that $(g_i^*\gamma)_t = 0$. If E_i is closed for all i, then $E = \bigcup g_i(E_i)$ is closed as well. This reduces us to the case discussed in the next paragraph.

Assume X and S are quasi-compact and $\gamma = \sum n_i[Z_i/X/S]_r$ for a finite number of closed subschemes $Z_i \subset X$ flat and of relative dimension $\leq r$ over S. Set $X' = \bigcup Z_i$ (scheme theoretic union). Then $i : X' \to X$ is a closed immersion and X' has relative dimension $\leq r$ over S. Also $\gamma = i_*\gamma'$ where $\gamma' = \sum n_i[Z_i/X'/S]_r$. Since clearly $E = E' = \{s \in S : \gamma'_s = 0\}$ we reduce to the case discussed in the next paragraph.

Assume X has relative dimension $\leq r$ over S. Let $s \in S$, $s \notin E$. We will show that there exists an open neighbourhood $V \subset S$ of s such that $E \cap V$ is empty. The assumption $s \notin E$ means there exists an integral closed subscheme $Z \subset X_s$ of dimension r such that the coefficient n of $[Z]$ in γ_s is nonzero. Let $x \in Z$ be the generic point. Since $\dim(Z) = r$ we see that x is a generic point of an irreducible component (namely Z) of X_s. Thus after replacing X by an open neighbourhood of x, we may assume that Z is the only irreducible component of X_s. In particular, we have $\gamma_s = n[Z]$.

At this point we apply More on Morphisms, Lemma 37.47.1 and we obtain a diagram

$$
\begin{tikzcd}
X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\
& Y \arrow[d, "h"] \\
S \arrow[r, equal] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\
& y \arrow[d, mapsto] \\
s \arrow[r, equal] & s
\end{tikzcd}
$$

with all the properties listed there. Let $\gamma' = g^*\gamma$ be the flat pullback. Note that $E \subset E' = \{s \in S : \gamma'_s = 0\}$ and that $s \notin E'$ because the coefficient of Z' in γ'_s is nonzero, where $Z' \subset X'_s$ is the closure of x'. Similarly, set $\gamma'' = \pi_*\gamma'$. Then we have $E' \subset E'' = \{s \in S : \gamma''_s = 0\}$ and $s \notin E''$ because the coefficient of Z'' in γ''_s is nonzero, where $Z'' \subset Y_s$ is the closure of y. By Lemma 62.6.11 and openess of $Y \to S$ we see that an open neighbourhood of s is disjoint from E'' and the proof is complete. ☐

0H5D Lemma 62.6.13. Let $S = \lim_{i \in I} S_i$ be the limit of a directed inverse system of Noetherian schemes with affine transition morphisms. Let $0 \in I$ and let $X_0 \to S_0$ be a finite type morphism of schemes. For $i \geq 0$ set $X_i = S_i \times_{S_0} X_0$ and set $X = S \times_{S_0} X_0$. If S is Noetherian too, then

$$z(X/S, r) = \text{colim}_{i \geq 0}\, z(X_i/S_i, r)$$

where the transition maps are given by base change of relative r-cycles.

Proof. Suppose that $i \geq 0$ and $\alpha_i, \beta_i \in z(X_i/S_i, r)$ map to the same element of $z(X/S, r)$. Then $S \to S_i$ maps into the closed subset $E \subset S_i$ of Lemma 62.6.12. Hence for some $j \geq i$ the morphism $S_j \to S_i$ maps into E, see Limits, Lemma 32.4.10. It follows that the base change of α_i and β_i to S_j agree. Thus the map is injective.

Let $\alpha \in z(X/S, r)$. Applying Lemma 62.6.9 a completely decomposed proper morphism $g : S' \to S$ such that $g^*\alpha$ is in the image of (62.6.8.1). Set $X' = S' \times_S X$. We write $g^*\alpha = \sum n_a [Z_a/X'/S']_r$ for some $Z_a \subset X'$ closed subscheme flat and of relative dimension $\leq r$ over S'.

Now we bring the machinery of Limits, Section 32.10 ff to bear. We can find an $i \geq 0$ such that there exist

(1) a completely decomposed proper morphism $g_i : S'_i \to S_i$ whose base change to S is $g : S' \to S$,
(2) setting $X'_i = S'_i \times_{S_i} X_i$ closed subschemes $Z_{ai} \subset X'_i$ flat and of relative dimension $\leq r$ over S'_i whose base change to S' is Z_a.

To do this one uses Limits, Lemmas 32.10.1, 32.8.5, 32.8.7, 32.13.1, and 32.18.1 and More on Morphisms, Lemma 37.78.5. Consider $\alpha'_i = \sum n_a[Z_{ai}/X'_i/S'_i]_r \in z(X'_i/S'_i, r)$. The image of α'_i in $z(X'/S', r)$ agrees with the base change $g^*\alpha$ by construction.

Set $S''_i = S'_i \times_{S_i} S'_i$ and $X''_i = S''_i \times_{S_i} X_i$ and set $S'' = S' \times_S S'$ and $X'' = S'' \times_S X$. We denote $\text{pr}_1, \text{pr}_2 : S'' \to S'$ and $\text{pr}_1, \text{pr}_2 : S''_i \to S'_i$ the projections. The two base changes $\text{pr}_1^*\alpha'_i$ and $\text{pr}_1^*\alpha'_i$ map to the same element of $z(X''/S'', r)$ because $\text{pr}_1^*g^*\alpha = \text{pr}_1^*g^*\alpha$. Hence after increasing i we may assume that $\text{pr}_1^*\alpha'_i = \text{pr}_1^*\alpha'_i$ by the first paragraph of the proof. By Lemma 62.5.9 we obtain a unique family α_i of r-cycles on fibres of X_i/S_i with $g_i^*\alpha_i = \alpha'_i$ (this uses that $S'_i \to S_i$ is completely decomposed). By Lemma 62.6.3 we see that $\alpha_i \in z(X_i/S_i, r)$. The uniqueness in Lemma 62.5.9 implies that the image of α_i in $z(X/S, r)$ is α and the proof is complete. □

0H5E Lemma 62.6.14. Let S be a locally Noetherian scheme. Let $i : X \to X'$ be a thickening of schemes locally of finite type over S. Let $r \geq 0$. Then $i_* : z(X/S, r) \to z(X'/S, r)$ is a bijection.

Proof. Since $i_s : X_s \to X'_s$ is a thickening it is clear that i_* induces a bijection between families of r-cycles on the fibres of X/S and families of r-cycles on the fibres of X'/S. Also, given a family α of r-cycles on the fibres of X/S $\alpha \in z(X/S, r) \Leftrightarrow i_*\alpha \in z(X'/S, r)$ by Lemma 62.6.5. The lemma follows. □

0H5F Lemma 62.6.15. Let S be a locally Noetherian scheme. Let X be a scheme locally of finite type over S. Let $r \geq 0$. Let $U \subset X$ be an open such that $X \setminus U$ has relative dimension $< r$ over S, i.e., $\dim(X_s \setminus U_s) < r$ for all $s \in S$. Then restriction defines a bijection $z(X/S, r) \to z(U/S, r)$.

Proof. Since $Z_r(X_s) \to Z_r(U_s)$ is a bijection by the dimension assumption, we see that restriction induces a bijection between families of r-cycles on the fibres of X/S and families of r-cycles on the fibres of U/S. These restriction maps $Z_r(X_s) \to Z_r(U_s)$ are compatible with base change and with specializations, see Lemma 62.5.1 and 62.4.4. The lemma follows easily from this; details omitted. □

0H5G Lemma 62.6.16. Let $g : S' \to S$ be a universal homeomorphism of locally Noetherian schemes which induces isomorphisms of residue fields. Let $f : X \to S$ be locally of finite type. Set $X' = S' \times_S X$. Let $r \geq 0$. Then base change by g determines a bijection $z(X/S, r) \to z(X'/S', r)$.

Proof. By Lemma 62.5.10 we have a bijection between the group of families of r-cycles on fibres of X/S and the group of families of r-cycles on fibres of X'/S'. Say α is a families of r-cycles on fibres of X/S and $\alpha' = g^*\alpha$ is the base change. If R is a discrete valuation ring, then any morphism $h : \text{Spec}(R) \to S$ factors as $g \circ h'$ for some unique morphism $h' : \text{Spec}(R) \to S'$. Namely, the morphism $S' \times_S \text{Spec}(R) \to \text{Spec}(R)$ is a univeral homomorphism inducing bijections on residue fields, and hence has a section (for example because R is a seminormal ring, see Morphisms, Section 29.47). Thus the condition that α is compatible with specializations (i.e., is a relative r-cycle) is equivalent to the condition that α' is compatible with specializations. □

62.7. Equidimensional relative cycles

0H5H Here is the definition.

0H5I **Definition 62.7.1.** Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S *equidimensional* if the support of α (Remark 62.5.6) is contained in a closed subset $W \subset X$ whose relative dimension over S is $\leq r$. The group of all equidimensional relative r-cycles on X/S is denoted $z_{equi}(X/S, r)$.

0H5J **Example 62.7.2.** There exist relative r-cycles which are not equidimensional. Namely, let k be a field and let $X = \text{Spec}(k[x, y, t])$ over $S = \text{Spec}(k[x, y])$. Let s be a point of S and denote $a, b \in \kappa(s)$ the images of x and y. Consider the family α of 0-cycles on X/S defined by [SV00, Example 3.1.9]

1. $\alpha_s = 0$ if $b = 0$ and otherwise
2. $\alpha_s = [p] - [q]$ where p, resp. q is the $\kappa(s)$-rational point of $\text{Spec}(\kappa(s)[t])$ with $t = a/b$, resp. $t = (a + b^2)/b$.

We leave it to the reader to show that this is compatible with specializations; the idea is that a/b and $(a + b^2)/b = a/b + b$ limit to the same point in $\mathbf{P}^1$ over the residue field of any valuation v on $\kappa(s)$ with $v(b) > 0$. On the other hand, the closure of the support of α containes the whole fibre over $(0, 0)$.

0H5K **Lemma 62.7.3.** Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is equidimensional, then any restriction, base change, or flat pullback of α is equidimensional.

Proof. Omitted. $\square$

0H5L **Lemma 62.7.4.** Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Then to check that α is equidimensional we may work Zariski locally on X and S.

Proof. Namely, the condition that α is equidimensional just means that the closure of the support of α has relative dimension $\leq r$ over S. Since taking closures commutes with restriction to opens, the lemma follows (small detail omitted). $\square$

0H5M **Lemma 62.7.5.** Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $\{g_i : S_i \to S\}$ be an fppf covering. Then α is equidimensional if and only if each base change $g_i^*\alpha$ is equidimensional.

Proof. If α is equidimensional, then each $g_i^*\alpha$ is too by Lemma 62.7.3. Assume each $g_i^*\alpha$ is equidimensional. Denote W the closure of $\text{Supp}(\alpha)$ in X. Since $g_i : S_i \to S$ is universally open (being flat and locally of finite presentation), so is the morphism $f_i : X_i = S_i \times_S X \to X$. Denote $\alpha_i = g_i^*\alpha$. We have $\text{Supp}(\alpha_i) = f_i^{-1}(\text{Supp}(\alpha))$ by Lemma 62.5.7. Since f_i is open, we see that $W_i = f_i^{-1}(W)$ is the closure of $\text{Supp}(\alpha_i)$. Hence by assumption the morphism $W_i \to S_i$ has relative dimension $\leq r$. By Morphisms, Lemma 29.28.3 (and the fact that the morphisms $S_i \to S$ are jointly surjective) we conclude that $W \to S$ has relative dimension $\leq r$. $\square$

0H5N Lemma 62.7.6. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is equidimensional if and only if each flat pullback $f_i^*\alpha$ is equidimensional.

Proof. Omitted. Hint: As in the proof of Lemma 62.7.5 one shows that the inverse image by f_i of the closure W of the support of α is the closure W_i of the support of $f_i^*\alpha$. Then $W \to S$ has relative dimension $\leq r$ holds if $W_i \to S$ has relative dimension $\leq r + e$ for all i. □

Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally quasi-finite morphism of schemes. Then we have $z(X/S, 0) = z_{equi}(X/S, 0)$ and $z(X/S, r) = 0$ for $r > 0$. Given $\alpha \in z(X/S, 0)$ let us define a map

$$w_\alpha : X \longrightarrow \mathbf{Z}, \quad x \mapsto \alpha(x)[\kappa(x) : \kappa(s)]_i \quad \text{where } s = f(x)$$

Here $\alpha(x)$ denotes the coefficient of x in the 0-cycle α_s on the fibre X_s and $[K : k]_i$ denotes the inseparable degree of a finite field extension. The following lemma shows that this map is a weighting of f (More on Morphisms, Definition 37.75.2) and that every weighting is of this form up to taking a multiple.

0H5P Lemma 62.7.7. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally quasi-finite morphism of schemes. Let $\alpha \in z(X/S, 0)$. The map $w_\alpha : X \to \mathbf{Z}$ constructed above is a weighting. Conversely, if X is quasi-compact, then given a weighting $w : X \to \mathbf{Z}$ there exists an integer $n > 0$ such that $nw = w_\alpha$ for some $\alpha \in z(X/S, 0)$. Finally, the integer n may be chosen to be a power of the prime p if S is a scheme over $\mathbf{F}_p$.

Proof. First, let us show that the construction is compatible with base change: if $g : S' \to S$ is a morphism of locally Noetherian schemes, then $w_{g^*\alpha} = w_\alpha \circ g'$ where $g' : X' \to X$ is the projection $X' = S' \times_S X \to X$. Namely, let $x' \in X'$ with images s', s, x in S', S, X. Then the coefficient of $[x']$ in the base change of $[x]$ by $\kappa(s')/\kappa(s)$ is the length of the local ring $(\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q}$. Here $\mathfrak{q}$ is the prime ideal corresponding to x'. Thus compatibility with base change follows if

$$[\kappa(x) : \kappa(s)]_i = \text{length}((\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q})[\kappa(x') : \kappa(s')]_i$$

Let $k/\kappa(s')$ be an algebraically closure. Choose a prime $\mathfrak{p} \subset k \otimes_{\kappa(s)} \kappa(x)$ lying over $\mathfrak{q}$. Suppose we can show that

$$[\kappa(x) : \kappa(s)]_i = \text{length}((k \otimes_{\kappa(s)} \kappa(x))_\mathfrak{p}) \quad \text{and} \quad [\kappa(x') : \kappa(s')]_i = \text{length}((k \otimes_{\kappa(s')} \kappa(x'))_\mathfrak{p})$$

Then we win because

$$\text{length}((\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q})\text{length}((k \otimes_{\kappa(s')} \kappa(x'))_\mathfrak{p}) = \text{length}((k \otimes_{\kappa(s)} \kappa(x))_\mathfrak{p})$$

by Algebra, Lemma 10.52.13 and flatness of $\kappa(s') \otimes_{\kappa(s)} \kappa(x) \to k \otimes_{\kappa(s)} \kappa(x)$. To show the two equalities, it suffices to prove the first. Let $\kappa(x)/\kappa/\kappa(s)$ be the subfield constructed in Fields, Lemma 9.14.6. Then we see that

$$k \otimes_{\kappa(s)} \kappa(x) = \prod\nolimits_{\sigma : \kappa \to k} k \otimes_{\sigma, \kappa} \kappa(x)$$

and each of the factors is local of degree $[\kappa(x) : \kappa] = [\kappa(x) : \kappa(s)]_i$ as desired.

Let $\alpha \in z(X/S, 0)$ and choose a diagram

$$\begin{tikzcd} X & U \arrow[l, "h"'] \arrow[d, "\pi"] \\ Y \arrow[u, leftarrow, "f"] & V \arrow[l, "g"] \end{tikzcd}$$

as in More on Morphisms, Definition 37.75.2. Denote $\beta \in z(U/V, 0)$ the restriction of the base change $g^*\alpha$. By the compatibility with base change above we have $w_\beta = w_\alpha \circ h$ and it suffices to show that $\int_\pi w_\beta$ is locally constant on V. Next, note that

$$\begin{aligned} \left(\int_\pi w_\beta\right)(v) &= \sum\nolimits_{u \in U, \pi(u) = v} \beta(u)[\kappa(u) : \kappa(v)]_i[\kappa(u) : \kappa(v)]_s \\ &= \sum\nolimits_{u \in U, \pi(u) = v} \beta(u)[\kappa(u) : \kappa(v)] \end{aligned}$$

This last expression is the coefficient of v in $\pi_*\beta \in z(V/V, 0)$. By Lemma 62.6.11 this function is locally constand on V.

Conversely, let $w : X \to S$ be a weighting and X quasi-compact. Choose a sufficiently divisible integer n. Let α be the family of 0-cycles on fibres of X/S such that for $s \in S$ we have

$$\alpha_s = \sum\nolimits_{f(x) = s} \frac{nw(x)}{[\kappa(x) : \kappa(s)]_i}[x]$$

as a zero cycle on X_s. This makes sense since the fibres of f are universally bounded (Morphisms, Lemma 29.57.9) hence we can find n such that the right hand side is an integer for all $s \in S$. The final statement of the lemma also follows, provided we show α is a relative 0-cycle. To do this we have to show that α is compatible with specializations along discrete valuation rings. By the first paragraph of the proof our construction is compatible with base change (small detail omitted; it is the "inverse" construction we are discussing here). Also, the base change of a weighting is a weighting, see More on Morphisms, Lemma 37.75.3. Thus we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with generic point η and closed point 0. Let $w : X \to S$ be a weighting with X quasi-finite over S. Let α be the family of 0-cycles on fibres of X/S constructed in the previous paragraph (for a suitable n). We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Let $\beta \in z(X/S, 0)$ be the relative 0-cycle on X/S with $\beta_\eta = \alpha_\eta$ and $\beta_0 = sp_{X/S}(\alpha_\eta)$. Then $w' = w_\beta - nw : X \to \mathbf{Z}$ is a weighting (using the result above) and zero in the points of X which map to η. Now it is easy to see that a weighting which is zero on all points of X mapping to η has to be zero; details omitted. Hence $w' = 0$, i.e., $w_\beta = nw$, hence $\alpha = \beta$ as desired. □

62.8. Effective relative cycles

0H5Q Here is the definition.

0H5R Definition 62.8.1. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S effective if α_s is an effective cycle (Chow Homology, Definition

42.8.4) for all $s \in S$. The monoid of all effective relative r-cycles on X/S is denoted $z^{eff}(X/S, r)$.

Below we will show that an effective relative cycle is equidimensional, see Lemma 62.8.7.

0H5S Lemma 62.8.2. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is effective, then any restriction, base change, flat pullback, or proper pushforward of α is effective.

Proof. Omitted. □

0H5T Lemma 62.8.3. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Then to check that α is effective we may work Zariski locally on X and S.

Proof. Omitted. □

0H5U Lemma 62.8.4. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $g : S' \to S$ be a surjective morphism. Then α is effective if and only if the base change $g^*\alpha$ is effective.

Proof. Omitted. □

0H5V Lemma 62.8.5. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is effective if and only if each flat pullback $f_i^*\alpha$ is effective.

Proof. Omitted. □

0H5W Lemma 62.8.6. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. If α is effective, then $\text{Supp}(\alpha)$ is closed in X.

Proof. Let $g : S' \to S$ be the inclusion of an irreducible component viewed as an integral closed subscheme. By Lemmas 62.8.2 and 62.5.7 it suffices to show that the support of the base change $g^*\alpha$ is closed in $S' \times_S S$. Thus we may assume S is an integral scheme with generic point η. We will show that $\text{Supp}(\alpha)$ is the closure of $\text{Supp}(\alpha_\eta)$. To do this, pick any $s \in S$. We can find a morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring mapping the generic point $\eta' \in S'$ to η and the closed point $0 \in S'$ to s, see Properties, Lemma 28.5.10. Then it suffices to prove that the support of $g^*\alpha$ is equal to the closure of $\text{Supp}((g^\alpha)_{\eta'})$. This reduces us to the case discussed in the next paragraph.

Here S is the spectrum of a discrete valuation ring with generic point η and closed point 0. We have to show that $\text{Supp}(\alpha)$ is the closure of $\text{Supp}(\alpha_\eta)$. Since α is effective we may write $\alpha_\eta = \sum n_i[Z_i]$ with $n_i > 0$ and $Z_i \subset X_\eta$ integral closed of dimension r. Since $\alpha_0 = sp_{X/S}(\alpha_\eta)$ we know that $\alpha_0 = \sum n_i[\overline{Z}_{i,0}]_r$ where $\overline{Z}_i$ is the closure of Z_i. By Varieties, Lemma 33.19.2 we see that $\overline{Z}_{i,0}$ is equidimensional of dimension r. Since $n_i > 0$ we conclude that $\text{Supp}(\alpha_0)$ is equal to the union of the $\overline{Z}_{i,0}$ which is the fibre over 0 of $\bigcup \overline{Z}_i$ which in turn is the closure of $\bigcup Z_i$ as desired. $\square$

0H5X Lemma 62.8.7. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. If α is effective, then α is equidimensional.

Proof. Assume α is effective. By Lemma 62.8.6 the support $\text{Supp}(\alpha)$ is closed in X. Thus α is equidimensional as the fibres of $\text{Supp}(\alpha) \to S$ are the supports of the cycles α_s and hence have dimension r. $\square$

0H5Y Remark 62.8.8. Let $f : X \to S$ be a morphism of schemes with S locally Noetherian and f locally of finite type. We can ask if the contravariant functor

$$\begin{matrix} \text{schemes } S' \text{ locally} \\ \text{of finite type over } S \end{matrix} \longrightarrow z^{eff}(X'/S', r) \text{ where } X' = S' \times_S X$$

is representable. Since $z(X'/S', r) = z(X'_{red}/S'_{red}, r)$ this cannot be true (we leave it to the reader to make an actual counter example). A better question would be if we can find a subcategory of the left hand side on which the functor is representable. Lemma 62.6.16 suggests we should restrict at least to the category of seminormal schemes over S.

If $S/\text{Spec}(\mathbf{Q})$ is Nagata and f is a projective morphism, then it turns out that $S' \mapsto z^{eff}(X'/S', r)$ is representable on the category of seminormal S'. Roughly speaking this is the content of [Kol96, Theorem 3.21].

If S has points of positive characteristic, then this no longer works even if we replace seminormality with weak normality; a locally Noetherian scheme T is weakly normal if any birational universal homeomorphism $T' \to T$ has a section. An example is to consider 0-cycles of degree 2 on $X = \mathbf{A}^2_k$ over $S = \text{Spec}(k)$ where k is a field of characteristic 2. Namely, over $W = X \times_S X$ we have a canonical relative 0-cycle $\alpha \in z^{eff}(X_W/W, 0)$: for $w = (x_1, x_2) \in W = X^2$ we have the cycle $\alpha_w = [x_1] + [x_2]$. This cycle is invariant under the involution $\sigma : W \to W$ switching the factors. Since W is smooth (hence normal, hence weakly normal), if $z(-/-, r)$ was representable by M on the category of weakly normal schemes of finite type over k we would get a σ-invariant morphism from W to M. This in turn would define a morphism from the quotient scheme $\text{Sym}^2_S(X) = W/\langle\sigma\rangle$ to M. Since $\text{Sym}^2_S(X)$ is normal, we would by the moduli property of M obtain a relative 0-cycle β on $X \times_S \text{Sym}^2_S(X)/\text{Sym}^2_S(X)$ whose pullback to W is α. However, there is no such cycle β. Namely, writing $X = \text{Spec}(k[u, v])$ the scheme $\text{Sym}^2_S(X)$ is the spectrum of

$$k[u_1 + u_2, u_1u_2, v_1 + v_2, v_1v_2, u_1v_1 + u_2v_2] \subset k[u_1, u_2, v_1, v_2]$$

The image of the diagonal $u_1 = u_2, v_1 = v_2$ in $\text{Sym}^2_S(X)$ is the closed subscheme $V = \text{Spec}(k[u_1^2, v_1^2])$; here we use that the characteristic of k is 2. Looking at the generic point η of V, the cycle β_η would be a zero cycle of degree 2 on $\mathbf{A}^2_{k(u_1^2, v_1^2)}$ whose pullback to $\mathbf{A}^2_{k(u_1, u_2)}$ whould be 2[the point with coordinates(u_1, v_2)]. This is clearly impossible.

The discussion above does not contradict [Kol96, Theorem 4.13] as the Chow variety in that theorem only coarsely represents a functor (in fact 2 distinct functors, only one of which agrees with ours for projective X as one can see with some work). Similarly, in [SV00, Section 4.4] it is shown that for projective X/S the h-sheafification of the presheaf $S' \mapsto z^{eff}(S' \times_S X/S', r)$ is equal to the h-sheafification of a representable functor.

0H5Z Remark 62.8.9. Let $f : X \to S$ be a morphism of schemes. Let $r \geq 0$. Let $Z \subset X$ be a closed subscheme. Assume

(1) S is Noetherian and geometrically unibranch,
(2) f is of finite type, and
(3) $Z \to S$ has relative dimension $\leq r$.

Then for all sufficiently divisible integers $n \geq 1$ there exists a unique effective relative r-cycle α on X/S such that $\alpha_\eta = n[Z_\eta]_r$ for every generic point η of S. This is a reformulation of [SV00, Theorem 3.4.2]. If we ever need this result, we will precisely state and prove it here.

62.9. Proper relative cycles

0H60 In our setting, the following is probably the correct definition.

0H61 Definition 62.9.1. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S is a proper relative cycle if the support of α (Remark 62.5.6) is contained in a closed subset $W \subset X$ proper over S (Cohomology of Schemes, Definition 30.26.2). The group of all proper relative r-cycles on X/S is denoted $c(X/S, r)$.

By Cohomology of Schemes, Lemma 30.26.3 this just means that the closure of the support is proper over the base. To see that these form a group, use Cohomology of Schemes, Lemma 30.26.6.

0H62 Lemma 62.9.2. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is proper, then any base change α is proper.

Proof. Omitted. □

0H63 Lemma 62.9.3. Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $\{g_i : S_i \to S\}$ be a h covering. Then α is proper if and only if each base change $g_i^*\alpha$ is proper.

Proof. If α is proper, then each $g_i^*\alpha$ is too by Lemma 62.9.2. Assume each $g_i^*\alpha$ is proper. To prove that α is proper, it clearly suffices to work affine locally on S. Thus we may and do assume that S is affine. Then we can refine our covering $\{S_i \to S\}$ by a family $\{T_j \to S\}$ where $g : T \to S$ is a proper surjective morphism and $T = \bigcup T_j$ is an open covering. It follows that $\beta = g^*\alpha$ is proper on $Y = T \times_S X$ over T. By Lemma 62.5.7 we find that the support of β is the inverse image of the support of α by the morphism $f : Y \to X$. Hence the closure $W \subset Y$ of $f^{-1}\text{Supp}(\alpha)$ is proper over T. Since the morphism $T \to S$ is proper, it follows that W is proper over S. Then by Cohomology of Schemes, Lemma 30.26.5 the image $f(W) \subset X$ is a closed subset proper over S. Since $f(W)$ contains $\text{Supp}(\alpha)$ we conclude α is proper. □

62.10. Proper and equidimensional relative cycles

0H64 Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S is a proper and equidimensional relative cycle if α is both equidimensional (Definition 62.7.1) and proper (Definition 62.9.1). The group of all proper, equidimensional relative r-cycles on X/S is denoted $c_{equi}(X/S, r)$.

Similarly we say a relative r-cycle α on X/S is a proper and effective relative cycle if α is both effective (Definition 62.8.1) and proper (Definition 62.9.1). The monoid of all proper, effective relative r-cycles on X/S is denoted $c^{eff}(X/S, r)$. Observe that these are equidimensional by Lemma 62.8.7.

Thus we have the following diagram of inclusion maps

$$\begin{tikzcd} c^{eff}(X/S, r) \arrow[r] \arrow[d] & c_{equi}(X/S, r) \arrow[r] \arrow[d] & c(X/S, r) \arrow[d] \\ z^{eff}(X/S, r) \arrow[r] & z_{equi}(X/S, r) \arrow[r] & z(X/S, r) \end{tikzcd}$$

62.11. Action on cycles

0H65 Let S be a locally Noetherian, universally catenary scheme endowed with a dimension function δ, see Chow Homology, Section 42.7. Let $X \to Y$ be a morphism of schemes over S, both locally of finite type over S. Let $r \geq 0$. Finally, let α be a family of r-cycles on fibres of X/Y. For $e \in \mathbf{Z}$ we will construct an operation

$$\alpha \cap - : Z_e(Y) \longrightarrow Z_{r+e}(X)$$

Namely, given $\beta \in Z_e(Y)$ write $\beta = \sum n_i[Z_i]$ where $Z_i \subset Y$ is an integral closed subscheme of δ-dimension e and the family Z_i is locally finite in the scheme Y. Let $y_i \in Z_i$ be the generic point. Write $\alpha_{y_i} = \sum m_{ij}[V_{ij}]$. Thus $V_{ij} \subset X_{y_i}$ is an integral closed subscheme of dimension r and the family V_{ij} is locally finite in the scheme X_{y_i}. Then we set

$$\alpha \cap \beta = \sum n_i m_{ij}[\overline{V}_{ij}] \quad \in \quad Z_{r+e}(X)$$

Here $\overline{V}_{ij} \subset X$ is the scheme theoretic image of the morphism $V_{ij} \to X_{y_i} \to X$ or equivalently, $\overline{V}_{ij} \subset X$ is an integral closed subscheme mapping dominantly to $Z_i \subset Y$ whose generic fibre is V_{ij}. It follows readily that $\dim_\delta(\overline{V}_{ij}) = r + e$ and that the family of closed subschemes $\overline{V}_{ij} \subset X$ is locally finite (we omit the verifications). Hence $\alpha \cap \beta$ is indeed an element of $Z_{r+e}(X)$.

0H66 **Lemma 62.11.1.** *The construction above is bilinear, i.e., we have $(\alpha_1 + \alpha_2) \cap \beta = \alpha_1 \cap \beta + \alpha_2 \cap \beta$ and $\alpha \cap (\beta_1 + \beta_2) = \alpha \cap \beta_1 + \alpha \cap \beta_2$.*

Proof. Omitted. □

0H67 **Lemma 62.11.2.** *If $U \subset X$ and $V \subset Y$ are open and $f(U) \subset V$, then $(\alpha \cap \beta)|_U$ is equal to $\alpha|_U \cap \beta|_V$.*

Proof. Immediate from the explict description of $\alpha \cap \beta$ given above. □

0H68 **Lemma 62.11.3.** *Forming $\alpha \cap \beta$ is compatible with flat base change and flat pullback (see proof for elucidation).*

Proof. Let (S, δ), (S', δ'), $g : S' \to S$, and $c \in \mathbf{Z}$ be as in Chow Homology, Situation 42.67.1. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Denote $X' \to Y'$ the base change of $X \to Y$ by g. Let α be a family of r-cycles on the fibres of X/Y. Let $\beta \in Z_e(Y)$. Denote α' the base change of α by $Y' \to Y$. Denote $\beta' = g^*\beta \in Z_{e+c}(Y')$ the pullback of β by g, see Chow Homology, Section 42.67. Compatibility with base change means $\alpha' \cap \beta'$ is the base change of $\alpha \cap \beta$.

Proof of compatibility with base change. Since we are proving an equality of cycles on X', we may work locally on Y, see Lemma 62.11.2. Thus we may assume Y is affine. In particular β is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 62.11.1), it suffices to prove the equality when $\beta = [Z]$ for some integral closed subscheme $Z \subset Y$ of δ-dimension e.

Let $y \in Z$ be the generic point. Write $\alpha_y = \sum m_j[V_j]$. Let $\overline{V}_j$ be the closure of V_j in X. Then we have

$$\alpha \cap \beta = \sum m_j[\overline{V}_j]$$

The base change of β is $\beta' = \sum [Z \times_S S']_{e+c}$ as a cycle on $Y' = Y \times_S S'$. Let $Z'_a \subset Z \times_S S'$ be the irreducible components, denote $y'_a \in Z'_a$ their generic points, and denote n_a the multiplicity of Z'_a in $Z \times_S S'$. We have

$$\beta' = \sum [Z \times_S S']_{e+c} = \sum n_a[Z'_a]$$

We have $\alpha'_{y'_a} = \sum m_j[V_{j,\kappa(y'_a)}]_r$ because α' is the base change of α by $Y' \to Y$. Let $V'_{jab} \subset V_{j,\kappa(y'_a)}$ be the irreducible components and denote m_{jab} the multiplicity of V'_{jab} in $V_{j,\kappa(y'_a)}$. We have

$$\alpha'_{y'_a} = \sum m_j[V_{j,\kappa(y'_a)}]_r = \sum m_j m_{jab}[V'_{jab}]$$

Thus we we have

$$\alpha' \cap \beta' = \sum n_a m_j m_{jab}[\overline{V}'_{jab}]$$

where $\overline{V}'_{jab}$ is the closure of V'_{jab} in X'. Thus to prove the desired equality it suffices to prove

(1) the irreducible components of $\overline{V}_j \times_S S'$ are the schemes $\overline{V}'_{jab}$ and
(2) the multiplicity of $\overline{V}'_{jab}$ in $\overline{V}_j \times_S S'$ is equal to $n_a m_{jab}$.

Note that $V_j \to \overline{V}_j$ is a birational morphism of integral schemes. The morphisms $V_j \times_S S' \to V_j$ and $\overline{V}_j \times_S S' \to \overline{V}_j$ are flat and hence map generic points of irreducible components to the (unique) generic points of V_j and $\overline{V}_j$. It follows that $V_j \times_S S' \to \overline{V}_j \times_S S'$ is a birational morphisms hence induces a bijection on irreducible components and identifies their multiplicities. This means that it suffices to prove that the irreducible components of $V_j \times_S S'$ are the schemes V'_{jab} and the multiplicity of V'_{jab} in $V_j \times_S S'$ is equal to $n_a m_{jab}$. However, then we are just saying that the diagram

$$\begin{tikzcd}
Z_r(V_j) \arrow[r] & Z_{r+c}(V_j \times_S S') \\
Z_0(\mathrm{Spec}(\kappa(y))) \arrow[r] \arrow[u] & Z_c(\mathrm{Spec}(\kappa(y)) \times_S S') \arrow[u]
\end{tikzcd}$$

is commutative where the horizontal arrows are base change by $\mathrm{Spec}(\kappa(y)) \times_S S' \to \mathrm{Spec}(\kappa(y))$ and the vertical arrows are flat pullback. This was shown in Chow Homology, Lemma 42.67.5.

The statement in the lemma on flat pullback means the following. Let (S, δ), $X \to Y$, α, and β be as in the constuction of $\alpha \cap \beta$ above. Let $Y' \to Y$ be a flat morphism, locally of finite type, and of relative dimension c. Then we can let α' be the base change of α by $Y' \to Y$ and β' the flat pullback of β. Compatibility with flat pullback means $\alpha' \cap \beta'$ is the flat pullback of $\alpha \cap \beta$ by $X \times_Y Y' \to Y$. This is actually a special case of the discussion above if we set $S = Y$ and $S' = Y'$. □

0H69 Lemma 62.11.4. Let (S, δ) and $f : X \to Y$ be as above. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with $\dim(\mathrm{Supp}(\mathcal{F}_y)) \leq r$ for all $y \in Y$. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module with $\dim_\delta(\mathrm{Supp}(\mathcal{G})) \leq e$. Set $\alpha = [\mathcal{F}/X/Y]_r$ (Example 62.5.2) and $\beta = [\mathcal{G}]_e$ (Chow Homology, Definition 42.10.2). If $\mathcal{F}$ is flat over Y, then $\alpha \cap \beta = [\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}]_{r+e}$.

Proof. Observe that

$$\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \mathrm{Supp}(\mathcal{F}) \cap f^{-1}\mathrm{Supp}(\mathcal{G}) = \bigcup\nolimits_{y \in \mathrm{Supp}(\mathcal{G})} \mathrm{Supp}(\mathcal{F}_y)$$

It follows that this is a closed subset of δ-dimension $\leq r+e$. Whence the expression $[\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}]_{r+e}$ makes sense.

We will use the notation $\beta = \sum n_i[Z_i]$, $y_i \in Z_i$, $\alpha_{y_i} = \sum m_{ij}[V_{ij}]$, and $\overline{V}_{ij}$ introduced in the construction of $\alpha \cap \beta$. Since $\beta = [\mathcal{G}]_e$ we see that the Z_i are the irreducible components of $\mathrm{Supp}(\mathcal{G})$ which have δ-dimension e. Similarly, the V_{ij} are the irreducible components of $\mathrm{Supp}(\mathcal{F}_{y_i})$ having dimension r. It follows from this and the equation in the first paragraph that $\overline{V}_{ij}$ are the irreducible components of $\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ having δ-dimension $r + e$. Thus to prove the lemma it now suffices to show that

$$\mathrm{length}_{\mathcal{O}_{X,\xi_{ij}}}((\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_{\xi_{ij}}) = \mathrm{length}_{\mathcal{O}_{X_{y_i},\xi_{ij}}}((\mathcal{F}_{y_i})_{\xi_{ij}}) \cdot \mathrm{length}_{\mathcal{O}_{Y,y_i}}(\mathcal{G}_{y_i})$$

By the first paragraph of the proof the left hand side is equal to the lenth of the $B = \mathcal{O}_{X,\xi_{ij}}$-module

$$\mathcal{G}_{y_i} \otimes_{\mathcal{O}_{Y,y_i}} \mathcal{F}_{\xi_{ij}} = M \otimes_A N$$

Here $M = \mathcal{G}_{y_i}$ is a finite length $A = \mathcal{O}_{Y,y_i}$-module and $N = \mathcal{F}_{\xi_{ij}}$ is a finite B-module such that $N/\mathfrak{m}_A N$ has finite length. Since $\mathcal{F}$ is flat over Y the module N is A-flat. The right hand side of the formula is equal to

$$\mathrm{length}_B(N/\mathfrak{m}_A N) \cdot \mathrm{length}_A(M)$$

Thus the right and left hand side of the formula are additive in M (use flatness of N over A). Thus it suffices to prove the formula with $M = \kappa_A$ is the residue field in which case it is immediate. □

0H6A Lemma 62.11.5. Let (S, δ) and $f : X \to Y$ be as above. Let $Z \subset X$ be a closed subscheme of relative dimension $\leq r$ over Y. Set $\alpha = [Z/X/Y]_r$ (Example 62.5.4). Let $W \subset Y$ be a closed subscheme of δ-dimension $\leq e$. Set $\beta = [W]_e$ (Chow Homology, Definition 42.9.2). If Z is flat over Y, then $\alpha \cap \beta = [Z \times_Y W]_{r+e}$.

Proof. This is a special case of Lemma 62.11.4 if we take $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{F} = \mathcal{O}_W$. □

0H6B Lemma 62.11.6. Let (S, δ) be as above. Let

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of schemes locally of finite type over S with g proper. Let $r, e \geq 0$. Let α be a family of r-cycles on the fibres of X/Y. Let $\beta' \in Z_e(Y')$. Then we have $f_*(g^*\alpha \cap \beta') = \alpha \cap g_*\beta'$.

Proof. Since we are proving an equality of cycles on X, we may work locally on Y, see Lemma 62.11.2. Thus we may assume Y is affine. Thus Y' is quasi-compact. In particular β' is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 62.11.1), it suffices to prove the equality when $\beta' = [Z']$ for some integral closed subscheme $Z' \subset Y'$ of δ-dimension e. Set $Z = g(Z')$. This is an integral closed subscheme of Y of δ-dimension $\leq e$. For simplicity we are going to assume Z has δ-dimension equal to e and leave the other case (which is easier) to the reader. Let $y \in Z$ and $y' \in Z'$ be the generic points. Write $\alpha_y = \sum m_j[V_j]$ with $V_j \subset X_y$ integral closed subschemes of dimension r.

Assume first g is a closed immersion. Then $g_*\beta' = [Z]$ and $(g^*\alpha)_{y'} = \sum n_j[V_j]$; this makes sense because V_j is contained in the closed subscheme $X'_{y'}$ of X_y. Thus in this case the equality is obvious: in both cases we obtain $\sum m_j[\overline{V}_j]$ where $\overline{V}_j$ is the closure of V_j in the closed subscheme $X' \subset X$.

Back to the general case with $\beta' = [Z']$ as above. Set $W = Z \times_X Y$ and $W' = Z' \times_{X'} Y'$. Consider the cartesian squares

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd} \quad \begin{tikzcd} W' \arrow[r] \arrow[d] & X' \arrow[d] \\ Z' \arrow[r] & Y' \end{tikzcd} \quad \begin{tikzcd} W' \arrow[r] \arrow[d] & W \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

Since we know the result for the first two squares with by the previous paragraph, a formal argument shows that it suffices to prove the result for the last square and the element $\beta' = [Z'] \in Z_e(Z')$. This reduces us to the case discussed in the next paragraph.

Assume $Y' \to Y$ is a generically finite morphism of integral schemes of δ-dimension e and $\beta' = [Y']$. In this case both $f_*(g^*\alpha \cap \beta')$ and $\alpha \cap g_*\beta'$ are cycles which can be written as a sum of prime cycles dominant over Y. Thus we may replace Y by a nonempty open subscheme in order to check the equality. After such a replacement we may assume g is finite and flat, say of degree $d \geq 1$. Of course, this means that $g_*\beta' = g_*[Y'] = d[Y]$. Also $\beta' = [Y'] = g^*[Y]$. Hence

$$f_*(g^*\alpha \cap \beta') = f_*(g^*\alpha \cap g^*[Y]) = f_*f^*(\alpha \cap [Y]) = d(\alpha \cap [Y]) = \alpha \cap g_*\beta')$$

as desired. The second equality is Lemma 62.11.3 and the third equality is Chow Homology, Lemma 42.15.2. $\square$

62.12. Action on chow groups

0H6C When α is a relative r-cycle, the operation $\alpha \cap -$ of Section 62.11 factors through rational equivalence and defines a bivariant class.

0H6D Lemma 62.12.1. Let (S, δ) be as in Section 62.11. Let $f : X' \to X$ be a proper morphism of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Chow Homology, Definition 42.29.1. Form the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Chow Homology, Remark 42.29.7. If $\mathcal{L}|_D \cong \mathcal{O}_D$, then $i^* f_* \alpha' = g_*(i')^*\alpha'$ in $Z_k(D)$ for any $\alpha' \in Z_{k+1}(X')$.

Proof. The statement makes sense as all operations are defined on the level of cycles, see Chow Homology, Remark 42.29.6 for the gysin maps. Suppose $\alpha = [W']$ for some integral closed subscheme $W' \subset X'$. Let $W = f(W') \subset X$. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \text{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \text{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Chow Homology, Lemma 42.26.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and both sides are zero by construction. $\square$

0H6E Lemma 62.12.2. Let (S, δ) be as in Section 62.11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$ and let $\alpha \in z(X/Y, r)$ be a relative r-cycle on X/Y. Let $(\mathcal{L}, s, i : D \to Y)$ be as in Chow Homology, Definition 42.29.1. Form the cartesian diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d] & X \arrow[d] \\ D \arrow[r, "i"] & Y \end{tikzcd}$$

See Chow Homology, Remark 42.29.7. If $\mathcal{L}|_D \cong \mathcal{O}_D$, then for $e \in \mathbf{Z}$ the diagram

$$\begin{tikzcd} Z_e(D) \arrow[r, "i^*\alpha\cap -"] & Z_{e+r}(E) \\ Z_{e+1}(Y) \arrow[r, "\alpha\cap -"] \arrow[u, "i^*"] & Z_{r+e+1}(X) \arrow[u, "j^*"] \end{tikzcd}$$

commutes where the vertical arrows i^* and j^* are the gysin maps on cycles as in Chow Homology, Remark 42.29.6.

Proof. Preliminary remark. Suppose that $g : Y' \to Y$ is an envelope (Chow Homology, Definition 42.22.1). Denote $D', i', E', j', X', \alpha'$ the base changes of D, i, E, j, X, α by g and denote $f : X' \to X$ the projection. Assume the lemma holds for $D', i', E', j', X', Y', \alpha'$. Then, if $\beta' \in Z_{e+1}(Y')$, we have

$$\begin{aligned} i^*\alpha \cap i^* g_*\beta' &= i^*\alpha \cap f_*(i')^*\beta' \\ &= f_*(f^*i^*\alpha \cap (i')^*\beta') \\ &= f_*((i')^*\alpha' \cap (i')^*\beta') \\ &= f_*((j')^*(\alpha' \cap \beta')) \\ &= j^*(f_*(f^*\alpha \cap \beta')) \\ &= j^*(\alpha \cap g_*\beta') \end{aligned}$$

Here the first equality is Lemma 62.12.1, the second equality is Lemma 62.11.6, the third equality is the definition of α', the fourth equality is the assumption that our lemma holds for $D', i', E', j', X', \alpha'$, the fifth equality is Lemma 62.12.1, and the sixth equality is Lemma 62.11.6. Thus we see that our lemma holds for the image of $g_* : Z_{e+1}(Y') \to Z_e(Y)$. However, since g is completely decomposed this map is surjective and we conclude the lemma holds for D, i, E, j, X, Y, α.

Let $\beta \in Z_{e+1}(Y)$. We have to show that $(D \to Y)^*\alpha \cap i^*\beta = j^*(\alpha \cap \beta)$ as cycles on E. This question is local on E hence we can replace X and Y by open subschemes. (This uses that formation of the operators i^*, j^*, $\alpha \cap -$ and $(D \to Y)^*\alpha \cap -$ commute with localization. This is obvious for the gysin maps and follows from Lemma 62.11.2 for the others.) Thus we may assume that X and Y are affine and we reduce to the case discussed in the next paragraph.

Assume X and Y are quasi-compact. By the first paragraph of the proof and Lemma 62.6.9 we may in addition assume that α is in the image of (62.6.8.1). By linearity of the operations in question, we may assume that $\alpha = [Z/X/Y]_r$ for some closed subscheme $Z \subset X$ which is flat and of relative dimension $\leq r$ over Y. Also, as Y is quasi-compact, the cycle β is a finite linear combination of prime cycles. Since the operations in question are linear, it suffices to prove the equality when $\beta = [W]$ for some integral closed subscheme $W \subset Y$ of δ-dimension $e + 1$.

If $W \subset D$, then on the one hand $i^*[W] = 0$ and on the other hand $\alpha \cap [W]$ is supported on E so also $j^*(\alpha \cap [W]) = 0$. Thus the equality holds in this case.

Say $W \not\subset D$. Then $i^*[W] = [D \cap W]_e$. Note that the pullback $i^*\alpha$ of $\alpha = [Z/X/Y]_r$ by i is $[(E \cap Z)/E/D]_r$ and that $(E \cap Z) = E \times_Y Z = D \times_Y Z$ is flat over D. Hence by Lemma 62.11.5 used twice we have

$$i^*\alpha \cap i^*[W] = [(E \cap Z) \times_D (D \cap W)]_{r+e} = [E \cap (Z \times_Y W)]_{r+e} = j^*(\alpha \cap [W])$$

as desired. $\square$

0H6F Proposition 62.12.3. Let (S, δ) be as in Section 62.11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$ and let $\alpha \in z(X/Y, r)$ be a relative r-cycle on X/Y. The rule that to every morphism $g : Y' \to Y$ locally of finite type and every $e \in \mathbf{Z}$ associates the operation

$$g^*\alpha \cap - : Z_e(Y') \to Z_{r+e}(X')$$

where $X' = Y' \times_Y X$ factors through rational equivalence to define a bivariant class $c(\alpha) \in A^{-r}(X \to Y)$.

Proof. The operation factors through rational equivalence by Lemma 62.12.2 and Chow Homology, Lemma 42.35.1. The resulting operation on chow groups is a bivariant class by Chow Homology, Lemma 42.35.2 and Lemmas 62.11.6, 62.11.3, and 62.12.2. $\square$

0H6G Remark 62.12.4. Let (S, δ) be as in Section 62.11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$. Let c be a rule that to every morphism $g : Y' \to Y$ locally of finite type and every $e \in \mathbf{Z}$ associates an operation

$$c \cap - : Z_e(Y') \to Z_{r+e}(X')$$

compatible with proper pushforward, flat pullback, and gysin maps as in Lemma 62.12.2. Then we claim there is a relative r-cycle α on X/Y such that $c\cap = g^*\alpha\cap -$

for every g as above. If we ever need this, we will carefully state and prove this here.

62.13. Composition of families of cycles on fibres

0H6H Let $X \to Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let α be a family of r-cycles on fibres of X/Y and let β be a family of e-cycles on fibres of Y/S. Then we obtain a family of of $(r+e)$-cycles $\alpha \circ \beta$ on the fibres of X/S by setting

$$(\alpha \circ \beta)_s = (Y_s \to Y)^*\alpha \cap \beta_s$$

More precisely, the expression $(Y_s \to Y)^*\alpha$ denotes the base change of α by $Y_s \to Y$ to a family of r-cycles on the fibres of X_s/Y_s and the operation $- \cap -$ was defined and studied in Section 62.11[3].

0H6I **Lemma 62.13.1.** *The construction above is bilinear, i.e., we have $(\alpha_1 + \alpha_2) \circ \beta \alpha_1 \circ \beta + \alpha_1 \circ \beta$ and $\alpha \circ (\beta_1 + \beta_2) = \alpha \circ \beta_1 + \alpha \circ \beta_2$.*

Proof. Omitted. Hint: on fibres the construction is bilinear by Lemma 62.11.1. □

0H6J **Lemma 62.13.2.** *If $U \subset X$ and $V \subset Y$ are open and $f(U) \subset V$, then $(\alpha \circ \beta)|_U$ is equal to $\alpha|_U \circ \beta|_V$.*

Proof. Omitted. Hint: on fibres use Lemma 62.11.2. □

0H6K **Lemma 62.13.3.** *The formation of $\alpha \circ \beta$ is compatible with base change.*

Proof. Let $g : S' \to S$ be a morphism of schemes. Denote $X' \to Y'$ the base change of $X \to Y$ by g. Denote α' the base change of α with respect to $Y' \to Y$. Denote β' the base change of β with respect to $S' \to S$. The assertion means that $\alpha' \circ \beta'$ is the base change of $\alpha \circ \beta$ by $g : S' \to S$.

Let $s' \in S'$ be a point with image $s \in S$. Then

$$(\alpha' \circ \beta')_{s'} = (Y'_{s'} \to Y')^*\alpha' \cap \beta'_{s'}$$

We observe that

$$(Y'_{s'} \to Y')^*\alpha' = (Y'_{s'} \to Y')^*(Y' \to Y)^*\alpha = (Y'_{s'} \to Y_s)^*(Y_s \to Y)^*\alpha$$

and that $\beta'_{s'}$ is the base change of β_s by $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$. Hence the result follows from Lemma 62.11.3 applied to $(Y_s \to Y)^*\alpha$, β_s, $X_s \to Y_s \to s$, and base change by $s' \to s$. □

0H6L **Lemma 62.13.4.** *Let $f : X \to Y$ and $Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type, with $\dim(\mathrm{Supp}(\mathcal{F}_y)) \leq r$ for all $y \in Y$. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type, with $\dim(\mathrm{Supp}(\mathcal{G}_s)) \leq e$ for all $s \in S$. If $\alpha = [\mathcal{F}/X/Y]_r$ and $\beta = [\mathcal{G}/Y/S]_e$ (Example 62.5.2) and $\mathcal{F}$ is flat over Y, then $\alpha \circ \beta = [\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}/X/S]_{r+e}$.*

Proof. First we observe that $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Observe that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_s = \mathcal{F}_s \otimes_{\mathcal{O}_{X_s}} f_s^*\mathcal{G}_s$$

by right exactness of tensor products. Moreover $\mathcal{F}_s$ is flat over Y_s as a base change of a flat module. Thus the equality $(\alpha \circ \beta)_s = [(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_s]_{r+e}$ follows from Lemma 62.11.4. □

[3]To be sure, we use $s = \mathrm{Spec}(\kappa(s))$ as the base scheme with $\delta(s) = 0$.

0H6M **Lemma 62.13.5.** Let $f : X \to Y$ and $Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let $Z \subset X$ be a closed subscheme of relative dimension $\leq r$ over Y. Let $W \subset Y$ be a closed subscheme of relative dimension $\leq e$ over S. If $\alpha = [Z/X/Y]_r$ and $\beta = [W/Y/S]_e$ (Example 62.5.4) and Z is flat over Y, then $\alpha \circ \beta = [Z \times_Y W/X/S]_{r+e}$.

Proof. This is a special case of Lemma 62.13.4 if we take $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{F} = \mathcal{O}_W$. $\square$

0H6N **Lemma 62.13.6.** Let S be a scheme. Let

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of schemes locally of finite type over S with g proper. Let $r, e \geq 0$. Let α be a family of r-cycles on the fibres of X/Y. Let β' be a family of e-cycles on the fibres of Y'/S. Then we have $f_*(g^*(\alpha) \circ \beta') = \alpha \circ g_*\beta'$.

Proof. Unwinding the definitions, this follows from Lemma 62.11.6. $\square$

0H6P **Lemma 62.13.7.** Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let $X \to Y \to Z$ be morphisms of schemes locally of finite type over S. Let $r, s, e \geq 0$. Then

$$(\alpha \circ \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma) \quad \text{in} \quad Z_{r+s+e}(X)$$

where α is a family of r-cycles on fibres of X/Y, β is a family of s-cycles on fibres of Y/Z, and $\gamma \in Z_e(Z)$.

Proof. Since we are proving an equality of cycles on X, we may work locally on Z, see Lemma 62.11.2. Thus we may assume Z is affine. In particular γ is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 62.11.1), it suffices to prove the equality when $\gamma = [W]$ for some integral closed subscheme $W \subset Z$ of δ-dimension e.

Let $z \in W$ be the generic point. Write $\beta_z = \sum m_j[V_j]$ in $Z_s(Y_z)$. Then $\beta \cap \gamma$ is equal to $\sum m_j[\overline{V}_j]$ where $\overline{V}_j \subset Y$ is an integral closed subscheme mapped by $Y \to Z$ into W with generic fibre V_j. Let $y_j \in V_j$ be the generic point. We may and do view also as the generic point of $\overline{V}_j$ (mapping to z in W). Write $\alpha_{y_j} = \sum n_{jk}[W_{jk}]$ in $Z_r(X_{y_j})$. Then $\alpha \cap (\beta \cap \gamma)$ is equal to

$$\sum m_j n_{jk}[\overline{W}_{jk}]$$

where $\overline{W}_{jk} \subset X$ is an integral closed subscheme mapped by $X \to Y$ into $\overline{V}_j$ with generic fibre W_{jk}.

On the other hand, let us consider

$$(\alpha \circ \beta)_z = (Y_z \to Y)^*\alpha \cap \beta_z = (Y_z \to Y)^*\alpha \cap (\sum m_j[V_j])$$

By the construction of $- \cap -$ this is equal to the cycle

$$\sum m_j n_{jk}[(\overline{W}_{jk})_z]$$

on X_z. Thus by definition we obtain

$$(\alpha \circ \beta) \cap [W] = \sum m_j n_{jk}[\widetilde{W}_{jk}]$$

where $\widetilde{W}_{jk} \subset X$ is an integral closed subscheme which is mapped by $X \to Z$ into W with generic fibre $(\overline{W}_{jk})_z$. Clearly, we must have $\widetilde{W}_{jk} = \overline{W}_{jk}$ and the proof is complete. □

62.14. Composition of relative cycles

0H6Q Let S be a locally Noetherian scheme. Let $X \to Y$ be a morphism of schemes locally of finite type over S. We are going to define a map

$$z(X/Y, r) \otimes_{\mathbf{Z}} z(Y/S, e) \longrightarrow z(X/S, r+e), \quad \alpha \otimes \beta \longmapsto \alpha \circ \beta$$

using the construction in Section 62.13. We already know the construction is bilinear (Lemma 62.13.1) hence we obtain the displayed arrow once we show the following.

0H6R **Lemma 62.14.1.** *If α and β are relative cycles, then so is $\alpha \circ \beta$.*

Proof. The formation of $\alpha \circ \beta$ is compatible with base change by Lemma 62.13.3. Thus we may assume S is the spectrum of a discrete valuation ring with generic point η and closed point 0 and we have to show that $sp_{X/S}((\alpha \circ \beta)_\eta) = (\alpha \circ \beta)_0$. Since we are trying to prove an equality of cycles, we may work locally on Y and X (this uses Lemmas 62.13.2 and 62.4.4 to see that the constructions commute with restriction). Thus we may assume X and Y are affine. By Lemma 62.6.9 we can find a completely decomposed proper morphism $g : Y' \to Y$ such that $g^*\alpha$ is in the image of (62.6.8.1).

Since the family of morphisms $g_\eta : Y'_\eta \to Y_\eta$ is completely decomposed, we can find $\beta'_\eta \in Z_e(Y'_\eta)$ such that $\beta_\eta = \sum g_{\eta,*}\beta'_\eta$, see Chow Homology, Lemma 42.22.4. Set $\beta'_0 = sp_{Y'/S}(\beta'_\eta)$ so that $\beta' = (\beta'_\eta, \beta'_0)$ is a relative e-cycle on Y'/S. Then $g_*\beta'$ and β are relative e-cycles on Y/S (Lemma 62.6.2) which have the same value at η and hence are equal (Lemma 62.6.6). By linearity (Lemma 62.13.1) it suffices to show that $\alpha \circ g_*\beta'$ is a relative $(r+e)$-cycle.

Set $X' = X \times_Y Y'$ and denote $f : X' \to X$ the projection. By Lemma 62.13.6 we see that $\alpha \circ g_*\beta' = f_*(g^*\alpha \circ \beta')$. By Lemma 62.6.2 it suffices to show that $g^*\alpha \circ \beta'$ is a relative $(r+e)$-cycle. Using Lemma 62.6.10 and bilinearity this reduces us to the case discussed in the next paragraph.

Assume $\alpha = [Z/X/Y]_r$ and $\beta = [W/Y/S]$ where $Z \subset X$ is a closed subscheme flat and of relative dimension $\leq r$ over Y and $W \subset Y$ is a closed subscheme flat and of relative dimension $\leq e$ over S. By Lemma 62.13.5 we see that

$$\alpha \circ \beta = [Z \times_X W/X/S]_{r+e}$$

and $Z \times_X W \subset X$ is a closed subscheme flat over S of relative dimension $\leq r + e$. This is a relative $(r+e)$-cycle by Lemma 62.6.8. □

0H6S **Lemma 62.14.2.** *Let $f : X \to Y$ and $g : Y \to S$ be a morphisms of schemes. Assume S locally Noetherian, g locally of finite type and flat of relative dimension $e \geq 0$, and f locally of finite type and flat of relative dimension $r \geq 0$. Then $[X/X/Y]_r \circ [Y/Y/S]_e = [X/X/S]_{r+e}$ in $z(X/S, r+e)$.*

Proof. Special case of Lemma 62.13.5. □

62.15. Comparison with Suslin and Voevodsky

0H6T We have tried to use the same notation as in [SV00], except that our notation for cycles is taken from Chow Homology, Section 42.8 ff. Here is a comparison:

(1) In [SV00, Section 3.1] there is a notion of a "relative cycle", of a "relative cycle of dimension r", and of a "equidimensional relative cycle of dimension r". There is no corresponding notion in this chapter. Consequently, the groups $Cycl(X/S, r)$, $Cycl_{equi}(X/S, r)$, $PropCycl(X/S, r)$, and $PropCycl_{equi}(X/S, r)$, have no counter parts in this chapter.
(2) On the bottom of [SV00, page 36] the groups $z(X/S, r)$, $c(X/S, r)$, $z_{equi}(X/S, r)$, $c_{equi}(X/S, r)$ are defined. These agree with our notions when S is separated Noetherian and $X \to S$ is separated and of finite type.
(3) In [SV00] the symbol $z(X/S, r)$ is sometimes used for the presheaf $S' \mapsto z(S' \times_S X/S', r)$ on the category of schemes of finite type over S. Similarly for $c(X/S, r)$, $z_{equi}(X/S, r)$, and $c_{equi}(X/S, r)$.
(4) Base change, flat pullback, and proper pushforward as defined in [SV00] agrees with ours when both apply.
(5) For $\alpha \in z(X/S, r)$ the operation $\alpha \cap - : Z_e(S) \to Z_{e+r}(X)$ defined in Section 62.11 agrees with the operation $Cor(\alpha, -)$ in [SV00, Section 3.7] when both are defined.
(6) For $X \to Y \to S$ the composition law $z(X/Y, r) \otimes_{\mathbf{Z}} z(Y/S, e) \longrightarrow z(X/S, r+e)$ defined in Section 62.14 agrees with the opration $Cor_{X/Y}(-, -)$ in [SV00, Corollary 3.7.5].

62.16. Relative cycles in the non-Noetherian case

0H6U We urge the reader to skip this section.

Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $r \geq 0$. Denote $Hilb(X/S, r)$ the set of closed subschemes $Z \subset X$ such that $Z \to S$ is flat, of finite presentation, and of relative dimension $\leq r$. We consider the group homomorphism

0H6V (62.16.0.1)
$$\begin{matrix}\text{free abelian group} \\ \text{on } Hilb(X/S, r)\end{matrix} \longrightarrow \begin{matrix}\text{families of } r\text{-cycles} \\ \text{on fibres of } X/S\end{matrix}$$

sending $\sum n_i[Z_i]$ to $\sum n_i[Z_i/X/S]_r$.

0H6W **Lemma 62.16.1.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $r \geq 0$ and let α be a family of r-cycles on fibres of X/S. The following are equivalent

(1) there exists a cartesian diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$
where $X_0 \to S_0$ is a finite type morphism of Noetherian schemes and $\alpha_0 \in z(X_0/S_0, r)$ such that α is the base change of α_0 by $S \to S_0$
(2) there exists a completely decomposed proper morphism $g : S' \to S$ of finite presentation such that $g^*\alpha$ is in the image of (62.16.0.1).

Proof. Let a diagram and $\alpha_0 \in z(X_0/S_0, r)$ as in (1) be given. By Lemma 62.6.9 there exists a proper surjective morphism $g_0 : S'_0 \to S_0$ such that $g_0^*\alpha_0$ is in the image of (62.16.0.1). Namely, since S'_0 is Noetherian, every closed subscheme of $S'_0 \times_{S_0} X_0$ is of finite presentation over S'_0. Setting $S' = S \times_{S_0} S'_0$ and using base change by $S' \to S'_0$ we see that (2) holds.

Conversely, assume that (2) holds. Choose a surjective proper morphism $g : S' \to S$ of finite presentation such that $g^*\alpha$ is in the image of (62.16.0.1). Set $X' = S' \times_S X$. Write $g^*\alpha = \sum n_a[Z_a/X'/S']_r$ for some $Z_a \subset X'$ closed subscheme flat, of finite presentation, and of relative dimension $\leq r$ over S'.

Write $S = \lim S_i$ as a directed limit with affine transition morphisms with S_i of finite type over $\mathbf{Z}$, see Limits, Proposition 32.5.4. We can find an i large enough such that there exist

(1) a completely decomposed proper morphism $g_i : S'_i \to S_i$ whose base change to S is $g : S' \to S$,
(2) setting $X'_i = S'_i \times_{S_i} X_i$ closed subschemes $Z_{ai} \subset X'_i$ flat and of relative dimension $\leq r$ over S'_i whose base change to S' is Z_a.

To do this one uses Limits, Lemmas 32.10.1, 32.8.5, 32.8.7, 32.8.15, 32.13.1, and 32.18.1 and and More on Morphisms, Lemma 37.78.5. Consider $\alpha'_i = \sum n_a[Z_{ai}/X'_i/S_i]_r \in z(X'_i/S'_i, r)$. The base change of α'_i to a family of r-cycles on fibres of X'/S' agrees with the base change $g^*\alpha$ by construction.

Set $S''_i = S'_i \times_{S_i} S'_i$ and $X''_i = S''_i \times_{S_i} X_i$ and set $S'' = S' \times_S S'$ and $X'' = S'' \times_S X$. We denote $\text{pr}_1, \text{pr}_2 : S'' \to S'$ and $\text{pr}_1, \text{pr}_2 : S''_i \to S'_i$ the projections. The relative r-cycles $\text{pr}_1^*\alpha'_i$ and $\text{pr}_1^*\alpha'_i$ on X''_i/S''_i base change to the same family of r-cycles on fibres of X''/S'' because $\text{pr}_1^*g^*\alpha = \text{pr}_1^*g^*\alpha$. Hence the morphism $S'' \to S''_i$ maps into $E = \{s \in S''_i : (\text{pr}_1^*\alpha'_i)_s = (\text{pr}_1^*\alpha'_i)_s\}$. By Lemma 62.6.12 this is a closed subset. Since $S'' = \lim_{i' \geq i} S''_{i'}$ we see from Limits, Lemma 32.4.10 that for some $i' \geq i$ the morphism $S''_{i'} \to S''_i$ maps into E. Therefore, after replacing i by i', we may assume that $\text{pr}_1^*\alpha'_i = \text{pr}_1^*\alpha'_i$. By Lemma 62.5.9 we obtain a unique family α_i of r-cycles on fibres of X_i/S_i with $g_i^*\alpha_i = \alpha'_i$ (this uses that $S'_i \to S_i$ is completely decomposed). By Lemma 62.6.3 we see that $\alpha_i \in z(X_i/S_i, r)$. The uniqueness in Lemma 62.5.9 implies that the base change of α_i is α and we see (1) holds. $\square$

Discussion. If $f : X \to S$, r, and α are as in Lemma 62.16.1, then it makes sense to say that α is a relative r-cycle on X/S if the equivalent conditions (1) and (2) of Lemma 62.16.1 hold. This definition has many good properties; for example it doesn't conflict with the earlier definition in case S is Noetherian and most of the results of Section 62.6 generalize to this setting.

We may still generalize further as follows. Assume S is arbitrary and $f : X \to S$ is locally of finite presentation. Let $r \geq 0$ and let α be a family of r-cycles α on fibres of X/S. Then α is an relative r-cycle on X/S if for $U \subset X$ and $V \subset S$ affine open with $f(U) \subset V$ the restriction $\alpha|_U$ is a relative r-cycle on U/V as defined in the previous paragraph. Again many of the earlier results generalize to this setting.

If we ever need these generalizations we will carefully state and prove them here.

62.17. Other chapters

CHAPTER 63

More Étale Cohomology

0F4U

63.1. Introduction

0F4V This chapter is the second in a series of chapter on the étale cohomology of schemes. To read the first chapter, please visit Étale Cohomology, Section 59.1.

The split with the previous chapter is roughly speaking that anything concerning "shriek functors" (cohomology with compact support and its right adjoint) and anything using this material goes into this chapter.

63.2. Growing sections

0F71 In this section we discuss results of the following type.

0F6F Lemma 63.2.1. Let X be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $\varphi : U' \to U$ be a morphism of $X_{étale}$. Let $Z' \subset U'$ be a closed subscheme such that $Z' \to U' \to U$ is a closed immersion with image $Z \subset U$. Then there is a canonical bijection

$$\{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\} = \{s' \in \mathcal{F}(U') \mid \text{Supp}(s') \subset Z'\}$$

which is given by restriction if $\varphi^{-1}(Z) = Z'$.

Proof. Consider the closed subscheme $Z'' = \varphi^{-1}(Z)$ of U'. Then $Z' \subset Z''$ is closed because Z' is closed in U'. On the other hand, $Z' \to Z''$ is an étale morphism (as a morphism between schemes étale over Z) and hence open. Thus $Z'' = Z' \amalg T$ for some closed subset T. The open covering $U' = (U' \setminus T) \cup (U' \setminus Z')$ shows that

$$\{s' \in \mathcal{F}(U') \mid \text{Supp}(s') \subset Z'\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \text{Supp}(s') \subset Z'\}$$

and the étale covering $\{U' \setminus T \to U, U \setminus Z \to U\}$ shows that

$$\{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \text{Supp}(s') \subset Z'\}$$

This finishes the proof. $\square$

0F6G Lemma 63.2.2. Let X be a scheme. Let $Z \subset X$ be a locally closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Given $U, U' \subset X$ open containing Z as a closed subscheme, there is a canonical bijection

$$\{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\} = \{s \in \mathcal{F}(U') \mid \text{Supp}(s) \subset Z\}$$

which is given by restriction if $U' \subset U$.

Proof. Since Z is a closed subscheme of $U \cap U'$, it suffices to prove the lemma when $U' \subset U$. Then it is a special case of Lemma 63.2.1. $\square$

Let us introduce a bit of nonstandard notation which will stand us in good stead later. Namely, in the situation of Lemma 63.2.2 above, let us denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\}$$

where $U \subset X$ is any choice of open subscheme containing Z as a closed subscheme. The reader who is troubled by the lack of precision this entails may choose $U = X \setminus \partial Z$ where $\partial Z = \overline{Z} \setminus Z$ is the "boundary" of Z in X. However, in many of the arguments below the flexibility of choosing different opens will play a role. Here are some properties of this construction:

0F6H (1) If $Z \subset Z'$ are locally closed subschemes of X and Z is closed in Z', then there is a natural injective map

$$H_Z(\mathcal{F}) \to H_{Z'}(\mathcal{F}).$$

0F6I (2) If $f : Y \to X$ is a morphism of schemes and $Z \subset X$ is a locally closed subscheme, then there is a natural pullback map $f^* : H_Z(\mathcal{F}) \to H_{f^{-1}Z}(f^{-1}\mathcal{F})$.

It will be convenient to extend our notation to the following situation: suppose that we have $W \in X_{\acute{e}tale}$ and a locally closed subscheme $Z \subset W$. Then we will denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\} = H_Z(\mathcal{F}|_{W_{\acute{e}tale}})$$

where $U \subset W$ is any choice of open subscheme containing Z as a closed subscheme, exactly as above[1].

63.3. Sections with compact support

0F4W A reference for this section is [AGV71, Exposee XVII, Section 6]. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. In this section we define a functor $f_! : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(Y_{\acute{e}tale})$ by taking $f_!\mathcal{F} \subset f_*\mathcal{F}$ to be the subsheaf of sections which have proper support relative to Y (suitably defined).

Warning: The functor $f_!$ is the zeroth cohomology sheaf of a functor $Rf_!$ on the derived category (insert future reference), but $Rf_!$ is not the derived functor of $f_!$.

0F4X **Lemma 63.3.1.** Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. The rule

$$Y_{\acute{e}tale} \longrightarrow \text{Ab}, \quad V \longmapsto \{s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V) \mid \text{Supp}(s) \subset X_V \text{ is proper over } V\}$$

is an abelian subsheaf of $f_*\mathcal{F}$.

Warning: This sheaf isn't the "correct one" if f is not separated.

Proof. Recall that the support of a section is closed (Étale Cohomology, Lemma 59.31.4) hence the material in Cohomology of Schemes, Section 30.26 applies. By the lemma above and Cohomology of Schemes, Lemma 30.26.6 we find that our subset of $f_*\mathcal{F}(V)$ is a subgroup. By Cohomology of Schemes, Lemma 30.26.4 we see that our rule defines a sub presheaf. Finally, suppose that we have $s \in f_*\mathcal{F}(V)$ and an étale covering $\{V_i \to V\}$ such that $s|_{V_i}$ has support proper over V_i. Observe that the support of $s|_{V_i}$ is the inverse image of the support of $s|_V$ (use the characterization of the support in terms of stalks and Étale Cohomology, Lemma 59.36.2). Whence the support of s is proper over V by Descent, Lemma 35.25.5. This proves that our rule satisfies the sheaf condition. $\square$

[1]In fact, Lemma 63.2.1 shows, given Z over X which is isomorphic to a locally closed subscheme of some object W of $X_{\acute{e}tale}$, that the choice of W is irrelevant.

0F4Y Lemma 63.3.2. Let $j : U \to X$ be a separated étale morphism. Let $\mathcal{F}$ be an abelian sheaf on $U_{étale}$. The image of the injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ of Étale Cohomology, Lemma 59.70.6 is the subsheaf of Lemma 63.3.1.

An alternative would be to move this lemma later and prove this using the descrition of the stalks of both sheaves.

Proof. The construction of $j_!\mathcal{F} \to j_*\mathcal{F}$ in the proof of Étale Cohomology, Lemma 59.70.6 is via the construction of a map $j_{p!}\mathcal{F} \to j_*\mathcal{F}$ of presheaves whose image is clearly contained in the subsheaf of Lemma 63.3.1. Hence since $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$ we conclude the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ is contained in this subsheaf. Conversely, let $s \in j_*\mathcal{F}(V)$ have support Z proper over V. Then $Z \to V$ is finite with closed image $Z' \subset V$, see More on Morphisms, Lemma 37.44.1. The restriction of s to $V \setminus Z'$ is zero and the zero section is contained in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$. On the other hand, if $v \in Z'$, then we can find an étale neighbourhood $(V', v') \to (V, v)$ such that we have a decomposition $U_{V'} = W \amalg U'_1 \amalg \ldots \amalg U'_n$ into open and closed subschemes with $U'_i \to V'$ an isomorphism and with $T_{V'} \subset U'_1 \amalg \ldots \amalg U'_n$, see Étale Morphisms, Lemma 41.18.2. Inverting the isomorphisms $U'_i \to V'$ we obtain n morphisms $\varphi'_i : V' \to U$ and sections s'_i over V' by pulling back s. Then the section $\sum(\varphi'_i, s'_i)$ of $j_{p!}\mathcal{F}$ over V', see formula for $j_{p!}\mathcal{F}(V')$ in proof of Étale Cohomology, Lemma 59.70.6, maps to the restriction of s to V' by construction. We conclude that s is étale locally in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ and the proof is complete. □

0F4Z Definition 63.3.3. Let $f : X \to Y$ be a morphism of schemes which is separated (!) and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. The subsheaf $f_!\mathcal{F} \subset f_*\mathcal{F}$ constructed in Lemma 63.3.1 is called the direct image with compact support.

By Lemma 63.3.2 this does not conflict with Étale Cohomology, Definition 59.70.1 as we have agreement when both definitions apply. Here is a sanity check.

0F51 Lemma 63.3.4. Let $f : X \to Y$ be a proper morphism of schemes. Then $f_! = f_*$.

Proof. Immediate from the construction of $f_!$. □

A very useful observation is the following.

0F53 Remark 63.3.5 (Covariance with respect to open embeddings). Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $X' \subset X$ be an open subscheme. Denote $f' : X' \to Y$ the restriction of f. There is a canonical injective map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, let $V \in Y_{étale}$ and consider a section $s' \in f'_*(\mathcal{F}|_{X'})(V) = \mathcal{F}(X' \times_Y V)$ with support Z' proper over V. Then Z' is closed in $X \times_Y V$ as well, see Cohomology of Schemes, Lemma 30.26.5. Thus there is a unique section $s \in \mathcal{F}(X \times_Y V) = f_*\mathcal{F}(V)$ whose restriction to $X' \times_Y V$ is s' and whose restriction to $X \times_Y V \setminus Z'$ is zero, see Lemma 63.2.2. This construction is compatible with restriction maps and hence induces the desired map of sheaves $f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$ which is clearly injective. By construction we obtain a commutative diagram

$$\begin{tikzcd} f'_!(\mathcal{F}|_{X'}) \arrow[r] \arrow[d] & f_!\mathcal{F} \arrow[d] \\ f'_*(\mathcal{F}|_{X'}) & f_*\mathcal{F} \arrow[l] \end{tikzcd}$$

functorial in $\mathcal{F}$. It is clear that for $X'' \subset X'$ open with $f'' = f|_{X''} : X'' \to Y$ the composition of the canonical maps $f''_!\mathcal{F}|_{X''} \to f'_!\mathcal{F}|_{X'} \to f_!\mathcal{F}$ just constructed is the canonical map $f''_!\mathcal{F}|_{X''} \to f_!\mathcal{F}$.

0F52 Lemma 63.3.6. Let Y be a scheme. Let $j : X \to \overline{X}$ be an open immersion of schemes over Y with $\overline{X}$ proper over Y. Denote $f : X \to Y$ and $\overline{f} : \overline{X} \to Y$ the structure morphisms. For $\mathcal{F} \in \textit{Ab}(X_{\acute{e}tale})$ there is a canonical isomorphism (see proof)

$$f_!\mathcal{F} \longrightarrow \overline{f}_!j_!\mathcal{F}$$

As we have $\overline{f}_! = \overline{f}_*$ by Lemma 63.3.4 we obtain $\overline{f}_* \circ j_! = f_!$ as functors $\textit{Ab}(X_{\acute{e}tale}) \to \textit{Ab}(Y_{\acute{e}tale})$.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 59.70.4. Thus the displayed arrow is the injective map $f_!(\mathcal{G}|_X) \to \overline{f}_!\mathcal{G}$ of Remark 63.3.5 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $V \in Y_{\acute{e}tale}$ and $s \in \overline{f}_!\mathcal{G}(V) = \overline{f}_*\mathcal{G}(V) = \mathcal{G}(\overline{X}_V)$ is a section, then the support of s is contained in the open $X_V \subset \overline{X}_V$. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. $\square$

We want to relate the stalks of $f_!\mathcal{F}$ to sections with compact support on fibres. In order to state this, we need a definition.

0F72 Definition 63.3.7. Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. We let $H^0_c(X, \mathcal{F}) \subset H^0(X, \mathcal{F})$ be the set of sections whose support is proper over k. Elements of $H^0_c(X, \mathcal{F})$ are called sections with compact support.

Warning: This definition isn't the "correct one" if X isn't separated over k.

0F73 Lemma 63.3.8. Let X be a proper scheme over a field k. Then $H^0_c(X, \mathcal{F}) = H^0(X, \mathcal{F})$.

Proof. Immediate from the construction of H^0_c. $\square$

0F74 Remark 63.3.9 (Open embeddings and compactly supported sections). Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Exactly as in Remark 63.3.5 for $X' \subset X$ open there is an injective map

$$H^0_c(X', \mathcal{F}|_{X'}) \longrightarrow H^0_c(X, \mathcal{F})$$

and these maps turn H^0_c into a "cosheaf" on the Zariski site of X.

0F75 Lemma 63.3.10. Let k be a field. Let $j : X \to \overline{X}$ be an open immersion of schemes over k with $\overline{X}$ proper over k. For $\mathcal{F} \in \textit{Ab}(X_{\acute{e}tale})$ there is a canonical isomorphism (see proof)

$$H^0_c(X, \mathcal{F}) \longrightarrow H^0_c(\overline{X}, j_!\mathcal{F}) = H^0(\overline{X}, j_!\mathcal{F})$$

where we have the equality on the right by Lemma 63.3.8.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 59.70.4. Thus the displayed arrow is the injective map $H^0_c(X, \mathcal{G}|_X) \to H^0_c(\overline{X}, \mathcal{G})$ of Remark 63.3.9 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $s \in H^0(\overline{X}, \mathcal{G})$ is a section, then the support of s is contained in the open X. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. ☐

0F76 Lemma 63.3.11. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then there is a canonical isomorphism

$$(f_!\mathcal{F})_{\overline{y}} \longrightarrow H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Recall that $(f_*\mathcal{F})_{\overline{y}} = \mathrm{colim}\, f_*\mathcal{F}(V)$ where the colimit is over the étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$. If $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$, then we can pullback s to a section of $\mathcal{F}$ over $(X_V)_{\overline{v}} = X_{\overline{y}}$. Thus we obtain a canonical map

$$c_{\overline{y}} : (f_*\mathcal{F})_{\overline{y}} \longrightarrow H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

We claim that this map induces a bijection between the subgroups $(f_!\mathcal{F})_{\overline{y}}$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. The claim implies the lemma, but is a little bit more precise in that it describes the identification of the lemma as given by pullbacks of sections of $\mathcal{F}$ to the geometric fibre of f.

Observe that any element $s \in (f_!\mathcal{F})_{\overline{y}} \subset (f_*\mathcal{F})_{\overline{y}}$ is mapped by $c_{\overline{y}}$ to an element of $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \subset H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. This is true because taking the support of a section commutes with pullback and because properness is preserved by base change. This at least produces the map in the statement of the lemma. To prove that it is an isomorphism we may work Zariski locally on Y and hence we may and do assume Y is affine.

An observation that we will use below is that given an open subscheme $X' \subset X$ and if $f' = f|_{X'}$, then we obtain a commutative diagram

$$\begin{tikzcd}
(f'_!(\mathcal{F}|_{X'}))_{\overline{y}} \arrow[r] \arrow[d] & H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \arrow[d] \\
(f_!\mathcal{F})_{\overline{y}} \arrow[r] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})
\end{tikzcd}$$

where the horizontal arrows are the maps constructed above and the vertical arrows are given in Remarks 63.3.5 and 63.3.9. The reason is that given an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$ and a section $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ whose support Z happens to be contained in X'_V and is proper over V, so that s gives rise to an element of both $(f'_!(\mathcal{F}|_{X'}))_{\overline{y}}$ and $(f_!\mathcal{F})_{\overline{y}}$ which correspond via the vertical arrow of the diagram, then these elements are mapped via the horizontal arrows to the pullback $s|_{X_{\overline{y}}}$ of s to $X_{\overline{y}}$ whose support $Z_{\overline{y}}$ is contained in $X'_{\overline{y}}$ and hence this restriction gives rise to a compatible pair of elements of $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$.

Suppose $s \in (f_!\mathcal{F})_{\overline{y}}$ maps to zero in $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Say s corresponds to $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ with support Z proper over V. We may assume that V is affine and hence Z is quasi-compact. Then we may choose a quasi-compact open $X' \subset X$ containing the image of Z. Then Z is contained in X'_V and hence s is the image of an element $s' \in f'_!(\mathcal{F}|_{X'})(V)$ where $f' = f|_{X'}$ as in the previous paragraph. Then s' maps to zero in $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$. Hence in order to prove injectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

Suppose that $t \in H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Then the support of t is contained in a quasi-compact open subscheme $W \subset X_{\overline{y}}$. Hence we can find a quasi-compact open subscheme $X' \subset X$ such that $X'_{\overline{y}}$ contains W. Then it is clear that t is contained in the image of the injective map $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \to H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Hence in order to show surjectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

In this last paragraph of the proof we prove the lemma in case X is quasi-compact and Y is affine. By More on Flatness, Theorem 38.33.8 there exists a compactification $j : X \to \overline{X}$ over Y. Set $\mathcal{G} = j_!\mathcal{F}$ so that $\mathcal{F} = \mathcal{G}|_X$ by Étale Cohomology, Lemma 59.70.4. By the disussion above we get a commutative diagram

$$\begin{tikzcd}
(f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[d] \\
(\overline{f}_!\mathcal{G})_{\overline{y}} \arrow[r] & H^0_c(\overline{X}_{\overline{y}}, \mathcal{G}|_{\overline{X}_{\overline{y}}})
\end{tikzcd}$$

By Lemmas 63.3.6 and 63.3.10 the vertical maps are isomorphisms. This reduces us to the case of the proper morphism $\overline{X} \to Y$. For a proper morphism our map is an isomorphism by Lemmas 63.3.4 and 63.3.8 and proper base change for pushforwards, see Étale Cohomology, Lemma 59.91.4. □

0F55 Lemma 63.3.12. Consider a cartesian square

$$\begin{tikzcd}
X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

of schemes with f separated and locally of finite type. For any abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $f'_!(g')^{-1}\mathcal{F} = g^{-1}f_!\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 7.45. We claim that this map sends $g^{-1}f_!\mathcal{F}$ into the subsheaf $f'_!(g')^{-1}\mathcal{F}$ and induces the isomorphism in the lemma.

Choose a geometric point $\overline{y}' : \mathrm{Spec}(k) \to Y'$ and denote $\overline{y} = g \circ \overline{y}'$ the image in Y. There is a commutative diagram

$$\begin{tikzcd}
(f_*\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[d] \\
(f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[r] & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}})
\end{tikzcd}$$

where the horizontal maps were used in the proof of Lemma 63.3.11 and the vertical maps are the pullback maps above. The diagram commutes because each of the four maps in question is given by pulling back local sections along a morphism of schemes and the underlying diagram of morphisms of schemes commutes. Since the diagram in the statement of the lemma is cartesian we have $X'_{\overline{y}'} = X_{\overline{y}}$. Hence by Lemma 63.3.11 and its proof we obtain a commutative diagram

$$
\begin{tikzcd}
(f_*\mathcal{F})_{\overline{y}} \arrow[rrr] \arrow[ddd] & & & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[ddd] \\
& (f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[ul] \arrow[d, dotted] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[ur] \arrow[d] & \\
& (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[r] \arrow[dl] & H^0_c(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}}) \arrow[dr] & \\
(f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[rrr] & & & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}})
\end{tikzcd}
$$

where the horizontal arrows of the inner square are isomorphisms and the two right vertical arrows are equalities. Also, the se, sw, ne, nw arrows are injective. It follows that there is a unique bijective dotted arrow fitting into the diagram. We conclude that $g^{-1}f_!\mathcal{F} \subset g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$ is mapped into the subsheaf $f'_!(g')^{-1}\mathcal{F} \subset f'_*(g')^{-1}\mathcal{F}$ because this is true on stalks, see Étale Cohomology, Theorem 59.29.10. The same theorem then implies that the induced map is an isomorphism and the proof is complete. □

0F50 Lemma 63.3.13. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of schemes which are separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Then $g_!f_!\mathcal{F} = (g \circ f)_!\mathcal{F}$ as subsheaves of $(g \circ f)_*\mathcal{F}$.

Proof. We strongly urge the reader to prove this for themselves. Let $W \in Z_{\acute{e}tale}$ and $s \in (g \circ f)_*\mathcal{F}(W) = \mathcal{F}(X_W)$. Denote $T \subset X_W$ the support of s; this is a closed subset. Observe that s is a section of $(g \circ f)_!\mathcal{F}$ if and only if T is proper over W. We have $f_!\mathcal{F} \subset f_*\mathcal{F}$ and hence $g_!f_!\mathcal{F} \subset g_!f_*\mathcal{F} \subset g_*f_*\mathcal{F}$. On the other hand, s is a section of $g_!f_!\mathcal{F}$ if and only if (a) T is proper over Y_W and (b) the support T' of s viewed as section of $f_!\mathcal{F}$ is proper over W. If (a) holds, then the image of T in Y_W is closed and since $f_!\mathcal{F} \subset f_*\mathcal{F}$ we see that $T' \subset Y_W$ is the image of T (details omitted; look at stalks).

The conclusion is that we have to show a closed subset $T \subset X_W$ is proper over W if and only if T is proper over Y_W and the image of T in Y_W is proper over W. Let us endow T with the reduced induced closed subscheme structure. If T is proper over W, then $T \to Y_W$ is proper by Morphisms, Lemma 29.41.7 and the image of T in Y_W is proper over W by Cohomology of Schemes, Lemma 30.26.5. Conversely, if T is proper over Y_W and the image of T in Y_W is proper over W, then the morphism $T \to W$ is proper as a composition of proper morphisms (here we endow the closed image of T in Y_W with its reduced induced scheme structure to turn the question into one about morphisms of schemes), see Morphisms, Lemma 29.41.4. □

0F77 Remark 63.3.14. The isomorphisms between functors constructed above satisfy the following two properties:

(1) Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to T$ be composable morphisms of schemes which are separated and locally of finite type. Then the diagram

$$\begin{tikzcd} (h \circ g \circ f)_! \arrow[r] \arrow[d] & (h \circ g)_! \circ f_! \arrow[d] \\ h_! \circ (g \circ f)_! \arrow[r] & h_! \circ g_! \circ f_! \end{tikzcd}$$

commutes where the arrows are those of Lemma 63.3.13.

(2) Suppose that we have a diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "c"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "a"] & Z \end{tikzcd}$$

with both squares cartesian and f and g separated and locally of finite type. Then the diagram

$$\begin{tikzcd} a^{-1} \circ (g \circ f)_! \arrow[rr] \arrow[d] & & (g' \circ f')_! \circ c^{-1} \arrow[d] \\ a^{-1} \circ g_! \circ f_! \arrow[r] & g'_! \circ b^{-1} \circ f_! \arrow[r] & g'_! \circ f'_! \circ c^{-1} \end{tikzcd}$$

commutes where the horizontal arrows are those of Lemma 63.3.12 the arrows are those of Lemma 63.3.13.

Part (1) holds true because we have a similar commutative diagram for pushforwards. Part (2) holds by the very general compatibility of base change maps for pushforwards (Sites, Remark 7.45.3) and the fact that the isomorphisms in Lemmas 63.3.12 and 63.3.13 are constructed using the corresponding maps fo pushforwards.

0F54 Lemma 63.3.15. Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $X = \bigcup_{i \in I} X_i$ be an open covering such that for all $i, j \in I$ there exists a k with $X_i \cup X_j \subset X_k$. Denote $f_i : X_i \to Y$ the restriction of f. Then

$$f_!\mathcal{F} = \text{colim}_{i \in I} f_{i,!}(\mathcal{F}|_{X_i})$$

functorially in $\mathcal{F} \in \text{Ab}(X_{\acute{e}tale})$ where the transition maps are the ones constructed in Remark 63.3.5.

Proof. It suffices to show that the canonical map from right to left is a bijection when evaluated on a quasi-compact object V of $Y_{\acute{e}tale}$. Observe that the colimit on the right hand side is directed and has injective transition maps. Thus we can use Sites, Lemma 7.17.7 to evaluate the colimit. Hence, the statement comes down to the observation that a closed subset $Z \subset X_V$ proper over V is quasi-compact and hence is contained in $X_{i,V}$ for some i. $\square$

0F56 Lemma 63.3.16. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Then functor $f_!$ commutes with direct sums.

Proof. Let $\mathcal{F} = \bigoplus \mathcal{F}_i$. To show that the map $\bigoplus f_!\mathcal{F}_i \to f_!\mathcal{F}$ is an isomorphism, it suffices to show that these sheaves have the same sections over a quasi-compact object V of $Y_{étale}$. Replacing Y by V it suffices to show $H^0(Y, f_!\mathcal{F}) \subset H^0(X, \mathcal{F})$ is equal to $\bigoplus H^0(Y, f_!\mathcal{F}_i) \subset \bigoplus H^0(X, \mathcal{F}_i) \subset H^0(X, \bigoplus \mathcal{F}_i)$. In this case, by writing X as the union of its quasi-compact opens and using Lemma 63.3.15 we reduce to the case where X is quasi-compact as well. Then $H^0(X, \mathcal{F}) = \bigoplus H^0(X, \mathcal{F}_i)$ by Étale Cohomology, Theorem 59.51.3. Looking at supports of sections the reader easily concludes. □

0F57 Lemma 63.3.17. Let $f : X \to Y$ be a morphism of schemes which is separated and locally quasi-finite. Then

(1) for $\mathcal{F}$ in $\textit{Ab}(X_{étale})$ and a geometric point $\overline{y} : \text{Spec}(k) \to Y$ we have
$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$
functorially in $\mathcal{F}$, and
(2) the functor $f_!$ is exact.

Proof. The functor $f_!$ is left exact by construction. Right exactness may be checked on stalks (Étale Cohomology, Theorem 59.29.10). Thus it suffices to prove part (1).

Let $\overline{y} : \text{Spec}(k) \to Y$ be a geometric point. The scheme $X_{\overline{y}}$ has a discrete underlying topological space (Morphisms, Lemma 29.20.8) and all the residue fields at the points are equal to k (as finite extensions of k). Hence $\{\overline{x} : \text{Spec}(k) \to X : f(\overline{x}) = \overline{y}\}$ is equal to the set of points of $X_{\overline{y}}$. Thus the computation of the stalk follows from the more general Lemma 63.3.11. □

63.4. Sections with finite support

0F6E In this section we extend the construction of Section 63.3 to not necessarily separated locally quasi-finite morphisms.

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Given V in $Y_{étale}$ denote $X_V = X \times_Y V$ the base change. We are going to consider the group of finite formal sums

0F6J (63.4.0.1)
$$s = \sum\nolimits_{i=1,\ldots,n} (Z_i, s_i)$$

where $Z_i \subset X_V$ is a locally closed subscheme such that the morphism $Z_i \to V$ is finite[2] and where $s_i \in H_{Z_i}(\mathcal{F})$. Here, as in Section 63.2, we set
$$H_{Z_i}(\mathcal{F}) = \{s_i \in \mathcal{F}(U_i) \mid \text{Supp}(s_i) \subset Z_i\}$$
where $U_i \subset X_V$ is an open subscheme containing Z_i as a closed subscheme. We are going to consider these formal sums modulo the following relations

0F6K (1) $(Z, s) + (Z, s') = (Z, s + s')$,
0F6L (2) $(Z, s) = (Z', s)$ if $Z \subset Z'$.

Observe that the second relation makes sense: since $Z \to V$ is finite and $Z' \to V$ is separated, the inclusion $Z \to Z'$ is closed and we can use the map discussed in (1).

Let us denote $f_{p!}\mathcal{F}(V)$ the quotient of the abelian group of formal sums (63.4.0.1) by these relations. The first relation tells us that $f_{p!}\mathcal{F}(V)$ is a quotient of the direct

[2]Since f is locally quasi-finite, the morphism $Z_i \to V$ is finite if and only if it is proper.

sum of the abelian groups $H_Z(\mathcal{F})$ over all locally closed subschemes $Z \subset X_V$ finite over V. The second relation tells us that we are really taking the colimit

0F6M (63.4.0.2)
$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F})$$

This formula will be a convenient abstract way to think about our construction.

Next, we observe that there is a natural way to turn this construction into a presheaf $f_{p!}\mathcal{F}$ of abelian groups on $Y_{\acute{e}tale}$. Namely, given $V' \to V$ in $Y_{\acute{e}tale}$ we obtain the base change morphism $X_{V'} \to X_V$. If $Z \subset X_V$ is a locally closed subscheme finite over V, then the scheme theoretic inverse image $Z' \subset X_{V'}$ is finite over V'. Moreover, if $U \subset X_V$ is an open such that Z is closed in U, then the inverse image $U' \subset X_{V'}$ is an open such that Z' is closed in U'. Hence the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ of $\mathcal{F}$ sends $H_Z(\mathcal{F})$ into $H_{Z'}(\mathcal{F})$; this is a special case of the functoriality discussed in (2) above. Clearly, these maps are compatible with inclusions $Z_1 \subset Z_2$ of such locally closed subschemes of X_V and we obtain a map

$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F}) \longrightarrow \mathrm{colim}_{Z'}\, H_{Z'}(\mathcal{F}) = f_{p!}\mathcal{F}(V')$$

These maps indeed turn $f_{p!}\mathcal{F}$ into a presheaf of abelian groups on $Y_{\acute{e}tale}$. We omit the details.

A final observation is that the construction of $f_{p!}\mathcal{F}$ is functorial in $\mathcal{F}$ in $\mathrm{Ab}(X_{\acute{e}tale})$. We conclude that given a locally quasi-finite morphism $f : X \to Y$ we have constructed a functor

$$f_{p!} : \mathrm{Ab}(X_{\acute{e}tale}) \longrightarrow \mathrm{PAb}(Y_{\acute{e}tale})$$

from the category of abelian sheaves on $X_{\acute{e}tale}$ to the category of abelian presheaves on $Y_{\acute{e}tale}$. Before we define $f_!$ as the sheafification of this functor, let us check that it agrees with the construction in Section 63.3 and with the construction in Étale Cohomology, Section 59.70 when both apply.

0F6N **Lemma 63.4.1.** Let $f : X \to Y$ be a separated and locally quasi-finite morphism of schemes. Functorially in $\mathcal{F} \in \mathrm{Ab}(X_{\acute{e}tale})$ there is a canonical isomorphism(!)

$$f_{p!}\mathcal{F} \longrightarrow f_!\mathcal{F}$$

of abelian presheaves which identifies the sheaf $f_!\mathcal{F}$ of Definition 63.3.3 with the presheaf $f_{p!}\mathcal{F}$ constructed above.

Proof. Let V be an object of $Y_{\acute{e}tale}$. If $Z \subset X_V$ is locally closed and finite over V, then, since f is separated, we see that the morphism $Z \to X_V$ is a closed immersion. Moreover, if Z_i, $i = 1, \ldots, n$ are closed subschemes of X_V finite over V, then $Z_1 \cup \ldots \cup Z_n$ (scheme theoretic union) is a closed subscheme finite over V. Hence in this case the colimit (63.4.0.2) defining $f_{p!}\mathcal{F}(V)$ is directed and we find that $f_{!p}\mathcal{F}(V)$ is simply equal to the set of sections of $\mathcal{F}(X_V)$ whose support is finite over V. Since any closed subset of X_V which is proper over V is actually finite over V (as f is locally quasi-finite) we conclude that this is equal to $f_!\mathcal{F}(V)$ by its very definition. $\square$

0F6P **Lemma 63.4.2.** Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Functorially in $\mathcal{F}$ in $\mathrm{Ab}(X_{\acute{e}tale})$ we have

$$(f_{p!}\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$

Proof. Recall that the stalk at $\overline{y}$ of a presheaf is defined by the usual colimit over étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$, see Étale Cohomology, Definition 59.29.6. Accordingly suppose $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (63.4.0.1) is an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Then since

$$X_{\overline{y}} = (X_V)_{\overline{v}} \supset Z_{i,\overline{v}}$$

and since s_i is a section of $\mathcal{F}$ on an open neighbourhood of Z_i in X_V we can send s to

$$\sum\nolimits_{i=1,\ldots,n} \sum\nolimits_{\overline{x} \in Z_{i,\overline{v}}} (\text{class of } s_i \text{ in } \mathcal{F}_{\overline{x}}) \quad \in \quad \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$

We omit the verification that this is compatible with restriction maps and that the relations (1) $(Z, s) + (Z, s') - (Z, s + s')$ and (2) $(Z, s) - (Z', s)$ if $Z \subset Z'$ are sent to zero. Thus we obtain a map

$$(f_{p!}\mathcal{F})_{\overline{y}} \longrightarrow \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$

Let us prove this arrow is surjective. For this it suffices to pick an $\overline{x}$ with $f(\overline{x}) = \overline{y}$ and prove that an element s in the summand $\mathcal{F}_{\overline{x}}$ is in the image. Let s correspond to the element $s \in \mathcal{F}(U)$ where $(U, \overline{u})$ is an étale neighbourhood of $\overline{x}$. Since f is locally quasi-finite, the morphism $U \to Y$ is locally quasi-finite too. By More on Morphisms, Lemma 37.41.3 we can find an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$, an open subscheme

$$W \subset U \times_Y V,$$

and a geometric point $\overline{w}$ mapping to $\overline{u}$ and $\overline{v}$ such that $W \to V$ is finite and $\overline{w}$ is the only geometric point of W mapping to $\overline{v}$. (We omit the translation between the language of geometric points we are currently using and the language of points and residue field extensions used in the statement of the lemma.) Observe that $W \to X_V = X \times_Y V$ is étale. Choose an affine open neighbourhood $W' \subset X_V$ of the image $\overline{w}'$ of $\overline{w}$. Since $\overline{w}$ is the only point of W over $\overline{v}$ and since $W \to V$ is closed, after replacing V by an open neighbourhood of $\overline{v}$, we may assume $W \to X_V$ maps into W'. Then $W \to W'$ is finite and étale and there is a unique geometric point $\overline{w}$ of W lying over $\overline{w}'$. It follows that $W \to W'$ is an open immersion over an open neighbourhood of $\overline{w}'$ in W', see Étale Morphisms, Lemma 41.14.2. Shrinking V and W' we may assume $W \to W'$ is an isomorphism. Thus s may be viewed as a section s' of $\mathcal{F}$ over the open subscheme $W' \subset X_V$ which is finite over V. Hence by definition (W', s') defines an element of $j_{p!}\mathcal{F}(V)$ which maps to s as desired.

Let us prove the arrow is injective. To do this, let $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (63.4.0.1) be an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Assume s maps to zero under the map constructed above. First, after replacing $(V, \overline{v})$ by an étale neighbourhood of itself, we may assume there exist decompositions $Z_i = Z_{i,1} \amalg \ldots \amalg Z_{i,m_i}$ into open and closed subschemes such that each $Z_{i,j}$ has exactly one geometric point over $\overline{v}$. Say under the obvious direct sum decomposition

$$H_{Z_i}(\mathcal{F}) = \bigoplus H_{Z_{i,j}}(\mathcal{F})$$

the element s_i corresponds to $\sum s_{i,j}$. We may use relations (1) and (2) to replace s by $\sum_{i=1,\ldots,n} \sum_{j=1,\ldots,m_i}(Z_{i,j}, s_{i,j})$. In other words, we may assume Z_i has a unique geometric point lying over $\overline{v}$. Let $\overline{x}_1, \ldots, \overline{x}_m$ be the geometric points of X over $\overline{y}$ corresponding to the geometric points of our Z_i over $\overline{v}$; note that for one $j \in \{1, \ldots, m\}$ there may be multiple indices i for which $\overline{x}_j$ corresponds to a point of Z_i. By More on Morphisms, Lemma 37.41.3 applied to both $X_V \to V$ after replacing $(V, \overline{v})$ by an étale neighbourhood of itself we may assume there exist open subschemes

$$W_j \subset X \times_Y V, \quad j = 1, \ldots, m$$

and a geometric point $\overline{w}_j$ of W_j mapping to $\overline{x}_j$ and $\overline{v}$ such that $W_j \to V$ is finite and $\overline{w}_j$ is the only geometric point of W_j mapping to $\overline{v}$. After shrinking V we may assume $Z_i \subset W_j$ for some j and we have the map $H_{Z_i}(\mathcal{F}) \to H_{W_j}(\mathcal{F})$. Thus by the relation (2) we see that our element is equivalent to an element of the form

$$\sum\nolimits_{j=1,\ldots,m} (W_j, t_j)$$

for some $t_j \in H_{W_j}(\mathcal{F})$. Clearly, this element is mapped simply to the class of t_j in the summand $\mathcal{F}_{\overline{x}_j}$. Since s maps to zero, we find that t_j maps to zero in $\mathcal{F}_{\overline{x}_j}$. This implies that t_j restricts to zero on an open neighbourhood of $\overline{w}_j$ in W_j, see Étale Cohomology, Lemma 59.31.2. Shrinking V once more we obtain $t_j = 0$ for all j as desired. □

0F6Q Lemma 63.4.3. Let $f = j : U \to X$ be an étale of schemes. Denote $j_{p!}$ the construction of Étale Cohomology, Equation (59.70.1.1) and denote $f_{p!}$ the construction above. Functorially in $\mathcal{F} \in \mathrm{Ab}(X_{\acute{e}tale})$ there is a canonical map

$$j_{p!}\mathcal{F} \longrightarrow f_{p!}\mathcal{F}$$

of abelian presheaves which identifies the sheaf $j_!\mathcal{F} = (j_{p!}\mathcal{F})^\#$ of Étale Cohomology, Definition 59.70.1 with $(f_{p!}\mathcal{F})^\#$.

Proof. Please read the proof of Étale Cohomology, Lemma 59.70.6 before reading the proof of this lemma. Let V be an object of $X_{\acute{e}tale}$. Recall that

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U)$$

Given φ we obtain an open subscheme $Z_\varphi \subset U_V = U \times_X V$, namely, the image of the graph of φ. Via φ we obtain an isomorphism $V \to Z_\varphi$ over U and we can think of an element

$$s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U) = \mathcal{F}(Z_\varphi) = H_{Z_\varphi}(\mathcal{F})$$

as a section of $\mathcal{F}$ over Z_φ. Since $Z_\varphi \subset U_V$ is open, we actually have $H_{Z_\varphi}(\mathcal{F}) = \mathcal{F}(Z_\varphi)$ and we can think of s_φ as an element of $H_{Z_\varphi}(\mathcal{F})$. Having said this, our map $j_{p!}\mathcal{F} \to f_{p!}\mathcal{F}$ is defined by the rule

$$\sum\nolimits_{i=1,\ldots,n} s_{\varphi_i} \longmapsto \sum\nolimits_{i=1,\ldots,n} (Z_{\varphi_i}, s_{\varphi_i})$$

with right hand side a sum as in (63.4.0.1). We omit the verification that this is compatible with restriction mappings and functorial in $\mathcal{F}$.

To finish the proof, we claim that given a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ there is a commutative diagram

$$\begin{tikzcd}
(j_{p!}\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & \bigoplus_{j(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}} \arrow[d, equal] \\
(f_{p!}\mathcal{F})_{\overline{y}} \arrow[r] & \bigoplus_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}
\end{tikzcd}$$

where the top horizontal arrow is constructed in the proof of Étale Cohomology, Proposition 59.70.3, the bottom horizontal arrow is constructed in the proof of Lemma 63.4.2, the right vertical arrow is the obvious equality, and the left veritical arrow is the map defined in the previous paragraph on stalks. The claim follows in a straightforward manner from the explicit description of all of the arrows involved here and in the references given. Since the horizontal arrows are isomorphisms we conclude so is the left vertical arrow. Hence we find that our map induces an isomorphism on sheafifications by Étale Cohomology, Theorem 59.29.10. □

0F6R Definition 63.4.4. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. We define the direct image with compact support to be the functor

$$f_! : \mathrm{Ab}(X_{\acute{e}tale}) \longrightarrow \mathrm{Ab}(Y_{\acute{e}tale})$$

defined by the formula $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$, i.e., $f_!\mathcal{F}$ is the sheafification of the presheaf $f_{p!}\mathcal{F}$ constructed above.

By Lemma 63.4.1 this does not conflict with Definition 63.3.3 (when both definitions apply) and by Lemma 63.4.3 this does not conflict with Étale Cohomology, Definition 59.70.1 (when both definitions apply).

0F5F Lemma 63.4.5. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then

(1) for $\mathcal{F}$ in $\mathrm{Ab}(X_{\acute{e}tale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have
$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$
functorially in $\mathcal{F}$, and
(2) the functor $f_! : \mathrm{Ab}(X_{\acute{e}tale}) \to \mathrm{Ab}(Y_{\acute{e}tale})$ is exact and commutes with direct sums.

Proof. The formula for the stalks is immediate (and in fact equivalent) to Lemma 63.4.2. The exactness of the functor follows immediately from this and the fact that exactness may be checked on stalks, see Étale Cohomology, Theorem 59.29.10. □

0F6S Remark 63.4.6 (Covariance with respect to open embeddings). Let $f : X \to Y$ be locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $X' \subset X$ be an open subscheme and denote $f' : X' \to Y$ the restriction of f. We claim there is a canonical map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, this map will be the sheafification of a canonical map

$$f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$$

constructed as follows. Let $V \in Y_{\acute{e}tale}$ and consider a section $s' = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (63.4.0.1) defining an element of $f'_{p!}(\mathcal{F}|_{X'})(V)$. Then $Z'_i \subset X'_V$ may also be viewed as a locally closed subscheme of X_V and we have $H_{Z'_i}(\mathcal{F}|_{X'}) = H_{Z'_i}(\mathcal{F})$. We will map s' to the exact same sum $s = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ but now viewed as an element of $f_{p!}\mathcal{F}(V)$. We omit the verification that this construction is compatible with restriction mappings and functorial in $\mathcal{F}$. This construction has the following properties:

(1) The maps $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ and $f'_!\mathcal{F}' \to f_!\mathcal{F}$ are compatible with the description of stalks given in Lemmas 63.4.2 and 63.4.5.

(2) If f is separated, then the map $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ is the same as the map constructed in Remark 63.3.5 via the isomorphism in Lemma 63.4.1.
(3) If $X'' \subset X'$ is another open, then the composition of $f''_{p!}(\mathcal{F}|_{X''}) \to f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$ is the map $f''_{p!}(\mathcal{F}|_{X''}) \to f_{p!}\mathcal{F}$ for the inclusion $X'' \subset X$. Sheafifying we conclude the same holds true for $f''_!(\mathcal{F}|_{X''}) \to f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$.
(4) The map $f'_!\mathcal{F}' \to f_!\mathcal{F}$ is injective because we can check this on stalks.

All of these statements are easily proven by representing elements as finite sums as above and considering what happens to these elements.

0F5H Lemma 63.4.7. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $X = \bigcup_{i \in I} X_i$ be an open covering. Then there exists an exact complex

$$\ldots \to \bigoplus\nolimits_{i_0,i_1,i_2} f_{i_0i_1i_2,!}\mathcal{F}|_{X_{i_0i_1i_2}} \to \bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}} \to f_!\mathcal{F} \to 0$$

functorial in $\mathcal{F} \in \textit{Ab}(X_{\acute{e}tale})$, see proof for details.

Proof. Here as usual we set $X_{i_0 \ldots i_p} = X_{i_0} \cap \ldots \cap X_{i_p}$ and we denote $f_{i_0 \ldots i_p}$ the restriction of f to $X_{i_0 \ldots i_p}$. The maps in the complex are the maps constructed in Remark 63.4.6 with sign rules as in the Čech complex. Exactness follows easily from the description of stalks in Lemma 63.4.5. Details omitted. □

0F5I Remark 63.4.8 (Alternative construction). Lemma 63.4.7 gives an alternative construction of the functor $f_!$ for locally quasi-finite morphisms f. Namely, given a locally quasi-finite morphism $f : X \to Y$ of schemes we can choose an open covering $X = \bigcup_{i \in I} X_i$ such that each $f_i : X_i \to Y$ is separated. For example choose an affine open covering of X. Then we can define $f_!\mathcal{F}$ as the cokernel of the penultimate map of the complex of the lemma, i.e.,

$$f_!\mathcal{F} = \text{Coker}\left(\bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}}\right)$$

where we can use the construction of $f_{i_0,!}$ and $f_{i_0i_1,!}$ in Section 63.3 because the morphisms f_{i_0} and $f_{i_0i_1}$ are separated. One can then compute the stalks of $f_!$ (using the separated case, namely Lemma 63.3.17) and obtain the result of Lemma 63.4.5. Having done so all the other results of this section can be deduced from this as well.

0F78 Remark 63.4.9. Let $g : Y' \to Y$ be a morphism of schemes. For an abelian presheaf $\mathcal{G}'$ on $Y'_{\acute{e}tale}$ let us denote $g_*\mathcal{G}'$ the presheaf $V \mapsto \mathcal{G}'(Y' \times_Y V)$. If $\alpha : \mathcal{G} \to g_*\mathcal{G}'$ is a map of abelian presheaves on $Y_{\acute{e}tale}$, then there is a unique map $\alpha^\# : \mathcal{G}^\# \to g_*((\mathcal{G}')^\#)$ of abelian sheaves on $Y_{\acute{e}tale}$ such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"'] \arrow[d] & g_*\mathcal{G}' \arrow[d] \\ \mathcal{G}^\# \arrow[r, "\alpha^\#"] & g_*((\mathcal{G}')^\#) \end{tikzcd}$$

is commutative where the vertical maps come from the canonical maps $\mathcal{G} \to \mathcal{G}^\#$ and $\mathcal{G}' \to (\mathcal{G}')^\#$. If $\alpha' : g^{-1}\mathcal{G}^\# \to (\mathcal{G}')^\#$ is the map adjoint to $\alpha^\#$, then for a geometric point $\overline{y}' : \text{Spec}(k) \to Y'$ with image $\overline{y} = g \circ \overline{y}'$ in Y, the map

$$\alpha'_{\overline{y}'} : \mathcal{G}_{\overline{y}} = (\mathcal{G}^\#)_{\overline{y}} = (g^{-1}\mathcal{G}^\#)_{\overline{y}'} \longrightarrow (\mathcal{G}')^\#_{\overline{y}'} = \mathcal{G}'_{\overline{y}'}$$

is given by mapping the class in the stalk of a section s of $\mathcal{G}$ over an étale neighbourhood $(V, \overline{v})$ to the class of the section $\alpha(s)$ in $g_*\mathcal{G}'(V) = \mathcal{G}'(Y' \times_Y V)$ over the étale neighbourhood $(Y' \times_Y V, (\overline{y}', \overline{v}))$ in the stalk of $\mathcal{G}'$ at $\overline{y}'$.

0F5J Lemma 63.4.10. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f locally quasi-finite. There is an isomorphism $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$ functorial for $\mathcal{F}$ in $\textit{Ab}(X_{\acute{e}tale})$ which is compatible with the descriptions of stalks given in Lemma 63.4.5 (see proof for the precise statement).

Proof. With conventions as in Remark 63.4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_*f'_{p!}(g')^{-1}\mathcal{F}$$

of abelian presheaves on $Y_{\acute{e}tale}$. By the discussion in Remark 63.4.9 this will determine a canonical map $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$. Finally, we will show this map induces isomorphisms on stalks and conclude by Étale Cohomology, Theorem 59.29.10.

Construction of the map c. Let $V \in Y_{\acute{e}tale}$ and consider a section $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (63.4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. The value of $g_*f'_{p!}(g')^{-1}\mathcal{F}$ at V is $f'_{p!}(g')^{-1}\mathcal{F}(V')$ where $V' = V \times_Y Y'$. Denote $Z'_i \subset X'_{V'}$ the base change of Z_i to V'. By (2) there is a pullback map $H_{Z_i}(\mathcal{F}) \to H_{Z'_i}((g')^{-1}\mathcal{F})$. Denoting $s'_i \in H_{Z'_i}((g')^{-1}\mathcal{F})$ the image of s_i under pullback, we set $c(s) = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (63.4.0.1) defining an element of $f'_{p!}(g')^{-1}\mathcal{F}(V')$. We omit the verification that this construction is compatible the relations (1) and (2) and compatible with restriction mappings. The construction is clearly functorial in $\mathcal{F}$.

Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image $\overline{y} = g \circ \overline{y}'$ in Y. Observe that $X'_{\overline{y}'} = X_{\overline{y}}$ by transitivity of fibre products. Hence g' produces a bijection $\{f'(\overline{x}') = \overline{y}'\} \to \{f(\overline{x}) = \overline{y}\}$ and if $\overline{x}'$ maps to $\overline{x}$, then $((g')^{-1}\mathcal{F})_{\overline{x}'} = \mathcal{F}_{\overline{x}}$ by Étale Cohomology, Lemma 59.36.2. Now we claim that the diagram

$$\begin{tikzcd} (g^{-1}f_!\mathcal{F})_{\overline{y}'} \arrow[r, equal] \arrow[d] & (f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[dl] & \bigoplus_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \arrow[d] \\ (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[rr] & & \bigoplus_{f'(\overline{x}')=\overline{y}'} (g')^{-1}\mathcal{F}_{\overline{x}'} \end{tikzcd}$$

commutes where the horizontal arrows are given in the proof of Lemma 63.4.2 and where the right vertical arrow is an equality by what we just said above. The southwest arrow is described in Remark 63.4.9 as the pullback map, i.e., simply given by our construction c above. Then the simple description of the image of a sum $\sum(Z_i, z_i)$ in the stalk at $\overline{x}$ given in the proof of Lemma 63.4.2 immediately shows the diagram commutes. This finishes the proof of the lemma. $\square$

0F79 Lemma 63.4.11. Let $f' : X \to Y'$ and $g : Y' \to Y$ be composable morphisms of schemes with f' and $f = g \circ f'$ locally quasi-finite and g separated and locally of finite type. Then there is a canonical isomorphism of functors $g_! \circ f'_! = f_!$. This isomorphism is compatible with

(a) covariance with respect to open embeddings as in Remarks 63.3.5 and 63.4.6,
(b) the base change isomorphisms of Lemmas 63.4.10 and 63.3.12, and
(c) equal to the isomorphism of Lemma 63.3.13 via the identifications of Lemma 63.4.1 in case f' is separated.

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. With conventions as in Remark 63.4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_* f'_{p!}\mathcal{F}$$

of abelian presheaves on $Y_{étale}$. By the discussion in Remark 63.4.9 this will determine a canonical map $c^\# : f_!\mathcal{F} \to g_* f'_!\mathcal{F}$. We will show that $c^\#$ has image contained in the subsheaf $g_! f'_!\mathcal{F}$, thereby obtaining a map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$. Next, we will prove (a), (b), and (c) that. Finally, part (b) will allow us to show that c' is an isomorphism.

Construction of the map c. Let $V \in Y_{étale}$ and let $s = \sum(Z_i, s_i)$ be a sum as in (63.4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. Recall that $Z_i \subset X_V = X \times_Y V$ is a locally closed subscheme finite over V. Setting $V' = Y' \times_Y V$ we get $X_{V'} = X \times_{Y'} V' = X_V$. Hence $Z_i \subset X_{V'}$ is locally closed and Z_i is finite over V' because g is separated (Morphisms, Lemma 29.44.14). Hence we may set $c(s) = \sum(Z_i, s_i)$ but now viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_* f'_{p!}\mathcal{F})(V)$. The construction is clearly compatible with relations (1) and (2) and compatible with restriction mappings and hence we obtain the map c.

Observe that in the discussion above our section $c(s) = \sum(Z_i, s_i)$ of $f'_!\mathcal{F}$ over V' restricts to zero on $V' \setminus \mathrm{Im}(\coprod Z_i \to V')$. Since $\mathrm{Im}(\coprod Z_i \to V')$ is proper over V (for example by Morphisms, Lemma 29.41.10) we conclude that $c(s)$ defines a section of $g_! f'_!\mathcal{F} \subset g_* f'_!\mathcal{F}$ over V. Since every local section of $f_!\mathcal{F}$ locally comes from a local section of $f_{p!}\mathcal{F}$ we conclude that the image of $c^\#$ is contained in $g_! f'_!\mathcal{F}$. Thus we obtain an induced map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$ factoring $c^\#$ as predicted in the first paragraph of the proof.

Proof of (a). Let $Y'_1 \subset Y'$ be an open subscheme and set $X_1 = (f')^{-1}(W')$. We obtain a diagram

$$\begin{tikzcd}
X_1 \arrow[r, "a"'] \arrow[d, "f'_1"'] \arrow[dd, bend right, "f_1"'] & X \arrow[d, "f'"] \arrow[dd, bend left, "f"] \\
Y'_1 \arrow[r, "b'"'] \arrow[d, "g_1"'] & Y' \arrow[d, "g"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

where the horizontal arrows are open immersions. Then our claim is that the diagram

$$\begin{tikzcd}
f_{1,!}\mathcal{F}|_{X_1} \arrow[r, "c_1'"'] \arrow[ddd] & g_{1,!}f'_{1,!}\mathcal{F}|_{X_1} \arrow[d, equal] & \\
& g_{1,!}(f'_!\mathcal{F})|_{Y_1'} \arrow[dd] & \\
& & \\
f_!\mathcal{F} \arrow[r, "c'"] & g_!f'_!\mathcal{F} \arrow[r] & g_*f'_!\mathcal{F}
\end{tikzcd}$$

commutes where the left vertical arrow is Remark 63.4.6 and the right vertical arrow is Remark 63.3.5. The equality sign in the diagram comes about because f_1' is the restriction of f' to Y_1' and our construction of $f'_!$ is local on the base. Finally, to prove the commutativity we choose an object V of $Y_{étale}$ and a formal sum $s_1 = \sum(Z_{1,i}, s_{1,i})$ as in (63.4.0.1) defining an element of $f_{1,p!}\mathcal{F}|_{X_1}(V)$. Recall this means $Z_{1,i} \subset X_1 \times_Y V$ is locally closed finite over V and $s_{1,i} \in H_{Z_{1,i}}(\mathcal{F})$. Then we chase this section across the maps involved, but we only need to show we end up with the same element of $g_*f'_!\mathcal{F}(V) = f'_!\mathcal{F}(Y' \times_Y V)$. Going around both sides of the diagram the reader immediately sees we end up with the element $\sum(Z_{1,i}, s_{1,i})$ where now $Z_{1,i}$ is viewed as a locally closed subscheme of $X \times_{Y'} (Y' \times_Y V) = X \times_Y V$ finite over $Y' \times_Y V$.

Proof of (b). Let $b : Y_1 \to Y$ be a morphism of schemes. Let us form the commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "a"'] \arrow[d, "f_1'"] \arrow[dd, bend right, "f_1"'] & X \arrow[d, "f'"'] \arrow[dd, bend left, "f"] \\
Y_1' \arrow[r, "b'"'] \arrow[d, "g_1"] & Y' \arrow[d, "g"'] \\
Y_1 \arrow[r, "b"] & Y
\end{tikzcd}$$

with cartesian squares. We claim that our construction is compatible with the base change maps of Lemmas 63.4.10 and 63.3.12, i.e., that the top rectangle of the diagram

$$\begin{tikzcd}
b^{-1}f_!\mathcal{F} \arrow[rr] \arrow[d, "b^{-1}c'"'] & & f_{1,!}a^{-1}\mathcal{F} \arrow[d, "c_1'"] \\
b^{-1}g_!f'_!\mathcal{F} \arrow[r] \arrow[d] & g_{1,!}(b')^{-1}f'_!\mathcal{F} \arrow[r] \arrow[d] & g_{1,!}f'_{1,!}a^{-1}\mathcal{F} \arrow[d] \\
b^{-1}g_*f'_!\mathcal{F} \arrow[r] & g_{1,*}(b')^{-1}f'_!\mathcal{F} \arrow[r] & g_{1,*}f'_{1,!}a^{-1}\mathcal{F}
\end{tikzcd}$$

commutes. The verification of this is completely routine and we urge the reader to skip it. Since the arrows going from the middle row down to the bottom row are injective, it suffices to show that the outer diagram commutes. To show this it suffices to take a local section of $b^{-1}f_!\mathcal{F}$ and show we end up with the same local section of $g_{1,*}f'_{1,!}a^{-1}\mathcal{F}$ going around either way. However, in fact it suffices to check this for local sections which are of the the pullback by b of a section $s = \sum(Z_i, s_i)$ of $f_{p!}\mathcal{F}(V)$ as above (since such pullbacks generate the abelian sheaf $b^{-1}f_!\mathcal{F}$). Denote V_1, V_1', and $Z_{1,i}$ the base change of V, $V' = Y' \times_Y V$, Z_i by $Y_1 \to Y$. Recall that Z_i is a locally closed subscheme of $X_V = X_{V'}$ and hence $Z_{1,i}$ is a locally closed subscheme of $(X_1)_{V_1} = (X_1)_{V_1'}$. Then $b^{-1}c'$ sends the pullback of s to the pullback of the local section $c(s)\sum(Z_i, s_i)$ viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_* f'_{p!}\mathcal{F})(V)$. The composition of the bottom two base change maps simply maps this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V_1') = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$. On the other hand, the base change map at the top of the diagram sends the pullback of s to $\sum(Z_{1,i}, s_{1,i})$ viewed as an element of $f_{1,!}a^{-1}\mathcal{F}(V_1)$. Then finally c_1' by its very construction does indeed map this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V_1') = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$ and the commutativity has been verified.

Proof of (c). This follows from comparing the definitions for both maps; we omit the details.

To finish the proof it suffices to show that the pullback of c' via any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ is an isomorphism. Namely, pulling back by $\overline{y}$ is the same thing as taking stalks and $\overline{y}$ (Étale Cohomology, Remark 59.56.6) and hence we can invoke Étale Cohomology, Theorem 59.29.10. By the compatibility (b) just shown, we conclude that we may assume Y is the spectrum of k and we have to show that c' is an isomorphism. To do this it suffices to show that the induced map

$$\bigoplus\nolimits_{x \in X} \mathcal{F}_x = H^0(Y, f_!\mathcal{F}) \longrightarrow H^0(Y, g_! f'_!\mathcal{F}) = H^0_c(Y', f'_!\mathcal{F})$$

is an isomorphism. The equalities hold by Lemmas 63.4.5 and 63.3.11. Recall that X is a disjoint union of spectra of Artinian local rings with residue field k, see Varieties, Lemma 33.20.2. Since the left and right hand side commute with direct sums (details omitted) we may assume that $\mathcal{F}$ is a skyscraper sheaf x_*A supported at some $x \in X$. Then $f'_!\mathcal{F}$ is the skyscraper sheaf at the image y' of x in Y by Lemma 63.4.5. In this case it is obvious that our construction produces the identity map $A \to H^0_c(Y', y'_*A) = A$ as desired. □

0F6T Lemma 63.4.12. Let $f : X \to Y$ and $g : Y \to Z$ be composable locally quasi-finite morphisms of schemes. Then there is a canonical isomorphism of functors

$$(g \circ f)_! \longrightarrow g_! \circ f_!$$

These isomorphisms satisfy the following properties:

(1) If f and g are separated, then the isomorphism agrees with Lemma 63.3.13.
(2) If g is separated, then the isomorphism agrees with Lemma 63.4.11.
(3) For a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$ the diagram

$$\begin{tikzcd}
((g \circ f)_!\mathcal{F})_{\overline{z}} \arrow[rr] \arrow[d] & & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}} \arrow[d, equal] \\
(g_! f_!\mathcal{F})_{\overline{z}} \arrow[r] & \bigoplus_{g(\overline{y})=\overline{z}} (f_!\mathcal{F})_{\overline{y}} \arrow[r] & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}}
\end{tikzcd}$$

is commutative where the horizontal arrows are given by Lemma 63.4.5.

(4) Let $h : Z \to T$ be a third locally quasi-finite morphism of schemes. Then the diagram

$$\begin{tikzcd}
(h \circ g \circ f)_! \arrow[r] \arrow[d] & (h \circ g)_! \circ f_! \arrow[d] \\
h_! \circ (g \circ f)_! \arrow[r] & h_! \circ g_! \circ f_!
\end{tikzcd}$$

commutes.

(5) Suppose that we have a diagram of schemes

$$\begin{tikzcd}
X' \arrow[r, "c"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "b"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z' \arrow[r, "a"] & Z
\end{tikzcd}$$

with both squares cartesian and f and g locally quasi-finite. Then the diagram

$$\begin{tikzcd}
a^{-1} \circ (g \circ f)_! \arrow[rr] \arrow[d] & & (g' \circ f')_! \circ c^{-1} \arrow[d] \\
a^{-1} \circ g_! \circ f_! \arrow[r] & g'_! \circ b^{-1} \circ f_! \arrow[r] & g'_! \circ f'_! \circ c^{-1}
\end{tikzcd}$$

commutes where the horizontal arrows are those of Lemma 63.4.10.

Proof. If f and g are separated, then this is a special case of Lemma 63.3.13. If g is separated, then this is a special case of Lemma 63.4.11 which moreover agrees with the case where f and g are separated.

Construction in the general case. Choose an open covering $Y = \bigcup Y_i$ such that the restriction $g_i : Y_i \to Z$ of g is separated. Set $X_i = f^{-1}(Y_i)$ and denote $f_i : X_i \to Y_i$ the restriction of f. Also denote $h = g \circ f$ and $h_i : X_i \to Z$ the restriction of h. Consider the following diagram

$$\begin{tikzcd}
\bigoplus_{i_0,i_1} h_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \arrow[r] \arrow[d] & \bigoplus_{i_0} h_{i_0,!}\mathcal{F}|_{X_{i_0}} \arrow[r] \arrow[d] & h_!\mathcal{F} \arrow[r] \arrow[dd, dotted] & 0 \\
\bigoplus_{i_0,i_1} g_{i_0i_1,!}f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \arrow[r] \arrow[d] & \bigoplus_{i_0} g_{i_0,!}f_{i_0,!}\mathcal{F}|_{X_{i_0}} \arrow[d] & & \\
\bigoplus_{i_0,i_1} g_{i_0i_1,!}(f_!\mathcal{F})|_{Y_{i_0i_1}} \arrow[r] & \bigoplus_{i_0} g_{i_0,!}(f_!\mathcal{F})|_{Y_{i_0}} \arrow[r] & g_!f_!\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

By Lemma 63.4.7 the top and bottom row in the diagram are exact. By Lemma 63.4.11 the top left square commutes. The vertical arrows in the lower left square come about because $(f_!\mathcal{F})|_{Y_{i_0i_1}} = f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}}$ and $(f_!\mathcal{F})|_{Y_{i_0}} = f_{i_0,!}\mathcal{F}|_{X_{i_0}}$ as the construction of $f_!$ is local on the base. Moreover, these equalities are (of course) compatible with the identifications $((f_!\mathcal{F})|_{Y_{i_0}})|_{Y_{i_0i_1}} = (f_!\mathcal{F})|_{Y_{i_0i_1}}$ and $(f_{i_0,!}\mathcal{F}|_{X_{i_0}})|_{Y_{i_0i_1}} = f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}}$ which are used (together with the covariance for open embeddings for

$Y_{i_0 i_1} \subset Y_{i_0}$) to define the horizontal maps of the lower left square. Thus this square commutes as well. In this way we conclude there is a unique dotted arrow as indicated in the diagram and moreover this arrow is an isomorphism.

Proof of properties (1) – (5). Fix the open covering $Y = \bigcup Y_i$. Observe that if $Y \to Z$ happens to be separated, then we get a dotted arrow fitting into the huge diagram above by using the map of Lemma 63.4.11 (by the very properties of that lemma). This proves (2) and hence also (1) by the compatibility of the maps of Lemma 63.4.11 and Lemma 63.3.13. Next, for any scheme Z' over Z, we obtain the compatibility in (5) for the map $(g' \circ f')_! \to g'_! \circ f'_!$ constructed using the open covering $Y' = \bigcup b^{-1}(Y_i)$. This is clear from the corresponding compatibility of the maps constructed in Lemma 63.4.11. In particular, we can consider a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$. Since $X_{\overline{z}} \to Y_{\overline{z}} \to \mathrm{Spec}(k)$ are separated maps, we find that the base change of $(g \circ f)_!\mathcal{F} \to g_!f_!\mathcal{F}$ by $\overline{z}$ is equal to the map of Lemma 63.3.13. The reader then immediately sees that we obtain property (3). Of course, property (3) guarantees that our transformation of functors $(g \circ f)_! \to g_! \circ f_!$ constructed using the open covering $Y = \bigcup Y_i$ doesn't depend on the choice of this open covering. Finally, property (4) follows by looking at what happens on stalks using the already proven property (3). $\square$

63.5. Weightings and trace maps for locally quasi-finite morphisms

0GKE A reference for this section is [AGV71, Exposee XVII, Proposition 6.2.5].

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f, see More on Morphisms, Definition 37.75.2. Let $\mathcal{F}$ be an abelian sheaf on $Y_{\acute{e}tale}$. In this section we will show that there exists map

$$\mathrm{Tr}_{f,w,\mathcal{F}} : f_!f^{-1}\mathcal{F} \longrightarrow \mathcal{F}$$

of abelian sheaves on $Y_{\acute{e}tale}$ characterized by the following property: on stalks at a geometric point $\overline{y}$ of Y we obtain the map

$$\bigoplus\nolimits_{f(\overline{x})=\overline{y}} w(\overline{x}) : (f_!f^{-1}\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{y}} \longrightarrow \mathcal{F}_{\overline{y}}$$

Here as indicated the arrow is given by multiplication by the integer $w(\overline{x})$ on the summand corresponding to $\overline{x}$. The equality on the left of the arrow follows from Lemma 63.4.5 combined with Étale Cohomology, Lemma 59.36.2.

If the morphism $f : X \to Y$ is flat, locally quasi-finite, and locally of finite presentation, then there exists a canonical weighting and we obtain a canonical trace map whose formation is compatible with base change, see Example 63.5.5. If Y is a locally Noetherian unibranch scheme and $f : X \to Y$ is locally quasi-finite, then we can also define a (natural) weighting for f and we have trace maps in this case as well, see Example 63.5.7.

0GKF **Lemma 63.5.1.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{\acute{e}tale}$ and let $\mathcal{G}$ be a sheaf of Λ-modues on $Y_{\acute{e}tale}$. There is a canonical isomorphism

$$can : f_!\mathcal{F} \otimes_\Lambda \mathcal{G} \longrightarrow f_!(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

of sheaves of Λ-modules on $Y_{\acute{e}tale}$.

Proof. Recall that $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$ by Definition 63.4.4 where $f_{p!}\mathcal{F}$ is the presheaf constructed in Section 63.4. Thus in order to construct the arrow it suffices to construct a map

$$f_{p!}\mathcal{F} \otimes_{p,\Lambda} \mathcal{G} \longrightarrow f_{p!}(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

of presheaves on $Y_{\acute{e}tale}$. Here the symbol $\otimes_{p,\Lambda}$ denotes the presheaf tensor product, see Modules on Sites, Section 18.26. Let V be an object of $Y_{\acute{e}tale}$. Recall that

$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F}) \quad \text{and} \quad f_{p!}(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

See Section 63.4. Our map will be defined on pure tensors by the rule

$$(Z, s) \otimes t \longmapsto (Z, s \otimes f^{-1}t)$$

(for notation see below) and extended by linearity to all of $(f_{p!}\mathcal{F} \otimes_{p,\Lambda} \mathcal{G})(V) = f_{p!}\mathcal{F}(V) \otimes_\Lambda \mathcal{G}(V)$. Here the notation used is as follows

(1) $Z \subset X_V$ is a locally closed subscheme finite over V,
(2) $s \in H_Z(\mathcal{F})$ which means that $s \in \mathcal{F}(U)$ with $\mathrm{Supp}(s) \subset Z$ for some $U \subset X_V$ open such that $Z \subset U$ is closed, and
(3) $t \in \mathcal{G}(V)$ with image $f^{-1}t \in f^{-1}\mathcal{G}(U)$.

Since the support of $s \in \mathcal{F}(U)$ is contained in Z it is clear that the support of $s \otimes f^{-1}t$ is contained in Z as well. Thus considering the pair $(Z, s \otimes f^{-1}t)$ makes sense. It is immediate that the construction commutes with the transition maps in the colimit $\mathrm{colim}_Z\, H_Z(\mathcal{F})$ and that it is compatible with restriction mappings. Finally, it is equally clear that the construction is compatible with the identifications of stalks of $f_!$ in Lemma 63.4.5. In other words, the map *can* we've produced on stalks at a geometric point $\overline{y}$ fits into a commutative diagram

$$\begin{tikzcd}
(f_!\mathcal{F} \otimes_\Lambda \mathcal{G})_{\overline{y}} \arrow[r, "can_{\overline{y}}"'] \arrow[d] & f_!(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})_{\overline{y}} \arrow[d] \\
(\bigoplus \mathcal{F}_{\overline{x}}) \otimes_\Lambda \mathcal{G}_{\overline{y}} \arrow[r] & \bigoplus (\mathcal{F}_{\overline{x}} \otimes_\Lambda \mathcal{G}_{\overline{y}})
\end{tikzcd}$$

where the direct sums are over the geometric points $\overline{x}$ lying over $\overline{y}$, where the vertical arrows are the identifications of Lemma 63.4.5, and where the lower horizontal arrow is the obvious isomorphism. We conclude that *can* is an isomorphism as desired. □

0GKG **Lemma 63.5.2.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f. For any abelian sheaf $\mathcal{F}$ on Y there exists a unique trace map $\mathrm{Tr}_{f,w,\mathcal{F}} : f_!f^{-1}\mathcal{F} \to \mathcal{F}$ having the prescribed behaviour on stalks.

Proof. By Lemma 63.5.1 we have an identification $f_!f^{-1}\mathcal{F} = f_!\underline{\mathbf{Z}} \otimes \mathcal{F}$ compatible with the description of stalks of these sheaves at geometric points. Hence it suffices to produce the map

$$\mathrm{Tr}_{f,w,\underline{\mathbf{Z}}} : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}$$

having the prescribed behaviour on stalks. By Definition 63.4.4 we have $f_!\underline{\mathbf{Z}} = (f_{p!}\underline{\mathbf{Z}})^\#$ where $f_{p!}\underline{\mathbf{Z}}$ is the presheaf constructed in Section 63.4. Thus it suffices to construct a map

$$f_{p!}\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}$$

of presheaves on $Y_{\acute{e}tale}$. Let V be an object of $Y_{\acute{e}tale}$. Recall from Section 63.4 that

$$f_{p!}\underline{\mathbf{Z}}(V) = \mathrm{colim}_Z\, H_Z(\underline{\mathbf{Z}})$$

Here the colimit is over the (partially ordered) collection of locally closed subschemes $Z \subset X_V$ which are finite over V. For each such Z we will define a map

$$H_Z(\underline{\mathbf{Z}}) \longrightarrow \underline{\mathbf{Z}}(V)$$

compatible with the maps defining the colimit.

Let $Z \subset X_V$ be locally closed and finite over V. Choose an open $U \subset X_V$ containing Z as a closed subset. An element s of $H_Z(\underline{\mathbf{Z}})$ is a section $s \in \underline{\mathbf{Z}}(U)$ whose support is contained in Z. Let $U_n \subset U$ be the open and closed subset where the value of s is $n \in \mathbf{Z}$. By the support condition we see that $Z \cap U_n = U_n$ for $n \neq 0$. Hence for $n \neq 0$, the open U_n is also closed in Z (as the complement of all the others) and we conclude that $U_n \to V$ is finite as Z is finite over V. By the very definition of a weighting this means the function $\int_{U_n \to V} w|_{U_n}$ is locally constant on V and we may view it as an element of $\underline{\mathbf{Z}}(V)$. Our construction sends (Z, s) to the element

$$\sum\nolimits_{n \in \mathbf{Z},\ n \neq 0} n \left(\int_{U_n \to V} w|_{U_n} \right) \quad \in \quad \underline{\mathbf{Z}}(V)$$

The sum is locally finite on V and hence makes sense; details omitted (in the whole discussion the reader may first choose affine opens and make sure all the schemes occuring in the argument are quasi-compact so the sum is finite). We omit the verification that this construction is compatible with the maps in the colimit and with the restriction mappings defining $f_{p!}\underline{\mathbf{Z}}$.

Let $\overline{y}$ be a geometric point of Y lying over the point $y \in Y$. Taking stalks at $\overline{y}$ the construction above determines a map

$$(f_!\underline{\mathbf{Z}})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{y}}$$

To finish the proof we will show this map is given by multiplication by $w(\overline{x})$ on the summand corresponding to $\overline{x}$. Namely, pick $\overline{x}$ lying over $\overline{y}$. We can find an étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ such that X_V contains an open U finite over V such that only the geometric point $\overline{x}$ is in U and not the other geometric points of X lifting $\overline{y}$. This follows from More on Morphisms, Lemma 37.41.3; some details omitted. Then $(U, 1)$ defines a section of $f_!\underline{\mathbf{Z}}$ over V which maps to 1 in the summand corresponding to $\overline{x}$ and zero in the other summands (see proof of Lemma 63.4.2) and our construction above sends $(U, 1)$ to $\int_{U \to V} w|_U$ which is constant with value $w(\overline{x})$ in a neighbourhood of $\overline{v}$ as desired. □

0GKH Lemma 63.5.3. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f. The trace maps constructed above have the following properties:

(1) $\mathrm{Tr}_{f,w,\mathcal{F}}$ is functorial in $\mathcal{F}$,
(2) $\mathrm{Tr}_{f,w,\mathcal{F}}$ is compatible with arbitrary base change,
(3) given a ring Λ and K in $D(Y_{étale}, \Lambda)$ we obtain $\mathrm{Tr}_{f,w,K} : f_! f^{-1} K \to K$ functorial in K and compatible with arbitrary base change.

Proof. Part (1) either follows from the construction of the trace map in the proof of Lemma 63.5.2 or more simply because the characterization of the map forces it to be true on all stalks. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of schemes. Then the function $w' = w \circ g' : X' \to \mathbf{Z}$ is a weighting of f' by More on Morphisms, Lemma 37.75.3. Statement (2) means that the diagram

$$\begin{tikzcd} g^{-1}f_!f^{-1}\mathcal{F} \arrow[r, "g^{-1}\mathrm{Tr}_{f,w,\mathcal{F}}"] \arrow[d, equal] & g^{-1}\mathcal{F} \arrow[d, equal] \\ f'_!(f')^{-1}g^{-1}\mathcal{F} \arrow[r, "\mathrm{Tr}_{f',w',g^{-1}\mathcal{F}}"] & g^{-1}\mathcal{F} \end{tikzcd}$$

is commutative where the left vertical equality is given by

$$g^{-1}f_!f^{-1}\mathcal{F} = f'_!(g')^{-1}f^{-1}\mathcal{F} = f'_!(f')^{-1}g^{-1}\mathcal{F}$$

with first equality sign given by Lemma 63.4.10 (base change for lower shriek). The commutativity of this diagram follows from the characterization of the action of our trace maps on stalks and the fact that the base change map of Lemma 63.4.10 respects the descriptions of stalks.

Given parts (1) and (2), part (3) follows as the functors $f^{-1} : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ and $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ are obtained by applying f^{-1} and $f_!$ to any complexes of modules representing the objects in question. □

0GL3 Lemma 63.5.4. Let $f : X \to Y$ and $g : Y \to Z$ be locally quasi-finite morphisms. Let $w_f : X \to \mathbf{Z}$ be a weighting of f and let $w_g : Y \to \mathbf{Z}$ be a weighting of g. For $K \in D(Z_{étale}, \Lambda)$ the composition

$$(g \circ f)_!(g \circ f)^{-1}K = g_!f_!f^{-1}g^{-1}K \xrightarrow{g_!\mathrm{Tr}_{f,w_f,g^{-1}K}} g_!g^{-1}K \xrightarrow{\mathrm{Tr}_{g,w_g,K}} K$$

is equal to $\mathrm{Tr}_{g\circ f, w_{g\circ f}, K}$ where $w_{g\circ f}(x) = w_f(x)w_g(f(x))$.

Proof. We have $(g \circ f)_! = g_! \circ f_!$ by Lemma 63.4.12. In More on Morphisms, Lemma 37.75.5 we have seen that $w_{g\circ f}$ is a weighting for $g \circ f$ so the statement makes sense. To check equality compute on stalks. Details omitted. □

0GKI Example 63.5.5 (Trace for flat quasi-finite). Let $f : X \to Y$ be a morphism of schemes which is flat, locally quasi-finite, and locally of finite presentation. Then we obtain a canonical positive weighting $w : X \to \mathbf{Z}$ by setting

$$w(x) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x})[\kappa(x) : \kappa(f(x))]_i$$

See More on Morphisms, Lemma 37.75.7. Thus by Lemmas 63.5.2 and 63.5.3 for f we obtain trace maps

$$\mathrm{Tr}_{f,K} : f_!f^{-1}K \longrightarrow K$$

functorial for K in $D(Y_{étale}, \Lambda)$ and compatible with arbitrary base change. Note that any base change $f' : X' \to Y'$ of f satisfies the same properties and that w restricts to the canonical weighting for f'.

0GL4 Remark 63.5.6. Let $j : U \to X$ be an étale morphism of schemes. Then the trace map $\text{Tr} : j_!j^{-1}K \to K$ of Example 63.5.5 is equal to the counit for the adjunction between $j_!$ and j^{-1}. We already used the terminology "trace" for this counit in Étale Cohomology, Section 59.66.

0GKJ Example 63.5.7 (Trace for quasi-finite over normal). Let Y be a geometrically unibranch and locally Noetherian scheme, for example Y could be a normal variety. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then there exists a positive weighting $w : X \to \mathbf{Z}$ for f which is roughly defined by sending x to the "generic separable degree" of $\mathcal{O}^{sh}_{X,x}$ over $\mathcal{O}^{sh}_{Y,f(x)}$. See More on Morphisms, Lemma 37.75.8. Thus by Lemmas 63.5.2 and 63.5.3 for f and w we obtain trace maps

$$\text{Tr}_{f,w,K} : f_!f^{-1}K \longrightarrow K$$

functorial for K in $D(Y_{\acute{e}tale}, \Lambda)$ and compatible with arbitrary base change. However, in this case, given a base change $f' : X' \to Y'$ of f the restriction of w to X' in general does not have a "natural" interpretation in terms of the morphism f'.

63.6. Upper shriek for locally quasi-finite morphisms

0F58 For a locally quasi-finite morphism $f : X \to Y$ of schemes, the functor $f_! : \textit{Ab}(X_{\acute{e}tale}) \to \textit{Ab}(Y_{\acute{e}tale})$ commutes with direct sums and is exact, see Lemma 63.4.5. This suggests that it has a right adjoint which we will denote $f^!$.

Warning: This functor is the non-derived version!

0F59 Lemma 63.6.1. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes.

(1) The functor $f_! : \textit{Ab}(X_{\acute{e}tale}) \to \textit{Ab}(Y_{\acute{e}tale})$ has a right adjoint $f^! : \textit{Ab}(Y_{\acute{e}tale}) \to \textit{Ab}(X_{\acute{e}tale})$.
(2) We have $f^!(\overline{y}_*A) = \prod_{f(\overline{x})=\overline{y}} \overline{x}_*A$.
(3) If Λ is a ring, then the functor $f_! : \textit{Mod}(X_{\acute{e}tale}, \Lambda) \to \textit{Mod}(Y_{\acute{e}tale}, \Lambda)$ has a right adjoint $f^! : \textit{Mod}(Y_{\acute{e}tale}, \Lambda) \to \textit{Mod}(X_{\acute{e}tale}, \Lambda)$ which agrees with $f^!$ on underlying abelian sheaves.

Proof. Proof of (1). Let $E \subset \text{Ob}(\textit{Ab}(Y_{\acute{e}tale}))$ be the class consisting of products of skyscraper sheaves. We claim that

(a) every $\mathcal{G}$ in $\textit{Ab}(Y_{\acute{e}tale})$ is a subsheaf of an element of E, and
(b) for every $\mathcal{G} \in E$ there exists an object $\mathcal{H}$ of $\textit{Ab}(X_{\acute{e}tale})$ such that $\text{Hom}(f_!\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{H})$ functorially in $\mathcal{F}$.

Once the claim has been verified, the dual of Homology, Lemma 12.29.6 produces the adjoint functor $f^!$.

Part (a) is true because we can map $\mathcal{G}$ to the sheaf $\prod \overline{y}_*\mathcal{G}_{\overline{y}}$ where the product is over all geometric points of Y. This is an injection by Étale Cohomology, Theorem 59.29.10. (This is the first step in the Godement resolution when done in the setting of abelian sheaves on topological spaces.)

Part (b) and part (2) of the lemma can be seen as follows. Suppose that $\mathcal{G} = \prod \overline{y}_*A_{\overline{y}}$ for some abelian groups $A_{\overline{y}}$. Then

$$\text{Hom}(f_!\mathcal{F}, \mathcal{G}) = \prod \text{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$$

Thus it suffices to find abelian sheaves $\mathcal{H}_{\overline{y}}$ on $X_{\acute{e}tale}$ representing the functors $\mathcal{F} \mapsto \text{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$ and to take $\mathcal{H} = \prod \mathcal{H}_{\overline{y}}$. This reduces us to the case $\mathcal{H} = \overline{y}_*A$

for some fixed geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ and some fixed abelian group A. We claim that in this case $\mathcal{H} = \prod_{f(\overline{x}) = \overline{y}} \overline{x}_* A$ works. This will finish the proof of parts (1) and (2) of the lemma. Namely, we have

$$\mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A) = \mathrm{Hom}_{\mathrm{Ab}}((f_!\mathcal{F})_{\overline{y}}, A) = \mathrm{Hom}_{\mathrm{Ab}}(\bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}, A)$$

by the description of stalks in Lemma 63.4.5 on the one hand and on the other hand we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}) = \prod\nolimits_{f(\overline{x}) = \overline{y}} \mathrm{Hom}(\mathcal{F}, \overline{x}_*A) = \prod\nolimits_{f(\overline{x}) = \overline{y}} \mathrm{Hom}_{\mathrm{Ab}}(\mathcal{F}_{\overline{x}}, A)$$

We leave it to the reader to identify these as functors of $\mathcal{F}$.

Proof of part (3). Observe that an object $\mathrm{Mod}(X_{\acute{e}tale}, \Lambda)$ is the same thing as an object $\mathcal{F}$ of $\mathrm{Ab}(X_{\acute{e}tale})$ together with a map $\Lambda \to \mathrm{End}(\mathcal{F})$. Hence the functors $f_!$ and $f^!$ in (1) define functors $f_!$ and $f^!$ as in (3). A straightforward computation shows that they are adjoints. □

0F5A Lemma 63.6.2. Let $j : U \to X$ be an étale morphism. Then $j^! = j^{-1}$.

Proof. This is true because $j_!$ as defined in Section 63.4 agrees with $j_!$ as defined in Étale Cohomology, Section 59.70, see Lemma 63.4.3. Finally, in Étale Cohomology, Section 59.70 the functor $j_!$ is defined as the left adjoint of j^{-1} and hence we conclude by uniqueness of adjoint functors. □

0F5B Lemma 63.6.3. Let $f : X \to Y$ and $g : Y \to Z$ be separated and locally quasi-finite morphisms. There is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$. Given a third locally quasi-finite morphism $h : Z \to T$ the diagram

$$\begin{tikzcd}
(h \circ g \circ f)^! \arrow[r] \arrow[d] & f^! \circ (h \circ g)^! \arrow[d] \\
(g \circ f)^! \circ h^! \arrow[r] & f^! \circ g^! \circ h^!
\end{tikzcd}$$

commutes.

Proof. By uniqueness of adjoint functors, this immediately translates into the corresponding (dual) statement for the functors $f_!$. See Lemma 63.4.12. □

0F5C Lemma 63.6.4. Let $j : U \to X$ and $j' : V \to U$ be étale morphisms. The isomorphism $(j \circ j')^{-1} = (j')^{-1} \circ j^{-1}$ and the isomorphism $(j \circ j')^! = (j')^! \circ j^!$ of Lemma 63.6.3 agree via the isomorphism of Lemma 63.6.2.

Proof. Omitted. □

0F6U Lemma 63.6.5. Consider a cartesian square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

of schemes with f locally quasi-finite. For any abelian sheaf $\mathcal{F}$ on $Y'_{\acute{e}tale}$ we have $(g')_*(f')^!\mathcal{F} = f^! g_* \mathcal{F}$.

Proof. By uniqueness of adjoint functors, this follows from the corresponding (dual) statement for the functors $f_!$. See Lemma 63.4.10. □

0F5L Remark 63.6.6. The material in this section can be generalized to sheaves of pointed sets. Namely, for a site $\mathcal{C}$ denote $Sh^*(\mathcal{C})$ the category of sheaves of pointed sets. The constructions in this and the preceding section apply, mutatis mutandis, to sheaves of pointed sets. Thus given a locally quasi-finite morphism $f : X \to Y$ of schemes we obtain an adjoint pair of functors

$$f_! : Sh^*(X_{étale}) \longrightarrow Sh^*(Y_{étale}) \quad \text{and} \quad f^! : Sh^*(Y_{étale}) \longrightarrow Sh^*(X_{étale})$$

such that for every geometric point $\overline{y}$ of Y there are isomorphisms

$$(f_!\mathcal{F})_{\overline{y}} = \coprod\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

(coproduct taken in the category of pointed sets) functorial in $\mathcal{F} \in Sh^*(X_{étale})$ and isomorphisms

$$f^!(\overline{y}_*S) = \prod\nolimits_{f(\overline{x})=\overline{y}} \overline{x}_*S$$

functorial in the pointed set S. If $F : \text{Ab}(X_{étale}) \to Sh^*(X_{étale})$ and $F : \text{Ab}(Y_{étale}) \to Sh^*(Y_{étale})$ denote the forgetful functors, compatibility between the constructions will guarantee the existence of canonical maps

$$f_!F(\mathcal{F}) \longrightarrow F(f_!\mathcal{F})$$

functorial in $\mathcal{F} \in \text{Ab}(X_{étale})$ and

$$F(f^!\mathcal{G}) \longrightarrow f^!F(\mathcal{G})$$

functorial in $\mathcal{G} \in \text{Ab}(Y_{étale})$ which produce the obvious maps on stalks, resp. skyscraper sheaves. In fact, the transformation $F \circ f^! \to f^! \circ F$ is an isomorphism (because $f^!$ commutes with products).

63.7. Derived upper shriek for locally quasi-finite morphisms

0F5M We can take the derived versions of the functors in Section 63.6 and obtain the following.

0F5N Lemma 63.7.1. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. The functors $f_!$ and $f^!$ of Definition 63.4.4 and Lemma 63.6.1 induce adjoint functors $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ and $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ on derived categories.

In the separated case the functor $f_!$ is defined in Section 63.3.

Proof. This follows immediately from Derived Categories, Lemma 13.30.3, the fact that $f_!$ is exact (Lemma 63.4.5) and hence $Lf_! = f_!$ and the fact that we have enough K-injective complexes of Λ-modules on $Y_{étale}$ so that $Rf^!$ is defined. □

0GJX Remark 63.7.2. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. The functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ of Lemma 63.7.1 sends complexes with torsion cohomology sheaves to complexes with torsion cohomology sheaves. This is immediate from the description of the stalks of $f_!$, see Lemma 63.4.5.

0GKK Lemma 63.7.3. Let X be a scheme. Let $X = U \cup V$ with U and V open. Let Λ be a ring. Let $K \in D(X_{étale}, \Lambda)$. There is a distinguished triangle

$$j_{U \cap V!}K|_{U \cap V} \to j_{U!}K|_U \oplus j_{V!}K|_V \to K \to j_{U \cap V!}K|_{U \cap V}[1]$$

in $D(X_{étale}, \Lambda)$ with obvious notation.

Proof. Since the restriction functors and the lower shriek functors we use are exact, it suffices to show for any abelian sheaf $\mathcal{F}$ on $X_{étale}$ the sequence

$$0 \to j_{U \cap V!}\mathcal{F}|_{U \cap V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$

is exact. This can be seen by looking at stalks. $\square$

0GKL Lemma 63.7.4. Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Let Λ be a ring. Let $K \in D(X_{étale}, \Lambda)$. There is a distinguished triangle

$$j_!j^{-1}K \to K \to i_*i^{-1}K \to j_!j^{-1}K[1]$$

in $D(X_{étale}, \Lambda)$.

Proof. Immediate consequence of Étale Cohomology, Lemma 59.70.8 and the fact that the functors $j_!$, j^{-1}, i_*, i^{-1} are exact and hence their derived versions are computed by applying these functors to any complex of sheaves representing K. $\square$

63.8. Preliminaries to derived lower shriek via compactifications

0F7A In this section we prove some lemmas on the existence of certain natural isomorphisms of functors which follow immediately from proper base change.

0F7B Lemma 63.8.1. Consider a commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f and f' proper and g and g' separated and locally quasi-finite. Let Λ be a ring. Functorially in $K \in D(X'_{étale}, \Lambda)$ there is a canonical map

$$g_!Rf'_*K \longrightarrow Rf_*(g'_!K)$$

in $D(Y_{étale}, \Lambda)$. This map is an isomorphism if (a) K is bounded below and has torsion cohomology sheaves, or (b) Λ is a torsion ring.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{étale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. Then we can consider the map

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

where the first and third equality come from Lemma 63.3.4 and the second equality comes from Lemma 63.3.13 which tells us that both $g_! \circ f'_!$ and $f_! \circ g'_!$ are equal to $(g \circ f')_! = (f \circ g')_!$ as subsheaves of $(g \circ f')_* = (f \circ g')_*$.

Assume Λ is torsion, i.e., we are in case (b). With notation as above, it suffices to show that $f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$ is an isomorphism. The question is local on Y. Hence we may assume that the dimension of fibres of f is bounded, see Morphisms, Lemma 29.28.5. Then we see that Rf_* has finite cohomological dimension, see Étale Cohomology, Lemma 59.92.2. Hence by Derived Categories, Lemma 13.32.2, if we show that $R^qf_*(g'_!\mathcal{J}) = 0$ for $q > 0$ and any injective sheaf of Λ-modules $\mathcal{J}$ on $X'_{\acute{e}tale}$, then the result follows.

The stalk of $R^qf_*(g'_!\mathcal{J})$ at a geometric point $\overline{y}$ is equal to $H^q(X_{\overline{y}}, (g'_!\mathcal{J})|_{X_{\overline{y}}})$ by Étale Cohomology, Lemma 59.91.13. Since formation of $g'_!$ commutes with base change (Lemma 63.3.12) this is equal to

$$H^q(X_{\overline{y}}, g'_{\overline{y},!}(\mathcal{J}|_{X'_{\overline{y}}}))$$

where $g'_{\overline{y}} : X'_{\overline{y}} \to X_{\overline{y}}$ is the induced morphism between geometric fibres. Since $Y' \to Y$ is locally quasi-finite, we see that $X'_{\overline{y}}$ is a disjoint union of the fibres $X'_{\overline{y}'}$ at geometric points $\overline{y}'$ of Y' lying over $\overline{y}$. Denote $g'_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ the restriction of $g'_{\overline{y}}$ to $X'_{\overline{y}'}$. Thus the previous cohomology group is equal to

$$H^q(X_{\overline{y}}, \bigoplus\nolimits_{\overline{y}'/\overline{y}} g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

for example by Lemma 63.3.15 (but it is also obvious from the definition of $g'_{\overline{y},!}$ in Section 63.3). Since taking étale cohomology over $X_{\overline{y}}$ commutes with direct sums (Étale Cohomology, Theorem 59.51.3) we conclude it suffices to show that

$$H^q(X_{\overline{y}}, g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

is zero. Observe that $g_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ is a morphism between proper scheme over $\overline{y}$ and hence is proper itself. As it is locally quasi-finite as well we conclude that $g_{\overline{y}'}$ is finite. Thus we see that $g'_{\overline{y}',!} = g'_{\overline{y}',*} = Rg'_{\overline{y}',*}$. By Leray we conlude that we have to show

$$H^q(X'_{\overline{y}'}, \mathcal{J}|_{X'_{\overline{y}'}})$$

is zero. As Λ is torsion, this follows from proper base change (Étale Cohomology, Lemma 59.91.13) as the higher direct images of $\mathcal{J}$ under f' are zero.

Proof in case (a). We will deduce this from case (b) by standard arguments. We will show that the induced map $g_!R^pf'_*K \to R^pf_*(g'_!K)$ is an isomorphism for all $p \in \mathbf{Z}$. Fix an integer $p_0 \in \mathbf{Z}$. Let a be an integer such that $H^j(K) = 0$ for $j < a$. We will prove $g_!R^pf'_*K \to R^pf_*(g'_!K)$ is an isomorphism for $p \leq p_0$ by descending induction on a. If $a > p_0$, then we see that the left and right hand side of the map are zero for $p \leq p_0$ by trivial vanishing, see Derived Categories, Lemma 13.16.1 (and use that $g_!$ and $g'_!$ are exact functors). Assume $a \leq p_0$. Consider the distinguished triangle

$$H^a(K)[-a] \to K \to \tau_{\geq a+1}K$$

By induction we have the result for $\tau_{\geq a+1}K$. In the next paragraph, we will prove the result for $H^a(K)[-a]$. Then five lemma applied to the map between long exact sequence of cohomology sheaves associated to the map of distinguished triangles

$$\begin{tikzcd}
g_!Rf'_*(H^a(K)[-a]) \arrow[r] \arrow[d] & g_!Rf'_*K \arrow[r] \arrow[d] & g_!Rf'_*\tau_{\geq a+1}K \arrow[d] \\
Rf_*(g'_!(H^a(K)[-a])) \arrow[r] & Rf_*(g'_!K) \arrow[r] & Rf_*(g'_!\tau_{|geqa+1}K)
\end{tikzcd}$$

gives the result for K. Some details omitted.

Let $\mathcal{F}$ be a torsion abelian sheaf on $X'_{étale}$. To finish the proof we show that $g_!Rf'_*\mathcal{F} \to R^pf_*(g'_!\mathcal{F})$ is an isomorphism for all p. We can write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n] = \mathrm{Ker}(n : \mathcal{F} \to \mathcal{F})$. We have the isomorphism for $\mathcal{F}[n]$ by case (b). Since the functors $g_!$, $g'_!$, R^pf_*, $R^pf'_*$ commute with filtered colimits (follows from Lemma 63.3.17 and Étale Cohomology, Lemma 59.51.8) the proof is complete. ☐

0F7C Lemma 63.8.2. Consider a commutative diagram of schemes

$$\begin{tikzcd}
X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "l"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z' \arrow[r, "m"] & Z
\end{tikzcd}$$

with f, f', g and g' proper and k, l, and m separated and locally quasi-finite. Then the isomorphisms of Lemma 63.8.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $R(g\circ f)_* = Rg_*\circ Rf_*$ and $R(g'\circ f')_* = Rg'_*\circ Rf'_*$, then the isomorphism $m_!\circ Rg'_*\circ Rf'_* \to Rg_*\circ Rf_*\circ k_!$ of the outer rectangle is equal to the composition

$$m_! \circ Rg'_* \circ Rf'_* \to Rg_* \circ l_! \circ Rf'_* \to Rg_* \circ Rf_* \circ k_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{étale}$ and a quasi-isomorphism $k_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X_{étale}$. The proof of Lemma 63.8.1 shows that the canonical map

$$a : l_!f'_*\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism produces the second arrow on applying Rg_*. By Cohomology on Sites, Lemma 21.20.10 the complex $f_*\mathcal{I}^\bullet$, resp. $f'_*\mathcal{J}^\bullet$ is a K-injective complex of Λ-modules on $Y_{étale}$, resp. $Y'_{étale}$. (Using this is cheating and could be avoided.) In particular, the same reasoning gives that the canonical map

$$b : m_!g'_*f'_*\mathcal{J}^\bullet \to g_*f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism represents the first arrow. Finally, the proof of Lemma 63.8.1 show that $g_*l_!f'_!\mathcal{J}^\bullet$ represents $Rg_*(l_!f'_*\mathcal{J}^\bullet)$ because $f'_*\mathcal{J}^\bullet$ is K-injective. Hence $Rg_*(a) = g_*(a)$ and the composition $g_*(a)\circ b$ is the arrow of Lemma 63.8.1 for the rectangle. ☐

0F7D Lemma 63.8.3. Consider a commutative diagram of schemes

$$\begin{tikzcd}
X'' \arrow[r, "g'"'] \arrow[d, "f''"'] & X' \arrow[r, "g"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

with f, f', and f'' proper and g, g', h, and h' separated and locally quasi-finite. Then the isomorphisms of Lemma 63.8.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $(h \circ h')_! = h_! \circ h'_!$ and $(g \circ g')_! = g_! \circ g'_!$ using the equalities of Lemma 63.3.13, then the isomorphism $h_! \circ h'_! \circ Rf''_* \to Rf_* \circ g_! \circ g'_!$ of the outer rectangle is equal to the composition

$$h_! \circ h'_! \circ Rf''_* \to h_! \circ Rf'_* \circ g'_! \to Rf_* \circ g_! \circ g'_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X''_{\acute{e}tale}$ and a quasi-isomorphism $g'_!\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ to a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{\acute{e}tale}$. Next, choose a quasi-isomorphism $g_!\mathcal{J}^\bullet \to \mathcal{K}^\bullet$ to a K-injective complex $\mathcal{K}^\bullet$ of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 63.8.1 shows that the canonical maps

$$h'_! f''_* \mathcal{I}^\bullet \to f'_* \mathcal{J}^\bullet \quad \text{and} \quad h_! f'_* \mathcal{J}^\bullet \to f_* \mathcal{K}^\bullet$$

are quasi-isomorphisms and these quasi-isomorphisms define the first and second arrow above. Since $g_!$ is an exact functor (Lemma 63.3.17) we find that $g_! g'_! \mathcal{I}^\bullet \to \mathcal{K}^\bullet$ is a quasi-ismorphism and hence the canonical map

$$h_! h'_! f''_* \mathcal{I}^\bullet \to f_* \mathcal{K}^\bullet$$

is a quasi-isomorphism and represents the map for the outer rectangle in the derived category. Clearly this map is the composition of the other two and the proof is complete. □

0F7E Remark 63.8.4. Consider a commutative diagram

$$\begin{tikzcd}
X'' \arrow[r, "k'"'] \arrow[d, "f''"'] & X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y'' \arrow[r, "l'"] \arrow[d, "g''"'] & Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z'' \arrow[r, "m'"] & Z' \arrow[r, "m"] & Z
\end{tikzcd}$$

of schemes whose vertical arrows are proper and whose horizontal arrows are separated and locally quasi-finite. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

Then the maps of Lemma 63.8.1 for the squares are (where we use $Rf_* = f_*$, etc)

$$\begin{matrix} \gamma_A : l'_! \circ f''_* \to f'_* \circ k'_! & \gamma_B : l_! \circ f'_* \to f_* \circ k_! \\ \gamma_C : m'_! \circ g''_* \to g'_* \circ l'_! & \gamma_D : m_! \circ g'_* \to g_* \circ l_! \end{matrix}$$

For the 2×1 and 1×2 rectangles we have four further maps

$$\begin{matrix} \gamma_{A+B} : (l \circ l')_! \circ f''_* \to f_* \circ (k \circ k')_* \\ \gamma_{C+D} : (m \circ m')_! \circ g''_* \to g_* \circ (l \circ l')_! \\ \gamma_{A+C} : m'_! \circ (g'' \circ f'')_* \to (g' \circ f')_* \circ k'_! \\ \gamma_{B+D} : m_! \circ (g' \circ f')_* \to (g \circ f)_* \circ k_! \end{matrix}$$

By Lemma 63.8.3 we have

$$\gamma_{A+B} = \gamma_B \circ \gamma_A, \quad \gamma_{C+D} = \gamma_D \circ \gamma_C$$

and by Lemma 63.8.2 we have

$$\gamma_{A+C} = \gamma_A \circ \gamma_C, \quad \gamma_{B+D} = \gamma_B \circ \gamma_D$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_B \star \mathrm{id}_{k'_!}) \circ (\mathrm{id}_{l_!} \star \gamma_A)$ with notation as in Categories, Section 4.28 and similarly for the others. Having said all of this we find (a priori) two transformations

$$m_! \circ m'_! \circ g''_* \circ f''_* \longrightarrow g_* \circ f_* \circ k_! \circ k'_!$$

namely

$$\gamma_B \circ \gamma_D \circ \gamma_A \circ \gamma_C = \gamma_{B+D} \circ \gamma_{A+C}$$

and

$$\gamma_B \circ \gamma_A \circ \gamma_D \circ \gamma_C = \gamma_{A+B} \circ \gamma_{C+D}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{tikzcd} m_! \circ g'_* \circ l'_! \circ f''_* \arrow[r, "\gamma_D"'] \arrow[d, "\gamma_A"'] & g_* \circ l_! \circ l'_! \circ f''_* \arrow[d, "\gamma_A"] \\ m_! \circ g'_* \circ f'_* \circ k'_! \arrow[r, "\gamma_D"] & g_* \circ l_! \circ f'_* \circ k'_! \end{tikzcd}$$

commutes. This is true because the squares A and D meet in only one point, more precisely by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1.

0F7F Lemma 63.8.5. Let $b : Y_1 \to Y$ be a morphism of schemes. Consider a commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \quad \text{and let} \quad \begin{tikzcd} X'_1 \arrow[r, "g'_1"'] \arrow[d, "f'_1"'] & X_1 \arrow[d, "f_1"] \\ Y'_1 \arrow[r, "g_1"] & Y_1 \end{tikzcd}$$

be the base change by b. Assume f and f' proper and g and g' separated and locally quasi-finite. For a ring Λ and K in $D(X'_{\acute{e}tale}, \Lambda)$ there is commutative diagram

$$\begin{tikzcd} b^{-1}g_!Rf'_*K \arrow[r] \arrow[d] & g_{1,!}(b')^{-1}Rf'_*K \arrow[r] & g_{1,!}Rf'_{1,*}(a')^{-1}K \arrow[d] \\ b^{-1}Rf_*g'_!K \arrow[r] & Rf_{1,*}a^{-1}g'_!K \arrow[r] & Rf_{1,*}g'_{1,!}(a')^{-1}K \end{tikzcd}$$

in $D(Y_{1,\acute{e}tale}, \Lambda)$ where $a : X_1 \to X$, $a' : X'_1 \to X'$, $b' : Y'_1 \to Y'$ are the projections, the vertical maps are the arrows of Lemma 63.8.1 and the horizontal arrows are the base change map (from Étale Cohomology, Section 59.86) and the base change map of Lemma 63.3.12.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{\acute{e}tale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 63.8.1 constructs $g_!Rf'_*K \to Rf_*g'_!K$ as

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

Choose a quasi-isomorphism $(a')^{-1}\mathcal{J}^\bullet \to \mathcal{J}_1^\bullet$ to a K-injective complex $\mathcal{J}_1^\bullet$ of sheaves of Λ-modules on $X'_{1,\acute{e}tale}$. Then we can pick a diagram of complexes

$$\begin{tikzcd} g'_{1,!}\mathcal{J}_1^\bullet \arrow[rr] & & \mathcal{I}_1^\bullet \\ g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[u] \arrow[r, equal] & a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] & a^{-1}\mathcal{I}^\bullet \arrow[u] \end{tikzcd}$$

commuting up to homotopy where all arrows are quasi-isomorphisms, the equality comes from Lemma 63.3.4, and $\mathcal{I}_1^\bullet$ is a K-injective complex of sheaves of Λ-modules on $X_{1,\acute{e}tale}$. The map $g_{1,!}Rf'_{1,*}(a')^{-1}K \to Rf_{1,*}g'_{1,!}(a')^{-1}K$ is given by

$$g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet = g_{1,!}f'_{1,!}\mathcal{J}_1^\bullet = f_{1,!}g'_{1,!}\mathcal{J}_1^\bullet = f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \to f_{1,*}\mathcal{I}_1^\bullet$$

The identifications across the 3 equal signs in both arrows are compatible with pullback maps, i.e., the diagram

$$\begin{tikzcd} b^{-1}g_!f'_*\mathcal{J}^\bullet \arrow[r] \arrow[d, equal] & g_{1,!}(b')^{-1}f'_*\mathcal{J}^\bullet \arrow[r] & g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \arrow[d, equal] \\ b^{-1}f_*g'_!\mathcal{J}^\bullet \arrow[r] & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \end{tikzcd}$$

of complexes of abelian sheaves commutes. To show this it is enough to show the diagram commutes with $g_!, g_{1,!}, g'_!, g'_{1,!}$ replaced by $g_*, g_{1,*}, g'_*, g'_{1,*}$ (because the shriek functors are defined as subfunctors of the $*$ functors and the base change maps are defined in a manner compatible with this, see proof of Lemma 63.3.12). For this new diagram the commutativity follows from the compatibility of pullback maps with horizontal and vertical stacking of diagrams, see Sites, Remarks 7.45.3 and 7.45.4 so that going around the diagram in either direction is the pullback map for the base change of $f \circ g' = g \circ f'$ by b. Since of course

$$\begin{tikzcd} g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \arrow[r] \arrow[d, equal] & g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet \arrow[d, equal] \\ f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \end{tikzcd}$$

commutes, to finish the proof it suffices to show that

$$\begin{tikzcd} b^{-1}f_*g'_!\mathcal{J}^\bullet \arrow[r] \arrow[d] & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] \arrow[d] & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \arrow[d] \\ b^{-1}f_*\mathcal{I}^\bullet \arrow[r] & f_{1,*}a^{-1}\mathcal{I}^\bullet \arrow[rr] & & f_{1,*}\mathcal{I}_1^\bullet \end{tikzcd}$$

commutes in the derived category, which holds by our choice of maps earlier. $\square$

0F7G **Lemma 63.8.6.** Consider a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

with f and g locally quasi-finite and h proper. Let Λ be a ring. Funtorially in $K \in D(X_{\acute{e}tale}, \Lambda)$ there is a canonical map

$$g_! K \longrightarrow Rh_*(f_! K)$$

in $D(Z_{\acute{e}tale}, \Lambda)$. This map is an isomorphism if (a) K is bounded below and has torsion cohomology sheaves, or (b) Λ is a torsion ring.

Proof. This is a special case of Lemma 63.8.1 if f and g are separated. We urge the reader to skip the proof in the general case as we'll mainly use the case where f and g are separated.

Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$. Choose a quasi-isomorphism $f_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $Y_{\acute{e}tale}$. Consider the map

$$g_!\mathcal{K}^\bullet = h_! f_! \mathcal{K}^\bullet = h_* f_! \mathcal{K}^\bullet \longrightarrow h_* \mathcal{I}^\bullet$$

where the equalities are Lemmas 63.4.11 and 63.3.4. This map of complexes determines the map $g_! K \to Rh_*(f_! K)$ of the statement of the lemma.

Assume Λ is torsion, i.e., we are in case (b). To check the map is an isomorphism we may work locally on Z. Hence we may assume that the dimension of fibres of h is bounded, see Morphisms, Lemma 29.28.5. Then we see that Rh_* has finite cohomological dimension, see Étale Cohomology, Lemma 59.92.2. Hence by Derived Categories, Lemma 13.32.2, if we show that $R^q h_*(f_!\mathcal{F}) = 0$ for $q > 0$ and any sheaf $\mathcal{F}$ of Λ-modules on $X_{\acute{e}tale}$, then $h_* f_! \mathcal{K}^\bullet \to h_* \mathcal{I}^\bullet$ is a quasi-isomorphism.

Observe that $\mathcal{G} = f_!\mathcal{F}$ is a sheaf of Λ-modules on Y whose stalks are nonzero only at points $y \in Y$ such that $\kappa(y)/\kappa(h(y))$ is a finite extension. This follows from the description of stalks of $f_!\mathcal{F}$ in Lemma 63.4.5 and the fact that both f and g are locally quasi-finite. Hence by the proper base change theorem (Étale Cohomology, Lemma 59.91.13) it suffices to show that $H^q(Y_{\overline{z}}, \mathcal{H}) = 0$ where $\mathcal{H}$ is a sheaf on the proper scheme $Y_{\overline{z}}$ over $\kappa(\overline{z})$ whose support is contained in the set of closed points. Thus the required vanishing by Étale Cohomology, Lemma 59.97.3.

Case (a) follows from case (b) by the exact same argument as used in the proof of Lemma 63.8.1 (using Lemma 63.4.5 instead of Lemma 63.3.17). □

63.9. Derived lower shriek via compactifications

0F7H Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Choose a compactification $j : X \to \overline{X}$ over Y, see More on Flatness, Theorem 38.33.8. Let Λ be a ring. Denote $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ the strictly full saturated triangulated subcategory of $D(X_{\acute{e}tale}, \Lambda)$ consisting of objects K which are bounded below and whose cohomology sheaves are torsion. We will consider the functor

$$Rf_! = R\overline{f}_* \circ j_! : D^+_{tors}(X_{\acute{e}tale}, \Lambda) \longrightarrow D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$$

where $\overline{f} : \overline{X} \to Y$ is the structure morphism. This makes sense: the functor $j_!$ sends $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ into $D^+_{tors}(\overline{X}_{\acute{e}tale}, \Lambda)$ by Remark 63.7.2 and $R\overline{f}_*$ sends $D^+_{tors}(\overline{X}_{\acute{e}tale}, \Lambda)$ into $D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$ by Étale Cohomology, Lemma 59.78.2. If Λ is a torsion ring, then we define

$$Rf_! = R\overline{f}_* \circ j_! : D(X_{\acute{e}tale}, \Lambda) \longrightarrow D(Y_{\acute{e}tale}, \Lambda)$$

Here is the obligatory lemma.

0F7I Lemma 63.9.1. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. The functors $Rf_!$ constructed above are, up to canonical isomorphism, independent of the choice of the compactification.

Proof. We will prove this for the functor $Rf_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$ when Λ is a torsion ring; the case of the functor $Rf_! : D^+_{tors}(X_{\acute{e}tale}, \Lambda) \to D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$ is proved in exactly the same way.

Consider the category of compactifications of X over Y, which is cofiltered according to More on Flatness, Theorem 38.33.8 and Lemmas 38.32.1 and 38.32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $R\overline{f}_* \circ j_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$. Let's be a little more explicit. Given a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$, we choose a quasi-isomorphism $j_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of sheaves of Λ-modules on $\overline{X}_{\acute{e}tale}$. Then our functor sends $\mathcal{K}^\bullet$ to $\overline{f}_*\mathcal{I}^\bullet$.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Then we get an isomorphism

$$R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

using Lemma 63.8.6 in the first equality.

To finish the proof, since the category of compactifications of X over Y is cofiltered, it suffices to show compositions of morphisms of compactifications of X over Y are turned into compositions of isomorphisms of functors[3]. To do this, suppose that $j_3 : X \to \overline{X}_3$ is a third compactification and that $h : \overline{X}_2 \to \overline{X}_3$ is a morphism of compactifications. Then we have to show that the composition

$$R\overline{f}_{3,*} \circ j_{3,!} = R\overline{f}_{3,*} \circ Rh_* \circ j_{2,!} = R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

is equal to the isomorphism of functors constructed using simply j_3, $g \circ h$, and j_1. A calculation shows that it suffices to prove that the composition of the maps

$$j_{3,!} \to Rh_* \circ j_{2,!} \to Rh_* \circ Rg_* \circ j_{1,!}$$

of Lemma 63.8.6 agrees with the corresponding map $j_{3,!} \to R(h \circ g)_* \circ j_{1,!}$ via the identification $R(h \circ g)_* = Rh_* \circ Rg_*$. Since the map of Lemma 63.8.6 is a special case of the map of Lemma 63.8.1 (as j_1 and j_2 are separated) this follows immediately from Lemma 63.8.2. $\square$

0F7J Lemma 63.9.2. Let $f : X \to Y$ and $g : Y \to Z$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then there is a canonical isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$.

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and

[3] Namely, if $\alpha, \beta : F \to G$ are morphisms of functors and $\gamma : G \to H$ is an isomorphism of functors such that $\gamma \circ \alpha = \gamma \circ \beta$, then we conclude $\alpha = \beta$.

Lemma 38.32.2 twice. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j"'] \arrow[d, "f"'] & U \arrow[r, "j'"'] \arrow[dl, "f'"] & \overline{X} \arrow[dl, "\overline{f}"] \\
Y \arrow[r, "i"'] \arrow[d, "g"'] & \overline{Y} \arrow[dl, "\overline{g}"] & \\
Z & &
\end{tikzcd}$$

Then we have

$$\begin{aligned}
R(g \circ f)_! &= R(\overline{g} \circ \overline{f})_* \circ (j' \circ j)_! \\
&= R\overline{g}_* \circ R\overline{f}_* \circ j'_! \circ j_! \\
&= R\overline{g}_* \circ i_! \circ Rf'_* \circ j_! \\
&= Rg_! \circ Rf_!
\end{aligned}$$

The first equality is the definition of $R(g \circ f)_!$. The second equality uses the identifications $R(\overline{g} \circ \overline{f})_* = R\overline{g}_* \circ R\overline{f}_*$ and $(j' \circ j)_! = j'_! \circ j_!$ of Lemma 63.3.13. The identification $i_! \circ Rf'_* \to R\overline{f}_* \circ j_!$ used in the third equality is Lemma 63.8.1. The final fourth equality is the definition of $Rg_!$ and $Rf_!$. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$$\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d] & U_1 \arrow[r, "j'_1"'] \arrow[dl, "f_1"] & \overline{X}_1 \arrow[dl, "\overline{f}_1"] \\
Y \arrow[r, "i_1"'] \arrow[d] & \overline{Y}_1 \arrow[dl, "\overline{g}_1"] & \\
Z & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "j_2"'] \arrow[d] & U_2 \arrow[r, "j'_2"'] \arrow[dl, "f_2"] & \overline{X}_2 \arrow[dl, "\overline{f}_2"] \\
Y \arrow[r, "i_2"'] \arrow[d] & \overline{Y}_2 \arrow[dl, "\overline{g}_2"] & \\
Z & &
\end{tikzcd}$$

We can first choose a compactification $i : Y \to \overline{Y}$ of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 38.32.1. By More on Flatness, Lemma 38.32.3 and Categories, Lemmas 4.27.13 and 4.27.14 we can choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when we have a commutative diagram as follows

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, equal] & U_1 \arrow[r, "j'_1"'] \arrow[d, "h'"] \arrow[ddl] & \overline{X}_1 \arrow[d, "h"] \arrow[ddl] \\
X \arrow[r, "j_2"] \arrow[d] & U_2 \arrow[r, "j'_2"] \arrow[ddl] & \overline{X}_2 \arrow[ddl] \\
Y \arrow[r, "i_1"] \arrow[d, equal] & \overline{Y}_1 \arrow[d, "k"] & \\
Y \arrow[r, "i_2"] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\
Z & &
\end{tikzcd}
$$

Each of the squares

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "A"] & U_1 \arrow[d, "h'"] \\
X \arrow[r, "j_2"] & U_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_2 \arrow[r, "j'_2"'] \arrow[d, "f_2"'] \arrow[dr, phantom, "B"] & \overline{X}_2 \arrow[d, "\overline{f}_2"] \\
Y \arrow[r, "i_2"] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_1 \arrow[r, "j'_1"'] \arrow[d, "f_1"'] \arrow[dr, phantom, "C"] & \overline{X}_1 \arrow[d, "\overline{f}_1"] \\
Y \arrow[r, "i_1"] & \overline{Y}_1
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, "i_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "D"] & \overline{Y}_1 \arrow[d, "k"] \\
Y \arrow[r, "i_2"] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "j'_1 \circ j_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "E"] & \overline{X}_1 \arrow[d, "h"] \\
X \arrow[r, "j_2"] & \overline{X}_2
\end{tikzcd}
$$

gives rise to an isomorphism as follows

$$
\begin{aligned}
\gamma_A &: j_{2,!} \to Rh'_* \circ j_{1,!} \\
\gamma_B &: i_{2,!} \circ Rf_{2,*} \to R\overline{f}_{2,*} \circ j'_{2,!} \\
\gamma_C &: i_{1,!} \circ Rf_{1,*} \to R\overline{f}_{1,*} \circ j'_{1,!} \\
\gamma_D &: i_{2,!} \to Rk_* \circ i_{1,!} \\
\gamma_E &: j_{2,!} \to Rh_* \circ (j'_1 \circ j_1)_!
\end{aligned}
$$

by applying the map from Lemma 63.8.1 (which is the same as the map in Lemma 63.8.6 in case the left vertical arrow is the identity). Let us write

$$
\begin{aligned}
F_1 &= Rf_{1,*} \circ j_{1,!} \\
F_2 &= Rf_{2,*} \circ j_{2,!} \\
G_1 &= R\overline{g}_{1,*} \circ i_{1,!} \\
G_2 &= R\overline{g}_{2,*} \circ i_{2,!} \\
C_1 &= R(\overline{g}_1 \circ \overline{f}_1)_* \circ (j'_1 \circ j_1)_! \\
C_2 &= R(\overline{g}_2 \circ \overline{f}_2)_* \circ (j'_2 \circ j_2)_!
\end{aligned}
$$

The construction given in the first paragraph of the proof and in Lemma 63.9.1 uses

(1) γ_C for the map $G_1 \circ F_1 \to C_1$,
(2) γ_B for the map $G_2 \circ F_2 \to C_2$,
(3) γ_A for the map $F_2 \to F_1$,
(4) γ_D for the map $G_2 \to G_1$, and
(5) γ_E for the map $C_2 \to C_1$.

This implies that we have to show that the diagram

$$\begin{tikzcd} C_2 \arrow[r, "\gamma_E"'] & C_1 \\ G_2 \circ F_2 \arrow[u, "\gamma_B"] \arrow[r, "\gamma_D \circ \gamma_A"] & G_1 \circ F_1 \arrow[u, "\gamma_C"'] \end{tikzcd}$$

is commutative. We will use Lemmas 63.8.2 and 63.8.3 and with (abuse of) notation as in Remark 63.8.4 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_F \circ \gamma_A$ where

$$\begin{tikzcd} U_1 \arrow[r, "j_1'"'] \arrow[d, "h'"'] \arrow[dr, phantom, "F"] & \overline{X}_1 \arrow[d, "h"] \\ U_2 \arrow[r, "j_2'"] & \overline{X}_2 \end{tikzcd}$$

Thus we see that

$$\gamma_E \circ \gamma_B = \gamma_F \circ \gamma_A \circ \gamma_B = \gamma_F \circ \gamma_B \circ \gamma_A$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 63.8.4). Thus it suffices to prove that $\gamma_C \circ \gamma_D = \gamma_F \circ \gamma_B$. Since both of these are equal to the map for the square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & \overline{X}_1 \arrow[d] \\ Y \arrow[r] & \overline{Y}_2 \end{tikzcd}$$

we conclude. □

0F7K Lemma 63.9.3. Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then the diagram

$$\begin{tikzcd} Rh_! \circ Rg_! \circ Rf_! \arrow[r, "\gamma_C"'] \arrow[d, "\gamma_A"] & R(h \circ g)_! \circ Rf_! \arrow[d, "\gamma_{A+B}"'] \\ Rh_! \circ R(g \circ f)_! \arrow[r, "\gamma_{B+C}"] & R(h \circ g \circ f)_! \end{tikzcd}$$

of isomorphisms of Lemma 63.9.2 commutes (for the meaning of the γ's see proof).

Proof. To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under

$\overline{X} \to \overline{Y}$. This produces the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X} \arrow[d] \\ Y \arrow[r] \arrow[d, "g"'] & Y \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{Y} \arrow[d] \\ Z \arrow[r] \arrow[d, "h"'] & Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\ T \arrow[r] & T \arrow[r] & T \arrow[r] & T \end{tikzcd}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 63.9.2 we see that the maps in the first displayed diagram use the maps of Lemma 63.8.1 for the rectangles $A+B$, $B+C$, A, and C as indicated in the diagram in the statement of the lemma. Since by Lemmas 63.8.2 and 63.8.3 we have $\gamma_{A+B} = \gamma_B \circ \gamma_A$ and $\gamma_{B+C} = \gamma_B \circ \gamma_C$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 63.8.4). $\square$

0F7L Lemma 63.9.4. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes with f separated and of finite type. Then there is a canonical isomorphism

$$g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$$

Moreover, these isomorphisms are compatible with the isomorphisms of Lemma 63.9.2.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $j' : X' \to \overline{X}'$ and $\overline{f}' : \overline{X}' \to Y'$ denote the base changes of j and $\overline{f}$. Since $Rf_! = R\overline{f}_* \circ j_!$ and $Rf'_! = R\overline{f}'_* \circ j'_!$ the isomorphism can be constructed via

$$g^{-1} \circ R\overline{f}_* \circ j_! \to R\overline{f}'_* \circ (\overline{g}')^{-1} \circ j_! \to R\overline{f}'_* \circ j'_! \circ (g')^{-1}$$

where the first arrow is the isomorphism given to us by the proper base change theorem (Étale Cohomology, Lemma 59.91.12 in the bounded below torsion case and Étale Cohomology, Lemma 59.92.3 in the case that Λ is torsion) and the second arrow is the isomorphism of Lemma 63.3.12.

To finish the proof we have to show two things: first we have to show that the isomorphism of functors so obtained does not depend on the choice of the compactification and second we have to show that if we vertically stack two base change diagrams as in the lemma, then these base change isomorphisms are compatible with the isomorphisms of Lemma 63.9.2. A straightforward argument which we omit shows that both follow if we can show that the isomorphisms

(1) $Rg_* \circ Rf_* = R(g \circ f)_*$ for $f : X \to Y$ and $g : Y \to Z$ proper,
(2) $g_! \circ f_! = (g \circ f)_!$ for $f : X \to Y$ and $g : Y \to Z$ separated and quasi-finite, and
(3) $g_! \circ Rf'_* = Rf_* \circ g'_!$ for $f : X \to Y$ and $f' : X' \to Y'$ proper and $g : Y' \to Y$ and $g' : X' \to X$ separated and quasi-finite with $f \circ g' = g \circ f'$

are compatible with base change. This holds for (1) by Cohomology on Sites, Remark 21.19.4, for (2) by Remark 63.3.14, and (3) by Lemma 63.8.5. □

0H6X Remark 63.9.5. Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Below we will construct a map

$$Rf_!K \longrightarrow Rf_*K$$

functorial for K in $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or $D(X_{\acute{e}tale}, \Lambda)$ if Λ is torsion. This transformation of functors in both cases is compatible with

(1) the isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$ of Lemma 63.9.2 and the isomorphism $Rg_* \circ Rf_* \to R(g \circ f)_*$ of Cohomology on Sites, Lemma 21.19.2 and
(2) the isomorphism $g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$ of Lemma 63.9.4 and the base change map of Cohomology on Sites, Remark 21.19.3.

Namely, choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Since $Rf_! = R\overline{f}_* \circ j_!$ and $Rf_* = R\overline{f}_* \circ Rj_*$ it suffices to construct a transformation of functors $j_! \to Rj_*$. For this we use the canonical transformation $j_! \to j_*$ of Étale Cohomology, Lemma 59.70.6. We omit the proof that the resulting transformation is independent of the choice of compactification and we omit the proof of the compatibilities (1) and (2).

63.10. Properties of derived lower shriek

0G28 Here are some properties of derived lower shriek.

0G29 Lemma 63.10.1. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a ring.

(1) Let $K_i \in D^+_{tors}(X_{\acute{e}tale}, \Lambda)$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $Rf_!(\bigoplus_i K_i) = \bigoplus_i Rf_!K_i$.
(2) If Λ is torsion, then the functor $Rf_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$ commutes with direct sums.

Proof. By construction it suffices to prove this when f is an open immersion and when f is a proper morphism. For any open immersion $j : U \to X$ of schemes, the functor $j_! : D(U_{\acute{e}tale}) \to D(X_{\acute{e}tale})$ is a left adjoint to pullback $j^{-1} : D(X_{\acute{e}tale}) \to D(U_{\acute{e}tale})$ and hence commutes with direct sums, see Cohomology on Sites, Lemma 21.20.8. In the proper case we have $Rf_! = Rf_*$ and we get the result from Étale Cohomology, Lemma 59.52.6 in the bounded belo case and from Étale Cohomology, Lemma 59.96.4 in the case that our coefficient ring Λ is a torsion ring. □

0G2A Lemma 63.10.2. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a ring. The functors $Rf_!$ constructed in Section 63.9 are bounded in the following sense: There exists an integer N such that for $E \in D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or $E \in D(X_{\acute{e}tale}, \Lambda)$ if Λ is torsion, we have

(1) $H^i(Rf_!(\tau_{\leq a}E) \to H^i(Rf_!(E))$ is an isomorphism for $i \leq a$,

(2) $H^i(Rf_!(E)) \to H^i(Rf_!(\tau_{\geq b-N}E))$ is an isomorphism for $i \geq b$,
(3) if $H^i(E) = 0$ for $i \not\in [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(Rf_!(E)) = 0$ for $i \not\in [a, b + N]$.

Proof. Assume Λ is torsion and consider the functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$. By construction it suffices to prove this when f is an open immersion and when f is a proper morphism. For any open immersion $j : U \to X$ of schemes, the functor $j_! : D(U_{étale}) \to D(X_{étale})$ is exact and hence the statement holds with $N = 0$ in this case. If f is proper then $Rf_! = Rf_*$, i.e., it is a right derived functor. Hence the bound on the left by Derived Categories, Lemma 13.16.1. Moreover in this case $f_* : \text{Mod}(X_{étale}, \Lambda) \to \text{Mod}(Y_{étale}, \Lambda)$ has bounded cohomological dimension by Morphisms, Lemma 29.28.5 and Étale Cohomology, Lemma 59.92.2. Thus we conclude by Derived Categories, Lemma 13.32.2.

Next, assume Λ is arbitrary and let us consider the functor $Rf_! : D^+_{tors}(X_{étale}, \Lambda) \to D^+_{tors}(Y_{étale}, \Lambda)$. Again we immediately reduce to the case where f is proper and $Rf_! = Rf_*$. Again part (1) is immediate. To show part (3) we can use induction on $b - a$, the distinguished triangles of trunctions, and Étale Cohomology, Lemma 59.92.2. Part (2) follows from (3). Details omitted. □

0GKM Lemma 63.10.3. Let $f : X \to Y$ be a quasi-finite separated morphism of quasi-compact and quasi-separated schemes. Then the functors $Rf_!$ constructed in Section 63.9 agree with the restriction of the functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ constructed in Section 63.7 to their common domains of definition.

Proof. By Zariski's main theorem (More on Morphisms, Lemma 37.43.3) we can find an open immersion $j : X \to \overline{X}$ and a finite morphism $\overline{f} : \overline{X} \to Y$ with $f = \overline{f} \circ j$. By construction we have $Rf_! = R\overline{f}_* \circ j_!$. Since $\overline{f}$ is finite, we have $R\overline{f}_* = \overline{f}_*$ by Étale Cohomology, Proposition 59.55.2. The lemma follows because $\overline{f}_* \circ j_! = f_!$ for example by Lemma 63.3.6. □

0GKN Lemma 63.10.4. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let U and V be quasi-compact opens of X such that $X = U \cup V$. Denote $a : U \to Y$, $b : V \to Y$ and $c : U \cap V \to Y$ the restrictions of f. Let Λ be a ring. For K in $D^+_{tors}(X_{étale}, \Lambda)$ or $K \in D(X_{étale}, \Lambda)$ if Λ is torsion, we have a distinguished triangle

$$Rc_!(K|_{U\cap V}) \to Ra_!(K|_U) \oplus Rb_!(K|_V) \to Rf_!K \to Rc_!(K|_{U\cap V})[1]$$

in $D(Y_{étale}, \Lambda)$.

Proof. This follows from Lemma 63.7.3, the fact that $Rf_! \circ Rj_{U!} = Ra_!$ by Lemma 63.9.2, and the fact that $Rj_{U!} = j_{U!}$ by Lemma 63.10.3. □

0GKP Lemma 63.10.5. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let U be a quasi-compact open of X with complement $Z \subset X$. Denote $g : U \to Y$ and $h : Z \to Y$ the restrictions of f. Let Λ be a ring. For K in $D^+_{tors}(X_{étale}, \Lambda)$ or $K \in D(X_{étale}, \Lambda)$ if Λ is torsion, we have a distinguished triangle

$$Rg_!(K|_U) \to Rf_!K \to Rh_!(K|_Z) \to Rg_!(K|_U)[1]$$

in $D(Y_{étale}, \Lambda)$.

Proof. This follows from Lemma 63.7.4, the fact that $Rf_! \circ Rj_! = Rg_!$ and $Rf_! \circ Ri_!$ by Lemma 63.9.2, and the fact that $Rj_! = j_!$ and $Ri_! = i_! = i_*$ by Lemma 63.10.3. $\square$

0GKQ Lemma 63.10.6. Let $f' : X' \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let $i : X \to X'$ be a thickening and denote $f = f' \circ i$. Let Λ be a ring. For K' in $D^+_{tors}(X'_{\acute{e}tale}, \Lambda)$ or $K' \in D(X'_{\acute{e}tale}, \Lambda)$ if Λ is torsion, we have $Rf_! i^{-1} K' = Rf'_! K'$.

Proof. This is true because i^{-1} and $i_* = i_!$ inverse equivalences of categories by the topological invariance of the small étale topos (Étale Cohomology, Theorem 59.45.2) and we can apply Lemma 63.9.2. $\square$

0GL5 Lemma 63.10.7. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. Let $E \in D(X_{\acute{e}tale}, \Lambda)$ and $K \in D(Y_{\acute{e}tale}, \Lambda)$. Then

$$Rf_! E \otimes^{\mathbf{L}}_\Lambda K = Rf_!(E \otimes^{\mathbf{L}}_\Lambda f^{-1}K)$$

in $D(Y_{\acute{e}tale}, \Lambda)$.

Proof. Choose $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ as in the construction of $Rf_!$. We have $j_! E \otimes^{\mathbf{L}}_\Lambda \overline{f}^{-1} K = j_!(E \otimes^{\mathbf{L}}_\Lambda f^{-1}K)$ by Cohomology on Sites, Lemma 21.20.9. Then we get the result by applying Étale Cohomology, Lemma 59.96.6 and using that $f^{-1} = j^{-1} \circ \overline{f}^{-1}$ and $Rf_! = R\overline{f}_* j_!$. $\square$

0GL6 Remark 63.10.8. Let $\Lambda_1 \to \Lambda_2$ be a homomorphism of torsion rings. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. The diagram

$$\begin{tikzcd} D(X_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"'] \arrow[d, "Rf_!"'] & D(X_{\acute{e}tale}, \Lambda_1) \arrow[d, "Rf_!"] \\ D(Y_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"] & D(Y_{\acute{e}tale}, \Lambda_1) \end{tikzcd}$$

commutes where *res* is the "restriction" functor which turns a Λ_2-module into a Λ_1-module using the given ring map. Writing $Rf_! = R\overline{f}_* \circ j_!$ for a factorization $f = \overline{f} \circ j$ as in Section 63.9, we see that the result holds for $j_!$ by inspection and for $R\overline{f}_*$ by Cohomology on Sites, Lemma 21.20.7. On the other hand, also the diagram

$$\begin{tikzcd} D(X_{\acute{e}tale}, \Lambda_1) \arrow[r, "- \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2"'] \arrow[d, "Rf_!"'] & D(X_{\acute{e}tale}, \Lambda_2) \arrow[d, "Rf_!"] \\ D(Y_{\acute{e}tale}, \Lambda_1) \arrow[r, "- \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2"] & D(Y_{\acute{e}tale}, \Lambda_2) \end{tikzcd}$$

is commutative as follows from Lemma 63.10.7.

0GL7 Remark 63.10.9. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring and let K and L be objects of $D(X_{\acute{e}tale}, \Lambda)$. We claim there is a canonical map

$$\alpha : Rf_* R\mathcal{H}om_\Lambda(K, L) \longrightarrow R\mathcal{H}om_\Lambda(Rf_!K, Rf_!L)$$

functorial in K and L. Namely, choose $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ as in the construction of $Rf_!$. We first define a map

$$\beta : Rj_* R\mathcal{H}om_\Lambda(K, L) \longrightarrow R\mathcal{H}om_\Lambda(j_!K, j_!L)$$

By the construction of internal hom in the derived category, this is the same thing as defining a map

$$\beta' : Rj_*R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda j_!K \longrightarrow j_!L$$

See Cohomology on Sites, Section 21.35. The source of β' is equal to

$$j_!\left(R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda K\right)$$

by Cohomology on Sites, Lemma 21.20.9. Hence we can set $\beta' = j_!\beta''$ where $\beta'' : R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda K \to L$ corresponds to the identity on $R\mathcal{H}om_\Lambda(K, L)$ via the universal property of internal hom mentioned above. By Cohomology on Sites, Remark 21.35.10 we have a canonical map

$$\gamma : R\overline{f}_*R\mathcal{H}om_\Lambda(j_!K, j_!L) \longrightarrow R\mathcal{H}om_\Lambda(R\overline{f}_*j_!K, R\overline{f}_*j_!L)$$

Since $Rf_! = R\overline{f}_*j_!$ and $Rf_* = R\overline{f}_*Rj_*$ (by Leray) we obtain the desired map $\alpha = \gamma \circ R\overline{f}_*\beta$.

63.11. Derived upper shriek

0G2B We obtain $Rf^!$ by a Brown representability theorem.

0G2C Lemma 63.11.1. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ has a right adjoint $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$.

Proof. This follows from Injectives, Proposition 19.15.2 and Lemma 63.10.1 above. □

0GL8 Lemma 63.11.2. Let $f : X \to Y$ be a separated quasi-finite morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ of Lemma 63.11.1 is the same as the functor $Rf^!$ of Lemma 63.7.1.

Proof. Follows from uniqueness of adjoints as $Rf_! = f_!$ by Lemma 63.10.3. □

0GL9 Lemma 63.11.3. Let $j : U \to X$ be a separated étale morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rj^! : D(X_{étale}, \Lambda) \to D(U_{étale}, \Lambda)$ is equal to j^{-1}.

Proof. This is true because both $Rj^!$ and j^{-1} are right adjoints to $Rj_! = j_!$. See for example Lemmas 63.11.2 and 63.6.2. □

0GLA Lemma 63.11.4. Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. The functor $Rf^!$ sends $D^+(Y_{étale}, \Lambda)$ into $D^+(X_{étale}, \Lambda)$. More precisely, there exists an integer $N \geq 0$ such that if $K \in D(Y_{étale}, \Lambda)$ has $H^i(K) = 0$ for $i < a$ then $H^i(Rf^!K) = 0$ for $i < a - N$.

Proof. Let N be the integer found in Lemma 63.10.2. By construction, for $K \in D(Y_{étale}, \Lambda)$ and $L \in\in D(X_{étale}, \Lambda)$ we have $\mathrm{Hom}_X(L, Rf^!K) = \mathrm{Hom}_Y(Rf_!L, K)$. Suppose $H^i(K) = 0$ for $i < a$. Then we take $L = \tau_{\leq a-N-1}Rf^!K$. By Lemma 63.10.2 the complex $Rf_!L$ has vanishing cohomology sheaves in degrees $\leq a - 1$. Hence $\mathrm{Hom}_Y(Rf_!L, K) = 0$ by Derived Categories, Lemma 13.27.3. Hence the canonical map $\tau_{\leq a-N-1}Rf^!K \to Rf^!K$ is zero which implies $H^i(Rf^!K) = 0$ for $i \leq a - N - 1$. □

Let $f : X \to Y$ be a separated finite type morphism of quasi-separated and quasi-compact schemes. Let Λ be a torsion coefficient ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ we obtain a canonical map

0GLB (63.11.4.1)
$$Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K) \longrightarrow R\mathcal{H}om_\Lambda(Rf_!L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K) \to R\mathcal{H}om_\Lambda(Rf_!L, Rf_!Rf^!K) \to R\mathcal{H}om_\Lambda(Rf_!L, K)$$

where the first arrow is Remark 63.10.9 and the second arrow is the counit $Rf_!Rf^!K \to K$ of the adjunction.

0GLC Lemma 63.11.5. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ the map (63.11.4.1)

$$Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K) \longrightarrow R\mathcal{H}om_\Lambda(Rf_!L, K)$$

is an isomorphism.

Proof. To prove the lemma we have to show that for any $M \in D(Y_{étale}, \Lambda)$ the map (63.11.4.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K)) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_\Lambda(Rf_!L, K))$$

To see this we use the following string of equalities

$$\begin{aligned}
\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K)) &= \mathrm{Hom}_X(f^{-1}M, R\mathcal{H}om_\Lambda(L, Rf^!K)) \\
&= \mathrm{Hom}_X(f^{-1}M \otimes_\Lambda^{\mathbf{L}} L, Rf^!K) \\
&= \mathrm{Hom}_Y(Rf_!(f^{-1}M \otimes_\Lambda^{\mathbf{L}} L), K) \\
&= \mathrm{Hom}_Y(M \otimes_\Lambda^{\mathbf{L}} Rf_!L, K) \\
&= \mathrm{Hom}_Y(M, R\mathcal{H}om_\Lambda(Rf_!L, K))
\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.19.1. The second equality by Cohomology on Sites, Lemma 21.35.2. The third equality by construction of $Rf^!$. The fourth equality by Lemma 63.10.7 (this is the important step). The fifth by Cohomology on Sites, Lemma 21.35.2. □

0GLD Lemma 63.11.6. Let $f : X \to Y$ be a separated finite type morphism of quasi-separated and quasi-compact schemes. Let Λ be a torsion ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ the map (63.11.4.1) induces an isomorphism

$$R\,\mathrm{Hom}_X(L, Rf^!K) \longrightarrow R\,\mathrm{Hom}_Y(Rf_!L, K)$$

of global derived homs.

Proof. By the construction in Cohomology on Sites, Section 21.36 we have

$$R\,\mathrm{Hom}_X(L, Rf^!K) = R\Gamma(X, R\mathcal{H}om_\Lambda(L, Rf^!K)) = R\Gamma(Y, Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K))$$

(the second equality by Leray) and

$$R\,\mathrm{Hom}_Y(Rf_!L, K) = R\Gamma(Y, R\mathcal{H}om_\Lambda(Rf_!L, K))$$

Thus the lemma is a consequence of Lemma 63.11.5. □

0GLE Lemma 63.11.7. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes with f separated and of finite type. Then we have $Rf^! \circ Rg_* = Rg'_* \circ R(f')^!$.

Proof. By uniqueness of adjoint functors this follows from base change for derived lower shriek: we have $g^{-1} \circ Rf_! = Rf'_! \circ (g')^{-1}$ by Lemma 63.9.4. $\square$

0GLF Remark 63.11.8. Let $\Lambda_1 \to \Lambda_2$ be a homomorphism of torsion rings. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. The diagram

$$\begin{tikzcd} D(X_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"] & D(X_{\acute{e}tale}, \Lambda_1) \\ D(Y_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"] \arrow[u, "Rf^!"] & D(Y_{\acute{e}tale}, \Lambda_1) \arrow[u, "Rf^!"'] \end{tikzcd}$$

commutes where res is the "restriction" functor which turns a Λ_2-module into a Λ_1-module using the given ring map. This holds by uniquenss of adjoints, the second commutative diagram of Remark 63.10.8 and because we have

$$\mathrm{Hom}_{\Lambda_2}(K_1 \otimes_{\Lambda_1}^{\mathbf{L}} \Lambda_2, K_2) = \mathrm{Hom}_{\Lambda_1}(K_1, res(K_2))$$

This equality either for objects living over $X_{\acute{e}tale}$ or on $Y_{\acute{e}tale}$ is a very special case of Cohomology on Sites, Lemma 21.19.1.

63.12. Compactly supported cohomology

0GJY Let k be a field. Let Λ be a ring. Let X be a separated scheme of finite type over k with structure morphism $f : X \to \mathrm{Spec}(k)$. In Section 63.9 we have defined the functor $Rf_! : D^+_{tors}(X_{\acute{e}tale}, \Lambda) \to D^+_{tors}(\mathrm{Spec}(k), \Lambda)$ and the functor $Rf_! : D(X_{\acute{e}tale}, \Lambda) \to D(\mathrm{Spec}(k), \Lambda)$ if Λ is a torsion ring. Composing with the global sections functor on $\mathrm{Spec}(k)$ we obtain what we will call the compactly supported cohomology.

0GJZ Definition 63.12.1. Let X be a separated scheme of finite type over a field k. Let Λ be a ring. Let K be an object of $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or of $D(X_{\acute{e}tale}, \Lambda)$ in case Λ is torsion. The cohomology of K with compact support or the compactly supported cohomology of K is

$$R\Gamma_c(X, K) = R\Gamma(\mathrm{Spec}(k), Rf_!K)$$

where $f : X \to \mathrm{Spec}(k)$ is the structure morphism. We will write $H^i_c(X, K) = H^i(R\Gamma_c(X, K))$.

We will check that this definition doesn't conflict with Definition 63.3.7 by Lemma 63.12.3. The utility of this definition lies in the following result.

0GK0 Lemma 63.12.2. Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Let K be an object of $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or of $D(X_{\acute{e}tale}, \Lambda)$ in case Λ is torsion. Then there is a canonical isomorphism

$$(Rf_!K)_{\overline{y}} \longrightarrow R\Gamma_c(X_{\overline{y}}, K|_{X_{\overline{y}}})$$

in $D(\Lambda)$ for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Immediate consequence of Lemma 63.9.4 and the definitions. $\square$

0GK1 Lemma 63.12.3. Let X be a separated scheme of finite type over a field k. If $\mathcal{F}$ is a torsion abelian sheaf, then the abelian group $H^0_c(X, \mathcal{F})$ defined in Definition 63.3.7 agrees with the abelian group $H^0_c(X, \mathcal{F})$ defined in Definition 63.12.1.

Proof. Choose a compactification $j : X \to \overline{X}$ over k. In both cases the group is defined as $H^0(\overline{X}, j_!\mathcal{F})$. This is true for the first version by Lemma 63.3.10 and for the second version by construction. $\square$

0GKR Lemma 63.12.4. Let k be an algebraically closed field. Let X be a separated scheme of finite type type over k of dimension ≤ 1. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on X which is torsion. Then $H^q_c(X, \mathcal{F})$ is a finite Λ-module.

Proof. This is a consequence of Étale Cohomology, Theorem 59.84.7. Namely, choose a compactification $j : X \to \overline{X}$. After replacing $\overline{X}$ by the scheme theoretic closure of X, we see that we may assume $\dim(\overline{X}) \leq 1$. Then $H^q_c(X, \mathcal{F}) = H^q(\overline{X}, j_!\mathcal{F})$ and the theorem applies. $\square$

0GKS Remark 63.12.5 (Covariance of compactly supported cohomology). Let k be a field. Let $f : X \to Y$ be a morphism of separated schemes of finite type over k. If X, Y, and f satisfies one of the following conditions

1. f is étale, or
2. f is flat and quasi-finite, or
3. f is quasi-finite and Y is geometrically unibranch, or
4. f is quasi-finite and there exists a weighting $w : X \to \mathbf{Z}$ of f

then compactly supported cohomology is covariant with respect to f. More precisely, let Λ be a ring. Let K be an object of $D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$ or of $D(Y_{\acute{e}tale}, \Lambda)$ in case Λ is torsion. Under one of the assumptions (1) – (4) there is a canonical map

$$\mathrm{Tr}_{f,w,K} : f_! f^{-1} K \longrightarrow K$$

See Section 63.5 for the existence of the trace map and Examples 63.5.5 and 63.5.7 for cases (2) and (3). If $p : X \to \mathrm{Spec}(k)$ and $q : Y \to \mathrm{Spec}(k)$ denote the structure morphisms, then we have $Rq_! \circ f_! = Rp_!$ by Lemma 63.9.2 and the fact that $Rf_! = f_!$ for the quasi-finite separated morphism f by Lemma 63.10.3. Hence we can look at the map

$$\begin{aligned} R\Gamma_c(X, f^{-1}K) &= R\Gamma(\mathrm{Spec}(k), Rp_! f^{-1}K) \\ &= R\Gamma(\mathrm{Spec}(k), Rq_! f_! f^{-1}K) \\ &\xrightarrow{Rq_! \mathrm{Tr}_{f,w,K}} R\Gamma(\mathrm{Spec}(k), Rq_! K) \\ &= R\Gamma_c(Y, K) \end{aligned}$$

In particular, if Λ is a torsion ring, then we obtain an arrow

$$\mathrm{Tr}_f : R\Gamma_c(X, \Lambda) \longrightarrow R\Gamma_c(Y, \Lambda)$$

This map has lots of additional properties, for example it is compatible with taking ground field extensions.

63.13. A constructibility result

0GKT We "compute" the cohomology of a smooth projective family of curves with constant coefficients.

0GKU Lemma 63.13.1. Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module viewed as an $\mathcal{O}_S$-module on $S_{\acute{e}tale}$. Let $F : \mathcal{E} \to \mathcal{E}$ be a homomorphism of abelian sheaves on $S_{\acute{e}tale}$ such that $F(ae) = a^pF(e)$ for local sections a, e of $\mathcal{O}_S$, $\mathcal{E}$ on $S_{\acute{e}tale}$. Then

$$\mathrm{Coker}(F - 1 : \mathcal{E} \to \mathcal{E})$$

is zero and

$$\mathrm{Ker}(F - 1 : \mathcal{E} \to \mathcal{E})$$

is a constructible abelian sheaf on $S_{\acute{e}tale}$.

This lemma is a generalization of Étale Cohomology, Lemma 59.63.2.

Proof. We may assume $S = \mathrm{Spec}(A)$ where A is an $\mathbf{F}_p$-algebra and that $\mathcal{E}$ is the quasi-coherent module associated to the free A-module $Ae_1 \oplus \ldots \oplus Ae_n$. We write $F(e_i) = \sum a_{ij}e_j$.

Surjectivity of $F - 1$. It suffices to show that any element $\sum a_ie_i$, $a_i \in A$ is in the image of $F - 1$ after replacing A by a faithfully flat étale extension. Observe that

$$F(\sum x_ie_i) - \sum x_ie_i = \sum x_i^pa_{ij}e_j - \sum x_ie_i$$

Consider the A-algebra

$$A' = A[x_1, \ldots, x_n]/(a_i + x_i - \sum\nolimits_j a_{ji}x_j^p)$$

A computation shows that $\mathrm{d}x_i$ is zero in $\Omega_{A'/A}$ and hence $\Omega_{A'/A} = 0$. Since A' is of finite type over A, this implies that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is unramified and hence is quasi-finite. Since A' is generated by n elements and cut out by n equations, we conclude that A' is a global relative complete intersection over A. Thus A' is flat over A and we conclude that $A \to A'$ is étale (as a flat and unramified ring map). Finally, the reader can show that $A \to A'$ is faithfully flat by verifying directly that all geometric fibres of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ are nonempty, however this also follows from Étale Cohomology, Lemma 59.63.2. Finally, the element $\sum x_ie_i \in A'e_1 \oplus \ldots \oplus A'e_n$ maps to $\sum a_ie_i$ by $F - 1$.

Constructibility of the kernel. The calculations above show that $\mathrm{Ker}(F - 1)$ is represented by the scheme

$$\mathrm{Spec}(A[x_1, \ldots, x_n]/(x_i - \sum\nolimits_j a_{ji}x_j^p))$$

over $S = \mathrm{Spec}(A)$. Since this is a scheme affine and étale over S we obtain the result from Étale Cohomology, Lemma 59.73.1. $\square$

0GKV Lemma 63.13.2. Let $f : X \to S$ be a proper smooth morphism of schemes with geometrically connected fibres of dimension 1. Let ℓ be a prime number. Then $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ is a constructible.

Proof. We may assume S is affine. Say $S = \mathrm{Spec}(A)$. Then, if we write $A = \bigcup A_i$ as the union of its finite type $\mathbf{Z}$-subalgebras, we can find an i and a morphism $f_i : X_i \to S_i = \mathrm{Spec}(A_i)$ of finite type whose base change to S is $f : X \to S$, see Limits, Lemma 32.10.1. After increasing i we may assume $f_i : X_i \to S_i$ is smooth, proper, and of relative dimension 1, see Limits, Lemmas 32.13.1 32.8.9, and 32.18.4. By More on Morphisms, Lemma 37.53.8 we obtain an open subscheme $U_i \subset S_i$ such that the fibres of $f_i : X_i \to S_i$ over U_i are geometrically connected. Then $S \to S_i$ maps into U_i. We may replace $X \to S$ by $f_i : f_i^{-1}(U_i) \to U_i$ to reduce to the case discussed in the next paragraph.

Assume S is Noetherian. We may write $S = U \cup Z$ where U is the open subscheme defined by the nonvanishing of ℓ and $Z = V(\ell) \subset S$. Since the formation of $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ commutes with arbtrary base change (Étale Cohomology, Theorem 59.91.11), it suffices to prove the result over U and over Z. Thus we reduce to the following two cases: (a) ℓ is invertible on S and (b) ℓ is zero on S.

Case (a). We claim that in this case the sheaves $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ are finite locally constant on S. First, by proper base change (in the form of Étale Cohomology, Lemma 59.91.13) and by finiteness (Étale Cohomology, Theorem 59.83.10) we see that the stalks of $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ are finite. By Étale Cohomology, Lemma 59.94.4 all specialization maps are isomorphisms. We conclude the claim holds by Étale Cohomology, Lemma 59.75.6.

Case (b). Here $\ell = p$ is a prime and S is a scheme over $\mathrm{Spec}(\mathbf{F}_p)$. By the same references as above we already know that the stalks of $R^qf_*\underline{\mathbf{Z}/p\mathbf{Z}}$ are finite and zero for $q \geq 2$. It follows from Étale Cohomology, Lemma 59.39.3 that $f_*\underline{\mathbf{Z}/p\mathbf{Z}} = \underline{\mathbf{Z}/p\mathbf{Z}}$. It remains to prove that $R^1f_*\underline{\mathbf{Z}/p\mathbf{Z}}$ is constructible. Consider the Artin-Schreyer sequence

$$0 \to \underline{\mathbf{Z}/p\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{F-1} \mathcal{O}_X \to 0$$

See Étale Cohomology, Section 59.63. Recall that $f_*\mathcal{O}_X = \mathcal{O}_S$ and $R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module of rank equal to the genera of the fibres of $X \to S$, see Algebraic Curves, Lemma 53.20.13. We conclude that we have a short exact sequence

$$0 \to \mathrm{Coker}(F-1 : \mathcal{O}_S \to \mathcal{O}_S) \to R^1f_*\underline{\mathbf{Z}/p\mathbf{Z}} \to \mathrm{Ker}(F-1 : R^1f_*\mathcal{O}_X \to R^1f_*\mathcal{O}_X) \to 0$$

Applying Lemma 63.13.1 we win. $\square$

0GKW **Lemma 63.13.3.** *Let $f : X \to S$ be a proper smooth morphism of schemes with geometrically connected fibres of dimension 1. Let Λ be a Noetherian ring. Let M be a finite Λ-module annihilated by an integer $n > 0$. Then $R^qf_*\underline{M}$ is a constructible sheaf of Λ-modules on S.*

Proof. If $n = \ell n'$ for some prime number ℓ, then we get a short exact sequence $0 \to M[\ell] \to M \to M' \to 0$ of finite Λ-modules and M' is annihilated by n'. This produces a corresponding short exact sequence of constant sheaves, which in turn gives rise to an exact sequence

$$R^{q-1}f_*\underline{M'} \to R^qf_*\underline{M[n]} \to R^qf_*\underline{M} \to R^qf_*\underline{M'} \to R^{q+1}f_*\underline{M[n]}$$

Thus, if we can show the result in case M is annihilated by a prime number, then by induction on n we win by Étale Cohomology, Lemma 59.71.6.

Let ℓ be a prime number such that ℓ annihilates M. Then we can replace Λ by the $\mathbf{F}_\ell$-algebra $\Lambda/\ell\Lambda$. Namely, the sheaf $R^qf_*\underline{M}$ where $\underline{M}$ is viewed as a sheaf of Λ-modules is the same as the sheaf $R^qf_*\underline{M}$ computed by viewing $\underline{M}$ as a sheaf of $\Lambda/\ell\Lambda$-modules, see Cohomology on Sites, Lemma 21.20.7.

Assume ℓ be a prime number such that ℓ annihilates M and Λ. Let us reduce to the case where M is a finite free Λ-module. Namely, choose a resolution

$$\ldots \to \Lambda^{\oplus m_2} \to \Lambda^{\oplus m_1} \to \Lambda^{\oplus m_0} \to M \to 0$$

Recall that f_* has finite cohomological dimension on sheaves of Λ-modules, see Étale Cohomology, Lemma 59.92.2 and Derived Categories, Lemma 13.32.2. Thus we see that $R^qf_*\underline{M}$ is the qth cohomology sheaf of the object

$$Rf_*(\underline{\Lambda^{\oplus m_a}} \to \ldots \to \underline{\Lambda^{\oplus m_0}})$$

in $D(S_{\acute{e}tale}, \Lambda)$ for some integer a large enough. Using the first spectral sequence of Derived Categories, Lemma 13.21.3 (or alternatively using an argument with truncations) we conclude that it suffices to prove that $R^qf_*\underline{\Lambda})$ is constructible.

At this point we can finally use that

$$(Rf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}) \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell\mathbf{Z}} \underline{\Lambda} = Rf_*\underline{\Lambda}$$

by Étale Cohomology, Lemma 59.96.6. Since any module over the field $\mathbf{Z}/\ell\mathbf{Z}$ is flat we obtain

$$(R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}) \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \underline{\Lambda} = R^qf_*\underline{\Lambda}$$

Hence it suffices to prove the result for $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ by Étale Cohomology, Lemma 59.71.10. This case is Lemma 63.13.2. □

63.14. Complexes with constructible cohomology

0GK2 We continue the discussion started in Étale Cohomology, Section 59.76. In particular, for a scheme X and a Noetherian ring Λ we denote $D_c(X_{\acute{e}tale}, \Lambda)$ the strictly full saturated triangulated subcategory of $D(X_{\acute{e}tale}, \Lambda)$ consisting of objects whose cohomology sheaves are constructible sheaves of Λ-modules.

0GK3 **Lemma 63.14.1.** Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite and of finite presentation. The functor $f_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$ of Lemma 63.7.1 sends $D_c(X_{\acute{e}tale}, \Lambda)$ into $D_c(Y_{\acute{e}tale}, \Lambda)$.

Proof. Since the functor $f_!$ is exact, it suffices to show that $f_!\mathcal{F}$ is constructible for any constructible sheaf $\mathcal{F}$ of Λ-modules on $X_{\acute{e}tale}$. The question is local on Y and hence we may and do assume Y is affine. Then X is quasi-compact and quasi-separated, see Morphisms, Definition 29.21.1. Say $X = \bigcup_{i=1,\ldots,n} X_i$ is a finite affine open covering. By Lemma 63.4.7 we see that it suffices to show that $f_{i,!}\mathcal{F}|_{X_i}$ and $f_{ii',!}\mathcal{F}|_{X_i \cap X_{i'}}$ are constructible where $f_i : X_i \to Y$ and $f_{ii'} : X_i \cap X_{i'} \to Y$ are the restrictions of f. Since X_i and $X_i \cap X_{i'}$ are quasi-compact and separated this means we may assume f is separated. By Zariski's main theorem (in the form of More on Morphisms, Lemma 37.43.4) we can choose a factorization $f = g \circ j$ where $j : X \to X'$ is an open immersion and $g : X' \to Y$ is finite and of finite presentation. Then $f_! = g_! \circ j_!$ by Lemma 63.3.13. By Étale Cohomology, Lemma 59.73.1 we see that $j_!\mathcal{F}$ is constructible on X'. The morphism g is finite hence $g_! = g_*$ by Lemma 63.3.4. Thus $f_!\mathcal{F} = g_!j_!\mathcal{F} = g_*j_!\mathcal{F}$ is constructible by Étale Cohomology, Lemma 59.73.9. □

0GKX **Lemma 63.14.2.** Let S be a Noetherian affine scheme of finite dimension. Let $f : X \to S$ be a separated, affine, smooth morphism of relative dimension 1. Let Λ be a Noetherian ring which is torsion. Let M be a finite Λ-module. Then $Rf_!\underline{M}$ has constructible cohomology sheaves.

Proof. We will prove the result by induction on $d = \dim(S)$.

Base case. If $d = 0$, then the only thing to show is that the stalks of $R^q f_!\underline{M}$ are finite Λ-modules. If $\overline{s}$ is a geometric point of S, then we have $(R^q f_!\underline{M})_{\overline{s}} = H^q_c(X_{\overline{s}}, \underline{M})$ by Lemma 63.12.2. This is a finite Λ-module by Lemma 63.12.4.

Induction step. It suffices to find a dense open $U \subset S$ such that $Rf_!\underline{M}|_U$ has constructible cohomology sheaves. Namely, the restriction of $Rf_!\underline{M}$ to the complement $S \setminus U$ will have constructible cohomology sheaves by induction and the fact that formation of $Rf_!\underline{M}$ commutes with all base change (Lemma 63.9.4). In fact, let $\eta \in S$ be a generic point of an irreducible component of S. Then it suffices to find an open neighbourhood U of η such that the restriction of $Rf_!\underline{M}$ to U is constructible. This is what we will do in the next paragraph.

Given a generic point $\eta \in S$ we choose a diagram

$$\begin{tikzcd}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[rd] & Y_1 \amalg \ldots \amalg Y_n \arrow[l, "j"'] \arrow[r, "\nu"'] \arrow[d] & X_V \arrow[r] \arrow[d] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & T_1 \amalg \ldots \amalg T_n \arrow[r] & V \arrow[r] & U \arrow[r] & S
\end{tikzcd}$$

as in More on Morphisms, Lemma 37.56.1. We will show that $Rf_!\underline{M}|_U$ is constructible. First, since $V \to U$ is finite and surjective, it suffices to show that the pullback to V is constructible, see Étale Cohomology, Lemma 59.73.3. Since formation of $Rf_!$ commutes with base change, we see that it suffices to show that $R(X_V \to V)_!\underline{M}$ is constructible. Let $W \subset X_V$ be the open subscheme given to us by More on Morphisms, Lemma 37.56.1 part (4). Let $Z \subset X_V$ be the reduced induced scheme structure on the complement of W in X_V. Then the fibres of $Z \to V$ have dimension 0 (as W is dense in the fibres) and hence $Z \to V$ is quasi-finite. From the distinguished triangle

$$R(W \to V)_!\underline{M} \to R(X_V \to V)_!\underline{M} \to R(Z \to V)_!\underline{M} \to \ldots$$

of Lemma 63.10.5 and from Lemma 63.14.1 we conclude that it suffices to show that $R(W \to V)_!\underline{M}$ has constructible cohomology sheaves. Next, we have

$$R(W \to V)_!\underline{M} = R(\nu^{-1}(W) \to V)_!\underline{M}$$

because the morphism $\nu : \nu^{-1}(W) \to W$ is a thickening and we may apply Lemma 63.10.6. Next, we let $Z' \subset \coprod \overline{Y}_i$ denote the complement of the open $j(\nu^{-1}(W))$. Again $Z' \to V$ is quasi-finite. Again use the distinguished triangle

$$R(\nu^{-1}(W) \to V)_!\underline{M} \to R(\coprod \overline{Y}_i \to V)_!\underline{M} \to R(Z' \to V)_!\underline{M} \to \ldots$$

to conclude that it suffices to prove

$$R(\coprod \overline{Y}_i \to V)_!\underline{M} = \bigoplus\nolimits_i R(\overline{Y}_i \to V)_!\underline{M} = \bigoplus\nolimits_i R(T_i \to V)_! R(\overline{Y}_i \to T_i)_!\underline{M}$$

has constructible cohomology sheaves (second equality by Lemma 63.9.2). The result for $R(\overline{Y}_i \to T_i)_!\underline{M}$ is Lemma 63.13.3 and we win because $T_i \to V$ is finite étale and we can apply Lemma 63.14.1. $\square$

0GKY Lemma 63.14.3. Let Y be a Noetherian affine scheme of finite dimension. Let Λ be a Noetherian ring which is torsion. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on an open subscheme $U \subset \mathbf{A}^1_Y$. Then $Rf_!\mathcal{F}$ has constructible cohomology sheaves where $f : U \to Y$ is the structure morphism.

Proof. We may decompose Λ as a product $\Lambda = \Lambda_1 \times \ldots \times \Lambda_r$ where Λ_i is ℓ_i-primary for some prime ℓ_i. Thus we may assume there exists a prime ℓ and an integer $n > 0$ such that ℓ^n annihilates Λ (and hence $\mathcal{F}$).

Since U is Noetherian, we see that U has finitely many connected components. Thus we may assume U is connected. Let $g : U' \to U$ be the finite étale covering constructed in Étale Cohomology, Lemma 59.66.4. The discussion in Étale Cohomology, Section 59.66 gives maps

$$\mathcal{F} \to g_*g^{-1}\mathcal{F} \to \mathcal{F}$$

whose composition is an isomorphism. Hence it suffices to prove the result for $g_*g^{-1}\mathcal{F}$. On the other hand, we have $Rf_!g_*g^{-1}\mathcal{F} = R(f \circ g)_!g^{-1}\mathcal{F}$ by Lemma 63.9.2. Since $g^{-1}\mathcal{F}$ has a finite filtration by constant sheaves of Λ-modules of the form $\underline{M}$ for some finite Λ-module M (by our choice of g) this reduces us to the case proved in Lemma 63.14.2. $\square$

0GKZ Lemma 63.14.4. Let Y be an affine scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $\mathbf{A}^1_Y$ which is torsion. Then $Rf_!\mathcal{F}$ has constructible cohomology sheaves where $f : \mathbf{A}^1_Y \to Y$ is the structure morphism.

Proof. Say $\mathcal{F}$ is annihilated by $n > 0$. Then we can replace Λ by $\Lambda/n\Lambda$ without changing $Rf_!\mathcal{F}$. Thus we may and do assume Λ is a torsion ring.

Say $Y = \text{Spec}(R)$. Then, if we write $R = \bigcup R_i$ as the union of its finite type $\mathbf{Z}$-subalgebras, we can find an i such that $\mathcal{F}$ is the pullback of a constructible sheaf of Λ-modules on $\mathbf{A}^1_{R_i}$, see Étale Cohomology, Lemma 59.73.10. Hence we may assume Y is a Noetherian scheme of finite dimension.

Assume Y is a Noetherian scheme of finite dimension $d = \dim(Y)$ and Λ is torsion. We will prove the result by induction on d.

Base case. If $d = 0$, then the only thing to show is that the stalks of $R^qf_!\mathcal{F}$ are finite Λ-modules. If $\overline{y}$ is a geometric point of Y, then we have $(R^qf_!\mathcal{F})_{\overline{y}} = H^q_c(X_{\overline{y}}, \mathcal{F})$ by Lemma 63.12.2. This is a finite Λ-module by Lemma 63.12.4.

Induction step. It suffices to find a dense open $V \subset Y$ such that $Rf_!\mathcal{F}|_V$ has constructible cohomology sheaves. Namely, the restriction of $Rf_!\mathcal{F}$ to the complement $Y \setminus V$ will have constructible cohomology sheaves by induction and the fact that formation of $Rf_!\mathcal{F}$ commutes with all base change (Lemma 63.9.4). By definition of constructible sheaves of Λ-modules, there is a dense open subscheme $U \subset \mathbf{A}^1_Y$ such that $\mathcal{F}|_U$ is a finite type, locally constant sheaf of Λ-modules. Denote $Z \subset \mathbf{A}^1_Y$ the complement (viewed as a reduced closed subscheme). Note that U contains all the generic points of the fibres of $\mathbf{A}^1_Y \to Y$ over the generic points $\xi_1, \ldots, \xi_n$ of the irreducible components of Y. Hence $Z \to Y$ has finite fibres over $\xi_1, \ldots, \xi_n$. After replacing Y by a dense open (which is allowed), we may assume $Z \to Y$ is finite, see Morphisms, Lemma 29.51.1. By the distinguished triangle of Lemma 63.10.5 and the result for $Z \to Y$ (Lemma 63.14.1) we reduce to showing that $R(U \to Y)_!\mathcal{F}$ has constructible cohomology sheaves. This is Lemma 63.14.3. $\square$

0GL0 Theorem 63.14.5. Let $f : X \to Y$ be a separated morphism of finite presentation of quasi-compact and quasi-separated schemes. Let Λ be a Noetherian ring. Let K be an object of $D_{tors,c}^+(X_{étale}, \Lambda)$ or of $D_c(X_{étale}, \Lambda)$ in case Λ is torsion. Then $Rf_!K$ has constructible cohomology sheaves, i.e., $Rf_!K$ is in $D_{tors,c}^+(Y_{étale}, \Lambda)$ or in $D_c(Y_{étale}, \Lambda)$ in case Λ is torsion.

Proof. The question is local on Y hence we may and do assume Y is affine. By the induction principle and Lemma 63.10.4 we reduce to the case where X is also affine.

Assume X and Y are affine. Since X is of finite presentation, we can choose a closed immersion $i : X \to \mathbf{A}^n_Y$ which is of finite presentation. If $p : \mathbf{A}^n_Y \to Y$ denotes the structure morphism, then we see that $Rf_! = Rp_! \circ Ri_!$ by Lemma 63.9.2. By Lemma 63.14.1 we have the result for $Ri_! = i_!$. Hence we may assume f is the projection morphism $\mathbf{A}^n_Y \to Y$. Since we can view f as the composition

$$X = \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \mathbf{A}^{n-2}_S \to \ldots \to \mathbf{A}^1_Y \to Y$$

we may assume $n = 1$.

Assume Y is affine and $X = \mathbf{A}^1_Y$. Since $Rf_!$ has finite cohomological dimension (Lemma 63.10.2) we may assume K is bounded below. Using the first spectral sequence of Derived Categories, Lemma 13.21.3 (or alternatively using an argument with truncations), we reduce to showing the result of Lemma 63.14.4. □

63.15. Applications

0GLG In this section we give some applications of Theorem 63.14.5.

0GLH Lemma 63.15.1. Let k be an algebraically closed field. Let X be a finite type separated scheme over k. Let Λ be a Noetherian ring. Let K be an object of $D_{tors,c}^+(X_{étale}, \Lambda)$ or of $D_c(X_{étale}, \Lambda)$ in case Λ is torsion. Then $H^i_c(X, K)$ is a finite Λ-module for all $i \in \mathbf{Z}$.

Proof. Immediate consequence of Theorem 63.14.5 and the definition of compactly supported cohomology in Section 63.12. □

0GLI Proposition 63.15.2. Let $f : X \to S$ be a smooth proper morphism of schemes. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $X_{étale}$ such that for every geometric point $\overline{x}$ of X the stalk $\mathcal{F}_{\overline{x}}$ is annihilated by an integer $n > 0$ prime to the residue characteristic of $\overline{x}$. Then $R^if_*\mathcal{F}$ is a finite type, locally constant sheaf of Λ-modules on $S_{étale}$ for all $i \in \mathbf{Z}$.

Proof. The question is local on S and hence we may assume S is affine. For a point x of X denote $n_x \geq 1$ the smallest integer annihilating $\mathcal{F}_{\overline{x}}$ for some (equivalently any) geometric point $\overline{x}$ of X lying over x. Since X is quasi-compact (being proper over affine) there exists a finite étale covering $\{U_j \to X\}_{j=1,\ldots,m}$ such that $\mathcal{F}|_{U_j}$ is constant. Since $U_j \to X$ is open, we conclude that the function $x \mapsto n_x$ is locally constant and takes finitely many values. Accordingly we obtain a finite decomposition $X = X_1 \amalg \ldots \amalg X_N$ into open and closed subschemes such that $n_x = n$ if and only if $x \in X_n$. Then it suffices to prove the lemma for the induced morphisms $X_n \to S$ and the restriction of $\mathcal{F}$ to X_n. Thus we may and do assume there exists an integer $n > 0$ such that $\mathcal{F}$ is annihilated by n and such that n is prime to the residue characteristics of all residue fields of X.

Since f is smooth and proper the image $f(X) \subset S$ is open and closed. Hence we may replace S by $f(X)$ and assume $f(X) = S$. In particular, we see that we may assume n is invertible in the ring defining the affine scheme S.

In this paragraph we reduce to the case where S is Noetherian. Write $S = \text{Spec}(A)$ for some $\mathbf{Z}[1/n]$-algebra A. Write $A = \bigcup A_i$ as the union of its finite type $\mathbf{Z}[1/n]$-subalgebras. We can find an i and a morphism $f_i : X_i \to S_i = \text{Spec}(A_i)$ of finite type whose base change to S is $f : X \to S$, see Limits, Lemma 32.10.1. After increasing i we may assume $f_i : X_i \to S_i$ is smooth and proper, see Limits, Lemmas 32.13.1 32.8.9, and 32.18.4. By Étale Cohomology, Lemma 59.73.11 we see that there exists an i and a finite type, locally constant sheaf of Λ-modules $\mathcal{F}_i$ whose pullback to X is isomorphic to $\mathcal{F}$. As $\mathcal{F}$ is annihilated by n, we may replace $\mathcal{F}_i$ by $\text{Ker}(n : \mathcal{F}_i \to \mathcal{F}_i)$ and assume the same thing is true for $\mathcal{F}_i$. This reduces us to the case discussed in the next paragraph.

Assume we have an integer $n \geq 1$, the base scheme S is Noetherian and lives over $\mathbf{Z}[1/n]$, and $\mathcal{F}$ is n-torsion. By Theorem 63.14.5 the sheaves $R^if_*\mathcal{F}$ are constructible sheaves of Λ-modules. By Étale Cohomology, Lemma 59.94.3 the specialization maps of $R^if_*\mathcal{F}$ are always isomorphisms. We conclude by Étale Cohomology, Lemma 59.75.6. $\square$

63.16. More on derived upper shriek

0GLJ Let Λ be a torsion ring. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr, "g"'] & & U' \arrow[dl, "g'"] \\ & Y & \end{tikzcd}$$

of quasi-compact and quasi-separated schemes with g and g' separated and of finite type and with j étale. This induces a canonical map

$$Rg_!\Lambda \longrightarrow Rg'_!\Lambda$$

in $D(Y_{étale}, \Lambda)$. Namely, by Lemmas 63.9.2 and 63.10.3 we have $Rg_! = Rg'_! \circ j_!$. On the other hand, since $j_!$ is left adjoint to j^{-1} we have the counit $\text{Tr}_j : j_!\Lambda = j_!j^{-1}\Lambda \to \Lambda$; we also call this the trace map for j, see Remark 63.5.6. The map above is constructed as the composition

$$Rg_!\Lambda = Rg'_!j_!\Lambda \xrightarrow{Rg'_!\text{Tr}_j} Rg'_!\Lambda$$

Given a second étale morphism $j' : U' \to U''$ for some $g'' : U'' \to Y$ separated and of finite type the composition

$$Rg_!\Lambda \longrightarrow Rg'_!\Lambda \longrightarrow Rg''_!\Lambda$$

of the maps for j and j' is equal to the map $Rg_!\Lambda \longrightarrow Rg''_!\Lambda$ constructed for $j' \circ j$. This follows from the corresponding statement on trace maps, see Lemma 63.5.4 for a more general case.

Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Then we obtain a functor

$$X_{affine,étale} \longrightarrow \left\{ \begin{matrix} \text{schemes separated of finite type over } Y \\ \text{with étale morphisms between them} \end{matrix} \right\}$$

Thus the construction above determines a functor $X_{affine,étale}^{opp} \to D(Y_{étale}, \Lambda)$ sending U to $R(U \to Y)_!\Lambda$.

0GLK **Lemma 63.16.1.** Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. Let $K \in D(Y_{étale}, \Lambda)$. For $n \in \mathbf{Z}$ the cohomology sheaf $H^n(Rf^!K)$ restricted to $X_{affine,étale}$ is the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_Y(R(U \to Y)_!\Lambda, K[n])$$

See discussion above for the functorial nature of $R(U \to Y)_!\Lambda$.

Proof. Let $j : U \to X$ be an object of $X_{affine,étale}$ and set $g = f \circ j$. Recall that $\mathrm{Hom}_X(j_!\Lambda, M[n]) = H^n(U, M)$ for any M in $D(X_{étale}, \Lambda)$. Then $H^n(Rf^!K)$ is the sheaf associated to the presheaf

$$U \mapsto H^n(U, Rf^!K) = \mathrm{Hom}_X(j_!\Lambda, Rf^!K[n]) = \mathrm{Hom}_Y(Rf_!j_!\Lambda, K[n] = \mathrm{Hom}_Y(Rg_!\Lambda, K[n])$$

We omit the verification that the transition maps are given by the transition maps between the objects $Rg_!\Lambda = R(U \to Y)_!\Lambda$ we constructed above. $\square$

63.17. Other chapters

CHAPTER 64

The Trace Formula

0F5P

64.1. Introduction

0F5Q These are the notes of the second part of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. Over time we will add references to background material in the rest of the Stacks project and provide rigorous proofs of all the statements.

64.2. The trace formula

03SJ A typical course in étale cohomology would normally state and prove the proper and smooth base change theorems, purity and Poincaré duality. All of these can be found in [Del77, Arcata]. Instead, we are going to study the trace formula for the frobenius, following the account of Deligne in [Del77, Rapport]. We will only look at dimension 1, but using proper base change this is enough for the general case. Since all the cohomology groups considered will be étale, we drop the subscript $_{étale}$. Let us now describe the formula we are after. Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

03SK (64.2.0.1)
$$\sum\nolimits_{x\in X(k)} \mathrm{Tr}(\mathrm{Frob}|\mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2}(-1)^i \mathrm{Tr}(\pi_X^*|H_c^i(X\otimes_k \bar{k}, \mathcal{F}))$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. As we will see, this formulation is slightly wrong as stated. Let us nevertheless describe the symbols that occur therein.

64.3. Frobenii

03SL In this section we will prove a “baffling” theorem. A topological analogue of the baffling theorem is the following.

03SO Exercise 64.3.1. Let X be a topological space and $g : X \to X$ a continuous map such that $g^{-1}(U) = U$ for all opens U of X. Then g induces the identity on cohomology on X (for any coefficients).

We now turn to the statement for the étale site.

03SP Lemma 64.3.2. Let X be a scheme and $g : X \to X$ a morphism. Assume that for all $\varphi : U \to X$ étale, there is an isomorphism

$$\begin{tikzcd} U \arrow[rr, "\sim"] \arrow[dr, "\varphi"'] & & U\times_{\varphi,X,g} X \arrow[dl, "\mathrm{pr}_2"] \\ & X & \end{tikzcd}$$

functorial in U. Then g induces the identity on cohomology (for any sheaf).

Proof. The proof is formal and without difficulty. ☐

Please see Varieties, Section 33.36 for a discussion of different variants of the Frobenius morphism.

03SN Theorem 64.3.3 (The Baffling Theorem). Let X be a scheme in characteristic $p > 0$. Then the absolute frobenius induces (by pullback) the trivial map on cohomology, i.e., for all integers $j \geq 0$,

$$F_X^* : H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \longrightarrow H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}})$$

is the identity.

This theorem is purely formal. It is a good idea, however, to review how to compute the pullback of a cohomology class. Let us simply say that in the case where cohomology agrees with Čech cohomology, it suffices to pull back (using the fiber products on a site) the Čech cocycles. The general case is quite technical, see Hypercoverings, Theorem 25.10.1. To prove the theorem, we merely verify that the assumption of Lemma 64.3.2 holds for the frobenius.

Proof of Theorem 64.3.3. We need to verify the existence of a functorial isomorphism as above. For an étale morphism $\varphi : U \to X$, consider the diagram

$$\begin{tikzcd}
U \arrow[rrd, bend left, "F_U"] \arrow[rd, dashed] \arrow[rdd, bend right, "\varphi"'] & & \\
& U \times_{\varphi, X, F_X} X \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & U \arrow[d, "\varphi"] \\
& X \arrow[r, "F_X"] & X.
\end{tikzcd}$$

The dotted arrow is an étale morphism and a universal homeomorphism, so it is an isomorphism. See Étale Morphisms, Lemma 41.14.3. ☐

03SQ Definition 64.3.4. Let k be a finite field with $q = p^f$ elements. Let X be a scheme over k. The geometric frobenius of X is the morphism $\pi_X : X \to X$ over $\mathrm{Spec}(k)$ which equals F_X^f.

Since π_X is a morphism over k, we can base change it to any scheme over k. In particular we can base change it to the algebraic closure $\bar{k}$ and get a morphism $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$. Using π_X also for this base change should not be confusing as $X_{\bar{k}}$ does not have a geometric frobenius of its own.

03SR Lemma 64.3.5. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then there are canonical isomorphisms $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ and $\mathcal{F} \cong \pi_{X*}\mathcal{F}$.

This is false for the fppf site.

Proof. Let $\varphi : U \to X$ be étale. Recall that $\pi_{X*}\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi, X, \pi_X} X)$. Since $\pi_X = F_X^f$, it follows from the proof of Theorem 64.3.3 that there is a functorial isomorphism

$$\begin{tikzcd} U \arrow[rr, "\gamma_U"] \arrow[dr, "\varphi"'] & & U \times_{\varphi, X, \pi_X} X \arrow[dl, "\mathrm{pr}_2"] \\ & X & \end{tikzcd}$$

where $\gamma_U = (\varphi, F_U^f)$. Now we define an isomorphism

$$\mathcal{F}(U) \longrightarrow \pi_{X*}\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi, X, \pi_X} X)$$

by taking the restriction map of $\mathcal{F}$ along γ_U^{-1}. The other isomorphism is analogous. □

03SS Remark 64.3.6. It may or may not be the case that F_U^f equals π_U.

We continue discussion cohomology of sheaves on our scheme X over the finite field k with $q = p^f$ elements. Fix an algebraic closure $\bar{k}$ of k and write $G_k = \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois group of k. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. We will define a left G_k-module structure cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as follows: if $\sigma \in G_k$, the diagram

$$\begin{tikzcd} X_{\bar{k}} \arrow[rr, "\mathrm{Spec}(\sigma) \times \mathrm{id}_X"] \arrow[dr] & & X_{\bar{k}} \arrow[dl] \\ & X & \end{tikzcd}$$

commutes. Thus we can set, for $\xi \in H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$

$$\sigma \cdot \xi := (\mathrm{Spec}(\sigma) \times \mathrm{id}_X)^*\xi \in H^j(X_{\bar{k}}, (\mathrm{Spec}(\sigma) \times \mathrm{id}_X)^{-1}\mathcal{F}|_{X_{\bar{k}}}) = H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}),$$

where the last equality follows from the commutativity of the previous diagram. This endows the latter group with the structure of a G_k-module.

03ST Lemma 64.3.7. In the situation above denote $\alpha : X \to \mathrm{Spec}(k)$ the structure morphism. Consider the stalk $(R^j\alpha_*\mathcal{F})_{\mathrm{Spec}(\bar{k})}$ endowed with its natural Galois action as in Étale Cohomology, Section 59.56. Then the identification

$$(R^j\alpha_*\mathcal{F})_{\mathrm{Spec}(\bar{k})} \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$$

from Étale Cohomology, Theorem 59.53.1 is an isomorphism of G_k-modules.

A similar result holds comparing $(R^j\alpha_!\mathcal{F})_{\mathrm{Spec}(\bar{k})}$ with $H_c^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$.

Proof. Omitted. □

03SU Definition 64.3.8. The arithmetic frobenius is the map $\mathrm{frob}_k : \bar{k} \to \bar{k}$, $x \mapsto x^q$ of G_k.

03SV Theorem 64.3.9. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Then for all $j \geq 0$, frob_k acts on the cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as the inverse of the map π_X^*.

The map π_X^* is defined by the composition

$$H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}) \xrightarrow{\pi_{X_{\bar{k}}}^*} H^j(X_{\bar{k}}, (\pi_X^{-1}\mathcal{F})|_{X_{\bar{k}}}) \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}).$$

where the last isomorphism comes from the canonical isomorphism $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ of Lemma 64.3.5.

Proof. The composition $X_{\bar{k}} \xrightarrow{\text{Spec}(\text{frob}_k)} X_{\bar{k}} \xrightarrow{\pi_X} X_{\bar{k}}$ is equal to $F^f_{X_{\bar{k}}}$, hence the result follows from the baffling theorem suitably generalized to nontrivial coefficients. Note that the previous composition commutes in the sense that $F^f_{X_{\bar{k}}} = \pi_X \circ \text{Spec}(\text{frob}_k) = \text{Spec}(\text{frob}_k) \circ \pi_X$. □

03SW Definition 64.3.10. If $x \in X(k)$ is a rational point and $\bar{x} : \text{Spec}(\bar{k}) \to X$ the geometric point lying over x, we let $\pi_x : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ denote the action by frob_k^{-1} and call it the geometric frobenius[1]

We can now make a more precise statement (albeit a false one) of the trace formula (64.2.0.1). Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

03SX (64.3.10.1)
$$\sum\nolimits_{x \in X(k)} \text{Tr}(\pi_x | \mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2} (-1)^i \text{Tr}(\pi_X^* | H^i_c(X_{\bar{k}}, \mathcal{F}))$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. The reason this equation is wrong is that the trace in the right-hand side does not make sense for the kind of sheaves considered. Before addressing this issue, we try to motivate the appearance of the geometric frobenius (apart from the fact that it is a natural morphism!).

Let us consider the case where $X = \mathbf{P}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. For any point, the Galois module $\mathcal{F}_{\bar{x}}$ is trivial, hence for any morphism φ acting on $\mathcal{F}_{\bar{x}}$, the left-hand side is

$$\sum\nolimits_{x \in X(k)} \text{Tr}(\varphi | \mathcal{F}_{\bar{x}}) = \#\mathbf{P}^1_k(k) = q + 1.$$

Now $\mathbf{P}^1_k$ is proper, so compactly supported cohomology equals standard cohomology, and so for a morphism $\pi : \mathbf{P}^1_k \to \mathbf{P}^1_k$, the right-hand side equals

$$\text{Tr}(\pi^* | H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) + \text{Tr}(\pi^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})).$$

The Galois module $H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}}) = \mathbf{Z}/\ell\mathbf{Z}$ is trivial, since the pullback of the identity is the identity. Hence the first trace is 1, regardless of π. For the second trace, we need to compute the pullback $\pi^* : H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}}))$ for a map $\pi : \mathbf{P}^1_{\bar{k}} \to \mathbf{P}^1_{\bar{k}}$. This is a good exercise and the answer is multiplication by the degree of π (for a proof see Étale Cohomology, Lemma 59.69.2). In other words, this works as in the familiar situation of complex cohomology. In particular, if π is the geometric frobenius we get

$$\text{Tr}(\pi_X^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q$$

and if π is the arithmetic frobenius then we get

$$\text{Tr}(\text{frob}_k^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q^{-1}.$$

The latter option is clearly wrong.

03SY Remark 64.3.11. The computation of the degrees can be done by lifting (in some obvious sense) to characteristic 0 and considering the situation with complex coefficients. This method almost never works, since lifting is in general impossible for schemes which are not projective space.

[1]This notation is not standard. This operator is denoted F_x in [Del77]. We will likely change this notation in the future.

The question remains as to why we have to consider compactly supported cohomology. In fact, in view of Poincaré duality, it is not strictly necessary for smooth varieties, but it involves adding in certain powers of q. For example, let us consider the case where $X = \mathbf{A}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. The action on stalks is again trivial, so we only need look at the action on cohomology. But then π_X^* acts as the identity on $H^0(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$ and as multiplication by q on $H^2_c(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$.

64.4. Traces

03SZ We now explain how to take the trace of an endomorphism of a module over a noncommutative ring. Fix a finite ring Λ with cardinality prime to p. Typically, Λ is the group ring $(\mathbf{Z}/\ell^n\mathbf{Z})[G]$ for some finite group G. By convention, all the Λ-modules considered will be left Λ-modules.

We introduce the following notation: We set $\Lambda^\natural$ to be the quotient of Λ by its additive subgroup generated by the commutators (i.e., the elements of the form $ab - ba$, $a, b \in \Lambda$). Note that $\Lambda^\natural$ is not a ring.

For instance, the module $(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural$ is the dual of the class functions, so

$$(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural = \bigoplus\nolimits_{\text{conjugacy classes of } G} \mathbf{Z}/\ell^n\mathbf{Z}.$$

For a free Λ-module, we have $\text{End}_\Lambda(\Lambda^{\oplus m}) = \text{Mat}_n(\Lambda)$. Note that since the modules are left modules, representation of endomorphism by matrices is a right action: if $a \in \text{End}(\Lambda^{\oplus m})$ has matrix A and $v \in \Lambda$, then $a(v) = vA$.

03T0 Definition 64.4.1. The trace of the endomorphism a is the sum of the diagonal entries of a matrix representing it. This defines an additive map $\text{Tr} : \text{End}_\Lambda(\Lambda^{\oplus m}) \to \Lambda^\natural$.

03T1 Exercise 64.4.2. Given maps

$$\Lambda^{\oplus m} \xrightarrow{a} \Lambda^{\oplus n} \quad \text{and} \quad \Lambda^{\oplus n} \xrightarrow{b} \Lambda^{\oplus m}$$

show that $\text{Tr}(ab) = \text{Tr}(ba)$.

We extend the definition of the trace to a finite projective Λ-module P and an endomorphism φ of P as follows. Write P as the summand of a free Λ-module, i.e., consider maps $P \xrightarrow{a} \Lambda^{\oplus n} \xrightarrow{b} P$ with

1. $\Lambda^{\oplus n} = \text{Im}(a) \oplus \text{Ker}(b)$; and
2. $b \circ a = \text{id}_P$.

Then we set $\text{Tr}(\varphi) = \text{Tr}(a\varphi b)$. It is easy to check that this is well-defined, using the previous exercise.

64.5. Why derived categories?

03T2 With this definition of the trace, let us now discuss another issue with the formula as stated. Let C be a smooth projective curve over k. Then there is a correspondence between finite locally constant sheaves $\mathcal{F}$ on $C_{étale}$ whose stalks are isomorphic to $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$ on the one hand, and continuous representations $\rho : \pi_1(C, \bar{c}) \to \text{GL}_m(\mathbf{Z}/\ell^n\mathbf{Z}))$ (for some fixed choice of $\bar{c}$) on the other hand. We denote $\mathcal{F}_\rho$ the sheaf corresponding to ρ. Then $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$ is the group of coinvariants for the action of $\rho(\pi_1(C, \bar{c}))$ on $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$, and there is a short exact sequence

$$0 \longrightarrow \pi_1(C_{\bar{k}}, \bar{c}) \longrightarrow \pi_1(C, \bar{c}) \longrightarrow G_k \longrightarrow 0.$$

For instance, let $\mathbf{Z} = \mathbf{Z}\sigma$ act on $\mathbf{Z}/\ell^2\mathbf{Z}$ via $\sigma(x) = (1+\ell)x$. The coinvariants are $(\mathbf{Z}/\ell^2\mathbf{Z})_\sigma = \mathbf{Z}/\ell\mathbf{Z}$, which is not a flat $\mathbf{Z}/\ell^2\mathbf{Z}$-module. Hence we cannot take the trace of some action on $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$, at least not in the sense of the previous section.

In fact, our goal is to consider a trace formula for ℓ-adic coefficients. But $\mathbf{Q}_\ell = \mathbf{Z}_\ell[1/\ell]$ and $\mathbf{Z}_\ell = \lim \mathbf{Z}/\ell^n\mathbf{Z}$, and even for a flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf, the individual cohomology groups may not be flat, so we cannot compute traces. One possible remedy is consider the total derived complex $R\Gamma(C_{\bar{k}}, \mathcal{F}_\rho)$ in the derived category $D(\mathbf{Z}/\ell^n\mathbf{Z})$ and show that it is a perfect object, which means that it is quasi-isomorphic to a finite complex of finite free module. For such complexes, we can define the trace, but this will require an account of derived categories.

64.6. Derived categories

03T3 To set up notation, let $\mathcal{A}$ be an abelian category. Let $\text{Comp}(\mathcal{A})$ be the abelian category of complexes in $\mathcal{A}$. Let $K(\mathcal{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathcal{A}$ and morphisms equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(A)$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\text{Comp}(\mathcal{A})$ or, equivalently, in $K(\mathcal{A})$. Moreover, we can define $\text{Comp}^+(\mathcal{A}), K^+(\mathcal{A}), D^+(\mathcal{A})$ analogously using only bounded below complexes. Similarly, we can define $\text{Comp}^-(\mathcal{A}), K^-(\mathcal{A}), D^-(\mathcal{A})$ using bounded above complexes, and we can define $\text{Comp}^b(\mathcal{A}), K^b(\mathcal{A}), D^b(\mathcal{A})$ using bounded complexes.

03T4 Remark 64.6.1. Notes on derived categories.

(1) There are some set-theoretical problems when $\mathcal{A}$ is somewhat arbitrary, which we will happily disregard.
(2) The categories $K(A)$ and $D(A)$ are endowed with the structure of a triangulated category.
(3) The categories $\text{Comp}(\mathcal{A})$ and $K(\mathcal{A})$ can also be defined when $\mathcal{A}$ is an additive category.

The homology functor $H^i : \text{Comp}(\mathcal{A}) \to \mathcal{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathcal{A}) \to \mathcal{A}$ and $H^i : D(\mathcal{A}) \to \mathcal{A}$.

03T5 Lemma 64.6.2. An object E of $D(\mathcal{A})$ is contained in $D^+(\mathcal{A})$ if and only if $H^i(E) = 0$ for all $i \ll 0$. Similar statements hold for D^- and D^+.

Proof. Hint: use truncation functors. See Derived Categories, Lemma 13.11.5. □

03T6 Lemma 64.6.3. Morphisms between objects in the derived category.

(1) Let $I^\bullet \in \text{Comp}^+(\mathcal{A})$ with I^n injective for all $n \in \mathbf{Z}$. Then
$$\text{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet).$$
(2) Let $P^\bullet \in \text{Comp}^-(\mathcal{A})$ with P^n is projective for all $n \in \mathbf{Z}$. Then
$$\text{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = \text{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet).$$
(3) If $\mathcal{A}$ has enough injectives and $\mathcal{I} \subset \mathcal{A}$ is the additive subcategory of injectives, then $D^+(\mathcal{A}) \cong K^+(\mathcal{I})$ (as triangulated categories).
(4) If $\mathcal{A}$ has enough projectives and $\mathcal{P} \subset \mathcal{A}$ is the additive subcategory of projectives, then $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$.

Proof. Omitted. □

03T7 Definition 64.6.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. We define the total right derived functor of F as the functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ fitting into the diagram

$$\begin{tikzcd} D^+(\mathcal{A}) \arrow[r, "RF"] & D^+(\mathcal{B}) \\ K^+(\mathcal{I}) \arrow[u] \arrow[r, "F"] & K^+(\mathcal{B}). \arrow[u] \end{tikzcd}$$

This is possible since the left vertical arrow is invertible by the previous lemma. Similarly, let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. We define the total left derived functor of G as the functor $LG : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ fitting into the diagram

$$\begin{tikzcd} D^-(\mathcal{A}) \arrow[r, "LG"] & D^-(\mathcal{B}) \\ K^-(\mathcal{P}) \arrow[u] \arrow[r, "G"] & K^-(\mathcal{B}). \arrow[u] \end{tikzcd}$$

This is possible since the left vertical arrow is invertible by the previous lemma.

03T8 Remark 64.6.5. In these cases, it is true that $R^iF(K^\bullet) = H^i(RF(K^\bullet))$, where the left hand side is defined to be ith homology of the complex $F(K^\bullet)$.

64.7. Filtered derived category

03T9 It turns out we have to do it all again and build the filtered derived category also.

03TA Definition 64.7.1. Let $\mathcal{A}$ be an abelian category.

(1) Let $\text{Fil}(\mathcal{A})$ be the category of filtered objects (A, F) of $\mathcal{A}$, where F is a filtration of the form
$$A \supset \ldots \supset F^nA \supset F^{n+1}A \supset \ldots \supset 0.$$
This is an additive category.
(2) We denote $\text{Fil}^f(\mathcal{A})$ the full subcategory of $\text{Fil}(\mathcal{A})$ whose objects (A, F) have finite filtration. This is also an additive category.
(3) An object $I \in \text{Fil}^f(\mathcal{A})$ is called filtered injective (respectively projective) provided that $\text{gr}^p(I) = \text{gr}^p_F(I) = F^pI/F^{p+1}I$ is injective (resp. projective) in $\mathcal{A}$ for all p.
(4) The category of complexes $\text{Comp}(\text{Fil}^f(\mathcal{A})) \supset \text{Comp}^+(\text{Fil}^f(\mathcal{A}))$ and its homotopy category $K(\text{Fil}^f(\mathcal{A})) \supset K^+(\text{Fil}^f(\mathcal{A}))$ are defined as before.
(5) A morphism $\alpha : K^\bullet \to L^\bullet$ of complexes in $\text{Comp}(\text{Fil}^f(\mathcal{A}))$ is called a filtered quasi-isomorphism provided that
$$\text{gr}^p(\alpha) : \text{gr}^p(K^\bullet) \to \text{gr}^p(L^\bullet)$$
is a quasi-isomorphism for all $p \in \mathbf{Z}$.
(6) We define $DF(\mathcal{A})$ (resp. $DF^+(\mathcal{A})$) by inverting the filtered quasi-isomorphisms in $K(\text{Fil}^f(\mathcal{A}))$ (resp. $K^+(\text{Fil}^f(\mathcal{A}))$).

03TB Lemma 64.7.2. If $\mathcal{A}$ has enough injectives, then $DF^+(\mathcal{A}) \cong K^+(\mathcal{I})$, where $\mathcal{I}$ is the full additive subcategory of $\text{Fil}^f(\mathcal{A})$ consisting of filtered injective objects.

Similarly, if $\mathcal{A}$ has enough projectives, then $DF^-(\mathcal{A}) \cong K^+(\mathcal{P})$, where $\mathcal{P}$ is the full additive subcategory of $\text{Fil}^f(\mathcal{A})$ consisting of filtered projective objects.

Proof. Omitted. $\square$

64.8. Filtered derived functors

03TC And then there are the filtered derived functors.

03TD Definition 64.8.1. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. Define $RT : DF^+(\mathcal{A}) \to DF^+(\mathcal{B})$ to fit in the diagram

$$\begin{tikzcd} DF^+(\mathcal{A}) \arrow[r, "RT"] & DF^+(\mathcal{B}) \\ K^+(\mathcal{I}) \arrow[u] \arrow[r, "T"] & K^+(\text{Fil}^f(\mathcal{B})). \arrow[u] \end{tikzcd}$$

This is well-defined by the previous lemma. Let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. Define $LG : DF^+(\mathcal{A}) \to DF^+(\mathcal{B})$ to fit in the diagram

$$\begin{tikzcd} DF^-(\mathcal{A}) \arrow[r, "LG"] & DF^-(\mathcal{B}) \\ K^-(\mathcal{P}) \arrow[u] \arrow[r, "G"] & K^-(\text{Fil}^f(\mathcal{B})). \arrow[u] \end{tikzcd}$$

Again, this is well-defined by the previous lemma. The functors RT, resp. LG, are called the filtered derived functor of T, resp. G.

03TE Proposition 64.8.2. In the situation above, we have

$$\text{gr}^p \circ RT = RT \circ \text{gr}^p$$

where the RT on the left is the filtered derived functor while the one on the right is the total derived functor. That is, there is a commuting diagram

$$\begin{tikzcd} DF^+(\mathcal{A}) \arrow[r, "RT"] \arrow[d, "\text{gr}^p"'] & DF^+(\mathcal{B}) \arrow[d, "\text{gr}^p"] \\ D^+(\mathcal{A}) \arrow[r, "RT"] & D^+(\mathcal{B}). \end{tikzcd}$$

Proof. Omitted. $\square$

Given $K^\bullet \in DF^+(\mathcal{B})$, we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(\text{gr}^p K^\bullet) \Rightarrow H^{p+q}(\text{forget filt}(K^\bullet)).$$

64.9. Application of filtered complexes

03TF Let $\mathcal{A}$ be an abelian category with enough injectives, and $0 \to L \to M \to N \to 0$ a short exact sequence in $\mathcal{A}$. Consider $\widetilde{M} \in \text{Fil}^f(\mathcal{A})$ to be M along with the filtration defined by

$$F^1M = L, \ \ F^nM = M \text{ for } n \leq 0, \text{ and } F^nM = 0 \text{ for } n \geq 2.$$

By definition, we have

$$\text{forget filt}(\widetilde{M}) = M, \quad \text{gr}^0(\widetilde{M}) = N, \quad \text{gr}^1(\widetilde{M}) = L$$

and $\text{gr}^n(\widetilde{M}) = 0$ for all other $n \neq 0, 1$. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. Then $RT(\widetilde{M}) \in DF^+(\mathcal{B})$ is a filtered complex with

$$\text{gr}^p(RT(\widetilde{M})) \overset{\text{qis}}{=} \begin{cases} 0 & \text{if} \quad p \neq 0, 1, \\ RT(N) & \text{if} \quad p = 0, \\ RT(L) & \text{if} \quad p = 1. \end{cases}$$

and $\text{forget filt}(RT(\widetilde{M})) \overset{\text{qis}}{=} RT(M)$. The spectral sequence applied to $RT(\widetilde{M})$ gives

$$E_1^{p,q} = R^{p+q}T(\text{gr}^p(\widetilde{M})) \Rightarrow R^{p+q}T(\text{forget filt}(\widetilde{M})).$$

Unwinding the spectral sequence gives us the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T(L) \arrow[r] & T(M) \arrow[r] & T(N) \arrow[dll, bend left] \\ & R^1T(L) \arrow[r] & R^1T(M) \arrow[r] & \ldots \end{tikzcd}$$

This will be used as follows. Let X/k be a scheme of finite type. Let $\mathcal{F}$ be a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module. Then we want to show that the trace

$$\text{Tr}(\pi_X^* | R\Gamma_c(X_{\bar{k}}, \mathcal{F})) \in \mathbf{Z}/\ell^n\mathbf{Z}$$

is additive on short exact sequences. To see this, it will not be enough to work with $R\Gamma_c(X_{\bar{k}}, -) \in D^+(\mathbf{Z}/\ell^n\mathbf{Z})$, but we will have to use the filtered derived category.

64.10. Perfectness

03TG Let Λ be a (possibly noncommutative) ring, Mod_Λ the category of left Λ-modules, $K(\Lambda) = K(\text{Mod}_\Lambda)$ its homotopy category, and $D(\Lambda) = D(\text{Mod}_\Lambda)$ the derived category.

03TH Definition 64.10.1. We denote by $K_{perf}(\Lambda)$ the category whose objects are bounded complexes of finite projective Λ-modules, and whose morphisms are morphisms of complexes up to homotopy. The functor $K_{perf}(\Lambda) \to D(\Lambda)$ is fully faithful (Derived Categories, Lemma 13.19.8). Denote $D_{perf}(\Lambda)$ its essential image. An object of $D(\Lambda)$ is called perfect if it is in $D_{perf}(\Lambda)$.

03TI Proposition 64.10.2. Let $K \in D_{perf}(\Lambda)$ and $f \in \text{End}_{D(\Lambda)}(K)$. Then the trace $\text{Tr}(f) \in \Lambda^\natural$ is well defined.

Proof. We will use Derived Categories, Lemma 13.19.8 without further mention in this proof. Let $P^\bullet$ be a bounded complex of finite projective Λ-modules and let $\alpha : P^\bullet \to K$ be an isomorphism in $D(\Lambda)$. Then $\alpha^{-1} \circ f \circ \alpha$ corresponds to a morphism of complexes $f^\bullet : P^\bullet \to P^\bullet$ well defined up to homotopy. Set

$$\text{Tr}(f) = \sum_i (-1)^i \text{Tr}(f^i : P^i \to P^i) \in \Lambda^\natural.$$

Given $P^\bullet$ and α, this is independent of the choice of $f^\bullet$. Namely, any other choice is of the form $\tilde{f}^\bullet = f^\bullet + dh + hd$ for some $h^i : P^i \to P^{i-1} (i \in \mathbf{Z})$. But

$$\begin{aligned} \mathrm{Tr}(dh) &= \sum_i (-1)^i \mathrm{Tr}(P^i \xrightarrow{dh} P^i) \\ &= \sum_i (-1)^i \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\sum_i (-1)^{i-1} \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\mathrm{Tr}(hd) \end{aligned}$$

and so $\sum_i (-1)^i \mathrm{Tr}((dh+hd)|_{P^i}) = 0$. Furthermore, this is independent of the choice of $(P^\bullet, \alpha)$: suppose $(Q^\bullet, \beta)$ is another choice. The compositions

$$Q^\bullet \xrightarrow{\beta} K \xrightarrow{\alpha^{-1}} P^\bullet \quad \text{and} \quad P^\bullet \xrightarrow{\alpha} K \xrightarrow{\beta^{-1}} Q^\bullet$$

are representable by morphisms of complexes $\gamma_1^\bullet$ and $\gamma_2^\bullet$ respectively, such that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity. Thus, the morphism of complexes $\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet : Q^\bullet \to Q^\bullet$ represents the morphism $\beta^{-1} \circ f \circ \beta$ in $D(\Lambda)$. Now

$$\begin{aligned} \mathrm{Tr}(\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet|_{Q^\bullet}) &= \mathrm{Tr}(\gamma_1^\bullet \circ \gamma_2^\bullet \circ f^\bullet|_{P^\bullet}) \\ &= \mathrm{Tr}(f^\bullet|_{P^\bullet}) \end{aligned}$$

by the fact that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity and the independence of the choice of $f^\bullet$ we saw above. □

64.11. Filtrations and perfect complexes

03TJ We now present a filtered version of the category of perfect complexes. An object (M, F) of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ is called filtered finite projective if for all p, $\mathrm{gr}^p_F(M)$ is finite and projective. We then consider the homotopy category $KF_{\text{perf}}(\Lambda)$ of bounded complexes of filtered finite projective objects of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$. We have a diagram of categories

$$\begin{matrix} KF(\Lambda) & \supset & KF_{\text{perf}}(\Lambda) \\ \downarrow & & \downarrow \\ DF(\Lambda) & \supset & DF_{\text{perf}}(\Lambda) \end{matrix}$$

where the vertical functor on the right is fully faithful and the category $DF_{\text{perf}}(\Lambda)$ is its essential image, as before.

03TK **Lemma 64.11.1** (Additivity). Let $K \in DF_{\text{perf}}(\Lambda)$ and $f \in \mathrm{End}_{DF}(K)$. Then

$$\mathrm{Tr}(f|_K) = \sum\nolimits_{p \in \mathbf{Z}} \mathrm{Tr}(f|_{\mathrm{gr}^p K}).$$

Proof. By Proposition 64.10.2, we may assume we have a bounded complex $P^\bullet$ of filtered finite projectives of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ and a map $f^\bullet : P^\bullet \to P^\bullet$ in $\mathrm{Comp}(\mathrm{Fil}^f(\mathrm{Mod}_\Lambda))$. So the lemma follows from the following result, which proof is left to the reader. □

03TL **Lemma 64.11.2.** Let $P \in \mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ be filtered finite projective, and $f : P \to P$ an endomorphism in $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$. Then

$$\mathrm{Tr}(f|_P) = \sum\nolimits_p \mathrm{Tr}(f|_{\mathrm{gr}^p(P)}).$$

Proof. Omitted. □

64.12. Characterizing perfect objects

03TM For the commutative case see More on Algebra, Sections 15.64, 15.66, and 15.74.

03TN **Definition 64.12.1.** Let Λ be a (possibly noncommutative) ring. An object $K \in D(\Lambda)$ has *finite Tor-dimension* if there exist $a, b \in \mathbf{Z}$ such that for any right Λ-module N, we have $H^i(N \otimes_\Lambda^{\mathbf{L}} K) = 0$ for all $i \not\in [a, b]$.

This in particular means that $K \in D^b(\Lambda)$ as we see by taking $N = \Lambda$.

03TO **Lemma 64.12.2.** Let Λ be a left Noetherian ring and $K \in D(\Lambda)$. Then K is perfect if and only if the two following conditions hold:

(1) K has finite Tor-dimension, and
(2) for all $i \in \mathbf{Z}$, $H^i(K)$ is a finite Λ-module.

Proof. See More on Algebra, Lemma 15.74.2 for the proof in the commutative case. $\square$

The reader is strongly urged to try and prove this. The proof relies on the fact that a finite module on a finitely left-presented ring is flat if and only if it is projective.

03TP **Remark 64.12.3.** A variant of this lemma is to consider a Noetherian scheme X and the category $D_{perf}(\mathcal{O}_X)$ of complexes which are locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules. Objects K of $D_{perf}(\mathcal{O}_X)$ can be characterized by having coherent cohomology sheaves and bounded tor dimension.

64.13. Cohomology of nice complexes

0964 The following is a special case of a more general result about compactly supported cohomology of objects of $D_{ctf}(X, \Lambda)$.

03TV **Proposition 64.13.1.** Let X be a projective curve over a field k, Λ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then $R\Gamma(X_{\bar k}, K) \in D_{perf}(\Lambda)$.

Sketch of proof. The first step is to show:

(1) The cohomology of $R\Gamma(X_{\bar k}, K)$ is bounded.

Consider the spectral sequence

$$H^i(X_{\bar k}, \underline{H}^j(K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar k}, K)).$$

Since K is bounded and Λ is finite, the sheaves $\underline{H}^j(K)$ are torsion. Moreover, $X_{\bar k}$ has finite cohomological dimension, so the left-hand side is nonzero for finitely many i and j only. Therefore, so is the right-hand side.

(2) The cohomology groups $H^{i+j}(R\Gamma(X_{\bar k}, K))$ are finite.

Since the sheaves $\underline{H}^j(K)$ are constructible, the groups $H^i(X_{\bar k}, \underline{H}^j(K))$ are finite (Étale Cohomology, Section 59.83) so it follows by the spectral sequence again.

(3) $R\Gamma(X_{\bar k}, K)$ has finite Tor-dimension.

Let N be a right Λ-module (in fact, since Λ is finite, it suffices to assume that N is finite). By the projection formula (change of module),

$$N \otimes_\Lambda^{\mathbf{L}} R\Gamma(X_{\bar k}, K) = R\Gamma(X_{\bar k}, N \otimes_\Lambda^{\mathbf{L}} K).$$

Therefore,

$$H^i(N \otimes_\Lambda^{\mathbf{L}} R\Gamma(X_{\bar k}, K)) = H^i(R\Gamma(X_{\bar k}, N \otimes_\Lambda^{\mathbf{L}} K)).$$

Now consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(N \otimes_\Lambda^{\mathbf{L}} K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}}, N \otimes_\Lambda^{\mathbf{L}} K)).$$

Since K has finite Tor-dimension, $\underline{H}^j(N \otimes_\Lambda^{\mathbf{L}} K)$ vanishes universally for j small enough, and the left-hand side vanishes whenever $i < 0$. Therefore $R\Gamma(X_{\bar{k}}, K)$ has finite Tor-dimension, as claimed. So it is a perfect complex by Lemma 64.12.2. □

64.14. Lefschetz numbers

03TW The fact that the total cohomology of a constructible complex of finite tor dimension is a perfect complex is the key technical reason why cohomology behaves well, and allows us to define rigorously the traces occurring in the trace formula.

03TX Definition 64.14.1. Let Λ be a finite ring, X a projective curve over a finite field k and $K \in D_{ctf}(X, \Lambda)$ (for instance $K = \underline{\Lambda}$). There is a canonical map $c_K : \pi_X^{-1}K \to K$, and its base change $c_K|_{X_{\bar{k}}}$ induces an action denoted π_X^* on the perfect complex $R\Gamma(X_{\bar{k}}, K|_{X_{\bar{k}}})$. The global Lefschetz number of K is the trace $\text{Tr}(\pi_X^*|_{R\Gamma(X_{\bar{k}}, K)})$ of that action. It is an element of $\Lambda^\natural$.

03TY Definition 64.14.2. With Λ, X, k, K as in Definition 64.14.1. Since $K \in D_{ctf}(X, \Lambda)$, for any geometric point $\bar{x}$ of X, the complex $K_{\bar{x}}$ is a perfect complex (in $D_{perf}(\Lambda)$). As we have seen in Section 64.3, the Frobenius π_X acts on $K_{\bar{x}}$. The local Lefschetz number of K is the sum

$$\sum\nolimits_{x \in X(k)} \text{Tr}(\pi_X|_{K_{\bar{x}}})$$

which is again an element of $\Lambda^\natural$.

At last, we can formulate precisely the trace formula.

03TZ Theorem 64.14.3 (Lefschetz Trace Formula). Let X be a projective curve over a finite field k, Λ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then the global and local Lefschetz numbers of K are equal, i.e.,

03U0 (64.14.3.1)
$$\text{Tr}(\pi_X^*|_{R\Gamma(X_{\bar{k}}, K)}) = \sum\nolimits_{x \in X(k)} \text{Tr}(\pi_X|_{K_{\bar{x}}})$$

in $\Lambda^\natural$.

Proof. See discussion below. □

We will use, rather than prove, the trace formula. Nevertheless, we will give quite a few details of the proof of the theorem as given in [Del77] (some of the things that are not adequately explained are listed in Section 64.21).

We only stated the formula for curves, and in some weak sense it is a consequence of the following result.

03U1 Theorem 64.14.4 (Weil). Let C be a nonsingular projective curve over an algebraically closed field k, and $\varphi : C \to C$ a k-endomorphism of C distinct from the identity. Let $V(\varphi) = \Delta_C \cdot \Gamma_\varphi$, where Δ_C is the diagonal, Γ_φ is the graph of φ, and the intersection number is taken on $C \times C$. Let $J = \underline{\text{Pic}}^0_{C/k}$ be the jacobian of C and denote $\varphi^* : J \to J$ the action induced by φ by taking pullbacks. Then

$$V(\varphi) = 1 - \text{Tr}_J(\varphi^*) + \deg \varphi.$$

Proof. The number $V(\varphi)$ is the number of fixed points of φ, it is equal to

$$V(\varphi) = \sum\nolimits_{c \in |C| : \varphi(c) = c} m_{\mathrm{Fix}(\varphi)}(c)$$

where $m_{\mathrm{Fix}(\varphi)}(c)$ is the multiplicity of c as a fixed point of φ, namely the order or vanishing of the image of a local uniformizer under $\varphi - \mathrm{id}_C$. Proofs of this theorem can be found in [Lan02] and [Wei48]. □

03U2 Example 64.14.5. Let $C = E$ be an elliptic curve and $\varphi = [n]$ be multiplication by n. Then $\varphi^* = \varphi^t$ is multiplication by n on the jacobian, so it has trace $2n$ and degree n^2. On the other hand, the fixed points of φ are the points $p \in E$ such that $np = p$, which is the $(n-1)$-torsion, which has cardinality $(n-1)^2$. So the theorem reads

$$(n-1)^2 = 1 - 2n + n^2.$$

Jacobians. We now discuss without proofs the correspondence between a curve and its jacobian which is used in Weil's proof. Let C be a nonsingular projective curve over an algebraically closed field k and choose a base point $c_0 \in C(k)$. Denote by $A^1(C \times C)$ (or $\mathrm{Pic}(C \times C)$, or $\mathrm{CaCl}(C \times C)$) the abelian group of codimension 1 divisors of $C \times C$. Then

$$A^1(C \times C) = \mathrm{pr}_1^*(A^1(C)) \oplus \mathrm{pr}_2^*(A^1(C)) \oplus R$$

where

$$R = \{Z \in A^1(C \times C) \mid Z|_{C \times \{c_0\}} \sim_{\mathrm{rat}} 0 \text{ and } Z|_{\{c_0\} \times C} \sim_{\mathrm{rat}} 0\}.$$

In other words, R is the subgroup of line bundles which pull back to the trivial one under either projection. Then there is a canonical isomorphism of abelian groups $R \cong \mathrm{End}(J)$ which maps a divisor Z in R to the endomorphism

$$\begin{array}{ccc} J & \to & J \\ [\mathcal{O}_C(D)] & \mapsto & (\mathrm{pr}_1|_Z)_*(\mathrm{pr}_2|_Z)^*(D). \end{array}$$

The aforementioned correspondence is the following. We denote by σ the automorphism of $C \times C$ that switches the factors.

$\mathrm{End}(J)$	R
composition of α, β	$\mathrm{pr}_{13*}(\mathrm{pr}_{12}{}^*(\alpha) \circ \mathrm{pr}_{23}{}^*(\beta))$
id_J	$\Delta_C - \{c_0\} \times C - C \times \{c_0\}$
φ^*	$\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C$
the trace form $\alpha, \beta \mapsto \mathrm{Tr}(\alpha\beta)$	$\alpha, \beta \mapsto -\int_{C \times C} \alpha.\sigma^*\beta$
the Rosati involution $\alpha \mapsto \alpha^\dagger$	$\alpha \mapsto \sigma^*\alpha$
positivity of Rosati $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$	Hodge index theorem on $C \times C$ $-\int_{C \times C} \alpha\sigma^*\alpha > 0.$

In fact, in light of the Kunneth formula, the subgroup R corresponds to the $1,1$ hodge classes in $H^1(C) \otimes H^1(C)$.

Weil's proof. Using this correspondence, we can prove the trace formula. We have

$$\begin{aligned} V(\varphi) &= \int_{C \times C} \Gamma_\varphi.\Delta \\ &= \int_{C \times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) + \int_{C \times C} \Gamma_\varphi.\left(\{c_0\} \times C + C \times \{c_0\}\right). \end{aligned}$$

Now, on the one hand

$$\int_{C \times C} \Gamma_\varphi.\left(\{c_0\} \times C + C \times \{c_0\}\right) = 1 + \deg \varphi$$

and on the other hand, since R is the orthogonal of the ample divisor $\{c_0\} \times C + C \times \{c_0\}$,

$$\begin{aligned} & \int_{C \times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) \\ = & \int_{C \times C} \left(\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C\right).\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) \\ = & -\mathrm{Tr}_J(\varphi^* \circ \mathrm{id}_J). \end{aligned}$$

Recapitulating, we have

$$V(\varphi) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg \varphi$$

which is the trace formula.

03U3 Lemma 64.14.6. Consider the situation of Theorem 64.14.4 and let ℓ be a prime number invertible in k. Then

$$\sum\nolimits_{i=0}^{2}(-1)^i\mathrm{Tr}(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = V(\varphi) \mod \ell^n.$$

Sketch of proof. Observe first that the assumption makes sense because $H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})$ is a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module for all i. The trace of φ^* on the 0th degree cohomology is 1. The choice of a primitive ℓ^nth root of unity in k gives an isomorphism

$$H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}}) \cong H^i(C,\mu_{\ell^n})$$

compatibly with the action of the geometric Frobenius. On the other hand, $H^1(C,\mu_{\ell^n}) = J[\ell^n]$. Therefore,

$$\begin{aligned}\mathrm{Tr}(\varphi^*|_{H^1(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) &= \mathrm{Tr}_J(\varphi^*) \mod \ell^n \\ &= \mathrm{Tr}_{\mathbf{Z}/\ell^n\mathbf{Z}}(\varphi^* : J[\ell^n] \to J[\ell^n]).\end{aligned}$$

Moreover, $H^2(C,\mu_{\ell^n}) = \mathrm{Pic}(C)/\ell^n \mathrm{Pic}(C) \cong \mathbf{Z}/\ell^n\mathbf{Z}$ where φ^* is multiplication by $\deg\varphi$. Hence

$$\mathrm{Tr}(\varphi^*|_{H^2(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = \deg\varphi.$$

Thus we have

$$\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi \mod \ell^n$$

and the corollary follows from Theorem 64.14.4. □

An alternative way to prove this corollary is to show that

$$X \mapsto H^*(X,\mathbf{Q}_\ell) = \mathbf{Q}_\ell \otimes \lim\nolimits_n H^*(X,\mathbf{Z}/\ell^n\mathbf{Z})$$

defines a Weil cohomology theory on smooth projective varieties over k. Then the trace formula

$$V(\varphi) = \sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\varphi^*|_{H^i(C,\mathbf{Q}_\ell)})$$

is a formal consequence of the axioms (it's an exercise in linear algebra, the proof is the same as in the topological case).

64.15. Preliminaries and sorites

03U4 Notation: We fix the notation for this section. We denote by A a commutative ring, Λ a (possibly noncommutative) ring with a ring map $A \to \Lambda$ which image lies in the center of Λ. We let G be a finite group, Γ a monoid extension of G by $\mathbf{N}$, meaning that there is an exact sequence

$$1 \to G \to \tilde{\Gamma} \to \mathbf{Z} \to 1$$

and Γ consists of those elements of $\tilde{\Gamma}$ which image is nonnegative. Finally, we let P be an $A[\Gamma]$-module which is finite and projective as an $A[G]$-module, and M a $\Lambda[\Gamma]$-module which is finite and projective as a Λ-module.

Our goal is to compute the trace of $1 \in \mathbf{N}$ acting over Λ on the coinvariants of G on $P \otimes_A M$, that is, the number

$$\mathrm{Tr}_\Lambda\left(1; (P \otimes_A M)_G\right) \in \Lambda^\natural.$$

The element $1 \in \mathbf{N}$ will correspond to the Frobenius.

03U5 Lemma 64.15.1. Let $e \in G$ denote the neutral element. The map

$$\begin{array}{ccc} \Lambda[G] & \longrightarrow & \Lambda^\natural \\ \sum \lambda_g \cdot g & \longmapsto & \lambda_e \end{array}$$

factors through $\Lambda[G]^\natural$. We denote $\varepsilon : \Lambda[G]^\natural \to \Lambda^\natural$ the induced map.

Proof. We have to show the map annihilates commutators. One has

$$\left(\sum \lambda_g g\right)\left(\sum \mu_g g\right) - \left(\sum \mu_g g\right)\left(\sum \lambda_g g\right) = \sum_g \left(\sum_{g_1 g_2 = g} \lambda_{g_1}\mu_{g_2} - \mu_{g_1}\lambda_{g_2}\right) g$$

The coefficient of e is

$$\sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_g \lambda_{g^{-1}}\right) = \sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_{g^{-1}} \lambda_g\right)$$

which is a sum of commutators, hence it zero in $\Lambda^\natural$. □

03U6 Definition 64.15.2. Let $f : P \to P$ be an endomorphism of a finite projective $\Lambda[G]$-module P. We define

$$\mathrm{Tr}^G_\Lambda(f; P) := \varepsilon\left(\mathrm{Tr}_{\Lambda[G]}(f; P)\right)$$

to be the G-trace of f on P.

03U7 Lemma 64.15.3. Let $f : P \to P$ be an endomorphism of the finite projective $\Lambda[G]$-module P. Then

$$\mathrm{Tr}_\Lambda(f; P) = \#G \cdot \mathrm{Tr}^G_\Lambda(f; P).$$

Proof. By additivity, reduce to the case $P = \Lambda[G]$. In that case, f is given by right multiplication by some element $\sum \lambda_g \cdot g$ of $\Lambda[G]$. In the basis $(g)_{g\in G}$, the matrix of f has coefficient $\lambda_{g_2^{-1}g_1}$ in the (g_1, g_2) position. In particular, all diagonal coefficients are λ_e, and there are $\#G$ such coefficients. □

03U8 Lemma 64.15.4. The map $A \to \Lambda$ defines an A-module structure on $\Lambda^\natural$.

Proof. This is clear. □

03U9 Lemma 64.15.5. Let P be a finite projective $A[G]$-module and M a $\Lambda[G]$-module, finite projective as a Λ-module. Then $P \otimes_A M$ is a finite projective $\Lambda[G]$-module, for the structure induced by the diagonal action of G.

Note that $P \otimes_A M$ is naturally a Λ-module since M is. Explicitly, together with the diagonal action this reads

$$\left(\sum \lambda_g g\right)(p \otimes m) = \sum gp \otimes \lambda_g gm.$$

Proof. For any $\Lambda[G]$-module N one has

$$\mathrm{Hom}_{\Lambda[G]}\left(P \otimes_A M, N\right) = \mathrm{Hom}_{A[G]}\left(P, \mathrm{Hom}_\Lambda(M, N)\right)$$

where the G-action on $\mathrm{Hom}_\Lambda(M, N)$ is given by $(g \cdot \varphi)(m) = g\varphi(g^{-1}m)$. Now it suffices to observe that the right-hand side is a composition of exact functors, because of the projectivity of P and M. □

03UA Lemma 64.15.6. With assumptions as in Lemma 64.15.5, let $u \in \mathrm{End}_{A[G]}(P)$ and $v \in \mathrm{End}_{\Lambda[G]}(M)$. Then

$$\mathrm{Tr}^G_\Lambda\left(u \otimes v; P \otimes_A M\right) = \mathrm{Tr}^G_A(u; P) \cdot \mathrm{Tr}_\Lambda(v; M).$$

Sketch of proof. Reduce to the case $P = A[G]$. In that case, u is right multiplication by some element $a = \sum a_g g$ of $A[G]$, which we write $u = R_a$. There is an isomorphism of $\Lambda[G]$-modules

$$\begin{array}{cccc} \varphi: & A[G] \otimes_A M & \cong & (A[G] \otimes_A M)' \\ & g \otimes m & \longmapsto & g \otimes g^{-1}m \end{array}$$

where $(A[G] \otimes_A M)'$ has the module structure given by the left G-action, together with the Λ-linearity on M. This transport of structure changes $u \otimes v$ into $\sum_g a_g R_g \otimes g^{-1}v$. In other words,

$$\varphi \circ (u \otimes v) \circ \varphi^{-1} = \sum_g a_g R_g \otimes g^{-1}v.$$

Working out explicitly both sides of the equation, we have to show

$$\mathrm{Tr}_\Lambda^G \left(\sum_g a_g R_g \otimes g^{-1}v \right) = a_e \cdot \mathrm{Tr}_\Lambda(v; M).$$

This is done by showing that

$$\mathrm{Tr}_\Lambda^G \left(a_g R_g \otimes g^{-1}v \right) = \begin{cases} 0 & \text{if } g \neq e \\ a_e \mathrm{Tr}_\Lambda(v; M) & \text{if } g = e \end{cases}$$

by reducing to $M = \Lambda$. □

Notation: Consider the monoid extension $1 \to G \to \Gamma \to \mathbf{N} \to 1$ and let $\gamma \in \Gamma$. Then we write $Z_\gamma = \{g \in G | g\gamma = \gamma g\}$.

03UB Lemma 64.15.7. Let P be a $\Lambda[\Gamma]$-module, finite and projective as a $\Lambda[G]$-module, and $\gamma \in \Gamma$. Then

$$\mathrm{Tr}_\Lambda(\gamma, P) = \# Z_\gamma \cdot \mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P).$$

Proof. This follows readily from Lemma 64.15.3. □

03UC Lemma 64.15.8. Let P be an $A[\Gamma]$-module, finite projective as $A[G]$-module. Let M be a $\Lambda[\Gamma]$-module, finite projective as a Λ-module. Then

$$\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P \otimes_A M) = \mathrm{Tr}_A^{Z_\gamma}(\gamma, P) \cdot \mathrm{Tr}_\Lambda(\gamma, M).$$

Proof. This follows directly from Lemma 64.15.6. □

03UD Lemma 64.15.9. Let P be a $\Lambda[\Gamma]$-module, finite projective as $\Lambda[G]$-module. Then the coinvariants $P_G = \Lambda \otimes_{\Lambda[G]} P$ form a finite projective Λ-module, endowed with an action of $\Gamma/G = \mathbf{N}$. Moreover, we have

$$\mathrm{Tr}_\Lambda(1; P_G) = \sum\nolimits'_{\gamma \mapsto 1} \mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P)$$

where $\sum'_{\gamma \mapsto 1}$ means taking the sum over the G-conjugacy classes in Γ.

Sketch of proof. We first prove this after multiplying by $\#G$.

$$\# G \cdot \mathrm{Tr}_\Lambda(1; P_G) = \mathrm{Tr}_\Lambda(\sum\nolimits_{\gamma \mapsto 1} \gamma, P_G) = \mathrm{Tr}_\Lambda(\sum\nolimits_{\gamma \mapsto 1} \gamma, P)$$

where the second equality follows by considering the commutative triangle

$$\begin{tikzcd} P^G \arrow[dr, "a"'] & & P_G \arrow[ll, "c"'] \\ & P \arrow[ur, "b"'] & \end{tikzcd}$$

where a is the canonical inclusion, b the canonical surjection and $c = \sum_{\gamma\mapsto 1}\gamma$. Then we have

$$(\sum\nolimits_{\gamma\mapsto 1}\gamma)|_P = a\circ c\circ b \quad\text{and}\quad (\sum\nolimits_{\gamma\mapsto 1}\gamma)|_{P_G} = b\circ a\circ c$$

hence they have the same trace. We then have

$$\#G\cdot \mathrm{Tr}_\Lambda(1; P_G) = \sum_{\gamma\mapsto 1}{}' \frac{\#G}{\#Z_\gamma}\mathrm{Tr}_\Lambda(\gamma, P) = \#G\sum_{\gamma\mapsto 1}{}'\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P).$$

To finish the proof, reduce to case Λ torsion-free by some universality argument. See [Del77] for details. □

03UE Remark 64.15.10. Let us try to illustrate the content of the formula of Lemma 64.15.8. Suppose that Λ, viewed as a trivial Γ-module, admits a finite resolution $0\to P_r\to\ldots\to P_1\to P_0\to\Lambda\to 0$ by some $\Lambda[\Gamma]$-modules P_i which are finite and projective as $\Lambda[G]$-modules. In that case

$$H_*\left((P_\bullet)_G\right) = \mathrm{Tor}_*^{\Lambda[G]}(\Lambda,\Lambda) = H_*(G,\Lambda)$$

and

$$\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P_\bullet) = \frac{1}{\#Z_\gamma}\mathrm{Tr}_\Lambda(\gamma, P_\bullet) = \frac{1}{\#Z_\gamma}\mathrm{Tr}(\gamma,\Lambda) = \frac{1}{\#Z_\gamma}.$$

Therefore, Lemma 64.15.8 says

$$\mathrm{Tr}_\Lambda(1, P_G) = \mathrm{Tr}\left(1|_{H_*(G,\Lambda)}\right) = \sum_{\gamma\mapsto 1}{}'\frac{1}{\#Z_\gamma}.$$

This can be interpreted as a point count on the stack BG. If $\Lambda = \mathbf{F}_\ell$ with ℓ prime to $\#G$, then $H_*(G,\Lambda)$ is $\mathbf{F}_\ell$ in degree 0 (and 0 in other degrees) and the formula reads

$$1 = \sum\nolimits_{\substack{\sigma\text{-conjugacy}\\ \text{classes}\langle\gamma\rangle}}\frac{1}{\#Z_\gamma} \mod \ell.$$

This is in some sense a "trivial" trace formula for G. Later we will see that (64.14.3.1) can in some cases be viewed as a highly nontrivial trace formula for a certain type of group, see Section 64.30.

64.16. Proof of the trace formula

03UF

03UG Theorem 64.16.1. Let k be a finite field and X a finite type, separated scheme of dimension at most 1 over k. Let Λ be a finite ring whose cardinality is prime to that of k, and $K\in D_{ctf}(X,\Lambda)$. Then

03UH (64.16.1.1) $$\mathrm{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar k},K)}) = \sum\nolimits_{x\in X(k)}\mathrm{Tr}(\pi_x|_{K_{\bar x}})$$

in $\Lambda^\natural$.

Please see Remark 64.16.2 for some remarks on the statement. Notation: For short, we write

$$T'(X, K) = \sum\nolimits_{x \in X(k)} \text{Tr}(\pi_x|_{K_{\bar{x}}})$$

for the right-hand side of (64.16.1.1) and

$$T''(X, K) = \text{Tr}(\pi_x^*|_{R\Gamma_c(X_{\bar{k}}, K)})$$

for the left-hand side.

Proof of Theorem 64.16.1. The proof proceeds in a number of steps.

Step 1. Let $j : \mathcal{U} \hookrightarrow X$ be an open immersion with complement $Y = X - \mathcal{U}$ and $i : Y \hookrightarrow X$. Then $T''(X, K) = T''(\mathcal{U}, j^{-1}K) + T''(Y, i^{-1}K)$ and $T'(X, K) = T'(\mathcal{U}, j^{-1}K) + T'(Y, i^{-1}K)$.

This is clear for T'. For T'' use the exact sequence

$$0 \to j_!j^{-1}K \to K \to i_*i^{-1}K \to 0$$

to get a filtration on K. This gives rise to an object $\widetilde{K} \in DF(X, \Lambda)$ whose graded pieces are $j_!j^{-1}K$ and $i_*i^{-1}K$, both of which lie in $D_{ctf}(X, \Lambda)$. Then, by filtered derived abstract nonsense (INSERT REFERENCE), $R\Gamma_c(X_{\bar{k}}, K) \in DF_{perf}(\Lambda)$, and it comes equipped with π_x^* in $DF_{perf}(\Lambda)$. By the discussion of traces on filtered complexes (INSERT REFERENCE) we get

$$\begin{aligned} \text{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar{k}}, K)}) &= \text{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar{k}}, j_!j^{-1}K)}) + \text{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar{k}}, i_*i^{-1}K)}) \\ &= T''(U, i^{-1}K) + T''(Y, i^{-1}K). \end{aligned}$$

Step 2. The theorem holds if $\dim X \leq 0$.

Indeed, in that case

$$R\Gamma_c(X_{\bar{k}}, K) = R\Gamma(X_{\bar{k}}, K) = \Gamma(X_{\bar{k}}, K) = \bigoplus\nolimits_{\bar{x} \in X_{\bar{k}}} K_{\bar{x}} \leftarrow \pi_X * .$$

Since the fixed points of $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$ are exactly the points $\bar{x} \in X_{\bar{k}}$ which lie over a k-rational point $x \in X(k)$ we get

$$\text{Tr}\big(\pi_X^*|_{R\Gamma_c(X_{\bar{k}}, K)}\big) = \sum\nolimits_{x \in X(k)} \text{Tr}(\pi_{\bar{x}}|_{K_{\bar{x}}}).$$

Step 3. It suffices to prove the equality $T'(\mathcal{U}, \mathcal{F}) = T''(\mathcal{U}, \mathcal{F})$ in the case where

- $\mathcal{U}$ is a smooth irreducible affine curve over k,
- $\mathcal{U}(k) = \emptyset$,
- $K = \mathcal{F}$ is a finite locally constant sheaf of Λ-modules on $\mathcal{U}$ whose stalk(s) are finite projective Λ-modules, and
- Λ is killed by a power of a prime ℓ and $\ell \in k^*$.

Indeed, because of Step 2, we can throw out any finite set of points. But we have only finitely many rational points, so we may assume there are none[2]. We may assume that $\mathcal{U}$ is smooth irreducible and affine by passing to irreducible components and throwing away the bad points if necessary. The assumptions of $\mathcal{F}$ come from unwinding the definition of $D_{ctf}(X, \Lambda)$ and those on Λ from considering its primary decomposition.

[2]At this point, there should be an evil laugh in the background.

For the remainder of the proof, we consider the situation

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "\bar{f}"] \\ \mathcal{U} \arrow[r] & X \end{tikzcd}$$

where $\mathcal{U}$ is as above, f is a finite étale Galois covering, $\mathcal{V}$ is connected and the horizontal arrows are projective completions. Denoting $G = \text{Aut}(\mathcal{V}|\mathcal{U})$, we also assume (as we may) that $f^{-1}\mathcal{F} = \underline{M}$ is constant, where the module $M = \Gamma(\mathcal{V}, f^{-1}\mathcal{F})$ is a $\Lambda[G]$-module which is finite and projective over Λ. This corresponds to the trivial monoid extension

$$1 \to G \to \Gamma = G \times \mathbf{N} \to \mathbf{N} \to 1.$$

In that context, using the reductions above, we need to show that $T''(\mathcal{U}, \mathcal{F}) = 0$.

Step 4. There is a natural action of G on $f_*f^{-1}\mathcal{F}$ and the trace map $f_*f^{-1}\mathcal{F} \to \mathcal{F}$ defines an isomorphism

$$(f_*f^{-1}\mathcal{F}) \otimes_{\Lambda[G]} \Lambda = (f_*f^{-1}\mathcal{F})_G \cong \mathcal{F}.$$

To prove this, simply unwind everything at a geometric point.

Step 5. Let $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$. Then $f_*f^{-1}\mathcal{F} \cong (f_*\underline{A}) \otimes_{\underline{A}} \underline{M}$ with diagonal G-action.

Step 6. There is a canonical isomorphism $(f_*\underline{A} \otimes_{\underline{A}} \underline{M}) \otimes_{\Lambda[G]} \underline{\Lambda} \cong \mathcal{F}$.

In fact, this is a derived tensor product, because of the projectivity assumption on $\mathcal{F}$.

Step 7. There is a canonical isomorphism

$$R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F}) = (R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A) \otimes^{\mathbf{L}}_A M) \otimes^{\mathbf{L}}_{\Lambda[G]} \Lambda,$$

compatible with the action of $\pi_{\mathcal{U}}^*$.

This comes from the universal coefficient theorem, i.e., the fact that $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$, and the flatness of $\mathcal{F}$ as a Λ-module.

We have

$$\begin{aligned} \text{Tr}(\pi_{\mathcal{U}}^*|_{R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F})}) &= \sum_{g \in G}{}' \text{Tr}_\Lambda^{Z_g}\left((g, \pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A) \otimes^{\mathbf{L}}_A M}\right) \\ &= \sum_{g \in G}{}' \text{Tr}_A^{Z_g}((g, \pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A)}) \cdot \text{Tr}_\Lambda(g|_M) \end{aligned}$$

where Γ acts on $R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F})$ by G and $(e, 1)$ acts via $\pi_{\mathcal{U}}^*$. So the monoidal extension is given by $\Gamma = G \times \mathbf{N} \to \mathbf{N}$, $\gamma \mapsto 1$. The first equality follows from Lemma 64.15.9 and the second from Lemma 64.15.8.

Step 8. It suffices to show that $\text{Tr}_A^{Z_g}((g, \pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A)}) \in A$ maps to zero in Λ.

Recall that

$$\begin{aligned} \#Z_g \cdot \text{Tr}_A^{Z_g}((g, \pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A)}) &= \text{Tr}_A((g, \pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A)}) \\ &= \text{Tr}_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}}, A)}). \end{aligned}$$

The first equality is Lemma 64.15.7, the second is the Leray spectral sequence, using the finiteness of f and the fact that we are only taking traces over A. Now since $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$ and $\#Z_g = \ell^a$ for some (fixed) a, it suffices to show the following result.

Step 9. We have $\mathrm{Tr}_A((g^{-1}\pi_\mathcal{V})^*|_{R\Gamma_c(\mathcal{V},A)}) = 0$ in A.

By additivity again, we have

$$\begin{aligned}\mathrm{Tr}_A((g^{-1}\pi_\mathcal{V})^*|_{R\Gamma_c(\mathcal{V}_{\bar k}A)}) + \mathrm{Tr}_A((g^{-1}\pi_\mathcal{V})^*|_{R\Gamma_c(Y-\mathcal{V})_{\bar k},A)}) \\ = \mathrm{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma(Y_{\bar k},A)})\end{aligned}$$

The latter trace is the number of fixed points of $g^{-1}\pi_Y$ on Y, by Weil's trace formula Theorem 64.14.4. Moreover, by the 0-dimensional case already proven in step 2,

$$\mathrm{Tr}_A((g^{-1}\pi_\mathcal{V})^*|_{R\Gamma_c(Y-\mathcal{V})_{\bar k},A)})$$

is the number of fixed points of $g^{-1}\pi_Y$ on $(Y-\mathcal{V})_{\bar k}$. Therefore,

$$\mathrm{Tr}_A((g^{-1}\pi_\mathcal{V})^*|_{R\Gamma_c(\mathcal{V}_{\bar k},A)})$$

is the number of fixed points of $g^{-1}\pi_Y$ on $\mathcal{V}_{\bar k}$. But there are no such points: if $\bar y \in Y_{\bar k}$ is fixed under $g^{-1}\pi_Y$, then $\bar f(\bar y) \in X_{\bar k}$ is fixed under π_X. But $\mathcal{U}$ has no k-rational point, so we must have $\bar f(\bar y) \in (X-\mathcal{U})_{\bar k}$ and so $\bar y \notin \mathcal{V}_{\bar k}$, a contradiction. This finishes the proof. $\square$

03UI Remark 64.16.2. Remarks on Theorem 64.16.1.

(1) This formula holds in any dimension. By a dévissage lemma (which uses proper base change etc.) it reduces to the current statement – in that generality.
(2) The complex $R\Gamma_c(X_{\bar k}, K)$ is defined by choosing an open immersion $j : X \hookrightarrow \bar X$ with $\bar X$ projective over k of dimension at most 1 and setting
$$R\Gamma_c(X_{\bar k}, K) := R\Gamma(\bar X_{\bar k}, j_!K).$$
This is independent of the choice of $\bar X$ follows from (insert reference here). We define $H^i_c(X_{\bar k}, K)$ to be the ith cohomology group of $R\Gamma_c(X_{\bar k}, K)$.

03UJ Remark 64.16.3. Even though all we did are reductions and mostly algebra, the trace formula Theorem 64.16.1 is much stronger than Weil's geometric trace formula (Theorem 64.14.4) because it applies to coefficient systems (sheaves), not merely constant coefficients.

64.17. Applications

03UK OK, having indicated the proof of the trace formula, let's try to use it for something.

64.18. On l-adic sheaves

03UL

03UM Definition 64.18.1. Let X be a Noetherian scheme. A $\mathbf{Z}_\ell$-sheaf on X, or simply an ℓ-adic sheaf $\mathcal{F}$ is an inverse system $\{\mathcal{F}_n\}_{n\geq 1}$ where

(1) $\mathcal{F}_n$ is a constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, and
(2) the transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong \mathcal{F}_n$.

We say that $\mathcal{F}$ is lisse if each $\mathcal{F}_n$ is locally constant. A morphism of such is merely a morphism of inverse systems.

03UN Lemma 64.18.2. Let $\{\mathcal{G}_n\}_{n\geq 1}$ be an inverse system of constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. Suppose that for all $k \geq 1$, the maps

$$\mathcal{G}_{n+1}/\ell^k\mathcal{G}_{n+1} \to \mathcal{G}_n/\ell^k\mathcal{G}_n$$

are isomorphisms for all $n \gg 0$ (where the bound possibly depends on k). In other words, assume that the system $\{\mathcal{G}_n/\ell^k\mathcal{G}_n\}_{n\geq 1}$ is eventually constant, and call $\mathcal{F}_k$ the corresponding sheaf. Then the system $\{\mathcal{F}_k\}_{k\geq 1}$ forms a $\mathbf{Z}_\ell$-sheaf on X.

Proof. The proof is obvious. $\square$

03UO Lemma 64.18.3. The category of $\mathbf{Z}_\ell$-sheaves on X is abelian.

Proof. Let $\Phi = \{\varphi_n\}_{n\geq 1} : \{\mathcal{F}_n\} \to \{\mathcal{G}_n\}$ be a morphism of $\mathbf{Z}_\ell$-sheaves. Set

$$\mathrm{Coker}(\Phi) = \left\{\mathrm{Coker}\left(\mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{G}_n\right)\right\}_{n\geq 1}$$

and $\mathrm{Ker}(\Phi)$ is the result of Lemma 64.18.2 applied to the inverse system

$$\left\{\bigcap_{m\geq n} \mathrm{Im}\left(\mathrm{Ker}(\varphi_m) \to \mathrm{Ker}(\varphi_n)\right)\right\}_{n\geq 1}.$$

That this defines an abelian category is left to the reader. $\square$

03UP Example 64.18.4. Let $X = \mathrm{Spec}(\mathbf{C})$ and $\Phi : \mathbf{Z}_\ell \to \mathbf{Z}_\ell$ be multiplication by ℓ. More precisely,

$$\Phi = \left\{\mathbf{Z}/\ell^n\mathbf{Z} \xrightarrow{\ell} \mathbf{Z}/\ell^n\mathbf{Z}\right\}_{n\geq 1}.$$

To compute the kernel, we consider the inverse system

$$\ldots \to \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z}.$$

Since the images are always zero, $\mathrm{Ker}(\Phi)$ is zero as a system.

03UQ Remark 64.18.5. If $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X and $\bar{x}$ is a geometric point then $M_n = \{\mathcal{F}_{n,\bar{x}}\}$ is an inverse system of finite $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that $M_{n+1} \to M_n$ is surjective and $M_n = M_{n+1}/\ell^n M_{n+1}$. It follows that

$$M = \lim_n M_n = \lim \mathcal{F}_{n,\bar{x}}$$

is a finite $\mathbf{Z}_\ell$-module. Indeed, $M/\ell M = M_1$ is finite over $\mathbf{F}_\ell$, so by Nakayama M is finite over $\mathbf{Z}_\ell$. Therefore, $M \cong \mathbf{Z}_\ell^{\oplus r} \oplus \oplus_{i=1}^t \mathbf{Z}_\ell/\ell^{e_i}\mathbf{Z}_\ell$ for some $r, t \geq 0$, $e_i \geq 1$. The module $M = \mathcal{F}_{\bar{x}}$ is called the stalk of $\mathcal{F}$ at $\bar{x}$.

03UR Definition 64.18.6. A $\mathbf{Z}_\ell$-sheaf $\mathcal{F}$ is torsion if $\ell^n : \mathcal{F} \to \mathcal{F}$ is the zero map for some n. The abelian category of $\mathbf{Q}_\ell$-sheaves on X is the quotient of the abelian category of $\mathbf{Z}_\ell$-sheaves by the Serre subcategory of torsion sheaves. In other words, its objects are $\mathbf{Z}_\ell$-sheaves on X, and if $\mathcal{F}, \mathcal{G}$ are two such, then

$$\mathrm{Hom}_{\mathbf{Q}_\ell}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We denote by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathbf{Q}_\ell$ the quotient functor (right adjoint to the inclusion). If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}'$ is a $\mathbf{Z}_\ell$-sheaf and $\bar{x}$ is a geometric point, then the stalk of $\mathcal{F}$ at $\bar{x}$ is $\mathcal{F}_{\bar{x}} = \mathcal{F}'_{\bar{x}} \otimes \mathbf{Q}_\ell$.

03US Remark 64.18.7. Since a $\mathbf{Z}_\ell$-sheaf is only defined on a Noetherian scheme, it is torsion if and only if its stalks are torsion.

03UT Definition 64.18.8. If X is a separated scheme of finite type over an algebraically closed field k and $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X, then we define

$$H^i(X, \mathcal{F}) := \lim\nolimits_n H^i(X, \mathcal{F}_n) \quad \text{and} \quad H^i_c(X, \mathcal{F}) := \lim\nolimits_n H^i_c(X, \mathcal{F}_n).$$

If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ for a $\mathbf{Z}_\ell$-sheaf $\mathcal{F}'$ then we set

$$H^i_c(X, \mathcal{F}) := H^i_c(X, \mathcal{F}') \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We call these the ℓ-adic cohomology of X with coefficients $\mathcal{F}$.

64.19. L-functions

03UU

03UV Definition 64.19.1. Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{\acute{e}tale}$. Then we set

$$L(X, \mathcal{F}) := \prod\nolimits_{x \in |X|} \det(1 - \pi_x^* T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} \in \Lambda[[T]]$$

where $|X|$ is the set of closed points of X, $\deg x = [\kappa(x) : k]$ and $\bar{x}$ is a geometric point lying over x. This definition clearly generalizes to the case where $\mathcal{F}$ is replaced by a $K \in D_{ctf}(X, \Lambda)$. We call this the L-function of $\mathcal{F}$.

03UW Remark 64.19.2. Intuitively, T should be thought of as $T = t^f$ where $p^f = \#k$. The definitions are then independent of the size of the ground field.

03UX Definition 64.19.3. Now assume that $\mathcal{F}$ is a $\mathbf{Q}_\ell$-sheaf on X. In this case we define

$$L(X, \mathcal{F}) := \prod\nolimits_{x \in |X|} \det(1 - \pi_x^* T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} \in \mathbf{Q}_\ell[[T]].$$

Note that this product converges since there are finitely many points of a given degree. We call this the L-function of $\mathcal{F}$.

64.20. Cohomological interpretation

03UY This is how Grothendieck interpreted the L-function.

03UZ Theorem 64.20.1 (Finite Coefficients). Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{\acute{e}tale}$. Then

$$L(X, \mathcal{F}) = \det(1 - \pi_X^*\ T|_{R\Gamma_c(X_{\bar{k}}, \mathcal{F})})^{-1} \in \Lambda[[T]].$$

Proof. Omitted. □

Thus far, we don't even know whether each cohomology group $H^i_c(X_{\bar{k}}, \mathcal{F})$ is free.

03V0 Theorem 64.20.2 (Adic sheaves). Let X be a scheme of finite type over a finite field k, and $\mathcal{F}$ a $\mathbf{Q}_\ell$-sheaf on X. Then

$$L(X, \mathcal{F}) = \prod\nolimits_i \det(1 - \pi_X^* T|_{H^i_c(X_{\bar{k}}, \mathcal{F})})^{(-1)^{i+1}} \in \mathbf{Q}_\ell[[T]].$$

Proof. This is sketched below. □

03V1 Remark 64.20.3. Since we have only developed some theory of traces and not of determinants, Theorem 64.20.1 is harder to prove than Theorem 64.20.2. We will only prove the latter, for the former see [Del77]. Observe also that there is no version of this theorem more general for $\mathbf{Z}_\ell$ coefficients since there is no ℓ-torsion.

We reduce the proof of Theorem 64.20.2 to a trace formula. Since $\mathbf{Q}_\ell$ has characteristic 0, it suffices to prove the equality after taking logarithmic derivatives. More precisely, we apply $T\frac{d}{dT}\log$ to both sides. We have on the one hand

$$\begin{aligned} T\frac{d}{dT}\log L(X,\mathcal{F}) &= T\frac{d}{dT}\log\prod_{x\in|X|}\det(1-\pi_x^*T^{\deg x}|_{\mathcal{F}_{\bar x}})^{-1} \\ &= \sum_{x\in|X|}T\frac{d}{dT}\log(\det(1-\pi_x^*T^{\deg x}|_{\mathcal{F}_{\bar x}})^{-1}) \\ &= \sum_{x\in|X|}\deg x\sum_{n\geq 1}\mathrm{Tr}((\pi_x^n)^*|_{\mathcal{F}_{\bar x}})T^{n\deg x} \end{aligned}$$

where the last equality results from the formula

$$T\frac{d}{dT}\log\left(\det\left(1-fT|_M\right)^{-1}\right) = \sum_{n\geq 1}\mathrm{Tr}(f^n|_M)T^n$$

which holds for any commutative ring Λ and any endomorphism f of a finite projective Λ-module M. On the other hand, we have

$$\begin{aligned} & T\frac{d}{dT}\log\left(\prod\nolimits_i\det(1-\pi_X^*T|_{H_c^i(X_{\bar k},\mathcal{F})})^{(-1)^{i+1}}\right) \\ & = \sum\nolimits_i(-1)^i\sum\nolimits_{n\geq 1}\mathrm{Tr}\left((\pi_X^n)^*|_{H_c^i(X_{\bar k},\mathcal{F})}\right)T^n \end{aligned}$$

by the same formula again. Now, comparing powers of T and using the Mobius inversion formula, we see that Theorem 64.20.2 is a consequence of the following equality

$$\sum_{d|n}d\sum_{\substack{x\in|X|\\ \deg x=d}}\mathrm{Tr}((\pi_X^{n/d})^*|_{\mathcal{F}_{\bar x}}) = \sum_i(-1)^i\mathrm{Tr}((\pi_X^n)^*|_{H_c^i(X_{\bar k},\mathcal{F})}).$$

Writing k_n for the degree n extension of k, $X_n = X\times_{\mathrm{Spec}\,k}\mathrm{Spec}(k_n)$ and ${}_n\mathcal{F} = \mathcal{F}|_{X_n}$, this boils down to

$$\sum_{x\in X_n(k_n)}\mathrm{Tr}(\pi_X^*|_{{}_n\mathcal{F}_{\bar x}}) = \sum_i(-1)^i\mathrm{Tr}((\pi_X^n)^*|_{H_c^i((X_n)_{\bar k},{}_n\mathcal{F})})$$

which is a consequence of Theorem 64.20.5.

03V3 Theorem 64.20.4. Let X/k be as above, let Λ be a finite ring with $\#\Lambda\in k^*$ and $K\in D_{ctf}(X,\Lambda)$. Then $R\Gamma_c(X_{\bar k},K)\in D_{perf}(\Lambda)$ and

$$\sum_{x\in X(k)}\mathrm{Tr}\left(\pi_x|_{K_{\bar x}}\right) = \mathrm{Tr}\left(\pi_X^*|_{R\Gamma_c(X_{\bar k},K)}\right).$$

Proof. Note that we have already proved this (REFERENCE) when $\dim X\leq 1$. The general case follows easily from that case together with the proper base change theorem. □

03V2 Theorem 64.20.5. Let X be a separated scheme of finite type over a finite field k and $\mathcal{F}$ be a $\mathbf{Q}_\ell$-sheaf on X. Then $\dim_{\mathbf{Q}_\ell}H_c^i(X_{\bar k},\mathcal{F})$ is finite for all i, and is nonzero for $0\leq i\leq 2\dim X$ only. Furthermore, we have

$$\sum_{x\in X(k)}\mathrm{Tr}\left(\pi_x|_{\mathcal{F}_{\bar x}}\right) = \sum_i(-1)^i\mathrm{Tr}\left(\pi_X^*|_{H_c^i(X_{\bar k},\mathcal{F})}\right).$$

Proof. We explain how to deduce this from Theorem 64.20.4. We first use some étale cohomology arguments to reduce the proof to an algebraic statement which we subsequently prove.

Let $\mathcal{F}$ be as in the theorem. We can write $\mathcal{F}$ as $\mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}' = \{\mathcal{F}'_n\}$ is a $\mathbf{Z}_\ell$-sheaf without torsion, i.e., $\ell : \mathcal{F}' \to \mathcal{F}'$ has trivial kernel in the category of $\mathbf{Z}_\ell$-sheaves. Then each $\mathcal{F}'_n$ is a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{\acute{e}tale}$, so $\mathcal{F}'_n \in D_{ctf}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ and $\mathcal{F}'_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = \mathcal{F}'_n$. Note that the last equality holds also for standard (non-derived) tensor product, since $\mathcal{F}'_n$ is flat (it is the same equality). Therefore,

(1) the complex $K_n = R\Gamma_c(X_{\bar{k}}, \mathcal{F}'_n)$ is perfect, and it is endowed with an endomorphism $\pi_n : K_n \to K_n$ in $D(\mathbf{Z}/\ell^n\mathbf{Z})$,
(2) there are identifications
$$K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = K_n$$
in $D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, compatible with the endomorphisms π_{n+1} and π_n (see [Del77, Rapport 4.12]),
(3) the equality $\text{Tr}(\pi_X^*|_{K_n}) = \sum_{x \in X(k)} \text{Tr}\left(\pi_x|_{(\mathcal{F}'_n)_{\bar{x}}}\right)$ holds, and
(4) for each $x \in X(k)$, the elements $\text{Tr}(\pi_x|_{\mathcal{F}'_{n,\bar{x}}}) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element of $\mathbf{Z}_\ell$ which is equal to $\text{Tr}(\pi_x|_{\mathcal{F}_{\bar{x}}}) \in \mathbf{Q}_\ell$.

It thus suffices to prove the following algebra lemma. $\square$

03V4 **Lemma 64.20.6.** Suppose we have $K_n \in D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, $\pi_n : K_n \to K_n$ and isomorphisms $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n$ compatible with π_{n+1} and π_n. Then

(1) the elements $t_n = \text{Tr}(\pi_n|_{K_n}) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element $t_\infty = \{t_n\}$ of $\mathbf{Z}_\ell$,
(2) the $\mathbf{Z}_\ell$-module $H^i_\infty = \lim_n H^i(k_n)$ is finite and is nonzero for finitely many i only, and
(3) the operators $H^i(\pi_n) : H^i(K_n) \to H^i(K_n)$ are compatible and define $\pi^i_\infty : H^i_\infty \to H^i_\infty$ satisfying
$$\sum (-1)^i \text{Tr}(\pi^i_\infty|_{H^i_\infty \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}) = t_\infty.$$

Proof. Since $\mathbf{Z}/\ell^n\mathbf{Z}$ is a local ring and K_n is perfect, each K_n can be represented by a finite complex $K_n^\bullet$ of finite free $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that the map $K_n^p \to K_n^{p+1}$ has image contained in ℓK_n^{p+1}. It is a fact that such a complex is unique up to isomorphism. Moreover π_n can be represented by a morphism of complexes $\pi_n^\bullet : K_n^\bullet \to K_n^\bullet$ (which is unique up to homotopy). By the same token the isomorphism $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n$ is represented by a map of complexes
$$\varphi_n^\bullet : K_{n+1}^\bullet \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n^\bullet.$$
In fact, $\varphi_n^\bullet$ is an isomorphism of complexes, thus we see that

- there exist $a, b \in \mathbf{Z}$ independent of n such that $K_n^i = 0$ for all $i \notin [a, b]$, and
- the rank of K_n^i is independent of n.

Therefore, the module $K_\infty^i = \lim_n \{K_n^i, \varphi_n^i\}$ is a finite free $\mathbf{Z}_\ell$-module and $K_\infty^\bullet$ is a finite complex of finite free $\mathbf{Z}_\ell$-modules. By induction on the number of nonzero terms, one can prove that $H^i(K_\infty^\bullet) = \lim_n H^i(K_n^\bullet)$ (this is not true for unbounded complexes). We conclude that $H^i_\infty = H^i(K_\infty^\bullet)$ is a finite $\mathbf{Z}_\ell$-module. This proves

ii. To prove the remainder of the lemma, we need to overcome the possible non-commutativity of the diagrams

$$\begin{tikzcd} K^\bullet_{n+1} \arrow[r, "\varphi^\bullet_n"] \arrow[d, "\pi^\bullet_{n+1}"'] & K^\bullet_n \arrow[d, "\pi^\bullet_n"] \\ K^\bullet_{n+1} \arrow[r, "\varphi^\bullet_n"'] & K^\bullet_n. \end{tikzcd}$$

However, this diagram does commute in the derived category, hence it commutes up to homotopy. We inductively replace $\pi^\bullet_n$ for $n \geq 2$ by homotopic maps of complexes making these diagrams commute. Namely, if $h^i : K^i_{n+1} \to K^{i-1}_n$ is a homotopy, i.e.,

$$\pi^\bullet_n \circ \varphi^\bullet_n - \varphi^\bullet_n \circ \pi^\bullet_{n+1} = dh + hd,$$

then we choose $\tilde{h}^i : K^i_{n+1} \to K^{i-1}_{n+1}$ lifting h^i. This is possible because K^i_{n+1} free and $K^{i-1}_{n+1} \to K^{i-1}_n$ is surjective. Then replace $\pi^\bullet_n$ by $\tilde{\pi}^\bullet_n$ defined by

$$\tilde{\pi}^\bullet_{n+1} = \pi^\bullet_{n+1} + d\tilde{h} + \tilde{h}d.$$

With this choice of $\{\pi^\bullet_n\}$, the above diagrams commute, and the maps fit together to define an endomorphism $\pi^\bullet_\infty = \lim_n \pi^\bullet_n$ of $K^\bullet_\infty$. Then part i is clear: the elements $t_n = \sum (-1)^i \text{Tr}\left(\pi^i_n|_{K^i_n}\right)$ fit into an element t_∞ of $\mathbf{Z}_\ell$. Moreover

$$\begin{aligned} t_\infty &= \sum (-1)^i \text{Tr}_{\mathbf{Z}_\ell}(\pi^i_\infty|_{K^i_\infty}) \\ &= \sum (-1)^i \text{Tr}_{\mathbf{Q}_\ell}(\pi^i_\infty|_{K^i_\infty \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}) \\ &= \sum (-1)^i \text{Tr}(\pi_\infty|_{H^i(K^\bullet_\infty \otimes \mathbf{Q}_\ell)}) \end{aligned}$$

where the last equality follows from the fact that $\mathbf{Q}_\ell$ is a field, so the complex $K^\bullet_\infty \otimes \mathbf{Q}_\ell$ is quasi-isomorphic to its cohomology $H^i(K^\bullet_\infty \otimes \mathbf{Q}_\ell)$. The latter is also equal to $H^i(K^\bullet_\infty) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell = H^i_\infty \otimes \mathbf{Q}_\ell$, which finishes the proof of the lemma, and also that of Theorem 64.20.5. □

64.21. List of things which we should add above

03V5 What did we skip the proof of in the lectures so far:

(1) curves and their Jacobians,
(2) proper base change theorem,
(3) inadequate discussion of $R\Gamma_c$,
(4) more generally, given $f : X \to S$ finite type, separated S quasi-projective, discussion of $Rf_!$ on étale sheaves.
(5) discussion of $\otimes^{\mathbf{L}}$
(6) discussion of why $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$

64.22. Examples of L-functions

03V6 We use Theorem 64.20.2 for curves to give examples of L-functions

64.23. Constant sheaves

03V7 Let k be a finite field, X a smooth, geometrically irreducible curve over k and $\mathcal{F} = \underline{\mathbf{Q}_\ell}$ the constant sheaf. If $\bar{x}$ is a geometric point of X, the Galois module $\mathcal{F}_{\bar{x}} = \underline{\mathbf{Q}_\ell}$ is trivial, so

$$\det(1 - \pi_x^* \, T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} = \frac{1}{1 - T^{\deg x}}.$$

Applying Theorem 64.20.2, we get

$$\begin{aligned} L(X, \mathcal{F}) &= \prod_{i=0}^{2} \det(1 - \pi_X^* T|_{H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell)})^{(-1)^{i+1}} \\ &= \frac{\det(1 - \pi_X^* T|_{H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell)})}{\det(1 - \pi_X^* T|_{H_c^0(X_{\bar{k}}, \mathbf{Q}_\ell)}) \cdot \det(1 - \pi_X^* T|_{H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell)})}. \end{aligned}$$

To compute the latter, we distinguish two cases.

Projective case. Assume that X is projective, so $H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell) = H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$, and we have

$$H^i(X_{\bar{k}}, \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & \pi_X^* = 1 \quad \text{if } i = 0, \\ \mathbf{Q}_\ell^{2g} & \pi_X^* = ? \quad \text{if } i = 1, \\ \mathbf{Q}_\ell & \pi_X^* = q \quad \text{if } i = 2. \end{cases}$$

The identification of the action of π_X^* on H^2 comes from Étale Cohomology, Lemma 59.69.2 and the fact that the degree of π_X is $q = \#(k)$. We do not know much about the action of π_X^* on the degree 1 cohomology. Let us call $\alpha_1, \ldots, \alpha_{2g}$ its eigenvalues in $\bar{\mathbf{Q}}_\ell$. Putting everything together, Theorem 64.20.2 yields the equality

$$\prod\nolimits_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi_X^* T|_{H^1(X_{\bar{k}}, \mathbf{Q}_\ell)})}{(1 - T)(1 - qT)} = \frac{(1 - \alpha_1 T) \ldots (1 - \alpha_{2g} T)}{(1 - T)(1 - qT)}$$

from which we deduce the following result.

03V8 **Lemma 64.23.1.** Let X be a smooth, projective, geometrically irreducible curve over a finite field k. Then

(1) the L-function $L(X, \mathbf{Q}_\ell)$ is a rational function,
(2) the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of π_X^* on $H^1(X_{\bar{k}}, \mathbf{Q}_\ell)$ are algebraic integers independent of ℓ,
(3) the number of rational points of X on k_n, where $[k_n : k] = n$, is

$$\#X(k_n) = 1 - \sum\nolimits_{i=1}^{2g} \alpha_i^n + q^n,$$

(4) for each i, $|\alpha_i| < q$.

Proof. Part (3) is Theorem 64.20.5 applied to $\mathcal{F} = \underline{\mathbf{Q}_\ell}$ on $X \otimes k_n$. For part (4), use the following result. $\square$

03V9 **Exercise 64.23.2.** Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$. Then for any conic sector containing the positive real axis of the form $C_\varepsilon = \{z \in \mathbf{C} \mid |\arg z| < \varepsilon\}$ with $\varepsilon > 0$, there exists an integer $k \geq 1$ such that $\alpha_1^k, \ldots, \alpha_n^k \in C_\varepsilon$.

Then prove that $|\alpha_i| \leq q$ for all i. Then, use elementary considerations on complex numbers to prove (as in the proof of the prime number theorem) that $|\alpha_i| < q$. In fact, the Riemann hypothesis says that for all $|\alpha_i| = \sqrt{q}$ for all i. We will come back to this later.

Affine case. Assume now that X is affine, say $X = \bar{X} - \{x_1, \ldots, x_n\}$ where $j : X \hookrightarrow \bar{X}$ is a projective nonsingular completion. Then $H_c^0(X_{\bar{k}}, \mathbf{Q}_\ell) = 0$ and $H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell) = H^2(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$ so Theorem 64.20.2 reads

$$L(X, \mathbf{Q}_\ell) = \prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi_X^* T|_{H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell)})}{1 - qT}.$$

On the other hand, the previous case gives

$$\begin{aligned} L(X, \mathbf{Q}_\ell) &= L(\bar{X}, \mathbf{Q}_\ell) \prod_{i=1}^{n} \left(1 - T^{\deg x_i}\right) \\ &= \frac{\prod_{i=1}^{n}(1 - T^{\deg x_i}) \prod_{j=1}^{2g}(1 - \alpha_j T)}{(1 - T)(1 - qT)}. \end{aligned}$$

Therefore, we see that $\dim H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell) = 2g + \sum_{i=1}^{n} \deg(x_i) - 1$, and the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi_{\bar{X}}^*$ acting on the degree 1 cohomology are roots of unity. More precisely, each x_i gives a complete set of $\deg(x_i)$th roots of unity, and one occurrence of 1 is omitted. To see this directly using coherent sheaves, consider the short exact sequence on $\bar{X}$

$$0 \to j_! \mathbf{Q}_\ell \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^{n} \mathbf{Q}_{\ell, x_i} \to 0.$$

The long exact cohomology sequence reads

$$0 \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^{n} \mathbf{Q}_\ell^{\oplus \deg x_i} \to H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell) \to H_c^1(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell) \to 0$$

where the action of Frobenius on $\bigoplus_{i=1}^{n} \mathbf{Q}_\ell^{\oplus \deg x_i}$ is by cyclic permutation of each term; and $H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell) = H_c^2(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$.

64.24. The Legendre family

03VA Let k be a finite field of odd characteristic, $X = \mathrm{Spec}(k[\lambda, \frac{1}{\lambda(\lambda - 1)}])$, and consider the family of elliptic curves $f : E \to X$ on $\mathbf{P}_X^2$ whose affine equation is $y^2 = x(x-1)(x-\lambda)$. We set $\mathcal{F} = Rf_*^1 \mathbf{Q}_\ell = \left\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\right\}_{n \geq 1} \otimes \mathbf{Q}_\ell$. In this situation, the following is true

- for each $n \geq 1$, the sheaf $R^1 f_*(\mathbf{Z}/\ell^n\mathbf{Z})$ is finite locally constant – in fact, it is free of rank 2 over $\mathbf{Z}/\ell^n\mathbf{Z}$,
- the system $\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\}_{n \geq 1}$ is a lisse ℓ-adic sheaf, and
- for all $x \in |X|$, $\det(1 - \pi_x\ T^{\deg x}|_{\mathcal{F}_{\bar{x}}}) = (1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})$ where α_x, β_x are the eigenvalues of the geometric frobenius of E_x acting on $H^1(E_{\bar{x}}, \mathbf{Q}_\ell)$.

Note that E_x is only defined over $\kappa(x)$ and not over k. The proof of these facts uses the proper base change theorem and the local acyclicity of smooth morphisms. For details, see [Del77]. It follows that

$$L(E/X) := L(X, \mathcal{F}) = \prod_{x \in |X|} \frac{1}{(1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})}.$$

Applying Theorem 64.20.2 we get

$$L(E/X) = \prod_{i=0}^{2} \det\left(1 - \pi_X^* T|_{H_c^i(X_{\bar k}, \mathcal{F})}\right)^{(-1)^{i+1}},$$

and we see in particular that this is a rational function. Furthermore, it is relatively easy to show that $H_c^0(X_{\bar k}, \mathcal{F}) = H_c^2(X_{\bar k}, \mathcal{F}) = 0$, so we merely have

$$L(E/X) = \det(1 - \pi_X^* T|_{H_c^1(X, \mathcal{F})}).$$

To compute this determinant explicitly, consider the Leray spectral sequence for the proper morphism $f : E \to X$ over $\mathbf{Q}_\ell$, namely

$$H_c^i(X_{\bar k}, R^j f_* \mathbf{Q}_\ell) \Rightarrow H_c^{i+j}(E_{\bar k}, \mathbf{Q}_\ell)$$

which degenerates. We have $f_*\mathbf{Q}_\ell = \mathbf{Q}_\ell$ and $R^1 f_* \mathbf{Q}_\ell = \mathcal{F}$. The sheaf $R^2 f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell(-1)$ is the Tate twist of $\mathbf{Q}_\ell$, i.e., it is the sheaf $\mathbf{Q}_\ell$ where the Galois action is given by multiplication by $\#\kappa(x)$ on the stalk at $\bar x$. It follows that, for all $n \geq 1$,

$$\begin{aligned}
\#E(k_n) &= \sum\nolimits_i (-1)^i \mathrm{Tr}({\pi_E^n}^*|_{H_c^i(E_{\bar k}, \mathbf{Q}_\ell)}) \\
&= \sum\nolimits_{i,j} (-1)^{i+j} \mathrm{Tr}({\pi_X^n}^*|_{H_c^i(X_{\bar k}, R^j f_* \mathbf{Q}_\ell)}) \\
&= (q^n - 2) + \mathrm{Tr}({\pi_X^n}^*|_{H_c^1(X_{\bar k}, \mathcal{F})}) + q^n(q^n - 2) \\
&= q^{2n} - q^n - 2 + \mathrm{Tr}({\pi_X^n}^*|_{H_c^1(X_{\bar k}, \mathcal{F})})
\end{aligned}$$

where the first equality follows from Theorem 64.20.5, the second one from the Leray spectral sequence and the third one by writing down the higher direct images of $\mathbf{Q}_\ell$ under f. Alternatively, we could write

$$\#E(k_n) = \sum_{x \in X(k_n)} \#E_x(k_n)$$

and use the trace formula for each curve. We can also find the number of k_n-rational points simply by counting. The zero section contributes $q^n - 2$ points (we omit the points where $\lambda = 0, 1$) hence

$$\#E(k_n) = q^n - 2 + \#\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1\}.$$

Now we have

$$\begin{aligned}
&\#\{y^2 = x(x-1)(x-\lambda),\ \lambda \neq 0, 1\} \\
&= \#\{y^2 = x(x-1)(x-\lambda) \text{ in } \mathbf{A}^3\} - \#\{y^2 = x^2(x-1)\} - \#\{y^2 = x(x-1)^2\} \\
&= \#\{\lambda = \tfrac{-y^2}{x(x-1)} + x,\ x \neq 0, 1\} + \#\{y^2 = x(x-1)(x-\lambda), x = 0, 1\} - 2(q^n - \varepsilon_n) \\
&= q^n(q^n - 2) + 2q^n - 2(q^n - \varepsilon_n) \\
&= q^{2n} - 2q^n + 2\varepsilon_n
\end{aligned}$$

where $\varepsilon_n = 1$ if -1 is a square in k_n, 0 otherwise, i.e.,

$$\varepsilon_n = \frac{1}{2}\left(1 + \left(\frac{-1}{k_n}\right)\right) = \frac{1}{2}\left(1 + (-1)^{\frac{q^n - 1}{2}}\right).$$

Thus $\#E(k_n) = q^{2n} - q^n - 2 + 2\varepsilon_n$. Comparing with the previous formula, we find

$$\mathrm{Tr}({\pi_X^n}^*|_{H_c^1(X_{\bar k}, \mathcal{F})}) = 2\varepsilon_n = 1 + (-1)^{\frac{q^n-1}{2}},$$

which implies, by elementary algebra of complex numbers, that if -1 is a square in k_n^*, then $\dim H^1_c(X_{\bar{k}}, \mathcal{F}) = 2$ and the eigenvalues are 1 and 1. Therefore, in that case we have

$$L(E/X) = (1-T)^2.$$

64.25. Exponential sums

03VB A standard problem in number theory is to evaluate sums of the form

$$S_{a,b}(p) = \sum_{x \in \mathbf{F}_p - \{0,1\}} e^{\frac{2\pi i x^a (x-1)^b}{p}}.$$

In our context, this can be interpreted as a cohomological sum as follows. Consider the base scheme $S = \mathrm{Spec}(\mathbf{F}_p[x, \frac{1}{x(x-1)}])$ and the affine curve $f : X \to \mathbf{P}^1 - \{0, 1, \infty\}$ over S given by the equation $y^{p-1} = x^a(x-1)^b$. This is a finite étale Galois cover with group $\mathbf{F}_p^*$ and there is a splitting

$$f_*(\bar{\mathbf{Q}}_\ell^*) = \bigoplus_{\chi : \mathbf{F}_p^* \to \bar{\mathbf{Q}}_\ell^*} \mathcal{F}_\chi$$

where χ varies over the characters of $\mathbf{F}_p^*$ and $\mathcal{F}_\chi$ is a rank 1 lisse $\mathbf{Q}_\ell$-sheaf on which $\mathbf{F}_p^*$ acts via χ on stalks. We get a corresponding decomposition

$$H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell) = \bigoplus_\chi H^1(\mathbf{P}^1_{\bar{k}} - \{0, 1, \infty\}, \mathcal{F}_\chi)$$

and the cohomological interpretation of the exponential sum is given by the trace formula applied to $\mathcal{F}_\chi$ over $\mathbf{P}^1 - \{0, 1, \infty\}$ for some suitable χ. It reads

$$S_{a,b}(p) = -\mathrm{Tr}(\pi_X^*|_{H^1(\mathbf{P}^1_{\bar{k}} - \{0,1,\infty\}, \mathcal{F}_\chi)}).$$

The general yoga of Weil suggests that there should be some cancellation in the sum. Applying (roughly) the Riemann-Hurwitz formula, we see that

$$2g_X - 2 \approx -2(p-1) + 3(p-2) \approx p$$

so $g_X \approx p/2$, which also suggests that the χ-pieces are small.

64.26. Trace formula in terms of fundamental groups

03VC In the following sections we reformulate the trace formula completely in terms of the fundamental group of a curve, except if the curve happens to be $\mathbf{P}^1$.

64.27. Fundamental groups

03VD This material is discussed in more detail in the chapter on fundamental groups. See Fundamental Groups, Section 58.1. Let X be a connected scheme and let $\overline{x} \to X$ be a geometric point. Consider the functor

$$\begin{array}{cccc} F_{\overline{x}} : & \substack{\text{finite étale} \\ \text{schemes over } X} & \longrightarrow & \text{finite sets} \\ & Y/X & \longmapsto & F_{\overline{x}}(Y) = \left\{ \substack{\text{geom points } \overline{y} \\ \text{of } Y \text{ lying over } \overline{x}} \right\} = Y_{\overline{x}} \end{array}$$

Set

$$\pi_1(X, \overline{x}) = Aut(F_{\overline{x}}) = \text{ set of automorphisms of the functor } F_{\overline{x}}$$

Note that for every finite étale $Y \to X$ there is an action

$$\pi_1(X, \overline{x}) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y)$$

03VE Definition 64.27.1. A subgroup of the form $\text{Stab}(\overline{y} \in F_{\overline{x}}(Y)) \subset \pi_1(X, \overline{x})$ is called open.

03VF Theorem 64.27.2 (Grothendieck). Let X be a connected scheme.

(1) There is a topology on $\pi_1(X, \overline{x})$ such that the open subgroups form a fundamental system of open nbhds of $e \in \pi_1(X, \overline{x})$.
(2) With topology of (1) the group $\pi_1(X, \overline{x})$ is a profinite group.
(3) The functor

$$\begin{matrix} \text{schemes finite} \\ \text{étale over } X \end{matrix} \rightarrow \begin{matrix} \text{finite discrete continuous} \\ \pi_1(X,\overline{x})\text{-sets} \end{matrix}$$
$$Y/X \mapsto F_{\overline{x}}(Y) \text{ with its natural action}$$

is an equivalence of categories.

Proof. See [Gro71]. □

03VG Proposition 64.27.3. Let X be an integral normal Noetherian scheme. Let $\overline{y} \to X$ be an algebraic geometric point lying over the generic point $\eta \in X$. Then

$$\pi_x(X, \overline{\eta}) = Gal(M/\kappa(\eta))$$

($\kappa(\eta)$, function field of X) where

$$\kappa(\overline{\eta}) \supset M \supset \kappa(\eta) = k(X)$$

is the max sub-extension such that for every finite sub extension $M \supset L \supset \kappa(\eta)$ the normalization of X in L is finite étale over X.

Proof. Omitted. □

Change of base point. For any $\overline{x}_1, \overline{x}_2$ geom. points of X there exists an isom. of fibre functions

$$\mathcal{F}_{\overline{x}_1} \cong \mathcal{F}_{\overline{x}_2}$$

(This is a path from $\overline{x}_1$ to $\overline{x}_2$.) Conjugation by this path gives isom

$$\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)$$

well defined up to inner actions.

Functoriality. For any morphism $X_1 \to X_2$ of connected schemes any $\overline{x} \in X_1$ there is a canonical map

$$\pi_1(X_1, \overline{x}) \to \pi_1(X_2, \overline{x})$$

(Why? because the fibre functor ...)

Base field. Let X be a variety over a field k. Then we get

$$\pi_1(X, \overline{x}) \to \pi_1(Spec(k), \overline{x}) =^{\text{prop}} Gal(k^{\text{sep}}/k)$$

This map is surjective if and only if X is geometrically connected over k. So in the geometrically connected case we get s.e.s. of profinite groups

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, \overline{x}) \to Gal(k^{\text{sep}}/k) \to 1$$

($\pi_1(X_{\overline{k}}, \overline{x})$: geometric fundamental group of X, $\pi_1(X, \overline{x})$: arithmetic fundamental group of X)

Comparison. If X is a variety over $\mathbf{C}$ then

$$\pi_1(X, \overline{x}) = \text{ profinite completion of } \pi_1(X(\mathbf{C})(\text{ usual topology}), x)$$

(have $x \in X(\mathbf{C})$)

Frobenii. X variety over k, $\#k < \infty$. For any $x \in X$ closed point, let

$$F_x \in \pi_1(x, \overline{x}) = \mathrm{Gal}(\kappa(x)^{\mathrm{sep}}/\kappa(x))$$

be the geometric frobenius. Let $\overline{\eta}$ be an alg. geom. gen. pt. Then

$$\pi_1(X, \overline{\eta}) \leftarrow^{\cong} \pi_1(X, \overline{x}) \overset{\text{functoriality}}{\leftarrow} \pi_1(x, \overline{x})$$

Easy fact:

$$\begin{array}{ccc} \pi_1(X, \overline{\eta}) & \to^{\deg} \pi_1(\mathrm{Spec}(k), \overline{\eta})* & = Gal(k^{sep}/k) \\ & & || \\ & & \widehat{\mathbf{Z}} \cdot F_{\mathrm{Spec}(k)} \\ F_x & \mapsto & \deg(x) \cdot F_{\mathrm{Spec}(k)} \end{array}$$

Recall: $\deg(x) = [\kappa(x) : k]$

Fundamental groups and lisse sheaves. Let X be a connected scheme, $\overline{x}$ geom. pt. There are equivalences of categories

(Λ finite ring)	fin. loc. const. sheaves of Λ-modules of $X_{\acute{e}tale}$	$\leftrightarrow$	finite (discrete) Λ-modules with continuous $\pi_1(X,\overline{x})$-action
(ℓ a prime)	lisse ℓ-adic sheaves	$\leftrightarrow$	finitely generated $\mathbf{Z}_\ell$-modules M with continuous $\pi_1(X,\overline{x})$-action where we use ℓ-adic topology on M

In particular lisse $\mathbf{Q}_l$-sheaves correspond to continuous homomorphisms

$$\pi_1(X, \overline{x}) \to \mathrm{GL}_r(\mathbf{Q}_l), \quad r \geq 0$$

Notation: A module with action (M, ρ) corresponds to the sheaf $\mathcal{F}_\rho$.

Trace formulas. X variety over k, $\#k < \infty$.

(1) Λ finite ring $(\#\Lambda, \#k) = 1$

$$\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(\Lambda)$$

continuous. For every $n \geq 1$ we have

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X|, \\ \deg(x) = d}} \mathrm{Tr}(\rho(F_x^{n/d})) \right) = \mathrm{Tr}\left((\pi_x^n)^*|_{R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

(2) $l \neq char(k)$ prime, $\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(\mathbf{Q}_l)$. For any $n \geq 1$

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X| \\ \deg(x) = d}} \mathrm{Tr}\left(\rho(F_x^{n/d})\right) \right) = \sum_{i=0}^{2\dim X} (-1)^i \mathrm{Tr}\left(\pi_X^*|_{H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

Weil conjectures. (Deligne-Weil I, 1974) X smooth proj. over k, $\#k = q$, then the eigenvalues of π_X^* on $H^i(X_{\overline{k}}, \mathbf{Q}_l)$ are algebraic integers α with $|\alpha| = q^{1/2}$.

Deligne's conjectures. (almost completely proved by Lafforgue + ...) Let X be a normal variety over k finite

$$\rho : \pi_1(X, \overline{x}) \longrightarrow \mathrm{GL}_r(\mathbf{Q}_l)$$

continuous. Assume: ρ irreducible $\det(\rho)$ of finite order. Then

(1) there exists a number field E such that for all $x \in |X|$(closed points) the char. poly of $\rho(F_x)$ has coefficients in E.
(2) for any $x \in |X|$ the eigenvalues $\alpha_{x,i}$, $i = 1, \ldots, r$ of $\rho(F_x)$ have complex absolute value 1. (these are algebraic numbers not necessary integers)

(3) for every finite place $\lambda($ not dividing $p)$, of E (maybe after enlarging E a bit) there exists

$$\rho\lambda : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(E_\lambda)$$

compatible with ρ. (some char. polys of F_x's)

03VH Theorem 64.27.4 (Deligne, Weil II). For a sheaf $\mathcal{F}_\rho$ with ρ satisfying the conclusions of the conjecture above then the eigenvalues of π_X^* on $H^i_c(X_{\overline{k}}, \mathcal{F}_\rho)$ are algebraic numbers α with absolute values

$$|\alpha| = q^{w/2}, \text{ for } w \in \mathbf{Z},\ w \leq i$$

Moreover, if X smooth and proj. then $w = i$.

Proof. See [Del80]. □

64.28. Profinite groups, cohomology and homology

03VI Let G be a profinite group.

Cohomology. Consider the category of discrete modules with continuous G-action. This category has enough injectives and we can define

$$H^i(G, M) = R^i H^0(G, M) = R^i(M \mapsto M^G)$$

Also there is a derived version $RH^0(G, -)$.

Homology. Consider the category of compact abelian groups with continuous G-action. This category has enough projectives and we can define

$$H_i(G, M) = L_i H_0(G, M) = L_i(M \mapsto M_G)$$

and there is also a derived version.

Trivial duality. The functor $M \mapsto M^\wedge = \mathrm{Hom}_{cont}(M, S^1)$ exchanges the categories above and

$$H^i(G, M)^\wedge = H_i(G, M^\wedge)$$

Moreover, this functor maps torsion discrete G-modules to profinite continuous G-modules and vice versa, and if M is either a discrete or profinite continuous G-module, then $M^\wedge = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$.

Notes on Homology.

(1) If we look at Λ-modules for a finite ring Λ then we can identify

$$H_i(G, M) = Tor_i^{\Lambda[[G]]}(M, \Lambda)$$

where $\Lambda[[G]]$ is the limit of the group algebras of the finite quotients of G.

(2) If G is a normal subgroup of Γ, and Γ is also profinite then
- $H^0(G, -)$: discrete Γ-module→ discrete Γ/G-modules
- $H_0(G, -)$: compact Γ-modules → compact Γ/G-modules

and hence the profinite group Γ/G acts on the cohomology groups of G with values in a Γ-module. In other words, there are derived functors

$$RH^0(G, -) : D^+(\text{discrete } \Gamma\text{-modules}) \longrightarrow D^+(\text{discrete } \Gamma/G\text{-modules})$$

and similarly for $LH_0(G, -)$.

64.29. Cohomology of curves, revisited

03VJ Let k be a field, X be geometrically connected, smooth curve over k. We have the fundamental short exact sequence

$$1 \to \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \mathrm{Gal}(k^{^{sep}}/k) \to 1$$

If Λ is a finite ring with $\#\Lambda \in k^*$ and M a finite Λ-module, and we are given

$$\rho : \pi_1(X, \overline{\eta}) \to \mathrm{Aut}_\Lambda(M)$$

continuous, then $\mathcal{F}_\rho$ denotes the associated sheaf on $X_{\acute{e}tale}$.

03VK **Lemma 64.29.1.** There is a canonical isomorphism

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) = (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$

as $\mathrm{Gal}(k^{^{sep}}/k)$-modules.

Here the subscript $_{\pi_1(X_{\overline{k}}, \overline{\eta})}$ indicates co-invariants, and (-1) indicates the Tate twist i.e., $\sigma \in \mathrm{Gal}(k^{^{sep}}/k)$ acts via

$$\chi_{cycl}(\sigma)^{-1}.\sigma \text{ on RHS}$$

where

$$\chi_{cycl} : \mathrm{Gal}(k^{^{sep}}/k) \to \prod\nolimits_{l \neq char(k)} \mathbf{Z}_l^*$$

is the cyclotomic character.

Reformulation (Deligne, Weil II, page 338). For any finite locally constant sheaf $\mathcal{F}$ on X there is a maximal quotient $\mathcal{F} \to \mathcal{F}''$ with $\mathcal{F}''/X_{\overline{k}}$ a constant sheaf, hence

$$\mathcal{F}'' = (X \to \mathrm{Spec}(k))^{-1} F''$$

where F'' is a sheaf $\mathrm{Spec}(k)$, i.e., a $\mathrm{Gal}(k^{^{sep}}/k)$-module. Then

$$H^2_c(X_{\overline{k}}, \mathcal{F}) \to H^2_c(X_{\overline{k}}, \mathcal{F}'') \to F''(-1)$$

is an isomorphism.

Proof of Lemma 64.29.1. Let $Y \to^{\varphi} X$ be the finite étale Galois covering corresponding to $\mathrm{Ker}(\rho) \subset \pi_1(X, \overline{\eta})$. So

$$\mathrm{Aut}(Y/X) = Ind(\rho)$$

is Galois group. Then $\varphi^*\mathcal{F}_\rho = \underline{M}_Y$ and

$$\varphi_*\varphi^*\mathcal{F}_\rho \to \mathcal{F}_\rho$$

which gives

$$\begin{aligned} & H^2_c(X_{\overline{k}}, \varphi_*\varphi^*\mathcal{F}_\rho) \to H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \\ & = H^2_c(Y_{\overline{k}}, \varphi^*\mathcal{F}_\rho) \\ & = H^2_c(Y_{\overline{k}}, \underline{M}) = \bigoplus\nolimits_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \end{aligned}$$

$$\mathrm{Im}(\rho) \to H^2_c(Y_{\overline{k}}, \underline{M}) = \bigoplus\nolimits_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \to_{\mathrm{Im}(\rho)\text{equivalent}} H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \to^{\substack{\text{trivial Im}(\rho) \\ \text{action}}}$$

irreducible curve $C/\overline{k}$, $H^2_c(C, \underline{M}) = M$.

Since

$$\begin{array}{c}\text{set of irreducible} \\ \text{components of } Y_k\end{array} = \frac{Im(\rho)}{Im(\rho|_{\pi_1(X_{\overline{k}}, \overline{\eta})})}$$

We conclude that $H^2_c(X_{\overline{k}}, \mathcal{F}_\rho)$ is a quotient of $M_{\pi_1(X_{\overline{k}},\overline{\eta})}$. On the other hand, there is a surjection

$$\mathcal{F}_\rho \to \mathcal{F}'' = \begin{array}{c}\text{sheaf on } X \text{ associated to} \\ (M)_{\pi_1(X_{\overline{k}},\overline{\eta})} \leftarrow \pi_1(X, \overline{\eta})\end{array}$$

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \to M_{\pi_1(X_{\overline{k}},\overline{\eta})}$$

The twist in Galois action comes from the fact that $H^2_c(X_{\overline{k}}, \mu_n) =^{\text{can}} \mathbf{Z}/n\mathbf{Z}$. □

03VL Remark 64.29.2. Thus we conclude that if X is also projective then we have functorially in the representation ρ the identifications

$$H^0(X_{\overline{k}}, \mathcal{F}_\rho) = M^{\pi_1(X_{\overline{k}},\overline{\eta})}$$

and

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) = M_{\pi_1(X_{\overline{k}},\overline{\eta})}(-1)$$

Of course if X is not projective, then $H^0_c(X_{\overline{k}}, \mathcal{F}_\rho) = 0$.

03VM Proposition 64.29.3. Let X/k as before but $X_{\overline{k}} \neq \mathbf{P}^1_{\overline{k}}$ The functors $(M, \rho) \mapsto H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho)$ are the left derived functor of $(M, \rho) \mapsto H^2_c(X_{\overline{k}}, \mathcal{F}_\rho)$ so

$$H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho) = H_i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)(-1)$$

Moreover, there is a derived version, namely

$$R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho) = LH_0(\pi_1(X_{\overline{k}}, \overline{\eta}), M(-1)) = M(-1) \otimes^{\mathbf{L}}_{\Lambda[[\pi_1(X_{\overline{k}},\overline{\eta})]]} \Lambda$$

in $D(\Lambda[[\widehat{\mathbf{Z}}]])$. Similarly, the functors $(M, \rho) \mapsto H^i(X_{\overline{k}}, \mathcal{F}_\rho)$ are the right derived functor of $(M, \rho) \mapsto M^{\pi_1(X_{\overline{k}},\overline{\eta})}$ so

$$H^i(X_{\overline{k}}, \mathcal{F}_\rho) = H^i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)$$

Moreover, in this case there is a derived version too.

Proof. (Idea) Show both sides are universal δ-functors. □

03VN Remark 64.29.4. By the proposition and Trivial duality then you get

$$H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho) \times H^i(X_{\overline{k}}, \mathcal{F}^\wedge_\rho(1)) \to \mathbf{Q}/\mathbf{Z}$$

a perfect pairing. If X is projective then this is Poincare duality.

64.30. Abstract trace formula

03VO Suppose given an extension of profinite groups,

$$1 \to G \to \Gamma \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

We say Γ has an abstract trace formula if and only if there exist

(1) an integer $q \geq 1$, and
(2) for every $d \geq 1$ a finite set S_d and for each $x \in S_d$ a conjugacy class $F_x \in \Gamma$ with $\deg(F_x) = d$

such that the following hold

(1) for all ℓ not dividing q have $\mathrm{cd}_\ell(G) < \infty$, and

(2) for all finite rings Λ with $q \in \Lambda^*$, for all finite projective Λ-modules M with continuous Γ-action, for all $n > 0$ we have
$$\sum\nolimits_{d|n} d\left(\sum\nolimits_{x \in S_d} \mathrm{Tr}(F_x^{n/d}|_M)\right) = q^n \mathrm{Tr}(F^n|_{M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda})$$
in $\Lambda^\natural$.

Here $M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda = LH_0(G, M)$ denotes derived homology, and $F = 1$ in $\Gamma/G = \widehat{\mathbf{Z}}$.

03VP Remark 64.30.1. Here are some observations concerning this notion.

(1) If modeling projective curves then we can use cohomology and we don't need factor q^n.
(2) The only examples I know are $\Gamma = \pi_1(X, \overline{\eta})$ where X is smooth, geometrically irreducible and $K(\pi, 1)$ over finite field. In this case $q = (\#k)^{\dim X}$. Modulo the proposition, we proved this for curves in this course.
(3) Given the integer q then the sets S_d are uniquely determined. (You can multiple q by an integer m and then replace S_d by m^d copies of S_d without changing the formula.)

03VQ Example 64.30.2. Fix an integer $q \geq 1$
$$\begin{array}{ccccccccc} 1 & \to & G = \widehat{\mathbf{Z}}^{(q)} & \to & \Gamma & \to & \widehat{\mathbf{Z}} & \to & 1 \\ & & = \prod_{l \nmid q} \mathbf{Z}_l & & F & \mapsto & 1 & & \end{array}$$
with $FxF^{-1} = ux$, $u \in (\widehat{\mathbf{Z}}^{(q)})^*$. Just using the trivial modules $\mathbf{Z}/m\mathbf{Z}$ we see
$$q^n - (qu)^n \equiv \sum\nolimits_{d|n} d\#S_d$$
in $\mathbf{Z}/m\mathbf{Z}$ for all $(m, q) = 1$ (up to $u \to u^{-1}$) this implies $qu = a \in \mathbf{Z}$ and $|a| < q$. The special case $a = 1$ does occur with
$$\Gamma = \pi_1^t(\mathbf{G}_{m,\mathbf{F}_p}, \overline{\eta}), \quad \#S_1 = q - 1, \quad \text{and} \quad \#S_2 = \frac{(q^2 - 1) - (q - 1)}{2}$$

64.31. Automorphic forms and sheaves

03VR References: See especially the amazing papers [Dri83], [Dri84] and [Dri80] by Drinfeld.

Unramified cusp forms. Let k be a finite field of characteristic p. Let X geometrically irreducible projective smooth curve over k. Set $K = k(X)$ equal to the function field of X. Let v be a place of K which is the same thing as a closed point $x \in X$. Let K_v be the completion of K at v, which is the same thing as the fraction field of the completion of the local ring of X at x. Denote $O_v \subset K_v$ the ring of integers. We further set
$$O = \prod\nolimits_v O_v \subset \mathbf{A} = \prod_v{}' K_v$$
and we let Λ be any ring with p invertible in Λ.

03VS Definition 64.31.1. An unramified cusp form on $\mathrm{GL}_2(\mathbf{A})$ with values in Λ[3] is a function
$$f : \mathrm{GL}_2(\mathbf{A}) \to \Lambda$$
such that

[3] This is likely nonstandard notation.

(1) $f(x\gamma) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbf{A})$ and all $\gamma \in \mathrm{GL}_2(K)$
(2) $f(ux) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbf{A})$ and all $u \in \mathrm{GL}_2(O)$
(3) for all $x \in \mathrm{GL}_2(\mathbf{A})$,
$$\int_{\mathbf{A} \mod K} f\left(x\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) dz = 0$$
see [dJ01, Section 4.1] for an explanation of how to make sense out of this for a general ring Λ in which p is invertible.

Hecke Operators. For v a place of K and f an unramified cusp form we set
$$T_v(f)(x) = \int_{g \in M_v} f(g^{-1}x)dg,$$
and
$$U_v(f)(x) = f\left(\begin{pmatrix}\pi_v^{-1} & 0\\ 0 & \pi_v^{-1}\end{pmatrix}x\right)$$
Notations used: here $\pi_v \in O_v$ is a uniformizer
$$M_v = \{h \in Mat(2 \times 2, O_v) | \det h = \pi_v O_v^*\}$$
and $dg =$ is the Haar measure on $\mathrm{GL}_2(K_v)$ with $\int_{\mathrm{GL}_2(O_v)} dg = 1$. Explicitly we have
$$T_v(f)(x) = f\left(\begin{pmatrix}\pi_v^{-1} & 0\\ 0 & 1\end{pmatrix}x\right) + \sum_{i=1}^{q_v} f\left(\begin{pmatrix}1 & 0\\ -\pi_v^{-1}\lambda_i & \pi_v^{-1}\end{pmatrix}x\right)$$
with $\lambda_i \in O_v$ a set of representatives of $O_v/(\pi_v) = \kappa_v$, $q_v = \#\kappa_v$.

Eigenforms. An eigenform f is an unramified cusp form such that some value of f is a unit and $T_v f = t_v f$ and $U_v f = u_v f$ for some (uniquely determined) $t_v, u_v \in \Lambda$.

03VT Theorem 64.31.2. Given an eigenform f with values in $\overline{\mathbf{Q}}_l$ and eigenvalues $u_v \in \overline{\mathbf{Z}}_l^*$ then there exists
$$\rho : \pi_1(X) \to \mathrm{GL}_2(E)$$
continuous, absolutely irreducible where E is a finite extension of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_l$ such that $t_v = \mathrm{Tr}(\rho(F_v))$, and $u_v = q_v^{-1}\det(\rho(F_v))$ for all places v.

Proof. See [Dri80]. □

03VU Theorem 64.31.3. Suppose $\mathbf{Q}_l \subset E$ finite, and
$$\rho : \pi_1(X) \to \mathrm{GL}_2(E)$$
absolutely irreducible, continuous. Then there exists an eigenform f with values in $\overline{\mathbf{Q}}_l$ whose eigenvalues t_v, u_v satisfy the equalities $t_v = \mathrm{Tr}(\rho(F_v))$ and $u_v = q_v^{-1}\det(\rho(F_v))$.

Proof. See [Dri83]. □

03VV Remark 64.31.4. We now have, thanks to Lafforgue and many other mathematicians, complete theorems like this two above for GL_n and allowing ramification! In other words, the full global Langlands correspondence for GL_n is known for function fields of curves over finite fields. At the same time this does not mean there aren't a lot of interesting questions left to answer about the fundamental groups of curves over finite fields, as we shall see below.

Central character. If f is an eigenform then

$$\chi_f : \begin{array}{ccc} O^*\backslash\mathbf{A}^*/K^* & \to & \Lambda^* \\ (1, \ldots, \pi_v, 1, \ldots, 1) & \mapsto & u_v^{-1} \end{array}$$

is called the central character. If corresponds to the determinant of ρ via normalizations as above. Set

$$C(\Lambda) = \left\{ \begin{array}{c} \text{unr. cusp forms } f \text{ with coefficients in } \Lambda \\ \text{such that } U_v f = \varphi_v^{-1} f \forall v \end{array} \right\}$$

03VW Proposition 64.31.5. If Λ is Noetherian then $C(\Lambda)$ is a finitely generated Λ-module. Moreover, if Λ is a field with prime subfield $\mathbf{F} \subset \Lambda$ then

$$C(\Lambda) = (C(\mathbf{F})) \otimes_{\mathbf{F}} \Lambda$$

compatibly with T_v acting.

Proof. See [dJ01, Proposition 4.7]. $\square$

This proposition trivially implies the following lemma.

03VX Lemma 64.31.6. Algebraicity of eigenvalues. If Λ is a field then the eigenvalues t_v for $f \in C(\Lambda)$ are algebraic over the prime subfield $\mathbf{F} \subset \Lambda$.

Proof. Follows from Proposition 64.31.5. $\square$

Combining all of the above we can do the following very useful trick.

03VY Lemma 64.31.7. Switching l. Let E be a number field. Start with

$$\rho : \pi_1(X) \to SL_2(E_\lambda)$$

absolutely irreducible continuous, where λ is a place of E not lying above p. Then for any second place λ' of E not lying above p there exists a finite extension $E'_{\lambda'}$ and a absolutely irreducible continuous representation

$$\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$$

which is compatible with ρ in the sense that the characteristic polynomials of all Frobenii are the same.

Note how this is an instance of Deligne's conjecture!

Proof. To prove the switching lemma use Theorem 64.31.3 to obtain $f \in C(\overline{\mathbf{Q}}_l)$ eigenform ass. to ρ. Next, use Proposition 64.31.5 to see that we may choose $f \in C(E')$ with $E \subset E'$ finite. Next we may complete E' to see that we get $f \in C(E'_{\lambda'})$ eigenform with $E'_{\lambda'}$ a finite extension of $E_{\lambda'}$. And finally we use Theorem 64.31.2 to obtain $\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$ abs. irred. and continuous after perhaps enlarging $E'_{\lambda'}$ a bit again. $\square$

Speculation: If for a (topological) ring Λ we have

$$\left(\begin{array}{c} \rho : \pi_1(X) \to SL_2(\Lambda) \\ \text{abs irred} \end{array} \right) \leftrightarrow \text{ eigen forms in } C(\Lambda)$$

then all eigenvalues of $\rho(F_v)$ algebraic (won't work in an easy way if Λ is a finite ring. Based on the speculation that the Langlands correspondence works more generally than just over fields one arrives at the following conjecture.

Conjecture. (See [dJ01]) For any continuous

$$\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbf{F}_l[[t]])$$

we have $\#\rho(\pi_1(X_{\overline{k}})) < \infty$.

A rephrasing in the language of sheaves: "For any lisse sheaf of $\overline{\mathbf{F}_l((t))}$-modules the geom monodromy is finite."

03VZ Theorem 64.31.8. The Conjecture holds if $n \leq 2$.

Proof. See [dJ01]. □

03W0 Theorem 64.31.9. Conjecture holds if $l > 2n$ modulo some unproven things.

Proof. See [Gai07]. □

It turns out the conjecture is useful for something. See work of Drinfeld on Kashiwara's conjectures. But there is also the much more down to earth application as follows.

03W1 Theorem 64.31.10. (See [dJ01, Theorem 3.5]) Suppose

$$\rho_0 : \pi_1(X) \to \mathrm{GL}_n(\mathbf{F}_l)$$

is a continuous, $l \neq p$. Assume

(1) Conj. holds for X,
(2) $\rho_0|_{\pi_1(X_{\overline{k}})}$ abs. irred., and
(3) l does not divide n.

Then the universal deformation ring R_{univ} of ρ_0 is finite flat over $\mathbf{Z}_l$.

Explanation: There is a representation $\rho_{\text{univ}} : \pi_1(X) \to \mathrm{GL}_n(R_{\text{univ}})$ (Univ. Defo ring) R_{univ} loc. complete, residue field $\mathbf{F}_l$ and $(R_{\text{univ}} \to \mathbf{F}_l) \circ \rho_{\text{univ}} \cong \rho_0$. And given any $R \to \mathbf{F}_l$, R local complete and $\rho : \pi_1(X) \to \mathrm{GL}_n(R)$ then there exists $\psi : R_{\text{univ}} \to R$ such that $\psi \circ \rho_{\text{univ}} \cong \rho$. The theorem says that the morphism

$$\mathrm{Spec}(R_{\text{univ}}) \longrightarrow \mathrm{Spec}(\mathbf{Z}_l)$$

is finite and flat. In particular, such a ρ_0 lifts to a $\rho : \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$.

Notes:

(1) The theorem on deformations is easy.
(2) Any result towards the conjecture seems hard.
(3) It would be interesting to have more conjectures on $\pi_1(X)$!

64.32. Counting points

03W2 Let X be a smooth, geometrically irreducible, projective curve over k and $q = \#k$. The trace formula gives: there exists algebraic integers $w_1, \ldots, w_{2g}$ such that

$$\#X(k_n) = q^n - \sum\nolimits_{i=1}^{2g_X} w_i^n + 1.$$

If $\sigma \in \mathrm{Aut}(X)$ then for all i, there exists j such that $\sigma(w_i) = w_j$.

Riemann-Hypothesis. For all i we have $|\omega_i| = \sqrt{q}$.

This was formulated by Emil Artin, in 1924, for hyperelliptic curves. Proved by Weil 1940. Weil gave two proofs

- using intersection theory on $X \times X$, using the Hodge index theorem, and

- using the Jacobian of X.

There is another proof whose initial idea is due to Stephanov, and which was given by Bombieri: it uses the function field $k(X)$ and its Frobenius operator (1969). The starting point is that given $f \in k(X)$ one observes that $f^q - f$ is a rational function which vanishes in all the $\mathbf{F}_q$-rational points of X, and that one can try to use this idea to give an upper bound for the number of points.

64.33. Precise form of Chebotarev

03W3 As a first application let us prove a precise form of Chebotarev for a finite étale Galois covering of curves. Let $\varphi : Y \to X$ be a finite étale Galois covering with group G. This corresponds to a homomorphism

$$\pi_1(X) \longrightarrow G = \text{Aut}(Y/X)$$

Assume $Y_{\overline{k}}$ = irreducible. If $C \subset G$ is a conjugacy class then for all $n > 0$, we have

$$|\#\{x \in X(k_n) \mid F_x \in C\} - \frac{\#C}{\#G} \cdot \#X(k_n)| \leq (\#C)(2g-2)\sqrt{q^n}$$

(Warning: Please check the coefficient $\#C$ on the right hand side carefuly before using.)

Sketch. Write

$$\varphi_*(\overline{\mathbf{Q}_l}) = \oplus_{\pi \in \widehat{G}} \mathcal{F}_\pi$$

where $\widehat{G}$ is the set of isomorphism classes of irred representations of G over $\overline{\mathbf{Q}}_l$. For $\pi \in \widehat{G}$ let $\chi_\pi : G \to \overline{\mathbf{Q}}_l$ be the character of π. Then

$$H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l) = \oplus_{\pi \in \widehat{G}} H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l)_\pi =_{(\varphi \text{ finite })} \oplus_{\pi \in \widehat{G}} H^*(X_{\overline{k}}, \mathcal{F}_\pi)$$

If $\pi \neq 1$ then we have

$$H^0(X_{\overline{k}}, \mathcal{F}_\pi) = H^2(X_{\overline{k}}, \mathcal{F}_\pi) = 0, \quad \dim H^1(X_{\overline{k}}, \mathcal{F}_\pi) = (2g_X - 2)d_\pi^2$$

(can get this from trace formula for acting on ...) and we see that

$$|\sum_{x \in X(k_n)} \chi_\pi(\mathcal{F}_x)| \leq (2g_X - 2)d_\pi^2\sqrt{q^n}$$

Write $1_C = \sum_\pi a_\pi \chi_\pi$, then $a_\pi = \langle 1_C, \chi_\pi \rangle$, and $a_1 = \langle 1_C, \chi_1 \rangle = \frac{\#C}{\#G}$ where

$$\langle f, h \rangle = \frac{1}{\#G} \sum_{g \in G} f(g)\overline{h(g)}$$

Thus we have the relation

$$\frac{\#C}{\#G} = ||1_C||^2 = \sum |a_\pi|^2$$

Final step:

$$\begin{aligned}
\#\left\{x \in X(k_n) \mid F_x \in C\right\} &= \sum_{x \in X(k_n)} 1_C(x) \\
&= \sum_{x \in X(k_n)} \sum_{\pi} a_\pi \chi_\pi(F_x) \\
&= \underbrace{\frac{\#C}{\#G} \#X(k_n)}_{\text{term for } \pi=1} + \underbrace{\sum_{\pi \neq 1} a_\pi \sum_{x \in X(k_n)} \chi_\pi(F_x)}_{\text{error term (to be bounded by } E)}
\end{aligned}$$

We can bound the error term by

$$\begin{aligned}
|E| &\leq \sum_{\substack{\pi \in \widehat{G}, \\ \pi \neq 1}} |a_\pi|(2g-2)d_\pi^2 \sqrt{q^n} \\
&\leq \sum_{\pi \neq 1} \frac{\#C}{\#G}(2g_X - 2)d_\pi^3 \sqrt{q^n}
\end{aligned}$$

By Weil's conjecture, $\#X(k_n) \sim q^n$. $\square$

64.34. How many primes decompose completely?

03W4 This section gives a second application of the Riemann Hypothesis for curves over a finite field. For number theorists it may be nice to look at the paper by Ihara, entitled "How many primes decompose completely in an infinite unramified Galois extension of a global field?", see [Iha83]. Consider the fundamental exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

03W5 Proposition 64.34.1. There exists a finite set $x_1, \ldots, x_n$ of closed points of X such that set of all frobenius elements corresponding to these points topologically generate $\pi_1(X)$.

Another way to state this is: There exist $x_1, \ldots, x_n \in |X|$ such that the smallest normal closed subgroup Γ of $\pi_1(X)$ containing 1 frobenius element for each x_i is all of $\pi_1(X)$. i.e., $\Gamma = \pi_1(X)$.

Proof. Pick $N \gg 0$ and let

$$\{x_1, \ldots, x_n\} = \begin{array}{l}\text{set of all closed points of} \\ X \text{ of degree} \leq N \text{ over } k\end{array}$$

Let $\Gamma \subset \pi_1(X)$ be as in the variant statement for these points. Assume $\Gamma \neq \pi_1(X)$. Then we can pick a normal open subgroup U of $\pi_1(X)$ containing Γ with $U \neq \pi_1(X)$. By R.H. for X our set of points will have some x_{i_1} of degree N, some x_{i_2} of degree $N-1$. This shows $\deg : \Gamma \to \widehat{\mathbf{Z}}$ is surjective and so the same holds for U. This exactly means if $Y \to X$ is the finite étale Galois covering corresponding to U, then $Y_{\overline{k}}$ irreducible. Set $G = \mathrm{Aut}(Y/X)$. Picture

$$Y \to^G X, \quad G = \pi_1(X)/U$$

By construction all points of X of degree $\leq N$, split completely in Y. So, in particular

$$\#Y(k_N) \geq (\#G)\#X(k_N)$$

Use R.H. on both sides. So you get

$$q^N + 1 + 2g_Y q^{N/2} \geq \#G\#X(k_N) \geq \#G(q^N + 1 - 2g_X q^{N/2})$$

Since $2g_Y - 2 = (\#G)(2g_X - 2)$, this means

$$q^N + 1 + (\#G)(2g_X - 1) + 1)q^{N/2} \geq \#G(q^N + 1 - 2g_X q^{N/2})$$

Thus we see that G has to be the trivial group if N is large enough. □

Weird Question. Set $W_X = \deg^{-1}(\mathbf{Z}) \subset \pi_1(X)$. Is it true that for some finite set of closed points $x_1, \ldots, x_n$ of X the set of all frobenii corresponding to these points algebraically generate W_X?

By a Baire category argument this translates into the same question for all Frobenii.

64.35. How many points are there really?

03W6 If the genus of the curve is large relative to q, then the main term in the formula $\#X(k) = q - \sum \omega_i + 1$ is not q but the second term $\sum \omega_i$ which can (a priori) have size about $2g_X\sqrt{q}$. In the paper [VD83] the authors Drinfeld and Vladut show that this maximum is (as predicted by Ihara earlier) actually at most about $g\sqrt{q}$.

Fix q and let k be a field with k elements. Set

$$A(q) = \limsup_{g_X \to \infty} \frac{\#X(k)}{g_X}$$

where X runs over geometrically irreducible smooth projective curves over k. With this definition we have the following results:

- RH $\Rightarrow A(q) \leq 2\sqrt{q}$
- Ihara $\Rightarrow A(q) \leq \sqrt{2q}$
- DV $\Rightarrow A(q) \leq \sqrt{q} - 1$ (actually this is sharp if q is a square)

Proof. Given X let $w_1, \ldots, w_{2g}$ and $g = g_X$ be as before. Set $\alpha_i = \frac{w_i}{\sqrt{q}}$, so $|\alpha_i| = 1$. If α_i occurs then $\overline{\alpha}_i = \alpha_i^{-1}$ also occurs. Then

$$N = \#X(k) \leq X(k_r) = q^r + 1 - (\sum_i \alpha_i^r)q^{r/2}$$

Rewriting we see that for every $r \geq 1$

$$-\sum_i \alpha_i^r \geq Nq^{-r/2} - q^{r/2} - q^{-r/2}$$

Observe that

$$0 \leq |\alpha_i^n + \alpha_i^{n-1} + \ldots + \alpha_i + 1|^2 = (n+1) + \sum_{j=1}^{n}(n+1-j)(\alpha_i^j + \alpha_i^{-j})$$

So

$$\begin{aligned} 2g(n+1) &\geq -\sum_i \left(\sum_{j=1}^{n}(n+1-j)(\alpha_i^j + \alpha_i^{-j})\right) \\ &= -\sum_{j=1}^{n}(n+1-j)\left(\sum_i \alpha_i^j + \sum_i \alpha_i^{-j}\right) \end{aligned}$$

Take half of this to get

$$\begin{aligned} g(n+1) &\geq -\sum_{j=1}^{n}(n+1-j)(\sum_i \alpha_i^j) \\ &\geq N\sum_{j=1}^{n}(n+1-j)q^{-j/2} - \sum_{j=1}^{n}(n+1-j)(q^{j/2}+q^{-j/2}) \end{aligned}$$

This gives

$$\frac{N}{g} \leq \left(\sum_{j=1}^{n}\frac{n+1-j}{n+1}q^{-j/2}\right)^{-1} \cdot \left(1 + \frac{1}{g}\sum_{j=1}^{n}\frac{n+1-j}{n+1}(q^{j/2}+q^{-j/2})\right)$$

Fix n let $g \to \infty$

$$A(q) \leq \left(\sum_{j=1}^{n}\frac{n+1-j}{n+1}q^{-j/2}\right)^{-1}$$

So

$$A(q) \leq \lim_{n\to\infty}(\ldots) = \left(\sum_{j=1}^{\infty}q^{-j/2}\right)^{-1} = \sqrt{q}-1$$

□

64.36. Other chapters

Part 4

Algebraic Spaces

CHAPTER 65

Algebraic Spaces

025R 65.1. Introduction

025S Algebraic spaces were first introduced by Michael Artin, see [Art69b], [Art70], [Art73], [Art71b], [Art71a], [Art69a], [Art69c], and [Art74]. Some of the foundational material was developed jointly with Knutson, who produced the book [Knu71]. Artin defined (see [Art69c, Definition 1.3]) an algebraic space as a sheaf for the étale topology which is locally in the étale topology representable. In most of Artin's work the categories of schemes considered are schemes locally of finite type over a fixed excellent Noetherian base.

Our definition is slightly different from Artin's original definition. Namely, our algebraic spaces are sheaves for the fppf topology whose diagonal is representable and which have an étale "cover" by a scheme. Working with the fppf topology instead of the étale topology is just a technical point and scarcely makes any difference; we will show in Bootstrap, Section 80.12 that we would have gotten the same category of algebraic spaces if we had worked with the étale topology. In that same chapter we will prove that the condition on the diagonal can in some sense be removed, see Bootstrap, Section 80.6.

After defining algebraic spaces we make some foundational observations. The main result in this chapter is that with our definitions an algebraic space is the same thing as an étale equivalence relation, see the discussion in Section 65.9 and Theorem 65.10.5. The analogue of this theorem in Artin's setting is [Art69c, Theorem 1.5], or [Knu71, Proposition II.1.7]. In other words, the sheaf defined by an étale equivalence relation has a representable diagonal. It follows that our definition agrees with Artin's original definition in a broad sense. It also means that one can give examples of algebraic spaces by simply writing down an étale equivalence relation.

In Section 65.13 we introduce various separation axioms on algebraic spaces that we have found in the literature. Finally in Section 65.14 we give some weird and not so weird examples of algebraic spaces.

65.2. General remarks

025T We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. In Section 65.15 we discuss what changes if you change the big fppf site.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \text{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points over S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 35.13.7. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

65.3. Representable morphisms of presheaves

025U Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$. Let $a : F \to G$ be a representable transformation of functors, see Categories, Definition 4.8.2. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the fiber product $h_U \times_{\xi, G} F$ is representable. Choose a representing object V_ξ and an isomorphism $h_{V_\xi} \to h_U \times_G F$. By the Yoneda lemma, see Categories, Lemma 4.3.5, the projection $h_{V_\xi} \to h_U \times_G F \to h_U$ comes from a unique morphism of schemes $a_\xi : V_\xi \to U$. Suggestively we could represent this by the diagram

$$\begin{tikzcd} V_\xi \arrow[r, squiggly] \arrow[d, "a_\xi"'] & h_{V_\xi} \arrow[r] \arrow[d] & F \arrow[d, "a"] \\ U \arrow[r, squiggly] & h_U \arrow[r, "\xi"] & G \end{tikzcd}$$

where the squiggly arrows represent the Yoneda embedding. Here are some lemmas about this notion that work in great generality.

02W9 **Lemma 65.3.1.** Let S be a scheme contained in Sch_{fppf} and let X, Y be objects of $(Sch/S)_{fppf}$. Let $f : X \to Y$ be a morphism of schemes. Then

$$h_f : h_X \longrightarrow h_Y$$

is a representable transformation of functors.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. $\square$

02WA **Lemma 65.3.2.** Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. Then

$$b \circ a : F \longrightarrow H$$

is a representable transformation of functors.

Proof. This is entirely formal and works in any category. $\square$

02WB **Lemma 65.3.3.** Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$. Let $a : F \to G$ be a representable transformation of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

Then the base change a' is a representable transformation of functors.

Proof. This is entirely formal and works in any category. $\square$

02WC Lemma 65.3.4. Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to$ Sets, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Then

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable transformation of functors.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 65.3.2 and 65.3.3. $\square$

02WD Lemma 65.3.5. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to$ Sets. Let $a : F \to G$ be a representable transformation of functors. If G is a sheaf, then so is F.

Proof. Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma, G, a} F$ is a representable presheaf and hence (see remarks in Section 65.2) a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. $\square$

05L9 Lemma 65.3.6. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to$ Sets. Let $a : F \to G$ be a representable transformation of functors. Then $\Delta_{F/G} : F \to F \times_G F$ is representable.

Proof. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist a scheme V and a morphism $V \to U$ representing the fibre product $h_U \times_{\xi', G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $h_U \times_{\xi', G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $h_U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is a scheme. $\square$

65.4. Lists of useful properties of morphisms of schemes

02WE For ease of reference we list in the following remarks the properties of morphisms which possess some of the properties required of them in later results.

02WF Remark 65.4.1. Here is a list of properties/types of morphisms which are stable under arbitrary base change:

(1) closed, open, and locally closed immersions, see Schemes, Lemma 26.18.2,
(2) quasi-compact, see Schemes, Lemma 26.19.3,
(3) universally closed, see Schemes, Definition 26.20.1,
(4) (quasi-)separated, see Schemes, Lemma 26.21.12,
(5) monomorphism, see Schemes, Lemma 26.23.5
(6) surjective, see Morphisms, Lemma 29.9.4,
(7) universally injective, see Morphisms, Lemma 29.10.2,
(8) affine, see Morphisms, Lemma 29.11.8,

(9) quasi-affine, see Morphisms, Lemma 29.13.5,
(10) (locally) of finite type, see Morphisms, Lemma 29.15.4,
(11) (locally) quasi-finite, see Morphisms, Lemma 29.20.13,
(12) (locally) of finite presentation, see Morphisms, Lemma 29.21.4,
(13) locally of finite type of relative dimension d, see Morphisms, Lemma 29.29.2,
(14) universally open, see Morphisms, Definition 29.23.1,
(15) flat, see Morphisms, Lemma 29.25.8,
(16) syntomic, see Morphisms, Lemma 29.30.4,
(17) smooth, see Morphisms, Lemma 29.34.5,
(18) unramified (resp. G-unramified), see Morphisms, Lemma 29.35.5,
(19) étale, see Morphisms, Lemma 29.36.4,
(20) proper, see Morphisms, Lemma 29.41.5,
(21) H-projective, see Morphisms, Lemma 29.43.8,
(22) (locally) projective, see Morphisms, Lemma 29.43.9,
(23) finite or integral, see Morphisms, Lemma 29.44.6,
(24) finite locally free, see Morphisms, Lemma 29.48.4,
(25) universally submersive, see Morphisms, Lemma 29.24.2,
(26) universal homeomorphism, see Morphisms, Lemma 29.45.2.

Add more as needed.

02WG Remark 65.4.2. Of the properties of morphisms which are stable under base change (as listed in Remark 65.4.1) the following are also stable under compositions:

(1) closed, open and locally closed immersions, see Schemes, Lemma 26.24.3,
(2) quasi-compact, see Schemes, Lemma 26.19.4,
(3) universally closed, see Morphisms, Lemma 29.41.4,
(4) (quasi-)separated, see Schemes, Lemma 26.21.12,
(5) monomorphism, see Schemes, Lemma 26.23.4,
(6) surjective, see Morphisms, Lemma 29.9.2,
(7) universally injective, see Morphisms, Lemma 29.10.5,
(8) affine, see Morphisms, Lemma 29.11.7,
(9) quasi-affine, see Morphisms, Lemma 29.13.4,
(10) (locally) of finite type, see Morphisms, Lemma 29.15.3,
(11) (locally) quasi-finite, see Morphisms, Lemma 29.20.12,
(12) (locally) of finite presentation, see Morphisms, Lemma 29.21.3,
(13) universally open, see Morphisms, Lemma 29.23.3,
(14) flat, see Morphisms, Lemma 29.25.6,
(15) syntomic, see Morphisms, Lemma 29.30.3,
(16) smooth, see Morphisms, Lemma 29.34.4,
(17) unramified (resp. G-unramified), see Morphisms, Lemma 29.35.4,
(18) étale, see Morphisms, Lemma 29.36.3,
(19) proper, see Morphisms, Lemma 29.41.4,
(20) H-projective, see Morphisms, Lemma 29.43.7,
(21) finite or integral, see Morphisms, Lemma 29.44.5,
(22) finite locally free, see Morphisms, Lemma 29.48.3,
(23) universally submersive, see Morphisms, Lemma 29.24.3,
(24) universal homeomorphism, see Morphisms, Lemma 29.45.3.

Add more as needed.

02WH Remark 65.4.3. Of the properties mentioned which are stable under base change (as listed in Remark 65.4.1) the following are also fpqc local on the base (and a fortiori fppf local on the base):

(1) for immersions we have this for
   (a) closed immersions, see Descent, Lemma 35.23.19,
   (b) open immersions, see Descent, Lemma 35.23.16, and
   (c) quasi-compact immersions, see Descent, Lemma 35.23.21,
(2) quasi-compact, see Descent, Lemma 35.23.1,
(3) universally closed, see Descent, Lemma 35.23.3,
(4) (quasi-)separated, see Descent, Lemmas 35.23.2, and 35.23.6,
(5) monomorphism, see Descent, Lemma 35.23.31,
(6) surjective, see Descent, Lemma 35.23.7,
(7) universally injective, see Descent, Lemma 35.23.8,
(8) affine, see Descent, Lemma 35.23.18,
(9) quasi-affine, see Descent, Lemma 35.23.20,
(10) (locally) of finite type, see Descent, Lemmas 35.23.10, and 35.23.12,
(11) (locally) quasi-finite, see Descent, Lemma 35.23.24,
(12) (locally) of finite presentation, see Descent, Lemmas 35.23.11, and 35.23.13,
(13) locally of finite type of relative dimension d, see Descent, Lemma 35.23.25,
(14) universally open, see Descent, Lemma 35.23.4,
(15) flat, see Descent, Lemma 35.23.15,
(16) syntomic, see Descent, Lemma 35.23.26,
(17) smooth, see Descent, Lemma 35.23.27,
(18) unramified (resp. G-unramified), see Descent, Lemma 35.23.28,
(19) étale, see Descent, Lemma 35.23.29,
(20) proper, see Descent, Lemma 35.23.14,
(21) finite or integral, see Descent, Lemma 35.23.23,
(22) finite locally free, see Descent, Lemma 35.23.30,
(23) universally submersive, see Descent, Lemma 35.23.5,
(24) universal homeomorphism, see Descent, Lemma 35.23.9.

Note that the property of being an “immersion” may not be fpqc local on the base, but in Descent, Lemma 35.24.1 we proved that it is fppf local on the base.

65.5. Properties of representable morphisms of presheaves

02WI Here is the definition that makes this work.

025V Definition 65.5.1. With S, and $a : F \to G$ representable as above. Let $\mathcal{P}$ be a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 26.18.3, and
(2) is fppf local on the base, see Descent, Definition 35.22.1.

In this case we say that a has property $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the resulting morphism of schemes $V_\xi \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

02YN Remark 65.5.2. Consider the property $\mathcal{P}$ ="surjective". In this case there could be some ambiguity if we say "let $F \to G$ be a surjective map". Namely, we could mean the notion defined in Definition 65.5.1 above, or we could mean a surjective map of presheaves, see Sites, Definition 7.3.1, or, if both F and G are sheaves, we could mean a surjective map of sheaves, see Sites, Definition 7.11.1. If not mentioned otherwise when discussing morphisms of algebraic spaces we will always mean the first. See Lemma 65.5.9 for a case where surjectivity implies surjectivity as a map of sheaves.

Here is a sanity check.

02WJ Lemma 65.5.3. Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{P}$ be as in Definition 65.5.1. Then $h_X \longrightarrow h_Y$ has property $\mathcal{P}$ if and only if f has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 65.3.1. Proof omitted. $\square$

02WK Lemma 65.5.4. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 65.5.1 which is stable under composition. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. If a and b have property $\mathcal{P}$ so does $b \circ a : F \longrightarrow H$.

Proof. Note that the lemma makes sense by Lemma 65.3.2. Proof omitted. $\square$

02WL Lemma 65.5.5. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 65.5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

If a has property $\mathcal{P}$ then also the base change a' has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 65.3.3. Proof omitted. $\square$

03KD Lemma 65.5.6. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 65.5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

Assume that b induces a surjective map of fppf sheaves $H^\# \to G^\#$. In this case, if a' has property $\mathcal{P}$, then also a has property $\mathcal{P}$.

Proof. First we remark that by Lemma 65.3.3 the transformation a' is representable. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and let $\xi \in G(U)$. By assumption there exists an fppf covering $\{U_i \to U\}_{i \in I}$ and elements $\xi_i \in H(U_i)$ mapping to $\xi|_U$ via

b. From general category theory it follows that for each i we have a fibre product diagram

$$\begin{tikzcd} U_i \times_{\xi_i, H, a'} (H \times_{b,G,a} F) \arrow[r] \arrow[d] & U \times_{\xi, G, a} F \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd}$$

By assumption the left vertical arrow is a morphism of schemes which has property $\mathcal{P}$. Since $\mathcal{P}$ is local in the fppf topology this implies that also the right vertical arrow has property $\mathcal{P}$ as desired. □

02WM Lemma 65.5.7. Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Let $\mathcal{P}$ be a property as in Definition 65.5.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.

Proof. Note that the lemma makes sense by Lemma 65.3.4. Proof omitted. □

02YO Lemma 65.5.8. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 65.5.1. Suppose that for any morphism of schemes $f : X \to Y$ we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$ then a has property $\mathcal{P}'$.

Proof. Formal. □

05VM Lemma 65.5.9. Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be sheaves. Let $a : F \to G$ be representable, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is (representable by) a scheme and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective. Hence $\{T' \to T\}$ is an fppf covering such that $g|_{T'} \in G(T')$ comes from an element of $F(T')$, namely the map $T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 7.11.1. □

Here is a characterization of those functors for which the diagonal is representable.

025W Lemma 65.5.10. Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. The following are equivalent:

(1) the diagonal $F \to F \times F$ is representable,
(2) for $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $a \in F(U)$ the map $a : h_U \to F$ is representable,
(3) for every pair $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $a \in F(U)$, $b \in F(V)$ the fibre product $h_U \times_{a,F,b} h_V$ is representable.

Proof. This is completely formal, see Categories, Lemma 4.8.4. It depends only on the fact that the category $(Sch/S)_{fppf}$ has products of pairs of objects and fibre products, see Topologies, Lemma 34.7.10. □

In the situation of the lemma, for any morphism $\xi : h_U \to F$ as in the lemma, it makes sense to say that ξ has property $\mathcal{P}$, for any property as in Definition 65.5.1. In particular this holds for $\mathcal{P}$ = "surjective" and $\mathcal{P}$ = "étale", see Remark 65.4.3 above. We will use this remark in the definition of algebraic spaces below.

0CB7 **Lemma 65.5.11.** Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 65.5.1. If for every $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and $a \in F(U)$, $b \in F(V)$ we have

1. $h_U \times_{a,F,b} h_V$ is representable, say by the scheme W, and
2. the morphism $W \to U \times_S V$ corresponding to $h_U \times_{a,F,b} h_V \to h_U \times h_V$ has property $\mathcal{P}$,

then $\Delta : F \to F \times F$ is representable and has property $\mathcal{P}$.

Proof. Observe that Δ is representable by Lemma 65.5.10. We can formulate condition (2) as saying that the transformation $h_U \times_{a,F,b} h_V \to h_{U\times_S V}$ has property $\mathcal{P}$, see Lemma 65.5.3. Consider $T \in \mathrm{Ob}((Sch/S)_{fppf})$ and $(a, b) \in (F \times F)(T)$. Observe that we have the commutative diagram

$$\begin{tikzcd}
F \times_{\Delta, F\times F,(a,b)} h_T \arrow[r] \arrow[d] & h_T \arrow[d, "\Delta_{T/S}"] \\
h_T \times_{a,F,b} h_T \arrow[r] \arrow[d] & h_{T\times_S T} \arrow[d, "(a,b)"] \\
F \arrow[r, "\Delta"] & F \times F
\end{tikzcd}$$

both of whose squares are cartesian. In this way we see that the morphism $F \times_{F\times F} h_T \to h_T$ is the base change of a morphism having property $\mathcal{P}$ by $\Delta_{T/S}$. Since $\mathcal{P}$ is preserved under base change this finishes the proof. $\square$

65.6. Algebraic spaces

025X Here is the definition.

025Y **Definition 65.6.1.** Let S be a scheme contained in Sch_{fppf}. An *algebraic space over S* is a presheaf

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow \text{Sets}$$

with the following properties

1. The presheaf F is a sheaf.
2. The diagonal morphism $F \to F \times F$ is representable.
3. There exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a map $h_U \to F$ which is surjective, and étale.

There are two differences with the "usual" definition, for example the definition in Knutson's book [Knu71].

The first is that we require F to be a sheaf in the fppf topology. One reason for doing this is that many natural examples of algebraic spaces satisfy the sheaf condition for the fppf coverings (and even for fpqc coverings). Also, one of the reasons that algebraic spaces have been so useful is via Michael Artin's results on algebraic spaces. Built into his method is a condition which guarantees the result is locally of finite presentation over S. Combined it somehow seems to us that the fppf topology is the natural topology to work with. In the end the category of algebraic spaces ends up being the same. See Bootstrap, Section 80.12.

The second is that we only require the diagonal map for F to be representable, whereas in [Knu71] it is required that it also be quasi-compact. If $F = h_U$ for some scheme U over S this corresponds to the condition that U be quasi-separated. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of F be representable, and simply add an additional hypothesis wherever this is necessary. In any case it has the pleasing consequence that the following lemma is true.

025Z Lemma 65.6.2. A scheme is an algebraic space. More precisely, given a scheme $T \in \mathrm{Ob}((Sch/S)_{fppf})$ the representable functor h_T is an algebraic space.

Proof. The functor h_T is a sheaf by our remarks in Section 65.2. The diagonal $h_T \to h_T \times h_T = h_{T\times T}$ is representable because $(Sch/S)_{fppf}$ has fibre products. The identity map $h_T \to h_T$ is surjective étale. □

0260 Definition 65.6.3. Let F, F' be algebraic spaces over S. A morphism $f : F \to F'$ of algebraic spaces over S is a transformation of functors from F to F'.

The category of algebraic spaces over S contains the category $(Sch/S)_{fppf}$ as a full subcategory via the Yoneda embedding $T/S \mapsto h_T$. From now on we no longer distinguish between a scheme T/S and the algebraic space it represents. Thus when we say "Let $f : T \to F$ be a morphism from the scheme T to the algebraic space F", we mean that $T \in \mathrm{Ob}((Sch/S)_{fppf})$, that F is an algebraic space over S, and that $f : h_T \to F$ is a morphism of algebraic spaces over S.

65.7. Fibre products of algebraic spaces

04T8 The category of algebraic spaces over S has both products and fibre products.

02X0 Lemma 65.7.1. Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Then $F \times G$ is an algebraic space, and is a product in the category of algebraic spaces over S.

Proof. It is clear that $H = F\times G$ is a sheaf. The diagonal of H is simply the product of the diagonals of F and G. Hence it is representable by Lemma 65.3.4. Finally, if $U \to F$ and $V \to G$ are surjective étale morphisms, with $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, then $U \times V \to F \times G$ is surjective étale by Lemma 65.5.7. □

04T9 Lemma 65.7.2. Let S be a scheme contained in Sch_{fppf}. Let H be a sheaf on $(Sch/S)_{fppf}$ whose diagonal is representable. Let F, G be algebraic spaces over S. Let $F \to H$, $G \to H$ be maps of sheaves. Then $F \times_H G$ is an algebraic space.

Proof. We check the 3 conditions of Definition 65.6.1. A fibre product of sheaves is a sheaf, hence $F \times_H G$ is a sheaf. The diagonal of $F \times_H G$ is the left vertical arrow in

$$\begin{tikzcd} F \times_H G \arrow[r] \arrow[d, "\Delta"'] & F \times G \arrow[d, "\Delta_F\times\Delta_G"] \\ (F \times F) \times_{(H\times H)} (G \times G) \arrow[r] & (F \times F) \times (G \times G) \end{tikzcd}$$

which is cartesian. Hence Δ is representable as the base change of the morphism on the right which is representable, see Lemmas 65.3.4 and 65.3.3. Finally, let $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and $a : U \to F$, $b : V \to G$ be surjective and étale. As Δ_H is representable, we see that $U \times_H V$ is a scheme. The morphism

$$U \times_H V \longrightarrow F \times_H G$$

is surjective and étale as a composition of the base changes $U \times_H V \to U \times_H G$ and $U \times_H G \to F \times_H G$ of the étale surjective morphisms $U \to F$ and $V \to G$, see Lemmas 65.3.2 and 65.3.3. This proves the last condition of Definition 65.6.1 holds and we conclude that $F \times_H G$ is an algebraic space. $\square$

02X2 Lemma 65.7.3. Let S be a scheme contained in Sch_{fppf}. Let $F \to H$, $G \to H$ be morphisms of algebraic spaces over S. Then $F \times_H G$ is an algebraic space, and is a fibre product in the category of algebraic spaces over S.

Proof. It follows from the stronger Lemma 65.7.2 that $F \times_H G$ is an algebraic space. It is clear that $F \times_H G$ is a fibre product in the category of algebraic spaces over S since that is a full subcategory of the category of (pre)sheaves of sets on $(Sch/S)_{fppf}$. $\square$

65.8. Glueing algebraic spaces

02WN In this section we really start abusing notation and not distinguish between schemes and the spaces they represent.

0F15 Lemma 65.8.1. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F and G be sheaves on $(Sch/S)_{fppf}^{opp}$ and denote $F \amalg G$ the coproduct in the category of sheaves. The map $F \to F \amalg G$ is representable by open and closed immersions.

Proof. Let U be a scheme and let $\xi \in (F \amalg G)(U)$. Recall the coproduct in the category of sheaves is the sheafification of the coproduct presheaf (Sites, Lemma 7.10.13). Thus there exists an fppf covering $\{g_i : U_i \to U\}_{i \in I}$ and a disjoint union decomposition $I = I' \amalg I''$ such that $U_i \to U \to F \amalg G$ factors through F, resp. G if and only if $i \in I'$, resp. $i \in I''$. Since F and G have empty intersection in $F \amalg G$ we conclude that $U_i \times_U U_j$ is empty if $i \in I'$ and $j \in I''$. Hence $U' = \bigcup_{i \in I'} g_i(U_i)$ and $U'' = \bigcup_{i \in I''} g_i(U_i)$ are disjoint open (Morphisms, Lemma 29.25.10) subschemes of U with $U = U' \amalg U''$. We omit the verification that $U' = U \times_{F \amalg G} F$. $\square$

02WO Lemma 65.8.2. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Given a set I and sheaves F_i on $\mathrm{Ob}((Sch/S)_{fppf})$, if $U \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is representable by an open and closed subscheme U_i and $U \cong \coprod U_i$ as schemes.

Proof. By Lemma 65.8.1 the map $F_i \to U$ is representable by open and closed immersions. Hence F_i is representable by an open and closed subscheme U_i of U. We have $U = \coprod U_i$ because we have $U \cong \coprod F_i$ as sheaves and we can test the equality on points. $\square$

02WP Lemma 65.8.3. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be an algebraic space over S. Given a set I and sheaves F_i on $\mathrm{Ob}((Sch/S)_{fppf})$, if $F \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is an algebraic space over S.

Proof. The representability of $F \to F \times F$ implies that each diagonal morphism $F_i \to F_i \times F_i$ is representable (immediate from the definitions and the fact that $F \times_{(F \times F)} (F_i \times F_i) = F_i$). Choose a scheme U in $(Sch/S)_{fppf}$ and a surjective étale morphism $U \to F$ (this exist by hypothesis). The base change $U \times_F F_i \to F_i$ is surjective and étale by Lemma 65.5.5. On the other hand, $U \times_F F_i$ is a scheme by Lemma 65.8.1. Thus we have verified all the conditions in Definition 65.6.1 and F_i is an algebraic space. $\square$

The condition on the size of I and the F_i in the following lemma may be ignored by those not worried about set theoretic questions.

02WQ Lemma 65.8.4. Let $S \in \text{Ob}(Sch_{fppf})$. Suppose given a set I and algebraic spaces F_i, $i \in I$. Then $F = \coprod_{i \in I} F_i$ is an algebraic space provided I, and the F_i are not too "large": for example if we can choose surjective étale morphisms $U_i \to F_i$ such that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$, then F is an algebraic space.

Proof. By construction F is a sheaf. We omit the verification that the diagonal morphism of F is representable. Finally, if U is an object of $(Sch/S)_{fppf}$ isomorphic to $\coprod_{i \in I} U_i$ then it is straightforward to verify that the resulting map $U \to \coprod F_i$ is surjective and étale. $\square$

Here is the analogue of Schemes, Lemma 26.15.4.

02WR Lemma 65.8.5. Let $S \in \text{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume

(1) F is a sheaf,
(2) there exists an index set I and subfunctors $F_i \subset F$ such that
    (a) each F_i is an algebraic space,
    (b) each $F_i \to F$ is representable,
    (c) each $F_i \to F$ is an open immersion (see Definition 65.5.1),
    (d) the map $\coprod F_i \to F$ is surjective as a map of sheaves, and
    (e) $\coprod F_i$ is an algebraic space (set theoretic condition, see Lemma 65.8.4).

Then F is an algebraic space.

Proof. Let T be an object of $(Sch/S)_{fppf}$. Let $T \to F$ be a morphism. By assumption (2)(b) and (2)(c) the fibre product $F_i \times_F T$ is representable by an open subscheme $V_i \subset T$. It follows that $(\coprod F_i) \times_F T$ is represented by the scheme $\coprod V_i$ over T. By assumption (2)(d) there exists an fppf covering $\{T_j \to T\}_{j \in J}$ such that $T_j \to T \to F$ factors through F_i, $i = i(j)$. Hence $T_j \to T$ factors through the open subscheme $V_{i(j)} \subset T$. Since $\{T_j \to T\}$ is jointly surjective, it follows that $T = \bigcup V_i$ is an open covering. In particular, the transformation of functors $\coprod F_i \to F$ is representable and surjective in the sense of Definition 65.5.1 (see Remark 65.5.2 for a discussion).

Next, let $T' \to F$ be a second morphism from an object in $(Sch/S)_{fppf}$. Write as above $T' = \bigcup V'_i$ with $V'_i = T' \times_F F_i$. To show that the diagonal $F \to F \times F$ is representable we have to show that $G = T \times_F T'$ is representable, see Lemma 65.5.10. Consider the subfunctors $G_i = G \times_F F_i$. Note that $G_i = V_i \times_{F_i} V'_i$, and hence is representable as F_i is an algebraic space. By the above the G_i form a Zariski covering of G. Hence by Schemes, Lemma 26.15.4 we see G is representable.

Choose a scheme $U \in \text{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to \coprod F_i$ (this exists by hypothesis). We may write $U = \coprod U_i$ with U_i the inverse image of F_i, see Lemma 65.8.2. We claim that $U \to F$ is surjective and étale. Surjectivity follows as $\coprod F_i \to F$ is surjective (see first paragraph of the proof) by applying Lemma 65.5.4. Consider the fibre product $U \times_F T$ where $T \to F$ is as above. We have to show that $U \times_F T \to T$ is étale. Since $U \times_F T = \coprod U_i \times_F T$ it suffices to show each $U_i \times_F T \to T$ is étale. Since $U_i \times_F T = U_i \times_{F_i} V_i$ this follows from the fact that $U_i \to F_i$ is étale and $V_i \to T$ is an open immersion (and Morphisms, Lemmas 29.36.9 and 29.36.3). $\square$

65.9. Presentations of algebraic spaces

0261 Given an algebraic space we can find a "presentation" of it.

0262 Lemma 65.9.1. Let F be an algebraic space over S. Let $f : U \to F$ be a surjective étale morphism from a scheme to F. Set $R = U \times_F U$. Then

(1) $j : R \to U \times_S U$ defines an equivalence relation on U over S (see Groupoids, Definition 39.3.1).
(2) the morphisms $s, t : R \to U$ are étale, and
(3) the diagram
$$R \rightrightarrows U \longrightarrow F$$
is a coequalizer diagram in $Sh((Sch/S)_{fppf})$.

Proof. Let T/S be an object of $(Sch/S)_{fppf}$. Then $R(T) = \{(a, b) \in U(T) \times U(T) \mid f \circ a = f \circ b\}$ which defines an equivalence relation on $U(T)$. The morphisms $s, t : R \to U$ are étale because the morphism $U \to F$ is étale.

To prove (3) we first show that $U \to F$ is a surjection of sheaves, see Sites, Definition 7.11.1. Let $\xi \in F(T)$ with T as above. Let $V = T \times_{\xi, F, f} U$. By assumption V is a scheme and $V \to T$ is surjective étale. Hence $\{V \to T\}$ is a covering for the fppf topology. Since $\xi|_V$ factors through U by construction we conclude $U \to F$ is surjective. Surjectivity implies that F is the coequalizer of the diagram by Sites, Lemma 7.11.3. $\square$

This lemma suggests the following definitions.

02WS Definition 65.9.2. Let S be a scheme. Let U be a scheme over S. An étale equivalence relation on U over S is an equivalence relation $j : R \to U \times_S U$ such that $s, t : R \to U$ are étale morphisms of schemes.

0263 Definition 65.9.3. Let F be an algebraic space over S. A presentation of F is given by a scheme U over S and an étale equivalence relation R on U over S, and a surjective étale morphism $U \to F$ such that $R = U \times_F U$.

Equivalently we could ask for the existence of an isomorphism
$$U/R \cong F$$
where the quotient U/R is as defined in Groupoids, Section 39.20. To construct algebraic spaces we will study the converse question, namely, for which equivalence relations the quotient sheaf U/R is an algebraic space. It will finally turn out this is always the case if R is an étale equivalence relation on U over S, see Theorem 65.10.5.

65.10. Algebraic spaces and equivalence relations

0264 Suppose given a scheme U over S and an étale equivalence relation R on U over S. We would like to show this defines an algebraic space. We will produce a series of lemmas that prove the quotient sheaf U/R (see Groupoids, Definition 39.20.1) has all the properties required of it in Definition 65.6.1.

02WT Lemma 65.10.1. Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Let $U' \to U$ be an étale morphism. Let R' be the restriction of R to U', see Groupoids, Definition 39.3.3. Then $j' : R' \to U' \times_S U'$ is an étale equivalence relation also.

Proof. It is clear from the description of s', t' in Groupoids, Lemma 39.18.1 that $s', t' : R' \to U'$ are étale as compositions of base changes of étale morphisms (see Morphisms, Lemma 29.36.4 and 29.36.3). $\square$

We will often use the following lemma to find open subspaces of algebraic spaces. A slight improvement (with more general hypotheses) of this lemma is Bootstrap, Lemma 80.7.1.

02WU Lemma 65.10.2. Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism. Assume

(1) j is an equivalence relation,
(2) $s, t : R \to U$ are surjective, flat and locally of finite presentation,
(3) g is flat and locally of finite presentation.

Let $R' = R|_{U'}$ be the restriction of R to U'. Then $U'/R' \to U/R$ is representable, and is an open immersion.

Proof. By Groupoids, Lemma 39.3.2 the morphism $j' = (s', t') : R' \to U' \times_S U'$ defines an equivalence relation. Since g is flat and locally of finite presentation we see that g is universally open as well (Morphisms, Lemma 29.25.10). For the same reason s, t are universally open as well. Let $W^1 = g(U') \subset U$, and let $W = t(s^{-1}(W^1))$. Then W^1 and W are open in U. Moreover, as j is an equivalence relation we have $t(s^{-1}(W)) = W$ (see Groupoids, Lemma 39.19.2 for example).

By Groupoids, Lemma 39.20.5 the map of sheaves $F' = U'/R' \to F = U/R$ is injective. Let $a : T \to F$ be a morphism from a scheme into U/R. We have to show that $T \times_F F'$ is representable by an open subscheme of T.

The morphism a is given by the following data: an fppf covering $\{\varphi_j : T_j \to T\}_{j \in J}$ of T and morphisms $a_j : T_j \to U$ such that the maps

$$a_j \times a_{j'} : T_j \times_T T_{j'} \longrightarrow U \times_S U$$

factor through $j : R \to U \times_S U$ via some (unique) maps $r_{jj'} : T_j \times_T T_{j'} \to R$. The system (a_j) corresponds to a in the sense that the diagrams

$$\begin{tikzcd} T_j \arrow[r, "a_j"] \arrow[d] & U \arrow[d] \\ T \arrow[r, "a"] & F \end{tikzcd}$$

commute.

Consider the open subsets $W_j = a_j^{-1}(W) \subset T_j$. Since $t(s^{-1}(W)) = W$ we see that

$$W_j \times_T T_{j'} = r_{jj'}^{-1}(t^{-1}(W)) = r_{jj'}^{-1}(s^{-1}(W)) = T_j \times_T W_{j'}.$$

By Descent, Lemma 35.13.6 this means there exists an open $W_T \subset T$ such that $\varphi_j^{-1}(W_T) = W_j$ for all $j \in J$. We claim that $W_T \to T$ represents $T \times_F F' \to T$.

First, let us show that $W_T \to T \to F$ is an element of $F'(W_T)$. Since $\{W_j \to W_T\}_{j \in J}$ is an fppf covering of W_T, it is enough to show that each $W_j \to U \to F$ is an element of $F'(W_j)$ (as F' is a sheaf for the fppf topology). Consider the

commutative diagram

$$\begin{tikzcd}
W'_j \arrow[rr] \arrow[dd] \arrow[dr] & & U' \arrow[d, "g"] \\
& s^{-1}(W^1) \arrow[r, "s"'] \arrow[d, "t"] & W^1 \arrow[d] \\
W_j \arrow[r, "a_j|_{W_j}"] & W \arrow[r] & F
\end{tikzcd}$$

where $W'_j = W_j \times_W s^{-1}(W^1) \times_{W^1} U'$. Since t and g are surjective, flat and locally of finite presentation, so is $W'_j \to W_j$. Hence the restriction of the element $W_j \to U \to F$ to W'_j is an element of F' as desired.

Suppose that $f : T' \to T$ is a morphism of schemes such that $a|_{T'} \in F'(T')$. We have to show that f factors through the open W_T. Since $\{T' \times_T T_j \to T'\}$ is an fppf covering of T' it is enough to show each $T' \times_T T_j \to T$ factors through W_T. Hence we may assume f factors as $\varphi_j \circ f_j : T' \to T_j \to T$ for some j. In this case the condition $a|_{T'} \in F'(T')$ means that there exists some fppf covering $\{\psi_i : T'_i \to T'\}_{i \in I}$ and some morphisms $b_i : T'_i \to U'$ such that

$$\begin{tikzcd}
T'_i \arrow[r, "b_i"'] \arrow[d, "f_j \circ \psi_i"'] & U' \arrow[r, "g"'] & U \arrow[d] \\
T_j \arrow[r, "a_j"] & U \arrow[r] & F
\end{tikzcd}$$

is commutative. This commutativity means that there exists a morphism $r'_i : T'_i \to R$ such that $t \circ r'_i = a_j \circ f_j \circ \psi_i$, and $s \circ r'_i = g \circ b_i$. This implies that $\text{Im}(f_j \circ \psi_i) \subset W_j$ and we win. □

The following lemma is not completely trivial although it looks like it should be trivial.

02WV Lemma 65.10.3. Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. If the quotient U/R is an algebraic space, then $U \to U/R$ is étale and surjective. Hence $(U, R, U \to U/R)$ is a presentation of the algebraic space U/R.

Proof. Denote $c : U \to U/R$ the morphism in question. Let T be a scheme and let $a : T \to U/R$ be a morphism. We have to show that the morphism (of schemes) $\pi : T \times_{a,U/R,c} U \to T$ is étale and surjective. The morphism a corresponds to an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = a \circ \varphi_i$. Hence

$$T_i \times_{\varphi_i,T} T \times_{a,U/R,c} U = T_i \times_{c\circ a_i,U/R,c} U = T_i \times_{a_i,U} U \times_{c,U/R,c} U = T_i \times_{a_i,U,t} R.$$

Since t is étale and surjective we conclude that the base change of π to T_i is surjective and étale. Since the property of being surjective and étale is local on the base in the fpqc topology (see Remark 65.4.3) we win. □

0265 Lemma 65.10.4. Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Assume that U is affine. Then the quotient $F = U/R$ is an algebraic space, and $U \to F$ is étale and surjective.

Proof. Since $j : R \to U \times_S U$ is a monomorphism we see that j is separated (see Schemes, Lemma 26.23.3). Since U is affine we see that $U \times_S U$ (which comes equipped with a monomorphism into the affine scheme $U \times U$) is separated. Hence we see that R is separated. In particular the morphisms s, t are separated as well as étale.

Since the composition $R \to U \times_S U \to U$ is locally of finite type we conclude that j is locally of finite type (see Morphisms, Lemma 29.15.8). As j is also a monomorphism it has finite fibres and we see that j is locally quasi-finite by Morphisms, Lemma 29.20.7. Altogether we see that j is separated and locally quasi-finite.

Our first step is to show that the quotient map $c : U \to F$ is representable. Consider a scheme T and a morphism $a : T \to F$. We have to show that the sheaf $G = T \times_{a,F,c} U$ is representable. As seen in the proofs of Lemmas 65.10.2 and 65.10.3 there exists an fppf covering $\{\varphi_i : T_i \to T\}_{i \in I}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = a \circ \varphi_i$. As in the proof of Lemma 65.10.3 we see that

$$\begin{aligned} T_i \times_{\varphi_i,T} G &= T_i \times_{\varphi_i,T} T \times_{a,U/R,c} U \\ &= T_i \times_{c \circ a_i,U/R,c} U \\ &= T_i \times_{a_i,U} U \times_{c,U/R,c} U \\ &= T_i \times_{a_i,U,t} R \end{aligned}$$

Since t is separated and étale, and in particular separated and locally quasi-finite (by Morphisms, Lemmas 29.35.10 and 29.36.16) we see that the restriction of G to each T_i is representable by a morphism of schemes $X_i \to T_i$ which is separated and locally quasi-finite. By Descent, Lemma 35.39.1 we obtain a descent datum $(X_i, \varphi_{ii'})$ relative to the fppf-covering $\{T_i \to T\}$. Since each $X_i \to T_i$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 37.57.1 that this descent datum is effective. Hence by Descent, Lemma 35.39.1 (2) we conclude that G is representable as desired.

The second step of the proof is to show that $U \to F$ is surjective and étale. This is clear from the above since in the first step above we saw that $G = T \times_{a,F,c} U$ is a scheme over T which base changes to schemes $X_i \to T_i$ which are surjective and étale. Thus $G \to T$ is surjective and étale (see Remark 65.4.3). Alternatively one can reread the proof of Lemma 65.10.3 in the current situation.

The third and final step is to show that the diagonal map $F \to F \times F$ is representable. We first observe that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & F \arrow[d, "\Delta"] \\ U \times_S U \arrow[r] & F \times F \end{tikzcd}$$

is a fibre product square. By Lemma 65.3.4 the morphism $U \times_S U \to F \times F$ is representable (note that $h_U \times h_U = h_{U \times_S U}$). Moreover, by Lemma 65.5.7 the morphism $U \times_S U \to F \times F$ is surjective and étale (note also that étale and surjective occur in the lists of Remarks 65.4.3 and 65.4.2). It follows either from

Lemma 65.3.3 and the diagram above, or by writing $R \to F$ as $R \to U \to F$ and Lemmas 65.3.1 and 65.3.2 that $R \to F$ is representable as well. Let T be a scheme and let $a : T \to F \times F$ be a morphism. We have to show that $G = T \times_{a, F \times F, \Delta} F$ is representable. By what was said above the morphism (of schemes)

$$T' = (U \times_S U) \times_{F \times F, a} T \longrightarrow T$$

is surjective and étale. Hence $\{T' \to T\}$ is an étale covering of T. Note also that

$$T' \times_T G = T' \times_{U \times_S U, j} R$$

as can be seen contemplating the following cube

$$\begin{tikzcd}
 & R \arrow[rr] \arrow[dd] & & F \arrow[dd] \\
T' \times_T G \arrow[ur] \arrow[rr] \arrow[dd] & & G \arrow[ur] \arrow[dd] & \\
 & U \times_S U \arrow[rr] & & F \times F \\
T' \arrow[ur] \arrow[rr] & & T \arrow[ur] &
\end{tikzcd}$$

Hence we see that the restriction of G to T' is representable by a scheme X, and moreover that the morphism $X \to T'$ is a base change of the morphism j. Hence $X \to T'$ is separated and locally quasi-finite (see second paragraph of the proof). By Descent, Lemma 35.39.1 we obtain a descent datum (X, φ) relative to the fppf-covering $\{T' \to T\}$. Since $X \to T'$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 37.57.1 that this descent datum is effective. Hence by Descent, Lemma 35.39.1 (2) we conclude that G is representable as desired. $\square$

02WW Theorem 65.10.5. Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Then the quotient U/R is an algebraic space, and $U \to U/R$ is étale and surjective, in other words $(U, R, U \to U/R)$ is a presentation of U/R.

Proof. By Lemma 65.10.3 it suffices to prove that U/R is an algebraic space. Let $U' \to U$ be a surjective, étale morphism. Then $\{U' \to U\}$ is in particular an fppf covering. Let R' be the restriction of R to U', see Groupoids, Definition 39.3.3. According to Groupoids, Lemma 39.20.6 we see that $U/R \cong U'/R'$. By Lemma 65.10.1 R' is an étale equivalence relation on U'. Thus we may replace U by U'.

We apply the previous remark to $U' = \coprod U_i$, where $U = \bigcup U_i$ is an affine open covering of U. Hence we may and do assume that $U = \coprod U_i$ where each U_i is an affine scheme.

Consider the restriction R_i of R to U_i. By Lemma 65.10.1 this is an étale equivalence relation. Set $F_i = U_i/R_i$ and $F = U/R$. It is clear that $\coprod F_i \to F$ is surjective. By Lemma 65.10.2 each $F_i \to F$ is representable, and an open immersion. By Lemma 65.10.4 applied to (U_i, R_i) we see that F_i is an algebraic space. Then by Lemma 65.10.3 we see that $U_i \to F_i$ is étale and surjective. From Lemma 65.8.4 it follows that $\coprod F_i$ is an algebraic space. Finally, we have verified all hypotheses of Lemma 65.8.5 and it follows that $F = U/R$ is an algebraic space. $\square$

65.11. Algebraic spaces, retrofitted

02WX We start building our arsenal of lemmas dealing with algebraic spaces. The first result says that in Definition 65.6.1 we can weaken the condition on the diagonal as follows.

0BGQ **Lemma 65.11.1.** *Let S be a scheme contained in Sch_{fppf}. Let F be a sheaf on $(Sch/S)_{fppf}$ such that there exists $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a map $U \to F$ which is representable, surjective, and étale. Then F is an algebraic space.*

Proof. Set $R = U \times_F U$. This is a scheme as $U \to F$ is assumed representable. The projections $s, t : R \to U$ are étale as $U \to F$ is assumed étale. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 65.10.5 the quotient sheaf $F' = U/R$ is an algebraic space and $U \to F'$ is surjective and étale. Again since $R = U \times_F U$ we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of sheaves by Lemma 65.5.9. Thus $F' \to F$ is also surjective and we conclude $F' = F$ is an algebraic space. $\square$

0BGR **Lemma 65.11.2.** *Let S be a scheme contained in Sch_{fppf}. Let G be an algebraic space over S, let F be a sheaf on $(Sch/S)_{fppf}$, and let $G \to F$ be a representable transformation of functors which is surjective and étale. Then F is an algebraic space.*

Proof. Pick a scheme U and a surjective étale morphism $U \to G$. Since G is an algebraic space $U \to G$ is representable. Hence the composition $U \to G \to F$ is representable, surjective, and étale. See Lemmas 65.3.2 and 65.5.4. Thus F is an algebraic space by Lemma 65.11.1. $\square$

02WY **Lemma 65.11.3.** *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $G \to F$ be a representable transformation of functors. Then G is an algebraic space.*

Proof. By Lemma 65.3.5 we see that G is a sheaf. The diagram

$$\begin{tikzcd} G \times_F G \arrow[r] \arrow[d] & F \arrow[d, "\Delta_F"] \\ G \times G \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. Hence we see that $G \times_F G \to G \times G$ is representable by Lemma 65.3.3. By Lemma 65.3.6 we see that $G \to G \times_F G$ is representable. Hence $\Delta_G : G \to G \times G$ is representable as a composition of representable transformations, see Lemma 65.3.2. Finally, let U be an object of $(Sch/S)_{fppf}$ and let $U \to F$ be surjective and étale. By assumption $U \times_F G$ is representable by a scheme U'. By Lemma 65.5.5 the morphism $U' \to G$ is surjective and étale. This verifies the final condition of Definition 65.6.1 and we win. $\square$

02WZ **Lemma 65.11.4.** *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $G \to F$ be a representable morphism. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and $q : U \to F$ surjective and étale. Set $V = G \times_F U$. Finally, let $\mathcal{P}$ be a property of morphisms of schemes as in Definition 65.5.1. Then $G \to F$ has property $\mathcal{P}$ if and only if $V \to U$ has property $\mathcal{P}$.*

Proof. (This lemma follows from Lemmas 65.5.5 and 65.5.6, but we give a direct proof here also.) It is clear from the definitions that if $G \to F$ has property $\mathcal{P}$, then $V \to U$ has property $\mathcal{P}$. Conversely, assume $V \to U$ has property $\mathcal{P}$. Let $T \to F$ be a morphism from a scheme to F. Let $T' = T \times_F G$ which is a scheme since $G \to F$ is representable. We have to show that $T' \to T$ has property $\mathcal{P}$. Consider the commutative diagram of schemes

$$\begin{tikzcd} V \arrow[d] & T \times_F V \arrow[l] \arrow[r] \arrow[d] & T \times_F G \arrow[r, equal] \arrow[d] & T' \\ U & T \times_F U \arrow[l] \arrow[r] & T \end{tikzcd}$$

where both squares are fibre product squares. Hence we conclude the middle arrow has property $\mathcal{P}$ as a base change of $V \to U$. Finally, $\{T \times_F U \to T\}$ is a fppf covering as it is surjective étale, and hence we conclude that $T' \to T$ has property $\mathcal{P}$ as it is local on the base in the fppf topology. $\square$

03I2 Lemma 65.11.5. Let S be a scheme contained in Sch_{fppf}. Let $G \to F$ be a transformation of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of schemes. Assume

(1) $\mathcal{P}$ is preserved under any base change, fppf local on the base, and morphisms of type $\mathcal{P}$ satisfy descent for fppf coverings, see Descent, Definition 35.36.1,
(2) G is a sheaf,
(3) F is an algebraic space,
(4) there exists a $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$ such that $V = G \times_F U$ is representable, and
(5) $V \to U$ has $\mathcal{P}$.

Then G is an algebraic space, $G \to F$ is representable and has property $\mathcal{P}$.

Proof. Let $R = U \times_F U$, and denote $t, s : R \to U$ the projection morphisms as usual. Let T be a scheme and let $T \to F$ be a morphism. Then $U \times_F T \to T$ is surjective étale, hence $\{U \times_F T \to T\}$ is a covering for the étale topology. Consider

$$W = G \times_F (U \times_F T) = V \times_F T = V \times_U (U \times_F T).$$

It is a scheme since F is an algebraic space. The morphism $W \to U \times_F T$ has property $\mathcal{P}$ since it is a base change of $V \to U$. There is an isomorphism

$$\begin{aligned} W \times_T (U \times_F T) &= (G \times_F (U \times_F T)) \times_T (U \times_F T) \\ &= (U \times_F T) \times_T (G \times_F (U \times_F T)) \\ &= (U \times_F T) \times_T W \end{aligned}$$

over $(U \times_F T) \times_T (U \times_F T)$. The middle equality maps $((g, (u_1, t)), (u_2, t))$ to $((u_1, t), (g, (u_2, t)))$. This defines a descent datum for $W/U \times_F T/T$, see Descent, Definition 35.34.1. This follows from Descent, Lemma 35.39.1. Namely we have a sheaf $G \times_F T$, whose base change to $U \times_F T$ is represented by W and the isomorphism above is the one from the proof of Descent, Lemma 35.39.1. By assumption on $\mathcal{P}$ the descent datum above is representable. Hence by the last statement of Descent, Lemma 35.39.1 we see that $G \times_F T$ is representable. This proves that $G \to F$ is a representable transformation of functors.

As $G \to F$ is representable, we see that G is an algebraic space by Lemma 65.11.3. The fact that $G \to F$ has property $\mathcal{P}$ now follows from Lemma 65.11.4. $\square$

02X1 Lemma 65.11.6. Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $a : F \to G$ be a morphism. Given any $V \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $q : V \to G$ there exists a $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ F \arrow[r, "a"] & G \end{tikzcd}$$

with p surjective and étale.

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ with surjective étale morphism $W \to F$. Next, put $U = W \times_G V$. Since G is an algebraic space we see that U is isomorphic to an object of $(Sch/S)_{fppf}$. As q is surjective étale, we see that $U \to W$ is surjective étale (see Lemma 65.5.5). Thus $U \to F$ is surjective étale as a composition of surjective étale morphisms (see Lemma 65.5.4). $\square$

65.12. Immersions and Zariski coverings of algebraic spaces

02YT At this point an interesting phenomenon occurs. We have already defined the notion of an open immersion of algebraic spaces (through Definition 65.5.1) but we have yet to define the notion of a point[1]. Thus the Zariski topology of an algebraic space has already been defined, but there is no space yet!

Perhaps superfluously we formally introduce immersions as follows.

02YU Definition 65.12.1. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S.

(1) A morphism of algebraic spaces over S is called an open immersion if it is representable, and an open immersion in the sense of Definition 65.5.1.
(2) An open subspace of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an open immersion.
(3) A morphism of algebraic spaces over S is called a closed immersion if it is representable, and a closed immersion in the sense of Definition 65.5.1.
(4) A closed subspace of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is a closed immersion.
(5) A morphism of algebraic spaces over S is called an immersion if it is representable, and an immersion in the sense of Definition 65.5.1.
(6) A locally closed subspace of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an immersion.

We note that these definitions make sense since an immersion is in particular a monomorphism (see Schemes, Lemma 26.23.8 and Lemma 65.5.8), and hence the image of an immersion $G \to F$ of algebraic spaces is a subfunctor $F' \subset F$ which is (canonically) isomorphic to G. Thus some of the discussion of Schemes, Section 26.10 carries over to the setting of algebraic spaces.

[1]We will associate a topological space to an algebraic space in Properties of Spaces, Section 66.4, and its opens will correspond exactly to the open subspaces defined below.

02YV Lemma 65.12.2. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. A composition of (closed, resp. open) immersions of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.

Proof. See Lemma 65.5.4 and Remarks 65.4.3 (see very last line of that remark) and 65.4.2. $\square$

02YW Lemma 65.12.3. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. A base change of a (closed, resp. open) immersion of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.

Proof. See Lemma 65.5.5 and Remark 65.4.3 (see very last line of that remark). $\square$

02YX Lemma 65.12.4. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. Let F_1, F_2 be locally closed subspaces of F. If $F_1 \subset F_2$ as subfunctors of F, then F_1 is a locally closed subspace of F_2. Similarly for closed and open subspaces.

Proof. Let $T \to F_2$ be a morphism with T a scheme. Since $F_2 \to F$ is a monomorphism, we see that $T \times_{F_2} F_1 = T \times_F F_1$. The lemma follows formally from this. $\square$

Let us formally define the notion of a Zariski open covering of algebraic spaces. Note that in Lemma 65.8.5 we have already encountered such open coverings as a method for constructing algebraic spaces.

02YY Definition 65.12.5. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A Zariski covering $\{F_i \subset F\}_{i \in I}$ of F is given by a set I and a collection of open subspaces $F_i \subset F$ such that $\coprod F_i \to F$ is a surjective map of sheaves.

Note that if T is a schemes, and $a : T \to F$ is a morphism, then each of the fibre products $T \times_F F_i$ is identified with an open subscheme $T_i \subset T$. The final condition of the definition signifies exactly that $T = \bigcup_{i \in I} T_i$.

It is clear that the collection F_{Zar} of open subspaces of F is a set (as $(Sch/S)_{fppf}$ is a site, hence a set). Moreover, we can turn F_{Zar} into a category by letting the morphisms be inclusions of subfunctors (which are automatically open immersions by Lemma 65.12.4). Finally, Definition 65.12.5 provides the notion of a Zariski covering $\{F_i \to F'\}_{i \in I}$ in the category F_{Zar}. Hence, just as in the case of a topological space (see Sites, Example 7.6.4) by suitably choosing a set of coverings we may obtain a Zariski site of the algebraic space F.

02YZ Definition 65.12.6. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A small Zariski site F_{Zar} of an algebraic space F is one of the sites described above.

Hence this gives a notion of what it means for something to be true Zariski locally on an algebraic space, which is how we will use this notion. In general the Zariski topology is not fine enough for our purposes. For example we can consider the category of Zariski sheaves on an algebraic space. It will turn out that this is not the correct thing to consider, even for quasi-coherent sheaves. One only gets the desired result when using the étale or fppf site of F to define quasi-coherent sheaves.

65.13. Separation conditions on algebraic spaces

02X3 A separation condition on an algebraic space F is a condition on the diagonal morphism $F \to F \times F$. Let us first list the properties the diagonal has automatically. Since the diagonal is representable by definition the following lemma makes sense (through Definition 65.5.1).

02X4 **Lemma 65.13.1.** Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism. Then

(1) Δ is locally of finite type,
(2) Δ is a monomorphism,
(3) Δ is separated, and
(4) Δ is locally quasi-finite.

Proof. Let $F = U/R$ be a presentation of F. As in the proof of Lemma 65.10.4 the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & F \arrow[d, "\Delta"] \\ U \times_S U \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. Hence according to Lemma 65.11.4 it suffices to show that j has the properties listed in the lemma. (Note that each of the properties (1) – (4) occur in the lists of Remarks 65.4.1 and 65.4.3.) Since j is an equivalence relation it is a monomorphism. Hence it is separated by Schemes, Lemma 26.23.3. As R is an étale equivalence relation we see that $s, t : R \to U$ are étale. Hence s, t are locally of finite type. Then it follows from Morphisms, Lemma 29.15.8 that j is locally of finite type. Finally, as it is a monomorphism its fibres are finite. Thus we conclude that it is locally quasi-finite by Morphisms, Lemma 29.20.7. □

Here are some common types of separation conditions, relative to the base scheme S. There is also an absolute notion of these conditions which we will discuss in Properties of Spaces, Section 66.3. Moreover, we will discuss separation conditions for a morphism of algebraic spaces in Morphisms of Spaces, Section 67.4.

02X5 **Definition 65.13.2.** Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism.

(1) We say F is *separated over S* if Δ is a closed immersion.
(2) We say F is *locally separated over S*[2] if Δ is an immersion.
(3) We say F is *quasi-separated over S* if Δ is quasi-compact.
(4) We say F is *Zariski locally quasi-separated over S*[3] if there exists a Zariski covering $F = \bigcup_{i \in I} F_i$ such that each F_i is quasi-separated.

Note that if the diagonal is quasi-compact (when F is separated or quasi-separated) then the diagonal is actually quasi-finite and separated, hence quasi-affine (by More on Morphisms, Lemma 37.43.2).

[2]In the literature this often refers to quasi-separated and locally separated algebraic spaces.
[3]This definition was suggested by B. Conrad.

65.14. Examples of algebraic spaces

02Z0 In this section we construct some examples of algebraic spaces. Some of these were suggested by B. Conrad. Since we do not yet have a lot of theory at our disposal the discussion is a bit awkward in some places.

02Z1 **Example 65.14.1.** Let k be a field of characteristic $\neq 2$. Let $U = \mathbf{A}^1_k$. Set

$$j : R = \Delta \amalg \Gamma \longrightarrow U \times_k U$$

where $\Delta = \{(x, x) \mid x \in \mathbf{A}^1_k\}$ and $\Gamma = \{(x, -x) \mid x \in \mathbf{A}^1_k, x \neq 0\}$. It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space by Theorem 65.10.5. Since R is quasi-compact we see that X is quasi-separated. On the other hand, X is not locally separated because the morphism j is not an immersion.

03FN **Example 65.14.2.** Let k be a field. Let k'/k be a degree 2 Galois extension with $\text{Gal}(k'/k) = \{1, \sigma\}$. Let $S = \text{Spec}(k[x])$ and $U = \text{Spec}(k'[x])$. Note that

$$U \times_S U = \text{Spec}((k' \otimes_k k')[x]) = \Delta(U) \amalg \Delta'(U)$$

where $\Delta' = (1, \sigma) : U \to U \times_S U$. Take

$$R = \Delta(U) \amalg \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ denotes the k'-rational point whose x-coordinate is zero. It is easy to see that R is an étale equivalence relation on U over S and hence $X = U/R$ is an algebraic space by Theorem 65.10.5. Here are some properties of X (some of which will not make sense until later):

1. $X \to S$ is an isomorphism over $S \setminus \{0_S\}$,
2. the morphism $X \to S$ is étale (see Properties of Spaces, Definition 66.16.2)
3. the fibre 0_X of $X \to S$ over 0_S is isomorphic to $\text{Spec}(k') = 0_U$,
4. X is not a scheme because if it were, then $\mathcal{O}_{X,0_X}$ would be a local domain $(\mathcal{O}, \mathfrak{m}, \kappa)$ with fraction field $k(x)$, with $x \in \mathfrak{m}$ and residue field $\kappa = k'$ which is impossible,
5. X is not separated, but it is locally separated and quasi-separated,
6. there exists a surjective, finite, étale morphism $S' \to S$ such that the base change $X' = S' \times_S X$ is a scheme (namely, if we base change to $S' = \text{Spec}(k'[x])$ then U splits into two copies of S' and X' becomes isomorphic to the affine line with 0 doubled, see Schemes, Example 26.14.3), and
7. if we think of X as a finite type algebraic space over $\text{Spec}(k)$, then similarly the base change $X_{k'}$ is a scheme but X is not a scheme.

In particular, this gives an example of a descent datum for schemes relative to the covering $\{\text{Spec}(k') \to \text{Spec}(k)\}$ which is not effective.

See also Examples, Lemma 110.65.1, which shows that descent data need not be effective even for a projective morphism of schemes. That example gives a smooth separated algebraic space of dimension 3 over $\mathbf{C}$ which is not a scheme.

We will use the following lemma as a convenient way to construct algebraic spaces as quotients of schemes by free group actions.

02Z2 **Lemma 65.14.3.** Let $U \to S$ be a morphism of Sch_{fppf}. Let G be an abstract group. Let $G \to \text{Aut}_S(U)$ be a group homomorphism. Assume

(*) if $u \in U$ is a point, and $g(u) = u$ for some non-identity element $g \in G$, then g induces a nontrivial automorphism of $\kappa(u)$.

Then

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is an étale equivalence relation and hence

$$F = U/R$$

is an algebraic space by Theorem 65.10.5.

Proof. In the statement of the lemma the symbol $\mathrm{Aut}_S(U)$ denotes the group of automorphisms of U over S. Assume $(*)$ holds. Let us show that

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is a monomorphism. This signifies that if T is a nonempty scheme, and $h : T \to U$ is a T-valued point such that $g \circ h = g' \circ h$ then $g = g'$. Suppose $T \neq \emptyset$, $h : T \to U$ and $g \circ h = g' \circ h$. Let $t \in T$. Consider the composition $\mathrm{Spec}(\kappa(t)) \to \mathrm{Spec}(\kappa(h(t))) \to U$. Then we conclude that $g^{-1} \circ g'$ fixes $u = h(t)$ and acts as the identity on its residue field. Hence $g = g'$ by $(*)$.

Thus if $(*)$ holds we see that j is a relation (see Groupoids, Definition 39.3.1). Moreover, it is an equivalence relation since on T-valued points for a connected scheme T we see that $R(T) = G \times U(T) \to U(T) \times U(T)$ (recall that we always work over S). Moreover, the morphisms $s, t : R \to U$ are étale since R is a disjoint product of copies of U. This proves that $j : R \to U \times_S U$ is an étale equivalence relation. $\square$

Given a scheme U and an action of a group G on U we say the action of G on U is free if condition $(*)$ of Lemma 65.14.3 holds. This is equivalent to the notion of a free action of the constant group scheme G_S on U as defined in Groupoids, Definition 39.10.2. The lemma can be interpreted as saying that quotients of schemes by free actions of groups exist in the category of algebraic spaces.

02Z3 Definition 65.14.4. Notation $U \to S$, G, R as in Lemma 65.14.3. If the action of G on U satisfies $(*)$ we say G acts freely on the scheme U. In this case the algebraic space U/R is denoted U/G and is called the quotient of U by G.

This notation is consistent with the notation U/G introduced in Groupoids, Definition 39.20.1. We will later make sense of the quotient as an algebraic stack without any assumptions on the action whatsoever; when we do this we will use the notation $[U/G]$. Before we discuss the examples we prove some more lemmas to facilitate the discussion. Here is a lemma discussing the various separation conditions for this quotient when G is finite.

02Z4 Lemma 65.14.5. Notation and assumptions as in Lemma 65.14.3. Assume G is finite. Then

(1) if $U \to S$ is quasi-separated, then U/G is quasi-separated over S, and
(2) if $U \to S$ is separated, then U/G is separated over S.

Proof. In the proof of Lemma 65.13.1 we saw that it suffices to prove the corresponding properties for the morphism $j : R \to U \times_S U$. If $U \to S$ is quasi-separated, then for every affine open $V \subset U$ which maps into an affine of S the opens $g(V) \cap V$ are quasi-compact. It follows that j is quasi-compact. If $U \to S$ is separated, the diagonal $\Delta_{U/S}$ is a closed immersion. Hence $j : R \to U \times_S U$ is a finite coproduct of closed immersions with disjoint images. Hence j is a closed immersion. $\square$

02Z5 Lemma 65.14.6. Notation and assumptions as in Lemma 65.14.3. If $\mathrm{Spec}(k) \to U/G$ is a morphism, then there exist

(1) a finite Galois extension k'/k,
(2) a finite subgroup $H \subset G$,
(3) an isomorphism $H \to \mathrm{Gal}(k'/k)$, and
(4) an H-equivariant morphism $\mathrm{Spec}(k') \to U$.

Conversely, such data determine a morphism $\mathrm{Spec}(k) \to U/G$.

Proof. Consider the fibre product $V = \mathrm{Spec}(k) \times_{U/G} U$. Here is a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & U/G \end{tikzcd}$$

Then V is a nonempty scheme étale over $\mathrm{Spec}(k)$ and hence is a disjoint union $V = \coprod_{i \in I} \mathrm{Spec}(k_i)$ of spectra of fields k_i finite separable over k (Morphisms, Lemma 29.36.7). We have

$$\begin{aligned} V \times_{\mathrm{Spec}(k)} V &= (\mathrm{Spec}(k) \times_{U/G} U) \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_{U/G} U) \\ &= \mathrm{Spec}(k) \times_{U/G} U \times_{U/G} U \\ &= \mathrm{Spec}(k) \times_{U/G} U \times G \\ &= V \times G \end{aligned}$$

The action of G on U induces an action of $a : G \times V \to V$. The displayed equality means that $G \times V \to V \times_{\mathrm{Spec}(k)} V$, $(g, v) \mapsto (a(g, v), v)$ is an isomorphism. In particular we see that for every i we have an isomorphism $H_i \times \mathrm{Spec}(k_i) \to \mathrm{Spec}(k_i \otimes_k k_i)$ where $H_i \subset G$ is the subgroup of elements fixing $i \in I$. Thus H_i is finite and is the Galois group of k_i/k. We omit the converse construction. $\square$

It follows from this lemma for example that if k'/k is a finite Galois extension, then $\mathrm{Spec}(k')/\mathrm{Gal}(k'/k) \cong \mathrm{Spec}(k)$. What happens if the extension is infinite? Here is an example.

02Z6 Example 65.14.7. Let $S = \mathrm{Spec}(\mathbf{Q})$. Let $U = \mathrm{Spec}(\overline{\mathbf{Q}})$. Let $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with obvious action on U. Then by construction property $(*)$ of Lemma 65.14.3 holds and we obtain an algebraic space

$$X = \mathrm{Spec}(\overline{\mathbf{Q}})/G \longrightarrow S = \mathrm{Spec}(\mathbf{Q}).$$

Of course this is totally ridiculous as an approximation of S! Namely, by the Artin-Schreier theorem, see [Jac64, Theorem 17, page 316], the only finite subgroups of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are $\{1\}$ and the conjugates of the order two group $\mathrm{Gal}(\overline{\mathbf{Q}}/\overline{\mathbf{Q}} \cap \mathbf{R})$. Hence, if $\mathrm{Spec}(k) \to X$ is a morphism with k algebraic over $\mathbf{Q}$, then it follows from Lemma 65.14.6 and the theorem just mentioned that either k is $\overline{\mathbf{Q}}$ or isomorphic to $\overline{\mathbf{Q}} \cap \mathbf{R}$.

What is wrong with the example above is that the Galois group comes equipped with a topology, and this should somehow be part of any construction of a quotient of $\text{Spec}(\overline{\mathbf{Q}})$. The following example is much more reasonable in my opinion and may actually occur in "nature".

02Z7 Example 65.14.8. Let k be a field of characteristic zero. Let $U = \mathbf{A}^1_k$ and let $G = \mathbf{Z}$. As action we take $n(x) = x + n$, i.e., the action of $\mathbf{Z}$ on the affine line by translation. The only fixed point is the generic point and it is clearly the case that $\mathbf{Z}$ injects into the automorphism group of the field $k(x)$. (This is where we use the characteristic zero assumption.) Consider the morphism

$$\gamma : \text{Spec}(k(x)) \longrightarrow X = \mathbf{A}^1_k/\mathbf{Z}$$

of the generic point of the affine line into the quotient. We claim that this morphism does not factor through any monomorphism $\text{Spec}(L) \to X$ of the spectrum of a field to X. (Contrary to what happens for schemes, see Schemes, Section 26.13.) In fact, since $\mathbf{Z}$ does not have any nontrivial finite subgroups we see from Lemma 65.14.6 that for any such factorization $k(x) = L$. Finally, γ is not a monomorphism since

$$\text{Spec}(k(x)) \times_{\gamma, X, \gamma} \text{Spec}(k(x)) \cong \text{Spec}(k(x)) \times \mathbf{Z}.$$

This example suggests that in order to define points of an algebraic space X we should consider equivalence classes of morphisms from spectra of fields into X and not the set of monomorphisms from spectra of fields.

We finish with a truly awful example.

02Z8 Example 65.14.9. Let k be a field. Let $A = \prod_{n \in \mathbf{N}} k$ be the infinite product. Set $U = \text{Spec}(A)$ seen as a scheme over $S = \text{Spec}(k)$. Note that the projection maps $\text{pr}_n : A \to k$ define open and closed immersions $f_n : S \to U$. Set

$$R = U \amalg \coprod\nolimits_{(n,m) \in \mathbf{N}^2,\ n \neq m} S$$

with morphism j equal to $\Delta_{U/S}$ on the component U and $j = (f_n, f_m)$ on the component S corresponding to (n, m). It is clear from the remark above that s, t are étale. It is also clear that j is an equivalence relation. Hence we obtain an algebraic space

$$X = U/R.$$

To see what this means we specialize to the case where the field k is finite with q elements. Let us first discuss the topological space $|U|$ associated to the scheme U a little bit. All elements of A satisfy $x^q = x$. Hence every residue field of A is isomorphic to k, and all points of U are closed. But the topology on U isn't the discrete topology. Let $u_n \in |U|$ be the point corresponding to f_n. As mentioned above the points u_n are the open points (and hence isolated). This implies there have to be other points since we know U is quasi-compact, see Algebra, Lemma 10.17.10 (hence not equal to an infinite discrete set). Another way to see this is because the (proper) ideal

$$I = \{x = (x_n) \in A \mid \text{all but a finite number of } x_n \text{ are zero}\}$$

is contained in a maximal ideal. Note also that every element x of A is of the form $x = ue$ where u is a unit and e is an idempotent. Hence a basis for the topology of A consists of open and closed subsets (see Algebra, Lemma 10.21.1.) So the topology on $|U|$ is totally disconnected, but nontrivial. Finally, note that $\{u_n\}$ is dense in $|U|$.

We will later define a topological space $|X|$ associated to X, see Properties of Spaces, Section 66.4. What can we say about $|X|$? It turns out that the map $|U| \to |X|$ is surjective and continuous. All the points u_n map to the same point x_0 of $|X|$, and none of the other points get identified. Since $\{u_n\}$ is dense in $|U|$ we conclude that the closure of x_0 in $|X|$ is $|X|$. In other words $|X|$ is irreducible and x_0 is a generic point of $|X|$. This seems bizarre since also x_0 is the image of a section $S \to X$ of the structure morphism $X \to S$ (and in the case of schemes this would imply it was a closed point, see Morphisms, Lemma 29.20.2).

Whatever you think is actually going on in this example, it certainly shows that some care has to be exercised when defining irreducible components, connectedness, etc of algebraic spaces.

65.15. Change of big site

03FO In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. Here is a precise statement of the result.

03FP **Lemma 65.15.1.** Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 34.12. Let S be an object of Sch_{fppf}. Let

$$g : Sh((Sch/S)_{fppf}) \longrightarrow Sh((Sch'/S)_{fppf}),$$
$$f : Sh((Sch'/S)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

be the morphisms of topoi of Topologies, Lemma 34.12.2. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then

(1) if F is representable by a scheme $X \in \mathrm{Ob}((Sch/S)_{fppf})$ over S, then $f^{-1}F$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$, and
(2) if F is an algebraic space over S, then $f^{-1}F$ is an algebraic space over S also.

Proof. Let $X \in \mathrm{Ob}((Sch/S)_{fppf})$. Let us write h_X for the representable sheaf on $(Sch/S)_{fppf}$ associated to X, and h'_X for the representable sheaf on $(Sch'/S)_{fppf}$ associated to X. By the description of f^{-1} in Topologies, Section 34.12 we see that $f^{-1}h_X = h'_X$. This proves (1).

Next, suppose that F is an algebraic space over S. By Lemma 65.9.1 this means that $F = h_U/h_R$ for some étale equivalence relation $R \to U \times_S U$ in $(Sch/S)_{fppf}$. Since f^{-1} is an exact functor we conclude that $f^{-1}F = h'_U/h'_R$. Hence $f^{-1}F$ is an algebraic space over S by Theorem 65.10.5. $\square$

Note that this lemma is purely set theoretical and has virtually no content. Moreover, it is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

04W1 **Lemma 65.15.2.** Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote Spaces$/S$ the category of algebraic spaces over S defined using

Sch_{fppf}. Similarly, denote Spaces'/S the category of algebraic spaces over S defined using Sch'_{fppf}. The construction of Lemma 65.15.1 defines a fully faithful functor

$$\text{Spaces}/S \longrightarrow \text{Spaces}'/S$$

whose essential image consists of those $X' \in \text{Ob}(\text{Spaces}'/S)$ such that there exist $U, R \in \text{Ob}((Sch/S)_{fppf})$[4] and morphisms

$$U \longrightarrow X' \quad \text{and} \quad R \longrightarrow U \times_{X'} U$$

in $Sh((Sch'/S)_{fppf})$ which are surjective as maps of sheaves (for example if the displayed morphisms are surjective and étale).

Proof. In Sites, Lemma 7.21.8 we have seen that the functor $f^{-1} : Sh((Sch/S)_{fppf}) \to Sh((Sch'/S)_{fppf})$ is fully faithful (see discussion in Topologies, Section 34.12). Hence we see that the displayed functor of the lemma is fully faithful.

Suppose that $X' \in \text{Ob}(\text{Spaces}'/S)$ such that there exists $U \in \text{Ob}((Sch/S)_{fppf})$ and a map $U \to X'$ in $Sh((Sch'/S)_{fppf})$ which is surjective as a map of sheaves. Let $U' \to X'$ be a surjective étale morphism with $U' \in \text{Ob}((Sch'/S)_{fppf})$. Let $\kappa = \text{size}(U)$, see Sets, Section 3.9. Then U has an affine open covering $U = \bigcup_{i \in I} U_i$ with $|I| \leq \kappa$. Observe that $U' \times_{X'} U \to U$ is étale and surjective. For each i we can pick a quasi-compact open $U'_i \subset U'$ such that $U'_i \times_{X'} U_i \to U_i$ is surjective (because the scheme $U' \times_{X'} U_i$ is the union of the Zariski opens $W \times_{X'} U_i$ for $W \subset U'$ affine and because $U' \times_{X'} U_i \to U_i$ is étale hence open). Then $\coprod_{i \in I} U'_i \to X$ is surjective étale because of our assumption that $U \to X$ and hence $\coprod U_i \to X$ is a surjection of sheaves (details omitted). Because $U'_i \times_{X'} U \to U'_i$ is a surjection of sheaves and because U'_i is quasi-compact, we can find a quasi-compact open $W_i \subset U'_i \times_{X'} U$ such that $W_i \to U'_i$ is surjective as a map of sheaves (details omitted). Then $W_i \to U$ is étale and we conclude that $\text{size}(W_i) \leq \text{size}(U)$, see Sets, Lemma 3.9.7. By Sets, Lemma 3.9.11 we conclude that $\text{size}(U'_i) \leq \text{size}(U)$. Hence $\coprod_{i \in I} U'_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 3.9.5.

Now let X', $U \to X'$ and $R \to U \times_{X'} U$ be as in the statement of the lemma. In the previous paragraph we have seen that we can find $U' \in \text{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U' \to X'$ in $Sh((Sch'/S)_{fppf})$. Then $U' \times_{X'} U \to U'$ is a surjection of sheaves, i.e., we can find an fppf covering $\{U'_i \to U'\}$ such that $U'_i \to U'$ factors through $U' \times_{X'} U \to U'$. By Sets, Lemma 3.9.12 we can find $\tilde{U} \to U'$ which is surjective, flat, and locally of finite presentation, with $\text{size}(\tilde{U}) \leq \text{size}(U')$, such that $\tilde{U} \to U'$ factors through $U' \times_{X'} U \to U'$. Then we consider

$$\begin{tikzcd} U' \times_{X'} U' \arrow[d] & \tilde{U} \times_{X'} \tilde{U} \arrow[l] \arrow[r] \arrow[d] & U \times_{X'} U \arrow[d] \\ U' \times_S U' & \tilde{U} \times_S \tilde{U} \arrow[l] \arrow[r] & U \times_S U \end{tikzcd}$$

The squares are cartesian. We know the objects of the bottom row are represented by objects of $(Sch/S)_{fppf}$. By the result of the argument of the previous paragraph,

[4]Requiring the existence of R is necessary because of our choice of the function *Bound* in Sets, Equation (3.9.1.1). The size of the fibre product $U \times_{X'} U$ can grow faster than *Bound* in terms of the size of U. We can illustrate this by setting $S = \text{Spec}(A)$, $U = \text{Spec}(A[x_i, i \in I])$ and $R = \coprod_{(\lambda_i) \in A^I} \text{Spec}(A[x_i, y_i]/(x_i - \lambda_i y_i))$. In this case the size of R grows like κ^κ where κ is the size of U.

the same is true for $U \times_{X'} U$ (as we have the surjection of sheaves $R \to U \times_{X'} U$ by assumption). Since $(Sch/S)_{fppf}$ is closed under fibre products (by construction), we see that $\tilde{U} \times_{X'} \tilde{U}$ is represented by an object of $(Sch/S)_{fppf}$. Finally, the map $\tilde{U} \times_{X'} \tilde{U} \to U' \times_{X'} U'$ is a surjection of fppf sheaves as $\tilde{U} \to U'$ is so. Thus we can once more apply the result of the previous paragraph to conclude that $R' = U' \times_{X'} U'$ is represented by an object of $(Sch/S)_{fppf}$. At this point Lemma 65.9.1 and Theorem 65.10.5 imply that $X = h_{U'}/h_{R'}$ is an object of Spaces$/S$ such that $f^{-1}X \cong X'$ as desired. $\square$

65.16. Change of base scheme

03I3 In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic space over S can be viewed as an algebraic space over S'. And, given an algebraic space F' over S' there is a base change F'_S which is an algebraic space over S. We explain only what happens in case $S \to S'$ is a morphism of the big fppf site under consideration, if only S or S' is contained in the big site, then one first enlarges the big site as in Section 65.15.

03I4 **Lemma 65.16.1.** *Suppose given a big site Sch_{fppf}. Let $g : S \to S'$ be morphism of Sch_{fppf}. Let $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ be the corresponding localization functor. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then*

(1) *for a scheme T' over S' we have $j_!F(T'/S') = \coprod_{\varphi: T' \to S} F(T' \xrightarrow{\varphi} S)$,*
(2) *if F is representable by a scheme $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $j_!F$ is representable by $j(X)$ which is X viewed as a scheme over S', and*
(3) *if F is an algebraic space over S, then $j_!F$ is an algebraic space over S', and if $F = U/R$ is a presentation, then $j_!F = j(U)/j(R)$ is a presentation.*

Let F' be a sheaf of sets on $(Sch/S')_{fppf}$. Then

(4) *for a scheme T over S we have $j^{-1}F'(T/S) = F'(T/S')$,*
(5) *if F' is representable by a scheme $X' \in \mathrm{Ob}((Sch/S')_{fppf})$, then $j^{-1}F'$ is representable, namely by $X'_S = S \times_{S'} X'$, and*
(6) *if F' is an algebraic space, then $j^{-1}F'$ is an algebraic space, and if $F' = U'/R'$ is a presentation, then $j^{-1}F' = U'_S/R'_S$ is a presentation.*

Proof. The functors $j_!$, j_* and j^{-1} are defined in Sites, Lemma 7.25.8 where it is also shown that $j = j_{S/S'}$ is the localization of $(Sch/S')_{fppf}$ at the object S/S'. Hence all of the material on localization functors is available for j. The formula in (1) is Sites, Lemma 7.27.1. By definition $j_!$ is the left adjoint to restriction j^{-1}, hence $j_!$ is right exact. By Sites, Lemma 7.25.5 it also commutes with fibre products and equalizers. By Sites, Lemma 7.25.3 we see that $j_!h_X = h_{j(X)}$ hence (2) holds. If F is an algebraic space over S, then we can write $F = U/R$ (Lemma 65.9.1) and we get

$$j_!F = j(U)/j(R)$$

because $j_!$ being right exact commutes with coequalizers, and moreover $j(R) = j(U) \times_{j_!F} j(U)$ as $j_!$ commutes with fibre products. Since the morphisms $j(s), j(t) : j(R) \to j(U)$ are simply the morphisms $s, t : R \to U$ (but viewed as morphisms of schemes over S'), they are still étale. Thus $(j(U), j(R), s, t)$ is an étale equivalence relation. Hence by Theorem 65.10.5 we conclude that $j_!F$ is an algebraic space.

Proof of (4), (5), and (6). The description of j^{-1} is in Sites, Section 7.25. The restriction of the representable sheaf associated to X'/S' is the representable sheaf associated to $X'_S = S \times_{S'} Y'$ by Sites, Lemma 7.27.2. The restriction functor j^{-1} is exact, hence $j^{-1}F' = U'_S/R'_S$. Again by exactness the sheaf R'_S is still an equivalence relation on U'_S. Finally the two maps $R'_S \to U'_S$ are étale as base changes of the étale morphisms $R' \to U'$. Hence $j^{-1}F' = U'_S/R'_S$ is an algebraic space by Theorem 65.10.5 and we win. □

Note how the presentation $j_!F = j(U)/j(R)$ is just the presentation of F but viewed as a presentation by schemes over S'. Hence the following definition makes sense.

03I5 Definition 65.16.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site.

(1) If F' is an algebraic space over S', then the base change of F' to S is the algebraic space $j^{-1}F'$ described in Lemma 65.16.1. We denote it F'_S.
(2) If F is an algebraic space over S, then F viewed as an algebraic space over S' is the algebraic space $j_!F$ over S' described in Lemma 65.16.1. We often simply denote this F; if not then we will write $j_!F$.

The algebraic space $j_!F$ comes equipped with a canonical morphism $j_!F \to S$ of algebraic spaces over S'. This is true simply because the sheaf $j_!F$ maps to h_S (see for example the explicit description in Lemma 65.16.1). In fact, in Sites, Lemma 7.25.4 we have seen that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of pairs $(\mathcal{F}', \mathcal{F}' \to h_S)$ consisting of a sheaf on $(Sch/S')_{fppf}$ and a map of sheaves $\mathcal{F}' \to h_S$. The equivalence assigns to the sheaf $\mathcal{F}$ the pair $(j_!\mathcal{F}, j_!\mathcal{F} \to h_S)$. This, combined with the above, leads to the following result for categories of algebraic spaces.

04SG Lemma 65.16.3. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The construction above give an equivalence of categories

$$\left\{\begin{matrix}\text{category of algebraic}\\ \text{spaces over } S\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{category of pairs } (F', F' \to S) \text{ consisting}\\ \text{of an algebraic space } F' \text{ over } S' \text{ and a}\\ \text{morphism } F' \to S \text{ of algebraic spaces over } S'\end{matrix}\right\}$$

Proof. Let F be an algebraic space over S. The functor from left to right assigns the pair $(j_!F, j_!F \to S)$ ot F which is an object of the right hand side by Lemma 65.16.1. Since this defines an equivalence of categories of sheaves by Sites, Lemma 7.25.4 to finish the proof it suffices to show: if F is a sheaf and $j_!F$ is an algebraic space, then F is an algebraic space. To do this, write $j_!F = U'/R'$ as in Lemma 65.9.1 with $U', R' \in \mathrm{Ob}((Sch/S')_{fppf})$. Then the compositions $U' \to j_!F \to S$ and $R' \to j_!F \to S$ are morphisms of schemes over S'. Denote U, R the corresponding objects of $(Sch/S)_{fppf}$. The two morphisms $R' \to U'$ are morphisms over S and hence correspond to morphisms $R \to U$. Since these are simply the same morphisms (but viewed over S) we see that we get an étale equivalence relation over S. As $j_!$ defines an equivalence of categories of sheaves (see reference above) we see that $F = U/R$ and by Theorem 65.10.5 we see that F is an algebraic space. □

The following lemma is a slight rephrasing of the above.

04SH Lemma 65.16.4. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F' be a sheaf on $(Sch/S')_{fppf}$. The following are equivalent:

(1) The restriction $F'|_{(Sch/S)_{fppf}}$ is an algebraic space over S, and

(2) the sheaf $h_S \times F'$ is an algebraic space over S'.

Proof. The restriction and the product match under the equivalence of categories of Sites, Lemma 7.25.4 so that Lemma 65.16.3 above gives the result. □

We finish this section with a lemma on a compatibility.

03I6 Lemma 65.16.5. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F be an algebraic space over S. Let T be a scheme over S and let $f : T \to F$ be a morphism over S. Let $f' : T' \to F'$ be the morphism over S' we get from f by applying the equivalence of categories described in Lemma 65.16.3. For any property $\mathcal{P}$ as in Definition 65.5.1 we have $\mathcal{P}(f') \Leftrightarrow \mathcal{P}(f)$.

Proof. Suppose that U is a scheme over S, and $U \to F$ is a surjective étale morphism. Denote U' the scheme U viewed as a scheme over S'. In Lemma 65.16.1 we have seen that $U' \to F'$ is surjective étale. Since

$$j(T \times_{f,F} U) = T' \times_{f',F'} U'$$

the morphism of schemes $T \times_{f,F} U \to U$ is identified with the morphism of schemes $T' \times_{f',F'} U' \to U'$. It is the same morphism, just viewed over different base schemes. Hence the lemma follows from Lemma 65.11.4. □

65.17. Other chapters

CHAPTER 66

Properties of Algebraic Spaces

03BO 66.1. Introduction

03BP Please see Spaces, Section 65.1 for a brief introduction to algebraic spaces, and please read some of that chapter for our basic definitions and conventions concerning algebraic spaces. In this chapter we start introducing some basic notions and properties of algebraic spaces. A fundamental reference for the case of quasi-separated algebraic spaces is [Knu71].

The discussion is somewhat awkward at times since we made the design decision to first talk about properties of algebraic spaces by themselves, and only later about properties of morphisms of algebraic spaces. We make an exception for this rule regarding étale morphisms of algebraic spaces, which we introduce in Section 66.16. But until that section whenever we say a morphism has a certain property, it automatically means the source of the morphism is a scheme (or perhaps the morphism is representable).

Some of the material in the chapter (especially regarding points) will be improved upon in the chapter on decent algebraic spaces.

66.2. Conventions

03BQ The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$. The reason is that we want to avoid confusion when changing base schemes, as in Spaces, Section 65.16.

66.3. Separation axioms

03BR In this section we collect all the "absolute" separation conditions of algebraic spaces. Since in our language any algebraic space is an algebraic space over some definite base scheme, any absolute property of X over S corresponds to a conditions imposed on X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Here is the precise formulation.

03BS **Definition 66.3.1.** (Compare Spaces, Definition 65.13.2.) Consider a big fppf site $Sch_{fppf} = (Sch/\mathrm{Spec}(\mathbf{Z}))_{fppf}$. Let X be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let $\Delta : X \to X \times X$ be the diagonal morphism.

(1) We say X is *separated* if Δ is a closed immersion.
(2) We say X is *locally separated*[1] if Δ is an immersion.

[1] In the literature this often refers to quasi-separated and locally separated algebraic spaces.

(3) We say X is quasi-separated if Δ is quasi-compact.
(4) We say X is Zariski locally quasi-separated[2] if there exists a Zariski covering $X = \bigcup_{i \in I} X_i$ (see Spaces, Definition 65.12.5) such that each X_i is quasi-separated.

Let S is a scheme contained in Sch_{fppf}, and let X be an algebraic space over S. Then we say X is separated, locally separated, quasi-separated, or Zariski locally quasi-separated if X viewed as an algebraic space over $\text{Spec}(\mathbf{Z})$ (see Spaces, Definition 65.16.2) has the corresponding property.

It is true that an algebraic space X over S which is separated (in the absolute sense above) is separated over S (and similarly for the other absolute separation properties above). This will be discussed in great detail in Morphisms of Spaces, Section 67.4. We will see in Lemma 66.6.6 that being Zariski locally separated is independent of the base scheme (hence equivalent to the absolute notion).

03DY Lemma 66.3.2. Let S be a scheme. Let X be an algebraic space over S. We have the following implications among the separation axioms of Definition 66.3.1:

(1) separated implies all the others,
(2) quasi-separated implies Zariski locally quasi-separated.

Proof. Omitted. $\square$

0AHR Lemma 66.3.3. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent

(1) X is a quasi-separated algebraic space,
(2) for $U \to X$, $V \to X$ with U, V quasi-compact schemes the fibre product $U \times_X V$ is quasi-compact,
(3) for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is quasi-compact.

Proof. Using Spaces, Lemma 65.16.3 we see that we may assume $S = \text{Spec}(\mathbf{Z})$. Since $U \times_X V = X \times_{X \times X} (U \times V)$ and since $U \times V$ is quasi-compact if U and V are so, we see that (1) implies (2). It is clear that (2) implies (3). Assume (3). Choose a scheme W and a surjective étale morphism $W \to X$. Then $W \times W \to X \times X$ is surjective étale. Hence it suffices to show that

$$j : W \times_X W = X \times_{(X \times X)} (W \times W) \to W \times W$$

is quasi-compact, see Spaces, Lemma 65.5.6. If $U \subset W$ and $V \subset W$ are affine opens, then $j^{-1}(U \times V) = U \times_X V$ is quasi-compact by assumption. Since the affine opens $U \times V$ form an affine open covering of $W \times W$ (Schemes, Lemma 26.17.4) we conclude by Schemes, Lemma 26.19.2. $\square$

0AHS Lemma 66.3.4. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent

(1) X is a separated algebraic space,
(2) for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is affine and
$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$
is surjective.

[2] This notion was suggested by B. Conrad.

Proof. Using Spaces, Lemma 65.16.3 we see that we may assume $S = \text{Spec}(\mathbf{Z})$. Since $U \times_X V = X \times_{X \times X} (U \times V)$ and since $U \times V$ is affine if U and V are so, we see that (1) implies (2). Assume (2). Choose a scheme W and a surjective étale morphism $W \to X$. Then $W \times W \to X \times X$ is surjective étale. Hence it suffices to show that

$$j : W \times_X W = X \times_{(X \times X)} (W \times W) \to W \times W$$

is a closed immersion, see Spaces, Lemma 65.5.6. If $U \subset W$ and $V \subset W$ are affine opens, then $j^{-1}(U \times V) = U \times_X V$ is affine by assumption and the map $U \times_X V \to U \times V$ is a closed immersion because the corresponding ring map is surjective. Since the affine opens $U \times V$ form an affine open covering of $W \times W$ (Schemes, Lemma 26.17.4) we conclude by Morphisms, Lemma 29.2.1. □

66.4. Points of algebraic spaces

03BT As is clear from Spaces, Example 65.14.8 a point of an algebraic space should not be defined as a monomorphism from the spectrum of a field. Instead we define them as equivalence classes of morphisms of spectra of fields exactly as explained in Schemes, Section 26.13.

Let S be a scheme. Let F be a presheaf on $(Sch/S)_{fppf}$. Let K be a field. Consider a morphism

$$\text{Spec}(K) \longrightarrow F.$$

By the Yoneda Lemma this is given by an element $p \in F(\text{Spec}(K))$. We say that two such pairs $(\text{Spec}(K), p)$ and $(\text{Spec}(L), q)$ are equivalent if there exists a third field Ω and a commutative diagram

$$\begin{tikzcd} \text{Spec}(\Omega) \arrow[r] \arrow[d] & \text{Spec}(L) \arrow[d, "q"] \\ \text{Spec}(K) \arrow[r, "p"] & F. \end{tikzcd}$$

In other words, there are field extensions $K \to \Omega$ and $L \to \Omega$ such that p and q map to the same element of $F(\text{Spec}(\Omega))$. We omit the verification that this defines an equivalence relation.

03BU Definition 66.4.1. Let S be a scheme. Let X be an algebraic space over S. A point of X is an equivalence class of morphisms from spectra of fields into X. The set of points of X is denoted $|X|$.

Note that if $f : X \to Y$ is a morphism of algebraic spaces over S, then there is an induced map $|f| : |X| \to |Y|$ which maps a representative $x : \text{Spec}(K) \to X$ to the representative $f \circ x : \text{Spec}(K) \to Y$.

03BV Lemma 66.4.2. Let S be a scheme. Let X be a scheme over S. The points of X as a scheme are in canonical 1-1 correspondence with the points of X as an algebraic space.

Proof. This is Schemes, Lemma 26.13.3. □

03H4 Lemma 66.4.3. Let S be a scheme. Let

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Then the map of sets of points

$$|Z \times_Y X| \longrightarrow |Z| \times_{|Y|} |X|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to X$, $\mathrm{Spec}(L) \to Z$, then the assumption that they agree as elements of $|Y|$ means that there is a common extension M/K and M/L such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to X \to Y$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to Z \to Y$ agree. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to Z \times_Y X$. □

03H5 Lemma 66.4.4. Let S be a scheme. Let X be an algebraic space over S. Let $f : T \to X$ be a morphism from a scheme to X. The following are equivalent

(1) $f : T \to X$ is surjective (according to Spaces, Definition 65.5.1), and
(2) $|f| : |T| \to |X|$ is surjective.

Proof. Assume (1). Let $x : \mathrm{Spec}(K) \to X$ be a morphism from the spectrum of a field into X. By assumption the morphism of schemes $\mathrm{Spec}(K) \times_X T \to \mathrm{Spec}(K)$ is surjective. Hence there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_X T$ such that the left square in the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \times_X T \arrow[r] \arrow[d] & T \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, equal] & \mathrm{Spec}(K) \arrow[r, "x"] & X \end{tikzcd}$$

is commutative. This shows that $|f| : |T| \to |X|$ is surjective.

Assume (2). Let $Z \to X$ be a morphism where Z is a scheme. We have to show that the morphism of schemes $Z \times_X T \to T$ is surjective, i.e., that $|Z \times_X T| \to |Z|$ is surjective. This follows from (2) and Lemma 66.4.3. □

03BW Lemma 66.4.5. Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X, see Spaces, Definition 65.9.3. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|X|$ is the quotient of $|U|$ by this equivalence relation.

Proof. The assumption means that U is a scheme, $p : U \to X$ is a surjective, étale morphism, $R = U \times_X U$ is a scheme and defines an étale equivalence relation on U such that $X = U/R$ as sheaves. By Lemma 66.4.4 we see that $|U| \to |X|$ is surjective. By Lemma 66.4.3 the map

$$|R| \longrightarrow |U| \times_{|X|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|X|$. Combining these two statements we get the result of the lemma. □

03BX Lemma 66.4.6. Let S be a scheme. There exists a unique topology on the sets of points of algebraic spaces over S with the following properties:

(1) if X is a scheme over S, then the topology on $|X|$ is the usual one (via the identification of Lemma 66.4.2),
(2) for every morphism of algebraic spaces $X \to Y$ over S the map $|X| \to |Y|$ is continuous, and
(3) for every étale morphism $U \to X$ with U a scheme the map of topological spaces $|U| \to |X|$ is continuous and open.

Proof. Let X be an algebraic space over S. Let $p : U \to X$ be a surjective étale morphism where U is a scheme over S. We define $W \subset |X|$ is open if and only if $|p|^{-1}(W)$ is an open subset of $|U|$. This is a topology on $|X|$ (it is the quotient topology on $|X|$, see Topology, Lemma 5.6.2).

Let us prove that the topology is independent of the choice of the presentation. To do this it suffices to show that if U' is a scheme, and $U' \to X$ is an étale morphism, then the map $|U'| \to |X|$ (with topology on $|X|$ defined using $U \to X$ as above) is open and continuous; which in addition will prove that (3) holds. Set $U'' = U \times_X U'$, so that we have the commutative diagram

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

As $U \to X$ and $U' \to X$ are étale we see that both $U'' \to U$ and $U'' \to U'$ are étale morphisms of schemes. Moreover, $U'' \to U'$ is surjective. Hence we get a commutative diagram of maps of sets

$$\begin{tikzcd} |U''| \arrow[r] \arrow[d] & |U'| \arrow[d] \\ |U| \arrow[r] & |X| \end{tikzcd}$$

The lower horizontal arrow is surjective (see Lemma 66.4.4 or Lemma 66.4.5) and continuous by definition of the topology on $|X|$. The top horizontal arrow is surjective, continuous, and open by Morphisms, Lemma 29.36.13. The left vertical arrow is continuous and open (by Morphisms, Lemma 29.36.13 again.) Hence it follows formally that the right vertical arrow is continuous and open.

To finish the proof we prove (2). Let $a : X \to Y$ be a morphism of algebraic spaces. According to Spaces, Lemma 65.11.6 we can find a diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

where U and V are schemes, and p and q are surjective and étale. This gives rise to the diagram

$$\begin{tikzcd} |U| \arrow[r, "\alpha"'] \arrow[d, "p"'] & |V| \arrow[d, "q"] \\ |X| \arrow[r, "a"] & |Y| \end{tikzcd}$$

where all but the lower horizontal arrows are known to be continuous and the two vertical arrows are surjective and open. It follows that the lower horizontal arrow is continuous as desired. □

03BY Definition 66.4.7. Let S be a scheme. Let X be an algebraic space over S. The underlying topological space of X is the set of points $|X|$ endowed with the topology constructed in Lemma 66.4.6.

It turns out that this topological space carries the same information as the small Zariski site X_{Zar} of Spaces, Definition 65.12.6.

03BZ Lemma 66.4.8. Let S be a scheme. Let X be an algebraic space over S.

(1) The rule $X' \mapsto |X'|$ defines an inclusion preserving bijection between open subspaces X' (see Spaces, Definition 65.12.1) of X, and opens of the topological space $|X|$.
(2) A family $\{X_i \subset X\}_{i \in I}$ of open subspaces of X is a Zariski covering (see Spaces, Definition 65.12.5) if and only if $|X| = \bigcup |X_i|$.

In other words, the small Zariski site X_{Zar} of X is canonically identified with a site associated to the topological space $|X|$ (see Sites, Example 7.6.4).

Proof. In order to prove (1) let us construct the inverse of the rule. Namely, suppose that $W \subset |X|$ is open. Choose a presentation $X = U/R$ corresponding to the surjective étale map $p : U \to X$ and étale maps $s, t : R \to U$. By construction we see that $|p|^{-1}(W)$ is an open of U. Denote $W' \subset U$ the corresponding open subscheme. It is clear that $R' = s^{-1}(W') = t^{-1}(W')$ is a Zariski open of R which defines an étale equivalence relation on W'. By Spaces, Lemma 65.10.2 the morphism $X' = W'/R' \to X$ is an open immersion. Hence X' is an algebraic space by Spaces, Lemma 65.11.3. By construction $|X'| = W$, i.e., X' is a subspace of X corresponding to W. Thus (1) is proved.

To prove (2), note that if $\{X_i \subset X\}_{i \in I}$ is a collection of open subspaces, then it is a Zariski covering if and only if the $U = \bigcup U \times_X X_i$ is an open covering. This follows from the definition of a Zariski covering and the fact that the morphism $U \to X$ is surjective as a map of presheaves on $(Sch/S)_{fppf}$. On the other hand, we see that $|X| = \bigcup |X_i|$ if and only if $U = \bigcup U \times_X X_i$ by Lemma 66.4.5 (and the fact that the projections $U \times_X X_i \to X_i$ are surjective and étale). Thus the equivalence of (2) follows. □

03IE Lemma 66.4.9. Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X' \subset X$ be an open subspace. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then f factors through X' if and only if $|f| : |Y| \to |X|$ factors through $|X'| \subset |X|$.

Proof. By Spaces, Lemma 65.12.3 we see that $Y' = Y \times_X X' \to Y$ is an open immersion. If $|f|(|Y|) \subset |X'|$, then clearly $|Y'| = |Y|$. Hence $Y' = Y$ by Lemma 66.4.8. □

06NF Lemma 66.4.10. Let S be a scheme. Let X be an algebraic spaces over S. Let U be a scheme and let $f : U \to X$ be an étale morphism. Let $X' \subset X$ be the open subspace corresponding to the open $|f|(|U|) \subset |X|$ via Lemma 66.4.8. Then f factors through a surjective étale morphism $f' : U \to X'$. Moreover, if $R = U \times_X U$, then $R = U \times_{X'} U$ and X' has the presentation $X' = U/R$.

Proof. The existence of the factorization follows from Lemma 66.4.9. The morphism f' is surjective according to Lemma 66.4.4. To see f' is étale, suppose that $T \to X'$ is a morphism where T is a scheme. Then $T \times_X U = T \times_{X'} U$ as $X' \to X$ is a monomorphism of sheaves. Thus the projection $T \times_{X'} U \to T$ is étale as we assumed f étale. We have $U \times_X U = U \times_{X'} U$ as $X' \to X$ is a monomorphism. Then $X' = U/R$ follows from Spaces, Lemma 65.9.1. ☐

0H2X Lemma 66.4.11. Let S be a scheme. Let X be an algebraic space over S. Let $p : \mathrm{Spec}(K) \to X$ and $q : \mathrm{Spec}(L) \to X$ be morphisms where K and L are fields. Assume p and q determine the same point of $|X|$ and p is a monomorphism. Then q factors uniquely through p.

Proof. Since p and q define the same point of $|X|$, we see that the scheme

$$Y = \mathrm{Spec}(K) \times_{p,X,q} \mathrm{Spec}(L)$$

is nonempty. Since the base change of a monomorphism is a monomorphism this means that the projection morphism $Y \to \mathrm{Spec}(L)$ is a monomorphism. Hence $Y = \mathrm{Spec}(L)$, see Schemes, Lemma 26.23.11. We conclude that q factors through p. Uniqueness comes from the fact that p is a monomorphism. ☐

03E1 Lemma 66.4.12. Let S be a scheme. Let X be an algebraic space over S. Consider the map

$$\{\mathrm{Spec}(k) \to X \text{ monomorphism where } k \text{ is a field}\} \longrightarrow |X|$$

This map is injective.

Proof. This follows from Lemma 66.4.11. ☐

We will see in Decent Spaces, Lemma 68.11.1 that the map of Lemma 66.4.12 is a bijection when X is decent.

66.5. Quasi-compact spaces

03E2

03E3 Definition 66.5.1. Let S be a scheme. Let X be an algebraic space over S. We say X is quasi-compact if there exists a surjective étale morphism $U \to X$ with U quasi-compact.

03E4 Lemma 66.5.2. Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if $|X|$ is quasi-compact.

Proof. Choose a scheme U and an étale surjective morphism $U \to X$. We will use Lemma 66.4.4. If U is quasi-compact, then since $|U| \to |X|$ is surjective we conclude that $|X|$ is quasi-compact. If $|X|$ is quasi-compact, then since $|U| \to |X|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still étale). Hence we win. ☐

040T Lemma 66.5.3. A finite disjoint union of quasi-compact algebraic spaces is a quasi-compact algebraic space.

Proof. This is clear from Lemma 66.5.2 and the corresponding topological fact. ☐

03IO Example 66.5.4. The space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ is a quasi-compact algebraic space.

04NN Lemma 66.5.5. Let S be a scheme. Let X be an algebraic space over S. Every point of $|X|$ has a fundamental system of open quasi-compact neighbourhoods. In particular $|X|$ is locally quasi-compact in the sense of Topology, Definition 5.13.1.

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |X|$ of topological spaces. To be a bit more precise, if $u \in U$ maps to $x \in |X|$, then the images of the affine neighbourhoods of u will give a fundamental system of quasi-compact open neighbourhoods of x. ☐

66.6. Special coverings

03FW In this section we collect some straightforward lemmas on the existence of étale surjective coverings of algebraic spaces.

03FX Lemma 66.6.1. Let S be a scheme. Let X be an algebraic space over S. There exists a surjective étale morphism $U \to X$ where U is a disjoint union of affine schemes. We may in addition assume each of these affines maps into an affine open of S.

Proof. Let $V \to X$ be a surjective étale morphism. Let $V = \bigcup_{i \in I} V_i$ be a Zariski open covering such that each V_i maps into an affine open of S. Then set $U = \coprod_{i \in I} V_i$ with induced morphism $U \to V \to X$. This is étale and surjective as a composition of étale and surjective representable transformations of functors (via the general principle Spaces, Lemma 65.5.4 and Morphisms, Lemmas 29.9.2 and 29.36.3). ☐

03FY Lemma 66.6.2. Let S be a scheme. Let X be an algebraic space over S. There exists a Zariski covering $X = \bigcup X_i$ such that each algebraic space X_i has a surjective étale covering by an affine scheme. We may in addition assume each X_i maps into an affine open of S.

Proof. By Lemma 66.6.1 we can find a surjective étale morphism $U = \coprod U_i \to X$, with U_i affine and mapping into an affine open of S. Let $X_i \subset X$ be the open subspace of X such that $U_i \to X$ factors through an étale surjective morphism $U_i \to X_i$, see Lemma 66.4.10. Since $U = \bigcup U_i$ we see that $X = \bigcup X_i$. As $U_i \to X_i$ is surjective it follows that $X_i \to S$ maps into an affine open of S. ☐

03H6 Lemma 66.6.3. Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if there exists an étale surjective morphism $U \to X$ with U an affine scheme.

Proof. If there exists an étale surjective morphism $U \to X$ with U affine then X is quasi-compact by Definition 66.5.1. Conversely, if X is quasi-compact, then $|X|$ is quasi-compact. Let $U = \coprod_{i \in I} U_i$ be a disjoint union of affine schemes with an étale and surjective map $\varphi : U \to X$ (Lemma 66.6.1). Then $|X| = \bigcup \varphi(|U_i|)$ and by quasi-compactness there is a finite subset $i_1, \ldots, i_n$ such that $|X| = \bigcup \varphi(|U_{i_j}|)$. Hence $U_{i_1} \cup \ldots \cup U_{i_n}$ is an affine scheme with a finite surjective morphism towards X. ☐

The following lemma will be obsoleted by the discussion of separated morphisms in the chapter on morphisms of algebraic spaces.

03FZ Lemma 66.6.4. Let S be a scheme. Let X be an algebraic space over S. Let U be a separated scheme and $U \to X$ étale. Then $U \to X$ is separated, and $R = U \times_X U$ is a separated scheme.

Proof. Let $X' \subset X$ be the open subscheme such that $U \to X$ factors through an étale surjection $U \to X'$, see Lemma 66.4.10. If $U \to X'$ is separated, then so is $U \to X$, see Spaces, Lemma 65.5.4 (as the open immersion $X' \to X$ is separated by Spaces, Lemma 65.5.8 and Schemes, Lemma 26.23.8). Moreover, since $U \times_{X'} U = U \times_X U$ it suffices to prove the result after replacing X by X', i.e., we may assume $U \to X$ surjective. Consider the commutative diagram

$$\begin{tikzcd} R = U \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

In the proof of Spaces, Lemma 65.13.1 we have seen that $j : R \to U \times_S U$ is separated. The morphism of schemes $U \to S$ is separated as U is a separated scheme, see Schemes, Lemma 26.21.13. Hence $U \times_S U \to U$ is separated as a base change, see Schemes, Lemma 26.21.12. Hence the scheme $U \times_S U$ is separated (by the same lemma). Since j is separated we see in the same way that R is separated. Hence $R \to U$ is a separated morphism (by Schemes, Lemma 26.21.13 again). Thus by Spaces, Lemma 65.11.4 and the diagram above we conclude that $U \to X$ is separated. $\square$

07S4 Lemma 66.6.5. Let S be a scheme. Let X be an algebraic space over S. If there exists a quasi-separated scheme U and a surjective étale morphism $U \to X$ such that either of the projections $U \times_X U \to U$ is quasi-compact, then X is quasi-separated.

Proof. We may think of X as an algebraic space over $\mathbf{Z}$. Consider the cartesian diagram

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d, "j"'] & X \arrow[d, "\Delta"] \\ U \times U \arrow[r] & X \times X \end{tikzcd}$$

Since U is quasi-separated the projection $U \times U \to U$ is quasi-separated (as a base change of a quasi-separated morphism of schemes, see Schemes, Lemma 26.21.12). Hence the assumption in the lemma implies j is quasi-compact by Schemes, Lemma 26.21.14. By Spaces, Lemma 65.11.4 we see that Δ is quasi-compact as desired. $\square$

03W7 Lemma 66.6.6. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent

(1) X is Zariski locally quasi-separated over S,
(2) X is Zariski locally quasi-separated,
(3) there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and
(4) there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i which maps into an affine open of S and a quasi-compact surjective étale morphism $U_i \to X_i$.

Proof. Assume $U_i \to X_i \subset X$ are as in (3). To prove (4) choose for each i a finite affine open covering $U_i = U_{i1} \cup \ldots \cup U_{in_i}$ such that each U_{ij} maps into an affine open of S. The compositions $U_{ij} \to U_i \to X_i$ are étale and quasi-compact (see Spaces, Lemma 65.5.4). Let $X_{ij} \subset X_i$ be the open subspace corresponding to the image of $|U_{ij}| \to |X_i|$, see Lemma 66.4.10. Note that $U_{ij} \to X_{ij}$ is quasi-compact as $X_{ij} \subset X_i$ is a monomorphism and as $U_{ij} \to X$ is quasi-compact. Then $X = \bigcup X_{ij}$ is a covering as in (4). The implication (4) $\Rightarrow$ (3) is immediate.

Assume (4). To show that X is Zariski locally quasi-separated over S it suffices to show that X_i is quasi-separated over S. Hence we may assume there exists an affine scheme U mapping into an affine open of S and a quasi-compact surjective étale morphism $U \to X$. Consider the fibre product square

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d] & U \times_S U \arrow[d] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X \end{tikzcd}$$

The right vertical arrow is surjective étale (see Spaces, Lemma 65.5.7) and $U \times_S U$ is affine (as U maps into an affine open of S, see Schemes, Section 26.17), and $U \times_X U$ is quasi-compact because the projection $U \times_X U \to U$ is quasi-compact as a base change of $U \to X$. It follows from Spaces, Lemma 65.11.4 that $\Delta_{X/S}$ is quasi-compact as desired.

Assume (1). To prove (3) there is an immediate reduction to the case where X is quasi-separated over S. By Lemma 66.6.2 we can find a Zariski open covering $X = \bigcup X_i$ such that each X_i maps into an affine open of S, and such that there exist affine schemes U_i and surjective étale morphisms $U_i \to X_i$. Since $U_i \to S$ maps into an affine open of S we see that $U_i \times_S U_i$ is affine, see Schemes, Section 26.17. As X is quasi-separated over S, the morphisms

$$R_i = U_i \times_{X_i} U_i = U_i \times_X U_i \longrightarrow U_i \times_S U_i$$

as base changes of $\Delta_{X/S}$ are quasi-compact. Hence we conclude that R_i is a quasi-compact scheme. This in turn implies that each projection $R_i \to U_i$ is quasi-compact. Hence, applying Spaces, Lemma 65.11.4 to the covering $U_i \to X_i$ and the morphism $U_i \to X_i$ we conclude that the morphisms $U_i \to X_i$ are quasi-compact as desired.

At this point we see that (1), (3), and (4) are equivalent. Since (3) does not refer to the base scheme we conclude that these are also equivalent with (2). $\square$

The following lemma will turn out to be quite useful.

03IJ Lemma 66.6.7. Let S be a scheme. Let X be an algebraic space over S. Let U be a scheme. Let $\varphi : U \to X$ be an étale morphism such that the projections $R = U \times_X U \to U$ are quasi-compact; for example if φ is quasi-compact. Then the fibres of

$$|U| \to |X| \quad \text{and} \quad |R| \to |X|$$

are finite.

Proof. Denote $R = U \times_X U$, and $s, t : R \to U$ the projections. Let $u \in U$ be a point, and let $x \in |X|$ be its image. The fibre of $|U| \to |X|$ over x is equal to $s(t^{-1}(\{u\}))$ by Lemma 66.4.3, and the fibre of $|R| \to |X|$ over x is $t^{-1}(s(t^{-1}(\{u\})))$. Since $t : R \to$

U is étale and quasi-compact, it has finite fibres (as its fibres are disjoint unions of spectra of fields by Morphisms, Lemma 29.36.7 and quasi-compact). Hence we win. □

66.7. Properties of Spaces defined by properties of schemes

03E5 Any étale local property of schemes gives rise to a corresponding property of algebraic spaces via the following lemma.

03E8 **Lemma 66.7.1.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{P}$ be a property of schemes which is local in the étale topology, see Descent, Definition 35.15.1. The following are equivalent

(1) for some scheme U and surjective étale morphism $U \to X$ the scheme U has property $\mathcal{P}$, and
(2) for every scheme U and every étale morphism $U \to X$ the scheme U has property $\mathcal{P}$.

If X is representable this is equivalent to $\mathcal{P}(X)$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. For the converse, choose a surjective étale morphism $U \to X$ with U a scheme that has $\mathcal{P}$ and let V be an étale X-scheme. Then $U \times_X V \to V$ is an étale surjection of schemes, so V inherits $\mathcal{P}$ from $U \times_X V$, which in turn inherits $\mathcal{P}$ from U (see discussion following Descent, Definition 35.15.1). The last claim is clear from (1) and Descent, Definition 35.15.1. □

03E6 **Definition 66.7.2.** Let $\mathcal{P}$ be a property of schemes which is local in the étale topology. Let S be a scheme. Let X be an algebraic space over S. We say X has *property $\mathcal{P}$* if any of the equivalent conditions of Lemma 66.7.1 hold.

03E7 **Remark 66.7.3.** Here is a list of properties which are local for the étale topology (keep in mind that the fpqc, fppf, syntomic, and smooth topologies are stronger than the étale topology):

(1) locally Noetherian, see Descent, Lemma 35.16.1,
(2) Jacobson, see Descent, Lemma 35.16.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 35.17.1,
(4) Cohen-Macaulay, see Descent, Lemma 35.17.2,
(5) Gorenstein, see Duality for Schemes, Lemma 48.24.6,
(6) reduced, see Descent, Lemma 35.18.1,
(7) normal, see Descent, Lemma 35.18.2,
(8) locally Noetherian and (R_k), see Descent, Lemma 35.18.3,
(9) regular, see Descent, Lemma 35.18.4,
(10) Nagata, see Descent, Lemma 35.18.5.

Any étale local property of germs of schemes gives rise to a corresponding property of algebraic spaces. Here is the obligatory lemma.

04N2 **Lemma 66.7.4.** Let $\mathcal{P}$ be a property of germs of schemes which is étale local, see Descent, Definition 35.21.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. Consider étale morphisms $a : U \to X$ where U is a scheme. The following are equivalent

(1) for any $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$, and
(2) for some $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$.

If X is representable, then this is equivalent to $\mathcal{P}(X, x)$.

Proof. Omitted. □

04RC Definition 66.7.5. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{P}$ be a property of germs of schemes which is étale local. We say X has property $\mathcal{P}$ at x if any of the equivalent conditions of Lemma 66.7.4 hold.

0BBL Remark 66.7.6. Let P be a property of local rings. Assume that for any étale ring map $A \to B$ and $\mathfrak{q}$ is a prime of B lying over the prime $\mathfrak{p}$ of A, then $P(A_\mathfrak{p}) \Leftrightarrow P(B_\mathfrak{q})$. Then we obtain an étale local property of germs (U, u) of schemes by setting $\mathcal{P}(U, u) = P(\mathcal{O}_{U,u})$. In this situation we will use the terminology "the local ring of X at x has P" to mean X has property $\mathcal{P}$ at x. Here is a list of such properties P:

(1) Noetherian, see More on Algebra, Lemma 15.44.1,
(2) dimension d, see More on Algebra, Lemma 15.44.2,
(3) regular, see More on Algebra, Lemma 15.44.3,
(4) discrete valuation ring, follows from (2), (3), and Algebra, Lemma 10.119.7,
(5) reduced, see More on Algebra, Lemma 15.45.4,
(6) normal, see More on Algebra, Lemma 15.45.6,
(7) Noetherian and depth k, see More on Algebra, Lemma 15.45.8,
(8) Noetherian and Cohen-Macaulay, see More on Algebra, Lemma 15.45.9,
(9) Noetherian and Gorenstein, see Dualizing Complexes, Lemma 47.21.8.

There are more properties for which this holds, for example G-ring and Nagata. If we every need these we will add them here as well as references to detailed proofs of the corresponding algebra facts.

66.8. Constructible sets

0ECS

0ECT Lemma 66.8.1. Let S be a scheme. Let X be an algebraic space over S. Let $E \subset |X|$ be a subset. The following are equivalent

(1) for every étale morphism $U \to X$ where U is a scheme the inverse image of E in U is a locally constructible subset of U,
(2) for every étale morphism $U \to X$ where U is an affine scheme the inverse image of E in U is a constructible subset of U,
(3) for some surjective étale morphism $U \to X$ where U is a scheme the inverse image of E in U is a locally constructible subset of U.

Proof. By Properties, Lemma 28.2.1 we see that (1) and (2) are equivalent. It is immediate that (1) implies (3). Thus we assume we have a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that $\varphi^{-1}(E)$ is locally constructible. Let $\varphi' : U' \to X$ be another étale morphism where U' is a scheme. Then we have

$$E'' = \text{pr}_1^{-1}(\varphi^{-1}(E)) = \text{pr}_2^{-1}((\varphi')^{-1}(E))$$

where $\text{pr}_1 : U \times_X U' \to U$ and $\text{pr}_2 : U \times_X U' \to U'$ are the projections. By Morphisms, Lemma 29.22.1 we see that E'' is locally constructible in $U \times_X U'$. Let $W' \subset U'$ be an affine open. Since pr_2 is étale and hence open, we can choose a quasi-compact open $W'' \subset U \times_X U'$ with $\text{pr}_2(W'') = W'$. Then $\text{pr}_2|_{W''} : W'' \to W'$ is quasi-compact. We have $W' \cap (\varphi')^{-1}(E) = \text{pr}_2(E'' \cap W'')$ as φ is surjective, see Lemma 66.4.3. Thus $W' \cap (\varphi')^{-1}(E) = \text{pr}_2(E'' \cap W'')$ is locally constructible by Morphisms, Theorem 29.22.3 as desired. □

0ECU Definition 66.8.2. Let S be a scheme. Let X be an algebraic space over S. Let $E \subset |X|$ be a subset. We say E is étale locally constructible if the equivalent conditions of Lemma 66.8.1 are satisfied.

Of course, if X is representable, i.e., X is a scheme, then this just means E is a locally constructible subset of the underlying topological space.

66.9. Dimension at a point

04N3 We can use Descent, Lemma 35.21.2 to define the dimension of an algebraic space X at a point x. This will give us a different notion than the topological one (i.e., the dimension of $|X|$ at x).

04N5 Definition 66.9.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. We define the dimension of X at x to be the element $\dim_x(X) \in \{0, 1, 2, \ldots, \infty\}$ such that $\dim_x(X) = \dim_u(U)$ for any (equivalently some) pair $(a : U \to X, u)$ consisting of an étale morphism $a : U \to X$ from a scheme to X and a point $u \in U$ with $a(u) = x$. See Definition 66.7.5, Lemma 66.7.4, and Descent, Lemma 35.21.2.

Warning: It is not the case that $\dim_x(X) = \dim_x(|X|)$ in general. A counter example is the algebraic space X of Spaces, Example 65.14.9. Namely, let $x \in |X|$ be a point not equal to the generic point x_0 of $|X|$. Then we have $\dim_x(X) = 0$ but $\dim_x(|X|) = 1$. In particular, the dimension of X (as defined below) is different from the dimension of $|X|$.

04N6 Definition 66.9.2. Let S be a scheme. Let X be an algebraic space over S. The dimension $\dim(X)$ of X is defined by the rule

$$\dim(X) = \sup_{x \in |X|} \dim_x(X)$$

By Properties, Lemma 28.10.2 we see that this is the usual notion if X is a scheme. There is another integer that measures the dimension of a scheme at a point, namely the dimension of the local ring. This invariant is compatible with étale morphisms also, see Section 66.10.

66.10. Dimension of local rings

04N7 The dimension of the local ring of an algebraic space is a well defined concept.

0BAM Lemma 66.10.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent

(1) for some scheme U and étale morphism $a : U \to X$ and point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$,
(2) for any scheme U, any étale morphism $a : U \to X$, and any point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$.

If X is a scheme, this is equivalent to $\dim(\mathcal{O}_{X,x}) = d$.

Proof. Combine Lemma 66.7.4 and Descent, Lemma 35.21.3. □

04NA Definition 66.10.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The dimension of the local ring of X at x is the element $d \in \{0, 1, 2, \ldots, \infty\}$ satisfying the equivalent conditions of Lemma 66.10.1. In this case we will also say x is a point of codimension d on X.

Besides the lemma below we also point the reader to Lemmas 66.22.4 and 66.22.5.

0BAN Lemma 66.10.3. Let S be a scheme. Let X be an algebraic space over S. The following quantities are equal:

(1) The dimension of X.
(2) The supremum of the dimensions of the local rings of X.
(3) The supremum of $\dim_x(X)$ for $x \in |X|$.

Proof. The numbers in (1) and (3) are equal by Definition 66.9.2. Let $U \to X$ be a surjective étale morphism from a scheme U. The supremum of $\dim_x(X)$ for $x \in |X|$ is the same as the supremum of $\dim_u(U)$ for points u of U by definition. This is the same as the supremum of $\dim(\mathcal{O}_{U,u})$ by Properties, Lemma 28.10.2. This in turn is the same as (2) by definition. $\square$

66.11. Generic points

0BAP Let T be a topological space. According to the second edition of EGA I, a maximal point of T is a generic point of an irreducible component of T. If $T = |X|$ is the topological space associated to an algebraic space X, there are at least two notions of maximal points: we can look at maximal points of T viewed as a topological space, or we can look at images of maximal points of U where $U \to X$ is an étale morphism and U is a scheme. The second notion corresponds to the set of points of codimension 0 (Lemma 66.11.1). The codimension 0 points are easier to work with for general algebraic spaces; the two notions agree for quasi-separated and more generally decent algebraic spaces (Decent Spaces, Lemma 68.20.1).

0BAQ Lemma 66.11.1. Let S be a scheme and let X be an algebraic space over S. Let $x \in |X|$. Consider étale morphisms $a : U \to X$ where U is a scheme. The following are equivalent

(1) x is a point of codimension 0 on X,
(2) for some $U \to X$ as above and $u \in U$ with $a(u) = x$, the point u is the generic point of an irreducible component of U, and
(3) for any $U \to X$ as above and any $u \in U$ mapping to x, the point u is the generic point of an irreducible component of U.

If X is representable, this is equivalent to x being a generic point of an irreducible component of $|X|$.

Proof. Observe that a point u of a scheme U is a generic point of an irreducible component of U if and only if $\dim(\mathcal{O}_{U,u}) = 0$ (Properties, Lemma 28.10.4). Hence this follows from the definition of the codimension of a point on X (Definition 66.10.2). $\square$

0BAR Lemma 66.11.2. Let S be a scheme and let X be an algebraic space over S. The set of codimension 0 points of X is dense in $|X|$.

Proof. If U is a scheme, then the set of generic points of irreducible components is dense in U (holds for any quasi-sober topological space). Thus if $U \to X$ is a surjective étale morphism, then the set of codimension 0 points of X is the image of a dense subset of $|U|$ (Lemma 66.11.1). Since $|X|$ has the quotient topology for $|U| \to |X|$ we conclude. $\square$

66.12. Reduced spaces

03IP We have already defined reduced algebraic spaces in Section 66.7. Here we just prove some simple lemmas regarding reduced algebraic spaces.

0ABJ Lemma 66.12.1. Let S be a scheme. Let $Z \to X$ be an immersion of algebraic spaces. Then $|Z| \to |X|$ is a homeomorphism of $|Z|$ onto a locally closed subset of $|X|$.

Proof. Let U be a scheme and $U \to X$ a surjective étale morphism. Then $Z \times_X U \to U$ is an immersion of schemes, hence gives a homeomorphism of $|Z \times_X U|$ with a locally closed subset T' of $|U|$. By Lemma 66.4.3 the subset T' is the inverse image of the image T of $|Z| \to |X|$. The map $|Z| \to |X|$ is injective because the transformation of functors $Z \to X$ is injective, see Spaces, Section 65.12. By Topology, Lemma 5.6.4 we see that T is locally closed in $|X|$. Moreover, the continuous map $|Z| \to T$ is a homeomorphism as the map $|Z \times_X U| \to T'$ is a homeomorphism and $|Z \times_Y U| \to |Z|$ is submersive. □

The following lemma will help us construct (locally) closed subspaces.

07TW Lemma 66.12.2. Let S be a scheme. Let $j : R \to U \times_S U$ be an étale equivalence relation. Let $X = U/R$ be the associated algebraic space (Spaces, Theorem 65.10.5). There is a canonical bijection

$$R\text{-invariant locally closed subschemes } Z' \text{ of } U \leftrightarrow \text{locally closed subspaces } Z \text{ of } X$$

Moreover, if $Z \to X$ is closed (resp. open) if and only if $Z' \to U$ is closed (resp. open).

Proof. Denote $\varphi : U \to X$ the canonical map. The bijection sends $Z \to X$ to $Z' = Z \times_X U \to U$. It is immediate from the definition that $Z' \to U$ is an immersion, resp. closed immersion, resp. open immersion if $Z \to X$ is so. It is also clear that Z' is R-invariant (see Groupoids, Definition 39.19.1).

Conversely, assume that $Z' \to U$ is an immersion which is R-invariant. Let R' be the restriction of R to Z', see Groupoids, Definition 39.18.2. Since $R' = R \times_{s,U} Z' = Z' \times_{U,t} R$ in this case we see that R' is an étale equivalence relation on Z'. By Spaces, Theorem 65.10.5 we see $Z = Z'/R'$ is an algebraic space. By construction we have $U \times_X Z = Z'$, so $U \times_X Z \to Z$ is an immersion. Note that the property "immersion" is preserved under base change and fppf local on the base (see Spaces, Section 65.4). Moreover, immersions are separated and locally quasi-finite (see Schemes, Lemma 26.23.8 and Morphisms, Lemma 29.20.16). Hence by More on Morphisms, Lemma 37.57.1 immersions satisfy descent for fppf covering. This means all the hypotheses of Spaces, Lemma 65.11.5 are satisfied for $Z \to X$, $\mathcal{P} =$"immersion", and the étale surjective morphism $U \to X$. We conclude that $Z \to X$ is representable and an immersion, which is the definition of a subspace (see Spaces, Definition 65.12.1).

It is clear that these constructions are inverse to each other and we win. □

03IQ Lemma 66.12.3. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. There exists a unique closed subspace $Z \subset X$ with the following properties: (a) we have $|Z| = T$, and (b) Z is reduced.

Proof. Let $U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$, so that $X = U/R$, see Spaces, Lemma 65.9.1. As usual we denote

$s, t : R \to U$ the two projection morphisms. By Lemma 66.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $s^{-1}(T') = t^{-1}(T')$. Let $Z' \subset U$ be the reduced induced scheme structure on T'. In this case the fibre products $Z' \times_{U,t} R$ and $Z' \times_{U,s} R$ are closed subschemes of R (Schemes, Lemma 26.18.2) which are étale over Z' (Morphisms, Lemma 29.36.4), and hence reduced (because being reduced is local in the étale topology, see Remark 66.7.3). Since they have the same underlying topological space (see above) we conclude that $Z' \times_{U,t} R = Z' \times_{U,s} R$. Thus we can apply Lemma 66.12.2 to obtain a closed subspace $Z \subset X$ whose pullback to U is Z'. By construction $|Z| = T$ and Z is reduced. This proves existence. We omit the proof of uniqueness. □

03JJ Lemma 66.12.4. Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume Y is reduced. A morphism $f : Y \to X$ factors through Z if and only if $f(|Y|) \subset |Z|$.

Proof. Assume $f(|Y|) \subset |Z|$. Choose a diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & U \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{f} & X \end{array}$$

where U, V are schemes, and the vertical arrows are surjective and étale. The scheme V is reduced, see Lemma 66.7.1. Hence h factors through $a^{-1}(Z)$ by Schemes, Lemma 26.12.7. So $a \circ h$ factors through Z. As $Z \subset X$ is a subsheaf, and $V \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ we conclude that $X \to Y$ factors through Z. □

047X Definition 66.12.5. Let S be a scheme, and let X be an algebraic space over S. Let $Z \subset |X|$ be a closed subset. An algebraic space structure on Z is given by a closed subspace Z' of X with $|Z'|$ equal to Z. The reduced induced algebraic space structure on Z is the one constructed in Lemma 66.12.3. The reduction X_{red} of X is the reduced induced algebraic space structure on $|X|$.

66.13. The schematic locus

03JG Every algebraic space has a largest open subspace which is a scheme; this is more or less clear but we also write out the proof below. Of course this subspace may be empty, for example if $X = \mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ (the universal counter example). On the other hand, if X is for example quasi-separated, then this largest open subscheme is actually dense in X!

03JH Lemma 66.13.1. Let S be a scheme. Let X be an algebraic space over S. There exists a largest open subspace $X' \subset X$ which is a scheme.

Proof. Let $U \to X$ be an étale surjective morphism, where U is a scheme. Let $R = U \times_X U$. The open subspaces of X correspond $1-1$ with open subschemes of U which are R-invariant. Hence there is a set of them. Let X_i, $i \in I$ be the set of open subspaces of X which are schemes, i.e., are representable. Consider the open subspace $X' \subset X$ whose underlying set of points is the open $\bigcup |X_i|$ of $|X|$. By Lemma 66.4.4 we see that

$$\coprod X_i \longrightarrow X'$$

is a surjective map of sheaves on $(Sch/S)_{fppf}$. But since each $X_i \to X'$ is representable by open immersions we see that in fact the map is surjective in the Zariski topology. Namely, if $T \to X'$ is a morphism from a scheme into X', then $X_i \times_{X'} T$ is an open subscheme of T. Hence we can apply Schemes, Lemma 26.15.4 to see that X' is a scheme. $\square$

In the rest of this section we say that an open subspace X' of an algebraic space X is dense if the corresponding open subset $|X'| \subset |X|$ is dense.

0BAS Lemma 66.13.2. Let S be a scheme. Let X be an algebraic space over S. If there exists a finite, étale, surjective morphism $U \to X$ where U is a quasi-separated scheme, then there exists a dense open subspace X' of X which is a scheme. More precisely, every point $x \in |X|$ of codimension 0 in X is contained in X'.

Proof. Let $X' \subset X$ be the maximal open subspace which is a scheme (Lemma 66.13.1). Let $x \in |X|$ be a point of codimension 0 on X. By Lemma 66.11.2 it suffices to show $x \in X'$. Let $U \to X$ be as in the statement of the lemma. Write $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, finite and étale. By Lemma 66.6.7 the fibre of $|U| \to |X|$ over x is finite, say $\{\eta_1, \ldots, \eta_n\}$. By Lemma 66.11.1 each η_i is the generic point of an irreducible component of U. By Properties, Lemma 28.29.1 we can find an affine open $W \subset U$ containing $\{\eta_1, \ldots, \eta_n\}$ (this is where we use that U is quasi-separated). By Groupoids, Lemma 39.24.1 we may assume that W is R-invariant. Since $W \subset U$ is an R-invariant affine open, the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 39.19.1 and discussion following it). In particular the maps $R_W \to W$ are finite étale also. It follows that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 39.23.9. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 65.10.2 and it contains x by construction. $\square$

We will improve the following proposition to the case of decent algebraic spaces in Decent Spaces, Theorem 68.10.2.

06NH Proposition 66.13.3. Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated (for example if X is quasi-separated), then there exists a dense open subspace X' of X which is a scheme. More precisely, every point $x \in |X|$ of codimension 0 on X is contained in X'.

Proof. The question is local on X by Lemma 66.13.1. Thus by Lemma 66.6.6 we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Moreover $U \to X$ is separated (Lemma 66.6.4). Set $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Then s, t are surjective, quasi-compact, separated, and étale. Hence s, t are also quasi-finite and have finite fibres (Morphisms, Lemmas 29.36.6, 29.20.9, and 29.20.10). By Morphisms, Lemma 29.51.1 for every $\eta \in U$ which is the generic point of an irreducible component of U, there exists an open neighbourhood $V \subset U$ of η such that $s^{-1}(V) \to V$ is finite. By Descent, Lemma 35.23.23 being finite is fpqc (and in particular étale) local on the target. Hence we may apply More on Groupoids, Lemma 40.6.4 which says that the largest open $W \subset U$ over which s is finite is R-invariant. By the above W contains every generic point of an irreducible component of U. The restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 39.19.1 and discussion following it). By construction $s_W, t_W : R_W \to W$ are finite étale. Consider the open subspace $X' = W/R_W \subset X$ (see Spaces, Lemma 65.10.2). By construction the inclusion map $X' \to X$ induces a bijection on points of codimension 0. This reduces us to Lemma 66.13.2. $\square$

66.14. Obtaining a scheme

07S5 We have used in the previous section that the quotient U/R of an affine scheme U by an equivalence relation R is a scheme if the morphisms $s, t : R \to U$ are finite étale. This is a special case of the following result.

07S6 Proposition 66.14.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume

(1) $s, t : R \to U$ finite locally free,
(2) $j = (t, s)$ is an equivalence, and
(3) for a dense set of points $u \in U$ the R-equivalence class $t(s^{-1}(\{u\}))$ is contained in an affine open of U.

Then there exists a finite locally free morphism $U \to M$ of schemes over S such that $R = U \times_M U$ and such that M represents the quotient sheaf U/R in the fppf topology.

Proof. By assumption (3) and Groupoids, Lemma 39.24.1 we can find an open covering $U = \bigcup U_i$ such that each U_i is an R-invariant affine open of U. Set $R_i = R|_{U_i}$. Consider the fppf sheaves $F = U/R$ and $F_i = U_i/R_i$. By Spaces, Lemma 65.10.2 the morphisms $F_i \to F$ are representable and open immersions. By Groupoids, Proposition 39.23.9 the sheaves F_i are representable by affine schemes. If T is a scheme and $T \to F$ is a morphism, then $V_i = F_i \times_F T$ is open in T and we claim that $T = \bigcup V_i$. Namely, fppf locally on T we can lift $T \to F$ to a morphism $f : T \to U$ and in that case $f^{-1}(U_i) \subset V_i$. Hence we conclude that F is representable by a scheme, see Schemes, Lemma 26.15.4. $\square$

For example, if U is isomorphic to a locally closed subscheme of an affine scheme or isomorphic to a locally closed subscheme of $\text{Proj}(A)$ for some graded ring A, then the third assumption holds by Properties, Lemma 28.29.5. In particular we can apply this to free actions of finite groups and finite group schemes on quasi-affine or quasi-projective schemes. For example, the quotient X/G of a quasi-projective variety X by a free action of a finite group G is a scheme. Here is a detailed statement.

07S7 Lemma 66.14.2. Let S be a scheme. Let $G \to S$ be a group scheme. Let $X \to S$ be a morphism of schemes. Let $a : G \times_S X \to X$ be an action. Assume that

(1) $G \to S$ is finite locally free,
(2) the action a is free,
(3) $X \to S$ is affine, or quasi-affine, or projective, or quasi-projective, or X is isomorphic to an open subscheme of an affine scheme, or X is isomorphic to an open subscheme of $\text{Proj}(A)$ for some graded ring A, or $G \to S$ is radicial.

Then the fppf quotient sheaf X/G is a scheme and $X \to X/G$ is an fppf G-torsor.

Proof. We first show that X/G is a scheme. Since the action is free the morphism $j = (a, \text{pr}) : G \times_S X \to X \times_S X$ is a monomorphism and hence an equivalence relation, see Groupoids, Lemma 39.10.3. The maps $s, t : G \times_S X \to X$ are finite locally free as we've assumed that $G \to S$ is finite locally free. To conclude it now suffices to prove the last assumption of Proposition 66.14.1 holds. Since the action of G is over S it suffices to prove that any finite set of points in a fibre of $X \to S$ is contained in an affine open of X. If X is isomorphic to an open subscheme of an affine scheme or isomorphic to an open subscheme of $\text{Proj}(A)$ for some graded ring A this follows from Properties, Lemma 28.29.5. If $X \to S$ is affine, or quasi-affine, or projective, or quasi-projective, we may replace S by an affine open and we get back to the case we just dealt with. If $G \to S$ is radicial, then the orbits of points on X under the action of G are singletons and the condition trivially holds. Some details omitted.

To see that $X \to X/G$ is an fppf G-torsor (Groupoids, Definition 39.11.3) we have to show that $G \times_S X \to X \times_{X/G} X$ is an isomorphism and that $X \to X/G$ fppf locally has sections. The second part is clear from the fact that $X \to X/G$ is surjective as a map of fppf sheaves (by construction). The first part follows from the isomorphism $R = U \times_M U$ in the conclusion of Proposition 66.14.1 (note that $R = G \times_S X$ in our case). □

0BBM Lemma 66.14.3. Notation and assumptions as in Proposition 66.14.1. Then

(1) if U is quasi-separated over S, then U/R is quasi-separated over S,
(2) if U is quasi-separated, then U/R is quasi-separated,
(3) if U is separated over S, then U/R is separated over S,
(4) if U is separated, then U/R is separated, and
(5) add more here.

Similar results hold in the setting of Lemma 66.14.2.

Proof. Since M represents the quotient sheaf we have a cartesian diagram

$$\begin{tikzcd} R \arrow[r, "j"'] \arrow[d] & U \times_S U \arrow[d] \\ M \arrow[r] & M \times_S M \end{tikzcd}$$

of schemes. Since $U \times_S U \to M \times_S M$ is surjective finite locally free, to show that $M \to M \times_S M$ is quasi-compact, resp. a closed immersion, it suffices to show that $j : R \to U \times_S U$ is quasi-compact, resp. a closed immersion, see Descent, Lemmas 35.23.1 and 35.23.19. Since $j : R \to U \times_S U$ is a morphism over U and since R is finite over U, we see that j is quasi-compact as soon as the projection $U \times_S U \to U$ is quasi-separated (Schemes, Lemma 26.21.14). Since j is a monomorphism and locally of finite type, we see that j is a closed immersion as soon as it is proper (Étale Morphisms, Lemma 41.7.2) which will be the case as soon as the projection $U \times_S U \to U$ is separated (Morphisms, Lemma 29.41.7). This proves (1) and (3). To prove (2) and (4) we replace S by $\text{Spec}(\mathbf{Z})$, see Definition 66.3.1. Since Lemma 66.14.2 is proved through an application of Proposition 66.14.1 the final statement is clear too. □

66.15. Points on quasi-separated spaces

06NI Points can behave very badly on algebraic spaces in the generality introduced in the Stacks project. However, for quasi-separated spaces their behaviour is mostly like the behaviour of points on schemes. We prove a few results on this in this section; the chapter on decent spaces contains many more results on this, see for example Decent Spaces, Section 68.12.

06NJ Lemma 66.15.1. Let S be a scheme. Let X be a Zariski locally quasi-separated algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.8.6).

Proof. Combining Topology, Lemma 5.8.8 and Lemma 66.6.6 we see that we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$ with projection maps $s, t : R \to U$. Applying Lemma 66.6.7 we see that the fibres of s, t are finite. It follows all the assumptions of Topology, Lemma 5.19.8 are met, and we conclude that $|X|$ is Kolmogorov[3].

It remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Lemma 66.12.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. Note that $U \times_X Z \to Z$ is a quasi-compact, surjective, étale morphism from an affine scheme to Z, hence Z is Zariski locally quasi-separated by Lemma 66.6.6. By Proposition 66.13.3 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 26.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in |Z'| \subset T$ and hence also $T = \overline{\{\eta\}}$, and we win. $\square$

0A4G Lemma 66.15.2. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The topological space $|X|$ is a spectral space.

Proof. By Topology, Definition 5.23.1 we have to check that $|X|$ is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. By Lemma 66.15.1 we see that $|X|$ is sober. By Lemma 66.5.2 we see that $|X|$ is quasi-compact. By Lemma 66.6.3 there exists an affine scheme U and a surjective étale morphism $f : U \to X$. Since $|f| : |U| \to |X|$ is open and continuous and since $|U|$ has a basis of quasi-compact opens, we conclude that $|X|$ has a basis of quasi-compact opens. Finally, suppose that $A, B \subset |X|$ are quasi-compact open. Then $A = |X'|$ and $B = |X''|$ for some open subspaces $X', X'' \subset X$ (Lemma 66.4.8) and we can choose affine schemes V and W and surjective étale morphisms $V \to X'$ and $W \to X''$ (Lemma 66.6.3). Then $A \cap B$ is the image of $|V \times_X W| \to |X|$ (Lemma 66.4.3). Since $V \times_X W$ is quasi-compact as X is quasi-separated (Lemma 66.3.3) we conclude that $A \cap B$ is quasi-compact and the proof is finished. $\square$

The following lemma can be used to prove that an algebraic space is isomorphic to the spectrum of a field.

03DZ Lemma 66.15.3. Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \to X$. If X is quasi-separated, then $X \cong \mathrm{Spec}(k')$ where k/k' is a finite separable extension.

[3]Actually we use here also Schemes, Lemma 26.11.1 (soberness schemes), Morphisms, Lemmas 29.36.12 and 29.25.9 (generalizations lift along étale morphisms), Lemma 66.4.5 (points on an algebraic space in terms of a presentation), and Lemma 66.4.6 (openness quotient map).

Proof. Set $R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k)$, so that we have a fibre product diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & \mathrm{Spec}(k) \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$

By Spaces, Lemma 65.9.1 we know $X = \mathrm{Spec}(k)/R$ is the quotient sheaf. Because $\mathrm{Spec}(k) \to X$ is étale, the morphisms s and t are étale. Hence $R = \coprod_{i \in I} \mathrm{Spec}(k_i)$ is a disjoint union of spectra of fields, and both s and t induce finite separable field extensions $s, t : k \subset k_i$, see Morphisms, Lemma 29.36.7. Because

$$R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) = (\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)) \times_{X \times_S X, \Delta} X$$

and since Δ is quasi-compact by assumption we conclude that $R \to \mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$ is quasi-compact. Hence R is quasi-compact as $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$ is affine. We conclude that I is finite. This implies that s and t are finite locally free morphisms. Hence by Groupoids, Proposition 39.23.9 we conclude that $\mathrm{Spec}(k)/R$ is represented by $\mathrm{Spec}(k')$, with $k' \subset k$ finite locally free where

$$k' = \{x \in k \mid s_i(x) = t_i(x) \text{ for all } i \in I\}$$

It is easy to see that k' is a field. □

03E0 Remark 66.15.4. Lemma 66.15.3 holds for decent algebraic spaces, see Decent Spaces, Lemma 68.12.8. In fact a decent algebraic space with one point is a scheme, see Decent Spaces, Lemma 68.14.2. This also holds when X is locally separated, because a locally separated algebraic space is decent, see Decent Spaces, Lemma 68.15.2.

66.16. Étale morphisms of algebraic spaces

03FQ This section really belongs in the chapter on morphisms of algebraic spaces, but we need the notion of an algebraic space étale over another in order to define the small étale site of an algebraic space. Thus we need to do some preliminary work on étale morphisms from schemes to algebraic spaces, and étale morphisms between algebraic spaces. For more about étale morphisms of algebraic spaces, see Morphisms of Spaces, Section 67.39.

03EC Lemma 66.16.1. Let S be a scheme. Let X be an algebraic space over S. Let U, U' be schemes over S.

(1) If $U \to U'$ is an étale morphism of schemes, and if $U' \to X$ is an étale morphism from U' to X, then the composition $U \to X$ is an étale morphism from U to X.
(2) If $\varphi : U \to X$ and $\varphi' : U' \to X$ are étale morphisms towards X, and if $\chi : U \to U'$ is a morphism of schemes such that $\varphi = \varphi' \circ \chi$, then χ is an étale morphism of schemes.
(3) If $\chi : U \to U'$ is a surjective étale morphism of schemes and $\varphi' : U' \to X$ is a morphism such that $\varphi = \varphi' \circ \chi$ is étale, then φ' is étale.

Proof. Recall that our definition of an étale morphism from a scheme into an algebraic space comes from Spaces, Definition 65.5.1 via the fact that any morphism from a scheme into an algebraic space is representable.

Part (1) of the lemma follows from this, the fact that étale morphisms are preserved under composition (Morphisms, Lemma 29.36.3) and Spaces, Lemmas 65.5.4 and 65.5.3 (which are formal).

To prove part (2) choose a scheme W over S and a surjective étale morphism $W \to X$. Consider the base change $\chi_W : W \times_X U \to W \times_X U'$ of χ. As $W \times_X U$ and $W \times_X U'$ are étale over W, we conclude that χ_W is étale, by Morphisms, Lemma 29.36.18. On the other hand, in the commutative diagram

$$\begin{tikzcd} W \times_X U \arrow[r] \arrow[d] & W \times_X U' \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

the two vertical arrows are étale and surjective. Hence by Descent, Lemma 35.14.4 we conclude that $U \to U'$ is étale.

To prove part (3) choose a scheme W over S and a morphism $W \to X$. As above we consider the diagram

$$\begin{tikzcd} W \times_X U \arrow[r] \arrow[d] & W \times_X U' \arrow[r] \arrow[d] & W \arrow[d] \\ U \arrow[r] & U' \arrow[r] & X \end{tikzcd}$$

Now we know that $W \times_X U \to W \times_X U'$ is surjective étale (as a base change of $U \to U'$) and that $W \times_X U \to W$ is étale. Thus $W \times_X U' \to W$ is étale by Descent, Lemma 35.14.4. By definition this means that φ' is étale. $\square$

03FR Definition 66.16.2. Let S be a scheme. A morphism $f : X \to Y$ between algebraic spaces over S is called étale if and only if for every étale morphism $\varphi : U \to X$ where U is a scheme, the composition $f \circ \varphi$ is étale also.

If X and Y are schemes, then this agree with the usual notion of an étale morphism of schemes. In fact, whenever $X \to Y$ is a representable morphism of algebraic spaces, then this agrees with the notion defined via Spaces, Definition 65.5.1. This follows by combining Lemma 66.16.3 below and Spaces, Lemma 65.11.4.

03FS Lemma 66.16.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is étale,
(2) there exists a surjective étale morphism $\varphi : U \to X$, where U is a scheme, such that the composition $f \circ \varphi$ is étale (as a morphism of algebraic spaces),
(3) there exists a surjective étale morphism $\psi : V \to Y$, where V is a scheme, such that the base change $V \times_X Y \to V$ is étale (as a morphism of algebraic spaces),
(4) there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and the left vertical arrow is surjective such that the horizontal arrow is étale.

Proof. Let us prove that (4) implies (1). Assume a diagram as in (4) given. Let $W \to X$ be an étale morphism with W a scheme. Then we see that $W \times_X U \to U$ is étale. Hence $W \times_X U \to V$ is étale as the composition of the étale morphisms of schemes $W \times_X U \to U$ and $U \to V$. Therefore $W \times_X U \to Y$ is étale by Lemma 66.16.1 (1). Since also the projection $W \times_X U \to W$ is surjective and étale, we conclude from Lemma 66.16.1 (3) that $W \to Y$ is étale.

Let us prove that (1) implies (4). Assume (1). Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 65.11.6. By assumption the morphism $U \to Y$ is étale, and hence $U \to V$ is étale by Lemma 66.16.1 (2).

We omit the proof that (2) and (3) are also equivalent to (1). □

03FT Lemma 66.16.4. The composition of two étale morphisms of algebraic spaces is étale.

Proof. This is immediate from the definition. □

03FU Lemma 66.16.5. The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \to Y$ be a morphism of algebraic spaces. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme W and a surjective étale morphism $W \to Z$. Then $U \to Y$ is étale, hence in the diagram

$$\begin{tikzcd} W \times_Y U \arrow[r] \arrow[d] & W \arrow[d] \\ Z \times_Y X \arrow[r] & Z \end{tikzcd}$$

the top horizontal arrow is étale. Moreover, the left vertical arrow is surjective and étale (verification omitted). Hence we conclude that the lower horizontal arrow is étale by Lemma 66.16.3. □

03FV Lemma 66.16.6. Let S be a scheme. Let X, Y, Z be algebraic spaces. Let $g : X \to Z$, $h : Y \to Z$ be étale morphisms and let $f : X \to Y$ be a morphism such that $h \circ f = g$. Then f is étale.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\chi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 65.11.6. By assumption the morphisms $\varphi : U \to X \to Z$ and $\psi : V \to Y \to Z$ are étale. Moreover, $\psi \circ \chi = \varphi$ by our assumption on f, g, h. Hence $U \to V$ is étale by Lemma 66.16.1 part (2). □

03IR Lemma 66.16.7. Let S be a scheme. If $X \to Y$ is an étale morphism of algebraic spaces over S, then the associated map $|X| \to |Y|$ of topological spaces is open.

Proof. This is clear from the diagram in Lemma 66.16.3 and Lemma 66.4.6. $\square$

Finally, here is a fun lemma. It is not true that an algebraic space with an étale morphism towards a scheme is a scheme, see Spaces, Example 65.14.2. But it is true if the target is the spectrum of a field.

03KX Lemma 66.16.8. Let S be a scheme. Let $X \to \mathrm{Spec}(k)$ be étale morphism over S, where k is a field. Then X is a scheme.

Proof. Let U be an affine scheme, and let $U \to X$ be an étale morphism. By Definition 66.16.2 we see that $U \to \mathrm{Spec}(k)$ is an étale morphism. Hence $U = \coprod_{i=1,\ldots,n} \mathrm{Spec}(k_i)$ is a finite disjoint union of spectra of finite separable extensions k_i of k, see Morphisms, Lemma 29.36.7. The $R = U \times_X U \to U \times_{\mathrm{Spec}(k)} U$ is a monomorphism and $U \times_{\mathrm{Spec}(k)} U$ is also a finite disjoint union of spectra of finite separable extensions of k. Hence by Schemes, Lemma 26.23.11 we see that R is similarly a finite disjoint union of spectra of finite separable extensions of k. This U and R are affine and both projections $R \to U$ are finite locally free. Hence U/R is a scheme by Groupoids, Proposition 39.23.9. By Spaces, Lemma 65.10.2 it is also an open subspace of X. By Lemma 66.13.1 we conclude that X is a scheme. $\square$

66.17. Spaces and fpqc coverings

03W8 Let S be a scheme. An algebraic space over S is defined as a sheaf in the fppf topology with additional properties. Hence it is not immediately clear that it satisfies the sheaf property for the fpqc topology (see Topologies, Definition 34.9.12). In this section we give Gabber's argument showing this is true. However, when we say that the algebraic space X satisfies the sheaf property for the fpqc topology we really only consider fpqc coverings $\{f_i : T_i \to T\}_{i \in I}$ such that T, T_i are objects of the big site $(Sch/S)_{fppf}$ (as per our conventions, see Section 66.2).

0APL Proposition 66.17.1 (Gabber). Let S be a scheme. Let X be an algebraic space over S. Then X satisfies the sheaf property for the fpqc topology.

Proof. Since X is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines $f : T' \to T$ we have: $X(T)$ is the equalizer of the two maps $X(T') \to X(T' \times_T T')$. See Topologies, Lemma 34.9.13 (there is a little argument omitted here because the lemma cited is formulated for functors defined on the category of all schemes).

Let $a, b : T \to X$ be two morphisms such that $a \circ f = b \circ f$. We have to show $a = b$. Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Spaces, Lemma 65.13.1 the morphism $\Delta_{X/S}$ is a representable monomorphism. Hence $E \to T$ is a monomorphism of schemes. Our assumption that $a \circ f = b \circ f$ implies that $T' \to T$ factors (uniquely) through E. Consider the commutative diagram

$$\begin{tikzcd} T' \times_T E \arrow[r] \arrow[d] & E \arrow[d] \\ T' \arrow[r] \arrow[ur] \arrow[u, bend left] & T \end{tikzcd}$$

Since the projection $T' \times_T E \to T'$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $E \to T$ is an isomorphism by Descent, Lemma 35.23.17. This means $a = b$ as desired.

Next, let $c : T' \to X$ be a morphism such that the two compositions $T' \times_T T' \to T' \to X$ are the same. We have to find a morphism $a : T \to X$ whose composition with $T' \to T$ is c. Choose an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |X|$ contains the image of $|c| : |T'| \to |X|$. This is possible by Lemmas 66.4.6 and 66.6.1, the fact that a finite disjoint union of affines is affine, and the fact that $|T'|$ is quasi-compact (small argument omitted). Since $U \to X$ is separated (Lemma 66.6.4), we see that

$$V = U \times_{X,c} T' \longrightarrow T'$$

is a surjective, étale, separated morphism of schemes (to see that it is surjective use Lemma 66.4.3 and our choice of $U \to X$). The fact that $c \circ \text{pr}_0 = c \circ \text{pr}_1$ means that we obtain a descent datum on $V/T'/T$ (Descent, Definition 35.34.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, c\circ\text{pr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{c\circ\text{pr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism $V \to T'$ is ind-quasi-affine by More on Morphisms, Lemma 37.66.8 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 29.36.6). By More on Groupoids, Lemma 40.15.3 the descent datum is effective. Say $W \to T$ is a morphism such that there is an isomorphism $\alpha : T' \times_T W \to V$ compatible with the given descent datum on V and the canonical descent datum on $T' \times_T W$. Then $W \to T$ is surjective and étale (Descent, Lemmas 35.23.7 and 35.23.29). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions $b' \circ (\text{pr}_0, 1), b' \circ (\text{pr}_1, 1) : (T' \times_T T') \times_T W \to T' \times_T W \to U$ agree by our choice of α and the corresponding property of c (computation omitted). Hence b' descends to a morphism $b : W \to U$ by Descent, Lemma 35.13.7. The diagram

$$\begin{tikzcd} T' \times_T W \arrow[r] \arrow[d] & W \arrow[r, "b"'] & U \arrow[d] \\ T' \arrow[rr, "c"] & & X \end{tikzcd}$$

is commutative. What this means is that we have proved the existence of a étale locally on T, i.e., we have an $a' : W \to X$. However, since we have proved uniqueness in the first paragraph, we find that this étale local solution satisfies the glueing condition, i.e., we have $\text{pr}_0^*a' = \text{pr}_1^*a'$ as elements of $X(W \times_T W)$. Since X is an étale sheaf we find a unique $a \in X(T)$ restricting to a' on W. □

66.18. The étale site of an algebraic space

03EB In this section we define the small étale site of an algebraic space. This is the analogue of the small étale site $S_{étale}$ of a scheme. Lemma 66.16.1 implies that in the definition below any morphism between objects of the étale site of X is étale, and that any scheme étale over an object of $X_{étale}$ is also an object of $X_{étale}$.

03ED Definition 66.18.1. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The small étale site $X_{étale}$ of X is defined as follows:

(1) An object of $X_{étale}$ is a morphism $\varphi : U \to X$ where $U \in \mathrm{Ob}((Sch/S)_{étale})$ is a scheme and φ is an étale morphism,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ is given by a morphism of schemes $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{(U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{étale}$ is a covering if and only if $\{U_i \to U\}_{i \in I}$ is a covering of $(Sch/S)_{étale}$.

A consequence of our choice is that the étale site of an algebraic space in general does not have a final object! On the other hand, if X happens to be a scheme, then the definition above agrees with Topologies, Definition 34.4.8.

The above is our default site, but there are a couple of variants which we will also use. Namely, we can consider all algebraic spaces U which are étale over X and this produces the site $X_{spaces,étale}$ we define below or we can consider all affine schemes U which are étale over X and this produces the site $X_{affine,étale}$ we define below. The first of these two notions is used when discussing functoriality of the small étale site, see Lemma 66.18.8.

03G0 Definition 66.18.2. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The site $X_{spaces,étale}$ of X is defined as follows:

(1) An object of $X_{spaces,étale}$ is a morphism $\varphi : U \to X$ where U is an algebraic space over S and φ is an étale morphism of algebraic spaces over S,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ of $X_{spaces,étale}$ is given by a morphism of algebraic spaces $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{\varphi_i : (U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{spaces,étale}$ is a covering if and only if $|U| = \bigcup \varphi_i(|U_i|)$.

As usual we choose a set of coverings of this type, including at least the coverings in $X_{étale}$, as in Sets, Lemma 3.11.1 to turn $X_{spaces,étale}$ into a site.

Since the identity morphism of X is étale it is clear that $X_{spaces,étale}$ does have a final object. Let us show right away that the corresponding topos equals the small étale topos of X.

03G1 Lemma 66.18.3. The functor

$$X_{étale} \longrightarrow X_{spaces,étale}, \quad U/X \longmapsto U/X$$

is a special cocontinuous functor (Sites, Definition 7.29.2) and hence induces an equivalence of topoi $Sh(X_{étale}) \to Sh(X_{spaces,étale})$.

Proof. We have to show that the functor satisfies the assumptions (1) – (5) of Sites, Lemma 7.29.1. It is clear that the functor is continuous and cocontinuous, which proves assumptions (1) and (2). Assumptions (3) and (4) hold simply because the functor is fully faithful. Assumption (5) holds, because an algebraic space by definition has a covering by a scheme. □

03H7 Remark 66.18.4. Let us explain the meaning of Lemma 66.18.3. Let S be a scheme, and let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on the small étale site $X_{étale}$ of X. The lemma says that there exists a unique sheaf $\mathcal{F}'$ on $X_{spaces,étale}$ which restricts back to $\mathcal{F}$ on the subcategory $X_{étale}$. If $U \to X$ is an étale morphism of algebraic spaces, then how do we compute $\mathcal{F}'(U)$? Well, by definition of an algebraic space there exists a scheme U' and a surjective étale morphism $U' \to U$. Then $\{U' \to U\}$ is a covering in $X_{spaces,étale}$ and hence we get an equalizer diagram

$$\mathcal{F}'(U) \longrightarrow \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U').$$

Note that $U' \times_U U'$ is a scheme, and hence we may write $\mathcal{F}$ and not $\mathcal{F}'$. Thus we see how to compute $\mathcal{F}'$ when given the sheaf $\mathcal{F}$.

0H01 Definition 66.18.5. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The site $X_{affine,étale}$ of X is defined as follows:

(1) An object of $X_{affine,étale}$ is a morphism $\varphi : U \to X$ where $U \in \text{Ob}((Sch/S)_{étale})$ is an affine scheme and φ is an étale morphism,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ of $X_{affine,étale}$ is given by a morphism of schemes $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{\varphi_i : (U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{affine,étale}$ is a covering if and only if $\{U_i \to U\}$ is a standard étale covering, see Topologies, Definition 34.4.5.

As usual we choose a set of coverings of this type, as in Sets, Lemma 3.11.1 to turn $X_{affine,étale}$ into a site.

04JS Lemma 66.18.6. Let S be a scheme. Let X be an algebraic space over S. The functor $X_{affine,étale} \to X_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh(X_{affine,étale})$ to $Sh(X_{étale})$.

Proof. Omitted. Hint: compare with the proof of Topologies, Lemma 34.4.11. □

04JT Definition 66.18.7. Let S be a scheme. Let X be an algebraic space over S. The étale topos of X, or more precisely the small étale topos of X is the category $Sh(X_{étale})$ of sheaves of sets on $X_{étale}$.

By Lemma 66.18.3 we have $Sh(X_{étale}) = Sh(X_{spaces,étale})$, so we can also think of this as the category of sheaves of sets on $X_{spaces,étale}$. Similarly, by Lemma 66.18.6 we see that $Sh(X_{étale}) = Sh(X_{affine,étale})$. It turns out that the topos is functorial with respect to morphisms of algebraic spaces. Here is a precise statement.

03G2 Lemma 66.18.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) The continuous functor
$$Y_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V \longmapsto X \times_Y V$$
induces a morphism of sites
$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$
(2) The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 7.14.5).

(3) The morphism of topoi associated to $f_{spaces,étale}$ induces, via Lemma 66.18.3, a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.
(4) If f is a representable morphism of algebraic spaces, then f_{small} comes from a morphism of sites $X_{étale} \to Y_{étale}$, corresponding to the continuous functor $V \mapsto X \times_Y V$.

Proof. Let us show that the functor described in (1) satisfies the assumptions of Sites, Proposition 7.14.7. Thus we have to show that $Y_{spaces,étale}$ has a final object (namely Y) and that the functor transforms this into a final object in $X_{spaces,étale}$ (namely X). This is clear as $X \times_Y Y = X$ in any category. Next, we have to show that $Y_{spaces,étale}$ has fibre products. This is true since the category of algebraic spaces has fibre products, and since $V \times_Y V'$ is étale over Y if V and V' are étale over Y (see Lemmas 66.16.4 and 66.16.5 above). OK, so the proposition applies and we see that we get a morphism of sites as described in (1).

Part (2) you get by unwinding the definitions. Part (3) is clear by using the equivalences for X and Y from Lemma 66.18.3 above. Part (4) follows, because if f is representable, then the functors above fit into a commutative diagram

$$\begin{tikzcd} X_{étale} \arrow[r] & X_{spaces,étale} \\ Y_{étale} \arrow[r] \arrow[u] & Y_{spaces,étale} \arrow[u] \end{tikzcd}$$

of categories. □

We can do a little bit better than the lemma above in describing the relationship between sheaves on X and sheaves on Y. Namely, we can formulate this in turns of f-maps, compare Sheaves, Definition 6.21.7, as follows.

03G3 Definition 66.18.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. An f-map $\varphi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\varphi_{(U,V,g)} : \mathcal{G}(V) \to \mathcal{F}(U)$ indexed by commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ such that whenever given an extended diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "g'"'] & U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V' \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

with $V' \to V$ and $U' \to U$ étale morphisms of schemes the diagram

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\varphi_{(U,V,g)}"'] \arrow[d, "\text{restriction of } \mathcal{G}"'] & \mathcal{F}(U) \arrow[d, "\text{restriction of } \mathcal{F}"] \\ \mathcal{G}(V') \arrow[r, "\varphi_{(U',V',g')}"] & \mathcal{F}(U') \end{tikzcd}$$

commutes.

03G4 Lemma 66.18.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. There are canonical bijections between the following three sets:

(1) The set of maps $\mathcal{G} \to f_{small,*}\mathcal{F}$.
(2) The set of maps $f_{small}^{-1}\mathcal{G} \to \mathcal{F}$.
(3) The set of f-maps $\varphi : \mathcal{G} \to \mathcal{F}$.

Proof. Note that (1) and (2) are the same because the functors $f_{small,*}$ and f^{-1}_{small} are a pair of adjoint functors. Suppose that $\alpha : f^{-1}_{small}\mathcal{G} \to \mathcal{F}$ is a map of sheaves on $Y_{étale}$. Let a diagram

$$\begin{tikzcd} U \arrow[r, "j_U"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "j_V"] & Y \end{tikzcd}$$

as in Definition 66.18.9 be given. By the commutativity of the diagram we also get a map $g^{-1}_{small}(j_V)^{-1}\mathcal{G} \to (j_U)^{-1}\mathcal{F}$ (compare Sites, Section 7.25 for the description of the localization functors). Hence we certainly get a map $\varphi_{(V,U,g)} : \mathcal{G}(V) = (j_V)^{-1}\mathcal{G}(V) \to (j_U)^{-1}\mathcal{F}(U) = \mathcal{F}(U)$. We omit the verification that this rule is compatible with further restrictions and defines an f-map from $\mathcal{G}$ to $\mathcal{F}$.

Conversely, suppose that we are given an f-map $\varphi = (\varphi_{(U,V,g)})$. Let $\mathcal{G}'$ (resp. $\mathcal{F}'$) denote the extension of $\mathcal{G}$ (resp. $\mathcal{F}$) to $Y_{spaces,étale}$ (resp. $X_{spaces,étale}$), see Lemma 66.18.3. Then we have to construct a map of sheaves

$$\mathcal{G}' \longrightarrow (f_{spaces,étale})_*\mathcal{F}'$$

To do this, let $V \to Y$ be an étale morphism of algebraic spaces. We have to construct a map of sets

$$\mathcal{G}'(V) \to \mathcal{F}'(X \times_Y V)$$

Choose an étale surjective morphism $V' \to V$ with V' a scheme, and after that choose an étale surjective morphism $U' \to X \times_U V'$ with U' a scheme. We get a morphism of schemes $g' : U' \to V'$ and also a morphism of schemes

$$g'' : U' \times_{X \times_Y V} U' \longrightarrow V' \times_V V'$$

Consider the following diagram

$$\begin{tikzcd} \mathcal{F}'(X \times_Y V) \arrow[r] & \mathcal{F}(U') \arrow[r, shift left] \arrow[r, shift right] & \mathcal{F}(U' \times_{X \times_Y V} U') \\ \mathcal{G}'(X \times_Y V) \arrow[r] \arrow[u, dotted] & \mathcal{G}(V') \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\varphi_{(U',V',g')}"'] & \mathcal{G}(V' \times_V V') \arrow[u, "\varphi_{(U'',V'',g'')}"'] \end{tikzcd}$$

The compatibility of the maps $\varphi_{...}$ with restriction shows that the two right squares commute. The definition of coverings in $X_{spaces,étale}$ shows that the horizontal rows are equalizer diagrams. Hence we get the dotted arrow. We leave it to the reader to show that these arrows are compatible with the restriction mappings. $\square$

If the morphism of algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

03LP Lemma 66.18.11. Let S be a scheme, and let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is étale. In this case there is a functor

$$j : X_{étale} \to Y_{étale}, \quad (\varphi : U \to X) \mapsto (f \circ \varphi : U \to Y)$$

which is cocontinuous. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 7.21.1. Moreover, j is continuous as well, hence Sites, Lemma 7.21.5 applies. In particular $f_{small}^{-1}\mathcal{G}(U) = \mathcal{G}(jU)$ for all sheaves $\mathcal{G}$ on $Y_{étale}$.

Proof. Note that by our very definition of an étale morphism of algebraic spaces (Definition 66.16.2) it is indeed the case that the rule given defines a functor j as indicated. It is clear that j is cocontinuous and continuous, simply because a covering $\{U_i \to U\}$ of $j(\varphi : U \to X)$ in $Y_{étale}$ is the same thing as a covering of $(\varphi : U \to X)$ in $X_{étale}$. It remains to show that j induces the same morphism of topoi as f_{small}. To see this we consider the diagram

$$\begin{tikzcd} X_{étale} \arrow[r] \arrow[d, "j"] & X_{spaces,étale} \arrow[d, bend right, "j_{spaces}"'] \\ Y_{étale} \arrow[r] & Y_{spaces,étale} \arrow[u, bend right, "v:V \mapsto X \times_Y V"'] \end{tikzcd}$$

of categories. Here the functor j_{spaces} is the obvious extension of j to the category $X_{spaces,étale}$. Thus the inner square is commutative. In fact j_{spaces} can be identified with the localization functor $j_X : Y_{spaces,étale}/X \to Y_{spaces,étale}$ discussed in Sites, Section 7.25. Hence, by Sites, Lemma 7.27.2 the cocontinuous functor j_{spaces} and the functor v of the diagram induce the same morphism of topoi. By Sites, Lemma 7.21.2 the commutativity of the inner square (consisting of cocontinuous functors between sites) gives a commutative diagram of associated morphisms of topoi. Hence, by the construction of f_{small} in Lemma 66.18.8 we win. □

The lemma above says that the pullback of $\mathcal{G}$ via an étale morphism $f : X \to Y$ of algebraic spaces is simply the restriction of $\mathcal{G}$ to the category $X_{étale}$. We will often use the short hand

03LQ (66.18.11.1) $$\mathcal{G}|_{X_{étale}} = f_{small}^{-1}\mathcal{G}$$

to indicate this. Note that the functor $j : X_{étale} \to Y_{étale}$ of the lemma in this situation is faithful, but not fully faithful in general. We will discuss this in a more technical fashion in Section 66.27.

03LR Lemma 66.18.12. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. If g is étale, then

(1) $f'_{small,*}(\mathcal{F}|_{X'}) = (f_{small,*}\mathcal{F})|_{Y'}$ in $Sh(Y'_{étale})$[4], and
(2) if $\mathcal{F}$ is an abelian sheaf, then $R^i f'_{small,*}(\mathcal{F}|_{X'}) = (R^i f_{small,*}\mathcal{F})|_{Y'}$.

[4]Also $(f')^{-1}_{small}(\mathcal{G}|_{Y'}) = (f^{-1}_{small}\mathcal{G})|_{X'}$ because of commutativity of the diagram and (66.18.11.1)

Proof. Consider the following diagram of functors

$$\begin{tikzcd} X'_{spaces,étale} \arrow[r, "j"'] & X_{spaces,étale} \\ Y'_{spaces,étale} \arrow[r, "j"] \arrow[u, "V' \mapsto V' \times_{Y'} X'"] & Y_{spaces,étale} \arrow[u, "V \mapsto V \times_Y X"'] \end{tikzcd}$$

The horizontal arrows are localizations and the vertical arrows induce morphisms of sites. Hence the last statement of Sites, Lemma 7.28.1 gives (1). To see (2) apply (1) to an injective resolution of $\mathcal{F}$ and use that restriction is exact and preserves injectives (see Cohomology on Sites, Lemma 21.7.1). □

The following lemma says that you can think of a sheaf on the small étale site of an algebraic space as a compatible collection of sheaves on the small étale sites of schemes étale over the space. Please note that all the comparison mappings c_f in the lemma are isomorphisms, which is compatible with Topologies, Lemma 34.4.20 and the fact that all morphisms between objects of $X_{étale}$ are étale.

03LS Lemma 66.18.13. Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ on $X_{étale}$ is given by the following data:

(1) for every $U \in \mathrm{Ob}(X_{étale})$ a sheaf $\mathcal{F}_U$ on $U_{étale}$,
(2) for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f_{small}^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $c_g \circ g_{small}^{-1}c_f$ is equal to $c_{f\circ g}$.

Proof. We may interpret g_{small}^{-1} as in Lemma 66.18.11. Then the lemma follows from a general fact about sites, see Sites, Lemma 7.26.6. □

Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 65.9.3. In particular, we obtain a groupoid (U, R, s, t, c, e, i) such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 39.13.3.

05YY Lemma 66.18.14. With S, $\varphi : U \to X$, and (U, R, s, t, c, e, i) as above. For any sheaf $\mathcal{F}$ on $X_{étale}$ the sheaf[5] $\mathcal{G} = \varphi^{-1}\mathcal{F}$ comes equipped with a canonical isomorphism

$$\alpha : t^{-1}\mathcal{G} \longrightarrow s^{-1}\mathcal{G}$$

such that the diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{pr}_1^{-1}t^{-1}\mathcal{G} \arrow[r, "\mathrm{pr}_1^{-1}\alpha"'] \arrow[dl, equal] & \mathrm{pr}_1^{-1}s^{-1}\mathcal{G} \arrow[dr, equal] & \\ \mathrm{pr}_0^{-1}s^{-1}\mathcal{G} & & & c^{-1}s^{-1}\mathcal{G} \\ & \mathrm{pr}_0^{-1}t^{-1}\mathcal{G} \arrow[ul, "\mathrm{pr}_0^{-1}\alpha"] \arrow[r, equal] & c^{-1}t^{-1}\mathcal{G} \arrow[ur, "c^{-1}\alpha"'] & \end{tikzcd}$$

is a commutative. The functor $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ defines an equivalence of categories between sheaves on $X_{étale}$ and pairs $(\mathcal{G}, \alpha)$ as above.

[5]In this lemma and its proof we write simply φ^{-1} instead of φ_{small}^{-1} and similarly for all the other pullbacks.

First proof of Lemma 66.18.14. Let $\mathcal{C} = X_{spaces,étale}$. By Lemma 66.18.11 and its proof we have $U_{spaces,étale} = \mathcal{C}/U$ and the pullback functor φ^{-1} is just the restriction functor. Moreover, $\{U \to X\}$ is a covering of the site $\mathcal{C}$ and $R = U \times_X U$. The isomorphism α is just the canonical identification

$$\left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U \times_X U} = \left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U \times_X U}$$

and the commutativity of the diagram is the cocycle condition for glueing data. Hence this lemma is a special case of glueing of sheaves, see Sites, Section 7.26. $\square$

Second proof of Lemma 66.18.14. The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see Lemma 66.18.8. In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism α fits into the commutative diagram. The construction $\mathcal{F} \mapsto (\varphi^{-1}\mathcal{F}, \alpha)$ is clearly functorial in the sheaf $\mathcal{F}$. Hence we obtain the functor.

Conversely, suppose that $(\mathcal{G}, \alpha)$ is a pair. Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Set $\mathcal{G}' = (V' \to V)^{-1}\mathcal{G}$. Since $R = U \times_X U$ with $t = \text{pr}_0$ and $s = \text{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $s', t' : V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : (t')^{-1}\mathcal{G}' \to (s')^{-1}\mathcal{G}'$. Having said this we simply define

$$\mathcal{F}(V) =\!=\!=\!= \text{Equalizer}(\mathcal{G}(V') \rightrightarrows \mathcal{G}(V' \times_V V')).$$

We omit the verification that this defines a sheaf. To see that $\mathcal{G}(V) = \mathcal{F}(V)$ if there exists a morphism $V \to U$ note that in this case the equalizer is $H^0(\{V' \to V\}, \mathcal{G}) = \mathcal{G}(V)$. $\square$

66.19. Points of the small étale site

04JU This section is the analogue of Étale Cohomology, Section 59.29.

0486 **Definition 66.19.1.** Let S be a scheme. Let X be an algebraic space over S.

(1) A *geometric point* of X is a morphism $\overline{x} : \text{Spec}(k) \to X$, where k is an algebraically closed field. We often abuse notation and write $\overline{x} = \text{Spec}(k)$.
(2) For every geometric point $\overline{x}$ we have the corresponding "image" point $x \in |X|$. We say that $\overline{x}$ is a geometric point *lying over* x.

It turns out that we can take stalks of sheaves on $X_{étale}$ at geometric points exactly in the same way as was done in the case of the small étale site of a scheme. In order to do this we define the notion of an étale neighbourhood as follows.

04JV **Definition 66.19.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.

(1) An *étale neighborhood* of $\overline{x}$ of X is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \bar{x} \arrow[ur, "\bar{u}"] \arrow[r, "\bar{x}"] & X \end{tikzcd}$$

where φ is an étale morphism of algebraic spaces over S. We will use the notation $\varphi : (U, \overline{u}) \to (X, \overline{x})$ to indicate this situation.

(2) A morphism of étale neighborhoods $(U, \overline{u}) \to (U', \overline{u}')$ is an X-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

Note that we allow U to be an algebraic space. When we take stalks of a sheaf on $X_{étale}$ we have to restrict to those U which are in $X_{étale}$, and so in this case we will only consider the case where U is a scheme. Alternately we can work with the site $X_{space,étale}$ and consider all étale neighbourhoods. And there won't be any difference because of the last assertion in the following lemma.

04JW Lemma 66.19.3. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. The category of étale neighborhoods is cofiltered. More precisely:

(1) Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{x}$ in X. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.
(2) Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.

Moreover, given any étale neighbourhood $(U, \overline{u}) \to (X, \overline{x})$ there exists a morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ where U' is a scheme.

Proof. For part (1), consider the fibre product $U = U_1 \times_X U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change and composition, see Lemmas 66.16.5 and 66.16.4. The map $\overline{u} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2.

For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_X U'. \end{tikzcd}$$

Since $\overline{u}$ and $\overline{u}'$ agree over X with $\overline{x}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over X, so is the fibre product $U' \times_X U'$ (as seen above in the case of $U_1 \times_X U_2$). Hence the vertical arrow (h_1, h_2) is étale by Lemma 66.16.6. Therefore U'' is étale over U' by base change, and hence also étale over X (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem posed by (2).

To see the final assertion, choose any surjective étale morphism $U' \to U$ where U' is a scheme. Then $U' \times_U \overline{u}$ is a scheme surjective and étale over $\overline{u} = \mathrm{Spec}(k)$ with k algebraically closed. It follows (see Morphisms, Lemma 29.36.7) that $U' \times_U \overline{u} \to \overline{u}$ has a section which gives us the desired $\overline{u}'$. $\square$

05VN Lemma 66.19.4. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point of X lying over $x \in |X|$. Let $\varphi : U \to X$ be an étale morphism of algebraic spaces and let $u \in |U|$ with $\varphi(u) = x$. Then there exists a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = \varphi \circ \overline{u}$.

Proof. Choose an affine scheme U' with $u' \in U'$ and an étale morphism $U' \to U$ which maps u' to u. If we can prove the lemma for $(U', u') \to (X, x)$ then the lemma follows. Hence we may assume that U is a scheme, in particular that $U \to X$ is representable. Then look at the cartesian diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \times_{\overline{x}, X, \varphi} U \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & U \arrow[d, "\varphi"] \\ \mathrm{Spec}(k) \arrow[r, "\overline{x}"] & X \end{tikzcd}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Lemma 66.16.5. Therefore, the scheme $\mathrm{Spec}(k) \times_{\overline{x}, X, \varphi} U$ is a disjoint union of finite separable extensions of k, see Morphisms, Lemma 29.36.7. But k is algebraically closed, so all these extensions are trivial, so $\mathrm{Spec}(k) \times_{\overline{x}, X, \varphi} U$ is a disjoint union of copies of $\mathrm{Spec}(k)$ and each of these corresponds to a geometric point $\overline{u}$ with $\varphi \circ \overline{u} = \overline{x}$. By Lemma 66.4.3 the map

$$|\mathrm{Spec}(k) \times_{\overline{x}, X, \varphi} U| \longrightarrow |\mathrm{Spec}(k)| \times_{|X|} |U|$$

is surjective, hence we can pick $\overline{u}$ to lie over u. ☐

04JX Lemma 66.19.5. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ an étale neighborhood of $\overline{x}$. Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering in $X_{spaces,étale}$. Then there exist $i \in I$ and $\overline{u}_i : \overline{x} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.

Proof. Let $u \in |U|$ be the image of $\overline{u}$. As $|U| = \bigcup_{i \in I} \varphi_i(|U_i|)$ there exists an i and a point $u_i \in U_i$ mapping to x. Apply Lemma 66.19.4 to $(U_i, u_i) \to (U, u)$ and $\overline{u}$ to get the desired geometric point. ☐

04JY Definition 66.19.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $X_{étale}$. Let $\overline{x}$ be a geometric point of X. The stalk of $\mathcal{F}$ at $\overline{x}$ is

$$\mathcal{F}_{\overline{x}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{x}$ in X with $U \in \mathrm{Ob}(X_{étale})$.

By Lemma 66.19.3, this colimit is over a filtered index category, namely the opposite of the category of étale neighborhoods in $X_{étale}$. More precisely Lemma 66.19.3 says the opposite of the category of all étale neighbourhoods is filtered, and the full subcategory of those which are in $X_{étale}$ is a cofinal subcategory hence also filtered.

This means an element of $\mathcal{F}_{\overline{x}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $U \in \mathrm{Ob}(X_{étale})$ and $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$, $U'' \in \mathrm{Ob}(X_{étale})$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.19.

This also implies that if $\mathcal{F}'$ is the sheaf on $X_{spaces,étale}$ corresponding to $\mathcal{F}$ on $X_{étale}$, then

04JZ (66.19.6.1)
$$\mathcal{F}_{\overline{x}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}'(U)$$

where now the colimit is over all the étale neighbourhoods of $\overline{x}$. We will often jump between the point of view of using $X_{étale}$ and $X_{spaces,étale}$ without further mention.

In particular this means that if $\mathcal{F}$ is a presheaf of abelian groups, rings, etc then $\mathcal{F}_{\overline{x}}$ is an abelian group, ring, etc simply by the usual way of defining the group structure on a directed colimit of abelian groups, rings, etc.

04K0 **Lemma 66.19.7.** Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Consider the functor

$$u : X_{étale} \longrightarrow \text{Sets}, \quad U \longmapsto |U_{\overline{x}}|$$

Then u defines a point p of the site $X_{étale}$ (Sites, Definition 7.32.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 7.32.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ defined above.

Proof. In the proof of Lemma 66.19.5 we have seen that the scheme $U_{\overline{x}}$ is a disjoint union of schemes isomorphic to $\overline{x}$. Thus we can also think of $|U_{\overline{x}}|$ as the set of geometric points of U lying over $\overline{x}$, i.e., as the collection of morphisms $\overline{u} : \overline{x} \to U$ fitting into the diagram of Definition 66.19.2. From this it follows that $u(X)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $X_{étale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $X_{étale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 66.19.5. Hence Sites, Proposition 7.33.3 applies, so p is a point of the site $X_{étale}$. Finally, the our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 7.32.1.1 which proves the final assertion. $\square$

04K1 **Lemma 66.19.8.** Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.

(1) The stalk functor $\text{PAb}(X_{étale}) \to \text{Ab}$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.
(2) We have $(\mathcal{F}^\#)_{\overline{x}} = \mathcal{F}_{\overline{x}}$ for any presheaf of sets $\mathcal{F}$ on $X_{étale}$.
(3) The functor $\text{Ab}(X_{étale}) \to \text{Ab}$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.
(4) Similarly the functors $\text{PSh}(X_{étale}) \to \text{Sets}$ and $Sh(X_{étale}) \to \text{Sets}$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.23.1) and commute with arbitrary colimits.

Proof. This result follows from the general material in Modules on Sites, Section 18.36. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ comes from a point of the small étale site of X, see Lemma 66.19.7. See the proof of Étale Cohomology, Lemma 59.29.9 for a direct proof of some of these statements in the setting of the small étale site of a scheme. $\square$

We will see below that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is really the pullback along the morphism $\overline{x}$. In that sense the following lemma is a generalization of the lemma above.

04K2 **Lemma 66.19.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) The functor $f_{small}^{-1} : \text{Ab}(Y_{étale}) \to \text{Ab}(X_{étale})$ is exact.
(2) The functor $f_{small}^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.23.1.
(3) For any étale morphism $V \to Y$ of algebraic spaces we have $f_{small}^{-1} h_V = h_{X \times_Y V}$.
(4) Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification

$$(f_{small}^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$

where $\overline{y} = f \circ \overline{x}$.

Proof. Recall that f_{small} is defined via $f_{spaces,small}$ in Lemma 66.18.8. Parts (1), (2) and (3) are general consequences of the fact that $f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ is a morphism of sites, see Sites, Definition 7.14.1 for (2), Modules on Sites, Lemma 18.31.2 for (1), and Sites, Lemma 7.13.5 for (3).

Proof of (4). This statement is a special case of Sites, Lemma 7.34.2 via Lemma 66.19.7. We also provide a direct proof. Note that by Lemma 66.19.8. taking stalks commutes with sheafification. Let $\mathcal{G}'$ be the sheaf on $Y_{spaces,étale}$ whose restriction to $Y_{étale}$ is $\mathcal{G}$. Recall that $f_{spaces,étale}^{-1}\mathcal{G}'$ is the sheaf associated to the presheaf

$$U \longrightarrow \operatorname{colim}_{U \to X \times_Y V} \mathcal{G}'(V),$$

see Sites, Sections 7.13 and 7.5. Thus we have

$$\begin{aligned}(f_{spaces,étale}^{-1}\mathcal{G}')_{\overline{x}} &= \operatorname{colim}_{(U,\overline{u})} f_{spaces,étale}^{-1}\mathcal{G}'(U) \\ &= \operatorname{colim}_{(U,\overline{u})} \operatorname{colim}_{a : U \to X \times_Y V} \mathcal{G}'(V) \\ &= \operatorname{colim}_{(V,\overline{v})} \mathcal{G}'(V) \\ &= \mathcal{G}'_{\overline{y}}\end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$. Since the stalk of $\mathcal{G}'$ (resp. $f_{spaces,étale}^{-1}\mathcal{G}'$) agrees with the stalk of $\mathcal{G}$ (resp. $f_{small}^{-1}\mathcal{G}$), see Equation (66.19.6.1) the result follows. □

04K3 Remark 66.19.10. This remark is the analogue of Étale Cohomology, Remark 59.56.6. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : \operatorname{Spec}(k) \to X$ be a geometric point of X. By Étale Cohomology, Theorem 59.56.3 the category of sheaves on $\operatorname{Spec}(k)_{étale}$ is equivalent to the category of sets (by taking a sheaf to its global sections). Hence it follows from Lemma 66.19.9 part (4) applied to the morphism $\overline{x}$ that the functor

$$Sh(X_{étale}) \longrightarrow \textit{Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{x}}$$

is isomorphic to the functor

$$Sh(X_{étale}) \longrightarrow Sh(\operatorname{Spec}(k)_{étale}) = \textit{Sets}, \quad \mathcal{F} \longmapsto \overline{x}^*\mathcal{F}$$

Hence we may view the stalk functors as pullback functors along geometric morphisms (and not just some abstract morphisms of topoi as in the result of Lemma 66.19.7).

04K4 Remark 66.19.11. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We claim that for any pair of geometric points $\overline{x}$ and $\overline{x}'$ lying over x the stalk functors are isomorphic. By definition of $|X|$ we can find a third geometric point $\overline{x}''$ so that there exists a commutative diagram

$$\begin{tikzcd} \overline{x}'' \arrow[r] \arrow[d] \arrow[dr, "\overline{x}''"] & \overline{x}' \arrow[d, "\overline{x}'"] \\ \overline{x} \arrow[r, "\overline{x}"] & X. \end{tikzcd}$$

Since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is given by pullback along the morphism $\overline{x}$ (and similarly for the others) we conclude by functoriality of pullbacks.

The following theorem says that the small étale site of an algebraic space has enough points.

04K5 Theorem 66.19.12. Let S be a scheme. Let X be an algebraic space over S. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{x}} : \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective (resp. surjective) for all geometric points of X. A sequence of abelian sheaves on $X_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.

Proof. We know the theorem is true if X is a scheme, see Étale Cohomology, Theorem 59.29.10. Choose a surjective étale morphism $f : U \to X$ where U is a scheme. Since $\{U \to X\}$ is a covering (in $X_{spaces,étale}$) we can check whether a map of sheaves is injective, or surjective by restricting to U. Now if $\overline{u} : \mathrm{Spec}(k) \to U$ is a geometric point of U, then $(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}}$ where $\overline{x} = f \circ \overline{u}$. (This is clear from the colimits defining the stalks at $\overline{u}$ and $\overline{x}$, but it also follows from Lemma 66.19.9.) Hence the result for U implies the result for X and we win. $\square$

The following lemma should be skipped on a first reading.

04K6 Lemma 66.19.13. Let S be a scheme. Let X be an algebraic space over S. Let $p : Sh(pt) \to Sh(X_{étale})$ be a point of the small étale topos of X. Then there exists a geometric point $\overline{x}$ of X such that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$.

Proof. By Sites, Lemma 7.32.7 there is a one to one correspondence between points of the site and points of the associated topos. Hence we may assume that p is given by a functor $u : X_{étale} \to \mathrm{Sets}$ which defines a point of the site $X_{étale}$. Let $U \in \mathrm{Ob}(X_{étale})$ be an object whose structure morphism $j : U \to X$ is surjective. Note that h_U is a sheaf which surjects onto the final sheaf. Since taking stalks is exact we see that $(h_U)_p = u(U)$ is not empty (use Sites, Lemma 7.32.3). Pick $x \in u(U)$. By Sites, Lemma 7.35.1 we obtain a point $q : Sh(pt) \to Sh(U_{étale})$ such that $p = j_{small} \circ q$, so that $\mathcal{F}_p = (\mathcal{F}|_U)_q$ functorially. By Étale Cohomology, Lemma 59.29.12 there is a geometric point $\overline{u}$ of U and a functorial isomorphism $\mathcal{G}_q = \mathcal{G}_{\overline{u}}$ for $\mathcal{G} \in Sh(U_{étale})$. Set $\overline{x} = j \circ \overline{u}$. Then we see that $\mathcal{F}_{\overline{x}} \cong (\mathcal{F}|_U)_{\overline{u}}$ functorially in $\mathcal{F}$ on $X_{étale}$ by Lemma 66.19.9 and we win. $\square$

66.20. Supports of abelian sheaves

04K7 First we talk about supports of local sections.

04K8 Lemma 66.20.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of X (see Sites, Example 7.10.2). Then there exists a unique open $W \subset X$ such that $\mathcal{F} = h_W$.

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to X$ in $\mathrm{Ob}(X_{spaces,étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\varphi(U) \subset X$ is an open subspace (Lemma 66.16.7) and $\{\varphi : U \to \varphi(U)\}$ is a covering in $X_{spaces,étale}$. Take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$ to conclude. $\square$

04K9 Lemma 66.20.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{spaces,étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subspace $W \subset U$ such that

(1) $W \subset U$ is the largest open subspace of U such that $\sigma|_W = 0$,
(2) for every $\varphi : V \to U$ in $X_{étale}$ we have

$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$

(3) for every geometric point $\overline{u}$ of U we have
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 17.5.2. Let $\varphi : V \to U$ be an arrow of $X_{étale}$. Note that $\varphi(V) \subset U$ is an open subspace (Lemma 66.16.7) and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. By Remark 66.19.11 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over x is well defined.

04KA Definition 66.20.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$.

(1) The support of $\mathcal{F}$ is the set of points $x \in |X|$ such that $\mathcal{F}_{\overline{x}} \neq 0$ for any (some) geometric point $\overline{x}$ lying over x.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The support of σ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 66.20.2).

04KB Lemma 66.20.4. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $U \in \mathrm{Ob}(X_{étale})$ and $\sigma \in \mathcal{F}(U)$.

(1) The support of σ is closed in $|X|$.
(2) The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(X)$.
(3) If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $X_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.
(4) The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.
(5) If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.
(6) If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 17.5.2. Part (4) is a direct consequence of Lemma 66.20.2 part (3). Parts (5) and (6) follow from the other parts. □

04KC Lemma 66.20.5. The support of a sheaf of rings on the small étale site of an algebraic space is closed.

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

66.21. The structure sheaf of an algebraic space

04KD The structure sheaf of an algebraic space is the sheaf of rings of the following lemma.

03G6 **Lemma 66.21.1.** Let S be a scheme. Let X be an algebraic space over S. The rule $U \mapsto \Gamma(U, \mathcal{O}_U)$ defines a sheaf of rings on $X_{étale}$.

Proof. Immediate from the definition of a covering and Descent, Lemma 35.8.1. □

03G7 **Definition 66.21.2.** Let S be a scheme. Let X be an algebraic space over S. The *structure sheaf* of X is the sheaf of rings $\mathcal{O}_X$ on the small étale site $X_{étale}$ described in Lemma 66.21.1.

According to Lemma 66.18.13 the sheaf $\mathcal{O}_X$ corresponds to a system of étale sheaves $(\mathcal{O}_X)_U$ for U ranging through the objects of $X_{étale}$. It is clear from the proof of that lemma and our definition that we have simply $(\mathcal{O}_X)_U = \mathcal{O}_U$ where $\mathcal{O}_U$ is the structure sheaf of $U_{étale}$ as introduced in Descent, Definition 35.8.2. In particular, if X is a scheme we recover the sheaf $\mathcal{O}_X$ on the small étale site of X.

Via the equivalence $Sh(X_{étale}) = Sh(X_{spaces,étale})$ of Lemma 66.18.3 we may also think of $\mathcal{O}_X$ as a sheaf of rings on $X_{spaces,étale}$. It is explained in Remark 66.18.4 how to compute $\mathcal{O}_X(Y)$, and in particular $\mathcal{O}_X(X)$, when $Y \to X$ is an object of $X_{spaces,étale}$.

03G8 **Lemma 66.21.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there is a canonical map $f^\sharp : f_{small}^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ such that

$$(f_{small}, f^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

is a morphism of ringed topoi. Furthermore,

1. The construction $f \mapsto (f_{small}, f^\sharp)$ is compatible with compositions.
2. If f is a morphism of schemes, then $f^\sharp$ is the map described in Descent, Remark 35.8.4.

Proof. By Lemma 66.18.10 it suffices to give an f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$. In other words, for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ we have to give a map of rings $(f^\sharp)_{(U,V,g)} : \Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$. Of course we just take $(f^\sharp)_{(U,V,g)} = g^\sharp$. It is clear that this is compatible with restriction mappings and hence indeed gives an f-map. We omit checking compatibility with compositions and agreement with the construction in Descent, Remark 35.8.4. □

0BGS **Lemma 66.21.4.** Let S be a scheme. Let X be an algebraic space over S. The following are equivalent

1. X is reduced,
2. for every $x \in |X|$ the local ring of X at x is reduced (Remark 66.7.6).

In this case $\Gamma(X, \mathcal{O}_X)$ is a reduced ring and if $f \in \Gamma(X, \mathcal{O}_X)$ has $X = V(f)$, then $f = 0$.

Proof. The equivalence of (1) and (2) follows from Properties, Lemma 28.3.2 applied to affine schemes étale over X. The final statements follow the cited lemma and fact that $\Gamma(X, \mathcal{O}_X)$ is a subring of $\Gamma(U, \mathcal{O}_U)$ for some reduced scheme U étale over X. □

66.22. Stalks of the structure sheaf

04KE This section is the analogue of Étale Cohomology, Section 59.33.

04KF Lemma 66.22.1. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{x}$ where U is a scheme. Then we have

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,\overline{u}} = \mathcal{O}_{U,u}^{sh}$$

where the left hand side is the stalk of the structure sheaf of X, and the right hand side is the strict henselization of the local ring of U at the point u at which $\overline{u}$ is centered.

Proof. We know that the structure sheaf $\mathcal{O}_U$ on $U_{étale}$ is the restriction of the structure sheaf of X. Hence the first equality follows from Lemma 66.19.9 part (4). The second equality is explained in Étale Cohomology, Lemma 59.33.1. □

04KG Definition 66.22.2. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$.

(1) The étale local ring of X at $\overline{x}$ is the stalk of the structure sheaf $\mathcal{O}_X$ on $X_{étale}$ at $\overline{x}$. Notation: $\mathcal{O}_{X,\overline{x}}$.
(2) The strict henselization of X at $\overline{x}$ is the scheme $\text{Spec}(\mathcal{O}_{X,\overline{x}})$.

The isomorphism type of the strict henselization of X at $\overline{x}$ (as a scheme over X) depends only on the point $x \in |X|$ and not on the choice of the geometric point lying over x, see Remark 66.19.11.

04KH Lemma 66.22.3. Let S be a scheme. Let X be an algebraic space over S. The small étale site $X_{étale}$ endowed with its structure sheaf $\mathcal{O}_X$ is a locally ringed site, see Modules on Sites, Definition 18.40.4.

Proof. This follows because the stalks $\mathcal{O}_{X,\overline{x}}$ are local, and because $S_{étale}$ has enough points, see Lemmas 66.22.1 and Theorem 66.19.12. See Modules on Sites, Lemma 18.40.2 and 18.40.3 for the fact that this implies the small étale site is locally ringed. □

04N9 Lemma 66.22.4. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent

(1) the dimension of the local ring of X at x (Definition 66.10.2) is d,
(2) $\dim(\mathcal{O}_{X,\overline{x}}) = d$ for some geometric point $\overline{x}$ lying over x, and
(3) $\dim(\mathcal{O}_{X,\overline{x}}) = d$ for any geometric point $\overline{x}$ lying over x.

Proof. The equivalence of (2) and (3) follows from the fact that the isomorphism type of $\mathcal{O}_{X,\overline{x}}$ only depends on $x \in |X|$, see Remark 66.19.11. Using Lemma 66.22.1 the equivalence of (1) and (2)+(3) comes down to the following statement: Given any local ring R we have $\dim(R) = \dim(R^{sh})$. This is More on Algebra, Lemma 15.45.7. □

0A4H **Lemma 66.22.5.** Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Let $x \in X$. Then (1) $\dim_x(X) = \dim_{f(x)}(Y)$ and (2) the dimension of the local ring of X at x equals the dimension of the local ring of Y at $f(x)$. If f is surjective, then (3) $\dim(X) = \dim(Y)$.

Proof. Choose a scheme U and a point $u \in U$ and an étale morphism $U \to X$ which maps u to x. Then the composition $U \to Y$ is also étale and maps u to $f(x)$. Thus the statements (1) and (2) follow as the relevant integers are defined in terms of the behaviour of the scheme U at u. See Definition 66.9.1 for (1). Part (3) is an immediate consequence of (1), see Definition 66.9.2. □

0E01 **Lemma 66.22.6.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent

(1) the local ring of X at x is reduced (Remark 66.7.6),
(2) $\mathcal{O}_{X,\overline{x}}$ is reduced for some geometric point $\overline{x}$ lying over x, and
(3) $\mathcal{O}_{X,\overline{x}}$ is reduced for any geometric point $\overline{x}$ lying over x.

Proof. The equivalence of (2) and (3) follows from the fact that the isomorphism type of $\mathcal{O}_{X,\overline{x}}$ only depends on $x \in |X|$, see Remark 66.19.11. Using Lemma 66.22.1 the equivalence of (1) and (2)+(3) comes down to the following statement: a local ring is reduced if and only if its strict henselization is reduced. This is More on Algebra, Lemma 15.45.4. □

66.23. Local irreducibility

06DJ A point on an algebraic space has a well defined étale local ring, which corresponds to the strict henselization of the local ring in the case of a scheme. In general we cannot see how many irreducible components of a scheme or an algebraic space pass through the given point from the étale local ring. We can only count the number of geometric branches.

06DK **Lemma 66.23.1.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent

(1) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,
(2) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ there is a unique irreducible component of U through u,
(3) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ is unibranch,
(4) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ is geometrically unibranch,
(5) $\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime for any geometric point $\overline{x}$ lying over x.

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. Let $a : U \to X$ and $u \in U$ be as in (1). Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ by Lemma 66.22.1. In particular (4) and (5) are equivalent by More on Algebra, Lemma 15.106.5. The equivalence of (2), (3), and (4) follows from More on Morphisms, Lemma 37.36.2. □

06DL **Definition 66.23.2.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We say that X is *geometrically unibranch at x* if the equivalent conditions of Lemma 66.23.1 hold. We say that X is geometrically unibranch if X is geometrically unibranch at every $x \in |X|$.

This is consistent with the definition for schemes (Properties, Definition 28.15.1).

0DQ3 Lemma 66.23.3. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $n \in \{1, 2, \ldots\}$ be an integer. The following are equivalent

(1) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number of minimal primes of the local ring $\mathcal{O}_{U,u}$ is $\leq n$ and for at least one choice of U, a, u it is n,
(2) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number irreducible components of U passing through u is $\leq n$ and for at least one choice of U, a, u it is n,
(3) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number of branches of U at u is $\leq n$ and for at least one choice of U, a, u it is n,
(4) for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number of geometric branches of U at u is n, and
(5) the number of minimal prime ideals of $\mathcal{O}_{X,\overline{x}}$ is n.

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. Let $a : U \to X$ and $u \in U$ be as in (1). Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ by Lemma 66.22.1. Recall that the (geometric) number of branches of U at u is the number of minimal prime ideals of the (strict) henselization of $\mathcal{O}_{U,u}$. In particular (4) and (5) are equivalent. The equivalence of (2), (3), and (4) follows from More on Morphisms, Lemma 37.36.2. □

0DQ4 Definition 66.23.4. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The number of geometric branches of X at x is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 66.23.3 hold, or else ∞.

66.24. Noetherian spaces

03E9 We have already defined locally Noetherian algebraic spaces in Section 66.7.

03EA Definition 66.24.1. Let S be a scheme. Let X be an algebraic space over S. We say X is Noetherian if X is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic space X is not just quasi-compact and locally Noetherian, but also quasi-separated. This does not conflict with the definition of a Noetherian scheme, as a locally Noetherian scheme is quasi-separated, see Properties, Lemma 28.5.4. This does not hold for algebraic spaces. Namely, $X = \mathbf{A}^1_k/\mathbf{Z}$, see Spaces, Example 65.14.8 is locally Noetherian and quasi-compact but not quasi-separated (hence not Noetherian according to our definitions).

A consequence of the choice made above is that an algebraic space of finite type over a Noetherian algebraic space is not automatically Noetherian, i.e., the analogue of Morphisms, Lemma 29.15.6 does not hold. The correct statement is that an algebraic space of finite presentation over a Noetherian algebraic space is Noetherian (see Morphisms of Spaces, Lemma 67.28.6).

A Noetherian algebraic space X is very close to being a scheme. In the rest of this section we collect some lemmas to illustrate this.

04ZF Lemma 66.24.2. Let S be a scheme. Let X be an algebraic space over S.

(1) If X is locally Noetherian then $|X|$ is a locally Noetherian topological space.
(2) If X is quasi-compact and locally Noetherian, then $|X|$ is a Noetherian topological space.

Proof. Assume X is locally Noetherian. Choose a scheme U and a surjective étale morphism $U \to X$. As X is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 28.5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |X|$ is open and surjective we conclude that $|X|$ is locally Noetherian by Topology, Lemma 5.9.3. This proves (1). If X is quasi-compact and locally Noetherian, then $|X|$ is quasi-compact and locally Noetherian. Hence $|X|$ is Noetherian by Topology, Lemma 5.12.14. □

04ZG Lemma 66.24.3. Let S be a scheme. Let X be an algebraic space over S. If X is Noetherian, then $|X|$ is a sober Noetherian topological space.

Proof. A quasi-separated algebraic space has an underlying sober topological space, see Lemma 66.15.1. It is Noetherian by Lemma 66.24.2. □

08AH Lemma 66.24.4. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\overline{x}$ be a geometric point of X. Then $\mathcal{O}_{X,\overline{x}}$ is a Noetherian local ring.

Proof. Choose an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ where U is a scheme. Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of the local ring of U at u, see Lemma 66.22.1. By our definition of Noetherian spaces the scheme U is locally Noetherian. Hence we conclude by More on Algebra, Lemma 15.45.3. □

66.25. Regular algebraic spaces

06LP We have already defined regular algebraic spaces in Section 66.7.

06LQ Lemma 66.25.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The following are equivalent

(1) X is regular, and
(2) every étale local ring $\mathcal{O}_{X,\overline{x}}$ is regular.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption U is locally Noetherian. Moreover, every étale local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of a local ring on U and conversely, see Lemma 66.22.1. Thus by More on Algebra, Lemma 15.45.10 we see that (2) is equivalent to every local ring of U being regular, i.e., U being a regular scheme (see Properties, Lemma 28.9.2). This equivalent to (1) by Definition 66.7.2. □

We can use Descent, Lemma 35.21.4 to define what it means for an algebraic space X to be regular at a point x.

0AH9 Definition 66.25.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. We say X is regular at x if $\mathcal{O}_{U,u}$ is a regular local ring for any (equivalently some) pair $(a : U \to X, u)$ consisting of an étale morphism $a : U \to X$ from a scheme to X and a point $u \in U$ with $a(u) = x$.

See Definition 66.7.5, Lemma 66.7.4, and Descent, Lemma 35.21.4.

0AHA Lemma 66.25.3. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent

(1) X is regular at x, and
(2) the étale local ring $\mathcal{O}_{X,\overline{x}}$ is regular for any (equivalently some) geometric point $\overline{x}$ lying over x.

Proof. Let U be a scheme, $u \in U$ a point, and let $a : U \to X$ be an étale morphism mapping u to x. For any geometric point $\overline{x}$ of X lying over x, the étale local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of a local ring on U at u, see Lemma 66.22.1. Thus we conclude by More on Algebra, Lemma 15.45.10. $\square$

0BGT Lemma 66.25.4. A regular algebraic space is normal.

Proof. This follows from the definitions and the case of schemes See Properties, Lemma 28.9.4. $\square$

66.26. Sheaves of modules on algebraic spaces

03LT If X is an algebraic space, then a sheaf of modules on X is a sheaf of $\mathcal{O}_X$-modules on the small étale site of X where $\mathcal{O}_X$ is the structure sheaf of X. The category of sheaves of modules is denoted $\text{Mod}(\mathcal{O}_X)$.

Given a morphism $f : X \to Y$ of algebraic spaces, by Lemma 66.21.3 we get a morphism of ringed topoi and hence by Modules on Sites, Definition 18.13.1 we get well defined pullback and direct image functors

03LU (66.26.0.1) $$f^* : \text{Mod}(\mathcal{O}_Y) \longrightarrow \text{Mod}(\mathcal{O}_X), \quad f_* : \text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_Y)$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of algebraic spaces over S, then we have $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$ simply because the morphisms of ringed topoi compose in the corresponding way (by the lemma).

03LV Lemma 66.26.1. Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Then $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$, and $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$ for any sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$. In particular, $f^* : \text{Mod}(\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X)$ is exact.

Proof. By the description of inverse image in Lemma 66.18.11 and the definition of the structure sheaves it is clear that $f_{small}^{-1}\mathcal{O}_Y = \mathcal{O}_X$. Since the pullback

$$f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \otimes_{f_{small}^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

by definition we conclude that $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$. The exactness is clear because f_{small}^{-1} is exact, as f_{small} is a morphism of topoi. $\square$

We continue our abuse of notation introduced in Equation (66.18.11.1) by writing

03LW (66.26.1.1) $$\mathcal{G}|_{X_{\acute{e}tale}} = f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$$

in the situation of the lemma above. We will discuss this in a more technical fashion in Section 66.27.

03LX Lemma 66.26.2. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$. If g is étale, then $f'_*(\mathcal{F}|_{X'}) = (f_*\mathcal{F})|_{Y'}$[6] and $R^if'_*(\mathcal{F}|_{X'}) = (R^if_*\mathcal{F})|_{Y'}$ in $\text{Mod}(\mathcal{O}_{Y'})$.

Proof. This is a reformulation of Lemma 66.18.12 in the case of modules. □

03LY Lemma 66.26.3. Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is given by the following data:

(1) for every $U \in \text{Ob}(X_{\acute{e}tale})$ a sheaf $\mathcal{F}_U$ of $\mathcal{O}_U$-modules on $U_{\acute{e}tale}$,
(2) for every $f : U' \to U$ in $X_{\acute{e}tale}$ an isomorphism $c_f : f^*_{small}\mathcal{F}_U \to \mathcal{F}_{U'}$.

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{\acute{e}tale}$ the composition $c_g \circ g^*_{small}c_f$ is equal to $c_{f\circ g}$.

Proof. Combine Lemmas 66.26.1 and 66.18.13, and use the fact that any morphism between objects of $X_{\acute{e}tale}$ is an étale morphism of schemes. □

66.27. Étale localization

04LX Reading this section should be avoided at all cost.

Let $X \to Y$ be an étale morphism of algebraic spaces. Then X is an object of $Y_{spaces,\acute{e}tale}$ and it is immediate from the definitions, see also the proof of Lemma 66.18.11, that

04LY (66.27.0.1) $$X_{spaces,\acute{e}tale} = Y_{spaces,\acute{e}tale}/X$$

where the right hand side is the localization of the site $Y_{spaces,\acute{e}tale}$ at the object X, see Sites, Definition 7.25.1. Moreover, this identification is compatible with the structure sheaves by Lemma 66.26.1. Hence the ringed site $(X_{spaces,\acute{e}tale}, \mathcal{O}_X)$ is identified with the localization of the ringed site $(Y_{spaces,\acute{e}tale}, \mathcal{O}_Y)$ at the object X:

04LZ (66.27.0.2) $$(X_{spaces,\acute{e}tale}, \mathcal{O}_X) = (Y_{spaces,\acute{e}tale}/X, \mathcal{O}_Y|_{Y_{spaces,\acute{e}tale}/X})$$

The localization of a ringed site used on the right hand side is defined in Modules on Sites, Definition 18.19.1.

Assume now $X \to Y$ is an étale morphism of algebraic spaces and X is a scheme. Then X is an object of $Y_{\acute{e}tale}$ and it follows that

04M0 (66.27.0.3) $$X_{\acute{e}tale} = Y_{\acute{e}tale}/X$$

and

04M1 (66.27.0.4) $$(X_{\acute{e}tale}, \mathcal{O}_X) = (Y_{\acute{e}tale}/X, \mathcal{O}_Y|_{Y_{\acute{e}tale}/X})$$

as above.

Finally, if $X \to Y$ is an étale morphism of algebraic spaces and X is an affine scheme, then X is an object of $Y_{affine,\acute{e}tale}$ and

04M2 (66.27.0.5) $$X_{affine,\acute{e}tale} = Y_{affine,\acute{e}tale}/X$$

and

04M3 (66.27.0.6) $$(X_{affine,\acute{e}tale}, \mathcal{O}_X) = (Y_{affine,\acute{e}tale}/X, \mathcal{O}_Y|_{Y_{affine,\acute{e}tale}/X})$$

as above.

Next, we show that these localizations are compatible with morphisms.

[6] Also $(f')^*(\mathcal{G}|_{Y'}) = (f^*\mathcal{G})|_{X'}$ by commutativity of the diagram and (66.26.1.1)

04M4 **Lemma 66.27.1.** Let S be a scheme. Let

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S with p and q étale. Via the identifications (66.27.0.2) for $U \to X$ and $V \to Y$ the morphism of ringed topoi

$$(g_{spaces,étale}, g^\sharp) : (Sh(U_{spaces,étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{spaces,étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{spaces,étale,c}, f_c^\sharp)$ constructed in Modules on Sites, Lemma 18.20.2 starting with the morphism of ringed sites $(f_{spaces,étale}, f^\sharp)$ and the map $c : U \to V \times_Y X$ corresponding to g.

Proof. The morphism $(f_{spaces,étale,c}, f_c^\sharp)$ is defined as a composition $f' \circ j$ of a localization and a base change map. Similarly g is a composition $U \to V \times_Y X \to V$. Hence it suffices to prove the lemma in the following two cases: (1) $f = \text{id}$, and (2) $U = X \times_Y V$. In case (1) the morphism $g : U \to V$ is étale, see Lemma 66.16.6. Hence $(g_{spaces,étale}, g^\sharp)$ is a localization morphism by the discussion surrounding Equations (66.27.0.1) and (66.27.0.2) which is exactly the content of the lemma in this case. In case (2) the morphism $g_{spaces,étale}$ comes from the morphism of ringed sites given by the functor $V_{spaces,étale} \to U_{spaces,étale}$, $V'/V \mapsto V' \times_V U/U$ which is also what the morphism f' is defined by, see Sites, Lemma 7.28.1. We omit the verification that $(f')^\sharp = g^\sharp$ in this case (both are the restriction of $f^\sharp$ to $U_{spaces,étale}$). $\square$

04M5 **Lemma 66.27.2.** Same notation and assumptions as in Lemma 66.27.1 except that we also assume U and V are schemes. Via the identifications (66.27.0.4) for $U \to X$ and $V \to Y$ the morphism of ringed topoi

$$(g_{small}, g^\sharp) : (Sh(U_{étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{small,s}, f_s^\sharp)$ constructed in Modules on Sites, Lemma 18.22.3 starting with $(f_{small}, f^\sharp)$ and the map $s : h_U \to f_{small}^{-1} h_V$ corresponding to g.

Proof. Note that $(g_{small}, g^\sharp)$ is 2-isomorphic as a morphism of ringed topoi to the morphism of ringed topoi associated to the morphism of ringed sites $(g_{spaces,étale}, g^\sharp)$. Hence we conclude by Lemma 66.27.1 and Modules on Sites, Lemma 18.22.4. $\square$

Finally, we discuss the relationship between sheaves of sets on the small étale site $Y_{étale}$ of an algebraic space Y and algebraic spaces étale over Y. Let S be a scheme and let Y be an algebraic space over S. Let $\mathcal{F}$ be an object of $Sh(Y_{étale})$. Consider the functor

$$X : (Sch/S)_{fppf}^{opp} \longrightarrow \text{Sets}$$

defined by the rule

$$X(T) = \{(y, s) \mid y : T \to Y \text{ is a morphism over } S \text{ and } s \in \Gamma(T, y_{small}^{-1}\mathcal{F})\}$$

Given a morphism $g : T' \to T$ the restriction map sends (y, s) to $(y \circ g, g_{small}^{-1} s)$. This makes sense as $y_{small} \circ g_{small} = (y \circ g)_{small}$ by Lemma 66.18.8.

0GF6 **Lemma 66.27.3.** Let S be a scheme and let Y be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of sets on $Y_{étale}$. Provided a set theoretic condition is satisfied (see proof) the functor X associated to $\mathcal{F}$ above is an algebraic space and there is an étale morphism $f : X \to Y$ of algebraic spaces such that $\mathcal{F} = f_{small,*}*$ where $*$ is the final object of the category $Sh(X_{étale})$ (constant sheaf with value a singleton).

Proof. Let us prove that X is a sheaf for the fppf topology. Namely, suppose that $\{g_i : T_i \to T\}$ is a covering of $(Sch/S)_{fppf}$ and $(y_i, s_i) \in X(T_i)$ satisfy the glueing condition, i.e., the restriction of (y_i, s_i) and (y_j, s_j) to $T_i \times_T T_j$ agree. Then since Y is a sheaf for the fppf topology, we see that the y_i give rise to a unique morphism $y : T \to Y$ such that $y_i = y \circ g_i$. Then we see that $y_{i,small}^{-1}\mathcal{F} = g_{i,small}^{-1}y_{small}^{-1}\mathcal{F}$. Hence the sections s_i glue uniquely to a section of $y_{small}^{-1}\mathcal{F}$ by Étale Cohomology, Lemma 59.39.2.

The construction that sends $\mathcal{F} \in \text{Ob}(Sh(Y_{étale}))$ to $X \in \text{Ob}((Sch/S)_{fppf})$ preserves finite limits and all colimits since each of the functors y_{small}^{-1} have this property. Of course, if $V \in \text{Ob}(Y_{étale})$, then the construction sends the representable sheaf h_V on $Y_{étale}$ to the representable functor represented by V.

By Sites, Lemma 7.12.5 we can find a set I, for each $i \in I$ an object V_i of $Y_{étale}$ and a surjective map of sheaves

$$\coprod h_{V_i} \longrightarrow \mathcal{F}$$

on $Y_{étale}$. The set theoretic condition we need is that the index set I is not too large[7]. Then $V = \coprod V_i$ is an object of $(Sch/S)_{fppf}$ and therefore an object of $Y_{étale}$ and we have a surjective map $h_V \to \mathcal{F}$.

Observe that the product of h_V with itself in $Sh(Y_{étale})$ is $h_{V \times_Y V}$. Consider the fibre product

$$h_V \times_{\mathcal{F}} h_V \subset h_{V \times_Y V}$$

There is an open subscheme R of $V \times_Y V$ such that $h_V \times_{\mathcal{F}} h_V = h_R$, see Lemma 66.20.1 (small detail omitted). By the Yoneda lemma we obtain two morphisms $s, t : R \to V$ in $Y_{étale}$ and we find a coequalizer diagram

$$\begin{tikzcd} h_R \arrow[r, shift left] \arrow[r, shift right] & h_V \arrow[r] & \mathcal{F} \end{tikzcd}$$

in $Sh(Y_{étale})$. Of course the morphisms s, t are étale and define an étale equivalence relation $(t, s) : R \to V \times_S V$.

By the discussion in the preceding two paragraphs we find a coequalizer diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & V \arrow[r] & X \end{tikzcd}$$

in $(Sch/S)_{fppf}$. Thus $X = V/R$ is an algebraic space by Spaces, Theorem 65.10.5. The other statements follow readily from this; details omitted. $\square$

[7]It suffices if the supremum of the cardinalities of the stalks of $\mathcal{F}$ at geometric points of Y is bounded by the size of some object of $(Sch/S)_{fppf}$.

66.28. Recovering morphisms

04KI In this section we prove that the rule which associates to an algebraic space its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 66.28.4.

04KJ **Lemma 66.28.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The morphism of ringed topoi $(f_{small}, f^\sharp)$ associated to f is a morphism of locally ringed topoi, see Modules on Sites, Definition 18.40.9.

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 66.22.3. Moreover, we know that $X_{étale}$ has enough points, see Theorem 66.19.12. Hence it suffices to prove that $(f_{small}, f^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 18.40.8. To see this take a point p of $X_{étale}$. By Lemma 66.19.13 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 66.19.9 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of étale local rings

$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is a local ring map. You can prove this directly, but instead we deduce it from the corresponding result for schemes. To do this choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes, and the vertical arrows are surjective étale (see Spaces, Lemma 65.11.6). Choose a lift $\overline{u} : \overline{x} \to U$ (possible by Lemma 66.19.5). Set $\overline{v} = \psi \circ \overline{u}$. We obtain a commutative diagram of étale local rings

$$\begin{tikzcd} \mathcal{O}_{U,\overline{u}} & \mathcal{O}_{V,\overline{v}} \arrow[l] \\ \mathcal{O}_{X,\overline{x}} \arrow[u] & \mathcal{O}_{Y,\overline{y}}. \arrow[l] \arrow[u] \end{tikzcd}$$

By Étale Cohomology, Lemma 59.40.1 the top horizontal arrow is a local ring map. Finally by Lemma 66.22.1 the vertical arrows are isomorphisms. Hence we win. $\square$

04KK **Lemma 66.28.2.** Let S be a scheme. Let X, Y be algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let t be a 2-morphism from $(f_{small}, f^\sharp)$ to itself, see Modules on Sites, Definition 18.8.1. Then $t = \text{id}$.

Proof. Let X', resp. Y' be X viewed as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2. It is clear from the construction that $(X_{small}, \mathcal{O})$ is equal to $(X'_{small}, \mathcal{O})$ and similarly for Y. Hence we may work with X' and Y'. In other words we may assume that $S = \text{Spec}(\mathbf{Z})$.

Assume $S = \mathrm{Spec}(\mathbf{Z})$, $f : X \to Y$ and t are as in the lemma. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{tikzcd} f_{small}^{-1}\mathcal{O}_Y \arrow[dr, "f^\sharp"'] & & f_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "f^\sharp"] \\ & \mathcal{O}_X & \end{tikzcd}$$

is commutative. Suppose $V \to Y$ is étale with V affine. Write $V = \mathrm{Spec}(B)$. Choose generators $b_j \in B$, $j \in J$ for B as a $\mathbf{Z}$-algebra. Set $T = \mathrm{Spec}(\mathbf{Z}[\{x_j\}_{j \in J}])$. In the following we will use that $\mathrm{Mor}_{Sch}(U, T) = \prod_{j \in J} \Gamma(U, \mathcal{O}_U)$ for any scheme U without further mention. The surjective ring map $\mathbf{Z}[x_j] \to B$, $x_j \mapsto b_j$ corresponds to a closed immersion $V \to T$. We obtain a monomorphism

$$i : V \longrightarrow T_Y = T \times Y$$

of algebraic spaces over Y. In terms of sheaves on $Y_{\acute{e}tale}$ the morphism i induces an injection $h_i : h_V \to \prod_{j \in J} \mathcal{O}_Y$ of sheaves. The base change $i' : X \times_Y V \to T_X$ of i to X is a monomorphism too (Spaces, Lemma 65.5.5). Hence $i' : X \times_Y V \to T_X$ is a monomorphism, which in turn means that $h_{i'} : h_{X \times_Y V} \to \prod_{j \in J} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1} h_V = h_{X \times_Y V}$ of Lemma 66.19.9 the map $h_{i'}$ is equal to

$$f_{small}^{-1} h_V \xrightarrow{f^{-1}h_i} \prod\nolimits_{j \in J} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod\nolimits_{j \in J} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1} h_V \to f_{small}^{-1} h_V$ fits into the commutative diagram

$$\begin{tikzcd} f_{small}^{-1} h_V \arrow[r, "f^{-1}h_i"] \arrow[d, "t"] & \prod_{j \in J} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] \arrow[d, "\prod t"] & \prod_{j \in J} \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\ f_{small}^{-1} h_V \arrow[r, "f^{-1}h_i"] & \prod_{j \in J} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] & \prod_{j \in J} \mathcal{O}_X \end{tikzcd}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{\acute{e}tale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 66.18.6 with Sites, Lemma 7.12.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. $\square$

04M6 Lemma 66.28.3. Let S be a scheme. Let X, Y be algebraic spaces over S. Any two morphisms $a, b : X \to Y$ of algebraic spaces over S for which there exists a 2-isomorphism $(a_{small}, a^\sharp) \cong (b_{small}, b^\sharp)$ in the 2-category of ringed topoi are equal.

Proof. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. We may equivalently think of t as a transformation $t : a_{spaces,\acute{e}tale}^{-1} \to b_{spaces,\acute{e}tale}^{-1}$ since there is not difference between sheaves on $X_{\acute{e}tale}$ and sheaves on $X_{spaces,\acute{e}tale}$. Choose a commutative

diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

where U and V are schemes, and p and q are surjective étale. Consider the diagram

$$\begin{tikzcd} h_U \arrow[r, "\alpha"'] \arrow[d, equal] & a_{spaces,étale}^{-1}h_V \arrow[d, "t"] \\ h_U \arrow[r, dotted] & b_{spaces,étale}^{-1}h_V \end{tikzcd}$$

Since the sheaf $b_{spaces,étale}^{-1}h_V$ is isomorphic to $h_{V \times_{Y,b} X}$ we see that the dotted arrow comes from a morphism of schemes $\beta : U \to V$ fitting into a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\beta"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "b"] & Y \end{tikzcd}$$

We claim that there exists a sequence of 2-isomorphisms

$$\begin{aligned} (\alpha_{small}, \alpha^\sharp) & \cong (\alpha_{spaces,étale}, \alpha^\sharp) \\ & \cong (a_{spaces,étale,c}, a_c^\sharp) \\ & \cong (b_{spaces,étale,d}, b_d^\sharp) \\ & \cong (\beta_{spaces,étale}, \beta^\sharp) \\ & \cong (\beta_{small}, \beta^\sharp) \end{aligned}$$

The first and the last 2-isomorphisms come from the identifications between sheaves on $U_{spaces,étale}$ and sheaves on $U_{étale}$ and similarly for V. The second and fourth 2-isomorphisms are those of Lemma 66.27.1 with $c : U \to X \times_{a,Y} V$ induced by α and $d : U \to X \times_{b,Y} V$ induced by β. The middle 2-isomorphism comes from the transformation t. Namely, the functor $a_{spaces,étale,c}^{-1}$ corresponds to the functor

$$(\mathcal{H} \to h_V) \longmapsto (a_{spaces,étale}^{-1}\mathcal{H} \times_{a_{spaces,étale}^{-1}h_V,\alpha} h_U \to h_U)$$

and similarly for $b_{spaces,étale,d}^{-1}$, see Sites, Lemma 7.28.3. This uses the identification of sheaves on $Y_{spaces,étale}/V$ as arrows $(\mathcal{H} \to h_V)$ in $Sh(Y_{spaces,étale})$ and similarly for U/X, see Sites, Lemma 7.25.4. Via this identification the structure sheaf $\mathcal{O}_V$ corresponds to the pair $(\mathcal{O}_Y \times h_V \to h_V)$ and similarly for $\mathcal{O}_U$, see Modules on Sites, Lemma 18.21.3. Since t switches α and β we see that t induces an isomorphism

$$t : a_{spaces,étale}^{-1}\mathcal{H} \times_{a_{spaces,étale}^{-1}h_V,\alpha} h_U \longrightarrow b_{spaces,étale}^{-1}\mathcal{H} \times_{b_{spaces,étale}^{-1}h_V,\beta} h_U$$

over h_U functorially in $(\mathcal{H} \to h_V)$. Also, t is compatible with $a_c^\sharp$ and $b_d^\sharp$ as t is compatible with $a^\sharp$ and $b^\sharp$ by our description of the structure sheaves $\mathcal{O}_U$ and $\mathcal{O}_V$ above. Hence, the morphisms of ringed topoi $(\alpha_{small}, \alpha^\sharp)$ and $(\beta_{small}, \beta^\sharp)$ are 2-isomorphic. By Étale Cohomology, Lemma 59.40.3 we conclude $\alpha = \beta$! Since $p : U \to X$ is a surjection of sheaves it follows that $a = b$. $\square$

Here is the main result of this section.

04KL Theorem 66.28.4. Let X, Y be algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$. Let

$$(g, g^\sharp) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of algebraic spaces $f : X \to Y$ such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$. In other words, the construction

$$\text{Spaces}/\mathrm{Spec}(\mathbf{Z}) \longrightarrow \text{Locally ringed topoi}, \quad X \longrightarrow (X_{\acute{e}tale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. The uniqueness we have seen in Lemma 66.28.3. Thus it suffices to prove existence. In this proof we will freely use the identifications of Equation (66.27.0.4) as well as the result of Lemma 66.27.2.

Let $U \in \mathrm{Ob}(X_{\acute{e}tale})$, let $V \in \mathrm{Ob}(Y_{\acute{e}tale})$ and let $s \in g^{-1}h_V(U)$ be a section. We may think of s as a map of sheaves $s : h_U \to g^{-1}h_V$. By Modules on Sites, Lemma 18.22.3 we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{tikzcd}
(Sh(X_{\acute{e}tale}/U), \mathcal{O}_U) \arrow[r, "(j, j^\sharp)"'] \arrow[d, "(g_s, g_s^\sharp)"'] & (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \arrow[d, "(g, g^\sharp)"] \\
(Sh(V_{\acute{e}tale}), \mathcal{O}_V) \arrow[r] & (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y).
\end{tikzcd}$$

By Étale Cohomology, Theorem 59.40.5 we obtain a unique morphism of schemes $f_s : U \to V$ such that $(g_s, g_s^\sharp)$ is 2-isomorphic to $(f_{s,small}, f_s^\sharp)$. The construction $(U, V, s) \rightsquigarrow f_s$ just explained satisfies the following functoriality property: Suppose given morphisms $a : U' \to U$ in $X_{\acute{e}tale}$ and $b : V' \to V$ in $Y_{\acute{e}tale}$ and a map $s' : h_{U'} \to g^{-1}h_{V'}$ such that the diagram

$$\begin{tikzcd}
h_{U'} \arrow[r, "s'"'] \arrow[d, "a"'] & g^{-1}h_{V'} \arrow[d, "g^{-1}b"] \\
h_U \arrow[r, "s"] & g^{-1}h_V
\end{tikzcd}$$

commutes. Then the diagram

$$\begin{tikzcd}
U' \arrow[r, "f_{s'}"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\
U \arrow[r, "f_s"] & u(V)
\end{tikzcd}$$

of schemes commutes. The reason this is true is that the same condition holds for the morphisms $(g_s, g_s^\sharp)$ constructed in Modules on Sites, Lemma 18.22.3 and the uniqueness in Étale Cohomology, Theorem 59.40.5.

The problem is to glue the morphisms f_s to a morphism of algebraic spaces. To do this first choose a scheme V and a surjective étale morphism $V \to Y$. This means that $h_V \to *$ is surjective and hence $g^{-1}h_V \to *$ is surjective too. This means there exists a scheme U and a surjective étale morphism $U \to X$ and a morphism $s : h_U \to g^{-1}h_V$. Next, set $R = V \times_Y V$ and $R' = U \times_X U$. Then we get $g^{-1}h_R = g^{-1}h_V \times g^{-1}h_V$ as g^{-1} is exact. Thus s induces a morphism

$s \times s : h_{R'} \to g^{-1}h_R$. Applying the constructions above we see that we get a commutative diagram of morphisms of schemes

$$\begin{tikzcd} R' \arrow[r, "f_{s\times s}"] \arrow[d, shift left] \arrow[d, shift right] & R \arrow[d, shift left] \arrow[d, shift right] \\ U \arrow[r, "f_s"] & V \end{tikzcd}$$

Since we have $X = U/R'$ and $Y = V/R$ (see Spaces, Lemma 65.9.1) we conclude that this diagram defines a morphism of algebraic spaces $f : X \to Y$ fitting into an obvious commutative diagram. Now we still have to show that $(f_{small}, f^\sharp)$ is 2-isomorphic to $(g, g^\sharp)$. Let $t_V : f_{s,small}^{-1} \to g_s^{-1}$ and $t_R : f_{s\times s,small}^{-1} \to g_{s\times s}^{-1}$ be the 2-isomorphisms which are given to us by the construction above. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then we see that t_V defines an isomorphism

$$f_{small}^{-1}\mathcal{G}|_{U_{étale}} = f_{s,small}^{-1}\mathcal{G}|_{V_{étale}} \xrightarrow{t_V} g_s^{-1}\mathcal{G}|_{V_{étale}} = g^{-1}\mathcal{G}|_{U_{étale}}.$$

Moreover, this isomorphism pulled back to R' via either projection $R' \to U$ is the isomorphism

$$f_{small}^{-1}\mathcal{G}|_{R'_{étale}} = f_{s\times s,small}^{-1}\mathcal{G}|_{R_{étale}} \xrightarrow{t_R} g_{s\times s}^{-1}\mathcal{G}|_{R_{étale}} = g^{-1}\mathcal{G}|_{R'_{étale}}.$$

Since $\{U \to X\}$ is a covering in the site $X_{spaces,étale}$ this means the first displayed isomorphism descends to an isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$ of sheaves (small detail omitted). The isomorphism is functorial in $\mathcal{G}$ since t_V and t_R are transformations of functors. Finally, t is compatible with $f^\sharp$ and $g^\sharp$ as t_V and t_R are (some details omitted). This finishes the proof of the theorem. $\square$

05YZ Lemma 66.28.5. Let X, Y be algebraic spaces over $\mathbf{Z}$. If

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

is an isomorphism of ringed topoi, then there exists a unique morphism $f : X \to Y$ of algebraic spaces such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$ and moreover f is an isomorphism of algebraic spaces.

Proof. By Theorem 66.28.4 it suffices to show that $(g, g^\sharp)$ is a morphism of locally ringed topoi. By Modules on Sites, Lemma 18.40.8 (and since the site $X_{étale}$ has enough points) it suffices to check that the map $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ induced by $g^\sharp$ is a local ring map where $q = f \circ p$ and p is any point of $X_{étale}$. As it is an isomorphism this is clear. $\square$

66.29. Quasi-coherent sheaves on algebraic spaces

03G5 In Descent, Sections 35.8, 35.9, and 35.10 we have seen that for a scheme U, there is no difference between a quasi-coherent $\mathcal{O}_U$-module on U, or a quasi-coherent $\mathcal{O}$-module on the small étale site of U. Hence the following definition is compatible with our original notion of a quasi-coherent sheaf on a scheme (Schemes, Section 26.24), when applied to a representable algebraic space.

03G9 Definition 66.29.1. Let S be a scheme. Let X be an algebraic space over S. A quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent module on the ringed site $(X_{étale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 18.23.1. The category of quasi-coherent sheaves on X is denoted $QCoh(\mathcal{O}_X)$.

Note that as being quasi-coherent is an intrinsic notion (see Modules on Sites, Lemma 18.23.2) this is equivalent to saying that the corresponding $\mathcal{O}_X$-module on $X_{spaces,\acute{e}tale}$ is quasi-coherent.

As usual, quasi-coherent sheaves behave well with respect to pullback.

03GA Lemma 66.29.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The pullback functor $f^* : \text{Mod}(\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X)$ preserves quasi-coherent sheaves.

Proof. This is a general fact, see Modules on Sites, Lemma 18.23.4. □

Note that this pullback functor agrees with the usual pullback functor between quasi-coherent sheaves of modules if X and Y happen to be schemes, see Descent, Proposition 35.9.4. Here is the obligatory lemma comparing this with quasi-coherent sheaves on the objects of the small étale site of X.

03LZ Lemma 66.29.3. Let S be a scheme. Let X be an algebraic space over S. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is given by the following data:

(1) for every $U \in \text{Ob}(X_{\acute{e}tale})$ a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_U$ on $U_{\acute{e}tale}$,
(2) for every $f : U' \to U$ in $X_{\acute{e}tale}$ an isomorphism $c_f : f_{small}^*\mathcal{F}_U \to \mathcal{F}_{U'}$.

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{\acute{e}tale}$ the composition $c_g \circ g_{small}^* c_f$ is equal to $c_{f\circ g}$.

Proof. Combine Lemmas 66.29.2 and 66.26.3. □

05VP Lemma 66.29.4. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and let $\overline{x}$ be a geometric point lying over x. Finally, let $\varphi : (U, \overline{u}) \to (X, \overline{x})$ be an étale neighbourhood where U is a scheme. Then

$$(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$$

where $u \in U$ is the image of $\overline{u}$.

Proof. Note that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,u}^{sh}$ by Lemma 66.22.1 hence the tensor product makes sense. Moreover, from Definition 66.19.6 it is clear that

$$\mathcal{F}_{\overline{u}} = \text{colim}(\varphi^*\mathcal{F})_u$$

where the colimit is over $\varphi : (U, \overline{u}) \to (X, \overline{x})$ as in the lemma. Hence there is a canonical map from left to right in the statement of the lemma. We have a similar colimit description for $\mathcal{O}_{X,\overline{x}}$ and by Lemma 66.29.3 we have

$$((\varphi')^*\mathcal{F})_{u'} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U',u'}$$

whenever $(U', \overline{u}') \to (U, \overline{u})$ is a morphism of étale neighbourhoods. To complete the proof we use that $\otimes$ commutes with colimits. □

05VQ Lemma 66.29.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\overline{x}$ be a geometric point of X and let $\overline{y} = f \circ \overline{x}$ be the image in Y. Then there is a canonical isomorphism

$$(f^*\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}}$$

of the stalk of the pullback with the tensor product of the stalk with the local ring of X at $\overline{x}$.

Proof. Since $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \otimes_{f_{small}^{-1}\mathcal{O}_Y} \mathcal{O}_X$ this follows from the description of stalks of pullbacks in Lemma 66.19.9 and the fact that taking stalks commutes with tensor products. A more direct way to see this is as follows. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

where U and V are schemes, and p and q are surjective étale. By Lemma 66.19.4 we can choose a geometric point $\overline{u}$ of U such that $\overline{x} = p \circ \overline{u}$. Set $\overline{v} = \alpha \circ \overline{u}$. Then we see that

$$\begin{aligned} (f^*\mathcal{G})_{\overline{x}} &= (p^*f^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (\alpha^*q^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{Y,\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \\ &= \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \end{aligned}$$

Here we have used Lemma 66.29.4 (twice) and the corresponding result for pullbacks of quasi-coherent sheaves on schemes, see Sheaves, Lemma 6.26.4. □

03M0 Lemma 66.29.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The following are equivalent

(1) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module,
(2) there exists an étale morphism $f : Y \to X$ of algebraic spaces over S with $|f| : |Y| \to |X|$ surjective such that $f^*\mathcal{F}$ is quasi-coherent on Y,
(3) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and
(4) for every affine scheme U and étale morphism $\varphi : U \to X$ the restriction $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module.

Proof. It is clear that (1) implies (2) by considering id_X. Assume $f : Y \to X$ is as in (2), and let $V \to Y$ be a surjective étale morphism from a scheme towards Y. Then the composition $V \to X$ is surjective étale as well and by Lemma 66.29.2 the pullback of $\mathcal{F}$ to V is quasi-coherent as well. Hence we see that (2) implies (3).

Let $U \to X$ be as in (3). Let us use the abuse of notation introduced in Equation (66.26.1.1). As $\mathcal{F}|_{U_{étale}}$ is quasi-coherent there exists an étale covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_{i,étale}}$ has a global presentation, see Modules on Sites, Definition 18.17.1 and Lemma 18.23.3. Let $V \to X$ be an object of $X_{étale}$. Since $U \to X$ is surjective and étale, the family of maps $\{U_i \times_X V \to V\}$ is an étale covering of V. Via the morphisms $U_i \times_X V \to U_i$ we can restrict the global presentations of $\mathcal{F}|_{U_{i,étale}}$ to get a global presentation of $\mathcal{F}|_{(U_i \times_X V)_{étale}}$ Hence the sheaf $\mathcal{F}$ on $X_{étale}$ satisfies the condition of Modules on Sites, Definition 18.23.1 and hence is quasi-coherent.

The equivalence of (3) and (4) comes from the fact that any scheme has an affine open covering. □

03M1 **Lemma 66.29.7.** Let S be a scheme. Let X be an algebraic space over S. The category $QCoh(\mathcal{O}_X)$ of quasi-coherent sheaves on X has the following properties:

(1) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(2) Any colimit of quasi-coherent sheaves is quasi-coherent.
(3) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(4) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(5) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent.
(6) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F}$ is of finite presentation (see Section 66.30), then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.

Proof. If X is a scheme, then this is Descent, Lemma 35.10.3. We will reduce the lemma to this case by étale localization.

Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Our notation will be that $\text{Mod}(\mathcal{O}_U) = \text{Mod}(U_{étale}, \mathcal{O}_U)$ and $QCoh(\mathcal{O}_U) = QCoh(U_{étale}, \mathcal{O}_U)$; in other words, even though U is a scheme we think of quasi-coherent modules on U as modules on the small étale site of U. By Lemma 66.29.2 we have a commutative diagram

$$\begin{tikzcd} QCoh(\mathcal{O}_X) \arrow[r, "\varphi^*"'] \arrow[d] & QCoh(\mathcal{O}_U) \arrow[d] \\ \text{Mod}(\mathcal{O}_X) \arrow[r, "\varphi^*"] & \text{Mod}(\mathcal{O}_U) \end{tikzcd}$$

The bottom horizontal arrow is the restriction functor (66.26.1.1) $\mathcal{G} \mapsto \mathcal{G}|_{U_{étale}}$. This functor has both a left adjoint and a right adjoint, see Modules on Sites, Section 18.19, hence commutes with all limits and colimits. Moreover, we know that an object of $\text{Mod}(\mathcal{O}_X)$ is in $QCoh(\mathcal{O}_X)$ if and only if its restriction to U is in $QCoh(\mathcal{O}_U)$, see Lemma 66.29.6. With these preliminaries out of the way we can start the proof.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of quasi-coherent $\mathcal{O}_X$-modules. By the discussion above we have

$$\left(\bigoplus \mathcal{F}_i\right)|_{U_{étale}} = \bigoplus \mathcal{F}_i|_{U_{étale}}$$

Each of the modules $\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent. Hence the direct sum is quasi-coherent by the case of schemes. Hence $\bigoplus \mathcal{F}_i$ is quasi-coherent as a module restricting to a quasi-coherent module on U.

Proof of (2). Let $\mathcal{I} \to QCoh(\mathcal{O}_X)$, $i \mapsto \mathcal{F}_i$ be a diagram. Then

$$(\operatorname{colim} \mathcal{F}_i)|_{U_{étale}} = \operatorname{colim} \mathcal{F}_i|_{U_{étale}}$$

by the discussion above and we conclude in the same manner.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh(\mathcal{O}_X)$. Then we have $\text{Ker}(a)|_{U_{étale}} = \text{Ker}(a|_{U_{étale}})$ and $\text{Coker}(a)|_{U_{étale}} = \text{Coker}(a|_{U_{étale}})$ and we conclude in the same manner.

Proof of (4). The restriction $0 \to \mathcal{F}_1|_{U_{étale}} \to \mathcal{F}_2|_{U_{étale}} \to \mathcal{F}_3|_{U_{étale}} \to 0$ is short exact. Hence we have the 2-out-of-3 property for this sequence and we conclude as before.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_X)$. Then we have

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_{U_{\acute{e}tale}} = \mathcal{F}|_{U_{\acute{e}tale}} \otimes_{\mathcal{O}_U} \mathcal{G}|_{U_{\acute{e}tale}}$$

and we conclude as before.

Proof of (6). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_X)$ with $\mathcal{F}$ of finite presentation. We have

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_{U_{\acute{e}tale}} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$$

Namely, restriction is a localization, see Section 66.27, especially formula (66.27.0.4)) and formation of internal hom commutes with localization, see Modules on Sites, Lemma 18.27.2. Thus we conclude as before. □

It is in general not the case that the pushforward of a quasi-coherent sheaf along a morphism of algebraic spaces is quasi-coherent. We will return to this issue in Morphisms of Spaces, Section 67.11.

66.30. Properties of modules

05VR In Modules on Sites, Sections 18.17, 18.23, and Definition 18.28.1 we have defined a number of intrinsic properties of modules of $\mathcal{O}$-module on any ringed topos. If X is an algebraic space, we will apply these notions freely to modules on the ringed site $(X_{\acute{e}tale}, \mathcal{O}_X)$, or equivalently on the ringed site $(X_{spaces,\acute{e}tale}, \mathcal{O}_X)$.

Global properties $\mathcal{P}$:

(a) free,
(b) finite free,
(c) generated by global sections,
(d) generated by finitely many global sections,
(e) having a global presentation, and
(f) having a global finite presentation.

Local properties $\mathcal{P}$:

(g) locally free,
(f) finite locally free,
(h) locally generated by sections,
(i) locally generated by r sections,
(j) finite type,
(k) quasi-coherent (see Section 66.29),
(l) of finite presentation,
(m) coherent, and
(n) flat.

Here are some results which follow immediately from the definitions:

(1) In each case, except for $\mathcal{P}$ =“coherent”, the property is preserved under pullback, see Modules on Sites, Lemmas 18.17.2, 18.23.4, and 18.39.1.
(2) Each of the properties above (including coherent) are preserved under pullbacks by étale morphisms of algebraic spaces (because in this case pullback is given by restriction, see Lemma 66.18.11).
(3) Assume $f : Y \to X$ is a surjective étale morphism of algebraic spaces. For each of the local properties (g) – (m), the fact that $f^*\mathcal{F}$ has $\mathcal{P}$ implies that $\mathcal{F}$ has $\mathcal{P}$. This follows as $\{Y \to X\}$ is a covering in $X_{spaces,\acute{e}tale}$ and Modules on Sites, Lemma 18.23.3.

(4) If X is a scheme, $\mathcal{F}$ is a quasi-coherent module on $X_{étale}$, and $\mathcal{P}$ any property except "coherent" or "locally free", then $\mathcal{P}$ for $\mathcal{F}$ on $X_{étale}$ is equivalent to the corresponding property for $\mathcal{F}|_{X_{Zar}}$, i.e., it corresponds to $\mathcal{P}$ for $\mathcal{F}$ when we think of it as a quasi-coherent sheaf on the scheme X. See Descent, Lemma 35.8.10.

(5) If X is a locally Noetherian scheme, $\mathcal{F}$ is a quasi-coherent module on $X_{étale}$, then $\mathcal{F}$ is coherent on $X_{étale}$ if and only if $\mathcal{F}|_{X_{Zar}}$ is coherent, i.e., it corresponds to the usual notion of a coherent sheaf on the scheme X being coherent. See Descent, Lemma 35.8.10.

66.31. Locally projective modules

060P Recall that in Properties, Section 28.21 we defined the notion of a locally projective quasi-coherent module.

060Q Lemma 66.31.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

(1) for some scheme U and surjective étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U, and
(2) for any scheme U and any étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U.

Proof. Let $U \to X$ be as in (1) and let $V \to X$ be étale where V is a scheme. Then $\{U \times_X V \to V\}$ is an fppf covering of schemes. Hence if $\mathcal{F}|_U$ is locally projective, then $\mathcal{F}|_{U \times_X V}$ is locally projective (see Properties, Lemma 28.21.3) and hence $\mathcal{F}|_V$ is locally projective, see Descent, Lemma 35.7.7. $\square$

060R Definition 66.31.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is locally projective if the equivalent conditions of Lemma 66.31.1 are satisfied.

060S Lemma 66.31.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Denote $\psi : U \to V$ the induced morphism. Then

$$f^*\mathcal{G}|_U = \psi^*(\mathcal{G}|_V)$$

Hence the lemma follows from the definition and the result in the case of schemes, see Properties, Lemma 28.21.3. $\square$

66.32. Quasi-coherent sheaves and presentations

03M2 Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 65.9.3. In particular, we obtain a groupoid (U, R, s, t, c), such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 39.13.3. In Groupoids, Definition 39.14.1 we have the defined the notion of a quasi-coherent sheaf on an arbitrary groupoid. With these notions in place we have the following observation.

03M3 **Proposition 66.32.1.** With S, $\varphi : U \to X$, and (U, R, s, t, c) as above. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf $\varphi^*\mathcal{F}$ comes equipped with a canonical isomorphism

$$\alpha : t^*\varphi^*\mathcal{F} \longrightarrow s^*\varphi^*\mathcal{F}$$

which satisfies the conditions of Groupoids, Definition 39.14.1 and therefore defines a quasi-coherent sheaf on (U, R, s, t, c). The functor $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ defines an equivalence of categories

$$\begin{matrix} \text{Quasi-coherent} \\ \mathcal{O}_X\text{-modules} \end{matrix} \longleftrightarrow \begin{matrix} \text{Quasi-coherent modules} \\ \text{on } (U, R, s, t, c) \end{matrix}$$

Proof. In the statement of the proposition, and in this proof we think of a quasi-coherent sheaf on a scheme as a quasi-coherent sheaf on the small étale site of that scheme. This is permissible by the results of Descent, Sections 35.8, 35.9, and 35.10.

The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see discussion surrounding Equation (66.26.0.1). In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism α satisfies condition (1) of Groupoids, Definition 39.14.1. To see condition (2) of the definition it suffices to see that α is an isomorphism which is clear. The construction $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ is clearly functorial in the quasi-coherent sheaf $\mathcal{F}$. Hence we obtain the functor from left to right in the displayed formula of the lemma.

Conversely, suppose that $(\mathcal{F}, \alpha)$ is a quasi-coherent sheaf on (U, R, s, t, c). Let $V \to X$ be an object of $X_{\acute{e}tale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Moreover, the quasi-coherent sheaf $\mathcal{F}$ pulls back to a quasi-coherent sheaf $\mathcal{F}'$ on V'. Since $R = U \times_X U$ with $t = \text{pr}_0$ and $s = \text{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : \text{pr}_0^*\mathcal{F}' \to \text{pr}_1^*\mathcal{F}'$, and the pair $(\mathcal{F}', \alpha')$ is a descend datum for quasi-coherent sheaves with respect to $\{V' \to V\}$. By Descent, Proposition 35.5.2 this descent datum is effective, and we obtain a quasi-coherent $\mathcal{O}_V$-module $\mathcal{F}_V$ on $V_{\acute{e}tale}$. To see that this gives a quasi-coherent sheaf on $X_{\acute{e}tale}$ we have to show (by Lemma 66.29.3) that for any morphism $f : V_1 \to V_2$ in $X_{\acute{e}tale}$ there is a canonical isomorphism $c_f : \mathcal{F}_{V_1} \to \mathcal{F}_{V_2}$ compatible with compositions of morphisms. We omit the verification. We also omit the verification that this defines a functor from the category on the right to the category on the left which is inverse to the functor described above. $\square$

077V **Proposition 66.32.2.** Let S be a scheme. Let X be an algebraic space over S.

(1) The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.
(2) The inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ has a right adjoint[8]

$$Q : \text{Mod}(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

[8]This functor is sometimes called the coherator.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 28.23.4. We advise the reader to read that proof first.

Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Lemma 66.29.7 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $\text{Mod}(\mathcal{O}_X)$. By Modules on Sites, Lemma 18.14.2 filtered colimits are exact. Hence (a) and (b) hold.

To construct a generator, choose a presentation $X = U/R$ so that (U, R, s, t, c) is an étale groupoid scheme and in particular s and t are flat morphisms of schemes. Pick a cardinal κ as in Groupoids, Lemma 39.15.7. Pick a collection $(\mathcal{E}_t, \alpha_t)_{t \in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) as in Groupoids, Lemma 39.15.6. Let $\mathcal{F}_t$ be the quasi-coherent module on X which corresponds to the quasi-coherent module $(\mathcal{E}_t, \alpha_t)$ via the equivalence of categories of Proposition 66.32.1. Then we see that every quasi-coherent module $\mathcal{H}$ is the directed colimit of its quasi-coherent submodules which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.7 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $\text{Mod}(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow \text{Sets}, \quad \mathcal{G} \longmapsto \text{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\text{Hom}_X(\mathcal{G}, \mathcal{F}) = \text{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ is fully faithful. $\square$

66.33. Morphisms towards schemes

05Z0 Here is the analogue of Schemes, Lemma 26.6.4.

05Z1 Lemma 66.33.1. Let X be an algebraic space over $\mathbf{Z}$. Let T be an affine scheme. The map

$$\text{Mor}(X, T) \longrightarrow \text{Hom}(\Gamma(T, \mathcal{O}_T), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. We construct the inverse of the map. Let $\varphi : \Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. Choose a presentation $X = U/R$, see Spaces, Definition 65.9.3. By Schemes, Lemma 26.6.4 the composition

$$\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$$

corresponds to a unique morphism of schemes $g : U \to T$. By the same lemma the two compositions $R \to U \to T$ are equal. Hence we obtain a morphism $f : X =$

$U/R \to T$ such that $U \to X \to T$ equals g. By construction the diagram

$$\begin{tikzcd} \Gamma(U, \mathcal{O}_U) & \Gamma(X, \mathcal{O}_X) \arrow[l] \\ & \Gamma(T, \mathcal{O}_T) \arrow[ul, "g^\sharp"] \arrow[u, "\varphi", "f^\sharp"'] \end{tikzcd}$$

commutes. Hence $f^\sharp$ equals φ because $U \to X$ is an étale covering and $\mathcal{O}_X$ is a sheaf on $X_{étale}$. The uniqueness of f follows from the uniqueness of g. □

66.34. Quotients by free actions

071R Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group. Let $a : G \to \text{Aut}(X)$ be a homomorphism, i.e., a is an action of G on X. We will say the action is free if for every scheme T over S the map

$$G \times X(T) \longrightarrow X(T)$$

is free. (We cannot use a criterion as in Spaces, Lemma 65.14.3 because points may not have well defined residue fields.) In case the action is free we're going to construct the quotient X/G as an algebraic space. This is a special case of the general Bootstrap, Lemma 80.11.7 that we will prove later.

071S Lemma 66.34.1. Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group with a free action on X. Then the quotient sheaf X/G is an algebraic space.

Proof. The statement means that the sheaf F associated to the presheaf

$$T \longmapsto X(T)/G$$

is an algebraic space. To see this we will construct a presentation. Namely, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Set $V = \coprod_{g \in G} U$ and set $\psi : V \to X$ equal to $a(g) \circ \varphi$ on the component corresponding to $g \in G$. Let G act on V by permuting the components, i.e., $g_0 \in G$ maps the component corresponding to g to the component corresponding to $g_0 g$ via the identity morphism of U. Then ψ is a G-equivariant morphism, i.e., we reduce to the case dealt with in the next paragraph.

Assume that there exists a G-action on U and that $U \to X$ is surjective, étale and G-equivariant. In this case there is an induced action of G on $R = U \times_X U$ compatible with the projection mappings $t, s : R \to U$. Now we claim that

$$X/G = U/\coprod\nolimits_{g \in G} R$$

where the map

$$j : \coprod\nolimits_{g \in G} R \longrightarrow U \times_S U$$

is given by $(r, g) \mapsto (t(r), g(s(r)))$. Note that j is a monomorphism: If $(t(r), g(s(r))) = (t(r'), g'(s(r')))$, then $t(r) = t(r')$, hence r and r' have the same image in X under both s and t, hence $g = g'$ (as G acts freely on X), hence $s(r) = s(r')$, hence $r = r'$ (as R is an equivalence relation on U). Moreover j is an equivalence relation (details omitted). Both projections $\coprod_{g \in G} R \to U$ are étale, as s and t are étale. Thus

j is an étale equivalence relation and $U/\coprod_{g \in G} R$ is an algebraic space by Spaces, Theorem 65.10.5. There is a map

$$U/\coprod\nolimits_{g \in G} R \longrightarrow X/G$$

induced by the map $U \to X$. We omit the proof that it is an isomorphism of sheaves. □

66.35. Other chapters

CHAPTER 67

Morphisms of Algebraic Spaces

03H8 67.1. Introduction

03H9 In this chapter we introduce some types of morphisms of algebraic spaces. A reference is [Knu71].

The goal is to extend the definition of each of the types of morphisms of schemes defined in the chapters on schemes, and on morphisms of schemes to the category of algebraic spaces. Each case is slightly different and it seems best to treat them all separately.

67.2. Conventions

040V The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

67.3. Properties of representable morphisms

03HA Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces. In Spaces, Section 65.5 we defined what it means for f to have property $\mathcal{P}$ in case $\mathcal{P}$ is a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 26.18.3, and
(2) is fppf local on the base, see Descent, Definition 35.22.1.

Namely, in this case we say f has property $\mathcal{P}$ if and only if for every scheme U and any morphism $U \to Y$ the morphism of schemes $X \times_Y U \to U$ has property $\mathcal{P}$.

According to the lists in Spaces, Section 65.4 this applies to the following properties: (1)(a) closed immersions, (1)(b) open immersions, (1)(c) quasi-compact immersions, (2) quasi-compact, (3) universally-closed, (4) (quasi-)separated, (5) monomorphism, (6) surjective, (7) universally injective, (8) affine, (9) quasi-affine, (10) (locally) of finite type, (11) (locally) quasi-finite, (12) (locally) of finite presentation, (13) locally of finite type of relative dimension d, (14) universally open, (15) flat, (16) syntomic, (17) smooth, (18) unramified (resp. G-unramified), (19) étale, (20) proper, (21) finite or integral, (22) finite locally free, (23) universally submersive, (24) universal homeomorphism, and (25) immersion.

In this chapter we will redefine these notions for not necessarily representable morphisms of algebraic spaces. Whenever we do this we will make sure that the new definition agrees with the old one, in order to avoid ambiguity.

Note that the definition above applies whenever X is a scheme, since a morphism from a scheme to an algebraic space is representable. And in particular it applies when both X and Y are schemes. In Spaces, Lemma 65.5.3 we have seen that in this case the definitions match, and no ambiguity arise.

Furthermore, in Spaces, Lemma 65.5.5 we have seen that the property of representable morphisms of algebraic spaces so defined is stable under arbitrary base change by a morphism of algebraic spaces. And finally, in Spaces, Lemmas 65.5.4 and 65.5.7 we have seen that if $\mathcal{P}$ is stable under compositions, which holds for the properties (1)(a), (1)(b), (1)(c), (2) – (25), except (13) above, then taking products of representable morphisms preserves property $\mathcal{P}$ and compositions of representable morphisms preserves property $\mathcal{P}$.

We will use these facts below, and whenever we do we will simply refer to this section as a reference.

67.4. Separation axioms

03HJ It makes sense to list some a priori properties of the diagonal of a morphism of algebraic spaces.

03HK **Lemma 67.4.1.** Let S be a scheme contained in Sch_{fppf}. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism. Then

1. $\Delta_{X/Y}$ is representable,
2. $\Delta_{X/Y}$ is locally of finite type,
3. $\Delta_{X/Y}$ is a monomorphism,
4. $\Delta_{X/Y}$ is separated, and
5. $\Delta_{X/Y}$ is locally quasi-finite.

Proof. We are going to use the fact that $\Delta_{X/S}$ is representable (by definition of an algebraic space) and that it satisfies properties (2) – (5), see Spaces, Lemma 65.13.1. Note that we have a factorization

$$X \longrightarrow X \times_Y X \longrightarrow X \times_S X$$

of the diagonal $\Delta_{X/S} : X \to X \times_S X$. Since $X \times_Y X \to X \times_S X$ is a monomorphism, and since $\Delta_{X/S}$ is representable, it follows formally that $\Delta_{X/Y}$ is representable. In particular, the rest of the statements now make sense, see Section 67.3.

Choose a surjective étale morphism $U \to X$, with U a scheme. Consider the diagram

$$\begin{tikzcd}
R = U \times_X U \arrow[r] \arrow[d] & U \times_Y U \arrow[r] \arrow[d] & U \times_S U \arrow[d] \\
X \arrow[r] & X \times_Y X \arrow[r] & X \times_S X
\end{tikzcd}$$

Both squares are cartesian, hence so is the outer rectangle. The top row consists of schemes, and the vertical arrows are surjective étale morphisms. By Spaces, Lemma 65.11.4 the properties (2) – (5) for $\Delta_{X/Y}$ are equivalent to those of $R \to U \times_Y U$. In the proof of Spaces, Lemma 65.13.1 we have seen that $R \to U \times_S U$ has properties (2) – (5). The morphism $U \times_Y U \to U \times_S U$ is a monomorphism of schemes. These facts imply that $R \to U \times_Y U$ have properties (2) – (5).

Namely: For (3), note that $R \to U \times_Y U$ is a monomorphism as the composition $R \to U \times_S U$ is a monomorphism. For (2), note that $R \to U \times_Y U$ is locally of finite type, as the composition $R \to U \times_S U$ is locally of finite type (Morphisms, Lemma 29.15.8). A monomorphism which is locally of finite type is locally quasi-finite because it has finite fibres (Morphisms, Lemma 29.20.7), hence (5). A monomorphism is separated (Schemes, Lemma 26.23.3), hence (4). $\square$

03HL Definition 67.4.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is separated if $\Delta_{X/Y}$ is a closed immersion.
(2) We say f is locally separated[1] if $\Delta_{X/Y}$ is an immersion.
(3) We say f is quasi-separated if $\Delta_{X/Y}$ is quasi-compact.

This definition makes sense since $\Delta_{X/Y}$ is representable, and hence we know what it means for it to have one of the properties described in the definition. We will see below (Lemma 67.4.13) that this definition matches the ones we already have for morphisms of schemes and representable morphisms.

03KK Lemma 67.4.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f is locally separated and f is quasi-separated.

Proof. This is true, via the general principle Spaces, Lemma 65.5.8, because a closed immersion of schemes is an immersion and is quasi-compact. $\square$

03KL Lemma 67.4.4. All of the separation axioms listed in Definition 67.4.2 are stable under base change.

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. By the results of Section 67.3 each of the properties of the diagonal used in Definition 67.4.2 is stable under base change. Hence the lemma is true. $\square$

03KN Lemma 67.4.5. Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then

(1) i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,
(2) if $Z \to T$ is locally separated, then i is an immersion,
(3) if $Z \to T$ is separated, then i is a closed immersion, and
(4) if $Z \to T$ is quasi-separated, then i is quasi-compact.

Proof. By general category theory the following diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "i"'] \arrow[d] & X \times_T Y \arrow[d] \\ Z \arrow[r, "\Delta_{Z/T}"] & Z \times_T Z \end{tikzcd}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 67.4.1, and the material in Section 67.3. $\square$

[1]In the literature this term often refers to quasi-separated and locally separated morphisms.

03KO Lemma 67.4.6. Let S be a scheme. Let T be an algebraic space over S. Let $g : X \to Y$ be a morphism of algebraic spaces over T. Consider the graph $i : X \to X \times_T Y$ of g. Then

(1) i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,
(2) if $Y \to T$ is locally separated, then i is an immersion,
(3) if $Y \to T$ is separated, then i is a closed immersion, and
(4) if $Y \to T$ is quasi-separated, then i is quasi-compact.

Proof. This is a special case of Lemma 67.4.5 applied to the morphism $X = X \times_Y Y \to X \times_T Y$. $\square$

03KP Lemma 67.4.7. Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces over S. Let $s : T \to X$ be a section of f (in a formula $f \circ s = \mathrm{id}_T$). Then

(1) s is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,
(2) if f is locally separated, then s is an immersion,
(3) if f is separated, then s is a closed immersion, and
(4) if f is quasi-separated, then s is quasi-compact.

Proof. This is a special case of Lemma 67.4.6 applied to $g = s$ so the morphism $i = s : T \to T \times_T X$. $\square$

03KQ Lemma 67.4.8. All of the separation axioms listed in Definition 67.4.2 are stable under composition of morphisms.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 67.4.5 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see Section 67.3. $\square$

04ZH Lemma 67.4.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) If Y is separated and f is separated, then X is separated.
(2) If Y is quasi-separated and f is quasi-separated, then X is quasi-separated.
(3) If Y is locally separated and f is locally separated, then X is locally separated.
(4) If Y is separated over S and f is separated, then X is separated over S.
(5) If Y is quasi-separated over S and f is quasi-separated, then X is quasi-separated over S.
(6) If Y is locally separated over S and f is locally separated, then X is locally separated over S.

Proof. Parts (4), (5), and (6) follow immediately from Lemma 67.4.8 and Spaces, Definition 65.13.2. Parts (1), (2), and (3) reduce to parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 66.3.1. $\square$

03KR Lemma 67.4.10. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.

(1) If $g \circ f$ is separated then so is f.
(2) If $g \circ f$ is locally separated then so is f.
(3) If $g \circ f$ is quasi-separated then so is f.

Proof. Consider the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal morphism of $g \circ f$. In any case the last morphism is a monomorphism. Hence for any scheme T and morphism $T \to X \times_Y X$ we have the equality

$$X \times_{(X \times_Y X)} T = X \times_{(X \times_Z X)} T.$$

Hence the result is clear. □

04ZI Lemma 67.4.11. Let S be a scheme. Let X be an algebraic space over S.

(1) If X is separated then X is separated over S.
(2) If X is locally separated then X is locally separated over S.
(3) If X is quasi-separated then X is quasi-separated over S.

Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(4) If X is separated over S then f is separated.
(5) If X is locally separated over S then f is locally separated.
(6) If X is quasi-separated over S then f is quasi-separated.

Proof. Parts (4), (5), and (6) follow immediately from Lemma 67.4.10 and Spaces, Definition 65.13.2. Parts (1), (2), and (3) follow from parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 66.3.1. □

03KM Lemma 67.4.12. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 67.4.2. The following are equivalent

(1) f is $\mathcal{P}$,
(2) for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,
(3) for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,
(4) for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Properties of Spaces, Definition 66.3.1),
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$, and
(6) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.

Proof. We will repeatedly use Lemma 67.4.4 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Let us prove that (3) and (4) are equivalent. Note that if Z is an affine scheme, then the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is a separated morphism as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$. If $Z \times_Y X \to Z$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$ as a composition (see Lemma 67.4.8). Hence the algebraic space $Z \times_Y X$ is $\mathcal{P}$.

Conversely, if the algebraic space $Z \times_Y X$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$, and hence by Lemma 67.4.10 we see that $Z \times_Y X \to Z$ is $\mathcal{P}$.

Let us prove that (3) implies (5). Assume (3). Let V be a scheme and let $V \to Y$ be étale surjective. We have to show that $V \times_Y X \to V$ has property $\mathcal{P}$. In other words, we have to show that the morphism

$$V \times_Y X \longrightarrow (V \times_Y X) \times_V (V \times_Y X) = V \times_Y X \times_Y X$$

has the corresponding property (i.e., is a closed immersion, immersion, or quasi-compact). Let $V = \bigcup V_j$ be an affine open covering of V. By assumption we know that each of the morphisms

$$V_j \times_Y X \longrightarrow V_j \times_Y X \times_Y X$$

does have the corresponding property. Since being a closed immersion, immersion, quasi-compact immersion, or quasi-compact is Zariski local on the target, and since the V_j cover V we get the desired conclusion.

Let us prove that (5) implies (1). Let $V \to Y$ be as in (5). Then we have the fibre product diagram

$$\begin{tikzcd} V \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ V \times_Y X \times_Y X \arrow[r] & X \times_Y X \end{tikzcd}$$

By assumption the left vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact. It follows from Spaces, Lemma 65.5.6 that also the right vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact.

It is clear that (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Choose schemes V_i and surjective étale morphisms $V_i \to Y_i$. Note that the morphisms $V_i \times_Y X \to V_i$ have $\mathcal{P}$ as they are base changes of the morphisms $f^{-1}(Y_i) \to Y_i$. Set $V = \coprod V_i$. Then $V \to Y$ is a morphism as in (5) (details omitted). Hence (6) implies (5) and we are done. $\square$

03KY **Lemma 67.4.13.** Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S.

(1) The morphism f is locally separated.
(2) The morphism f is (quasi-)separated in the sense of Definition 67.4.2 above if and only if f is (quasi-)separated in the sense of Section 67.3.

In particular, if $f : X \to Y$ is a morphism of schemes over S, then f is (quasi-)separated in the sense of Definition 67.4.2 if and only if f is (quasi-)separated as a morphism of schemes.

Proof. This is the equivalence of (1) and (2) of Lemma 67.4.12 combined with the fact that any morphism of schemes is locally separated, see Schemes, Lemma 26.21.2. $\square$

67.5. Surjective morphisms

03MC We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be surjective.

03MD **Lemma 67.5.1.** Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is surjective (in the sense of Section 67.3) if and only if $|f| : |X| \to |Y|$ is surjective.

Proof. Namely, if $f : X \to Y$ is representable, then it is surjective if and only if for every scheme T and every morphism $T \to Y$ the base change $f_T : T \times_Y X \to T$ of f is a surjective morphism of schemes, in other words, if and only if $|f_T|$ is surjective. By Properties of Spaces, Lemma 66.4.3 the map $|T \times_Y X| \to |T| \times_{|Y|} |X|$ is always surjective. Hence $|f_T| : |T \times_Y X| \to |T|$ is surjective if $|f| : |X| \to |Y|$ is surjective. Conversely, if $|f_T|$ is surjective for every $T \to Y$ as above, then by taking T to be the spectrum of a field we conclude that $|X| \to |Y|$ is surjective. $\square$

This clears the way for the following definition.

03ME **Definition 67.5.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *surjective* if the map $|f| : |X| \to |Y|$ of associated topological spaces is surjective.

03MF **Lemma 67.5.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is surjective,
(2) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,
(3) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a surjective morphism,
(5) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is surjective,
(6) there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are surjective étale such that the top horizontal arrow is surjective, and
(7) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is surjective.

Proof. Omitted. $\square$

03MG **Lemma 67.5.4.** The composition of surjective morphisms is surjective.

Proof. This is immediate from the definition. $\square$

03MH **Lemma 67.5.5.** The base change of a surjective morphism is surjective.

Proof. Follows immediately from Properties of Spaces, Lemma 66.4.3. $\square$

67.6. Open morphisms

03Z0 For a representable morphism of algebraic spaces we have already defined (in Section 67.3) what it means to be universally open. Hence before we give the natural definition we check that it agrees with this in the representable case.

03Z1 Lemma 67.6.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent

(1) f is universally open (in the sense of Section 67.3), and
(2) for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is open.

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally open. By Properties of Spaces, Section 66.4 in the commutative diagram

$$\begin{tikzcd} |V \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[d] \\ |V| \arrow[r] & |Z| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) ⇒ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

03Z2 Definition 67.6.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is open if the map of topological spaces $|f| : |X| \to |Y|$ is open.
(2) We say f is universally open if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is open, i.e., the base change $Z \times_Y X \to Z$ is open.

Note that an étale morphism of algebraic spaces is universally open, see Properties of Spaces, Definition 66.16.2 and Lemmas 66.16.7 and 66.16.5.

03Z3 Lemma 67.6.3. The base change of a universally open morphism of algebraic spaces by any morphism of algebraic spaces is universally open.

Proof. This is immediate from the definition. □

03Z4 Lemma 67.6.4. The composition of a pair of (universally) open morphisms of algebraic spaces is (universally) open.

Proof. Omitted. □

03Z5 Lemma 67.6.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is universally open,

(2) for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open,
(3) for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open, and
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces, and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally open.

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. Then we have the following commutative diagram

$$\begin{tikzcd} \coprod_i |W_i \times_Y X| \arrow[r, equal] \arrow[d] & |W \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[r, equal] \arrow[d] & |Z \times_V (V \times_Y X)| \arrow[ld] \\ \coprod |W_i| \arrow[r, equal] & |W| \arrow[r] & |Z| & \end{tikzcd}$$

We have to show the south-east arrow is open. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 66.16.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are open. Hence it follows that the right vertical arrow is open.

Assume $V \to Y$ is as in (4). We will show that f is universally open. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{tikzcd} |(V \times_Y Z) \times_V (V \times_Y X)| \arrow[r, equal] \arrow[rd] & |V \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[d] \\ & |V \times_Y Z| \arrow[r] & |Z| \end{tikzcd}$$

The south-west arrow is open by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 66.16.7). It follows that the right vertical arrow is open.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is open. By a simple topological argument this implies that $Z \times_Y X \to Z$ is open. Hence (1) holds. □

06DN Lemma 67.6.6. Let S be a scheme. Let $p : X \to \mathrm{Spec}(k)$ be a morphism of algebraic spaces over S where k is a field. Then $p : X \to \mathrm{Spec}(k)$ is universally open.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The composition $U \to \mathrm{Spec}(k)$ is universally open (as a morphism of schemes) by Morphisms, Lemma 29.23.4. Let $Z \to \mathrm{Spec}(k)$ be a morphism of schemes. Then $U \times_{\mathrm{Spec}(k)} Z \to X \times_{\mathrm{Spec}(k)} Z$ is surjective, see Lemma 67.5.5. Hence the first of the maps

$$|U \times_{\mathrm{Spec}(k)} Z| \to |X \times_{\mathrm{Spec}(k)} Z| \to |Z|$$

is surjective. Since the composition is open by the above we conclude that the second map is open as well. Whence p is universally open by Lemma 67.6.5. □

67.7. Submersive morphisms

0411 For a representable morphism of algebraic spaces we have already defined (in Section 67.3) what it means to be universally submersive. Hence before we give the natural definition we check that it agrees with this in the representable case.

0CFQ **Lemma 67.7.1.** Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent

(1) f is universally submersive (in the sense of Section 67.3), and
(2) for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is submersive.

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally submersive. By Properties of Spaces, Section 66.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_Y X|} \arrow[r] \arrow[d] & {|Z \times_Y X|} \arrow[d] \\ {|V|} \arrow[r] & {|Z|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is submersive it follows that the right vertical arrow is submersive. This proves (2). The implication (2) ⇒ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

0412 **Definition 67.7.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *submersive*[2] if the continuous map $|X| \to |Y|$ is submersive, see Topology, Definition 5.6.3.
(2) We say f is *universally submersive* if for every morphism of algebraic spaces $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

0CFR **Lemma 67.7.3.** The base change of a universally submersive morphism of algebraic spaces by any morphism of algebraic spaces is universally submersive.

Proof. This is immediate from the definition. □

0CFS **Lemma 67.7.4.** The composition of a pair of (universally) submersive morphisms of algebraic spaces is (universally) submersive.

Proof. Omitted. □

[2] This is very different from the notion of a submersion of differential manifolds.

67.8. Quasi-compact morphisms

03HC By Section 67.3 we know what it means for a representable morphism of algebraic spaces to be quasi-compact. In order to formulate the definition for a general morphism of algebraic spaces we make the following observation.

03HD **Lemma 67.8.1.** Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent:

(1) f is quasi-compact (in the sense of Section 67.3), and
(2) for every quasi-compact algebraic space Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact.

Proof. Assume (1), and let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. By Properties of Spaces, Definition 66.5.1 there exists a quasi-compact scheme U and a surjective étale morphism $U \to Z$. Since f is representable and quasi-compact we see by definition that $U \times_Y X$ is a scheme, and that $U \times_Y X \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact scheme. The morphism $U \times_Y X \to Z \times_Y X$ is étale and surjective (as the base change of the representable étale and surjective morphism $U \to Z$, see Section 67.3). Hence by definition $Z \times_Y X$ is quasi-compact.

Assume (2). Let $Z \to Y$ be a morphism, where Z is a scheme. We have to show that $p : Z \times_Y X \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_Y Z$ and the scheme $U \times_Y Z$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Schemes, Section 26.19. □

This motivates the following definition.

03HE **Definition 67.8.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-compact* if for every quasi-compact algebraic space Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is quasi-compact.

By Lemma 67.8.1 above this agrees with the already existing notion for representable morphisms of algebraic spaces.

0EMK **Lemma 67.8.3.** Let S be a scheme. If $f : X \to Y$ is a quasi-compact morphism of algebraic spaces over S, then the underlying map $|f| : |X| \to |Y|$ of topological space is quasi-compact.

Proof. Let $V \subset |Y|$ be quasi-compact open. By Properties of Spaces, Lemma 66.4.8 there is an open subspace $Y' \subset Y$ with $V = |Y'|$. Then Y' is a quasi-compact algebraic space by Properties of Spaces, Lemma 66.5.2 and hence $X' = Y' \times_Y X$ is a quasi-compact algebraic space by Definition 67.8.2. On the other hand, $X' \subset X$ is an open subspace (Spaces, Lemma 65.12.3) and $|X'| = |f|^{-1}(|X'|) = |f|^{-1}(V)$ by Properties of Spaces, Lemma 66.4.3. We conclude using Properties of Spaces, Lemma 66.5.2 again that $|X'|$ is a quasi-compact open of $|X|$ as desired. □

03HF **Lemma 67.8.4.** The base change of a quasi-compact morphism of algebraic spaces by any morphism of algebraic spaces is quasi-compact.

Proof. Omitted. Hint: Transitivity of fibre products. □

03HG **Lemma 67.8.5.** The composition of a pair of quasi-compact morphisms of algebraic spaces is quasi-compact.

Proof. Omitted. Hint: Transitivity of fibre products. ☐

040W Lemma 67.8.6. Let S be a scheme.

(1) If $X \to Y$ is a surjective morphism of algebraic spaces over S, and X is quasi-compact then Y is quasi-compact.
(2) If
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$
is a commutative diagram of morphisms of algebraic spaces over S and f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Assume X is quasi-compact and $X \to Y$ is surjective. By Definition 67.5.2 the map $|X| \to |Y|$ is surjective, hence we see Y is quasi-compact by Properties of Spaces, Lemma 66.5.2 and the topological fact that the image of a quasi-compact space under a continuous map is quasi-compact, see Topology, Lemma 5.12.7. Let f, p, q be as in (2). Let $T \to Z$ be a morphism whose source is a quasi-compact algebraic space. By assumption $T \times_Z X$ is quasi-compact. By Lemma 67.5.5 the morphism $T \times_Z X \to T \times_Z Y$ is surjective. Hence by part (1) we see $T \times_Z Y$ is quasi-compact too. Thus q is quasi-compact. ☐

04ZJ Lemma 67.8.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a universally open and surjective morphism of algebraic spaces such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.

Proof. Let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. As g is universally open and surjective, we see that $Y' \times_Y Z \to Z$ is open and surjective. As every point of $|Y' \times_Y Z|$ has a fundamental system of quasi-compact open neighbourhoods (see Properties of Spaces, Lemma 66.5.5) we can find a quasi-compact open $W \subset |Y' \times_Y Z|$ which surjects onto Z. Denote $f'' : W \times_Y X \to W$ the base change of f' by $W \to Y'$. By assumption $W \times_Y X$ is quasi-compact. As $W \to Z$ is surjective we see that $W \times_Y X \to Z \times_Y X$ is surjective. Hence $Z \times_Y X$ is quasi-compact by Lemma 67.8.6. Thus f is quasi-compact. ☐

03KG Lemma 67.8.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is quasi-compact,
(2) for every scheme Z and any morphism $Z \to Y$ the morphism of algebraic spaces $Z \times_Y X \to Z$ is quasi-compact,
(3) for every affine scheme Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a quasi-compact morphism of algebraic spaces, and
(5) there exists a surjective étale morphism $Y' \to Y$ of algebraic spaces such that $Y' \times_Y X \to Y'$ is a quasi-compact morphism of algebraic spaces, and
(6) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-compact.

Proof. We will use Lemma 67.8.4 without further mention. It is clear that (1) implies (2) and that (2) implies (3). Assume (3). Let Z be a quasi-compact algebraic space over S, and let $Z \to Y$ be a morphism. By Properties of Spaces, Lemma 66.6.3 there exists an affine scheme U and a surjective étale morphism $U \to Z$. Then $U \times_Y X \to Z \times_Y X$ is a surjective morphism of algebraic spaces, see Lemma 67.5.5. By assumption $|U \times_Y X|$ is quasi-compact. It surjects onto $|Z \times_Y X|$, hence we conclude that $|Z \times_Y X|$ is quasi-compact, see Topology, Lemma 5.12.7. This proves that (3) implies (1).

The implications (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) are clear. The implication (5) $\Rightarrow$ (1) follows from Lemma 67.8.7 and the fact that an étale morphism of algebraic spaces is universally open (see discussion following Definition 67.6.2).

Of course (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Let Z be affine and let $Z \to Y$ be a morphism. Then there exists a finite standard affine covering $Z = Z_1 \cup \ldots \cup Z_n$ such that each $Z_j \to Y$ factors through Y_{i_j} for some i_j. Hence the algebraic space

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} f^{-1}(Y_{i_j})$$

is quasi-compact. Since $Z \times_Y X = \bigcup_{j=1,\ldots,n} Z_j \times_Y X$ is a Zariski covering we see that $|Z \times_Y X| = \bigcup_{j=1,\ldots,n} |Z_j \times_Y X|$ (see Properties of Spaces, Lemma 66.4.8) is a finite union of quasi-compact spaces, hence quasi-compact. Thus we see that (6) implies (3). $\square$

The following (and the next) lemma guarantees in particular that a morphism $X \to \mathrm{Spec}(A)$ is quasi-compact as soon as X is a quasi-compact algebraic space

03KS Lemma 67.8.9. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 67.4.7 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 67.4.4). The second map is quasi-compact as it is the base change of f, see Lemma 67.8.4. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 67.8.5. $\square$

073B Lemma 67.8.10. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S.

(1) If X is quasi-compact and Y is quasi-separated, then f is quasi-compact.
(2) If X is quasi-compact and quasi-separated and Y is quasi-separated, then f is quasi-compact and quasi-separated.
(3) A fibre product of quasi-compact and quasi-separated algebraic spaces is quasi-compact and quasi-separated.

Proof. Part (1) follows from Lemma 67.8.9 with $Z = S = \mathrm{Spec}(\mathbf{Z})$. Part (2) follows from (1) and Lemma 67.4.10. For (3) let $X \to Y$ and $Z \to Y$ be morphisms of quasi-compact and quasi-separated algebraic spaces. Then $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of $X \to Y$ using (2) and Lemmas 67.8.4 and 67.4.4. Hence $X \times_Y Z$ is quasi-compact and quasi-separated as an algebraic space quasi-compact and quasi-separated over Z, see Lemmas 67.4.9 and 67.8.5. $\square$

67.9. Universally closed morphisms

03HH For a representable morphism of algebraic spaces we have already defined (in Section 67.3) what it means to be universally closed. Hence before we give the natural definition we check that it agrees with this in the representable case.

03XD **Lemma 67.9.1.** Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent

(1) f is universally closed (in the sense of Section 67.3), and
(2) for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is closed.

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally closed. By Properties of Spaces, Section 66.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_Y X|} \arrow[r] \arrow[d] & {|Z \times_Y X|} \arrow[d] \\ {|V|} \arrow[r] & {|Z|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. $\square$

Thus we may use the following natural definition.

03HI **Definition 67.9.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *closed* if the map of topological spaces $|X| \to |Y|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces

$$|Z \times_Y X| \to |Z|$$

is closed, i.e., the base change $Z \times_Y X \to Z$ is closed.

03IS **Lemma 67.9.3.** The base change of a universally closed morphism of algebraic spaces by any morphism of algebraic spaces is universally closed.

Proof. This is immediate from the definition. $\square$

03IU **Lemma 67.9.4.** The composition of a pair of (universally) closed morphisms of algebraic spaces is (universally) closed.

Proof. Omitted. $\square$

03IT **Lemma 67.9.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is universally closed,
(2) for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,

(3) for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces, and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally closed.

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. Then we have the following commutative diagram

$$\begin{tikzcd} \coprod_i |W_i \times_Y X| \arrow[r, equal] \arrow[d] & |W \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[r, equal] \arrow[d] & |Z \times_V (V \times_Y X)| \arrow[dl] \\ \coprod |W_i| \arrow[r, equal] & |W| \arrow[r] & |Z| & \end{tikzcd}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 66.16.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{tikzcd} |(V \times_Y Z) \times_V (V \times_Y X)| \arrow[r, equal] \arrow[dr] & |V \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[d] \\ & |V \times_Y Z| \arrow[r] & |Z| \end{tikzcd}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 66.16.7). It follows that the right vertical arrow is closed.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is closed. By a simple topological argument this implies that $Z \times_Y X \to Z$ is closed. Hence (1) holds. $\square$

03IV Example 67.9.6. Strange example of a universally closed morphism. Let $\mathbf{Q} \subset k$ be a field of characteristic zero. Let $X = \mathbf{A}^1_k/\mathbf{Z}$ as in Spaces, Example 65.14.8. We claim the structure morphism $p : X \to \mathrm{Spec}(k)$ is universally closed. Namely, if Z/k is a scheme, and $T \subset |X \times_k Z|$ is closed, then T corresponds to a $\mathbf{Z}$-invariant closed subset of $T' \subset |\mathbf{A}^1 \times Z|$. It is easy to see that this implies that T' is the inverse image of a subset T'' of Z. By Morphisms, Lemma 29.25.12 we have that $T'' \subset Z$ is closed. Of course T'' is the image of T. Hence p is universally closed by Lemma 67.9.5.

04XW **Lemma 67.9.7.** Let S be a scheme. A universally closed morphism of algebraic spaces over S is quasi-compact.

Proof. This proof is a repeat of the proof in the case of schemes, see Morphisms, Lemma 29.41.8. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Lemma 67.8.8 there exists an affine scheme Z and a morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is not quasi-compact. To achieve our goal it suffices to show that $Z \times_Y X \to Z$ is not universally closed, hence we may assume that $Y = \mathrm{Spec}(B)$ for some ring B.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are quasi-compact open subspaces of X. For example, choose a surjective étale morphism $U \to X$ where U is a scheme, choose an affine open covering $U = \bigcup U_i$ and let $X_i \subset X$ be the image of U_i. We will use later that the morphisms $X_i \to Y$ are quasi-compact, see Lemma 67.8.9. Let $T = \mathrm{Spec}(B[a_i; i \in I])$. Let $T_i = D(a_i) \subset T$. Let $Z \subset T \times_Y X$ be the reduced closed subspace whose underlying closed set of points is $|T \times_Y Z| \setminus \bigcup_{i \in I} |T_i \times_Y X_i|$, see Properties of Spaces, Lemma 66.12.3. (Note that $T_i \times_Y X_i$ is an open subspace of $T \times_Y X$ as $T_i \to T$ and $X_i \to X$ are open immersions, see Spaces, Lemmas 65.12.3 and 65.12.2.) Here is a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[rd] & T \times_Y X \arrow[r, "q"] \arrow[d, "f_T"] & X \arrow[d, "f"] \\ & T \arrow[r, "p"] & Y \end{tikzcd}$$

It suffices to prove that the image $f_T(|Z|)$ is not closed in $|T|$.

We claim there exists a point $y \in Y$ such that there is no affine open neighborhood V of y in Y such that X_V is quasi-compact. If not then we can cover Y with finitely many such V and for each V the morphism $Y_V \to V$ is quasi-compact by Lemma 67.8.9 and then Lemma 67.8.8 implies f quasi-compact, a contradiction. Fix a $y \in Y$ as in the claim.

Let $t \in T$ be the point lying over y with $\kappa(t) = \kappa(y)$ such that $a_i = 1$ in $\kappa(t)$ for all i. Suppose $z \in |Z|$ with $f_T(z) = t$. Then $q(t) \in X_i$ for some i. Hence $f_T(z) \notin T_i$ by construction of Z, which contradicts the fact that $t \in T_i$ by construction. Hence we see that $t \in |T| \setminus f_T(|Z|)$.

Assume $f_T(|Z|)$ is closed in $|T|$. Then there exists an element $g \in B[a_i; i \in I]$ with $f_T(|Z|) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(y)$. Hence this coefficient is invertible on some affine open neighborhood V of y. Let J be the finite set of $j \in I$ such that the variable a_j appears in g. Since X_V is not quasi-compact and each $X_{i,V}$ is quasi-compact, we may choose a point $x \in |X_V| \setminus \bigcup_{j \in J} |X_{j,V}|$. In other words, $x \in |X| \setminus \bigcup_{j \in J} |X_j|$ and x lies above some $v \in V$. Since g has a coefficient that is invertible on V, we can find a point $t' \in T$ lying above v such that $t' \notin V(g)$ and $t' \in V(a_i)$ for all $i \notin J$. This is true because $V(a_i; i \in I \setminus J) = \mathrm{Spec}(B[a_j; j \in J])$ and the set of points of this scheme lying over v is bijective with $\mathrm{Spec}(\kappa(v)[a_j; j \in J])$ and g restricts to a nonzero element of this polynomial ring by construction. In other words $t' \notin T_i$ for each $i \notin J$. By Properties of Spaces, Lemma 66.4.3 we can find a point z of $X \times_Y T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin |X_j|$ for $j \in J$

and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in |Z|$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(|Z|)$ closed" is wrong and we conclude indeed that f_T is not closed as desired. □

The target of a separated algebraic space under a surjective universally closed morphism is separated.

05Z2 Lemma 67.9.8. Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a surjective universally closed morphism of algebraic spaces over B.

(1) If X is quasi-separated, then Y is quasi-separated.
(2) If X is separated, then Y is separated.
(3) If X is quasi-separated over B, then Y is quasi-separated over B.
(4) If X is separated over B, then Y is separated over B.

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = B = \mathrm{Spec}(\mathbf{Z})$ (see Properties of Spaces, Definition 66.3.1). Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/B}"'] \arrow[d] & X \times_B X \arrow[d] \\ Y \arrow[r, "\Delta_{Y/B}"] & Y \times_B Y \end{tikzcd}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_B X \to X \times_B Y \to Y \times_B Y$. Hence it is also quasi-compact, see Lemma 67.9.7.

Assume X is quasi-separated over B, i.e., $\Delta_{X/B}$ is quasi-compact. Then if Z is quasi-compact and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/B}$ is quasi-compact, i.e., Y is quasi-separated over B.

Assume X is separated over B, i.e., $\Delta_{X/B}$ is a closed immersion. Then if Z is affine, and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X \to Z$ is universally closed by our remarks above. We conclude that $\Delta_{Y/B}$ is universally closed. It follows that $\Delta_{Y/B}$ is representable, locally of finite type, a monomorphism (see Lemma 67.4.1) and universally closed, hence a closed immersion, see Étale Morphisms, Lemma 41.7.2 (and also the abstract principle Spaces, Lemma 65.5.8). Thus Y is separated over B. □

67.10. Monomorphisms

042K A representable morphism $X \to Y$ of algebraic spaces is a monomorphism according to Section 67.3 if for every scheme Z and morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is representable by a monomorphism of schemes. This means exactly that $Z \times_Y X \to Z$ is an injective map of sheaves on $(Sch/S)_{fppf}$. Since this is supposed to hold for all Z and all maps $Z \to Y$ this is in turn equivalent to the map $X \to Y$ being an injective map of sheaves on $(Sch/S)_{fppf}$. Thus we may define a monomorphism of a (possibly nonrepresentable[3]) morphism of algebraic spaces as follows.

[3]We do not know whether any monomorphism of algebraic spaces is representable. For a discussion see More on Morphisms of Spaces, Section 76.4.

042L **Definition 67.10.1.** Let S be a scheme. A morphism of algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves, i.e., a monomorphism in the category of sheaves on $(Sch/S)_{fppf}$.

The following lemma shows that this also means that it is a monomorphism in the category of algebraic spaces over S.

042M **Lemma 67.10.2.** Let S be a scheme. Let $j : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) j is a monomorphism (as in Definition 67.10.1),
(2) j is a monomorphism in the category of algebraic spaces over S, and
(3) the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.

Proof. Note that $X \times_Y X$ is both the fibre product in the category of sheaves on $(Sch/S)_{fppf}$ and the fibre product in the category of algebraic spaces over S, see Spaces, Lemma 65.7.3. The equivalence of (1) and (3) is a general characterization of injective maps of sheaves on any site. The equivalence of (2) and (3) is a characterization of monomorphisms in any category with fibre products. □

042N **Lemma 67.10.3.** A monomorphism of algebraic spaces is separated.

Proof. This is true because an isomorphism is a closed immersion, and Lemma 67.10.2 above. □

042O **Lemma 67.10.4.** A composition of monomorphisms is a monomorphism.

Proof. True because a composition of injective sheaf maps is injective. □

042P **Lemma 67.10.5.** The base change of a monomorphism is a monomorphism.

Proof. This is a general fact about fibre products in a category of sheaves. □

042Q **Lemma 67.10.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is a monomorphism,
(2) for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,
(3) for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ is a monomorphism, and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a monomorphism.

Proof. We will use without further mention that a base change of a monomorphism is a monomorphism, see Lemma 67.10.5. In particular it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) (by taking V to be a disjoint union of affine schemes étale over Y, see Properties of Spaces, Lemma 66.6.1). Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"] \arrow[d, "b"] & \mathcal{G} \arrow[d, "c"] \\ \mathcal{H} \arrow[r, "d"] & \mathcal{I} \end{tikzcd} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. Thus (4) implies (1). Proof of the equivalence of (5) and (1) is omitted. $\square$

042R Lemma 67.10.7. An immersion of algebraic spaces is a monomorphism. In particular, any immersion is separated.

Proof. Let $f : X \to Y$ be an immersion of algebraic spaces. For any morphism $Z \to Y$ with Z representable the base change $Z \times_Y X \to Z$ is an immersion of schemes, hence a monomorphism, see Schemes, Lemma 26.23.8. Hence f is representable, and a monomorphism. $\square$

We will improve on the following lemma in Decent Spaces, Lemma 68.19.1.

06MG Lemma 67.10.8. Let S be a scheme. Let k be a field and let $Z \to \mathrm{Spec}(k)$ be a monomorphism of algebraic spaces over S. Then either $Z = \emptyset$ or $Z = \mathrm{Spec}(k)$.

Proof. By Lemmas 67.10.3 and 67.4.9 we see that Z is a separated algebraic space. Hence there exists an open dense subspace $Z' \subset Z$ which is a scheme, see Properties of Spaces, Proposition 66.13.3. By Schemes, Lemma 26.23.11 we see that either $Z' = \emptyset$ or $Z' \cong \mathrm{Spec}(k)$. In the first case we conclude that $Z = \emptyset$ and in the second case we conclude that $Z' = Z = \mathrm{Spec}(k)$ as $Z \to \mathrm{Spec}(k)$ is a monomorphism which is an isomorphism over Z'. $\square$

06RV Lemma 67.10.9. Let S be a scheme. If $X \to Y$ is a monomorphism of algebraic spaces over S, then $|X| \to |Y|$ is injective.

Proof. Immediate from the definitions. $\square$

67.11. Pushforward of quasi-coherent sheaves

03M7 We first prove a simple lemma that relates pushforward of sheaves of modules for a morphism of algebraic spaces to pushforward of sheaves of modules for a morphism of schemes.

03M8 Lemma 67.11.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \to X$ be a surjective étale morphism from a scheme to X. Set $R = U \times_X U$ and denote $t, s : R \to U$ the projection morphisms as usual. Denote $a : U \to Y$ and $b : R \to Y$ the induced morphisms. For any object $\mathcal{F}$ of $\mathrm{Mod}(\mathcal{O}_X)$ there exists an exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R)$$

where the second arrow is the difference $t^* - s^*$.

Proof. We denote $\mathcal{F}$ also its extension to a sheaf of modules on $X_{spaces,étale}$, see Properties of Spaces, Remark 66.18.4. Let $V \to Y$ be an object of $Y_{étale}$. Then $V \times_Y X$ is an object of $X_{spaces,étale}$, and by definition $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. Since $U \to X$ is surjective étale, we see that $\{V \times_Y U \to V \times_Y X\}$ is a covering. Also, we have $(V \times_Y U) \times_X (V \times_Y U) = V \times_Y R$. Hence, by the sheaf condition of $\mathcal{F}$ on $X_{spaces,étale}$ we have a short exact sequence

$$0 \to \mathcal{F}(V \times_Y X) \to \mathcal{F}(V \times_Y U) \to \mathcal{F}(V \times_Y R)$$

where the second arrow is the difference of restricting via t or s. This exact sequence is functorial in V and hence we obtain the lemma. $\square$

Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. By Descent, Proposition 35.9.4 the functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ agrees with the usual functor if we think of X and Y as schemes.

More generally, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

04CF (67.11.1.1)
$$f'_*(\mathcal{F}|_U) = (f_*\mathcal{F})|_V,$$

see Properties of Spaces, Lemma 66.26.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

The next level of generality is to consider an arbitrary quasi-compact and quasi-separated morphism of algebraic spaces.

03M9 **Lemma 67.11.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. We have to show that $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y. For this it suffices to show that for any affine scheme V and étale morphism $V \to Y$ the restriction of $f_*\mathcal{F}$ to V is quasi-coherent, see Properties of Spaces, Lemma 66.29.6. Let $f' : V \times_Y X \to V$ be the base change of f by $V \to Y$. Note that f' is also quasi-compact and quasi-separated, see Lemmas 67.8.4 and 67.4.4. By (67.11.1.1) we know that the restriction of $f_*\mathcal{F}$ to V is f'_* of the restriction of $\mathcal{F}$ to $V \times_Y X$. Hence we may replace f by f', and assume that Y is an affine scheme.

Assume Y is an affine scheme. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 66.6.3. By Lemma 67.11.1 we get an exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R).$$

where $R = U \times_X U$. As $X \to Y$ is quasi-separated we see that $R \to U \times_Y U$ is a quasi-compact monomorphism. This implies that R is a quasi-compact separated scheme (as U and Y are affine at this point). Hence $a : U \to Y$ and $b : R \to Y$ are quasi-compact and quasi-separated morphisms of schemes. Thus by Descent, Proposition 35.9.4 the sheaves $a_*(\mathcal{F}|_U)$ and $b_*(\mathcal{F}|_R)$ are quasi-coherent (see also the discussion preceding this lemma). This implies that $f_*\mathcal{F}$ is a kernel of quasi-coherent modules, and hence itself quasi-coherent, see Properties of Spaces, Lemma 66.29.7. $\square$

Higher direct images are discussed in Cohomology of Spaces, Section 69.3.

67.12. Immersions

03HB Open, closed and locally closed immersions of algebraic spaces were defined in Spaces, Section 65.12. Namely, a morphism of algebraic spaces is a closed immersion (resp. open immersion, resp. immersion) if it is representable and a closed immersion (resp. open immersion, resp. immersion) in the sense of Section 67.3.

In particular these types of morphisms are stable under base change and compositions of morphisms in the category of algebraic spaces over S, see Spaces, Lemmas 65.12.2 and 65.12.3.

03M4 **Lemma 67.12.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is a closed immersion (resp. open immersion, resp. immersion),
(2) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),
(3) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a closed immersion (resp. open immersion, resp. immersion), and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a closed immersion (resp. open immersion, resp. immersion).

Proof. Using that a base change of a closed immersion (resp. open immersion, resp. immersion) is another one it is clear that (1) implies (2) and (2) implies (3). Also (3) implies (4) since we can take V to be a disjoint union of affines, see Properties of Spaces, Lemma 66.6.1.

Assume $V \to Y$ is as in (4). Let $\mathcal{P}$ be the property closed immersion (resp. open immersion, resp. immersion) of morphisms of schemes. Note that property $\mathcal{P}$ is preserved under any base change and fppf local on the base (see Section 67.3). Moreover, morphisms of type $\mathcal{P}$ are separated and locally quasi-finite (in each of the three cases, see Schemes, Lemma 26.23.8, and Morphisms, Lemma 29.20.16). Hence by More on Morphisms, Lemma 37.57.1 the morphisms of type $\mathcal{P}$ satisfy descent for fppf covering. Thus Spaces, Lemma 65.11.5 applies and we see that $X \to Y$ is representable and has property $\mathcal{P}$, in other words (1) holds.

The equivalence of (1) and (5) follows from the fact that $\mathcal{P}$ is Zariski local on the target (since we saw above that $\mathcal{P}$ is in fact fppf local on the target). $\square$

0AGC **Lemma 67.12.2.** Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S.

(1) If $Z \to X$ is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism, then $Z \to Y$ is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism.
(2) If $Z \to X$ is an immersion and $Y \to X$ is locally separated, then $Z \to Y$ is an immersion.
(3) If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[rd] & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence representable, locally of finite type, locally quasi-finite, separated, and a monomorphism by Lemma 67.4.7. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism (as each of these properties of morphisms of schemes is stable under base change, see Spaces, Remark 65.4.1). Hence the same is true for the composition (as each of these properties of morphisms of schemes is stable under composition, see Spaces, Remark 65.4.2). This proves (1). The other results are proved in exactly the same manner. □

04CD Lemma 67.12.3. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then $|i| : |Z| \to |X|$ is a homeomorphism onto a locally closed subset, and i is a closed immersion if and only if the image $|i|(|Z|) \subset |X|$ is a closed subset.

Proof. The first statement is Properties of Spaces, Lemma 66.12.1. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption $T = U \times_X Z$ is a scheme and the morphism $j : T \to U$ is an immersion of schemes. By Lemma 67.12.1 the morphism i is a closed immersion if and only if j is a closed immersion. By Schemes, Lemma 26.10.4 this is true if and only if $j(T)$ is closed in U. However, the subset $j(T) \subset U$ is the inverse image of $|i|(|Z|) \subset |X|$, see Properties of Spaces, Lemma 66.4.3. This finishes the proof. □

04CE Remark 67.12.4. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Since i is a monomorphism we may think of $|Z|$ as a subset of $|X|$; in the rest of this remark we do so. Let $\partial|Z|$ be the boundary of $|Z|$ in the topological space $|X|$. In a formula

$$\partial|Z| = \overline{|Z|} \setminus |Z|.$$

Let ∂Z be the reduced closed subspace of X with $|\partial Z| = \partial|Z|$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 66.12.5. By construction we see that $|Z|$ is closed in $|X| \setminus |\partial Z| = |X \setminus \partial Z|$. Hence it is true that any immersion of algebraic spaces can be factored as a closed immersion followed by an open immersion (but not the other way in general, see Morphisms, Example 29.3.4).

06EC Remark 67.12.5. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|X|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $U \subset X$ be the open subspace of X with $|U| = |X| \setminus \partial T$, see Properties of Spaces, Lemma 66.4.8. Let Z be the reduced closed subspace of U with $|Z| = T$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 66.12.5. By construction $Z \to U$ is a closed immersion of algebraic spaces and $U \to X$ is an open immersion, hence $Z \to X$ is an immersion of algebraic spaces over S (see Spaces, Lemma 65.12.2). Note that Z is a reduced algebraic space and that $|Z| = T$ as subsets of $|X|$. We sometimes say Z is the reduced induced subspace structure on T.

081U Lemma 67.12.6. Let S be a scheme. Let $Z \to X$ be an immersion of algebraic spaces over S. Assume $Z \to X$ is quasi-compact. There exists a factorization $Z \to \overline{Z} \to X$ where $Z \to \overline{Z}$ is an open immersion and $\overline{Z} \to X$ is a closed immersion.

Proof. Let U be a scheme and let $U \to X$ be surjective étale. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Set $T = Z \times_U X$. Let $\overline{T} \subset U$ be the scheme theoretic image of $T \to U$. Note that $s^{-1}\overline{T} = t^{-1}\overline{T}$ as taking scheme theoretic images of quasi-compact morphisms commute with flat base change, see Morphisms, Lemma 29.25.16. Hence we obtain a closed subspace $\overline{Z} \subset X$ whose pullback to U is $\overline{T}$, see Properties of Spaces, Lemma 66.12.2. By Morphisms, Lemma 29.7.7 the morphism $T \to \overline{T}$ is an open immersion. It follows that $Z \to \overline{Z}$ is an open immersion and we win. □

67.13. Closed immersions

03MA In this section we elucidate some of the results obtained previously on immersions of algebraic spaces. See Spaces, Section 65.12 and Section 67.12 in this chapter. This section is the analogue of Morphisms, Section 29.2 for algebraic spaces.

03MB Lemma 67.13.1. Let S be a scheme. Let X be an algebraic space over S. For every closed immersion $i : Z \to X$ the sheaf $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective and its kernel is a quasi-coherent sheaf of ideals. The rule $Z \mapsto \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ defines an inclusion reversing bijection

$$\begin{matrix} \text{closed subspaces} \\ Z \subset X \end{matrix} \longrightarrow \begin{matrix} \text{quasi-coherent sheaves} \\ \text{of ideals } \mathcal{I} \subset \mathcal{O}_X \end{matrix}$$

Moreover, given a closed subscheme Z corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ a morphism of algebraic spaces $h : Y \to X$ factors through Z if and only if the map $h^*\mathcal{I} \to h^*\mathcal{O}_X = \mathcal{O}_Y$ is zero.

Proof. Let $U \to X$ be a surjective étale morphism whose source is a scheme. Consider the diagram

$$\begin{tikzcd} U \times_X Z \arrow[r] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ U \arrow[r] & X \end{tikzcd}$$

By Lemma 67.12.1 we see that i is a closed immersion if and only if i' is a closed immersion. By Properties of Spaces, Lemma 66.26.2 we see that $i'_*\mathcal{O}_{U \times_X Z}$ is the restriction of $i_*\mathcal{O}_Z$ to U. Hence the assertions on $\mathcal{O}_X \to i_*\mathcal{O}_Z$ are equivalent to the corresponding assertions on $\mathcal{O}_U \to i'_*\mathcal{O}_{U \times_X Z}$. And since i' is a closed immersion of schemes, these results follow from Morphisms, Lemma 29.2.1.

Let us prove that given a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ the formula

$$Z(T) = \{h : T \to X \mid h^*\mathcal{I} \to \mathcal{O}_T \text{ is zero}\}$$

defines a closed subspace of X. It is clearly a subfunctor of X. To show that $Z \to X$ is representable by closed immersions, let $\varphi : U \to X$ be a morphism from a scheme towards X. Then $Z \times_X U$ is represented by the analogous subfunctor of U corresponding to the sheaf of ideals $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. By Properties of Spaces, Lemma 66.29.2 the $\mathcal{O}_U$-module $\varphi^*\mathcal{I}$ is quasi-coherent on U, and hence $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$ is a quasi-coherent sheaf of ideals on U. By Schemes, Lemma 26.4.6 we conclude that $Z \times_X U$ is represented by the closed subscheme of U associated to $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. Thus Z is a closed subspace of X.

In the formula for Z above the inputs T are schemes since algebraic spaces are sheaves on $(Sch/S)_{fppf}$. We omit the verification that the same formula remains true if T is an algebraic space. □

083Q Definition 67.13.2. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. The inverse image $f^{-1}(Z)$ of the closed subspace Z is the closed subspace $Z \times_X Y$ of Y.

This definition makes sense by Lemma 67.12.1. If $\mathcal{I} \subset \mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to Z via Lemma 67.13.1 then $f^{-1}\mathcal{I}\mathcal{O}_Y = \mathrm{Im}(f^*\mathcal{I} \to \mathcal{O}_Y)$ is the sheaf of ideals corresponding to $f^{-1}(Z)$.

04CG Lemma 67.13.3. A closed immersion of algebraic spaces is quasi-compact.

Proof. This follows from Schemes, Lemma 26.19.5 by general principles, see Spaces, Lemma 65.5.8. □

04CH Lemma 67.13.4. A closed immersion of algebraic spaces is separated.

Proof. This follows from Schemes, Lemma 26.23.8 by general principles, see Spaces, Lemma 65.5.8. □

04E5 Lemma 67.13.5. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S.

(1) The functor
$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) the functor
$$i_{small,*} : \mathrm{Ab}(Z_{étale}) \longrightarrow \mathrm{Ab}(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in $|Z|$.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let U be a scheme and let $U \to X$ be surjective étale. Set $V = Z \times_X U$. Then V is a scheme and $i' : V \to U$ is a closed immersion of schemes. By Properties of Spaces, Lemma 66.18.12 for any sheaf $\mathcal{G}$ on Z we have
$$(i_{small}^{-1} i_{small,*}\mathcal{G})|_V = (i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V)$$
By Étale Cohomology, Proposition 59.46.4 the map $(i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V) \to \mathcal{G}|_V$ is an isomorphism. Since $V \to Z$ is surjective and étale this implies that $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism. This clearly implies that $i_{small,*}$ is fully faithful, see Sites, Lemma 7.41.1. To prove the statement on the essential image, consider a sheaf of

sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$. As in the proof of Étale Cohomology, Proposition 59.46.4 we consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*}i_{small}^{-1}\mathcal{F}.$$

As in the first part we see that the restriction of this map to U is an isomorphism by the corresponding result for the case of schemes. Since U is an étale covering of X we conclude it is an isomorphism. □

0DK1 Lemma 67.13.6. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\overline{z}$ be a geometric point of Z with image $\overline{x}$ in X. Then $(i_{small,*}\mathcal{F})_{\overline{z}} = \mathcal{F}_{\overline{x}}$ for any sheaf $\mathcal{F}$ on $Z_{étale}$.

Proof. Choose an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$. Then the stalk $(i_{small,*}\mathcal{F})_{\overline{z}}$ is the stalk of $i_{small,*}\mathcal{F}|_U$ at $\overline{u}$. By Properties of Spaces, Lemma 66.18.12 we may replace X by U and Z by $Z \times_X U$. Then $Z \to X$ is a closed immersion of schemes and the result is Étale Cohomology, Lemma 59.46.3. □

The following lemma holds more generally in the setting of a closed immersion of topoi (insert future reference here).

04G0 Lemma 67.13.7. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Z_{étale}$. Let $\varphi : \mathcal{A} \to i_{small,*}\mathcal{B}$ be a homomorphism of sheaves of rings so that we obtain a morphism of ringed topoi

$$f : (Sh(Z_{étale}), \mathcal{B}) \longrightarrow (Sh(X_{étale}), \mathcal{A}).$$

For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ and a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ the canonical map

$$\mathcal{F} \otimes_{\mathcal{A}} f_*\mathcal{G} \longrightarrow f_*(f^*\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}).$$

is an isomorphism.

Proof. The map is the map adjoint to the map

$$f^*\mathcal{F} \otimes_{\mathcal{B}} f^*f_*\mathcal{G} = f^*(\mathcal{F} \otimes_{\mathcal{A}} f_*\mathcal{G}) \longrightarrow f^*\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}$$

coming from $\mathrm{id} : f^*\mathcal{F} \to f^*\mathcal{F}$ and the adjunction map $f^*f_*\mathcal{G} \to \mathcal{G}$. To see this map is an isomorphism, we may check on stalks (Properties of Spaces, Theorem 66.19.12). Let $\overline{z} : \mathrm{Spec}(k) \to Z$ be a geometric point with image $\overline{x} = i \circ \overline{z} : \mathrm{Spec}(k) \to X$. Working out what our maps does on stalks, we see that we have to show

$$\mathcal{F}_{\overline{x}} \otimes_{\mathcal{A}_{\overline{x}}} \mathcal{G}_{\overline{z}} = (\mathcal{F}_{\overline{x}} \otimes_{\mathcal{A}_{\overline{x}}} \mathcal{B}_{\overline{z}}) \otimes_{\mathcal{B}_{\overline{z}}} \mathcal{G}_{\overline{z}}$$

which holds true. Here we have used that taking tensor products commutes with taking stalks, the behaviour of stalks under pullback Properties of Spaces, Lemma 66.19.9, and the behaviour of stalks under pushforward along a closed immersion Lemma 67.13.6. □

67.14. Closed immersions and quasi-coherent sheaves

04CI This section is the analogue of Morphisms, Section 29.4.

04CJ Lemma 67.14.1. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z.

(1) For any $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ induces an isomorphism $\mathcal{F}/\mathcal{I}\mathcal{F} \cong i_*i^*\mathcal{F}$.
(2) The functor i^* is a left inverse to i_*, i.e., for any $\mathcal{O}_Z$-module $\mathcal{G}$ the adjunction map $i^*i_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.
(3) The functor

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{I}\mathcal{F} = 0$.

Proof. During this proof we work exclusively with sheaves on the small étale sites, and we use $i_*, i^{-1}, \ldots$ to denote pushforward and pullback of sheaves of abelian groups instead of $i_{small,*}, i_{small}^{-1}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By Lemma 67.13.7 applied with $\mathcal{A} = \mathcal{O}_X$ and $\mathcal{G} = \mathcal{B} = \mathcal{O}_Z$ we see that $i_*i^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. By Lemma 67.13.1 we see that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

It follows from properties of the tensor product that $\mathcal{F} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Z = \mathcal{F}/\mathcal{I}\mathcal{F}$. This proves (1) (except that we omit the verification that the map is induced by the adjunction mapping).

Let $\mathcal{G}$ be any $\mathcal{O}_Z$-module. By Lemma 67.13.5 we see that $i^{-1}i_*\mathcal{G} = \mathcal{G}$. Hence to prove (2) we have to show that the canonical map $\mathcal{G} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{G}$ is an isomorphism. This follows from general properties of tensor products if we can show that $i^{-1}\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. By Lemma 67.13.5 it suffices to prove that $i_*i^{-1}\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Since the surjective map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ factors through this map we see that (2) holds.

Finally we prove the most interesting part of the lemma, namely part (3). A closed immersion is quasi-compact and separated, see Lemmas 67.13.3 and 67.13.4. Hence Lemma 67.11.2 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X. Thus we obtain our functor $i_*^{QCoh} : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$. It is clear from part (2) that i_*^{QCoh} is fully faithful since it has a left inverse, namely i^*.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{G}) = 0$ for any $\mathcal{O}_Z$-module, since $\mathcal{I}$ is the kernel of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ which is the map we use to put an $\mathcal{O}_X$-module structure on $i_*\mathcal{G}$. Next, suppose that $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{F} = 0$. Then we see that $\mathcal{F}$ is an $i_*\mathcal{O}_Z$-module because $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. Hence in particular its support is contained in $|Z|$. We apply Lemma 67.13.5 to see that $\mathcal{F} \cong i_*\mathcal{G}$ for some $\mathcal{O}_Z$-module $\mathcal{G}$. The only small detail left over is to see why $\mathcal{G}$ is quasi-coherent. This is true because $\mathcal{G} \cong i^*\mathcal{F}$ by part (2) and Properties of Spaces, Lemma 66.29.2. $\square$

Let $i : Z \to X$ be a closed immersion of algebraic spaces. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

04CK Lemma 67.14.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 66.29.7. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. $\square$

04CL Lemma 67.14.3. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. There is a functor[4] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 17.6.3.)

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 67.14.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. $\square$

Using the 1-to-1 corresponding between quasi-coherent sheaves of ideals and closed subspaces (see Lemma 67.13.1) we can define scheme theoretic intersections and unions of closed subschemes.

0CYZ Definition 67.14.4. Let S be a scheme. Let X be an algebraic space over S. Let $Z, Y \subset X$ be closed subspaces corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The scheme theoretic intersection of Z and Y is the closed subspace of X cut out by $\mathcal{I} + \mathcal{J}$. Then scheme theoretic union of Z and Y is the closed subspace of X cut out by $\mathcal{I} \cap \mathcal{J}$.

It is clear that formation of scheme theoretic intersection commutes with étale localization and the same is true for scheme theoretic union.

0CZ0 Lemma 67.14.5. Let S be a scheme. Let X be an algebraic space over S. Let $Z, Y \subset X$ be closed subspaces. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and

$$\begin{tikzcd} Z \cap Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

[4] This is likely nonstandard notation.

is a cartesian diagram of algebraic spaces over S, i.e., $Z \cap Y = Z \times_X Y$.

Proof. The morphisms $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions by Lemma 67.13.1. Since formation of the scheme theoretic intersection commutes with étale localization we conclude the diagram is cartesian by the case of schemes. See Morphisms, Lemma 29.4.5. □

0CZ1 **Lemma 67.14.6.** Let S be a scheme. Let X be an algebraic space over S. Let $Y, Z \subset X$ be closed subspaces. Let $Y \cup Z$ be the scheme theoretic union of Y and Z. Let $Y \cap Z$ be the scheme theoretic intersection of Y and Z. Then $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions, there is a short exact sequence

$$0 \to \mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y \times \mathcal{O}_Z \to \mathcal{O}_{Y \cap Z} \to 0$$

of $\mathcal{O}_Z$-modules, and the diagram

$$\begin{tikzcd} Y \cap Z \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Y \cup Z \end{tikzcd}$$

is cocartesian in the category of algebraic spaces over S, i.e., $Y \cup Z = Y \amalg_{Y \cap Z} Z$.

Proof. The morphisms $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions by Lemma 67.13.1. In the short exact sequence we use the equivalence of Lemma 67.14.1 to think of quasi-coherent modules on closed subspaces of X as quasi-coherent modules on X. For the first map in the sequence we use the canonical maps $\mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y$ and $\mathcal{O}_{Y \cup Z} \to \mathcal{O}_Z$ and for the second map we use the canonical map $\mathcal{O}_Y \to \mathcal{O}_{Y \cap Z}$ and the negative of the canonical map $\mathcal{O}_Z \to \mathcal{O}_{Y \cap Z}$. Then to check exactness we may work étale locally and deduce exactness from the case of schemes (Morphisms, Lemma 29.4.6).

To show the diagram is cocartesian, suppose we are given an algebraic space T over S and morphisms $f : Y \to T$, $g : Z \to T$ agreeing as morphisms $Y \cap Z \to T$. Goal: Show there exists a unique morphism $h : Y \cup Z \to T$ agreeing with f and g. To construct h we may work étale locally on $Y \cup Z$ (as $Y \cup Z$ is an étale sheaf being an algebraic space). Hence we may assume that X is a scheme. In this case we know that $Y \cup Z$ is the pushout of Y and Z along $Y \cap Z$ in the category of schemes by Morphisms, Lemma 29.4.6. Choose a scheme T' and a surjective étale morphism $T' \to T$. Set $Y' = T' \times_{T,f} Y$ and $Z' = T' \times_{T,g} Z$. Then Y' and Z' are schemes and we have a canonical isomorphism $\varphi : Y' \times_Y (Y \cap Z) \to Z' \times_Z (Y \cap Z)$ of schemes. By More on Morphisms, Lemma 37.67.8 the pushout $W' = Y' \amalg_{Y' \times_Y (Y \cap Z), \varphi} Z'$ exists in the category of schemes. The morphism $W' \to Y \cup Z$ is étale by More on Morphisms, Lemma 37.67.9. It is surjective as $Y' \to Y$ and $Z' \to Z$ are surjective. The morphisms $f' : Y' \to T'$ and $g' : Z' \to T'$ glue to a unique morphism of schemes $h' : W' \to T'$. By uniqueness the composition $W' \to T' \to T$ descends to the desired morphism $h : Y \cup Z \to T$. Some details omitted. □

67.15. Supports of modules

07TX In this section we collect some elementary results on supports of quasi-coherent modules on algebraic spaces. Let X be an algebraic space. The support of an abelian sheaf on $X_{étale}$ has been defined in Properties of Spaces, Section 66.20. We use the same definition for supports of modules. The following lemma tells us this agrees with the notion as defined for quasi-coherent modules on schemes.

07TY Lemma 67.15.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. Then

$$\text{Supp}(\varphi^*\mathcal{F}) = |\varphi|^{-1}(\text{Supp}(\mathcal{F}))$$

where the left hand side is the support of $\varphi^*\mathcal{F}$ as a quasi-coherent module on the scheme U.

Proof. Let $u \in U$ be a (usual) point and let $\overline{x}$ be a geometric point lying over u. By Properties of Spaces, Lemma 66.29.4 we have $(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is the strict henselization by Properties of Spaces, Lemma 66.22.1 we see that it is faithfully flat (see More on Algebra, Lemma 15.45.1). Thus we see that $(\varphi^*\mathcal{F})_u = 0$ if and only if $\mathcal{F}_{\overline{x}} = 0$. This proves the lemma. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

07TZ Lemma 67.15.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then

(1) The support of $\mathcal{F}$ is closed.
(2) For a geometric point $\overline{x}$ lying over $x \in |X|$ we have

$$x \in \text{Supp}(\mathcal{F}) \Leftrightarrow \mathcal{F}_{\overline{x}} \neq 0 \Leftrightarrow \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \kappa(\overline{x}) \neq 0.$$

(3) For any morphism of algebraic spaces $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $\text{Supp}(f^*\mathcal{F}) = f^{-1}(\text{Supp}(\mathcal{F}))$.

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. By Lemma 67.15.1 the inverse image of the support of $\mathcal{F}$ is the support of $\varphi^*\mathcal{F}$ which is closed by Morphisms, Lemma 29.5.3. Thus (1) follows from the definition of the topology on $|X|$.

The first equivalence in (2) is the definition of support. The second equivalence follows from Nakayama's lemma, see Algebra, Lemma 10.20.1.

Let $f : Y \to X$ be as in (3). Note that $f^*\mathcal{F}$ is of finite type by Properties of Spaces, Section 66.30. For the final assertion, let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ on X. Recall that

$$(f^*\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}},$$

see Properties of Spaces, Lemma 66.29.5. Hence $(f^*\mathcal{F})_{\overline{y}} \otimes \kappa(\overline{y})$ is nonzero if and only if $\mathcal{F}_{\overline{x}} \otimes \kappa(\overline{x})$ is nonzero. By (2) this implies $x \in \text{Supp}(\mathcal{F})$ if and only if $y \in \text{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

Our next task is to show that the scheme theoretic support of a finite type quasi-coherent module (see Morphisms, Definition 29.5.5) also makes sense for finite type quasi-coherent modules on algebraic spaces.

07U0 Lemma 67.15.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. There exists a smallest closed subspace $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:

(1) If U is a scheme and $\varphi : U \to X$ is an étale morphism then $Z \times_X U$ is the scheme theoretic support of $\varphi^*\mathcal{F}$.
(2) The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.
(3) The quasi-coherent sheaf $\mathcal{G}$ is of finite type.
(4) The support of $\mathcal{G}$ and of $\mathcal{F}$ is $|Z|$.

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Let $R = U \times_X U$ with projections $s, t : R \to U$. Let $i' : Z' \to U$ be the scheme theoretic support of $\varphi^*\mathcal{F}$ and let $\mathcal{G}'$ be the (unique up to unique isomorphism) finite type quasi-coherent $\mathcal{O}_{Z'}$-module with $i'_*\mathcal{G}' = \varphi^*\mathcal{F}$, see Morphisms, Lemma 29.5.4. As $s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F}$ we see that $R' = s^{-1}Z' = t^{-1}Z'$ as closed subschemes of R by Morphisms, Lemma 29.25.14. Thus we may apply Properties of Spaces, Lemma 66.12.2 to find a closed subspace $i : Z \to X$ whose pullback to U is Z'. Writing $s', t' : R' \to Z'$ the projections and $j' : R' \to R$ the given closed immersion, we see that

$$j'_*(s')^*\mathcal{G}' = s^*i'_*\mathcal{G}' = s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F} = t^*i'_*\mathcal{G}' = j'_*(t')^*\mathcal{G}'$$

(the first and the last equality by Cohomology of Schemes, Lemma 30.5.2). Hence the uniqueness of Morphisms, Lemma 29.25.14 applied to $R' \to R$ gives an isomorphism $\alpha : (t')^*\mathcal{G}' \to (s')^*\mathcal{G}'$ compatible with the canonical isomorphism $t^*\varphi^*\mathcal{F} = s^*\varphi^*\mathcal{F}$ via j'_*. Clearly α satisfies the cocycle condition, hence we may apply Properties of Spaces, Proposition 66.32.1 to obtain a quasi-coherent module $\mathcal{G}$ on Z whose restriction to Z' is $\mathcal{G}'$ compatible with α. Again using the equivalence of the proposition mentioned above (this time for X) we conclude that $i_*\mathcal{G} \cong \mathcal{F}$.

This proves existence. The other properties of the lemma follow by comparing with the result for schemes using Lemma 67.15.1. Detailed proofs omitted. ☐

07U1 Definition 67.15.4. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. The scheme theoretic support of $\mathcal{F}$ is the closed subspace $Z \subset X$ constructed in Lemma 67.15.3.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 67.14.1).

67.16. Scheme theoretic image

082W Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

082X Lemma 67.16.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There exists a closed subspace $Z \subset Y$ such that f factors through Z and such that for any other closed subspace $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.

Proof. Let $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 67.13.1. In general the lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 67.14.2. ☐

Suppose that in the situation of Lemma 67.16.1 above X and Y are representable. Then the closed subspace $Z \subset Y$ found in the lemma agrees with the closed subscheme $Z \subset Y$ found in Morphisms, Lemma 29.6.1. The reason is that closed subspaces (or subschemes) are in a inclusion reversing correspondence with quasi-coherent ideal sheaves on $X_{étale}$ and X. As the category of quasi-coherent modules on $X_{étale}$ and X are the same (Properties of Spaces, Section 66.29) we conclude. Thus the following definition agrees with the earlier definition for morphisms of schemes.

082Y Definition 67.16.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The scheme theoretic image of f is the smallest closed subspace $Z \subset Y$ through which f factors, see Lemma 67.16.1 above.

We often just denote $f : X \to Z$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subspaces of Y.

082Z Lemma 67.16.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then

(1) the sheaf of ideals $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,
(2) the scheme theoretic image Z is the closed subspace corresponding to $\mathcal{I}$,
(3) for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$, and
(4) the image $|f|(|X|) \subset |Z|$ is a dense subset of $|Z|$.

Proof. To prove (3) it suffices to prove (1) and (2) since the formation of $\mathcal{I}$ commutes with étale localization. If (1) holds then in the proof of Lemma 67.16.1 we showed (2). Let us prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is étale local we may assume Y is an affine scheme. As f is quasi-compact, we can find an affine scheme U and a surjective étale morphism $U \to X$. Denote f' the composition $U \to X \to Y$. Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_U$, and hence $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. By Lemma 67.11.2 the sheaf $f'_*\mathcal{O}_U$ is quasi-coherent on Y. Hence $\mathcal{I}$ is quasi-coherent as a kernel of a map between coherent modules. Finally, part (4) follows from parts (1), (2), and (3) as the ideal $\mathcal{I}$ will be the unit ideal in any point of $|Y|$ which is not contained in the closure of $|f|(|X|)$. $\square$

0830 Lemma 67.16.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X is reduced. Then

(1) the scheme theoretic image Z of f is the reduced induced algebraic space structure on $\overline{|f|(|X|)}$, and
(2) for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$.

Proof. Part (1) is true because the reduced induced algebraic space structure on $\overline{|f|(|X|)}$ is the smallest closed subspace of Y through which f factors, see Properties of Spaces, Lemma 66.12.4. Part (2) follows from (1), the fact that $|V| \to |Y|$ is open, and the fact that being reduced is preserved under étale localization. $\square$

089B Lemma 67.16.5. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. Let Z be the scheme theoretic image of f. Let $z \in |Z|$.

There exists a valuation ring A with fraction field K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[rr] \arrow[d] & & X \arrow[dl] \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z.

Proof. Choose an affine scheme V with a point $z' \in V$ and an étale morphism $V \to Y$ mapping z' to z. Let $Z' \subset V$ be the scheme theoretic image of $X \times_Y V \to V$. By Lemma 67.16.3 we have $Z' = Z \times_Y V$. Thus $z' \in Z'$. Since f is quasi-compact and V is affine we see that $X \times_Y V$ is quasi-compact. Hence there exists an affine scheme W and a surjective étale morphism $W \to X \times_Y V$. Then $Z' \subset V$ is also the scheme theoretic image of $W \to V$. By Morphisms, Lemma 29.6.5 we can choose a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & X \times_Y V \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Z' \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z'. Composing with $Z' \to Z$ and $W \to X \times_Y V \to X$ we obtain a solution. $\square$

0CP2 Lemma 67.16.6. Let S be a scheme. Let

$$\begin{tikzcd} X_1 \arrow[r, "f_1"'] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r] & Z_2 \arrow[r] & Y_2 \end{tikzcd}$$

Proof. The scheme theoretic inverse image of Z_2 in Y_1 is a closed subspace of Y_1 through which f_1 factors. Hence Z_1 is contained in this. This proves the lemma. $\square$

0CP3 Lemma 67.16.7. Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace such that $V \to Y$ is quasi-compact. Let $s : V \to X$ be a morphism such that $f \circ s = \mathrm{id}_V$. Let Y' be the scheme theoretic image of s. Then $Y' \to Y$ is an isomorphism over V.

Proof. By Lemma 67.8.9 the morphism $s : V \to X$ is quasi-compact. Hence the construction of the scheme theoretic image Y' of s commutes with restriction to opens by Lemma 67.16.3. In particular, we see that $Y' \cap f^{-1}(V)$ is the scheme theoretic image of a section of the separated morphism $f^{-1}(V) \to V$. Since a section of a separated morphism is a closed immersion (Lemma 67.4.7), we conclude that $Y' \cap f^{-1}(V) \to V$ is an isomorphism as desired. $\square$

67.17. Scheme theoretic closure and density

0831 This section is the analogue of Morphisms, Section 29.7.

0832 **Lemma 67.17.1.** Let S be a scheme. Let $W \subset S$ be a scheme theoretically dense open subscheme (Morphisms, Definition 29.7.1). Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Then $f^{-1}(W)$ is scheme theoretically dense in X.

Proof. We will use the characterization of Morphisms, Lemma 29.7.5. Assume $V \subset X$ is an open and $g \in \Gamma(V, \mathcal{O}_V)$ is a function which restricts to zero on $f^{-1}(W) \cap V$. We have to show that $g = 0$. Assume $g \neq 0$ to get a contradiction. By More on Morphisms, Lemma 37.45.6 we may shrink V, find an open $U \subset S$ fitting into a commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ U \arrow[r] & S, \end{tikzcd}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$. Say $(g_1, \ldots, g_r) \in \Gamma(U, \mathcal{F}^{\oplus r})$ maps to g. Then we see that $g_i|_{W \cap U} = 0$ because $g|_{f^{-1}W \cap V} = 0$. Hence $g_i = 0$ because $\mathcal{F} \subset \mathcal{O}_U$ and W is scheme theoretically dense in S. This implies $g = 0$ which is the desired contradiction. $\square$

0833 **Lemma 67.17.2.** Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. The following are equivalent

(1) for every étale morphism $\varphi : V \to X$ (of algebraic spaces) the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,
(2) there exists a scheme V and a surjective étale morphism $\varphi : V \to X$ such that the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,

Proof. Observe that if $V \to V'$ is a morphism of algebraic spaces étale over X, and $Z \subset V$, resp. $Z' \subset V'$ is the scheme theoretic closure of $U \times_X V$, resp. $U \times_X V'$ in V, resp. V', then Z maps into Z'. Thus if $V \to V'$ is surjective and étale then $Z = V$ implies $Z' = V'$. Next, note that an étale morphism is flat, locally of finite presentation, and locally quasi-finite (see Morphisms, Section 29.36). Thus Lemma 67.17.1 implies that if V and V' are schemes, then $Z' = V'$ implies $Z = V$. A formal argument using that every algebraic space has an étale covering by a scheme shows that (1) and (2) are equivalent. $\square$

It follows from Lemma 67.17.2 that the following definition is compatible with the definition in the case of schemes.

0834 **Definition 67.17.3.** Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace.

(1) The scheme theoretic image of the morphism $U \to X$ is called the scheme theoretic closure of U in X.
(2) We say U is scheme theoretically dense in X if the equivalent conditions of Lemma 67.17.2 are satisfied.

With this definition it is not the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X. This is somewhat inelegant. But with suitable finiteness conditions we will see that it does hold.

0835 Lemma 67.17.4. Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. If $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.

Proof. Follows from Lemma 67.16.3 part (3). □

0836 Lemma 67.17.5. Let S be a scheme. Let $j : U \to X$ be an open immersion of algebraic spaces over S. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any algebraic space V étale over X. Hence the scheme theoretic closure of $U \times_X V$ in V is equal to V, see proof of Lemma 67.16.1. Conversely, assume the scheme theoretic closure of $U \times_X V$ is equal to V for all V étale over X. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine, say $V = \text{Spec}(A)$, étale over X and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \times_X U, \mathcal{O})$. In this case the scheme theoretic closure of $V \times_X U$ in V is clearly contained in $\text{Spec}(A/(f))$ a contradiction. □

0837 Lemma 67.17.6. Let S be a scheme. Let X be an algebraic space over S. If U, V are scheme theoretically dense open subspaces of X, then so is $U \cap V$.

Proof. Let $W \to X$ be any étale morphism. Consider the map $\mathcal{O}(W) \to \mathcal{O}(W \times_X V) \to \mathcal{O}(W \times_X (V \cap U))$. By Lemma 67.17.5 both maps are injective. Hence the composite is injective. Hence by Lemma 67.17.5 $U \cap V$ is scheme theoretically dense in X. □

088G Lemma 67.17.7. Let S be a scheme. Let $h : Z \to X$ be an immersion of algebraic spaces over S. Assume either $Z \to X$ is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subspace of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.

Proof. In both cases the formation of $\overline{Z}$ commutes with étale localization, see Lemmas 67.16.3 and 67.16.4. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 29.7.7. □

084N Lemma 67.17.8. Let S be a scheme. Let B be an algebraic space over S. Let $f, g : X \to Y$ be morphisms of algebraic spaces over B. Let $U \subset X$ be an open subspace such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to B$ is separated, then $f = g$.

Proof. As $Y \to B$ is separated the fibre product $Y \times_{\Delta, Y \times_B Y, (f,g)} X$ is a closed subspace $Z \subset X$. As $f|_U = g|_U$ we see that $U \subset Z$. Hence $Z = X$ as U is assumed scheme theoretically dense in X. □

67.18. Dominant morphisms

0ABK We copy the definition of a dominant morphism of schemes to get the notion of a dominant morphism of algebraic spaces. We caution the reader that this definition is not well behaved unless the morphism is quasi-compact and the algebraic spaces satisfy some separation axioms.

0ABL Definition 67.18.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is called dominant if the image of $|f| : |X| \to |Y|$ is dense in $|Y|$.

67.19. Universally injective morphisms

03MT We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be universally injective. For a field K over S (recall this means that we are given a structure morphism $\mathrm{Spec}(K) \to S$) and an algebraic space X over S we write $X(K) = \mathrm{Mor}_S(\mathrm{Spec}(K), X)$. We first translate the condition for representable morphisms into a condition on the functor of points.

03MU Lemma 67.19.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is universally injective (in the sense of Section 67.3) if and only if for all fields K the map $X(K) \to Y(K)$ is injective.

Proof. We are going to use Morphisms, Lemma 29.10.2 without further mention. Suppose that f is universally injective. Then for any field K and any morphism $\mathrm{Spec}(K) \to Y$ the morphism of schemes $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$ is universally injective. Hence there exists at most one section of the morphism $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$. Hence the map $X(K) \to Y(K)$ is injective. Conversely, suppose that for every field K the map $X(K) \to Y(K)$ is injective. Let $T \to Y$ be a morphism from a scheme into Y, and consider the base change $f_T : T \times_Y X \to T$. For any field K we have

$$(T \times_Y X)(K) = T(K) \times_{Y(K)} X(K)$$

by definition of the fibre product, and hence the injectivity of $X(K) \to Y(K)$ guarantees the injectivity of $(T \times_Y X)(K) \to T(K)$ which means that f_T is universally injective as desired. $\square$

Next, we translate the property that the transformation between field valued points is injective into something more geometric.

040X Lemma 67.19.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) the map $X(K) \to Y(K)$ is injective for every field K over S
(2) for every morphism $Y' \to Y$ of algebraic spaces over S the induced map $|Y' \times_Y X| \to |Y'|$ is injective, and
(3) the diagonal morphism $X \to X \times_Y X$ is surjective.

Proof. Assume (1). Let $g : Y' \to Y$ be a morphism of algebraic spaces, and denote $f' : Y' \times_Y X \to Y'$ the base change of f. Let K_i, $i = 1, 2$ be fields and let $\varphi_i : \mathrm{Spec}(K_i) \to Y' \times_Y X$ be morphisms such that $f' \circ \varphi_1$ and $f' \circ \varphi_2$ define the same element of $|Y'|$. By definition this means there exists a field Ω and embeddings $\alpha_i : K_i \subset \Omega$ such that the two morphisms $f' \circ \varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y'$ are equal.

Here is the corresponding commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\Omega) \arrow[r, "\alpha_2"'] \arrow[dr, "\alpha_1"] \arrow[ddrr, bend right] & \mathrm{Spec}(K_2) \arrow[dr, "\varphi_2"] & & \\
& \mathrm{Spec}(K_1) \arrow[r, "\varphi_1"] & Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& & Y' \arrow[r, "g"] & Y.
\end{tikzcd}$$

In particular the compositions $g \circ f' \circ \varphi_i \circ \alpha_i$ are equal. By assumption (1) this implies that the morphism $g' \circ \varphi_i \circ \alpha_i$ are equal, where $g' : Y' \times_Y X \to X$ is the projection. By the universal property of the fibre product we conclude that the morphisms $\varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y' \times_Y X$ are equal. In other words φ_1 and φ_2 define the same point of $Y' \times_Y X$. We conclude that (2) holds.

Assume (2). Let K be a field over S, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. Denote $c : \mathrm{Spec}(K) \to Y$ the common value. By assumption $|\mathrm{Spec}(K) \times_{c,Y} X| \to |\mathrm{Spec}(K)|$ is injective. This means there exists a field Ω and embeddings $\alpha_i : K \to \Omega$ such that

$$\begin{tikzcd}
\mathrm{Spec}(\Omega) \arrow[r, "\alpha_1"'] \arrow[d, "\alpha_2"'] & \mathrm{Spec}(K) \arrow[d, "a"] \\
\mathrm{Spec}(K) \arrow[r, "b"] & \mathrm{Spec}(K) \times_{c,Y} X
\end{tikzcd}$$

is commutative. Composing with the projection to $\mathrm{Spec}(K)$ we see that $\alpha_1 = \alpha_2$. Denote the common value α. Then we see that $\{\alpha : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)\}$ is a fpqc covering of $\mathrm{Spec}(K)$ such that the two morphisms a, b become equal on the members of the covering. By Properties of Spaces, Proposition 66.17.1 we conclude that $a = b$. We conclude that (1) holds.

Assume (3). Let $x, x' \in |X|$ be a pair of points such that $f(x) = f(x')$ in $|Y|$. By Properties of Spaces, Lemma 66.4.3 we see there exists a $x'' \in |X \times_Y X|$ whose projections are x and x'. By assumption and Properties of Spaces, Lemma 66.4.4 there exists a $x''' \in |X|$ with $\Delta_{X/Y}(x''') = x''$. Thus $x = x'$. In other words f is injective. Since condition (3) is stable under base change we see that f satisfies (2).

Assume (2). Then in particular $|X \times_Y X| \to |X|$ is injective which implies immediately that $|\Delta_{X/Y}| : |X| \to |X \times_Y X|$ is surjective, which implies that $\Delta_{X/Y}$ is surjective by Properties of Spaces, Lemma 66.4.4. □

By the two lemmas above the following definition does not conflict with the already defined notion of a universally injective representable morphism of algebraic spaces.

03MV Definition 67.19.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is universally injective if for every morphism $Y' \to Y$ the induced map $|Y' \times_Y X| \to |Y'|$ is injective.

To be sure this means that any or all of the equivalent conditions of Lemma 67.19.2 hold.

05VS Remark 67.19.4. A universally injective morphism of schemes is separated, see Morphisms, Lemma 29.10.3. This is not the case for morphisms of algebraic spaces. Namely, the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$ constructed in Spaces, Example 65.14.1 comes equipped with a morphism $X \to \mathbf{A}^1_k$ which maps the point with coordinate x to the point with coordinate x^2. This is an isomorphism away from 0, and there is a unique point of X lying above 0. As X isn't separated this is a universally injective morphism of algebraic spaces which is not separated.

03MW Lemma 67.19.5. The base change of a universally injective morphism is universally injective.

Proof. Omitted. Hint: This is formal. $\square$

03MX Lemma 67.19.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is universally injective,
(2) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,
(3) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,
(4) there exists a scheme Z and a surjective morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is universally injective, and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally injective.

Proof. We will use that being universally injective is preserved under base change (Lemma 67.19.5) without further mention in this proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Assume $g : Z \to Y$ as in (4). Let $y : \mathrm{Spec}(K) \to Y$ be a morphism from the spectrum of a field into Y. By assumption we can find an extension field $\alpha : K \subset K'$ and a morphism $z : \mathrm{Spec}(K') \to Z$ such that $y \circ \alpha = g \circ z$ (with obvious abuse of notation). By assumption the morphism $Z \times_Y X \to Z$ is universally injective, hence there is at most one lift of $g \circ z : \mathrm{Spec}(K') \to Y$ to a morphism into X. Since $\{\alpha : \mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is a fpqc covering this implies there is at most one lift of $y : \mathrm{Spec}(K) \to Y$ to a morphism into X, see Properties of Spaces, Proposition 66.17.1. Thus we see that (1) holds.

We omit the verification that (5) is equivalent to (1). $\square$

03MY Lemma 67.19.7. A composition of universally injective morphisms is universally injective.

Proof. Omitted. $\square$

67.20. Affine morphisms

03WD We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be affine.

03WE Lemma 67.20.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is affine (in the sense of Section 67.3) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine.

Proof. This follows directly from the definition of an affine morphism of schemes (Morphisms, Definition 29.11.1). $\square$

This clears the way for the following definition.

03WF Definition 67.20.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is affine if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme.

03WG Lemma 67.20.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is representable and affine,
(2) f is affine,
(3) for every affine scheme V and étale morphism $V \to Y$ the scheme $X \times_Y V$ is affine,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is affine, and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is affine.

Proof. It is clear that (1) implies (2), that (2) implies (3), and that (3) implies (4) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (4). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 65.11.5 because the class of affine morphisms satisfies all the required properties (see Morphisms, Lemmas 29.11.8 and Descent, Lemmas 35.23.18 and 35.37.1). The conclusion of applying this lemma is that f is representable and affine, i.e., (1) holds.

The equivalence of (1) and (5) follows from the fact that being affine is Zariski local on the target (the reference above shows that being affine is in fact fpqc local on the target). $\square$

03WH Lemma 67.20.4. The composition of affine morphisms is affine.

Proof. Omitted. Hint: Transitivity of fibre products. $\square$

03WI Lemma 67.20.5. The base change of an affine morphism is affine.

Proof. Omitted. Hint: Transitivity of fibre products. $\square$

07U2 Lemma 67.20.6. A closed immersion is affine.

Proof. Follows immediately from the corresponding statement for morphisms of schemes, see Morphisms, Lemma 29.11.9. $\square$

081V Lemma 67.20.7. Let S be a scheme. Let X be an algebraic space over S. There is an anti-equivalence of categories

$$\begin{matrix} \text{algebraic spaces} \\ \text{affine over } X \end{matrix} \longleftrightarrow \begin{matrix} \text{quasi-coherent sheaves} \\ \text{of } \mathcal{O}_X\text{-algebras} \end{matrix}$$

which associates to $f : Y \to X$ the sheaf $f_*\mathcal{O}_Y$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. This lemma is the analogue of Morphisms, Lemma 29.11.5. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We will construct an affine morphism of algebraic spaces $\pi : Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ with $\pi_*\mathcal{O}_Y \cong \mathcal{A}$. To do this, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Denote $\psi : R \to X$ the composition $\psi = \varphi \circ s = \varphi \circ t$. By the aforementioned lemma there exists an affine morphisms of schemes $\pi_0 : V \to U$ and $\pi_1 : W \to R$ with $\pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$ and $\pi_{1,*}\mathcal{O}_W \cong \psi^*\mathcal{A}$. Since the construction is compatible with base change there exist morphisms $s', t' : W \to V$ such that the diagrams

$$\begin{tikzcd} W \arrow[r, "s'"'] \arrow[d] & V \arrow[d] \\ R \arrow[r, "s"] & U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W \arrow[r, "t'"'] \arrow[d] & V \arrow[d] \\ R \arrow[r, "t"] & U \end{tikzcd}$$

are cartesian. It follows that s', t' are étale. It is a formal consequence of the above that $(t', s') : W \to V \times_S V$ is a monomorphism. We omit the verification that $W \to V \times_S V$ is an equivalence relation (hint: think about the pullback of $\mathcal{A}$ to $U \times_X U \times_X U = R \times_{s,U,t} R$). The quotient sheaf $Y = V/W$ is an algebraic space, see Spaces, Theorem 65.10.5. By Groupoids, Lemma 39.20.7 we see that $Y \times_X U \cong V$. Hence $Y \to X$ is affine by Lemma 67.20.3. Finally, the isomorphism of

$$(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U} = \pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$$

is compatible with glueing isomorphisms, whence $(Y \to X)_*\mathcal{O}_Y \cong \mathcal{A}$ by Properties of Spaces, Proposition 66.32.1. We omit the verification that this construction is compatible with base change. $\square$

081W **Definition 67.20.8.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *relative spectrum of $\mathcal{A}$ over X*, or simply the *spectrum of $\mathcal{A}$ over X* is the affine morphism $\underline{\mathrm{Spec}}(\mathcal{A}) \to X$ corresponding to $\mathcal{A}$ under the equivalence of categories of Lemma 67.20.7.

Forming the relative spectrum commutes with arbitrary base change.

081X **Remark 67.20.9.** Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Then f has a canonical factorization

$$Y \longrightarrow \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y) \longrightarrow X$$

This makes sense because $f_*\mathcal{O}_Y$ is quasi-coherent by Lemma 67.11.2. The morphism $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ comes from the canonical $\mathcal{O}_Y$-algebra map $f^*f_*\mathcal{O}_Y \to \mathcal{O}_Y$ which corresponds to a canonical morphism $Y \to Y \times_X \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ over Y (see Lemma 67.20.7) whence a factorization of f as above.

08AI **Lemma 67.20.10.** Let S be a scheme. Let $f : Y \to X$ be an affine morphism of algebraic spaces over S. Let $\mathcal{A} = f_*\mathcal{O}_Y$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories

$$\left\{\begin{matrix}\text{category of quasi-coherent} \\ \mathcal{O}_Y\text{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{category of quasi-coherent} \\ \mathcal{A}\text{-modules}\end{matrix}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_X$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. $\square$

08GB Lemma 67.20.11. Let S be a scheme. Let B be an algebraic space over S. Suppose $g : X \to Y$ is a morphism of algebraic spaces over B.

(1) If X is affine over B and $\Delta : Y \to Y \times_B Y$ is affine, then g is affine.
(2) If X is affine over B and Y is separated over B, then g is affine.
(3) A morphism from an affine scheme to an algebraic space with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1) is affine.
(4) A morphism from an affine scheme to a separated algebraic space is affine.

Proof. Proof of (1). The base change $X \times_B Y \to Y$ is affine by Lemma 67.20.5. The morphism $(1, g) : X \to X \times_B Y$ is the base change of $Y \to Y \times_B Y$ by the morphism $X \times_B Y \to Y \times_B Y$. Hence it is affine by Lemma 67.20.5. The composition of affine morphisms is affine (see Lemma 67.20.4) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 67.20.6) and Y/B separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). $\square$

09TF Lemma 67.20.12. Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to X$ is affine.

Proof. Let $U \to X$ be an étale morphism with U affine. To prove the lemma we have to show that $\mathrm{Spec}(A) \times_X U$ is affine, see Lemma 67.20.3. Since X is quasi-separated the scheme $\mathrm{Spec}(A) \times_X U$ is quasi-compact. Moreover, the projection morphism $\mathrm{Spec}(A) \times_X U \to \mathrm{Spec}(A)$ is étale. Hence this morphism has finite discrete fibers and moreover the topology on $\mathrm{Spec}(A)$ is discrete. Thus $\mathrm{Spec}(A) \times_X U$ is a scheme whose underlying topological space is a finite discrete set. We are done by Schemes, Lemma 26.11.8. $\square$

67.21. Quasi-affine morphisms

03WJ We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be quasi-affine.

03WK Lemma 67.21.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is quasi-affine (in the sense of Section 67.3) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is quasi-affine.

Proof. This follows directly from the definition of a quasi-affine morphism of schemes (Morphisms, Definition 29.13.1). $\square$

This clears the way for the following definition.

03WL Definition 67.21.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is quasi-affine if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by a quasi-affine scheme.

03WM Lemma 67.21.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is representable and quasi-affine,
(2) f is quasi-affine,
(3) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is quasi-affine, and
(4) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-affine.

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is a quasi-affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is quasi-affine. This means we can apply Spaces, Lemma 65.11.5 because the class of quasi-affine morphisms satisfies all the required properties (see Morphisms, Lemmas 29.13.5 and Descent, Lemmas 35.23.20 and 35.38.1). The conclusion of applying this lemma is that f is representable and quasi-affine, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being quasi-affine is Zariski local on the target (the reference above shows that being quasi-affine is in fact fpqc local on the target). $\square$

03WN **Lemma 67.21.4.** *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Omitted. $\square$

03WO **Lemma 67.21.5.** *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Omitted. $\square$

086S **Lemma 67.21.6.** *Let S be a scheme. A quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ is quasi-affine if and only if the canonical factorization $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (Remark 67.20.9) is an open immersion.*

Proof. Let $U \to X$ be a surjective morphism where U is a scheme. Since we may check whether f is quasi-affine after base change to U (Lemma 67.21.3), since $f_*\mathcal{O}_Y|_V$ is equal to $(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U}$ (Properties of Spaces, Lemma 66.26.2), and since formation of relative spectrum commutes with base change (Lemma 67.20.7), we see that the assertion reduces to the case that X is a scheme. If X is a scheme and either f is quasi-affine or $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ is an open immersion, then Y is a scheme as well. Thus we reduce to Morphisms, Lemma 29.13.3. $\square$

67.22. Types of morphisms étale local on source-and-target

03MI Given a property of morphisms of schemes which is étale local on the source-and-target, see Descent, Definition 35.32.3 we may use it to define a corresponding property of morphisms of algebraic spaces, namely by imposing either of the equivalent conditions of the lemma below.

03MJ **Lemma 67.22.1.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale. The following are equivalent

(1) *for any diagram as above the morphism h has property $\mathcal{P}$, and*
(2) *for some diagram as above with $a : U \to X$ surjective the morphism h has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for representable morphisms f this is also equivalent to f having property $\mathcal{P}$ in the sense of Section 67.3.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate (taking into account Spaces, Lemma 65.11.6). Assume

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "h'"'] \arrow[d] & V' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are two diagrams as in the lemma. Assume $U \to X$ is surjective and h has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the diagram

$$\begin{tikzcd} U \arrow[d, "h"'] & U \times_X U' \arrow[l] \arrow[r] \arrow[d, "{(h,h')}"] & U' \arrow[d, "h'"] \\ V & V \times_Y V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By Descent, Lemma 35.32.5 we see that h has $\mathcal{P}$ implies (h, h') has $\mathcal{P}$ and since $U \times_X U' \to U'$ is surjective this implies (by the same lemma) that h' has $\mathcal{P}$.

If X and Y are representable, then Descent, Lemma 35.32.5 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to X$ the base change $Z \times_Y X \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} Z \times_Y U \arrow[r] \arrow[d] & Z \times_Y V \arrow[d] \\ Z \times_Y X \arrow[r] & Z \end{tikzcd}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is étale and surjective and the right vertical arrow is étale. Thus Descent, Lemma 35.32.5 once again kicks in and shows that $Z \times_Y X \to Z$ has property $\mathcal{P}$. $\square$

04RD Definition 67.22.2. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. We say a morphism $f : X \to Y$ of algebraic spaces over S has property $\mathcal{P}$ if the equivalent conditions of Lemma 67.22.1 hold.

Here are a couple of obvious remarks.

0AML Remark 67.22.3. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Suppose that moreover $\mathcal{P}$ is stable under compositions. Then the class of morphisms of algebraic spaces having property $\mathcal{P}$ is stable under composition.

0AMM Remark 67.22.4. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Suppose that moreover $\mathcal{P}$ is stable under base change. Then the class of morphisms of algebraic spaces having property $\mathcal{P}$ is stable under base change.

Given a property of morphisms of germs of schemes which is étale local on the source-and-target, see Descent, Definition 35.33.1 we may use it to define a corresponding property of morphisms of algebraic spaces at a point, namely by imposing either of the equivalent conditions of the lemma below.

04NC Lemma 67.22.5. Let $\mathcal{Q}$ be a property of morphisms of germs which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point of X. Consider the diagrams

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

where U and V are schemes, a, b are étale, and u, v, x, y are points of the corresponding spaces. The following are equivalent

(1) for any diagram as above we have $\mathcal{Q}((U, u) \to (V, v))$, and
(2) for some diagram as above we have $\mathcal{Q}((U, u) \to (V, v))$.

If X and Y are representable, then this is also equivalent to $\mathcal{Q}((X, x) \to (Y, y))$.

Proof. Omitted. Hint: Very similar to the proof of Lemma 67.22.1. □

04RE Definition 67.22.6. Let $\mathcal{Q}$ be a property of morphisms of germs of schemes which is étale local on the source-and-target. Let S be a scheme. Given a morphism $f : X \to Y$ of algebraic spaces over S and a point $x \in |X|$ we say that f has property $\mathcal{Q}$ at x if the equivalent conditions of Lemma 67.22.5 hold.

The following lemma should not be used blindly to go from a property of morphisms to a property of morphisms at a point. For example if $\mathcal{P}$ is the property of being flat, then the property Q in the following lemma means "f is flat in an open neighbourhood of x" which is not the same as "f is flat at x".

04RF Lemma 67.22.7. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs associated to $\mathcal{P}$ in Descent, Lemma 35.33.2. Then

(1) $\mathcal{Q}$ is étale local on the source-and-target.
(2) given a morphism of algebraic spaces $f : X \to Y$ and $x \in |X|$ the following are equivalent
   (a) f has $\mathcal{Q}$ at x, and
   (b) there is an open neighbourhood $X' \subset X$ of x such that $X' \to Y$ has $\mathcal{P}$.
(3) given a morphism of algebraic spaces $f : X \to Y$ the following are equivalent:
   (a) f has $\mathcal{P}$,
   (b) for every $x \in |X|$ the morphism f has $\mathcal{Q}$ at x.

Proof. See Descent, Lemma 35.33.2 for (1). The implication (1)(a) $\Rightarrow$ (2)(b) follows on letting $X' = a(U) \subset X$ given a diagram as in Lemma 67.22.5. The implication

(2)(b) $\Rightarrow$ (1)(a) is clear. The equivalence of (3)(a) and (3)(b) follows from the corresponding result for morphisms of schemes, see Descent, Lemma 35.33.3. $\square$

04RG Remark 67.22.8. We will apply Lemma 67.22.7 above to all cases listed in Descent, Remark 35.32.7 except "flat". In each case we will do this by defining f to have property $\mathcal{P}$ at x if f has $\mathcal{P}$ in a neighbourhood of x.

67.23. Morphisms of finite type

03XE The property "locally of finite type" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.15.4, and Descent, Lemmas 35.23.10. Hence, by Lemma 67.22.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03XF Definition 67.23.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f locally of finite type if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ locally of finite type.
(2) Let $x \in |X|$. We say f is of finite type at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite type.
(3) We say f is of finite type if it is locally of finite type and quasi-compact.

Consider the algebraic space $\mathbf{A}^1_k/\mathbf{Z}$ of Spaces, Example 65.14.8. The morphism $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is of finite type.

03XG Lemma 67.23.2. The composition of finite type morphisms is of finite type. The same holds for locally of finite type.

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.15.3. $\square$

03XH Lemma 67.23.3. A base change of a finite type morphism is finite type. The same holds for locally of finite type.

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.15.4. $\square$

040Y Lemma 67.23.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is locally of finite type,
(2) for every $x \in |X|$ the morphism f is of finite type at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite type,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite type,

(7) for every commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite type,

(8) there exists a commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are schemes, the vertical arrows are étale, $U \to X$ is surjective, and the top horizontal arrow is locally of finite type, and

(9) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite type.

Proof. Each of the conditions (2), (3), (4), (5), (6), (7), and (9) imply condition (8) in a straightforward manner. For example, if (5) holds, then we can choose a scheme V which is a disjoint union of affines and a surjective morphism $V \to Y$ (see Properties of Spaces, Lemma 66.6.1). Then $V \times_Y X \to V$ is locally of finite type by (5). Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is locally of finite type by Lemma 67.23.2. Hence (8) is true.

The conditions (1), (7), and (8) are equivalent by definition.

To finish the proof, we show that (1) implies all of the conditions (2), (3), (4), (5), (6), and (9). For (2) this is immediate. For (3), (4), (5), and (9) this follows from the fact that being locally of finite type is preserved under base change, see Lemma 67.23.3. For (6) we can take $U = X$ and we're done. $\square$

04ZK Lemma 67.23.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and Y is locally Noetherian, then X is locally Noetherian.

Proof. Let
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
be a commutative diagram where U, V are schemes and the vertical arrows are surjective étale. If f is locally of finite type, then $U \to V$ is locally of finite type. If Y is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 29.15.6 we see that U is locally Noetherian, which means that X is locally Noetherian. $\square$

0462 Lemma 67.23.6. Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.

Proof. We can find a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

where U, V, W are schemes, the vertical arrows are étale and surjective, see Spaces, Lemma 65.11.6. At this point we can use Lemma 67.23.4 and Morphisms, Lemma 29.15.8 to conclude. ☐

06ED Lemma 67.23.7. An immersion is locally of finite type.

Proof. Follows from the general principle Spaces, Lemma 65.5.8 and Morphisms, Lemmas 29.15.5. ☐

67.24. Points and geometric points

0485 In this section we make some remarks on points and geometric points (see Properties of Spaces, Definition 66.19.1). One way to think about a geometric point of X is to consider a geometric point $\overline{s} : \mathrm{Spec}(k) \to S$ of S and a lift of $\overline{s}$ to a morphism $\overline{x}$ into X. Here is a diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "\overline{x}"'] \arrow[rd, "\overline{s}"'] & X \arrow[d] \\ & S. \end{tikzcd}$$

We often say "let k be an algebraically closed field over S" to indicate that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to S$. In this situation we write

$$X(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), X) = \{\overline{x} \in X \text{ lying over } \overline{s}\}$$

for the set of k-valued points of X. In this case the map $X(k) \to |X|$ maps into the subset $|X_s| \subset |X|$. Here $X_s = \mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. As $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism, also the base change $X_s \to X$ is a monomorphism, and $|X_s|$ is indeed a subset of $|X|$.

0487 Lemma 67.24.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. The following are equivalent:

(1) f is surjective, and
(2) for every algebraically closed field k over S the induced map $X(k) \to Y(k)$ is surjective.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 65.11.6. Since f is locally of finite type we see that $U \to V$ is locally of finite type.

Assume (1) and let $\overline{y} \in Y(k)$. Then $U \to Y$ is surjective and locally of finite type by Lemmas 67.5.4 and 67.23.2. Let $Z = U \times_{Y,\overline{y}} \mathrm{Spec}(k)$. This is a scheme. The projection $Z \to \mathrm{Spec}(k)$ is surjective and locally of finite type by Lemmas 67.5.5

and 67.23.3. It follows from Varieties, Lemma 33.14.1 that Z has a k valued point $\overline{z}$. The image $\overline{x} \in X(k)$ of $\overline{z}$ maps to $\overline{y}$ as desired.

Assume (2). By Properties of Spaces, Lemma 66.4.4 it suffices to show that $|X| \to |Y|$ is surjective. Let $y \in |Y|$. Choose a $u \in U$ mapping to y. Let $k \supset \kappa(u)$ be an algebraic closure. Denote $\overline{u} \in U(k)$ the corresponding point and $\overline{y} \in Y(k)$ its image. By assumption there exists a $\overline{x} \in X(k)$ mapping to $\overline{y}$. Then it is clear that the image $x \in |X|$ of $\overline{x}$ maps to y. $\square$

In order to state the next lemma we introduce the following notation. Given a scheme T we denote

$$\lambda(T) = \sup\{\aleph_0, |\kappa(t)|; t \in T\}.$$

In words $\lambda(T)$ is the smallest infinite cardinal bounding all the cardinalities of residue fields ot T. Note that if R is a ring then the cardinality of any residue field $\kappa(\mathfrak{p})$ of R is bounded by the cardinality of R (details omitted). This implies that $\lambda(T) \leq \text{size}(T)$ where $\text{size}(T)$ is the size of the scheme T as introduced in Sets, Section 3.9. If L/K is a finitely generated field extension then $|K| \leq |L| \leq \max\{\aleph_0, |K|\}$. It follows that if $T' \to T$ is a morphism of schemes which is locally of finite type then $\lambda(T') \leq \lambda(T)$, and if $T' \to T$ is also surjective then equality holds. Next, suppose that S is a scheme and that X is an algebraic space over S. In this case we define

$$\lambda(X) := \lambda(U)$$

where U is any scheme over S which has a surjective étale morphism towards X. The reason that this is independent of the choice of U is that given a pair of such schemes U and U' the fibre product $U \times_X U'$ is a scheme which admits a surjective étale morphism to both U and U', whence $\lambda(U) = \lambda(U \times_X U') = \lambda(U')$ by the discussion above.

0488 Lemma 67.24.2. Let S be a scheme. Let X, Y be algebraic spaces over S.

(1) As k ranges over all algebraically closed fields over S the collection of geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.
(2) As k ranges over all algebraically closed fields over S with $|k| \geq \lambda(Y)$ and $|k| > \lambda(X)$ the geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.
(3) For any geometric point $\overline{s} : \text{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map
$$X(k) \longrightarrow |X_s|$$
is surjective.
(4) Let $X \to Y$ be a morphism of algebraic spaces over S. For any geometric point $\overline{s} : \text{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map
$$X(k) \longrightarrow |X| \times_{|Y|} Y(k)$$
is surjective.
(5) Let $X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:
    (a) the map $X \to Y$ is surjective,
    (b) for all algebraically closed fields k over S with $|k| > \lambda(X)$, and $|k| \geq \lambda(Y)$ the map $X(k) \to Y(k)$ is surjective.

Proof. To prove part (1) choose a surjective étale morphism $V \to Y$ where V is a scheme. For each $v \in V$ choose an algebraic closure $\kappa(v) \subset k_v$. Consider the morphisms $\overline{x} : \mathrm{Spec}(k_v) \to V \to Y$. By construction of $|Y|$ these cover $|Y|$.

To prove part (2) we will use the following two facts whose proofs we omit: (i) If K is a field and $\overline{K}$ is algebraic closure then $|\overline{K}| \leq \max\{\aleph_0, |K|\}$. (ii) For any algebraically closed field k and any cardinal $\aleph$, $\aleph \geq |k|$ there exists an extension of algebraically closed fields k'/k with $|k'| = \aleph$. Now we set $\aleph = \max\{\lambda(X), \lambda(Y)\}^+$. Here $\lambda^+ > \lambda$ indicates the next bigger cardinal, see Sets, Section 3.6. Now (i) implies that the fields k_u constructed in the first paragraph of the proof all have cardinality bounded by $\lambda(X)$. Hence by (ii) we can find extensions $k_u \subset k'_u$ such that $|k'_u| = \aleph$. The morphisms $\overline{x}' : \mathrm{Spec}(k'_u) \to X$ cover $|X|$ as desired. To really finish the proof of (2) we need to show that the schemes $\mathrm{Spec}(k'_u)$ are (isomorphic to) objects of Sch_{fppf} because our conventions are that all schemes are objects of Sch_{fppf}; the rest of this paragraph should be skipped by anyone who is not interested in set theoretical considerations. By construction there exists an object T of Sch_{fppf} such that $\lambda(X)$ and $\lambda(Y)$ are bounded by $\text{size}(T)$. By our construction of the category Sch_{fppf} in Topologies, Definitions 34.7.6 as the category Sch_α constructed in Sets, Lemma 3.9.2 we see that any scheme whose size is $\leq \text{size}(T)^+$ is isomorphic to an object of Sch_{fppf}. See the expression for the function *Bound* in Sets, Equation (3.9.1.1). Since $\aleph \leq \text{size}(T)^+$ we conclude.

The notation X_s in part (3) means the fibre product $\mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. Hence part (2) follows from (4) with $Y = \mathrm{Spec}(\kappa(s))$.

Let us prove (4). Let $X \to Y$ be a morphism of algebraic spaces over S. Let k be an algebraically closed field over S of cardinality $> \lambda(X)$. Let $\overline{y} \in Y(k)$ and $x \in |X|$ which map to the same element y of $|Y|$. We have to find $\overline{x} \in X(k)$ mapping to x and $\overline{y}$. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 65.11.6. Choose a $u \in |U|$ which maps to x, and denote $v \in |V|$ the image. We will think of $u = \mathrm{Spec}(\kappa(u))$ and $v = \mathrm{Spec}(\kappa(v))$ as schemes. Note that $V \times_Y \mathrm{Spec}(k)$ is a scheme étale over k. Hence it is a disjoint union of spectra of finite separable extensions of k, see Morphisms, Lemma 29.36.7. As v maps to y we see that $v \times_Y \mathrm{Spec}(k)$ is a nonempty scheme. As $v \to V$ is a monomorphism, we see that $v \times_Y \mathrm{Spec}(k) \to V \times_Y \mathrm{Spec}(k)$ is a monomorphism. Hence $v \times_Y \mathrm{Spec}(k)$ is a disjoint union of spectra of finite separable extensions of k, by Schemes, Lemma 26.23.11. We conclude that the morphism $v \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ has a section, i.e., we can find a morphism $\overline{v} : \mathrm{Spec}(k) \to V$ lying over v and over $\overline{y}$. Finally we consider the scheme

$$u \times_{V,\overline{v}} \mathrm{Spec}(k) = \mathrm{Spec}(\kappa(u) \otimes_{\kappa(v)} k)$$

where $\kappa(v) \to k$ is the field map defining the morphism $\overline{v}$. Since the cardinality of k is larger than the cardinality of $\kappa(u)$ by assumption we may apply Algebra, Lemma 10.35.12 to see that any maximal ideal $\mathfrak{m} \subset \kappa(u) \otimes_{\kappa(v)} k$ has a residue field which is algebraic over k and hence equal to k. Such a maximal ideal will hence produce a morphism $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u and mapping to $\overline{v}$. The composition $\mathrm{Spec}(k) \to U \to X$ will be the desired geometric point $\overline{x} \in X(k)$. This concludes the proof of part (4).

Part (5) is a formal consequence of parts (2) and (4) and Properties of Spaces, Lemma 66.4.4. $\square$

67.25. Points of finite type

06EE Let S be a scheme. Let X be an algebraic space over S. A finite type point $x \in |X|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to X$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

06EF **Lemma 67.25.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

1. *There exists a morphism $\mathrm{Spec}(k) \to X$ which is locally of finite type and represents x.*
2. *There exists a scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to X$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to X$ is representable and locally of finite type (by the general principle Spaces, Lemma 65.5.8 and Morphisms, Lemmas 29.36.11 and 29.21.8). Hence we see (1) holds by Lemma 67.23.2.

Conversely, assume $\mathrm{Spec}(k) \to X$ is locally of finite type and represents x. Let $U \to X$ be a surjective étale morphism where U is a scheme. By assumption $U \times_X \mathrm{Spec}(k) \to U$ is locally of finite type. Pick a finite type point v of $U \times_X \mathrm{Spec}(k)$ (there exists at least one, see Morphisms, Lemma 29.16.4). By Morphisms, Lemma 29.16.5 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 29.16.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. $\square$

06EG **Definition 67.25.2.** Let S be a scheme. Let X be an algebraic space over S. We say a point $x \in |X|$ is a *finite type point*[5] if the equivalent conditions of Lemma 67.25.1 are satisfied. We denote $X_{\text{ft-pts}}$ the set of finite type points of X.

We can describe the set of finite type points as follows.

06EH **Lemma 67.25.3.** *Let S be a scheme. Let X be an algebraic space over S. We have*

$$X_{\text{ft-pts}} = \bigcup\nolimits_{\varphi : U \to X \text{ étale}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes étale over X or over all affine schemes étale over X.

Proof. Immediate from Lemma 67.25.1. $\square$

06EI **Lemma 67.25.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $f(X_{\text{ft-pts}}) \subset Y_{\text{ft-pts}}$.*

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[5] This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

Proof. Take $x \in X_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \text{Spec}(k) \to X$. Then $f \circ x$ is locally of finite type by Lemma 67.23.2. Hence $f(x) \in Y_{\text{ft-pts}}$. $\square$

06EJ Lemma 67.25.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and surjective, then $f(X_{\text{ft-pts}}) = Y_{\text{ft-pts}}$.

Proof. We have $f(X_{\text{ft-pts}}) \subset Y_{\text{ft-pts}}$ by Lemma 67.25.4. Let $y \in |Y|$ be a finite type point. Represent y by a morphism $\text{Spec}(k) \to Y$ which is locally of finite type. As f is surjective the algebraic space $X_k = \text{Spec}(k) \times_Y X$ is nonempty, therefore has a finite type point $x \in |X_k|$ by Lemma 67.25.3. Now $X_k \to X$ is a morphism which is locally of finite type as a base change of $\text{Spec}(k) \to Y$ (Lemma 67.23.3). Hence the image of x in X is a finite type point by Lemma 67.25.4 which maps to y by construction. $\square$

06EK Lemma 67.25.6. Let S be a scheme. Let X be an algebraic space over S. For any locally closed subset $T \subset |X|$ we have

$$T \neq \emptyset \Rightarrow T \cap X_{\text{ft-pts}} \neq \emptyset.$$

In particular, for any closed subset $T \subset |X|$ we see that $T \cap X_{\text{ft-pts}}$ is dense in T.

Proof. Let $i : Z \to X$ be the reduced induce subspace structure on T, see Remark 67.12.5. Any immersion is locally of finite type, see Lemma 67.23.7. Hence by Lemma 67.25.4 we see $Z_{\text{ft-pts}} \subset X_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with an étale morphism towards Z has at least one closed point. Hence Z has at least one finite type point by Lemma 67.25.3. The lemma follows. $\square$

Here is another, more technical, characterization of a finite type point on an algebraic space.

06EL Lemma 67.25.7. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:

(1) x is a finite type point,
(2) there exists an algebraic space Z whose underlying topological space $|Z|$ is a singleton, and a morphism $f : Z \to X$ which is locally of finite type such that $\{x\} = |f|(|Z|)$, and
(3) there exists an algebraic space Z and a morphism $f : Z \to X$ with the following properties:
  (a) there is a surjective étale morphism $z : \text{Spec}(k) \to Z$ where k is a field,
  (b) f is locally of finite type,
  (c) f is a monomorphism, and
  (d) $x = f(z)$.

Proof. Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ with $\varphi(u) = x$, see Lemma 67.25.3. Set $u = \text{Spec}(\kappa(u))$ as usual. The projection morphisms $u \times_X u \to u$ are the compositions

$$u \times_X u \to u \times_X U \to u \times_X X = u$$

where the first arrow is a closed immersion (a base change of $u \to U$) and the second arrow is étale (a base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of finite separable extensions of k (see Morphisms, Lemma

29.36.7) and therefore the closed subscheme $u \times_X u$ is a disjoint union of finite separable extension of k, i.e., $u \times_X u \to u$ is étale. By Spaces, Theorem 65.10.5 we see that $Z = u/u \times_X u$ is an algebraic space. By construction the diagram

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is commutative with étale vertical arrows. Hence $Z \to X$ is locally of finite type (see Lemma 67.23.4). By construction the morphism $Z \to X$ is a monomorphism and the image of z is x. Thus (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of X by Lemma 67.25.4 (and Lemma 67.25.6 to see that $Z_{\text{ft-pts}}$ is nonempty, i.e., the unique point of Z is a finite type point of Z). □

67.26. Nagata spaces

0BAT See Properties of Spaces, Section 66.7 for the definition of a Nagata algebraic space.

0BAU Lemma 67.26.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is Nagata and f locally of finite type then X is Nagata.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let U be a scheme and let $U \to X \times_Y V$ be a surjective étale morphism. If Y is Nagata, then V is a Nagata scheme. If $X \to Y$ is locally of finite type, then $U \to V$ is locally of finite type. Hence V is a Nagata scheme by Morphisms, Lemma 29.18.1. Then X is Nagata by definition. □

0BAV Lemma 67.26.2. The following types of algebraic spaces are Nagata.

(1) Any algebraic space locally of finite type over a Nagata scheme.
(2) Any algebraic space locally of finite type over a field.
(3) Any algebraic space locally of finite type over a Noetherian complete local ring.
(4) Any algebraic space locally of finite type over **Z**.
(5) Any algebraic space locally of finite type over a Dedekind ring of characteristic zero.
(6) And so on.

Proof. The first property holds by Lemma 67.26.1. Thus the others hold as well, see Morphisms, Lemma 29.18.2. □

67.27. Quasi-finite morphisms

03XI The property "locally quasi-finite" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.20.13, and Descent, Lemma 35.23.24. Hence, by Lemma 67.22.1 above, we may define what it means for a morphism of algebraic spaces to be locally quasi-finite as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03XJ Definition 67.27.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is locally quasi-finite if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ locally quasi-finite.
(2) Let $x \in |X|$. We say f is quasi-finite at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally quasi-finite.
(3) A morphism of algebraic spaces $f : X \to Y$ is quasi-finite if it is locally quasi-finite and quasi-compact.

The last part is compatible with the notion of quasi-finiteness for morphisms of schemes by Morphisms, Lemma 29.20.9.

0ABM Lemma 67.27.2. Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of algebraic spaces over S. Denote $f' : X' \to Y'$ the base change of f by g. Denote $g' : X' \to X$ the projection. Assume f is locally of finite type. Let $W \subset |X|$, resp. $W' \subset |X'|$ be the set of points where f, resp. f' is quasi-finite.

(1) $W \subset |X|$ and $W' \subset |X'|$ are open,
(2) $W' = (g')^{-1}(W)$, i.e., formation of the locus where f is quasi-finite commutes with base change,
(3) the base change of a locally quasi-finite morphism is locally quasi-finite, and
(4) the base change of a quasi-finite morphism is quasi-finite.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to Y' \times_Y V$. Set $U' = V' \times_V U$ so that $U' \to X'$ is a surjective étale morphism as well. Picture

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] & \text{lying over} & X' \arrow[r] \arrow[d] & X \arrow[d] \\ V' \arrow[r] & V & & Y' \arrow[r] & Y \end{tikzcd}$$

Choose $u \in |U|$ with image $x \in |X|$. The property of being "locally quasi-finite" is étale local on the source-and-target, see Descent, Remark 35.32.7. Hence Lemmas 67.22.5 and 67.22.7 apply and we see that $f : X \to Y$ is quasi-finite at x if and only if $U \to V$ is quasi-finite at u. Similarly for $f' : X' \to Y'$ and the morphism $U' \to V'$. Hence parts (1), (2), and (3) reduce to Morphisms, Lemmas 29.20.13 and 29.56.2. Part (4) follows from (3) and Lemma 67.8.4. □

03XK Lemma 67.27.3. The composition of quasi-finite morphisms is quasi-finite. The same holds for locally quasi-finite.

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.20.12. □

03XL Lemma 67.27.4. A base change of a quasi-finite morphism is quasi-finite. The same holds for locally quasi-finite.

Proof. Immediate consequence of Lemma 67.27.2. □

The following lemma characterizes locally quasi-finite morphisms as those morphisms which are locally of finite type and have “discrete fibres”. However, this is not the same thing as asking $|X| \to |Y|$ to have discrete fibres as the discussion in Examples, Section 110.50 shows.

06RW **Lemma 67.27.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces. Assume f is locally of finite type. The following are equivalent

(1) f is locally quasi-finite,
(2) for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field the space $|X_k|$ is discrete. Here $X_k = \mathrm{Spec}(k) \times_Y X$.

Proof. Assume f is locally quasi-finite. Let $\mathrm{Spec}(k) \to Y$ be as in (2). Choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_k = \mathrm{Spec}(k) \times_Y U \to X_k$ is an étale morphism of algebraic spaces by Properties of Spaces, Lemma 66.16.5. By Lemma 67.27.4 we see that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. By definition this means that $U_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Hence $|U_k|$ is discrete by Morphisms, Lemma 29.20.8. Since $|U_k| \to |X_k|$ is surjective and open we conclude that $|X_k|$ is discrete.

Conversely, assume (2). Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Note that $U \to V$ is locally of finite type as f is locally of finite type. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

If f is not locally quasi-finite then $U \to V$ is not locally quasi-finite. Hence there exists a specialization $u \rightsquigarrow u'$ for some $u, u' \in U$ lying over the same point $v \in V$, see Morphisms, Lemma 29.20.6. We claim that u, u' do not have the same image in $X_v = \mathrm{Spec}(\kappa(v)) \times_Y X$ which will contradict the assumption that $|X_v|$ is discrete as desired. Let $d = \mathrm{trdeg}_{\kappa(v)}(\kappa(u))$ and $d' = \mathrm{trdeg}_{\kappa(v)}(\kappa(u'))$. Then we see that $d > d'$ by Morphisms, Lemma 29.28.7. Note that U_v (the fibre of $U \to V$ over v) is the fibre product of U and X_v over $X \times_Y V$, hence $U_v \to X_v$ is étale (as a base change of the étale morphism $U \to X \times_Y V$). If $u, u' \in U_v$ map to the same element of $|X_v|$ then there exists a point $r \in R_v = U_v \times_{X_v} U_v$ with $t(r) = u$ and $s(r) = u'$, see Properties of Spaces, Lemma 66.4.3. Note that $s, t : R_v \to U_v$ are étale morphisms of schemes over $\kappa(v)$, hence $\kappa(u) \subset \kappa(r) \supset \kappa(u')$ are finite separable extensions of fields over $\kappa(v)$ (see Morphisms, Lemma 29.36.7). We conclude that the transcendence degrees are equal. This contradiction finishes the proof. $\square$

040Z **Lemma 67.27.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is locally quasi-finite,
(2) for every $x \in |X|$ the morphism f is quasi-finite at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally quasi-finite,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally quasi-finite,

(7) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally quasi-finite,

(8) there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally quasi-finite, and

(9) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally quasi-finite.

Proof. Omitted. □

03XM Lemma 67.27.7. An immersion is locally quasi-finite.

Proof. Omitted. □

03XN Lemma 67.27.8. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is locally quasi-finite, then $X \to Y$ is locally quasi-finite.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 65.11.6.) Apply Morphisms, Lemma 29.20.17 to the top row. □

0ABN Lemma 67.27.9. Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let $y \in |Y|$. There are at most finitely many points of $|X|$ lying over y at which f is quasi-finite.

Proof. Choose a field k and a morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class determined by y. The fibre $X_k = \mathrm{Spec}(k) \times_Y X$ is an algebraic space of finite type over a field, in particular quasi-compact. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 66.4.3). Moreover, the set of points where $X_k \to \mathrm{Spec}(k)$ is quasi-finite maps onto the set of points lying over y where f is quasi-finite by Lemma 67.27.2. Choose an affine scheme U and a surjective étale morphism $U \to X_k$ (Properties of Spaces, Lemma 66.6.3). Then $U \to \mathrm{Spec}(k)$ is a morphism of finite type and there are at most a finite number of points where this morphism is quasi-finite, see Morphisms, Lemma 29.20.14. Since $X_k \to \mathrm{Spec}(k)$ is quasi-finite at a point x' if and only if it is the image of a point of U where $U \to \mathrm{Spec}(k)$ is quasi-finite, we conclude. □

0463 Lemma 67.27.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.

Proof. A monomorphism is separated, see Lemma 67.10.3. By Lemma 67.27.6 it suffices to prove the lemma after performing a base change by $Z \to Y$ with Z affine. Hence we may assume that Y is an affine scheme. Choose an affine scheme U and an étale morphism $U \to X$. Since $X \to Y$ is locally of finite type the morphism of affine schemes $U \to Y$ is of finite type. Since $X \to Y$ is a monomorphism we have $U \times_X U = U \times_Y U$. In particular the maps $U \times_Y U \to U$ are étale. Let $y \in Y$. Then either U_y is empty, or $\mathrm{Spec}(\kappa(u)) \times_{\mathrm{Spec}(\kappa(y))} U_y$ is isomorphic to the fibre of $U \times_Y U \to U$ over u for some $u \in U$ lying over y. This implies that the fibres of $U \to Y$ are finite discrete sets (as $U \times_Y U \to U$ is an étale morphism of affine schemes, see Morphisms, Lemma 29.36.7). Hence $U \to Y$ is quasi-finite, see Morphisms, Lemma 29.20.6. As $U \to X$ was an arbitrary étale morphism with U affine this implies that $X \to Y$ is locally quasi-finite. □

67.28. Morphisms of finite presentation

03XO The property "locally of finite presentation" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.21.4, and Descent, Lemma 35.23.11. Hence, by Lemma 67.22.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03XP Definition 67.28.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is locally of finite presentation if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P}$ ="locally of finite presentation".
(2) Let $x \in |X|$. We say f is of finite presentation at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite presentation[6].
(3) A morphism of algebraic spaces $f : X \to Y$ is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is not just a quasi-compact morphism which is locally of finite presentation.

03XQ Lemma 67.28.2. The composition of morphisms of finite presentation is of finite presentation. The same holds for locally of finite presentation.

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.21.3. Also use the result for quasi-compact and for quasi-separated morphisms (Lemmas 67.8.5 and 67.4.8). □

03XR Lemma 67.28.3. A base change of a morphism of finite presentation is of finite presentation. The same holds for locally of finite presentation.

[6] It seems awkward to use "locally of finite presentation at x", but the current terminology may be misleading in the sense that "of finite presentation at x" does not mean that there is an open neighbourhood $X' \subset X$ such that $f|_{X'}$ is of finite presentation.

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.21.4. Also use the result for quasi-compact and for quasi-separated morphisms (Lemmas 67.8.4 and 67.4.4). □

0410 Lemma 67.28.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is locally of finite presentation,
(2) for every $x \in |X|$ the morphism f is of finite presentation at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite presentation,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite presentation,
(7) for every commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite presentation,
(8) there exists a commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally of finite presentation, and
(9) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite presentation.

Proof. Omitted. □

0464 Lemma 67.28.5. A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite presentation. This means there exists a diagram as in Lemma 67.22.1 with h locally of finite presentation and surjective vertical arrow a. By Morphisms, Lemma 29.21.8 h is locally of finite type. Hence $X \to Y$ is locally of finite type by definition. If f is of finite presentation then it is quasi-compact and it follows that f is of finite type. □

04ZL Lemma 67.28.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is of finite presentation and Y is Noetherian, then X is Noetherian.

Proof. Assume f is of finite presentation and Y Noetherian. By Lemmas 67.28.5 and 67.23.5 we see that X is locally Noetherian. As f is quasi-compact and Y is quasi-compact we see that X is quasi-compact. As f is of finite presentation it is quasi-separated (see Definition 67.28.1) and as Y is Noetherian it is quasi-separated (see Properties of Spaces, Definition 66.24.1). Hence X is quasi-separated by Lemma 67.4.9. Hence we have checked all three conditions of Properties of Spaces, Definition 66.24.1 and we win. □

06G4 Lemma 67.28.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) If Y is locally Noetherian and f locally of finite type then f is locally of finite presentation.
(2) If Y is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.

Proof. Assume $f : X \to Y$ locally of finite type and Y locally Noetherian. This means there exists a diagram as in Lemma 67.22.1 with h locally of finite type and surjective vertical arrow a. By Morphisms, Lemma 29.21.9 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

06G5 Lemma 67.28.8. Let S be a scheme. Let Y be an algebraic space over S which is quasi-compact and quasi-separated. If X is of finite presentation over Y, then X is quasi-compact and quasi-separated.

Proof. Omitted. □

05WT Lemma 67.28.9. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. If X is locally of finite presentation over Z, and Y is locally of finite type over Z, then f is locally of finite presentation.

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Then choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Finally choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By definition U is locally of finite presentation over W and V is locally of finite type over W. By Morphisms, Lemma 29.21.11 the morphism $U \to V$ is locally of finite presentation. Hence f is locally of finite presentation. □

084P Lemma 67.28.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.

Proof. Note that Δ is a morphism over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is of finite type over X (by Lemma 67.23.3). Thus the first statement holds by Lemma 67.28.9. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is separated (Lemma 67.4.1). □

06CN Lemma 67.28.11. An open immersion of algebraic spaces is locally of finite presentation.

Proof. An open immersion is by definition representable, hence we can use the general principle Spaces, Lemma 65.5.8 and Morphisms, Lemma 29.21.5. $\square$

084Q Lemma 67.28.12. A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By Lemma 67.28.4 we see that $i' : Z \times_X U \to U$ is of finite presentation if and only if i is. By Properties of Spaces, Section 66.30 we see that $\mathcal{I}$ is of finite type if and only if $\mathcal{I}|_U = \mathrm{Ker}(\mathcal{O}_U \to i'_*\mathcal{O}_{Z\times_X U})$ is. Hence the result follows from the case of schemes, see Morphisms, Lemma 29.21.7. $\square$

67.29. Constructible sets

0ECV This section is the continuation of Properties of Spaces, Section 66.8.

0ECW Lemma 67.29.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $E \subset |Y|$ be a subset. If E is étale locally constructible in Y, then $f^{-1}(E)$ is étale locally constructible in X.

Proof. Choose a scheme V and a surjective étale morphism $\varphi : V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to X$ is surjective étale and the inverse image of $f^{-1}(E)$ in U is the inverse image of $\varphi^{-1}(E)$ by $U \to V$. Thus the lemma follows from the case of schemes for $U \to V$ (Morphisms, Lemma 29.22.1) and the definition (Properties of Spaces, Definition 66.8.2). $\square$

0ECX Theorem 67.29.2 (Chevalley's Theorem). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and locally of finite presentation. Then the image of every étale locally constructible subset of $|X|$ is an étale locally constructible subset of $|Y|$.

Proof. Let $E \subset |X|$ be étale locally constructible. Let $V \to Y$ be an étale morphism with V affine. It suffices to show that the inverse image of $f(E)$ in V is constructible, see Properties of Spaces, Definition 66.8.2. Since f is quasi-compact $V \times_Y X$ is a quasi-compact algebraic space. Choose an affine scheme U and a surjective étale morphism $U \to V \times_Y X$ (Properties of Spaces, Lemma 66.6.3). By Properties of Spaces, Lemma 66.4.3 the inverse image of $f(E)$ in V is the image under $U \to V$ of the inverse image of E in U. Thus the result follows from the case of schemes, see Morphisms, Lemma 29.22.2. $\square$

67.30. Flat morphisms

03MK The property "flat" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.25.8 and Descent, Lemma 35.23.15. Hence, by Lemma 67.22.1 above, we may define the notion of a flat morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03ML Definition 67.30.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is flat if the equivalent conditions of Lemma 67.22.1 with $\mathcal{P} =$"flat".

(2) Let $x \in |X|$. We say f is flat at x if the equivalent conditions of Lemma 67.22.5 hold with $\mathcal{Q}$ ="induced map local rings is flat".

Note that the second part makes sense by Descent, Lemma 35.33.4.

We do a quick sanity check.

08EW Lemma 67.30.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if f is flat at all points of $|X|$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. By definition f is flat if and only if h is flat (Definition 67.22.2). By definition f is flat at $x \in |X|$ if and only if h is flat at some (equivalently any) $u \in U$ which maps to x (Definition 67.22.6). Thus the lemma follows from the fact that a morphism of schemes is flat if and only if it is flat at all points of the source (Morphisms, Definition 29.25.1). □

03MN Lemma 67.30.3. The composition of flat morphisms is flat.

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.25.6. □

03MO Lemma 67.30.4. The base change of a flat morphism is flat.

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.25.8. □

03MM Lemma 67.30.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is flat,
(2) for every $x \in |X|$ the morphism f is flat at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is flat,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is flat,
(7) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is flat,

(8) there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is flat, and

(9) there exists a Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is flat.

Proof. Omitted. ☐

042S Lemma 67.30.6. A flat morphism locally of finite presentation is universally open.

Proof. Let $f : X \to Y$ be a flat morphism locally of finite presentation of algebraic spaces over S. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. By Lemmas 67.30.5 and 67.28.4 the morphism α is flat and locally of finite presentation. Hence by Morphisms, Lemma 29.25.10 we see that α is universally open. Hence $X \to Y$ is universally open according to Lemma 67.6.5. ☐

0413 Lemma 67.30.7. Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, surjective morphism of algebraic spaces over S. A subset $T \subset |Y|$ is open (resp. closed) if and only $f^{-1}(|T|)$ is open (resp. closed) in $|X|$. In other words f is submersive, and in fact universally submersive.

Proof. Choose affine schemes V_i and étale morphisms $V_i \to Y$ such that $V = \coprod V_i \to Y$ is surjective, see Properties of Spaces, Lemma 66.6.1. For each i the algebraic space $V_i \times_Y X$ is quasi-compact. Hence we can find an affine scheme U_i and a surjective étale morphism $U_i \to V_i \times_Y X$, see Properties of Spaces, Lemma 66.6.3. Then the composition $U_i \to V_i \times_Y X \to V_i$ is a surjective, flat morphism of affines. Of course then $U = \coprod U_i \to X$ is surjective and étale and $U = V \times_Y X$. Moreover, the morphism $U \to V$ is the disjoint union of the morphisms $U_i \to V_i$. Hence $U \to V$ is surjective, quasi-compact and flat. Consider the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

By definition of the topology on $|Y|$ the set T is closed (resp. open) if and only if $g^{-1}(T) \subset |V|$ is closed (resp. open). The same holds for $f^{-1}(T)$ and its inverse image in $|U|$. Since $U \to V$ is quasi-compact, surjective, and flat we win by Morphisms, Lemma 29.25.12. ☐

04NG Lemma 67.30.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$. Let $\overline{y} = f \circ \overline{x}$. The following are equivalent

(1) f is flat at x, and
(2) the map on étale local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x. We can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 66.19.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 66.22.1. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{U,u} \arrow[r] & \mathcal{O}_{X,\overline{x}} \\ \mathcal{O}_{V,v} \arrow[r] \arrow[u] & \mathcal{O}_{Y,\overline{y}} \arrow[u] \end{tikzcd}$$

of local rings with flat horizontal arrows. We have to show that the left vertical arrow is flat if and only if the right vertical arrow is. Algebra, Lemma 10.39.9 tells us $\mathcal{O}_{U,u}$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{O}_{X,\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 38.2.5. □

073C Lemma 67.30.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if the morphism of sites $(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$ associated to f is flat.

Proof. Flatness of $(f_{small}, f^\sharp)$ is defined in terms of flatness of $\mathcal{O}_X$ as a $f_{small}^{-1}\mathcal{O}_Y$-module. This can be checked at stalks, see Modules on Sites, Lemma 18.39.3 and Properties of Spaces, Theorem 66.19.12. But we've already seen that flatness of f can be checked on stalks, see Lemma 67.30.8. □

089C Lemma 67.30.10. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.

Proof. Using the characterization of the scheme theoretic support as given in Lemma 67.15.3 and using the characterization of flat morphisms in terms of étale coverings in Lemma 67.30.5 we reduce to the case of schemes which is Morphisms, Lemma 29.25.14. □

089D Lemma 67.30.11. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $V \to Y$ be a quasi-compact open immersion. If V is scheme theoretically dense in Y, then $f^{-1}V$ is scheme theoretically dense in X.

Proof. Using the characterization of scheme theoretically dense opens in Lemma 67.17.2 and using the characterization of flat morphisms in terms of étale coverings in Lemma 67.30.5 we reduce to the case of schemes which is Morphisms, Lemma 29.25.15. □

089E Lemma 67.30.12. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $g : V \to Y$ be a quasi-compact morphism of algebraic spaces. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.

Proof. Let $Y' \to Y$ be a surjective étale morphism such that Y' is a disjoint union of affine schemes (Properties of Spaces, Lemma 66.6.1). Let $X' \to X \times_Y Y'$ be a surjective étale morphism such that X' is a disjoint union of affine schemes. By Lemma 67.30.5 the morphism $X' \to Y'$ is flat. Set $V' = V \times_Y Y'$. By Lemma 67.16.3 the inverse image of Z in Y' is the scheme theoretic image of $V' \to Y'$ and the inverse image of Z' in X' is the scheme theoretic image of $V' \times_{Y'} X' \to X'$. Since $X' \to X$ is surjective étale, it suffices to prove the result in the case of the morphisms $X' \to Y'$ and $V' \to Y'$. Thus we may assume X and Y are affine schemes. In this case V is a quasi-compact algebraic space. Choose an affine scheme W and a surjective étale morphism $W \to V$ (Properties of Spaces, Lemma 66.6.3). It is clear that the scheme theoretic image of $V \to Y$ agrees with the scheme theoretic image of $W \to Y$ and similarly for $V \times_Y X \to Y$ and $W \times_Y X \to X$. Thus we reduce to the case of schemes which is Morphisms, Lemma 29.25.16. □

67.31. Flat modules

05VT In this section we define what it means for a module to be flat at a point. To do this we will use the notion of the stalk of a sheaf on the small étale site $X_{\acute{e}tale}$ of an algebraic space, see Properties of Spaces, Definition 66.19.6.

05VU Lemma 67.31.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$. The following are equivalent

(1) for some commutative diagram
$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
where U and V are schemes, a, b are étale, and $u \in U$ mapping to x the module $a^*\mathcal{F}$ is flat at u over V,

(2) the stalk $\mathcal{F}_{\overline{x}}$ is flat over the étale local ring $\mathcal{O}_{Y,\overline{y}}$ where $\overline{x}$ is any geometric point lying over x and $\overline{y} = f \circ \overline{x}$.

Proof. During this proof we fix a geometric proof $\overline{x} : \text{Spec}(k) \to X$ over x and we denote $\overline{y} = f \circ \overline{x}$ its image in Y. Given a diagram as in (1) we can find a geometric point $\overline{u} : \text{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 66.19.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that
$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 66.22.1. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{U,u} \arrow[r] & \mathcal{O}_{X,\overline{x}} \\ \mathcal{O}_{V,v} \arrow[r] \arrow[u] & \mathcal{O}_{Y,\overline{y}} \arrow[u] \end{tikzcd}$$

of local rings. Finally, we have

$$\mathcal{F}_{\overline{x}} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$

by Properties of Spaces, Lemma 66.29.4. Thus Algebra, Lemma 10.39.9 tells us $(\varphi^*\mathcal{F})_u$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 38.2.5. □

05VV Definition 67.31.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) Let $x \in |X|$. We say $\mathcal{F}$ is flat at x over Y if the equivalent conditions of Lemma 67.31.1 hold.
(2) We say $\mathcal{F}$ is flat over Y if $\mathcal{F}$ is flat over Y at all $x \in |X|$.

Having defined this we have the obligatory base change lemma. This lemma implies that formation of the flat locus of a quasi-coherent sheaf commutes with flat base change.

05VW Lemma 67.31.3. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $x' \in |X'|$ with image $x \in |X|$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and denote $\mathcal{F}' = (g')^*\mathcal{F}$.

(1) If $\mathcal{F}$ is flat at x over Y then $\mathcal{F}'$ is flat at x' over Y'.
(2) If g is flat at $f'(x')$ and $\mathcal{F}'$ is flat at x' over Y', then $\mathcal{F}$ is flat at x over Y.

In particular, if $\mathcal{F}$ is flat over Y, then $\mathcal{F}'$ is flat over Y'.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme endowed with a surjective étale morphism $U' = V' \times_V U \to Y' \times_Y X = X'$. Pick $u' \in U'$ mapping to $x' \in |X'|$. Then we can check flatness of $\mathcal{F}'$ at x' over Y' in terms of flatness of $\mathcal{F}'|_{U'}$ at u' over V'. Hence the lemma follows from More on Morphisms, Lemma 37.15.2. □

The following lemma discusses "composition" of flat morphisms in terms of modules. It also shows that flatness satisfies a kind of top down descent.

05VX Lemma 67.31.4. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$.

(1) If $\mathcal{F}$ is flat at x over Y and Y is flat at y over Z, then $\mathcal{F}$ is flat at x over Z.

(2) Let $x : \mathrm{Spec}(K) \to X$ be a representative of x. If
   (a) $\mathcal{F}$ is flat at x over Y,
   (b) $x^*\mathcal{F} \neq 0$, and
   (c) $\mathcal{F}$ is flat at x over Z,

   then Y is flat at y over Z.
(3) Let $\overline{x}$ be a geometric point of X lying over x with image $\overline{y}$ in Y. If $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module and $\mathcal{F}$ is flat at x over Z, then Y is flat at y over Z.

Proof. Pick $\overline{x}$ and $\overline{y}$ as in part (3) and denote $\overline{z}$ the induced geometric point of Z. Via the characterization of flatness in Lemmas 67.31.1 and 67.30.8 the lemma reduces to a purely algebraic question on the local ring map $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}}$ and the module $\mathcal{F}_{\overline{x}}$. Part (1) follows from Algebra, Lemma 10.39.4. We remark that condition (2)(b) guarantees that $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{F}_{\overline{x}}$ is nonzero. Hence (2)(a) + (2)(b) imply that $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module, see Algebra, Lemma 10.39.15. Thus (2) is a special case of (3). Finally, (3) follows from Algebra, Lemma 10.39.10. □

Sometimes the base change happens "up on top". Here is a precise statement.

05VY Lemma 67.31.5. Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in |X|$ with image $y \in |Y|$. If f is flat at x, then

$$\mathcal{G} \text{ flat over } Z \text{ at } y \Leftrightarrow f^*\mathcal{G} \text{ flat over } Z \text{ at } x.$$

In particular: If f is surjective and flat, then $\mathcal{G}$ is flat over Z, if and only if $f^*\mathcal{G}$ is flat over Z.

Proof. Pick a geometric point $\overline{x}$ of X and denote $\overline{y}$ the image in Y and $\overline{z}$ the image in Z. Via the characterization of flatness in Lemmas 67.31.1 and 67.30.8 and the description of the stalk of $f^*\mathcal{G}$ at $\overline{x}$ of Properties of Spaces, Lemma 66.29.5 the lemma reduces to a purely algebraic question on the local ring maps $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ and the module $\mathcal{G}_{\overline{y}}$. This algebraic statement is Algebra, Lemma 10.39.9. □

0CVU Lemma 67.31.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f locally finite presentation, $\mathcal{F}$ of finite type, $X = \mathrm{Supp}(\mathcal{F})$, and $\mathcal{F}$ flat over Y. Then f is universally open.

Proof. Choose a surjective étale morphism $\varphi : V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Then it suffices to prove the lemma for $U \to V$ and the quasi-coherent $\mathcal{O}_V$-module $\varphi^*\mathcal{F}$. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 29.25.11. □

67.32. Generic flatness

06QR This section is the analogue of Morphisms, Section 29.27.

06QS Proposition 67.32.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume
(1) Y is reduced,
(2) f is of finite type, and
(3) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let $X_V = V \times_Y X$ and let $\mathcal{F}_V$ be the restriction of $\mathcal{F}$ to X_V. Suppose that the result holds for the morphism $X_V \to V$ and the sheaf $\mathcal{F}_V$. Then there exists an open subscheme $V' \subset V$ such that $X_{V'} \to V'$ is flat and of finite presentation and $\mathcal{F}_{V'}$ is an $\mathcal{O}_{X_{V'}}$-module of finite presentation flat over V'. Let $W \subset Y$ be the image of the étale morphism $V' \to Y$, see Properties of Spaces, Lemma 66.4.10. Then $V' \to W$ is a surjective étale morphism, hence we see that $X_W \to W$ is flat, locally of finite presentation, and quasi-compact by Lemmas 67.28.4, 67.30.5, and 67.8.8. By the discussion in Properties of Spaces, Section 66.30 we see that $\mathcal{F}_W$ is of finite presentation as a $\mathcal{O}_{X_W}$-module and by Lemma 67.31.3 we see that $\mathcal{F}_W$ is flat over W. This argument reduces the proposition to the case where Y is a scheme.

Suppose we can prove the proposition when Y is an affine scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S with Y a scheme, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Choose an affine open covering $Y = \bigcup V_j$. By assumption we can find dense open $W_j \subset V_j$ such that $X_{W_j} \to W_j$ is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_{W_j}}$ is flat over W_j and of finite presentation as an $\mathcal{O}_{X_{W_j}}$-module. In this situation we simply take $W = \bigcup W_j$ and we win. Hence we reduce the proposition to the case where Y is an affine scheme.

Let Y be an affine scheme over S, let $f : X \to Y$ be a finite type morphism of algebraic spaces over S, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Since f is of finite type it is quasi-compact, hence X is quasi-compact. Thus we can find an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 66.6.3. Note that $U \to Y$ is of finite type (this is what it means for f to be of finite type in this case). Hence we can apply Morphisms, Proposition 29.27.2 to see that there exists a dense open $W \subset Y$ such that $U_W \to W$ is flat and of finite presentation and such that $\mathcal{F}|_{U_W}$ is flat over W and of finite presentation as an $\mathcal{O}_{U_W}$-module. According to our definitions this means that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$. $\square$

We cannot improve the result of the lemma above to requiring $X_W \to W$ to be of finite presentation as $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z} \to \mathrm{Spec}(\mathbf{Q})$ gives a counter example. The problem is that the diagonal morphism $\Delta_{X/Y}$ may not be quasi-compact, i.e., f may not be quasi-separated. Clearly, this is also the only problem.

06QT Proposition 67.32.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume

(1) Y is reduced,
(2) f is quasi-separated,
(3) f is of finite type, and
(4) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat and of finite presentation and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. This follows immediately from Proposition 67.32.1 and the fact that "of finite presentation" = "locally of finite presentation" + "quasi-compact" + "quasi-separated". ☐

67.33. Relative dimension

04NH In this section we define the relative dimension of a morphism of algebraic spaces at a point, and some closely related properties.

04NM Definition 67.33.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Let $d, r \in \{0, 1, 2, \ldots, \infty\}$.

(1) We say the dimension of the local ring of the fibre of f at x is d if the equivalent conditions of Lemma 67.22.5 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 35.33.6.
(2) We say the transcendence degree of $x/f(x)$ is r if the equivalent conditions of Lemma 67.22.5 hold for the property $\mathcal{P}_r$ described in Descent, Lemma 35.33.7.
(3) We say f has relative dimension d at x if the equivalent conditions of Lemma 67.22.5 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 35.33.8.

Let us spell out what this means. Namely, choose some diagrams

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

as in Lemma 67.22.5. Then we have

$$\begin{array}{ccc} \text{relative dimension of } f \text{ at } x & = & \dim_u(U_v) \\ \text{dimension of local ring of the fibre of } f \text{ at } x & = & \dim(\mathcal{O}_{U_v,u}) \\ \text{transcendence degree of } x/f(x) & = & \text{trdeg}_{\kappa(v)}(\kappa(u)) \end{array}$$

Note that if $Y = \text{Spec}(k)$ is the spectrum of a field, then the relative dimension of X/Y at x is the same as $\dim_x(X)$, the transcendence degree of $x/f(x)$ is the transcendence degree over k, and the dimension of the local ring of the fibre of f at x is just the dimension of the local ring at x, i.e., the relative notions become absolute notions in that case.

06LR Definition 67.33.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $d \in \{0, 1, 2, \ldots\}$.

(1) We say f has relative dimension $\leq d$ if f has relative dimension $\leq d$ at all $x \in |X|$.
(2) We say f has relative dimension d if f has relative dimension d at all $x \in |X|$.

Having relative dimension equal to d means roughly speaking that all nonempty fibres are equidimensional of dimension d.

06RX Lemma 67.33.3. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and let $y \in |Y|$, $z \in |Z|$ be the images. Assume $X \to Y$ is locally quasi-finite and $Y \to Z$ locally of finite type. Then the transcendence degree of x/z is equal to the transcendence degree of y/z.

Proof. We can choose commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[r] \arrow[d] & w \arrow[d] \\ x \arrow[r] & y \arrow[r] & z \end{tikzcd}$$

where U, V, W are schemes and the vertical arrows are étale. By definition the morphism $U \to V$ is locally quasi-finite which implies that $\kappa(v) \subset \kappa(u)$ is finite, see Morphisms, Lemma 29.20.5. Hence the result is clear. $\square$

0ECY Lemma 67.33.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, Y is Jacobson (Properties of Spaces, Remark 66.7.3), and $x \in |X|$ is a finite type point of X, then the transcendence degree of $x/f(x)$ is 0.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. By Lemma 67.25.5 we can find a finite type point $u \in U$ mapping to x. After shrinking U we may assume $u \in U$ is closed (Morphisms, Lemma 29.16.4). Let $v \in V$ be the image of u. By Morphisms, Lemma 29.16.8 the extension $\kappa(u)/\kappa(v)$ is finite. This finishes the proof. $\square$

0AFH Lemma 67.33.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian algebraic spaces over S which is flat, locally of finite type and of relative dimension d. For every point x in $|X|$ with image y in $|Y|$ we have $\dim_x(X) = \dim_y(Y) + d$.

Proof. By definition of the dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1) and by definition of having relative dimension d, this reduces to the corresponding statement for schemes (Morphisms, Lemma 29.29.6). $\square$

67.34. Morphisms and dimensions of fibres

04NP This section is the analogue of Morphisms, Section 29.28. The formulations in this section are a bit awkward since we do not have local rings of algebraic spaces at points.

04NQ Lemma 67.34.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume f is locally of finite type. Then we have

$$\begin{matrix} \text{relative dimension of } f \text{ at } x \\ = \\ \text{dimension of local ring of the fibre of } f \text{ at } x \\ + \\ \text{transcendence degree of } x/f(x) \end{matrix}$$

where the notation is as in Definition 67.33.1.

Proof. This follows immediately from Morphisms, Lemma 29.28.1 applied to $h : U \to V$ and $u \in U$ as in Lemma 67.22.5. $\square$

04NR Lemma 67.34.2. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and set $y = f(x)$. Assume f and g locally of finite type. Then

(1)
$$\begin{matrix}\text{relative dimension of } g \circ f \text{ at } x \\ \leq \\ \text{relative dimension of } f \text{ at } x \\ + \\ \text{relative dimension of } g \text{ at } y\end{matrix}$$

(2) equality holds in (1) if for some morphism $\mathrm{Spec}(k) \to Z$ from the spectrum of a field in the class of $g(f(x)) = g(y)$ the morphism $X_k \to Y_k$ is flat at x, for example if f is flat at x,

(3)
$$\begin{matrix}\text{transcendence degree of } x/g(f(x)) \\ = \\ \text{transcendence degree of } x/f(x) \\ + \\ \text{transcendence degree of } f(x)/g(f(x))\end{matrix}$$

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with U, V, W schemes and vertical arrows étale and surjective. (See Spaces, Lemma 65.11.6.) Choose $u \in U$ mapping to x. Set v, w equal to the images of u in V, W. Apply Morphisms, Lemma 29.28.2 to the top row and the points u, v, w. Details omitted. $\square$

04NS Lemma 67.34.3. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S. Let $x' \in |X'|$. Set $x = g'(x')$. Assume f locally of finite type. Then

(1)
$$\begin{matrix}\text{relative dimension of } f \text{ at } x \\ = \\ \text{relative dimension of } f' \text{ at } x'\end{matrix}$$

(2) we have
$$\begin{matrix}\text{dimension of local ring of the fibre of } f' \text{ at } x' \\ - \\ \text{dimension of local ring of the fibre of } f \text{ at } x \\ = \\ \text{transcendence degree of } x/f(x) \\ - \\ \text{transcendence degree of } x'/f'(x')\end{matrix}$$
and the common value is ≥ 0,

(3) given x and $y' \in |Y'|$ mapping to the same $y \in |Y|$ there exists a choice of x' such that the integer in (2) is 0.

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Choose a surjective étale morphism $V' \to V \times_Y Y'$ with V' a scheme. Set $U' = V' \times_V U$. Then the induced morphism $U' \to X'$ is also surjective and étale (argument omitted). Choose $u' \in U'$ mapping to x'. At this point parts (1) and (2) follow by applying Morphisms, Lemma 29.28.3 to the diagram of schemes involving U', U, V', V and the point u'. To prove (3) first choose $v \in V$ mapping to y. Then using Properties of Spaces, Lemma 66.4.3 we can choose $v' \in V'$ mapping to y' and v and $u \in U$ mapping to x and v. Finally, according to Morphisms, Lemma 29.28.3 we can choose $u' \in U'$ mapping to v' and u such that the integer is zero. Then taking $x' \in |X'|$ the image of u' works. $\square$

04NT Lemma 67.34.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $n \geq 0$. Assume f is locally of finite type. The set

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

is open in $|X|$.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. By Morphisms, Lemma 29.28.4 the set U_n of points where h has relative dimension $\leq n$ is open in U. By our definition of relative dimension for morphisms of algebraic spaces at points we see that $U_n = a^{-1}(W_n)$. The lemma follows by definition of the topology on $|X|$. $\square$

04NU Lemma 67.34.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S Let $n \geq 0$. Assume f is locally of finite presentation. The open

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

of Lemma 67.34.4 is retrocompact in $|X|$. (See Topology, Definition 5.12.1.)

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. In the proof of Lemma 67.34.4 we have seen that $a^{-1}(W_n) = U_n$ is the corresponding set for the morphism h. By Morphisms, Lemma 29.28.6 we see that U_n is retrocompact in U. The lemma follows by definition of the topology on $|X|$, compare with Properties of Spaces, Lemma 66.5.5 and its proof. $\square$

04NV **Lemma 67.34.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then f is locally quasi-finite if and only if f has relative dimension 0 at each $x \in |X|$.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. The definitions imply that h is locally quasi-finite if and only if f is locally quasi-finite, and that f has relative dimension 0 at all $x \in |X|$ if and only if h has relative dimension 0 at all $u \in U$. Hence the result follows from the result for h which is Morphisms, Lemma 29.29.5. $\square$

04NW **Lemma 67.34.7.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then there exists a canonical open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is locally quasi-finite, and such that the relative dimension of f at any $x \in |X|$, $x \notin |X'|$ is ≥ 1. Formation of X' commutes with arbitrary base change.

Proof. Combine Lemmas 67.34.4, 67.34.6, and 67.34.3. $\square$

06LS **Lemma 67.34.8.** Let S be a scheme. Consider a cartesian diagram

$$\begin{tikzcd} X \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ Y & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is locally of finite type and where k is a field over S. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$X \longrightarrow Y$$

is locally quasi-finite.

Proof. Let $X' \subset X$ be the open subspace over which f is locally quasi-finite found in Lemma 67.34.7. Since the formation of X' commutes with arbitrary base change we see that $z \in X' \times_Y \mathrm{Spec}(k)$. Hence the lemma is clear. $\square$

67.35. The dimension formula

0BAW The analog of the dimension formula (Morphisms, Lemma 29.52.1) is a bit tricky to formulate, because we would have to define integral algebraic spaces (we do this later) as well as universally catenary algebraic spaces. However, the following version is straightforward.

0BAX **Lemma 67.35.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. Let

$x \in |X|$ with image $y \in |Y|$. Then we have

$$\begin{aligned} &\text{the dimension of the local ring of } X \text{ at } x \leq \\ &\text{the dimension of the local ring of } Y \text{ at } y + E - \\ &\text{ the transcendence degree of } x/y \end{aligned}$$

Here E is the maximum of the transcendence degrees of $\xi/f(\xi)$ where $\xi \in |X|$ runs over the points specializing to x at which the local ring of X has dimension 0.

Proof. Choose an affine scheme V, an étale morphism $V \to Y$, and a point $v \in V$ mapping to y. Choose an affine scheme U , an étale morphism $U \to X \times_Y V$ and a point $u \in U$ mapping to v in V and x in X. Unwinding Definition 67.33.1 and Properties of Spaces, Definition 66.10.2 we have to show that

$$\dim(\mathcal{O}_{U,u}) \leq \dim(\mathcal{O}_{V,v}) + E - \text{trdeg}_{\kappa(v)}(\kappa(u))$$

Let $\xi_U \in U$ be a generic point of an irreducible component of U which contains u. Then ξ_U maps to a point $\xi \in |X|$ which is in the list used to define the quantity E and in fact every ξ used in the definition of E occurs in this manner (small detail omitted). In particular, there are only a finite number of these ξ and we can take the maximum (i.e., it really is a maximum and not a supremum). The transcendence degree of ξ over $f(\xi)$ is $\text{trdeg}_{\kappa(\xi_V)}(\kappa(\xi_U))$ where $\xi_V \in V$ is the image of ξ_U. Thus the lemma follows from Morphisms, Lemma 29.52.2. □

0BAY Lemma 67.35.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f is locally of finite type. Then

$$\dim(X) \leq \dim(Y) + E$$

where E is the supremum of the transcendence degrees of $\xi/f(\xi)$ where ξ runs through the points at which the local ring of X has dimension 0.

Proof. Immediate consequence of Lemma 67.35.1 and Properties of Spaces, Lemma 66.10.3. □

67.36. Syntomic morphisms

03Z6 The property "syntomic" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.30.4 and Descent, Lemma 35.23.26. Hence, by Lemma 67.22.1 above, we may define the notion of a syntomic morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03Z7 Definition 67.36.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is syntomic if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P}$ ="syntomic".
(2) Let $x \in |X|$. We say f is syntomic at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is syntomic.

03Z8 Lemma 67.36.2. The composition of syntomic morphisms is syntomic.

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.30.3. □

03Z9 Lemma 67.36.3. The base change of a syntomic morphism is syntomic.

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.30.4. □

03ZA Lemma 67.36.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is syntomic,
(2) for every $x \in |X|$ the morphism f is syntomic at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a syntomic morphism,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is syntomic,
(7) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is syntomic,
(8) there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is syntomic, and
(9) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is syntomic.

Proof. Omitted. □

0DEY Lemma 67.36.5. A syntomic morphism is locally of finite presentation.

Proof. Follows immediately from the case of schemes (Morphisms, Lemma 29.30.6). □

0DEZ Lemma 67.36.6. A syntomic morphism is flat.

Proof. Follows immediately from the case of schemes (Morphisms, Lemma 29.30.7). □

0DF0 Lemma 67.36.7. A syntomic morphism is universally open.

Proof. Combine Lemmas 67.36.5, 67.36.6, and 67.30.6. □

67.37. Smooth morphisms

03ZB The property "smooth" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.34.5 and Descent, Lemma 35.23.27. Hence, by Lemma 67.22.1 above, we may define the notion of a smooth morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03ZC **Definition 67.37.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

1. We say f is *smooth* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$"smooth".
2. Let $x \in |X|$. We say f is *smooth at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is smooth.

03ZD **Lemma 67.37.2.** *The composition of smooth morphisms is smooth.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.34.4. □

03ZE **Lemma 67.37.3.** *The base change of a smooth morphism is smooth.*

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.34.5. □

03ZF **Lemma 67.37.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is smooth,*
2. *for every $x \in |X|$ the morphism f is smooth at x,*
3. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
4. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
5. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a smooth morphism,*
6. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is smooth,*
7. *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is smooth,

8. *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is smooth, and

(9) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is smooth.

Proof. Omitted. □

04AJ **Lemma 67.37.5.** A smooth morphism of algebraic spaces is locally of finite presentation.

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 67.22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 29.34.8 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. □

06MH **Lemma 67.37.6.** A smooth morphism of algebraic spaces is locally of finite type.

Proof. Combine Lemmas 67.37.5 and 67.28.5. □

04TA **Lemma 67.37.7.** A smooth morphism of algebraic spaces is flat.

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 67.22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 29.34.8 h is flat. Hence $X \to Y$ is flat by definition. □

06CP **Lemma 67.37.8.** A smooth morphism of algebraic spaces is syntomic.

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 67.22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 29.34.7 h is syntomic. Hence $X \to Y$ is syntomic by definition. □

0DZI **Lemma 67.37.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There is a maximal open subspace $U \subset X$ such that $f|_U : U \to Y$ is smooth. Moreover, formation of this open commutes with base change by

(1) morphisms which are flat and locally of finite presentation,
(2) flat morphisms provided f is locally of finite presentation.

Proof. The existence of U follows from the fact that the property of being smooth is Zariski (and even étale) local on the source, see Lemma 67.37.4. Moreover, this lemma allows us to translate properties (1) and (2) into the case of morphisms of schemes. The case of schemes is Morphisms, Lemma 29.34.15. Some details omitted. □

0AFI **Lemma 67.37.10.** Let X and Y be locally Noetherian algebraic spaces over a scheme S, and let $f : X \to Y$ be a smooth morphism. For every point $x \in |X|$ with image $y \in |Y|$,

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y)$$

where $\dim_x(X_y)$ is the relative dimension of f at x as in Definition 67.33.1.

Proof. By definition of the dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1), this reduces to the corresponding statement for schemes (Morphisms, Lemma 29.34.21). □

67.38. Unramified morphisms

03ZG The property "unramified" (resp. "G-unramified") of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.35.5 and Descent, Lemma 35.23.28. Hence, by Lemma 67.22.1 above, we may define the notion of an unramified morphism (resp. G-unramified morphism) of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03ZH **Definition 67.38.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *unramified* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ unramified.
(2) Let $x \in |X|$. We say f is *unramified at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is unramified.
(3) We say f is *G-unramified* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ G-unramified.
(4) Let $x \in |X|$. We say f is *G-unramified at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is G-unramified.

Because of the following lemma, from here on we will only develop theory for unramified morphisms, and whenever we want to use a G-unramified morphism we will simply say "an unramified morphism locally of finite presentation".

04G1 **Lemma 67.38.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is G-unramified if and only if f is unramified and locally of finite presentation.

Proof. Consider any diagram as in Lemma 67.22.1. Then all we are saying is that the morphism h is G-unramified if and only if it is unramified and locally of finite presentation. This is clear from Morphisms, Definition 29.35.1. $\square$

03ZI **Lemma 67.38.3.** The composition of unramified morphisms is unramified.

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.35.4. $\square$

03ZJ **Lemma 67.38.4.** The base change of an unramified morphism is unramified.

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.35.5. $\square$

03ZK **Lemma 67.38.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is unramified,
(2) for every $x \in |X|$ the morphism f is unramified at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an unramified morphism,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is unramified,

(7) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is unramified,

(8) there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is unramified, and

(9) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is unramified.

Proof. Omitted. $\square$

05VZ Lemma 67.38.6. An unramified morphism of algebraic spaces is locally of finite type.

Proof. Via a diagram as in Lemma 67.22.1 this translates into Morphisms, Lemma 29.35.9. $\square$

05W0 Lemma 67.38.7. If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.

Proof. Via a diagram as in Lemma 67.22.1 this translates into Morphisms, Lemma 29.35.10. $\square$

06CQ Lemma 67.38.8. An immersion of algebraic spaces is unramified.

Proof. Let $i : X \to Y$ be an immersion of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is an immersion of schemes, hence unramified (see Morphisms, Lemmas 29.35.7 and 29.35.8). Thus by definition i is unramified. $\square$

05W1 Lemma 67.38.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) If f is unramified, then the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion.
(2) If f is locally of finite type and $\Delta_{X/Y}$ is an open immersion, then f is unramified.

Proof. We know in any case that $\Delta_{X/Y}$ is a representable monomorphism, see Lemma 67.4.1. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the

commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Delta_{U/V}"'] \arrow[d] & U \times_V U \arrow[r] \arrow[d] & V \arrow[d, "\Delta_{V/Y}"] \\ X \arrow[r, "\Delta_{X/Y}"] & X \times_Y X \arrow[r] & V \times_Y V \end{tikzcd}$$

with cartesian right square. The left vertical arrow is surjective étale. The right vertical arrow is étale as a morphism between schemes étale over Y, see Properties of Spaces, Lemma 66.16.6. Hence the middle vertical arrow is étale too (but it need not be surjective).

Assume f is unramified. Then $U \to V$ is unramified, hence $\Delta_{U/V}$ is an open immersion by Morphisms, Lemma 29.35.13. Looking at the left square of the diagram above we conclude that $\Delta_{X/Y}$ is an étale morphism, see Properties of Spaces, Lemma 66.16.3. Hence $\Delta_{X/Y}$ is a representable étale monomorphism, which implies that it is an open immersion by Étale Morphisms, Theorem 41.14.1. (See also Spaces, Lemma 65.5.8 for the translation from schemes language into the language of functors.)

Assume that f is locally of finite type and that $\Delta_{X/Y}$ is an open immersion. This implies that $U \to V$ is locally of finite type too (by definition of a morphism of algebraic spaces which is locally of finite type). Looking at the displayed diagram above we conclude that $\Delta_{U/V}$ is étale as a morphism between schemes étale over $X \times_Y X$, see Properties of Spaces, Lemma 66.16.6. But since $\Delta_{U/V}$ is the diagonal of a morphism between schemes we see that it is in any case an immersion, see Schemes, Lemma 26.21.2. Hence it is an open immersion, and we conclude that $U \to V$ is unramified by Morphisms, Lemma 29.35.13. This in turn means that f is unramified by definition. □

05W2 Lemma 67.38.10. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces over S. Assume that $X \to Z$ is locally of finite type. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is unramified. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of Morphisms, Lemma 29.35.15 and in fact we will deduce the lemma from it. By Definition 67.38.1 the set $\{x \in |X| : f \text{ is unramified at } x\}$ is open in X. Hence we only need to prove the final statement. By Lemma 67.23.6 the morphism $X \to Y$ is locally of finite type. By Lemma 67.23.3 the morphism $X' \to Y'$ is locally of finite type.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 66.16.7). This combined with Lemma 67.38.5 implies that $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism $U \to V$ is unramified. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism $U' \to V'$ is unramified. Now, by construction the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is cartesian (in the category of schemes). Hence the aforementioned Morphisms, Lemma 29.35.15 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

06G6 Lemma 67.38.11. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is unramified, then $X \to Y$ is unramified.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 65.11.6.) Apply Morphisms, Lemma 29.35.16 to the top row. □

67.39. Étale morphisms

03XS The notion of an étale morphism of algebraic spaces was defined in Properties of Spaces, Definition 66.16.2. Here is what it means for a morphism to be étale at a point.

04RH Definition 67.39.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. We say f is étale at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is étale.

03XT Lemma 67.39.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is étale,
(2) for every $x \in |X|$ the morphism f is étale at x,
(3) for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,
(4) for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,
(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an étale morphism,
(6) there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is étale,

(7) for every commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are schemes and the vertical arrows are étale the top horizontal arrow is étale,

(8) there exists a commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are schemes, the vertical arrows are étale, and $U \to X$ surjective such that the top horizontal arrow is étale, and

(9) there exist Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is étale.

Proof. Combine Properties of Spaces, Lemmas 66.16.3, 66.16.5 and 66.16.4. Some details omitted. □

0465 Lemma 67.39.3. The composition of two étale morphisms of algebraic spaces is étale.

Proof. This is a copy of Properties of Spaces, Lemma 66.16.4. □

0466 Lemma 67.39.4. The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.

Proof. This is a copy of Properties of Spaces, Lemma 66.16.5. □

03XU Lemma 67.39.5. An étale morphism of algebraic spaces is locally quasi-finite.

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces, see Properties of Spaces, Definition 66.16.2. By Properties of Spaces, Lemma 66.16.3 we see this means there exists a diagram as in Lemma 67.22.1 with h étale and surjective vertical arrow a. By Morphisms, Lemma 29.36.6 h is locally quasi-finite. Hence $X \to Y$ is locally quasi-finite by definition. □

04XX Lemma 67.39.6. An étale morphism of algebraic spaces is smooth.

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses the fact that an étale morphism of schemes is smooth (by definition of an étale morphism of schemes). □

0467 Lemma 67.39.7. An étale morphism of algebraic spaces is flat.

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses Morphisms, Lemma 29.36.12. □

0468 Lemma 67.39.8. An étale morphism of algebraic spaces is locally of finite presentation.

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses Morphisms, Lemma 29.36.11. □

06LT Lemma 67.39.9. An étale morphism of algebraic spaces is locally of finite type.

Proof. An étale morphism is locally of finite presentation and a morphism locally of finite presentation is locally of finite type, see Lemmas 67.39.8 and 67.28.5. □

06CR Lemma 67.39.10. An étale morphism of algebraic spaces is unramified.

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses Morphisms, Lemma 29.36.5. □

05W3 Lemma 67.39.11. Let S be a scheme. Let X, Y be algebraic spaces étale over an algebraic space Z. Any morphism $X \to Y$ over Z is étale.

Proof. This is a copy of Properties of Spaces, Lemma 66.16.6. □

06LU Lemma 67.39.12. A locally finitely presented, flat, unramified morphism of algebraic spaces is étale.

Proof. Let $X \to Y$ be a locally finitely presented, flat, unramified morphism of algebraic spaces. By Properties of Spaces, Lemma 66.16.3 we see this means there exists a diagram as in Lemma 67.22.1 with h locally finitely presented, flat, unramified and surjective vertical arrow a. By Morphisms, Lemma 29.36.16 h is étale. Hence $X \to Y$ is étale by definition. □

67.40. Proper morphisms

03ZL The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic spaces.

03ZM Definition 67.40.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is proper if f is separated, finite type, and universally closed.

083R Lemma 67.40.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is proper,
(2) for every scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,
(3) for every affine scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is proper, and
(5) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is proper.

Proof. Combine Lemmas 67.4.12, 67.23.4, 67.8.8, and 67.9.5. □

04WP Lemma 67.40.3. A base change of a proper morphism is proper.

Proof. See Lemmas 67.4.4, 67.23.3, and 67.9.3. □

04XY Lemma 67.40.4. A composition of proper morphisms is proper.

Proof. See Lemmas 67.4.8, 67.23.2, and 67.9.4. □

04XZ Lemma 67.40.5. A closed immersion of algebraic spaces is a proper morphism of algebraic spaces.

Proof. As a closed immersion is by definition representable this follows from Spaces, Lemma 65.5.8 and the corresponding result for morphisms of schemes, see Morphisms, Lemma 29.41.6. □

04NX Lemma 67.40.6. Let S be a scheme. Consider a commutative diagram of algebraic spaces

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

over S.

(1) If $X \to B$ is universally closed and $Y \to B$ is separated, then the morphism $X \to Y$ is universally closed. In particular, the image of $|X|$ in $|Y|$ is closed.
(2) If $X \to B$ is proper and $Y \to B$ is separated, then the morphism $X \to Y$ is proper.

Proof. Assume $X \to B$ is universally closed and $Y \to B$ is separated. We factor the morphism as $X \to X \times_B Y \to Y$. The first morphism is a closed immersion, see Lemma 67.4.6 hence universally closed. The projection $X \times_B Y \to Y$ is the base change of a universally closed morphism and hence universally closed, see Lemma 67.9.3. Thus $X \to Y$ is universally closed as the composition of universally closed morphisms, see Lemma 67.9.4. This proves (1). To deduce (2) combine (1) with Lemmas 67.4.10, 67.8.9, and 67.23.6. □

08AJ Lemma 67.40.7. Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces over B. If X is universally closed over B and f is surjective then Y is universally closed over B. In particular, if also Y is separated and of finite type over B, then Y is proper over B.

Proof. Assume X is universally closed and f surjective. Denote $p : X \to B$, $q : Y \to B$ the structure morphisms. Let $B' \to B$ be a morphism of algebraic spaces over S. The base change $f' : X_{B'} \to Y_{B'}$ is surjective (Lemma 67.5.5), and the base change $p' : X_{B'} \to B'$ is closed. If $T \subset Y_{B'}$ is closed, then $(f')^{-1}(T) \subset X_{B'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

0AGD Lemma 67.40.8. Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[rr, "h"'] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphism of algebraic spaces over S. Assume

(1) $X \to B$ is a proper morphism,
(2) $Y \to B$ is separated and locally of finite type,

Then the scheme theoretic image $Z \subset Y$ of h is proper over B and $X \to Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 67.16. Observe that h is quasi-compact (Lemma 67.8.10) hence $|h|(|X|) \subset |Z|$ is dense (Lemma

67.16.3). On the other hand $|h|(|X|)$ is closed in $|Y|$ (Lemma 67.40.6) hence $X \to Z$ is surjective. Thus $Z \to B$ is a proper (Lemma 67.40.7). $\square$

04Y0 **Lemma 67.40.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is separated,
(2) $\Delta_{X/Y} : X \to X \times_Y X$ is universally closed, and
(3) $\Delta_{X/Y} : X \to X \times_Y X$ is proper.

Proof. The implication (1) $\Rightarrow$ (3) follows from Lemma 67.40.5. We will use Spaces, Lemma 65.5.8 without further mention in the rest of the proof. Recall that $\Delta_{X/Y}$ is a representable monomorphism which is locally of finite type, see Lemma 67.4.1. Since proper $\Rightarrow$ universally closed for morphisms of schemes we conclude that (3) implies (2). If $\Delta_{X/Y}$ is universally closed then Étale Morphisms, Lemma 41.7.2 implies that it is a closed immersion. Thus (2) $\Rightarrow$ (1) and we win. $\square$

67.41. Valuative criteria

03IW The section introduces the basics on valuative criteria for morphisms of algebraic spaces. Here is a list of references to further results

(1) the valuative criterion for universal closedness can be found in Section 67.42,
(2) the valuative criterion of separatedness can be found in Section 67.43,
(3) the valuative criterion for properness can be found in Section 67.44,
(4) additional converse statements can be found in Decent Spaces, Section 68.16 and Decent Spaces, Lemma 68.17.11, and
(5) in the Noetherian case it is enough to check the criterion for discrete valuation rings as is shown in Cohomology of Spaces, Section 69.19 and Limits of Spaces, Section 70.21, and
(6) refined versions of the valuative criteria in the Noetherian case can be found in Limits of Spaces, Section 70.22.

We first formally state the definition and then we discuss how this differs from the case of morphisms of schemes.

03IX **Definition 67.41.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f satisfies the *uniqueness part of the valuative criterion* if given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists at most one dotted arrow (without requiring existence). We say f satisfies the *existence part of the valuative criterion* if given any solid diagram as above there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to$

X such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

We say f satisfies the valuative criterion if f satisfies both the existence and uniqueness part.

The formulation of the existence part of the valuative criterion is slightly different for morphisms of algebraic spaces, since it may be necessary to extend the fraction field of the valuation ring. In practice this difference almost never plays a role.

(1) Checking the uniqueness part of the valuative criterion never involves any fraction field extensions, hence this is exactly the same as in the case of schemes.
(2) It is necessary to allow for field extensions in general, see Example 67.41.6.
(3) For morphisms of algebraic spaces it always suffices to take a finite separable extensions K'/K in the existence part of the valuative criterion, see Lemma 67.41.3.
(4) If $f : X \to Y$ is a separated morphism of algebraic spaces, then we can always take $K = K'$ when we check the existence part of the valuative criterion, see Lemma 67.41.5.
(5) For a quasi-compact and quasi-separated morphism $f : X \to Y$, we get an equivalence between "f is separated and universally closed" and "f satisfies the usual valuative criterion", see Lemma 67.43.3. The valuative criterion for properness is the usual one, see Lemma 67.44.1.

As a first step in the theory, we show that the criterion is identical to the criterion as formulated for morphisms of schemes in case the morphism of algebraic spaces is representable.

03K8 Lemma 67.41.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable. The following are equivalent

(1) f satisfies the existence part of the valuative criterion as in Definition 67.41.1,
(2) given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 26.20.3.

Proof. It suffices to show that given a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[rru, "\varphi"] & \mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

as in Definition 67.41.1, then we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram too. Set $X_A = \mathrm{Spec}(A) \times_Y Y$. As f is representable we see that X_A is a scheme. The morphism φ gives a morphism $\varphi' : \mathrm{Spec}(A') \to X_A$. Let $x \in X_A$ be the image of the closed point of $\varphi' : \mathrm{Spec}(A') \to X_A$. Then we have the following commutative diagram of rings

$$\begin{tikzcd}
K' & K \arrow[l] & \mathcal{O}_{X_A,x} \arrow[l] \arrow[lld] \\
A' \arrow[u] & A \arrow[l] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

Since A is a valuation ring, and since A' dominates A, we see that $K \cap A' = A$. Hence the ring map $\mathcal{O}_{X_A,x} \to K$ has image contained in A. Whence a morphism $\mathrm{Spec}(A) \to X_A$ (see Schemes, Section 26.13) as desired. $\square$

03KH **Lemma 67.41.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f satisfies the existence part of the valuative criterion as in Definition 67.41.1,
(2) f satisfies the existence part of the valuative criterion as in Definition 67.41.1 modified by requiring the extension K'/K to be finite separable.

Proof. We have to show that (1) implies (2). Suppose given a diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

as in Definition 67.41.1 with $K \subset K'$ arbitrary. Choose a scheme U and a surjective étale morphism $U \to X$. Then

$$\mathrm{Spec}(A') \times_X U \longrightarrow \mathrm{Spec}(A')$$

is surjective étale. Let p be a point of $\mathrm{Spec}(A') \times_X U$ mapping to the closed point of $\mathrm{Spec}(A')$. Let $p' \rightsquigarrow p$ be a generalization of p mapping to the generic point of $\mathrm{Spec}(A')$. Such a generalization exists because generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 29.25.9. Then p' corresponds to a point of the scheme $\mathrm{Spec}(K') \times_X U$. Note that

$$\mathrm{Spec}(K') \times_X U = \mathrm{Spec}(K') \times_{\mathrm{Spec}(K)} (\mathrm{Spec}(K) \times_X U)$$

Hence p' maps to a point $q' \in \mathrm{Spec}(K) \times_X U$ whose residue field is a finite separable extension of K. Finally, $p' \rightsquigarrow p$ maps to a specialization $u' \rightsquigarrow u$ on the scheme U.

With all this notation we get the following diagram of rings

$$\begin{tikzcd}
\kappa(p') & & \kappa(q') \arrow[ll] & \kappa(u') \arrow[l] \\
& \mathcal{O}_{\mathrm{Spec}(A') \times_X U, p} \arrow[ul] & & \mathcal{O}_{U,u} \arrow[ll] \arrow[u] \\
K' \arrow[uu] & A' \arrow[l] \arrow[u] & A \arrow[l] \arrow[uu]
\end{tikzcd}$$

This means that the ring $B \subset \kappa(q')$ generated by the images of A and $\mathcal{O}_{U,u}$ maps to a subring of $\kappa(p')$ contained in the image B' of $\mathcal{O}_{\mathrm{Spec}(A') \times_X U, p} \to \kappa(p')$. Note that B' is a local ring. Let $\mathfrak{m} \subset B$ be the maximal ideal. By construction $A \cap \mathfrak{m}$, (resp. $\mathcal{O}_{U,u} \cap \mathfrak{m}$, resp. $A' \cap \mathfrak{m}$) is the maximal ideal of A (resp. $\mathcal{O}_{U,u}$, resp. A'). Set $\mathfrak{q} = B \cap \mathfrak{m}$. This is a prime ideal such that $A \cap \mathfrak{q}$ is the maximal ideal of A. Hence $B_\mathfrak{q} \subset \kappa(q')$ is a local ring dominating A. By Algebra, Lemma 10.50.2 we can find a valuation ring $A_1 \subset \kappa(q')$ with field of fractions $\kappa(q')$ dominating $B_\mathfrak{q}$. The (local) ring map $\mathcal{O}_{U,u} \to A_1$ gives a morphism $\mathrm{Spec}(A_1) \to U \to X$ such that the diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(q')) \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A_1) \arrow[r] \arrow[urr] & \mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

is commutative. Since the fraction field of A_1 is $\kappa(q')$ and since $\kappa(q')/K$ is finite separable by construction the lemma is proved. $\square$

0ARH **Lemma 67.41.4.** Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Suppose given a diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[urr] & \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

as in Definition 67.41.1 with $K \subset K'$ arbitrary. Then the dotted arrow exists making the diagram commute.

Proof. We have to show that we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram.

Consider the base change $X_A = \mathrm{Spec}(A) \times_Y X$ of X. Then $X_A \to \mathrm{Spec}(A)$ is a separated morphism of algebraic spaces (Lemma 67.4.4). Base changing all the morphisms of the diagram above we obtain

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X_A \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[urr] & \mathrm{Spec}(A) \arrow[r, equal] & \mathrm{Spec}(A)
\end{tikzcd}$$

Thus we may replace X by X_A, assume that $Y = \mathrm{Spec}(A)$ and that we have a diagram as above. We may and do replace X by a quasi-compact open subspace containing the image of $|\mathrm{Spec}(A')| \to |X|$.

The morphism $\mathrm{Spec}(A') \to X$ is quasi-compact by Lemma 67.8.9. Let $Z \subset X$ be the scheme theoretic image of $\mathrm{Spec}(A') \to X$. Then Z is a reduced (Lemma 67.16.4), quasi-compact (as a closed subspace of X), separated (as a closed subspace of X) algebraic space over A. Consider the base change

$$\mathrm{Spec}(K') = \mathrm{Spec}(A') \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) \to X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) = X_K$$

of the morphism $\mathrm{Spec}(A') \to X$ by the flat morphism of schemes $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$. By Lemma 67.30.12 we see that the scheme theoretic image of this morphism is the base change Z_K of Z. On the other hand, by assumption (i.e., the commutative diagram above) this morphism factors through a morphism $\mathrm{Spec}(K) \to Z_K$ which is a section to the structure morphism $Z_K \to \mathrm{Spec}(K)$. As Z_K is separated, this section is a closed immersion (Lemma 67.4.7). We conclude that $Z_K = \mathrm{Spec}(K)$.

Let $V \to Z$ be a surjective étale morphism with V an affine scheme (Properties of Spaces, Lemma 66.6.3). Say $V = \mathrm{Spec}(B)$. Then $V \times_Z \mathrm{Spec}(A') = \mathrm{Spec}(C)$ is affine as Z is separated. Note that $B \to C$ is injective as V is the scheme theoretic image of $V \times_Z \mathrm{Spec}(A') \to V$ by Lemma 67.16.3. On the other hand, $A' \to C$ is étale as corresponds to the base change of $V \to Z$. Since A' is a torsion free A-module, the flatness of $A' \to C$ implies C is a torsion free A-module, hence B is a torsion free A-module. Note that being torsion free as an A-module is equivalent to being flat (More on Algebra, Lemma 15.22.10). Next, we write

$$V \times_Z V = \mathrm{Spec}(B')$$

Note that the two ring maps $B \to B'$ are étale as $V \to Z$ is étale. The canonical surjective map $B \otimes_A B \to B'$ becomes an isomorphism after tensoring with K over A because $Z_K = \mathrm{Spec}(K)$. However, $B \otimes_A B$ is torsion free as an A-module by our remarks above. Thus $B' = B \otimes_A B$. It follows that the base change of the ring map $A \to B$ by the faithfully flat ring map $A \to B$ is étale (note that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective as $X \to \mathrm{Spec}(A)$ is surjective). Hence $A \to B$ is étale (Descent, Lemma 35.23.29), in other words, $V \to X$ is étale. Since we have $V \times_Z V = V \times_{\mathrm{Spec}(A)} V$ we conclude that $Z = \mathrm{Spec}(A)$ as algebraic spaces (for example by Spaces, Lemma 65.9.1) and the proof is complete. □

0A3W Lemma 67.41.5. Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. The following are equivalent

(1) f satisfies the existence part of the valuative criterion as in Definition 67.41.1,
(2) given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 26.20.3.

Proof. We have to show that (1) implies (2). Suppose given a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

as in part (2). By (1) there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[urr] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

as in Definition 67.41.1 with $K \subset K'$ arbitrary. By Lemma 67.41.4 we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram, i.e., (2) holds. $\square$

03KI Example 67.41.6. Consider the algebraic space X constructed in Spaces, Example 65.14.2. Recall that it is Galois twist of the affine line with zero doubled. The Galois twist is with respect to a degree two Galois extension k'/k of fields. As such it comes with a morphism

$$\pi : X \longrightarrow S = \mathbf{A}^1_k$$

which is quasi-compact. We claim that π is universally closed. Namely, after base change by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ the morphism π is identified with the morphism

$$\text{affine line with zero doubled} \longrightarrow \text{affine line}$$

which is universally closed (some details omitted). Since the morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is universally closed and surjective, a diagram chase shows that π is universally closed. On the other hand, consider the diagram

$$\begin{tikzcd} \mathrm{Spec}(k((x))) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \mathrm{Spec}(k[[x]]) \arrow[r] \arrow[ur, dotted] & \mathbf{A}^1_k \end{tikzcd}$$

Since the unique point of X above $0 \in \mathbf{A}^1_k$ corresponds to a monomorphism $\mathrm{Spec}(k') \to X$ it is clear there cannot exist a dotted arrow! This shows that a finite separable field extension is needed in general.

03IY Lemma 67.41.7. The base change of a morphism of algebraic spaces which satisfies the existence part of (resp. uniqueness part of) the valuative criterion by any morphism of algebraic spaces satisfies the existence part of (resp. uniqueness part of) the valuative criterion.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over the scheme S. Let $Z \to Y$ be any morphism of algebraic spaces over S. Consider a solid commutative diagram of the following shape

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & Z \arrow[r] & Y \end{tikzcd}$$

Then the set of north-west dotted arrows making the diagram commute is in 1-1 correspondence with the set of west-north-west dotted arrows making the diagram commute. This proves the lemma in the case of "uniqueness". For the existence part, assume f satisfies the existence part of the valuative criterion. If we are given a solid commutative diagram as above, then by assumption there exists an extension K'/K of fields and a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ fitting into the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & Z \times_Y X \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[rrru] & \mathrm{Spec}(A) \arrow[r] & Z \arrow[r] & Y
\end{tikzcd}$$

And by the remarks above the skew arrow corresponds to an arrow $\mathrm{Spec}(A') \to Z \times_Y X$ as desired. □

03IZ **Lemma 67.41.8.** The composition of two morphisms of algebraic spaces which satisfy the (existence part of, resp. uniqueness part of) the valuative criterion satisfies the (existence part of, resp. uniqueness part of) the valuative criterion.

Proof. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Consider a solid commutative diagram of the following shape

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[dd] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[uur, dashed] \arrow[ur, dashed] & Z
\end{tikzcd}$$

If we have the uniqueness part for g, then there exists at most one north-west dotted arrow making the diagram commute. If we also have the uniqueness part for f, then we have at most one north-north-west dotted arrow making the diagram commute. The proof in the existence case comes from contemplating the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(K'') \arrow[r] \arrow[dd] & \mathrm{Spec}(K') \arrow[r] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d, "f"] \\
 & & & Y \arrow[d, "g"] \\
\mathrm{Spec}(A'') \arrow[r] \arrow[rrruu] & \mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Z
\end{tikzcd}$$

Namely, the existence part for g gives us the extension K', the valuation ring A' and the arrow $\mathrm{Spec}(A') \to Y$, whereupon the existence part for f gives us the extension K'', the valuation ring A'' and the arrow $\mathrm{Spec}(A'') \to X$. □

67.42. Valuative criterion for universal closedness

03K9 The existence part of the valuative criterion implies universal closedness for quasi-compact morphisms, see Lemma 67.42.1. In the case of schemes, this is an "if and only if" statement, but for morphisms of algebraic spaces this is wrong. Example 67.9.6 shows that $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is universally closed, but it is easy to see that the existence part of the valuative criterion fails. We revisit this topic in Decent Spaces, Section 68.16 and show the converse holds if the source of the morphism is a decent space (see also Decent Spaces, Lemma 68.17.11 for a relative version).

03KA Lemma 67.42.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is quasi-compact, and
(2) f satisfies the existence part of the valuative criterion.

Then f is universally closed.

Proof. By Lemmas 67.8.4 and 67.41.7 properties (1) and (2) are preserved under any base change. By Lemma 67.9.5 we only have to show that $|T \times_Y X| \to |T|$ is closed, whenever T is an affine scheme over S mapping into Y. Hence it suffices to prove: If Y is an affine scheme, $f : X \to Y$ is quasi-compact and satisfies the existence part of the valuative criterion, then $f : |X| \to |Y|$ is closed. In this situation X is a quasi-compact algebraic space. By Properties of Spaces, Lemma 66.6.3 there exists an affine scheme U and a surjective étale morphism $\varphi : U \to X$. Let $T \subset |X|$ closed. The inverse image $\varphi^{-1}(T) \subset U$ is closed, and hence is the set of points of an affine closed subscheme $Z \subset U$. Thus, by Algebra, Lemma 10.41.5 we see that $f(T) = f(\varphi(|Z|)) \subset |Y|$ is closed if it is closed under specialization.

Let $y' \rightsquigarrow y$ be a specialization in Y with $y' \in f(T)$. Choose a point $x' \in T \subset |X|$ mapping to y' under f. We may represent x' by a morphism $\mathrm{Spec}(K) \to X$ for some field K. Thus we have the following diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x'"'] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y, \end{tikzcd}$$

see Schemes, Section 26.13 for the existence of the left vertical map. Choose a valuation ring $A \subset K$ dominating the image of the ring map $\mathcal{O}_{Y,y} \to K$ (this is possible since the image is a local ring and not a field as $y' \neq y$, see Algebra, Lemma 10.50.2). By assumption there exists a field extension K'/K and a valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ fitting into the commutative diagram. Since A' dominates A, and A dominates $\mathcal{O}_{Y,y}$ we see that the closed point of $\mathrm{Spec}(A')$ maps to a point $x \in X$ with $f(x) = y$ which is a specialization of x'. Hence $x \in T$ as T is closed, and hence $y \in f(T)$ as desired. $\square$

The following lemma will be generalized in Decent Spaces, Lemma 68.17.11.

0A3X Lemma 67.42.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) If f is quasi-separated and universally closed, then f satisfies the existence part of the valuative criterion.
(2) If f is quasi-compact and quasi-separated, then f is universally closed if and only if the existence part of the valuative criterion holds.

Proof. If (1) is true then combined with Lemma 67.42.1 we obtain (2). Assume f is quasi-separated and universally closed. Assume given a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

as in Definition 67.41.1. A formal argument shows that the existence of the desired diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

follows from existence in the case of the morphism $X_A \to \mathrm{Spec}(A)$. Since being quasi-separated and universally closed are preserved by base change, the lemma follows from the result in the next paragraph.

Consider a solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, equal] \arrow[ru, dotted] & \mathrm{Spec}(A) \end{tikzcd}$$

where A is a valuation ring with field of fractions K. By Lemma 67.8.9 and the fact that f is quasi-separated we have that the morphism x is quasi-compact. Since f is universally closed, we have in particular that $|f|(\overline{\{x\}})$ is closed in $\mathrm{Spec}(A)$. Since this image contains the generic point of $\mathrm{Spec}(A)$ there exists a point $x' \in |X|$ in the closure of x mapping to the closed point of $\mathrm{Spec}(A)$. By Lemma 67.16.5 we can find a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] & X \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A')$ maps to $x' \in |X|$. It follows that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ maps the closed point to the closed point, i.e., A' dominates A and this finishes the proof. □

0A3Y **Lemma 67.42.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and separated. Then the following are equivalent

(1) f is universally closed,
(2) the existence part of the valuative criterion holds as in Definition 67.41.1, and

(3) given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 26.20.3.

Proof. Since f is separated parts (2) and (3) are equivalent by Lemma 67.41.5. The equivalence of (3) and (1) follows from Lemma 67.42.2. □

089F Lemma 67.42.4. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $\mathrm{Spec}(A) \to Y$ be a morphism where A is a valuation ring. If the closed point of $\mathrm{Spec}(A)$ maps to a point of $|Y|$ in the image of $|X| \to |Y|$, then there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(A') \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where $A \to A'$ is an extension of valuation rings (More on Algebra, Definition 15.123.1).

Proof. The base change $X_A \to \mathrm{Spec}(A)$ is flat (Lemma 67.30.4) and the closed point of $\mathrm{Spec}(A)$ is in the image of $|X_A| \to |\mathrm{Spec}(A)|$ (Properties of Spaces, Lemma 66.4.3). Thus we may assume $Y = \mathrm{Spec}(A)$. Let $U \to X$ be a surjective étale morphism where U is a scheme. Let $u \in U$ map to the closed point of $\mathrm{Spec}(A)$. Consider the flat local ring map $A \to B = \mathcal{O}_{U,u}$. By Algebra, Lemma 10.39.16 there exists a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{q}$ lies over $(0) \subset A$. By Algebra, Lemma 10.50.2 we can find a valuation ring $A' \subset \kappa(\mathfrak{q})$ dominating $B/\mathfrak{q}$. The induced morphism $\mathrm{Spec}(A') \to U \to X$ is a solution to the problem posed by the lemma. □

089G Lemma 67.42.5. Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. If

(1) f and h are quasi-compact,
(2) $|h|(|U|)$ is dense in $|X|$, and

given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K

(3) there exists at most one dotted arrow making the diagram commute, and

(4) there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & U \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rrru] & \mathrm{Spec}(A) \arrow[rr] & & Y \end{tikzcd}$$

then f is universally closed. If moreover

(5) f is quasi-separated

then f is separated and universally closed.

Proof. Assume (1), (2), (3), and (4). We will verify the existence part of the valuative criterion for f which will imply f is universally closed by Lemma 67.42.1. To do this, consider a commutative diagram

089H (67.42.5.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Properties of Spaces, Lemma 66.12.4. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 67.16.4.

Reduction to the case Y affine. Choose an étale morphism $\mathrm{Spec}(R) \to Y$ such that the closed point of $\mathrm{Spec}(A)$ maps to an element of $\mathrm{Im}(|\mathrm{Spec}(R)| \to |Y|)$. By Lemma 67.42.4 we can find a local ring map $A \to A'$ of valuation rings and a morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(R)$ fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(A') \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

Since in Definition 67.41.1 we allow for extensions of valuation rings it is clear that we may replace A by A', Y by $\mathrm{Spec}(R)$, X by $X \times_Y \mathrm{Spec}(R)$ and U by $U \times_Y \mathrm{Spec}(R)$.

From now on we assume that $Y = \mathrm{Spec}(R)$ is an affine scheme. Let $\mathrm{Spec}(B) \to X$ be an étale morphism from an affine scheme such that the morphism $\mathrm{Spec}(K) \to X$ is in the image of $|\mathrm{Spec}(B)| \to |X|$. Since we may replace K by an extension $K' \supset K$ and A by a valuation ring $A' \subset K'$ dominating A (which exists by Algebra, Lemma 10.50.2), we may assume the morphism $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(B)$ (by definition of $|X|$). In other words, we may think of K as a B-algebra. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat as a composition $\mathrm{Spec}(P) \to \mathrm{Spec}(B) \to X$. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma

67.30.12. By Lemma 67.16.5 we can find a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \times_X \mathrm{Spec}(P) \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] & \mathrm{Spec}(P) \end{tikzcd}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 10.50.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 10.50.10. As C is an R-algebra with fraction field K', we obtain a solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K'_1) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(C_1) \arrow[r, dashed] \arrow[urrr, dashed] & \mathrm{Spec}(C) \arrow[rr] & & Y \end{tikzcd}$$

as in the statement of the lemma. Thus assumption (4) produces $C \to C_1$ and the dotted arrows making the diagram commute. Let $A'_1 = (C_1)_\mathfrak{p}$ be the localization of C_1 at a prime $\mathfrak{p} \subset C_1$ lying over $\mathfrak{m}_{A'} \subset C$. Since $C \to C_1$ is flat by More on Algebra, Lemma 15.22.10 such a prime $\mathfrak{p}$ exists by Algebra, Lemmas 10.39.17 and 10.39.16. Note that A' is the localization of C at $\mathfrak{m}_{A'}$ and that A'_1 is a valuation ring (Algebra, Lemma 10.50.9). In other words, $A' \to A'_1$ is a local ring map of valuation rings. Assumption (3) implies

$$\begin{tikzcd} \mathrm{Spec}(A'_1) \arrow[r] \arrow[d] & \mathrm{Spec}(C_1) \arrow[r] & X \\ \mathrm{Spec}(A') \arrow[r] & \mathrm{Spec}(P) \arrow[r] & \mathrm{Spec}(B) \arrow[u] \end{tikzcd}$$

commutes. Hence the restriction of the morphism $\mathrm{Spec}(C_1) \to X$ to $\mathrm{Spec}(C_1/\mathfrak{p})$ restricts to the composition

$$\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(A'/\mathfrak{m}_{A'}) = \mathrm{Spec}(K'') \to \mathrm{Spec}(K) \to X$$

on the generic point of $\mathrm{Spec}(C_1/\mathfrak{p})$. Moreover, $C_1/\mathfrak{p}$ is a valuation ring (Algebra, Lemma 10.50.9) dominating A'' which dominates A. Thus the morphism $\mathrm{Spec}(C_1/\mathfrak{p}) \to X$ witnesses the existence part of the valuative criterion for the diagram (67.42.5.1) as desired.

Next, suppose that (5) is satisfied as well, i.e., the morphism $\Delta : X \to X \times_S X$ is quasi-compact. In this case assumptions (1) – (4) hold for h and Δ. Hence the first part of the proof shows that Δ is universally closed. By Lemma 67.40.9 we conclude that f is separated. $\square$

67.43. Valuative criterion of separatedness

03KT First we prove a converse and then we state the criterion.

03KU **Lemma 67.43.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f satisfies the uniqueness part of the valuative criterion.

Proof. Let a diagram as in Definition 67.41.1 be given. Suppose there are two distinct morphisms $a, b : \mathrm{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \mathrm{Spec}(A)$ be the equalizer of a and b. Then $Z = \mathrm{Spec}(A) \times_{(a,b), X \times_Y X, \Delta} X$. If f is separated, then Δ is a closed immersion, and this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. $\square$

03KV **Lemma 67.43.2** (Valuative criterion separatedness)**.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) the morphism f is quasi-separated, and
(2) the morphism f satisfies the uniqueness part of the valuative criterion.

Then f is separated.

Proof. Assumption (1) means $\Delta_{X/Y}$ is quasi-compact. We claim the morphism $\Delta_{X/Y} : X \to X \times_Y X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & X \times_Y X \end{tikzcd}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over Y. By assumption (2) we see that $a = b$. Hence using a as the dotted arrow works. Hence Lemma 67.42.1 applies, and we see that $\Delta_{X/Y}$ is universally closed. Since always $\Delta_{X/Y}$ is locally of finite type and separated, we conclude from More on Morphisms, Lemma 37.44.1 that $\Delta_{X/Y}$ is a finite morphism (also, use the general principle of Spaces, Lemma 65.5.8). At this point $\Delta_{X/Y}$ is a representable, finite monomorphism, hence a closed immersion by Morphisms, Lemma 29.44.15. $\square$

0A3Z **Lemma 67.43.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and quasi-separated. Then the following are equivalent

(1) f is separated and universally closed,
(2) the valuative criterion holds as in Definition 67.41.1,
(3) given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 26.20.3.

Proof. Since f is quasi-separated, the uniqueness part of the valutative criterion implies f is separated (Lemma 67.43.2). Conversely, if f is separated, then it satisfies the uniqueness part of the valuative criterion (Lemma 67.43.1). Having said this, we see that in each of the three cases the morphism f is separated and satisfies the uniqueness part of the valuative criterion. In this case the lemma is a formal consequence of Lemma 67.42.3. □

67.44. Valuative criterion of properness

0CKZ Here is a statement.

0A40 Lemma 67.44.1 (Valuative criterion for properness). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is of finite type and quasi-separated. Then the following are equivalent

(1) f is proper,
(2) the valuative criterion holds as in Definition 67.41.1,
(3) given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 26.20.3.

Proof. Formal consequence of Lemma 67.43.3 and the definitions. □

67.45. Integral and finite morphisms

03ZN We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be integral (resp. finite).

03ZO Lemma 67.45.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is integral, resp. finite (in the sense of Section 67.3), if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine and integral, resp. finite, over Z.

Proof. This follows directly from the definition of an integral (resp. finite) morphism of schemes (Morphisms, Definition 29.44.1). □

This clears the way for the following definition.

03ZP Definition 67.45.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say that f is integral if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme integral over Z.
(2) We say that f is finite if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme finite over Z.

03ZQ **Lemma 67.45.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is representable and integral (resp. finite),
(2) f is integral (resp. finite),
(3) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is integral (resp. finite), and
(4) there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is integral (resp. finite).

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 65.11.5 because the class of integral (resp. finite) morphisms satisfies all the required properties (see Morphisms, Lemmas 29.44.6 and Descent, Lemmas 35.23.22, 35.23.23, and 35.37.1). The conclusion of applying this lemma is that f is representable and integral (resp. finite), i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being integral (resp. finite) is Zariski local on the target (the reference above shows that being integral or finite is in fact fpqc local on the target). $\square$

03ZR **Lemma 67.45.4.** The composition of integral (resp. finite) morphisms is integral (resp. finite).

Proof. Omitted. $\square$

03ZS **Lemma 67.45.5.** The base change of an integral (resp. finite) morphism is integral (resp. finite).

Proof. Omitted. $\square$

0414 **Lemma 67.45.6.** A finite morphism of algebraic spaces is integral. An integral morphism of algebraic spaces which is locally of finite type is finite.

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 67.45.3. Hence this lemma follows from the same lemma for the case of schemes, see Morphisms, Lemma 29.44.4. $\square$

0415 **Lemma 67.45.7.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is integral, and
(2) f is affine and universally closed.

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 67.45.3, 67.20.3, and 67.9.5. Hence the result follows from Morphisms, Lemma 29.44.7. $\square$

04NY **Lemma 67.45.8.** A finite morphism of algebraic spaces is quasi-finite.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. By Definition 67.45.2 and Lemmas 67.8.8 and 67.27.6 both properties may be checked after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 29.44.10. $\square$

04NZ Lemma 67.45.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is finite, and
(2) f is affine and proper.

Proof. In both cases the morphism is representable, and you can check the condition after base change to an affine scheme mapping into Y, see Lemmas 67.45.3, 67.20.3, and 67.40.2. Hence the result follows from Morphisms, Lemma 29.44.11. $\square$

081Y Lemma 67.45.10. A closed immersion is finite (and a fortiori integral).

Proof. Omitted. $\square$

0CZ2 Lemma 67.45.11. Let S be a scheme. Let $X_i \to Y$, $i = 1, \ldots, n$ be finite morphisms of algebraic spaces over S. Then $X_1 \amalg \ldots \amalg X_n \to Y$ is finite too.

Proof. Follows from the case of schemes (Morphisms, Lemma 29.44.13) by étale localization. $\square$

081Z Lemma 67.45.12. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.

(1) If $g \circ f$ is finite and g separated then f is finite.
(2) If $g \circ f$ is integral and g separated then f is integral.

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 67.45.5. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Lemma 67.4.7. A closed immersion is finite (resp. integral), see Lemma 67.45.10. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 67.45.4. Thus we win. $\square$

67.46. Finite locally free morphisms

03ZT We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be finite locally free.

03ZU Lemma 67.46.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is finite locally free (in the sense of Section 67.3) if and only if f is affine and the sheaf $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module.

Proof. Assume f is finite locally free (as defined in Section 67.3). This means that for every morphism $V \to Y$ whose source is a scheme the base change $f' : V \times_Y X \to V$ is a finite locally free morphism of schemes. This in turn means (by the definition of a finite locally free morphism of schemes) that $f'_*\mathcal{O}_{V \times_Y X}$ is a finite locally free $\mathcal{O}_V$-module. We may choose $V \to Y$ to be surjective and étale. By Properties of Spaces, Lemma 66.26.2 we conclude the restriction of $f_*\mathcal{O}_X$ to V is finite locally free. Hence by Modules on Sites, Lemma 18.23.3 applied to the sheaf $f_*\mathcal{O}_X$ on $Y_{spaces,étale}$ we conclude that $f_*\mathcal{O}_X$ is finite locally free.

Conversely, assume f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. Again by Properties of Spaces, Lemma 66.26.2 we see that $f'_*\mathcal{O}_{V \times_Y X}$ is finite locally free. Hence $f' : V \times_Y X \to V$ is finite locally free (as it is also affine). By Spaces, Lemma 65.11.5 we conclude that f is finite locally free (use Morphisms, Lemma 29.48.4 Descent, Lemmas 35.23.30 and 35.37.1). Thus we win. $\square$

This clears the way for the following definition.

03ZV Definition 67.46.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that f is finite locally free if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. In this case we say f is has rank or degree d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of rank d.

03ZW Lemma 67.46.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is representable and finite locally free,
(2) f is finite locally free,
(3) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is finite locally free, and
(4) there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ is finite locally free.

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 65.11.5 because the class of finite locally free morphisms satisfies all the required properties (see Morphisms, Lemma 29.48.4 Descent, Lemmas 35.23.30 and 35.37.1). The conclusion of applying this lemma is that f is representable and finite locally free, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being finite locally free is Zariski local on the target (the reference above shows that being finite locally free is in fact fpqc local on the target). $\square$

03ZX Lemma 67.46.4. The composition of finite locally free morphisms is finite locally free.

Proof. Omitted. $\square$

03ZY Lemma 67.46.5. The base change of a finite locally free morphism is finite locally free.

Proof. Omitted. $\square$

0416 Lemma 67.46.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is finite locally free,
(2) f is finite, flat, and locally of finite presentation.

If Y is locally Noetherian these are also equivalent to

(3) f is finite and flat.

Proof. In each of the three cases the morphism is representable and you can check the property after base change by a surjective étale morphism $V \to Y$, see Lemmas 67.45.3, 67.46.3, 67.30.5, and 67.28.4. If Y is locally Noetherian, then V is locally Noetherian. Hence the result follows from the corresponding result in the schemes case, see Morphisms, Lemma 29.48.2. □

67.47. Rational maps

0EML This section is the analogue of Morphisms, Section 29.49. We will use without further mention that the intersection of dense opens of a topological space is a dense open.

0EMM Definition 67.47.1. Let S be a scheme. Let X, Y be algebraic spaces over S.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of algebraic spaces over S defined on dense open subspaces U, V of X. We say that f is equivalent to g if $f|_W = g|_W$ for some dense open subspace $W \subset U \cap V$.
(2) A rational map from X to Y is an equivalence class for the equivalence relation defined in (1).
(3) Given morphisms $X \to B$ and $Y \to B$ of algebraic spaces over S we say that a rational map from X to Y is a B-rational map from X to Y if there exists a representative $f : U \to Y$ of the equivalence class which is a morphism over B.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent. In many cases we will consider in the future, the algebraic spaces X and Y will contain a dense open subspaces X' and Y' which are schemes. In that case a rational map from X to Y is the same as an S-rational map from X' to Y' in the sense of Morphisms, Definition 67.47.1. Then all of the theory developed for schemes can be brought to bear.

0EMN Definition 67.47.2. Let S be a scheme. Let X be an algebraic space over S. A rational function on X is a rational map from X to $\mathbf{A}^1_S$.

Looking at the discussion following Morphisms, Definition 29.49.3 we find that this is the same as the notion defined there in case X happens to be a scheme.

Recall that we have the canonical identification

$$\mathrm{Mor}_S(T, \mathbf{A}^1_S) = \mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$$

for any scheme T over S, see Schemes, Example 26.15.2. Hence $\mathbf{A}^1_S$ is a ring-object in the category of schemes over S. In other words, addition and multiplication define morphisms

$$+ : \mathbf{A}^1_S \times_S \mathbf{A}^1_S \to \mathbf{A}^1_S \quad \text{and} \quad * : \mathbf{A}^1_S \times_S \mathbf{A}^1_S \to \mathbf{A}^1_S$$

satisfying the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_S$ has a natural ring structure.

0EMP Definition 67.47.3. Let S be a scheme. Let X be an algebraic space over S. The ring of rational functions on X is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

We will define function fields for integral algebraic spaces later, see Spaces over Fields, Section 72.4.

0EMQ Definition 67.47.4. Let S be a scheme. Let φ be a rational map between two algebraic spaces X and Y over S. We say φ is defined in a point $x \in |X|$ if there exists a representative (U, f) of φ with $x \in |U|$. The domain of definition of φ is the set of all points where φ is defined.

The domain of definition is viewed as an open subspace of X via Properties of Spaces, Lemma 66.4.8. With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

0EMR Lemma 67.47.5. Let S be a scheme. Let X and Y be algebraic spaces over S. Assume X is reduced and Y is separated over S. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \to Y$ of algebraic spaces representing φ.

Proof. Let (V, g) and (V', g') be representatives of φ. Then g, g' agree on a dense open subspace $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V \cap V'}$ and $g'|_{V \cap V'}$ is a closed subspace of $V \cap V'$ because it is the base change of $\Delta : Y \to Y \times_S Y$ by the morphism $V \cap V' \to Y \times_S Y$ given by $g|_{V \cap V'}$ and $g'|_{V \cap V'}$. Now $W \subset E$ implies that $|E| = |V \cap V'|$. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V \cap V'} = g'|_{V \cap V'}$, see Properties of Spaces, Lemma 66.12.4. It follows that we can glue the representatives $g : V \to Y$ of φ to a morphism $f : U \to Y$ because $\coprod V \to U$ is a surjection of fppf sheaves and $\coprod_{V, V'} V \cap V' = (\coprod V) \times_U (\coprod V)$. $\square$

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our spaces are irreducible and we look at dominant rational maps, then we can compose rational maps.

0EMS Definition 67.47.6. Let S be a scheme. Let X and Y be algebraic spaces over S. Assume $|X|$ and $|Y|$ are irreducible. A rational map from X to Y is called dominant if any representative $f : U \to Y$ is a dominant morphism in the sense of Definition 67.18.1.

We can compose a dominant rational map φ between irreducible algebraic spaces X and Y with an arbitrary rational map ψ from Y to Z. Namely, choose representatives $f : U \to Y$ with $|U| \subset |X|$ open dense and $g : V \to Z$ with $|V| \subset |Y|$ open dense. Then $W = |f|^{-1}(V) \subset |X|$ is open nonempty (because the image of $|f|$ is dense and hence must meet the nonempty open V) and hence dense as $|X|$ is irreducible. We define $\psi \circ \varphi$ as the equivalence class of $g \circ f|_W : W \to Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible algebraic spaces over S and whose morphisms are dominant rational maps.

0EMT Definition 67.47.7. Let S be a scheme. Let X and Y be algebraic spaces over S with $|X|$ and $|Y|$ irreducible. We say X and Y are birational if X and Y are isomorphic in the category of irreducible algebraic spaces over S and dominant rational maps.

If X and Y are birational irreducible algebraic spaces, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all algebraic spaces Z (functorially in Z). For "general" irreducible algebraic spaces this is just one possible definition. Another would be to require X and Y have isomorphic rings of rational functions; sometimes these two notions are equivalent (insert future reference here).

0EMU Lemma 67.47.8. Let S be a scheme. Let X and Y be algebraic space over S with $|X|$ and $|Y|$ irreducible. Then X and Y are birational if and only if there are nonempty open subspaces $U \subset X$ and $V \subset Y$ which are isomorphic as algebraic spaces over S.

Proof. Assume X and Y are birational. Let $f : U \to Y$ and $g : V \to X$ define inverse dominant rational maps from X to Y and from Y to X. After shrinking U we may assume $f : U \to Y$ factors through V. As $g \circ f$ is the identity as a dominant rational map, we see that the composition $U \to V \to X$ is the identity on a dense open of U. Thus after replacing U by a smaller open we may assume that $U \to V \to X$ is the inclusion of U into X. By symmetry we find there exists an open subspace $V' \subset V$ such that $g|_{V'} : V' \to X$ factors through $U \subset X$ and such that $V' \to U \to Y$ is the identity. The inverse image of $|V'|$ by $|U| \to |V|$ is an open of $|U|$ and hence equal to $|U'|$ for some open subspace $U' \subset U$, see Properties of Spaces, Lemma 66.4.8. Then $U' \subset U \to V$ factors as $U' \to V'$. Similarly $V' \to U$ factors as $V' \to U'$. The reader finds that $U' \to V'$ and $V' \to U'$ are mutually inverse morphisms of algebraic spaces over S and the proof is complete. □

67.48. Relative normalization of algebraic spaces

0BAZ This section is the analogue of Morphisms, Section 29.53.

0820 Lemma 67.48.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. There exists a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}' \subset \mathcal{A}$ such that for any affine object U of $X_{\acute{e}tale}$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

Proof. Let U be an object of $X_{\acute{e}tale}$. Then U is a scheme. Denote $\mathcal{A}|_U$ the restriction to the Zariski site. Then $\mathcal{A}|_U$ is a quasi-coherent sheaf of $\mathcal{O}_U$-algebras hence we can apply Morphisms, Lemma 29.53.1 to find a quasi-coherent subalgebra $\mathcal{A}'_U \subset \mathcal{A}|_U$ such that the value of $\mathcal{A}'_U$ on any affine open $W \subset U$ is as given in the statement of the lemma. If $f : U' \to U$ is a morphism in $X_{\acute{e}tale}$, then $\mathcal{A}|_{U'} = f^*(\mathcal{A}|_U)$ where f^* means pullback by the morphism f in the Zariski topology; this holds because $\mathcal{A}$ is quasi-coherent (see introduction to Properties of Spaces, Section 66.29 and the references to the discussion in the chapter on descent on schemes). Since f is étale we find that More on Morphisms, Lemma 37.19.1 says that we get a canonical isomorphism $f^*(\mathcal{A}'_U) = \mathcal{A}'_{U'}$. This immediately tells us that we obtain a sub presheaf $\mathcal{A}' \subset \mathcal{A}$ of $\mathcal{O}_X$-algebras over $X_{\acute{e}tale}$ which is a sheaf for the Zariski topology and has the right values on affine objects. But the fact that each $\mathcal{A}'_U$ is quasi-coherent on the scheme U and that for $f : U' \to U$ étale we have $\mathcal{A}'_{U'} = f^*(\mathcal{A}'_U)$ implies that $\mathcal{A}'$ is quasi-coherent on $X_{\acute{e}tale}$ as well (as this is a local property and we have the references above describing quasi-coherent modules on $U_{\acute{e}tale}$ in exactly this manner). □

0821 Definition 67.48.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_X$ in $\mathcal{A}$ is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 67.48.1 above.

We will apply this in particular when $\mathcal{A} = f_*\mathcal{O}_Y$ for a quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ (see Lemma 67.11.2). We can then take the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra (Lemma 67.20.7) to obtain the normalization of X in Y.

0822 Definition 67.48.3. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The normalization of X in Y is the morphism of algebraic spaces

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over S. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

To get the factorization, use Remark 67.20.9 and functoriality of the $\underline{\mathrm{Spec}}$ construction.

0ABP Lemma 67.48.4. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $Y \to X' \to X$ be the normalization of X in Y.

(1) If $W \to X$ is an étale morphism of algebraic spaces over S, then $W \times_X X'$ is the normalization of W in $W \times_X Y$.
(2) If Y and X are representable, then Y' is representable and is canonically isomorphic to the normalization of the scheme X in the scheme Y as constructed in Morphisms, Section 29.54.

Proof. It is immediate from the construction that the formation of the normalization of X in Y commutes with étale base change, i.e., part (1) holds. On the other hand, if X and Y are schemes, then for $U \subset X$ affine open, $f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U))$ and hence $\nu^{-1}(U)$ is the spectrum of exactly the same ring as we get in the corresponding construction for schemes. $\square$

Here is a characterization of this construction.

0823 Lemma 67.48.5. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:

(1) the morphism ν is integral, and
(2) for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[d, "\pi"] \\ X' \arrow[ur, "h"] \arrow[r, "\nu"] & X \end{tikzcd}$$

for a unique morphism $h : X' \to Z$.

Moreover, in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 67.48.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 67.45.2 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : X \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine U étale over X (by Definition 67.45.2) we see from Lemma 67.48.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 67.20.7 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 67.45.12. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

0AYF Lemma 67.48.6. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $X' \to X$ be the normalization of X in Y. If Y is reduced, so is X'.

Proof. This follows from the fact that a subring of a reduced ring is reduced. Some details omitted. □

0AYG Lemma 67.48.7. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. If $x' \in |X'|$ is a point of codimension 0 (Properties of Spaces, Definition 66.10.2), then x' is the image of some $y \in |Y|$ of codimension 0.

Proof. By Lemma 67.48.4 and the definitions, we may assume that $X = \text{Spec}(A)$ is affine. Then $X' = \text{Spec}(A')$ where A' is the integral closure of A in $\Gamma(Y, \mathcal{O}_Y)$ and x' corresponds to a minimal prime of A'. Choose a surjective étale morphism $V \to Y$ where $V = \text{Spec}(B)$ is affine. Then $A' \to B$ is injective, hence every minimal prime of A' is the image of a minimal prime of B, see Algebra, Lemma 10.30.5. The lemma follows. □

0824 Lemma 67.48.8. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two algebraic spaces. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \amalg X_2'$ is the normalization of X in Y.

Proof. Omitted. □

0A0Q Lemma 67.48.9. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact, quasi-separated and universally closed morphisms of algebraic spaces over S. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_Y$. In other words, the normalization of Y in X is equal to the factorization

$$X \longrightarrow \underline{\text{Spec}}_Y(f_*\mathcal{O}_X) \longrightarrow Y$$

of Remark 67.20.9.

Proof. The question is étale local on Y, hence we may reduce to the case where $Y = \text{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathbf{A}^1_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

Let $Z \subset \mathbf{A}^1_Y$ be the scheme theoretic image of h, see Definition 67.16.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_Y \to Y$ is separated, see Lemma 67.8.9. By Lemma 67.16.3 the morphism $X \to Z$ has dense image on underlying topological spaces. By Lemma 67.40.6 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 67.40.7 to conclude that $Z \to Y$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_Y$, we see that $Z \to Y$ is affine and proper, hence integral by Lemma 67.45.7. Writing $\mathbf{A}^1_Y = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. □

0825 Lemma 67.48.10. Let S be a scheme. Let $f : Y \to X$ be an integral morphism of algebraic spaces over S. Then the integral closure of X in Y is equal to Y.

Proof. By Lemma 67.45.7 this is a special case of Lemma 67.48.9. □

0BB0 Lemma 67.48.11. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that

(1) Y is Nagata,
(2) f is quasi-separated of finite type,
(3) X is reduced.

Then the normalization $\nu : Y' \to Y$ of Y in X is finite.

Proof. The question is étale local on Y, see Lemma 67.48.4. Thus we may assume $Y = \mathrm{Spec}(R)$ is affine. Then R is a Noetherian Nagata ring and we have to show that the integral closure of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is quasi-compact we see that X is quasi-compact. Choose an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.3). Then $\Gamma(X, \mathcal{O}_X) \subset \Gamma(U, \mathcal{O}_X)$. Since R is Noetherian it suffices to show that the integral closure of R in $\Gamma(U, \mathcal{O}_U)$ is finite over R. As $U \to Y$ is of finite type this follows from Morphisms, Lemma 29.53.15. □

67.49. Normalization

07U3 This section is the analogue of Morphisms, Section 29.54.

0BB1 Lemma 67.49.1. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent

(1) there is a surjective étale morphism $U \to X$ where U is a scheme such that every quasi-compact open of U has finitely many irreducible components,
(2) for every scheme U and every étale morphism $U \to X$ every quasi-compact open of U has finitely many irreducible components,
(3) for every quasi-compact algebraic space Y étale over X the set of codimension 0 points of Y (Properties of Spaces, Definition 66.10.2) is finite, and

(4) for every quasi-compact algebraic space Y étale over X the space $|Y|$ has finitely many irreducible components.

If X is representable this means that every quasi-compact open of X has finitely many irreducible components.

Proof. The equivalence of (1) and (2) and the final statement follow from Descent, Lemma 35.16.3 and Properties of Spaces, Lemma 66.7.1. It is clear that (4) implies (1) and (2) by considering only those Y which are schemes. Similarly, (3) implies (1) and (2) since for a scheme the codimension 0 points are the generic points of its irreducible components, see for example Properties of Spaces, Lemma 66.11.1.

Conversely, assume (2) and let $Y \to X$ be an étale morphism of algebraic spaces with Y quasi-compact. Then we can choose an affine scheme V and a surjective étale morphism $V \to Y$ (Properties of Spaces, Lemma 66.6.3). Since V has finitely many irreducible components by (2) and since $|V| \to |Y|$ is surjective and continuous, we conclude that $|Y|$ has finitely many irreducible components by Topology, Lemma 5.8.5. Thus (4) holds. Similarly, by Properties of Spaces, Lemma 66.11.1 the images of the generic points of the irreducible components of V are the codimension 0 points of Y and we conclude that there are finitely many, i.e., (3) holds. □

0GMB Lemma 67.49.2. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Then X satisfies the equivalent conditions of Lemma 67.49.1.

Proof. If $U \to X$ is étale and U is a scheme, then U is a locally Noetherian scheme, see Properties of Spaces, Section 66.7. A locally Noetherian scheme has a locally finite set of irreducible components (Divisors, Lemma 31.26.1). Thus we conclude that X passes condition (2) of the lemma. □

0GMC Lemma 67.49.3. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Then for $x \in |X|$ we have: x has codimension 0 in $X \Rightarrow f(x)$ has codimension 0 in Y.

Proof. Via Properties of Spaces, Lemma 66.11.1 and étale localization this translates into the case of a morphism of schemes and generic points of irreducible components. Here the result follows as generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 29.25.9. □

0GMD Lemma 67.49.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is flat and locally of finite type and assume Y satisfies the equivalent conditions of Lemma 67.49.1. Then X satisfies the equivalent conditions of Lemma 67.49.1 and for $x \in |X|$ we have: x has codimension 0 in $X \Rightarrow f(x)$ has codimension 0 in Y.

Proof. The last statement follows from Lemma 67.49.3. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. It suffices to show that every quasi-compact open of U has finitely many irreducible components. We will use the results of Properties of Spaces, Lemma 66.11.1 without further mention. By what we've already shown, the codimension 0 points of U lie above codimension 0 points in U and these are locally finite by assumption. Hence it suffices to show that for $v \in V$ of codimension 0 the codimension 0 points of the scheme theoretic fibre $U_v = U \times_V v$ are locally finite. This is true because U_v is a scheme locally of finite type over $\kappa(v)$, hence locally Noetherian and we can apply Lemma 67.49.2 for example. □

07U4 **Lemma 67.49.5.** Let S be a scheme. For every algebraic space X over S satisfying the equivalent conditions of Lemma 67.49.1 there exists a morphism of algebraic spaces

$$\nu_X : X^\nu \longrightarrow X$$

with the following properties

(1) if X satisfies the equivalent conditions of Lemma 67.49.1 then X^ν is normal and ν_X is integral,
(2) if X is a scheme such that every quasi-compact open has finitely many irreducible components, then $\nu_X : X^\nu \to X$ is the normalization of X constructed in Morphisms, Section 29.54,
(3) if $f : X \to Y$ is a morphism of algebraic spaces over S which both satisfy the equivalent conditions of Lemma 67.49.1 and every codimension 0 point of X is mapped by f to a codimension 0 point of Y, then there is a unique morphism $f^\nu : X^\nu \to Y^\nu$ of algebraic spaces over S such that $\nu_Y \circ f^\nu = f \circ \nu_X$, and
(4) if $f : X \to Y$ is an étale or smooth morphism of algebraic spaces and Y satisfies the equivalent conditions of Lemma 67.49.1, then the hypotheses of (3) hold and the morphism f^ν induces an isomorphism $X^\nu \to X \times_Y Y^\nu$.

Proof. Consider the category $\mathcal{C}$ whose objects are the schemes U over S such that every quasi-compact open of U has finitely many irreducible components and whose morphisms are those morphisms $g : U \to V$ of schemes over S such that every generic point of an irreducible component of U is mapped to the generic point of an irreducible component of V. We have already shown that

(a) for $U \in \mathrm{Ob}(\mathcal{C})$ we have a normalization morphism $\nu_U : U^\nu \to U$ as in Morphisms, Definition 29.54.1,
(b) for $U \in \mathrm{Ob}(\mathcal{C})$ the morphism ν_U is integral and U^ν is a normal scheme, see Morphisms, Lemma 29.54.5,
(c) for every $g : U \to V \in \mathrm{Arrows}(\mathcal{C})$ there is a unique morphism $g^\nu : U^\nu \to V^\nu$ such that $\nu_V \circ g^\nu = g \circ \nu_U$, see Morphisms, Lemma 29.54.5 part (4) applied to the composition $X^\nu \to X \to Y$,
(d) if $V \in \mathrm{Ob}(\mathcal{C})$ and $g : U \to V$ is étale or smooth, then $U \in \mathrm{Ob}(\mathcal{C})$ and $g \in \mathrm{Arrows}(\mathcal{C})$ and the morphism g^ν induces an isomorphism $U^\nu \to U \times_V V^\nu$, see Lemma 67.49.4 and More on Morphisms, Lemma 37.19.3.

Our task is to extend this construction to the corresponding category of algebraic spaces X over S.

Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1. Let $U \to X$ be a surjective étale morphism where U is a scheme. Set $R = U \times_X U$ with projections $s, t : R \to U$ and $j = (t, s) : R \to U \times_S U$ so that $X = U/R$, see Spaces, Lemma 65.9.1. Observe that U and R are objects of $\mathcal{C}$ by our assumptions on X and that the morphisms s and t are étale morphisms of schemes over S. By (a) we have the normalization morphisms $\nu_U : U^\nu \to U$ and $\nu_R : R^\nu \to R$, by (d) we have morphisms $s^\nu : R^\nu \to U^\nu$, $t^\nu : R^\nu \to U^\nu$ which define isomorphisms $R^\nu \to R \times_{s,U} U^\nu$ and $R^\nu \to U^\nu \times_{U,t} R$. It follows that s^ν and t^ν are étale (as they are isomorphic to base changes of étale morphisms). The induced morphism $j^\nu = (t^\nu, s^\nu) : R^\nu \to U^\nu \times_S U^\nu$ is a monomorphism as it is equal to the composition

$$\begin{aligned} R^\nu & \to (U^\nu \times_{U,t} R) \times_R (R \times_{s,U} U^\nu) \\ & = U^\nu \times_{U,t} R \times_{s,U} U^\nu \\ & \xrightarrow{j} U^\nu \times_U (U \times_S U) \times_U U^\nu \\ & = U^\nu \times_S U^\nu \end{aligned}$$

The first arrow is the diagonal morphism of ν_R. (This tells us that R^ν is a subscheme of the restriction of R to U^ν.) A formal computation with fibre products using property (d) shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the étale morphism $c : R \times_{s,U,t} R \to R$ extends uniquely to c^ν by (d). The morphism c^ν is compatible with the projection $\text{pr}_{13} : U^\nu \times_S U^\nu \times_S U^\nu \to U^\nu \times_S U^\nu$. Similarly, there are morphisms $i^\nu : R^\nu \to R^\nu$ compatible with the morphism $U^\nu \times_S U^\nu \to U^\nu \times_S U^\nu$ which switches factors and there is a morphism $e^\nu : U^\nu \to R^\nu$ compatible with the diagonal morphism $U^\nu \to U^\nu \times_S U^\nu$. All in all it follows that $j^\nu : R^\nu \to U^\nu \times_S U^\nu$ is an étale equivalence relation. At this point we may and do set $X^\nu = U^\nu / R^\nu$ (Spaces, Theorem 65.10.5). Then we see that we have $U^\nu = X^\nu \times_X U$ by Groupoids, Lemma 39.20.7.

What have we shown in the previous paragraph is this: for every algebraic space X over S satisfying the equivalent conditions of Lemma 67.49.1 if we choose a surjective étale morphism $g : U \to X$ where U is a scheme, then we obtain a cartesian diagram

$$\begin{tikzcd} X^\nu \arrow[d, "\nu_X"'] & U^\nu \arrow[l, "g^\nu"] \arrow[d, "\nu_U"] \\ X & U \arrow[l, "g"] \end{tikzcd}$$

of algebraic spaces. This immediately implies that X^ν is a normal algebraic space and that ν_X is a integral morphism. This gives part (1) of the lemma.

We will show below that the morphism $\nu_X : X^\nu \to X$ up to unique isomorphism is independent of the choice of g, but for now, if X is a scheme, we choose $\text{id} : X \to X$ so that it is clear that we have part (2) of the lemma.

We still have to prove parts (3) and (4). Let $g : U \to X$ and $\nu_X : X^\nu \to X$ and $g^\nu : U^\nu \to X^\nu$ be as above. Let Z be a normal scheme and let $h : Z \to U$ and $a : Z \to X^\nu$ be morphisms over S such that $g \circ h = \nu_X \circ a$ and such that every irreducible compoent of Z dominates an irreducible component of U (via h). By Morphisms, Lemma 29.54.5 part (4) we obtain a unique morphism $h^\nu : Z \to U^\nu$ such that $h = \nu_U \circ h^\nu$. Picture:

$$\begin{tikzcd} X^\nu \arrow[d, "\nu_X"'] & U^\nu \arrow[l, "g^\nu"] \arrow[d, "\nu_U"] & Z \arrow[l, "h^\nu"] \arrow[ll, bend right, "a"'] \arrow[dl, "h"] \\ X & U \arrow[l, "g"] & \end{tikzcd}$$

Observe that $a = g^\nu \circ h^\nu$. Namely, since the square with corners X^ν, X, U^ν, U is cartesian, this follows immediately from the fact that h^ν is unique (given h). In other words, given $h : Z \to U$ as above (and not a) there is a unique morphism $a : Z \to X^\nu$ with $\nu_X \circ a = g \circ h$.

Let $f : X \to Y$ be as in part (3) of the statement of the lemma. Suppose we have chosen surjective étale morphisms $U \to X$ and $V \to Y$ where U and V are schemes such that f lifts to a morphism $g : U \to V$. Then $g \in \text{Arrows}(\mathcal{C})$ and we obtain a unique morphism $g^\nu : U^\nu \to V^\nu$ compatible with ν_U and ν_V. However, then the two morphisms

$$R^\nu = U^\nu \times_{X^\nu} U^\nu \to U^\nu \to V^\nu \to Y^\nu$$

must be the same by our comments in the previous paragraph (applied with Y in stead of X). Since X^ν is constructed by taking the quotient of U^ν by R^ν it follows that we obtain a (unique) morphism $f^\nu : X^\nu \to Y^\nu$ as stated in (3).

To see that the construction of X^ν is independent of the choice of $g : U \to X$ surjective étale, apply the construction in the previous paragraph to $\text{id} : X \to X$ and a morphism $U' \to U$ between étale coverings of X. This is enough because given any two étale coverings of X there is a third one which dominates both. The reader shows that the morphism between the two normalizations constructed using either $U' \to X$ or $U \to X$ becomes an isomorphism after base change to U' and hence was an isomorphism. We omit the details.

We omit the proof of (4) which is similar; hint use part (d) above. $\square$

This leads us to the following definition.

0BB2 Definition 67.49.6. Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1. We define the normalization of X as the morphism

$$\nu_X : X^\nu \longrightarrow X$$

constructed in Lemma 67.49.5.

The definition applies to locally Noetherian algebraic spaces, see Lemma 67.49.2. Usually the normalization is defined only for reduced algebraic spaces. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

0BB3 Lemma 67.49.7. Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.

Proof. We may check this étale locally on X and hence reduce to the case of schemes which is Morphisms, Lemma 29.54.2. Some details omitted. $\square$

0BB4 Lemma 67.49.8. Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1.

(1) The normalization X^ν is normal.
(2) The morphism $\nu : X^\nu \to X$ is integral and surjective.
(3) The map $|\nu| : |X^\nu| \to |X|$ induces a bijection between the sets of points of codimension 0 (Properties of Spaces, Definition 66.10.2).
(4) Let $Z \to X$ be a morphism. Assume Z is a normal algebraic space and that for $z \in |Z|$ we have: z has codimension 0 in $Z \Rightarrow f(z)$ has codimension 0 in X. Then there exists a unique factorization $Z \to X^\nu \to X$.

Proof. Properties (1), (2), and (3) follow from the corresponding results for schemes (Morphisms, Lemma 29.54.5) combined with the fact that a point of a scheme is a generic point of an irreducible component if and only if the dimension of the local ring is zero (Properties, Lemma 28.10.4).

Let $Z \to X$ be a morphism as in (4). Let U be a scheme and let $U \to X$ be a surjective étale morphism. Choose a scheme V and a surjective étale morphism $V \to U \times_X Z$. The condition on codimension 0 points assures us that $V \to U$ maps generic points of irreducible components of V to generic points of irreducible components of U. Thus we obtain a unique factorization $V \to U^\nu \to U$ by Morphisms, Lemma 29.54.5. The uniqueness guarantees us that the two maps

$$V \times_{U \times_X Z} V \to V \to U^\nu$$

agree because these maps are the unique factorization of the map $V \times_{U \times_X Z} V \to V \to U$. Since the algebraic space $U \times_X Z$ is equal to the quotient $V/V \times_{U \times_X Z} V$ (see Spaces, Section 65.9) we find a canonical morphism $U \times_X Z \to U^\nu$. Picture

$$\begin{tikzcd} U \times_X Z \arrow[r] \arrow[d] & U^\nu \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r, dotted] \arrow[rr, bend right] & X^\nu \arrow[r] & X \end{tikzcd}$$

To obtain the dotted arrow we note that the construction of the arrow $U \times_X Z \to U^\nu$ is functorial in the étale morphism $U \to X$ (precise formulation and proof omitted). Hence if we set $R = U \times_X U$ with projections $s, t : R \to U$, then we obtain a morphism $R \times_X Z \to R^\nu$ commuting with $s, t : R \to U$ and $s^\nu, t^\nu : R^\nu \to U^\nu$. Recall that $X^\nu = U^\nu / R^\nu$, see proof of Lemma 67.49.5. Since $X = U/R$ a simple sheaf theoretic argument shows that $Z = (U \times_X Z)/(R \times_X Z)$. Thus the morphisms $U \times_X Z \to U^\nu$ and $R \times_X Z \to R^\nu$ define a morphism $Z \to X^\nu$ as desired. $\square$

0BB5 Lemma 67.49.9. Let S be a scheme. Let X be a Nagata algebraic space over S. The normalization $\nu : X^\nu \to X$ is a finite morphism.

Proof. Since X being Nagata is locally Noetherian, Definition 67.49.6 applies. By construction of X^ν in Lemma 67.49.5 we immediately reduce to the case of schemes which is Morphisms, Lemma 29.54.10. $\square$

67.50. Separated, locally quasi-finite morphisms

0417 In this section we prove that an algebraic space which is locally quasi-finite and separated over a scheme, is representable. This implies that a separated and locally quasi-finite morphism is representable (see Lemma 67.51.1). But first... a lemma (which will be obsoleted by Proposition 67.50.2).

03XW Lemma 67.50.1. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} V' \arrow[r] \arrow[dr] & T' \times_T X \arrow[r] \arrow[d] & X \arrow[d] \\ & T' \arrow[r] & T \end{tikzcd}$$

of algebraic spaces over S. Assume

(1) $T' \to T$ is an étale morphism of affine schemes,
(2) $X \to T$ is a separated, locally quasi-finite morphism,
(3) V' is an open subspace of $T' \times_T X$, and
(4) $V' \to T'$ is quasi-affine.

In this situation the image U of V' in X is a quasi-compact open subspace of X which is representable.

Proof. We first make some trivial observations. Note that V' is representable by Lemma 67.21.3. It is also quasi-compact (as a quasi-affine scheme over an affine scheme, see Morphisms, Lemma 29.13.2). Since $T' \times_T X \to X$ is étale (Properties of Spaces, Lemma 66.16.5) the map $|T' \times_T X| \to |X|$ is open, see Properties of Spaces, Lemma 66.16.7. Let $U \subset X$ be the open subspace corresponding to the image of $|V'|$, see Properties of Spaces, Lemma 66.4.8. As $|V'|$ is quasi-compact we see that $|U|$ is quasi-compact, hence U is a quasi-compact algebraic space, by Properties of Spaces, Lemma 66.5.2.

By Morphisms, Lemma 29.57.9 the morphism $T' \to T$ is universally bounded. Hence we can do induction on the integer n bounding the degree of the fibres of $T' \to T$, see Morphisms, Lemma 29.57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $T' \to T$ is an open immersion (see Étale Morphisms, Theorem 41.14.1), and the result is clear. Assume $n > 1$.

Consider the affine scheme $T'' = T' \times_T T'$. As $T' \to T$ is étale we have a decomposition (into open and closed affine subschemes) $T'' = \Delta(T') \amalg T^*$. Namely $\Delta = \Delta_{T'/T}$ is open by Morphisms, Lemma 29.35.13 and closed because $T' \to T$ is separated as a morphism of affines. As a base change the degrees of the fibres of the second projection $\text{pr}_1 : T' \times_T T' \to T'$ are bounded by n, see Morphisms, Lemma 29.57.5. On the other hand, $\text{pr}_1|_{\Delta(T')} : \Delta(T') \to T'$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 29.57.8 we conclude the degrees of the fibres of the restriction $\text{pr}_1|_{T^*} : T^* \to T'$ are bounded by $n - 1$. Hence the induction hypothesis applied to the diagram

$$\begin{tikzcd} p_0^{-1}(V') \cap X^* \arrow[r] \arrow[rd] & X^* \arrow[r, "p_1|_{X^*}"'] \arrow[d] & X' \arrow[d] \\ & T^* \arrow[r, "\text{pr}_1|_{T^*}"] & T' \end{tikzcd}$$

gives that $p_1(p_0^{-1}(V') \cap X^*)$ is a quasi-compact scheme. Here we set $X'' = T'' \times_T X$, $X^* = T^* \times_T X$, and $X' = T' \times_T X$, and $p_0, p_1 : X'' \to X'$ are the base changes of pr_0, pr_1. Most of the hypotheses of the lemma imply by base change the corresponding hypothesis for the diagram above. For example $p_0^{-1}(V') = T'' \times_{T'} V'$ is a scheme quasi-affine over T'' as a base change. Some verifications omitted.

By Properties of Spaces, Lemma 66.13.1 we conclude that

$$p_1(p_0^{-1}(V')) = V' \cup p_1(p_0^{-1}(V') \cap X^*)$$

is a quasi-compact scheme. Moreover, it is clear that $p_1(p_0^{-1}(V'))$ is the inverse image of the quasi-compact open subspace $U \subset X$ discussed in the first paragraph of the proof. In other words, $T' \times_T U$ is a scheme! Note that $T' \times_T U$ is quasi-compact and separated and locally quasi-finite over T', as $T' \times_T X \to T'$ is locally quasi-finite and separated being a base change of the original morphism $X \to T$ (see Lemmas 67.4.4 and 67.27.4). This implies by More on Morphisms, Lemma 37.43.2 that $T' \times_T U \to T'$ is quasi-affine.

By Descent, Lemma 35.39.1 this gives a descent datum on $T' \times_T U/T'$ relative to the étale covering $\{T' \to W\}$, where $W \subset T$ is the image of the morphism $T' \to T$.

Because U' is quasi-affine over T' we see from Descent, Lemma 35.38.1 that this datum is effective, and by the last part of Descent, Lemma 35.39.1 this implies that U is a scheme as desired. Some minor details omitted. □

03XX Proposition 67.50.2. Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces over S. Assume

(1) T is representable,
(2) f is locally quasi-finite, and
(3) f is separated.

Then X is representable.

Proof. Let $T = \bigcup T_i$ be an affine open covering of the scheme T. If we can show that the open subspaces $X_i = f^{-1}(T_i)$ are representable, then X is representable, see Properties of Spaces, Lemma 66.13.1. Note that $X_i = T_i \times_T X$ and that locally quasi-finite and separated are both stable under base change, see Lemmas 67.4.4 and 67.27.4. Hence we may assume T is an affine scheme.

By Properties of Spaces, Lemma 66.6.2 there exists a Zariski covering $X = \bigcup X_i$ such that each X_i has a surjective étale covering by an affine scheme. By Properties of Spaces, Lemma 66.13.1 again it suffices to prove the proposition for each X_i. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$. This reduces us to the situation in the next paragraph.

Assume we have

$$U \longrightarrow X \longrightarrow T$$

where U and T are affine schemes, $U \to X$ is étale surjective, and $X \to T$ is separated and locally quasi-finite. By Lemmas 67.39.5 and 67.27.3 the morphism $U \to T$ is locally quasi-finite. Since U and T are affine it is quasi-finite. Set $R = U \times_X U$. Then $X = U/R$, see Spaces, Lemma 65.9.1. As $X \to T$ is separated the morphism $R \to U \times_T U$ is a closed immersion, see Lemma 67.4.5. In particular R is an affine scheme also. As $U \to X$ is étale the projection morphisms $t, s : R \to U$ are étale as well. In particular s and t are quasi-finite, flat and of finite presentation (see Morphisms, Lemmas 29.36.6, 29.36.12 and 29.36.11).

Let (U, R, s, t, c) be the groupoid associated to the étale equivalence relation R on U. Let $u \in U$ be a point, and denote $p \in T$ its image. We are going to use More on Groupoids, Lemma 40.13.2 for the groupoid (U, R, s, t, c) over the scheme T with points p and u as above. By the discussion in the previous paragraph all the assumptions (1) – (7) of that lemma are satisfied. Hence we get an étale neighbourhood $(T', p') \to (T, p)$ and disjoint union decompositions

$$U_{T'} = U' \amalg W, \quad R_{T'} = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), (g), and (h) of the aforementioned More on Groupoids, Lemma 40.13.2. We may and do assume that T' is affine (after possibly shrinking T'). Conclusion (h) implies that $R' = U' \times_{X_{T'}} U'$ with projection mappings identified with the restrictions of s' and t'. Thus $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t', U', s'} R'})$ of conclusion (g) is an étale equivalence relation. By Spaces, Lemma 65.10.2 we conclude that U'/R' is an open subspace of $X_{T'}$. By conclusion (d) the schemes U', R' are affine and the morphisms $s'|_{R'}, t'|_{R'}$ are finite étale. Hence Groupoids, Proposition 39.23.9 kicks in and we see that U'/R' is an affine scheme.

We conclude that for every pair of points (u, p) as above we can find an étale neighbourhood $(T', p') \to (T, p)$ with $\kappa(p) = \kappa(p')$ and a point $u' \in U_{T'}$ mapping to u such that the image x' of u' in $|X_{T'}|$ has an open neighbourhood V' in $X_{T'}$ which is an affine scheme. We apply Lemma 67.50.1 to obtain an open subspace $W \subset X$ which is a scheme, and which contains x (the image of u in $|X|$). Since this works for every x we see that X is a scheme by Properties of Spaces, Lemma 66.13.1. This ends the proof. □

67.51. Applications

05W4 An alternative proof of the following lemma is to see it as a consequence of Zariski's main theorem for (nonrepresentable) morphisms of algebraic spaces as discussed in More on Morphisms of Spaces, Section 76.34. Namely, More on Morphisms of Spaces, Lemma 76.34.2 implies that a quasi-finite and separated morphism of algebraic spaces is quasi-affine and therefore representable.

0418 Lemma 67.51.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally quasi-finite and separated, then f is representable.

Proof. This is immediate from Proposition 67.50.2 and the fact that being locally quasi-finite and separated is preserved under any base change, see Lemmas 67.27.4 and 67.4.4. □

05W5 Lemma 67.51.2. Let S be a scheme. Let $f : X \to Y$ be an étale and universally injective morphism of algebraic spaces over S. Then f is an open immersion.

Proof. Let $T \to Y$ be a morphism from a scheme into Y. If we can show that $X \times_Y T \to T$ is an open immersion, then we are done. Since being étale and being universally injective are properties of morphisms stable under base change (see Lemmas 67.39.4 and 67.19.5) we may assume that Y is a scheme. Note that the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is étale, a monomorphism, and surjective by Lemma 67.19.2. Hence we see that $\Delta_{X/Y}$ is an isomorphism (see Spaces, Lemma 65.5.9), in particular we see that X is separated over Y. It follows that X is a scheme too, by Proposition 67.50.2. Finally, $X \to Y$ is an open immersion by the fundamental theorem for étale morphisms of schemes, see Étale Morphisms, Theorem 41.14.1. □

67.52. Zariski's Main Theorem (representable case)

0ABQ This is the version you can prove using that normalization commutes with étale localization. Before we can prove more powerful versions (for non-representable morphisms) we need to develop more tools. See More on Morphisms of Spaces, Section 76.34.

0ABR Lemma 67.52.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is representable, of finite type, and separated. Let Y' be the normalization of Y in X. Picture:

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\nu"] \\ & Y & \end{tikzcd}$$

Then there exists an open subspace $U' \subset Y'$ such that

(1) $(f')^{-1}(U') \to U'$ is an isomorphism, and
(2) $(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.

Proof. Let $W \to Y$ be a surjective étale morphism where W is a scheme. Then $W \times_Y X$ is a scheme as well. By Lemma 67.48.4 the algebraic space $W \times_Y Y'$ is representable and is the normalization of the scheme W in the scheme $W \times_Y X$. Picture

$$\begin{tikzcd} W \times_Y X \arrow[rr, "(1,f')"] \arrow[dr, "(1,f)"'] & & W \times_Y Y' \arrow[dl, "(1,\nu)"] \\ & W & \end{tikzcd}$$

By More on Morphisms, Lemma 37.43.1 the result of the lemma holds over W. Let $V' \subset W \times_Y Y'$ be the open subscheme such that

(1) $(1, f')^{-1}(V') \to V'$ is an isomorphism, and
(2) $(1, f')^{-1}(V') \subset W \times_Y X$ is the set of points at which $(1, f)$ is quasi-finite.

By Lemma 67.34.7 there is a maximal open set of points $U \subset X$ where f is quasi-finite and $W \times_Y U = (1, f')^{-1}(V')$. The morphism $f'|_U : U \to Y'$ is an open immersion by Lemma 67.12.1 as its base change to W is the isomorphism $(1, f')^{-1}(V') \to V'$ followed by the open immersion $V' \to W \times_Y Y'$. Setting $U' = \mathrm{Im}(U \to Y')$ finishes the proof (omitted: the verification that $(f')^{-1}(U') = U$). $\square$

In the following lemma we can drop the assumption of being representable as we've shown that a locally quasi-finite separated morphism is representable.

0ABS Lemma 67.52.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\nu"] \\ & Y & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. By Lemma 67.51.1 the morphism f is representable. Hence we may apply Lemma 67.52.1. Thus there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Lemma 67.8.9). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. $\square$

67.53. Universal homeomorphisms

05Z3 The class of universal homeomorphisms of schemes is closed under composition and arbitrary base change and is fppf local on the base. See Morphisms, Lemmas 29.45.3 and 29.45.2 and Descent, Lemma 35.23.9. Thus, if we apply the discussion in Section 67.3 to this notion we see that we know what it means for a representable morphism of algebraic spaces to be a universal homeomorphism.

05Z4 Lemma 67.53.1. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is a universal homeomorphism (in the sense of Section 67.3) if and only if for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.

Proof. If for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$, then the same is true whenever Z is a scheme, which formally implies that f is a universal homeomorphism in the sense of Section 67.3. Conversely, if f is a universal homeomorphism in the sense of Section 67.3 then $X \to Y$ is integral, universally injective and surjective (by Spaces, Lemma 65.5.8 and Morphisms, Lemma 29.45.5). Hence f is universally closed, see Lemma 67.45.7 and universally injective and (universally) surjective, i.e., f is a universal homeomorphism. □

05Z5 Definition 67.53.2. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is called a universal homeomorphism if and only if for every morphism of algebraic spaces $Z \to Y$ the base change $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.

This definition does not clash with the pre-existing definition for representable morphisms of algebraic spaces by our Lemma 67.53.1. For morphisms of algebraic spaces it is not the case that universal homeomorphisms are always integral.

05Z6 Example 67.53.3. This is a continuation of Remark 67.19.4. Consider the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$. There are morphisms

$$\mathbf{A}^1_k \longrightarrow X \longrightarrow \mathbf{A}^1_k$$

such that the first arrow is étale surjective, the second arrow is universally injective, and the composition is the map $x \mapsto x^2$. Hence the composition is universally closed. Thus it follows that the map $X \to \mathbf{A}^1_k$ is a universal homeomorphism, but $X \to \mathbf{A}^1_k$ is not separated.

Let S be a scheme. Let $f : X \to Y$ be a universal homeomorphism of algebraic spaces over S. Then f is universally closed, hence is quasi-compact, see Lemma 67.9.7. But f need not be separated (see example above), and not even quasi-separated: an example is to take infinite dimensional affine space $\mathbf{A}^\infty = \mathrm{Spec}(k[x_1, x_2, \ldots])$ modulo the equivalence relation given by flipping finitely many signs of nonzero coordinates (details omitted).

First we state the obligatory lemmas.

0CFT Lemma 67.53.4. The base change of a universal homeomorphism of algebraic spaces by any morphism of algebraic spaces is a universal homeomorphism.

Proof. This is immediate from the definition. □

0CFU Lemma 67.53.5. The composition of a pair of universal homeomorphisms of algebraic spaces is a universal homeomorphism.

Proof. Omitted. □

08AK Lemma 67.53.6. Let S be a scheme. Let X be an algebraic space over S. The canonical closed immersion $X_{red} \to X$ (see Properties of Spaces, Definition 66.12.5) is a universal homeomorphism.

Proof. Omitted. ☐

We put the following result here as we do not currently have a better place to put it.

0AEH Lemma 67.53.7. Let S be a scheme. Let $f : Y \to X$ be a universally injective, integral morphism of algebraic spaces over S.

(1) The functor
$$f_{small,*} : Sh(Y_{\acute{e}tale}) \longrightarrow Sh(X_{\acute{e}tale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ whose restriction to $|X| \setminus f(|Y|)$ is isomorphic to $*$, and
(2) the functor
$$f_{small,*} : \textit{Ab}(Y_{\acute{e}tale}) \longrightarrow \textit{Ab}(X_{\acute{e}tale})$$
is fully faithful and its essential image is those abelian sheaves on $Y_{\acute{e}tale}$ whose support is contained in $f(|Y|)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. Since f is integral it is universally closed (Lemma 67.45.7). In particular, $f(|Y|)$ is a closed subset of $|X|$ and the statements make sense. The rest of the proof is identical to the proof of Lemma 67.13.5 except that we use Étale Cohomology, Proposition 59.47.1 instead of Étale Cohomology, Proposition 59.46.4. ☐

67.54. Other chapters

CHAPTER 68

Decent Algebraic Spaces

06NK 68.1. Introduction

06NL In this chapter we study "local" properties of general algebraic spaces, i.e., those algebraic spaces which aren't quasi-separated. Quasi-separated algebraic spaces are studied in [Knu71]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn't worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.

68.2. Conventions

06NM The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

68.3. Universally bounded fibres

03JK We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 29.57 for similar definitions and results on morphisms of schemes.

03JL Definition 68.3.1. Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $f : U \to X$ be a morphism over S. We say the fibres of f are universally bounded[1] if there exists an integer n such that for all fields k and all morphisms $\mathrm{Spec}(k) \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k whose degree over k is $\leq n$.

This definition makes sense because the fibre product $\mathrm{Spec}(k) \times_Y X$ is a scheme. Moreover, if Y is a scheme we recover the notion of Morphisms, Definition 29.57.1 by virtue of Morphisms, Lemma 29.57.2.

03JM Lemma 68.3.2. Let S be a scheme. Let X be an algebraic space over S. Let $V \to U$ be a morphism of schemes over S, and let $U \to X$ be a morphism from U to X. If the fibres of $V \to U$ and $U \to X$ are universally bounded, then so are the fibres of $V \to X$.

[1]This is probably nonstandard notation.

Proof. Let n be an integer which works for $V \to U$, and let m be an integer which works for $U \to X$ in Definition 68.3.1. Let $\mathrm{Spec}(k) \to X$ be a morphism, where k is a field. Consider the morphisms

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U \longrightarrow \mathrm{Spec}(k).$$

By assumption the scheme $\mathrm{Spec}(k) \times_X U$ is finite of degree at most m over k, and n is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 29.57.4 we conclude that $\mathrm{Spec}(k) \times_X V$ is finite over k of degree at most nm. $\square$

03JN **Lemma 68.3.3.** Let S be a scheme. Let $Y \to X$ be a representable morphism of algebraic spaces over S. Let $U \to X$ be a morphism from a scheme to X. If the fibres of $U \to X$ are universally bounded, then the fibres of $U \times_X Y \to Y$ are universally bounded.

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \to X$ representable, so the definition applies.) $\square$

03JO **Lemma 68.3.4.** Let S be a scheme. Let $g : Y \to X$ be a representable morphism of algebraic spaces over S. Let $f : U \to X$ be a morphism from a scheme towards X. Let $f' : U \times_X Y \to Y$ be the base change of f. If

$$\mathrm{Im}(|f| : |U| \to |X|) \subset \mathrm{Im}(|g| : |Y| \to |X|)$$

and f' has universally bounded fibres, then f has universally bounded fibres.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $\mathrm{Spec}(k) \times_Y (U \times_X Y)$ as in Definition 68.3.1 for the morphism f'. We claim that n works for f also. Namely, suppose that $x : \mathrm{Spec}(k) \to X$ is a morphism from the spectrum of a field. Then either $\mathrm{Spec}(k) \times_X U$ is empty (and there is nothing to prove), or x is in the image of $|f|$. By Properties of Spaces, Lemma 66.4.3 and the assumption of the lemma we see that this means there exists a field extension k'/k and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$

Hence we see that

$$\mathrm{Spec}(k') \times_Y (U \times_X Y) = \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

Since the scheme $\mathrm{Spec}(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over k' it follows that also $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k as desired. (Some details omitted.) $\square$

03JP **Lemma 68.3.5.** Let S be a scheme. Let X be an algebraic space over S. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "g"'] & & V \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

where U and V are schemes. If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $\mathrm{Spec}(k) \times_X U$ as in Definition 68.3.1. We claim n also works for h. Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field to X. Consider the morphism of schemes

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $\mathrm{Spec}(k)$. It follows from Morphisms, Lemma 29.57.10 that the degree of the scheme on the right is also bounded by n as desired. □

03JQ Lemma 68.3.6. Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $\varphi : U \to X$ be a morphism over S. If the fibres of φ are universally bounded, then there exists an integer n such that each fibre of $|U| \to |X|$ has at most n elements.

Proof. The integer n of Definition 68.3.1 works. Namely, pick $x \in |X|$. Represent x by a morphism $x : \mathrm{Spec}(k) \to X$. Then we get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "x"] & X \end{tikzcd}$$

which shows (via Properties of Spaces, Lemma 66.4.3) that the inverse image of x in $|U|$ is the image of the top horizontal arrow. Since $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k it has at most n points. □

68.4. Finiteness conditions and points

03JR In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

03II Remark 68.4.1. Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

(1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 29.36.12 and 29.25.9).
(2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 29.36.16, 29.35.10, and 29.20.6).
(3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 29.20.9 and 29.20.10).
(4) An étale scheme over a field k is a disjoint union of spectra of finite separable field extension of k (Morphisms, Lemma 29.36.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 41.11.

03JS Lemma 68.4.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:

(1) there exists a family of schemes U_i and étale morphisms $\varphi_i : U_i \to X$ such that $\coprod \varphi_i : \coprod U_i \to X$ is surjective, and such that for each i the fibre of $|U_i| \to |X|$ over x is finite, and
(2) for every affine scheme U and étale morphism $\varphi : U \to X$ the fibre of $|U| \to |X|$ over x is finite.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let $\varphi_i : U_i \to X$ be a family of étale morphisms as in (1). Let $\varphi : U \to X$ be an étale morphism from an affine scheme towards X. Consider the fibre product diagrams

$$\begin{tikzcd} U \times_X U_i \arrow[r, "p_i"] \arrow[d, "q_i"] & U_i \arrow[d, "\varphi_i"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd} \qquad \begin{tikzcd} \coprod U \times_X U_i \arrow[r, "\coprod p_i"] \arrow[d, "\coprod q_i"] & \coprod U_i \arrow[d, "\coprod \varphi_i"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

Since q_i is étale it is open (see Remark 68.4.1). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices $i_1, \ldots, i_n$ and a quasi-compact opens $W_{i_j} \subset U \times_X U_{i_j}$ which surject onto U. The morphism p_i is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 29.57.9 to see the fibres of $p_{i_j}|_{W_{i_j}} : W_{i_j} \to U_i$ are finite. Hence by Properties of Spaces, Lemma 66.4.3 and the assumption on φ_i we conclude that the fibre of φ over x is finite. In other words (2) holds. $\square$

03JU Lemma 68.4.3. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:

(1) there exists a scheme U, an étale morphism $\varphi : U \to X$, and points $u, u' \in U$ mapping to x such that setting $R = U \times_X U$ the fibre of
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(2) for every scheme U, étale morphism $\varphi : U \to X$ and any points $u, u' \in U$ mapping to x setting $R = U \times_X U$ the fibre of
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(3) there exists a morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x such that the projections $\mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact, and
(4) there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x.

Proof. Assume (1), i.e., let $\varphi : U \to X$ be an étale morphism from a scheme towards X, and let u, u' be points of U lying over x such that the fibre of $|R| \to |U| \times_{|X|} |U|$ over (u, u') is a finite set. In this proof we think of a point $u = \mathrm{Spec}(\kappa(u))$ as a scheme. Note that $u \to U$, $u' \to U$ are monomorphisms (see Schemes, Lemma 26.23.7), hence $u \times_X u' \to R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \to X$ be an étale morphism from a scheme towards X. Let $w, w' \in W$ be points of W mapping to x. We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$\begin{tikzcd} W \times_X U \arrow[r, "p"'] \arrow[d, "q"'] & U \arrow[d, "\varphi"] \\ W \arrow[r, "\psi"] & X \end{tikzcd}$$

As x is the image of u and u' we may pick points $\tilde{w}, \tilde{w}'$ in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma 66.4.3. As p, q are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark 68.4.1. Then we get a commutative diagram

$$\begin{tikzcd} w \times_X w' \arrow[d] & \tilde{w} \times_X \tilde{w}' \arrow[l] \arrow[r] \arrow[d] & u \times_X u' \arrow[d] \\ w \times_X w' & \tilde{w} \times_S \tilde{w}' \arrow[l] \arrow[r] & u \times_S u' \end{tikzcd}$$

where the squares are fibre product squares. The lower horizontal morphisms are étale and quasi-compact, as any scheme of the form $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X w'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \to X$ be an étale morphism from a scheme U such that x is in the image of $|U| \to |X|$. Let $u \in U$ be a point mapping to x. Then we have seen in the previous paragraph that $u = \mathrm{Spec}(\kappa(u)) \to X$ has the property that $u \times_X u$ has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \to u$ are the composition

$$u \times_X u \longrightarrow u \times_X U \longrightarrow u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \to U$) by an étale morphism (the base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 68.4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 26.23.11 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \to u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field into X, in the equivalence class of x such that the two projections $t, s : R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are quasi-compact and étale. This means in particular that R is an étale equivalence relation on $\mathrm{Spec}(k)$. By Spaces, Theorem 65.10.5 we know that the quotient sheaf $X' = \mathrm{Spec}(k)/R$ is an algebraic space. By Groupoids, Lemma 39.20.6 the map $X' \to X$ is a monomorphism. Since s, t are quasi-compact, we see that R is quasi-compact and hence Properties of Spaces, Lemma 66.15.3 applies to X', and we see that $X' = \mathrm{Spec}(k')$ for some field k'. Hence we get a factorization

$$\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k') \longrightarrow X$$

which shows that $\mathrm{Spec}(k') \to X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $\mathrm{Spec}(k) \to X$ be a monomorphism with k a field in the equivalence class of x. Let $U \to X$ be a surjective étale morphism from a scheme U to X. Let $u \in U$ be a point over x. Since $\mathrm{Spec}(k) \times_X u$ is nonempty, and since $\mathrm{Spec}(k) \times_X u \to u$ is a monomorphism we conclude that $\mathrm{Spec}(k) \times_X u = u$ (see Schemes, Lemma 26.23.11). Hence $u \to U \to X$ factors through $\mathrm{Spec}(k) \to X$, here is a picture

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$

Since the right vertical arrow is étale this implies that $\kappa(u)/k$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{\mathrm{Spec}(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. $\square$

040U **Lemma 68.4.4.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. The following are equivalent:

(1) x is in the image of $|U| \to |X|$, and setting $R = U \times_X U$ the fibres of both
$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$
over x are finite,

(2) there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite nonempty scheme over k.

Proof. Assume (1). This clearly implies the first condition of Lemma 68.4.3 and hence we obtain a monomorphism $\mathrm{Spec}(k) \to X$ in the class of x. Taking the fibre product we see that $\mathrm{Spec}(k) \times_X U \to \mathrm{Spec}(k)$ is a scheme étale over $\mathrm{Spec}(k)$ with finitely many points, hence a finite nonempty scheme over k, i.e., (2) holds.

Assume (2). By assumption x is in the image of $|U| \to |X|$. The finiteness of the fibre of $|U| \to |X|$ over x is clear since this fibre is equal to $|\mathrm{Spec}(k) \times_X U|$ by Properties of Spaces, Lemma 66.4.3. The finiteness of the fibre of $|R| \to |X|$ above x is also clear since it is equal to the set underlying the scheme

$$(\mathrm{Spec}(k) \times_X U) \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

which is finite over k. Thus (1) holds. $\square$

03JV **Lemma 68.4.5.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:

(1) for every affine scheme U, any étale morphism $\varphi : U \to X$ setting $R = U \times_X U$ the fibres of both
$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$
over x are finite,

(2) there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and for each i, setting $R_i = U_i \times_X U_i$ the fibres of both
$$|U_i| \longrightarrow |X| \quad \text{and} \quad |R_i| \longrightarrow |X|$$
over x are finite,

(3) there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and for any affine scheme U and étale morphism $U \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k,

(4) there exists a quasi-compact monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x,

(5) there exists a quasi-compact morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and

(6) every morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x is quasi-compact.

Proof. The equivalence of (1) and (3) follows on applying Lemma 68.4.4 to every étale morphism $U \to X$ with U affine. It is clear that (3) implies (2). Assume $U_i \to X$ and R_i are as in (2). We conclude from Lemma 68.4.2 that for any affine scheme U and étale morphism $U \to X$ the fibre of $|U| \to |X|$ over x is finite. Say this fibre is $\{u_1, \ldots, u_n\}$. Then, as Lemma 68.4.3 (1) applies to $U_i \to X$ for some i such that x is in the image of $|U_i| \to |X|$, we see that the fibre of $|R = U \times_X U| \to |U| \times_{|X|} |U|$ is finite over (u_a, u_b), $a, b \in \{1, \ldots, n\}$. Hence the fibre of $|R| \to |X|$ over x is finite. In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme U and étale morphism $U \to X$ the scheme $\mathrm{Spec}(k) \times_X U$ is on the one hand étale over k (hence a disjoint union of spectra of finite separable extensions of k by Remark 68.4.1) and on the other hand quasi-compact over U (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \to X$ is as in (2) and $\mathrm{Spec}(k) \to X$ is a monomorphism as in (3), then
$$\coprod \mathrm{Spec}(k) \times_X U_i \longrightarrow \coprod U_i$$
is quasi-compact (because over each U_i we see that $\mathrm{Spec}(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $\mathrm{Spec}(k) \to X$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.8.

It is immediate that (4) implies (5). Conversely, let $\mathrm{Spec}(k) \to X$ be a quasi-compact morphism in the equivalence class of x. Let $U \to X$ be an étale morphism with U affine. Consider the fibre product
$$\begin{tikzcd} F \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$
Then $F \to U$ is quasi-compact, hence F is quasi-compact. On the other hand, $F \to \mathrm{Spec}(k)$ is étale, hence F is a finite disjoint union of spectra of finite separable extensions of k (Remark 68.4.1). Since the image of $|F| \to |U|$ is the fibre of $|U| \to |X|$ over x (Properties of Spaces, Lemma 66.4.3), we conclude that the fibre of $|U| \to |X|$ over x is finite. The scheme $F \times_{\mathrm{Spec}(k)} F$ is also a finite union of spectra of fields because it is also quasi-compact and étale over $\mathrm{Spec}(k)$. There is

a monomorphism $F \times_X F \to F \times_{\mathrm{Spec}(k)} F$, hence $F \times_X F$ is a finite disjoint union of spectra of fields (Schemes, Lemma 26.23.11). Thus the image of $F \times_X F \to U \times_X U = R$ is finite. Since this image is the fibre of $|R| \to |X|$ over x by Properties of Spaces, Lemma 66.4.3 we conclude that (1) holds. At this point we know that (1) – (5) are equivalent.

It is clear that (6) implies (5). Conversly, assume $\mathrm{Spec}(k) \to X$ is as in (4) and let $\mathrm{Spec}(k') \to X$ be another morphism with k' a field in the equivalence class of x. By Properties of Spaces, Lemma 66.4.11 we have a factorization $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to X$ of the given morphism. This is a composition of quasi-compact morphisms and hence quasi-compact (Morphisms of Spaces, Lemma 67.8.5) as desired. □

03JT Lemma 68.4.6. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:

(1) there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and each $U_i \to X$ has universally bounded fibres, and
(2) for every affine scheme U and étale morphism $\varphi : U \to X$ the fibres of $U \to X$ are universally bounded.

Proof. The implication (2) ⇒ (1) is trivial. Assume (1). Let $(\varphi_i : U_i \to X)_{i \in I}$ be a collection of étale morphisms from schemes towards X, covering X, such that each φ_i has universally bounded fibres. Let $\psi : U \to X$ be an étale morphism from an affine scheme towards X. For each i consider the fibre product diagram

$$\begin{tikzcd} U \times_X U_i \arrow[r, "p_i"'] \arrow[d, "q_i"'] & U_i \arrow[d, "\varphi_i"] \\ U \arrow[r, "\psi"] & X \end{tikzcd}$$

Since q_i is étale it is open (see Remark 68.4.1). Moreover, we have $U = \bigcup \mathrm{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since U is affine, hence quasi-compact we can finite finitely many $i_1, \ldots, i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_{i_j}$ such that $U = \bigcup p_{i_j}(W_j)$. The morphism p_{i_j} is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 29.57.9 to see the fibres of $p_{i_j}|_{W_j} : W_j \to U_{i_j}$ are universally bounded. Hence by Lemma 68.3.2 we see that the fibres of $W_j \to X$ are universally bounded. Thus also $\coprod_{j=1,\ldots,n} W_j \to X$ has universally bounded fibres. Since $\coprod_{j=1,\ldots,n} W_j \to X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1,\ldots,n} W_j \to U$ we see that the fibres of $U \to X$ are universally bounded by Lemma 68.3.5. In other words (2) holds. □

03IH Lemma 68.4.7. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:

(1) there exists a Zariski covering $X = \bigcup X_i$ and for each i a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and
(2) there exist schemes U_i and étale morphisms $U_i \to X$ such that the projections $U_i \times_X U_i \to U_i$ are quasi-compact and $\coprod U_i \to X$ is surjective.

Proof. If (1) holds then the morphisms $U_i \to X_i \to X$ are étale (combine Morphisms, Lemma 29.36.3 and Spaces, Lemmas 65.5.4 and 65.5.3). Moreover, as $U_i \times_X U_i = U_i \times_{X_i} U_i$, both projections $U_i \times_X U_i \to U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \to |X|$, see Properties of Spaces, Lemma 66.4.10. The morphisms $U_i \to X_i$ are surjective. Hence $U_i \to X_i$ is surjective étale, and the projections $U_i \times_{X_i} U_i \to U_i$ are quasi-compact, because $U_i \times_{X_i} U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 65.11.4 the morphisms $U_i \to X_i$ are quasi-compact. □

68.5. Conditions on algebraic spaces

03JW In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 68.6 to get a feeling for the meaning of these conditions.

03JX **Lemma 68.5.1.** Let S be a scheme. Let X be an algebraic space over S. Consider the following conditions on X:

(α) For every $x \in |X|$, the equivalent conditions of Lemma 68.4.2 hold.
(β) For every $x \in |X|$, the equivalent conditions of Lemma 68.4.3 hold.
(γ) For every $x \in |X|$, the equivalent conditions of Lemma 68.4.5 hold.
(δ) The equivalent conditions of Lemma 68.4.6 hold.
(ϵ) The equivalent conditions of Lemma 68.4.7 hold.
(ζ) The space X is Zariski locally quasi-separated.
(η) The space X is quasi-separated
(θ) The space X is representable, i.e., X is a scheme.
(ι) The space X is a quasi-separated scheme.

We have

$$\begin{tikzcd}[column sep=small] & (\theta) \arrow[rd, Rightarrow] & \\ (\iota) \arrow[ru, Rightarrow] \arrow[rd, Rightarrow] & & (\zeta) \arrow[r, Rightarrow] & (\epsilon) \arrow[r, Rightarrow] & (\delta) \arrow[r, Rightarrow] & (\gamma) \arrow[r, Leftrightarrow] & (\alpha) + (\beta) \\ & (\eta) \arrow[ru, Rightarrow] & \end{tikzcd}$$

Proof. The implication $(\gamma) \Leftrightarrow (\alpha) + (\beta)$ is immediate. The implications in the diamond on the left are clear from the definitions.

Assume (ζ), i.e., that X is Zariski locally quasi-separated. Then (ϵ) holds by Properties of Spaces, Lemma 66.6.6.

Assume (ϵ). By Lemma 68.4.7 there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$. Choose an i and an affine open subscheme $W \subset U_i$. It suffices to show that $W \to X$ has universally bounded fibres, since then the family of all these morphisms $W \to X$ covers X. To do this we consider the diagram

$$\begin{tikzcd} W \times_X U_i \arrow[r, "p"'] \arrow[d, "q"'] & U_i \arrow[d] \\ W \arrow[r] & X \end{tikzcd}$$

Since $W \to X$ factors through X_i we see that $W \times_X U_i = W \times_{X_i} U_i$, and hence q is quasi-compact. Since W is affine this implies that the scheme $W \times_X U_i$ is quasi-compact. Thus we may apply Morphisms, Lemma 29.57.9 and we conclude that p has universally bounded fibres. From Lemma 68.3.4 we conclude that $W \to X$ has universally bounded fibres as well.

Assume (δ). Let U be an affine scheme, and let $U \to X$ be an étale morphism. By assumption the fibres of the morphism $U \to X$ are universally bounded. Thus also the fibres of both projections $R = U \times_X U \to U$ are universally bounded, see Lemma 68.3.3. And by Lemma 68.3.2 also the fibres of $R \to X$ are universally bounded. Hence for any $x \in X$ the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite, see Lemma 68.3.6. In other words, the equivalent conditions of Lemma 68.4.5 hold. This proves that (δ) $\Rightarrow$ (γ). $\square$

03KE Lemma 68.5.2. Let S be a scheme. Let $\mathcal{P}$ be one of the properties (α), (β), (γ), (δ), (ϵ), (ζ), or (θ) of algebraic spaces listed in Lemma 68.5.1. Then if X is an algebraic space over S, and $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$, then X has $\mathcal{P}$.

Proof. Let X be an algebraic space over S, and let $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$.

The case $\mathcal{P} = (\alpha)$. The condition (α) for X_i means that for every $x \in |X_i|$ and every affine scheme U, and étale morphism $\varphi : U \to X_i$ the fibre of $\varphi : |U| \to |X_i|$ over x is finite. Consider $x \in X$, an affine scheme U and an étale morphism $U \to X$. Since $X = \bigcup X_i$ is a Zariski open covering there exits a finite affine open covering $U = U_1 \cup \ldots \cup U_n$ such that each $U_j \to X$ factors through some X_{i_j}. By assumption the fibres of $|U_j| \to |X_{i_j}|$ over x are finite for $j = 1, \ldots, n$. Clearly this means that the fibre of $|U| \to |X|$ over x is finite. This proves the result for (α).

The case $\mathcal{P} = (\beta)$. The condition (β) for X_i means that every $x \in |X_i|$ is represented by a monomorphism from the spectrum of a field towards X_i. Hence the same follows for X as $X_i \to X$ is a monomorphism and $X = \bigcup X_i$.

The case $\mathcal{P} = (\gamma)$. Note that $(\gamma) = (\alpha) + (\beta)$ by Lemma 68.5.1 hence the lemma for (γ) follows from the cases treated above.

The case $\mathcal{P} = (\delta)$. The condition (δ) for X_i means there exist schemes U_{ij} and étale morphisms $U_{ij} \to X_i$ with universally bounded fibres which cover X_i. These schemes also give an étale surjective morphism $\coprod U_{ij} \to X$ and $U_{ij} \to X$ still has universally bounded fibres.

The case $\mathcal{P} = (\epsilon)$. The condition (ϵ) for X_i means we can find a set J_i and morphisms $\varphi_{ij} : U_{ij} \to X_i$ such that each φ_{ij} is étale, both projections $U_{ij} \times_{X_i} U_{ij} \to U_{ij}$ are quasi-compact, and $\coprod_{j \in J_i} U_{ij} \to X_i$ is surjective. In this case the compositions $U_{ij} \to X_i \to X$ are étale (combine Morphisms, Lemmas 29.36.3 and 29.36.9 and Spaces, Lemmas 65.5.4 and 65.5.3). Since $X_i \subset X$ is a subspace we see that $U_{ij} \times_{X_i} U_{ij} = U_{ij} \times_X U_{ij}$, and hence the condition on fibre products is preserved. And clearly $\coprod_{i,j} U_{ij} \to X$ is surjective. Hence X satisfies (ϵ).

The case $\mathcal{P} = (\zeta)$. The condition (ζ) for X_i means that X_i is Zariski locally quasi-separated. It is immediately clear that this means X is Zariski locally quasi-separated.

For (θ), see Properties of Spaces, Lemma 66.13.1. $\square$

03KF Lemma 68.5.3. Let S be a scheme. Let $\mathcal{P}$ be one of the properties (β), (γ), (δ), (ϵ), or (θ) of algebraic spaces listed in Lemma 68.5.1. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y has property $\mathcal{P}$, so does X.

Proof. Assume $f : X \to Y$ is a representable morphism of algebraic spaces, and assume that Y has $\mathcal{P}$. Let $x \in |X|$, and set $y = f(x) \in |Y|$.

The case $\mathcal{P} = (\beta)$. Condition (β) for Y means there exists a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a scheme, and x corresponds to a point of X_y, i.e., to a monomorphism $\mathrm{Spec}(k') \to X_y$. As $X_y \to X$ is a monomorphism also we see that x is represented by the monomorphism $\mathrm{Spec}(k') \to X_y \to X$. In other words (β) holds for X.

The case $\mathcal{P} = (\gamma)$. Since $(\gamma) \Rightarrow (\beta)$ we have seen in the preceding paragraph that y and x can be represented by monomorphisms as in the following diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "x"'] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "y"] & Y \end{tikzcd}$$

Also, by definition of property (γ) via Lemma 68.4.5 (2) there exist schemes V_i and étale morphisms $V_i \to Y$ such that $\coprod V_i \to Y$ is surjective and for each i, setting $R_i = V_i \times_Y V_i$ the fibres of both

$$|V_i| \longrightarrow |Y| \quad \text{and} \quad |R_i| \longrightarrow |Y|$$

over y are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = \mathrm{Spec}(k)$. As $X \to Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Finally, for each i we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

and

$$(U_i \times_X U_i)_x = ((V_i \times_Y X) \times_X (V_i \times_Y X))_x = (R_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

hence these are finite over k' as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that (γ) holds for X, again via the second condition of Lemma 68.4.5.

The case $\mathcal{P} = (\delta)$. Let $V \to Y$ be an étale morphism with V an affine scheme. Since Y has property (δ) this morphism has universally bounded fibres. By Lemma 68.3.3 the base change $V \times_Y X \to X$ also has universally bounded fibres. Hence the first part of Lemma 68.4.6 applies and we see that Y also has property (δ).

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 65.5.5. Let $V_i \to Y$ be as in Lemma 68.4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \to U_i$ are base changes of the projections $V_i \times_Y V_i \to V_i$, and so quasi-compact as well. Hence X satisfies Lemma 68.4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 4.8.3. $\square$

68.6. Reasonable and decent algebraic spaces

03I7 In Lemma 68.5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into X and related to the existence of special étale coverings of X by schemes. We tabulate the different types of conditions here:

(α)	fibres of étale morphisms from affines are finite
(β)	points come from monomorphisms of spectra of fields
(γ)	points come from quasi-compact monomorphisms of spectra of fields
(δ)	fibres of étale morphisms from affines are universally bounded
(ϵ)	cover by étale morphisms from schemes quasi-compact onto their image

The conditions in the following definition are not exactly conditions on the diagonal of X, but they are in some sense separation conditions on X.

03I8 Definition 68.6.1. Let S be a scheme. Let X be an algebraic space over S.

(1) We say X is decent if for every point $x \in X$ the equivalent conditions of Lemma 68.4.5 hold, in other words property (γ) of Lemma 68.5.1 holds.
(2) We say X is reasonable if the equivalent conditions of Lemma 68.4.6 hold, in other words property (δ) of Lemma 68.5.1 holds.
(3) We say X is very reasonable if the equivalent conditions of Lemma 68.4.7 hold, i.e., property (ϵ) of Lemma 68.5.1 holds.

We have the following implications among these conditions on algebraic spaces:

$$\begin{tikzcd}
\text{representable} \arrow[dr, Rightarrow] & & & \\
& \text{very reasonable} \arrow[r, Rightarrow] & \text{reasonable} \arrow[r, Rightarrow] & \text{decent} \\
\text{quasi-separated} \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

The notion of a very reasonable algebraic space is obsolete. It was introduced because the assumption was needed to prove some results which are now proven for the class of decent spaces. The class of decent spaces is the largest class of spaces X where one has a good relationship between the topology of $|X|$ and properties of X itself.

03ID Example 68.6.2. The algebraic space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ constructed in Spaces, Example 65.14.8 is not decent as its "generic point" cannot be represented by a monomorphism from the spectrum of a field.

03JY Remark 68.6.3. Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if $X \to Y$ is a quasi-compact étale surjective morphism of algebraic spaces and X is reasonable, then so is Y, see Lemma 68.17.8 but we don't know if this is true for the property "very reasonable". Below we give another technical property enjoyed by reasonable algebraic spaces.

03K0 Lemma 68.6.4. Let S be a scheme. Let X be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated algebraic spaces X_i such that $X = \operatorname{colim}_i X_i$ (colimit in the category of sheaves). Moreover we can arrange it such that

(1) for every quasi-compact scheme T over S we have $\operatorname{colim} X_i(T) = X(T)$,
(2) the transition morphisms $X_i \to X_{i'}$ of the system and the coprojections $X_i \to X$ are surjective and étale, and
(3) if X is a scheme, then the algebraic spaces X_i are schemes and the transition morphisms $X_i \to X_{i'}$ and the coprojections $X_i \to X$ are local isomorphisms.

Proof. We sketch the proof. By Properties of Spaces, Lemma 66.6.3 we have $X = U/R$ with U affine. In this case, reasonable means $U \to X$ is universally bounded. Hence there exists an integer N such that the "fibres" of $U \to X$ have degree at most N, see Definition 68.3.1. Denote $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ the groupoid structural maps.

Claim: for every quasi-compact open $A \subset R$ there exists an open $R' \subset R$ such that

(1) $A \subset R'$,
(2) R' is quasi-compact, and
(3) $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$ is a groupoid scheme.

Note that $e : U \to R$ is open as it is a section of the étale morphism $s : R \to U$, see Étale Morphisms, Proposition 41.6.1. Moreover U is affine hence quasi-compact. Hence we may replace A by $A \cup e(U) \subset R$, and assume that A contains $e(U)$. Next, we define inductively $A^1 = A$, and

$$A^n = c(A^{n-1} \times_{s,U,t} A) \subset R$$

for $n \geq 2$. Arguing inductively, we see that A^n is quasi-compact for all $n \geq 2$, as the image of the quasi-compact fibre product $A^{n-1} \times_{s,U,t} A$. If k is an algebraically closed field over S, and we consider k-points then

$$A^n(k) = \left\{ (u, u') \in U(k) : \begin{matrix} \text{there exist } u = u_1, u_2, \ldots, u_n \in U(k) \text{ with} \\ (u_i, u_{i+1}) \in A \text{ for all } i = 1, \ldots, n-1. \end{matrix} \right\}$$

But as the fibres of $U(k) \to X(k)$ have size at most N we see that if $n > N$ then we get a repeat in the sequence above, and we can shorten it proving $A^N = A^n$ for all $n \geq N$. This implies that $R' = A^N$ gives a groupoid scheme $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$, proving the claim above.

Consider the map of sheaves on $(Sch/S)_{fppf}$

$$\operatorname{colim}_{R' \subset R} U/R' \longrightarrow U/R$$

where $R' \subset R$ runs over the quasi-compact open subschemes of R which give étale equivalence relations as above. Each of the quotients U/R' is an algebraic space (see Spaces, Theorem 65.10.5). Since R' is quasi-compact, and U affine the morphism $R' \to U \times_{\mathrm{Spec}(\mathbf{Z})} U$ is quasi-compact, and hence U/R' is quasi-separated. Finally, if T is a quasi-compact scheme, then

$$\operatorname{colim}_{R' \subset R} U(T)/R'(T) \longrightarrow U(T)/R(T)$$

is a bijection, since every morphism from T into R ends up in one of the open subrelations R' by the claim above. This clearly implies that the colimit of the sheaves U/R' is U/R. In other words the algebraic space $X = U/R$ is the colimit of the quasi-separated algebraic spaces U/R'.

Properties (1) and (2) follow from the discussion above. If X is a scheme, then if we choose U to be a finite disjoint union of affine opens of X we will obtain (3). Details omitted. □

0ABT Lemma 68.6.5. Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y is decent (resp. reasonable), then so is X.

Proof. Translation of Lemma 68.5.3. □

0ABU Lemma 68.6.6. Let S be a scheme. Let $X \to Y$ be an étale morphism of algebraic spaces over S. If Y is decent, resp. reasonable, then so is X.

Proof. Let U be an affine scheme and $U \to X$ an étale morphism. Set $R = U \times_X U$ and $R' = U \times_Y U$. Note that $R \to R'$ is a monomorphism.

Let $x \in |X|$. To show that X is decent, we have to show that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite. But if Y is decent, then the fibres of $|U| \to |Y|$ and $|R'| \to |Y|$ are finite. Hence the result for "decent".

To show that X is reasonable, we have to show that the fibres of $U \to X$ are universally bounded. However, if Y is reasonable, then the fibres of $U \to Y$ are universally bounded, which immediately implies the same thing for the fibres of $U \to X$. Hence the result for "reasonable". □

68.7. Points and specializations

03K1 There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization between points in a fibre of $|f| : |X| \to |Y|$, see Examples, Lemma 110.50.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on Y.

03IM Lemma 68.7.1. Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If the pair (X, x) satisfies the equivalent conditions of Lemma 68.4.2 then $u = u'$.

Proof. Assume the pair (X, x) satisfies the equivalent conditions for Lemma 68.4.2. Let U be a scheme, $U \to X$ étale, and let $u, u' \in |U|$ map to x of $|X|$, and $u' \rightsquigarrow u$. We may and do replace U by an affine neighbourhood of u. Let $t, s : R = U \times_X U \to U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 66.4.5. Because generalizations lift along the étale morphism t (Remark 68.4.1) we can find a specialization $r' \rightsquigarrow r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \rightsquigarrow u'$. Thus we may repeat and find $r'' \rightsquigarrow r'$ with $t(r'') = u''$. Set

$u''' = s(r'')$, and so on. Here is a picture:

$$
\begin{tikzcd}
 & r'' \arrow[ld, "t"'] \arrow[d, rightsquigarrow] \arrow[rd, "s"] & \\
u'' \arrow[d, rightsquigarrow] & r' \arrow[ld, "t"'] \arrow[d, rightsquigarrow] \arrow[rd, "s"] & u''' \arrow[d, rightsquigarrow] \\
u' \arrow[d, rightsquigarrow] & r \arrow[ld, "t"'] \arrow[rd, "s"] & u'' \arrow[d, rightsquigarrow] \\
u & & u'
\end{tikzcd}
$$

In Remark 68.4.1 we have seen that there are no specializations among points in the fibres of the étale morphism s. Hence if $u^{(n+1)} = u^{(n)}$ for some n, then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking t) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \ldots\}$ is an infinite set of points in U which map to the point x in $|X|$. As we chose U affine this contradicts the second part of Lemma 68.4.2, as desired. $\square$

0H1Q Lemma 68.7.2. Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If X is locally Noetherian, then $u = u'$.

Proof. The discussion in Schemes, Section 26.13 shows that $\mathcal{O}_{U,u'}$ is a localization of the Noetherian local ring $\mathcal{O}_{U,u}$. By Properties of Spaces, Lemma 66.10.1 we have $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{U,u'})$. By dimension theory for Noetherian local rings we conclude $u = u'$. $\square$

03K2 Lemma 68.7.3. Let S be a scheme. Let X be an algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 68.4.5. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.

Proof. We may replace U by an affine open neighbourhood of u. Hence we may assume that U is affine. As x is in the image of the open map $|U| \to |X|$, so is x'. Thus we may replace X by the Zariski open subspace corresponding to the image of $|U| \to |X|$, see Properties of Spaces, Lemma 66.4.10. In other words we may assume that $U \to X$ is surjective and étale. Let $s, t : R = U \times_X U \to U$ be the projections. By our assumption that (X, x') satisfies the equivalent conditions of Lemma 68.4.5 we see that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x' are finite. Say $\{u'_1, \ldots, u'_n\} \subset U$ and $\{r'_1, \ldots, r'_m\} \subset R$ form the complete inverse image of $\{x'\}$. Consider the closed sets

$$T = \overline{\{u'_1\}} \cup \ldots \cup \overline{\{u'_n\}} \subset |U|, \quad T' = \overline{\{r'_1\}} \cup \ldots \cup \overline{\{r'_m\}} \subset |R|.$$

Trivially we have $s(T') \subset T$. Because R is an equivalence relation we also have $t(T') = s(T')$ as the set $\{r'_j\}$ is invariant under the inverse of R by construction. Let $w \in T$ be any point. Then $u'_i \rightsquigarrow w$ for some i. Choose $r \in R$ with $s(r) = w$. Since generalizations lift along $s : R \to U$, see Remark 68.4.1, we can find $r' \rightsquigarrow r$

with $s(r') = u'_i$. Then $r' = r'_j$ for some j and we conclude that $w \in s(T')$. Hence $T = s(T') = t(T')$ is an $|R|$-invariant closed set in $|U|$. This means T is the inverse image of a closed (!) subset $T'' = \varphi(T)$ of $|X|$, see Properties of Spaces, Lemmas 66.4.5 and 66.4.6. Hence $T'' = \overline{\{x'\}}$. Thus T contains some point u_1 mapping to x as $x \in T''$. I.e., we see that for some i there exists a specialization $u'_i \rightsquigarrow u_1$ which maps to the given specialization $x' \rightsquigarrow x$.

To finish the proof, choose a point $r \in R$ such that $s(r) = u$ and $t(r) = u_1$ (using Properties of Spaces, Lemma 66.4.3). As generalizations lift along t, and $u'_i \rightsquigarrow u_1$ we can find a specialization $r' \rightsquigarrow r$ such that $t(r') = u'_i$. Set $u' = s(r')$. Then $u' \rightsquigarrow u$ and $\varphi(u') = x'$ as desired. □

0B7W Lemma 68.7.4. Let S be a scheme. Let $f : Y \to X$ be a flat morphism of algebraic spaces over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 68.4.5 (for example if X is decent, X is quasi-separated, or X is representable). Then for every $y \in |Y|$ with $f(y) = x$, there exists a point $y' \in |Y|$, $y' \rightsquigarrow y$ with $f(y') = x'$.

Proof. (The parenthetical statement holds by the definition of decent spaces and the implications between the different separation conditions mentioned in Section 68.6.) Choose a scheme V and a surjective étale morphism $V \to Y$. Choose $v \in V$ mapping to y. Then we see that it suffices to prove the lemma for $V \to X$. Thus we may assume Y is a scheme. Choose a scheme U and a surjective étale morphism $U \to X$. Choose $u \in U$ mapping to x. By Lemma 68.7.3 we may choose $u' \rightsquigarrow u$ mapping to x'. By Properties of Spaces, Lemma 66.4.3 we may choose $z \in U \times_X Y$ mapping to y and u. Thus we reduce to the case of the flat morphism of schemes $U \times_X Y \to U$ which is Morphisms, Lemma 29.25.9. □

68.8. Stratifying algebraic spaces by schemes

0A4I In this section we prove that a quasi-compact and quasi-separated algebraic space has a finite stratification by locally closed subspaces each of which is a scheme and such that the glueing of the parts is by elementary distinguished squares. We first prove a slightly weaker result for reasonable algebraic spaces.

07S8 Lemma 68.8.1. Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. There exist reduced closed subspaces

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = X$$

such that with $X_r = Z_r \setminus Z_{r-1}$ the stratification $X = \coprod_{r=0,\ldots,n} X_r$ is characterized by the following universal property: Given $g : T \to X$ the projection $W \times_X T \to T$ is finite locally free of degree r if and only if $g(|T|) \subset |X_r|$.

Proof. Let n be an integer bounding the degrees of the fibres of $W \to X$. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 37.45.3 to $W \times_X U \to U$. We obtain closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset Y_2 \subset \ldots \subset Y_n = U$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting

$R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(Y_r) = t^{-1}(Y_r)$$

as closed subsets of R. In other words the closed subsets $Y_r \subset U$ are R-invariant. This means that $|Y_r|$ is the inverse image of a closed subset $Z_r \subset |X|$. Denote $Z_r \subset X$ also the reduced induced algebraic space structure, see Properties of Spaces, Definition 66.12.5.

Let $g : T \to X$ be a morphism of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to T$. To prove the final assertion of the lemma it suffices to prove the assertion for the composition $V \to X$ (by our definition of finite locally free morphisms, see Morphisms of Spaces, Section 67.46). Similarly, the morphism of schemes $W \times_X V \to V$ is finite locally free of degree r if and only if the morphism of schemes

$$W \times_X (U \times_X V) \longrightarrow U \times_X V$$

is finite locally free of degree r (see Descent, Lemma 35.23.30). By construction this happens if and only if $|U \times_X V| \to |U|$ maps into $|Y_r|$, which is true if and only if $|V| \to |X|$ maps into $|Z_r|$. □

086T Lemma 68.8.2. Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, and locally quasi-finite. Then there exist open subspaces

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to X$ where k is a field factors through X_d if and only if $W \times_X \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 37.45.5 to $W \times_X U \to U$. We obtain open subschemes

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(U_d) = t^{-1}(U_d)$$

as open subschemes of R. In other words the open subschemes $U_d \subset U$ are R-invariant. This means that U_d is the inverse image of an open subspace $X_d \subset X$ (Properties of Spaces, Lemma 66.12.2). □

0BBN Lemma 68.8.3. Let S be a scheme. Let X be a quasi-compact algebraic space over S. There exist open subspaces

$$\ldots \subset U_4 \subset U_3 \subset U_2 \subset U_1 = X$$

with the following properties:

(1) setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism,
(2) if $x \in |X|$ can be represented by a quasi-compact morphism $\mathrm{Spec}(k) \to X$ from a field, then $x \in T_p$ for some p.

Proof. By Properties of Spaces, Lemma 66.6.3 we can choose an affine scheme U and a surjective étale morphism $U \to X$. For $p \geq 0$ set

$$W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals}$$

where the fibre product has p factors. Since U is separated, the morphism $U \to X$ is separated and all fibre products $U \times_X \ldots \times_X U$ are separated schemes. Since $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 67.39.10 and 67.38.9. Similarly, all the diagonal morphisms are open and closed immersions and W_p is an open and closed subscheme of $U \times_X \ldots \times_X U$. Moreover, the morphism

$$U \times_X \ldots \times_X U \longrightarrow U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$$

is locally quasi-finite and separated (Morphisms of Spaces, Lemma 67.4.5) and its target is an affine scheme. Hence every finite set of points of $U \times_X \ldots \times_X U$ is contained in an affine open, see More on Morphisms, Lemma 37.45.1. Therefore, the same is true for W_p. There is a free action of the symmetric group S_p on W_p over X (because we threw out the fix point locus from $U \times_X \ldots \times_X U$). By the above and Properties of Spaces, Proposition 66.14.1 the quotient $V_p = W_p/S_p$ is a scheme. Since the action of S_p on W_p was over X, there is a morphism $V_p \to X$. Since $W_p \to X$ is étale and since $W_p \to V_p$ is surjective étale, it follows that also $V_p \to X$ is étale, see Properties of Spaces, Lemma 66.16.3. Observe that V_p is a separated scheme by Properties of Spaces, Lemma 66.14.3.

We let $U_p \subset X$ be the open subspace which is the image of $V_p \to X$. By construction a morphism $\mathrm{Spec}(k) \to X$ with k algebraically closed, factors through U_p if and only if $U \times_X \mathrm{Spec}(k)$ has $\geq p$ points; as usual observe that $U \times_X \mathrm{Spec}(k)$ is scheme theoretically a disjoint union of (possibly infinitely many) copies of $\mathrm{Spec}(k)$, see Remark 68.4.1. It follows that the U_p give a filtration of X as stated in the lemma. Moreover, our morphism $\mathrm{Spec}(k) \to X$ factors through T_p if and only if $U \times_X \mathrm{Spec}(k)$ has exactly p points. In this case we see that $V_p \times_X \mathrm{Spec}(k)$ has exactly one point. Set $Z_p = f_p^{-1}(T_p) \subset V_p$. This is a closed subscheme of V_p. Then $Z_p \to T_p$ is an étale morphism between algebraic spaces which induces a bijection on k-valued points for any algebraically closed field k. To be sure this implies that $Z_p \to T_p$ is universally injective, whence an open immersion by Morphisms of Spaces, Lemma 67.51.2 hence an isomorphism and (1) has been proved.

Let $x : \mathrm{Spec}(k) \to X$ be a quasi-compact morphism where k is a field. Then the composition $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k) \to X$ is quasi-compact as well (Morphisms of Spaces, Lemma 67.8.5). In this case the scheme $U \times_X \mathrm{Spec}(\overline{k})$ is quasi-compact. In view of the fact (seen above) that it is a disjoint union of copies of $\mathrm{Spec}(\overline{k})$ we find that it has finitely many points. If the number of points is p, then we see that indeed $x \in T_p$ and the proof is finished. □

07S9 Lemma 68.8.4. Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Lemma 68.8.3. Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 68.6.1. Then we see that $U_{n+1} = \emptyset$ and the proof is complete. $\square$

07SA Lemma 68.8.5. Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

such that each $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) is a scheme.

Proof. Immediate consequence of Lemma 68.8.4. $\square$

The following result is almost identical to [GR71, Proposition 5.7.8].

07ST Lemma 68.8.6. Let X be a quasi-compact and quasi-separated algebraic space over $\mathrm{Spec}(\mathbf{Z})$. There exist an integer n and open subspaces This result is almost identical to [GR71, Proposition 5.7.8].

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Lemma 68.8.3. Observe that a quasi-separated space is reasonable, see Lemma 68.5.1 and Definition 68.6.1. Hence we find that $U_{n+1} = \emptyset$ as in Lemma 68.8.4. At the end of the argument we add that since X is quasi-separated the schemes $U \times_X \ldots \times_X U$ are all quasi-compact. Hence the schemes W_p are quasi-compact. Hence the quotients $V_p = W_p/S_p$ by the symmetric group S_p are quasi-compact schemes. $\square$

The following lemma probably belongs somewhere else.

0ECZ Lemma 68.8.7. Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let $E \subset |X|$ be a subset. Then E is étale locally constructible (Properties of Spaces, Definition 66.8.2) if and only if E is a locally constructible subset of the topological space $|X|$ (Topology, Definition 5.15.1).

Proof. Assume $E \subset |X|$ is a locally constructible subset of the topological space $|X|$. Let $f : U \to X$ be an étale morphism where U is a scheme. We have to show that $f^{-1}(E)$ is locally constructible in U. The question is local on U and X, hence we may assume that X is quasi-compact, $E \subset |X|$ is constructible, and U is affine. In this case $U \to X$ is quasi-compact, hence $f : |U| \to |X|$ is quasi-compact. Observe that retrocompact opens of $|X|$, resp. U are the same thing as quasi-compact opens of $|X|$, resp. U, see Topology, Lemma 5.27.1. Thus $f^{-1}(E)$ is constructible by Topology, Lemma 5.15.3.

Conversely, assume E is étale locally constructible. We want to show that E is locally constructible in the topological space $|X|$. The question is local on X, hence we may assume that X is quasi-compact as well as quasi-separated. We will show that in this case E is constructible in $|X|$. Choose open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and surjective étale morphisms $f_p : V_p \to U_p$ inducing isomorphisms $f_p^{-1}(T_p) \to T_p = U_p \setminus U_{p+1}$ where V_p is a quasi-compact separated scheme as in Lemma 68.8.6. By definition the inverse image $E_p \subset V_p$ of E is locally constructible in V_p. Then E_p is constructible in V_p by Properties, Lemma 28.2.5. Thus $E_p \cap |f_p^{-1}(T_p)| = E \cap |T_p|$ is constructible in $|T_p|$ by Topology, Lemma 5.15.7 (observe that $V_p \setminus f_p^{-1}(T_p)$ is quasi-compact as it is the inverse image of the quasi-compact space U_{p+1} by the quasi-compact morphism f_p). Thus

$$E = (|T_n| \cap E) \cup (|T_{n-1}| \cap E) \cup \ldots \cup (|T_1| \cap E)$$

is constructible by Topology, Lemma 5.15.14. Here we use that $|T_p|$ is constructible in $|X|$ which is clear from what was said above. □

68.9. Integral cover by a scheme

0D2T Here we prove that given any quasi-compact and quasi-separated algebraic space X, there is a scheme Y and a surjective, integral morphism $Y \to X$. After we develop some theory about limits of algebraic spaces, we will prove that one can do this with a finite morphism, see Limits of Spaces, Section 70.16.

0G2D **Lemma 68.9.1.** *Let S be a scheme. Let $j : V \to Y$ be a quasi-compact open immersion of algebraic spaces over S. Let $\pi : Z \to V$ be an integral morphism. Then there exists an integral morphism $\nu : Y' \to Y$ such that Z is V-isomorphic to the inverse image of V in Y'.*

Proof. Since both j and π are quasi-compact and separated, so is $j \circ \pi$. Let $\nu : Y' \to Y$ be the normalization of Y in Z, see Morphisms of Spaces, Section 67.48. Of course ν is integral, see Morphisms of Spaces, Lemma 67.48.5. The final statement follows formally from Morphisms of Spaces, Lemmas 67.48.4 and 67.48.10. □

09YB **Lemma 68.9.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S.*

1. *There exists a surjective integral morphism $Y \to X$ where Y is a scheme,*
2. *given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.*

Proof. Part (1) is the special case of part (2) where $U = X$. Choose a surjective étale morphism $U' \to U$ where U' is a scheme. It is clear that we may replace U by U' and hence we may assume U is a scheme. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U' again, we see that we may assume U is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Lemma 68.5.1). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Morphisms of Spaces, Lemma 67.52.2 and we obtain a factorization

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Then $U \times_X Y$ is a quasi-compact, separated scheme (being finite over U) and we have

$$U \times_X Y = U \amalg W$$

Here the first summand is the image of $U \to U \times_X Y$ (which is closed by Morphisms of Spaces, Lemma 67.4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subspace containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $|U| \subset |Y|$ is dense, it holds for all geometric points of Y by Lemma 68.8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 68.9.1). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced closed subspace structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have an open neighbourhood of z in Z (which is also an open neighbourhood of z in Z') which factors through $W \subset U \times_X Y$ and hence through U. □

0GUL Lemma 68.9.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S such that $|X|$ has finitely many irreducible components.

(1) There exists a surjective integral morphism $Y \to X$ where Y is a scheme such that f is finite étale over a quasi-compact dense open $U \subset X$,
(2) given a surjective étale morphism $V \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $W \subset Y$ such that $W \to X$ factors through V.

Proof. The proof is the (roughly) same as the proof of Lemma 68.9.2 with additional technical comments to obtain the dense quasi-compact open U (and unfortunately changes in notation to keep track of U).

Part (1) is the special case of part (2) where $V = X$.

Proof of (2). Choose a surjective étale morphism $V' \to V$ where V' is a scheme. It is clear that we may replace V by V' and hence we may assume V is a scheme. Since X is quasi-compact, there exist finitely many affine opens $V_i \subset V$ such that $V' = \coprod V_i \to X$ is surjective. After replacing V by V' again, we see that we may assume V is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $V \to X$ (see Lemma 68.5.1).

By induction on $d \geq 1$ we will prove the following induction hypothesis (H_d):

- for any quasi-compact and quasi-separated algebraic space X with finitely many irreducible components, for any $m \geq 0$, for any quasi-compact and separated schemes V_j, $j = 1, \ldots, m$, for any étale morphisms $\varphi_j : V_j \to X$, $j = 1, \ldots, m$ such that d bounds the degree of the geometric fibres of $\varphi_j : V_j \to X$ and $\varphi = \coprod \varphi_j : V = \coprod V_j \to X$ is surjective, the statement of the lemma holds for $\varphi : V \to X$.

If $d = 1$, then each φ_j is an open immersion. Hence X is a scheme and the result holds with $Y = V$. Assume $d > 1$, assume (H_{d-1}) and let m, $\varphi : V_j \to X$, $j = 1, \ldots, m$ be as in (H_d).

Let $\eta_1, \ldots, \eta_n \in |X|$ be the generic points of the irreducible components of $|X|$. By Properties of Spaces, Proposition 66.13.3 there is an open subscheme $U \subset X$ with $\eta_1, \ldots, \eta_n \in U$. By shrinking U we may assume U affine and by Morphisms, Lemma 29.51.1 we may assume each $\varphi_j : V_j \to X$ is finite étale over U. Of course, we see that U is quasi-compact and dense in X and that $\varphi_j^{-1}(U)$ is dense in V_j. In particular each V_j has finitely many irreducible components.

Fix $j \in \{1, \ldots, m\}$. As in Morphisms of Spaces, Lemma 67.52.2 we let Y_j be the normalization of X in V_j. We obtain a factorization

$$\begin{tikzcd} V_j \arrow[rr] \arrow[dr, "\varphi_j"'] & & Y_j \arrow[dl, "\pi_j"] \\ & X & \end{tikzcd}$$

with π_j integral and $V_j \to Y_j$ a quasi-compact open immersion. Since Y_j is the normalization of X in V_j, we see from Morphisms of Spaces, Lemmas 67.48.4 and 67.48.10 that $\varphi_j^{-1}(U) \to \pi_j^{-1}(U)$ is an isomorphism. Thus π_j is finite étale over U. Observe that V_j is scheme theoretically dense in Y_j because Y_j is the normalization of X in V_j (follows from the characterization of relative normalization in Morphisms of Spaces, Lemma 67.48.5). Since V_j is quasi-compact we see that $|V_j| \subset |Y_j|$ is dense, see Morphisms of Spaces, Section 67.17 (and especially Morphisms of Spaces, Lemma 67.17.7). It follows that $|Y_j|$ has finitely many irreducible components. Then $V_j \times_X Y_j$ is a quasi-compact, separated scheme (being finite over V_j) and

$$V_j \times_X Y_j = V_j \amalg W_j$$

Here the first summand is the image of $V_j \to V_j \times_X Y_j$ (which is closed by Morphisms of Spaces, Lemma 67.4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement.

The étale morphism $W_j \to Y_j$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $V_j \subset Y_j$ by inspection. Since $|V_j| \subset |Y_j|$ is dense, it holds for all geometric points of Y_j by Lemma 68.8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). By (H_{d-1}) applied to $V_j \amalg W_j \to Y_j$ we find a surjective integral morphism $Y'_j \to Y_j$ with Y'_j a scheme, which Zariski locally factors through $V_j \amalg W_j$, and which is finite étale over a quasi-compact dense open $U_j \subset Y_j$. After shrinking U we may and do assume that $\pi_j^{-1}(U) \subset U_j$ (we may and do choose the same U for all j; some details omitted).

We claim that

$$Y = \coprod\nolimits_{j=1,\ldots,m} Y_j' \longrightarrow X$$

is the solution to our problem. First, this morphism is integral as on each summand we have the composition $Y_j' \to Y \to X$ of integral morphisms (Morphisms of Spaces, Lemma 67.45.4). Second, this morphism Zariski locally factors through $V = \coprod V_j$ because we saw above that each $Y_j' \to Y_j$ factors Zariski locally through $V_j \amalg W_j = V_j \times_X Y_j$. Finally, since both $Y_j' \to Y_j$ and $Y_j \to X$ are finite étale over U, so is the composition. This finishes the proof. $\square$

68.10. Schematic locus

06NN In this section we prove that a decent algebraic space has a dense open subspace which is a scheme. We first prove this for reasonable algebraic spaces.

03JI Proposition 68.10.1. Let S be a scheme. Let X be an algebraic space over S. If X is reasonable, then there exists a dense open subspace of X which is a scheme.

Proof. By Properties of Spaces, Lemma 66.13.1 the question is local on X. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.1). Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 68.6.1. We will argue by induction on n that whenever

(1) $U \to X$ is a surjective étale morphism whose fibres have degree $\leq n$, and
(2) U is isomorphic to a locally closed subscheme of an affine scheme

then the schematic locus is dense in X.

Let $X_n \subset X$ be the open subspace which is the complement of the closed subspace $Z_{n-1} \subset X$ constructed in Lemma 68.8.1 using the morphism $U \to X$. Let $U_n \subset U$ be the inverse image of X_n. Then $U_n \to X_n$ is finite locally free of degree n. Hence X_n is a scheme by Properties of Spaces, Proposition 66.14.1 (and the fact that any finite set of points of U_n is contained in an affine open of U_n, see Properties, Lemma 28.29.5).

Let $X' \subset X$ be the open subspace such that $|X'|$ is the interior of $|Z_{n-1}|$ in $|X|$ (see Topology, Definition 5.21.1). Let $U' \subset U$ be the inverse image. Then $U' \to X'$ is surjective étale and has degrees of fibres bounded by $n - 1$. By induction we see that the schematic locus of X' is an open dense $X'' \subset X'$. By elementary topology we see that $X'' \cup X_n \subset X$ is open and dense and we win. $\square$

086U Theorem 68.10.2 (David Rydh). Let S be a scheme. Let X be an algebraic space over S. If X is decent, then there exists a dense open subspace of X which is a scheme.

Proof. Assume X is a decent algebraic space for which the theorem is false. By Properties of Spaces, Lemma 66.13.1 there exists a largest open subspace $X' \subset X$ which is a scheme. Since X' is not dense in X, there exists an open subspace $X'' \subset X$ such that $|X''| \cap |X'| = \emptyset$. Replacing X by X'' we get a nonempty decent algebraic space X which does not contain any open subspace which is a scheme.

Choose a nonempty affine scheme U and an étale morphism $U \to X$. We may and do replace X by the open subscheme corresponding to the image of $|U| \to |X|$.

Consider the sequence of open subspaces

$$X = X_0 \supset X_1 \supset X_2 \ldots$$

constructed in Lemma 68.8.2 for the morphism $U \to X$. Note that $X_0 = X_1$ as $U \to X$ is surjective. Let $U = U_0 = U_1 \supset U_2 \ldots$ be the induced sequence of open subschemes of U.

Choose a nonempty open affine $V_1 \subset U_1$ (for example $V_1 = U_1$). By induction we will construct a sequence of nonempty affine opens $V_1 \supset V_2 \supset \ldots$ with $V_n \subset U_n$. Namely, having constructed $V_1, \ldots, V_{n-1}$ we can always choose V_n unless $V_{n-1} \cap U_n = \emptyset$. But if $V_{n-1} \cap U_n = \emptyset$, then the open subspace $X' \subset X$ with $|X'| = \text{Im}(|V_{n-1}| \to |X|)$ is contained in $|X| \setminus |X_n|$. Hence $V_{n-1} \to X'$ is an étale morphism whose fibres have degree bounded by $n - 1$. In other words, X' is reasonable (by definition), hence X' contains a nonempty open subscheme by Proposition 68.10.1. This is a contradiction which shows that we can pick V_n.

By Limits, Lemma 32.4.3 the limit $V_\infty = \lim V_n$ is a nonempty scheme. Pick a morphism $\text{Spec}(k) \to V_\infty$. The composition $\text{Spec}(k) \to V_\infty \to U \to X$ has image contained in all X_d by construction. In other words, the fibred $U \times_X \text{Spec}(k)$ has infinite degree which contradicts the definition of a decent space. This contradiction finishes the proof of the theorem. □

0BA1 Lemma 68.10.3. Let S be a scheme. Let $X \to Y$ be a surjective finite locally free morphism of algebraic spaces over S. For $y \in |Y|$ the following are equivalent

(1) y is in the schematic locus of Y, and
(2) there exists an affine open $U \subset X$ containing the preimage of y.

Proof. If $y \in Y$ is in the schematic locus, then it has an affine open neighbourhood $V \subset Y$ and the inverse image U of V in X is an open finite over V, hence affine. Thus (1) implies (2).

Conversely, assume that $U \subset X$ as in (2) is given. Set $R = X \times_Y X$ and denote the projections $s, t : R \to X$. Consider $Z = R \setminus s^{-1}(U) \cap t^{-1}(U)$. This is a closed subset of R. The image $t(Z)$ is a closed subset of X which can loosely be described as the set of points of X which are R-equivalent to a point of $X \setminus U$. Hence $U' = X \setminus t(Z)$ is an R-invariant, open subspace of X contained in U which contains the fibre of $X \to Y$ over y. Since $X \to Y$ is open (Morphisms of Spaces, Lemma 67.30.6) the image of U' is an open subspace $V' \subset Y$. Since U' is R-invariant and $R = X \times_Y X$, we see that U' is the inverse image of V' (use Properties of Spaces, Lemma 66.4.3). After replacing Y by V' and X by U' we see that we may assume X is a scheme isomorphic to an open subscheme of an affine scheme.

Assume X is a scheme isomorphic to an open subscheme of an affine scheme. In this case the fppf quotient sheaf X/R is a scheme, see Properties of Spaces, Proposition 66.14.1. Since Y is a sheaf in the fppf topology, obtain a canonical map $X/R \to Y$ factoring $X \to Y$. Since $X \to Y$ is surjective finite locally free, it is surjective as a map of sheaves (Spaces, Lemma 65.5.9). We conclude that $X/R \to Y$ is surjective as a map of sheaves. On the other hand, since $R = X \times_Y X$ as sheaves we conclude that $X/R \to Y$ is injective as a map of sheaves. Hence $X/R \to Y$ is an isomorphism and we see that Y is representable. □

At this point we have several different ways for proving the following lemma.

06NG **Lemma 68.10.4.** Let S be a scheme. Let X be an algebraic space over S. If there exists a finite, étale, surjective morphism $U \to X$ where U is a scheme, then there exists a dense open subspace of X which is a scheme.

First proof. The morphism $U \to X$ is finite locally free. Hence there is a decomposition of X into open and closed subspaces $X_d \subset X$ such that $U \times_X X_d \to X_d$ is finite locally free of degree d. Thus we may assume $U \to X$ is finite locally free of degree d. In this case, let $U_i \subset U$, $i \in I$ be the set of affine opens. For each i the morphism $U_i \to X$ is étale and has universally bounded fibres (namely, bounded by d). In other words, X is reasonable and the result follows from Proposition 68.10.1. $\square$

Second proof. The question is local on X (Properties of Spaces, Lemma 66.13.1), hence may assume X is quasi-compact. Then U is quasi-compact. Then there exists a dense open subscheme $W \subset U$ which is separated (Properties, Lemma 28.29.3). Set $Z = U \setminus W$. Let $R = U \times_X U$ and $s, t : R \to U$ the projections. Then $t^{-1}(Z)$ is nowhere dense in R (Topology, Lemma 5.21.6) and hence $\Delta = s(t^{-1}(Z))$ is an R-invariant closed nowhere dense subset of U (Morphisms, Lemma 29.48.7). Let $u \in U \setminus \Delta$ be a generic point of an irreducible component. Since these points are dense in $U \setminus \Delta$ and since Δ is nowhere dense, it suffices to show that the image $x \in X$ of u is in the schematic locus of X. Observe that $t(s^{-1}(\{u\})) \subset W$ is a finite set of generic points of irreducible components of W (compare with Properties of Spaces, Lemma 66.11.1). By Properties, Lemma 28.29.1 we can find an affine open $V \subset W$ such that $t(s^{-1}(\{u\})) \subset V$. Since $t(s^{-1}(\{u\}))$ is the fibre of $|U| \to |X|$ over x, we conclude by Lemma 68.10.3. $\square$

Third proof. (This proof is essentially the same as the second proof, but uses fewer references.) Assume X is an algebraic space, U a scheme, and $U \to X$ is a finite étale surjective morphism. Write $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, finite and étale. Claim: The union of the R-invariant affine opens of U is topologically dense in U.

Proof of the claim. Let $W \subset U$ be an affine open. Set $W' = t(s^{-1}(W)) \subset U$. Since $s^{-1}(W)$ is affine (hence quasi-compact) we see that $W' \subset U$ is a quasi-compact open. By Properties, Lemma 28.29.3 there exists a dense open $W'' \subset W'$ which is a separated scheme. Set $\Delta' = W' \setminus W''$. This is a nowhere dense closed subset of W''. Since $t|_{s^{-1}(W)} : s^{-1}(W) \to W'$ is open (because it is étale) we see that the inverse image $(t|_{s^{-1}(W)})^{-1}(\Delta') \subset s^{-1}(W)$ is a nowhere dense closed subset (see Topology, Lemma 5.21.6). Hence, by Morphisms, Lemma 29.48.7 we see that

$$\Delta = s\left((t|_{s^{-1}(W)})^{-1}(\Delta')\right)$$

is a nowhere dense closed subset of W. Pick any point $\eta \in W$, $\eta \notin \Delta$ which is a generic point of an irreducible component of W (and hence of U). By our choices above the finite set $t(s^{-1}(\{\eta\})) = \{\eta_1, \ldots, \eta_n\}$ is contained in the separated scheme W''. Note that the fibres of s is are finite discrete spaces, and that generalizations lift along the étale morphism t, see Morphisms, Lemmas 29.36.12 and 29.25.9. In this way we see that each η_i is a generic point of an irreducible component of W''. Thus, by Properties, Lemma 28.29.1 we can find an affine open $V \subset W''$ such that $\{\eta_1, \ldots, \eta_n\} \subset V$. By Groupoids, Lemma 39.24.1 this implies that η is contained in an R-invariant affine open subscheme of U. The claim follows as W was chosen as an arbitrary affine open of U and because the set of generic points of irreducible components of $W \setminus \Delta$ is dense in W.

Using the claim we can finish the proof. Namely, if $W \subset U$ is an R-invariant affine open, then the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 39.19.1 and discussion following it). In particular the maps $R_W \to W$ are finite étale also. It follows in particular that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 39.23.9. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 65.10.2. Hence having a dense collection of points contained in R-invariant affine open of U certainly implies that the schematic locus of X (see Properties of Spaces, Lemma 66.13.1) is open dense in X. $\square$

68.11. Residue fields and henselian local rings

0EMV For a decent algebraic space we can define the residue field and the henselian local ring at a point. For example, the following lemma tells us the residue field of a point on a decent space is defined.

03K4 **Lemma 68.11.1.** Let S be a scheme. Let X be an algebraic space over S. Consider the map

$$\{\mathrm{Spec}(k) \to X \text{ monomorphism where } k \text{ is a field}\} \longrightarrow |X|$$

This map is always injective. If X is decent then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 66.4.12 that the map is an injection in general. By Lemma 68.5.1 it is surjective when X is decent (actually one can say this is part of the definition of being decent). $\square$

Let S be a scheme. Let X be an algebraic space over S. If a point $x \in |X|$ can be represented by a monomorphism $\mathrm{Spec}(k) \to X$, then the field k is unique up to unique isomorphism. For a decent algebraic space such a monomorphism exists for every point by Lemma 68.11.1 and hence the following definition makes sense.

0EMW **Definition 68.11.2.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The *residue field of X at x* is the unique field $\kappa(x)$ which comes equipped with a monomorphism $\mathrm{Spec}(\kappa(x)) \to X$ representing x.

Let S be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over S. Let $x \in |X|$ be a point. Set $y = f(x) \in |Y|$. Then the composition $\mathrm{Spec}(\kappa(x)) \to Y$ is in the equivalence class defining y and hence factors through $\mathrm{Spec}(\kappa(y)) \to Y$. In other words we get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa(x)) \arrow[r, "x"'] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\kappa(y)) \arrow[r, "y"] & Y \end{tikzcd}$$

The left vertical morphism corresponds to a homomorphism $\kappa(y) \to \kappa(x)$ of fields. We will often simply call this the homomorphism induced by f.

0EMX **Lemma 68.11.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over S. Let $x \in |X|$ be a point with image $y = f(x) \in |Y|$. The following are equivalent

(1) f induces an isomorphism $\kappa(y) \to \kappa(x)$, and
(2) the induced morphism $\mathrm{Spec}(\kappa(x)) \to Y$ is a monomorphism.

Proof. Immediate from the discussion above. □

The following lemma tells us that the henselian local ring of a point on a decent algebraic space is defined.

0BBP Lemma 68.11.4. Let S be a scheme. Let X be a decent algebraic space over S. For every point $x \in |X|$ there exists an étale morphism

$$(U, u) \longrightarrow (X, x)$$

where U is an affine scheme, u is the only point of U lying over x, and the induced homomorphism $\kappa(x) \to \kappa(u)$ is an isomorphism.

Proof. We may assume that X is quasi-compact by replacing X with a quasi-compact open containing x. Recall that x can be represented by a quasi-compact (mono)morphism from the spectrum a field (by definition of decent spaces). Thus the lemma follows from Lemma 68.8.3. □

0BGU Definition 68.11.5. Let S be a scheme. Let X be an algebraic space over S. Let $x \in X$ be a point. An elementary étale neighbourhood is an étale morphism $(U, u) \to (X, x)$ where U is a scheme, $u \in U$ is a point mapping to x, and the morphism $u = \mathrm{Spec}(\kappa(u)) \to X$ is a monomorphism. A morphism of elementary étale neighbourhoods $(U, u) \to (U', u')$ is defined as a morphism $U \to U'$ over X mapping u to u'.

If X is not decent then the category of elementary étale neighbourhoods may be empty.

0BGV Lemma 68.11.6. Let S be a scheme. Let X be a decent algebraic space over S. Let x be a point of X. The category of elementary étale neighborhoods of (X, x) is cofiltered (see Categories, Definition 4.20.1).

Proof. The category is nonempty by Lemma 68.11.4. Suppose that we have two elementary étale neighbourhoods $(U_i, u_i) \to (X, x)$. Then consider $U = U_1 \times_X U_2$. Since $\mathrm{Spec}(\kappa(u_i)) \to X$, $i = 1, 2$ are both monomorphisms in the class of x (Lemma 68.11.3) , we see that

$$u = \mathrm{Spec}(\kappa(u_1)) \times_X \mathrm{Spec}(\kappa(u_2))$$

is the spectrum of a field $\kappa(u)$ such that the induced maps $\kappa(u_i) \to \kappa(u)$ are isomorphisms. Then $u \to U$ is a point of U and we see that $(U, u) \to (X, x)$ is an elementary étale neighbourhood dominating (U_i, u_i). If $a, b : (U_1, u_1) \to (U_2, u_2)$ are two morphisms between our elementary étale neighbourhoods, then we consider the scheme

$$U = U_1 \times_{(a,b),(U_2 \times_X U_2),\Delta} U_2$$

Using Properties of Spaces, Lemma 66.16.6 we see that $U \to X$ is étale. Moreover, in exactly the same manner as before we see that U has a point u such that $(U, u) \to (X, x)$ is an elementary étale neighbourhood. Finally, $U \to U_1$ equalizes a and b and the proof is finished. □

0BGW Definition 68.11.7. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The henselian local ring of X at x, is

$$\mathcal{O}^h_{X,x} = \operatorname{colim} \Gamma(U, \mathcal{O}_U)$$

where the colimit is over the elementary étale neighbourhoods $(U, u) \to (X, x)$.

Here is the analogue of Properties of Spaces, Lemma 66.22.1.

0EMY Lemma 68.11.8. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. Let $(U, u) \to (X, x)$ be an elementary étale neighbourhood. Then

$$\mathcal{O}^h_{X,x} = \mathcal{O}^h_{U,u}$$

In words: the henselian local ring of X at x is equal to the henselization $\mathcal{O}^h_{U,u}$ of the local ring $\mathcal{O}_{U,u}$ of U at u.

Proof. Since the category of elementary étale neighbourhood of (X, x) is cofiltered (Lemma 68.11.6) we see that the category of elementary étale neighbourhoods of (U, u) is initial in the category of elementary étale neighbourhood of (X, x). Then the equality follows from More on Morphisms, Lemma 37.35.5 and Categories, Lemma 4.17.2 (initial is turned into cofinal because the colimit defininining henselian local rings is over the opposite of the category of elementary étale neighbourhoods). $\square$

0EMZ Lemma 68.11.9. Let S be a scheme. Let X be a decent algebraic space over S. Let $\overline{x}$ be a geometric point of X lying over $x \in |X|$. The étale local ring $\mathcal{O}_{X,\overline{x}}$ of X at $\overline{x}$ (Properties of Spaces, Definition 66.22.2) is the strict henselization of the henselian local ring $\mathcal{O}^h_{X,x}$ of X at x.

Proof. Follows from Lemma 68.11.8, Properties of Spaces, Lemma 66.22.1 and the fact that $(R^h)^{sh} = R^{sh}$ for a local ring $(R, \mathfrak{m}, \kappa)$ and a given separable algebraic closure κ^{sep} of κ. This equality follows from Algebra, Lemma 10.154.7. $\square$

0EN0 Lemma 68.11.10. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The residue field of the henselian local ring of X at x (Definition 68.11.7) is the residue field of X at x (Definition 68.11.2).

Proof. Choose an elementary étale neighbourhood $(U, u) \to (X, x)$. Then $\kappa(u) = \kappa(x)$ and $\mathcal{O}^h_{X,x} = \mathcal{O}^h_{U,u}$ (Lemma 68.11.8). The residue field of $\mathcal{O}^h_{U,u}$ is $\kappa(u)$ by Algebra, Lemma 10.155.1 (the output of this lemma is the construction/definition of the henselization of a local ring, see Algebra, Definition 10.155.3). $\square$

0EPL Remark 68.11.11. Let S be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Choose an elementary étale neighbourhood $(V, v) \to (Y, y)$ (possible by Lemma 68.11.4). Then $V \times_Y X$ is an algebraic space étale over X which has a unique point x' mapping to x in X and to v in V. (Details omitted; use that all points can be represented by monomorphisms from spectra of fields.) Choose an elementary étale neighbourhood $(U, u) \to (V \times_Y X, x')$. Then we obtain the following commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{O}_{X,\overline{x}}) \arrow[r] \arrow[d] & \operatorname{Spec}(\mathcal{O}^h_{X,x}) \arrow[r] \arrow[d] & \operatorname{Spec}(\mathcal{O}_{U,u}) \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{Y,\overline{y}}) \arrow[r] & \operatorname{Spec}(\mathcal{O}^h_{Y,y}) \arrow[r] & \operatorname{Spec}(\mathcal{O}_{V,v}) \arrow[r] & V \arrow[r] & Y
\end{tikzcd}$$

This comes from the identifications $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$, $\mathcal{O}^h_{X,x} = \mathcal{O}^h_{U,u}$, $\mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$, $\mathcal{O}^h_{Y,y} = \mathcal{O}^h_{V,v}$ see in Lemma 68.11.8 and Properties of Spaces, Lemma 66.22.1 and the functoriality of the (strict) henselization discussed in Algebra, Sections 10.154 and 10.155.

68.12. Points on decent spaces

03IG In this section we prove some properties of points on decent algebraic spaces. The following lemma shows that specialization of points behaves well on decent algebraic spaces. Spaces, Example 65.14.9 shows that this is not true in general.

03K5 **Lemma 68.12.1.** Let S be a scheme. Let X be a decent algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \leadsto u$, then $u = u'$.

Proof. Combine Lemmas 68.5.1 and 68.7.1. □

03IL **Lemma 68.12.2.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x, x' \in |X|$ and assume $x' \leadsto x$, i.e., x is a specialization of x'. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \leadsto u$ with $\varphi(u') = x'$.

Proof. Combine Lemmas 68.5.1 and 68.7.3. □

03K3 **Lemma 68.12.3.** Let S be a scheme. Let X be a decent algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.8.6).

Proof. Let $x_1, x_2 \in |X|$ with $x_1 \leadsto x_2$ and $x_2 \leadsto x_1$. We have to show that $x_1 = x_2$. Pick a scheme U and an étale morphism $U \to X$ such that x_1, x_2 are both in the image of $|U| \to |X|$. By Lemma 68.12.2 we can find a specialization $u_1 \leadsto u_2$ in U mapping to $x_1 \leadsto x_2$. By Lemma 68.12.2 we can find $u_2' \leadsto u_1$ mapping to $x_2 \leadsto x_1$. This means that $u_2' \leadsto u_2$ is a specialization between points of U mapping to the same point of X, namely x_2. This is not possible, unless $u_2' = u_2$, see Lemma 68.12.1. Hence also $u_1 = u_2$ as desired. □

03K6 **Proposition 68.12.4.** Let S be a scheme. Let X be a decent algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.8.6).

Proof. We have seen in Lemma 68.12.3 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 66.12.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \to X$ is a representable morphism of algebraic spaces. Hence Z is a decent algebraic space by Lemma 68.5.3. By Theorem 68.10.2 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 26.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

For decent algebraic spaces dimension works as expected.

0A4J **Lemma 68.12.5.** Let S be a scheme. Dimension as defined in Properties of Spaces, Section 66.9 behaves well on decent algebraic spaces X over S.

(1) If $x \in |X|$, then $\dim_x(|X|) = \dim_x(X)$, and

(2) $\dim(|X|) = \dim(X)$.

Proof. Proof of (1). Choose a scheme U with a point $u \in U$ and an étale morphism $h : U \to X$ mapping u to x. By definition the dimension of X at x is $\dim_u(|U|)$. Thus we may pick U such that $\dim_x(X) = \dim(|U|)$. Let d be an integer. If $\dim(U) \geq d$, then there exists a sequence of nontrivial specializations $u_d \rightsquigarrow \ldots \rightsquigarrow u_0$ in U. Taking the image we find a corresponding sequence $h(u_d) \rightsquigarrow \ldots \rightsquigarrow h(u_0)$ each of which is nontrivial by Lemma 68.12.1. Hence we see that the image of $|U|$ in $|X|$ has dimension at least d. Conversely, suppose that $x_d \rightsquigarrow \ldots \rightsquigarrow x_0$ is a sequence of specializations in $|X|$ with x_0 in the image of $|U| \to |X|$. Then we can lift this to a sequence of specializations in U by Lemma 68.12.2.

Part (2) is an immediate consequence of part (1), Topology, Lemma 5.10.2, and Properties of Spaces, Section 66.9. $\square$

0ABW Lemma 68.12.6. Let S be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Then the dimension of the local ring of Y at y is $\geq$ to the dimension of the local ring of X at x.

Proof. The definition of the dimension of the local ring of a point on an algebraic space is given in Properties of Spaces, Definition 66.10.2. Choose an étale morphism $(V, v) \to (Y, y)$ where V is a scheme. Choose an étale morphism $U \to V \times_Y X$ and a point $u \in U$ mapping to $x \in |X|$ and $v \in V$. Then $U \to V$ is locally quasi-finite and we have to prove that

$$\dim(\mathcal{O}_{V,v}) \geq \dim(\mathcal{O}_{U,u})$$

This is Algebra, Lemma 10.125.4. $\square$

0ED0 Lemma 68.12.7. Let S be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over S. Then $\dim(X) \leq \dim(Y)$.

Proof. This follows from Lemma 68.12.6 and Properties of Spaces, Lemma 66.10.3. $\square$

The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 66.15.3. We will improve this lemma in Lemma 68.14.2.

03IK Lemma 68.12.8. Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \to X$. If X is decent, then $X \cong \mathrm{Spec}(k')$ where k/k' is a finite separable extension.

Proof. The assumption implies that $|X| = \{x\}$ is a singleton. Since X is decent we can find a quasi-compact monomorphism $\mathrm{Spec}(k') \to X$ whose image is x. Then the projection $U = \mathrm{Spec}(k') \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a monomorphism, whence $U = \mathrm{Spec}(k)$, see Schemes, Lemma 26.23.11. Hence the projection $\mathrm{Spec}(k) = U \to \mathrm{Spec}(k')$ is étale and we win. $\square$

68.13. Reduced singleton spaces

06QU A singleton space is an algebraic space X such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 65.14.7. We develop a tiny bit of machinery to be able to talk about these.

06QV **Lemma 68.13.1.** Let S be a scheme. Let Z be an algebraic space over S. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to Z$ where k' is a field is surjective and flat.

Proof. Consider the fibre square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ \mathrm{Spec}(k') \arrow[r] & Z \end{tikzcd}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 67.31.5 that $\mathrm{Spec}(k') \to Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. $\square$

06QW **Lemma 68.13.2.** Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent

(1) Z is reduced and $|Z|$ is a singleton,
(2) there exists a surjective flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field, and
(3) there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.

Proof. Assume (1). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to Z$ is flat (see Morphisms of Spaces, Lemma 67.30.3). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k. Hence $W \times_Z \mathrm{Spec}(k)$ is a disjoint union of spectra of fields (see Remark 68.4.1), in particular reduced. Since $W \times_Z \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent, Lemma 35.19.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 67.23.2 and 67.39.9. It is also flat and surjective by Lemma 68.13.1. Hence (3) holds. $\square$

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

06QX **Lemma 68.13.3.** Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent

(1) Z is reduced, locally Noetherian, and $|Z|$ is a singleton, and

(2) there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.

Proof. Assume (2) holds. By Lemma 68.13.2 we see that Z is reduced and $|Z|$ is a singleton. Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k, hence a disjoint union of spectra of fields (see Remark 68.4.1), hence locally Noetherian. Since $W \times_Z \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent, Lemma 35.16.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 29.21.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 67.28.2 and 67.39.8. It is also flat and surjective by Lemma 68.13.1. Hence (2) holds. $\square$

06QY **Lemma 68.13.4.** Let S be a scheme. Let $Z' \to Z$ be a monomorphism of algebraic spaces over S. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then either Z' is empty or $Z' = Z$.

Proof. We may assume that Z' is nonempty. In this case the fibre product $T = Z' \times_Z \mathrm{Spec}(k)$ is nonempty, see Properties of Spaces, Lemma 66.4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 67.10.8. We conclude that $\mathrm{Spec}(k) \to Z$ factors through Z'. But as $\mathrm{Spec}(k) \to Z$ is surjective, flat and locally of finite presentation, we see that $\mathrm{Spec}(k) \to Z$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$ (see Spaces, Remark 65.5.2) and we conclude that $Z' = Z$. $\square$

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.

06QZ **Lemma 68.13.5.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Then there exists a unique monomorphism $Z \to X$ of algebraic spaces over S such that Z is an algebraic space which satisfies the equivalent conditions of Lemma 68.13.3 and such that the image of $|Z| \to |X|$ is $\{x\}$.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 65.9). Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \mathrm{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U').$$

Because $U' \to U$ is a monomorphism we see that the projections $s', t' : R' \to U'$ factor as a monomorphism followed by an étale morphism. Hence, as U' is a disjoint union of spectra of fields, using Remark 68.4.1, and using Schemes, Lemma 26.23.11 we conclude that R' is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \to U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces, Theorem 65.10.5. As R' is the restriction of R by $U' \to U$ we see $Z \to X$ is a monomorphism by Groupoids, Lemma 39.20.6. Since $Z \to X$ is a monomorphism we see that $|Z| \to |X|$ is injective, see Morphisms of Spaces, Lemma 67.10.9. By Properties of Spaces, Lemma 66.4.3 we see that

$$|U'| = |Z \times_X U'| \to |Z| \times_{|X|} |U'|$$

is surjective which implies (by our choice of U') that $|Z| \to |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., we see that Z satisfies the equivalent conditions of Lemma 68.13.3.

Let us prove uniqueness of $Z \to X$. Suppose that $Z' \to X$ is a second such monomorphism of algebraic spaces. Then the projections

$$Z' \longleftarrow Z' \times_X Z \longrightarrow Z$$

are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 66.4.3. Hence the two projections are isomorphisms by Lemma 68.13.4 and we win. □

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 100.11.8.

06R0 Definition 68.13.6. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The residual space of X at x[2] is the monomorphism $Z_x \to X$ constructed in Lemma 68.13.5.

In particular we know that Z_x is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow Z_x.$$

The residual space is often given by a monomorphism from the spectrum of a field.

0H1R Lemma 68.13.7. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The residual space Z_x of X at x is isomorphic to the spectrum of a field if and only if x can be represented by a monomorphism $\mathrm{Spec}(k) \to X$ where k is a field. If X is decent, this holds for all $x \in |X|$.

Proof. Since $Z_x \to X$ is a monomorphism, if $Z_x = \mathrm{Spec}(k)$ for some field k, then x is represented by the monomorphism $\mathrm{Spec}(k) = Z_x \to X$. Conversely, if $\mathrm{Spec}(k) \to X$ is a monomorphism which represents x, then $Z_x \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a monomorphism whose source is nonempty by Properties of Spaces, Lemma 66.4.3. Hence $Z_x \times_X \mathrm{Spec}(k) = \mathrm{Spec}(k)$ by Morphisms of Spaces, Lemma 67.10.8. Hence we get a monomorphism $\mathrm{Spec}(k) \to Z_x$. This is an isomorphism by Lemma 68.13.4. The final statement follows from Lemma 68.11.1. □

The residual space is a regular algebraic space by the following lemma.

06R1 Lemma 68.13.8. A reduced, locally Noetherian singleton algebraic space Z is regular.

[2]This is nonstandard notation.

Proof. Let Z be a reduced, locally Noetherian singleton algebraic space over a scheme S. Let $W \to Z$ be a surjective étale morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective, flat, and locally of finite presentation (see Lemma 68.13.3). The scheme $T = W \times_Z \mathrm{Spec}(k)$ is étale over k in particular regular, see Remark 68.4.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent, Lemma 35.19.2. By definition this means that Z is regular. □

0H1S **Lemma 68.13.9.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point. Assume

(1) $|f|(|Y|)$ is contained in $\{x\} \subset |X|$,
(2) Y is reduced, and
(3) X is locally Noetherian.

Then f factors through the residual space Z_x of X at x.

Proof. Preliminary remark: since $Z_x \to X$ is a monomorphism, it suffices to find a surjective étale morphism $Y' \to Y$ such that $Y' \to X$ factors through Z_x. A remark here is that Y' is reduced as well.

Let U be an affine scheme and let $U \to X$ be an étale morphism such that x is in the image of $|U| \to |X|$. Since X is locally Noetherian, U is a Noetherian affine scheme. By assumption (1) we see that $Y' = U \times_X Y \to Y$ is surjective as well as étale. Denote $E \subset |U|$ the set of points mapping to x. There are no nontrivial specializations between the elements of E, see Lemma 68.7.2. The morphism $Y' \to U$ maps $|Y'|$ into E. By our construction of Z_x in the proof of Lemma 68.13.5 we know that $\coprod_{u \in E} u \to X$ factors through Z_x. Hence it suffices to prove that $Y' \to U$ factors through $\coprod_{u \in E} u \to X$. After replacing Y' by an étale covering by a scheme (which we are allowed by our preliminary remark), this follows from Morphisms, Lemma 29.58.2. □

0H1T **Lemma 68.13.10.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point. Assume

(1) $|f|(|Y|)$ is contained in $\{x\} \subset |X|$,
(2) Y is reduced, and
(3) x can be represented by a quasi-compact monomorphism $x : \mathrm{Spec}(k) \to X$ where k is a field (for example if X is decent).

Then f factors through the residual space $Z_x = \mathrm{Spec}(k)$ of X at x.

Proof. By Lemma 68.13.7 we have $Z_x = \mathrm{Spec}(k)$.

Preliminary remark: since $\mathrm{Spec}(k) \to X$ is a monomorphism, it suffices to find a surjective étale morphism $Y' \to Y$ such that $Y' \to X$ factors through Z_x. A remark here is that Y' is reduced as well.

After replacing X by a quasi-compact open neighbourhood of x, we may assume X quasi-compact. By Lemma 68.8.3, x is a point of $T \subset U \subset X$ where $T \to U$ (resp. $U \to X$) is a closed (resp. open) immersion, and T is a scheme. By Properties of Spaces, Lemma 66.4.9, f factors through U, so we may assume $U = X$. Then f factors through T because Y is reduced, see Properties of Spaces, Lemma 66.12.4. So we may assume that $X = T$ is a scheme. By our preliminary remark we may assume Y is a scheme too. This reduces us to Morphisms, Lemma 29.58.1. □

0H2Y Example 68.13.11. Here is a counter example to Lemmas 68.13.9 and 68.13.10 in case X is neither locally Noetherian nor decent. Let k be a field. Let G be an infinite profinite group. Let Y be G viewed as a zero-dimensional affine k-group scheme, i.e., $Y = \text{Spec}(\text{locally constant maps } G \to k)$. Let Γ be G viewed as a discrete k-group scheme, acting on X by translations. Put $X = Y/\Gamma$. This is a one-point algebraic space, with projection $q : Y \to X$. Let $e \in G$ be the origin (any element would do), and view it as a k-point of Y. We get a k-point $x : \text{Spec}(k) \to X$ which is a monomorphism since it is a section of $X \to \text{Spec}(k)$. We claim that (although Y is affine and reduced and $|X| = \{x\}$), the morphism q does not factor through any morphism $\text{Spec}(K) \to X$, where K is a field. Otherwise it would factor through x by Properties of Spaces, Lemma 66.4.11. Now the pullback of q by x is $\Gamma \to \text{Spec}(k)$, with the projection $\Gamma \to Y$ being the orbit map $g \mapsto g \cdot e$. The latter has no section, whence the claim.

0H1U Lemma 68.13.12. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ with residual space $Z_x \subset X$. Assume X is locally Noetherian. Then x is a closed point of $|X|$ if and only if the morphism $Z_x \to X$ is a closed immersion.

Proof. If $Z_x \to X$ is a closed immersion, then x is a closed point of $|X|$, see Morphisms of Spaces, Lemma 67.12.3. Conversely, assume x is a closed point of $|X|$. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 66.12.3). Then Z is locally Noetherian by Morphisms of Spaces, Lemmas 67.23.7 and 67.23.5. Since also Z is reduced and $|Z| = \{x\}$ it $Z = Z_x$ is the residual space by definition. $\square$

68.14. Decent spaces

047Y In this section we collect some useful facts on decent spaces.

0BB6 Lemma 68.14.1. Any locally Noetherian decent algebraic space is quasi-separated.

Proof. Namely, let X be an algebraic space (over some base scheme, for example over $\mathbf{Z}$) which is decent and locally Noetherian. Let $U \to X$ and $V \to X$ be étale morphisms with U and V affine schemes. We have to show that $W = U \times_X V$ is quasi-compact (Properties of Spaces, Lemma 66.3.3). Since X is locally Noetherian, the schemes U, V are Noetherian and W is locally Noetherian. Since X is decent, the fibres of the morphism $W \to U$ are finite. Namely, we can represent any $x \in |X|$ by a quasi-compact monomorphism $\text{Spec}(k) \to X$. Then U_k and V_k are finite disjoint unions of spectra of finite separable extensions of k (Remark 68.4.1) and we see that $W_k = U_k \times_{\text{Spec}(k)} V_k$ is finite. Let n be the maximum degree of a fibre of $W \to U$ at a generic point of an irreducible component of U. Consider the stratification

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

associated to $W \to U$ in More on Morphisms, Lemma 37.45.5. By our choice of n above we conclude that U_{n+1} is empty. Hence we see that the fibres of $W \to U$ are universally bounded. Then we can apply More on Morphisms, Lemma 37.45.3 to find a stratification

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = U$$

by closed subsets such that with $S_r = Z_r \setminus Z_{r-1}$ the morphism $W \times_U S_r \to S_r$ is finite locally free. Since U is Noetherian, the schemes S_r are Noetherian, whence the schemes $W \times_U S_r$ are Noetherian, whence $W = \coprod W \times_U S_r$ is quasi-compact as desired. $\square$

047Z Lemma 68.14.2. Let S be a scheme. Let X be a decent algebraic space over S.

(1) If $|X|$ is a singleton then X is a scheme.
(2) If $|X|$ is a singleton and X is reduced, then $X \cong \mathrm{Spec}(k)$ for some field k.

Proof. Assume $|X|$ is a singleton. It follows immediately from Theorem 68.10.2 that X is a scheme, but we can also argue directly as follows. Choose an affine scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$. Then U and R have finitely many points by Lemma 68.4.5 (and the definition of a decent space). All of these points are closed in U and R by Lemma 68.12.1. It follows that U and R are affine schemes. We may shrink U to a singleton space. Then U is the spectrum of a henselian local ring, see Algebra, Lemma 10.153.10. The projections $R \to U$ are étale, hence finite étale because U is the spectrum of a 0-dimensional henselian local ring, see Algebra, Lemma 10.153.3. It follows that X is a scheme by Groupoids, Proposition 39.23.9.

Part (2) follows from (1) and the fact that a reduced singleton scheme is the spectrum of a field. $\square$

049D Remark 68.14.3. We will see in Limits of Spaces, Lemma 70.15.3 that an algebraic space whose reduction is a scheme is a scheme.

07U5 Lemma 68.14.4. Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ is a closed point and that $\kappa(s) \subset k$ is algebraic. Then the image x of $\mathrm{Spec}(k) \to X$ is a closed point of $|X|$.

Proof. Suppose that $x \leadsto x'$ for some $x' \in |X|$. Choose an étale morphism $U \to X$ where U is a scheme and a point $u' \in U'$ mapping to x'. Choose a specialization $u \leadsto u'$ in U with u mapping to x in X, see Lemma 68.12.2. Then u is the image of a point w of the scheme $W = \mathrm{Spec}(k) \times_X U$. Since the projection $W \to \mathrm{Spec}(k)$ is étale we see that $\kappa(w) \supset k$ is finite. Hence $\kappa(w) \supset \kappa(s)$ is algebraic. Hence $\kappa(u) \supset \kappa(s)$ is algebraic. Thus u is a closed point of U by Morphisms, Lemma 29.20.2. Thus $u = u'$, whence $x = x'$. $\square$

08AL Lemma 68.14.5. Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ is a closed point and that the field extension $k/\kappa(s)$ is finite. Then $\mathrm{Spec}(k) \to X$ is a finite morphism. If $\kappa(s) = k$ then $\mathrm{Spec}(k) \to X$ is a closed immersion.

Proof. By Lemma 68.14.4 the image point $x \in |X|$ is closed. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 66.12.3). Note that Z is a decent algebraic space by Lemma 68.6.5. By Lemma 68.14.2 we see that $Z = \mathrm{Spec}(k')$ for some field k'. Of course $k \supset k' \supset \kappa(s)$. Then $\mathrm{Spec}(k) \to Z$ is a finite morphism of schemes and $Z \to X$ is a finite morphism as it is a closed immersion. Hence $\mathrm{Spec}(k) \to X$ is finite (Morphisms of Spaces, Lemma 67.45.4). If $k = \kappa(s)$, then $\mathrm{Spec}(k) = Z$ and $\mathrm{Spec}(k) \to X$ is a closed immersion. $\square$

0AHB **Lemma 68.14.6.** *Let S be a scheme. Suppose X is a decent algebraic space over S. Let $x \in |X|$ be a closed point. Then x can be represented by a closed immersion $i : \mathrm{Spec}(k) \to X$ from the spectrum of a field.*

Proof. We know that x can be represented by a quasi-compact monomorphism $i : \mathrm{Spec}(k) \to X$ where k is a field (Definition 68.6.1). Let $U \to X$ be an étale morphism where U is an affine scheme. As x is closed and X decent, the fibre F of $|U| \to |X|$ over x consists of closed points (Lemma 68.12.1). As i is a monomorphism, so is $U_k = U \times_X \mathrm{Spec}(k) \to U$. In particular, the map $|U_k| \to F$ is injective. Since U_k is quasi-compact and étale over a field, we see that U_k is a finite disjoint union of spectra of fields (Remark 68.4.1). Say $U_k = \mathrm{Spec}(k_1) \amalg \ldots \amalg \mathrm{Spec}(k_r)$. Since $\mathrm{Spec}(k_i) \to U$ is a monomorphism, we see that its image u_i has residue field $\kappa(u_i) = k_i$. Since $u_i \in F$ is a closed point we conclude the morphism $\mathrm{Spec}(k_i) \to U$ is a closed immersion. As the u_i are pairwise distinct, $U_k \to U$ is a closed immersion. Hence i is a closed immersion (Morphisms of Spaces, Lemma 67.12.1). This finishes the proof. $\square$

68.15. Locally separated spaces

088H It turns out that a locally separated algebraic space is decent.

088I **Lemma 68.15.1.** *Let A be a ring. Let k be a field. Let $\mathfrak p_n$, $n \geq 1$ be a sequence of pairwise distinct primes of A. Moreover, for each n let $k \to \kappa(\mathfrak p_n)$ be an embedding. Then the closure of the image of*

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(\kappa(\mathfrak p_n) \otimes_k \kappa(\mathfrak p_m)) \longrightarrow \mathrm{Spec}(A \otimes A)$$

meets the diagonal.

Proof. Set $k_n = \kappa(\mathfrak p_n)$. We may assume that $A = \prod k_n$. Denote $x_n = \mathrm{Spec}(k_n)$ the open and closed point corresponding to $A \to k_n$. Then $\mathrm{Spec}(A) = Z \amalg \{x_n\}$ where Z is a nonempty closed subset. Namely, $Z = V(e_n; n \geq 1)$ where e_n is the idempotent of A corresponding to the factor k_n and Z is nonempty as the ideal generated by the e_n is not equal to A. We will show that the closure of the image contains $\Delta(Z)$. The kernel of the map

$$(\prod k_n) \otimes_k (\prod k_m) \longrightarrow \prod\nolimits_{n \neq m} k_n \otimes_k k_m$$

is the ideal generated by $e_n \otimes e_n$, $n \geq 1$. Hence the closure of the image of the map on spectra is $V(e_n \otimes e_n; n \geq 1)$ whose intersection with $\Delta(\mathrm{Spec}(A))$ is $\Delta(Z)$. Thus it suffices to show that

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(k_n \otimes_k k_m) \longrightarrow \mathrm{Spec}(\prod\nolimits_{n \neq m} k_n \otimes_k k_m)$$

has dense image. This follows as the family of ring maps $\prod_{n \neq m} k_n \otimes_k k_m \to k_n \otimes_k k_m$ is jointly injective. $\square$

088J Lemma 68.15.2 (David Rydh). A locally separated algebraic space is decent.

Proof. Let S be a scheme and let X be a locally separated algebraic space over S. We may assume $S = \mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 66.3.1. Unadorned fibre products will be over $\mathbf{Z}$. Let $x \in |X|$. Choose a scheme U, an étale morphism $U \to X$, and a point $u \in U$ mapping to x in $|X|$. As usual we identify $u = \mathrm{Spec}(\kappa(u))$. As X is locally separated the morphism

$$u \times_X u \to u \times u$$

is an immersion (Morphisms of Spaces, Lemma 67.4.5). Hence More on Groupoids, Lemma 40.11.5 tells us that it is a closed immersion (use Schemes, Lemma 26.10.4). As $u \times_X u \to u \times_X U$ is a monomorphism (base change of $u \to U$) and as $u \times_X U \to u$ is étale we conclude that $u \times_X u$ is a disjoint union of spectra of fields (see Remark 68.4.1 and Schemes, Lemma 26.23.11). Since it is also closed in the affine scheme $u \times u$ we conclude $u \times_X u$ is a finite disjoint union of spectra of fields. Thus x can be represented by a monomorphism $\mathrm{Spec}(k) \to X$ where k is a field, see Lemma 68.4.3.

Next, let $U = \mathrm{Spec}(A)$ be an affine scheme and let $U \to X$ be an étale morphism. To finish the proof it suffices to show that $F = U \times_X \mathrm{Spec}(k)$ is finite. Write $F = \coprod_{i \in I} \mathrm{Spec}(k_i)$ as the disjoint union of finite separable extensions of k. We have to show that I is finite. Set $R = U \times_X U$. As X is locally separated, the morphism $j : R \to U \times U$ is an immersion. Let $U' \subset U \times U$ be an open such that j factors through a closed immersion $j' : R \to U'$. Let $e : U \to R$ be the diagonal map. Using that e is a morphism between schemes étale over U such that $\Delta = j \circ e$ is a closed immersion, we conclude that $R = e(U) \amalg W$ for some open and closed subscheme $W \subset R$. Since j' is a closed immersion we conclude that $j'(W) \subset U'$ is closed and disjoint from $j'(e(U))$. Therefore $\overline{j(W)} \cap \Delta(U) = \emptyset$ in $U \times U$. Note that W contains $\mathrm{Spec}(k_i \otimes_k k_{i'})$ for all $i \neq i'$, $i, i' \in I$. By Lemma 68.15.1 we conclude that I is finite as desired. $\square$

68.16. Valuative criterion

06NP For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

03KJ Proposition 68.16.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact, and X is decent. Then f is universally closed if and only if the existence part of the valuative criterion holds.

Proof. In Morphisms of Spaces, Lemma 67.42.1 we have seen one of the implications. To prove the other, assume that f is universally closed. Let

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

be a diagram as in Morphisms of Spaces, Definition 67.41.1. Let $X_A = \mathrm{Spec}(A) \times_Y X$, so that we have

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[rd] & X_A \arrow[d] \\ & \mathrm{Spec}(A) \end{tikzcd}$$

By Morphisms of Spaces, Lemma 67.8.4 we see that $X_A \to \mathrm{Spec}(A)$ is quasi-compact. Since $X_A \to X$ is representable, we see that X_A is decent also, see Lemma 68.5.3. Moreover, as f is universally closed, we see that $X_A \to \mathrm{Spec}(A)$ is universally closed. Hence we may and do replace X by X_A and Y by $\mathrm{Spec}(A)$.

Let $x' \in |X|$ be the equivalence class of $\mathrm{Spec}(K) \to X$. Let $y \in |Y| = |\mathrm{Spec}(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $\mathrm{Spec}(A)$. Since f is universally closed we see that $f(\overline{\{x'\}})$ contains $\overline{\{y'\}}$, and hence contains y. Let $x \in \overline{\{x'\}}$ be a point such that $f(x) = y$. Let U be a scheme, and $\varphi : U \to X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 68.7.3 and our assumption that X is decent there exists a specialization $u' \rightsquigarrow u$ on U with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$ such that

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] \arrow[rd] & X \arrow[d] \\ & \mathrm{Spec}(A) \end{tikzcd}$$

is commutative. This gives the following commutative diagram of rings

$$\begin{tikzcd} K' & \mathcal{O}_{U,u} \arrow[l] \\ K \arrow[u] & \\ & A \arrow[lu] \arrow[uu] \end{tikzcd}$$

By Algebra, Lemma 10.50.2 we can find a valuation ring $A' \subset K'$ dominating the image of $\mathcal{O}_{U,u}$ in K'. Since by construction $\mathcal{O}_{U,u}$ dominates A we see that A' dominates A also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 67.41.1 and the proposition is proved. □

68.17. Relative conditions

03KW This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

03KZ Definition 68.17.1. Let S be a scheme. We say an algebraic space X over S has property (β) if X has the corresponding property of Lemma 68.5.1. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f has property (β) if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ has property (β).
(2) We say f is decent if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a decent algebraic space.
(3) We say f is reasonable if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a reasonable algebraic space.
(4) We say f is very reasonable if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a very reasonable algebraic space.

We refer to Remark 68.17.10 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.

03M5 Lemma 68.17.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We have the following implications among the conditions on f:

$$\begin{tikzcd} \text{representable} \arrow[rd, Rightarrow] & & & & \\ & \text{very reasonable} \arrow[r, Rightarrow] & \text{reasonable} \arrow[r, Rightarrow] & \text{decent} \arrow[r, Rightarrow] & (\beta) \\ \text{quasi-separated} \arrow[ru, Rightarrow] & & & & \end{tikzcd}$$

Proof. This is clear from the definitions, Lemma 68.5.1 and Morphisms of Spaces, Lemma 67.4.12. $\square$

Here is another sanity check.

0ABX Lemma 68.17.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If X is decent (resp. is reasonable, resp. has property (β) of Lemma 68.5.1), then f is decent (resp. reasonable, resp. has property (β)).

Proof. Let T be a scheme and let $T \to Y$ be a morphism. Then $T \to Y$ is representable, hence the base change $T \times_Y X \to X$ is representable. Hence if X is decent (or reasonable), then so is $T \times_Y X$, see Lemma 68.6.5. Similarly, for property (β), see Lemma 68.5.3. $\square$

03L0 Lemma 68.17.4. Having property (β), being decent, or being reasonable is preserved under arbitrary base change.

Proof. This is immediate from the definition. $\square$

0ABY Lemma 68.17.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\omega \in \{\beta, decent, reasonable\}$. Suppose that Y has property (ω) and $f : X \to Y$ has (ω). Then X has (ω).

Proof. Let us prove the lemma in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into X. Let $y = f(x) \in |Y|$. By assumption there exists a field k and a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. Then x corresponds to a point x' of $\mathrm{Spec}(k) \times_Y X$. By assumption x' is represented by a monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$. Clearly the composition $\mathrm{Spec}(k') \to X$ is a monomorphism representing x.

Let us prove the lemma in case $\omega = decent$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "x"'] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "y"] & Y \end{tikzcd}$$

whose horizontal arrows monomorphisms. As Y is decent the morphism y is quasi-compact. As f is decent the algebraic space $\mathrm{Spec}(k) \times_Y X$ is decent. Hence the monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : \mathrm{Spec}(k') \to X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 67.8.4 and 67.8.5). As the point x was arbitrary this implies X is decent.

Let us prove the lemma in case $\omega = reasonable$. Choose $V \to Y$ étale with V an affine scheme. Choose $U \to V \times_Y X$ étale with U an affine scheme. By assumption $V \to Y$ has universally bounded fibres. By Lemma 68.3.3 the morphism $V \times_Y X \to X$ has universally bounded fibres. By assumption on f we see that $U \to V \times_Y X$ has universally bounded fibres. By Lemma 68.3.2 the composition $U \to X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \to X$ from schemes with universally bounded fibres, and we conclude that X is reasonable. ☐

03L1 Lemma 68.17.6. Having property (β), being decent, or being reasonable is preserved under compositions.

Proof. Let $\omega \in \{\beta, decent, reasonable\}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Assume f and g both have property (ω). Then we have to show that for any scheme T and morphism $T \to Z$ the space $T \times_Z X$ has (ω). By Lemma 68.17.4 this reduces us to the following claim: Suppose that Y is an algebraic space having property (ω), and that $f : X \to Y$ is a morphism with (ω). Then X has (ω). This is the content of Lemma 68.17.5. ☐

0ABZ Lemma 68.17.7. Let S be a scheme. Let $f : X \to Y$, $g : Z \to Y$ be morphisms of algebraic spaces over S. If X and Z are decent (resp. reasonable, resp. have property (β) of Lemma 68.5.1), then so does $X \times_Y Z$.

Proof. Namely, by Lemma 68.17.3 the morphism $X \to Y$ has the property. Then the base change $X \times_Y Z \to Z$ has the property by Lemma 68.17.4. And finally this implies $X \times_Y Z$ has the property by Lemma 68.17.5. ☐

03L2 Lemma 68.17.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable\}$. Assume

(1) f is quasi-compact,
(2) f is étale,
(3) $|f| : |X| \to |Y|$ is surjective, and
(4) the algebraic space X has property $\mathcal{P}$.

Then Y has property $\mathcal{P}$.

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that y can be represented by a monomorphism from a field. Choose a point

$x \in |X|$ with $f(x) = y$. By assumption we may represent x by a monomorphism $\mathrm{Spec}(k) \to X$, with k a field. By Lemma 68.4.3 it suffices to show that the projections $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact. We can factor the first projection as

$$\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_Y X \longrightarrow \mathrm{Spec}(k)$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 66.16.8 we see that $\mathrm{Spec}(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of k. By Schemes, Lemma 26.23.11 we see that the first arrow identifies $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k)$ with a finite disjoint union of spectra of finite separable field extensions of k. Hence the projection morphism is étale and quasi-compact.

Let us prove this in case $\mathcal{P} = \textit{decent}$. We have already seen in the first paragraph of the proof that this implies that every $y \in |Y|$ can be represented by a monomorphism $y : \mathrm{Spec}(k) \to Y$. Pick such a y. Pick an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |Y|$ contains y. By Lemma 68.4.5 it suffices to show that U_y is a finite scheme over k. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a quasi-compact étale algebraic space over k. Hence by Properties of Spaces, Lemma 66.16.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of k. Say $X_y = \{x_1, \ldots, x_n\}$ so x_i is given by $x_i : \mathrm{Spec}(k_i) \to X$ with $[k_i : k] < \infty$. By assumption X is decent, so the schemes $U_{x_i} = \mathrm{Spec}(k_i) \times_X U$ are finite over k_i. Finally, we note that $U_y = \coprod U_{x_i}$ as a scheme and we conclude that U_y is finite over k as desired.

Let us prove this in case $\mathcal{P} = \textit{reasonable}$. Pick an affine scheme V and an étale morphism $V \to Y$. We have the show the fibres of $V \to Y$ are universally bounded. The algebraic space $V \times_Y X$ is quasi-compact. Thus we can find an affine scheme W and a surjective étale morphism $W \to V \times_Y X$, see Properties of Spaces, Lemma 66.6.3. Here is a picture (solid diagram)

$$\begin{tikzcd} W \arrow[r] \arrow[rd] & V \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"'] & \mathrm{Spec}(k) \arrow[l, dotted, "x"'] \arrow[ld, dotted, "y"] \\ & V \arrow[r] & Y & \end{tikzcd}$$

The morphism $W \to X$ is universally bounded by our assumption that the space X is reasonable. Let n be an integer bounding the degrees of the fibres of $W \to X$. We claim that the same integer works for bounding the fibres of $V \to Y$. Namely, suppose $y \in |Y|$ is a point. Then there exists a $x \in |X|$ with $f(x) = y$ (see above). This means we can find a field k and morphisms x, y given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

$$\mathrm{Spec}(k) \times_{x,X} W \to \mathrm{Spec}(k) \times_{x,X} (V \times_Y X) = \mathrm{Spec}(k) \times_{y,Y} V$$

which shows that the degree of $\mathrm{Spec}(k) \times_{y,Y} V$ over k is less than or equal to the degree of $\mathrm{Spec}(k) \times_{x,X} W$ over k, i.e., $\leq n$, and we win. (This last part of the argument is the same as the argument in the proof of Lemma 68.3.4. Unfortunately that lemma is not general enough because it only applies to representable morphisms.) $\square$

03L3 **Lemma 68.17.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable, very\ reasonable\}$. The following are equivalent

(1) f is $\mathcal{P}$,
(2) for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,
(3) for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$, and
(4) there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.

If $\mathcal{P} \in \{(\beta), decent, reasonable\}$, then this is also equivalent to

(5) there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. The implication (3) $\Rightarrow$ (1) can be seen as follows. Let $Z \to Y$ be a morphism whose source is a scheme over S. Consider the algebraic space $Z \times_Y X$. If we assume (3), then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property $\mathcal{P}$. Hence by Lemma 68.5.2 the space $Z \times_Y X$ has property $\mathcal{P}$, i.e., (1) holds. A similar argument (omitted) shows that (4) implies (1).

The implication (1) $\Rightarrow$ (5) is trivial. Let $V \to Y$ be an étale morphism from a scheme as in (5). Let Z be an affine scheme, and let $Z \to Y$ be a morphism. Consider the diagram

$$\begin{tikzcd} Z \times_Y V \arrow[r, "q"'] \arrow[d, "p"'] & V \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

Since p is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$\begin{tikzcd} V \times_Y X \arrow[d] & (\coprod W_i) \times_Y X \arrow[l] \arrow[r] \arrow[d] & Z \times_Y X \arrow[d] \\ V & \coprod W_i \arrow[l] \arrow[r] & Z \end{tikzcd}$$

We know $V \times_Y X$ has property $\mathcal{P}$. By Lemma 68.5.3 we see that $(\coprod W_i) \times_Y X$ has property $\mathcal{P}$. Note that the morphism $(\coprod W_i) \times_Y X \to Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \to Z$. Hence by Lemma 68.17.8 we conclude that $Z \times_Y X$ has property $\mathcal{P}$. $\square$

03L4 **Remark 68.17.10.** An informal description of the properties (β), decent, reasonable, very reasonable was given in Section 68.6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger and technically quite easy to work with.

Here is a lemma we promised earlier which uses decent morphisms.

03M6 **Lemma 68.17.11.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and decent. (For example if f is representable, or quasi-separated, see Lemma 68.17.2.) Then f is universally closed if and only if the existence part of the valuative criterion holds.

Proof. In Morphisms of Spaces, Lemma 67.42.1 we proved that any quasi-compact morphism which satisfies the existence part of the valuative criterion is universally closed. To prove the other, assume that f is universally closed. In the proof of Proposition 68.16.1 we have seen that it suffices to show, for any valuation ring A, and any morphism $\mathrm{Spec}(A) \to Y$, that the base change $f_A : X_A \to \mathrm{Spec}(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space X_A has property (γ) and hence Proposition 68.16.1 applies to the morphism f_A and we win. $\square$

68.18. Points of fibres

0AC0 Let S be a scheme. Consider a cartesian diagram

0AC1 (68.18.0.1)
$$\begin{tikzcd} W \arrow[r, "q"'] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of algebraic spaces over S. Let $x \in |X|$ and $z \in |Z|$ be points mapping to the same point $y \in |Y|$. We may ask: When is the set

0AC2 (68.18.0.2)
$$F_{x,z} = \{w \in |W| \text{ such that } p(w) = x \text{ and } q(w) = z\}$$

finite?

0AC3 **Example 68.18.1.** If X, Y, Z are schemes, then the set $F_{x,z}$ is equal to the spectrum of $\kappa(x) \otimes_{\kappa(y)} \kappa(z)$ (Schemes, Lemma 26.17.5). Thus we obtain a finite set if either $\kappa(y) \subset \kappa(x)$ is finite or if $\kappa(y) \subset \kappa(z)$ is finite. In particular, this is always the case if g is quasi-finite at z (Morphisms, Lemma 29.20.5).

0AC4 **Example 68.18.2.** Let K be a characteristic 0 field endowed with an automorphism σ of infinite order. Set $Y = \mathrm{Spec}(K)/\mathbf{Z}$ and $X = \mathbf{A}^1_K/\mathbf{Z}$ where $\mathbf{Z}$ acts on K via σ and on $\mathbf{A}^1_K = \mathrm{Spec}(K[t])$ via $t \mapsto t+1$. Let $Z = \mathrm{Spec}(K)$. Then $W = \mathbf{A}^1_K$. Picture

$$\begin{tikzcd} \mathbf{A}^1_K \arrow[r, "q"'] \arrow[d, "p"'] & \mathrm{Spec}(K) \arrow[d, "g"] \\ \mathbf{A}^1_K/\mathbf{Z} \arrow[r, "f"] & \mathrm{Spec}(K)/\mathbf{Z} \end{tikzcd}$$

Take x corresponding to $t = 0$ and z the unique point of $\mathrm{Spec}(K)$. Then we see that $F_{x,z} = \mathbf{Z}$ as a set.

0AC5 **Lemma 68.18.3.** In the situation of (68.18.0.1) if $Z' \to Z$ is a morphism and $z' \in |Z'|$ maps to z, then the induced map $F_{x,z'} \to F_{x,z}$ is surjective.

Proof. Set $W' = X \times_Y Z' = W \times_Z Z'$. Then $|W'| \to |W| \times_{|Z|} |Z'|$ is surjective by Properties of Spaces, Lemma 66.4.3. Hence the surjectivity of $F_{x,z'} \to F_{x,z}$. $\square$

0AC6 **Lemma 68.18.4.** In diagram (68.18.0.1) the set (68.18.0.2) is finite if f is of finite type and f is quasi-finite at x.

Proof. The morphism q is quasi-finite at every $w \in F_{x,z}$, see Morphisms of Spaces, Lemma 67.27.2. Hence the lemma follows from Morphisms of Spaces, Lemma 67.27.9. $\square$

0AC7 **Lemma 68.18.5.** In diagram (68.18.0.1) the set (68.18.0.2) is finite if y can be represented by a monomorphism $\mathrm{Spec}(k) \to Y$ where k is a field and g is quasi-finite at z. (Special case: Y is decent and g is étale.)

Proof. By Lemma 68.18.3 applied twice we may replace Z by $Z_k = \mathrm{Spec}(k) \times_Y Z$ and X by $X_k = \mathrm{Spec}(k) \times_Y X$. We may and do replace Y by $\mathrm{Spec}(k)$ as well. Note that $Z_k \to \mathrm{Spec}(k)$ is quasi-finite at z by Morphisms of Spaces, Lemma 67.27.2. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Z_k$ mapping v to z. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to X_k$ mapping u to x. Again by Lemma 68.18.3 it suffices to show $F_{u,v}$ is finite for the diagram

$$\begin{tikzcd} U \times_{\mathrm{Spec}(k)} V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

The morphism $V \to \mathrm{Spec}(k)$ is quasi-finite at v (follows from the general discussion in Morphisms of Spaces, Section 67.22 and the definition of being quasi-finite at a point). At this point the finiteness follows from Example 68.18.1. The parenthetical remark of the statement of the lemma follows from the fact that on decent spaces points are represented by monomorphisms from fields and from the fact that an étale morphism of algebraic spaces is locally quasi-finite. $\square$

0AC8 **Lemma 68.18.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$ and assume that y is represented by a quasi-compact monomorphism $\mathrm{Spec}(k) \to Y$. Then $|X_k| \to |X|$ is a homeomorphism onto $f^{-1}(\{y\}) \subset |X|$ with induced topology.

Proof. We will use Properties of Spaces, Lemma 66.16.7 and Morphisms of Spaces, Lemma 67.10.9 without further mention. Let $V \to Y$ be an étale morphism with V affine such that there exists a $v \in V$ mapping to y. Since $\mathrm{Spec}(k) \to Y$ is quasi-compact there are a finite number of points of V mapping to y (Lemma 68.4.5). After shrinking V we may assume v is the only one. Choose a scheme U and a surjective étale morphism $U \to X$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[d] & U_V \arrow[l] \arrow[d] & U_v \arrow[l] \arrow[d] \\ X \arrow[d] & X_V \arrow[l] \arrow[d] & X_v \arrow[l] \arrow[d] \\ Y & V \arrow[l] & v \arrow[l] \end{tikzcd}$$

Since $U_v \to U_V$ identifies U_v with a subset of U_V with the induced topology (Schemes, Lemma 26.18.5), and since $|U_V| \to |X_V|$ and $|U_v| \to |X_v|$ are surjective and open, we see that $|X_v| \to |X_V|$ is a homeomorphism onto its image (with induced topology). On the other hand, the inverse image of $f^{-1}(\{y\})$ under the open map $|X_V| \to |X|$ is equal to $|X_v|$. We conclude that $|X_v| \to f^{-1}(\{y\})$ is open.

The morphism $X_v \to X$ factors through X_k and $|X_k| \to |X|$ is injective with image $f^{-1}(\{y\})$ by Properties of Spaces, Lemma 66.4.3. Using $|X_v| \to |X_k| \to f^{-1}(\{y\})$ the lemma follows because $X_v \to X_k$ is surjective. $\square$

0AC9 Lemma 68.18.7. Let X be an algebraic space locally of finite type over a field k. Let $x \in |X|$. Consider the conditions

(1) $\dim_x(|X|) = 0$,
(2) x is closed in $|X|$ and if $x' \rightsquigarrow x$ in $|X|$ then $x' = x$,
(3) x is an isolated point of $|X|$,
(4) $\dim_x(X) = 0$,
(5) $X \to \mathrm{Spec}(k)$ is quasi-finite at x.

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) is equivalent to the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemmas 67.34.7 and 67.34.8.

Let $U \to X$ be an étale morphism where U is an affine scheme and let $u \in U$ be a point mapping to x. Moreover, if x is a closed point, e.g., in case (2) or (3), then we may and do assume that u is a closed point. Observe that $\dim_u(U) = \dim_x(X)$ by definition and that this is equal to $\dim(\mathcal{O}_{U,u})$ if u is a closed point, see Algebra, Lemma 10.114.6.

If $\dim_x(X) > 0$ and u is closed, by the arguments above we can choose a nontrivial specialization $u' \rightsquigarrow u$ in U. Then the transcendence degree of $\kappa(u')$ over k exceeds the transcendence degree of $\kappa(u)$ over k. It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k are well defined, see Morphisms of Spaces, Definition 67.33.1. This applies in particular in cases (2) and (3) and we conclude that (2) and (3) imply (4).

Conversely, if $X \to \mathrm{Spec}(k)$ is locally quasi-finite at x, then $U \to \mathrm{Spec}(k)$ is locally quasi-finite at u, hence u is an isolated point of U (Morphisms, Lemma 29.20.6). It follows that (5) implies (2) and (3) as $|U| \to |X|$ is continuous and open.

Assume X is decent and (1) holds. Then $\dim_x(X) = \dim_x(|X|)$ by Lemma 68.12.5 and the proof is complete. $\square$

0ACA Lemma 68.18.8. Let X be an algebraic space locally of finite type over a field k. Consider the conditions

(1) $|X|$ is a finite set,
(2) $|X|$ is a discrete space,
(3) $\dim(|X|) = 0$,
(4) $\dim(X) = 0$,
(5) $X \to \mathrm{Spec}(k)$ is locally quasi-finite,

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) implies the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemma 67.34.7.

Let $U \to X$ be a surjective étale morphism where U is a scheme.

If $\dim(U) > 0$, then choose a nontrivial specialization $u \rightsquigarrow u'$ in U and the transcendence degree of $\kappa(u)$ over k exceeds the transcendence degree of $\kappa(u')$ over k.

It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k is well defined, see Morphisms of Spaces, Definition 67.33.1. We conclude that (2) and (3) imply (4).

Conversely, if $X \to \mathrm{Spec}(k)$ is locally quasi-finite, then U is locally Noetherian (Morphisms, Lemma 29.15.6) of dimension 0 (Morphisms, Lemma 29.29.5) and hence is a disjoint union of spectra of Artinian local rings (Properties, Lemma 28.10.5). Hence U is a discrete topological space, and since $|U| \to |X|$ is continuous and open, the same is true for $|X|$. In other words, (4) implies (2) and (3).

Assume X is decent and (1) holds. Then we may choose U above to be affine. The fibres of $|U| \to |X|$ are finite (this is a part of the defining property of decent spaces). Hence U is a finite type scheme over k with finitely many points. Hence U is quasi-finite over k (Morphisms, Lemma 29.20.7) which by definition means that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. □

0ACB Lemma 68.18.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $F = f^{-1}(\{y\})$ with induced topology from $|X|$. Let k be a field and let $\mathrm{Spec}(k) \to Y$ be in the equivalence class defining y. Set $X_k = \mathrm{Spec}(k) \times_Y X$. Let $\tilde{x} \in |X_k|$ map to $x \in |X|$. Consider the following conditions

0ACC (1) $\dim_x(F) = 0$,
0ACD (2) x is isolated in F,
0ACE (3) x is closed in F and if $x' \rightsquigarrow x$ in F, then $x = x'$,
0ACF (4) $\dim_{\tilde{x}}(|X_k|) = 0$,
0ACG (5) $\tilde{x}$ is isolated in $|X_k|$,
0ACH (6) $\tilde{x}$ is closed in $|X_k|$ and if $\tilde{x}' \rightsquigarrow \tilde{x}$ in $|X_k|$, then $\tilde{x} = \tilde{x}'$,
0ACI (7) $\dim_{\tilde{x}}(X_k) = 0$,
0ACJ (8) f is quasi-finite at x.

Then we have

$$(4) \underset{f \text{ decent}}{\Longrightarrow} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then all conditions are equivalent.

Proof. By Lemma 68.18.7 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \mathrm{Spec}(k)$ is quasi-finite at $\tilde{x}$. Thus by Morphisms of Spaces, Lemma 67.27.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 68.18.7 shows that (4) implies (5).

If Y is decent, then we can pick a quasi-compact monomorphism $\mathrm{Spec}(k') \to Y$ in the equivalence class of y. In this case Lemma 68.18.6 tells us that $|X_{k'}| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. □

0ACK Lemma 68.18.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $y \in |Y|$. Let k be a field and let $\mathrm{Spec}(k) \to Y$ be in the equivalence class defining y. Set $X_k = \mathrm{Spec}(k) \times_Y X$ and let $F = f^{-1}(\{y\})$ with the induced topology from $|X|$. Consider the following conditions

0ACL (1) F is finite,
0ACM (2) F is a discrete topological space,
0ACN (3) $\dim(F) = 0$,
0ACP (4) $|X_k|$ is a finite set,
0ACQ (5) $|X_k|$ is a discrete space,
0ACR (6) $\dim(|X_k|) = 0$,
0ACS (7) $\dim(X_k) = 0$,
0ACT (8) f is quasi-finite at all points of $|X|$ lying over y.

Then we have

$$(1) \Longleftarrow (4) \underset{f \text{ decent}}{\Longrightarrow} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then (1) implies all the other conditions.

Proof. By Lemma 68.18.8 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Thus by Morphisms of Spaces, Lemma 67.27.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 68.18.8 shows that (4) implies (5).

The map $|X_k| \to F$ is surjective by Properties of Spaces, Lemma 66.4.3 and we see (4) $\Rightarrow$ (1).

If Y is decent, then we can pick a quasi-compact monomorphism $\mathrm{Spec}(k') \to Y$ in the equivalence class of y. In this case Lemma 68.18.6 tells us that $|X_{k'}| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. ☐

68.19. Monomorphisms

06RY Here is another case where monomorphisms are representable. Please see More on Morphisms of Spaces, Section 76.4 for more information.

06RZ Lemma 68.19.1. Let S be a scheme. Let Y be a disjoint union of spectra of zero dimensional local rings over S. Let $f : X \to Y$ be a monomorphism of algebraic spaces over S. Then f is representable, i.e., X is a scheme.

Proof. This immediately reduces to the case $Y = \mathrm{Spec}(A)$ where A is a zero dimensional local ring, i.e., $\mathrm{Spec}(A) = \{\mathfrak{m}_A\}$ is a singleton. If $X = \emptyset$, then there is nothing to prove. If not, choose a nonempty affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. As $|X|$ is a singleton (as a subset of $|Y|$, see Morphisms of Spaces, Lemma 67.10.9) we see that $U \to X$ is surjective. Note that $U \times_X U = U \times_Y U = \mathrm{Spec}(B \otimes_A B)$. Thus we see that the ring maps $B \to B \otimes_A B$ are étale. Since

$$(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)$$

we see that $B/\mathfrak{m}_A B \to (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B)$ is flat and in fact free of rank equal to the dimension of $B/\mathfrak{m}_A B$ as a $A/\mathfrak{m}_A$-vector space. Since $B \to B \otimes_A B$ is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 29.57.8 and 29.57.9). Every prime of B lies over $\mathfrak{m}_A$ (the unique prime of A). Hence $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}_A)$ as a topological space, and this space is a finite discrete set as $B/\mathfrak{m}_A B$ is an Artinian ring, see Algebra, Lemmas 10.53.2 and 10.53.6. Hence all prime ideals of B are maximal and $B = B_1 \times \ldots \times B_n$ is a product of finitely many local rings of dimension zero, see Algebra, Lemma 10.53.5. Thus $B \to B \otimes_A B$ is finite étale as all the local rings B_i are henselian by Algebra, Lemma 10.153.10. Thus X is an affine scheme by Groupoids, Proposition 39.23.9. □

68.20. Generic points

0BB7 This section is a continuation of Properties of Spaces, Section 66.11.

0ABV **Lemma 68.20.1.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The following are equivalent

1. x is a generic point of an irreducible component of $|X|$,
2. for any étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,
3. for some étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,
4. the dimension of the local ring of X at x is zero, and
5. x is a point of codimension 0 on X

Proof. Conditions (4) and (5) are equivalent for any algebraic space by definition, see Properties of Spaces, Definition 66.10.2. Observe that any Y as in (2) and (3) is decent by Lemma 68.6.6. Thus it suffices to prove the equivalence of (1) and (4) as then the equivalence with (2) and (3) follows since the dimension of the local ring of Y at y is equal to the dimension of the local ring of X at x. Let $f : U \to X$ be an étale morphism from an affine scheme and let $u \in U$ be a point mapping to x.

Assume (1). Let $u' \rightsquigarrow u$ be a specialization in U. Then $f(u') = f(u) = x$. By Lemma 68.12.1 we see that $u' = u$. Hence u is a generic point of an irreducible component of U. Thus $\dim(\mathcal{O}_{U,u}) = 0$ and we see that (4) holds.

Assume (4). The point x is contained in an irreducible component $T \subset |X|$. Since $|X|$ is sober (Proposition 68.12.4) we T has a generic point x'. Of course $x' \rightsquigarrow x$. Then we can lift this specialization to $u' \rightsquigarrow u$ in U (Lemma 68.12.2). This contradicts the assumption that $\dim(\mathcal{O}_{U,u}) = 0$ unless $u' = u$, i.e., $x' = x$. □

0ED1 **Lemma 68.20.2.** Let S be a scheme. Let X be a decent algebraic space over S. Let $T \subset |X|$ be an irreducible closed subset. Let $\xi \in T$ be the generic point (Proposition 68.12.4). Then $\text{codim}(T, |X|)$ (Topology, Definition 5.11.1) is the dimension of the local ring of X at ξ (Properties of Spaces, Definition 66.10.2).

Proof. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to X$ sending u to ξ. Then any sequence of nontrivial specializations $\xi_e \rightsquigarrow \ldots \rightsquigarrow \xi_0 = \xi$ can be lifted to a sequence $u_e \rightsquigarrow \ldots \rightsquigarrow u_0 = u$ in U by Lemma 68.12.2. Conversely, any sequence of nontrivial specializations $u_e \rightsquigarrow \ldots \rightsquigarrow u_0 = u$ in U maps to a sequence of nontrivial specializations $\xi_e \rightsquigarrow \ldots \rightsquigarrow \xi_0 = \xi$ by Lemma 68.12.1. Because $|X|$ and U are sober topological spaces we conclude that the codimension of T in $|X|$ and of $\overline{\{u\}}$ in U are the same. In this way the lemma reduces to the schemes case which is Properties, Lemma 28.10.3. □

0BB8 **Lemma 68.20.3.** Let S be a scheme. Let X be an algebraic space over S. Assume

(1) every quasi-compact scheme étale over X has finitely many irreducible components, and
(2) every $x \in |X|$ of codimension 0 on X can be represented by a monomorphism $\mathrm{Spec}(k) \to X$.

Then X is a reasonable algebraic space.

Proof. Let U be an affine scheme and let $a : U \to X$ be an étale morphism. We have to show that the fibres of a are universally bounded. By assumption (1) the scheme U has finitely many irreducible components. Let $u_1, \ldots, u_n \in U$ be the generic points of these irreducible components. Let $\{x_1, \ldots, x_m\} \subset |X|$ be the image of $\{u_1, \ldots, u_n\}$. Each x_j is a point of codimension 0. By assumption (2) we may choose a monomorphism $\mathrm{Spec}(k_j) \to X$ representing x_j. By Properties of Spaces, Lemma 66.11.1 we have

$$U \times_X \mathrm{Spec}(k_j) = \coprod\nolimits_{a(u_i) = x_j} \mathrm{Spec}(\kappa(u_i))$$

This is a scheme finite over $\mathrm{Spec}(k_j)$ of degree $d_j = \sum_{a(u_i) = x_j} [\kappa(u_i) : k_j]$. Set $n = \max d_j$.

Observe that a is separated (Properties of Spaces, Lemma 66.6.4). Consider the stratification

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

associated to $U \to X$ in Lemma 68.8.2. By our choice of n above we conclude that X_{n+1} is empty. Namely, if not, then $a^{-1}(X_{n+1})$ is a nonempty open of U and hence would contain one of the x_i. This would mean that X_{n+1} contains $x_j = a(u_i)$ which is impossible. Hence we see that the fibres of $U \to X$ are universally bounded (in fact by the integer n). □

0BB9 Lemma 68.20.4. Let S be a scheme. Let X be an algebraic space over S. The following are equivalent

(1) X is decent and $|X|$ has finitely many irreducible components,
(2) every quasi-compact scheme étale over X has finitely many irreducible components, there are finitely many $x \in |X|$ of codimension 0 on X, and each of these can be represented by a monomorphism $\mathrm{Spec}(k) \to X$,
(3) there exists a dense open $X' \subset X$ which is a scheme, X' has finitely many irreducible components with generic points $\{x'_1, \ldots, x'_m\}$, and the morphism $x'_j \to X$ is quasi-compact for $j = 1, \ldots, m$.

Moreover, if these conditions hold, then X is reasonable and the points $x'_j \in |X|$ are the generic points of the irreducible components of $|X|$.

Proof. In the proof we use Properties of Spaces, Lemma 66.11.1 without further mention. Assume (1). Then X has a dense open subscheme X' by Theorem 68.10.2. Since the closure of an irreducible component of $|X'|$ is an irreducible component of $|X|$, we see that $|X'|$ has finitely many irreducible components. Thus (3) holds.

Assume $X' \subset X$ is as in (3). Let $\{x'_1, \ldots, x'_m\}$ be the generic points of the irreducible components of X'. Let $a : U \to X$ be an étale morphism with U a quasi-compact scheme. To prove (2) it suffices to show that U has finitely many irreducible components whose generic points lie over $\{x'_1, \ldots, x'_m\}$. It suffices to prove this for the members of a finite affine open cover of U, hence we may and do assume U is affine. Note that $U' = a^{-1}(X') \subset U$ is a dense open. Since $U' \to X'$ is an étale morphism of schemes, we see the generic points of irreducible components of U' are the points lying over $\{x'_1, \ldots, x'_m\}$. Since $x'_j \to X$ is quasi-compact there are finitely many points of U lying over x'_j (Lemma 68.4.5). Hence U' has finitely many irreducible components, which implies that the closures of these irreducible components are the irreducible components of U. Thus (2) holds.

Assume (2). This implies (1) and the final statement by Lemma 68.20.3. (We also use that a reasonable algebraic space is decent, see discussion following Definition 68.6.1.) ☐

68.21. Generically finite morphisms

0BBA This section discusses for morphisms of algebraic spaces the material discussed in Morphisms, Section 29.51 and Varieties, Section 33.17 for morphisms of schemes.

0ACZ Lemma 68.21.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is quasi-separated of finite type. Let $y \in |Y|$ be a point of codimension 0 on Y. The following are equivalent:

(1) the space $|X_k|$ is finite where $\mathrm{Spec}(k) \to Y$ represents y,
(2) $X \to Y$ is quasi-finite at all points of $|X|$ over y,
(3) there exists an open subspace $Y' \subset Y$ with $y \in |Y'|$ such that $Y' \times_Y X \to Y'$ is finite.

If Y is decent these are also equivalent to

(4) the set $f^{-1}(\{y\})$ is finite.

Proof. The equivalence of (1) and (2) follows from Lemma 68.18.10 (and the fact that a quasi-separated morphism is decent by Lemma 68.17.2).

Assume the equivalent conditions of (1) and (2). Choose an affine scheme V and an étale morphism $V \to Y$ mapping a point $v \in V$ to y. Then v is a generic point of an irreducible component of V by Properties of Spaces, Lemma 66.11.1. Choose an affine scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is of finite type. The morphism $U \to V$ is quasi-finite at every point lying over v by (2). It follows that the fibre of $U \to V$ over v is finite (Morphisms, Lemma 29.20.14). By Morphisms, Lemma 29.51.1 after shrinking V we may assume that $U \to V$ is finite. Let

$$R = U \times_{V \times_Y X} U$$

Since f is quasi-separated, we see that $V \times_Y X$ is quasi-separated and hence R is a quasi-compact scheme. Moreover the morphisms $R \to V$ is quasi-finite as the composition of an étale morphism $R \to U$ and a finite morphism $U \to V$. Hence we may apply Morphisms, Lemma 29.51.1 once more and after shrinking V we may assume that $R \to V$ is finite as well. This of course implies that the two projections $R \to V$ are finite étale. It follows that $V/R = V \times_Y X$ is an affine scheme, see Groupoids, Proposition 39.23.9. By Morphisms, Lemma 29.41.9 we conclude that $V \times_Y X \to V$ is proper and by Morphisms, Lemma 29.44.11 we conclude that $V \times_Y X \to V$ is finite. Finally, we let $Y' \subset Y$ be the open subspace of Y corresponding to the image of $|V| \to |Y|$. By Morphisms of Spaces, Lemma 67.45.3 we conclude that $Y' \times_Y X \to Y'$ is finite as the base change to V is finite and as $V \to Y'$ is a surjective étale morphism.

If Y is decent and f is quasi-separated, then we see that X is decent too; use Lemmas 68.17.2 and 68.17.5. Hence Lemma 68.18.10 applies to show that (4) implies (1) and (2). On the other hand, we see that (2) implies (4) by Morphisms of Spaces, Lemma 67.27.9. □

0AD0 Lemma 68.21.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is quasi-separated and locally of finite type and Y quasi-separated. Let $y \in |Y|$ be a point of codimension 0 on Y. The following are equivalent:

(1) the set $f^{-1}(\{y\})$ is finite,
(2) the space $|X_k|$ is finite where $\text{Spec}(k) \to Y$ represents y,
(3) there exist open subspaces $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$, $y \in |Y'|$, and $f^{-1}(\{y\}) \subset |X'|$ such that $f|_{X'} : X' \to Y'$ is finite.

Proof. Since quasi-separated algebraic spaces are decent, the equivalence of (1) and (2) follows from Lemma 68.18.10. To prove that (1) and (2) imply (3) we may and do replace Y by a quasi-compact open containing y. Since $f^{-1}(\{y\})$ is finite, we can find a quasi-compact open subspace of $X' \subset X$ containing the fibre. The restriction $f|_{X'} : X' \to Y$ is quasi-compact and quasi-separated by Morphisms of Spaces, Lemma 67.8.10 (this is where we use that Y is quasi-separated). Applying Lemma 68.21.1 to $f|_{X'} : X' \to Y$ we see that (3) holds. We omit the proof that (3) implies (2). □

0BBB Lemma 68.21.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of X, resp. Y. Let $y \in Y^0$. The following are equivalent

(1) $f^{-1}(\{y\}) \subset X^0$,
(2) f is quasi-finite at all points lying over y,
(3) f is quasi-finite at all $x \in X^0$ lying over y.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let U be a scheme and let $U \to V \times_Y X$ be a surjective étale morphism. Then f is quasi-finite at the image x of a point $u \in U$ if and only if $U \to V$ is quasi-finite at u. Moreover, $x \in X^0$ if and only if u is the generic point of an irreducible component of U (Properties of Spaces, Lemma 66.11.1). Thus the lemma reduces to the case of the morphism $U \to V$, i.e., to Morphisms, Lemma 29.51.4. □

0BBC Lemma 68.21.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of X, resp. Y. Assume

(1) Y is decent,
(2) X^0 and Y^0 are finite and $f^{-1}(Y^0) = X^0$,
(3) either f is quasi-compact or f is separated.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. By Lemmas 68.20.4 and 68.20.1 we may assume Y is a scheme with finitely many irreducible components. Shrinking further we may assume Y is an irreducible affine scheme with generic point y. Then the fibre of f over y is finite.

Assume f is quasi-compact and Y affine irreducible. Then X is quasi-compact and we may choose an affine scheme U and a surjective étale morphism $U \to X$. Then $U \to Y$ is of finite type and the fibre of $U \to Y$ over y is the set U^0 of generic points of irreducible components of U (Properties of Spaces, Lemma 66.11.1). Hence U^0 is finite (Morphisms, Lemma 29.20.14) and after shrinking Y we may assume that $U \to Y$ is finite (Morphisms, Lemma 29.51.1). Next, consider $R = U \times_X U$. Since the projection $s : R \to U$ is étale we see that $R^0 = s^{-1}(U^0)$ lies over y. Since $R \to U \times_Y U$ is a monomorphism, we conclude that R^0 is finite as $U \times_Y U \to Y$ is finite. And R is separated (Properties of Spaces, Lemma 66.6.4). Thus we may shrink Y once more to reach the situation where R is finite over Y (Morphisms, Lemma 29.51.5). In this case it follows that $X = U/R$ is finite over Y by exactly the same arguments as given in the proof of Lemma 68.21.1 (or we can simply apply that lemma because it follows immediately that X is quasi-separated as well).

Assume f is separated and Y affine irreducible. Choose $V \subset Y$ and $U \subset X$ as in Lemma 68.21.2. Since $f|_U : U \to V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Morphisms of Spaces, Lemmas 67.40.6 and 67.45.9). Thus $f^{-1}(V) = U \amalg W$ for some open subspace W of X. However, since U contains all the codimension 0 points of X we conclude that $W = \emptyset$ (Properties of Spaces, Lemma 66.11.2) as desired. □

68.22. Birational morphisms

0ACU The following definition of a birational morphism of algebraic spaces seems to be the closest to our definition (Morphisms, Definition 29.50.1) of a birational morphism of schemes.

0ACV Definition 68.22.1. Let S be a scheme. Let X and Y algebraic spaces over S. Assume X and Y are decent and that $|X|$ and $|Y|$ have finitely many irreducible components. We say a morphism $f : X \to Y$ is birational if

(1) $|f|$ induces a bijection between the set of generic points of irreducible components of $|X|$ and the set of generic points of the irreducible components of $|Y|$, and
(2) for every generic point $x \in |X|$ of an irreducible component the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism (see clarification below).

Clarification: Since X and Y are decent the topological spaces $|X|$ and $|Y|$ are sober (Proposition 68.12.4). Hence condition (1) makes sense. Moreover, because we have assumed that $|X|$ and $|Y|$ have finitely many irreducible components, we see that the generic points $x_1, \ldots, x_n \in |X|$, resp. $y_1, \ldots, y_n \in |Y|$ are contained in any dense open of $|X|$, resp. $|Y|$. In particular, they are contained in the schematic locus of X, resp. Y by Theorem 68.10.2. Thus we can define $\mathcal{O}_{X,x_i}$, resp. $\mathcal{O}_{Y,y_i}$ to be the local ring of this scheme at x_i, resp. y_i.

We conclude that if the morphism $f : X \to Y$ is birational, then there exist dense open subspaces $X' \subset X$ and $Y' \subset Y$ such that

(1) $f(X') \subset Y'$,
(2) X' and Y' are representable, and
(3) $f|_{X'} : X' \to Y'$ is birational in the sense of Morphisms, Definition 29.50.1.

However, we do insist that X and Y are decent with finitely many irreducible components. Other ways to characterize decent algebraic spaces with finitely many irreducible components are given in Lemma 68.20.4. In most cases birational morphisms are isomorphisms over dense opens.

0ACW Lemma 68.22.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational then f is dominant.

Proof. Follows immediately from the definitions. See Morphisms of Spaces, Definition 67.18.1. $\square$

0BBD Lemma 68.22.3. Let S be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.

Proof. Let $X' \subset X$ and $Y' \subset Y$ be the maximal open subspaces which are representable, see Lemma 68.20.4. By Lemma 68.21.3 the fibre of f over y is consists of points of codimension 0 of X and is therefore contained in X'. Hence $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) = X' \times_{Y'} \mathrm{Spec}(\mathcal{O}_{Y',y})$ and the result follows from Morphisms, Lemma 29.50.3. $\square$

0BBE Lemma 68.22.4. Let S be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. Assume one of the following conditions is satisfied

(1) f is locally of finite type and Y reduced (i.e., integral),
(2) f is locally of finite presentation.

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism.

Proof. By Lemma 68.20.4 we may assume that X and Y are schemes. In this case the result is Morphisms, Lemma 29.50.5. $\square$

0BBF Lemma 68.22.5. Let S be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. Assume

(1) either f is quasi-compact or f is separated, and
(2) either f is locally of finite type and Y is reduced or f is locally of finite presentation.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

Proof. By Lemma 68.20.4 we may assume Y is a scheme. By Lemma 68.21.4 we may assume that f is finite. Then X is a scheme too and the result follows from Morphisms, Lemma 29.51.6. $\square$

0B4D Lemma 68.22.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational and $V \to Y$ is an étale morphism with V affine, then $X \times_Y V$ is decent with finitely many irreducible components and $X \times_Y V \to V$ is birational.

Proof. The algebraic space $U = X \times_Y V$ is decent (Lemma 68.6.6). The generic points of V and U are the elements of $|V|$ and $|U|$ which lie over generic points of $|Y|$ and $|X|$ (Lemma 68.20.1). Since Y is decent we conclude there are finitely many generic points on V. Let $\xi \in |X|$ be a generic point of an irreducible component.

By the discussion following Definition 68.22.1 we have a cartesian square

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{X,\xi}) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{Y,f(\xi)}) \arrow[r] & Y \end{tikzcd}$$

whose horizontal morphisms are monomorphisms identifying local rings and where the left vertical arrow is an isomorphism. It follows that in the diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{X,\xi}) \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{Y,f(\xi)}) \times_Y V \arrow[r] & V \end{tikzcd}$$

the vertical arrow on the left is an isomorphism. The horizonal arrows have image contained in the schematic locus of U and V and identify local rings (some details omitted). Since the image of the horizontal arrows are the points of $|U|$, resp. $|V|$ lying over ξ, resp. $f(\xi)$ we conclude. $\square$

0BBG Lemma 68.22.7. Let S be a scheme. Let $f : X \to Y$ be a birational morphism between algebraic spaces over S which are decent and have finitely many irreducible components. Then the normalizations $X^\nu \to X$ and $Y^\nu \to Y$ exist and there is a commutative diagram

$$\begin{tikzcd} X^\nu \arrow[r] \arrow[d] & Y^\nu \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

of algebraic spaces over S. The morphism $X^\nu \to Y^\nu$ is birational.

Proof. By Lemma 68.20.4 we see that X and Y satisfy the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1 and the normalizations are defined. By Morphisms of Spaces, Lemma 67.49.8 the algebraic space X^ν is normal and maps codimension 0 points to codimension 0 points. Since f maps codimension 0 points to codimension 0 points (this is the same as generic points on decent spaces by Lemma 68.20.1) we obtain from Morphisms of Spaces, Lemma 67.49.8 a factorization of the composition $X^\nu \to X \to Y$ through Y^ν.

Observe that X^ν and Y^ν are decent for example by Lemma 68.6.5. Moreover the maps $X^\nu \to X$ and $Y^\nu \to Y$ induce bijections on irreducible components (see references above) hence X^ν and Y^ν both have a finite number of irreducible components and the map $X^\nu \to Y^\nu$ induces a bijection between their generic points. To prove that $X^\nu \to Y^\nu$ is birational, it therefore suffices to show it induces an isomorphism on local rings at these points. To do this we may replace X and Y by open neighbourhoods of their generic points, hence we may assume X and Y are affine irreducible schemes with generic points x and y. Since f is birational the map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism. Let $x^\nu \in X^\nu$ and $y^\nu \in Y^\nu$ be the points lying over x and y. By construction of the normalization we see that $\mathcal{O}_{X^\nu,x^\nu} = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and similarly on Y. Thus the map $\mathcal{O}_{X^\nu,x^\nu} \to \mathcal{O}_{Y^\nu,y^\nu}$ is an isomorphism as well. $\square$

0B4E Lemma 68.22.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) X and Y are decent and have finitely many irreducible components,
(2) f is integral and birational,
(3) Y is normal, and
(4) X is reduced.

Then f is an isomorphism.

Proof. Let $V \to Y$ be an étale morphism with V affine. It suffices to show that $U = X \times_Y V \to V$ is an isomorphism. By Lemma 68.22.6 and its proof we see that U and V are decent and have finitely many irreducible components and that $U \to V$ is birational. By Properties, Lemma 28.7.5 V is a finite disjoint union of integral schemes. Thus we may assume V is integral. As f is birational, we see that U is irreducible and reduced, i.e., integral (note that U is a scheme as f is integral, hence representable). Thus we may assume that X and Y are integral schemes and the result follows from the case of schemes, see Morphisms, Lemma 29.54.8. □

0BBH Lemma 68.22.9. Let S be a scheme. Let $f : X \to Y$ be an integral birational morphism of decent algebraic spaces over S which have finitely many irreducible components. Then there exists a factorization $Y^\nu \to X \to Y$ and $Y^\nu \to X$ is the normalization of X.

Proof. Consider the map $X^\nu \to Y^\nu$ of Lemma 68.22.7. This map is integral by Morphisms of Spaces, Lemma 67.45.12. Hence it is an isomorphism by Lemma 68.22.8. □

68.23. Jacobson spaces

0BA2 We have defined the Jacobson property for algebraic spaces in Properties of Spaces, Remark 66.7.3. For representable algebraic spaces it agrees with the property discussed in Properties, Section 28.6. The relationship between the Jacobson property and the behaviour of the topological space $|X|$ is not evident for general algebraic spaces $|X|$. However, a decent (for example quasi-separated or locally separated) algebraic space X is Jacobson if and only if $|X|$ is Jacobson (see Lemma 68.23.4).

0BA3 Lemma 68.23.1. Let S be a scheme. Let X be a Jacobson algebraic space over S. Any algebraic space locally of finite type over X is Jacobson.

Proof. Let $U \to X$ be a surjective étale morphism where U is a scheme. Then U is Jacobson (by definition) and for a morphism of schemes $V \to U$ which is locally of finite type we see that V is Jacobson by the corresponding result for schemes (Morphisms, Lemma 29.16.9). Thus if $Y \to X$ is a morphism of algebraic spaces which is locally of finite type, then setting $V = U \times_X Y$ we see that Y is Jacobson by definition. □

0BA4 Lemma 68.23.2. Let S be a scheme. Let X be a Jacobson algebraic space over S. For $x \in X_{\text{ft-pts}}$ and $g : W \to X$ locally of finite type with W a scheme, if $x \in \text{Im}(|g|)$, then there exists a closed point of W mapping to x.

Proof. Let $U \to X$ be an étale morphism with U a scheme and with $u \in U$ closed mapping to x, see Morphisms of Spaces, Lemma 67.25.3. Observe that W, $W \times_X U$, and U are Jacobson schemes by Lemma 68.23.1. Hence finite type points on these schemes are the same thing as closed points by Morphisms, Lemma 29.16.8. The inverse image $T \subset W \times_X U$ of u is a nonempty (as x in the image of $W \to X$) closed subset. By Morphisms, Lemma 29.16.7 there is a closed point t of $W \times_X U$

which maps to u. As $W \times_X U \to W$ is locally of finite type the image of t in W is closed by Morphisms, Lemma 29.16.8. □

0BA5 Lemma 68.23.3. Let S be a scheme. Let X be a decent Jacobson algebraic space over S. Then $X_{\text{ft-pts}} \subset |X|$ is the set of closed points.

Proof. If $x \in |X|$ is closed, then we can represent x by a closed immersion $\text{Spec}(k) \to X$, see Lemma 68.14.6. Hence x is certainly a finite type point.

Conversely, let $x \in |X|$ be a finite type point. We know that x can be represented by a quasi-compact monomorphism $\text{Spec}(k) \to X$ where k is a field (Definition 68.6.1). On the other hand, by definition, there exists a morphism $\text{Spec}(k') \to X$ which is locally of finite type and represents x (Morphisms, Definition 29.16.3). We obtain a factorization $\text{Spec}(k') \to \text{Spec}(k) \to X$. Let $U \to X$ be any étale morphism with U affine and consider the morphisms

$$\text{Spec}(k') \times_X U \to \text{Spec}(k) \times_X U \to U$$

The quasi-compact scheme $\text{Spec}(k) \times_X U$ is étale over $\text{Spec}(k)$ hence is a finite disjoint union of spectra of fields (Remark 68.4.1). Moreover, the first morphism is surjective and locally of finite type (Morphisms, Lemma 29.15.8) hence surjective on finite type points (Morphisms, Lemma 29.16.6) and the composition (which is locally of finite type) sends finite type points to closed points as U is Jacobson (Morphisms, Lemma 29.16.8). Thus the image of $\text{Spec}(k) \times_X U \to U$ is a finite set of closed points hence closed. Since this is true for every affine U and étale morphism $U \to X$, we conclude that $x \in |X|$ is closed. □

0BA6 Lemma 68.23.4. Let S be a scheme. Let X be a decent algebraic space over S. Then X is Jacobson if and only if $|X|$ is Jacobson.

Proof. Assume X is Jacobson and that $T \subset |X|$ is a closed subset. By Morphisms of Spaces, Lemma 67.25.6 we see that $T \cap X_{\text{ft-pts}}$ is dense in T. By Lemma 68.23.3 we see that $X_{\text{ft-pts}}$ are the closed points of $|X|$. Thus $|X|$ is indeed Jacobson.

Assume $|X|$ is Jacobson. Let $f : U \to X$ be an étale morphism with U an affine scheme. We have to show that U is Jacobson. If $x \in |X|$ is closed, then the fibre $F = f^{-1}(\{x\})$ is a finite (by definition of decent) closed (by construction of the topology on $|X|$) subset of U. Since there are no specializations between points of F (Lemma 68.12.1) we conclude that every point of F is closed in U. If U is not Jacobson, then there exists a non-closed point $u \in U$ such that $\{u\}$ is locally closed (Topology, Lemma 5.18.3). We will show that $f(u) \in |X|$ is closed; by the above u is closed in U which is a contradiction and finishes the proof. To prove this we may replace U by an affine open neighbourhood of u. Thus we may assume that $\{u\}$ is closed in U. Let $R = U \times_X U$ with projections $s, t : R \to U$. Then $s^{-1}(\{u\}) = \{r_1, \ldots, r_m\}$ is finite (by definition of decent spaces). After replacing U by a smaller affine open neighbourhood of u we may assume that $t(r_j) = u$ for $j = 1, \ldots, m$. It follows that $\{u\}$ is an R-invariant closed subset of U. Hence $\{f(u)\}$ is a locally closed subset of X as it is closed in the open $|f|(|U|)$ of $|X|$. Since $|X|$ is Jacobson we conclude that $f(u)$ is closed in $|X|$ as desired. □

0ED2 Lemma 68.23.5. Let S be a scheme. Let X be a decent locally Noetherian algebraic space over S. Let $x \in |X|$. Then

$$W = \{x' \in |X| : x' \rightsquigarrow x,\ x' \neq x\}$$

is a Noetherian, spectral, sober, Jacobson topological space.

Proof. We may replace by any open subspace containing x. Thus we may assume that X is quasi-compact. Then $|X|$ is a Noetherian topological space (Properties of Spaces, Lemma 66.24.2). Thus W is a Noetherian topological space (Topology, Lemma 5.9.2).

Combining Lemma 68.14.1 with Properties of Spaces, Lemma 66.15.2 we see that $|X|$ is a spectral toplogical space. By Topology, Lemma 5.24.7 we see that $W \cup \{x\}$ is a spectral topological space. Now W is a quasi-compact open of $W \cup \{x\}$ and hence W is spectral by Topology, Lemma 5.23.5.

Let $E \subset W$ be an irreducible closed subset. Then if $Z \subset |X|$ is the closure of E we see that $x \in Z$. There is a unique generic point $\eta \in Z$ by Proposition 68.12.4. Of course $\eta \in W$ and hence $\eta \in E$. We conclude that E has a unique generic point, i.e., W is sober.

Let $x' \in W$ be a point such that $\{x'\}$ is locally closed in W. To finish the proof we have to show that x' is a closed point of W. If not, then there exists a nontrivial specialization $x' \rightsquigarrow x'_1$ in W. Let U be an affine scheme, $u \in U$ a point, and let $U \to X$ be an étale morphism mapping u to x. By Lemma 68.12.2 we can choose specializations $u' \rightsquigarrow u'_1 \rightsquigarrow u$ mapping to $x' \rightsquigarrow x'_1 \rightsquigarrow x$. Let $\mathfrak p' \subset \mathcal{O}_{U,u}$ be the prime ideal corresponding to u'. The existence of the specializations implies that $\dim(\mathcal{O}_{U,u}/\mathfrak p') \geq 2$. Hence every nonempty open of $\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak p')$ is infinite by Algebra, Lemma 10.61.1. By Lemma 68.12.1 we obtain a continuous map

$$\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak p') \setminus \{\mathfrak m_u/\mathfrak p'\} \longrightarrow W$$

Since the generic point of the LHS maps to x' the image is contained in $\overline{\{x'\}}$. We conclude the inverse image of $\{x'\}$ under the displayed arrow is nonempty open hence infinite. However, the fibres of $U \to X$ are finite as X is decent and we conclude that $\{x'\}$ is infinite. This contradiction finishes the proof. $\square$

68.24. Local irreducibility

0DQ5 We have already defined the geometric number of branches of an algebraic space at a point in Properties of Spaces, Section 66.23. The number of branches of an algebraic space at a point can only be defined for decent algebraic spaces.

0DQ6 Lemma 68.24.1. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent

(1) for any elementary étale neighbourhood $(U, u) \to (X, x)$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,
(2) for any elementary étale neighbourhood $(U, u) \to (X, x)$ there is a unique irreducible component of U through u,
(3) for any elementary étale neighbourhood $(U, u) \to (X, x)$ the local ring $\mathcal{O}_{U,u}$ is unibranch,
(4) the henselian local ring $\mathcal{O}^h_{X,x}$ has a unique minimal prime.

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. The ring $\mathcal{O}^h_{X,x}$ is the henselization of $\mathcal{O}_{U,u}$, see discussion following Definition 68.11.7. In particular (3) and (4) are equivalent by More on

Algebra, Lemma 15.106.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 37.36.2. ☐

0DQ7 Definition 68.24.2. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. We say that X is unibranch at x if the equivalent conditions of Lemma 68.24.1 hold. We say that X is unibranch if X is unibranch at every $x \in |X|$.

This is consistent with the definition for schemes (Properties, Definition 28.15.1).

0DQ8 Lemma 68.24.3. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a point. Let $n \in \{1, 2, \ldots\}$ be an integer. The following are equivalent

(1) for any elementary étale neighbourhood $(U, u) \to (X, x)$ the number of minimal primes of the local ring $\mathcal{O}_{U,u}$ is $\leq n$ and for at least one choice of (U, u) it is n,
(2) for any elementary étale neighbourhood $(U, u) \to (X, x)$ the number irreducible components of U passing through u is $\leq n$ and for at least one choice of (U, u) it is n,
(3) for any elementary étale neighbourhood $(U, u) \to (X, x)$ the number of branches of U at u is $\leq n$ and for at least one choice of (U, u) it is n,
(4) the number of minimal prime ideals of $\mathcal{O}^h_{X,x}$ is n.

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. The ring $\mathcal{O}_{X,x}$ is the henselization of $\mathcal{O}_{U,u}$, see discussion following Definition 68.11.7. In particular (3) and (4) are equivalent by More on Algebra, Lemma 15.106.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 37.36.2. ☐

0DQ9 Definition 68.24.4. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The number of branches of X at x is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 68.24.3 hold, or else ∞.

68.25. Catenary algebraic spaces

0ED3 This section extends the material in Properties, Section 28.11 and Morphisms, Section 29.17 to algebraic spaces.

0ED4 Definition 68.25.1. Let S be a scheme. Let X be a decent algebraic space over S. We say X is catenary if $|X|$ is catenary (Topology, Definition 5.11.4).

If X is representable, then this is equivalent to the corresponding notion for the scheme representing X.

0ED5 Lemma 68.25.2. Let S be a locally Noetherian and universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let X be a decent algebraic space over S such that the structure morphism $X \to S$ is locally of finite type. Let $\delta_X : |X| \to \mathbf{Z}$ be the map sending x to $\delta(f(x))$ plus the transcendence degree of $x/f(x)$. Then δ_X is a dimension function on $|X|$.

Proof. Let $\varphi : U \to X$ be a surjective étale morphism where U is a scheme. Then the similarly defined function δ_U is a dimension function on U by Morphisms, Lemma 29.52.3. On the other hand, by the definition of relative transcendence degree in (Morphisms of Spaces, Definition 67.33.1) we see that $\delta_U(u) = \delta_X(\varphi(u))$.

Let $x \rightsquigarrow x'$ be a specialization of points in $|X|$. by Lemma 68.12.2 we can find a specialization $u \rightsquigarrow u'$ of points of U with $\varphi(u) = x$ and $\varphi(u') = x'$. Moreover, we see that $x = x'$ if and only if $u = u'$, see Lemma 68.12.1. Thus the fact that δ_U is a dimension function implies that δ_X is a dimension function, see Topology, Definition 5.20.1. $\square$

0ED6 Lemma 68.25.3. Let S be a locally Noetherian and universally catenary scheme. Let X be an algebraic space over S such that X is decent and such that the structure morphism $X \to S$ is locally of finite type. Then X is catenary.

Proof. The question is local on S (use Topology, Lemma 5.11.5). Thus we may assume that S has a dimension function, see Topology, Lemma 5.20.4. Then we conclude that $|X|$ has a dimension function by Lemma 68.25.2. Since $|X|$ is sober (Proposition 68.12.4) we conclude that $|X|$ is catenary by Topology, Lemma 5.20.2. $\square$

By Lemma 68.25.3 the following definition is compatible with the already existing notion for representable algebraic spaces.

0ED7 Definition 68.25.4. Let S be a scheme. Let X be a decent and locally Noetherian algebraic space over S. We say X is universally catenary if for every morphism $Y \to X$ of algebraic spaces which is locally of finite type and with Y decent, the algebraic space Y is catenary.

If X is an algebraic space, then the condition "X is decent and locally Noetherian" is equivalent to "X is quasi-separated and locally Noetherian". This is Lemma 68.14.1. Thus another way to understand the definition above is that X is universally catenary if and only if Y is catenary for all morphisms $Y \to X$ which are quasi-separated and locally of finite type.

0ED8 Lemma 68.25.5. Let S be a scheme. Let X be a decent, locally Noetherian, and universally catenary algebraic space over S. Then any decent algebraic space locally of finite type over X is universally catenary.

Proof. This is formal from the definitions and the fact that compositions of morphisms locally of finite type are locally of finite type (Morphisms of Spaces, Lemma 67.23.2). $\square$

0ED9 Lemma 68.25.6. Let S be a scheme. Let $f : Y \to X$ be a surjective finite morphism of decent and locally Noetherian algebraic spaces. Let $\delta : |X| \to \mathbf{Z}$ be a function. If $\delta \circ |f|$ is a dimension function, then δ is a dimension function.

Proof. Let $x \mapsto x'$, $x \neq x'$ be a specialization in $|X|$. Choose $y \in |Y|$ with $|f|(y) = x$. Since $|f|$ is closed (Morphisms of Spaces, Lemma 67.45.9) we find a specialization $y \rightsquigarrow y'$ with $|f|(y') = x'$. Thus we conclude that $\delta(x) = \delta(|f|(y)) > \delta(|f|(y')) = \delta(x')$ (see Topology, Definition 5.20.1). If $x \rightsquigarrow x'$ is an immediate specialization, then $y \rightsquigarrow y'$ is an immediate specialization too: namely if $y \rightsquigarrow y'' \rightsquigarrow y'$, then $|f|(y'')$ must be either x or x' and there are no nontrivial specializations between points of fibres of $|f|$ by Lemma 68.18.10. $\square$

The discussion will be continued in More on Morphisms of Spaces, Section 76.32.

68.26. Other chapters

CHAPTER 69

Cohomology of Algebraic Spaces

071T 69.1. Introduction

071U In this chapter we write about cohomology of algebraic spaces. Although we prove some results on cohomology of abelian sheaves, we focus mainly on cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapter "Cohomology of Schemes". Some of the results in this chapter can be found in [Knu71].

An important missing ingredient in this chapter is the induction principle, i.e., the analogue for quasi-compact and quasi-separated algebraic spaces of Cohomology of Schemes, Lemma 30.4.1. This is formulated precisely and proved in detail in Derived Categories of Spaces, Section 75.9. Instead of the induction principle, in this chapter we use the alternating Čech complex, see Section 69.6. It is designed to prove vanishing statements such as Proposition 69.7.2, but in some cases the induction principle is a more powerful and perhaps more "standard" tool. We encourage the reader to take a look at the induction principle after reading some of the material in this section.

69.2. Conventions

071V The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

69.3. Higher direct images

071Y Let S be a scheme. Let X be a representable algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent module on X (see Properties of Spaces, Section 66.29). By Descent, Proposition 35.9.3 the cohomology groups $H^i(X, \mathcal{F})$ agree with the usual cohomology group computed in the Zariski topology of the corresponding quasi-coherent module on the scheme representing X.

More generally, let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y. Let $\mathcal{F}$ be a quasi-coherent module on X. By Descent, Lemma 35.9.5 the sheaf $R^if_*\mathcal{F}$ agrees with the usual higher direct image computed for the Zariski topology of the quasi-coherent module on the scheme representing X mapping to the scheme representing Y.

More generally still, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let

$V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

071Z (69.3.0.1) $$R^i f'_*(\mathcal{F}|_U) = (R^i f_* \mathcal{F})|_V,$$

see Properties of Spaces, Lemma 66.26.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $R^i f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

Next, we prove that higher direct images of quasi-coherent sheaves are quasi-coherent for any quasi-compact and quasi-separated morphism of algebraic spaces. In the proof we use a trick; a "better" proof would use a relative Čech complex, as discussed in Sheaves on Stacks, Sections 96.18 and 96.19 ff.

0720 Lemma 69.3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then $R^i f_*$ transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.

Proof. Let $V \to Y$ be an étale morphism where V is an affine scheme. Set $U = V \times_Y X$ and denote $f' : U \to V$ the induced morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. By Properties of Spaces, Lemma 66.26.2 we have $R^i f'_*(\mathcal{F}|_U) = (R^i f_* \mathcal{F})|_V$. Since the property of being a quasi-coherent module is local in the étale topology on Y (see Properties of Spaces, Lemma 66.29.6) we may replace Y by V, i.e., we may assume Y is an affine scheme.

Assume Y is affine. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $g : U \to X$, see Properties of Spaces, Lemma 66.6.3. Picture

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f \circ g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

The morphism $g : U \to X$ is representable, separated and quasi-compact because X is quasi-separated. Hence the lemma holds for g (by the discussion above the lemma). It also holds for $f \circ g : U \to Y$ (as this is a morphism of affine schemes).

In the situation described in the previous paragraph we will show by induction on n that IH_n: for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaves $R^i f \mathcal{F}$ are quasi-coherent for $i \leq n$. The case $n = 0$ follows from Morphisms of Spaces, Lemma 67.11.2. Assume IH_n. In the rest of the proof we show that IH_{n+1} holds.

Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_U$-module. Consider the Leray spectral sequence

$$E_2^{p,q} = R^p f_* R^q g_* \mathcal{H} \Rightarrow R^{p+q}(f \circ g)_* \mathcal{H}$$

Cohomology on Sites, Lemma 21.14.7. As $R^q g_* \mathcal{H}$ is quasi-coherent by IH_n all the sheaves $R^p f_* R^q g_* \mathcal{H}$ are quasi-coherent for $p \leq n$. The sheaves $R^{p+q}(f \circ g)_* \mathcal{H}$ are all quasi-coherent (in fact zero for $p + q > 0$ but we do not need this). Looking in degrees $\leq n + 1$ the only module which we do not yet know is quasi-coherent is $E_2^{n+1,0} = R^{n+1} f_* g_* \mathcal{H}$. Moreover, the differentials $d_r^{n+1,0} : E_r^{n+1,0} \to E_r^{n+1+r,1-r}$ are zero as the target is zero. Using that $QCoh(\mathcal{O}_X)$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_X)$ (Properties of Spaces, Lemma 66.29.7) it follows that $R^{n+1}f_*g_*\mathcal{H}$ is quasi-coherent (details omitted).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $\mathcal{H} = g^*\mathcal{F}$. The adjunction mapping $\mathcal{F} \to g_*g^*\mathcal{F} = g_*\mathcal{H}$ is injective as $U \to X$ is surjective étale. Consider the exact sequence

$$0 \to \mathcal{F} \to g_*\mathcal{H} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel of the first map and in particular quasi-coherent. Applying the long exact cohomology sequence we obtain

$$R^nf_*g_*\mathcal{H} \to R^nf_*\mathcal{G} \to R^{n+1}f_*\mathcal{F} \to R^{n+1}f_*g_*\mathcal{H} \to R^{n+1}f_*\mathcal{G}$$

The cokernel of the first arrow is quasi-coherent and we have seen above that $R^{n+1}f_*g_*\mathcal{H}$ is quasi-coherent. Thus $R^{n+1}f_*\mathcal{F}$ has a 2-step filtration where the first step is quasi-coherent and the second a submodule of a quasi-coherent sheaf. Since $\mathcal{F}$ is an arbitrary quasi-coherent $\mathcal{O}_X$-module, this result also holds for $\mathcal{G}$. Thus we can choose an exact sequence $0 \to \mathcal{A} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ with $\mathcal{A}$, $\mathcal{B}$ quasi-coherent $\mathcal{O}_Y$-modules. Then the kernel $\mathcal{K}$ of $R^{n+1}f_*g_*\mathcal{H} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ is quasi-coherent, whereupon we obtain a map $\mathcal{K} \to \mathcal{A}$ whose kernel $\mathcal{K}'$ is quasi-coherent too. Hence $R^{n+1}f_*\mathcal{F}$ sits in an exact sequence

$$R^nf_*g_*\mathcal{H} \to R^nf_*\mathcal{G} \to R^{n+1}f_*\mathcal{F} \to \mathcal{K}' \to 0$$

with all modules quasi-coherent except for possibly $R^{n+1}f_*\mathcal{F}$. We conclude that $R^{n+1}f_*\mathcal{F}$ is quasi-coherent, i.e., IH_{n+1} holds as desired. $\square$

08EX Lemma 69.3.2. Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any affine object V of $Y_{\acute{e}tale}$ we have

$$H^q(V \times_Y X, \mathcal{F}) = H^0(V, R^qf_*\mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Since formation of Rf_* commutes with étale localization (Properties of Spaces, Lemma 66.26.2) we may replace Y by V and assume $Y = V$ is affine. Consider the Leray spectral sequence $E_2^{p,q} = H^p(Y, R^qf_*\mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology on Sites, Lemma 21.14.5. By Lemma 69.3.1 we see that the sheaves $R^qf_*\mathcal{F}$ are quasi-coherent. By Cohomology of Schemes, Lemma 30.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. $\square$

69.4. Finite morphisms

0DK2 Here are some results which hold for all abelian sheaves (in particular also quasi-coherent modules). We warn the reader that these lemmas do not hold for finite morphisms of schemes and the Zariski topology.

0A4K Lemma 69.4.1. Let S be a scheme. Let $f : X \to Y$ be an integral (for example finite) morphism of algebraic spaces. Then $f_* : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(Y_{\acute{e}tale})$ is an exact functor and $R^pf_* = 0$ for $p > 0$.

Proof. By Properties of Spaces, Lemma 66.18.12 we may compute the higher direct images on an étale cover of Y. Hence we may assume Y is a scheme. This implies that X is a scheme (Morphisms of Spaces, Lemma 67.45.3). In this case we may apply Étale Cohomology, Lemma 59.43.5. For the finite case the reader may wish to consult the less technical Étale Cohomology, Proposition 59.55.2. □

0DK3 Lemma 69.4.2. Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y with lifts $\overline{x}_1, \ldots, \overline{x}_n$ in X. Then

$$(f_*\mathcal{F})_{\overline{y}} = \prod\nolimits_{i=1,\ldots,n} \mathcal{F}_{\overline{x}_i}$$

for any sheaf $\mathcal{F}$ on $X_{étale}$.

Proof. Choose an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$. Then the stalk $(f_*\mathcal{F})_{\overline{y}}$ is the stalk of $f_*\mathcal{F}|_V$ at $\overline{v}$. By Properties of Spaces, Lemma 66.18.12 we may replace Y by V and X by $X \times_Y V$. Then $Z \to X$ is a finite morphism of schemes and the result is Étale Cohomology, Proposition 59.55.2. □

0DK4 Lemma 69.4.3. Let S be a scheme. Let $\pi : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Y_{étale}$. Let $\varphi : \mathcal{B} \to \pi_*\mathcal{A}$ be a homomorphism of sheaves of rings so that we obtain a morphism of ringed topoi

$$f = (\pi, \varphi) : (Sh(X_{étale}), \mathcal{A}) \longrightarrow (Sh(Y_{étale}), \mathcal{B}).$$

For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ and a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ the canonical map

$$\mathcal{G} \otimes_{\mathcal{B}} f_*\mathcal{F} \longrightarrow f_*(f^*\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}).$$

is an isomorphism.

Proof. The map is the map adjoint to the map

$$f^*\mathcal{G} \otimes_{\mathcal{A}} f^*f_*\mathcal{F} = f^*(\mathcal{G} \otimes_{\mathcal{B}} f_*\mathcal{F}) \longrightarrow f^*\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}$$

coming from $\text{id} : f^*\mathcal{G} \to f^*\mathcal{G}$ and the adjunction map $f^*f_*\mathcal{F} \to \mathcal{F}$. To see this map is an isomorphism, we may check on stalks (Properties of Spaces, Theorem 66.19.12). Let $\overline{y}$ be a geometric point of Y and let $\overline{x}_1, \ldots, \overline{x}_n$ be the geometric points of X lying over $\overline{y}$. Working out what our maps does on stalks, we see that we have to show

$$\mathcal{G}_{\overline{y}} \otimes_{\mathcal{B}_{\overline{y}}} \left(\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{\overline{x}_i}\right) = \bigoplus\nolimits_{i=1,\ldots,n} (\mathcal{G}_{\overline{y}} \otimes_{\mathcal{B}_{\overline{x}}} \mathcal{A}_{\overline{x}_i}) \otimes_{\mathcal{A}_{\overline{x}_i}} \mathcal{F}_{\overline{x}_i}$$

which holds true. Here we have used that taking tensor products commutes with taking stalks, the behaviour of stalks under pullback Properties of Spaces, Lemma 66.19.9, and the behaviour of stalks under pushforward along a closed immersion Lemma 69.4.2. □

We end this section with an insanely general projection formula for finite morphisms.

0DK5 Lemma 69.4.4. With S, X, Y, π, $\mathcal{A}$, $\mathcal{B}$, φ, and f as in Lemma 69.4.3 we have

$$K \otimes_{\mathcal{B}}^{\mathbf{L}} Rf_*M = Rf_*(Lf^*K \otimes_{\mathcal{A}}^{\mathbf{L}} M)$$

in $D(\mathcal{B})$ for any $K \in D(\mathcal{B})$ and $M \in D(\mathcal{A})$.

Proof. Since f_* is exact (Lemma 69.4.1) the functor Rf_* is computed by applying f_* to any representative complex. Choose a complex $\mathcal{K}^\bullet$ of $\mathcal{B}$-modules representing K which is K-flat with flat terms, see Cohomology on Sites, Lemma 21.17.11. Then

$f^*\mathcal{K}^\bullet$ is K-flat with flat terms, see Cohomology on Sites, Lemma 21.18.1. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{A}$-modules representing M. Then we have to show

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{B} f_*\mathcal{M}^\bullet) = f_*\mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_\mathcal{A} \mathcal{M}^\bullet)$$

because by our choices these complexes represent the right and left hand side of the formula in the lemma. Since f_* commutes with direct sums (for example by the description of the stalks in Lemma 69.4.2), this reduces to the equalities

$$\mathcal{K}^n \otimes_\mathcal{B} f_*\mathcal{M}^m = f_*(f^*\mathcal{K}^n \otimes_\mathcal{A} \mathcal{M}^m)$$

which are true by Lemma 69.4.3. □

69.5. Colimits and cohomology

073D The following lemma in particular applies to diagrams of quasi-coherent sheaves.

073E Lemma 69.5.1. Let S be a scheme. Let X be an algebraic space over S. If X is quasi-compact and quasi-separated, then

$$\mathrm{colim}_i\, H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \mathrm{colim}_i\, \mathcal{F}_i)$$

is an isomorphism for every filtered diagram of abelian sheaves on $X_{étale}$.

Proof. This follows from Cohomology on Sites, Lemma 21.16.1. Namely, let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$ be the set of quasi-compact and quasi-separated spaces étale over X. Note that if $U \in \mathcal{B}$ then, because U is quasi-compact, the collection of finite coverings $\{U_i \to U\}$ with $U_i \in \mathcal{B}$ is cofinal in the set of coverings of U in $X_{spaces,étale}$. By Morphisms of Spaces, Lemma 67.8.10 the set $\mathcal{B}$ satisfies all the assumptions of Cohomology on Sites, Lemma 21.16.1. Since $X \in \mathcal{B}$ we win. □

07U6 Lemma 69.5.2. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $X_{étale}$. Then for any $p \geq 0$ we have

$$R^pf_*\mathcal{F} = \mathrm{colim}\, R^pf_*\mathcal{F}_i.$$

Proof. Recall that $R^pf_*\mathcal{F}$ is the sheaf on $Y_{spaces,étale}$ associated to $V \mapsto H^p(V \times_Y X, \mathcal{F})$, see Cohomology on Sites, Lemma 21.7.4 and Properties of Spaces, Lemma 66.18.8. Recall that the colimit is the sheaf associated to the presheaf colimit. Hence we can apply Lemma 69.5.1 to $H^p(V \times_Y X, -)$ where V is affine to conclude (because when V is affine, then $V \times_Y X$ is quasi-compact and quasi-separated). Strictly speaking this also uses Properties of Spaces, Lemma 66.18.6 to see that there exist enough affine objects. □

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic spaces.

07U7 Lemma 69.5.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let I be a directed set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation. Then we have

$$\mathrm{colim}_i\, \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \mathrm{Hom}_X(\mathcal{G}, \mathrm{colim}_i\, \mathcal{F}_i).$$

In particular, $\mathrm{Hom}_X(\mathcal{G}, -)$ commutes with filtered colimits in $QCoh(\mathcal{O}_X)$.

Proof. The displayed equality is a special case of Modules on Sites, Lemma 18.27.12. In order to apply it, we need to check the hypotheses of Sites, Lemma 7.17.8 part (4) for the site $X_{étale}$. In order to do this, we will check hypotheses (2)(a), (2)(b), (2)(c) of Sites, Remark 7.17.9. Namely, let $\mathcal{B} \subset \mathrm{Ob}(X_{étale})$ be the set of affine objects. Then

(1) Since X is quasi-compact, there exists a $U \in \mathcal{B}$ such that $U \to X$ is surjective (Properties of Spaces, Lemma 66.6.3), hence $h_U^\# \to *$ is surjective.
(2) For $U \in \mathcal{B}$ every étale covering $\{U_i \to U\}_{i \in I}$ of U can be refined by a finite étale covering $\{U_j \to U\}_{j=1,\ldots,m}$ with $U_j \in \mathcal{B}$ (Topologies, Lemma 34.4.4).
(3) For $U, U' \in \mathrm{Ob}(X_{étale})$ we have $h_U^\# \times h_{U'}^\# = h_{U \times_X U'}^\#$. If $U, U' \in \mathcal{B}$, then $U \times_X U'$ is quasi-compact because X is quasi-separated, see Morphisms of Spaces, Lemma 67.8.10 for example. Hence we can find a surjective étale morphism $U'' \to U \times_X U'$ with $U'' \in \mathcal{B}$ (Properties of Spaces, Lemma 66.6.3). In other words, we have morphisms $U'' \to U$ and $U'' \to U'$ such that the map $h_{U''}^\# \to h_U^\# \times h_{u'}^\#$ is surjective.

For the final statement, observe that the inclusion functor $QCoh(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$ commutes with colimits and that finitely presented modules are quasi-coherent. See Properties of Spaces, Lemma 66.29.7. □

69.6. The alternating Čech complex

0721 Let S be a scheme. Let $f : U \to X$ be an étale morphism of algebraic spaces over S. The functor

$$j : U_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V/U \longmapsto V/X$$

induces an equivalence of $U_{spaces,étale}$ with the localization $X_{spaces,étale}/U$, see Properties of Spaces, Section 66.27. Hence there exist functors

$$f_! : \mathrm{Ab}(U_{étale}) \longrightarrow \mathrm{Ab}(X_{étale}), \quad f_! : \mathrm{Mod}(\mathcal{O}_U) \longrightarrow \mathrm{Mod}(\mathcal{O}_X),$$

which are left adjoint to

$$f^{-1} : \mathrm{Ab}(X_{étale}) \longrightarrow \mathrm{Ab}(U_{étale}), \quad f^* : \mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{O}_U)$$

see Modules on Sites, Section 18.19. Warning: This functor, a priori, has nothing to do with cohomology with compact supports! We dubbed this functor "extension by zero" in the reference above. Note that the two versions of $f_!$ agree as $f^* = f^{-1}$ for sheaves of $\mathcal{O}_X$-modules.

As we are going to use this construction below let us recall some of its properties. Given an abelian sheaf $\mathcal{G}$ on $U_{étale}$ the sheaf $f_!$ is the sheafification of the presheaf

$$V/X \longmapsto f_!\mathcal{G}(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

see Modules on Sites, Lemma 18.19.2. Moreover, if $\mathcal{G}$ is an $\mathcal{O}_U$-module, then $f_!\mathcal{G}$ is the sheafification of the exact same presheaf of abelian groups which is endowed with an $\mathcal{O}_X$-module structure in an obvious way (see loc. cit.). Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point. Then there is a canonical identification

$$(f_!\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

where the sum is over all $\overline{u} : \mathrm{Spec}(k) \to U$ such that $f \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 18.38.1 and Properties of Spaces, Lemma 66.19.13. In the following we are going to study the sheaf $f_!\underline{\mathbf{Z}}$. Here $\underline{\mathbf{Z}}$ denotes the constant sheaf on $X_{étale}$ or $U_{étale}$.

0722 Lemma 69.6.1. Let S be a scheme. Let $f_i : U_i \to X$ be étale morphisms of algebraic spaces over S. Then there are isomorphisms

$$f_{1,!}\underline{\mathbf{Z}} \otimes_{\mathbf{Z}} f_{2,!}\underline{\mathbf{Z}} \longrightarrow f_{12,!}\underline{\mathbf{Z}}$$

where $f_{12} : U_1 \times_X U_2 \to X$ is the structure morphism and

$$(f_1 \amalg f_2)_!\underline{\mathbf{Z}} \longrightarrow f_{1,!}\underline{\mathbf{Z}} \oplus f_{2,!}\underline{\mathbf{Z}}$$

Proof. Once we have defined the map it will be an isomorphism by our description of stalks above. To define the map it suffices to work on the level of presheaves. Thus we have to define a map

$$\left(\bigoplus\nolimits_{\varphi_1 \in \mathrm{Mor}_X(V, U_1)} \mathbf{Z}\right) \otimes_{\mathbf{Z}} \left(\bigoplus\nolimits_{\varphi_2 \in \mathrm{Mor}_X(V, U_2)} \mathbf{Z}\right) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V, U_1 \times_X U_2)} \mathbf{Z}$$

We map the element $1_{\varphi_1} \otimes 1_{\varphi_2}$ to the element $1_{\varphi_1 \times \varphi_2}$ with obvious notation. We omit the proof of the second equality. $\square$

Another important feature is the trace map

$$\mathrm{Tr}_f : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}.$$

The trace map is adjoint to the map $\mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$ (which is an isomorphism). If $\overline{x}$ is above, then Tr_f on stalks at $\overline{x}$ is the map

$$(\mathrm{Tr}_f)_{\overline{x}} : (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{x}}$$

which sums the given integers. This is true because it is adjoint to the map $1 : \mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$. In particular, if f is surjective as well as étale then Tr_f is surjective.

Assume that $f : U \to X$ is a surjective étale morphism of algebraic spaces. Consider the Koszul complex associated to the trace map we discussed above

$$\ldots \to \wedge^3 f_!\underline{\mathbf{Z}} \to \wedge^2 f_!\underline{\mathbf{Z}} \to f_!\underline{\mathbf{Z}} \to \underline{\mathbf{Z}} \to 0$$

Here the exterior powers are over the sheaf of rings $\underline{\mathbf{Z}}$. The maps are defined by the rule

$$e_1 \wedge \ldots \wedge e_n \longmapsto \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \mathrm{Tr}_f(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$$

where $e_1, \ldots, e_n$ are local sections of $f_!\underline{\mathbf{Z}}$. Let $\overline{x}$ be a geometric point of X and set $M_{\overline{x}} = (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus_{\overline{u}} \mathbf{Z}$. Then the stalk of the complex above at $\overline{x}$ is the complex

$$\ldots \to \wedge^3 M_{\overline{x}} \to \wedge^2 M_{\overline{x}} \to M_{\overline{x}} \to \mathbf{Z} \to 0$$

which is exact because $M_{\overline{x}} \to \mathbf{Z}$ is surjective, see More on Algebra, Lemma 15.28.5. Hence if we let $K^\bullet = K^\bullet(f)$ be the complex with $K^i = \wedge^{i+1} f_!\underline{\mathbf{Z}}$, then we obtain a quasi-isomorphism

0723 (69.6.1.1) $$K^\bullet \longrightarrow \underline{\mathbf{Z}}[0]$$

We use the complex $K^\bullet$ to define what we call the alternating Čech complex associated to $f : U \to X$.

0724 Definition 69.6.2. Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $\textit{Ab}(X_{étale})$. The alternating Čech complex[1] $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ associated to $\mathcal{F}$ and f is the complex

$$\mathrm{Hom}(K^0, \mathcal{F}) \to \mathrm{Hom}(K^1, \mathcal{F}) \to \mathrm{Hom}(K^2, \mathcal{F}) \to \ldots$$

with Hom groups computed in $\textit{Ab}(X_{étale})$.

The reader may verify that if $U = \coprod U_i$ and $f|_{U_i} : U_i \to X$ is the open immersion of a subspace, then $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ agrees with the complex introduced in Cohomology, Section 20.23 for the Zariski covering $X = \bigcup U_i$ and the restriction of $\mathcal{F}$ to the Zariski site of X. What is more important however, is to relate the cohomology of the alternating Čech complex to the cohomology.

0725 Lemma 69.6.3. Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $\textit{Ab}(X_{étale})$. There exists a canonical map

$$\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D(\textit{Ab})$. Moreover, there is a spectral sequence with E_1-page

$$E_1^{p,q} = \mathrm{Ext}^q_{\textit{Ab}(X_{étale})}(K^p, \mathcal{F})$$

converging to $H^{p+q}(X, \mathcal{F})$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Recall that we have the quasi-isomorphism $K^\bullet \to \underline{\mathbf{Z}}[0]$, see (69.6.1.1). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $\textit{Ab}(X_{étale})$. Consider the double complex $\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet)$ with terms $\mathrm{Hom}(K^p, \mathcal{I}^q)$. The differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $K^{p+1} \to K^p$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $\mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))$ the associated total complex, see Homology, Section 12.18. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.25.

Because $K^\bullet$ is a resolution of $\underline{\mathbf{Z}}$ we see that the complexes

$$\mathrm{Hom}(K^\bullet, \mathcal{I}^q) : \mathrm{Hom}(K^0, \mathcal{I}^q) \to \mathrm{Hom}(K^1, \mathcal{I}^q) \to \mathrm{Hom}(K^2, \mathcal{I}^q) \to \ldots$$

are acyclic in positive degrees and have H^0 equal to $\Gamma(X, \mathcal{I}^q)$. Hence by Homology, Lemma 12.25.4 the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow \mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(\mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))) = H^n(X, \mathcal{F})$.

The map $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to R\Gamma(X, \mathcal{F})$ of the lemma is the composition of $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to \mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))$ with the inverse of the displayed quasi-isomorphism.

Finally, consider the spectral sequence $({}'E_r, {}'d_r)$. We have

$$E_1^{p,q} = q\text{th cohomology of } \mathrm{Hom}(K^p, \mathcal{I}^0) \to \mathrm{Hom}(K^p, \mathcal{I}^1) \to \mathrm{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

This proves the lemma. □

It follows from the lemma that it is important to understand the ext groups $\mathrm{Ext}_{\textit{Ab}(X_{étale})}(K^p, \mathcal{F})$, i.e., the right derived functors of $\mathcal{F} \mapsto \mathrm{Hom}(K^p, \mathcal{F})$.

[1] This may be nonstandard notation

0726 Lemma 69.6.4. Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set

$$W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals}$$

where the fibre product has $p+1$ factors. There is a free action of S_{p+1} on W_p over X and

$$\mathrm{Hom}(K^p, \mathcal{F}) = S_{p+1}\text{-anti-invariant elements of } \mathcal{F}(W_p)$$

functorially in $\mathcal{F}$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Because $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 67.39.10 and 67.38.9. Hence W_p is an open and closed subspace of $U^{p+1} = U \times_X \ldots \times_X U$. The action of S_{p+1} on W_p is free as we've thrown out the fixed points of the action. By Lemma 69.6.1 we see that

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} = f_!^{p+1}\underline{\mathbf{Z}} = (W_p \to X)_!\underline{\mathbf{Z}} \oplus \mathit{Rest}$$

where $f^{p+1} : U^{p+1} \to X$ is the structure morphism. Looking at stalks over a geometric point $\overline{x}$ of X we see that

$$\left(\bigoplus\nolimits_{\overline{u} \mapsto \overline{x}} \mathbf{Z}\right)^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}}_{\overline{x}}$$

is the quotient whose kernel is generated by all tensors $1_{\overline{u}_0} \otimes \ldots \otimes 1_{\overline{u}_p}$ where $\overline{u}_i = \overline{u}_j$ for some $i \neq j$. Thus the quotient map

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

factors through $(W_p \to X)_!\underline{\mathbf{Z}}$, i.e., we get

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

This already proves that $\mathrm{Hom}(K^p, \mathcal{F})$ is (functorially) a subgroup of

$$\mathrm{Hom}((W_p \to X)_!\underline{\mathbf{Z}}, \mathcal{F}) = \mathcal{F}(W_p)$$

To identify it with the S_{p+1}-anti-invariants we have to prove that the surjection $(W_p \to X)_!\underline{\mathbf{Z}} \to \wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the maximal S_{p+1}-anti-invariant quotient. In other words, we have to show that $\wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the quotient of $(W_p \to X)_!\underline{\mathbf{Z}}$ by the subsheaf generated by the local sections $s - \mathrm{sign}(\sigma)\sigma(s)$ where s is a local section of $(W_p \to X)_!\underline{\mathbf{Z}}$. This can be checked on the stalks, where it is clear. □

0727 Lemma 69.6.5. Let S be a scheme. Let W be an algebraic space over S. Let G be a finite group acting freely on W. Let $U = W/G$, see Properties of Spaces, Lemma 66.34.1. Let $\chi : G \to \{+1, -1\}$ be a character. Then there exists a rank 1 locally free sheaf of $\mathbf{Z}$-modules $\underline{\mathbf{Z}}(\chi)$ on $U_{\acute{e}tale}$ such that for every abelian sheaf $\mathcal{F}$ on $U_{\acute{e}tale}$ we have

$$H^0(W, \mathcal{F}|_W)^\chi = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi))$$

Proof. The quotient morphism $q : W \to U$ is a G-torsor, i.e., there exists a surjective étale morphism $U' \to U$ such that $W \times_U U' = \coprod_{g \in G} U'$ as spaces with G-action over U'. (Namely, $U' = W$ works.) Hence $q_*\underline{\mathbf{Z}}$ is a finite locally free $\mathbf{Z}$-module with an action of G. For any geometric point $\overline{u}$ of U, then we get G-equivariant isomorphisms

$$(q_*\underline{\mathbf{Z}})_{\overline{u}} = \bigoplus\nolimits_{\overline{w} \mapsto \overline{u}} \mathbf{Z} = \bigoplus\nolimits_{g \in G} \mathbf{Z} = \mathbf{Z}[G]$$

where the second = uses a geometric point $\overline{w}_0$ lying over $\overline{u}$ and maps the summand corresponding to $g \in G$ to the summand corresponding to $g(\overline{w}_0)$. We have

$$H^0(W, \mathcal{F}|_W) = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})$$

because $q_*\mathcal{F}|_W = \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}}$ as one can check by restricting to U'. Let

$$\underline{\mathbf{Z}}(\chi) = (q_*\underline{\mathbf{Z}})^\chi \subset q_*\underline{\mathbf{Z}}$$

be the subsheaf of sections that transform according to χ. For any geometric point $\overline{u}$ of U we have

$$\underline{\mathbf{Z}}(\chi)_{\overline{u}} = \mathbf{Z} \cdot \sum\nolimits_g \chi(g)g \subset \mathbf{Z}[G] = (q_*\underline{\mathbf{Z}})_{\overline{u}}$$

It follows that $\underline{\mathbf{Z}}(\chi)$ is locally free of rank 1 (more precisely, this should be checked after restricting to U'). Note that for any $\mathbf{Z}$-module M the χ-semi-invariants of $M[G]$ are the elements of the form $m \cdot \sum_g \chi(g)g$. Thus we see that for any abelian sheaf $\mathcal{F}$ on U we have

$$(\mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})^\chi = \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$$

because we have equality at all stalks. The result of the lemma follows by taking global sections. □

Now we can put everything together and obtain the following pleasing result.

0728 Lemma 69.6.6. Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set

$$W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals}$$

(with $p + 1$ factors) as in Lemma 69.6.4. Let $\chi_p : S_{p+1} \to \{+1, -1\}$ be the sign character. Let $U_p = W_p/S_{p+1}$ and $\underline{\mathbf{Z}}(\chi_p)$ be as in Lemma 69.6.5. Then the spectral sequence of Lemma 69.6.3 has E_1-page

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$$

and converges to $H^{p+q}(X, \mathcal{F})$.

Proof. Note that since the action of S_{p+1} on W_p is over X we do obtain a morphism $U_p \to X$. Since $W_p \to X$ is étale and since $W_p \to U_p$ is surjective étale, it follows that also $U_p \to X$ is étale, see Morphisms of Spaces, Lemma 67.39.2. Therefore an injective object of $\text{Ab}(X_{\acute{e}tale})$ restricts to an injective object of $\text{Ab}(U_{p,\acute{e}tale})$, see Cohomology on Sites, Lemma 21.7.1. Moreover, the functor $\mathcal{G} \mapsto \mathcal{G} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is an auto-equivalence of $\text{Ab}(U_p)$, whence transforms injective objects into injective objects and is exact (because $\underline{\mathbf{Z}}(\chi_p)$ is an invertible $\underline{\mathbf{Z}}$-module). Thus given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $\text{Ab}(X_{\acute{e}tale})$ the complex

$$\Gamma(U_p, \mathcal{I}^0|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^1|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^2|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \ldots$$

computes $H^*(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$. On the other hand, by Lemma 69.6.5 it is equal to the complex of S_{p+1}-anti-invariants in

$$\Gamma(W_p, \mathcal{I}^0) \to \Gamma(W_p, \mathcal{I}^1) \to \Gamma(W_p, \mathcal{I}^2) \to \ldots$$

which by Lemma 69.6.4 is equal to the complex

$$\text{Hom}(K^p, \mathcal{I}^0) \to \text{Hom}(K^p, \mathcal{I}^1) \to \text{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

which computes $\text{Ext}^*_{\text{Ab}(X_{\acute{e}tale})}(K^p, \mathcal{F})$. Putting everything together we win. □

69.7. Higher vanishing for quasi-coherent sheaves

0729 In this section we show that given a quasi-compact and quasi-separated algebraic space X there exists an integer $n = n(X)$ such that the cohomology of any quasi-coherent sheaf on X vanishes beyond degree n.

072A Lemma 69.7.1. With S, W, G, U, χ as in Lemma 69.6.5. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, then so is $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$.

Proof. The $\mathcal{O}_U$-module structure is clear. To check that $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$ is quasi-coherent it suffices to check étale locally. Hence the lemma follows as $\underline{\mathbf{Z}}(\chi)$ is finite locally free as a $\underline{\mathbf{Z}}$-module. □

The following proposition is interesting even if X is a scheme. It is the natural generalization of Cohomology of Schemes, Lemma 30.4.2. Before we state it, observe that given an étale morphism $f : U \to X$ from an affine scheme towards a quasi-separated algebraic space X the fibres of f are universally bounded, in particular there exists an integer d such that the fibres of $|U| \to |X|$ all have size at most d; this is the implication $(\eta) \Rightarrow (\delta)$ of Decent Spaces, Lemma 68.5.1.

072B Proposition 69.7.2. Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and separated. Let U be an affine scheme, and let $f : U \to X$ be a surjective étale morphism. Let d be an upper bound for the size of the fibres of $|U| \to |X|$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq d$.

Proof. We will use the spectral sequence of Lemma 69.6.6. The lemma applies since f is separated as U is separated, see Morphisms of Spaces, Lemma 67.4.10. Since X is separated the scheme $U \times_X \ldots \times_X U$ is a closed subscheme of $U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$ hence is affine. Thus W_p is affine. Hence $U_p = W_p/S_{p+1}$ is an affine scheme by Groupoids, Proposition 39.23.9. The discussion in Section 69.3 shows that cohomology of quasi-coherent sheaves on W_p (as an algebraic space) agrees with the cohomology of the corresponding quasi-coherent sheaf on the underlying affine scheme, hence vanishes in positive degrees by Cohomology of Schemes, Lemma 30.2.2. By Lemma 69.7.1 the sheaves $\mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)$ are quasi-coherent. Hence $H^q(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $q > 0$. By our definition of the integer d we see that $W_p = \emptyset$ for $p \geq d$. Hence also $H^0(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $p \geq d$. This proves the proposition. □

In the following lemma we establish that a quasi-compact and quasi-separated algebraic space has finite cohomological dimension for quasi-coherent modules. We are explicit about the bound only because we will use it later to prove a similar result for higher direct images.

072C Lemma 69.7.3. Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Then we can choose

(1) an affine scheme U,
(2) a surjective étale morphism $f : U \to X$,
(3) an integer d bounding the degrees of the fibres of $U \to X$,
(4) for every $p = 0, 1, \ldots, d$ a surjective étale morphism $V_p \to U_p$ from an affine scheme V_p where U_p is as in Lemma 69.6.6, and
(5) an integer d_p bounding the degree of the fibres of $V_p \to U_p$.

Moreover, whenever we have (1) – (5), then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq \max(d_p + p)$.

Proof. Since X is quasi-compact we can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 66.6.3. By Decent Spaces, Lemma 68.5.1 the fibres of f are universally bounded, hence we can find d. We have $U_p = W_p/S_{p+1}$ and $W_p \subset U \times_X \ldots \times_X U$ is open and closed. Since X is quasi-separated the schemes W_p are quasi-compact, hence U_p is quasi-compact. Since U is separated, the schemes W_p are separated, hence U_p is separated by (the absolute version of) Spaces, Lemma 65.14.5. By Properties of Spaces, Lemma 66.6.3 we can find the morphisms $V_p \to W_p$. By Decent Spaces, Lemma 68.5.1 we can find the integers d_p.

At this point the proof uses the spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

see Lemma 69.6.6. By definition of the integer d we see that $U_p = 0$ for $p \geq d$. By Proposition 69.7.2 and Lemma 69.7.1 we see that $H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero for $q \geq d_p$ for $p = 0, \ldots, d$. Whence the lemma. $\square$

69.8. Vanishing for higher direct images

073F We apply the results of Section 69.7 to obtain vanishing of higher direct images of quasi-coherent sheaves for quasi-compact and quasi-separated morphisms. This is useful because it allows one to argue by descending induction on the cohomological degree in certain situations.

073G Lemma 69.8.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that

(1) f is quasi-compact and quasi-separated, and
(2) Y is quasi-compact.

Then there exists an integer $n(X \to Y)$ such that for any algebraic space Y', any morphism $Y' \to Y$ and any quasi-coherent sheaf $\mathcal{F}'$ on $X' = Y' \times_Y X$ the higher direct images $R^i f'_* \mathcal{F}'$ are zero for $i \geq n(X \to Y)$.

Proof. Let $V \to Y$ be a surjective étale morphism where V is an affine scheme, see Properties of Spaces, Lemma 66.6.3. Suppose we prove the result for the base change $f_V : V \times_Y X \to V$. Then the result holds for f with $n(X \to Y) = n(X_V \to V)$. Namely, if $Y' \to Y$ and $\mathcal{F}'$ are as in the lemma, then $R^i f'_* \mathcal{F}'|_{V \times_Y Y'}$ is equal to $R^i f'_{V,*} \mathcal{F}'|_{X'_V}$ where $f'_V : X'_V = V \times_Y Y' \times_Y X \to V \times_Y Y' = Y'_V$, see Properties of Spaces, Lemma 66.26.2. Thus we may assume that Y is an affine scheme.

Moreover, to prove the vanishing for all $Y' \to Y$ and $\mathcal{F}'$ it suffices to do so when Y' is an affine scheme. In this case, $R^i f'_* \mathcal{F}'$ is quasi-coherent by Lemma 69.3.1. Hence it suffices to prove that $H^i(X', \mathcal{F}') = 0$, because $H^i(X', \mathcal{F}') = H^0(Y', R^i f'_* \mathcal{F}')$ by Cohomology on Sites, Lemma 21.14.6 and the vanishing of higher cohomology of quasi-coherent sheaves on affine algebraic spaces (Proposition 69.7.2).

Choose $U \to X$, d, $V_p \to U_p$ and d_p as in Lemma 69.7.3. For any affine scheme Y' and morphism $Y' \to Y$ denote $X' = Y' \times_Y X$, $U' = Y' \times_Y U$, $V'_p = Y' \times_Y V_p$. Then $U' \to X'$, $d' = d$, $V'_p \to U'_p$ and $d'_p = d$ is a collection of choices as in Lemma 69.7.3 for the algebraic space X' (details omitted). Hence we see that $H^i(X', \mathcal{F}') = 0$ for $i \geq \max(p + d_p)$ and we win. $\square$

073H Lemma 69.8.2. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $R^i f_*\mathcal{F} = 0$ for $i > 0$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Proof. Recall that an affine morphism of algebraic spaces is representable. Hence this follows from (69.3.0.1) and Cohomology of Schemes, Lemma 30.2.3. $\square$

0D2U Lemma 69.8.3. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for all $i \geq 0$.

Proof. Follows from Lemma 69.8.2 and the Leray spectral sequence. See Cohomology on Sites, Lemma 21.14.6. $\square$

69.9. Cohomology with support in a closed subspace

0A4L This section is the analogue of Cohomology, Sections 20.21 and 20.34 and Étale Cohomology, Section 59.79 for abelian sheaves on algebraic spaces.

Let S be a scheme. Let X be an algebraic space over S and let $Z \subset X$ be a closed subspace. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z\}$$

be the sections with support in Z (Properties of Spaces, Definition 66.20.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{étale}) \longrightarrow D(\text{Ab})$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Let $U \subset X$ be the open subspace which is the complement of Z. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 21.12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{étale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(\text{Ab})$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{étale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we can consider the subsheaf of sections with support in Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Properties of Spaces, Definition 66.20.3. Using the equivalence of Morphisms of Spaces, Lemma 67.13.5 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$\text{Ab}(X_{étale}) \longrightarrow \text{Ab}(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

0A4M Lemma 69.9.1. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\mathrm{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Lemma 69.4.1) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

0A4N Lemma 69.9.2. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.

Proof. This is true because the functor i_* is exact (Lemma 69.4.1) and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 21.14.2). □

0A4P Lemma 69.9.3. Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \subset Y$ be a closed subspace such that $f^{-1}(Z) \to Z$ is an isomorphism of algebraic spaces. Let $\mathcal{F}$ be an abelian sheaf on X. Then

$$\mathcal{H}^q_Z(\mathcal{F}) = \mathcal{H}^q_{f^{-1}(Z)}(f^{-1}\mathcal{F})$$

as abelian sheaves on $Z = f^{-1}(Z)$ and we have $H^q_Z(Y, \mathcal{F}) = H^q_{f^{-1}(Z)}(X, f^{-1}\mathcal{F})$.

Proof. Because f is étale an injective resolution of $\mathcal{F}$ pulls back to an injective resolution of $f^{-1}\mathcal{F}$. Hence it suffices to check the equality for $\mathcal{H}_Z(-)$ which follows from the definitions. The proof for cohomology with supports is the same. Some details omitted. □

Let S be a scheme and let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. We denote $D_T(X_{étale})$ the strictly full saturated triangulated subcategory of $D(X_{étale})$ consisting of objects whose cohomology sheaves are supported on T.

0AEI Lemma 69.9.4. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. The map $Ri_* = i_* : D(Z_{étale}) \to D(X_{étale})$ induces an equivalence $D(Z_{étale}) \to D_{|Z|}(X_{étale})$ with quasi-inverse

$$i^{-1}|_{D_Z(X_{étale})} = R\mathcal{H}_Z|_{D_{|Z|}(X_{étale})}$$

Proof. Recall that i^{-1} and i_* is an adjoint pair of exact functors such that $i^{-1}i_*$ is isomorphic to the identify functor on abelian sheaves. See Properties of Spaces, Lemma 66.19.9 and Morphisms of Spaces, Lemma 67.13.5. Thus $i_* : D(Z_{étale}) \to D_Z(X_{étale})$ is fully faithful and i^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X_{étale})$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of i_* and i^{-1} this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z_{étale}) \to D_Z(X_{étale})$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if K is an object of $D_Z(X_{étale})$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i^{-1}K$. Since i_* is the right adjoint to the exact functor i^{-1}, the complex $i_*\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.9). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. $\square$

69.10. Vanishing above the dimension

0A4Q Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. In this case $|X|$ is a spectral space, see Properties of Spaces, Lemma 66.15.2. Moreover, the dimension of X (as defined in Properties of Spaces, Definition 66.9.2) is equal to the Krull dimension of $|X|$, see Decent Spaces, Lemma 68.12.5. We will show that for quasi-coherent sheaves on X we have vanishing of cohomology above the dimension. This result is already interesting for quasi-separated algebraic spaces of finite type over a field.

0A4R **Lemma 69.10.1.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume $\dim(X) \leq d$ for some integer d. Let $\mathcal{F}$ be a quasi-coherent sheaf $\mathcal{F}$ on X.

(1) $H^q(X, \mathcal{F}) = 0$ for $q > d$,
(2) $H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for any quasi-compact open $U \subset X$,
(3) $H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ for any closed subspace $Z \subset X$ whose complement is quasi-compact.

Proof. By Properties of Spaces, Lemma 66.22.5 every algebraic space Y étale over X has dimension $\leq d$. If Y is quasi-separated, the dimension of Y is equal to the Krull dimension of $|Y|$ by Decent Spaces, Lemma 68.12.5. Also, if Y is a scheme, then étale cohomology of $\mathcal{F}$ over Y, resp. étale cohomology of $\mathcal{F}$ with support in a closed subscheme, agrees with usual cohomology of $\mathcal{F}$, resp. usual cohomology with support in the closed subscheme. See Descent, Proposition 35.9.3 and Étale Cohomology, Lemma 59.79.5. We will use these facts without further mention.

By Decent Spaces, Lemma 68.8.6 there exist an integer n and open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

As $U_n = V_n$ is a scheme, our initial remarks imply the cohomology of $\mathcal{F}$ over U_n vanishes in degrees $> d$ by Cohomology, Proposition 20.22.4. Suppose we have shown, by induction, that $H^q(U_{p+1}, \mathcal{F}|_{U_{p+1}}) = 0$ for $q > d$. It suffices to show $H^q_{T_p}(U_p, \mathcal{F})$ for $q > d$ is zero in order to conclude the vanishing of cohomology of $\mathcal{F}$ over U_p in degrees $> d$. However, we have

$$H^q_{T_p}(U_p, \mathcal{F}) = H^q_{f_p^{-1}(T_p)}(V_p, \mathcal{F})$$

by Lemma 69.9.3 and as V_p is a scheme we obtain the desired vanishing from Cohomology, Proposition 20.22.4. In this way we conclude that (1) is true.

To prove (2) let $U \subset X$ be a quasi-compact open subspace. Consider the open subspace $U' = U \cup U_n$. Let $Z = U' \setminus U$. Then $g : U_n \to U'$ is an étale morphism such that $g^{-1}(Z) \to Z$ is an isomorphism. Hence by Lemma 69.9.3 we have $H^q_Z(U', \mathcal{F}) = H^q_Z(U_n, \mathcal{F})$ which vanishes in degree $> d$ because U_n is a scheme and we can apply Cohomology, Proposition 20.22.4. We conclude that $H^d(U', \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective. Assume, by induction, that we have reduced our problem to the case where U contains U_{p+1}. Then we set $U' = U \cup U_p$, set $Z = U' \setminus U$, and we argue using the morphism $f_p : V_p \to U'$ which is étale and has the property that $f_p^{-1}(Z) \to Z$ is an isomorphism. In other words, we again see that

$$H^q_Z(U', \mathcal{F}) = H^q_{f_p^{-1}(Z)}(V_p, \mathcal{F})$$

and we again see this vanishes in degrees $> d$. We conclude that $H^d(U', \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective. Eventually we reach the stage where $U_1 = X \subset U$ which finishes the proof.

A formal argument shows that (2) implies (3). □

69.11. Cohomology and base change, I

073I Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : Y' \to Y$ is a morphism of algebraic spaces over S. Denote $X' = X_{Y'} = Y' \times_Y X$ the base change of X and denote $f' : X' \to Y'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

073J (69.11.0.1)
$$\begin{tikzcd} \mathcal{F}' = (g')^*\mathcal{F} & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] & \mathcal{F} \\ Rf'_*\mathcal{F}' & Y' \arrow[r, "g"] & Y & Rf_*\mathcal{F} \end{tikzcd}$$

Here is the simplest case of the base change property we have in mind.

07U8 **Lemma 69.11.1.** Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every diagram (69.11.0.1) we have

$$g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}.$$

Proof. By the discussion surrounding (69.3.0.1) this reduces to the case of an affine morphism of schemes which is treated in Cohomology of Schemes, Lemma 30.5.1. □

073K **Lemma 69.11.2** (Flat base change). Let S be a scheme. Consider a cartesian diagram of algebraic spaces

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{F}' = (g')^*\mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

(1) the base change map of Cohomology on Sites, Lemma 21.15.1 is an isomorphism
$$g^*R^if_*\mathcal{F} \longrightarrow R^if'_*\mathcal{F}',$$
(2) if $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.

Proof. The morphism g' is flat by Morphisms of Spaces, Lemma 67.30.4. Note that flatness of g and g' is equivalent to flatness of the morphisms of small étale ringed sites, see Morphisms of Spaces, Lemma 67.30.9. Hence we can apply Cohomology on Sites, Lemma 21.15.1 to obtain a base change map

$$g^*R^pf_*\mathcal{F} \longrightarrow R^pf'_*\mathcal{F}'$$

To prove this map is an isomorphism we can work locally in the étale topology on Y'. Thus we may assume that Y and Y' are affine schemes. Say $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$. In this case we are really trying to show that the map

$$H^p(X, \mathcal{F}) \otimes_A B \longrightarrow H^p(X_B, \mathcal{F}_B)$$

is an isomorphism where $X_B = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B. In other words, it suffices to prove (2).

Fix $A \to B$ a flat ring map and let X be a quasi-compact and quasi-separated algebraic space over A. Note that $g' : X_B \to X$ is affine as a base change of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Hence the higher direct images $R^i(g')_*\mathcal{F}_B$ are zero by Lemma 69.8.2. Thus $H^p(X_B, \mathcal{F}_B) = H^p(X, g'_*\mathcal{F}_B)$, see Cohomology on Sites, Lemma 21.14.6. Moreover, we have

$$g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$$

where $\underline{A}$, $\underline{B}$ denotes the constant sheaf of rings with value A, B. Namely, it is clear that there is a map from right to left. For any affine scheme U étale over X we have

$$\begin{aligned} g'_*\mathcal{F}_B(U) &= \mathcal{F}_B(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U) \\ &= \Gamma(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U, (\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U \to U)^*\mathcal{F}|_U) \\ &= B \otimes_A \mathcal{F}(U) \end{aligned}$$

hence the map is an isomorphism. Write $B = \mathrm{colim}\, M_i$ as a filtered colimit of finite free A-modules M_i using Lazard's theorem, see Algebra, Theorem 10.81.4. We deduce that

$$\begin{aligned} H^p(X, g'_*\mathcal{F}_B) &= H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{B}) \\ &= H^p(X, \mathrm{colim}_i\, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= \mathrm{colim}_i\, H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= \mathrm{colim}_i\, H^p(X, \mathcal{F}) \otimes_A M_i \\ &= H^p(X, \mathcal{F}) \otimes_A \mathrm{colim}_i\, M_i \\ &= H^p(X, \mathcal{F}) \otimes_A B \end{aligned}$$

The first equality because $g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$ as seen above. The second because $\otimes$ commutes with colimits. The third equality because cohomology on X commutes with colimits (see Lemma 69.5.1). The fourth equality because M_i is finite free (i.e., because cohomology commutes with finite direct sums). The fifth because $\otimes$ commutes with colimits. The sixth by choice of our system. $\square$

69.12. Coherent modules on locally Noetherian algebraic spaces

07U9 This section is the analogue of Cohomology of Schemes, Section 30.9. In Modules on Sites, Definition 18.23.1 we have defined coherent modules on any ringed topos. We use this notion to define coherent modules on locally Noetherian algebraic spaces.

Although it is possible to work with coherent modules more generally we resist the urge to do so.

07UA Definition 69.12.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. A quasi-coherent module $\mathcal{F}$ on X is called coherent if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on the site $X_{\acute{e}tale}$ in the sense of Modules on Sites, Definition 18.23.1.

This definition is compatible with the already existing notion of a coherent module on a locally Noetherian scheme; see assertion (5) of Properties of Spaces, Section 66.30 (or more directly Descent, Lemma 35.8.10). Thus from now on, if X is a locally Noetherian scheme over S, we will not distinguish between a coherent module on X viewed as a scheme or a coherent module on X viewed as an algebraic space; this is compatible with the corresponding identifications of categories of quasi-coherent modules discussed in Properties of Spaces, Section 66.29.

Having said the above, the following lemma gives an understandable characterization of coherent modules on locally Noetherian algebraic spaces.

07UB Lemma 69.12.2. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is coherent,
(2) $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,
(3) $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,
(4) for any étale morphism $\varphi : U \to X$ where U is a scheme the pullback $\varphi^*\mathcal{F}$ is a coherent module on U, and
(5) there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that the pullback $\varphi^*\mathcal{F}$ is a coherent module on U.

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is coherent.

Proof. To be sure, if X is a locally Noetherian algebraic space and $U \to X$ is an étale morphism, then U is locally Noetherian, see Properties of Spaces, Section 66.7. The lemma then follows from the points (1) – (5) made in Properties of Spaces, Section 66.30 and the corresponding result for coherent modules on locally Noetherian schemes, see Cohomology of Schemes, Lemma 30.9.1. $\square$

07UC Lemma 69.12.3. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (66.26.1.1). By Lemma 69.12.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent by checking whether $f^*\mathcal{F}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 30.9.2. $\square$

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_X$-modules. This does not hold for modules on a general ringed topos.

07UD Lemma 69.12.4. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (66.26.1.1). By Lemma 69.12.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{G}$ is coherent by checking whether $f^*\mathcal{H}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 30.9.3. $\square$

07UE Lemma 69.12.5. Let S be a scheme. Let X be a locally Noetherian algebraic space over S,. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.

Proof. Via Lemma 69.12.2 this follows from the result for schemes, see Cohomology of Schemes, Lemma 30.9.4. $\square$

07UF Lemma 69.12.6. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $\overline{x}$ be a geometric point of X lying over $x \in |X|$.

(1) If $\mathcal{F}_{\overline{x}} = 0$ then there exists an open neighbourhood $X' \subset X$ of x such that $\mathcal{F}|_{X'} = 0$.
(2) If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is injective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is injective.
(3) If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is surjective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is surjective.
(4) If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is bijective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is an isomorphism.

Proof. Let $\varphi : U \to X$ be an étale morphism where U is a scheme and let $u \in U$ be a point mapping to x. By Properties of Spaces, Lemmas 66.29.4 and 66.22.1 as well as More on Algebra, Lemma 15.45.1 we see that $\varphi_{\overline{x}}$ is injective, surjective, or bijective if and only if $\varphi_u : \varphi^*\mathcal{F}_u \to \varphi^*\mathcal{G}_u$ has the corresponding property. Thus we can apply the schemes version of this lemma to see that (after possibly shrinking U) the map $\varphi^*\mathcal{F} \to \varphi^*\mathcal{G}$ is injective, surjective, or an isomorphism. Let $X' \subset X$ be the open subspace corresponding to $|\varphi|(|U|) \subset |X|$, see Properties of Spaces, Lemma 66.4.8. Since $\{U \to X'\}$ is a covering for the étale topology, we conclude that $\varphi|_{X'}$ is injective, surjective, or an isomorphism as desired. Finally, observe that (1) follows from (2) by looking at the map $\mathcal{F} \to 0$. $\square$

07UG Lemma 69.12.7. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and $\mathcal{G}$ the quasi-coherent $\mathcal{O}_Z$-module such that $i_*\mathcal{G} = \mathcal{F}$, see Morphisms of Spaces, Definition 67.15.4. Then $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module.

Proof. The statement of the lemma makes sense as a coherent module is in particular of finite type. Moreover, as $Z \to X$ is a closed immersion it is locally of finite type and hence Z is locally Noetherian, see Morphisms of Spaces, Lemmas 67.23.7 and 67.23.5. Finally, as $\mathcal{G}$ is of finite type it is a coherent $\mathcal{O}_Z$-module by Lemma 69.12.2 $\square$

08AM Lemma 69.12.8. Let S be a scheme. Let $i : Z \to X$ be a closed immersion of locally Noetherian algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.

Proof. The functor is fully faithful by Morphisms of Spaces, Lemma 67.14.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms of Spaces, Lemma 67.14.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. To check that $\mathcal{G}$ is coherent we can work étale locally (Lemma 69.12.2). Choosing an étale covering by a scheme we conclude that $\mathcal{G}$ is coherent by the case of schemes (Cohomology of Schemes, Lemma 30.9.8). Hence the functor is fully faithful and the proof is done. □

07UH **Lemma 69.12.9.** Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is a finite morphism of locally Noetherian schemes. By (69.3.0.1) we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.9.9. □

69.13. Coherent sheaves on Noetherian spaces

07UI In this section we mention some properties of coherent sheaves on Noetherian algebraic spaces.

07UJ **Lemma 69.13.1.** Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 67.23.5). If $\mathcal{F}_n|_U = \mathcal{F}_{n+1}|_U = \ldots$ then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 30.10.1. □

07UK **Lemma 69.13.2.** Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subspace $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $\text{Supp}(\mathcal{F}) \subset Z$ (set theoretically).

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 67.23.5). Note that $\mathcal{I}^n\mathcal{F}|_U = 0$ if and only if $\mathcal{I}^n\mathcal{F} = 0$ and similarly for the condition on the support. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 30.10.2. □

07UL **Lemma 69.13.3 (Artin-Rees).** Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F} \cap \mathcal{G}$$

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 67.23.5). The equality of the lemma holds if and only if it holds after restricting to U. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 30.10.3. $\square$

07UM Lemma 69.13.4. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subspace and set $U = X \setminus Z$. There is a canonical isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. Let W be an affine scheme and let $W \to X$ be a surjective étale morphism (see Properties of Spaces, Lemma 66.6.3). Set $R = W \times_X W$. Then W and R are Noetherian schemes, see Morphisms of Spaces, Lemma 67.23.5. Hence the result hold for the restrictions of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{I}$, U, Z to W and R by Cohomology of Schemes, Lemma 30.10.5. It follows formally that the result holds over X. $\square$

69.14. Devissage of coherent sheaves

07UN This section is the analogue of Cohomology of Schemes, Section 30.12.

07UP Lemma 69.14.1. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\mathrm{Supp}(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $\mathrm{Supp}(\mathcal{G}') \subset Z'$ and $\mathrm{Supp}(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subspace structure on Z, see Properties of Spaces, Lemma 66.12.3. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every geometric point $\overline{x}$ of $Z' \setminus Z$ we have $\mathcal{I}_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$ and hence $\mathcal{G}_{n,\overline{x}} = 0$. Thus we see that $\mathrm{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian algebraic space. Hence by Lemma 69.13.2 there exists an n such that $\mathcal{G}'_n|_{X \setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X \setminus Z'} = 0$. For such an n we see that $\mathrm{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. $\square$

In the following we will freely use the scheme theoretic support of finite type modules as defined in Morphisms of Spaces, Definition 67.15.4.

07UQ Lemma 69.14.2. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that the scheme theoretic support of $\mathcal{F}$ is a reduced $Z \subset X$ with $|Z|$ irreducible. Then there exist an integer $r > 0$, a nonzero sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$, and an injective map of coherent sheaves

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

whose cokernel is supported on a proper closed subspace of Z.

Proof. By assumption there exists a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with support Z and $\mathcal{F} \cong i_*\mathcal{G}$, see Lemma 69.12.7. Hence it suffices to prove the lemma for the case $Z = X$ and $i = \text{id}$.

By Properties of Spaces, Proposition 66.13.3 there exists a dense open subspace $U \subset X$ which is a scheme. Note that U is a Noetherian integral scheme. After shrinking U we may assume that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ (for example by Cohomology of Schemes, Lemma 30.12.2 or by a direct algebra argument). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals whose associated closed subspace is the complement of U in X (see for example Properties of Spaces, Section 66.12). By Lemma 69.13.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our isomorphism over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective: namely, if σ is a nonzero section of $\mathcal{I}^{\oplus r}$ over a scheme W étale over X, then because X hence W is reduced the support of σ contains a nonempty open of W. But the kernel of $(\mathcal{I}^n)^{\oplus r} \to \mathcal{F}$ is zero over a dense open, hence σ cannot be a section of the kernel. $\square$

07UR Lemma 69.14.3. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists a reduced closed subspace $Z_j \subset X$ with $|Z_j|$ irreducible and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{matrix} T \subset |X| \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \text{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{matrix} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 66.24.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold. Clearly $T \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 69.14.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\text{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that T is irreducible.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 66.12.3. By Lemma 69.13.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Then the scheme theoretic support of $\mathcal{F}$ is T, see Morphisms of Spaces, Lemma 67.14.1. Hence we can apply Lemma 69.14.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is a proper closed subset of T. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of T. But then clearly $\mathcal{F}$ does too, which is our final contradiction. $\square$

07US Lemma 69.14.4. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume

(1) For any short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) For every reduced closed subspace $Z \subset X$ with $|Z|$ irreducible and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 69.14.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. $\square$

Here is a more useful variant of the lemma above.

07UT Lemma 69.14.5. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume

(1) For any short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.

(3) For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on Z such that
   (a) $\text{Supp}(\mathcal{G}) = Z$,
   (b) for every nonzero quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ such that $\text{Supp}(\mathcal{G}/\mathcal{G}')$ is proper closed in $|Z|$ and such that $\mathcal{P}$ holds for $i_*\mathcal{G}'$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{matrix} T \subset |X| \text{ nonempty closed such that there exists a coherent sheaf} \\ \mathcal{F} \text{ with } \text{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{matrix} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 66.24.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold.

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 69.14.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\text{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has $\mathcal{P}$. Hence $\mathcal{F}$ has property $\mathcal{P}$ by (1), a contradiction.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 66.12.3. By Lemma 69.13.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{J}^n\mathcal{F} \subset \mathcal{J}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{J}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does by (1). In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Denote $i : Z \to X$ the closed subspace corresponding to $\mathcal{J}$. Then $\mathcal{F} = i_*\mathcal{H}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}$, see Morphisms of Spaces, Lemma 67.14.1 and Lemma 69.12.7. Let $\mathcal{G}$ be the coherent sheaf on Z satisfying (3)(a) and (3)(b). We apply Lemma 69.14.2 to get injective maps

$$\mathcal{I}_1^{\oplus r_1} \to \mathcal{H} \quad \text{and} \quad \mathcal{I}_2^{\oplus r_2} \to \mathcal{G}$$

where the support of the cokernels are proper closed in Z. Hence we find an nonempty open $V \subset Z$ such that

$$\mathcal{H}_V^{\oplus r_2} \cong \mathcal{G}_V^{\oplus r_1}$$

Let $\mathcal{I} \subset \mathcal{O}_Z$ be a quasi-coherent ideal sheaf cutting out $Z \setminus V$ we obtain (Lemma 69.13.4) a map

$$\mathcal{I}^n\mathcal{G}^{\oplus r_1} \longrightarrow \mathcal{H}^{\oplus r_2}$$

which is an isomorphism over V. The kernel is supported on $Z \setminus V$ hence annihilated by some power of $\mathcal{I}$, see Lemma 69.13.2. Thus after increasing n we may assume the displayed map is injective, see Lemma 69.13.3. Applying (3)(b) we find $\mathcal{G}' \subset \mathcal{I}^n\mathcal{G}$ such that

$$(i_*\mathcal{G}')^{\oplus r_1} \longrightarrow i_*\mathcal{H}^{\oplus r_2} = \mathcal{F}^{\oplus r_2}$$

is injective with cokernel supported in a proper closed subset of Z and such that property $\mathcal{P}$ holds for $i_*\mathcal{G}'$. By (1) property $\mathcal{P}$ holds for $(i_*\mathcal{G}')^{\oplus r_1}$. By (1) and minimality of $T = |Z|$ property $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r_2}$. And finally by (2) property $\mathcal{P}$ holds for $\mathcal{F}$ which is the desired contradiction. $\square$

08AN Lemma 69.14.6. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume

(1) For any short exact sequence of coherent sheaves on X if two out of three have property $\mathcal{P}$ so does the third.
(2) If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.
(3) For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on X whose scheme theoretic support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. We will show that conditions (1) and (2) of Lemma 69.14.4 hold. This is clear for condition (1). To show that (2) holds, let

$$\mathcal{T} = \left\{ \begin{matrix} i : Z \to X \text{ reduced closed subspace with } |Z| \text{ irreducible such} \\ \text{that } i_*\mathcal{I} \text{ does not have } \mathcal{P} \text{ for some quasi-coherent } \mathcal{I} \subset \mathcal{O}_Z \end{matrix} \right\}$$

If $\mathcal{T}$ is nonempty, then since X is Noetherian, we can find an $i : Z \to X$ which is minimal in $\mathcal{T}$. We will show that this leads to a contradiction.

Let $\mathcal{G}$ be the sheaf whose scheme theoretic support is Z whose existence is assumed in assumption (3). Let $\varphi : i_*\mathcal{I}^{\oplus r} \to \mathcal{G}$ be as in Lemma 69.14.2. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \text{Coker}(\varphi)$$

be a filtration as in Lemma 69.14.3. By minimality of Z and assumption (1) we see that $\text{Coker}(\varphi)$ has property $\mathcal{P}$. As φ is injective we conclude using assumption (1) once more that $i_*\mathcal{I}^{\oplus r}$ has property $\mathcal{P}$. Using assumption (2) we conclude that $i_*\mathcal{I}$ has property $\mathcal{P}$.

Finally, if $\mathcal{J} \subset \mathcal{O}_Z$ is a second quasi-coherent sheaf of ideals, set $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$ and consider the short exact sequences

$$0 \to \mathcal{K} \to \mathcal{I} \to \mathcal{I}/\mathcal{K} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{J} \to \mathcal{J}/\mathcal{K} \to 0$$

Arguing as above, using the minimality of Z, we see that $i_*\mathcal{I}/\mathcal{K}$ and $i_*\mathcal{J}/\mathcal{K}$ satisfy $\mathcal{P}$. Hence by assumption (1) we conclude that $i_*\mathcal{K}$ and then $i_*\mathcal{J}$ satisfy $\mathcal{P}$. In other words, Z is not an element of $\mathcal{T}$ which is the desired contradiction. □

69.15. Limits of coherent modules

07UU A colimit of coherent modules (on a locally Noetherian algebraic space) is typically not coherent. But it is quasi-coherent as any colimit of quasi-coherent modules on an algebraic space is quasi-coherent, see Properties of Spaces, Lemma 66.29.7. Conversely, if the algebraic space is Noetherian, then every quasi-coherent module is a filtered colimit of coherent modules.

07UV **Lemma 69.15.1.** Let S be a scheme. Let X be a Noetherian algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules.

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_X$-submodule which contains both of them (see Lemmas 69.12.3 and 69.12.4). In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to X$ a surjective étale morphism. Set $R = U \times_X U$ so that $X = U/R$ as usual. By Properties of Spaces, Proposition 66.32.1 we see that $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$. Hence we reduce to showing the corresponding thing for $QCoh(U, R, s, t, c)$. Thus the result follows from the more general Groupoids, Lemma 39.15.4. □

07UW **Lemma 69.15.2.** Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S with Y Noetherian. Then every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Write $f_*\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ with $\mathcal{H}_i$ a coherent $\mathcal{O}_Y$-module, see Lemma 69.15.1. By Lemma 69.12.2 the modules $\mathcal{H}_i$ are $\mathcal{O}_Y$-modules of finite presentation. Hence $f^*\mathcal{H}_i$ is an $\mathcal{O}_X$-module of finite presentation, see Properties of Spaces, Section 66.30. We claim the map

$$\operatorname{colim} f^*\mathcal{H}_i = f^*f_*\mathcal{F} \to \mathcal{F}$$

is surjective as f is assumed affine, Namely, choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then U is a scheme, $f' : U \to V$ is affine, and $U \to X$ is surjective étale. By Properties of Spaces, Lemma 66.26.2 we see that $f'_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ and similarly for pullbacks. Thus the restriction of $f^*f_*\mathcal{F} \to \mathcal{F}$ to U is the map

$$f^*f_*\mathcal{F}|_U = (f')^*(f_*\mathcal{F})|_V) = (f')^*f'_*(\mathcal{F}|_U) \to \mathcal{F}|_U$$

which is surjective as f' is an affine morphism of schemes. Hence the claim holds.

We conclude that every quasi-coherent module on X is a quotient of a filtered colimit of finitely presented modules. In particular, we see that $\mathcal{F}$ is a cokernel of a map

$$\operatorname{colim}_{j \in J} \mathcal{G}_j \longrightarrow \operatorname{colim}_{i \in I} \mathcal{H}_i$$

with $\mathcal{G}_j$ and $\mathcal{H}_i$ finitely presented. Note that for every $j \in I$ there exist $i \in I$ and a morphism $\alpha : \mathcal{G}_j \to \mathcal{H}_i$ such that

$$\begin{tikzcd} \mathcal{G}_j \arrow[r, "\alpha"'] \arrow[d] & \mathcal{H}_i \arrow[d] \\ \operatorname{colim}_{j \in J} \mathcal{G}_j \arrow[r] & \operatorname{colim}_{i \in I} \mathcal{H}_i \end{tikzcd}$$

commutes, see Lemma 69.5.3. In this situation $\operatorname{Coker}(\alpha)$ is a finitely presented $\mathcal{O}_X$-module which comes endowed with a map $\operatorname{Coker}(\alpha) \to \mathcal{F}$. Consider the set K of triples (i, j, α) as above. We say that $(i, j, \alpha) \leq (i', j', \alpha')$ if and only if $i \leq i'$, $j \leq j'$, and the diagram

$$\begin{tikzcd} \mathcal{G}_j \arrow[r, "\alpha"'] \arrow[d] & \mathcal{H}_i \arrow[d] \\ \mathcal{G}_{j'} \arrow[r, "\alpha'"] & \mathcal{H}_{i'} \end{tikzcd}$$

commutes. It follows from the above that K is a directed partially ordered set,

$$\mathcal{F} = \operatorname{colim}_{(i,j,\alpha) \in K} \operatorname{Coker}(\alpha),$$

and we win. $\square$

69.16. Vanishing of cohomology

07UX In this section we show that a quasi-compact and quasi-separated algebraic space is affine if it has vanishing higher cohomology for all quasi-coherent sheaves. We do this in a sequence of lemmas all of which will become obsolete once we prove Proposition 69.16.7.

07UY Situation 69.16.1. Here S is a scheme and X is a quasi-compact and quasi-separated algebraic space over S with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. We set $A = \Gamma(X, \mathcal{O}_X)$.

We would like to show that the canonical morphism

$$p : X \longrightarrow \mathrm{Spec}(A)$$

(see Properties of Spaces, Lemma 66.33.1) is an isomorphism. If M is an A-module we denote $M \otimes_A \mathcal{O}_X$ the quasi-coherent module $p^*\widetilde{M}$.

07UZ **Lemma 69.16.2.** In Situation 69.16.1 for an A-module M we have $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ and $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$.

Proof. The equality $p_*(M\otimes_A\mathcal{O}_X) = \widetilde{M}$ follows from the equality $\Gamma(X, M\otimes_A\mathcal{O}_X) = M$ as $p_*(M\otimes_A\mathcal{O}_X)$ is a quasi-coherent module on $\mathrm{Spec}(A)$ by Morphisms of Spaces, Lemma 67.11.2. Observe that $\Gamma(X, \bigoplus_{i\in I} \mathcal{O}_X) = \bigoplus_{i\in I} A$ by Lemma 69.5.1. Hence the lemma holds for free modules. Choose a short exact sequence $F_1 \to F_0 \to M$ where F_0, F_1 are free A-modules. Since $H^1(X, -)$ is zero the global sections functor is right exact. Moreover the pullback p^* is right exact as well. Hence we see that

$$\Gamma(X, F_1 \otimes_A \mathcal{O}_X) \to \Gamma(X, F_0 \otimes_A \mathcal{O}_X) \to \Gamma(X, M \otimes_A \mathcal{O}_X) \to 0$$

is exact. The result follows. $\square$

The following lemma shows that Situation 69.16.1 is preserved by base change of $X \to \mathrm{Spec}(A)$ by $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$.

07V0 **Lemma 69.16.3.** In Situation 69.16.1.

(1) Given an affine morphism $X' \to X$ of algebraic spaces, we have $H^1(X', \mathcal{F}') = 0$ for every quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$.
(2) Given an A-algebra A' setting $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ the morphism $X' \to X$ is affine and $\Gamma(X', \mathcal{O}_{X'}) = A'$.

Proof. Part (1) follows from Lemma 69.8.2 and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.5). Let $A \to A'$ be as in (2). Then $X' \to X$ is affine because affine morphisms are preserved under base change (Morphisms of Spaces, Lemma 67.20.5) and the fact that a morphism of affine schemes is affine. The equality $\Gamma(X', \mathcal{O}_{X'}) = A'$ follows as $(X' \to X)_*\mathcal{O}_{X'} = A' \otimes_A \mathcal{O}_X$ by Lemma 69.11.1 and thus

$$\Gamma(X', \mathcal{O}_{X'}) = \Gamma(X, (X' \to X)_*\mathcal{O}_{X'}) = \Gamma(X, A' \otimes_A \mathcal{O}_X) = A'$$

by Lemma 69.16.2. $\square$

07V1 **Lemma 69.16.4.** In Situation 69.16.1. Let $Z_0, Z_1 \subset |X|$ be disjoint closed subsets. Then there exists an $a \in A$ such that $Z_0 \subset V(a)$ and $Z_1 \subset V(a - 1)$.

Proof. We may and do endow Z_0, Z_1 with the reduced induced subspace structure (Properties of Spaces, Definition 66.12.5) and we denote $i_0 : Z_0 \to X$ and $i_1 : Z_1 \to X$ the corresponding closed immersions. Since $Z_0 \cap Z_1 = \emptyset$ we see that the canonical map of quasi-coherent $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow i_{0,*}\mathcal{O}_{Z_0} \oplus i_{1,*}\mathcal{O}_{Z_1}$$

is surjective (look at stalks at geometric points). Since $H^1(X, -)$ is zero on the kernel of this map the induced map of global sections is surjective. Thus we can find $a \in A$ which maps to the global section $(0, 1)$ of the right hand side. $\square$

07V4 **Lemma 69.16.5.** In Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally injective.

Proof. Let $A \to k$ be a ring homomorphism where k is a field. It suffices to show that $\mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} X$ has at most one point (see Morphisms of Spaces, Lemma 67.19.6). Using Lemma 69.16.3 we may assume that A is a field and we have to show that $|X|$ has at most one point.

Let's think of X as an algebraic space over $\mathrm{Spec}(k)$ and let's use the notation $X(K)$ to denote K-valued points of X for any extension K/k, see Morphisms of Spaces, Section 67.24. If K/k is an algebraically closed field extension of large transcendence degree, then we see that $X(K) \to |X|$ is surjective, see Morphisms of Spaces, Lemma 67.24.2. Hence, after replacing k by K, we see that it suffices to prove that $X(k)$ is a singleton (in the case $A = k$).

Let $x, x' \in X(k)$. By Decent Spaces, Lemma 68.14.4 we see that x and x' are closed points of $|X|$. Hence x and x' map to distinct points of $\mathrm{Spec}(k)$ if $x \neq x'$ by Lemma 69.16.4. We conclude that $x = x'$ as desired. □

07V5 **Lemma 69.16.6.** *In Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is separated.*

Proof. By Decent Spaces, Lemma 68.9.2 we can find a scheme Y and a surjective integral morphism $Y \to X$. Since an integral morphism is affine, we can apply Lemma 69.16.3 to see that $H^1(Y, \mathcal{G}) = 0$ for every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$. Since $Y \to X$ is quasi-compact and X is quasi-compact, we see that Y is quasi-compact. Since Y is a scheme, we may apply Cohomology of Schemes, Lemma 30.3.1 to see that Y is affine. Hence Y is separated. Note that an integral morphism is affine and universally closed, see Morphisms of Spaces, Lemma 67.45.7. By Morphisms of Spaces, Lemma 67.9.8 we see that X is a separated algebraic space. □

07V6 **Proposition 69.16.7.** *A quasi-compact and quasi-separated algebraic space is affine if and only if all higher cohomology groups of quasi-coherent sheaves vanish. More precisely, any algebraic space as in Situation 69.16.1 is an affine scheme.*

Proof. Choose an affine scheme $U = \mathrm{Spec}(B)$ and a surjective étale morphism $\varphi : U \to X$. Set $R = U \times_X U$. As p is separated (Lemma 69.16.6) we see that R is a closed subscheme of $U \times_{\mathrm{Spec}(A)} U = \mathrm{Spec}(B \otimes_A B)$. Hence $R = \mathrm{Spec}(C)$ is affine too and the ring map

$$B \otimes_A B \longrightarrow C$$

is surjective. Let us denote the two maps $s, t : B \to C$ as usual. Pick $g_1, \ldots, g_m \in B$ such that $s(g_1), \ldots, s(g_m)$ generate C over $t : B \to C$ (which is possible as $t : B \to C$ is of finite presentation and the displayed map is surjective). Then $g_1, \ldots, g_m$ give global sections of $\varphi_*\mathcal{O}_U$ and the map

$$\mathcal{O}_X[z_1, \ldots, z_n] \longrightarrow \varphi_*\mathcal{O}_U, \quad z_j \longmapsto g_j$$

is surjective: you can check this by restricting to U. Namely, $\varphi^*\varphi_*\mathcal{O}_U = t_*\mathcal{O}_R$ (by Lemma 69.11.2) hence you get exactly the condition that $s(g_i)$ generate C over $t : B \to C$. By the vanishing of H^1 of the kernel we see that

$$\Gamma(X, \mathcal{O}_X[x_1, \ldots, x_n]) = A[x_1, \ldots, x_n] \longrightarrow \Gamma(X, \varphi_*\mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = B$$

is surjective. Thus we conclude that B is a finite type A-algebra. Hence $X \to \mathrm{Spec}(A)$ is of finite type and separated. By Lemma 69.16.5 and Morphisms of Spaces, Lemma 67.27.5 it is also locally quasi-finite. Hence $X \to \mathrm{Spec}(A)$ is representable by Morphisms of Spaces, Lemma 67.51.1 and X is a scheme. Finally

X is affine, hence equal to $\mathrm{Spec}(A)$, by an application of Cohomology of Schemes, Lemma 30.3.1. □

0D2V Lemma 69.16.8. Let S be a scheme. Let X be a Noetherian algebraic space over S. Assume that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. Then X is an affine scheme.

Proof. The assumption implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 69.15.1 and 69.5.1. Then X is affine by Proposition 69.16.7. □

0D2W Lemma 69.16.9. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$. Then X is a scheme and $\mathcal{L}$ is ample on X.

Proof. Let $s \in H^0(X, \mathcal{L}^{\otimes d})$ be a global section. Let $U \subset X$ be the open subspace over which s is a generator of $\mathcal{L}^{\otimes d}$. In particular we have $\mathcal{L}^{\otimes d}|_U \cong \mathcal{O}_U$. We claim that U is affine.

Proof of the claim. We will show that $H^1(U, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$. This will prove the claim by Proposition 69.16.7. Denote $j : U \to X$ the inclusion morphism. Since étale locally the morphism j is affine (by Morphisms, Lemma 29.11.10) we see that j is affine (Morphisms of Spaces, Lemma 67.20.3). Hence we have

$$H^1(U, \mathcal{F}) = H^1(X, j_*\mathcal{F})$$

by Lemma 69.8.2 (and Cohomology on Sites, Lemma 21.14.6). Write $j_*\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_X$-modules, see Lemma 69.15.1. Then

$$H^1(X, j_*\mathcal{F}) = \mathrm{colim}\, H^1(X, \mathcal{F}_i)$$

by Lemma 69.5.1. Thus it suffices to show that $H^1(X, \mathcal{F}_i)$ maps to zero in $H^1(U, j^*\mathcal{F}_i)$. By assumption there exists an $n \geq 1$ such that

$$H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} (\mathcal{O}_X \oplus \mathcal{L} \oplus \ldots \oplus \mathcal{L}^{\otimes d-1}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$$

Hence there exists an $a \geq 0$ such that $H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}) = 0$. On the other hand, the map

$$s^a : \mathcal{F}_i \longrightarrow \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}$$

is an isomorphism after restriction to U. Contemplating the commutative diagram

$$\begin{tikzcd}
H^1(X, \mathcal{F}_i) \arrow[r] \arrow[d, "s^a"'] & H^1(U, j^*\mathcal{F}_i) \arrow[d, "\cong"] \\
H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}) \arrow[r] & H^1(U, j^*(\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}))
\end{tikzcd}$$

we conclude that the map $H^1(X, \mathcal{F}_i) \to H^1(U, j^*\mathcal{F}_i)$ is zero and the claim holds.

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 68.14.6 we can represent x by a closed immersion $i : \mathrm{Spec}(k) \to X$ (this also uses that a quasi-separated algebraic space is decent, see Decent Spaces, Section 68.6). Thus $\mathcal{O}_X \to i_*\mathcal{O}_{\mathrm{Spec}(k)}$ is surjective. Let $\mathcal{I} \subset \mathcal{O}_X$ be the kernel and choose $d \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$. Then

$$H^0(X, \mathcal{L}^{\otimes d}) \to H^0(X, i_*\mathcal{O}_{\mathrm{Spec}(k)} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = H^0(\mathrm{Spec}(k), i^*\mathcal{L}^{\otimes d}) \cong k$$

is surjective by the long exact cohomology sequence. Hence there exists an $s \in H^0(X, \mathcal{L}^{\otimes d})$ such that $x \in U$ where U is the open subspace corresponding to s as above. Thus x is in the schematic locus (see Properties of Spaces, Lemma 66.13.1) of X by our claim.

To conclude that X is a scheme, it suffices to show that any open subset of $|X|$ which contains all the closed points is equal to $|X|$. This follows from the fact that $|X|$ is a Noetherian topological space, see Properties of Spaces, Lemma 66.24.3. Finally, if X is a scheme, then we can apply Cohomology of Schemes, Lemma 30.3.3 to conclude that $\mathcal{L}$ is ample. □

69.17. Finite morphisms and affines

07VN This section is the analogue of Cohomology of Schemes, Section 30.13.

0GF7 Lemma 69.17.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume f is finite, surjective and X locally Noetherian. Let $i : Z \to X$ be a closed immersion. Denote $i' : Z' \to Y$ the inverse image of Z (Morphisms of Spaces, Section 67.13) and $f' : Z' \to Z$ the induced morphism. Then $\mathcal{G} = f'_*\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_Z$-module whose support is Z.

Proof. Observe that f' is the base change of f and hence is finite and surjective by Morphisms of Spaces, Lemmas 67.5.5 and 67.45.5. Note that Y, Z, and Z' are locally Noetherian by Morphisms of Spaces, Lemma 67.23.5 (and the fact that closed immersions and finite morphisms are of finite type). By Lemma 69.12.9 we see that $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. The support of $\mathcal{G}$ is closed in $|Z|$, see Morphisms of Spaces, Lemma 67.15.2. Hence if the support of $\mathcal{G}$ is not equal to $|Z|$, then after replacing X by an open subspace we may assume $\mathcal{G} = 0$ but $Z \neq \emptyset$. This would mean that $f'_*\mathcal{O}_{Z'} = 0$. In particular the section $1 \in \Gamma(Z', \mathcal{O}_{Z'}) = \Gamma(Z, f'_*\mathcal{O}_{Z'})$ would be zero which would imply $Z' = \emptyset$ is the empty algebraic space. This is impossible as $Z' \to Z$ is surjective. □

0GF8 Lemma 69.17.2. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$ (with notation as explained in the proof).

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ on $Y_{\acute{e}tale}$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is étale local on X and hence we may assume X is an affine scheme. In this case the result is a consequence of the corresponding result for schemes, see Cohomology of Schemes, Lemma 30.13.2. □

07VP Lemma 69.17.3. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume

(1) f finite,
(2) f surjective,

(3) Y affine, and
(4) X Noetherian.

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 69.15.1 and 69.5.1. Then it follows that X is affine from Proposition 69.16.7.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 69.14.5. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $i' : Z' \to Y$ and $f' : Z' \to Z$ be as in Lemma 69.17.1 and set $\mathcal{G} = f'_*\mathcal{O}_{Z'}$. We claim that $\mathcal{G}$ satisfies properties (3)(a) and (3)(b) of Lemma 69.14.5 which will finish the proof. Property (3)(a) we have seen in Lemma 69.17.1. To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. Denote $\mathcal{I}' \subset \mathcal{O}_{Z'}$ the quasi-coherent ideal $(f')^{-1}\mathcal{I}\mathcal{O}_{Z'}$, i.e., the image of $(f')^*\mathcal{I} \to \mathcal{O}_{Z'}$. By Lemma 69.17.2 we have $f_*\mathcal{I}' = \mathcal{I}\mathcal{G}$. We claim the common value $\mathcal{G}' = \mathcal{I}\mathcal{G} = f'_*\mathcal{I}'$ satisfies the condition expressed in (3)(b). First, it is clear that the support of $\mathcal{G}/\mathcal{G}'$ is contained in the support of $\mathcal{O}_Z/\mathcal{I}$ which is a proper subspace of $|Z|$ as $\mathcal{I}$ is a nonzero ideal sheaf on the reduced and irreducible algebraic space Z. The morphism f' is affine, hence $R^1f'_*\mathcal{I}' = 0$ by Lemma 69.8.2. As Z' is affine (as a closed subscheme of an affine scheme) we have $H^1(Z', \mathcal{I}') = 0$. Hence the Leray spectral sequence (in the form Cohomology on Sites, Lemma 21.14.6) implies that $H^1(Z, f'_*\mathcal{I}') = 0$. Since $i : Z \to X$ is affine we conclude that $R^1i_*f'_*\mathcal{I}' = 0$ hence $H^1(X, i_*f'_*\mathcal{I}') = 0$ by Leray again. In other words, we have $H^1(X, i_*\mathcal{G}') = 0$ as desired. □

69.18. A weak version of Chow's lemma

089I In this section we quickly prove the following lemma in order to help us prove the basic results on cohomology of coherent modules on proper algebraic spaces.

089J Lemma 69.18.1. Let A be a ring. Let X be an algebraic space over $\text{Spec}(A)$ whose structure morphism $X \to \text{Spec}(A)$ is separated of finite type. Then there exists a proper surjective morphism $X' \to X$ where X' is a scheme which is H-quasi-projective over $\text{Spec}(A)$.

Proof. Let W be an affine scheme and let $f : W \to X$ be a surjective étale morphism. There exists an integer d such that all geometric fibres of f have $\leq d$ points (because X is a separated algebraic hence reasonable, see Decent Spaces, Lemma 68.5.1). Picking d minimal we get a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is finite étale of degree d, see Decent Spaces, Lemma 68.8.1. Let

$$V \subset W \times_X W \times_X \ldots \times_X W$$

(d factors in the fibre product) be the complement of all the diagonals. Because $W \to X$ is separated the diagonal $W \to W \times_X W$ is a closed immersion. Since $W \to X$ is étale the diagonal $W \to W \times_X W$ is an open immersion, see Morphisms of Spaces, Lemmas 67.39.10 and 67.38.9. Hence the diagonals are open and closed subschemes of the quasi-compact scheme $W \times_X \ldots \times_X W$. In particular we conclude V is a quasi-compact scheme. Choose an open immersion $W \subset Y$ with Y H-projective over A (this is possible as W is affine and of finite type over A; for example we can use Morphisms, Lemmas 29.39.2 and 29.43.11). Let

$$Z \subset Y \times_A Y \times_A \ldots \times_A Y$$

be the scheme theoretic image of the composition $V \to W \times_X \ldots \times_X W \to Y \times_A \ldots \times_A Y$. Observe that this morphism is quasi-compact since V is quasi-compact and $Y \times_A \ldots \times_A Y$ is separated. Note that $V \to Z$ is an open immersion as $V \to Y \times_A \ldots \times_A Y$ is an immersion, see Morphisms, Lemma 29.7.7. The projection morphisms give d morphisms $g_i : Z \to Y$. These morphisms g_i are projective as Y is projective over A, see material in Morphisms, Section 29.43. We set

$$X' = \bigcup g_i^{-1}(W) \subset Z$$

There is a morphism $X' \to X$ whose restriction to $g_i^{-1}(W)$ is the composition $g_i^{-1}(W) \to W \to X$. Namely, these morphisms agree over V hence agree over $g_i^{-1}(W) \cap g_j^{-1}(W)$ by Morphisms of Spaces, Lemma 67.17.8. Claim: the morphism $X' \to X$ is proper.

If the claim holds, then the lemma follows by induction on d. Namely, by construction X' is H-quasi-projective over $\mathrm{Spec}(A)$. The image of $X' \to X$ contains the open U as V surjects onto U. Denote T the reduced induced algebraic space structure on $X \setminus U$. Then $T \times_X W$ is a closed subscheme of W, hence affine. Moreover, the morphism $T \times_X W \to T$ is étale and every geometric fibre has $< d$ points. By induction hypothesis there exists a proper surjective morphism $T' \to T$ where T' is a scheme H-quasi-projective over $\mathrm{Spec}(A)$. Since T is a closed subspace of X we see that $T' \to X$ is a proper morphism. Thus the lemma follows by taking the proper surjective morphism $X' \amalg T' \to X$.

Proof of the claim. By construction the morphism $X' \to X$ is separated and of finite type. We will check conditions (1) – (4) of Morphisms of Spaces, Lemma 67.42.5 for the morphisms $V \to X'$ and $X' \to X$. Conditions (1) and (2) we have seen above. Condition (3) holds as $X' \to X$ is separated (as a morphism whose source is a separated algebraic space). Thus it suffices to check liftability to X' for diagrams

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & X \end{tikzcd}$$

where R is a valuation ring with fraction field K. Note that the top horizontal map is given by d pairwise distinct K-valued points $w_1, \ldots, w_d$ of W. In fact, this is a complete set of inverse images of the point $x \in X(K)$ coming from the diagram. Since $W \to X$ is surjective, we can, after possibly replacing R by an extension of valuation rings, lift the morphism $\mathrm{Spec}(R) \to X$ to a morphism $w : \mathrm{Spec}(R) \to W$, see Morphisms of Spaces, Lemma 67.42.4. Since $w_1, \ldots, w_d$ is a complete collection of inverse images of x we see that $w|_{\mathrm{Spec}(K)}$ is equal to one of them, say w_i. Thus we see that we get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Z \arrow[d, "g_i"] \\ \mathrm{Spec}(R) \arrow[r, "w"] & Y \end{tikzcd}$$

By the valuative criterion of properness for the projective morphism g_i we can lift w to $z : \mathrm{Spec}(R) \to Z$, see Morphisms, Lemma 29.43.5 and Schemes, Proposition 26.20.6. The image of z is in $g_i^{-1}(W) \subset X'$ and the proof is complete. □

69.19. Noetherian valuative criterion

0ARI We prove a version of the valuative criterion for properness using discrete valuation rings. More precise (and therefore more technical) versions can be found in Limits of Spaces, Section 70.21.

0ARJ **Lemma 69.19.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) Y is locally Noetherian,
(2) f is locally of finite type and quasi-separated,
(3) for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field, there is at most one dotted arrow making the diagram commute.

Then f is separated.

Proof. We have to show that the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. We already know Δ is representable, separated, a monomorphism, and locally of finite type, see Morphisms of Spaces, Lemma 67.4.1. Choose an affine scheme U and an étale morphism $U \to X \times_Y X$. Set $V = X \times_{\Delta, X \times_Y X} U$. It suffices to show that $V \to U$ is a closed immersion (Morphisms of Spaces, Lemma 67.12.1). Since $X \times_Y X$ is locally of finite type over Y we see that U is Noetherian (use Morphisms of Spaces, Lemmas 67.23.2, 67.23.3, and 67.23.5). Note that V is a scheme as Δ is representable. Also, V is quasi-compact because f is quasi-separated. Hence $V \to U$ is of finite type. Consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & U \end{tikzcd}$$

of morphisms of schemes where A is a discrete valuation ring with fraction field K. We can interpret the composition $\mathrm{Spec}(A) \to U \to X \times_Y X$ as a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ agreeing as morphisms into Y and equal when restricted to $\mathrm{Spec}(K)$. Hence our assumption (3) guarantees $a = b$ and we find the dotted arrow in the diagram. By Limits, Lemma 32.15.3 we conclude that $V \to U$ is proper. In other words, Δ is proper. Since Δ is a monomorphism, we find that Δ is a closed immersion (Étale Morphisms, Lemma 41.7.2) as desired. $\square$

0ARK **Lemma 69.19.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) Y is locally Noetherian,
(2) f is of finite type and quasi-separated,
(3) for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field, there is a unique dotted arrow making the diagram commute.

Then f is proper.

Proof. It suffices to prove f is universally closed because f is separated by Lemma 69.19.1. To do this we may work étale locally on Y (Morphisms of Spaces, Lemma 67.9.5). Hence we may assume $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme. Choose $X' \to X$ as in the weak form of Chow's lemma (Lemma 69.18.1). We claim that $X' \to \mathrm{Spec}(A)$ is universally closed. The claim implies the lemma by Morphisms of Spaces, Lemma 67.40.7. To prove this, according to Limits, Lemma 32.15.4 it suffices to prove that in every solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[ru, dashed, "a"] \arrow[rru, dashed, "b"'] & & Y \end{tikzcd}$$

where A is a dvr with fraction field K we can find the dotted arrow a. By assumption we can find the dotted arrow b. Then the morphism $X' \times_{X,b} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is a proper morphism of schemes and by the valuative criterion for morphisms of schemes we can lift b to the desired morphism a. $\square$

0ARL **Remark 69.19.3** (Variant for complete discrete valuation rings). In Lemmas 69.19.1 and 69.19.2 it suffices to consider complete discrete valuation rings. To be precise in Lemma 69.19.1 we can replace condition (3) by the following condition: Given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a complete discrete valuation ring with fraction field K there exists at most one dotted arrow making the diagram commute. Namely, given any diagram as in Lemma 69.19.1 (3) the completion $A^\wedge$ is a discrete valuation ring (More on Algebra, Lemma 15.43.5) and the uniqueness of the arrow $\mathrm{Spec}(A^\wedge) \to X$ implies the uniqueness of the arrow $\mathrm{Spec}(A) \to X$ for example by Properties of Spaces, Proposition 66.17.1. Similarly in Lemma 69.19.2 we can replace condition (3) by the following condition: Given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a complete discrete valuation ring with fraction field K there exists an extension $A \subset A'$ of complete discrete valuation rings inducing a fraction field extension $K \subset K'$ such that there exists a unique arrow $\mathrm{Spec}(A') \to X$ making the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

commute. Namely, given any diagram as in Lemma 69.19.2 part (3) the existence of any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

for any extension $A \subset B$ of discrete valuation rings will imply there exists an arrow $\mathrm{Spec}(A) \to X$ fitting into the diagram. This was shown in Morphisms of Spaces, Lemma 67.41.4. In fact, it follows from these considerations that it suffices to look for dotted arrows in diagrams for any class of discrete valuation rings such that, given any discrete valuation ring, there is an extension of it that is in the class. For example, we could take complete discrete valuation rings with algebraically closed residue field.

69.20. Higher direct images of coherent sheaves

08AP In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent. First we prove a helper lemma.

08AQ Lemma 69.20.1. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rd, "f"'] & \mathbf{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

of algebraic spaces over S. Assume i is a closed immersion and Y Noetherian. Set $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_Y}(1)$. Let $\mathcal{F}$ be a coherent module on X. Then there exists an integer d_0 such that for all $d \geq d_0$ we have $R^pf_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$.

Proof. Checking whether $R^pf_*(\mathcal{F} \otimes \mathcal{L}^{\otimes d})$ is zero can be done étale locally on Y, see Equation (69.3.0.1). Hence we may assume Y is the spectrum of a Noetherian ring.

In this case X is a scheme and the result follows from Cohomology of Schemes, Lemma 30.16.2. □

08AR Lemma 69.20.2. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^if_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $i \geq 0$.

Proof. We first remark that X is a locally Noetherian algebraic space by Morphisms of Spaces, Lemma 67.23.5. Hence the statement of the lemma makes sense. Moreover, computing $R^if_*\mathcal{F}$ commutes with étale localization on Y (Properties of Spaces, Lemma 66.26.2) and checking whether $R^if_*\mathcal{F}$ coherent can be done étale locally on Y (Lemma 69.12.2). Hence we may assume that $Y = \text{Spec}(A)$ is a Noetherian affine scheme.

Assume $Y = \text{Spec}(A)$ is an affine scheme. Note that f is locally of finite presentation (Morphisms of Spaces, Lemma 67.28.7). Thus it is of finite presentation, hence X is Noetherian (Morphisms of Spaces, Lemma 67.28.6). Thus Lemma 69.14.6 applies to the category of coherent modules of X. For a coherent sheaf $\mathcal{F}$ on X we say $\mathcal{P}$ holds if and only if $R^if_*\mathcal{F}$ is a coherent module on $\text{Spec}(A)$. We will show that conditions (1), (2), and (3) of Lemma 69.14.6 hold for this property thereby finishing the proof of the lemma.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1}f_*\mathcal{F}_3 \to R^pf_*\mathcal{F}_1 \to R^pf_*\mathcal{F}_2 \to R^pf_*\mathcal{F}_3 \to R^{p+1}f_*\mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemmas 69.12.3 and 69.12.4. Hence property $\mathcal{P}$ holds for the third as well.

Verification of condition (2). This follows immediately from the fact that $R^if_*(\mathcal{F}_1 \oplus \mathcal{F}_2) = R^if_*\mathcal{F}_1 \oplus R^if_*\mathcal{F}_2$ and that a summand of a coherent module is coherent (see lemmas cited above).

Verification of condition (3). Let $i : Z \to X$ be a closed immersion with Z reduced and $|Z|$ irreducible. Set $g = f \circ i : Z \to \text{Spec}(A)$. Let $\mathcal{G}$ be a coherent module on Z whose scheme theoretic support is equal to Z such that $R^pg_*\mathcal{G}$ is coherent for all p. Then $\mathcal{F} = i_*\mathcal{G}$ is a coherent module on X whose scheme theoretic support is Z such that $R^pf_*\mathcal{F} = R^pg_*\mathcal{G}$. To see this use the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7) and the fact that $R^qi_*\mathcal{G} = 0$ for $q > 0$ by Lemma 69.8.2 and the fact that a closed immersion is affine. (Morphisms of Spaces, Lemma 67.20.6). Thus we reduce to finding a coherent sheaf $\mathcal{G}$ on Z with support equal to Z such that $R^pg_*\mathcal{G}$ is coherent for all p.

We apply Lemma 69.18.1 to the morphism $Z \to \mathrm{Spec}(A)$. Thus we get a diagram

$$\begin{tikzcd} Z \arrow[dr, "g"'] & Z' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^n_A \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

with $\pi : Z' \to Z$ proper surjective and i an immersion. Since $Z \to \mathrm{Spec}(A)$ is proper we conclude that g' is proper (Morphisms of Spaces, Lemma 67.40.4). Hence i is a closed immersion (Morphisms of Spaces, Lemmas 67.40.6 and 67.12.3). It follows that the morphism $i' = (i, \pi) : \mathbf{P}^n_A \times_{\mathrm{Spec}(A)} Z' = \mathbf{P}^n_Z$ is a closed immersion (Morphisms of Spaces, Lemma 67.4.6). Set

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_A}(1) = (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

We may apply Lemma 69.20.1 to $\mathcal{L}$ and π as well as $\mathcal{L}$ and g'. Hence for all $d \gg 0$ we have $R^p\pi_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes d}$. By the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7) we have

$$E_2^{p,q} = R^pg_*R^q\pi_*\mathcal{L}^{\otimes d} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes d}$$

and by choice of d the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes d}$ are those with $p = q = 0$. This implies that $R^pg_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$. Applying Cohomology of Schemes, Lemma 30.16.3 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$ is coherent.

We still have to check that the support of $\mathcal{G}$ is Z. This follows from the fact that $\mathcal{L}^{\otimes d}$ has lots of global sections. We spell it out here. Note that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq 0$ because the same is true for $\mathcal{O}_{\mathbf{P}^n}(d)$. Pick a point $z \in Z'$ mapping to the generic point ξ of Z which we can do as π is surjective. (Observe that Z does indeed have a generic point as $|Z|$ is irreducible and Z is Noetherian, hence quasi-separated, hence $|Z|$ is a sober topological space by Properties of Spaces, Lemma 66.15.1.) Pick $s \in \Gamma(Z', \mathcal{L}^{\otimes d})$ which does not vanish at z. Since $\Gamma(Z, \mathcal{G}) = \Gamma(Z', \mathcal{L}^{\otimes d})$ we may think of s as a global section of $\mathcal{G}$. Choose a geometric point $\overline{z}$ of Z' lying over z and denote $\overline{\xi} = g' \circ \overline{z}$ the corresponding geometric point of Z. The adjunction map

$$(g')^*\mathcal{G} = (g')^*g'_*\mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d}$$

induces a map of stalks $\mathcal{G}_{\overline{\xi}} \to \mathcal{L}_{\overline{z}}$, see Properties of Spaces, Lemma 66.29.5. Moreover the adjunction map sends the pullback of s (viewed as a section of $\mathcal{G}$) to s (viewed as a section of $\mathcal{L}^{\otimes d}$). Thus the image of s in the vector space which is the source of the arrow

$$\mathcal{G}_{\overline{\xi}} \otimes \kappa(\overline{\xi}) \longrightarrow \mathcal{L}^{\otimes d}_{\overline{z}} \otimes \kappa(\overline{z})$$

isn't zero since by choice of s the image in the target of the arrow is nonzero. Hence ξ is in the support of $\mathcal{G}$ (Morphisms of Spaces, Lemma 67.15.2). Since $|Z|$ is irreducible and Z is reduced we conclude that the scheme theoretic support of $\mathcal{G}$ is all of Z as desired. $\square$

08AS Lemma 69.20.3. Let A be a Noetherian ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.

Proof. This is just the affine case of Lemma 69.20.2. Namely, by Lemma 69.3.1 we know that $R^if_*\mathcal{F}$ is a quasi-coherent sheaf. Hence it is the quasi-coherent sheaf associated to the A-module $\Gamma(\mathrm{Spec}(A), R^if_*\mathcal{F}) = H^i(X, \mathcal{F})$. The equality holds by Cohomology on Sites, Lemma 21.14.6 and vanishing of higher cohomology groups of quasi-coherent modules on affine schemes (Cohomology of Schemes, Lemma 30.2.2). By Lemma 69.12.2 we see $R^if_*\mathcal{F}$ is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. Hence Lemma 69.20.2 gives us the conclusion. □

08AT Lemma 69.20.4. Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Set $\mathcal{B} = f^*\widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type. For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.

Proof. To prove this we consider the fibre product diagram

$$\begin{tikzcd} X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X \arrow[r, "\pi"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Note that f' is a proper morphism, see Morphisms of Spaces, Lemma 67.40.3. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 10.31.1). This implies that X' is a Noetherian algebraic space (Morphisms of Spaces, Lemma 67.28.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Morphisms of Spaces, Lemma 67.20.7. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \mathcal{F}$, see Morphisms of Spaces, Lemma 67.20.10. Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 69.12.2). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 69.8.2 and Cohomology on Sites, Lemma 21.14.6. Thus the lemma follows from Lemma 69.20.3. □

69.21. Ample invertible sheaves and cohomology

0GF9 Here is a criterion for ampleness on proper algebraic spaces over affine bases in terms of vanishing of cohomology after twisting.

0GFA Lemma 69.21.1. Let R be a Noetherian ring. Let X be a proper algebraic space over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent

(1) X is a scheme and $\mathcal{L}$ is ample on X,
(2) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$, and
(3) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Cohomology of Schemes, Lemma 30.17.1. The implication (2) $\Rightarrow$ (3) is trivial. The implication (3) $\Rightarrow$ (1) is Lemma 69.16.9. □

0GFB **Lemma 69.21.2.** Let R be a Noetherian ring. Let $f : Y \to X$ be a morphism of algebraic spaces proper over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume f is finite and surjective. The following are equivalent

(1) X is a scheme and $\mathcal{L}$ is ample, and
(2) Y is a scheme and $f^*\mathcal{L}$ is ample.

Proof. Assume (1). Then Y is a scheme as a finite morphism is representable (by schemes), see Morphisms of Spaces, Lemma 67.45.3. Hence (2) follows from Cohomology of Schemes, Lemma 30.17.2.

Assume (2). Let P be the following property on coherent $\mathcal{O}_X$-modules $\mathcal{F}$: there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, which implies $\mathcal{L}$ is ample by Lemma 69.21.1. We are going to apply Lemma 69.14.5. Thus we have to verify (1), (2) and (3) of that lemma for P. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor.

To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $i' : Z' \to Y$ and $f' : Z' \to Z$ be as in Lemma 69.17.1 and set $\mathcal{G} = f'_*\mathcal{O}_{Z'}$. We claim that $\mathcal{G}$ satisfies properties (3)(a) and (3)(b) of Lemma 69.14.5 which will finish the proof. Property (3)(a) we have seen in Lemma 69.17.1. To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. Denote $\mathcal{I}' \subset \mathcal{O}_{Z'}$ the quasi-coherent ideal $(f')^{-1}\mathcal{I}\mathcal{O}_{Z'}$, i.e., the image of $(f')^*\mathcal{I} \to \mathcal{O}_{Z'}$. By Lemma 69.17.2 we have $f_*\mathcal{I}' = \mathcal{I}\mathcal{G}$. We claim the common value $\mathcal{G}' = \mathcal{I}\mathcal{G} = f'_*\mathcal{I}'$ satisfies the condition expressed in (3)(b). First, it is clear that the support of $\mathcal{G}/\mathcal{G}'$ is contained in the support of $\mathcal{O}_Z/\mathcal{I}$ which is a proper subspace of $|Z|$ as $\mathcal{I}$ is a nonzero ideal sheaf on the reduced and irreducible algebraic space Z. Recall that f'_*, i_*, and i'_* transform coherent modules into coherent modules, see Lemmas 69.12.9 and 69.12.8. As Y is a scheme and $\mathcal{L}$ is ample we see from Lemma 69.21.1 that there exists an n_0 such that

$$H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0$$

for $n \geq n_0$ and $p > 0$. Now we get

$$\begin{aligned} H^p(X, i_*\mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(Z, \mathcal{G}' \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}) \\ &= H^p(Z, f'_*\mathcal{I}' \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})) \\ &= H^p(Z, f'_*(\mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (f')^*i^*\mathcal{L}^{\otimes n})) \\ &= H^p(Z, f'_*(\mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (i')^*f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Z', \mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (i')^*f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0 \end{aligned}$$

Here we have used the projection formula and the Leray spectral sequence (see Cohomology on Sites, Sections 21.50 and 21.14) and Lemma 69.4.1. This verifies property (3)(b) of Lemma 69.14.5 as desired. $\square$

69.22. The theorem on formal functions

08AU This section is the analogue of Cohomology of Schemes, Section 30.20. We encourage the reader to read that section first.

08AV Situation 69.22.1. Here A is a Noetherian ring and $I \subset A$ is an ideal. Also, $f : X \to \mathrm{Spec}(A)$ is a proper morphism of algebraic spaces and $\mathcal{F}$ is a coherent sheaf on X.

In this situation we denote $I^n\mathcal{F}$ the quasi-coherent submodule of $\mathcal{F}$ generated as an $\mathcal{O}_X$-module by products of local sections of $\mathcal{F}$ and elements of I^n. In other words, it is the image of the map $f^*\widetilde{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$.

08AW Lemma 69.22.2. In Situation 69.22.1. Set $B = \bigoplus_{n \geq 0} I^n$. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n \geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 69.20.4. $\square$

08AX Lemma 69.22.3. In Situation 69.22.1. For every $p \geq 0$ there exists an integer $c \geq 0$ such that

(1) the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and
(2) the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.

Proof. By Lemma 69.22.2 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n \geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $B = \bigoplus_{n \geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n - c)$. Consider the commutative diagram of A-modules

$$\begin{tikzcd}
I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) \arrow[r] \arrow[d] & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) \arrow[r] & H^p(X, I^{n+m}\mathcal{F}) \arrow[d] \\
I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) \arrow[rr] & & H^p(X, I^n\mathcal{F})
\end{tikzcd}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n + m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). $\square$

08AY Lemma 69.22.4. In Situation 69.22.1. Fix $p \geq 0$.

(1) There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have
$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1}H^p(X, \mathcal{F}).$$
(2) The inverse system
$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n \in \mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 12.31.2).
(3) In fact for any p and n there exists a $c_2(n) \geq n$ such that
$$\mathrm{Im}(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 69.22.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence

$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

From the long exact cohomology sequence we see that

$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$

Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X, \mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/I^n\mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}/I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd}
H^p(X, I^n\mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r, "\delta"'] & H^{p+1}(X, I^n\mathcal{F}) \\
H^p(X, I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^p(X, \mathcal{F}) \arrow[r] \arrow[u, "1"] & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^{p+1}(X, I^{n+m}\mathcal{F}) \arrow[u, "a"]
\end{tikzcd}$$

If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X, I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 10.51.3) there exists an integer $c_3(n)$ such that

$$I^N H^{p+1}(X, I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X, \mathcal{F}/I^n\mathcal{F})\right)$$

for all $N \geq c_3(n)$. As $H^p(X, \mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then

$$\mathrm{Im}(H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$

In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. □

08AZ Theorem 69.22.5 (Theorem on formal functions). In Situation 69.22.1. Fix $p \geq 0$. The system of maps

$$H^p(X, \mathcal{F})/I^nH^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 10.96. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 69.22.4. We spell out the details. Set $M = H^p(X, \mathcal{F})$ and $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 12.31 we have

$$\lim_n M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2, 3, \ldots\}$$

Pick an element $x = (x_n) \in \lim_n M_n$. By Lemma 69.22.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 69.22.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence $y = (y_{n+c_1})$ defines an element of $\lim_n M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = \lim_n M/I^nM \longrightarrow \lim_n M_n$$

of the lemma. We also omit the verification on topologies. □

08B0 Lemma 69.22.6. Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent sheaf on X. Then

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 69.22.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 69.20.3) hence I-adically complete (Algebra, Lemma 10.97.1) and we see that completion on the left hand side is not necessary. □

08B1 Lemma 69.22.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S and let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Assume

(1) Y locally Noetherian,
(2) f proper, and
(3) $\mathcal{F}$ coherent.

Let $\overline{y}$ be a geometric point of Y. Consider the "infinitesimal neighbourhoods"

$$\begin{tikzcd} X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \times_Y X \arrow[r, "i_n"'] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \arrow[r, "c_n"] & Y \end{tikzcd}$$

of the fibre $X_1 = X_{\overline{y}}$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})_{\overline{y}}^\wedge \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}_{Y,\overline{y}}^\wedge$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 69.22.5. Let us spell it out. Note that $\mathcal{O}_{Y,\overline{y}}$ is a Noetherian local ring, see Properties of Spaces, Lemma 66.24.4. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to Y$. This is a flat morphism as it identifies local rings. Denote $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 69.11.2. Moreover, we have canonical identifications $X_n = X'_n$ for all $n \geq 1$.

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a strictly henselian Noetherian local ring A with maximal ideal $\mathfrak m$ and that $\overline{y} \to Y$ is equal to $\mathrm{Spec}(A/\mathfrak m) \to Y$. It follows that

$$(R^pf_*\mathcal{F})_{\overline{y}} = \Gamma(Y, R^pf_*\mathcal{F}) = H^p(X, \mathcal{F})$$

because $(Y, \overline{y})$ is an initial object in the category of étale neighbourhoods of $\overline{y}$. The morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak m^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 69.12.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak m^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

0A4S Lemma 69.22.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume

(1) Y locally Noetherian,
(2) f is proper, and
(3) $X_{\overline{y}}$ has discrete underlying topological space.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_{\overline{y}} = 0$ for all $p > 0$.

Proof. Let $\kappa(\overline{y})$ be the residue field of the local ring of $\mathcal{O}_{Y,\overline{y}}$. As in Lemma 69.22.7 we set $X_{\overline{y}} = X_1 = \mathrm{Spec}(\kappa(\overline{y})) \times_Y X$. By Morphisms of Spaces, Lemma 67.34.8 the morphism $f : X \to Y$ is quasi-finite at each of the points of the fibre of $X \to Y$ over $\overline{y}$. It follows that $X_{\overline{y}} \to \overline{y}$ is separated and quasi-finite. Hence $X_{\overline{y}}$ is a scheme by Morphisms of Spaces, Proposition 67.50.2. Since it is quasi-compact its underlying topological space is a finite discrete space. Then it is an affine scheme by Schemes, Lemma 26.11.8. By Lemma 69.17.3 it follows that the algebraic spaces X_n are affine schemes as well. Moreover, the underlying topological of each X_n is the same as that of X_1. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^pf_*\mathcal{F})^\wedge_{\overline{y}} = 0$ by Lemma 69.22.7. Note that $R^pf_*\mathcal{F}$ is coherent by Lemma 69.20.2 and hence $R^pf_*\mathcal{F}_{\overline{y}}$ is a finite $\mathcal{O}_{Y,\overline{y}}$-module. By Algebra, Lemma 10.97.1 this implies that $(R^pf_*\mathcal{F})_{\overline{y}} = 0$. □

0A4T Lemma 69.22.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume

(1) Y locally Noetherian,
(2) f is proper, and
(3) $\dim(X_{\overline{y}}) = d$.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_{\overline{y}} = 0$ for all $p > d$.

Proof. Let $\kappa(\overline{y})$ be the residue field of the local ring of $\mathcal{O}_{Y,\overline{y}}$. As in Lemma 69.22.7 we set $X_{\overline{y}} = X_1 = \mathrm{Spec}(\kappa(\overline{y})) \times_Y X$. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of $X_{\overline{y}}$. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Lemma 69.10.1. Hence we see that $(R^pf_*\mathcal{F})^\wedge_{\overline{y}} = 0$ by Lemma 69.22.7 for $p > d$. Note that $R^pf_*\mathcal{F}$ is coherent by Lemma 69.20.2 and hence $R^pf_*\mathcal{F}_{\overline{y}}$ is a finite $\mathcal{O}_{Y,\overline{y}}$-module. By Algebra, Lemma 10.97.1 this implies that $(R^pf_*\mathcal{F})_{\overline{y}} = 0$. □

69.23. Applications of the theorem on formal functions

0A4U We will add more here as needed.

0A4V **Lemma 69.23.1.** (For a more general version see More on Morphisms of Spaces, Lemma 76.35.1). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian. The following are equivalent

(1) f is finite, and
(2) f is proper and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field.

Proof. A finite morphism is proper according to Morphisms of Spaces, Lemma 67.45.9. A finite morphism is quasi-finite according to Morphisms of Spaces, Lemma 67.45.8. A quasi-finite morphism has discrete fibres X_k, see Morphisms of Spaces, Lemma 67.27.5. Hence a finite morphism is proper and has discrete fibres X_k.

Assume f is proper with discrete fibres X_k. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms of Spaces, Lemma 67.45.7 whereupon it follows from Morphisms of Spaces, Lemma 67.45.6 that f is finite.

To show that f is affine we may assume that Y is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. We will show that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 69.15.1 and 69.5.1. Then it follows that X is affine from Proposition 69.16.7. By Lemma 69.22.8 we conclude that the stalks of $R^1f_*\mathcal{F}$ are zero for all geometric points of Y. In other words, $R^1f_*\mathcal{F} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{F}) = H^1(Y, f_*\mathcal{F})$. Since Y is affine, and $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 67.11.2) we conclude $H^1(Y, f_*\mathcal{F}) = 0$ from Cohomology of Schemes, Lemma 30.2.2. Hence $H^1(X, \mathcal{F}) = 0$ as desired. $\square$

As a consequence we have the following useful result.

0A4W **Lemma 69.23.2.** (For a more general version see More on Morphisms of Spaces, Lemma 76.35.2). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume

(1) Y is locally Noetherian,
(2) f is proper, and
(3) $|X_{\overline{y}}|$ is finite.

Then there exists an open neighbourhood $V \subset Y$ of $\overline{y}$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the geometric points of X lying over $\overline{y}$ by Morphisms of Spaces, Lemma 67.34.8. By Morphisms of Spaces, Lemma 67.34.7 the set of points at which f is quasi-finite is an open subspace $U \subset X$. Let $Z = X \setminus U$. Then $\overline{y} \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of $\overline{y}$ with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ has discrete fibres X_k (Morphisms of Spaces, Lemma 67.27.5) which are quasi-compact hence finite. Thus $f^{-1}(V) \to V$ is finite by Lemma 69.23.1. $\square$

69.24. Other chapters

(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
(89) Resolution of Surfaces Revisited

Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
(93) Deformation Problems

Algebraic Stacks

(94) Algebraic Stacks
(95) Examples of Stacks
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CHAPTER 70

Limits of Algebraic Spaces

07SB 70.1. Introduction

07SC In this chapter we put material related to limits of algebraic spaces. A first topic is the characterization of algebraic spaces F locally of finite presentation over the base S as limit preserving functors. We continue with a study of limits of inverse systems over directed sets (Categories, Definition 4.21.1) with affine transition maps. We discuss absolute Noetherian approximation for quasi-compact and quasi-separated algebraic spaces following [CLO12]. Another approach is due to David Rydh (see [Ryd08]) whose results also cover absolute Noetherian approximation for certain algebraic stacks.

70.2. Conventions

07SD The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

70.3. Morphisms of finite presentation

049I In this section we generalize Limits, Proposition 32.6.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

049J Definition 70.3.1. Let S be a scheme.

(1) A functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ is said to be limit preserving or locally of finite presentation if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have
$$F(T) = \text{colim}\, F(T_i).$$
We sometimes say that F is locally of finite presentation over S.

(2) Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. A transformation of functors $a : F \to G$ is limit preserving or locally of finite presentation if for every scheme T over S and every $y \in G(T)$ the functor
$$F_y : (Sch/T)_{fppf}^{opp} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$
is locally of finite presentation over T[1]. We sometimes say that F is relatively limit preserving over G.

[1]The characterization (2) in Lemma 70.3.2 may be easier to parse.

The functor F_y is in some sense the fiber of $a : F \to G$ over y, except that it is a presheaf on the big fppf site of T. A formula for this functor is:

049K (70.3.1.1)
$$F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} *$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 7.10.2) and the map $* \to G|_{(Sch/T)_{fppf}}$ is given by y. Note that if $j : (Sch/T)_{fppf} \to (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 7.25, for information on localization, and especially Sites, Remark 7.25.10 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \to Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 67.28.1 and one using that $X \to Y$ is a transformation of functors so that Definition 70.3.1 applies (we will use the terminology "limit preserving" for this notion as much as possible). We will show in Proposition 70.3.10 that these two definitions agree.

06BC Lemma 70.3.2. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)_{fppf}^{opp} \to Sets$. The following are equivalent

(1) $a : F \to G$ is limit preserving, and
(2) for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets

$$\begin{tikzcd} \mathrm{colim}_i F(T_i) \arrow[r] \arrow[d, "a"'] & F(T) \arrow[d, "a"] \\ \mathrm{colim}_i G(T_i) \arrow[r] & G(T) \end{tikzcd}$$

is a fibre product diagram.

Proof. Assume (1). Consider $T = \lim_{i \in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $\mathrm{colim}_i G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i. Consider the functor F_{y_i} on $(Sch/T_i)_{fppf}$ as in Definition 70.3.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = \lim_{i' \geq i} T_{i'}$ is a directed system of affine schemes over T_i. Hence (1) implies that x_T the image of a unique element x of $\mathrm{colim}_{i' \geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $\mathrm{colim}\, F(T_i)$ which maps to the pair (y, x_T). This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = \lim_{i \in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T. Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $\mathrm{colim}_i G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $\mathrm{colim}_i F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $\mathrm{colim}_i F_y(T'_i)$ mapping to $x_{T'}$ and we win. $\square$

049L Lemma 70.3.3. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to$ Sets. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If a and b are limit preserving, then

$$b \circ a : F \longrightarrow H$$

is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 70.3.2. Consider the diagram of sets

$$\begin{tikzcd} \operatorname{colim}_i F(T_i) \arrow[r] \arrow[d, "a"'] & F(T) \arrow[d, "a"] \\ \operatorname{colim}_i G(T_i) \arrow[r] \arrow[d, "b"'] & G(T) \arrow[d, "b"] \\ \operatorname{colim}_i H(T_i) \arrow[r] & H(T) \end{tikzcd}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. □

0GDY Lemma 70.3.4. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If $b \circ a$ and b are limit preserving, then a is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 70.3.2. Consider the diagram of sets

$$\begin{tikzcd} \operatorname{colim}_i F(T_i) \arrow[r] \arrow[d, "a"'] & F(T) \arrow[d, "a"] \\ \operatorname{colim}_i G(T_i) \arrow[r] \arrow[d, "b"'] & G(T) \arrow[d, "b"] \\ \operatorname{colim}_i H(T_i) \arrow[r] & H(T) \end{tikzcd}$$

By assumption the lower square and the outer rectangle are fibre products of sets. Hence the upper square is a fibre product square too which proves the lemma. □

049M Lemma 70.3.5. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

If a is limit preserving, then the base change a' is limit preserving.

Proof. Omitted. Hint: This is formal. □

0GDZ Lemma 70.3.6. Let S be a scheme contained in Sch_{fppf}. Let $E, F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$, and $c : G \to E$ be transformations of functors. If c, $c \circ a$, and $c \circ b$ are limit preserving, then $F \times_G H \to E$ is too.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 70.3.2. Then we have

$$\operatorname{colim}(F \times_G H)(T_i) = \operatorname{colim} F(T_i) \times_{\operatorname{colim} G(T_i)} \operatorname{colim} H(T_i)$$

as filtered colimits commute with finite products. Our goal is thus to show that

$$\begin{tikzcd} \mathrm{colim}\, F(T_i) \times_{\mathrm{colim}\, G(T_i)} \mathrm{colim}\, H(T_i) \arrow[r] \arrow[d] & F(T) \times_{G(T)} H(T) \arrow[d] \\ \mathrm{colim}_i\, E(T_i) \arrow[r] & E(T) \end{tikzcd}$$

is a fibre product diagram. This follows from the observation that given maps of sets $E' \to E$, $F \to G$, $H \to G$, and $G \to E$ we have

$$E' \times_E (F \times_G H) = (E' \times_E F) \times_{(E' \times_E G)} (E' \times_E H)$$

Some details omitted. □

049O Lemma 70.3.7. Let S be a scheme contained in Sch_{fppf}. Let $F : (Sch/S)_{fppf}^{opp} \to$ Sets be a functor. If F is limit preserving then its sheafification $F^\#$ is limit preserving.

Proof. Assume F is limit preserving. It suffices to show that F^+ is limit preserving, since $F^\# = (F^+)^+$, see Sites, Theorem 7.10.10. Let T be an affine scheme over S, and let $T = \lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 34.7.4. Hence we can write

$$F^+(T) = \mathrm{colim}_{\mathcal{V} \text{ standard covering } T}\, \check{H}^0(\mathcal{V}, F).$$

Any $\mathcal{V} = \{T_k \to T\}_{k=1,\ldots,n}$ in the colimit may be written as $V_i \times_{T_i} T$ for some i and some standard fppf covering $\mathcal{V}_i = \{T_{i,k} \to T_i\}_{k=1,\ldots,n}$ of T_i. Denote $\mathcal{V}_{i'} = \{T_{i',k} \to T_{i'}\}_{k=1,\ldots,n}$ the base change for $i' \geq i$. Then we see that

$$\begin{aligned} \mathrm{colim}_{i' \geq i}\, \check{H}^0(\mathcal{V}_i, F) &= \mathrm{colim}_{i' \geq i}\, \mathrm{Equalizer}\big(\textstyle\prod F(T_{i',k}) \rightrightarrows \prod F(T_{i',k} \times_{T_{i'}} T_{i',l}) \big) \\ &= \mathrm{Equalizer}\big(\mathrm{colim}_{i' \geq i} \textstyle\prod F(T_{i',k}) \rightrightarrows \mathrm{colim}_{k' \geq k} \prod F(T_{i',k} \times_{T_{i'}} T_{i',l}) \big) \\ &= \mathrm{Equalizer}\big(\textstyle\prod F(T_k) \rightrightarrows \prod F(T_k \times_T T_l) \big) \\ &= \check{H}^0(\mathcal{V}, F) \end{aligned}$$

Here the second equality holds because filtered colimits are exact. The third equality holds because F is limit preserving and because $\lim_{i' \geq i} T_{i',k} = T_k$ and $\lim_{i' \geq i} T_{i',k} \times_{T_{i'}} T_{i',l} = T_k \times_T T_l$ by Limits, Lemma 32.2.3. If we use this for all coverings at the same time we obtain

$$\begin{aligned} F^+(T) &= \mathrm{colim}_{\mathcal{V} \text{ standard covering } T}\, \check{H}^0(\mathcal{V}, F) \\ &= \mathrm{colim}_{i \in I}\, \mathrm{colim}_{\mathcal{V}_i \text{ standard covering } T_i}\, \check{H}^0(T \times_{T_i} \mathcal{V}_i, F) \\ &= \mathrm{colim}_{i \in I}\, F^+(T_i) \end{aligned}$$

The switch of the order of the colimits is allowed by Categories, Lemma 4.14.10. □

049P Lemma 70.3.8. Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to$ Sets be a functor. Assume that

(1) F is a sheaf, and
(2) there exists an fppf covering $\{U_j \to S\}_{j \in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is limit preserving.

Then F is limit preserving.

Proof. Let T be an affine scheme over S. Let I be a directed set, and let T_i be an inverse system of affine schemes over S such that $T = \lim T_i$. We have to show that the canonical map $\text{colim}\, F(T_i) \to F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \to T_0\}_{k=1,\ldots,m}$ which refines the pullback $\{U_j \times_S T_0 \to T_0\}$ of the given fppf covering of S. For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = \lim_{i \geq 0} V_{i,k}$, see Limits, Lemma 32.2.3.

Suppose that $x, x' \in \text{colim}\, F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x'_i \in F(T_i)$ for some $i \in I$ (we may choose the same i for both as I is directed). By assumption (2) and the fact that x_i, x'_i map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x'_i|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \ldots, m\}$ ranges over a finite set. Since $\{V_{i',k} \to T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x'_i|_{T_{i'}}$ as desired. This proves that the map $\text{colim}\, F(T_i) \to F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k. By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k} \times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k} \times_{T_{i'}} V_{i',l}}$$

for some large enough i'. Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. $\square$

049Q Lemma 70.3.9. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be functors. If $a : F \to G$ is a transformation which is limit preserving, then the induced transformation of sheaves $F^\# \to G^\#$ is limit preserving.

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F^\#_y : (Sch/T)_{fppf}^{opp} \to Sets$ constructed from $F^\# \to G^\#$ and y as in Definition 70.3.1 is limit preserving. By Equation (70.3.1.1) we see that $F^\#_y$ is a sheaf. Choose an fppf covering $\{V_j \to T\}_{j \in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F^\#_{y_j}$. If we can show that $F^\#_{y_j}$ is limit preserving then Lemma 70.3.8 guarantees that $F^\#_y$ is limit preserving and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F^\#_y = (F_y)^\#$. This follows from Equation (70.3.1.1). Thus this case follows from Lemma 70.3.7. $\square$

04AK Proposition 70.3.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 67.28.1.
(2) The morphism $f : X \to Y$ is limit preserving as a transformation of functors, see Definition 70.3.1.

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_Y X$ is limit preserving over T in the sense of Definition 70.3.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is limit preserving as a functor $X : (Sch/S)_{fppf}^{opp} \to Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 65.9.3. It follows from Morphisms of Spaces, Definition 67.28.1 that both U and R are schemes which are locally of finite presentation over S. Hence by Limits, Proposition 32.6.1 we have

$$U(T) = \text{colim}\, U(T_i), \quad R(T) = \text{colim}\, R(T_i)$$

whenever $T = \lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)_{fppf}^{opp} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is limit preserving. Hence by Lemma 70.3.7 its sheafification $X = U/R$ is limit preserving too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Next, choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By Lemma 70.3.5 the transformation of functors $V \times_Y X \to V$ is limit preserving. By Morphisms of Spaces, Lemma 67.39.8 the morphism of algebraic spaces $U \to V \times_Y X$ is locally of finite presentation, hence limit preserving as a transformation of functors by the first part of the proof. By Lemma 70.3.3 the composition $U \to V \times_Y X \to V$ is limit preserving as a transformation of functors. Hence the morphism of schemes $U \to V$ is locally of finite presentation by Limits, Proposition 32.6.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \to V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 32.6.1 we characterize $U \to V$ as being locally of finite presentation if for all directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = \text{colim}\, V(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) $\Rightarrow$ (1) of Limits, Proposition 32.6.1 we see that the question reduces to the case that U and V are affine. Say $U = \text{Spec}(A)$ and $V = \text{Spec}(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 10.127.3 only A-algebras of cardinality $\leq |B|$ are used. □

05N0 Remark 70.3.11. Here is an important special case of Proposition 70.3.10. Let S be a scheme. Let X be an algebraic space over S. Then X is locally of finite presentation over S if and only if X, as a functor $(Sch/S)^{opp} \to Sets$, is limit preserving. Compare with Limits, Remark 32.6.2. In fact, we will see in Lemma 70.3.12 below that it suffices if the map

$$\text{colim}\, X(T_i) \longrightarrow X(T)$$

is surjective whenever $T = \lim T_i$ is a directed limit of affine schemes over S.

0CM6 **Lemma 70.3.12.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map

$$\operatorname{colim} X(T_i) \longrightarrow X(T) \times_{Y(T)} \operatorname{colim} Y(T_i)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 70.3.10 part (2) it suffices to check surjectivity in the criterion of Lemma 70.3.2.

Proof. Choose a scheme V and a surjective étale morphism $g : V \to Y$. Next, choose a scheme U and a surjective étale morphism $h : U \to V \times_Y X$. It suffices to show for $T = \lim T_i$ as in the lemma that the map

$$\operatorname{colim} U(T_i) \longrightarrow U(T) \times_{V(T)} \operatorname{colim} V(T_i)$$

is surjective, because then $U \to V$ will be locally of finite presentation by Limits, Lemma 32.6.3 (modulo a set theoretic remark exactly as in the proof of Proposition 70.3.10). Thus we take $a : T \to U$ and $b_i : T_i \to V$ which determine the same morphism $T \to V$. Picture

$$\begin{tikzcd}
T \arrow[rr, "p_i"'] \arrow[d, "a"'] & & T_i \arrow[d, "b_i"] \arrow[dl, dotted] \\
U \arrow[r, "h"] & X \times_Y V \arrow[r] \arrow[d] & V \arrow[d, "g"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

By the assumption of the lemma after increasing i we can find a morphism $c_i : T_i \to X$ such that $h \circ a = (b_i, c_i) \circ p_i : T_i \to V \times_Y X$ and such that $f \circ c_i = g \circ b_i$. Since h is an étale morphism of algebraic spaces (and hence locally of finite presentation), we have the surjectivity of

$$\operatorname{colim} U(T_i) \longrightarrow U(T) \times_{(X \times_Y V)(T)} \operatorname{colim}(X \times_Y V)(T_i)$$

by Proposition 70.3.10. Hence after increasing i again we can find the desired morphism $a_i : T_i \to U$ with $a = a_i \circ p_i$ and $b_i = (U \to V) \circ a_i$. $\square$

70.4. Limits of algebraic spaces

07SE The following lemma explains how we think of limits of algebraic spaces in this chapter. We will use (without further mention) that the base change of an affine morphism of algebraic spaces is affine (see Morphisms of Spaces, Lemma 67.20.5).

07SF **Lemma 70.4.1.** Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I in the category of algebraic spaces over S. If the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine, then the limit $X = \lim_i X_i$ (as an fppf sheaf) is an algebraic space. Moreover,

(1) each of the morphisms $f_i : X \to X_i$ is affine,
(2) for any $i \in I$ and any morphism of algebraic spaces $T \to X_i$ we have

$$X \times_{X_i} T = \lim_{i' \geq i} X_{i'} \times_{X_i} T.$$

as algebraic spaces over S.

Proof. Part (2) is a formal consequence of the existence of the limit $X = \lim X_i$ as an algebraic space over S. Choose an element $0 \in I$ (this is possible as a directed set is nonempty). Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $R_0 = U_0 \times_{X_0} U_0$ so that $X_0 = U_0/R_0$. For $i \geq 0$ set $U_i = X_i \times_{X_0} U_0$ and $R_i = X_i \times_{X_0} R_0 = U_i \times_{X_i} U_i$. By Limits, Lemma 32.2.2 we see that $U = \lim_{i \geq 0} U_i$ and $R = \lim_{i \geq 0} R_i$ are schemes. Moreover, the two morphisms $s, t : R \to U$ are the base change of the two projections $R_0 \to U_0$ by the morphism $U \to U_0$, in particular étale. The morphism $R \to U \times_S U$ defines an equivalence relation as directed a limit of equivalence relations is an equivalence relation. Hence the morphism $R \to U \times_S U$ is an étale equivalence relation. We claim that the natural map

07SG (70.4.1.1) $$U/R \longrightarrow \lim X_i$$

is an isomorphism of fppf sheaves on the category of schemes over S. The claim implies $X = \lim X_i$ is an algebraic space by Spaces, Theorem 65.10.5.

Let Z be a scheme and let $a : Z \to \lim X_i$ be a morphism. Then $a = (a_i)$ where $a_i : Z \to X_i$. Set $W_0 = Z \times_{a_0, X_0} U_0$. Note that $W_0 = Z \times_{a_i, X_i} U_i$ for all $i \geq 0$ by our choice of $U_i \to X_i$ above. Hence we obtain a morphism $W_0 \to \lim_{i \geq 0} U_i = U$. Since $W_0 \to Z$ is surjective and étale, we conclude that (70.4.1.1) is a surjective map of sheaves. Finally, suppose that Z is a scheme and that $a, b : Z \to U/R$ are two morphisms which are equalized by (70.4.1.1). We have to show that $a = b$. After replacing Z by the members of an fppf covering we may assume there exist morphisms $a', b' : Z \to U$ which give rise to a and b. The condition that a, b are equalized by (70.4.1.1) means that for each $i \geq 0$ the compositions $a'_i, b'_i : Z \to U \to U_i$ are equal as morphisms into $U_i/R_i = X_i$. Hence $(a'_i, b'_i) : Z \to U_i \times_S U_i$ factors through R_i, say by some morphism $c_i : Z \to R_i$. Since $R = \lim_{i \geq 0} R_i$ we see that $c = \lim c_i : Z \to R$ is a morphism which shows that a, b are equal as morphisms of Z into U/R.

Part (1) follows as we have seen above that $U_i \times_{X_i} X = U$ and $U \to U_i$ is affine by construction. □

07SH Lemma 70.4.2. Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S with affine transition maps. Let $X = \lim_i X_i$. Let $0 \in I$. Suppose that $T \to X_0$ is a morphism of algebraic spaces. Then

$$T \times_{X_0} X = \lim_{i \geq 0} T \times_{X_0} X_i$$

as algebraic spaces over S.

Proof. The limit X is an algebraic space by Lemma 70.4.1. The equality is formal, see Categories, Lemma 4.14.10. □

0CUH Lemma 70.4.3. Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i}) \to (Y_i, g_{i'i})$ be a morphism of inverse systems over I of algebraic spaces over S. Assume

(1) the morphisms $f_{i'i} : X_{i'} \to X_i$ are affine,
(2) the morphisms $g_{i'i} : Y_{i'} \to Y_i$ are affine,
(3) the morphisms $X_i \to Y_i$ are closed immersions.

Then $\lim X_i \to \lim Y_i$ is a closed immersion.

Proof. Observe that $\lim X_i$ and $\lim Y_i$ exist by Lemma 70.4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $V_0 \to Y_0$. Then the morphisms $V_i = Y_i \times_{Y_0} V_0 \to U_i = X_i \times_{Y_0} V_0$ are closed immersions of affine schemes. Hence the morphism $V = Y \times_{Y_0} V_0 \to U = X \times_{Y_0} V_0$ is a closed immersion because $V = \lim V_i$, $U = \lim U_i$ and because a limit of closed immersions of affine schemes is a closed immersion: a filtered colimit of surjective ring maps is surjective. Since the étale morphisms $V \to Y$ form an étale covering of Y as we vary our choice of $V_0 \to Y_0$ we see that the lemma is true. $\square$

0CUI Lemma 70.4.4. Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse systems over I of algebraic spaces over S. If X_i is reduced for all i, then X is reduced.

Proof. Observe that $\lim X_i$ exists by Lemma 70.4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $U_0 \to X_0$. Then the affine schemes $U_i = X_i \times_{X_0} U_0$ are reduced. Hence $U = X \times_{X_0} U_0$ is a reduced affine scheme as a limit of reduced affine schemes: a filtered colimit of reduced rings is reduced. Since the étale morphisms $U \to X$ form an étale covering of X as we vary our choice of $U_0 \to X_0$ we see that the lemma is true. $\square$

0CP4 Lemma 70.4.5. Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. The equivalent conditions (1) and (2) of Proposition 70.3.10 are also equivalent to

(3) for every directed limit $T = \lim T_i$ of quasi-compact and quasi-separated algebraic spaces T_i over S with affine transition morphisms the diagram of sets

$$\begin{tikzcd} \mathrm{colim}_i \mathrm{Mor}(T_i, X) \arrow[r] \arrow[d] & \mathrm{Mor}(T, X) \arrow[d] \\ \mathrm{colim}_i \mathrm{Mor}(T_i, Y) \arrow[r] & \mathrm{Mor}(T, Y) \end{tikzcd}$$

is a fibre product diagram.

Proof. It is clear that (3) implies (2). We will assume (2) and prove (3). The proof is rather formal and we encourage the reader to find their own proof.

Let us first prove that (3) holds when T_i is in addition assumed separated for all i. Choose $i \in I$ and choose a surjective étale morphism $U_i \to T_i$ where U_i is affine. Using Lemma 70.4.2 we see that with $U = U_i \times_{T_i} T$ and $U_{i'} = U_i \times_{T_i} T_{i'}$ we have $U = \lim_{i' \geq i} U_{i'}$. Of course U and $U_{i'}$ are affine (see Lemma 70.4.1). Since T_i is separated, the fibre product $V_i = U_i \times_{T_i} U_i$ is an affine scheme as well and we obtain affine schemes $V = V_i \times_{T_i} T$ and $V_{i'} = V_i \times_{T_i} T_{i'}$ with $V = \lim_{i' \geq i} V_{i'}$. Observe that $U \to T$ and $U_i \to T_i$ are surjective étale and that $V = U \times_T U$ and $V_{i'} = U_{i'} \times_{T_{i'}} U_{i'}$. Note that $\mathrm{Mor}(T, X)$ is the equalizer of the two maps $\mathrm{Mor}(U, X) \to \mathrm{Mor}(V, X)$; this is true for example because X as a sheaf on $(Sch/S)_{fppf}$ is the coequalizer of the two maps $h_V \to h_u$. Similarly $\mathrm{Mor}(T_{i'}, X)$ is the equalizer of the two maps $\mathrm{Mor}(U_{i'}, X) \to \mathrm{Mor}(V_{i'}, X)$. And of course the same thing is true with X replaced with Y. Condition (2) says that the diagrams of in (3) are fibre products in the case of $U = \lim U_i$ and $V = \lim V_i$. It follows formally that the same thing is true for $T = \lim T_i$.

In the general case, choose an affine scheme U, an $i \in I$, and a surjective étale morphism $U \to T_i$. Repeating the argument of the previous paragraph we still achieve the proof: the schemes $V_{i'}$, V are no longer affine, but they are still quasi-compact and separated and the result of the preceding paragraph applies. $\square$

70.5. Descending properties

0826 This section is the analogue of Limits, Section 32.4.

0CUJ **Lemma 70.5.1.** Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). If each X_i is decent (for example quasi-separated or locally separated) then $|X| = \lim_i |X_i|$ as sets.

Proof. There is a canonical map $|X| \to \lim |X_i|$. Choose $0 \in I$. If $W_0 \subset X_0$ is an open subspace, then we have $f_0^{-1}W_0 = \lim_{i \geq 0} f_{i0}^{-1}W_0$, see Lemma 70.4.1. Hence, if we can prove the lemma for inverse systems where X_0 is quasi-compact, then the lemma follows in general. Thus we may and do assume X_0 is quasi-compact.

Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$. Recall that $U = \lim U_i$ and $R = \lim R_i$, see proof of Lemma 70.4.1. Recall that $|X| = |U|/|R|$ and $|X_i| = |U_i|/|R_i|$. By Limits, Lemma 32.4.6 we have $|U| = \lim |U_i|$ and $|R| = \lim |R_i|$.

Surjectivity of $|X| \to \lim |X_i|$. Let $(x_i) \in \lim |X_i|$. Denote $S_i \subset |U_i|$ the inverse image of x_i. This is a finite nonempty set by the definition of decent spaces (Decent Spaces, Definition 68.6.1). Hence $\lim S_i$ is nonempty, see Categories, Lemma 4.21.7. Let $(u_i) \in \lim S_i \subset \lim |U_i|$. By the above this determines a point $u \in |U|$ which maps to an $x \in |X|$ mapping to the given element (x_i) of $\lim |X_i|$.

Injectivity of $|X| \to \lim |X_i|$. Suppose that $x, x' \in |X|$ map to the same point of $\lim |X_i|$. Choose lifts $u, u' \in |U|$ and denote $u_i, u_i' \in |U_i|$ the images. For each i let $T_i \subset |R_i|$ be the set of points mapping to $(u_i, u_i') \in |U_i| \times |U_i|$. This is a finite set by the definition of decent spaces (Decent Spaces, Definition 68.6.1). Moreover T_i is nonempty as we've assumed that x and x' map to the same point of X_i. Hence $\lim T_i$ is nonempty, see Categories, Lemma 4.21.7. As before let $r \in |R| = \lim |R_i|$ be a point corresponding to an element of $\lim T_i$. Then r maps to (u, u') in $|U| \times |U|$ by construction and we see that $x = x'$ in $|X|$ as desired.

Parenthetical statement: A quasi-separated algebraic space is decent, see Decent Spaces, Section 68.6 (the key observation to this is Properties of Spaces, Lemma 66.6.7). A locally separated algebraic space is decent by Decent Spaces, Lemma 68.15.2. $\square$

086V **Lemma 70.5.2.** With same notation and assumptions as in Lemma 70.5.1 we have $|X| = \lim_i |X_i|$ as topological spaces.

Proof. We will use the criterion of Topology, Lemma 5.14.3. We have seen that $|X| = \lim_i |X_i|$ as sets in Lemma 70.5.1. The maps $f_i : X \to X_i$ are morphisms of algebraic spaces hence determine continuous maps $|X| \to |X_i|$. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset |X_i|$. Finally, let $x \in |X|$ and let $x \in V \subset |X|$ be an open neighbourhood. We have to find an i and an open neighbourhood $W_i \subset |X_i|$ of the image x with $f_i^{-1}(W_i) \subset V$. Choose $0 \in I$. Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$ for $i \geq 0$. Then $U = \lim_{i \geq 0} U_i$ in the category of schemes by Lemma 70.4.1. Choose $u \in U$ mapping to x. By the result for schemes (Limits, Lemma 32.4.2) we can find an $i \geq 0$ and an open neighbourhood $E_i \subset U_i$ of the image of u whose inverse image in U is contained in the inverse image of V in U. Then we can set $W_i \subset |X_i|$ equal to the image of E_i. This works because $|U_i| \to |X_i|$ is open. $\square$

086W Lemma 70.5.3. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). If each X_i is quasi-compact and nonempty, then $|X|$ is nonempty.

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Then each U_i is a nonempty affine scheme. Hence $U = \lim U_i$ is nonempty (Limits, Lemma 32.4.3) and thus X is nonempty. $\square$

0CUK Lemma 70.5.4. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). Let $x \in |X|$ with images $x_i \in |X_i|$. If each X_i is decent, then $\overline{\{x\}} = \lim_i \overline{\{x_i\}}$ as sets and as algebraic spaces if endowed with reduced induced scheme structure.

Proof. Set $Z = \overline{\{x\}} \subset |X|$ and $Z_i = \overline{\{x_i\}} \subset |X_i|$. Since $|X| \to |X_i|$ is continuous we see that Z maps into Z_i for each i. Hence we obtain an injective map $Z \to \lim Z_i$ because $|X| = \lim |X_i|$ as sets (Lemma 70.5.1). Suppose that $x' \in |X|$ is not in Z. Then there is an open subset $U \subset |X|$ with $x' \in U$ and $x \notin U$. Since $|X| = \lim |X_i|$ as topological spaces (Lemma 70.5.2) we can write $U = \bigcup_{j \in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset |X_j|$, see Topology, Lemma 5.14.2. Then we see that for some $j \in J$ we have $f_j(x') \in U_j$ and $f_j(x) \notin U_j$. In other words, we see that $f_j(x') \notin Z_j$. Thus $Z = \lim Z_i$ as sets.

Next, endow Z and Z_i with their reduced induced scheme structures, see Properties of Spaces, Definition 66.12.5. The transition morphisms $X_{i'} \to X_i$ induce affine morphisms $Z_{i'} \to Z_i$ and the projections $X \to X_i$ induce compatible morphisms $Z \to Z_i$. Hence we obtain morphisms $Z \to \lim Z_i \to X$ of algebraic spaces. By Lemma 70.4.3 we see that $\lim Z_i \to X$ is a closed immersion. By Lemma 70.4.4 the algebraic space $\lim Z_i$ is reduced. By the above $Z \to \lim Z_i$ is bijective on points. By uniqueness of the reduced induced closed subscheme structure we find that this morphism is an isomorphism of algebraic spaces. $\square$

084R Situation 70.5.5. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

07SI Lemma 70.5.6. Notation and assumptions as in Situation 70.5.5. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on X_0. Set $\mathcal{F}_i = f_{0i}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then

$$\Gamma(X, \mathcal{F}) = \mathrm{colim}_{i \geq 0} \Gamma(X_i, \mathcal{F}_i)$$

Proof. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 66.6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. In the proof of Lemma 70.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Hence Limits, Lemma 32.4.7 implies that

$$\mathcal{F}(U) = \operatorname{colim} \mathcal{F}_i(U_i) \quad \text{and} \quad \mathcal{F}(R) = \operatorname{colim} \mathcal{F}_i(R_i).$$

The lemma follows as $\Gamma(X, \mathcal{F}) = \operatorname{Ker}(\mathcal{F}(U) \to \mathcal{F}(R))$ and similarly $\Gamma(X_i, \mathcal{F}_i) = \operatorname{Ker}(\mathcal{F}_i(U_i) \to \mathcal{F}_i(R_i))$ □

0827 Lemma 70.5.7. Notation and assumptions as in Situation 70.5.5. For any quasi-compact open subspace $U \subset X$ there exists an i and a quasi-compact open $U_i \subset X_i$ whose inverse image in X is U.

Proof. Follows formally from the construction of limits in Lemma 70.4.1 and the corresponding result for schemes: Limits, Lemma 32.4.11. □

The following lemma will be superseded by the stronger Lemma 70.6.10.

084S Lemma 70.5.8. Notation and assumptions as in Situation 70.5.5. Let $f_0 : Y_0 \to Z_0$ be a morphism of algebraic spaces over X_0. Assume (a) $Y_0 \to X_0$ and $Z_0 \to X_0$ are representable, (b) Y_0, Z_0 quasi-compact and quasi-separated, (c) f_0 locally of finite presentation, and (d) $Y_0 \times_{X_0} X \to Z_0 \times_{X_0} X$ an isomorphism. Then there exists an $i \geq 0$ such that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism.

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = U_0 \times_{X_0} X_i$ and $U = U_0 \times_{X_0} X$. Apply Limits, Lemma 32.8.11 to see that $Y_0 \times_{X_0} U_i \to Z_0 \times_{X_0} U_i$ is an isomorphism of schemes for some $i \geq 0$ (details omitted). As $U_i \to X_i$ is surjective étale, it follows that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism (details omitted). □

084T Lemma 70.5.9. Notation and assumptions as in Situation 70.5.5. If X is separated, then X_i is separated for some $i \in I$.

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. For $i \geq 0$ set $U_i = U_0 \times_{X_0} X_i$ and set $U = U_0 \times_{X_0} X$. Note that U_i and U are affine schemes which come equipped with surjective étale morphisms $U_i \to X_i$ and $U \to X$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$ with projections $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$. Note that R_i and R are quasi-compact separated schemes (as the algebraic spaces X_i and X are quasi-separated). The maps $s_i : R_i \to U_i$ and $s : R \to U$ are of finite type. By definition X_i is separated if and only if $(t_i, s_i) : R_i \to U_i \times U_i$ is a closed immersion, and since X is separated by assumption, the morphism $(t, s) : R \to U \times U$ is a closed immersion. Since $R \to U$ is of finite type, there exists an i such that the morphism $R \to U_i \times U$ is a closed immersion (Limits, Lemma 32.4.16). Fix such an $i \in I$. Apply Limits, Lemma 32.8.5 to the system of morphisms $R_{i'} \to U_i \times U_{i'}$ for $i' \geq i$ (this is permissible as indeed $R_{i'} = R_i \times_{U_i \times U_i} U_i \times U_{i'}$) to see that $R_{i'} \to U_i \times U_{i'}$ is a closed immersion for i' sufficiently large. This implies immediately that $R_{i'} \to U_{i'} \times U_{i'}$ is a closed immersion finishing the proof of the lemma. □

07SQ Lemma 70.5.10. Notation and assumptions as in Situation 70.5.5. If X is affine, then there exists an i such that X_i is affine.

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = U_0 \times_{X_0} X$ and $U_i = U_0 \times_{X_0} X_i$ for $i \geq 0$. Since the transition morphisms are affine, the algebraic spaces U_i and U are affine. Thus $U \to X$ is an étale morphism of affine schemes. Hence we can write $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$ and

$$B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$$

such that $\Delta = \det(\partial g_\lambda/\partial x_\mu)$ is invertible in B, see Algebra, Lemma 10.143.2. Set $A_i = \mathcal{O}_{X_i}(X_i)$. We have $A = \mathrm{colim}\, A_i$ by Lemma 70.5.6. After increasing 0 we may assume we have $g_{1,i}, \ldots, g_{n,i} \in A_i[x_1, \ldots, x_n]$ mapping to $g_1, \ldots, g_n$. Set

$$B_i = A_i[x_1, \ldots, x_n]/(g_{1,i}, \ldots, g_{n,i})$$

for all $i \geq 0$. Increasing 0 if necessary we may assume that $\Delta_i = \det(\partial g_{\lambda,i}/\partial x_\mu)$ is invertible in B_i for all $i \geq 0$. Thus $A_i \to B_i$ is an étale ring map. After increasing 0 we may assume also that $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)$ is surjective, see Limits, Lemma 32.8.15. Increasing 0 yet again we may choose elements $h_{1,i}, \ldots, h_{n,i} \in \mathcal{O}_{U_i}(U_i)$ which map to the classes of $x_1, \ldots, x_n$ in $B = \mathcal{O}_U(U)$ and such that $g_{\lambda,i}(h_{\nu,i}) = 0$ in $\mathcal{O}_{U_i}(U_i)$. Thus we obtain a commutative diagram

084U (70.5.10.1)
$$\begin{tikzcd} X_i \arrow[d] & U_i \arrow[l] \arrow[d] \\ \mathrm{Spec}(A_i) & \mathrm{Spec}(B_i) \arrow[l] \end{tikzcd}$$

By construction $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$. Consider the morphism

$$f_0 : U_0 \longrightarrow X_0 \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(B_0)$$

This is a morphism of quasi-compact and quasi-separated algebraic spaces representable, separated and étale over X_0. The base change of f_0 to X is an isomorphism by our choices. Hence Lemma 70.5.8 guarantees that there exists an i such that the base change of f_0 to X_i is an isomorphism, in other words the diagram (70.5.10.1) is cartesian. Thus Descent, Lemma 35.39.1 applied to the fppf covering $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)\}$ combined with Descent, Lemma 35.37.1 give that $X_i \to \mathrm{Spec}(A_i)$ is representable by a scheme affine over $\mathrm{Spec}(A_i)$ as desired. (Of course it then also follows that $X_i = \mathrm{Spec}(A_i)$ but we don't need this.) □

07SR Lemma 70.5.11. Notation and assumptions as in Situation 70.5.5. If X is a scheme, then there exists an i such that X_i is a scheme.

Proof. Choose a finite affine open covering $X = \bigcup W_j$. By Lemma 70.5.7 we can find an $i \in I$ and open subspaces $W_{j,i} \subset X_i$ whose base change to X is $W_j \to X$. By Lemma 70.5.10 we may assume that each $W_{j,i}$ is an affine scheme. This means that X_i is a scheme (see for example Properties of Spaces, Section 66.13). □

0828 Lemma 70.5.12. Let S be a scheme. Let B be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over B with affine transition morphisms. Let $Y \to X$ be a morphism of algebraic spaces over B.

(1) If $Y \to X$ is a closed immersion, X_i quasi-compact, and $Y \to B$ locally of finite type, then $Y \to X_i$ is a closed immersion for i large enough.
(2) If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.

(3) If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to B$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to B$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.
(4) If $Y \to X$ is a monomorphism, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is a monomorphism for i large enough.

Proof. Proof of (1). Choose $0 \in I$. As X_0 is quasi-compact, we can choose an affine scheme W and an étale morphism $W \to B$ such that the image of $|X_0| \to |B|$ is contained in $|W| \to |B|$. Choose an affine scheme U_0 and an étale morphism $U_0 \to X_0 \times_B W$ such that $U_0 \to X_0$ is surjective. (This is possible by our choice of W and the fact that X_0 is quasi-compact; details omitted.) Let $V \to Y$, resp. $U \to X$, resp. $U_i \to X_i$ be the base change of $U_0 \to X_0$ (for $i \geq 0$). It suffices to prove that $V \to U_i$ is a closed immersion for i sufficiently large. Thus we reduce to proving the result for $V \to U = \lim U_i$ over W. This follows from the case of schemes, which is Limits, Lemma 32.4.16.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open subspace $X'_0 \subset X_0$ such that $Y \to X_0$ factors through X'_0. After replacing X_i by the inverse image of X'_0 for $i \geq 0$ we may assume all X'_i are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 70.5.7 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0 = Y \times_{X_0} U_0$. Then $U = \lim U_i$ is a limit of affine schemes, the transition maps of the system are closed immersions, and $U \to U_0$ is of finite presentation (because $U \to B$ is locally of finite presentation and $U_0 \to B$ is locally of finite type and Morphisms of Spaces, Lemma 67.28.9). Thus we've reduced to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$.

Proof of (4). Set $Z_i = Y \times_{X_i} Y$. As the transition morphisms $X_{i'} \to X_i$ are affine hence separated, the transition morphisms $Z_{i'} \to Z_i$ are closed immersions, see Morphisms of Spaces, Lemma 67.4.5. We have $\lim Z_i = Y \times_X Y = Y$ as $Y \to X$ is a monomorphism. Choose $0 \in I$. Since $Y \to X_0$ is locally of finite type (Morphisms of Spaces, Lemma 67.23.6) the morphism $Y \to Z_0$ is locally of finite presentation (Morphisms of Spaces, Lemma 67.28.10). The morphisms $Z_i \to Z_0$ are locally of finite type (they are closed immersions). Finally, $Z_i = Y \times_{X_i} Y$ is quasi-compact as X_i is quasi-separated and Y is quasi-compact. Thus part (3) applies to $Y = \lim_{i \geq 0} Z_i$ over Z_0 and we conclude $Y = Z_i$ for some i. This proves (4) and the lemma. □

086X Lemma 70.5.13. Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume

(1) Y is quasi-separated,

(2) X_i is quasi-compact and quasi-separated,
(3) the morphism $X \to Y$ is separated.

Then $X_i \to Y$ is separated for all i large enough.

Proof. Let $0 \in I$. Choose an affine scheme W and an étale morphism $W \to Y$ such that the image of $|W| \to |Y|$ contains the image of $|X_0| \to |Y|$. This is possible as X_0 is quasi-compact. It suffices to check that $W \times_Y X_i \to W$ is separated for some $i \geq 0$ because the diagonal of $W \times_Y X_i$ over W is the base change of $X_i \to X_i \times_Y X_i$ by the surjective étale morphism $(X_i \times_Y X_i) \times_Y W \to X_i \times_Y X_i$. Since Y is quasi-separated the algebraic spaces $W \times_Y X_i$ are quasi-compact (as well as quasi-separated). Thus we may base change to W and assume Y is an affine scheme. When Y is an affine scheme, we have to show that X_i is a separated algebraic space for i large enough and we are given that X is a separated algebraic space. Thus this case follows from Lemma 70.5.9. □

0A0R Lemma 70.5.14. Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume

(1) Y quasi-compact and quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) $X \to Y$ affine.

Then $X_i \to Y$ is affine for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is affine and it suffices to check that $X_i \times_Y W$ is affine for some i (Morphisms of Spaces, Lemma 67.20.3). This follows from Lemma 70.5.10. □

0A0S Lemma 70.5.15. Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume

(1) Y quasi-compact and quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) the transition morphisms $X_{i'} \to X_i$ are finite,
(4) $X_i \to Y$ locally of finite type
(5) $X \to Y$ integral.

Then $X_i \to Y$ is finite for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is finite over W and it suffices to check that $X_i \times_Y W$ is finite over W for some i (Morphisms of Spaces, Lemma 67.45.3). By Lemma 70.5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 32.4.19. □

0A0T Lemma 70.5.16. Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume

(1) Y quasi-compact and quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) the transition morphisms $X_{i'} \to X_i$ are closed immersions,
(4) $X_i \to Y$ locally of finite type

(5) $X \to Y$ is a closed immersion.

Then $X_i \to Y$ is a closed immersion for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is a closed subspace of W and it suffices to check that $X_i \times_Y W$ is a closed subspace W for some i (Morphisms of Spaces, Lemma 67.12.1). By Lemma 70.5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 32.4.20. $\square$

70.6. Descending properties of morphisms

084V This section is the analogue of Section 70.5 for properties of morphisms. We will work in the following situation.

084W Situation 70.6.1. Let S be a scheme. Let $B = \lim B_i$ be a limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of algebraic spaces over B_0. Assume B_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to B_i and let $f : X \to Y$ be the base change of f_0 to B.

07SL Lemma 70.6.2. With notation and assumptions as in Situation 70.6.1. If

(1) f is étale,
(2) f_0 is locally of finite presentation,

then f_i is étale for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is étale if and only if $U_i \to V_i$ is étale and similarly $X \to Y$ is étale if and only if $U \to V$ is étale (Morphisms of Spaces, Lemma 67.39.2). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 32.8.10. $\square$

0CN2 Lemma 70.6.3. With notation and assumptions as in Situation 70.6.1. If

(1) f is smooth,
(2) f_0 is locally of finite presentation,

then f_i is smooth for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is smooth if and only if $U_i \to V_i$ is smooth and similarly $X \to Y$ is smooth if and only if $U \to V$ is smooth (Morphisms of Spaces, Definition 67.37.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 32.8.9. $\square$

07SN **Lemma 70.6.4.** With notation and assumptions as in Situation 70.6.1. If

(1) f is surjective,
(2) f_0 is locally of finite presentation,

then f_i is surjective for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, the limit of the morphisms $U_i \to V_i$ is $U \to V$, and the morphisms $U_i \to X_i \times_{Y_i} V_i$ and $U \to X \times_Y V$ are surjective (as base changes of $U_0 \to X_0 \times_{Y_0} V_0$). In particular, we see that $X_i \to Y_i$ is surjective if and only if $U_i \to V_i$ is surjective and similarly $X \to Y$ is surjective if and only if $U \to V$ is surjective. Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the case of schemes (Limits, Lemma 32.8.15). $\square$

084X **Lemma 70.6.5.** Notation and assumptions as in Situation 70.6.1. If

(1) f is universally injective,
(2) f_0 is locally of finite type,

then f_i is universally injective for some $i \geq 0$.

Proof. Recall that a morphism $X \to Y$ is universally injective if and only if the diagonal $X \to X \times_Y X$ is surjective (Morphisms of Spaces, Definition 67.19.3 and Lemma 67.19.2). Observe that $X_0 \to X_0 \times_{Y_0} X_0$ is of locally of finite presentation (Morphisms of Spaces, Lemma 67.28.10). Hence the lemma follows from Lemma 70.6.4 by considering the morphism $X_0 \to X_0 \times_{Y_0} X_0$. □

084Y Lemma 70.6.6. Notation and assumptions as in Situation 70.6.1. If f is affine, then f_i is affine for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is affine we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is affine. By Lemma 70.5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. For this i the morphism f_i is affine (Morphisms of Spaces, Lemma 67.20.3). □

084Z Lemma 70.6.7. Notation and assumptions as in Situation 70.6.1. If

(1) f is finite,
(2) f_0 is locally of finite type,

then f_i is finite for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is finite we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a scheme finite over V. By Lemma 70.5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is finite by Limits, Lemma 32.8.3. For this i the morphism f_i is finite (Morphisms of Spaces, Lemma 67.45.3). □

0850 Lemma 70.6.8. Notation and assumptions as in Situation 70.6.1. If

(1) f is a closed immersion,
(2) f_0 is locally of finite type,

then f_i is a closed immersion for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is a closed immersion we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a closed subscheme of the affine scheme V. By Lemma 70.5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is a closed immersion by Limits, Lemma 32.8.5. For this i the morphism f_i is a closed immersion (Morphisms of Spaces, Lemma 67.45.3). □

0851 Lemma 70.6.9. Notation and assumptions as in Situation 70.6.1. If f is separated, then f_i is separated for some $i \geq 0$.

Proof. Apply Lemma 70.6.8 to the diagonal morphism $\Delta_{X_0/Y_0} : X_0 \to X_0 \times_{Y_0} X_0$. (Diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated. Some details omitted.) □

0852 Lemma 70.6.10. Notation and assumptions as in Situation 70.6.1. If

(1) f is a isomorphism,
(2) f_0 is locally of finite presentation,

then f_i is a isomorphism for some $i \geq 0$.

Proof. Being an isomorphism is equivalent to being étale, universally injective, and surjective, see Morphisms of Spaces, Lemma 67.51.2. Thus the lemma follows from Lemmas 70.6.2, 70.6.4, and 70.6.5. $\square$

07SM Lemma 70.6.11. Notation and assumptions as in Situation 70.6.1. If

(1) f is a monomorphism,
(2) f_0 is locally of finite type,

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism is a monomorphism if and only if the diagonal is an isomorphism. The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms of Spaces, Lemma 67.28.10. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated we conclude from Lemma 70.6.10 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. $\square$

08K0 Lemma 70.6.12. Notation and assumptions as in Situation 70.6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module and denote $\mathcal{F}_i$ the pullback to X_i and $\mathcal{F}$ the pullback to X. If

(1) $\mathcal{F}$ is flat over Y,
(2) $\mathcal{F}_0$ is of finite presentation, and
(3) f_0 is locally of finite presentation,

then $\mathcal{F}_i$ is flat over Y_i for some $i \geq 0$. In particular, if f_0 is locally of finite presentation and f is flat, then f_i is flat for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $\mathcal{F}_i$ is flat over Y_i if and only if $\mathcal{F}_i|_{U_i}$ is flat over V_i and similarly $\mathcal{F}$ is flat over Y if and only if $\mathcal{F}|_U$ is flat over V (Morphisms of Spaces, Definition 67.30.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 32.10.4. $\square$

08K1 Lemma 70.6.13. Assumptions and notation as in Situation 70.6.1. If

(1) f is proper, and
(2) f_0 is locally of finite type,

then there exists an i such that f_i is proper.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = Y_i \times_{Y_0} V_0$ and $V = Y \times_{Y_0} V_0$. It suffices to prove that the base change of f_i to V_i is proper, see Morphisms of Spaces, Lemma 67.40.2. Thus we may assume Y_0 is affine.

By Lemma 70.6.9 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact. Thus f_0 is separated and of finite type. By Cohomology of Spaces, Lemma 69.18.1 we can choose a diagram

$$\begin{tikzcd} X_0 \arrow[dr] & X_0' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_0} \arrow[dl] \\ & Y_0 & \end{tikzcd}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms of Spaces, Lemmas 67.40.4 and 67.40.3 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemma 67.40.6). By Morphisms of Spaces, Lemma 67.40.7 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 70.6.8 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. $\square$

0D4K Lemma 70.6.14. Assumptions and notation as in Situation 70.6.1. Let $d \geq 0$. If

(1) f has relative dimension $\leq d$ (Morphisms of Spaces, Definition 67.33.2), and
(2) f_0 is locally of finite type,

then there exists an i such that f_i has relative dimension $\leq d$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. In this situation $X_i \to Y_i$ has relative dimension $\leq d$ if and only if $U_i \to V_i$ has relative dimension $\leq d$ (as defined in Morphisms, Definition 29.29.1). To see the equivalence, use that the definition for morphisms of algebraic spaces involves Morphisms of Spaces, Definition 67.33.1 which uses étale localization. The same is true for $X \to Y$ and $U \to V$. Since f_0 is locally of finite type, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the more general Limits, Lemma 32.18.1. ☐

70.7. Descending relative objects

07SJ The following lemma is typical of the type of results in this section.

07SK Lemma 70.7.1. Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S. Assume

(1) the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,
(2) the spaces X_i are quasi-compact and quasi-separated.

Let $X = \lim_i X_i$. Then the category of algebraic spaces of finite presentation over X is the colimit over I of the categories of algebraic spaces of finite presentation over X_i.

Proof. Pick $0 \in I$. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 66.6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 70.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Let Y be an algebraic space over S and let $Y \to X$ be a morphism of finite presentation. Set $V = U \times_X Y$. This is an algebraic space of finite presentation over U. Choose an affine scheme W and a surjective étale morphism $W \to V$. Then $W \to Y$ is surjective étale as well. Set $R' = W \times_Y W$ so that $Y = W/R'$ (see Spaces, Section 65.9). Note that W is a scheme of finite presentation over U and that R' is a scheme of finite presentation over R (details omitted). By Limits, Lemma 32.10.1 we can find an index i and a morphism of schemes $W_i \to U_i$ of finite presentation whose base change to U gives $W \to U$. Similarly we can find, after possibly increasing i, a scheme R'_i of finite presentation over R_i whose base change to R is R'. The projection morphisms $s', t' : R' \to W$ are morphisms over the projection morphisms $s, t : R \to U$. Hence we can view s', resp. t' as a morphism between schemes of finite presentation over U (with structure morphism $R' \to U$ given by $R' \to R$ followed by s, resp. t). Hence we can apply Limits, Lemma 32.10.1 again to see that, after possibly increasing i, there exist morphisms $s'_i, t'_i : R'_i \to W_i$, whose base change to U is S', t'. By Limits, Lemmas 32.8.10 and 32.8.14 we may assume that s'_i, t'_i are étale and that $j'_i : R'_i \to W_i \times_{X_i} W_i$ is a monomorphism (here we view j'_i as a morphism of schemes of finite presentation over U_i via one of the projections – it doesn't matter which one). Setting $Y_i = W_i/R'_i$ (see Spaces, Theorem 65.10.5) we obtain an algebraic space of finite presentation over X_i whose base change to X is isomorphic to Y.

This shows that every algebraic space of finite presentation over X comes from an algebraic space of finite presentation over some X_i, i.e., it shows that the functor of the lemma is essentially surjective. To show that it is fully faithful, consider an index $0 \in I$ and two algebraic spaces Y_0, Z_0 of finite presentation over X_0. Set $Y_i = X_i \times_{X_0} Y_0$, $Y = X \times_{X_0} Y_0$, $Z_i = X_i \times_{X_0} Z_0$, and $Z = X \times_{X_0} Z_0$. Let $\alpha : Y \to Z$ be a morphism of algebraic spaces over X. Choose a surjective étale morphism $V_0 \to Y_0$ where V_0 is an affine scheme. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$ which are affine schemes endowed with surjective étale morphisms to Y_i and Y. The composition $V \to Y \to Z \to Z_0$ comes from a (essentially unique) morphism

$V_i \to Z_0$ for some $i \geq 0$ by Proposition 70.3.10 (applied to $Z_0 \to X_0$ which is of finite presentation by assumption). After increasing i the two compositions

$$V_i \times_{Y_i} V_i \to V_i \to Z_0$$

are equal as this is true in the limit. Hence we obtain a (essentially unique) morphism $Y_i \to Z_0$. Since this is a morphism over X_0 it induces a morphism into $Z_i = Z_0 \times_{X_0} X_i$ as desired. □

07V7 Lemma 70.7.2. With notation and assumptions as in Lemma 70.7.1. The category of $\mathcal{O}_X$-modules of finite presentation is the colimit over I of the categories $\mathcal{O}_{X_i}$-modules of finite presentation.

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 70.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Moreover, it is also true that $R \times_{s,U,t} R = \operatorname{colim} R_i \times_{s_i,U_i,t_i} R_i$. Thus we know that $QCoh(\mathcal{O}_U) = \operatorname{colim} QCoh(\mathcal{O}_{U_i})$, $QCoh(\mathcal{O}_R) = \operatorname{colim} QCoh(\mathcal{O}_{R_i})$, and $QCoh(\mathcal{O}_{R \times_{s,U,t} R}) = \operatorname{colim} QCoh(\mathcal{O}_{R_i \times_{s_i,U_i,t_i} R_i})$ by Limits, Lemma 32.10.2. We have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $QCoh(\mathcal{O}_{X_i}) = QCoh(U_i, R_i, s_i, t_i, c_i)$, see Properties of Spaces, Proposition 66.32.1. Thus the result follows formally. □

0D2X Lemma 70.7.3. With notation and assumptions as in Lemma 70.7.1. Then

(1) any finite locally free $\mathcal{O}_X$-module is the pullback of a finite locally free $\mathcal{O}_{X_i}$-module for some i,
(2) any invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_{X_i}$-module for some i.

Proof. Proof of (2). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Since invertible modules are of finite presentation we can find an i and modules $\mathcal{L}_i$ and $\mathcal{N}_i$ of finite presentation over X_i such that $f_i^*\mathcal{L}_i \cong \mathcal{L}$ and $f_i^*\mathcal{N}_i \cong \mathcal{L}^{\otimes -1}$, see Lemma 70.7.2. Since pullback commutes with tensor product we see that $f_i^*(\mathcal{L}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i)$ is isomorphic to $\mathcal{O}_X$. Since the tensor product of finitely presented modules is finitely presented, the same lemma implies that $f_{i'i}^*\mathcal{L}_i \otimes_{\mathcal{O}_{X_{i'}}} f_{i'i}^*\mathcal{N}_i$ is isomorphic to $\mathcal{O}_{X_{i'}}$ for some $i' \geq i$. It follows that $f_{i'i}^*\mathcal{L}_i$ is invertible (Modules on Sites, Lemma 18.32.2) and the proof is complete.

Proof of (1). Omitted. Hint: argue as in the proof of (2) using that a module (on a locally ringed site) is finite locally free if and only if it has a dual, see Modules on Sites, Section 18.29. Alternatively, argue as in the proof for schemes, see Limits, Lemma 32.10.3. □

70.8. Absolute Noetherian approximation

07SS The following result is [CLO12, Theorem 1.2.2]. A key ingredient in the proof is Decent Spaces, Lemma 68.8.6.

07SU Proposition 70.8.1. Let X be a quasi-compact and quasi-separated algebraic space over $\operatorname{Spec}(\mathbf{Z})$. There exist a directed set I and an inverse system of algebraic spaces $(X_i, f_{ii'})$ over I such that

Our proof follows closely the proof given in [CLO12, Theorem 1.2.2].

(1) the transition morphisms $f_{ii'}$ are affine
(2) each X_i is quasi-separated and of finite type over $\mathbf{Z}$, and
(3) $X = \lim X_i$.

Proof. We apply Decent Spaces, Lemma 68.8.6 to get open subspaces $U_p \subset X$, schemes V_p, and morphisms $f_p : V_p \to U_p$ with properties as stated. Note that $f_n : V_n \to U_n$ is an étale morphism of algebraic spaces whose restriction to the inverse image of $T_n = (V_n)_{red}$ is an isomorphism. Hence f_n is an isomorphism, for example by Morphisms of Spaces, Lemma 67.51.2. In particular U_n is a quasi-compact and separated scheme. Thus we can write $U_n = \lim U_{n,i}$ as a directed limit of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms, see Limits, Proposition 32.5.4. Thus, applying descending induction on p, we see that we have reduced to the problem posed in the following paragraph.

Here we have $U \subset X$, $U = \lim U_i$, $Z \subset X$, and $f : V \to X$ with the following properties

(1) X is a quasi-compact and quasi-separated algebraic space,
(2) V is a quasi-compact and separated scheme,
(3) $U \subset X$ is a quasi-compact open subspace,
(4) $(U_i, g_{ii'})$ is a directed inverse system of quasi-separated algebraic spaces of finite type over $\mathbf{Z}$ with affine transition morphisms whose limit is U,
(5) $Z \subset X$ is a closed subspace such that $|X| = |U| \amalg |Z|$,
(6) $f : V \to X$ is a surjective étale morphism such that $f^{-1}(Z) \to Z$ is an isomorphism.

Problem: Show that the conclusion of the proposition holds for X.

Note that $W = f^{-1}(U) \subset V$ is a quasi-compact open subscheme étale over U. Hence we may apply Lemmas 70.7.1 and 70.6.2 to find an index $0 \in I$ and an étale morphism $W_0 \to U_0$ of finite presentation whose base change to U produces W. Setting $W_i = W_0 \times_{U_0} U_i$ we see that $W = \lim_{i \geq 0} W_i$. After increasing 0 we may assume the W_i are schemes, see Lemma 70.5.11. Moreover, W_i is of finite type over $\mathbf{Z}$.

Apply Limits, Lemma 32.5.3 to $W = \lim_{i \geq 0} W_i$ and the inclusion $W \subset V$. Replace I by the directed set J found in that lemma. This allows us to write V as a directed limit $V = \lim V_i$ of finite type schemes over $\mathbf{Z}$ with affine transition maps such that each V_i contains W_i as an open subscheme (compatible with transition morphisms). For each i we can form the push out

$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "\Delta"'] & V_i \arrow[d] \\ W_i \times_{U_i} W_i \arrow[r] & R_i \end{tikzcd}$$

in the category of schemes. Namely, the left vertical and upper horizontal arrows are open immersions of schemes. In other words, we can construct R_i as the glueing of V_i and $W_i \times_{U_i} W_i$ along the common open W_i (see Schemes, Section 26.14). Note that the étale projection maps $W_i \times_{U_i} W_i \to W_i$ extend to étale morphisms $s_i, t_i : R_i \to V_i$. It is clear that the morphism $j_i = (t_i, s_i) : R_i \to V_i \times V_i$ is an étale equivalence relation on V_i. Note that $W_i \times_{U_i} W_i$ is quasi-compact (as

U_i is quasi-separated and W_i quasi-compact) and V_i is quasi-compact, hence R_i is quasi-compact. For $i \geq i'$ the diagram

07SV (70.8.1.1)
$$\begin{tikzcd} R_i \arrow[r] \arrow[d, "s_i"'] & R_{i'} \arrow[d, "s_{i'}"] \\ V_i \arrow[r] & V_{i'} \end{tikzcd}$$

is cartesian because

$$(W_{i'} \times_{U_{i'}} W_{i'}) \times_{U_{i'}} U_i = W_{i'} \times_{U_{i'}} U_i \times_{U_i} U_i \times_{U_{i'}} W_{i'} = W_i \times_{U_i} W_i.$$

Consider the algebraic space $X_i = V_i/R_i$ (see Spaces, Theorem 65.10.5). As V_i is of finite type over $\mathbf{Z}$ and R_i is quasi-compact we see that X_i is quasi-separated and of finite type over $\mathbf{Z}$ (see Properties of Spaces, Lemma 66.6.5 and Morphisms of Spaces, Lemmas 67.8.6 and 67.23.4). As the construction of R_i above is compatible with transition morphisms, we obtain morphisms of algebraic spaces $X_i \to X_{i'}$ for $i \geq i'$. The commutative diagrams

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & V_{i'} \arrow[d] \\ X_i \arrow[r] & X_{i'} \end{tikzcd}$$

are cartesian as (70.8.1.1) is cartesian, see Groupoids, Lemma 39.20.7. Since $V_i \to V_{i'}$ is affine, this implies that $X_i \to X_{i'}$ is affine, see Morphisms of Spaces, Lemma 67.20.3. Thus we can form the limit $X' = \lim X_i$ by Lemma 70.4.1. We claim that $X \cong X'$ which finishes the proof of the proposition.

Proof of the claim. Set $R = \lim R_i$. By construction the algebraic space X' comes equipped with a surjective étale morphism $V \to X'$ such that

$$V \times_{X'} V \cong R$$

(use Lemma 70.4.1). By construction $\lim W_i \times_{U_i} W_i = W \times_U W$ and $V = \lim V_i$ so that R is the union of $W \times_U W$ and V glued along W. Property (6) implies the projections $V \times_X V \to V$ are isomorphisms over $f^{-1}(Z) \subset V$. Hence the scheme $V \times_X V$ is the union of the opens $\Delta_{V/X}(V)$ and $W \times_U W$ which intersect along $\Delta_{W/X}(W)$. We conclude that there exists a unique isomorphism $R \cong V \times_X V$ compatible with the projections to V. Since $V \to X$ and $V \to X'$ are surjective étale we see that

$$X = V/V \times_X V = V/R = V/V \times_{X'} V = X'$$

by Spaces, Lemma 65.9.1 and we win. □

70.9. Applications

07V8 The following lemma can also be deduced directly from Decent Spaces, Lemma 68.8.6 without passing through absolute Noetherian approximation.

07V9 **Lemma 70.9.1.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.

Proof. We may view X as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2 and Properties of Spaces, Definition 66.3.1. Thus we may apply Proposition 70.8.1 and write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$. Thus X_i is a Noetherian algebraic space, see Morphisms of Spaces, Lemma 67.28.6. The morphism $X \to X_i$ is affine, see Lemma 70.4.1. Conclusion by Cohomology of Spaces, Lemma 69.15.2. $\square$

The rest of this section consists of straightforward applications of Lemma 70.9.1.

0829 **Lemma 70.9.2.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 70.9.1. Then the images $\mathcal{G}_i = \mathrm{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. $\square$

086Y **Lemma 70.9.3.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ where each $\mathcal{F}_i$ is an $\mathcal{O}_X$-module of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.

Proof. Write $\mathcal{F} = \mathrm{colim}\, \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 70.9.1). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose an étale surjection $U \to X$ where U is an affine scheme. Choose finitely many sections $s_k \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since U is affine we see that s_k is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \mathrm{colim}\, \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 70.9.2). Then $\mathcal{F} = \mathrm{colim}\, \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. $\square$

Let X be an algebraic space. In the following lemma we use the notion of a finitely presented quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

082A **Lemma 70.9.4.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras.

Proof. First we write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 70.9.1. For each i let $\mathcal{B}_i = \mathrm{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \mathrm{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \mathrm{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 70.9.2. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \mathrm{colim}_{i,j} \mathcal{A}_{i,j}$. $\square$

Let X be an algebraic space. In the following lemma we use the notion of a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

082B Lemma 70.9.5. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.

Proof. Omitted. Hint: Compare with the proof of Lemma 70.9.2. $\square$

Let X be an algebraic space. In the following lemma we use the notion of a finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

086Z Lemma 70.9.6. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras with surjective transition maps.

Proof. By Lemma 70.9.3 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \mathrm{colim}\, \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 70.9.2). Set

$$\mathcal{A}_i = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjective, and $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an étale surjective map $U \to X$ where U is an affine scheme. Take generators $f_1, \ldots, f_m \in \Gamma(U, \mathcal{F})$. As $\mathcal{A}(U)$ is a finite $\mathcal{O}_X(U)$-algebra we see that for each j there exists a monic polynomial $P_j \in \mathcal{O}(U)[T]$ such that $P_j(f_j)$ is zero in $\mathcal{A}(U)$. Since $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ by construction, we have $P_j(f_j) = 0$ in $\mathcal{A}_i(U)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. $\square$

082C Lemma 70.9.7. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then

(1) $\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and
(2) $\mathcal{A}$ is a directed colimit of finite and finitely presented $\mathcal{O}_X$-algebras.

Proof. By Lemma 70.9.5 we have $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-sub algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (use Algebra, Lemma 10.36.5 on affine schemes étale over X). This proves (1).

To prove (2), write $\mathcal{A} = \mathrm{colim}\, \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 70.9.1. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{tikzcd} \mathcal{J}_i \arrow[r] & \mathcal{J}_{i'} \\ \mathcal{E}_{ik} \arrow[r] \arrow[u] & \mathcal{E}_{i'k'} \arrow[u] \end{tikzcd}$$

commute. This follows from Cohomology of Spaces, Lemma 69.5.3. This induces a map

$$\mathcal{A}_{ik} = \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 70.9.6). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 70.9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

0853 Lemma 70.9.8. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.

Proof. Denote $j : U \to X$ the inclusion morphism. As X is quasi-separated and U quasi-compact, the morphism j is quasi-compact. Hence $j_*\mathcal{G} \subset j_*\mathcal{F}|_U$ are quasi-coherent modules on X (Morphisms of Spaces, Lemma 67.11.2). Let $\mathcal{H} = \text{Ker}(j_*\mathcal{G} \oplus \mathcal{F} \to j_*\mathcal{F}|_U)$. Then $\mathcal{H}|_U = \mathcal{G}$. By Lemma 70.9.2 we can find a finite type quasi-coherent submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{H}|_U = \mathcal{G}$. Set $\mathcal{G}' = \text{Im}(\mathcal{H}' \to \mathcal{F})$ to conclude. □

70.10. Relative approximation

09NR We discuss variants of Proposition 70.8.1 over a base.

0GS3 Lemma 70.10.1. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over $\mathbf{Z}$. Then there exists a direct set I and an inverse system $(f_i : X_i \to Y_i)$ of morphisms algebraic spaces over I, such that the transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$ are affine, such that X_i and Y_i are quasi-separated and of finite type over $\mathbf{Z}$, and such that $(X \to Y) = \lim(X_i \to Y_i)$.

Proof. Write $X = \lim_{a \in A} X_a$ and $Y = \lim_{b \in B} Y_b$ as in Proposition 70.8.1, i.e., with X_a and Y_b quasi-separated and of finite type over $\mathbf{Z}$ and with affine transition morphisms.

Fix $b \in B$. By Lemma 70.4.5 applied to Y_b and $X = \lim X_a$ over $\mathbf{Z}$ we find there exists an $a \in A$ and a morphism $f_{a,b} : X_a \to Y_b$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X_a \arrow[r] & Y_b \end{tikzcd}$$

commute. Let I be the set of triples $(a, b, f_{a,b})$ we obtain in this manner.

Let $(a, b, f_{a,b})$ and $(a', b', f_{a',b'})$ be in I. Let $b'' \leq \min(b, b')$. By Lemma 70.4.5 again, there exists an $a'' \geq \max(a, a')$ such that the compositions $X_{a''} \to X_a \to Y_b \to Y_{b''}$ and $X_{a''} \to X_{a'} \to Y_{b'} \to Y_{b''}$ are equal. We endow I with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b \geq b', \text{ and } g_{b,b'} \circ f_{a,b} = f_{a',b'} \circ h_{a,a'}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b,b'} : Y_b \to Y_{b'}$ are the transition morphisms. The remarks above show that I is directed and that the maps $I \to A$, $(a, b, f_{a,b}) \mapsto a$ and $I \to B$, $(a, b, f_{a,b})$ are cofinal. If for $i = (a, b, f_{a,b})$ we set $X_i = X_a$, $Y_i = Y_b$, and $f_i = f_{a,b}$, then we get an inverse system of morphisms over I and we have

$$\lim_{i \in I} X_i = \lim_{a \in A} X_a = X \quad \text{and} \quad \lim_{i \in I} S_i = \lim_{b \in B} Y_b = Y$$

by Categories, Lemma 4.17.4 (recall that limits over I are really limits over the opposite category associated to I and hence cofinal turns into initial). This finishes the proof. $\square$

09NS Lemma 70.10.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that

(1) X is quasi-compact and quasi-separated, and
(2) Y is quasi-separated.

Then $X = \lim X_i$ is a limit of a directed inverse system of algebraic spaces X_i of finite presentation over Y with affine transition morphisms over Y.

Proof. Since $|f|(|X|)$ is quasi-compact we may replace Y by a quasi-compact open subspace whose set of points contains $|f|(|X|)$. Hence we may assume Y is quasi-compact as well. By Lemma 70.10.1 we can write $(X \to Y) = \lim(X_i \to Y_i)$ for some directed inverse system of morphisms of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. Since limits commute with limits (Categories, Lemma 4.14.10) we have $X = \lim X_i \times_{Y_i} Y$. For $i \geq i'$ the transition morphism $X_i \times_{Y_i} Y \to X_{i'} \times_{Y_{i'}} Y$ is affine as the composition

$$X_i \times_{Y_i} Y \to X_i \times_{Y_{i'}} Y \to X_{i'} \times_{Y_{i'}} Y$$

where the first morphism is a closed immersion (by Morphisms of Spaces, Lemma 67.4.5) and the second is a base change of an affine morphism (Morphisms of Spaces, Lemma 67.20.5) and the composition of affine morphisms is affine (Morphisms of Spaces, Lemma 67.20.4). The morphisms f_i are of finite presentation (Morphisms of Spaces, Lemmas 67.28.7 and 67.28.9) and hence the base changes $X_i \times_{f_i, Y_i} Y \to Y$ are of finite presentation (Morphisms of Spaces, Lemma 67.28.3). $\square$

70.11. Finite type closed in finite presentation

07SP This section is the analogue of Limits, Section 32.9.

0870 **Lemma 70.11.1.** Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. If Y quasi-compact and quasi-separated, then X is a directed limit $X = \lim X_i$ with each X_i affine and of finite presentation over Y.

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 70.9.4 we can write $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ as a directed colimit of finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is affine and of finite presentation and $X = \lim X_i$. $\square$

09YA **Lemma 70.11.2.** Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where X_i are finite and of finite presentation over Y.

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 70.9.7 we can write $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is finite and of finite presentation and $X = \lim X_i$. $\square$

07VR **Lemma 70.11.3.** Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over Y.

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 70.9.6 we can write $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ with surjective transition maps. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is finite and of finite presentation, the transition maps are closed immersions, and $X = \lim X_i$. $\square$

0A0U **Lemma 70.11.4.** Let S be a scheme. Let $f : X \to Y$ be a closed immersion of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the morphisms $X_i \to Y$ are closed immersions of finite presentation.

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subspace of Y. By Lemma 70.9.2 we can write $\mathcal{I} = \mathrm{colim}\, \mathcal{I}_i$ as the filtered colimit of its finite type quasi-coherent submodules. Let X_i be the closed subspace of X cut out by $\mathcal{I}_i$. Then $X_i \to Y$ is a closed immersion of finite presentation, and $X = \lim X_i$. Some details omitted. $\square$

0871 **Lemma 70.11.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is locally of finite type and quasi-affine, and
(2) Y is quasi-compact and quasi-separated.

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Morphisms of Spaces, Lemma 67.21.6 we can find a factorization $X \to Z \to Y$ where $X \to Z$ is a quasi-compact open immersion and $Z \to Y$ is affine. Write $Z = \lim Z_i$ with Z_i affine and of finite presentation over Y (Lemma 70.11.1). For some $0 \in I$ we can find a quasi-compact open $U_0 \subset Z_0$ such that X is isomorphic to the inverse image of U_0 in Z (Lemma 70.5.7). Let U_i be the inverse image of U_0 in Z_i, so $U = \lim U_i$. By Lemma 70.5.12 we see that $X \to U_i$ is a closed immersion for some i large enough. Setting $X' = U_i$ finishes the proof. $\square$

0872 Lemma 70.11.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume:

(1) f is of locally of finite type.
(2) X is quasi-compact and quasi-separated, and
(3) Y is quasi-compact and quasi-separated.

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ of algebraic spaces over Y.

Proof. By Proposition 70.8.1 we can write $X = \lim_i X_i$ with X_i quasi-separated of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rd] & X_{i,Y} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & Y \arrow[r] & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms of Spaces, Lemma 67.28.7. Hence the base change $X_{i,Y} \to Y$ is of finite presentation by Morphisms of Spaces, Lemma 67.28.3. Observe that $\lim X_{i,Y} = X \times Y$ and that $X \to X \times Y$ is a monomorphism. By Lemma 70.5.12 we see that $X \to X_{i,Y}$ is a monomorphism for i large enough. Fix such an i. Note that $X \to X_{i,Y}$ is locally of finite type (Morphisms of Spaces, Lemma 67.23.6) and a monomorphism, hence separated and locally quasi-finite (Morphisms of Spaces, Lemma 67.27.10). Hence $X \to X_{i,Y}$ is representable. Hence $X \to X_{i,Y}$ is quasi-affine because we can use the principle Spaces, Lemma 65.5.8 and the result for morphisms of schemes More on Morphisms, Lemma 37.43.2. Thus Lemma 70.11.5 gives a factorization $X \to X' \to X_{i,Y}$ with $X \to X'$ a closed immersion and $X' \to X_{i,Y}$ of finite presentation. Finally, $X' \to Y$ is of finite presentation as a composition of morphisms of finite presentation (Morphisms of Spaces, Lemma 67.28.2). $\square$

0873 Proposition 70.11.7. Let S be a scheme. $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is of finite type and separated, and
(2) Y is quasi-compact and quasi-separated.

Then there exists a separated morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Lemma 70.11.6 there is a closed immersion $X \to Z$ with Z/Y of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Lemma 70.9.2 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \mathrm{colim}_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Z$ be the closed subspace defined by $\mathcal{I}_a$. These form an inverse system indexed by A.

The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subspace of Z and Z is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_a X_a$ as follows directly from the fact that $\mathcal{I} = \mathrm{colim}_{a \in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Z$ is of finite presentation, see Morphisms, Lemma 29.21.7. Hence the morphisms $X_a \to Y$ are of finite presentation. Thus it suffices to show that $X_a \to Y$ is separated for some $a \in A$. This follows from Lemma 70.5.13 as we have assumed that $X \to Y$ is separated. □

70.12. Approximating proper morphisms

0A0V

0A0W Lemma 70.12.1. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of algebraic spaces X_i proper and of finite presentation over Y and with transition morphisms and morphisms $X \to X_i$ closed immersions.

Proof. By Proposition 70.11.7 we can find a closed immersion $X \to X'$ with X' separated and of finite presentation over Y. By Lemma 70.11.4 we can write $X = \lim X_i$ with $X_i \to X'$ a closed immersion of finite presentation. We claim that for all i large enough the morphism $X_i \to Y$ is proper which finishes the proof.

To prove this we may assume that Y is an affine scheme, see Morphisms of Spaces, Lemma 67.40.2. Next, we use the weak version of Chow's lemma, see Cohomology of Spaces, Lemma 69.18.1, to find a diagram

$$\begin{tikzcd} X' \arrow[dr] & X'' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X'' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X'' \to X'$ is proper and surjective. Denote $X'_i \subset X''$, resp. $\pi^{-1}(X)$ the scheme theoretic inverse image of $X_i \subset X'$, resp. $X \subset X'$. Then $\lim X'_i = \pi^{-1}(X)$. Since $\pi^{-1}(X) \to Y$ is proper (Morphisms of Spaces, Lemmas 67.40.4), we see that $\pi^{-1}(X) \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemmas 67.40.6 and 67.12.3). Hence for i large enough we find that $X'_i \to \mathbf{P}^n_Y$ is a closed immersion by Lemma 70.5.16. Thus X'_i is proper over Y. For such i the morphism $X_i \to Y$ is proper by Morphisms of Spaces, Lemma 67.40.7. □

0A0X Lemma 70.12.2. Let $f : X \to Y$ be a proper morphism of algebraic spaces over $\mathbf{Z}$ with Y quasi-compact and quasi-separated. Then there exists a directed set I, an inverse system $(f_i : X_i \to Y_i)$ of morphisms of algebraic spaces over I, such that the transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$ are affine, such that f_i is proper and of finite presentation, such that Y_i is of finite presentation over $\mathbf{Z}$, and such that $(X \to Y) = \lim(X_i \to Y_i)$.

Proof. By Lemma 70.12.1 we can write $X = \lim_{k \in K} X_k$ with $X_k \to Y$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 70.8.1) we can write $Y = \lim_{j \in J} Y_j$ with Y_j of finite presentation over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to Y_j$ of finite presentation with $X_k \cong Y \times_{Y_j} X_{k,j}$ as algebraic spaces over Y, see Lemma 70.7.1. After increasing

j we may assume $X_{k,j} \to Y_j$ is proper, see Lemma 70.6.13. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to Y_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $Y_{j'} \to Y_j$ whose base change to Y gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 70.7.1). These morphisms form the transition morphisms of the system. Some details omitted. $\square$

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms of Spaces, Definition 67.15.4.

08K2 Lemma 70.12.3. Assumptions and notation as in Situation 70.6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume

(1) f_0 is locally of finite type,
(2) $\mathcal{F}_0$ is of finite type,
(3) the scheme theoretic support of $\mathcal{F}$ is proper over Y.

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms of Spaces, Lemma 67.15.2 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 29.41.9. By Lemma 70.6.13 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms of Spaces, Lemmas 67.40.5 and 67.40.4. $\square$

70.13. Embedding into affine space

088K Some technical lemmas to be used in the proof of Chow's lemma later.

088L Lemma 70.13.1. Let S be a scheme. Let $f : U \to X$ be a morphism of algebraic spaces over S. Assume U is an affine scheme, f is locally of finite type, and X quasi-separated and locally separated. Then there exists an immersion $U \to \mathbf{A}^n_X$ over X.

Proof. Say $U = \operatorname{Spec}(A)$. Write $A = \operatorname{colim} A_i$ as a filtered colimit of finite type $\mathbf{Z}$-subalgebras. For each i the morphism $U \to U_i = \operatorname{Spec}(A_i)$ induces a morphism

$$U \longrightarrow X \times U_i$$

over X. In the limit the morphism $U \to X \times U$ is an immersion as X is locally separated, see Morphisms of Spaces, Lemma 67.4.6. By Lemma 70.5.12 we see that $U \to X \times U_i$ is an immersion for some i. Since U_i is isomorphic to a closed subscheme of $\mathbf{A}^n_{\mathbf{Z}}$ the lemma follows. $\square$

088M Remark 70.13.2. We have seen in Examples, Section 110.28 that Lemma 70.13.1 does not hold if we drop the assumption that X be locally separated. This raises the question: Does Lemma 70.13.1 hold if we drop the assumption that X be quasi-separated? If you know the answer, please email stacks.project@gmail.com.

088N Lemma 70.13.3. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume X Noetherian and f of finite presentation. Then there exists a dense open $V \subset Y$ and an immersion $V \to \mathbf{A}^n_X$.

Proof. The assumptions imply that Y is Noetherian (Morphisms of Spaces, Lemma 67.28.6). Then Y is quasi-separated, hence has a dense open subscheme (Properties of Spaces, Proposition 66.13.3). Thus we may assume that Y is a Noetherian scheme. By removing intersections of irreducible components of Y (use Topology, Lemma 5.9.2 and Properties, Lemma 28.5.5) we may assume that Y is a disjoint union of irreducible Noetherian schemes. Since there is an immersion

$$\mathbf{A}^n_X \amalg \mathbf{A}^m_X \longrightarrow \mathbf{A}^{\max(n,m)+1}_X$$

(details omitted) we see that it suffices to prove the result in case Y is irreducible.

Assume Y is an irreducible scheme. Let $T \subset |X|$ be the closure of the image of $f : Y \to X$. Note that since $|Y|$ and $|X|$ are sober topological spaces (Properties of Spaces, Lemma 66.15.1) T is irreducible with a unique generic point ξ which is the image of the generic point η of Y. Let $\mathcal{I} \subset X$ be a quasi-coherent sheaf of ideals cutting out the reduced induced space structure on T (Properties of Spaces, Definition 66.12.5). Since $\mathcal{O}_{Y,\eta}$ is an Artinian local ring we see that for some $n > 0$ we have $f^{-1}\mathcal{I}^n\mathcal{O}_{Y,\eta} = 0$. As $f^{-1}\mathcal{I}\mathcal{O}_Y$ is a finite type quasi-coherent ideal we conclude that $f^{-1}\mathcal{I}^n\mathcal{O}_V = 0$ for some nonempty open $V \subset Y$. Let $Z \subset X$ be the closed subspace cut out by $\mathcal{I}^n$. By construction $V \to Y \to X$ factors through Z. Because $\mathbf{A}^n_Z \to \mathbf{A}^n_X$ is an immersion, we may replace X by Z and Y by V. Hence we reach the situation where Y and X are irreducible and $Y \to X$ maps the generic point of Y onto the generic point of X.

Assume Y and X are irreducible, Y is a scheme, and $Y \to X$ maps the generic point of Y onto the generic point of X. By Properties of Spaces, Proposition 66.13.3 X has a dense open subscheme $U \subset X$. Choose a nonempty affine open $V \subset Y$ whose image in X is contained in U. By Morphisms, Lemma 29.39.2 we may factor $V \to U$ as $V \to \mathbf{A}^n_U \to U$. Composing with $\mathbf{A}^n_U \to \mathbf{A}^n_X$ we obtain the desired immersion. □

70.14. Sections with support in a closed subset

0854 This section is the analogue of Properties, Section 28.24.

0855 Lemma 70.14.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U \subset X$ be an open subspace. The following are equivalent:

(1) $U \to X$ is quasi-compact,
(2) U is quasi-compact, and
(3) there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $|X| \setminus |U| = |V(\mathcal{I})|$.

Proof. Let W be an affine scheme and let $\varphi : W \to X$ be a surjective étale morphism, see Properties of Spaces, Lemma 66.6.3. If (1) holds, then $\varphi^{-1}(U) \to W$ is quasi-compact, hence $\varphi^{-1}(U)$ is quasi-compact, hence U is quasi-compact (as $|\varphi^{-1}(U)| \to |U|$ is surjective). If (2) holds, then $\varphi^{-1}(U)$ is quasi-compact because φ is quasi-compact since X is quasi-separated (Morphisms of Spaces, Lemma 67.8.10). Hence $\varphi^{-1}(U) \to W$ is a quasi-compact morphism of schemes by Properties, Lemma 28.24.1. It follows that $U \to X$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.8. Thus (1) and (2) are equivalent.

Assume (1) and (2). By Properties of Spaces, Lemma 66.12.3 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subspace structure on $|X| \setminus |U|$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. As U is quasi-compact it follows from Lemma 70.9.2 that there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $|X| \setminus |U| = |V(\mathcal{I})|$ and we obtain (3). Conversely, if $\mathcal{I}$ is as in (3), then $\varphi^{-1}(U) \subset W$ is a quasi-compact open by the lemma for schemes (Properties, Lemma 28.24.1) applied to $\varphi^{-1}\mathcal{I}$ on W. Thus (2) holds. □

0856 Lemma 70.14.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object U of $X_{\acute{e}tale}$ the module

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

1. $\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,
2. for affine U in $X_{\acute{e}tale}$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and
3. $\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 28.24.2. □

0857 Definition 70.14.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 70.14.2 above is called the subsheaf of sections annihilated by $\mathcal{I}$.

0858 Lemma 70.14.4. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.

Proof. Omitted. Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 67.11.2) so that Lemma 70.14.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$. □

Next we come to the sheaf of sections supported in a closed subset. Again this isn't always a quasi-coherent sheaf, but if the complement of the closed is "retrocompact" in the given algebraic space, then it is.

0859 Lemma 70.14.5. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset and let $U \subset X$ be the open subspace such that $T \amalg |U| = |X|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object $\varphi : W \to X$ of $X_{\acute{e}tale}$ the module

$$\mathcal{F}'(W) = \{s \in \mathcal{F}(W) \mid \text{the support of } s \text{ is contained in } |\varphi|^{-1}(T)\}$$

If $U \to X$ is quasi-compact, then

1. for W affine there exist a finitely generated ideal $I \subset \mathcal{O}_X(W)$ such that $|\varphi|^{-1}(T) = V(I)$,
2. for W and I as in (1) we have $\mathcal{F}'(W) = \{x \in \mathcal{F}(W) \mid I^n x = 0 \text{ for some } n\}$,
3. $\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 28.24.5. $\square$

085A Definition 70.14.6. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset whose complement corresponds to an open subspace $U \subset X$ with quasi-compact inclusion morphism $U \to X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 70.14.5 above is called the subsheaf of sections supported on T.

085B Lemma 70.14.7. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $T \subset |Y|$ be a closed subset. Assume $|Y| \setminus T$ corresponds to an open subspace $V \subset Y$ such that $V \to Y$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported on $|f|^{-1}T$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported on T.

Proof. Omitted. Hints: $|X| \setminus |f|^{-1}T$ is the support of the open subspace $U = f^{-1}V \subset X$. Since $V \to Y$ is quasi-compact, so is $U \to X$ (by base change). The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 70.14.5 applies to T and $f_*\mathcal{F}$ as well as to $|f|^{-1}T$ and $\mathcal{F}$. The equality of the given quasi-coherent modules is immediate from the definitions. $\square$

70.15. Characterizing affine spaces

07VQ This section is the analogue of Limits, Section 32.11.

07VS Lemma 70.15.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and finite, and assume that X is affine. Then Y is affine.

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 65.16.2). Note that a finite morphism is affine and universally closed, see Morphisms of Spaces, Lemma 67.45.7. By Morphisms of Spaces, Lemma 67.9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

By Lemma 70.11.3 we can write $X = \lim X_a$ with each $X_a \to Y$ finite and of finite presentation. By Lemma 70.5.10 we see that X_a is affine for a large enough. Hence we may and do assume that $f : X \to Y$ is finite, surjective, and of finite presentation.

By Proposition 70.8.1 we may write $Y = \lim Y_i$ as a directed limit of algebraic spaces of finite presentation over $\mathbf{Z}$. By Lemma 70.7.1 we can find $0 \in I$ and a morphism $X_0 \to Y_0$ of finite presentation such that $X_i = X_0 \times_{Y_0} Y_i$ for $i \geq 0$ and such that $X = \lim_i X_i$. By Lemma 70.6.7 we see that $X_i \to Y_i$ is finite for i large enough. By Lemma 70.6.4 we see that $X_i \to Y_i$ is surjective for i large enough. By Lemma 70.5.10 we see that X_i is affine for i large enough. Hence for i large enough we can apply Cohomology of Spaces, Lemma 69.17.3 to conclude that Y_i is affine. This implies that Y is affine and we conclude. $\square$

07VT Proposition 70.15.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that X is affine and f is surjective and universally closed[2]. Then Y is affine.

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 65.16.2). By Morphisms of Spaces, Lemma 67.9.8 we see that Y is a separated algebraic space. Then by Morphisms of Spaces, Lemma 67.20.11 we find that f is affine. Whereupon by Morphisms of Spaces, Lemma 67.45.7 we see that f is integral.

By the preceding paragraph, we may assume $f : X \to Y$ is surjective and integral, X is affine, and Y is separated. Since f is surjective and X is quasi-compact we also deduce that Y is quasi-compact.

Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, see Morphisms of Spaces, Lemma 67.11.2. By Lemma 70.9.1 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a filtered colimit of finite type $\mathcal{O}_Y$-modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the $\mathcal{O}_Y$-subalgebra generated by $\mathcal{F}_i$. Since the map of algebras $\mathcal{O}_Y \to \mathcal{A}$ is integral, we see that each $\mathcal{A}_i$ is a finite quasi-coherent $\mathcal{O}_Y$-algebra. Hence

$$X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \longrightarrow Y$$

is a finite morphism of algebraic spaces. Here $\underline{\mathrm{Spec}}$ is the construction of Morphisms of Spaces, Lemma 67.20.7. It is clear that $X = \lim_i X_i$. Hence by Lemma 70.5.10 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to Y$ factors through each X_i we see that $X_i \to Y$ is surjective. Hence we conclude that Y is affine by Lemma 70.15.1. $\square$

The following corollary of the result above can be found in [CLO12].

07VU Lemma 70.15.3. Let S be a scheme. Let X be an algebraic space over S. If X_{red} is a scheme, then X is a scheme. [CLO12, 3.1.12]

Proof. Let $U' \subset X_{red}$ be an open affine subscheme. Let $U \subset X$ be the open subspace corresponding to the open $|U'| \subset |X_{red}| = |X|$. Then $U' \to U$ is surjective and integral. Hence U is affine by Proposition 70.15.2. Thus every point is contained in an open subscheme of X, i.e., X is a scheme. $\square$

07VV Lemma 70.15.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral and induces a bijection $|X| \to |Y|$. Then X is a scheme if and only if Y is a scheme.

Proof. An integral morphism is representable by definition, hence if Y is a scheme, so is X. Conversely, assume that X is a scheme. Let $U \subset X$ be an affine open. An integral morphism is closed and $|f|$ is bijective, hence $|f|(|U|) \subset |Y|$ is open as the complement of $|f|(|X| \setminus |U|)$. Let $V \subset Y$ be the open subspace with $|V| = |f|(|U|)$, see Properties of Spaces, Lemma 66.4.8. Then $U \to V$ is integral and surjective, hence V is an affine scheme by Proposition 70.15.2. This concludes the proof. $\square$

08B2 Lemma 70.15.5. Let S be a scheme. Let $f : X \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Assume

(1) $B' \to B$ is a closed immersion,
(2) $|B'| \to |B|$ is bijective,

[2] An integral morphism is universally closed, see Morphisms of Spaces, Lemma 67.45.7.

(3) $X \times_B B' \to B'$ is a closed immersion, and
(4) $X \to B$ is of finite type or $B' \to B$ is of finite presentation.

Then $f : X \to B$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $B_{red} = B'_{red}$. Set $X' = X \times_B B'$. Then $X' \to X$ is closed immersion and $X'_{red} = X_{red}$. Let $U \to B$ be an étale morphism with U affine. Then $X' \times_B U \to X \times_B U$ is a closed immersion of algebraic spaces inducing an isomorphism on underlying reduced spaces. Since $X' \times_B U$ is a scheme (as $B' \to B$ and $X' \to B'$ are representable) so is $X \times_B U$ by Lemma 70.15.3. Hence $X \to B$ is representable too. Thus we reduce to the case of schemes, see Morphisms, Lemma 29.45.7. □

70.16. Finite cover by a scheme

0ACX As an application of the limit results of this chapter, we prove that given any quasi-compact and quasi-separated algebraic space X, there is a scheme Y and a surjective, finite morphism $Y \to X$. We will rely on the already proven result that we can find a finite integral cover by a scheme, which was proved in Decent Spaces, Section 68.9.

09YC Proposition 70.16.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S.

(1) There exists a surjective finite morphism $Y \to X$ of finite presentation where Y is a scheme,
(2) given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.

Proof. Part (1) is the special case of (2) with $U = X$. Let $Y \to X$ be as in Decent Spaces, Lemma 68.9.2. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Lemma 70.11.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 70.5.11. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Lemma 70.5.7. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Proposition 70.3.10. This finishes the proof. □

0GUM Lemma 70.16.2. Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Let $V \subset Y$ be a quasi-compact open subspace such that $f^{-1}(V) \to V$ is finite and of finite presentation. Then X can be written as a directed limit $X = \lim X_i$ where $f_i : X_i \to Y$ are finite and of finite presentation such that $f^{-1}(V) \to f_i^{-1}(V)$ is an isomorphism for all i.

Proof. This lemma is a slight refinement of Proposition 70.16.1. Consider the integral quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A} = f_*\mathcal{O}_X$. In the next paragraph, we will write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ such that $\mathcal{A}_i|_V = \mathcal{A}|_V$. Having done this we set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is finite and of finite presentation, $X = \lim X_i$, and $f_i^{-1}(V) = f^{-1}(V)$.

The proof of the assertion on algebras is similar to the proof of part (2) of Lemma 70.9.7. First, write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_Y$-modules using Lemma 70.9.1. Since $\mathcal{A}|_V$ is a finite type $\mathcal{O}_V$-module we may and do assume that $\mathcal{F}_i|_V \to \mathcal{A}|_V$ is surjective for all i. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$. We have $\mathcal{A} = \operatorname{colim} \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (as finite type quasi-coherent subalgebras of the integral quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A}$ over $\mathcal{O}_X$). The restriction of $\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ to V is $\mathcal{A}|_V$ by the surjectivity above. Hence $\mathcal{J}_i|_V$ is finitely generated as an ideal sheaf of $\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)|_V$ due to the fact that $\mathcal{A}|_V$ is finitely presented as an $\mathcal{O}_Y$-algebra. Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. We may and do assume that $\mathcal{E}_{ik}|_V$ generates $\mathcal{J}_i|_V$ as a sheaf of ideal of $\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)|_V$ by the statement on finite generation above. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{tikzcd}
\mathcal{J}_i \arrow[r] & \mathcal{J}_{i'} \\
\mathcal{E}_{ik} \arrow[r] \arrow[u] & \mathcal{E}_{i'k'} \arrow[u]
\end{tikzcd}$$

commute. This follows from Cohomology of Spaces, Lemma 69.5.3. This induces a map

$$\mathcal{A}_{ik} = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 70.9.6). Moreover we have $\mathcal{A}_{ik}|_V = \mathcal{A}|_V$ by construction. Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 70.9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. $\square$

0GUN Lemma 70.16.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S such that $|X|$ has finitely many irreducible components.

(1) There exists a surjective finite morphism $f : Y \to X$ of finite presentation where Y is a scheme such that f is finite étale over a quasi-compact dense open $U \subset X$,
(2) given a surjective étale morphism $V \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $W \subset Y$ such that $W \to X$ factors through V.

Proof. Part (1) is the special case of (2) with $V = X$.

Proof of (2). Let $\pi : Y \to X$ be as in Decent Spaces, Lemma 68.9.3 and let $U \subset X$ be a quasi-compact dense open such that $\pi^{-1}(U) \to U$ is finite étale. Choose a finite affine open covering $Y = \bigcup W_j$ such that $W_j \to X$ factors through V. We can write $Y = \lim Y_i$ with $\pi_i : Y_i \to X$ finite and of finite presentation such that $\pi^{-1}(U) \to \pi_i^{-1}(U)$ is an isomorphism, see Lemma 70.16.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 70.5.11. For large enough i we can find affine opens $W_{i,j} \subset Y_i$ whose inverse image in Y recovers W_j, see Lemma 70.5.7.

For even larger i the morphisms $W_j \to V$ over X come from morphisms $W_{i,j} \to U$ over X, see Proposition 70.3.10. This finishes the proof. □

0GUP Lemma 70.16.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. There exists a $t \geq 0$ and closed subspaces

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is of finite presentation, $Z_0 \subset X$ is a thickening, and for each $i = 0, \ldots t-1$ there exists a scheme Y_i, a surjective, finite, and finitely presented morphism $Y_i \to Z_i$ which is finite étale over $Z_i \setminus Z_{i+1}$.

Proof. We may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2 and Properties of Spaces, Definition 66.3.1. Thus we may apply Proposition 70.8.1. It follows that we can find an affine morphism $X \to X_0$ with X_0 of finite presentation over $\mathbf{Z}$. If we can prove the lemma for X_0, then we can pull back the stratification and the morphisms to X and get the result for X; some details omitted. This reduces us to the case discussed in the next paragraph.

Assume X is of finite presentation over $\mathbf{Z}$. Then X is Noetherian and $|X|$ is a Noetherian topological space (with finitely many irreducible components) of finite dimension. Hence we may use induction on $\dim(|X|)$. Any finite morphism towards X is of finite presentation, so we can ignore that requirement in the rest of the proof. By Lemma 70.16.3 there exists a surjective finite morphism $Y \to X$ which is finite étale over a dense open $U \subset X$. Set $Z_0 = X$ and let $Z_1 \subset X$ be the reduced closed subspace with $|Z_1| = |X| \setminus |U|$. By induction we find an integer $t \geq 0$ and a filtration

$$Z_1 \supset Z_{1,0} \supset Z_{1,1} \supset \ldots \supset Z_{1,t} = \emptyset$$

by closed subspaces, where $Z_{1,0} \to Z_1$ is a thickening and there exist finite surjective morphisms $Y_{1,i} \to Z_{1,i}$ which are finite étale over $Z_{1,i} \setminus Z_{1,i+1}$. Since Z_1 is reduced, we have $Z_1 = Z_{1,0}$. Hence we can set $Z_i = Z_{1,i-1}$ and $Y_i = Y_{1,i-1}$ for $i \geq 1$ and the lemma is proved. □

70.17. Obtaining schemes

0B7X A few more techniques to show an algebraic space is a scheme. The first is that we can show there is a minimal closed subspace which is not a scheme.

0B7Y Lemma 70.17.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. If X is not a scheme, then there exists a closed subspace $Z \subset X$ such that Z is not a scheme, but every proper closed subspace $Z' \subset Z$ is a scheme.

Proof. We prove this by Zorn's lemma. Let $\mathcal{Z}$ be the set of closed subspaces Z which are not schemes ordered by inclusion. By assumption $\mathcal{Z}$ contains X, hence is nonempty. If Z_α is a totally ordered subset of $\mathcal{Z}$, then $Z = \bigcap Z_\alpha$ is in $\mathcal{Z}$. Namely,

$$Z = \lim Z_\alpha$$

and the transition morphisms are affine. Thus we may apply Lemma 70.5.11 to see that if Z were a scheme, then so would one of the Z_α. (This works even if $Z = \emptyset$, but note that by Lemma 70.5.3 this cannot happen.) Thus $\mathcal{Z}$ has minimal elements by Zorn's lemma. □

Now we can prove a little bit about these minimal non-schemes.

0B7Z Lemma 70.17.2. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume that every proper closed subspace $Z \subset X$ is a scheme, but X is not a scheme. Then X is reduced and irreducible.

Proof. We see that X is reduced by Lemma 70.15.3. Choose closed subsets $T_1 \subset |X|$ and $T_2 \subset |X|$ such that $|X| = T_1 \cup T_2$. If T_1 and T_2 are proper closed subsets, then the corresponding reduced induced closed subspaces $Z_1, Z_2 \subset X$ (Properties of Spaces, Definition 66.12.5) are schemes and so is $Z = Z_1 \times_X Z_2 = Z_1 \cap Z_2$ as a closed subscheme of either Z_1 or Z_2. Observe that the coproduct $Z_1 \amalg_Z Z_2$ exists in the category of schemes, see More on Morphisms, Lemma 37.67.8. One way to proceed, is to show that $Z_1 \amalg_Z Z_2$ is isomorphic to X, but we cannot use this here as the material on pushouts of algebraic spaces comes later in the theory. Instead we will use Lemma 70.15.1 to find an affine neighbourhood of every point. Namely, let $x \in |X|$. If $x \notin Z_1$, then x has a neighbourhood which is a scheme, namely, $X \setminus Z_1$. Similarly if $x \notin Z_2$. If $x \in Z = Z_1 \cap Z_2$, then we choose an affine open $U \subset Z_1 \amalg_Z Z_2$ containing z. Then $U_1 = Z_1 \cap U$ and $U_2 = Z_2 \cap U$ are affine opens whose intersections with Z agree. Since $|Z_1| = T_1$ and $|Z_2| = T_2$ are closed subsets of $|X|$ which intersect in $|Z|$, we find an open $W \subset |X|$ with $W \cap T_1 = |U_1|$ and $W \cap T_2 = |U_2|$. Let W denote the corresponding open subspace of X. Then $x \in |W|$ and the morphism $U_1 \amalg U_2 \to W$ is a surjective finite morphism whose source is an affine scheme. Thus W is an affine scheme by Lemma 70.15.1. $\square$

A key point in the following lemma is that we only need to check the condition in the images of points of X.

0B80 Lemma 70.17.3. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism from an algebraic space to a scheme S. If for every $x \in |X|$ with image $s = f(x) \in S$ the algebraic space $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is a scheme, then X is a scheme.

Proof. Let $x \in |X|$. It suffices to find an open neighbourhood U of $s = f(x)$ such that $X \times_S U$ is a scheme. As $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is a scheme, then, since $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_S(U)$ where the colimit is over affine open neighbourhoods of s in S we see that

$$X \times_S \mathrm{Spec}(\mathcal{O}_{S,s}) = \lim X \times_S U$$

By Lemma 70.5.11 we see that $X \times_S U$ is a scheme for some U. $\square$

Instead of restricting to local rings as in Lemma 70.17.3, we can restrict to closed subschemes of the base.

0B81 Lemma 70.17.4. Let $\varphi : X \to \mathrm{Spec}(A)$ be a quasi-compact and quasi-separated morphism from an algebraic space to an affine scheme. If X is not a scheme, then there exists an ideal $I \subset A$ such that the base change $X_{A/I}$ is not a scheme, but for every $I \subset I'$, $I \neq I'$ the base change $X_{A/I'}$ is a scheme.

Proof. We prove this by Zorn's lemma. Let $\mathcal{I}$ be the set of ideals I such that $X_{A/I}$ is not a scheme. By assumption $\mathcal{I}$ contains (0). If I_α is a chain of ideals in $\mathcal{I}$, then $I = \bigcup I_\alpha$ is in $\mathcal{I}$. Namely, $A/I = \mathrm{colim}\, A/I_\alpha$, hence

$$X_{A/I} = \lim X_{A/I_\alpha}$$

Thus we may apply Lemma 70.5.11 to see that if $X_{A/I}$ were a scheme, then so would be one of the X_{A/I_α}. Thus $\mathcal{I}$ has maximal elements by Zorn's lemma. $\square$

70.18. Glueing in closed fibres

0E8Y Applying our theory above to the spectrum of a local ring we obtain a few pleasing glueing results for relative algebraic spaces. We first prove a helper lemma (which will be vastly generalized in Bootstrap, Section 80.11).

0E8Z **Lemma 70.18.1.** Let $S = U \cup W$ be an open covering of a scheme. Then the functor

$$FP_S \longrightarrow FP_U \times_{FP_{U \cap W}} FP_W$$

given by base change is an equivalence where FP_T is the category of algebraic spaces of finite presentation over the scheme T.

Proof. First, since $S = U \cup W$ is a Zariski covering, we see that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of triples $(\mathcal{F}_U, \mathcal{F}_W, \varphi)$ where $\mathcal{F}_U$ is a sheaf on $(Sch/U)_{fppf}$, $\mathcal{F}_W$ is a sheaf on $(Sch/W)_{fppf}$, and

$$\varphi : \mathcal{F}_U|_{(Sch/U \cap W)_{fppf}} \longrightarrow \mathcal{F}_W|_{(Sch/U \cap W)_{fppf}}$$

is an isomorphism. See Sites, Lemma 7.26.5 (note that no other gluing data are necessary because $U \times_S U = U$, $W \times_S W = W$ and that the cocycle condition is automatic for the same reason). Now, if the sheaf $\mathcal{F}$ on $(Sch/S)_{fppf}$ maps to $(\mathcal{F}_U, \mathcal{F}_W, \varphi)$ via this equivalence, then $\mathcal{F}$ is an algebraic space if and only if $\mathcal{F}_U$ and $\mathcal{F}_W$ are algebraic spaces. This follows immediately from Algebraic Spaces, Lemma 65.8.5 as $\mathcal{F}_U \to \mathcal{F}$ and $\mathcal{F}_W \to \mathcal{F}$ are representable by open immersions and cover $\mathcal{F}$. Finally, in this case the algebraic space $\mathcal{F}$ is of finite presentation over S if and only if $\mathcal{F}_U$ is of finite presentation over U and $\mathcal{F}_W$ is of finite presentation over W by Morphisms of Spaces, Lemmas 67.8.8, 67.4.12, and 67.28.4. $\square$

0E90 **Lemma 70.18.2.** Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories

$$FP_S \longrightarrow FP_U \times_{FP_V} FP_{\mathrm{Spec}(\mathcal{O}_{S,s})}$$

where FP_T is the category of algebraic spaces of finite presentation over T.

Proof. Let $W \subset S$ be an open neighbourhood of s. The functor

$$FP_S \to FP_U \times_{FP_{W \setminus \{s\}}} FP_W$$

is an equivalence of categories by Lemma 70.18.1. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of algebraic spaces of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of algebraic spaces of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 70.7.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{s\}$ is a limit of quasi-compact and quasi-separated schemes (see Limits, Lemma 32.2.2). Thus also the category of algebraic spaces of finite presentation over V is the limit of the categories of algebraic spaces of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. $\square$

0E91 Lemma 70.18.3. Let S be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of U. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an equivalence of categories

$$\mathrm{colim}_{s \in U' \supset U \text{ open}} FP_{U'} \longrightarrow FP_U \times_{FP_V} FP_{\mathrm{Spec}(\mathcal{O}_{S,s})}$$

where FP_T is the category of algebraic spaces of finite presentation over T.

Proof. Let $W \subset S$ be an open neighbourhood of s. By Lemma 70.18.1 the functor

$$FP_{U \cup W} \longrightarrow FP_U \times_{FP_{U \cap W}} FP_W$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of algebraic spaces of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of algebraic spaces of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 70.7.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Limits, Lemma 32.2.2). Thus also the category of algebraic spaces of finite presentation over V is the limit of the categories of algebraic spaces of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. $\square$

0E92 Lemma 70.18.4. Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \ldots, s_n\} \to S$ is quasi-compact. With $S_i = \mathrm{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \ldots \times FP_{U_n})} (FP_{S_1} \times \ldots \times FP_{S_n})$$

where FP_T is the category of algebraic spaces of finite presentation over T.

Proof. For $n = 1$ this is Lemma 70.18.2. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \to S_i$ are objects of FP_{S_i} and $f : X \to U$ is an object of FP_U and we're given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 70.18.2 we can find a morphism $f' : X' \to U' = S \setminus \{s_1, \ldots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to X_i over S_i, which is isomorphic to X over U, and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_n} U_n = X \times_U U_n$. Then we can apply induction to $f_i : X_i \to S_i$, $i \leq n-1$, $f' : X' \to U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n-1$. This shows essential surjectivity. We omit the proof of fully faithfulness. $\square$

70.19. Application to modifications

0BGX Using limits we can describe the category of modifications of a decent algebraic space over a closed point in terms of the henselian local ring.

0BGY Lemma 70.19.1. Let S be a scheme. Consider a separated étale morphism $f : V \to W$ of algebraic spaces over S. Assume there exists a closed subspace $T \subset W$ such that $f^{-1}T \to T$ is an isomorphism. Then, with $W^0 = W \setminus T$ and $V^0 = f^{-1}W^0$ the base change functor

$$\left\{ \begin{matrix} g : X \to W \text{ morphism of algebraic spaces} \\ g^{-1}(W^0) \to W^0 \text{ is an isomorphism} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} h : Y \to V \text{ morphism of algebraic spaces} \\ h^{-1}(V^0) \to V^0 \text{ is an isomorphism} \end{matrix} \right\}$$

is an equivalence of categories.

Proof. Since $V \to W$ is separated we see that $V \times_W V = \Delta(V) \amalg U$ for some open and closed subspace U of $V \times_W V$. By the assumption that $f^{-1}T \to T$ is an isomorphism we see that $U \times_W T = \emptyset$, i.e., the two projections $U \to V$ maps into V^0.

Given $h : Y \to V$ in the right hand category, consider the contravariant functor X on $(Sch/S)_{fppf}$ defined by the rule

$$X(T) = \{(w, y) \mid w : T \to W,\ y : T \times_{w,W} V \to Y \text{ morphism over } V\}$$

Denote $g : X \to W$ the map sending $(w, y) \in X(T)$ to $w \in W(T)$. Since $h^{-1}V^0 \to V^0$ is an isomorphism, we see that if $w : T \to W$ maps into W^0, then there is a unique choice for h. In other words $X \times_{g,W} W^0 = W^0$. On the other hand, consider a T-valued point (w, y, v) of $X \times_{g,W,f} V$. Then $w = f \circ v$ and

$$y : T \times_{f \circ v,W} V \longrightarrow V$$

is a morphism over V. Consider the morphism

$$T \times_{f \circ v,W} V \xrightarrow{(v,\mathrm{id}_V)} V \times_W V = V \amalg U$$

The inverse image of V is T embedded via $(\mathrm{id}_T, v) : T \to T \times_{f \circ v,W} V$. The composition $y' = y \circ (\mathrm{id}_T, v) : T \to Y$ is a morphism with $v = h \circ y'$ which determines y because the restriction of y to the other part is uniquely determined as U maps into V^0 by the second projection. It follows that $X \times_{g,W,f} V \to Y$, $(w, y, v) \mapsto y'$ is an isomorphism.

Thus if we can show that X is an algebraic space, then we are done. Since $V \to W$ is separated and étale it is representable by Morphisms of Spaces, Lemma 67.51.1 (and Morphisms of Spaces, Lemma 67.39.5). Of course $W^0 \to W$ is representable and étale as it is an open immersion. Thus

$$W^0 \amalg Y = X \times_{g,W} W^0 \amalg X \times_{g,W,f} V = X \times_{g,W} (W^0 \amalg V) \longrightarrow X$$

is representable, surjective, and étale by Spaces, Lemmas 65.3.3 and 65.5.5. Thus X is an algebraic space by Spaces, Lemma 65.11.2. □

0BGZ Lemma 70.19.2. Notation and assumptions as in Lemma 70.19.1. Let $g : X \to W$ correspond to $h : Y \to V$ via the equivalence. Then g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, and add more here if and only if h is so.

Proof. If g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, finite, so is h as a base change of g by Morphisms of Spaces, Lemmas 67.8.4, 67.4.4, 67.28.3, 67.23.3, 67.40.3, 67.45.5. Conversely, let P be a property of morphisms of algebraic spaces which is étale local on the base and which holds for the identity morphism of any algebraic space. Since $\{W^0 \to W, V \to W\}$ is an étale covering, to prove that g has P it suffices to show that h has P. Thus we conclude using Morphisms of Spaces, Lemmas 67.8.8, 67.4.12, 67.28.4, 67.23.4, 67.40.2, 67.45.3. □

0BH0 Lemma 70.19.3. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a closed point such that $U = X \setminus \{x\} \to X$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}^h_{X,x}) \setminus \{\mathfrak{m}^h_x\}$ the base change functor

$$\left\{\begin{matrix} f : Y \to X \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} g : Y \to \mathrm{Spec}(\mathcal{O}^h_{X,x}) \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism}\end{matrix}\right\}$$

is an equivalence of categories.

Proof. Let $a : (W, w) \to (X, x)$ be an elementary étale neighbourhood of x with W affine as in Decent Spaces, Lemma 68.11.4. Since x is a closed point of X and w is the unique point of W lying over x, we see that w is a closed point of W. Since a is étale and identifies residue fields at x and w, it follows that a induces an isomorphism $a^{-1}x \to x$ (as closed subspaces of X and W). Thus we may apply Lemma 70.19.1 and 70.19.2 to reduce the problem to the case where X is an affine scheme.

Assume X is an affine scheme. Recall that $\mathcal{O}^h_{X,x}$ is the colimit of $\Gamma(U, \mathcal{O}_U)$ over affine elementary étale neighbourhoods $(U, u) \to (X, x)$. Recall that the category of these neighbourhoods is cofiltered, see Decent Spaces, Lemma 68.11.6 or More on Morphisms, Lemma 37.35.4. Then $\mathrm{Spec}(\mathcal{O}^h_{X,x}) = \lim U$ and $V = \lim U \setminus \{u\}$ (Lemma 70.4.1) where the limits are taken over the same category. Thus by Lemma 70.7.1 The category on the right is the colimit of the categories for the pairs (U, u). And by the material in the first paragraph, each of these categories is equivalent to the category for the pair (X, x). This finishes the proof. □

70.20. Universally closed morphisms

0CM7 In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

0CM8 Lemma 70.20.1. Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces over S. Let $z \in |Z|$ and let $T \subset |X \times_Y Z|$ be a closed subset with $z \notin \mathrm{Im}(T \to |Z|)$. If f is quasi-compact, then there exists an étale neighbourhood $(V, v) \to (Z, z)$, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

(1) the morphism $b : Z' \to Y$ is locally of finite presentation,
(2) with $z' = a(v)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \to |X \times_Y Z'|$.

Moreover, we may assume V and Z' are affine schemes and if Z is a scheme we may assume V is an affine open neighbourhood of z.

Proof. We will deduce this from the corresponding result for morphisms of schemes. Let $y \in |Y|$ be the image of z. First we choose an affine étale neighbourhood $(U, u) \to (Y, y)$ and then we choose an affine étale neighbourhood $(V, v) \to (Z, z)$ such that the morphism $V \to Y$ factors through U. Then we may replace

(1) $X \to Y$ by $X \times_Y U \to U$,
(2) $Z \to Y$ by $V \to U$,
(3) z by v, and
(4) T by its inverse image in $|(X \times_Y U) \times_U V| = |X \times_Y V|$.

In fact, below we will show that after replacing V by an affine open neighbourhood of v there will be a morphism $a : V \to Z'$ for some $Z' \to U$ of finite presentation and a closed subset T' of $|(X \times_Y U) \times_U Z'| = |X \times_Y Z'|$ such that T maps into T' and $a(v) \notin \mathrm{Im}(T' \to |Z'|)$. Thus we may and do assume that Z and Y are affine schemes with the proviso that we need to find a solution where V is an open neighbourhood of z.

Since f is quasi-compact and Y is affine, the algebraic space X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $W \to X$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T. Then z is not in the image of T_W. By the schemes case (Limits, Lemma 32.14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

(1) the morphism $b : Z' \to Y$ is locally of finite presentation,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to Z')$, and
(3) $T_1 = T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \to |W \times_Y Z'|$.

The commutative diagram

$$\begin{tikzcd} W \times_Y Z \arrow[d] & W \times_Y V \arrow[l] \arrow[r, "a_1"'] \arrow[d, "c"'] & W \times_Y Z' \arrow[d, "q"] \\ X \times_Y Z & X \times_Y V \arrow[l] \arrow[r, "a_2"] & X \times_Y Z' \end{tikzcd}$$

has cartesian squares and the vertical maps are, surjective, étale and a fortiori open. Looking at the left hand square we see that $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |X \times_Y V|$ by c. By Properties of Spaces, Lemma 66.4.3 we get $a_1(T_1) = q^{-1}(a_2(T_2))$. By Topology, Lemma 5.6.4 we get

$$q^{-1}\left(\overline{a_2(T_2)}\right) = \overline{q^{-1}(a_2(T_2))} = \overline{a_1(T_1)} \subset T'$$

As q is surjective the image of $\overline{a_2(T_2)} \to |Z'|$ does not contain z' since the same is true for T'. Thus we can take the diagram with Z', V, a, b above and the closed subset $\overline{a_2(T_2)} \subset |X \times_Y Z'|$ as a solution to the problem posed by the lemma. □

0CM9 **Lemma 70.20.2.** Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. The following are equivalent

(1) f is universally closed,
(2) for every morphism $Z \to Y$ which is locally of finite presentation the map $|X \times_Y Z| \to |Z|$ is closed, and
(3) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.

Proof. It is clear that (1) implies (2). Suppose that $|X \times_Y Z| \to |Z|$ is not closed for some morphism of algebraic spaces $Z \to Y$ over S. This means that there exists some closed subset $T \subset |X \times_Y Z|$ such that $\mathrm{Im}(T \to |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 70.20.1. We find an étale neighbourhood $(V, v) \to (Z, z)$, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

(1) the morphism $b : Z' \to Y$ is locally of finite presentation,
(2) with $z' = a(v)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \to |X \times_Y Z'|$.

We claim that z' is in the closure of $\mathrm{Im}(T' \to |Z'|)$ which implies that $|X \times_Y Z'| \to |Z'|$ is not closed. The claim shows that (2) implies (1). To see the claim is true we contemplate following commutative diagram

$$\begin{tikzcd} X \times_Y Z \arrow[d] & X \times_Y V \arrow[l] \arrow[r] \arrow[d] & X \times_Y Z' \arrow[d] \\ Z & V \arrow[l] \arrow[r, "a"] & Z' \end{tikzcd}$$

Let $T_V \subset |X \times_Y V|$ be the inverse image of T. By Properties of Spaces, Lemma 66.4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \to |Z|$ and since $|V| \to |Z|$ is open, we see that v is in the closure of the image of $T_V \to |V|$. Since the image of T_V in $|X \times_Y Z'|$ is contained in $|T'|$ it follows immediately that $z' = a(v)$ is in the closure of the image of T'.

It is clear that (1) implies (3). Let $V \to Y$ be as in (3). If we can show that $X \times_Y V \to V$ is universally closed, then f is universally closed by Morphisms of Spaces, Lemma 67.9.5. Thus it suffices to show that $f : X \to Y$ satisfies (2) if f is a quasi-compact morphism of algebraic spaces, Y is a scheme, and $|\mathbf{A}^n \times X| \to |\mathbf{A}^n \times Y|$ is closed for all n. Let $Z \to Y$ be locally of finite presentation. We have to show the map $|X \times_Y Z| \to |Z|$ is closed. This question is étale local on Z hence we may assume Z is affine (some details omitted). Since Y is a scheme, Z is affine, and $Z \to Y$ is locally of finite presentation we can find an immersion $Z \to \mathbf{A}^n \times Y$, see Morphisms, Lemma 29.39.2. Consider the cartesian diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & \mathbf{A}^n \times X \arrow[d] \\ Z \arrow[r] & \mathbf{A}^n \times Y \end{tikzcd} \quad \text{inducing the cartesian square} \quad \begin{tikzcd} {|X \times_Y Z|} \arrow[r] \arrow[d] & {|\mathbf{A}^n \times X|} \arrow[d] \\ {|Z|} \arrow[r] & {|\mathbf{A}^n \times Y|} \end{tikzcd}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Spaces, Lemma 66.12.1). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. $\square$

0CMA Lemma 70.20.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated and of finite type. The following are equivalent

(1) The morphism f is proper.
(2) For any morphism $Y \to Z$ which is locally of finite presentation the map $|X \times_Y Z| \to |Z|$ is closed, and
(3) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.

Proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 70.20.2. □

70.21. Noetherian valuative criterion

0CMB We have already proved some results in Cohomology of Spaces, Section 69.19. The corresponding section for schemes is Limits, Section 32.15.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CMC Lemma 70.21.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f finite type and Y locally Noetherian. Let $y \in |Y|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. Moreover, we can assume that the point $x \in |X|$ corresponding to $\mathrm{Spec}(K) \to X$ is a codimension 0 point[3] and that K is the residue field of a point on a scheme étale over X.

Proof. Choose an affine scheme V, a point $v \in V$ and an étale morphism $V \to Y$ mapping v to y. The map $|V| \to |Y|$ is open and by Properties of Spaces, Lemma 66.4.3 the image of $|X \times_Y V| \to |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|X \times_Y V| \to |V|$. If we prove the lemma for $X \times_Y V \to V$ and the point v, then the lemma follows for f and y. In this way we reduce to the situation described in the next paragraph.

Assume we have $f : X \to Y$ and $y \in |Y|$ as in the lemma where Y is an affine scheme. Since f is quasi-compact, we conclude that X is quasi-compact. Hence we can choose an affine scheme W and a surjective étale morphism $W \to X$. Then the image of $|f|$ is the same as the image of $W \to Y$. In this way we reduce to the case of schemes which is Limits, Lemma 32.15.1. □

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of algebraic spaces over a base scheme S having the following

[3]See discussion in Properties of Spaces, Section 66.11.

shape

0H1V (70.21.1.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

with A a valuation ring and K its field of fractions.

0H1W **Lemma 70.21.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-separated and locally of finite type and Y is locally Noetherian. The following are equivalent:

(1) The morphism f is separated.
(2) For any diagram (70.21.1.1) there is at most one dotted arrow.
(3) For all diagrams (70.21.1.1) with A a discrete valuation ring there is at most one dotted arrow.
(4) For all diagrams (70.21.1.1) where A is a discrete valuation ring and where the image of $\mathrm{Spec}(K) \to X$ is a point of codimension 0 on X there is at most one dotted arrow.

Proof. We have (1) $\Rightarrow$ (2) by Morphisms of Spaces, Lemma 67.43.1. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate. It remains to show (4) implies (1).

Assume (4). We have to show that the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. We already know Δ is representable, separated, a monomorphism, and locally of finite type, see Morphisms of Spaces, Lemma 67.4.1. Choose an affine scheme U and an étale morphism $U \to X \times_Y X$. Set $V = X \times_{\Delta, X \times_Y X} U$. It suffices to show that $V \to U$ is a closed immersion (Morphisms of Spaces, Lemma 67.12.1). Since $X \times_Y X$ is locally of finite type over Y we see that U is Noetherian (use Morphisms of Spaces, Lemmas 67.23.2, 67.23.3, and 67.23.5). Note that V is a scheme as Δ is representable. Also, V is quasi-compact because f is quasi-separated. Hence $V \to U$ is separated and of finite type. Consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & U \end{tikzcd}$$

of morphisms of schemes where A is a discrete valuation ring with fraction field K and where K is the residue field of a generic point of the Noetherian scheme V. Since $V \to X$ is étale (as a base change of the étale morphism $U \to X \times_Y X$) we see that the image of $\mathrm{Spec}(K) \to V \to X$ is a point of codimension 0, see Properties of Spaces, Section 66.10. We can interpret the composition $\mathrm{Spec}(A) \to U \to X \times_Y X$ as a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ agreeing as morphisms into Y and equal when restricted to $\mathrm{Spec}(K)$ and that this restriction maps to a point of codimension 0. Hence our assumption (4) guarantees $a = b$ and we find the dotted arrow in the diagram. By Limits, Lemma 32.15.3 we conclude that $V \to U$ is proper. In other words, Δ is proper. Since Δ is a monomorphism, we find that Δ is a closed immersion (Étale Morphisms, Lemma 41.7.2) as desired. $\square$

0H1X **Lemma 70.21.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-separated and of finite type and Y is locally Noetherian. The following are equivalent:

(1) f is proper,
(2) f satisfies the valuative criterion, see Morphisms of Spaces, Definition 67.41.1,
(3) for any diagram (70.21.1.1) there exists exactly one dotted arrow,
(4) for all diagrams (70.21.1.1) with A a discrete valuation ring there exists exactly one dotted arrow, and
(5) for all diagrams (70.21.1.1) where A is a discrete valuation ring and where the image of $\mathrm{Spec}(K) \to X$ is a point of codimension 0 on X there exists exactly one dotted arrow[4].

Proof. We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Morphisms of Spaces, Lemma 67.44.1. It is clear that (3) $\Rightarrow$ (4) $\Rightarrow$ (5). To finish the proof we will now show (5) implies (1).

Assume (5). By Lemma 70.21.2 we see that f is separated. To finish the proof it suffices to show that f is universally closed. Let $V \to Y$ be an étale morphism where V is an affine scheme. It suffices to show that the base change $V \times_Y X \to V$ is universally closed, see Morphisms of Spaces, Lemma 67.9.5. Let

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & V \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] \arrow[rru, dotted] & V \arrow[r] & Y
\end{tikzcd}$$

of algebraic spaces over S be a commutative diagram where A is a discrete valuation ring with fraction field K and where $\mathrm{Spec}(K) \to V \times_Y X$ maps to a point of codimension 0 of the algebraic space $V \times_Y X$. Since $V \times_Y X \to X$ is étale it follows that the image of $\mathrm{Spec}(K) \to X$ is a point of codimension 0 of X. Thus by (5) we obtain the longer of the two dotted arrows fitting into the diagram. Then of course we obtain the shorter one as well. It follows that our assumptions hold for the morphism $V \times_Y X \to V$ and we reduce to the case discussed in the next paragraph.

Aassume Y is a Noetherian affine scheme. In this case X is a separated Noetherian algebraic space (we already know f is separated) of finite type over Y. (In particular, the algebraic space X has a dense open subspace which is a scheme by Properties of Spaces, Proposition 66.13.3 although strictly speaking we will not need this.) Choose a quasi-projective scheme X' over Y and a proper surjective morphism $X' \to X$ as in the weak form of Chow's lemma (Cohomology of Spaces, Lemma 69.18.1). We may replace X' by the disjoint union of the irreducible components which dominate an irreducible component of X; details omitted. In particular, we may assume that generic points of the scheme X' map to points of codimension 0 of X (in this case these are exactly the generic points of X). We claim that $X' \to Y$ is proper. The claim implies X is proper over Y by Morphisms of Spaces, Lemma 67.40.7. To prove this, according to Limits, Lemma 32.15.3 it

[4]There is a sharper formulation where in the existence part one only requires the dotted arrow exists after an extension of discrete valuation rings.

suffices to prove that in every solid commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[ru, dashed, "a"] \arrow[rru, dashed, "b"'] \arrow[rr] & & Y
\end{tikzcd}$$

where A is a dvr with fraction field K and where K is the residue field of a generic point of X' we can find the dotted arrow a (we already know uniqueness as X' is separated). By assumption (5) we can find the dotted arrow b. Then the morphism $X' \times_{X,b} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is a proper morphism of schemes and by the valuative criterion for morphisms of schemes we can lift b to the desired morphism a. $\square$

0H1Y Lemma 70.21.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f is of finite type. Then the following are equivalent

(1) f is universally closed,
(2) f satisfies the existence part of the valuative criterion,
(3) there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times X \times_Y V| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$,
(4) for all diagrams (70.21.1.1) with A a discrete valuation ring there there exists a finite separable extension K'/K of fields, a discrete valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

(5) for all diagrams (70.21.1.1) with A a discrete valuation ring there there exists a field extension K'/K, a valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

Proof. Parts (1), (2), and (3) are equivalent by Lemma 70.20.2 and Morphisms of Spaces, Lemma 67.42.1. These equivalent conditions imply part (4) as Morphisms of Spaces, Lemma 67.41.3 tells us that we may always choose K'/K finite separable in the existence part of the valuative criterion and this automatically forces A' to be a discrete valuation ring by Krull-Akizuki (Algebra, Lemma 10.119.12). The implication (4) $\Rightarrow$ (5) is immediate. In the rest of the proof we show that (5) implies (1).

Assume (5). Chose an affine scheme V and an étale morphism $V \to Y$. It suffices to show that the base change of f to V is universally closed, see Morphisms of Spaces, Lemma 67.9.5. Exactly as in the proof of Lemma 70.21.3 we see that assumption

(5) is inherited by this base change; details omitted. This reduces us to the case discussed in the next paragraph.

Assume Y is a Noetherian affine scheme and we have (5). To prove that f is universally closed it suffices to show that $|X \times \mathbf{A}^n| \to |Y \times \mathbf{A}^n|$ is closed for all n (by the discussion above). Since assumption (5) is inherited by the product morphism $X \times \mathbf{A}^n \to Y \times \mathbf{A}^n$ (details omitted) we reduce to proving that $|X| \to |Y|$ is closed.

Assume Y is a Noetherian affine scheme and we have (5). Let $T \subset |X|$ be a closed subset. We have to show that the image of T in $|Y|$ is closed. We may replace X by the reduced induced closed subspace structure on T; we omit the verification that property (5) is preserved by this replacement. Thus we reduce to proving that the image of $|X| \to |Y|$ is closed.

Let $y \in |Y|$ be a point in the closure of the image of $|X| \to |Y|$. By Lemma 70.21.1 we may choose a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. It follows immediately from property (5) that y is in the image of $|X| \to |Y|$ and the proof is complete. □

70.22. Refined Noetherian valuative criteria

0H1Z This section is the analogue of Limits, Section 32.16. One usually does not have to consider all possible diagrams with valuation rings when checking valuative criteria.

0CMD **Lemma 70.22.1.** Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f and h are of finite type, that f is separated, and that the image of $|h| : |U| \to |X|$ is dense in $|X|$. If given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute, then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \to Y$ be an étale morphism where V is an affine scheme. By Morphisms of Spaces, Lemma 67.9.5 it suffices to prove that the base change $X \times_Y V \to V$ is universally closed. By Properties of Spaces, Lemma 66.4.3 the image I of $|U \times_Y V| \to |X \times_Y V|$ is the inverse image of the image of $|h|$. Since $|X \times_Y V| \to |X|$ is open (Properties of Spaces, Lemma 66.16.7) we conclude that I is dense in $|X \times_Y V|$. Therefore the assumptions of the lemma are satisfied for the morphisms $U \times_Y V \to X \times_Y V \to V$. Hence we may assume Y is an affine scheme.

Assume Y is an affine scheme. Then U is quasi-compact. Choose an affine scheme and a surjective étale morphism $W \to U$. Then we may and do replace U by W and assume that U is affine. By the weak version of Chow's lemma (Cohomology of Spaces, Lemma 69.18.1) we can choose a surjective proper morphism $X' \to X$ where X' is a scheme. Then $U' = X' \times_X U$ is a scheme and $U' \to X'$ is of finite type. We may replace X' by the scheme theoretic image of $h' : U' \to X'$ and hence $h'(U')$ is dense in X'. We claim that for every diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U' \arrow[r, "h"] & X' \arrow[d, "f'"] \\ \mathrm{Spec}(A) \arrow[rru, dashed] \arrow[rr] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute. Namely, we first get an arrow $\mathrm{Spec}(A) \to X$ by the assumption of the lemma and then we lift this to an arrow $\mathrm{Spec}(A) \to X'$ using the valuative criterion for properness (Morphisms of Spaces, Lemma 67.44.1). The morphism $X' \to Y$ is separated as a composition of a proper and a separated morphism. Thus by the case of schemes the morphism $X' \to Y$ is proper (Limits, Lemma 32.16.1). By Morphisms of Spaces, Lemma 67.40.7 we conclude that $X \to Y$ is proper. $\square$

0CME Lemma 70.22.2. Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |U| \to |X|$ is dense in $|X|$. If given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rru, dashed] \arrow[rr] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 70.22.1 to the morphisms $U \to X$ and $\Delta : X \to X \times_Y X$. We check the conditions. Observe that Δ is quasi-compact because f is quasi-separated. Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rru, dashed] \arrow[rr, "{(a,b)}"] & & X \times_Y X \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K. Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. $\square$

0CMF Lemma 70.22.3. Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f and h are of finite type, that f is quasi-separated, and that $h(U)$ is dense in X. If given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 70.22.2 and 70.22.1. $\square$

70.23. Descending finite type spaces

0CP5 This section continues the theme of Section 70.11 in the spirit of the results discussed in Section 70.7. It is also the analogue of Limits, Section 32.22 for algebraic spaces.

0CP6 Situation 70.23.1. Let S be a scheme, for example $\mathrm{Spec}(\mathbf{Z})$. Let $B = \lim_{i \in I} B_i$ be the limit of a directed inverse system of Noetherian spaces over S with affine transition morphisms $B_{i'} \to B_i$ for $i' \geq i$.

0CP7 Lemma 70.23.2. In Situation 70.23.1. Let $X \to B$ be a quasi-separated and finite type morphism of algebraic spaces. Then there exists an $i \in I$ and a diagram

0CP8 (70.23.2.1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ B \arrow[r] & B_i \end{tikzcd}$$

such that $W \to B_i$ is of finite type and such that the induced morphism $X \to B \times_{B_i} W$ is a closed immersion.

Proof. By Lemma 70.11.6 we can find a closed immersion $X \to X'$ over B where X' is an algebraic space of finite presentation over B. By Lemma 70.7.1 we can find an i and a morphism of finite presentation $X'_i \to B_i$ whose pull back is X'. Set $W = X'_i$. $\square$

0CP9 Lemma 70.23.3. In Situation 70.23.1. Let $X \to B$ be a quasi-separated and finite type morphism of algebraic spaces. Given $i \in I$ and a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ B \arrow[r] & B_i \end{tikzcd}$$

as in (70.23.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to B_{i'} \times_{B_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to B_{i'} \times_{B_i} W$ commutes with étale localization (Morphisms of Spaces, Lemma 67.16.3). Hence we may and do assume W is affine and maps into an affine U_i étale over B_i. Then

$$B_{i'} \times_{B_i} W = B_{i'} \times_{B_i} U_i \times_{U_i} W = U_{i'} \times_{U_i} W$$

where $U_{i'} = B_{i'} \times_{B_i} U_i$ is affine as the transition morphisms are affine. Thus the lemma follows from the case of schemes which is Limits, Lemma 32.22.3. $\square$

0CPA **Lemma 70.23.4.** In Situation 70.23.1. Let $f : X \to Y$ be a morphism of algebraic spaces quasi-separated and of finite type over B. Let

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ B \arrow[r] & B_{i_1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & V \arrow[d] \\ B \arrow[r] & B_{i_2} \end{tikzcd}$$

be diagrams as in (70.23.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 70.23.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$$

of inverse systems over $(B_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(B_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \gg i_0$.

Proof. Since $V \to B_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 70.3.10 as improved by Lemma 70.4.5 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \to V$ over B_{i_2} such that $X \to X_{i_0} \to V$ is equal to $X \to Y \to V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d, dotted] \arrow[dd, bend right] & X_{i_0} \arrow[d, "h"] \\ Y \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & V \arrow[d] \\ B \arrow[r] & B_i \arrow[r] & B_{i_0} \end{tikzcd}$$

Since $X \to X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \to B_i \times_{B_{i_2}} V$ we find that the morphism $X_i \to B_i \times_{B_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms of Spaces, Lemma 67.16.6). This proves existence.

Uniqueness. Let $E_i \to X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. We have $E_i = Y_i \times_{\Delta, Y_i \times_{B_i} Y_i, (f_i, g_i)} X_i$. Hence $E_i \to X_i$ is a monomorphism of finite presentation as a base change of the diagonal of Y_i over B_i, see Morphisms of Spaces, Lemmas 67.4.1 and 67.28.10. Since X_i is a closed subspace of $B_i \times_{B_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(B_i \times_{B_{i_0}} X_{i_0})} (B_i \times_{B_{i_0}} E_{i_0}) = X_i \times_{X_{i_0}} E_{i_0}$$

Similarly, we have $X = X \times_{X_{i_0}} E_{i_0}$. Hence we conclude that $E_i = X_i$ for i large enough by Lemma 70.6.10. $\square$

0CPB **Remark 70.23.5.** In Situation 70.23.1 Lemmas 70.23.2, 70.23.3, and 70.23.4 tell us that the category of algebraic spaces quasi-separated and of finite type over B is equivalent to certain types of inverse systems of algebraic spaces over $(B_i)_{i \in I}$, namely the ones produced by applying Lemma 70.23.3 to a diagram of the form

(70.23.2.1). For example, given $X \to B$ finite type and quasi-separated if we choose two different diagrams $X \to V_1 \to B_{i_1}$ and $X \to V_2 \to B_{i_2}$ as in (70.23.2.1), then applying Lemma 70.23.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

0CPC Lemma 70.23.6. Notation and assumptions as in Lemma 70.23.4. If f is flat and of finite presentation, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.

Proof. By Lemma 70.7.1 we can choose an $i \geq i_2$ and a morphism $U \to Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \to Y_i$ is flat, see Lemma 70.6.12. As discussed in Remark 70.23.5 we may and do replace the initial diagram used to define the system $(X_i)_{i \geq i_1}$ by the system corresponding to $X \to U \to B_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \to B_{i'} \times_{B_i} U$.

Because $U \to Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \to Y_i$ is Y_i, we see that the scheme theoretic image of $X \to U$ is U (Morphisms of Spaces, Lemma 67.30.12). Observe that $Y_{i'} \to B_{i'} \times_{B_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j. Then the same argument as above shows that the scheme theoretic image of $X \to B_{i'} \times_{B_i} U$ is equal to the closed subspace $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. $\square$

0CPD Lemma 70.23.7. Notation and assumptions as in Lemma 70.23.4. If f is smooth, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is smooth.

Proof. Combine Lemmas 70.23.6 and 70.6.3. $\square$

0CPE Lemma 70.23.8. Notation and assumptions as in Lemma 70.23.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.

Proof. By the discussion in Remark 70.23.5 the choice of i_1 and W fitting into a diagram as in (70.23.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \to X'$ with $X' \to Y$ proper and of finite presentation, see Lemma 70.12.1. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \to Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i \geq i_2} Y_i$ and Lemmas 70.10.2 and 70.6.13. With this choice of W it is immediate from the construction that for $i \geq i_3$ the algebraic space X_i is a closed subspace of $Y_i \times_{Y_{i_3}} W \subset B_i \times_{B_{i_3}} W$ and hence proper over Y_i. $\square$

0CPF Lemma 70.23.9. In Situation 70.23.1 suppose that we have a cartesian diagram

$$\begin{tikzcd} X^1 \arrow[r, "p"'] \arrow[d, "q"'] & X^3 \arrow[d, "a"] \\ X^2 \arrow[r, "b"] & X^4 \end{tikzcd}$$

of algebraic spaces quasi-separated and of finite type over B. For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{tikzcd} X^j \arrow[r] \arrow[d] & W^j \arrow[d] \\ B \arrow[r] & B_{i_j} \end{tikzcd}$$

as in (70.23.2.1). Let $X^j = \lim_{i \geq i_j} X_i^j$ be the corresponding limit descriptions as in Lemma 70.23.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding morphisms of inverse systems contructed in Lemma 70.23.4. Then there exists an $i_9 \geq \max(i_5, i_6, i_7, i_8)$ such that for $i \geq i_9$ we have $a_i \circ p_i = b_i \circ q_i$ and such that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion. If a and b are flat and of finite presentation, then there exists an $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ such that for $i \geq i_{10}$ the last displayed morphism is an isomorphism.

Proof. According to the discussion in Remark 70.23.5 the choice of W^1 fitting into a diagram as in (70.23.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X_i^1 that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b. Thus we can apply Lemma 70.23.6 to each of a, b, p, q, and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X_i^1 \to X_i^2 \times_{X_i^4} X_i^3$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X_i^4 \to X_{i_9}^4$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 70.6.10. $\square$

70.24. Other chapters

Preliminaries

(1) Introduction
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Schemes

(26) Schemes

CHAPTER 71

Divisors on Algebraic Spaces

0838 71.1. Introduction

0839 In this chapter we study divisors on algebraic spaces and related topics. A basic reference for algebraic spaces is [Knu71].

71.2. Associated and weakly associated points

0CTV In the case of schemes we have introduced two competing notions of associated points. Namely, the usual associated points (Divisors, Section 31.2) and the weakly associated points (Divisors, Section 31.5). For a general algebraic space the notion of an associated point is basically useless and we don't even bother to introduce it. If the algebraic space is locally Noetherian, then we allow ourselves to use the phrase "associated point" instead of "weakly associated point" as the notions are the same for Noetherian schemes (Divisors, Lemma 31.5.8). Before we make our definition, we need a lemma.

0CTW Lemma 71.2.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. The following are equivalent

(1) for some étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is weakly associated to $f^*\mathcal{F}$,
(2) for every étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is weakly associated to $f^*\mathcal{F}$,
(3) the maximal ideal of $\mathcal{O}_{X,\overline{x}}$ is a weakly associated prime of the stalk $\mathcal{F}_{\overline{x}}$.

If X is locally Noetherian, then these are also equivalent to

(4) for some étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is associated to $f^*\mathcal{F}$,
(5) for every étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is associated to $f^*\mathcal{F}$,
(6) the maximal ideal of $\mathcal{O}_{X,\overline{x}}$ is an associated prime of the stalk $\mathcal{F}_{\overline{x}}$.

Proof. Choose a scheme U with a point u and an étale morphism $f : U \to X$ mapping u to x. Lift $\overline{x}$ to a geometric point of U over u. Recall that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,u}^{sh}$ where the strict henselization is with respect to our chosen lift of $\overline{x}$, see Properties of Spaces, Lemma 66.22.1. Finally, we have

$$\mathcal{F}_{\overline{x}} = (f^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = (f^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U,u}^{sh}$$

by Properties of Spaces, Lemma 66.29.4. Hence the equivalence of (1), (2), and (3) follows from More on Flatness, Lemma 38.2.9. If X is locally Noetherian, then any U as above is locally Noetherian, hence we see that (1), resp. (2) are equivalent to (4), resp. (5) by Divisors, Lemma 31.5.8. On the other hand, in the locally Noetherian case the local ring $\mathcal{O}_{X,\overline{x}}$ is Noetherian too (Properties of Spaces, Lemma 66.24.4). Hence the equivalence of (3) and (6) by the same lemma (or by Algebra, Lemma 10.66.9). □

0CTX Definition 71.2.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$.

(1) We say x is weakly associated to $\mathcal{F}$ if the equivalent conditions (1), (2), and (3) of Lemma 71.2.1 are satisfied.
(2) We denote $\text{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.
(3) The weakly associated points of X are the weakly associated points of $\mathcal{O}_X$.

If X is locally Noetherian we will say x is associated to $\mathcal{F}$ if and only if x is weakly associated to $\mathcal{F}$ and we set $\text{Ass}(\mathcal{F}) = \text{WeakAss}(\mathcal{F})$. Finally (still assuming X is locally Noetherian), we will say x is an associated point of X if and only if x is a weakly associated point of X.

At this point we can prove the obligatory lemmas.

0CTY Lemma 71.2.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}(\mathcal{F}) \subset \text{Supp}(\mathcal{F})$.

Proof. This is immediate from the definitions. The support of an abelian sheaf on X is defined in Properties of Spaces, Definition 66.20.3. □

0CTZ Lemma 71.2.4. Let S be a scheme. Let X be an algebraic space over S. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $\text{WeakAss}(\mathcal{F}_2) \subset \text{WeakAss}(\mathcal{F}_1) \cup \text{WeakAss}(\mathcal{F}_3)$ and $\text{WeakAss}(\mathcal{F}_1) \subset \text{WeakAss}(\mathcal{F}_2)$.

Proof. For every geometric point $\overline{x} \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,\overline{x}} \to \mathcal{F}_{2,\overline{x}} \to \mathcal{F}_{3,\overline{x}} \to 0$ is a short exact sequence of $\mathcal{O}_{X,\overline{x}}$-modules. Hence the lemma follows from Algebra, Lemma 10.66.4. □

0CU0 Lemma 71.2.5. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

$$\mathcal{F} = (0) \Leftrightarrow \text{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Then $\mathcal{F}$ is zero if and only if $f^*\mathcal{F}$ is zero. Hence the lemma follows from the definition and the lemma in the case of schemes, see Divisors, Lemma 31.5.5. □

0CUL Lemma 71.2.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. If

(1) $x \in \text{Supp}(\mathcal{F})$
(2) x is a codimension 0 point of X (Properties of Spaces, Definition 66.10.2).

Then $x \in \text{WeakAss}(\mathcal{F})$. If $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module with scheme theoretic support Z (Morphisms of Spaces, Definition 67.15.4) and x is a codimension 0 point of Z, then $x \in \text{WeakAss}(\mathcal{F})$.

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the stalk $\mathcal{F}_{\overline{x}}$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_{\overline{x}})$ is nonempty by Algebra, Lemma 10.66.5. On the other hand, the spectrum of $\mathcal{O}_{X,\overline{x}}$ is a singleton. Hence x is a weakly associated point of $\mathcal{F}$ by definition. The final statement follows as $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Z,\overline{z}}$ is a surjection, the spectrum of $\mathcal{O}_{Z,\overline{z}}$ is a singleton, and $\mathcal{F}_{\overline{x}}$ is a nonzero module over $\mathcal{O}_{Z,\overline{z}}$. □

0CUM Lemma 71.2.7. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. If

(1) X is decent (for example quasi-separated or locally separated),
(2) $x \in \text{Supp}(\mathcal{F})$
(3) x is not a specialization of another point in $\text{Supp}(\mathcal{F})$.

Then $x \in \text{WeakAss}(\mathcal{F})$.

Proof. (A quasi-separated algebraic space is decent, see Decent Spaces, Section 68.6. A locally separated algebraic space is decent, see Decent Spaces, Lemma 68.15.2.) Choose a scheme U, a point $u \in U$, and an étale morphism $f : U \to X$ mapping u to x. By Decent Spaces, Lemma 68.12.1 if $u' \rightsquigarrow u$ is a nontrivial specialization, then $f(u') \neq x$. Hence we see that $u \in \text{Supp}(f^*\mathcal{F})$ is not a specialization of another point of $\text{Supp}(f^*\mathcal{F})$. Hence $u \in \text{WeakAss}(f^*\mathcal{F})$ by Divisors, Lemma 71.2.6. $\square$

0CUN Lemma 71.2.8. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\text{Ass}(\mathcal{F}) \cap W$ is finite for every quasi-compact open $W \subset |X|$.

Proof. Choose a quasi-compact scheme U and an étale morphism $U \to X$ such that W is the image of $|U| \to |X|$. Then U is a Noetherian scheme and we may apply Divisors, Lemma 31.2.5 to conclude. $\square$

0CUP Lemma 71.2.9. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \to X$ is an étale morphism such that $\text{WeakAss}(\mathcal{F}) \subset \text{Im}(|U| \to |X|)$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\mathcal{F}'|_U = 0$. This implies that $\text{WeakAss}(\mathcal{F}') \cap \text{Im}(|U| \to |X|) = \emptyset$ (by the definition of weakly associated points). On the other hand, $\text{WeakAss}(\mathcal{F}') \subset \text{WeakAss}(\mathcal{F})$ by Lemma 71.2.4. We conclude $\text{WeakAss}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 71.2.5. $\square$

0CUQ Lemma 71.2.10. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$ be a point which is not in the image of $|f|$. Then y is not weakly associated to $f_*\mathcal{F}$.

Proof. By Morphisms of Spaces, Lemma 67.11.2 the $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is quasi-coherent hence the lemma makes sense. Choose an affine scheme V, a point $v \in V$, and an étale morphism $V \to Y$ mapping v to y. We may replace $f : X \to Y$, $\mathcal{F}$, y by $X \times_Y V \to V$, $\mathcal{F}|_{X \times_Y V}$, v. Thus we may assume Y is an affine scheme. In this case X is quasi-compact, hence we can choose an affine scheme U and a surjective étale morphism $U \to X$. Denote $g : U \to Y$ the composition. Then $f_*\mathcal{F} \subset g_*(\mathcal{F}|_U)$. By Lemma 71.2.4 we reduce to the case of schemes which is Divisors, Lemma 31.5.9. $\square$

0CUR Lemma 71.2.11. Let S be a scheme. Let X be an algebraic space over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in |X|$ at least one of the following happens

(1) $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective, or
(2) $x \notin \text{WeakAss}(\mathcal{F})$.

Then φ is injective.

Proof. The assumptions imply that $\text{WeakAss}(\text{Ker}(\varphi)) = \emptyset$ and hence $\text{Ker}(\varphi) = 0$ by Lemma 71.2.5. □

0EN1 Lemma 71.2.12. Let S be a scheme. Let X be a reduced algebraic space over S. Then the weakly associated point of X are exactly the codimension 0 points of X.

Proof. Working étale locally this follows from Divisors, Lemma 31.5.12 and Properties of Spaces, Lemma 66.11.1. □

71.3. Morphisms and weakly associated points

0CU1

0CU2 Lemma 71.3.1. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have

$$\text{WeakAss}_S(f_*\mathcal{F}) \subset f(\text{WeakAss}_X(\mathcal{F}))$$

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then $U \to V$ is an affine morphism of schemes. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.1. □

0CU8 Lemma 71.3.2. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have

$$\text{WeakAss}_Y(f_*\mathcal{F}) = f(\text{WeakAss}_X(\mathcal{F}))$$

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then $U \to V$ is an affine morphism of schemes and U is locally Noetherian. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.2. □

0CU9 Lemma 71.3.3. Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}(f_*\mathcal{F}) = f(\text{WeakAss}(\mathcal{F}))$.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then $U \to V$ is a finite morphism of schemes. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.3. □

0CUA Lemma 71.3.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $x \in |X|$ and $y = f(x) \in |Y|$. If

(1) $y \in \text{WeakAss}_S(\mathcal{G})$,
(2) f is flat at x, and
(3) the dimension of the local ring of the fibre of f at x is zero (Morphisms of Spaces, Definition 67.33.1),

then $x \in \text{WeakAss}(f^*\mathcal{G})$.

Proof. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Y$ mapping v to y. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to V \times_Y X$ mapping v to a point lying over v and x. This is possible because there is a $t \in |V \times_Y X|$ mapping to (v, y) by Properties of Spaces, Lemma 66.4.3. By definition we see that the dimension of $\mathcal{O}_{U_v,u}$ is zero. Hence u is a generic point of the fiber U_v. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.4. $\square$

0CUS Lemma 71.3.5. Let K/k be a field extension. Let X be an algebraic space over k. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in X_K$ with image $x \in X$. If y is a weakly associated point of the pullback $\mathcal{F}_K$, then x is a weakly associated point of $\mathcal{F}$.

Proof. This is the translation of Divisors, Lemma 31.6.5 into the language of algebraic spaces. We omit the details of the translation. $\square$

0CUT Lemma 71.3.6. Let S be a scheme. Let $f : X \to Y$ be a finite flat morphism of algebraic spaces. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $x \in |X|$ be a point with image $y \in |Y|$. Then

$$x \in \text{WeakAss}(g^*\mathcal{G}) \Leftrightarrow y \in \text{WeakAss}(\mathcal{G})$$

Proof. Follows immediately from the case of schemes (More on Flatness, Lemma 38.2.7) by étale localization. $\square$

0CUU Lemma 71.3.7. Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $x \in |X|$ be a point with image $y \in |Y|$. Then

$$x \in \text{WeakAss}(f^*\mathcal{G}) \Leftrightarrow y \in \text{WeakAss}(\mathcal{G})$$

Proof. This is immediate from the definition of weakly associated points and in fact the corresponding lemma for the case of schemes (More on Flatness, Lemma 38.2.8) is the basis for our definition. $\square$

71.4. Relative weak assassin

0CUV We need a couple of lemmas to define this gadget.

0CUW Lemma 71.4.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. The following are equivalent

(1) for some scheme V, point $v \in V$, and étale morphism $V \to Y$ mapping v to y, the algebraic space X_v is locally Noetherian,
(2) for every scheme V, point $v \in V$, and étale morphism $V \to Y$ mapping v to y, the algebraic space X_v is locally Noetherian, and
(3) there exists a field k and a morphism $\text{Spec}(k) \to Y$ representing y such that X_k is locally Noetherian.

If there exists a field k_0 and a monomorphism $\text{Spec}(k_0) \to Y$ representing y, then these are also equivalent to

(4) the algebraic space X_{k_0} is locally Noetherian.

Proof. Observe that $X_v = v \times_Y X = \text{Spec}(\kappa(v)) \times_Y X$. Hence the implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are clear. Assume that $\text{Spec}(k) \to Y$ is a morphism from the spectrum of a field such that X_k is locally Noetherian. Let $V \to Y$ be an étale morphism from a scheme V and let $v \in V$ a point mapping to y. Then the scheme $v \times_Y \mathrm{Spec}(k)$ is nonempty. Choose a point $w \in v \times_Y \mathrm{Spec}(k)$. Consider the morphisms

$$X_v \longleftarrow X_w \longrightarrow X_k$$

Since $V \to Y$ is étale and since w may be viewed as a point of $V \times_Y \mathrm{Spec}(k)$, we see that $\kappa(w)/k$ is a finite separable extension of fields (Morphisms, Lemma 29.36.7). Thus $X_w \to X_k$ is a finite étale morphism as a base change of $w \to \mathrm{Spec}(k)$. Hence X_w is locally Noetherian (Morphisms of Spaces, Lemma 67.23.5). The morphism $X_w \to X_v$ is a surjective, affine, flat morphism as a base change of the surjective, affine, flat morphism $w \to v$. Then the fact that X_w is locally Noetherian implies that X_v is locally Noetherian. This can be seen by picking a surjective étale morphism $U \to X$ and then using that $U_w \to U_v$ is surjective, affine, and flat. Working affine locally on the scheme U_v we conclude that U_w is locally Noetherian by Algebra, Lemma 10.164.1.

Finally, it suffices to prove that (3) implies (4) in case we have a monomorphism $\mathrm{Spec}(k_0) \to Y$ in the class of y. Then $\mathrm{Spec}(k) \to Y$ factors as $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0) \to Y$. The argument given above then shows that X_k being locally Noetherian impies that X_{k_0} is locally Noetherian. □

0CUX Definition 71.4.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. We say the fibre of f over y is locally Noetherian if the equivalent conditions (1), (2), and (3) of Lemma 71.4.1 are satisfied. We say the fibres of f are locally Noetherian if this holds for every $y \in |Y|$.

Of course, the usual way to guarantee locally Noetherian fibres is to assume the morphism is locally of finite type.

0CUY Lemma 71.4.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then the fibres of f are locally Noetherian.

Proof. This follows from Morphisms of Spaces, Lemma 67.23.5 and the fact that the spectrum of a field is Noetherian. □

0CUZ Lemma 71.4.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ and $y = f(x) \in |Y|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider commutative diagrams

$$\begin{tikzcd} X \arrow[d] & X \times_Y V \arrow[l] \arrow[d] & X_v \arrow[l] \arrow[d] \\ Y & V \arrow[l] & v \arrow[l] \end{tikzcd} \quad \begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & U_v \arrow[l] \arrow[d] \\ Y & V \arrow[l] & v \arrow[l] \end{tikzcd} \quad \begin{tikzcd} x \arrow[d, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] & u \arrow[l, mapsto] \arrow[dl, mapsto] \\ y & v \arrow[l, mapsto] & \end{tikzcd}$$

where V and U are schemes, $V \to Y$ and $U \to X \times_Y V$ are étale, $v \in V$, $x' \in |X_v|$, $u \in U$ are points related as in the last diagram. Denote $\mathcal{F}|_{X_v}$ and $\mathcal{F}|_{U_v}$ the pullbacks of $\mathcal{F}$. The following are equivalent

(1) for some V, v, x' as above x' is a weakly associated point of $\mathcal{F}|_{X_v}$,
(2) for every $V \to Y, v, x'$ as above x' is a weakly associated point of $\mathcal{F}|_{X_v}$,
(3) for some U, V, u, v as above u is a weakly associated point of $\mathcal{F}|_{U_v}$,
(4) for every U, V, u, v as above u is a weakly associated point of $\mathcal{F}|_{U_v}$,
(5) for some field k and morphism $\mathrm{Spec}(k) \to Y$ representing y and some $t \in |X_k|$ mapping to x, the point t is a weakly associated point of $\mathcal{F}|_{X_k}$.

If there exists a field k_0 and a monomorphism $\mathrm{Spec}(k_0) \to Y$ representing y, then these are also equivalent to

(6) x_0 is a weakly associated point of $\mathcal{F}|_{X_{k_0}}$ where $x_0 \in |X_{k_0}|$ is the unique point mapping to x.

If the fibre of f over y is locally Noetherian, then in conditions (1), (2), (3), (4), and (6) we may replace "weakly associated" with "associated".

Proof. Observe that given V, v, x' as in the lemma we can find $U \to X \times_Y V$ and $u \in U$ mapping to x' and then the morphism $U_v \to X_v$ is étale. Thus it is clear that (1) and (3) are equivalent as well as (2) and (4). Each of these implies (5). We will show that (5) implies (2). Suppose given V, v, x' as well as $\mathrm{Spec}(k) \to X$ and $t \in |X_k|$ such that the point t is a weakly associated point of $\mathcal{F}|_{X_k}$. We can choose a point $w \in v \times_Y \mathrm{Spec}(k)$. Then we obtain the morphisms

$$X_v \longleftarrow X_w \longrightarrow X_k$$

Since $V \to Y$ is étale and since w may be viewed as a point of $V \times_Y \mathrm{Spec}(k)$, we see that $\kappa(w)/k$ is a finite separable extension of fields (Morphisms, Lemma 29.36.7). Thus $X_w \to X_k$ is a finite étale morphism as a base change of $w \to \mathrm{Spec}(k)$. Thus any point x'' of X_w lying over t is a weakly associated point of $\mathcal{F}|_{X_w}$ by Lemma 71.3.7. We may pick x'' mapping to x' (Properties of Spaces, Lemma 66.4.3). Then Lemma 71.3.5 implies that x' is a weakly associated point of $\mathcal{F}|_{X_v}$.

To finish the proof it suffices to show that the equivalent conditions (1) – (5) imply (6) if we are given $\mathrm{Spec}(k_0) \to Y$ as in (6). In this case the morphism $\mathrm{Spec}(k) \to Y$ of (5) factors uniquely as $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0) \to Y$. Then x_0 is the image of t under the morphism $X_k \to X_{k_0}$. Hence the same lemma as above shows that (6) is true. □

0CV0 Definition 71.4.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The relative weak assassin of $\mathcal{F}$ in X over Y is the set $\mathrm{WeakAss}_{X/Y}(\mathcal{F}) \subset |X|$ consisting of those $x \in |X|$ such that the equivalent conditions of Lemma 71.4.4 are satisfied. If the fibres of f are locally Noetherian (Definition 71.4.2) then we use the notation $\mathrm{Ass}_{X/Y}(\mathcal{F})$.

With this notation we can formulate some of the results already proven for schemes.

0CV1 Lemma 71.4.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Assume

(1) $\mathcal{F}$ is flat over Y,
(2) X and Y are locally Noetherian, and
(3) the fibres of f are locally Noetherian.

Then

$$\mathrm{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \{x \in \mathrm{Ass}_{X/Y}(\mathcal{F}) \text{ such that } f(x) \in \mathrm{Ass}_Y(\mathcal{G})\}$$

Proof. Via étale localization, this is an immediate consequence of the result for schemes, see Divisors, Lemma 31.3.1. The result for schemes is more general only because we haven't defined associated points for non-Noetherian algebraic spaces (hence we need to assume X and the fibres of $X \to Y$ are locally Noetherian to even be able to formulate this result). □

0CV2 **Lemma 71.4.7.** Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and set $\mathcal{F}' = (g')^*\mathcal{F}$. If f is locally of finite type, then

(1) $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}') \Rightarrow g'(x') \in \text{Ass}_{X/Y}(\mathcal{F})$
(2) if $x \in \text{Ass}_{X/Y}(\mathcal{F})$, then given $y' \in |Y'|$ with $f(x) = g(y')$, there exists an $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ with $g'(x') = x$ and $f'(x') = y'$.

Proof. This follows from the case of schemes by étale localization. We write out the details completely. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme and the morphism $U' \to X'$ is surjective and étale.

Proof of (1). Choose $u' \in U'$ mapping to x'. Denote $v' \in V'$ the image of u'. Then $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ is equivalent to $u' \in \text{Ass}(\mathcal{F}|_{U'_{v'}})$ by definition (writing Ass instead of WeakAss makes sense as $U'_{v'}$ is locally Noetherian). Applying Divisors, Lemma 31.7.3 we see that the image $u \in U$ of u' is in $\text{Ass}(\mathcal{F}|_{U_v})$ where $v \in V$ is the image of u. This in turn means $g'(x') \in \text{Ass}_{X/Y}(\mathcal{F})$.

Proof of (2). Choose $u \in U$ mapping to x. Denote $v \in V$ the image of u. Then $x \in \text{Ass}_{X/Y}(\mathcal{F})$ is equivalent to $u \in \text{Ass}(\mathcal{F}|_{U_v})$ by definition. Choose a point $v' \in V'$ mapping to $y' \in |Y'|$ and to $v \in V$ (possible by Properties of Spaces, Lemma 66.4.3). Let $t \in \text{Spec}(\kappa(v') \otimes_{\kappa(v)} \kappa(u))$ be a generic point of an irreducible component. Let $u' \in U'$ be the image of t. Applying Divisors, Lemma 31.7.3 we see that $u' \in \text{Ass}(\mathcal{F}'|_{U'_{v'}})$. This in turn means $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ where $x' \in |X'|$ is the image of u'. □

0CV3 **Lemma 71.4.8.** With notation and assumptions as in Lemma 71.4.7. Assume g is locally quasi-finite, or more generally that for every $y' \in |Y'|$ the transcendence degree of $y'/g(y')$ is 0. Then $\text{Ass}_{X'/Y'}(\mathcal{F}')$ is the inverse image of $\text{Ass}_{X/Y}(\mathcal{F})$.

Proof. The transcendence degree of a point over its image is defined in Morphisms of Spaces, Definition 67.33.1. Let $x' \in |X'|$ with image $x \in |X|$. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme and the morphism $U' \to X'$ is surjective and étale. Choose $u \in U$ mapping to x. Denote $v \in V$ the image of u. Then $x \in \text{Ass}_{X/Y}(\mathcal{F})$ is equivalent to $u \in \text{Ass}(\mathcal{F}|_{U_v})$ by definition. Choose a point $u' \in U'$ mapping to $x' \in |X'|$ and to $u \in U$ (possible by Properties of Spaces, Lemma 66.4.3). Let $v' \in V'$ be the image of u'. Then $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ is equivalent to $u' \in \text{Ass}(\mathcal{F}'|_{U'_{v'}})$ by definition. Now the lemma follows from the discussion in Divisors, Remark 31.7.4 applied to $u' \in \text{Spec}(\kappa(v') \otimes_{\kappa(v)} \kappa(u))$. □

0CV4 **Lemma 71.4.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $i : Z \to X$ be a finite morphism. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $\text{WeakAss}_{X/Y}(i_*\mathcal{G}) = i(\text{WeakAss}_{Z/Y}(\mathcal{G}))$.

Proof. Follows from the case of schemes (Divisors, Lemma 31.8.3) by étale localization. Details omitted. □

0CVV Lemma 71.4.10. Let Y be a scheme. Let X be an algebraic space of finite presentation over Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subspace such that $U \to Y$ is quasi-compact. Then the set

$$E = \{y \in Y \mid \text{Ass}_{X_y}(\mathcal{F}_y) \subset |U_y|\}$$

is locally constructible in Y.

Proof. Note that since Y is a scheme, it makes sense to take the fibres $X_y = \text{Spec}(\kappa(y)) \times_Y X$. (Also, by our definitions, the set $\text{Ass}_{X_y}(\mathcal{F}_y)$ is exactly the fibre of $\text{Ass}_{X/Y}(\mathcal{F}) \to Y$ over y, but we won't need this.) The question is local on Y, indeed, we have to show that E is constructible if Y is affine. In this case X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to X$. Then $\text{Ass}_{X_y}(\mathcal{F}_y)$ is the image of $\text{Ass}_{W_y}(\varphi^*\mathcal{F}_y)$ for all $y \in Y$. Hence the lemma follows from the case of schemes for the open $\varphi^{-1}(U) \subset W$ and the morphism $W \to Y$. The case of schemes is More on Morphisms, Lemma 37.25.5. □

71.5. Fitting ideals

0CZ3 This section is the continuation of the discussion in Divisors, Section 31.9. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. In this situation we can construct the Fitting ideals

$$0 = \text{Fit}_{-1}(\mathcal{F}) \subset \text{Fit}_0(\mathcal{F}) \subset \text{Fit}_1(\mathcal{F}) \subset \ldots \subset \mathcal{O}_X$$

as the sequence of quasi-coherent sheaves ideals characterized by the following property: for every affine $U = \text{Spec}(A)$ étale over X if $\mathcal{F}|_U$ corresponds to the A-module M, then $\text{Fit}_i(\mathcal{F})|_U$ corresponds to the ideal $\text{Fit}_i(M) \subset A$. This is well defined and a quasi-coherent sheaf of ideals because if $A \to B$ is an étale ring map, then the ith Fitting ideal of $M \otimes_A B$ over B is equal to $\text{Fit}_i(M)B$ by More on Algebra, Lemma 15.8.4 part (3). More precisely (perhaps), the existence of the quasi-coherent sheaves of ideals $\text{Fit}_0(\mathcal{O}_X)$ follows (for example) from the description of quasi-coherent sheaves in Properties of Spaces, Lemma 66.29.3 and the pullback property given in Divisors, Lemma 31.9.1.

The advantage of constructing the Fitting ideals in this way is that we see immediately that formation of Fitting ideals commutes with étale localization hence many properties of the Fitting ideals immediately reduce to the corresponding properties in the case of schemes. Often we will use the discussion in Properties of Spaces, Section 66.30 to do the translation between properties of quasi-coherent sheaves on schemes and on algebraic spaces.

0CZ4 Lemma 71.5.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_Y$-module. Then $f^{-1}\text{Fit}_i(\mathcal{F}) \cdot \mathcal{O}_X = \text{Fit}_i(f^*\mathcal{F})$.

Proof. Reduces to Divisors, Lemma 31.9.1 by étale localization. □

0CZ5 Lemma 71.5.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Then $\text{Fit}_r(\mathcal{F})$ is a quasi-coherent ideal of finite type.

Proof. Reduces to Divisors, Lemma 31.9.2 by étale localization. □

0CZ6 Lemma 71.5.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $Z_0 \subset X$ be the closed subspace cut out by $\text{Fit}_0(\mathcal{F})$. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then

(1) $Z \subset Z_0 \subset X$ as closed subspaces,
(2) $|Z| = |Z_0| = \text{Supp}(\mathcal{F})$ as closed subsets of $|X|$,
(3) there exists a finite type, quasi-coherent $\mathcal{O}_{Z_0}$-module $\mathcal{G}_0$ with

$$(Z_0 \to X)_*\mathcal{G}_0 = \mathcal{F}.$$

Proof. Recall that formation of Z commutes with étale localization, see Morphisms of Spaces, Definition 67.15.4 (which uses Morphisms of Spaces, Lemma 67.15.3 to define Z). Hence (1) and (2) follow from the case of schemes, see Divisors, Lemma 31.9.3. To get $\mathcal{G}_0$ as in part (3) we can use that we have $\mathcal{G}$ on Z as in Morphisms of Spaces, Lemma 67.15.3 and set $\mathcal{G}_0 = (Z \to Z_0)_*\mathcal{G}$. □

0CZ7 Lemma 71.5.4. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. Then $\mathcal{F}$ can be generated by r elements in an étale neighbourhood of x if and only if $\text{Fit}_r(\mathcal{F})_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$.

Proof. Reduces to Divisors, Lemma 31.9.4 by étale localization (as well as the description of the local ring in Properties of Spaces, Section 66.22 and the fact that the strict henselization of a local ring is faithfully flat to see that the equality over the strict henselization is equivalent to the equality over the local ring). □

0CZ8 Lemma 71.5.5. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $r \geq 0$. The following are equivalent

(1) $\mathcal{F}$ is finite locally free of rank r
(2) $\text{Fit}_{r-1}(\mathcal{F}) = 0$ and $\text{Fit}_r(\mathcal{F}) = \mathcal{O}_X$, and
(3) $\text{Fit}_k(\mathcal{F}) = 0$ for $k < r$ and $\text{Fit}_k(\mathcal{F}) = \mathcal{O}_X$ for $k \geq r$.

Proof. Reduces to Divisors, Lemma 31.9.5 by étale localization. □

0CZ9 Lemma 71.5.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. The closed subspaces

$$X = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \ldots$$

defined by the Fitting ideals of $\mathcal{F}$ have the following properties

(1) The intersection $\bigcap Z_r$ is empty.
(2) The functor $(Sch/X)^{opp} \to \textit{Sets}$ defined by the rule

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the open subspace $X \setminus Z_r$.
(3) The functor $F_r : (Sch/X)^{opp} \to \textit{Sets}$ defined by the rule

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subspace $Z_{r-1} \setminus Z_r$ of X.

If $\mathcal{F}$ is of finite presentation, then $Z_r \to X$, $X \setminus Z_r \to X$, and $Z_{r-1} \setminus Z_r \to X$ are of finite presentation.

Proof. Reduces to Divisors, Lemma 31.9.6 by étale localization. □

0CZA Lemma 71.5.7. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $X = Z_{-1} \subset Z_0 \subset Z_1 \subset \ldots$ be as in Lemma 71.5.6. Set $X_r = Z_{r-1} \setminus Z_r$. Then $X' = \coprod_{r \geq 0} X_r$ represents the functor

$$F_{flat} : Sch/X \longrightarrow \text{Sets}, \qquad T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ flat over } T \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, $\mathcal{F}|_{X_r}$ is locally free of rank r and the morphisms $X_r \to X$ and $X' \to X$ are of finite presentation.

Proof. Reduces to Divisors, Lemma 31.9.7 by étale localization. □

71.6. Effective Cartier divisors

083A For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else. Note that in Morphisms of Spaces, Section 67.13 we discussed the correspondence between closed subspaces and quasi-coherent sheaves of ideals. Moreover, in Properties of Spaces, Section 66.30, we discussed properties of quasi-coherent modules, in particular "locally generated by 1 element". These references show that the following definition is compatible with the definition for schemes.

083B Definition 71.6.1. Let S be a scheme. Let X be an algebraic space over S.

(1) A locally principal closed subspace of X is a closed subspace whose sheaf of ideals is locally generated by 1 element.
(2) An effective Cartier divisor on X is a closed subspace $D \subset X$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subspace, but the converse is not always true. Effective Cartier divisors are closed subspaces of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

083C Lemma 71.6.2. Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be a closed subspace. The following are equivalent:

(1) The subspace D is an effective Cartier divisor on X.
(2) For some scheme U and surjective étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.
(3) For every scheme U and every étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.
(4) For every $x \in |D|$ there exists an étale morphism $(U, u) \to (X, x)$ of pointed algebraic spaces such that $U = \text{Spec}(A)$ and $D \times_X U = \text{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.

Proof. The equivalence of (1) – (3) follows from Definition 71.6.1 and the references preceding it. Assume (1) and let $x \in |D|$. Choose a scheme W and a surjective étale morphism $W \to X$. Choose $w \in D \times_X W$ mapping to x. By (3) $D \times_X W$ is an effective Cartier divisor on W. Hence we can find affine étale neighbourhood U by choosing an affine open neighbourhood of w in W as in Divisors, Lemma 31.13.2.

Assume (4). Then we see that $\mathcal{I}_D|_U$ is invertible by Divisors, Lemma 31.13.2. Since we can find an étale covering of X by the collection of all such U and $X \setminus D$, we conclude that $\mathcal{I}_D$ is an invertible $\mathcal{O}_X$-module. □

083D Lemma 71.6.3. Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a locally principal closed subspace. Let $U = X \setminus Z$. Then $U \to X$ is an affine morphism.

Proof. The question is étale local on X, see Morphisms of Spaces, Lemmas 67.20.3 and Lemma 71.6.2. Thus this follows from the case of schemes which is Divisors, Lemma 31.13.3. $\square$

083S Lemma 71.6.4. Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $U = X \setminus D$. Then $U \to X$ is an affine morphism and U is scheme theoretically dense in X.

Proof. Affineness is Lemma 71.6.3. The density question is étale local on X by Morphisms of Spaces, Definition 67.17.3. Thus this follows from the case of schemes which is Divisors, Lemma 31.13.4. $\square$

083T Lemma 71.6.5. Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $x \in |D|$. If $\dim_x(X) < \infty$, then $\dim_x(D) < \dim_x(X)$.

Proof. Both the definition of an effective Cartier divisor and of the dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1) are étale local. Hence this lemma follows from the case of schemes which is Divisors, Lemma 31.13.5. $\square$

083U Definition 71.6.6. Let S be a scheme. Let X be an algebraic space over S. Given effective Cartier divisors D_1, D_2 on X we set $D = D_1 + D_2$ equal to the closed subspace of X corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the sum of the effective Cartier divisors D_1 and D_2.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

083V Lemma 71.6.7. The sum of two effective Cartier divisors is an effective Cartier divisor.

Proof. Omitted. Étale locally this reduces to the following simple algebra fact: if $f_1, f_2 \in A$ are nonzerodivisors of a ring A, then $f_1 f_2 \in A$ is a nonzerodivisor. $\square$

083W Lemma 71.6.8. Let S be a scheme. Let X be an algebraic space over S. Let Z, Y be two closed subspaces of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.

Proof. By Lemma 71.6.2 this reduces to the case of schemes which is Divisors, Lemma 31.13.9. $\square$

Recall that we have defined the inverse image of a closed subspace under any morphism of algebraic spaces in Morphisms of Spaces, Definition 67.13.2.

083X Lemma 71.6.9. Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a locally principal closed subspace. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subspace of X'.

Proof. Omitted. $\square$

083Y Definition 71.6.10. Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $D \subset X$ be an effective Cartier divisor. We say the pullback of D by f is defined if the closed subspace $f^{-1}(D) \subset X'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the pullback of the effective Cartier divisor.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice.

083Z Lemma 71.6.11. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:

(1) $f(x) \notin |D|$ for any weakly associated point x of X,
(2) f is flat, and
(3) add more here as needed.

Proof. Working étale locally this lemma reduces to the case of schemes, see Divisors, Lemma 31.13.13. □

0840 Lemma 71.6.12. Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let D_1, D_2 be effective Cartier divisors on X. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.

Proof. Omitted. □

71.7. Effective Cartier divisors and invertible sheaves

0CPG Since an effective Cartier divisor has an invertible ideal sheaf (Definition 71.6.1) the following definition makes sense.

0841 Definition 71.7.1. Let S be a scheme. Let X be an algebraic space over S and let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I}_D$.

(1) The invertible sheaf $\mathcal{O}_X(D)$ associated to D is defined by
$$\mathcal{O}_X(D) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X) = \mathcal{I}_D^{\otimes -1}.$$
(2) The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_X(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_X$.
(3) We write $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\otimes -1} = \mathcal{I}_D$.
(4) Given a second effective Cartier divisor $D' \subset X$ we define $\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D')$.

Some comments. We will see below that the assignment $D \mapsto \mathcal{O}_X(D)$ turns addition of effective Cartier divisors (Definition 71.6.6) into addition in the Picard group of X (Lemma 71.7.3). However, the expression $D - D'$ in the definition above does not have any geometric meaning. More precisely, we can think of the set of effective Cartier divisors on X as a commutative monoid $\text{EffCart}(X)$ whose zero element is the empty effective Cartier divisor. Then the assignment $(D, D') \mapsto \mathcal{O}_X(D - D')$ defines a group homomorphism
$$\text{EffCart}(X)^{gp} \longrightarrow \text{Pic}(X)$$
where the left hand side is the group completion of $\text{EffCart}(X)$. In other words, when we write $\mathcal{O}_X(D - D')$ we may think of $D - D'$ as an element of $\text{EffCart}(X)^{gp}$.

0B4F Lemma 71.7.2. Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Then for the conormal sheaf we have $\mathcal{C}_{D/X} = \mathcal{I}_D|D = \mathcal{O}_X(D)^{\otimes -1}|_D$.

Proof. Omitted. $\square$

0842 Lemma 71.7.3. Let S be a scheme. Let X be an algebraic space over S. Let D_1, D_2 be effective Cartier divisors on X. Let $D = D_1 + D_2$. Then there is a unique isomorphism

$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. $\square$

0843 Definition 71.7.4. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a regular section if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

0844 Lemma 71.7.5. Let S be a scheme. Let X be an algebraic space over S. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:

(1) f is a regular section, and
(2) for any $x \in X$ the image $f \in \mathcal{O}_{X,\overline{x}}$ is not a zerodivisor.
(3) for any affine $U = \operatorname{Spec}(A)$ étale over X the restriction $f|_U$ is a nonzerodivisor of A, and
(4) there exists a scheme U and a surjective étale morphism $U \to X$ such that $f|_U$ is a regular section of $\mathcal{O}_U$.

Proof. Omitted. $\square$

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules on Sites, Lemma 18.32.4 for the dual invertible sheaf.)

0845 Definition 71.7.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The zero scheme of s is the closed subspace $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

0846 Lemma 71.7.7. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$.

(1) Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.
(2) For any morphism of algebraic spaces $f : Y \to X$ over S we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.
(3) The zero scheme $Z(s)$ is a locally principal closed subspace of X.
(4) The zero scheme $Z(s)$ is an effective Cartier divisor on X if and only if s is a regular section of $\mathcal{L}$.

Proof. Omitted. $\square$

0847 Lemma 71.7.8. Let S be a scheme. Let X be an algebraic space over S.

(1) If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.

(2) Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\text{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{matrix}\text{pairs } (\mathcal{L}, s) \text{ consisting of an invertible} \\ \mathcal{O}_X\text{-module and a regular global section}\end{matrix}\right\}$$

Proof. Omitted. □

71.8. Effective Cartier divisors on Noetherian spaces

0CPH In the locally Noetherian setting most of the discussion of effective Cartier divisors and regular sections simplifies somewhat.

0B4G **Lemma 71.8.1.** *Let S be a scheme and let X be a locally Noetherian algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. If X is (S_k), then D is (S_{k-1}).*

Proof. By our definition of the property (S_k) for algebraic spaces (Properties of Spaces, Section 66.7) and Lemma 71.6.2 this follows from the case of schemes (Divisors, Lemma 31.15.5). □

0B4H **Lemma 71.8.2.** *Let S be a scheme and let X be a locally Noetherian normal algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Then D is (S_1).*

Proof. By our definition of normality for algebraic spaces (Properties of Spaces, Section 66.7) and Lemma 71.6.2 this follows from the case of schemes (Divisors, Lemma 31.15.6). □

The following lemma can sometimes be used to produce effective Cartier divisors.

0DML **Lemma 71.8.3.** *Let S be a scheme. Let X be a regular Noetherian separated algebraic space over S. Let $U \subset X$ be a dense affine open. Then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. We claim that the reduced induced algebraic space structure D on $X \setminus U$ (Properties of Spaces, Definition 66.12.5) is the desired effective Cartier divisor. The construction of D commutes with étale localization, see proof of Properties of Spaces, Lemma 66.12.3. Let $X' \to X$ be a surjective étale morphism with X' affine. Since X is separated, we see that $U' = X' \times_X U$ is affine. Since $|X'| \to |X|$ is open, we see that U' is dense in X'. Since $D' = X' \times_X D$ is the reduced induced scheme structure on $X' \setminus U'$, we conclude that D' is an effective Cartier divisor by Divisors, Lemma 31.16.6 and its proof. This is what we had to show. □

0DMM **Lemma 71.8.4.** *Let S be a scheme. Let X be a regular Noetherian separated algebraic space over S. Then every invertible $\mathcal{O}_X$-module is isomorphic to*

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')^{\otimes -1}$$

for some effective Cartier divisors D, D' in X.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Choose a dense affine open $U \subset X$ such that $\mathcal{L}|_U$ is trivial. This is possible because X has a dense open subspace which is a scheme, see Properties of Spaces, Proposition 66.13.3. Denote $s : \mathcal{O}_U \to \mathcal{L}|_U$ the trivialization. The complement of U is an effective Cartier divisor D. We claim that for some $n > 0$ the map s extends uniquely to a map

$$s : \mathcal{O}_X(-nD) \longrightarrow \mathcal{L}$$

The claim implies the lemma because it shows that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nD)$ has a regular global section hence is isomorphic to $\mathcal{O}_X(D')$ for some effective Cartier divisor D' by Lemma 71.7.8. To prove the claim we may work étale locally. Thus we may assume X is an affine Noetherian scheme. Since $\mathcal{O}_X(-nD) = \mathcal{I}^n$ where $\mathcal{I} = \mathcal{O}_X(-D)$ is the ideal sheaf of D in X, this case follows from Cohomology of Schemes, Lemma 30.10.5. □

The following lemma really belongs to a different section.

0DMC Lemma 71.8.5. Let R be a valuation ring with fraction field K. Let X be an algebraic space over R such that $X \to \mathrm{Spec}(R)$ is smooth. For every effective Cartier divisor $D \subset X_K$ there exists an effective Cartier divisor $D' \subset X$ with $D'_K = D$.

Proof. Let $D' \subset X$ be the scheme theoretic image of $D \to X_K \to X$. Since this morphism is quasi-compact, formation of D' commutes with flat base change, see Morphisms of Spaces, Lemma 67.30.12. In particular we find that $D'_K = D$. Hence, we may assume X is affine. Say $X = \mathrm{Spec}(A)$. Then $X_K = \mathrm{Spec}(A \otimes_R K)$ and D corresponds to an ideal $I \subset A \otimes_R K$. We have to show that $J = I \cap A$ cuts out an effective Cartier divisor in X. First, observe that A/J is flat over R (as a torsion free R-module, see More on Algebra, Lemma 15.22.10), hence J is finitely generated by More on Algebra, Lemma 15.25.6 and Algebra, Lemma 10.5.3. Thus it suffices to show that $J_\mathfrak{q} \subset A_\mathfrak{q}$ is generated by a single element for each prime $\mathfrak{q} \subset A$. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R_\mathfrak{p}$ is a valuation ring (Algebra, Lemma 10.50.9). Observe further that $A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}$ is a regular ring by Algebra, Lemma 10.140.3. Thus we may apply More on Algebra, Lemma 15.121.3 to see that $I(A_\mathfrak{q} \otimes_R K)$ is generated by a single element $f \in A_\mathfrak{p} \otimes_R K$. After clearing denominators we may assume $f \in A_\mathfrak{q}$. Let $\mathfrak{c} \subset R_\mathfrak{p}$ be the content ideal of f (see More on Algebra, Definition 15.24.1 and More on Flatness, Lemma 38.19.6). Since $R_\mathfrak{p}$ is a valuation ring and since $\mathfrak{c}$ is finitely generated (More on Algebra, Lemma 15.24.2) we see $\mathfrak{c} = (\pi)$ for some $\pi \in R_\mathfrak{p}$ (Algebra, Lemma 10.50.15). After relacing f by $\pi^{-1}f$ we see that $f \in A_\mathfrak{q}$ and $f \notin \mathfrak{p}A_\mathfrak{q}$. Claim: $I_\mathfrak{q} = (f)$ which finishes the proof. To see the claim, observe that $f \in I_\mathfrak{q}$. Hence we have a surjection $A_\mathfrak{q}/(f) \to A_\mathfrak{q}/I_\mathfrak{q}$ which is an isomorphism after tensoring over R with K. Thus we are done if $A_\mathfrak{q}/(f)$ is $R_\mathfrak{p}$-flat. This follows from Algebra, Lemma 10.128.5 and our choice of f. □

71.9. Relative effective Cartier divisors

0EPM The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

0EPN Lemma 71.9.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $D \subset X$ be a closed subspace. Assume

(1) D is an effective Cartier divisor, and
(2) $D \to Y$ is a flat morphism.

Then for every morphism of schemes $g : Y' \to Y$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = Y' \times_Y X$ where $g' : X' \to X$ is the projection.

Proof. Using Lemma 71.6.2 the property of being an effective Cartier divisor is étale local. Thus this lemmma immediately reduces to the case of schemes which is Divisors, Lemma 31.18.1. $\square$

This lemma is the motivation for the following definition.

0EPP Definition 71.9.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. A relative effective Cartier divisor on X/Y is an effective Cartier divisor $D \subset X$ such that $D \to Y$ is a flat morphism of algebraic spaces.

71.10. Meromorphic functions and sections

0EN2 This section is the analogue of Divisors, Section 31.23. Beware: it is even easier to make mistakes with this material in the case of algebraic space, than it is in the case of schemes!

Let S be a scheme. Let X be an algebraic space over S. For any scheme U étale over X we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 71.7.4. The restriction of a regular section to V/U étale is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

on $X_{étale}$ and its sheafification, see Modules on Sites, Section 18.44.

0EN3 Definition 71.10.1. Let S be a scheme. Let X be an algebraic space over S. The sheaf of meromorphic functions on X is the sheaf $\mathcal{K}_X$ on $X_{étale}$ associated to the presheaf displayed above. A meromorphic function on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective. Moreover, by the compatibility of sheafification and taking stalks we see that

$$\mathcal{K}_{X,\overline{x}} = \mathcal{S}_{\overline{x}}^{-1}\mathcal{O}_{X,\overline{x}}$$

for any geometric point $\overline{x}$ of X. The set $\mathcal{S}_{\overline{x}}$ is a subset of the set of nonzerodivisors of $\mathcal{O}_{X,\overline{x}}$, but in general not equal to this.

0EN4 Lemma 71.10.2. Let S be a scheme. Let X be an algebraic space over S. For U affine and étale over X the set $\mathcal{S}_X(U)$ is the set of nonzerodivisors in $\mathcal{O}_X(U)$.

Proof. Follows from Lemma 71.7.5. $\square$

Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X_{étale}$. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules on Sites, Lemma 18.44.2.

0EN5 Definition 71.10.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X_{étale}$.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).

(2) A meromorphic section of $\mathcal{F}$ is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{K}_{X,\overline{x}} = \mathcal{S}_{\overline{x}}^{-1}\mathcal{F}_{\overline{x}}$$

for any geometric point $\overline{x}$ of X. However, one has to be careful since it may not be the case that $\mathcal{S}_{\overline{x}}$ is the set of nonzerodivisors in the étale local ring $\mathcal{O}_{X,\overline{x}}$ as we pointed out above. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

0EN6 Lemma 71.10.4. Let S be a scheme. Let X be an algebraic space over S. Assume

(a) every weakly associated point of X is a point of codimension 0, and
(b) X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1.

Then

(1) $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,
(2) for $U \in X_{étale}$ affine $\mathcal{K}_X(U)$ is the total ring of fractions of $\mathcal{O}_X(U)$,
(3) for a geometric point $\overline{x}$ the set $\mathcal{S}_{\overline{x}}$ the set of nonzerodivisors of $\mathcal{O}_{X,\overline{x}}$, and
(4) for a geometric point $\overline{x}$ the ring $\mathcal{K}_{X,\overline{x}}$ is the total ring of fractions of $\mathcal{O}_{X,\overline{x}}$.

Proof. By Lemma 71.7.5 we see that $U \in X_{étale}$ affine $\mathcal{S}_X(U) \subset \mathcal{O}_X(U)$ is the set of nonzerodivisors in $\mathcal{O}_X(U)$. Thus the presheaf $\mathcal{S}^{-1}\mathcal{O}_X$ is equal to

$$U \longmapsto Q(\mathcal{O}_X(U))$$

on $X_{affine,étale}$, with notation as in Algebra, Example 10.9.8. Observe that the codimension 0 points of X correspond to the generic points of U, see Properties of Spaces, Lemma 66.11.1. Hence if $U = \mathrm{Spec}(A)$, then A is a ring with finitely many minimal primes such that any weakly associated prime of A is minimal. The same is true for any étale extension of A (because the spectrum of such is an affine scheme étale over X hence can play the role of A in the previous sentence). In order to show that our presheaf is a sheaf and quasi-coherent it suffices to show that

$$Q(A) \otimes_A B \longrightarrow Q(B)$$

is an isomorphism when $A \to B$ is an étale ring map, see Properties of Spaces, Lemma 66.29.3. (To define the displayed arrow, observe that since $A \to B$ is flat it maps nonzerodivisors to nonzerodivisors.) By Algebra, Lemmas 10.25.4 and 10.66.7. we have

$$Q(A) = \prod\nolimits_{\mathfrak{p} \subset A \text{ minimal}} A_{\mathfrak{p}} \quad \text{and} \quad Q(B) = \prod\nolimits_{\mathfrak{q} \subset B \text{ minimal}} B_{\mathfrak{q}}$$

Since $A \to B$ is étale, the minimal primes of B are exactly the primes of B lying over the minimal primes of A (for example by More on Algebra, Lemma 15.44.2). By Algebra, Lemmas 10.153.10, 10.153.3 (13), and 10.153.5 we see that $A_{\mathfrak{p}} \otimes_A B$ is a finite product of local rings finite étale over $A_{\mathfrak{p}}$. This cleary implies that $A_{\mathfrak{p}} \otimes_A B = \prod_{\mathfrak{q} \text{ lies over } \mathfrak{p}} B_{\mathfrak{q}}$ as desired.

At this point we know that (1) and (2) hold. Proof of (3). Let $s \in \mathcal{O}_{X,\overline{x}}$ be a nonzerodivisor. Then we can find an étale neighbourhood $(U, \overline{u}) \to (X, \overline{x})$ and $f \in \mathcal{O}_X(U)$ mapping to s. Let $u \in U$ be the point determined by $\overline{u}$. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is faithfully flat (as a strict henselization), we see that f maps to a nonzerodivisor in $\mathcal{O}_{U,u}$. By Divisors, Lemma 31.23.6 after shrinking U we find that

f is a nonzerodivisor and hence a section of $\mathcal{S}_X(U)$. Part (4) follows from (3) by computing stalks. □

0EN7 Lemma 71.10.5. Let S be a scheme. Let X be an algebraic space over S. Assume

(a) every weakly associated point of X is a point of codimension 0, and
(b) X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1.
(c) X is representable by a scheme X_0 (awkward but temporary notation).

Then the sheaf of meromorphic functions $\mathcal{K}_X$ is the quasi-coherent sheaf of $\mathcal{O}_X$-algebras associated to the quasi-coherent sheaf of meromorphic functions $\mathcal{K}_{X_0}$.

Proof. For the equivalence between $QCoh(\mathcal{O}_X)$ and $QCoh(\mathcal{O}_{X_0})$, please see Properties of Spaces, Section 66.29. The lemma is true because $\mathcal{K}_X$ and $\mathcal{K}_{X_0}$ are quasi-coherent and have the same value on corresponding affine opens of X and X_0 by Lemma 71.10.4 and Divisors, Lemma 31.23.6. □

0EN8 Definition 71.10.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that pullbacks of meromorphic functions are defined for f if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

with $U \in X_{étale}$ and $V \in Y_{étale}$ and any section $s \in \mathcal{S}_Y(V)$ the pullback $f^\sharp(s) \in \mathcal{O}_X(U)$ is an element of $\mathcal{S}_X(U)$.

In this case there is an induced map $f^\sharp : f_{small}^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{tikzcd} (Sh(X_{étale}), \mathcal{K}_X) \arrow[r] \arrow[d, "f_{small}"] & (Sh(X_{étale}), \mathcal{O}_X) \arrow[d, "f_{small}"] \\ (Sh(Y_{étale}), \mathcal{K}_Y) \arrow[r] & (Sh(Y_{étale}), \mathcal{O}_Y) \end{tikzcd}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

0EN9 Lemma 71.10.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Pullbacks of meromorphic sections are defined in each of the following cases

(1) weakly associated points of X are mapped to points of codimension 0 on Y,
(2) f is flat,
(3) add more here as needed.

Proof. Working étale locally, this translates into the case of schemes, see Divisors, Lemma 31.23.5. To do the translation use Lemma 71.7.5 (description of regular sections), Definition 71.2.2 (definition of weakly associated points), and Properties of Spaces, Lemma 66.11.1 (description of codimension 0 points). □

0ENA Lemma 71.10.8. Let S be a scheme. Let X be an algebraic space over S. Assume

(a) every weakly associated point of X is a point of codimension 0, and

(b) X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1,
(c) every codimension 0 point of X can be represented by a monomorphism $\text{Spec}(k) \to X$.

Let $X^0 \subset |X|$ be the set of codimension 0 points of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \text{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 26.13; this makes sense because X^0 is contained in the schematic locus of X. Similarly, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we obtain the formula

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$

for the sheaf of meromorphic sections of $\mathcal{F}$. Finally, the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.

Proof. By Decent Spaces, Lemma 68.20.3 and Section 68.6 we see that X is decent[1]. Thus $X^0 \subset |X|$ is the set of generic points of irreducible components (Decent Spaces, Lemma 68.20.1) and X^0 is locally finite in $|X|$ by (b). It follows that X^0 is contained in every dense open subset of $|X|$. In particular, X^0 is contained in the schematic locus (Decent Spaces, Theorem 68.10.2). Thus the local rings $\mathcal{O}_{X,\eta}$ and the morphisms j_η are defined.

Observe that a locally finite direct sum of sheaves of modules is equal to the product. This and the fact that X^0 is locally finite in $|X|$ explains the equalities between direct sums and products in the statement. Then since $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ we see that the second equality follows from the first.

Let $j : Y = \coprod_{\eta \in X^0} \text{Spec}(\mathcal{O}_{X,\eta}) \to X$ be the product of the morphisms j_η. We have to show that $\mathcal{K}_X = j_*\mathcal{O}_Y$. Observe that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of local rings of dimension 0: in a local ring of dimension zero any nonzerodivisor is a unit. Next, note that pullbacks of meromorphic functions are defined for j by Lemma 71.10.7. This gives a map

$$\mathcal{K}_X \longrightarrow j_*\mathcal{O}_Y.$$

Let $U \in X_{étale}$ be affine. By Lemma 71.10.4 the left hand side evaluates to total ring of fractions of $\mathcal{O}_X(U)$. On the other hand, the right hand side is equal to the product of the local rings of U at the codimension 0 points, i.e., the generic points of U. These two rings are equal (as we already saw in the proof of Lemma 71.10.4) by Algebra, Lemmas 10.25.4 and 10.66.7. Thus our map is an isomorphism.

Finally, we have to show that $R(X) = \Gamma(X, \mathcal{K}_X)$. This follows from the case of schemes (Divisors, Lemma 31.23.6) applied to the schematic locus $X' \subset X$. Namely, the ring of rational functions of X is by definition the same as the ring of rational functions on X' as it is a dense open subspace of X (see above). Certainly, $R(X')$ agrees with the ring of rational functions when X' is viewed as a scheme. On the other hand, by our description of $\mathcal{K}_X$ above, and the fact, seen above, that $X^0 \subset |X'|$ is contained in any dense open, we see that $\Gamma(X, \mathcal{K}_X) = \Gamma(X', \mathcal{K}_{X'})$. Finally, use the compatibility recorded in Lemma 71.10.5. $\square$

[1] Conversely, if X is decent, then condition (c) holds automatically.

0ENB Definition 71.10.9. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be regular if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective.

Let us spell out when (regular) meromorphic sections can be pulled back.

0ENC Lemma 71.10.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that pullbacks of meromorphic functions are defined for f (see Definition 71.10.6).

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.
(2) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.

Proof. Omitted. □

0EPQ Lemma 71.10.11. Let S be a scheme. Let X be an algebraic space over S satisfying (a), (b), and (c) of Lemma 71.10.8. Then every invertible $\mathcal{O}_X$-module $\mathcal{L}$ has a regular meromorphic section.

Proof. With notation as in Lemma 71.10.8 the stalk $\mathcal{L}_\eta$ of $\mathcal{L}$ at is defined for all $\eta \in X^0$ and it is a rank 1 free $\mathcal{O}_{X,\eta}$-module. Pick a generator $s_\eta \in \mathcal{L}_\eta$ for all $\eta \in X^0$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 71.10.8 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. □

71.11. Relative Proj

0848 This section revisits the construction of the relative proj in the setting of algebraic spaces. The material in this section corresponds to the material in Constructions, Section 27.16 and Divisors, Section 31.30 in the case of schemes.

0849 Situation 71.11.1. Here S is a scheme, X is an algebraic space over S, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra.

In Situation 71.11.1 we are going to define a functor $F : (Sch/S)_{fppf}^{opp} \to$ Sets which will turn out to be an algebraic space. We will follow (mutatis mutandis) the procedure of Constructions, Section 27.16. First, given a scheme T over S we define a quadruple over T to be a system $(d, f : T \to X, \mathcal{L}, \psi)$

(1) $d \geq 1$ is an integer,
(2) $f : T \to X$ is a morphism over S,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

We say two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are equivalent[2] if and only if we have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$. Given a quadruple $(d, f, \mathcal{L}, \psi)$ and a morphism $h : T' \to T$ we have the pullback $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$. Pullback

[2]This definition is motivated by Constructions, Lemma 27.16.4. The advantage of choosing this one is that it clearly defines an equivalence relation.

preserves the equivalence relation. Finally, for a quasi-compact scheme T over S we set

$$F(T) = \text{the set of equivalence classes of quadruples over } T$$

and for an arbitrary scheme T over S we set

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. Thus we have defined our functor

084A (71.11.1.1)
$$F : Sch^{opp} \longrightarrow \text{Sets}$$

There is a morphism $F \to X$ of functors sending the quadruple $(d, f, \mathcal{L}, \psi)$ to f.

084B **Lemma 71.11.2.** In Situation 71.11.1. The functor F above is an algebraic space. For any morphism $g : Z \to X$ where Z is a scheme there is a canonical isomorphism $\underline{\text{Proj}}_Z(g^*\mathcal{A}) = Z \times_X F$ compatible with further base change.

Proof. It suffices to prove the second assertion, see Spaces, Lemma 65.11.3. Let $g : Z \to X$ be a morphism where Z is a scheme. Let F' be the functor of quadruples associated to the graded quasi-coherent $\mathcal{O}_Z$-algebra $g^*\mathcal{A}$. Then there is a canonical isomorphism $F' = Z \times_X F$, sending a quadruple $(d, f : T \to Z, \mathcal{L}, \psi)$ for F' to $(d, g \circ f, \mathcal{L}, \psi)$ (details omitted, see proof of Constructions, Lemma 27.16.1). By Constructions, Lemmas 27.16.4, 27.16.5, and 27.16.6 and Definition 27.16.7 we see that F' is representable by $\underline{\text{Proj}}_Z(g^*\mathcal{A})$. $\square$

The lemma above tells us the following definition makes sense.

084C **Definition 71.11.3.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over X*, or the *homogeneous spectrum of $\mathcal{A}$ over X*, or the *relative Proj of $\mathcal{A}$ over X* is the algebraic space F over X of Lemma 71.11.2. We denote it $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \to X$.

In particular the structure morphism of the relative Proj is representable by construction. We can also think about the relative Proj via glueing. Let $\varphi : U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$ with projection morphisms $s, t : R \to U$. By Lemma 71.11.2 there exists a canonical isomorphism

$$\gamma : \underline{\text{Proj}}_U(\varphi^*\mathcal{A}) \longrightarrow \underline{\text{Proj}}_X(\mathcal{A}) \times_X U$$

over U. Let $\alpha : t^*\varphi^*\mathcal{A} \to s^*\varphi^*\mathcal{A}$ be the canonical isomorphism of Properties of Spaces, Proposition 66.32.1. Then the diagram

$$\begin{tikzcd}
 & \underline{\text{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,s} R \arrow[r, equal] & \underline{\text{Proj}}_R(s^*\varphi^*\mathcal{A}) \arrow[dd, "\text{induced by } \alpha"'] \\
\underline{\text{Proj}}_X(\mathcal{A}) \times_X R \arrow[ur, "s^*\gamma"] \arrow[dr, "t^*\gamma"'] & & \\
 & \underline{\text{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,t} R \arrow[r, equal] & \underline{\text{Proj}}_R(t^*\varphi^*\mathcal{A})
\end{tikzcd}$$

is commutative (the equal signs come from Constructions, Lemma 27.16.10). Thus, if we denote $\mathcal{A}_U$, $\mathcal{A}_R$ the pullback of $\mathcal{A}$ to U, R, then $P = \underline{\text{Proj}}_X(\mathcal{A})$ has an étale covering by the scheme $P_U = \underline{\text{Proj}}_U(\mathcal{A}_U)$ and $P_U \times_P P_U$ is equal to $P_R = \underline{\text{Proj}}_R(\mathcal{A}_R)$. Using these remarks we can argue in the usual fashion using étale localization to transfer results on the relative proj from the case of schemes to the case of algebraic spaces.

084D Lemma 71.11.4. In Situation 71.11.1. The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_X(\mathcal{A})}(n)$ and a canonical homomorphism of graded algebras

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\text{Proj}}_X(\mathcal{A})}(n)$$

whose base change to any scheme over X agrees with Constructions, Lemma 27.15.5.

Proof. As in the discussion following Definition 71.11.3 choose a scheme U and a surjective étale morphism $U \to X$, set $R = U \times_X U$ with projections $s, t : R \to U$, $\mathcal{A}_U = \mathcal{A}|_U$, $\mathcal{A}_R = \mathcal{A}|_R$, and $\pi : P = \underline{\text{Proj}}_X(\mathcal{A}) \to X$, $\pi_U : P_U = \underline{\text{Proj}}_U(\mathcal{A}_U)$ and $\pi_R : P_R = \underline{\text{Proj}}_U(\mathcal{A}_R)$. By the Constructions, Lemma 27.15.5 we have a quasi-coherent sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{P_U}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{P_U}(n)$ and a canonical map $\psi_U : \pi_U^*\mathcal{A}_U \to \bigoplus_{n\geq 0} \mathcal{O}_{P_U}(n)$ and similarly for P_R. By Constructions, Lemma 27.16.10 the pullback of $\mathcal{O}_{P_U}(n)$ and ψ_U by either projection $P_R \to P_U$ is equal to $\mathcal{O}_{P_R}(n)$ and ψ_R. By Properties of Spaces, Proposition 66.32.1 we obtain $\mathcal{O}_P(n)$ and ψ. We omit the verification of compatibility with pullback to arbitrary schemes over X. $\square$

Having constructed the relative Proj we turn to some basic properties.

085C Lemma 71.11.5. Let S be a scheme. Let $g : X' \to X$ be a morphism of algebraic spaces over S and let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then there is a canonical isomorphism

$$r : \underline{\text{Proj}}_{X'}(g^*\mathcal{A}) \longrightarrow X' \times_X \underline{\text{Proj}}_X(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^*\text{pr}_2^* \left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_X(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_{X'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\text{Proj}}_{X'}(g^*\mathcal{A})}$-algebras.

Proof. Let F be the functor (71.11.1.1) and let F' be the corresponding functor defined using $g^*\mathcal{A}$ on X'. We claim there is a canonical isomorphism $r : F' \to X' \times_X F$ of functors (and of course r is the isomorphism of the lemma). It suffices to construct the bijection $r : F'(T) \to X'(T) \times_{X(T)} F(T)$ for quasi-compact schemes T over S. First, if $\xi = (d', f', \mathcal{L}', \psi')$ is a quadruple over T for F', then we can set $r(\xi) = (f', (d', g \circ f', \mathcal{L}', \psi'))$. This makes sense as $(g \circ f')^*\mathcal{A}^{(d)} = (f')^*(g^*\mathcal{A})^{(d)}$. The inverse map sends the pair $(f', (d, f, \mathcal{L}, \psi))$ to the quadruple $(d, f', \mathcal{L}, \psi)$. We omit the proof of the final assertion (hint: reduce to the case of schemes by étale localization and apply Constructions, Lemma 27.16.10). $\square$

084E Lemma 71.11.6. In Situation 71.11.1 the morphism $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \to X$ is separated.

Proof. By Morphisms of Spaces, Lemma 67.4.12 and the construction of the relative Proj this follows from the case of schemes which is Constructions, Lemma 27.16.9. $\square$

084F Lemma 71.11.7. In Situation 71.11.1. If one of the following holds

(1) $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,
(2) $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,
(3) there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,

then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \to X$ is quasi-compact.

Proof. By Morphisms of Spaces, Lemma 67.8.8 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.1. $\square$

084G Lemma 71.11.8. In Situation 71.11.1. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_X$-algebras, then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \to X$ is of finite type.

Proof. By Morphisms of Spaces, Lemma 67.23.4 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.2. $\square$

084H Lemma 71.11.9. In Situation 71.11.1. If $\mathcal{O}_X \to \mathcal{A}_0$ is an integral algebra map[3] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \to X$ is universally closed.

Proof. By Morphisms of Spaces, Lemma 67.9.5 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.3. $\square$

084I Lemma 71.11.10. In Situation 71.11.1. The following conditions are equivalent

(1) $\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,
(2) $\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_X$-algebra.

If these conditions hold, then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \to X$ is proper.

Proof. By Morphisms of Spaces, Lemma 67.40.2 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.3. $\square$

085D Lemma 71.11.11. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. With $P = \underline{\text{Proj}}_X(\mathcal{A})$ we have

(1) P represents the functor F_1 which associates to T over S the set of isomorphism classes of triples $(f, \mathcal{L}, \psi)$, where $f : T \to X$ is a morphism over S, $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a map of graded $\mathcal{O}_T$-algebras inducing a surjection $f^*\mathcal{A}_1 \to \mathcal{L}$,
(2) the canonical map $\pi^*\mathcal{A}_1 \to \mathcal{O}_P(1)$ is surjective, and
(3) each $\mathcal{O}_P(n)$ is invertible and the multiplication maps induce isomorphisms $\mathcal{O}_P(n) \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) = \mathcal{O}_P(n + m)$.

Proof. Omitted. See Constructions, Lemma 27.16.11 for the case of schemes. $\square$

71.12. Functoriality of relative proj

085E This section is the analogue of Constructions, Section 27.18.

085F Lemma 71.12.1. Let S be a scheme. Let X be an algebraic space over S. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a map of quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{\text{Proj}}_X(\mathcal{A}) \to X$ and $Q = \underline{\text{Proj}}_X(\mathcal{B}) \to X$. There is a canonical open subspace $U(\psi) \subset Q$ and a canonical morphism of algebraic spaces

$$r_\psi : U(\psi) \longrightarrow P$$

[3]In other words, the integral closure of $\mathcal{O}_X$ in $\mathcal{A}_0$, see Morphisms of Spaces, Definition 67.48.2, equals $\mathcal{A}_0$.

over X and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_P(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any scheme W étale over X the triple

$$(U(\psi) \times_X W, \quad r_\psi|_{U(\psi)\times_X W} : U(\psi) \times_X W \to P \times_X W, \quad \theta|_{U(\psi)\times_X W})$$

is equal to the triple associated to $\psi : \mathcal{A}|_W \to \mathcal{B}|_W$ of Constructions, Lemma 27.18.1.

Proof. This lemma follows from étale localization and the case of schemes, see discussion following Definition 71.11.3. Details omitted. $\square$

085G Lemma 71.12.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{\text{Proj}}_X(\mathcal{A})$, $Q = \underline{\text{Proj}}_X(\mathcal{B})$ and $R = \underline{\text{Proj}}_X(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_X$-algebra maps. Then we have

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^*\theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. $\square$

085H Lemma 71.12.3. With hypotheses and notation as in Lemma 71.12.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then

(1) $U(\psi) = Q$,
(2) $r_\psi : Q \to R$ is a closed immersion, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).

Proof. Follows from the case of schemes (Constructions, Lemma 27.18.3) by étale localization. $\square$

085I Lemma 71.12.4. With hypotheses and notation as in Lemma 71.12.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then

(1) $U(\psi) = Q$,
(2) $r_\psi : Q \to P$ is an isomorphism, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.

Proof. Follows from the case of schemes (Constructions, Lemma 27.18.4) by étale localization. $\square$

085J Lemma 71.12.5. With hypotheses and notation as in Lemma 71.12.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then

(1) $U(\psi) = Q$,
(2) $r_\psi : Q \to P$ is a closed immersion, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.

Proof. Follows from the case of schemes (Constructions, Lemma 27.18.5) by étale localization. $\square$

71.13. Invertible sheaves and morphisms into relative Proj

0D2Y It seems that we may need the following lemma somewhere. The situation is the following:

(1) Let S be a scheme and Y an algebraic space over S.
(2) Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_Y$-algebra.
(3) Denote $\pi : \underline{\text{Proj}}_Y(\mathcal{A}) \to Y$ the relative Proj of $\mathcal{A}$ over Y.
(4) Let $f : X \to Y$ be a morphism of algebraic spaces over S.
(5) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
(6) Let $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data let $U(\psi) \subset X$ be the open subspace with

$$|U(\psi)| = \bigcup\nolimits_{d \geq 1} \{\text{locus where } f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d} \text{ is surjective}\}$$

Formation of $U(\psi) \subset X$ commutes with pullback by any morphism $X' \to X$.

0D2Z **Lemma 71.13.1.** With assumptions and notation as above. The morphism ψ induces a canonical morphism of algebraic spaces over Y

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{\text{Proj}}_Y(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta : r_{\mathcal{L},\psi}^*\left(\bigoplus\nolimits_{d \geq 0} \mathcal{O}_{\underline{\text{Proj}}_Y(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) For $V \to Y$ étale and $d \geq 0$ the diagram

$$\begin{tikzcd} \mathcal{A}_d(V) \arrow[rr, "\psi"'] \arrow[d, "\psi"'] & & \Gamma(V \times_Y X, \mathcal{L}^{\otimes d}) \arrow[d, "restrict"] \\ \Gamma(V \times_Y \underline{\text{Proj}}_Y(\mathcal{A}), \mathcal{O}_{\underline{\text{Proj}}_Y(\mathcal{A})}(d)) \arrow[rr, "\theta"] & & \Gamma(V \times_Y U(\psi), \mathcal{L}^{\otimes d}) \end{tikzcd}$$

is commutative.

(2) For any $d \geq 1$ and any morphism $W \to X$ where W is a scheme such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective we have (a) $W \to X$ factors through $U(\psi)$ and (b) composition of $W \to U(\psi)$ with $r_{\mathcal{L},\psi}$ agrees with the morphism $W \to \underline{\text{Proj}}_Y(\mathcal{A})$ which exists by the construction of $\underline{\text{Proj}}_Y(\mathcal{A})$, see Definition 71.11.3.

(3) Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where X' and Y' are schemes, set $\mathcal{A}' = g^*\mathcal{A}$ and $\mathcal{L}' = (g')^*\mathcal{L}$ and denote $\psi' : (f')^*\mathcal{A} \to \bigoplus_{d \geq 0} (\mathcal{L}')^{\otimes d}$ the pullback of ψ. Let $U(\psi')$, $r_{\psi',\mathcal{L}'}$, and θ' be the open, morphism, and homomorphism constructed in Constructions, Lemma 71.13.1. Then $U(\psi') = (g')^{-1}(U(\psi))$ and $r_{\psi',\mathcal{L}'}$ agrees with the base change of $r_{\psi,\mathcal{L}}$ via the isomorphism $\underline{\text{Proj}}_{Y'}(\mathcal{A}') = Y' \times_Y \underline{\text{Proj}}_Y(\mathcal{A})$ of Lemma 71.11.5. Moreover, θ' is the pullback of θ.

Proof. Omitted. Hints: First we observe that for a quasi-compact scheme W over X the following are equivalent

(1) $W \to X$ factors through $U(\psi)$, and
(2) there exists a d such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective.

This gives a description of $U(\psi)$ as a subfunctor of X on our base category $(Sch/S)_{fppf}$. For such a W and d we consider the quadruple $(d, W \to Y, \mathcal{L}|_W, \psi^{(d)}|_W)$. By definition of $\underline{\text{Proj}}_Y(\mathcal{A})$ we obtain a morphism $W \to \underline{\text{Proj}}_Y(\mathcal{A})$. By our notion of equivalence of quadruples one sees that this morphism is independent of the choice of d. This clearly defines a transformation of functors $r_{\psi,\mathcal{L}} : U(\psi) \to \underline{\text{Proj}}_Y(\mathcal{A})$, i.e., a morphism of algebraic spaces. By construction this morphism satisfies (2). Since the morphism constructed in Constructions, Lemma 27.19.1 satisfies the same property, we see that (3) is true.

To construct θ and check the compatibility (1) of the lemma, work étale locally on Y and X, arguing as in the discussion following Definition 71.11.3. □

71.14. Relatively ample sheaves

0D30 This section is the analogue of Morphisms, Section 29.37 for algebraic spaces. Our definition of a relatively ample invertible sheaf is as follows.

0D31 **Definition 71.14.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/Y*, or *f-ample* if $f : X \to Y$ is representable and for every morphism $Z \to Y$ where Z is a scheme, the pullback $\mathcal{L}_Z$ of $\mathcal{L}$ to $X_Z = Z \times_Y X$ is ample on X_Z/Z as in Morphisms, Definition 29.37.1.

We will almost always reduce questions about relatively ample invertible sheaves to the case of schemes. Thus in this section we have mainly sanity checks.

0D32 **Lemma 71.14.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume Y is a scheme. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/Y in the sense of Definition 71.14.1, and*
(2) *X is a scheme and $\mathcal{L}$ is ample on X/Y in the sense of Morphisms, Definition 29.37.1.*

Proof. This follows from the definitions and Morphisms, Lemma 29.37.9 (which says that being relatively ample for schemes is preserved under base change). □

0D33 **Lemma 71.14.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y' \to Y$ be a morphism of algebraic spaces over S. Let $f' : X' \to Y'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-ample, then $\mathcal{L}'$ is f'-ample.*

Proof. This follows immediately from the definition! (Hint: transitivity of base change.) □

0D34 **Lemma 71.14.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If there exists an f-ample invertible sheaf, then f is representable, quasi-compact, and separated.*

Proof. This is clear from the definitions and Morphisms, Lemma 29.37.3. (If in doubt, take a look at the principle of Algebraic Spaces, Lemma 65.5.8.) □

0D35 **Lemma 71.14.5.** Let $V \to U$ be a surjective étale morphism of affine schemes. Let X be an algebraic space over U. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y = V \times_U X$ and let $\mathcal{N}$ be the pullback of $\mathcal{L}$ to Y. The following are equivalent

(1) $\mathcal{L}$ is ample on X/U, and
(2) $\mathcal{N}$ is ample on Y/V.

Proof. The implication (1) ⇒ (2) follows from Lemma 71.14.3. Assume (2). This implies that $Y \to V$ is quasi-compact and separated (Lemma 71.14.4) and Y is a scheme. It follows that the morphism $f : X \to U$ is quasi-compact and separated (Morphisms of Spaces, Lemmas 67.8.8 and 67.4.12). Set $\mathcal{A} = \bigoplus_{d \geq 0} f_*\mathcal{L}^{\otimes d}$. This is a quasi-coherent sheaf of graded $\mathcal{O}_U$-algebras (Morphisms of Spaces, Lemma 67.11.2). By adjunction we have a map $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$. Applying Lemma 71.13.1 we obtain an open subspace $U(\psi) \subset X$ and a morphism

$$r_{\mathcal{L},\psi} : U(\psi) \to \underline{\text{Proj}}_U(\mathcal{A})$$

Since $h : V \to U$ is étale we have $\mathcal{A}|_V = (Y \to V)_*(\bigoplus_{d \geq 0} \mathcal{N}^{\otimes d})$, see Properties of Spaces, Lemma 66.26.2. It follows that the pullback ψ' of ψ to Y is the adjunction map for the situation $(Y \to V, \mathcal{N})$ as in Morphisms, Lemma 29.37.4 part (5). Since $\mathcal{N}$ is ample on Y/V we conclude from the lemma just cited that $U(\psi') = Y$ and that $r_{\mathcal{N},\psi'}$ is an open immersion. Since Lemma 71.13.1 tells us that the formation of $r_{\mathcal{L},\psi}$ commutes with base change, we conclude that $U(\psi) = X$ and that we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "r'"'] \arrow[d] & \underline{\text{Proj}}_V(\mathcal{A}|_V) \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "r"] & \underline{\text{Proj}}_U(\mathcal{A}) \arrow[r] & U \end{tikzcd}$$

whose squares are fibre products. We conclude that r is an open immersion by Morphisms of Spaces, Lemma 67.12.1. Thus X is a scheme. Then we can apply Morphisms, Lemma 29.37.4 part (5) to conclude that $\mathcal{L}$ is ample on X/U. □

0D36 **Lemma 71.14.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{L}$ is ample on X/Y,
(2) for every scheme Z and every morphism $Z \to Y$ the algebraic space $X_Z = Z \times_Y X$ is a scheme and the pullback $\mathcal{L}_Z$ is ample on X_Z/Z,
(3) for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $X_Z = Z \times_Y X$ is a scheme and the pullback $\mathcal{L}_Z$ is ample on X_Z/Z,
(4) there exists a scheme V and a surjective étale morphism $V \to Y$ such that the algebraic space $X_V = V \times_Y X$ is a scheme and the pullback $\mathcal{L}_V$ is ample on X_V/V.

Proof. Parts (1) and (2) are equivalent by definition. The implication (2) ⇒ (3) is immediate. If (3) holds and $Z \to Y$ is as in (2), then we see that $X_Z \to Z$ is affine locally on Z representable. Hence X_Z is a scheme for example by Properties of Spaces, Lemma 66.13.1. Then it follows that $\mathcal{L}_Z$ is ample on X_Z/Z because it holds locally on Z and we can use Morphisms, Lemma 29.37.4. Thus (1), (2), and (3) are equivalent. Clearly these conditions imply (4).

Assume (4). Let $Z \to Y$ be a morphism with Z affine. Then $U = V \times_Y Z \to Z$ is a surjective étale morphism such that the pullback of $\mathcal{L}_Z$ by $X_U \to X_Z$ is relatively ample on X_U/U. Of course we may replace U by an affine open. It follows that $\mathcal{L}_Z$ is ample on X_Z/Z by Lemma 71.14.5. Thus (4) $\Rightarrow$ (3) and the proof is complete. □

0GUQ Lemma 71.14.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is quasi-affine if and only if $\mathcal{O}_X$ is f-relatively ample.

Proof. Follows from the case of schemes, see Morphisms, Lemma 29.37.6. □

71.15. Relative ampleness and cohomology

0D37 This section contains some results related to the results in Cohomology of Schemes, Sections 30.21 and 30.17.

The following lemma is just an example of what we can do.

0D38 Lemma 71.15.1. Let R be a Noetherian ring. Let X be an algebraic space over R such that the structure morphism $f : X \to \mathrm{Spec}(R)$ is proper. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{L}$ is ample on X/R (Definition 71.14.1),
(2) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Cohomology of Schemes, Lemma 30.16.1 because assumption (1) implies that X is a scheme. The implication (2) $\Rightarrow$ (1) is Cohomology of Spaces, Lemma 69.16.9. □

0D39 Lemma 71.15.2. Let Y be a Noetherian scheme. Let X be an algebraic space over Y such that the structure morphism $f : X \to Y$ is proper. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that X_y is a scheme and $\mathcal{L}_y$ is ample on X_y. Then there exists a d_0 such that for all $d \geq d_0$ we have

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = 0 \text{ for } p > 0$$

and the map

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y \longrightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$$

is surjective.

Proof. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (26.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Cohomology of Spaces, Lemma 69.11.2. Moreover, the fibres X_y and X'_y are identified. Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ and $y \in Y$ corresponds to $\mathfrak{m}$. In this case $R^pf_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ for all $p \geq 0$. Denote $f_y : X_y \to \mathrm{Spec}(\kappa)$ the projection.

Let $B = \mathrm{Gr}_\mathfrak{m}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Consider the sheaf $\mathcal{B} = f_y^*\widetilde{B}$ of quasi-coherent graded $\mathcal{O}_{X_y}$-algebras. We will use notation as in Cohomology of Spaces, Section 69.22 with I replaced by $\mathfrak{m}$. Since X_y is the closed subspace of X cut out by $\mathfrak{m}\mathcal{O}_X$ we may think of $\mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ as a coherent $\mathcal{O}_{X_y}$-module, see Cohomology of Spaces, Lemma 69.12.8. Then $\bigoplus_{n \geq 0} \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ is a quasi-coherent graded $\mathcal{B}$-module of finite type because it is generated in degree zero over $\mathcal{B}$ abd because the degree zero part is $\mathcal{F}_y = \mathcal{F}/\mathfrak{m}\mathcal{F}$ which is a coherent $\mathcal{O}_{X_y}$-module. Hence by Cohomology of Schemes, Lemma 30.19.3 part (2) there exists a d_0 such that

$$H^p(X_y, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d}) = 0$$

for all $p > 0$, $d \geq d_0$, and $n \geq 0$. By Cohomology of Spaces, Lemma 69.8.3 this is the same as the statement that $H^p(X, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$.

Consider the short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes d}$ is an exact functor and we obtain short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to 0$$

Using the long exact cohomology sequence and the vanishing above we conclude (using induction) that

(1) $H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$, and
(2) $H^0(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective for all $d \geq d_0$ and $n \geq 1$.

By the theorem on formal functions (Cohomology of Spaces, Theorem 69.22.5) we find that the $\mathfrak{m}$-adic completion of $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. Since $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is a finite A-module by Cohomology of Spaces, Lemma 69.20.3 it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. For $p = 0$ we deduce from Cohomology of Spaces, Lemma 69.22.4 part (3) that $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective, which gives the final statement of the lemma. $\square$

0D3A **Lemma 71.15.3.** (For a more general version see Descent on Spaces, Lemma 74.13.2). Let Y be a Noetherian scheme. Let X be an algebraic space over Y such that the structure morphism $f : X \to Y$ is proper. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that X_y is a scheme and $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$ (as in Definition 71.14.1).

Proof. Pick d_0 as in Lemma 71.15.2 for $\mathcal{F} = \mathcal{O}_X$. Pick $d \geq d_0$ so that we can find $r \geq 0$ and sections $s_{y,0}, \ldots, s_{y,r} \in H^0(X_y, \mathcal{L}_y^{\otimes d})$ which define a closed immersion

$$\varphi_y = \varphi_{\mathcal{L}_y^{\otimes d}, (s_{y,0}, \ldots, s_{y,r})} : X_y \to \mathbf{P}^r_{\kappa(y)}.$$

This is possible by Morphisms, Lemma 29.39.4 but we also use Morphisms, Lemma 29.41.7 to see that φ_y is a closed immersion and Constructions, Section 27.13 for the description of morphisms into projective space in terms of invertible sheaves and sections. By our choice of d_0, after replacing Y by an open neighbourhood of y, we can choose $s_0, \ldots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ mapping to $s_{y,0}, \ldots, s_{y,r}$. Let $X_{s_i} \subset X$ be the open subspace where s_i is a generator of $\mathcal{L}^{\otimes d}$. Since the $s_{y,i}$ generate $\mathcal{L}_y^{\otimes d}$ we see that $|X_y| \subset U = \bigcup |X_{s_i}|$. Since $X \to Y$ is closed, we see that there is an open neighbourhood $y \in V \subset Y$ such that $|f|^{-1}(V) \subset U$. After replacing Y by V we may assume that the s_i generate $\mathcal{L}^{\otimes d}$. Thus we obtain a morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes d}, (s_0, \ldots, s_r)} : X \longrightarrow \mathbf{P}^r_Y$$

with $\mathcal{L}^{\otimes d} \cong \varphi^*\mathcal{O}_{\mathbf{P}^r_Y}(1)$ whose base change to y gives φ_y (strictly speaking we need to write out a proof that the construction of morphisms into projective space given in Constructions, Section 27.13 also works to describe morphisms of algebraic spaces into projective space; we omit the details).

We will finish the proof by a sleight of hand; the "correct" proof proceeds by directly showing that φ is a closed immersion after base changing to an open neighbourhood of y. Namely, by Cohomology of Spaces, Lemma 69.23.2 we see that φ is a finite over an open neighbourhood of the fibre $\mathbf{P}^r_{\kappa(y)}$ of $\mathbf{P}^r_Y \to Y$ above y. Using that $\mathbf{P}^r_Y \to Y$ is closed, after shrinking Y we may assume that φ is finite. In particular X is a scheme. Then $\mathcal{L}^{\otimes d} \cong \varphi^*\mathcal{O}_{\mathbf{P}^r_Y}(1)$ is ample by the very general Morphisms, Lemma 29.37.7. □

71.16. Closed subspaces of relative proj

085K Some auxiliary lemmas about closed subspaces of relative proj. This section is the analogue of Divisors, Section 31.31.

085L Lemma 71.16.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{\text{Proj}}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subspace. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} \pi_* \left((i_*\mathcal{O}_Z)(d)\right)$$

If π is quasi-compact, then there is an isomorphism $Z = \underline{\text{Proj}}_X(\mathcal{A}/\mathcal{I})$.

Proof. The morphism π is separated by Lemma 71.11.6. As π is quasi-compact, π_* transforms quasi-coherent modules into quasi-coherent modules, see Morphisms of Spaces, Lemma 67.11.2. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra. The functoriality morphism $Z' = \underline{\text{Proj}}_X(\mathcal{B}) \to \underline{\text{Proj}}_X(\mathcal{A})$ is everywhere defined and a closed immersion, see Lemma 71.12.3. Hence it suffices to prove $Z = Z'$ as closed subspaces of P.

Having said this, the question is étale local on the base and we reduce to the case of schemes (Divisors, Lemma 31.31.1) by étale localization. □

In case the closed subspace is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

085M Lemma 71.16.2. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{\text{Proj}}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subscheme. If π is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{\text{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$.

Proof. The reader can redo the arguments used in the case of schemes. However, we will show the lemma follows from the case of schemes by a trick. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 71.16.1. Choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 66.6.3. By the case of schemes (Divisors, Lemma 31.31.4) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_U$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_U \subset \mathcal{A}_d|_U$ such that $Z \times_X U$ is equal to $\underline{\text{Proj}}_U(\mathcal{A}|_U/\mathcal{F}'\mathcal{A}|_U)$. By Limits of Spaces, Lemma 70.9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_U$. Let

$Z' = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 71.12.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X U \subset Z \times_X U$ by our choice of $\mathcal{F}$. Thus $Z = Z'$ as desired. $\square$

085N Lemma 71.16.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subspace. Let $U \subset X$ be an open. Assume that

(1) π is quasi-compact,
(2) i of finite presentation,
(3) $|U| \cap |\pi|(|i|(|Z|)) = \emptyset$,
(4) U is quasi-compact,
(5) $\mathcal{A}_n$ is a finite type $\mathcal{O}_X$-module for all n.

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. We use the same trick as in the proof of Lemma 71.16.2 to reduce to the case of schemes. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 71.16.1. Choose an affine scheme W and a surjective étale morphism $W \to X$, see Properties of Spaces, Lemma 66.6.3. By the case of schemes (Divisors, Lemma 31.31.5) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_W$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_W \subset \mathcal{A}_d|_W$ such that (a) $Z \times_X W$ is equal to $\underline{\mathrm{Proj}}_W(\mathcal{A}|_W/\mathcal{F}'\mathcal{A}|_W)$ and (b) the support of $\mathcal{A}_d|_W/\mathcal{F}'$ is disjoint from $U \times_X W$. By Limits of Spaces, Lemma 70.9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_W$. Let $Z' = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 71.12.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X W \subset Z \times_X W$ by our choice of $\mathcal{F}$. Thus $Z = Z'$. Finally, we see that $\mathcal{A}_d/\mathcal{F}$ is supported on $X \setminus U$ as $\mathcal{A}_d|_W/\mathcal{F}|_W$ is a quotient of $\mathcal{A}_d|_W/\mathcal{F}'$ which is supported on $W \setminus U \times_X W$. Thus the lemma follows. $\square$

0B4I Lemma 71.16.4. Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. There is a bijection

$$\left\{\begin{matrix}\text{sections } \sigma \text{ of the}\\ \text{morphism } \mathbf{P}(\mathcal{E}) \to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{surjections } \mathcal{E} \to \mathcal{L} \text{ where}\\ \mathcal{L} \text{ is an invertible } \mathcal{O}_X\text{-module}\end{matrix}\right\}$$

In this case σ is a closed immersion and there is a canonical isomorphism

$$\mathrm{Ker}(\mathcal{E} \to \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \longrightarrow \mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}$$

Both the bijection and isomorphism are compatible with base change.

Proof. Because the constructions are compatible with base change, it suffices to check the statement étale locally on X. Thus we may assume X is a scheme and the result is Divisors, Lemma 31.31.6. $\square$

71.17. Blowing up

085P Blowing up is an important tool in algebraic geometry.

085Q Definition 71.17.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subspace corresponding to $\mathcal{I}$ (Morphisms of Spaces, Lemma 67.13.1). The blowing up of X along Z, or the blowing up of X in the ideal sheaf $\mathcal{I}$ is the morphism

$$b : \underline{\text{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

The exceptional divisor of the blowup is the inverse image $b^{-1}(Z)$. Sometimes Z is called the center of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" algebraic space over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blowup of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Lemma 71.11.11.

085R Lemma 71.17.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \text{Spec}(A)$ be an affine scheme étale over X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $X' \to X$ is the blowup of X in $\mathcal{I}$, then there is a canonical isomorphism

$$U \times_X X' = \text{Proj}(\bigoplus\nolimits_{d \geq 0} I^d)$$

of schemes over U, where the right hand side is the homogeneous spectrum of the Rees algebra of I in A. Moreover, $U \times_X X'$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. Note that the restriction $\mathcal{I}|_U$ is equal to the pullback of $\mathcal{I}$ via the morphism $U \to X$, see Properties of Spaces, Section 66.26. Thus the lemma follows on combining Lemma 71.11.2 with Divisors, Lemma 31.32.2. $\square$

085S Lemma 71.17.3. Let S be a scheme. Let $X_1 \to X_2$ be a flat morphism of algebraic spaces over S. Let $Z_2 \subset X_2$ be a closed subspace. Let Z_1 be the inverse image of Z_2 in X_1. Let X'_i be the blowup of Z_i in X_i. Then there exists a cartesian diagram

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X'_2 \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

of algebraic spaces over S.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (see Morphisms of Spaces, Definition 67.13.2 and discussion following the definition). By Lemma 71.11.5 we see that $X_1 \times_{X_2} X'_2$ is the relative Proj of $\bigoplus_{n \geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X'_2 = X'_1$. $\square$

085T Lemma 71.17.4. Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The blowing up $b : X' \to X$ of Z in X has the following properties:

(1) $b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,

(2) the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',
(3) there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 71.17.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 31.32.4. $\square$

085U Lemma 71.17.5 (Universal property blowing up). Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $\mathcal{C}$ be the full subcategory of (Spaces/X) consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 71.17.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By Lemma 71.11.11. the triple $(f : Y \to X, \mathcal{I}_D, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 71.6.4. Thus the morphism $Y \to X'$ is unique by Morphisms of Spaces, Lemma 67.17.8 (also b is separated by Lemma 71.11.6). $\square$

085V Lemma 71.17.6. Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.

Proof. Immediate from the universal property of blowups (Lemma 71.17.5). $\square$

085W Lemma 71.17.7. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X in $\mathcal{I}$ is reduced.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 71.17.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 31.32.8. $\square$

0BH1 Lemma 71.17.8. Let S be a scheme. Let X be an algebraic space over S. Let $b : X' \to X$ be the blowup of X in a closed subspace. If X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1 then so does X'.

Proof. Follows immediately from the lemma cited in the statement, the étale local description of blowing ups in Lemma 71.17.2, and Divisors, Lemma 31.32.10. $\square$

085X Lemma 71.17.9. Let S be a scheme. Let X be an algebraic space over S. Let $b : X' \to X$ be a blowup of X in a closed subspace. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 71.6.10).

Proof. By Lemmas 71.17.2 and 71.6.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. □

085Y Lemma 71.17.10. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J}$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 71.17.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 71.17.9. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 71.17.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 71.6.8. Thus a morphism $c' : Y \to X'$ over X by Lemma 71.17.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. □

085Z Lemma 71.17.11. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then $b : X' \to X$ is a proper morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 71.17.3) we can prove each of these statements after base change to U (see Morphisms of Spaces, Lemma 67.40.2). This reduces us to the case of schemes. In this case the morphism b is projective by Divisors, Lemma 31.32.13 hence proper by Morphisms, Lemma 29.43.5. □

0860 Lemma 71.17.12. Let S be a scheme and let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Let $Z \subset X$ be a closed subspace of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subspace of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subspace $Y \subset X$ of finite presentation, such that

(1) $|Y| = |Z| \cup |b|(|Z'|)$, and
(2) the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms of Spaces, Lemma 67.28.12. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\text{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open subspace of X by Limits of Spaces, Lemma 70.14.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 71.17.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open subspace in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open subspace in X. By Lemma 71.16.3 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $|X| \setminus |U|$ we see that $|V(\mathcal{J})|$ is contained in $|X| \setminus |U|$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $|Z| \subset |V(\mathcal{J})|$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $|V(\mathcal{J})|$ equals $|Z| \cup |b|(|Z'|)$. It follows that also $|V(\mathcal{I}\mathcal{J})| = |Z| \cup |b|(|Z'|)$. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blowup of X in $\mathcal{I}\mathcal{J}$ is the same as the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 71.17.10. Hence it suffices to show that the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blowup of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d\mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 71.17.6 and 71.17.10.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 71.17.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \text{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. $\square$

71.18. Strict transform

0861 This section is the analogue of Divisors, Section 31.33. Let S be a scheme, let B be an algebraic space over S, and let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of B in Z and denote $E \subset B'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider an algebraic space X over B and form the cartesian diagram

$$\begin{tikzcd} \text{pr}_{B'}^{-1}E \arrow[r] \arrow[d] & X \times_B B' \arrow[r, "\text{pr}_X"'] \arrow[d, "\text{pr}_{B'}"'] & X \arrow[d, "f"] \\ E \arrow[r] & B' \arrow[r] & B \end{tikzcd}$$

Since E is an effective Cartier divisor (Lemma 71.17.4) we see that $\text{pr}_{B'}^{-1}E \subset X \times_B B'$ is locally principal (Lemma 71.6.9). Thus the inclusion morphism of the complement of $\text{pr}_{B'}^{-1}E$ in $X \times_B B'$ is affine and in particular quasi-compact (Lemma 71.6.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_B B'}$-module $\mathcal{G}$ the subsheaf of sections supported on $|\text{pr}_{B'}^{-1}E|$ is a quasi-coherent submodule, see Limits of Spaces, Definition 70.14.6. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_B B'}$, then this subsheaf is an ideal of $\mathcal{G}$.

0862 Definition 71.18.1. With $Z \subset B$ and $f : X \to B$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform of $\mathcal{F}$ with respect to the blowup of B in Z is the quotient $\mathcal{F}'$ of $\text{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $|\text{pr}_{B'}^{-1}E|$.
(2) The strict transform of X is the closed subspace $X' \subset X \times_B B'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_B B'}$ supported on $|\text{pr}_{B'}^{-1}E|$.

Note that taking the strict transform along a blowup depends on the closed subspace used for the blowup (and not just on the morphism $B' \to B$).

0863 Lemma 71.18.2 (Étale localization and strict transform). In the situation of Definition 71.18.1. Let

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & B \end{tikzcd}$$

be a commutative diagram of morphisms with U and V schemes and étale horizontal arrows. Let $V' \to V$ be the blowup of V in $Z \times_B V$. Then

(1) $V' = V \times_B B'$ and the maps $V' \to B'$ and $U \times_V V' \to X \times_B B'$ are étale,
(2) the strict transform U' of U relative to $V' \to V$ is equal to $X' \times_X U$ where X' is the strict transform of X relative to $B' \to B$, and
(3) for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the restriction of the strict transform $\mathcal{F}'$ to $U \times_V V'$ is the strict transform of $\mathcal{F}|_U$ relative to $V' \to V$.

Proof. Part (1) follows from the fact that blowup commutes with flat base change (Lemma 71.17.3), the fact that étale morphisms are flat, and that the base change of an étale morphism is étale. Part (3) then follows from the fact that taking the sheaf of sections supported on a closed commutes with pullback by étale morphisms, see Limits of Spaces, Lemma 70.14.5. Part (2) follows from (3) applied to $\mathcal{F} = \mathcal{O}_X$. $\square$

0864 Lemma 71.18.3. In the situation of Definition 71.18.1.

(1) The strict transform X' of X is the blowup of X in the closed subspace $f^{-1}Z$ of X.
(2) For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_B B'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 71.17.5) there exists a commutative diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

whence a morphism $i : X'' \to X \times_B B'$. The first assertion of the lemma is that i is a closed immersion with image X'. The second assertion of the lemma is that $\mathcal{F}' = i_*\mathcal{F}''$ where $\mathcal{F}''$ is the strict transform of $\mathcal{F}$ with respect to the blowing up $X'' \to X$. We can check these assertions étale locally on X, hence we reduce to the case of schemes (Divisors, Lemma 31.33.2). Some details omitted. $\square$

0865 Lemma 71.18.4. In the situation of Definition 71.18.1.

(1) If X is flat over B at all points lying over Z, then the strict transform of X is equal to the base change $X \times_B B'$.
(2) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over B at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $\text{pr}_X^*\mathcal{F}$.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.3) by étale localization (Lemma 71.18.2). $\square$

0866 Lemma 71.18.5. Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of Z in B. Let $g : X \to Y$ be an affine morphism of spaces over B. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_B B' \to Y \times_B B'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.4) by étale localization (Lemma 71.18.2). $\square$

0867 Lemma 71.18.6. Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $D \subset B$ be an effective Cartier divisor. Let $Z' \subset B$ be the closed subspace cut out by the product of the ideal sheaves of Z and D. Let $B' \to B$ be the blowup of B in Z.

(1) The blowup of B in Z' is isomorphic to $B' \to B$.
(2) Let $f : X \to B$ be a morphism of algebraic spaces and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If the subsheaf of $\mathcal{F}$ of sections supported on $|f^{-1}D|$ is zero, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of B in Z'.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.5) by étale localization (Lemma 71.18.2). $\square$

0868 Lemma 71.18.7. Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up with center Z. Let $Z' \subset B'$ be a closed subspace. Let $B'' \to B'$ be the blowing up with center Z'. Let $Y \subset B$ be a closed subscheme such that $|Y| = |Z| \cup |b|(|Z'|)$ and the composition $B'' \to B$ is isomorphic to the blowing up of B in Y. In this situation, given any scheme X over B and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have

(1) the strict transform of $\mathcal{F}$ with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $B' \to B$ of B in Z, and
(2) the strict transform of X with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of X with respect to the blowup $B' \to B$ of B in Z.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.6) by étale localization (Lemma 71.18.2). $\square$

0869 Lemma 71.18.8. In the situation of Definition 71.18.1. Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to B$. Then the strict transforms of $\mathcal{F}_i'$ relative to any blowup $B' \to B$ form a short exact sequence $0 \to \mathcal{F}_1' \to \mathcal{F}_2' \to \mathcal{F}_3' \to 0$ too.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.7) by étale localization (Lemma 71.18.2). $\square$

0D0P Lemma 71.18.9. Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_B$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $\text{Fit}_k(\mathcal{F})$, see Section 71.5. Let $B' \to B$ be the blowup of B in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ can locally be generated by $\leq k$ sections.

Proof. Omitted. Follows from the case of schemes (Divisors, Lemma 31.35.1) by étale localization (Lemma 71.18.2). $\square$

0D0Q Lemma 71.18.10. Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_B$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $\text{Fit}_k(\mathcal{F})$, see Section 71.5. Assume that $\mathcal{F}$ is locally free of rank k on $B \setminus Z_k$. Let $B' \to B$ be the blowup of B in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ is locally free of rank k.

Proof. Omitted. Follows from the case of schemes (Divisors, Lemma 31.35.2) by étale localization (Lemma 71.18.2). $\square$

71.19. Admissible blowups

086A To have a bit more control over our blowups we introduce the following standard terminology.

086B Definition 71.19.1. Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. A morphism $X' \to X$ is called a U-admissible blowup if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blowup of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms of Spaces, Lemma 67.28.12. In particular, a U-admissible blowup is a proper morphism, see Lemma 71.17.11. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 71.17.4) we will often abuse notation and think of U as an open subspace of X' as well.

086C Lemma 71.19.2. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.

Proof. Immediate from the more precise Lemma 71.17.12. $\square$

086D Lemma 71.19.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U, V \subset X$ be quasi-compact open subspaces. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blowup of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$. By Limits of Spaces, Lemma 70.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. □

086E Lemma 71.19.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blowup of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 71.17.10. □

086F Lemma 71.19.5. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let U, V be quasi-compact disjoint open subspaces of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Limits of Spaces, Lemma 70.14.1. Then $|V(\mathcal{I}\mathcal{J})| = |X|$. Hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I}+\mathcal{J}$. This is $U \cup V$-admissible as $|V(\mathcal{I}+\mathcal{J})| = |X| \setminus |U| \cup |V|$. We will show that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ as in the statement of the lemma.

Since $|V(\mathcal{I} + \mathcal{J})|$ is the complement of $|U \cup V|$ we conclude that $V \cup U$ is scheme theoretically dense in X', see Lemmas 71.17.4 and 71.6.4. Thus if such a decomposition $X' = X'_1 \amalg X'_2$ into open and closed subspaces exists, then X'_1 is the scheme theoretic closure of U in X' and similarly X'_2 is the scheme theoretic closure of V in X'. Since $U \to X'$ and $V \to X'$ are quasi-compact taking scheme theoretic closures commutes with étale localization (Morphisms of Spaces, Lemma 67.16.3). Hence to verify the existence of X'_1 and X'_2 we may work étale locally on X. This reduces us to the case of schemes which is treated in the proof of Divisors, Lemma 31.34.5. □

71.20. Other chapters

CHAPTER 72

Algebraic Spaces over Fields

06DR

72.1. Introduction

06DS This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [Knu71].

72.2. Conventions

06LX The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

72.3. Generically finite morphisms

0ACY This section continues the discussion in Decent Spaces, Section 68.21 and the analogue for morphisms of algebraic spaces of Varieties, Section 33.17.

0AD1 Lemma 72.3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is locally Noetherian. Let $y \in |Y|$ be a point of codimension ≤ 1 on Y. Let $X^0 \subset |X|$ be the set of points of codimension 0 on X. Assume in addition one of the following conditions is satisfied

(1) for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,
(2) for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,
(3) f is quasi-finite at every $x \in X^0$,
(4) f is quasi-finite at a dense set of points of $|X|$,
(5) add more here.

Then f is quasi-finite at every point of X lying over y.

Proof. We want to reduce the proof to the case of schemes. To do this we choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and where the horizontal arrows are étale and surjective. Pick $v \in V$ mapping to y. Observe that V is locally Noetherian and that $\dim(\mathcal{O}_{V,v}) \leq 1$ (see Properties of Spaces, Definitions 66.10.2 and Remark 66.7.3). The fibre U_v of $U \to V$ over v surjects onto $f^{-1}(\{y\}) \subset |X|$. The inverse image of

X^0 in U is exactly the set of generic points of irreducible components of U (Properties of Spaces, Lemma 66.11.1). If $\eta \in U$ is such a point with image $x \in X^0$, then the transcendence degree of $x/f(x)$ is the transcendence degree of $\kappa(\eta)$ over $\kappa(g(\eta))$ (Morphisms of Spaces, Definition 67.33.1). Observe that $U \to V$ is quasi-finite at $u \in U$ if and only if f is quasi-finite at the image of u in X.

Case (1). Here case (1) of Varieties, Lemma 33.17.1 applies and we conclude that $U \to V$ is quasi-finite at all points of U_v. Hence f is quasi-finite at every point lying over y.

Case (2). Let $u \in U$ be a generic point of an irreducible component whose image in V specializes to v. Then the image $x \in X^0$ of u has the property that $f(x) \rightsquigarrow y$. Hence we see that case (2) of Varieties, Lemma 33.17.1 applies and we conclude as before.

Case (3) follows from case (3) of Varieties, Lemma 33.17.1.

In case (4), since $|U| \to |X|$ is open, we see that the set of points where $U \to V$ is quasi-finite is dense as well. Hence case (4) of Varieties, Lemma 33.17.1 applies. $\square$

0AD2 Lemma 72.3.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper and Y is locally Noetherian. Let $y \in Y$ be a point of codimension ≤ 1 in Y. Let $X^0 \subset |X|$ be the set of points of codimension 0 on X. Assume in addition one of the following conditions is satisfied

(1) for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,
(2) for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,
(3) f is quasi-finite at every $x \in X^0$,
(4) f is quasi-finite at a dense set of points of $|X|$,
(5) add more here.

Then there exists an open subspace $Y' \subset Y$ containing y such that $Y' \times_Y X \to Y'$ is finite.

Proof. By Lemma 72.3.1 the morphism f is quasi-finite at every point lying over y. Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point lying over y. Then $|X_{\overline{y}}|$ is a discrete space (Decent Spaces, Lemma 68.18.10). Since $X_{\overline{y}}$ is quasi-compact as f is proper we conclude that $|X_{\overline{y}}|$ is finite. Thus we can apply Cohomology of Spaces, Lemma 69.23.2 to conclude. $\square$

0BBQ Lemma 72.3.3. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $f : Y \to X$ be a birational proper morphism of algebraic spaces with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains

(1) every point of codimension 0 in X,
(2) every $x \in |X|$ of codimension 1 on X such that the local ring of X at x is normal (Properties of Spaces, Remark 66.7.6), and
(3) every $x \in |X|$ such that the fibre of $|Y| \to |X|$ over x is finite and such that the local ring of X at x is normal.

Proof. Part (1) follows from Decent Spaces, Lemma 68.22.5 (and the fact that the Noetherian algebraic spaces X and Y are quasi-separated and hence decent). Part (2) follows from part (3) and Lemma 72.3.2 (and the fact that finite morphisms have finite fibres). Let $x \in |X|$ be as in (3). By Cohomology of Spaces, Lemma 69.23.2 (which applies by Decent Spaces, Lemma 68.18.10) we may assume f is finite.

Choose an affine scheme X' and an étale morphism $X' \to X$ and a point $x' \in X$ mapping to x. It suffices to show there exists an open neighbourhood U' of $x' \in X'$ such that $Y \times_X X' \to X'$ is an isomorphism over U' (namely, then U contains the image of U' in X, see Spaces, Lemma 65.5.6). Then $Y \times_X X' \to X$ is a finite birational (Decent Spaces, Lemma 68.22.6) morphism. Since a finite morphism is affine we reduce to the case of a finite birational morphism of Noetherian affine schemes $Y \to X$ and $x \in X$ such that $\mathcal{O}_{X,x}$ is a normal domain. This is treated in Varieties, Lemma 33.17.3. □

72.4. Integral algebraic spaces

0AD3 We have not yet defined the notion of an integral algebraic space. The problem is that being integral is not an étale local property of schemes. We could use the property, that X is reduced and $|X|$ is irreducible, given in Properties, Lemma 28.3.4 to define integral algebraic spaces. In this case the algebraic space described in Spaces, Example 65.14.9 would be integral which does not seem right. To avoid this type of pathology we will in addition assume that X is a decent algebraic space, although perhaps a weaker alternative exists.

0AD4 **Definition 72.4.1.** Let S be a scheme. We say an algebraic space X over S is *integral* if it is reduced, decent, and $|X|$ is irreducible.

In this case the irreducible topological space $|X|$ is sober (Decent Spaces, Proposition 68.12.4). Hence it has a unique generic point x. In fact, in Decent Spaces, Lemma 68.20.4 we characterized decent algebraic spaces with finitely many irreducible components. Applying that lemma we see that an algebraic space X is integral if it is reduced, has an irreducible dense open subscheme X' with generic point x' and the morphism $x' \to X$ is quasi-compact.

0END **Lemma 72.4.2.** Let S be a scheme. Let X be an integral algebraic space over S. Let $\eta \in |X|$ be the generic point of X. There are canonical identifications

$$R(X) = \mathcal{O}^h_{X,\eta} = \kappa(\eta)$$

where $R(X)$ is the ring of rational functions defined in Morphisms of Spaces, Definition 67.47.3, $\kappa(\eta)$ is the residue field defined in Decent Spaces, Definition 68.11.2, and $\mathcal{O}^h_{X,\eta}$ is the henselian local ring defined in Decent Spaces, Definition 68.11.5. In particular, these rings are fields.

Proof. Since X is a scheme in an open neighbourhood of η (see discussion above), this follows immediately from the corresponding result for schemes, see Morphisms, Lemma 29.49.5. We also use: the henselianization of a field is itself and that our definitions of these objects for algebraic spaces are compatible with those for schemes. Details omitted. □

This leads to the following definition.

0ENE **Definition 72.4.3.** Let S be a scheme. Let X be an integral algebraic space over S. The *function field*, or the *field of rational functions* of X is the field $R(X)$ of Lemma 72.4.2.

We may occasionally indicate this field $k(X)$ instead of $R(X)$.

0BH2 **Lemma 72.4.4.** Let S be a scheme. Let X be an integral algebraic space over S. Then $\Gamma(X, \mathcal{O}_X)$ is a domain.

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. If $f, g \in R$ are nonzero and $fg = 0$ then $X = V(f) \cup V(g)$ where $V(f)$ denotes the closed subspace of X cut out by f. Since X is irreducible, we see that either $V(f) = X$ or $V(g) = X$. Then either $f = 0$ or $g = 0$ by Properties of Spaces, Lemma 66.21.4. $\square$

Here is a lemma about normal integral algebraic spaces.

0AYH Lemma 72.4.5. Let S be a scheme. Let X be a normal integral algebraic space over S. For every $x \in |X|$ there exists a normal integral affine scheme U and an étale morphism $U \to X$ such that x is in the image.

Proof. Choose an affine scheme U and an étale morphism $U \to X$ such that x is in the image. Let u_i, $i \in I$ be the generic points of irreducible components of U. Then each u_i maps to the generic point of X (Decent Spaces, Lemma 68.20.1). By our definition of a decent space (Decent Spaces, Definition 68.6.1), we see that I is finite. Hence $U = \mathrm{Spec}(A)$ where A is a normal ring with finitely many minimal primes. Thus $A = \prod_{i \in I} A_i$ is a product of normal domains by Algebra, Lemma 10.37.16. Then $U = \coprod U_i$ with $U_i = \mathrm{Spec}(A_i)$ and x is in the image of $U_i \to X$ for some i. This proves the lemma. $\square$

0BH3 Lemma 72.4.6. Let S be a scheme. Let X be a normal integral algebraic space over S. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Then R is a domain by Lemma 72.4.4. Let $f = a/b$ be an element of the fraction field of R which is integral over R. For any $U \to X$ étale with U a scheme there is at most one $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$. Namely, U is reduced and the generic points of U map to the generic point of X which implies that $b|_U$ is a nonzerodivisor. For every $x \in |X|$ we choose $U \to X$ as in Lemma 72.4.5. Then there is a unique $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$ because $\Gamma(U, \mathcal{O}_U)$ is a normal domain by Properties, Lemma 28.7.9. By the uniqueness mentioned above these f_U glue and define a global section f of the structure sheaf, i.e., of R. $\square$

0ENF Lemma 72.4.7. Let S be a scheme. Let X be a decent algebraic space over S. There are canonical bijections between the following sets:

(1) the set of points of X, i.e., $|X|$,
(2) the set of irreducible closed subsets of $|X|$,
(3) the set of integral closed subspaces of X.

The bijection from (1) to (2) sends x to $\overline{\{x\}}$. The bijection from (3) to (2) sends Z to $|Z|$.

Proof. Our map defines a bijection between (1) and (2) as $|X|$ is sober by Decent Spaces, Proposition 68.12.4. Given $T \subset |X|$ closed and irreducible, there is a unique reduced closed subspace $Z \subset X$ such that $|Z| = T$, namely, Z is the reduced induced subspace structure on T, see Properties of Spaces, Definition 66.12.5. This is an integral algebraic space because it is decent, reduced, and irreducible. $\square$

72.5. Morphisms between integral algebraic spaces

0ENG The following lemma characterizes dominant morphisms of finite degree between integral algebraic spaces.

0AD5 Lemma 72.5.1. Let S be a scheme. Let X, Y be integral algebraic spaces over S Let $x \in |X|$ and $y \in |Y|$ be the generic points. Let $f : X \to Y$ be locally of finite type. Assume f is dominant (Morphisms of Spaces, Definition 67.18.1). The following are equivalent:

(1) the transcendence degree of x/y is 0,
(2) the extension $\kappa(x)/\kappa(y)$ (see proof) is finite,
(3) there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite,
(4) f is quasi-finite at x, and
(5) x is the only point of $|X|$ mapping to y.

If f is separated or if f is quasi-compact, then these are also equivalent to

(6) there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. By elementary topology, we see that $f(x) = y$ as f is dominant. Let $Y' \subset Y$ be the schematic locus of Y and let $X' \subset f^{-1}(Y')$ be the schematic locus of $f^{-1}(Y')$. By the discussion above, using Decent Spaces, Proposition 68.12.4 and Theorem 68.10.2, we see that $x \in |X'|$ and $y \in |Y'|$. Then $f|_{X'} : X' \to Y'$ is a morphism of integral schemes which is locally of finite type. Thus we see that (1), (2), (3) are equivalent by Morphisms, Lemma 29.51.7.

Condition (4) implies condition (1) by Morphisms of Spaces, Lemma 67.33.3 applied to $X \to Y \to Y$. On the other hand, condition (3) implies condition (4) as a finite morphism is quasi-finite and as $x \in U$ because x is the generic point. Thus (1) – (4) are equivalent.

Assume the equivalent conditions (1) – (4). Suppose that $x' \mapsto y$. Then $x \rightsquigarrow x'$ is a specialization in the fibre of $|X| \to |Y|$ over y. If $x' \neq x$, then f is not quasi-finite at x by Decent Spaces, Lemma 68.18.9. Hence $x = x'$ and (5) holds. Conversely, if (5) holds, then (5) holds for the morphism of schemes $X' \to Y'$ (see above) and we can use Morphisms, Lemma 29.51.7 to see that (1) holds.

Observe that (6) implies the equivalent conditions (1) – (5) without any further assumptions on f. To finish the proof we have to show the equivalent conditions (1) – (5) imply (6). This follows from Decent Spaces, Lemma 68.21.4. $\square$

0AD6 Definition 72.5.2. Let S be a scheme. Let X and Y be integral algebraic spaces over S. Let $f : X \to Y$ be locally of finite type and dominant. Assume any of the equivalent conditions (1) – (5) of Lemma 72.5.1. Let $x \in |X|$ and $y \in |Y|$ be the generic points. Then the positive integer

$$\deg(X/Y) = [\kappa(x) : \kappa(y)]$$

is called the degree of X over Y.

0ENH Lemma 72.5.3. Let S be a scheme. Let X, Y, Z be integral algebraic spaces over S. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume any of the equivalent conditions (1) – (5) of Lemma 72.5.1 hold for f and g. Then

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 9.7.7. $\square$

72.6. Weil divisors

0ENI This section is the analogue of Divisors, Section 31.26.

We will introduce Weil divisors and rational equivalence of Weil divisors for locally Noetherian integral algebraic spaces. Since we are not assuming our algebraic spaces are quasi-compact we have to be a little careful when defining Weil divisors. We have to allow infinite sums of prime divisors because a rational function may have infinitely many poles for example. In the quasi-compact case our Weil divisors are finite sums as usual. Here is a basic lemma we will often use to prove collections of closed subspaces are locally finite.

0EE5 Lemma 72.6.1. Let S be a scheme and let X be a locally Noetherian algebraic space over S. If $T \subset |X|$ is a closed subset, then the collection of irreducible components of T is locally finite.

Proof. The topological space $|X|$ is locally Noetherian (Properties of Spaces, Lemma 66.24.2). A Noetherian topological space has a finite number of irreducible components and a subspace of a Noetherian space is Noetherian (Topology, Lemma 5.9.2). Thus the lemma follows from the definition of locally finite (Topology, Definition 5.28.4). $\square$

Let S be a scheme. Let X be a decent algebraic space over S. Let Z be an integral closed subspace of X and let $\xi \in |Z|$ be the generic point. Then the codimension of $|Z|$ in $|X|$ is equal to the dimension of the local ring of X at ξ by Decent Spaces, Lemma 68.20.2. Recall that we also indicate this by saying that ξ is a point of codimension 1 on X, see Properties of Spaces, Definition 66.10.2.

0ENJ Definition 72.6.2. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S.

(1) A prime divisor is an integral closed subspace $Z \subset X$ of codimension 1, i.e., the generic point of $|Z|$ is a point of codimension 1 on X.
(2) A Weil divisor is a formal sum $D = \sum n_Z Z$ where the sum is over prime divisors of X and the collection $\{|Z| : n_Z \neq 0\}$ is locally finite in $|X|$ (Topology, Definition 5.28.4).

The group of all Weil divisors on X is denoted $\text{Div}(X)$.

Our next task is to define the Weil divisor associated to a rational function. In order to do this we need to define the order of vanishing of a rational function on a locally Noetherian integral algebraic space X along a prime divisor Z. Let $\xi \in |Z|$ be the generic point. Here we run into the problem that the local ring $\mathcal{O}_{X,\xi}$ doesn't exist and the henselian local ring $\mathcal{O}^h_{X,\xi}$ may not be a domain, see Example 72.6.11. To get around this we use the following lemma.

0ENK Lemma 72.6.3. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $Z \subset X$ be a prime divisor and let $\xi \in |Z|$ be the generic point. Then the henselian local ring $\mathcal{O}^h_{X,\xi}$ is a reduced 1-dimensional Noetherian local ring and there is a canonical injective map

$$R(X) \longrightarrow Q(\mathcal{O}^h_{X,\xi})$$

from the function field $R(X)$ of X into the total ring of fractions.

Proof. We will use the results of Decent Spaces, Section 68.11. Let $(U, u) \to (X, \xi)$ be an elementary étale neighbourhood. Observe that U is locally Noetherian and reduced. Thus $\mathcal{O}_{U,u}$ is a 1-dimensional (by our definition of prime divisors) reduced Noetherian ring. After replacing U by an affine open neighbourhood of u we may assume U is Noetherian and affine. After replacing U by a smaller open, we may assume every irreducible component of U passes through u. Since $U \to X$ is open and X irreducible, $U \to X$ is dominant. Hence we obtain a ring map $R(X) \to R(U)$ by composing rational maps, see Morphisms of Spaces, Section 67.47. Since $R(X)$ is a field, this map is injective. By our choice of U we see that $R(U)$ is the total quotient ring $Q(\mathcal{O}_{U,u})$, see Morphisms, Lemma 29.49.5 and Algebra, Lemma 10.25.4.

At this point we have proved all the statements in the lemma with $\mathcal{O}_{U,u}$ in stead of $\mathcal{O}^h_{X,\xi}$. However, $\mathcal{O}^h_{X,\xi}$ is the henselization of $\mathcal{O}_{U,u}$. Thus $\mathcal{O}^h_{X,\xi}$ is a 1-dimensional reduced Noetherian ring, see More on Algebra, Lemmas 15.45.4, 15.45.7, and 15.45.3. Since $\mathcal{O}_{U,u} \to \mathcal{O}^h_{X,\xi}$ is faithfully flat by More on Algebra, Lemma 15.45.1 it sends nonzerodivisors to nonzerodivisors. Therefore we obtain a canonical map $Q(\mathcal{O}_{U,u}) \to Q(\mathcal{O}^h_{X,\xi})$ and we obtain our map. We omit the verification that the map is independent of the choice of $(U, u) \to (X, x)$; a slightly better approach would be to first observe that $\operatorname{colim} Q(\mathcal{O}_{U,u}) = Q(\mathcal{O}^h_{X,\xi})$. $\square$

0ENL Definition 72.6.4. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. For every prime divisor $Z \subset X$ we define the order of vanishing of f along Z as the integer

$$\operatorname{ord}_Z(f) = \operatorname{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi}) - \operatorname{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/b\mathcal{O}^h_{X,\xi})$$

where $a, b \in \mathcal{O}^h_{X,\xi}$ are nonzerodivisors such that the image of f in $Q(\mathcal{O}^h_{X,\xi})$ (Lemma 72.6.3) is equal to a/b. This is well defined by Algebra, Lemma 10.121.1.

If $\mathcal{O}^h_{X,\xi}$ happens to be a domain, then we obtain

$$\operatorname{ord}_Z(f) = \operatorname{ord}_{\mathcal{O}^h_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 10.121.2. Note that for $f, g \in R(X)^*$ we have

$$\operatorname{ord}_Z(fg) = \operatorname{ord}_Z(f) + \operatorname{ord}_Z(g).$$

Of course it can happen that $\operatorname{ord}_Z(f) < 0$. In this case we say that f has a pole along Z and that $-\operatorname{ord}_Z(f) > 0$ is the order of pole of f along Z. It is important to note that the condition $\operatorname{ord}_Z(f) \geq 0$ is not equivalent to the condition $f \in \mathcal{O}^h_{X,\xi}$ unless the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring.

0ENM Lemma 72.6.5. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. If the prime divisor $Z \subset X$ meets the schematic locus of X, then the order of vanishing $\operatorname{ord}_Z(f)$ of Definition 72.6.4 agrees with the order of vanishing of Divisors, Definition 31.26.3.

Proof. After shrinking X we may assume X is an integral Noetherian scheme. If $\xi \in Z$ denotes the generic point, then we find that $\mathcal{O}^h_{X,\xi}$ is the henselization of $\mathcal{O}_{X,\xi}$ (Decent Spaces, Lemma 68.11.8). To prove the lemma it suffices and is necessary to show that

$$\operatorname{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/a\mathcal{O}_{X,\xi}) = \operatorname{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi})$$

This follows immediately from Algebra, Lemma 10.52.13 (and the fact that $\mathcal{O}_{X,\xi} \to \mathcal{O}^h_{X,\xi}$ is a flat local ring homomorphism of local Noetherian rings). $\square$

0ENN Lemma 72.6.6. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. Then the collections

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } f \text{ not in } \mathcal{O}_{X,\xi}\}$$

and

$$\{Z \subset X \mid Z \text{ a prime divisor and } \text{ord}_Z(f) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subspace $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$. Hence the prime divisors which can occur in the sets of the lemma all correspond to irreducible components of $|X| \setminus |U|$. Hence Lemma 72.6.1 gives the desired result. $\square$

This lemma allows us to make the following definition.

0ENP Definition 72.6.7. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. The principal Weil divisor associated to f is the Weil divisor

$$\text{div}(f) = \text{div}_X(f) = \sum \text{ord}_Z(f)[Z]$$

where the sum is over prime divisors and $\text{ord}_Z(f)$ is as in Definition 72.6.4. This makes sense by Lemma 72.6.6.

0ENQ Lemma 72.6.8. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f, g \in R(X)^*$. Then

$$\text{div}_X(fg) = \text{div}_X(f) + \text{div}_X(g)$$

as Weil divisors on X.

Proof. This is clear from the additivity of the ord functions. $\square$

We see from the lemma above that the collection of principal Weil divisors form a subgroup of the group of all Weil divisors. This leads to the following definition.

0ENR Definition 72.6.9. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. The Weil divisor class group of X is the quotient of the group of Weil divisors by the subgroup of principal Weil divisors. Notation: $\text{Cl}(X)$.

By construction we obtain an exact complex

0ENS (72.6.9.1) $$R(X)^* \xrightarrow{\text{div}} \text{Div}(X) \to \text{Cl}(X) \to 0$$

which we can think of as a presentation of $\text{Cl}(X)$. Our next task is to relate the Weil divisor class group to the Picard group.

0ENT Example 72.6.10. This is a continuation of Morphisms of Spaces, Example 67.53.3. Consider the algebraic space $X = \mathbf{A}^1_k/\{t \sim -t \mid t \neq 0\}$. This is a smooth algebraic space over the field k. There is a universal homeomorphism

$$X \longrightarrow \mathbf{A}^1_k = \text{Spec}(k[t])$$

which is an isomorphism over $\mathbf{A}^1_k \setminus \{0\}$. We conclude that X is Noetherian and integral. Since $\dim(X) = 1$, we see that the prime divisors of X are the closed points of X. Consider the unique closed point $x \in |X|$ lying over $0 \in \mathbf{A}^1_k$. Since

$X \setminus \{x\}$ maps isomorphically to $\mathbf{A}^1 \setminus \{0\}$ we see that the classes in $\text{Cl}(X)$ of closed points different from x are zero. However, the divisor of t on X is $2[x]$. We conclude that $\text{Cl}(X) = \mathbf{Z}/2\mathbf{Z}$.

0ENU Example 72.6.11. Let k be a field. Let

$$U = \text{Spec}(k[x, y]/(xy))$$

be the union of the coordinate axes in $\mathbf{A}^2_k$. Denote $\Delta : U \to U \times_k U$ the diagonal and $\Delta' : U \to U \times_k U$ the map $u \mapsto (u, \sigma(u))$ where $\sigma : U \to U$, $(x, y) \mapsto (y, x)$ is the automorphism flipping the coordinate axes. Set

$$R = \Delta(U) \amalg \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ is the origin. It is easy to see that R is an étale equivalence relation on U. The quotient $X = U/R$ is an algebraic space. The morphism $U \to \mathbf{A}^1_k$, $(x, y) \mapsto x + y$ is R-invariant and hence defines a morphism

$$X \longrightarrow \mathbf{A}^1_k$$

This morphism is a universal homeomorphism and an isomorphism over $\mathbf{A}^1_k \setminus \{0\}$. It follows that X is integral and Noetherian. Exactly as in Example 72.6.10 the reader shows that $\text{Cl}(X) = \mathbf{Z}/2\mathbf{Z}$ with generator corresponding to the unique closed point $x \in |X|$ mapping to $0 \in \mathbf{A}^1_k$. However, in this case the henselian local ring of X at x isn't a domain, as it is the henselization of $\mathcal{O}_{U,0_U}$.

72.7. The Weil divisor class associated to an invertible module

0ENV In this section we go through exactly the same progression as in Section 72.6 to define a canonical map $\text{Pic}(X) \to \text{Cl}(X)$ on a locally Noetherian integral algebraic space.

Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. By Divisors on Spaces, Lemma 71.10.11 there exists a regular meromorphic section $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$. In fact, by Divisors on Spaces, Lemma 71.10.8 this is the same thing as a nonzero element in $\mathcal{L}_\eta$ where $\eta \in |X|$ is the generic point. The same lemma tells us that if $\mathcal{L} = \mathcal{O}_X$, then s is the same thing as a nonzero rational function on X (so what we will do below matches the construction in Section 72.6).

Let $Z \subset X$ be a prime divisor and let $\xi \in |Z|$ be the generic point. We are going to define the order of vanishing of s along Z. Consider the canonical morphism

$$c_\xi : \text{Spec}(\mathcal{O}^h_{X,\xi}) \longrightarrow X$$

whose source is the spectrum of the henselian local ring of X as ξ (Decent Spaces, Definition 68.11.7). The pullback $\mathcal{L}_\xi = c_\xi^*\mathcal{L}$ is an invertible module and hence trivial; choose a generator s_ξ of $\mathcal{L}_\xi$. Since c_ξ is flat, pullbacks of meromorphic functions and (regular) sections are defined for c_ξ, see Divisors on Spaces, Definition 71.10.6 and Lemmas 71.10.7 and 71.10.10. Thus we get

$$c_\xi^*(s) = f s_\xi$$

for some nonzerodivisor $f \in Q(\mathcal{O}^h_{X,\xi})$. Here we are using Divisors, Lemma 31.24.2 to identify the space of meromorphic sections of $\mathcal{L}_\xi \cong \mathcal{O}_{\text{Spec}(\mathcal{O}^h_{X,\xi})}$ in terms of the total ring of fractions of $\mathcal{O}^h_{X,\xi}$. Let us agree to denote this element

$$s/s_\xi = f \in Q(\mathcal{O}^h_{X,\xi})$$

Observe that $f = s/s_\xi$ is replaced by uf where $u \in \mathcal{O}^h_{X,\xi}$ is a unit if we change our choice of s_ξ.

0EPR Definition 72.7.1. Let S be a scheme. Let X be a locally Noetherian integral algebraic algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every prime divisor $Z \subset X$ with generic point $\xi \in |Z|$ we define the order of vanishing of s along Z as the integer

$$\text{ord}_{Z,\mathcal{L}}(s) = \text{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi}) - \text{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/b\mathcal{O}^h_{X,\xi})$$

where $a, b \in \mathcal{O}^h_{X,\xi}$ are nonzerodivisors such that the element s/s_ξ of $Q(\mathcal{O}^h_{X,\xi})$ constructed above is equal to a/b. This is well defined by the above and Algebra, Lemma 10.121.1.

As explained above, a regular meromorphic section s of $\mathcal{O}_X$ can be written $s = f \cdot 1$ where f is a nonzero rational function on X and we have $\text{ord}_Z(f) = \text{ord}_{Z,\mathcal{O}_X}(s)$. As in the case of principal divisors we have the following lemma.

0EPS Lemma 72.7.2. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the sets

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } s \text{ not in } \mathcal{L}_\xi\}$$

and

$$\{Z \subset X \mid Z \text{ is a prime divisor and } \text{ord}_{Z,\mathcal{L}}(s) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subspace $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the prime divisors which can occur in the sets of the lemma all correspond to irreducible components of $|X| \setminus |U|$. Hence Lemma 72.6.1. gives the desired result. $\square$

0EPT Lemma 72.7.3. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have

$$\sum \text{ord}_{Z,\mathcal{L}}(s)[Z] = \sum \text{ord}_{Z,\mathcal{L}}(s')[Z] + \text{div}(f)$$

as Weil divisors.

Proof. This is clear from the definitions. Note that Lemma 72.7.2 guarantees that the sums are indeed Weil divisors. $\square$

0EPU Definition 72.7.4. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the Weil divisor associated to s as
$$\text{div}_\mathcal{L}(s) = \sum \text{ord}_{Z,\mathcal{L}}(s)[Z] \in \text{Div}(X)$$
where the sum is over prime divisors. This is well defined by Lemma 72.7.2.

(2) We define Weil divisor class associated to $\mathcal{L}$ as the image of $\text{div}_\mathcal{L}(s)$ in $\text{Cl}(X)$ where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 72.7.3.

As expected this construction is additive in the invertible module.

0EPV Lemma 72.7.5. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ and

$$\operatorname{div}_{\mathcal{L}\otimes\mathcal{N}}(st) = \operatorname{div}_{\mathcal{L}}(s) + \operatorname{div}_{\mathcal{N}}(t)$$

in $\operatorname{Div}(X)$. In particular, the Weil divisor class of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sum of the Weil divisor classes of $\mathcal{L}$ and $\mathcal{N}$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$. Let $Z \subset X$ be a prime divisor. Let $\xi \in |Z|$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$ with notation as described earlier in this section. Then $s_\xi \otimes t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$ in $Q(\mathcal{O}^h_{X,\xi})$. Applying the additivity of Algebra, Lemma 10.121.1 we conclude that

$$\operatorname{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \operatorname{div}_{\mathcal{L},Z}(s) + \operatorname{div}_{\mathcal{N},Z}(t)$$

Some details omitted. □

Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. By the constructions and lemmas above we obtain a homomorphism of abelian groups

0EPW (72.7.5.1) $$\operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X)$$

which assigns to an invertible module its Weil divisor class.

0EPX Lemma 72.7.6. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. If X is normal, then the map (72.7.5.1) $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ is injective.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose associated Weil divisor class is trivial. Let s be a regular meromorphic section of $\mathcal{L}$. The assumption means that $\operatorname{div}_{\mathcal{L}}(s) = \operatorname{div}(f)$ for some $f \in R(X)^*$. Then we see that $t = f^{-1}s$ is a regular meromorphic section of $\mathcal{L}$ with $\operatorname{div}_{\mathcal{L}}(t) = 0$, see Lemma 72.7.3. We claim that t defines a trivialization of $\mathcal{L}$. The claim finishes the proof of the lemma. Our proof of the claim is a bit awkward as we don't yet have a lot of theory at our dispposal; we suggest the reader skip the proof.

We may check our claim étale locally. Let $U \in X_{\acute{e}tale}$ be affine such that $\mathcal{L}|_U$ is trivial. Say $s_U \in \Gamma(U, \mathcal{L}|_U)$ is a trivialization. By Properties, Lemma 28.7.5 we may also assume U is integral. Write $U = \operatorname{Spec}(A)$ as the spectrum of a normal Noetherian domain A with fraction field K. We may write $t|_U = fs_U$ for some element f of K, see Divisors on Spaces, Lemma 71.10.4 for example. Let $\mathfrak{p} \subset A$ be a height one prime corresponding to a codimension 1 point $u \in U$ which maps to a codimension 1 point $\xi \in |X|$. Choose a trivialization s_ξ of $\mathcal{L}_\xi$ as in the beginning of this section. Choose a geometric point $\overline{u}$ of U lying over u. Then

$$(\mathcal{O}^h_{X,\xi})^{sh} = \mathcal{O}_{X,\overline{u}} = \mathcal{O}^{sh}_{U,u} = (A_\mathfrak{p})^{sh}$$

see Decent Spaces, Lemmas 68.11.9 and Properties of Spaces, Lemma 66.22.1. The normality of X shows that all of these are discrete valuation rings. The trivializations s_U and s_ξ differ by a unit as sections of $\mathcal{L}$ pulled back to $\operatorname{Spec}(\mathcal{O}_{X,\overline{u}})$. Write $t = f_\xi s_\xi$ with $f_\xi \in Q(\mathcal{O}^h_{X,\xi})$. We conclude that f_ξ and f differ by a unit in

$Q(\mathcal{O}_{X,\overline{u}})$. If $Z \subset X$ denotes the prime divisor corresponding to ξ (Lemma 72.4.7), then $0 = \text{ord}_{Z,\mathcal{L}}(t) = \text{ord}_{\mathcal{O}^h_{X,\xi}}(f_\xi)$ and since $\mathcal{O}^h_{X,\xi}$ is a discrete valuation ring we see that f_ξ is a unit. Thus f is a unit in $\mathcal{O}_{X,\overline{u}}$ and hence in particular $f \in A_\mathfrak{p}$. This implies $f \in A$ by Algebra, Lemma 10.157.6. We conclude that $t \in \Gamma(X, \mathcal{L})$. Repeating the argument with t^{-1} viewed as a meromorphic section of $\mathcal{L}^{\otimes -1}$ finishes the proof. □

72.8. Modifications and alterations

0AD7 Using our notion of an integral algebraic space we can define a modification as follows.

0AD8 Definition 72.8.1. Let S be a scheme. Let X be an integral algebraic space over S. A modification of X is a birational proper morphism $f : X' \to X$ of algebraic spaces over S with X' integral.

For birational morphisms of algebraic spaces, see Decent Spaces, Definition 68.22.1.

0AD9 Lemma 72.8.2. Let $f : X' \to X$ be a modification as in Definition 72.8.1. There exists a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 72.5.1 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Morphisms of Spaces, Proposition 67.32.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Morphisms of Spaces, Lemma 67.46.6). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. □

0ADA Definition 72.8.3. Let S be a scheme. Let X be an integral algebraic space over S. An alteration of X is a proper dominant morphism $f : Y \to X$ of algebraic spaces over S with Y integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$.

If $f : Y \to X$ is a dominant and proper morphism between integral algebraic spaces, then it is an alteration as soon as the induced extension of residue fields in generic points is finite. Here is the precise statement.

0ADB Lemma 72.8.4. Let S be a scheme. Let $f : X \to Y$ be a proper dominant morphism of integral algebraic spaces over S. Then f is an alteration if and only if any of the equivalent conditions (1) – (6) of Lemma 72.5.1 hold.

Proof. Immediate consequence of the lemma referenced in the statement. □

0DMN Lemma 72.8.5. Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of algebraic spaces over S. Assume Y is integral. Then there exists an integral closed subspace $X' \subset X$ such that $f' = f|_{X'} : X' \to Y$ is an alteration.

Proof. Let $V \subset Y$ be a nonempty open affine (Decent Spaces, Theorem 68.10.2). Let $\eta \in V$ be the generic point. Then X_η is a nonempty proper algebraic space over η. Choose a closed point $x \in |X_\eta|$ (exists because $|X_\eta|$ is a quasi-compact, sober topological space, see Decent Spaces, Proposition 68.12.4 and Topology, Lemma 5.12.8.) Let X' be the reduced induced closed subspace structure on $\overline{\{x\}} \subset |X|$ (Properties of Spaces, Definition 66.12.5. Then $f' : X' \to Y$ is surjective as the image contains η. Also f' is proper as a composition of a closed immersion and a proper morphism. Finally, the fibre X'_η has a single point; to see this use Decent Spaces, Lemma 68.18.6 for both $X \to Y$ and $X' \to Y$ and the point η. Since Y is decent and $X' \to Y$ is separated we see that X' is decent (Decent Spaces, Lemmas 68.17.2 and 68.17.5). Thus f' is an alteration by Lemma 72.8.4. □

72.9. Schematic locus

06LY We have already proven a number of results on the schematic locus of an algebraic space. Here is a list of references:

(1) Properties of Spaces, Sections 66.13 and 66.14,
(2) Decent Spaces, Section 68.10,
(3) Properties of Spaces, Lemma 66.15.3 $\Leftarrow$ Decent Spaces, Lemma 68.12.8 $\Leftarrow$ Decent Spaces, Lemma 68.14.2,
(4) Limits of Spaces, Section 70.15, and
(5) Limits of Spaces, Section 70.17.

There are some cases where certain types of morphisms of algebraic spaces are automatically representable, for example separated, locally quasi-finite morphisms (Morphisms of Spaces, Lemma 67.51.1), and flat monomorphisms (More on Morphisms of Spaces, Lemma 76.4.1). In Section 72.10 we will study what happens with the schematic locus under extension of base field.

06LZ **Lemma 72.9.1.** Let S be a scheme. Let X be an algebraic space over S. Assume X satisfies at least one of the following conditions

(1) X is quasi-separated and $\dim(X) = 0$,
(2) X is locally of finite type over a field k and $\dim(X) = 0$,
(3) X is Noetherian and $\dim(X) = 0$, or
(4) add more here.

Then X is a separated scheme and any quasi-compact open of X is affine.

Proof. If we prove that any quasi-compact open of X is affine, then X is a separated scheme. Thus we may assume X is quasi-compact and we aim to show that X is affine. Cases (2) and (3) follow immediately from case (1) but we will give a separate proofs of (2) and (3) as these proofs use significantly less theory.

Proof of (3). Let U be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 66.9.2 we see that $\dim(U) = 0$ and $\dim(R) = 0$. Since R is a locally Noetherian scheme of dimension 0, we see that R is a disjoint union of spectra of Artinian local rings (Properties, Lemma 28.10.5). Since we assumed that X is Noetherian (so quasi-separated) we conclude that R is quasi-compact. Hence R is an affine scheme (use Schemes, Lemma 26.6.8). The étale morphisms $s, t : R \to U$ induce finite residue field extensions. Hence s and t are finite by Algebra, Lemma 10.54.4 (small detail omitted). Thus Groupoids, Proposition 39.23.9 shows that $X = U/R$ is an affine scheme.

Proof of (2) – almost identical to the proof of (3). Let U be an affine scheme and let $U \to X$ be a surjective étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 66.9.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \to \mathrm{Spec}(k)$ is locally of finite type as the composition of the étale morphism $U \to X$ and $X \to \mathrm{Spec}(k)$, see Morphisms of

Spaces, Lemmas 67.23.2 and 67.39.9. Similarly, $R \to \mathrm{Spec}(k)$ is locally of finite type. Hence by Varieties, Lemma 33.20.2 we see that U and R are disjoint unions of spectra of local Artinian k-algebras finite over k. The same thing is therefore true of $U \times_{\mathrm{Spec}(k)} U$. As

$$R = U \times_X U \longrightarrow U \times_{\mathrm{Spec}(k)} U$$

is a monomorphism, we see that R is a finite(!) union of spectra of finite k-algebras. It follows that R is affine, see Schemes, Lemma 26.6.8. Applying Varieties, Lemma 33.20.2 once more we see that R is finite over k. Hence s, t are finite, see Morphisms, Lemma 29.44.14. Thus Groupoids, Proposition 39.23.9 shows that $X = U/R$ is an affine scheme.

Cohomological proof of (1). By Cohomology of Spaces, Lemma 69.10.1 we have vanishing of higher cohomology groups for all quasi-coherent sheaves $\mathcal{F}$ on X. Hence X is affine (in particular a scheme) by Cohomology of Spaces, Proposition 69.16.7.

Geometric proof of (1). Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and étale morphisms $f_p : V_p \to U_p$ as in Decent Spaces, Lemma 68.8.6 (we will use all their properties below). Then $\dim(V_p) = 0$ by our definition of dimension of algebraic spaces. Thus Properties, Lemma 28.10.6 applies to each V_p. Then $f_p^{-1}(U_{p+1}) \subset V_p$ is quasi-compact open and hence is affine as well as closed. It follows that $|T_p| \subset |U_p|$ (see locus citatus) is open as well as closed. Hence X is a disjoint union of open and closed subspaces whose reduced structures are schemes. It follows that X is a scheme (Limits of Spaces, Lemma 70.15.3). Then the proof is finished by the case of schemes that we already referenced above. □

The following lemma tells us that a quasi-separated algebraic space is a scheme away from codimension 1.

0ADC Lemma 72.9.2. Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let $x \in |X|$. The following are equivalent

(1) x is a point of codimension 0 on X,
(2) the local ring of X at x has dimension 0, and
(3) x is a generic point of an irreducible component of $|X|$.

If true, then there exists an open subspace of X containing x which is a scheme.

Proof. The equivalence of (1), (2), and (3) follows from Decent Spaces, Lemma 68.20.1 and the fact that a quasi-separated algebraic space is decent (Decent Spaces, Section 68.6). However in the next paragraph we will give a more elementary proof of the equivalence.

Note that (1) and (2) are equivalent by definition (Properties of Spaces, Definition 66.10.2). To prove the equivalence of (1) and (3) we may assume X is quasi-compact. Choose

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and $f_i : V_i \to U_i$ as in Decent Spaces, Lemma 68.8.6. Say $x \in U_i$, $x \notin U_{i+1}$. Then $x = f_i(y)$ for a unique $y \in V_i$. If (1) holds, then y is a generic point of an irreducible component of V_i (Properties of Spaces, Lemma 66.11.1). Since $f_i^{-1}(U_{i+1})$ is a quasi-compact open of V_i not containing y, there is an open neighbourhood $W \subset V_i$ of

y disjoint from $f_i^{-1}(V_i)$ (see Properties, Lemma 28.2.2 or more simply Algebra, Lemma 10.26.4). Then $f_i|_W : W \to X$ is an isomorphism onto its image and hence $x = f_i(y)$ is a generic point of $|X|$. Conversely, assume (3) holds. Then f_i maps $\overline{\{y\}}$ onto the irreducible component $\overline{\{x\}}$ of $|U_i|$. Since $|f_i|$ is bijective over $\overline{\{x\}}$, it follows that $\overline{\{y\}}$ is an irreducible component of U_i. Thus x is a point of codimension 0.

The final statement of the lemma is Properties of Spaces, Proposition 66.13.3. $\square$

The following lemma says that a separated locally Noetherian algebraic space is a scheme in codimension 1, i.e., away from codimension 2.

0ADD Lemma 72.9.3. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. If X is separated, locally Noetherian, and the dimension of the local ring of X at x is ≤ 1 (Properties of Spaces, Definition 66.10.2), then there exists an open subspace of X containing x which is a scheme.

Proof. (Please see the remark below for a different approach avoiding the material on finite groupoids.) We can replace X by an quasi-compact neighbourhood of x, hence we may assume X is quasi-compact, separated, and Noetherian. There exists a scheme U and a finite surjective morphism $U \to X$, see Limits of Spaces, Proposition 70.16.1. Let $R = U \times_X U$. Then $j : R \to U \times_S U$ is an equivalence relation and we obtain a groupoid scheme (U, R, s, t, c) over S with s, t finite and U Noetherian and separated. Let $\{u_1, \ldots, u_n\} \subset U$ be the set of points mapping to x. Then $\dim(\mathcal{O}_{U,u_i}) \leq 1$ by Decent Spaces, Lemma 68.12.6.

By More on Groupoids, Lemma 40.14.10 there exists an R-invariant affine open $W \subset U$ containing the orbit $\{u_1, \ldots, u_n\}$. Since $U \to X$ is finite surjective the continuous map $|U| \to |X|$ is closed surjective, hence submersive by Topology, Lemma 5.6.5. Thus $f(W)$ is open and there is an open subspace $X' \subset X$ with $f : W \to X'$ a surjective finite morphism. Then X' is an affine scheme by Cohomology of Spaces, Lemma 69.17.3 and the proof is finished. $\square$

0ADE Remark 72.9.4. Here is a sketch of a proof of Lemma 72.9.3 which avoids using More on Groupoids, Lemma 40.14.10.

Step 1. We may assume X is a reduced Noetherian separated algebraic space (for example by Cohomology of Spaces, Lemma 69.17.3 or by Limits of Spaces, Lemma 70.15.3) and we may choose a finite surjective morphism $Y \to X$ where Y is a Noetherian scheme (by Limits of Spaces, Proposition 70.16.1).

Step 2. After replacing X by an open neighbourhood of x, there exists a birational finite morphism $X' \to X$ and a closed subscheme $Y' \subset X' \times_X Y$ such that $Y' \to X'$ is surjective finite locally free. Namely, because X is reduced there is a dense open subspace $U \subset X$ over which Y is flat (Morphisms of Spaces, Proposition 67.32.1). Then we can choose a U-admissible blowup $b : \tilde{X} \to X$ such that the strict transform $\tilde{Y}$ of Y is flat over $\tilde{X}$, see More on Morphisms of Spaces, Lemma 76.39.1. (An alternative is to use Hilbert schemes if one wants to avoid using the result on blowups). Then we let $X' \subset \tilde{X}$ be the scheme theoretic closure of $b^{-1}(U)$ and $Y' = X' \times_{\tilde{X}} \tilde{Y}$. Since x is a codimension 1 point, we see that $X' \to X$ is finite over a neighbourhood of x (Lemma 72.3.2).

Step 3. After shrinking X to a smaller neighbourhood of x we get that X' is a scheme. This holds because Y' is a scheme and $Y' \to X'$ being finite locally free and because every finite set of codimension 1 points of Y' is contained in an affine open. Use Properties of Spaces, Proposition 66.14.1 and Varieties, Proposition 33.42.7.

Step 4. There exists an affine open $W' \subset X'$ containing all points lying over x which is the inverse image of an open subspace of X. To prove this let $Z \subset X$ be the closure of the set of points where $X' \to X$ is not an isomorphism. We may assume $x \in Z$ otherwise we are already done. Then x is a generic point of an irreducible component of Z and after shrinking X we may assume Z is an affine scheme (Lemma 72.9.2). Then the inverse image $Z' \subset X'$ is an affine scheme as well. Say $x_1, \ldots, x_n \in Z'$ are the points mapping to x. Then we can find an affine open W' in X' whose intersection with Z' is the inverse image of a principal open of Z containing x. Namely, we first pick an affine open $W' \subset X'$ containing $x_1, \ldots, x_n$ using Varieties, Proposition 33.42.7. Then we pick a principal open $D(f) \subset Z$ containing x whose inverse image $D(f|_{Z'})$ is contained in $W' \cap Z'$. Then we pick $f' \in \Gamma(W', \mathcal{O}_{W'})$ restricting to $f|_{Z'}$ and we replace W' by $D(f') \subset W'$. Since $X' \to X$ is an isomorphism away from $Z' \to Z$ the choice of W' guarantees that the image $W \subset X$ of W' is open with inverse image W' in X'.

Step 5. Then $W' \to W$ is a finite surjective morphism and W is a scheme by Cohomology of Spaces, Lemma 69.17.3 and the proof is complete.

72.10. Schematic locus and field extension

0B82 It can happen that a nonrepresentable algebraic space over a field k becomes representable (i.e., a scheme) after base change to an extension of k. See Spaces, Example 65.14.2. In this section we address this issue.

0B83 Lemma 72.10.1. Let k be a field. Let X be an algebraic space over k. If there exists a purely inseparable field extension k'/k such that $X_{k'}$ is a scheme, then X is a scheme.

Proof. The morphism $X_{k'} \to X$ is integral, surjective, and universally injective. Hence this lemma follows from Limits of Spaces, Lemma 70.15.4. $\square$

0B84 Lemma 72.10.2. Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-separated algebraic space over k.

(1) If there exists a field extension K/k such that X_K is a scheme, then $X_{\overline{k}}$ is a scheme.
(2) If X is quasi-compact and there exists a field extension K/k such that X_K is a scheme, then $X_{k'}$ is a scheme for some finite separable extension k' of k.

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we see that the first part of the lemma follows from the second part (some details omitted). Thus we assume X is quasi-compact and we assume given an extension K/k with X_K representable. Write $K = \bigcup A$ as the colimit of finitely generated k-subalgebras A. By Limits of Spaces, Lemma 70.5.11 we see that X_A is a scheme for some A. Choose a maximal ideal $\mathfrak{m} \subset A$. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A/\mathfrak{m}$ is a finite extension of k. Thus we see that $X_{k'}$ is a scheme. If $k' \supset k$ is not separable, let $k'/k''/k$ be the subextension found in Fields, Lemma 9.14.6. Since k'/k'' is purely inseparable, by

Lemma 72.10.1 the algebraic space $X_{k''}$ is a scheme. Since $k''|k$ is separable the proof is complete. ☐

0B86 Lemma 72.10.3. Let k'/k be a finite Galois extension with Galois group G. Let X be an algebraic space over k. Then G acts freely on the algebraic space $X_{k'}$ and $X = X_{k'}/G$ in the sense of Properties of Spaces, Lemma 66.34.1.

Proof. Omitted. Hints: First show that $\text{Spec}(k) = \text{Spec}(k')/G$. Then use compatibility of taking quotients with base change. ☐

0B87 Lemma 72.10.4. Let S be a scheme. Let X be an algebraic space over S and let G be a finite group acting freely on X. Set $Y = X/G$ as in Properties of Spaces, Lemma 66.34.1. For $y \in |Y|$ the following are equivalent

(1) y is in the schematic locus of Y, and
(2) there exists an affine open $U \subset X$ containing the preimage of y.

Proof. It follows from the construction of $Y = X/G$ in Properties of Spaces, Lemma 66.34.1 that the morphism $X \to Y$ is surjective and étale. Of course we have $X \times_Y X = X \times G$ hence the morphism $X \to Y$ is even finite étale. It is also surjective. Thus the lemma follows from Decent Spaces, Lemma 68.10.3. ☐

0B85 Lemma 72.10.5. Let k be a field. Let X be a quasi-separated algebraic space over k. If there exists a purely transcendental field extension K/k such that X_K is a scheme, then X is a scheme.

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we may assume X is quasi-compact (some details omitted). Recall (Fields, Definition 9.26.1) that the assumption on the extension K/k signifies that K is the fraction field of a polynomial ring (in possibly infinitely many variables) over k. Thus $K = \bigcup A$ is the union of subalgebras each of which is a localization of a finite polynomial algebra over k. By Limits of Spaces, Lemma 70.5.11 we see that X_A is a scheme for some A. Write

$$A = k[x_1, \ldots, x_n][1/f]$$

for some nonzero $f \in k[x_1, \ldots, x_n]$.

If k is infinite then we can finish the proof as follows: choose $a_1, \ldots, a_n \in k$ with $f(a_1, \ldots, a_n) \neq 0$. Then $(a_1, \ldots, a_n)$ define an k-algebra map $A \to k$ mapping x_i to a_i and $1/f$ to $1/f(a_1, \ldots, a_n)$. Thus the base change $X_A \times_{\text{Spec}(A)} \text{Spec}(k) \cong X$ is a scheme as desired.

In this paragraph we finish the proof in case k is finite. In this case we write $X = \lim X_i$ with X_i of finite presentation over k and with affine transition morphisms (Limits of Spaces, Lemma 70.10.2). Using Limits of Spaces, Lemma 70.5.11 we see that $X_{i,A}$ is a scheme for some i. Thus we may assume $X \to \text{Spec}(k)$ is of finite presentation. Let $x \in |X|$ be a closed point. We may represent x by a closed immersion $\text{Spec}(\kappa) \to X$ (Decent Spaces, Lemma 68.14.6). Then $\text{Spec}(\kappa) \to \text{Spec}(k)$ is of finite type, hence κ is a finite extension of k (by the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1; some details omitted). Say $[\kappa : k] = d$. Choose an integer $n \gg 0$ prime to d and let k'/k be the extension of degree n. Then k'/k is Galois with $G = \text{Aut}(k'/k)$ cyclic of order n. If n is large enough there will be k-algebra homomorphism $A \to k'$ by the same reason as above. Then $X_{k'}$ is a scheme and $X = X_{k'}/G$ (Lemma 72.10.3). On the other hand, since n and d are relatively prime we see that

$$\mathrm{Spec}(\kappa) \times_X X_{k'} = \mathrm{Spec}(\kappa) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') = \mathrm{Spec}(\kappa \otimes_k k')$$

is the spectrum of a field. In other words, the fibre of $X_{k'} \to X$ over x consists of a single point. Thus by Lemma 72.10.4 we see that x is in the schematic locus of X as desired. $\square$

0BA7 Remark 72.10.6. Let k be a finite field. Let K/k be a geometrically irreducible field extension. Then K is the limit of geometrically irreducible finite type k-algebras A. Given A the estimates of Lang and Weil [LW54], show that for $n \gg 0$ there exists an k-algebra homomorphism $A \to k'$ with k'/k of degree n. Analyzing the argument given in the proof of Lemma 72.10.5 we see that if X is a quasi-separated algebraic space over k and X_K is a scheme, then X is a scheme. If we ever need this result we will precisely formulate it and prove it here.

0B88 Lemma 72.10.7. Let k be a field with algebraic closure $\overline{k}$. Let X be an algebraic space over k such that

(1) X is decent and locally of finite type over k,
(2) $X_{\overline{k}}$ is a scheme, and
(3) any finite set of $\overline{k}$-rational points of $X_{\overline{k}}$ is contained in an affine.

Then X is a scheme.

Proof. If K/k is an extension, then the base change X_K is decent (Decent Spaces, Lemma 68.6.5) and locally of finite type over K (Morphisms of Spaces, Lemma 67.23.3). By Lemma 72.10.1 it suffices to prove that X becomes a scheme after base change to the perfection of k, hence we may assume k is a perfect field (this step isn't strictly necessary, but makes the other arguments easier to think about). By covering X by quasi-compact opens we see that it suffices to prove the lemma in case X is quasi-compact (small detail omitted). In this case $|X|$ is a sober topological space (Decent Spaces, Proposition 68.12.4). Hence it suffices to show that every closed point in $|X|$ is contained in the schematic locus of X (use Properties of Spaces, Lemma 66.13.1 and Topology, Lemma 5.12.8).

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 68.14.6 we can find a closed immersion $\mathrm{Spec}(l) \to X$ representing x. Then $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ is of finite type (Morphisms of Spaces, Lemma 67.23.2) and we conclude that l is a finite extension of k by the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1). It is separable because k is perfect. Thus the scheme

$$\mathrm{Spec}(l) \times_X X_{\overline{k}} = \mathrm{Spec}(l) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) = \mathrm{Spec}(l \otimes_k \overline{k})$$

is the disjoint union of a finite number of $\overline{k}$-rational points. By assumption (3) we can find an affine open $W \subset X_{\overline{k}}$ containing these points.

By Lemma 72.10.2 we see that $X_{k'}$ is a scheme for some finite extension k'/k. After enlarging k' we may assume that there exists an affine open $U' \subset X_{k'}$ whose base change to $\overline{k}$ recovers W (use that $X_{\overline{k}}$ is the limit of the schemes $X_{k''}$ for $k' \subset k'' \subset \overline{k}$ finite and use Limits, Lemmas 32.4.11 and 32.4.13). We may assume that k'/k is a Galois extension (take the normal closure Fields, Lemma 9.16.3 and use that k is perfect). Set $G = \mathrm{Gal}(k'/k)$. By construction the G-invariant closed subscheme

$\mathrm{Spec}(l) \times_X X_{k'}$ is contained in U'. Thus x is in the schematic locus by Lemmas 72.10.3 and 72.10.4. $\square$

The following two lemmas should go somewhere else. Please compare the next lemma to Decent Spaces, Lemma 68.18.8.

06S0 Lemma 72.10.8. Let k be a field. Let X be an algebraic space over k. The following are equivalent

(1) X is locally quasi-finite over k,
(2) X is locally of finite type over k and has dimension 0,
(3) X is a scheme and is locally quasi-finite over k,
(4) X is a scheme and is locally of finite type over k and has dimension 0, and
(5) X is a disjoint union of spectra of Artinian local k-algebras A over k with $\dim_k(A) < \infty$.

Proof. Because we are over a field relative dimension of X/k is the same as the dimension of X. Hence by Morphisms of Spaces, Lemma 67.34.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 72.9.1 (and trivial implications) that (1) – (4) are equivalent. Finally, Varieties, Lemma 33.20.2 shows that (1) – (4) are equivalent with (5). $\square$

06S1 Lemma 72.10.9. Let k be a field. Let $f : X \to Y$ be a monomorphism of algebraic spaces over k. If Y is locally quasi-finite over k so is X.

Proof. Assume Y is locally quasi-finite over k. By Lemma 72.10.8 we see that $Y = \coprod \mathrm{Spec}(A_i)$ where each A_i is an Artinian local ring finite over k. By Decent Spaces, Lemma 68.19.1 we see that X is a scheme. Consider $X_i = f^{-1}(\mathrm{Spec}(A_i))$. Then X_i has either one or zero points. If X_i has zero points there is nothing to prove. If X_i has one point, then $X_i = \mathrm{Spec}(B_i)$ with B_i a zero dimensional local ring and $A_i \to B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i}B_i$ and we see that $A_i \to B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 10.20.1 (because $\mathfrak{m}_{A_i}$ is a nilpotent ideal!). Thus B_i is a finite local k-algebra, and we conclude by Lemma 72.10.8 that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. $\square$

72.11. Geometrically reduced algebraic spaces

0DMP If X is a reduced algebraic space over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced algebraic spaces.

0DMQ Definition 72.11.1. Let k be a field. Let X be an algebraic space over k.

(1) Let $x \in |X|$ be a point. We say X is geometrically reduced at x if $\mathcal{O}_{X,\overline{x}}$ is geometrically reduced over k.
(2) We say X is geometrically reduced over k if X is geometrically reduced at every point of X.

Observe that if X is geometrically reduced at x, then the local ring of X at x is reduced (Properties of Spaces, Lemma 66.22.6). Similarly, if X is geometrically reduced over k, then X is reduced (by Properties of Spaces, Lemma 66.21.4). The following lemma in particular implies this definition does not clash with the corresponding property for schemes over a field.

0DMR Lemma 72.11.2. Let k be a field. Let X be an algebraic space over k. Let $x \in |X|$. The following are equivalent

(1) X is geometrically reduced at x,
(2) for some étale neighbourhood $(U, u) \to (X, x)$ where U is a scheme, U is geometrically reduced at u,
(3) for any étale neighbourhood $(U, u) \to (X, x)$ where U is a scheme, U is geometrically reduced at u.

Proof. Recall that the local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$, see Properties of Spaces, Lemma 66.22.1. By Varieties, Lemma 33.6.2 we find that U is geometrically reduced at u if and only if $\mathcal{O}_{U,u}$ is geometrically reduced over k. Thus we have to show: if A is a local k-algebra, then A is geometrically reduced over k if and only if A^{sh} is geometrically reduced over k. We check this using the definition of geometrically reduced algebras (Algebra, Definition 10.43.1). Let K/k be a field extension. Since $A \to A^{sh}$ is faithfully flat (More on Algebra, Lemma 15.45.1) we see that $A \otimes_k K \to A^{sh} \otimes_k K$ is faithfully flat (Algebra, Lemma 10.39.7). Hence if $A^{sh} \otimes_k K$ is reduced, so is $A \otimes_k K$ by Algebra, Lemma 10.164.2. Conversely, recall that A^{sh} is a colimit of étale A-algebra, see Algebra, Lemma 10.155.2. Thus $A^{sh} \otimes_k K$ is a filtered colimit of étale $A \otimes_k K$-algebras. We conclude by Algebra, Lemma 10.163.7. □

0DMS Lemma 72.11.3. Let k be a field. Let X be an algebraic space over k. The following are equivalent

(1) X is geometrically reduced,
(2) for some surjective étale morphism $U \to X$ where U is a scheme, U is geometrically reduced,
(3) for any étale morphism $U \to X$ where U is a scheme, U is geometrically reduced.

Proof. Immediate from the definitions and Lemma 72.11.2. □

The notion isn't interesting in characteristic zero.

0E02 Lemma 72.11.4. Let X be an algebraic space over a perfect field k (for example k has characteristic zero).

(1) For $x \in |X|$, if $\mathcal{O}_{X,\overline{x}}$ is reduced, then X is geometrically reduced at x.
(2) If X is reduced, then X is geometrically reduced over k.

Proof. The first statement follows from Algebra, Lemma 10.43.6 and the definition of a perfect field (Algebra, Definition 10.45.1). The second statement follows from the first. □

0E03 Lemma 72.11.5. Let k be a field of characteristic $p > 0$. Let X be an algebraic space over k. The following are equivalent

(1) X is geometrically reduced over k,
(2) $X_{k'}$ is reduced for every field extension k'/k,
(3) $X_{k'}$ is reduced for every finite purely inseparable field extension k'/k,
(4) $X_{k^{1/p}}$ is reduced,
(5) $X_{k^{perf}}$ is reduced, and
(6) $X_{\bar{k}}$ is reduced.

Proof. Choose a surjective étale morphism $U \to X$ where U is a scheme. Via Lemma 72.11.3 the lemma follows from the result for U over k. See Varieties, Lemma 33.6.4. $\square$

0E04 Lemma 72.11.6. Let k be a field. Let X be an algebraic space over k. Let k'/k be a field extension. Let $x \in |X|$ be a point and let $x' \in |X_{k'}|$ be a point lying over x. The following are equivalent

(1) X is geometrically reduced at x,
(2) $X_{k'}$ is geometrically reduced at x'.

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. Choose an étale morphism $U \to X$ where U is a scheme and a point $u \in U$ mapping to $x \in |X|$. By Properties of Spaces, Lemma 66.4.3 we may choose a point $u' \in U_{k'} = U \times_X X_{k'}$ mapping to both u and x'. By Lemma 72.11.2 the lemma follows from the lemma for U, u, u' which is Varieties, Lemma 33.6.6. $\square$

0E05 Lemma 72.11.7. Let k be a field. Let $f : X \to Y$ be a morphism of algebraic spaces over k. Let $x \in |X|$ be a point with image $y \in |Y|$.

(1) if f is étale at x, then X is geometrically reduced at $x \Leftrightarrow Y$ is geometrically reduced at y,
(2) if f is surjective étale, then X is geometrically reduced $\Leftrightarrow Y$ is geometrically reduced.

Proof. Part (1) is clear because $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{Y,\overline{y}}$ if f is étale at x. Part (2) follows immediately from part (1). $\square$

72.12. Geometrically connected algebraic spaces

0A0Y If X is a connected algebraic space over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected algebraic spaces.

0A0Z Definition 72.12.1. Let X be an algebraic space over the field k. We say X is geometrically connected over k if the base change $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected algebraic spaces are nonempty.

0A10 Lemma 72.12.2. Let X be an algebraic space over the field k. Let k'/k be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension k''/k there exists a common field extension k'''/k' and k'''/k'. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. $\square$

0A11 **Lemma 72.12.3.** Let k be a field. Let X, Y be algebraic spaces over k. Assume X is geometrically connected over k. Then the projection morphism

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. Let $y \in |Y|$ be represented by a morphism $\mathrm{Spec}(K) \to Y$ where K is a field. The fibre of $|X \times_k Y| \to |Y|$ over y is the image of $|X_K| \to |X \times_k Y|$ by Properties of Spaces, Lemma 66.4.3. Thus these fibres are connected by our assumption that X is geometrically connected. By Morphisms of Spaces, Lemma 67.6.6 the map $|p|$ is open. Thus we may apply Topology, Lemma 5.7.6 to conclude. □

0A12 **Lemma 72.12.4.** Let k'/k be an extension of fields. Let X be an algebraic space over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 10.48.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Varieties, Lemma 33.7.5. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 72.12.3. □

0A13 **Lemma 72.12.5.** Let k be a field. Let X be an algebraic space over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.

Proof. Assume $X_{\overline{k}}$ is connected. Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 72.12.4 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. □

Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be an algebraic space over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

0A14 (72.12.5.1) $$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

0A15 **Lemma 72.12.6.** Let k be a field. Let X be an algebraic space over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then

(1) there exists a finite subextension $\overline{k}/k'/k$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,
(2) there exists an open subgroup $H \subset \mathrm{Gal}(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a quasi-compact open $W \subset U_{\overline{k}}$ whose image in $X_{\overline{k}}$ is V. This is possible because $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is continuous and because $|U_{\overline{k}}|$ has a basis of quasi-compact opens. We can apply Varieties, Lemma 33.7.9 to $W \subset U_{\overline{k}}$ to obtain the lemma. $\square$

0A16 Lemma 72.12.7. Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let X be an algebraic space over k. Let $\overline{T} \subset |X_{\overline{k}}|$ have the following properties

1. $\overline{T}$ is a closed subset of $|X_{\overline{k}}|$,
2. for every $\sigma \in \text{Gal}(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.

Then there exists a closed subset $T \subset |X|$ whose inverse image in $|X_{k'}|$ is $\overline{T}$.

Proof. Let $T \subset |X|$ be the image of $\overline{T}$. Since $|X_{\overline{k}}| \to |X|$ is surjective, the statement means that T is closed and that its inverse image is $\overline{T}$. Choose a scheme U and a surjective étale morphism $U \to X$. By the case of schemes (see Varieties, Lemma 33.7.10) there exists a closed subset $T' \subset |U|$ whose inverse image in $|U_{\overline{k}}|$ is the inverse image of $\overline{T}$. Since $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is surjective, we see that T' is the inverse image of T via $|U| \to |X|$. By our construction of the topology on $|X|$ this means that T is closed. In the same manner one sees that $\overline{T}$ is the inverse image of T. $\square$

0A17 Lemma 72.12.8. Let k be a field. Let X be an algebraic space over k. The following are equivalent

1. X is geometrically connected,
2. for every finite separable field extension k'/k the algebraic space $X_{k'}$ is connected.

Proof. This proof is identical to the proof of Varieties, Lemma 33.7.11 except that we replace Varieties, Lemma 33.7.7 by Lemma 72.12.5, we replace Varieties, Lemma 33.7.9 by Lemma 72.12.6, and we replace Varieties, Lemma 33.7.10 by Lemma 72.12.7. We urge the reader to read that proof in stead of this one.

It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 72.12.5 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty algebraic subspaces of $X_{\overline{k}}$.

Suppose that $W \subset X$ is any quasi-compact open subspace. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed subspaces of $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 72.12.6 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\text{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open subspace such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $\overline{k}/k'/k$ and a decomposition $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U'_0)_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V'_0)_{\overline{k}}$. Let $H = \text{Gal}(\overline{k}/k') \subset \text{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open subspace $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\text{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed subspaces. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open subspaces, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 72.12.7 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. □

72.13. Geometrically irreducible algebraic spaces

0DMT Spaces, Example 65.14.9 shows that it is best not to think about irreducible algebraic spaces in complete generality[1]. For decent (for example quasi-separated) algebraic spaces this kind of disaster doesn't happen. Thus we make the following definition only under the assumption that our algebraic space is decent.

0DMU Definition 72.13.1. Let k be a field. Let X be a decent algebraic space over k. We say X is geometrically irreducible if the topological space $|X_{k'}|$ is irreducible[2] for any field extension k' of k.

Observe that $X_{k'}$ is a decent algebraic space (Decent Spaces, Lemma 68.6.5). Hence the topological space $|X_{k'}|$ is sober. Decent Spaces, Proposition 68.12.4.

72.14. Geometrically integral algebraic spaces

0DMV Recall that integral algebraic spaces are by definition decent, see Section 72.4.

0DMW Definition 72.14.1. Let X be an algebraic space over the field k. We say X is geometrically integral over k if the algebraic space $X_{k'}$ is integral (Definition 72.4.1) for every field extension k' of k.

In particular X is a decent algebraic space. We can relate this to being geometrically reduced and geometrically irreducible as follows.

0DMX Lemma 72.14.2. Let k be a field. Let X be a decent algebraic space over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.

Proof. This is an immediate consequence of the definitions because our notion of integral (in the presence of decency) is equivalent to reduced and irreducible. □

0DMY Lemma 72.14.3. Let k be a field. Let X be a proper algebraic space over k.

(1) $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra,
(2) $A = \prod_{i=1,\ldots,n} A_i$ is a product of Artinian local k-algebras, one factor for each connected component of $|X|$,

[1] To be sure, if we say "the algebraic space X is irreducible", we probably mean to say "the topological space $|X|$ is irreducible".

[2] An irreducible space is nonempty.

(3) if X is reduced, then $A = \prod_{i=1,\ldots,n} k_i$ is a product of fields, each a finite extension of k,
(4) if X is geometrically reduced, then k_i is finite separable over k,
(5) if X is geometrically connected, then A is geometrically irreducible over k,
(6) if X is geometrically irreducible, then A is geometrically irreducible over k,
(7) if X is geometrically reduced and connected, then $A = k$, and
(8) if X is geometrically integral, then $A = k$.

Proof. By Cohomology of Spaces, Lemma 69.20.3 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra. This proves (1).

Then A is a product of local rings by Algebra, Lemma 10.53.2 and Algebra, Proposition 10.60.7. If $X = Y \amalg Z$ with Y and Z open subspaces of X, then we obtain an idempotent $e \in A$ by taking the section of $\mathcal{O}_X$ which is 1 on Y and 0 on Z. Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of $|X|$. Finally, as $|X|$ is a Noetherian topological space (by Morphisms of Spaces, Lemma 67.28.6 and Properties of Spaces, Lemma 66.24.2) its connected components are open. Hence the connected components of $|X|$ correspond 1-to-1 with primitive idempotents of A. This proves (2).

If X is reduced, then A is reduced (Properties of Spaces, Lemma 66.21.4). Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 10.25.1). This proves (3).

If X is geometrically reduced, then same thing is true for $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ (see Cohomology of Spaces, Lemma 69.11.2 for equality). This implies that $k_i \otimes_k \overline{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 10.44.1 and 10.44.3. This proves (4).

If X is geometrically connected, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5). Of course (5) implies (6).

If X is geometrically reduced and connected, then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \overline{k}$ is a product of $[k_1 : k]$ copies of $\overline{k}$ and we conclude that $k_1 = k$. This proves (7). Of course (7) implies (8). □

0DMZ Lemma 72.14.4. Let k be a field. Let X be a proper integral algebraic space over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are both nonzero, then $\mathcal{L} \cong \mathcal{O}_X$.

Proof. Let $s \in H^0(X, \mathcal{L})$ and $t \in H^0(X, \mathcal{L}^{\otimes -1})$ be nonzero sections. Let $x \in |X|$ be a point in the support of s. Choose an affine étale neighbourhood $(U, u) \to (X, x)$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $s|_U$ corresponds to a nonzero regular function on the reduced (because X is reduced) scheme U and hence is nonvanishing in a generic point of an irreducible component of U. By Decent Spaces, Lemma 68.20.1 we conclude that the generic point η of $|X|$ is in the support of s. The same is true for t. Then of course st must be nonzero because the local ring of X at η is a field (by aforementioned lemma the local ring has dimension zero, as X is reduced the local ring is reduced, and Algebra, Lemma 10.25.1). However, we have seen that $K = H^0(X, \mathcal{O}_X)$ is a field in Lemma 72.14.3. Thus st is everywhere nonzero and we see that $s : \mathcal{O}_X \to \mathcal{L}$ is an isomorphism. $\square$

72.15. Dimension

0EDA In this section we continue the discussion about dimension. Here is a list of previous material:

(1) dimension is defined in Properties of Spaces, Section 66.9,
(2) dimension of local ring is defined in Properties of Spaces, Section 66.10,
(3) a couple of results in Properties of Spaces, Lemmas 66.22.4 and 66.22.5,
(4) relative dimension is defined in Morphisms of Spaces, Section 67.33,
(5) results on dimension of fibres in Morphisms of Spaces, Section 67.34,
(6) a weak form of the dimension formula Morphisms of Spaces, Section 67.35,
(7) a result on smoothness and dimension Morphisms of Spaces, Lemma 67.37.10,
(8) dimension is $\dim(|X|)$ for decent spaces Decent Spaces, Lemma 68.12.5,
(9) quasi-finite maps and dimension Decent Spaces, Lemmas 68.12.6 and 68.12.7.

In More on Morphisms of Spaces, Section 76.31 we will discuss jumping of dimension in fibres of a finite type morphism.

0EDB Lemma 72.15.1. Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces. Then $\dim(X) \leq \dim(Y)$. If f is surjective then $\dim(X) = \dim(Y)$.

Proof. Choose $V \to Y$ surjective étale with V a scheme. Then $U = X \times_Y V$ is a scheme and $U \to V$ is integral (and surjective if f is surjective). By Properties of Spaces, Lemma 66.22.5 we have $\dim(X) = \dim(U)$ and $\dim(Y) = \dim(V)$. Thus the result follows from the case of schemes which is Morphisms, Lemma 29.44.9. $\square$

0EDC Lemma 72.15.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that

(1) Y is locally Noetherian,
(2) X and Y are integral algebraic spaces,
(3) f is dominant, and
(4) f is locally of finite type.

If $x \in |X|$ and $y \in |Y|$ are the generic points, then

$$\dim(X) \leq \dim(Y) + \text{transcendence degree of } x/y.$$

If f is proper, then equality holds.

Proof. Recall that $|X|$ and $|Y|$ are irreducible sober topological spaces, see discussion following Definition 72.4.1. Thus the fact that f is dominant means that $|f|$ maps x to y. Moreover, $x \in |X|$ is the unique point at which the local ring of X has dimension 0, see Decent Spaces, Lemma 68.20.1. By Morphisms of Spaces, Lemma 67.35.1 we see that the dimension of the local ring of X at any point $x' \in |X|$ is at most the dimension of the local ring of Y at $y' = f(x')$ plus the transcendence degree of x/y. Since the dimension of X, resp. dimension of Y is the supremum of the dimensions of the local rings at x', resp. y' (Properties of Spaces, Lemma 66.10.3) we conclude the inequality holds.

Assume f is proper. Let $V \subset Y$ be a nonempty quasi-compact open subspace. If we can prove the equality for the morphism $f^{-1}(V) \to V$, then we get the equality for $X \to Y$. Thus we may assume that X and Y are quasi-compact. Observe that X is quasi-separated as a locally Noetherian decent algebraic space, see Decent Spaces, Lemma 68.14.1. Thus we may choose $Y' \to Y$ finite surjective where Y' is a scheme, see Limits of Spaces, Proposition 70.16.1. After replacing Y' by a suitable closed subscheme, we may assume Y' is integral, see for example the more general Lemma 72.8.5. By the same lemma, we may choose a closed subspace $X' \subset X \times_Y Y'$ such that X' is integral and $X' \to X$ is finite surjective. Now X' is also locally Noetherian (Morphisms of Spaces, Lemma 67.23.5) and we can use Limits of Spaces, Proposition 70.16.1 once more to choose a finite surjective morphism $X'' \to X'$ with X'' a scheme. As before we may assume that X'' is integral. Picture

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

By Lemma 72.15.1 we have $\dim(X'') = \dim(X)$ and $\dim(Y') = \dim(Y)$. Since X and Y have open neighbourhoods of x, resp. y which are schemes, we readily see that the generic points $x'' \in X''$, resp. $y' \in Y'$ are the unique points mapping to x, resp. y and that the residue field extensions $\kappa(x'')/\kappa(x)$ and $\kappa(y')/\kappa(y)$ are finite. This implies that the transcendence degree of x''/y' is the same as the transcendence degree of x/y. Thus the equality follows from the case of schemes whicn is Morphisms, Lemma 29.52.4. □

72.16. Spaces smooth over fields

06M0 This section is the analogue of Varieties, Section 33.25.

06M1 **Lemma 72.16.1.** *Let k be a field. Let X be an algebraic space smooth over k. Then X is a regular algebraic space.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2). Hence U is regular by Varieties, Lemma 33.25.3. By Properties of Spaces, Definition 66.7.2 this means that X is regular. □

07W4 **Lemma 72.16.2.** *Let k be a field. Let X be an algebraic space smooth over $\mathrm{Spec}(k)$. The set of $x \in |X|$ which are image of morphisms $\mathrm{Spec}(k') \to X$ with $k' \supset k$ finite separable is dense in $|X|$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2). Hence we can apply Varieties, Lemma 33.25.6 to see that the closed points of U whose residue fields are finite separable over k are dense. This implies the lemma by our definition of the topology on $|X|$. □

72.17. Euler characteristics

0DN0 In this section we prove some elementary properties of Euler characteristics of coherent sheaves on algebraic spaces proper over fields.

0DN1 **Definition 72.17.1.** Let k be a field. Let X be a proper algebraic over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of $\mathcal{F}$* is the integer

$$\chi(X, \mathcal{F}) = \sum\nolimits_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Spaces, Lemma 69.7.3) and each of these spaces is finite dimensional (Cohomology of Spaces, Lemma 69.20.3). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

0DN2 **Lemma 72.17.2.** *Let k be a field. Let X be a proper algebraic space over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. □

0EDD **Lemma 72.17.3.** *Let k be a field. Let $f : Y \to X$ be a morphism of algebraic spaces proper over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then*

$$\chi(Y, \mathcal{G}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^i f_* \mathcal{G}$ are coherent and only a finite number of them are nonzero, see Cohomology of Spaces, Lemmas 69.20.2 and 69.8.1. By Cohomology on Sites, Lemma 21.14.5 there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{G})$$

converging to $H^{p+q}(Y, \mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum (-1)^{p+q} \dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y, \mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p + 1 = n$ (this is the meaning of convergence of the spectral sequence), we conclude. □

72.18. Numerical intersections

0DN3 In this section we play around with the Euler characteristic of coherent sheaves on proper algebraic spaces to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

0DN4 **Lemma 72.18.1.** Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. The map

$$(n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most the dimension of the scheme theoretic support of $\mathcal{F}$.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$ (Cohomology of Spaces, Lemma 69.12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes i^*\mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 21.50.1) and Cohomology of Spaces, Lemma 69.8.3. Since $|Z| = \text{Supp}(\mathcal{F})$ we see that it suffices to show

$$P_\mathcal{F}(n_1, \ldots, n_r) : (n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most $\dim(X)$. Let us say property $\mathcal{P}$ holds for the coherent $\mathcal{O}_X$-module $\mathcal{F}$ if the above is true.

We will prove this statement by devissage, more precisely we will check conditions (1), (2), and (3) of Cohomology of Spaces, Lemma 69.14.6 are satisfied.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. By Lemma 72.17.2 we have

$$P_{\mathcal{F}_2}(n_1, \ldots, n_r) = P_{\mathcal{F}_1}(n_1, \ldots, n_r) + P_{\mathcal{F}_3}(n_1, \ldots, n_r)$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then so does the third.

Condition (2) follows because $P_{\mathcal{F}^{\oplus m}}(n_1, \ldots, n_r) = mP_\mathcal{F}(n_1, \ldots, n_r)$.

Proof of (3). Let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. We have to find a coherent module $\mathcal{G}$ on X whose support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$. We will give two constructions: one using Chow's lemma and one using a finite cover by a scheme.

Proof existence $\mathcal{G}$ using a finite cover by a scheme. Choose $\pi : Z' \to Z$ finite surjective where Z' is a scheme, see Limits of Spaces, Proposition 70.16.1. Set $\mathcal{G} = i_*\pi_*\mathcal{O}_{Z'} = (i \circ \pi)_*\mathcal{O}_{Z'}$. Note that Z' is proper over k and that the support of $\mathcal{G}$ is Y (details omitted). We have

$$R(\pi \circ i)_*(\mathcal{O}_{Z'}) = \mathcal{G} \quad \text{and} \quad R(\pi \circ i)_*(\pi^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})) = \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$$

The first equality holds because $i \circ \pi$ is affine (Cohomology of Spaces, Lemma 69.8.2) and the second equality follows from the first and the projection formula (Cohomology on Sites, Lemma 21.50.1). Using Leray (Cohomology on Sites, Lemma 21.14.6) we obtain

$$P_\mathcal{G}(n_1, \ldots, n_r) = \chi(Z', \pi^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}))$$

By the case of schemes (Varieties, Lemma 33.45.1) this is a numerical polynomial in $n_1, \ldots, n_r$ of degree at most $\dim(Z')$. We conclude because $\dim(Z') \leq \dim(Z) \leq \dim(X)$. The first inequality follows from Decent Spaces, Lemma 68.12.7.

Proof existence $\mathcal{G}$ using Chow's lemma. We apply Cohomology of Spaces, Lemma 69.18.1 to the morphism $Z \to \mathrm{Spec}(k)$. Thus we get a surjective proper morphism $f : Y \to Z$ over $\mathrm{Spec}(k)$ where Y is a closed subscheme of $\mathbf{P}^m_k$ for some m. After replacing Y by a closed subscheme we may assume that Y is integral and $f : Y \to Z$ is an alteration, see Lemma 72.8.5. Denote $\mathcal{O}_Y(n)$ the pullback of $\mathcal{O}_{\mathbf{P}^m_k}(n)$. Pick $n > 0$ such that $R^pf_*\mathcal{O}_Y(n) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 69.20.1. We claim that $\mathcal{G} = i_*f_*\mathcal{O}_Y(n)$ satisfies $\mathcal{P}$. Namely, by the case of schemes (Varieties, Lemma 33.45.1) we know that

$$(n_1, \ldots, n_r) \longmapsto \chi(Y, \mathcal{O}_Y(n) \otimes f^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}))$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most $\dim(Y)$. On the other hand, by the projection formula (Cohomology on Sites, Lemma 21.50.1)

$$\begin{aligned} i_*Rf_*\left(\mathcal{O}_Y(n) \otimes f^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})\right) &= i_*Rf_*\mathcal{O}_Y(n) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r} \\ &= \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r} \end{aligned}$$

the last equality by our choice of n. By Leray (Cohomology on Sites, Lemma 21.14.6) we get

$$\chi(Y, \mathcal{O}_Y(n) \otimes f^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})) = P_{\mathcal{G}}(n_1, \ldots, n_r)$$

and we conclude because $\dim(Y) \leq \dim(Z) \leq \dim(X)$. The first inequality holds by Morphisms of Spaces, Lemma 67.35.2 and the fact that $Y \to Z$ is an alteration (and hence the induced extension of residue fields in generic points is finite). □

The following lemma roughly shows that the leading coefficient only depends on the length of the coherent module in the generic points of its support.

0EDE Lemma 72.18.2. Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. Let $d = \dim(\mathrm{Supp}(\mathcal{F}))$. Let $Z_i \subset X$ be the irreducible components of $\mathrm{Supp}(\mathcal{F})$ of dimension d. Let $\overline{x}_i$ be a geometric generic point of Z_i and set $m_i = \mathrm{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i})$. Then

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i\ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< d$.

Proof. We first prove a slightly weaker statement. Namely, say $\dim(X) = N$ and let $X_i \subset X$ be the irreducible components of dimension N. Let $\overline{x}_i$ be a geometric generic point of X_i. The étale local ring $\mathcal{O}_{X,\overline{x}_i}$ is Noetherian of dimension 0, hence for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ the length

$$m_i(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i})$$

is an integer ≥ 0. We claim that

$$E(\mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i(\mathcal{F})\ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< N$. We will prove this using Cohomology of Spaces, Lemma 69.14.6. For any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have $E(\mathcal{F}) = E(\mathcal{F}') + E(\mathcal{F}'')$. This follows from additivity of Euler characteristics (Lemma 72.17.2) and additivity of lengths (Algebra, Lemma

10.52.3). This immediately implies properties (1) and (2) of Cohomology of Spaces, Lemma 69.14.6. Finally, property (3) holds because for $\mathcal{G} = \mathcal{O}_Z$ for any $Z \subset X$ irreducible reduced closed subspace. Namely, if $Z = Z_{i_0}$ for some i_0, then $m_i(\mathcal{G}) = \delta_{i_0 i}$ and we conclude $E(\mathcal{G}) = 0$. If $Z \neq Z_i$ for any i, then $m_i(\mathcal{G}) = 0$ for all i, $\dim(Z) < N$ and we get the result from Lemma 72.18.1.

Proof of the statement as in the lemma. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$ (Cohomology of Spaces, Lemma 69.12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes i^*\mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 21.50.1) and Cohomology of Spaces, Lemma 69.8.3. Since $|Z| = \text{Supp}(\mathcal{F})$ we see that $Z_i \subset Z$ for all i and we see that these are the irreducible components of Z of dimension d. We may and do think of $\overline{x}_i$ as a geometric point of Z. The map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ determines a surjection

$$\mathcal{O}_{X,\overline{x}_i} \to \mathcal{O}_{Z,\overline{x}_i}$$

Via this map we have an isomorphism of modules $\mathcal{G}_{\overline{x}_i} = \mathcal{F}_{\overline{x}_i}$ as $\mathcal{F} = i_*\mathcal{G}$. This implies that

$$m_i = \text{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i}) = \text{length}_{\mathcal{O}_{Z,\overline{x}_i}}(\mathcal{G}_{\overline{x}_i})$$

Thus we see that the expression in the lemma is equal to

$$\chi(Z, \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

and the result follows from the discussion in the first paragraph (applied with Z in stead of X). $\square$

0EDF Definition 72.18.3. Let k be a field. Let X be a proper algebraic space over k. Let $i : Z \to X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. We define the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ as the coefficient of $n_1 \ldots n_d$ in the numerical polynomial

$$\chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

In the special case that $\mathcal{L}_1 = \ldots = \mathcal{L}_d = \mathcal{L}$ we write $(\mathcal{L}^d \cdot Z)$.

The displayed equality in the definition follows from the projection formula (Cohomology, Section 20.54) and Cohomology of Schemes, Lemma 30.2.4. We prove a few lemmas for these intersection numbers.

0EDG Lemma 72.18.4. In the situation of Definition 72.18.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is an integer.

Proof. Any numerical polynomial of degree e in $n_1, \ldots, n_d$ can be written uniquely as a $\mathbf{Z}$-linear combination of the functions $\binom{n_1}{k_1}\binom{n_2}{k_2} \ldots \binom{n_d}{k_d}$ with $k_1 + \ldots + k_d \leq e$. Apply this with $e = d$. Left as an exercise. $\square$

0EDH Lemma 72.18.5. In the situation of Definition 72.18.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is additive: if $\mathcal{L}_i = \mathcal{L}_i' \otimes \mathcal{L}_i''$, then we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_i \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_1 \cdots \mathcal{L}_i' \cdots \mathcal{L}_d \cdot Z) + (\mathcal{L}_1 \cdots \mathcal{L}_i'' \cdots \mathcal{L}_d \cdot Z)$$

Proof. This is true because by Lemma 72.18.1 the function

$$(n_1, \ldots, n_{i-1}, n'_i, n''_i, n_{i+1}, \ldots, n_d) \mapsto \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes (\mathcal{L}'_i)^{\otimes n'_i} \otimes (\mathcal{L}''_i)^{\otimes n''_i} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

is a numerical polynomial of total degree at most d in $d + 1$ variables. □

0EDI **Lemma 72.18.6.** In the situation of Definition 72.18.3 let $Z_i \subset Z$ be the irreducible components of dimension d. Let $m_i = \text{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{O}_{Z,\overline{x}_i})$ where $\overline{x}_i$ is a geometric generic point of Z_i. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \sum m_i (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z_i)$$

Proof. Immediate from Lemma 72.18.2 and the definitions. □

0EDJ **Lemma 72.18.7.** Let k be a field. Let $f : Y \to X$ be a morphism of algebraic spaces proper over k. Let $Z \subset Y$ be an integral closed subspace of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is as in Definition 72.5.2 or 0 if $\dim(f(Z)) < d$.

Proof. In the statement $f(Z) \subset X$ is the scheme theoretic image of f and it is also the reduced induced algebraic space structure on the closed subset $f(|Z|) \subset X$, see Morphisms of Spaces, Lemma 67.16.4. Then Z and $f(Z)$ are reduced, proper (hence decent) algebraic spaces over k, whence integral (Definition 72.4.1). The left hand side is computed using the coefficient of $n_1 \ldots n_d$ in the function

$$\chi(Y, \mathcal{O}_Z \otimes f^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes f^*\mathcal{L}_d^{\otimes n_d}) = \sum (-1)^i \chi(X, R^if_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$$

The equality follows from Lemma 72.17.3 and the projection formula (Cohomology, Lemma 20.54.2). If $f(Z)$ has dimension $< d$, then the right hand side is a polynomial of total degree $< d$ by Lemma 72.18.1 and the result is true. Assume $\dim(f(Z)) = d$. Then by dimension theory (Lemma 72.15.2) we find that the equivalent conditions (1) – (5) of Lemma 72.5.1 hold. Thus $\deg(Z \to f(Z))$ is well defined. By the already used Lemma 72.5.1 we find $f : Z \to f(Z)$ is finite over a nonempty open V of $f(Z)$; after possibly shrinking V we may assume V is a scheme. Let $\xi \in V$ be the generic point. Thus $\deg(f : Z \to f(Z))$ the length of the stalk of $f_*\mathcal{O}_Z$ at ξ over $\mathcal{O}_{X,\xi}$ and the stalk of $R^if_*\mathcal{O}_X$ at ξ is zero for $i > 0$ (for example by Cohomology of Spaces, Lemma 69.4.1). Thus the terms $\chi(X, R^if_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ with $i > 0$ have total degree $< d$ and

$$\chi(X, f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \deg(f : Z \to f(Z))\chi(f(Z), \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_{f(Z)})$$

modulo a polynomial of total degree $< d$ by Lemma 72.18.2. The desired result follows. □

0EDK **Lemma 72.18.8.** Let k be a field. Let X be a proper algebraic space over k. Let $Z \subset X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Assume there exists an effective Cartier divisor $D \subset Z$ such that $\mathcal{L}_1|_Z \cong \mathcal{O}_Z(D)$. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

Proof. We may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$. Thus we may assume $X = Z$ and $\mathcal{L}_1 = \mathcal{O}_X(D)$. Then $\mathcal{L}_1^{-1}$ is the ideal sheaf of D and we can consider the short exact sequence

$$0 \to \mathcal{L}_1^{\otimes -1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Set $P(n_1, \ldots, n_d) = \chi(X, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ and $Q(n_1, \ldots, n_d) = \chi(D, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_D)$. We conclude from additivity (Lemma 72.17.2) that

$$P(n_1, \ldots, n_d) - P(n_1 - 1, n_2, \ldots, n_d) = Q(n_1, \ldots, n_d)$$

Because the total degree of P is at most d, we see that the coefficient of $n_1 \ldots n_d$ in P is equal to the coefficient of $n_2 \ldots n_d$ in Q. $\square$

72.19. Other chapters

CHAPTER 73

Topologies on Algebraic Spaces

03Y4 73.1. Introduction

03Y5 In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [Gro71], [BLR90], [LMB00] and [Knu71]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 7.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

73.2. The general procedure

03Y6 In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 34.2.

Let S be a base scheme. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with S and any set of schemes over S you want to be included. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained, and let $(Sch/S)_{fppf}$ denote the corresponding big fppf site of S. (The above is entirely as prescribed in Topologies, Section 34.7.)

Given choices as above the category of algebraic spaces over S has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over S is of the form U/R for some étale equivalence relation $j : R \to U \times_S U$ with $U, R \in \text{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 65.9.1. Hence we can find a full subcategory Spaces/S of the category of algebraic spaces over S which has a set of objects such that each algebraic space is isomorphic to an object of Spaces/S. We fix a choice of such a category.

In the sections below, given a topology τ, the big site $(\text{Spaces}/S)_\tau$ (resp. the big site $(\text{Spaces}/X)_\tau$ of an algebraic space X over S) has as underlying category the category Spaces/S (resp. the subcategory Spaces/X of Spaces/S, see Categories, Example 4.2.13). The procedure for turning this into a site is as usual by defining a class of τ-coverings and using Sets, Lemma 3.11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the small étale site $X_{étale}$ of an algebraic space X has already been defined in Properties of Spaces, Definition 66.18.1. Its objects are schemes étale over X, of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 34.4.8) is the site $X_{spaces,étale}$ of Properties of Spaces, Definition 66.18.2.

These two sites define the same topos, see Properties of Spaces, Lemma 66.18.3. We will not redefine these in this chapter; instead we will simply use them.

73.3. Zariski topology

03YD In Spaces, Section 65.12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaced by open immersions.

041G Definition 73.3.1. Let S be a scheme, and let X be an algebraic space over S. A Zariski covering of X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is an open immersion and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

Although Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

041H Lemma 73.3.2. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a Zariski covering of X.
(2) If $\{X_i \to X\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a Zariski covering.
(3) If $\{X_i \to X\}_{i \in I}$ is a Zariski covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a Zariski covering.

Proof. Omitted. □

73.4. Étale topology

03YC In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 34.4.

041E Definition 73.4.1. Let S be a scheme, and let X be an algebraic space over S. An étale covering of X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is étale and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.4.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a étale covering of schemes.

0DF1 Lemma 73.4.2. Any Zariski covering is an étale covering.

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism (this follows from Morphisms, Lemma 29.36.9 via Spaces, Lemma 65.5.8 as immersions are representable). □

041F Lemma 73.4.3. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a étale covering of X.

(2) If $\{X_i \to X\}_{i \in I}$ is a étale covering and for each i we have a étale covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a étale covering.
(3) If $\{X_i \to X\}_{i \in I}$ is a étale covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a étale covering.

Proof. Omitted. $\square$

The following lemma tells us that the sites $(\text{Spaces}/X)_{\acute{e}tale}$ and $(\text{Spaces}/X)_{smooth}$ have the same categories of sheaves.

0CFV Lemma 73.4.4. Let S be a scheme. Let X be an algebraic space over S. Let $\{X_i \to X\}_{i \in I}$ be a smooth covering of X. Then there exists an étale covering $\{U_j \to X\}_{j \in J}$ of X which refines $\{X_i \to X\}_{i \in I}$.

Proof. First choose a scheme U and a surjective étale morphism $U \to X$. For each i choose a scheme W_i and a surjective étale morphism $W_i \to X_i$. Then $\{W_i \to X\}_{i \in I}$ is a smooth covering which refines $\{X_i \to X\}_{i \in I}$. Hence $\{W_i \times_X U \to U\}_{i \in I}$ is a smooth covering of schemes. By More on Morphisms, Lemma 37.38.7 we can choose an étale covering $\{U_j \to U\}$ which refines $\{W_i \times_X U \to U\}$. Then $\{U_j \to X\}_{j \in J}$ is an étale covering refining $\{X_i \to X\}_{i \in I}$. $\square$

0DBX Definition 73.4.5. Let S be a scheme. A big étale site $(\text{Spaces}/S)_{\acute{e}tale}$ is any site constructed as follows:

(1) Choose a big étale site $(Sch/S)_{\acute{e}tale}$ as in Topologies, Section 34.4.
(2) As underlying category take the category Spaces/S of algebraic spaces over S (see discussion in Section 73.2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Spaces/S and the class of étale coverings of Definition 73.4.1.

Having defined this, we can localize to get the étale site of an algebraic space.

0DBY Definition 73.4.6. Let S be a scheme. Let $(\text{Spaces}/S)_{\acute{e}tale}$ be as in Definition 73.4.5. Let X be an algebraic space over S, i.e., an object of $(\text{Spaces}/S)_{\acute{e}tale}$. Then the big étale site $(\text{Spaces}/X)_{\acute{e}tale}$ of X is the localization of the site $(\text{Spaces}/S)_{\acute{e}tale}$ at X introduced in Sites, Section 7.25.

Recall that given an algebraic space X over S as in the definition, we already have defined the small étale sites $X_{spaces,\acute{e}tale}$ and $X_{\acute{e}tale}$, see Properties of Spaces, Section 66.18. We will silently identify the corresponding topoi using the inclusion functor $X_{\acute{e}tale} \subset X_{spaces,\acute{e}tale}$ (Properties of Spaces, Lemma 66.18.3) and we will call it the small étale topos of X. Next, we establish some relationships between the topoi associated to these sites.

0DF2 Lemma 73.4.7. Let S be a scheme. Let $f : Y \to X$ be a morphism of $(\text{Spaces}/S)_{\acute{e}tale}$. The inclusion functor $Y_{spaces,\acute{e}tale} \to (\text{Spaces}/X)_{\acute{e}tale}$ is cocontinuous and induces a morphism of topoi

$$i_f : Sh(Y_{\acute{e}tale}) \longrightarrow Sh((\text{Spaces}/X)_{\acute{e}tale})$$

For a sheaf $\mathcal{G}$ on $(\text{Spaces}/X)_{\acute{e}tale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : Y_{spaces,étale} \to (\text{Spaces}/X)_{étale}$. In other words, given an étale morphism $j : U \to Y$ corresponding to an object of $Y_{spaces,étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. The category $Y_{spaces,étale}$ has fibre products and equalizers and u commutes with them. It is immediate that u cocontinuous. The functor u is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

0DF3 Lemma 73.4.8. Let S be a scheme. Let X be an object of $(\text{Spaces}/S)_{étale}$. The inclusion functor $X_{spaces,étale} \to (\text{Spaces}/X)_{étale}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites

$$\pi_X : (\text{Spaces}/X)_{étale} \longrightarrow X_{spaces,étale}$$

and a morphism of topoi

$$i_X : Sh(X_{étale}) \longrightarrow Sh((\text{Spaces}/X)_{étale})$$

such that $\pi_X \circ i_X = \text{id}$. Moreover, $i_X = i_{\text{id}_X}$ with i_{id_X} as in Lemma 73.4.7. In particular the functor $i_X^{-1} = \pi_{X,*}$ is described by the rule $i_X^{-1}(\mathcal{G})(U/X) = \mathcal{G}(U/X)$.

Proof. In this case the functor $u : X_{spaces,étale} \to (\text{Spaces}/X)_{étale}$, in addition to the properties seen in the proof of Lemma 73.4.7 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.21.8. □

0DF4 Definition 73.4.9. In the situation of Lemma 73.4.8 the functor $i_X^{-1} = \pi_{X,*}$ is often called the restriction to the small étale site, and for a sheaf $\mathcal{F}$ on the big étale site we often denote $\mathcal{F}|_{X_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\text{Mor}_{Sh(X_{étale})}(\mathcal{F}|_{X_{étale}}, \mathcal{G}) = \text{Mor}_{Sh((\text{Spaces}/X)_{étale})}(\mathcal{F}, i_{X,*}\mathcal{G})$$
$$\text{Mor}_{Sh(X_{étale})}(\mathcal{G}, \mathcal{F}|_{X_{étale}}) = \text{Mor}_{Sh((\text{Spaces}/X)_{étale})}(\pi_X^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{X,*}\mathcal{G})|_{X_{étale}} = \mathcal{G}$ and we have $(\pi_X^{-1}\mathcal{G})|_{X_{étale}} = \mathcal{G}$.

0DF5 Lemma 73.4.10. Let S be a scheme. Let $f : Y \to X$ be a morphism in $(\text{Spaces}/S)_{étale}$. The functor

$$u : (\text{Spaces}/Y)_{étale} \longrightarrow (\text{Spaces}/X)_{étale}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Spaces}/X)_{étale} \longrightarrow (\text{Spaces}/Y)_{étale}, \quad (U \to X) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((\text{Spaces}/Y)_{étale}) \longrightarrow Sh((\text{Spaces}/X)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 73.4.7). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/Y and

V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DF6 Lemma 73.4.11. Let S be a scheme. Let $f : Y \to X$ be a morphism in $(\mathrm{Spaces}/S)_{étale}$.

(1) We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 73.4.7 and i_T as in Lemma 73.4.8.
(2) The functor $X_{spaces,étale} \to T_{spaces,étale}$, $(U \to X) \mapsto (U \times_X Y \to Y)$ is continuous and induces a morphism of sites

$$f_{spaces,étale} : Y_{spaces,étale} \longrightarrow X_{spaces,étale}$$

The corresponding morphism of small étale topoi is denoted

$$f_{small} : Sh(Y_{étale}) \to Sh(X_{étale})$$

We have $f_{small,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$.
(3) We have a commutative diagram of morphisms of sites

$$\begin{tikzcd} Y_{spaces,étale} \arrow[d, "f_{spaces,étale}"'] & (\mathrm{Spaces}/Y)_{étale} \arrow[l, "\pi_Y"'] \arrow[d, "f_{big}"] \\ X_{spaces,étale} & (\mathrm{Spaces}/X)_{étale} \arrow[l, "\pi_X"] \end{tikzcd}$$

so that $f_{small} \circ \pi_Y = \pi_X \circ f_{big}$ as morphisms of topoi.
(4) We have $f_{small} = \pi_X \circ f_{big} \circ i_Y = \pi_X \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_Y$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : X_{spaces,étale} \to Y_{spaces,étale}$, $u(U \to X) = (U \times_X Y \to Y)$ was shown to give rise to a morphism of sites and correspong morphism of small étale topoi in Properties of Spaces, Lemma 66.18.8. The description of the pushforward is clear.

Part (3) follows because π_X and π_Y are given by the inclusion functors and $f_{spaces,étale}$ and f_{big} by the base change functors $U \mapsto U \times_X Y$.

Statement (4) follows from (3) by precomposing with i_Y. □

In the situation of the lemma, using the terminology of Definition 73.4.9 we have: for $\mathcal{F}$ a sheaf on the big étale site of Y

$$(f_{big,*}\mathcal{F})|_{X_{étale}} = f_{small,*}(\mathcal{F}|_{Y_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of Y, resp. X is given by $\pi_{Y,*}$, resp. $\pi_{X,*}$. A similar formula involving pullbacks and restrictions is false.

0DF7 Lemma 73.4.12. Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ in $(\mathrm{Spaces}/S)_{étale}$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 73.4.10. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 73.4.11. □

0DF8 **Lemma 73.4.13.** Let S be a scheme. Consider a cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

in $(Spaces/S)_{étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/X' that $U' \times_{X'} Y' = U' \times_X Y$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Spaces/Y)_{étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{X'} Y')$ on $X'_{étale}$ (use Lemmas 73.4.7 and 73.4.10). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. □

0DF9 **Remark 73.4.14.** The sites $(Spaces/X)_{étale}$ and $X_{spaces,étale}$ come with structure sheaves. For the small étale site we have seen this in Properties of Spaces, Section 66.21. The structure sheaf $\mathcal{O}$ on the big étale site $(Spaces/X)_{étale}$ is defined by assigning to an object U the global sections of the structure sheaf of U. This makes sense because after all U is an algebraic space itself hence has a structure sheaf. Since $\mathcal{O}_U$ is a sheaf on the étale site of U, the presheaf $\mathcal{O}$ so defined satisfies the sheaf condition for coverings of U, i.e., $\mathcal{O}$ is a sheaf. We can upgrade the morphisms i_f, π_X, i_X, f_{small}, and f_{big} defined above to morphisms of ringed sites, respectively topoi. Let us deal with these one by one.

(1) In Lemma 73.4.7 denote $\mathcal{O}$ the structure sheaf on $(Spaces/X)_{étale}$. We have $(i_f^{-1}\mathcal{O})(U/Y) = \mathcal{O}_U(U) = \mathcal{O}_Y(U)$ by construction. Hence an isomorphism $i_f^\sharp : i_f^{-1}\mathcal{O} \to \mathcal{O}_Y$.
(2) In Lemma 73.4.8 it was noted that i_X is a special case of i_f with $f = \mathrm{id}_X$ hence we are back in case (1).
(3) In Lemma 73.4.8 the morphism π_X satisfies $(\pi_{X,*}\mathcal{O})(U) = \mathcal{O}(U) = \mathcal{O}_X(U)$. Hence we can use this to define $\pi_X^\sharp : \mathcal{O}_X \to \pi_{X,*}\mathcal{O}$.
(4) In Lemma 73.4.11 the extension of f_{small} to a morphism of ringed topoi was discussed in Properties of Spaces, Lemma 66.21.3.
(5) In Lemma 73.4.11 the functor f_{big}^{-1} is simply the restriction via the inclusion functor $(Spaces/Y)_{étale} \to (Spaces/X)_{étale}$. Let $\mathcal{O}_1$ be the structure sheaf on $(Spaces/X)_{étale}$ and let $\mathcal{O}_2$ be the structure sheaf on $(Spaces/Y)_{étale}$. We obtain a canonical isomorphism $f_{big}^\sharp : f_{big}^{-1}\mathcal{O}_1 \to \mathcal{O}_2$.

Moreover, with these definitions compositions work out correctly too. We omit giving a detailed statement and proof.

73.5. Smooth topology

03YB In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 34.5.

041C **Definition 73.5.1.** Let S be a scheme, and let X be an algebraic space over S. A *smooth covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces over S such that each f_i is smooth and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.5.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a smooth covering of schemes.

0DFA Lemma 73.5.2. Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.

Proof. This is clear from the definitions, the fact that an étale morphism is smooth (Morphisms of Spaces, Lemma 67.39.6), and Lemma 73.4.2. □

041D Lemma 73.5.3. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a smooth covering of X.
(2) If $\{X_i \to X\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a smooth covering.
(3) If $\{X_i \to X\}_{i \in I}$ is a smooth covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a smooth covering.

Proof. Omitted. □

To be continued...

73.6. Syntomic topology

03YA In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 34.6.

041A Definition 73.6.1. Let S be a scheme, and let X be an algebraic space over S. A syntomic covering of X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is syntomic and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.6.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a syntomic covering of schemes.

0DFB Lemma 73.6.2. Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic (Morphisms of Spaces, Lemma 67.37.8), and Lemma 73.5.2. □

041B Lemma 73.6.3. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a syntomic covering of X.
(2) If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a syntomic covering.

(3) If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a syntomic covering.

Proof. Omitted. □

To be continued...

73.7. Fppf topology

03Y7 In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 34.7.

03Y8 Definition 73.7.1. Let S be a scheme, and let X be an algebraic space over S. An fppf covering of X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is flat and locally of finite presentation and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.7.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of an fppf covering of schemes.

0DFC Lemma 73.7.2. Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation (Morphisms of Spaces, Lemmas 67.36.5 and 67.36.6) and Lemma 73.6.2. □

03Y9 Lemma 73.7.3. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fppf covering of X.
(2) If $\{X_i \to X\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is an fppf covering.
(3) If $\{X_i \to X\}_{i \in I}$ is an fppf covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is an fppf covering.

Proof. Omitted. □

042T Lemma 73.7.4. Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i \in I}$ is an fppf covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fppf covering such that each T_i is a scheme.

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fppf covering. □

0469 Lemma 73.7.5. Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S. Then the map of sheaves

$$\coprod X_i \longrightarrow X$$

is surjective.

Proof. This follows from Spaces, Lemma 65.5.9. See also Spaces, Remark 65.5.2 in case you are confused about the meaning of this lemma. □

0DBV Definition 73.7.6. Let S be a scheme. A big fppf site $(\text{Spaces}/S)_{fppf}$ is any site constructed as follows:

(1) Choose a big fppf site $(Sch/S)_{fppf}$ as in Topologies, Section 34.7.
(2) As underlying category take the category Spaces/S of algebraic spaces over S (see discussion in Section 73.2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Spaces/S and the class of fppf coverings of Definition 73.7.1.

Having defined this, we can localize to get the fppf site of an algebraic space.

0DBW Definition 73.7.7. Let S be a scheme. Let $(\text{Spaces}/S)_{fppf}$ be as in Definition 73.7.6. Let X be an algebraic space over S, i.e., an object of $(\text{Spaces}/S)_{fppf}$. Then the big fppf site $(\text{Spaces}/X)_{fppf}$ of X is the localization of the site $(\text{Spaces}/S)_{fppf}$ at X introduced in Sites, Section 7.25.

Next, we establish some relationships between the topoi associated to these sites.

0DFD Lemma 73.7.8. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor

$$u : (\text{Spaces}/Y)_{fppf} \longrightarrow (\text{Spaces}/X)_{fppf}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Spaces}/X)_{fppf} \longrightarrow (\text{Spaces}/Y)_{fppf}, \quad (U \to Y) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((\text{Spaces}/Y)_{fppf}) \longrightarrow Sh((\text{Spaces}/X)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/X we have $\text{Mor}_X(u(U), V) = \text{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DFE Lemma 73.7.9. Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ of algebraic spaces over S we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 73.7.8. □

73.8. The ph topology

0DFF In this section we define the ph topology. This is the topology generated by étale coverings and proper surjective morphisms, see Lemma 73.8.7.

0DFG Definition 73.8.1. Let S be a scheme and let X be an algebraic space over S. A ph covering of X is a family of morphisms $\{X_i \to X\}_{i \in I}$ of algebraic spaces over S such that f_i is locally of finite type and such that for every $U \to X$ with U affine there exists a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{X_i \times_X U \to U\}_{i \in I}$.

In other words, there exists indices $i_1, \ldots, i_m \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a ph covering of schemes by Topologies, Definition 34.8.4.

0DFH Lemma 73.8.2. Any fppf covering is a ph covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is a ph covering.

Proof. We will show that an fppf covering is a ph covering, and then the rest follows from Lemma 73.7.2. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over a base scheme S. Let U be an affine scheme and let $U \to X$ be a morphism. We can refine the fppf covering $\{X_i \times_U U \to U\}_{i \in I}$ by an fppf covering $\{T_i \to U\}_{i \in I}$ where T_i is a scheme (Lemma 73.7.4). Then we can find a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining $\{T_i \to U\}_{i \in I}$ by More on Morphisms, Lemma 37.48.7 (and the definition of ph coverings for schemes). Thus $\{X_i \to X\}_{i \in I}$ is a ph covering by definition. $\square$

0DFI Lemma 73.8.3. Let S be a scheme. Let $f : Y \to X$ be a surjective proper morphism of algebraic spaces over S. Then $\{Y \to X\}$ is a ph covering.

Proof. Let $U \to X$ be a morphism with U affine. By Chow's lemma (in the weak form given as Cohomology of Spaces, Lemma 69.18.1) we see that there is a surjective proper morphism of schemes $V \to U$ which factors through $Y \times_X U \to U$. Taking any finite affine open cover of V we obtain a standard ph covering of U refining $\{X \times_Y U \to U\}$ as desired. $\square$

0DFJ Lemma 73.8.4. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a ph covering of X.
(2) If $\{X_i \to X\}_{i \in I}$ is a ph covering and for each i we have a ph covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a ph covering.
(3) If $\{X_i \to X\}_{i \in I}$ is a ph covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a ph covering.

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i \in I}$ a ph covering as in (3). By Morphisms of Spaces, Lemma 67.23.3 the morphisms $X' \times_X X_i \to X'$ are locally of finite type. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i \in I}$ means there exists a standard ph covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i \in I}$ is a ph covering. This proves (3).

Let $\{X_i \to X\}_{i \in I}$ and $\{X_{ij} \to X_i\}_{j \in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard ph covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard ph coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 34.8.3 the family $\{Z_{j,l} \to Z\}$ can be refined by a standard ph covering. Hence we conclude that $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a ph covering. $\square$

0DFK Definition 73.8.5. Let S be a scheme. A big ph site $(\text{Spaces}/S)_{ph}$ is any site constructed as follows:

(1) Choose a big ph site $(Sch/S)_{ph}$ as in Topologies, Section 34.8.
(2) As underlying category take the category Spaces$/S$ of algebraic spaces over S (see discussion in Section 73.2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Spaces$/S$ and the class of ph coverings of Definition 73.8.1.

Having defined this, we can localize to get the ph site of an algebraic space.

0DFL Definition 73.8.6. Let S be a scheme. Let $(\text{Spaces}/S)_{ph}$ be as in Definition 73.8.5. Let X be an algebraic space over S, i.e., an object of $(\text{Spaces}/S)_{ph}$. Then the big ph site $(\text{Spaces}/X)_{ph}$ of X is the localization of the site $(\text{Spaces}/S)_{ph}$ at X introduced in Sites, Section 7.25.

Here is the promised characterization of ph sheaves.

0DFM Lemma 73.8.7. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $(\text{Spaces}/X)_{ph}$. Then $\mathcal{F}$ is a sheaf if and only if

(1) $\mathcal{F}$ satisfies the sheaf condition for étale coverings, and
(2) if $f : V \to U$ is a proper surjective morphism of $(\text{Spaces}/X)_{ph}$, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(\text{Spaces}/X)_{ph}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}$ restricting to s_i over T_i. Let $\{U_j \to T\}$ be an étale covering with U_j affine. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties. $\square$

Next, we establish some relationships between the topoi associated to these sites.

0DFN Lemma 73.8.8. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor

$$u : (\text{Spaces}/Y)_{ph} \longrightarrow (\text{Spaces}/X)_{ph}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Spaces}/X)_{ph} \longrightarrow (\text{Spaces}/Y)_{ph}, \quad (U \to Y) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((\text{Spaces}/Y)_{ph}) \longrightarrow Sh((\text{Spaces}/X)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DFP Lemma 73.8.9. Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ of algebraic spaces over S we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 73.8.8. □

0DK6 Lemma 73.8.10. Let S be a scheme. Let X be an algebraic space over S. Let P be a property of objects in $(\mathrm{Spaces}/X)_{fppf}$ such that whenever $\{U_i \to U\}$ is a covering in $(\mathrm{Spaces}/X)_{fppf}$, then

$$P(U_{i_0} \times_U \ldots \times_U U_{i_p}) \text{ for all } p \geq 0,\ i_0, \ldots, i_p \in I \Rightarrow P(U)$$

If $P(U)$ for all U affine and flat, locally of finite presentation over X, then $P(X)$.

Proof. Let U be a separated algebraic space locally of finite presentation over X. Then we can choose an étale covering $\{U_i \to U\}_{i \in I}$ with V_i affine. Since U is separated, we conclude that $U_{i_0} \times_U \ldots \times_U U_{i_p}$ is always affine. Whence $P(U_{i_0} \times_U \ldots \times_U U_{i_p})$ always. Hence $P(U)$ holds. Choose a scheme U which is a disjoint union of affines and a surjective étale morphism $U \to X$. Then $U \times_X \ldots \times_X U$ (with $p+1$ factors) is a separated algebraic space étale over X. Hence $P(U \times_X \ldots \times_X U)$ by the above. We conclude that $P(X)$ is true. □

73.9. Fpqc topology

03MP We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 34.9. We will show in Descent on Spaces, Proposition 74.4.1 that quasi-coherent sheaves descent along these.

03MQ Definition 73.9.1. Let S be a scheme, and let X be an algebraic space over S. An fpqc covering of X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is flat and such that for every affine scheme Z and morphism $h : Z \to X$ there exists a standard fpqc covering $\{g_j : Z_j \to Z\}_{j=1,\ldots,m}$ which refines the family $\{X_i \times_X Z \to Z\}_{i \in I}$.

In other words, there exists indices $i_1, \ldots, i_m \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 34.9.11.

0DFQ Lemma 73.9.2. Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 73.7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering of algebraic spaces over S. By definition this means that the f_i are flat which checks the first condition of Definition 73.9.1. To check the second, let $V \to U$ be a morphism with V affine. We may choose an étale covering $\{V_{ij} \to V \times_U U_i\}$ with V_{ij} affine. Then the compositions $f_{ij} : V_{ij} \to V \times_U U_i \to V$ are flat and locally of finite presentation as compositions of such (Morphisms of Spaces, Lemmas 67.28.2, 67.30.3, 67.39.7, and 67.39.8). Hence these morphisms are open (Morphisms of Spaces, Lemma 67.30.6) and we see that $|V| = \bigcup_{i\in I}\bigcup_{j\in J_i} f_{ij}(|V_{ij}|)$ is an open covering of $|V|$. Since $|V|$ is quasi-compact, this covering has a finite refinement. Say $V_{i_1j_1}, \ldots, V_{i_Nj_N}$ do the job. Then $\{V_{i_kj_k} \to V\}_{k=1,\ldots,N}$ is a standard fpqc covering of V refinining the family $\{U_i \times_U V \to V\}$. This finishes the proof. $\square$

03MR Lemma 73.9.3. Let S be a scheme. Let X be an algebraic space over S.

(1) If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fpqc covering of X.
(2) If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering.
(3) If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering.

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i\in I}$ an fpqc covering as in (3). By Morphisms of Spaces, Lemma 67.30.4 the morphisms $X' \times_X X_i \to X'$ are flat. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i\in I}$ means there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering. This proves (3).

Let $\{X_i \to X\}_{i\in I}$ and $\{X_{ij} \to X_i\}_{j\in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard fpqc coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 34.9.10 the family $\{Z_{j,l} \to Z\}$ is a standard fpqc covering. Hence we conclude that $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering. $\square$

03MS Lemma 73.9.4. Let S be a scheme, and let X be an algebraic space over S. Suppose that $\{f_i : X_i \to X\}_{i\in I}$ is a family of morphisms of algebraic spaces with target X. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $\{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X if and only if $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering of U.

Proof. If $\{X_i \to X\}_{i\in I}$ is an fpqc covering, then so is $\{U \times_X X_i \to U\}_{i\in I}$ by Lemma 73.9.3. Assume that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering. Let $h : Z \to X$ be a morphism from an affine scheme towards X. Then we see that $U \times_X Z \to Z$ is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens $W_1, \ldots, W_t$ of $U \times_X Z$ whose images cover Z. For each j we may apply the condition that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering to the morphism $W_j \to U$, and obtain a standard fpqc covering $\{W_{jl} \to W_j\}$ which refines $\{W_j \times_X X_i \to W_j\}_{i\in I}$. Hence $\{W_{jl} \to Z\}$ is a standard fpqc covering of Z (see Topologies, Lemma 34.9.10) which refines $\{Z \times_X X_i \to X\}$ and we win. $\square$

0419 Lemma 73.9.5. Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fpqc covering such that each T_i is a scheme.

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fpqc covering. $\square$

To be continued...

73.10. Other chapters

CHAPTER 74

Descent and Algebraic Spaces

03YE 74.1. Introduction

03YF In the chapter on topologies on algebraic spaces (see Topologies on Spaces, Section 73.1) we introduced étale, fppf, smooth, syntomic and fpqc coverings of algebraic spaces. In this chapter we discuss what kind of structures over algebraic spaces can be descended through such coverings. See for example [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d].

74.2. Conventions

041I The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

74.3. Descent data for quasi-coherent sheaves

04W2 This section is the analogue of Descent, Section 35.2 for algebraic spaces. It makes sense to read that section first.

04W3 Definition 74.3.1. Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on X_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{X_i \times_X X_j}$-modules $\varphi_{ij} : \text{pr}_0^*\mathcal{F}_i \to \text{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd} \text{pr}_0^*\mathcal{F}_i \arrow[rr, "\text{pr}_{02}^*\varphi_{ik}"'] \arrow[dr, "\text{pr}_{01}^*\varphi_{ij}"'] & & \text{pr}_2^*\mathcal{F}_k \\ & \text{pr}_1^*\mathcal{F}_j \arrow[ur, "\text{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of $\mathcal{O}_{X_i \times_X X_j \times_X X_k}$-modules commutes. This is called the cocycle condition.

(2) A morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{X_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the

diagrams

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^*\mathcal{F}_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^*\mathcal{F}'_j \end{tikzcd}$$

commute.

04W4 Lemma 74.3.2. Let S be a scheme. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be families of morphisms of algebraic spaces over S with fixed targets. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j \in J}$. Then

(1) The system
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i \in I}$.
(2) This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.
(3) Given a second morphism $(g', \alpha' : I \to J, (g'_i))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g'_i)^*\mathcal{F}_{\alpha'(i)}, (g'_i \times g'_{i'})^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g'_i)^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g'_i) : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. $\square$

Let $g : U \to V$ be a morphism of algebraic spaces. The lemma above tells us that there is a well defined pullback functor between the categories of descent data relative to families of maps with target V and U provided there is a morphism between those families of maps which "lives over g".

04W5 Definition 74.3.3. Let S be a scheme. Let $\{U_i \to U\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{U \to U\}$ the trivial descent datum.
(2) The pullback of the trivial descent datum to $\{U_i \to U\}$ is called the canonical descent datum. Notation: $(\mathcal{F}|_{U_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is said to be effective if there exists a quasi-coherent sheaf $\mathcal{F}$ on U such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{U_i}, can)$.

04W6 Lemma 74.3.4. Let S be a scheme. Let U be an algebraic space over S. Let $\{U_i \to U\}$ be a Zariski covering of U, see Topologies on Spaces, Definition 73.3.1. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to U\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_U$-modules to the category of descent data with respect to $\{U_i \to U\}$ is fully faithful.

Proof. Omitted. $\square$

74.4. Fpqc descent of quasi-coherent sheaves

04W7 The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

04W8 Proposition 74.4.1. Let S be a scheme. Let $\{X_i \to X\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 73.9.1. Any descent datum on quasi-coherent sheaves for $\{X_i \to X\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data with respect to $\{X_i \to X\}$ is fully faithful.

Proof. This is more or less a formal consequence of the corresponding result for schemes, see Descent, Proposition 35.5.2. Here is a strategy for a proof:

(1) The fact that $\{X_i \to X\}$ is a refinement of the trivial covering $\{X \to X\}$ gives, via Lemma 74.3.2, a functor $QCoh(\mathcal{O}_X) \to DD(\{X_i \to X\})$ from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data for the given family.
(2) In order to prove the proposition we will construct a quasi-inverse functor $back : DD(\{X_i \to X\}) \to QCoh(\mathcal{O}_X)$.
(3) Applying again Lemma 74.3.2 we see that there is a functor $DD(\{X_i \to X\}) \to DD(\{T_j \to X\})$ if $\{T_j \to X\}$ is a refinement of the given family. Hence in order to construct the functor *back* we may assume that each X_i is a scheme, see Topologies on Spaces, Lemma 73.9.5. This reduces us to the case where all the X_i are schemes.
(4) A quasi-coherent sheaf on X is by definition a quasi-coherent $\mathcal{O}_X$-module on $X_{étale}$. Now for any $U \in \text{Ob}(X_{étale})$ we get an fppf covering $\{U_i \times_X X_i \to U\}$ by schemes and a morphism $g : \{U_i \times_X X_i \to U\} \to \{X_i \to X\}$ of coverings lying over $U \to X$. Given a descent datum $\xi = (\mathcal{F}_i, \varphi_{ij})$ we obtain a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_{\xi,U}$ corresponding to the pullback $g^*\xi$ of Lemma 74.3.2 to the covering of U and using effectivity for fppf covering of schemes, see Descent, Proposition 35.5.2.
(5) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is functorial in ξ. Omitted.
(6) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is compatible with morphisms $U \to U'$ of the site $X_{étale}$, so that the system of sheaves $\mathcal{F}_{\xi,U}$ corresponds to a quasi-coherent $\mathcal{F}_\xi$ on $X_{étale}$, see Properties of Spaces, Lemma 66.29.3. Details omitted.
(7) Check that $back : \xi \mapsto \mathcal{F}_\xi$ is quasi-inverse to the functor constructed in (1). Omitted.

This finishes the proof. □

74.5. Quasi-coherent modules and affines

0H02 Let S be a scheme. Let X be an algebraic space over S. Recall that $X_{affine,étale}$ is the full subcategory of $X_{étale}$ whose objects are affine turned into a site by declaring the coverings to be the standard étale coverings. See Properties of Spaces, Definition 66.18.5. By Properties of Spaces, Lemma 66.18.6 we have an equivalence of topoi $g : Sh(X_{affine,étale}) \to Sh(X_{étale})$ whose pullback functor is given by restriction. Recall that $\mathcal{O}_X$ denotes the structure sheaf on $X_{étale}$. Then we obtain an equivalence

0H03 (74.5.0.1)
$$(Sh(X_{affine,étale}), \mathcal{O}_X|_{X_{affine,étale}}) \longrightarrow (Sh(X_{étale}), \mathcal{O}_X)$$

of ringed topoi. We will often write $\mathcal{O}_X$ in stead of $\mathcal{O}_X|_{X_{affine,étale}}$. Having said this we can compare quasi-coherent modules as well.

0H04 Lemma 74.5.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules on $X_{affine,étale}$. The following are equivalent

(1) for every morphism $U \to U'$ of $X_{affine,étale}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}_X(U')} \mathcal{O}_X(U) \to \mathcal{F}(U)$ is an isomorphism,
(2) $\mathcal{F}$ is a quasi-coherent module on the ringed site $(X_{affine,étale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 18.23.1,
(3) $\mathcal{F}$ corresponds to a quasi-coherent module on X via the equivalence (74.5.0.1),

Proof. Assume (1) holds. To show that $\mathcal{F}$ is a sheaf, let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a covering of $X_{affine,étale}$. The sheaf condition for $\mathcal{F}$ and $\mathcal{U}$, by our assumption on $\mathcal{F}$, reduces to showing that

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i \times_U U_j)$$

is exact. This is true because $\mathcal{O}_X(U) \to \prod \mathcal{O}_X(U_i)$ is faithfully flat (by Descent, Lemma 35.9.1 and the fact that coverings in $X_{affine,étale}$ are standard étale coverings) and we may apply Descent, Lemma 35.3.6. Next, we show that $\mathcal{F}$ is quasi-coherent on $X_{affine,étale}$. Namely, for U in $X_{affine,étale}$, set $R = \mathcal{O}_X(U)$ and choose a presentation

$$\bigoplus\nolimits_{k\in K} R \longrightarrow \bigoplus\nolimits_{l\in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $X_{affine,étale}$ we obtain a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}_X(U') \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}_X(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $X_{affine,etale}/U$ has a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}_X|_{X_{affine,étale}/U} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}_X|_{X_{affine,étale}/U} \longrightarrow \mathcal{F}|_{X_{affine,étale}/U} \longrightarrow 0$$

as required to show that $\mathcal{F}$ is quasi-coherent. With apologies for the horrible notation, this finishes the proof that (1) implies (2).

Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 18.23.2) we see that the equivalence (74.5.0.1) induces an equivalence between categories of quasi-coherent modules. Thus we have the equivalence of (2) and (3).

Let us assume (3) and prove (1). Namely, let $\mathcal{G}$ be a quasi-coherent module on X corresponding to $\mathcal{F}$. Let $h : U \to U' \to X$ be a morphism of $X_{affine,étale}$. Denote $f : U \to X$ and $f' : U' \to X$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 26.7.3. ☐

74.6. Descent of finiteness properties of modules

060T This section is the analogue for the case of algebraic spaces of Descent, Section 35.7. The goal is to show that one can check a quasi-coherent module has a certain finiteness conditions by checking on the members of a covering. We will repeatedly use the following proof scheme. Suppose that X is an algebraic space, and that $\{X_i \to X\}$ is a fppf (resp. fpqc) covering. Let $U \to X$ be a surjective étale morphism such that U is a scheme. Then there exists an fppf (resp. fpqc) covering $\{Y_j \to X\}$ such that

(1) $\{Y_j \to X\}$ is a refinement of $\{X_i \to X\}$,
(2) each Y_j is a scheme, and
(3) each morphism $Y_j \to X$ factors though U, and
(4) $\{Y_j \to U\}$ is an fppf (resp. fpqc) covering of U.

Namely, first refine $\{X_i \to X\}$ by an fppf (resp. fpqc) covering such that each X_i is a scheme, see Topologies on Spaces, Lemma 73.7.4, resp. Lemma 73.9.5. Then set $Y_i = U \times_X X_i$. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type, of finite presentation, etc if and only if the quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}|_U$ is of finite type, of finite presentation, etc. Hence we can use the existence of the refinement $\{Y_j \to X\}$ to reduce the proof of the following lemmas to the case of schemes. We will indicate this by saying that "the result follows from the case of schemes by étale localization".

060U Lemma 74.6.1. Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.1, by étale localization. $\square$

060V Lemma 74.6.2. Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.3, by étale localization. $\square$

060W Lemma 74.6.3. Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.5, by étale localization. $\square$

060X Lemma 74.6.4. Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.6, by étale localization. $\square$

The definition of a locally projective quasi-coherent sheaf can be found in Properties of Spaces, Section 66.31. It is also proved there that this notion is preserved under pullback.

060Y Lemma 74.6.5. Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.7, by étale localization. $\square$

We also add here two results which are related to the results above, but are of a slightly different nature.

060Z Lemma 74.6.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 35.7.9. □

0610 Lemma 74.6.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 35.7.10. □

74.7. Fpqc coverings

04P0 This section is the analogue of Descent, Section 35.13. At the moment we do not know if all of the material for fpqc coverings of schemes holds also for algebraic spaces.

04P1 Lemma 74.7.1. Let S be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Suppose that for each i we have an open subspace $W_i \subset T_i$ such that for all $i, j \in I$ we have $\mathrm{pr}_0^{-1}(W_i) = \mathrm{pr}_1^{-1}(W_j)$ as open subspaces of $T_i \times_T T_j$. Then there exists a unique open subspace $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.

Proof. By Topologies on Spaces, Lemma 73.9.5 we may assume each T_i is a scheme. Choose a scheme U and a surjective étale morphism $U \to T$. Then $\{T_i \times_T U \to U\}$ is an fpqc covering of U and $T_i \times_T U$ is a scheme for each i. Hence we see that the collection of opens $W_i \times_T U$ comes from a unique open subscheme $W' \subset U$ by Descent, Lemma 35.13.6. As $U \to X$ is open we can define $W \subset X$ the Zariski open which is the image of W', see Properties of Spaces, Section 66.4. We omit the verification that this works, i.e., that W_i is the inverse image of W for each i. □

04P2 Lemma 74.7.2. Let S be a scheme. Let $\{T_i \to T\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 73.9.1. Then given an algebraic space B over S the sequence

$$\mathrm{Mor}_S(T, B) \longrightarrow \prod\nolimits_i \mathrm{Mor}_S(T_i, B) \rightrightarrows \prod\nolimits_{i,j} \mathrm{Mor}_S(T_i \times_T T_j, B)$$

is an equalizer diagram. In other words, every representable functor on the category of algebraic spaces over S satisfies the sheaf condition for fpqc coverings.

Proof. We know this is true if $\{T_i \to T\}$ is an fpqc covering of schemes, see Properties of Spaces, Proposition 66.17.1. This is the key fact and we encourage the reader to skip the rest of the proof which is formal. Choose a scheme U and a surjective étale morphism $U \to T$. Let U_i be a scheme and let $U_i \to T_i \times_T U$ be a surjective étale morphism. Then $\{U_i \to U\}$ is an fpqc covering. This follows from Topologies on Spaces, Lemmas 73.9.3 and 73.9.4. By the above we have the result for $\{U_i \to U\}$.

What this means is the following: Suppose that $b_i : T_i \to B$ is a family of morphisms with $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_j \to B$. Then we let $a_i : U_i \to B$ be the composition of $U_i \to T_i$ with b_i. By what was said above we find a unique morphism $a : U \to B$ such that a_i is the composition of a with $U_i \to U$. The uniqueness guarantees that $a \circ \mathrm{pr}_0 = a \circ \mathrm{pr}_1$ as morphisms $U \times_T U \to B$. Then since $T = U/(U \times_T U)$ as a sheaf, we find that a comes from a unique morphism $b : T \to B$. Chasing diagrams we find that b is the morphism we are looking for. $\square$

74.8. Descent of finiteness and smoothness properties of morphisms

06NQ The following type of lemma is occasionally useful.

06NR **Lemma 74.8.1.** *Let S be a scheme. Let $X \to Y \to Z$ be morphism of algebraic spaces. Let P be one of the following properties of morphisms of algebraic spaces over S: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $X \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$. Then $Y \to Z$ is P.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By assumption we can find an fppf covering $\{V_i \to V\}$ and lifts $V_i \to X$ of the morphism $V_i \to Y$. Since $U \to X$ is surjective étale we see that over the members of the fppf covering $\{V_i \times_X U \to V\}$ we have lifts into U. Hence $U \to V$ induces a surjection of sheaves on $(Sch/S)_{fppf}$. By our definition of what it means to have property P for a morphism of algebraic spaces (see Morphisms of Spaces, Definition 67.30.1, Definition 67.23.1, and Definition 67.28.1) we see that $U \to W$ has P and we have to show $V \to W$ has P. Thus we reduce the question to the case of morphisms of schemes which is treated in Descent, Lemma 35.14.8. $\square$

A more standard case of the above lemma is the following. (The version with "flat" follows from Morphisms of Spaces, Lemma 67.31.5.)

0AHC **Lemma 74.8.2.** *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that f is surjective, flat, and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. Since $\{X \to Y\}$ is an fppf covering, it induces a surjection of fppf sheaves (Topologies on Spaces, Lemma 73.7.5) and the lemma is a special case of Lemma 74.8.1. On the other hand, an easier argument is to deduce it from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 67.23.4 and 67.28.4). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.3. $\square$

0AHD Lemma 74.8.3. Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

(1) f is surjective, and syntomic (resp. smooth, resp. étale),
(2) p is syntomic (resp. smooth, resp. étale).

Then q is syntomic (resp. smooth, resp. étale).

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 67.36.4, 67.37.4, and 67.39.2). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.4. $\square$

Actually we can strengthen this result as follows.

0AHE Lemma 74.8.4. Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

(1) f is surjective, flat, and locally of finite presentation,
(2) p is smooth (resp. étale).

Then q is smooth (resp. étale).

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 67.37.4 and 67.39.2). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.5. $\square$

0AHF **Lemma 74.8.5.** Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

(1) f is surjective, flat, and locally of finite presentation,
(2) p is syntomic.

Then both q and f are syntomic.

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemma 67.36.4). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.7. □

74.9. Descending properties of spaces

06DP In this section we put some results of the following kind.

06DQ **Lemma 74.9.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. If f is flat at x and X is geometrically unibranch at x, then Y is geometrically unibranch at $f(x)$.

Proof. Consider the map of étale local rings $\mathcal{O}_{Y,f(\overline{x})} \to \mathcal{O}_{X,\overline{x}}$. By Morphisms of Spaces, Lemma 67.30.8 this is flat. Hence if $\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime, so does $\mathcal{O}_{Y,f(\overline{x})}$ (by going down, see Algebra, Lemma 10.39.19). □

06MI **Lemma 74.9.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat and surjective and X is reduced, then Y is reduced.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat, the morphism of schemes $U \to V$ is surjective and flat. In this way we reduce the problem to the case of schemes (as reducedness of X and Y is defined in terms of reducedness of U and V, see Properties of Spaces, Section 66.7). The case of schemes is Descent, Lemma 35.19.1. □

06MJ **Lemma 74.9.3.** Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is locally Noetherian, then Y is locally Noetherian.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective, flat, and locally of finite presentation the morphism of schemes $U \to V$ is surjective, flat, and locally of finite presentation. In this way we reduce the problem to the case of schemes (as being locally Noetherian for X and Y is defined in terms of being locally Noetherian of U and V, see Properties of Spaces, Section 66.7). In the case of schemes the result follows from Descent, Lemma 35.16.1. □

06MK Lemma 74.9.4. Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.

Proof. By Lemma 74.9.3 we know that Y is locally Noetherian. Choose a scheme V and a surjective étale morphism $V \to Y$. It suffices to prove that the local rings of V are all regular local rings, see Properties, Lemma 28.9.2. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat the morphism of schemes $U \to V$ is surjective and flat. By assumption U is a regular scheme in particular all of its local rings are regular (by the lemma above). Hence the lemma follows from Algebra, Lemma 10.110.9. □

0GB3 Lemma 74.9.5. Let $f : X \to Y$ be a smooth morphism of algebraic spaces. If Y is reduced, then X is reduced. If f is surjective and X is reduced, then Y is reduced.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are surjective and étale, and $U \to X \times_Y V$ is surjective étale. Observe that X is a reduced algebraic space if and only if U is a reduced scheme by our definition of reduced algebraic spaces in Properties of Spaces, Section 66.7. Similarly for Y and V. The morphism $U \to V$ is a smooth morphism of schemes, see Morphisms of Spaces, Lemma 67.37.4. Since being reduced is local for the smooth topology for schemes (Descent, Lemma 35.18.1) we see that U is reduced if V is reduced. On the other hand, if $X \to Y$ is surjective, then $U \to V$ is surjective and in this case if U is reduced, then V is reduced. □

74.10. Descending properties of morphisms

03YG In this section we introduce the notion of when a property of morphisms of algebraic spaces is local on the target in a topology. Please compare with Descent, Section 35.22.

03YH Definition 74.10.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. We say $\mathcal{P}$ is τ local on the base, or τ local on the target, or local on the base for the τ-topology if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ of algebraic spaces and any morphism of algebraic spaces $f : X \to Y$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

06EM Lemma 74.10.2. Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. $\square$

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subspace $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

06R2 Lemma 74.10.3. Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. For any morphism of algebraic spaces $f : X \to Y$ over S there exists a largest open subspace $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,

(1) if $g : Y' \to Y$ is a morphism of algebraic spaces which is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then g factors through $W(f)$,
(2) if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and
(3) if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.

Proof. Consider the union $W_{set} \subset |Y|$ of the images $g(|Y'|) \subset |Y|$ of morphisms $g : Y' \to Y$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms of Spaces, Lemma 67.30.6) we see that W_{set} is an open subset of $|Y|$. Denote $W \subset Y$ the open subspace whose underlying set of points is W_{set}, see Properties of Spaces, Lemma 66.4.8. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 74.10.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. $\square$

041J Lemma 74.10.4. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume

(1) if $X_i \to Y_i$, $i = 1, 2$ have property $\mathcal{P}$ so does $X_1 \amalg X_2 \to Y_1 \amalg Y_2$,
(2) a morphism of algebraic spaces $f : X \to Y$ has property $\mathcal{P}$ if and only if for every affine scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f has property $\mathcal{P}$, and
(3) for any surjective flat morphism of affine schemes $Z' \to Z$ over S and a morphism $f : X \to Z$ from an algebraic space to Z we have

$$f' : Z' \times_Z X \to Z' \text{ has } \mathcal{P} \Rightarrow f \text{ has } \mathcal{P}.$$

Then $\mathcal{P}$ is fpqc local on the base.

Proof. If $\mathcal{P}$ has property (2), then it is automatically stable under any base change. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $f_i : Y_i \times_Y X \to Y_i$ has property $\mathcal{P}$. Our goal is to show that f has $\mathcal{P}$. Let Z be an affine scheme, and let $Z \to Y$ be a morphism. By (2) it suffices to show that the morphism of algebraic spaces $Z \times_Y X \to Z$ has $\mathcal{P}$. Since $\{Y_i \to Y\}_{i \in I}$ is an fpqc covering we know there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to Y_{i_j}$ over Y for suitable indices $i_j \in I$. Since f_{i_j} has $\mathcal{P}$ we see that

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} (Y_{i_j} \times_Y X) \longrightarrow Z_j$$

has $\mathcal{P}$ as a base change of f_{i_j} (see first remark of the proof). Set $Z' = \coprod_{j=1,\ldots,n} Z_j$, so that $Z' \to Z$ is a flat and surjective morphism of affine schemes over S. By (1) we conclude that $Z' \times_Y X \to Z'$ has property $\mathcal{P}$. Since this is the base change of the morphism $Z \times_Y X \to Z$ by the morphism $Z' \to Z$ we conclude that $Z \times_Y X \to Z$ has property $\mathcal{P}$ as desired. $\square$

74.11. Descending properties of morphisms in the fpqc topology

041K In this section we find a large number of properties of morphisms of algebraic spaces which are local on the base in the fpqc topology. Please compare with Descent, Section 35.23 for the case of morphisms of schemes.

041L Lemma 74.11.1. Let S be a scheme. The property $\mathcal{P}(f) =$"f is quasi-compact" is fpqc local on the base on algebraic spaces over S.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.8.8. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-compact. We have to show that f is quasi-compact. To see this, using Morphisms of Spaces, Lemma 67.8.8 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the fibre product $Y \times_Z X$ is quasi-compact. Here is a picture:

041M (74.11.1.1)
$$\begin{tikzcd}
Y \times_Z Z' \times_Z X \arrow[rr] \arrow[dr] \arrow[dd] & & Z' \times_Z X \arrow[dr] \arrow[dd, "f'"] & \\
& Y \times_Z X \arrow[rr] \arrow[dd] & & X \arrow[dd, "f"] \\
Y \times_Z Z' \arrow[rr] \arrow[dr] & & Z' \arrow[dr] & \\
& Y \arrow[rr] & & Z
\end{tikzcd}$$

Note that all squares are cartesian and the bottom square consists of affine schemes. The assumption that f' is quasi-compact combined with the fact that $Y \times_Z Z'$ is affine implies that $Y \times_Z Z' \times_Z X$ is quasi-compact. Since

$$Y \times_Z Z' \times_Z X \longrightarrow Y \times_Z X$$

is surjective as a base change of $Z' \to Z$ we conclude that $Y \times_Z X$ is quasi-compact, see Morphisms of Spaces, Lemma 67.8.6. This finishes the proof. $\square$

041N Lemma 74.11.2. Let S be a scheme. The property $\mathcal{P}(f) =$"f is quasi-separated" is fpqc local on the base on algebraic spaces over S.

Proof. A base change of a quasi-separated morphism is quasi-separated, see Morphisms of Spaces, Lemma 67.4.4. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is quasi-separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is quasi-compact. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 73.9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.11.1 that Δ is quasi-compact, i.e., f is quasi-separated. $\square$

041O Lemma 74.11.3. Let S be a scheme. The property $\mathcal{P}(f) =$"f is universally closed" is fpqc local on the base on algebraic spaces over S.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.9.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is universally closed. We have to show that f is universally closed. To see this, using Morphisms of Spaces, Lemma 67.9.5 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the map $|Y \times_Z X| \to |Y|$ is closed. Consider the cube (74.11.1.1). The assumption that f' is universally closed implies that $|Y \times_Z Z' \times_Z X| \to |Y \times_Z Z'|$ is closed. As $Y \times_Z Z' \to Y$ is quasi-compact, surjective, and flat as a base change of $Z' \to Z$ we see the map $|Y \times_Z Z'| \to |Y|$ is submersive, see Morphisms, Lemma 29.25.12. Moreover the map

$$|Y \times_Z Z' \times_Z X| \longrightarrow |Y \times_Z Z'| \times_{|Y|} |Y \times_Z X|$$

is surjective, see Properties of Spaces, Lemma 66.4.3. It follows by elementary topology that $|Y \times_Z X| \to |Y|$ is closed. $\square$

041P Lemma 74.11.4. Let S be a scheme. The property $\mathcal{P}(f) =$"f is universally open" is fpqc local on the base on algebraic spaces over S.

Proof. The proof is the same as the proof of Lemma 74.11.3. $\square$

0CFW Lemma 74.11.5. The property $\mathcal{P}(f) =$"f is universally submersive" is fpqc local on the base.

Proof. The proof is the same as the proof of Lemma 74.11.3. $\square$

041Q Lemma 74.11.6. The property $\mathcal{P}(f) =$"f is surjective" is fpqc local on the base.

Proof. Omitted. (Hint: Use Properties of Spaces, Lemma 66.4.3.) $\square$

041R Lemma 74.11.7. The property $\mathcal{P}(f) =$"f is universally injective" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.9.5. Let $Z' \to Z$ be a flat surjective morphism of affine schemes over S and let $f : X \to Z$ be a morphism from an algebraic space to Z. Assume that the base change $f' : X' \to Z'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $Z' \to Z$ is surjective there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to Z'$ such that the following solid diagram commutes

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[d] \arrow[rd, dashed, "{a',b'}"'] \arrow[rrd] & & \\ & X' \arrow[r] \arrow[d] & Z' \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, "{a,b}"] & X \arrow[r] & Z \end{tikzcd}$$

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to Z'$ is universally injective we get $a' = b'$. This forces $a = b$ as $\{\mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is an fpqc covering, see Properties of Spaces, Proposition 66.17.1. Hence f is universally injective as desired. $\square$

0CFX Lemma 74.11.8. The property $\mathcal{P}(f)$ ="f is a universal homeomorphism" is fpqc local on the base.

Proof. This can be proved in exactly the same manner as Lemma 74.11.3. Alternatively, one can use that a map of topological spaces is a homeomorphism if and only if it is injective, surjective, and open. Thus a universal homeomorphism is the same thing as a surjective, universally injective, and universally open morphism. See Morphisms of Spaces, Lemma 67.5.5 and Morphisms of Spaces, Definitions 67.19.3, 67.5.2, 67.6.2, 67.53.2. Thus the lemma follows from Lemmas 74.11.6, 74.11.7, and 74.11.4. $\square$

041S Lemma 74.11.9. The property $\mathcal{P}(f)$ ="f is locally of finite type" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.23.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite type. We have to show that f is locally of finite type. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.23.4 again, it is enough to show that $U \to Z$ is locally of finite type. Since f' is locally of finite type, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite type. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite type by Descent, Lemma 35.23.10 as desired. $\square$

041T Lemma 74.11.10. The property $\mathcal{P}(f)$ ="f is locally of finite presentation" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.28.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite presentation. We have to show that f is locally of finite presentation. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.28.4 again, it is enough to show that $U \to Z$ is locally of finite presentation. Since f' is locally of finite presentation, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite presentation. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite presentation by Descent, Lemma 35.23.11 as desired. □

041U Lemma 74.11.11. The property $\mathcal{P}(f)$ =“f is of finite type” is fpqc local on the base.

Proof. Combine Lemmas 74.11.1 and 74.11.9. □

041V Lemma 74.11.12. The property $\mathcal{P}(f)$ =“f is of finite presentation” is fpqc local on the base.

Proof. Combine Lemmas 74.11.1, 74.11.2 and 74.11.10. □

041W Lemma 74.11.13. The property $\mathcal{P}(f)$ =“f is flat” is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.30.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is flat. We have to show that f is flat. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.30.5 again, it is enough to show that $U \to Z$ is flat. Since f' is flat, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is flat. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is flat by Descent, Lemma 35.23.15 as desired. □

041X Lemma 74.11.14. The property $\mathcal{P}(f)$ =“f is an open immersion” is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.12.1. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is an open immersion. We have to show that $X \to Z$ is an open immersion. Note that $|X'| \subset |Z'|$ corresponds to an open subscheme $U' \subset Z'$ (isomorphic to X') with the property that $\text{pr}_0^{-1}(U') = \text{pr}_1^{-1}(U')$ as open subschemes of $Z' \times_Z Z'$. Hence there exists an open subscheme $U \subset Z$ such that $X' = (Z' \to Z)^{-1}(U)$, see Descent, Lemma 35.13.6. By Properties of Spaces, Proposition 66.17.1 we see that X satisfies the sheaf condition for the fpqc topology. Now we have the fpqc covering $\mathcal{U} = \{U' \to U\}$ and the element $U' \to X' \to X \in \check{H}^0(\mathcal{U}, X)$. By the sheaf condition we obtain a morphism $U \to X$ such that

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "\cong"] \arrow[dd, bend right] & U \arrow[d] \arrow[dd, bend left] \\ X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

is commutative. On the other hand, we know that for any scheme T over S and T-valued point $T \to X$ the composition $T \to X \to Z$ is a morphism such that $Z' \times_Z T \to Z'$ factors through U'. Clearly this means that $T \to Z$ factors through U. In other words the map of sheaves $U \to X$ is bijective and we win. □

041Y Lemma 74.11.15. The property $\mathcal{P}(f)$ ="f is an isomorphism" is fpqc local on the base.

Proof. Combine Lemmas 74.11.6 and 74.11.14. □

041Z Lemma 74.11.16. The property $\mathcal{P}(f)$ ="f is affine" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.20.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is affine. Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 35.34.1 (verification omitted, compare with Descent, Lemma 35.39.1). Since $X' \to Z'$ is affine this descent datum is effective, see Descent, Lemma 35.37.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is affine (by construction or by Descent, Lemma 35.23.18). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X with respect to this covering (see Properties of Spaces, Proposition 66.17.1) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 74.11.15. This proves that X is representable by an affine scheme and we win. □

0420 Lemma 74.11.17. The property $\mathcal{P}(f)$ ="f is a closed immersion" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.12.1. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a closed immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is affine. Hence by Lemma 74.11.16 we see that X is a scheme and $X \to Z$ is affine. It follows from Descent, Lemma 35.23.19 that $X \to Z$ is a closed immersion as desired. □

0421 Lemma 74.11.18. The property $\mathcal{P}(f) =$"f is separated" is fpqc local on the base.

Proof. A base change of a separated morphism is separated, see Morphisms of Spaces, Lemma 67.4.4. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is a closed immersion. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 73.9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.11.17 that Δ is a closed immersion, i.e., f is separated. □

0422 Lemma 74.11.19. The property $\mathcal{P}(f) =$"f is proper" is fpqc local on the base.

Proof. The lemma follows by combining Lemmas 74.11.3, 74.11.18 and 74.11.11. □

0423 Lemma 74.11.20. The property $\mathcal{P}(f) =$"f is quasi-affine" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.21.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-affine. Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 35.34.1 (verification omitted, compare with Descent, Lemma 35.39.1). Since $X' \to Z'$ is quasi-affine this descent datum is effective, see Descent, Lemma 35.38.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is quasi-affine (by construction or by Descent, Lemma 35.23.20). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see Properties of Spaces, Proposition 66.17.1) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 74.11.15. This proves that X is representable by a quasi-affine scheme and we win. □

0424 Lemma 74.11.21. The property $\mathcal{P}(f) =$"f is a quasi-compact immersion" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemmas 67.12.1 and 67.8.8. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a quasi-compact immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is quasi-affine. Hence by Lemma 74.11.20 we see that X is a scheme and $X \to Z$ is quasi-affine. It follows from Descent, Lemma 35.23.21 that $X \to Z$ is a quasi-compact immersion as desired. $\square$

0425 Lemma 74.11.22. The property $\mathcal{P}(f) =$"f is integral" is fpqc local on the base.

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms of Spaces, Lemma 67.45.7. Hence the lemma follows on combining Lemmas 74.11.3 and 74.11.16. $\square$

0426 Lemma 74.11.23. The property $\mathcal{P}(f) =$"f is finite" is fpqc local on the base.

Proof. An finite morphism is the same thing as an integral, morphism which is locally of finite type. See Morphisms of Spaces, Lemma 67.45.6. Hence the lemma follows on combining Lemmas 74.11.9 and 74.11.22. $\square$

0427 Lemma 74.11.24. The properties $\mathcal{P}(f) =$"f is locally quasi-finite" and $\mathcal{P}(f) =$"f is quasi-finite" are fpqc local on the base.

Proof. We have already seen that "quasi-compact" is fpqc local on the base, see Lemma 74.11.1. Hence it is enough to prove the lemma for "locally quasi-finite". We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.27.6. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally quasi-finite. We have to show that f is locally quasi-finite. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.27.6 again, it is enough to show that $U \to Z$ is locally quasi-finite. Since f' is locally quasi-finite, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally quasi-finite. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally quasi-finite by Descent, Lemma 35.23.24 as desired. $\square$

0428 Lemma 74.11.25. The property $\mathcal{P}(f) =$"f is syntomic" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.36.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is syntomic. We have to show that f is syntomic. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.36.4 again, it is enough to show that $U \to Z$ is syntomic. Since f' is syntomic, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is syntomic. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is syntomic by Descent, Lemma 35.23.26 as desired. $\square$

0429 Lemma 74.11.26. The property $\mathcal{P}(f) =$"f is smooth" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.37.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is smooth. We have to show that f is smooth. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.37.4 again, it is enough to show that $U \to Z$ is smooth. Since f' is smooth, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is smooth. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is smooth by Descent, Lemma 35.23.27 as desired. $\square$

042A Lemma 74.11.27. The property $\mathcal{P}(f) =$"f is unramified" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.38.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is unramified. We have to show that f is unramified. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.38.5 again, it is enough to show that $U \to Z$ is unramified. Since f' is unramified, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is unramified. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is unramified by Descent, Lemma 35.23.28 as desired. $\square$

042B Lemma 74.11.28. The property $\mathcal{P}(f) =$"f is étale" is fpqc local on the base.

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.39.2. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is étale. We have to show that f is étale. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.39.2 again, it is enough to show that $U \to Z$ is étale. Since f' is étale, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is étale. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is étale by Descent, Lemma 35.23.29 as desired. $\square$

042C Lemma 74.11.29. The property $\mathcal{P}(f) =$"f is finite locally free" is fpqc local on the base.

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms of Spaces, Lemma 67.46.6). Hence this follows from Lemmas 74.11.23, 74.11.13, and 74.11.10. $\square$

042D Lemma 74.11.30. The property $\mathcal{P}(f) =$"f is a monomorphism" is fpqc local on the base.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to Y_i$ of f is a monomorphism. We have to show that f is a monomorphism.

First proof. Note that f is a monomorphism if and only if $\Delta : X \to X \times_Y X$ is an isomorphism. By applying this to f_i we see that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an isomorphism. The base change of an fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 73.9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.11.15 that Δ is an isomorphism, i.e., f is a monomorphism.

Second proof. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If we can show that $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"] \arrow[d, "b"] & \mathcal{G} \arrow[d, "c"] \\ \mathcal{H} \arrow[r, "d"] & \mathcal{I} \end{tikzcd} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. This reduces the problem to the case where Y is a scheme. Moreover, in this case we may assume that the algebraic spaces Y_i are schemes also, since we can always refine the covering to place ourselves in this situation, see Topologies on Spaces, Lemma 73.9.5.

Assume $\{Y_i \to Y\}$ is an fpqc covering of schemes. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : Y_i \times_Y T \to X_i$ are the base changes. In particular the compositions $Y_i \times_Y T \to T \to X$ are equal. Since $\{Y_i \times_Y T \to T\}$ is an fpqc covering we deduce that $a = b$ from Properties of Spaces, Proposition 66.17.1. $\square$

74.12. Descending properties of morphisms in the fppf topology

042E In this section we find some properties of morphisms of algebraic spaces for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

042U Lemma 74.12.1. The property $\mathcal{P}(f) =$"f is an immersion" is fppf local on the base.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Let $f_i : X_i \to Y_i$ be the base change of f.

If f is an immersion, then each f_i is an immersion by Spaces, Lemma 65.12.3. This proves the direct implication in Definition 74.10.1.

Conversely, assume each f_i is an immersion. By Morphisms of Spaces, Lemma 67.10.7 this implies each f_i is separated. By Morphisms of Spaces, Lemma 67.27.7 this implies each f_i is locally quasi-finite. Hence we see that f is locally quasi-finite and separated, by applying Lemmas 74.11.18 and 74.11.24. By Morphisms of Spaces, Lemma 67.51.1 this implies that f is representable!

By Morphisms of Spaces, Lemma 67.12.1 it suffices to show that for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ is an immersion. By Topologies on Spaces, Lemma 73.7.4 we can find an fppf covering $\{Z_i \to Z\}$ by schemes which refines the pullback of the covering $\{Y_i \to Y\}$ to Z. Hence we see that $Z \times_Y X \to Z$ (which is a morphism of schemes according to the result of the preceding paragraph) becomes an immersion after pulling back to the members of an fppf (by schemes) of Z. Hence $Z \times_Y X \to Z$ is an immersion by the result for schemes, see Descent, Lemma 35.24.1. □

042F Lemma 74.12.2. The property $\mathcal{P}(f) =$"f is locally separated" is fppf local on the base.

Proof. A base change of a locally separated morphism is locally separated, see Morphisms of Spaces, Lemma 67.4.4. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is locally separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an immersion. The base change of a fppf covering is an fppf covering, see Topologies on Spaces, Lemma 73.7.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fppf covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.12.1 that Δ is a immersion, i.e., f is locally separated. □

74.13. Application of descent of properties of morphisms

0D3B This section is the analogue of Descent, Section 35.25.

0D3C Lemma 74.13.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{g_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering. Let $f_i : X_i \to Y_i$ be the base change of f and let $\mathcal{L}_i$ be the pullback of $\mathcal{L}$ to X_i. The following are equivalent

(1) $\mathcal{L}$ is ample on X/Y, and
(2) $\mathcal{L}_i$ is ample on X_i/Y_i for every $i \in I$.

Proof. The implication (1) ⇒ (2) follows from Divisors on Spaces, Lemma 71.14.3. Assume (2). To check $\mathcal{L}$ is ample on X/Y we may work étale localy on Y, see Divisors on Spaces, Lemma 71.14.6. Thus we may assume that Y is a scheme and then we may in turn assume each Y_i is a scheme too, see Topologies on Spaces, Lemma 73.9.5. In other words, we may assume that $\{Y_i \to Y\}$ is an fpqc covering of schemes.

By Divisors on Spaces, Lemma 71.14.4 we see that $X_i \to Y_i$ is representable (i.e., X_i is a scheme), quasi-compact, and separated. Hence f is quasi-compact and separated by Lemmas 74.11.1 and 74.11.18. This means that $\mathcal{A} = \bigoplus_{d \geq 0} f_*\mathcal{L}^{\otimes d}$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra (Morphisms of Spaces, Lemma 67.11.2). Moreover, the formation of $\mathcal{A}$ commutes with flat base change by Cohomology of Spaces, Lemma 69.11.2. In particular, if we set $\mathcal{A}_i = \bigoplus_{d \geq 0} f_{i,*}\mathcal{L}_i^{\otimes d}$ then we have $\mathcal{A}_i = g_i^*\mathcal{A}$. It follows that the natural maps $\psi_d : f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ of $\mathcal{O}_X$ pullback to give the natural maps $\psi_{i,d} : f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ of $\mathcal{O}_{X_i}$-modules. Since $\mathcal{L}_i$ is ample on X_i/Y_i we see that for any point $x_i \in X_i$, there exists a $d \geq 1$ such that $f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ is surjective on stalks at x_i. This follows either directly from the definition of a relatively ample module or from Morphisms, Lemma 29.37.4. If $x \in |X|$, then we can choose an i and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x}_i}$ is flat hence faithfully flat, we conclude that for every $x \in |X|$ there exists a $d \geq 1$ such that $f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ is surjective on stalks at x. This implies that the open subset $U(\psi) \subset X$ of Divisors on Spaces, Lemma 71.13.1 corresponding to the map $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ of graded $\mathcal{O}_X$-algebras is equal to X. Consider the corresponding morphism

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\text{Proj}}_Y(\mathcal{A})$$

It is clear from the above that the base change of $r_{\mathcal{L},\psi}$ to Y_i is the morphism $r_{\mathcal{L}_i,\psi_i}$ which is an open immersion by Morphisms, Lemma 29.37.4. Hence $r_{\mathcal{L},\psi}$ is an open immersion by Lemma 74.11.14. Hence X is a scheme and we conclude $\mathcal{L}$ is ample on X/Y by Morphisms, Lemma 29.37.4. □

0D3D Lemma 74.13.2. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. There exists an open subspace $V \subset Y$ characterized by the following property: A morphism $Y' \to Y$ of algebraic spaces factors through V if and only if the pullback $\mathcal{L}'$ of $\mathcal{L}$ to $X' = Y' \times_Y X$ is ample on X'/Y' (as in Divisors on Spaces, Definition 71.14.1).

Proof. Suppose that the lemma holds whenever Y is a scheme. Let U be a scheme and let $U \to Y$ be a surjective étale morphism. Let $R = U \times_Y U$ with projections $t, s : R \to U$. Denote $X_U = U \times_Y X$ and $\mathcal{L}_U$ the pullback. Then we get an open subscheme $V' \subset U$ as in the lemma for $(X_U \to U, \mathcal{L}_U)$. By the functorial characterization we see that $s^{-1}(V') = t^{-1}(V')$. Thus there is an open subspace $V \subset Y$ such that V' is the inverse image of V in U. In particular $V' \to V$ is surjective étale and we conclude that $\mathcal{L}_V$ is ample on X_V/V (Divisors on Spaces, Lemma 71.14.6). Now, if $Y' \to Y$ is a morphism such that $\mathcal{L}'$ is ample on X'/Y', then $U \times_Y Y' \to Y'$ must factor through V' and we conclude that $Y' \to Y$ factors through V. Hence $V \subset Y$ is as in the statement of the lemma. In this way we reduce to the case dealt with in the next paragraph.

Assume Y is a scheme. Since the question is local on Y we may assume Y is an affine scheme. We will show the following:

(A) If $\text{Spec}(k) \to Y$ is a morphism such that $\mathcal{L}_k$ is ample on X_k/k, then there is an open neighbourhood $V \subset Y$ of the image of $\text{Spec}(k) \to Y$ such that $\mathcal{L}_V$ is ample on X_V/V.

It is clear that (A) implies the truth of the lemma.

Let $X \to Y$, $\mathcal{L}$, $\text{Spec}(k) \to Y$ be as in (A). By Lemma 74.13.1 we may assume that $k = \kappa(y)$ is the residue field of a point y of Y.

As Y is affine we can find a directed set I and an inverse system of morphisms $X_i \to Y_i$ of algebraic spaces with Y_i of finite presentation over $\mathbf{Z}$, with affine transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$, with $X_i \to Y_i$ proper and of finite presentation, and such that $X \to Y = \lim(X_i \to Y_i)$. See Limits of Spaces, Lemma 70.12.2. After shrinking I we may assume Y_i is an (affine) scheme for all i, see Limits of Spaces, Lemma 70.5.10. After shrinking I we can assume we have a compatible system of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$ pulling back to $\mathcal{L}$, see Limits of Spaces, Lemma 70.7.3.

Let $y_i \in Y_i$ be the image of y. Then $\kappa(y) = \text{colim}\, \kappa(y_i)$. Hence $X_y = \lim X_{i,y_i}$ and after shrinking I we may assume X_{i,y_i} is a scheme for all i, see Limits of Spaces, Lemma 70.5.11. Hence for some i we have $\mathcal{L}_{i,y_i}$ is ample on X_{i,y_i} by Limits, Lemma 32.4.15. By Divisors on Spaces, Lemma 71.15.3 we find an open neigbourhood $V_i \subset Y_i$ of y_i such that $\mathcal{L}_i$ restricted to $f_i^{-1}(V_i)$ is ample relative to V_i. Letting $V \subset Y$ be the inverse image of V_i finishes the proof (hints: use Morphisms, Lemma 29.37.9 and the fact that $X \to Y \times_{Y_i} X_i$ is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma 29.37.7). $\square$

74.14. Properties of morphisms local on the source

06EN In this section we define what it means for a property of morphisms of algebraic spaces to be local on the source. Please compare with Descent, Section 35.26.

06EP Definition 74.14.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. We say $\mathcal{P}$ is τ local on the source, or local on the source for the τ-topology if for any morphism $f : X \to Y$ of algebraic spaces over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ of algebraic spaces we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

06EQ Lemma 74.14.2. Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. smooth, resp. étale, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. $\square$

06ER Lemma 74.14.3. Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. Suppose that $\mathcal{P}$ is a property of morphisms of schemes over S which is étale local on the source-and-target. Denote $\mathcal{P}_{spaces}$ the corresponding property of morphisms of algebraic spaces over S, see Morphisms of Spaces, Definition 67.22.2. If $\mathcal{P}$ is local on the source for the τ-topology, then $\mathcal{P}_{spaces}$ is local on the source for the τ-topology.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a τ-covering of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. For each i choose a scheme U_i and a surjective étale morphism $U_i \to X_i \times_X U$.

Note that $\{X_i \times_X U \to U\}_{i \in I}$ is a τ-covering. Note that each $\{U_i \to X_i \times_X U\}$ is an étale covering, hence a τ-covering. Hence $\{U_i \to U\}_{i \in I}$ is a τ-covering of algebraic spaces over S. But since U and each U_i is a scheme we see that $\{U_i \to U\}_{i \in I}$ is a τ-covering of schemes over S.

Now we have

$$\begin{aligned} f \text{ has } \mathcal{P}_{spaces} &\Leftrightarrow U \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } U_i \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}_{spaces}. \end{aligned}$$

the first and last equivalence by the definition of $\mathcal{P}_{spaces}$ the middle equivalence because we assumed $\mathcal{P}$ is local on the source in the τ-topology. □

74.15. Properties of morphisms local in the fpqc topology on the source

06ES Here are some properties of morphisms that are fpqc local on the source.

06ET Lemma 74.15.1. The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the source.

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.30.1 and Descent, Lemma 35.27.1. □

74.16. Properties of morphisms local in the fppf topology on the source

06EU Here are some properties of morphisms that are fppf local on the source.

06EV Lemma 74.16.1. The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fppf local on the source.

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.28.1 and Descent, Lemma 35.28.1. □

06EW Lemma 74.16.2. The property $\mathcal{P}(f) =$"f is locally of finite type" is fppf local on the source.

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.23.1 and Descent, Lemma 35.28.2. □

06EX Lemma 74.16.3. The property $\mathcal{P}(f) =$"f is open" is fppf local on the source.

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.6.2 and Descent, Lemma 35.28.3. □

06EY Lemma 74.16.4. The property $\mathcal{P}(f) =$"f is universally open" is fppf local on the source.

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.6.2 and Descent, Lemma 35.28.4. □

74.17. Properties of morphisms local in the syntomic topology on the source

06EZ Here are some properties of morphisms that are syntomic local on the source.

06F0 Lemma 74.17.1. The property $\mathcal{P}(f) =$"f is syntomic" is syntomic local on the source.

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.36.1 and Descent, Lemma 35.29.1. □

74.18. Properties of morphisms local in the smooth topology on the source

06F1 Here are some properties of morphisms that are smooth local on the source.

06F2 **Lemma 74.18.1.** *The property $\mathcal{P}(f) =$"f is smooth" is smooth local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.37.1 and Descent, Lemma 35.30.1. □

74.19. Properties of morphisms local in the étale topology on the source

06F3 Here are some properties of morphisms that are étale local on the source.

06F4 **Lemma 74.19.1.** *The property $\mathcal{P}(f) =$"f is étale" is étale local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.39.1 and Descent, Lemma 35.31.1. □

06F5 **Lemma 74.19.2.** *The property $\mathcal{P}(f) =$"f is locally quasi-finite" is étale local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.27.1 and Descent, Lemma 35.31.2. □

06F6 **Lemma 74.19.3.** *The property $\mathcal{P}(f) =$"f is unramified" is étale local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.38.1 and Descent, Lemma 35.31.3. □

74.20. Properties of morphisms smooth local on source-and-target

06F7 Let $\mathcal{P}$ be a property of morphisms of algebraic spaces. There is an intuitive meaning to the phrase "$\mathcal{P}$ is smooth local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both smooth local on the source and smooth local on the target. We have discussed a similar phenomenon (for the étale topology and the category of schemes) in great detail in Descent, Section 35.32 (for a quick overview take a look at Descent, Remark 35.32.8). However, there is an important difference between the case of the smooth and the étale topology. To see this difference we encourage the reader to ponder the difference between Descent, Lemma 35.32.4 and Lemma 74.20.2 as well as the difference between Descent, Lemma 35.32.5 and Lemma 74.20.3. Namely, in the étale setting the choice of the étale "covering" of the target is immaterial, whereas in the smooth setting it is not.

06F8 **Definition 74.20.1.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *smooth local on source-and-target* if

(1) (stable under precomposing with smooth maps) if $f : X \to Y$ is smooth and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
    (a) f has $\mathcal{P}$,

(b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with smooth vertical arrows and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being smooth local on the target, smooth local on the source, and stable under post-composing by smooth morphisms.

06F9 Lemma 74.20.2. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Then

(1) $\mathcal{P}$ is smooth local on the source,
(2) $\mathcal{P}$ is smooth local on the target,
(3) $\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth, then $g \circ f$ has $\mathcal{P}$.

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a smooth covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 74.20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 74.20.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 74.20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 74.20.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth. For every $x \in |X|$ we can think of $(X, x) \to (X, x)$ as a smooth morphism of pairs, $Y \to Z$ is a smooth morphism, and $h = f$ is as in part (3) of Definition 74.20.1. Thus we see that $g \circ f$ has $\mathcal{P}$. $\square$

The following lemma is the analogue of Morphisms, Lemma 29.14.4.

06FA Lemma 74.20.3. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(a) f has property $\mathcal{P}$,
(b) for every $x \in |X|$ there exists a smooth morphism of pairs $a : (U, u) \to (X, x)$, a smooth morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,

(c) for some commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b smooth and a surjective the morphism h has $\mathcal{P}$,

(d) for any commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ smooth the morphism h has $\mathcal{P}$,

(e) there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,

(f) there exists a smooth covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,

(g) there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ a smooth covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.

Proof. The equivalence of (a) and (b) is part of Definition 74.20.1. The equivalence of (a) and (e) is Lemma 74.20.2 part (2). The equivalence of (a) and (f) is Lemma 74.20.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism of pairs $a_x : (U_x, u_x) \to (X, x)$, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with smooth morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. $\square$

06FB Lemma 74.20.4. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume

(1) $\mathcal{P}$ is smooth local on the source,
(2) $\mathcal{P}$ is smooth local on the target, and
(3) $\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $Y \to Z$ is a smooth morphism then $X \to Z$ has $\mathcal{P}$.

Then $\mathcal{P}$ is smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 74.14.2 we see that $\mathcal{P}$ is stable under precomposing with smooth morphisms. By Lemma 74.10.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 74.20.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 74.20.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $a_x : U_x \to X$, a point $u_x \in |U_x|$ mapping to x, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b smooth, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. Because a is surjective and $\mathcal{P}$ is smooth local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This follows as we assumed that $\mathcal{P}$ is stable under postcomposing with a smooth morphism and as b is smooth. $\square$

06FC Remark 74.20.5. Using Lemma 74.20.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is smooth local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 74.20.4 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 74.15.1 and 74.11.13,
(2) locally of finite presentation, see Lemmas 74.16.1 and 74.11.10,
(3) locally finite type, see Lemmas 74.16.2 and 74.11.9,
(4) universally open, see Lemmas 74.16.4 and 74.11.4,
(5) syntomic, see Lemmas 74.17.1 and 74.11.25,
(6) smooth, see Lemmas 74.18.1 and 74.11.26,
(7) add more here as needed.

74.21. Properties of morphisms étale-smooth local on source-and-target

0CFY This section is the analogue of Section 74.20 for properties of morphisms which are étale local on the source and smooth local on the target. We give this property a ridiculously long name in order to avoid using it too much.

0CFZ Definition 74.21.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is étale-smooth local on source-and-target if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
  (a) f has $\mathcal{P}$,

(b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ étale and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being étale local on the target, smooth local on the source, and stable under post-composing by étale morphisms.

0CG0 Lemma 74.21.2. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is étale-smooth local on source-and-target. Then

(1) $\mathcal{P}$ is étale local on the source,
(2) $\mathcal{P}$ is smooth local on the target,
(3) $\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and
(4) $\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 74.21.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 74.21.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $X_i \to X \times_Y Y_i$ is an étale morphism (because it is an isomorphism), $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 74.20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 74.20.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. The morphism $X \to Y \times_Z X$ is étale as a morphism between algebraic spaces étale over X (Properties of Spaces, Lemma 66.16.6). Also $Y \to Z$ is étale hence a smooth morphism. Thus the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "g \circ f"] & Z \end{tikzcd}$$

works for every $x \in |X|$ in part (3) of Definition 74.20.1 and we conclude that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 74.21.1 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$

has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\text{pr}_X : Y \times_Z X \to X$, see Morphisms of Spaces, Lemma 67.39.11. Hence $f = \text{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 74.21.1 part (1). $\square$

0CG1 Lemma 74.21.3. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is etale-smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(a) f has property $\mathcal{P}$,
(b) for every $x \in |X|$ there exists a smooth morphism $b : V \to Y$, an étale morphism $a : U \to V \times_Y X$, and a point $u \in |U|$ mapping to x such that $U \to V$ has $\mathcal{P}$,
(c) for some commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth, $U \to V \times_Y X$ étale, and a surjective the morphism h has $\mathcal{P}$,
(d) for any commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ étale, the morphism h has $\mathcal{P}$,
(e) there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,
(f) there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,
(g) there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.

Proof. The equivalence of (a) and (b) is part of Definition 74.21.1. The equivalence of (a) and (e) is Lemma 74.21.2 part (2). The equivalence of (a) and (f) is Lemma 74.21.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism a smooth morphism $b_x : V_x \to Y$, an étale morphism $U_x \to V_x \times_Y X$, and $u_x \in |U_x|$ mapping to x such that $U_x \to V_x$ has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with étale morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. $\square$

0CG2 **Lemma 74.21.4.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume

(1) $\mathcal{P}$ is étale local on the source,
(2) $\mathcal{P}$ is smooth local on the target, and
(3) $\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open embedding then $X \to Z$ has $\mathcal{P}$.

Then $\mathcal{P}$ is étale-smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 74.14.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 74.10.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 74.20.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 74.20.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $b_x : V_x \to Y$, an étale morphism $U_x \to V_x \times_Y X$, and a point $u_x \in |U_x|$ mapping to x such that $h_x : U_x \to V_x$ has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$.

Let $a_x : U_x \to X$ be the composition $U_x \to V_x \times_Y X \to X$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth, $U \to V \times_Y X$ étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. In the next paragraph we prove that we may assume U, V, X, Y are schemes; we encourage the reader to skip it.

Let X, Y, U, V, a, b, f, h be as in the previous paragraph. We have to show f has $\mathcal{P}$. Let $X' \to X$ be a surjective étale morphism with X_i a scheme. Set $U' = X' \times_X U$. Then $U' \to X'$ is surjective and $U' \to X' \times_Y V$ is étale. Since $\mathcal{P}$ is étale local on the source, we see that $U' \to V$ has $\mathcal{P}$ and that it suffices to show that $X' \to Y$ has $\mathcal{P}$. In other words, we may assume that X is a scheme. Next, choose a surjective étale morphism $Y' \to Y$ with Y' a scheme. Set $V' = V \times_Y Y'$, $X' = X \times_Y Y'$, and $U' = U \times_Y Y'$. Then $U' \to X'$ is surjective and $U' \to X' \times_{Y'} V'$ is étale. Since $\mathcal{P}$ is smooth local on the target, we see that $U' \to V'$ has $\mathcal{P}$ and that it suffices to prove $X' \to Y'$ has $\mathcal{P}$. Thus we may assume both X and Y are schemes. Choose a surjective étale morphism $V' \to V$ with V' a scheme. Set $U' = U \times_V V'$. Then $U' \to X$ is surjective and $U' \to X \times_Y V'$ is étale. Since $\mathcal{P}$ is smooth local on the source, we see that $U' \to V'$ has $\mathcal{P}$. Thus we may replace U, V by U', V' and assume X, Y, V are schemes. Finally, we replace U by a scheme surjective étale over U and we see that we may assume U, V, X, Y are all schemes.

If U, V, X, Y are schemes, then f has $\mathcal{P}$ by Descent, Lemma 35.32.11. $\square$

0CG3 **Remark 74.21.5.** Using Lemma 74.21.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is etale local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 74.21.4 is trivial to check, and we omit it. Here is the list:

(1) étale, see Lemmas 74.19.1 and 74.11.28,
(2) locally quasi-finite, see Lemmas 74.19.2 and 74.11.24,
(3) unramified, see Lemmas 74.19.3 and 74.11.27, and
(4) add more here as needed.

Of course any property listed in Remark 74.20.5 is a fortiori an example that could be listed here.

74.22. Descent data for spaces over spaces

0ADF This section is the analogue of Descent, Section 35.34 for algebraic spaces. Most of the arguments in this section are formal relying only on the definition of a descent datum.

0ADG Definition 74.22.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) Let $V \to Y$ be a morphism of algebraic spaces. A descent datum for $V/Y/X$ is an isomorphism $\varphi : V \times_X Y \to Y \times_X V$ of algebraic spaces over $Y \times_X Y$ satisfying the cocycle condition that the diagram

$$\begin{tikzcd} V \times_X Y \times_X Y \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & Y \times_X Y \times_X V \\ & Y \times_X V \times_X Y \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

commutes (with obvious notation).
(2) We also say that the pair $(V/Y, \varphi)$ is a descent datum relative to $Y \to X$.
(3) A morphism $f : (V/Y, \varphi) \to (V'/Y, \varphi')$ of descent data relative to $Y \to X$ is a morphism $f : V \to V'$ of algebraic spaces over Y such that the diagram

$$\begin{tikzcd} V \times_X Y \arrow[r, "\varphi"'] \arrow[d, "f \times \mathrm{id}_Y"'] & Y \times_X V \arrow[d, "\mathrm{id}_Y \times f"] \\ V' \times_X Y \arrow[r, "\varphi'"] & Y \times_X V' \end{tikzcd}$$

commutes.

0ADH Remark 74.22.2. Let S be a scheme. Let $Y \to X$ be a morphism of algebraic spaces over S. Let $(V/Y, \varphi)$ be a descent datum relative to $Y \to X$. We may think of the isomorphism φ as an isomorphism

$$(Y \times_X Y) \times_{\mathrm{pr}_0, Y} V \longrightarrow (Y \times_X Y) \times_{\mathrm{pr}_1, Y} V$$

of algebraic spaces over $Y \times_X Y$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^* V \to \mathrm{pr}_1^* V$[1]. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

[1]Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 74.22.1 and 74.22.3 we should have the opposite direction to what was done in Definition 74.3.1 by the general principle that "functions" and "spaces" are dual.

0ADI Definition 74.22.3. Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to X\}$ is given by an algebraic space V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_X X_j \to X_i \times_X V_j$ of algebraic spaces over $X_i \times_X X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd} V_i \times_X X_j \times_X X_k \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & X_i \times_X X_j \times_X V_k \\ & X_i \times_X V_j \times_X X_k \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of algebraic spaces over $X_i \times_X X_j \times_X X_k$ commutes (with obvious notation).

(2) A morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms $\psi_i : V_i \to V'_i$ of algebraic spaces over X_i such that all the diagrams

$$\begin{tikzcd} V_i \times_X X_j \arrow[r, "\varphi_{ij}"'] \arrow[d, "\psi_i \times \mathrm{id}"'] & X_i \times_X V_j \arrow[d, "\mathrm{id} \times \psi_j"] \\ V'_i \times_X X_j \arrow[r, "\varphi'_{ij}"] & X_i \times_X V'_j \end{tikzcd}$$

commute.

0ADJ Remark 74.22.4. Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to X\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_X X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_X X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of algebraic spaces over $X_i \times_X X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_X X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi_{ik} = \mathrm{pr}_{12}^*\varphi_{jk} \circ \mathrm{pr}_{01}^*\varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

0ADK Lemma 74.22.5. Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X. Set $Y = \coprod_{i \in I} X_i$. There is a canonical equivalence of categories

$$\begin{matrix} \text{category of descent data} \\ \text{relative to the family } \{X_i \to X\}_{i \in I} \end{matrix} \longrightarrow \begin{matrix} \text{category of descent data} \\ \text{relative to } Y/X \end{matrix}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i \in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $Y \times_X Y = \coprod_{ij} X_i \times_X X_j$ and similarly for higher fibre products. Giving a morphism $V \to Y$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. $\square$

0ADL Lemma 74.22.6. Pullback of descent data. Let S be a scheme.

(1) Let

$$\begin{tikzcd} Y' \arrow[r, "f"] \arrow[d, "a'"'] & Y \arrow[d, "a"] \\ X' \arrow[r, "h"] & X \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. The construction

$$(V \to Y, \varphi) \longmapsto f^*(V \to Y, \varphi) = (V' \to Y', \varphi')$$

where $V' = Y' \times_Y V$ and where φ' is defined as the composition

$$\begin{tikzcd} V' \times_{X'} Y' \arrow[r, equal] & (Y' \times_Y V) \times_{X'} Y' \arrow[r, equal] & (Y' \times_{X'} Y') \times_{Y \times_X Y} (V \times_X Y) \arrow[d, "\mathrm{id} \times \varphi"] \\ Y' \times_{X'} V' \arrow[r, equal] & Y' \times_{X'} (Y' \times_Y V) \arrow[r, equal] & (Y' \times_X Y') \times_{Y \times_X Y} (Y \times_X V) \end{tikzcd}$$

defines a functor from the category of descent data relative to $Y \to X$ to the category of descent data relative to $Y' \to X'$.

(2) Given two morphisms $f_i : Y' \to Y$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f \times f)^*\varphi$ in the notation introduced in Remark 74.22.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : Y' \to Y \times_X Y$ with $f_i = \mathrm{pr}_i \circ r$. Then we take

$$\begin{aligned} f_0^*V &= Y' \times_{f_0, Y} V \\ &= Y' \times_{\mathrm{pr}_0 \circ r, Y} V \\ &= Y' \times_{r, Y \times_X Y} (Y \times_X Y) \times_{\mathrm{pr}_0, Y} V \\ &\xrightarrow{\varphi} Y' \times_{r, Y \times_X Y} (Y \times_X Y) \times_{\mathrm{pr}_1, Y} V \\ &= Y' \times_{\mathrm{pr}_1 \circ r, Y} V \\ &= Y' \times_{f_1, Y} V \\ &= f_1^*V \end{aligned}$$

We omit the verification that this works. □

0ADM Definition 74.22.7. With $S, X, X', Y, Y', f, a, a', h$ as in Lemma 74.22.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the pullback functor on descent data.

0ADN Lemma 74.22.8. Let S be a scheme. Let $\mathcal{U}' = \{X'_i \to X'\}_{i \in I'}$ and $\mathcal{U} = \{X_j \to X\}_{i \in I}$ be families of morphisms with fixed target. Let $\alpha : I' \to I$, $g : X' \to X$ and $g_i : X'_i \to X_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.

(1) Let (V_i, φ_{ij}) be a descent datum relative to the family $\mathcal{U}$. The system

$$\left(g_i^* V_{\alpha(i)}, (g_i \times g_j)^* \varphi_{\alpha(i)\alpha(j)}\right)$$

(with notation as in Remark 74.22.4) is a descent datum relative to $\mathcal{U}'$.

(2) This construction defines a functor between the category of descent data relative to $\mathcal{U}$ and the category of descent data relative to $\mathcal{U}'$.
(3) Given a second $\beta : I' \to I$, $h : X' \to X$ and $h'_i : X'_i \to X_{\beta(i)}$ morphism of families of maps with fixed target, then if $g = h$ the two resulting functors between descent data are canonically isomorphic.
(4) These functors agree, via Lemma 74.22.5, with the pullback functors constructed in Lemma 74.22.6.

Proof. This follows from Lemma 74.22.6 via the correspondence of Lemma 74.22.5. ☐

0ADP Definition 74.22.9. With $\mathcal{U}' = \{X'_i \to X'\}_{i \in I'}$, $\mathcal{U} = \{X_i \to X\}_{i \in I}$, $\alpha : I' \to I$, $g : X' \to X$, and $g_i : X'_i \to X_{\alpha(i)}$ as in Lemma 74.22.8 the functor

$$(V_i, \varphi_{ij}) \longmapsto (g_i^* V_{\alpha(i)}, (g_i \times g_j)^* \varphi_{\alpha(i)\alpha(j)})$$

constructed in that lemma is called the pullback functor on descent data.

If $\mathcal{U}$ and $\mathcal{U}'$ have the same target X, and if $\mathcal{U}'$ refines $\mathcal{U}$ (see Sites, Definition 7.8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 74.22.8 that the choice of the pair does not matter (up to a canonical isomorphism).

0ADQ Definition 74.22.10. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) Given an algebraic space U over X we have the trivial descent datum of U relative to $\text{id} : X \to X$, namely the identity morphism on U.
(2) By Lemma 74.22.6 we get a canonical descent datum on $Y \times_X U$ relative to $Y \to X$ by pulling back the trivial descent datum via f. We often denote $(Y \times_X U, can)$ this descent datum.
(3) A descent datum (V, φ) relative to Y/X is called effective if (V, φ) is isomorphic to the canonical descent datum $(Y \times_X U, can)$ for some algebraic space U over X.

Thus being effective means there exists an algebraic space U over X and an isomorphism $\psi : V \to Y \times_X U$ over Y such that φ is equal to the composition

$$V \times_X Y \xrightarrow{\psi \times \text{id}_Y} Y \times_X U \times_S Y = Y \times_X Y \times_X U \xrightarrow{\text{id}_Y \times \psi^{-1}} Y \times_X V$$

There is a slight problem here which is that this definition (in spirit) conflicts with the definition given in Descent, Definition 35.34.10 in case Y and X are schemes. However, it will always be clear from context which version we mean.

0ADR Definition 74.22.11. Let S be a scheme. Let $\{X_i \to X\}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) Given an algebraic space U over X we have a canonical descent datum on the family of algebraic spaces $X_i \times_X U$ by pulling back the trivial descent datum for U relative to $\{\text{id} : S \to S\}$. We denote this descent datum $(X_i \times_X U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called effective if there exists an algebraic space U over X such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_X U, can)$.

74.23. Descent data in terms of sheaves

0ADS This section is the analogue of Descent, Section 35.39. It is slightly different as algebraic spaces are already sheaves.

0ADT **Lemma 74.23.1.** Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S (Topologies on Spaces, Definition 73.7.1). There is an equivalence of categories

$$\left\{\begin{matrix} \text{descent data } (V_i, \varphi_{ij}) \\ \text{relative to } \{X_i \to X\} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{sheaves } F \text{ on } (Sch/S)_{fppf} \text{ endowed} \\ \text{with a map } F \to X \text{ such that each} \\ X_i \times_X F \text{ is an algebraic space} \end{matrix}\right\}.$$

Moreover,

(1) the algebraic space $X_i \times_X F$ on the right hand side corresponds to V_i on the left hand side, and
(2) the sheaf F is an algebraic space[2] if and only if the corresponding descent datum (X_i, φ_{ij}) is effective.

Proof. Let us construct the functor from right to left. Let $F \to X$ be a map of sheaves on $(Sch/S)_{fppf}$ such that each $V_i = X_i \times_X F$ is an algebraic space. We have the projection $V_i \to X_i$. Then both $V_i \times_X X_j$ and $X_i \times_X V_j$ represent the sheaf $X_i \times_X F \times_X X_j$ and hence we obtain an isomorphism

$$\varphi_{ii'} : V_i \times_X X_j \to X_i \times_X V_j$$

It is straightforward to see that the maps φ_{ij} are morphisms over $X_i \times_X X_j$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (V_i, \varphi_{ij})$.

Let us construct a functor from left to right. The isomorphisms φ_{ij} give isomorphisms

$$\varphi_{ij} : V_i \times_X X_j \longrightarrow X_i \times_X V_j$$

over $X_i \times X_j$. Set F equal to the coequalizer in the following diagram

$$\begin{tikzcd} \coprod_{i,i'} V_i \times_X X_j \arrow[r, shift left, "\mathrm{pr}_0"] \arrow[r, shift right, "\mathrm{pr}_1 \circ \varphi_{ij}"'] & \coprod_i V_i \arrow[r] & F \end{tikzcd}$$

The cocycle condition guarantees that F comes with a map $F \to X$ and that $X_i \times_X F$ is isomorphic to V_i. The functor from left to right is given by this construction $(V_i, \varphi_{ij}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. □

74.24. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields

[2] We will see later that this is always the case if I is not too large, see Bootstrap, Lemma 80.11.3.

(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
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(40) More on Groupoid Schemes
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Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
(67) Morphisms of Algebraic Spaces
(68) Decent Algebraic Spaces
(69) Cohomology of Algebraic Spaces
(70) Limits of Algebraic Spaces
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(87) Formal Algebraic Spaces
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Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
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Algebraic Stacks

CHAPTER 75

Derived Categories of Spaces

08EY
75.1. Introduction

08EZ In this chapter we discuss derived categories of modules on algebraic spaces. There do not seem to be good introductory references addressing this topic; it is covered in the literature by referring to papers dealing with derived categories of modules on algebraic stacks, for example see [Ols07b].

75.2. Conventions

08F0 If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

75.3. Generalities

08GD In this section we put some general results on cohomology of unbounded complexes of modules on algebraic spaces.

08GE Lemma 75.3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $U = V \times_Y X$ and denote $g : U \to V$ the projection morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Cohomology on Sites, Lemma 21.20.1. Hence the result follows from Properties of Spaces, Lemma 66.26.2. $\square$

08GF Definition 75.3.2. Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset |X|$ be a closed subset. We say E is supported on T if the cohomology sheaves $H^i(E)$ are supported on T.

75.4. Derived category of quasi-coherent modules on the small étale site

071P Let X be a scheme. In this section we show that $D_{QCoh}(\mathcal{O}_X)$ can be defined in terms of the small étale site $X_{étale}$ of X. Denote $\mathcal{O}_{étale}$ the structure sheaf on $X_{étale}$. Consider the morphism of ringed sites

08H7 (75.4.0.1) $$\epsilon : (X_{étale}, \mathcal{O}_{étale}) \longrightarrow (X_{Zar}, \mathcal{O}_X).$$

denoted $\text{id}_{small,étale,Zar}$ in Descent, Lemma 35.8.5.

08H8 **Lemma 75.4.1.** The morphism ϵ of (75.4.0.1) is a flat morphism of ringed sites. In particular the functor $\epsilon^* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_{étale})$ is exact. Moreover, if $\epsilon^*\mathcal{F} = 0$, then $\mathcal{F} = 0$.

Proof. The flatness of the morphism ϵ is Descent, Lemma 35.10.1. Here is another proof. We have to show that $\mathcal{O}_{étale}$ is a flat $\epsilon^{-1}\mathcal{O}_X$-module. To do this it suffices to check $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is flat for any geometric point $\overline{x}$ of X, see Modules on Sites, Lemma 18.39.3, Sites, Lemma 7.34.2, and Étale Cohomology, Remarks 59.29.11. By Étale Cohomology, Lemma 59.33.1 we see that $\mathcal{O}_{étale,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x}$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is faithfully flat by More on Algebra, Lemma 15.45.1.

The exactness of ϵ^* follows from the flatness of ϵ by Modules on Sites, Lemma 18.31.2.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\epsilon^*\mathcal{F} = 0$, then with notation as above

$$0 = \epsilon^*\mathcal{F}_{\overline{x}} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{étale,\overline{x}}$$

(Modules on Sites, Lemma 18.36.4) for all geometric points $\overline{x}$. By faithful flatness of $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ we conclude $\mathcal{F}_x = 0$ for all $x \in X$. $\square$

Let X be a scheme. Notation as in (75.4.0.1). Recall that $\epsilon^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{étale})$ is an equivalence by Descent, Proposition 35.8.9 and Remark 35.8.6. Moreover, $QCoh(\mathcal{O}_{étale})$ forms a Serre subcategory of $\text{Mod}(\mathcal{O}_{étale})$ by Descent, Lemma 35.10.2. Hence we can let $D_{QCoh}(\mathcal{O}_{étale})$ be the triangulated subcategory of $D(\mathcal{O}_{étale})$ whose objects are the complexes with quasi-coherent cohomology sheaves, see Derived Categories, Section 13.17. The functor ϵ^* is exact (Lemma 75.4.1) hence induces $\epsilon^* : D(\mathcal{O}_X) \to D(\mathcal{O}_{étale})$ and since pullbacks of quasi-coherent modules are quasi-coherent also $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$.

071Q **Lemma 75.4.2.** Let X be a scheme. The functor $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$ defined above is an equivalence.

Proof. We will prove this by showing the functor $R\epsilon_* : D(\mathcal{O}_{étale}) \to D(\mathcal{O}_X)$ induces a quasi-inverse. We will use freely that ϵ_* is given by restriction to $X_{Zar} \subset X_{étale}$ and the description of $\epsilon^* = \text{id}^*_{small,étale,Zar}$ in Descent, Lemma 35.8.5.

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to \epsilon_*\epsilon^*\mathcal{F}$ is an isomorphism by the fact that $\mathcal{F}^a$ (Descent, Definition 35.8.2) is a sheaf as proved in Descent, Lemma 35.8.1. Conversely, every quasi-coherent $\mathcal{O}_{étale}$-module $\mathcal{H}$ is of the form $\epsilon^*\mathcal{F}$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, see Descent, Proposition 35.8.9. Then $\mathcal{F} = \epsilon_*\mathcal{H}$ by what we just said and we conclude that the adjunction map $\epsilon^*\epsilon_*\mathcal{H} \to \mathcal{H}$ is an isomorphism for all quasi-coherent $\mathcal{O}_{étale}$-modules $\mathcal{H}$.

Let E be an object of $D_{QCoh}(\mathcal{O}_{étale})$ and denote $\mathcal{H}^q = H^q(E)$ its qth cohomology sheaf. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^q) = 0$ for all $p > 0$, all $q \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. By Cohomology on Sites, Lemma 21.23.11 this means that

$$H^q(U, E) = H^0(U, \mathcal{H}^q)$$

for all $U \in \mathcal{B}$. In particular, we find that this holds for affine opens $U \subset X$. It follows that the qth cohomology of $R\epsilon_*E$ over U is the value of the sheaf $\epsilon_*\mathcal{H}^q$ over U. Applying sheafification we obtain

$$H^q(R\epsilon_*E) = \epsilon_*\mathcal{H}^q$$

which in particular shows that $R\epsilon_*$ induces a functor $D_{QCoh}(\mathcal{O}_{étale}) \to D_{QCoh}(\mathcal{O}_X)$. Since ϵ^* is exact we then obtain $H^q(\epsilon^* R\epsilon_* E) = \epsilon^*\epsilon_*\mathcal{H}^q = \mathcal{H}^q$ (by discussion above). Thus the adjunction map $\epsilon^* R\epsilon_* E \to E$ is an isomorphism.

Conversely, for $F \in D_{QCoh}(\mathcal{O}_X)$ the adjunction map $F \to R\epsilon_*\epsilon^* F$ is an isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_*\epsilon^*F$ are isomorphic to $\epsilon_* H^m(\epsilon^*F) = \epsilon_*\epsilon^* H^m(F) = H^m(F)$. □

75.5. Derived category of quasi-coherent modules

071W Let S be a scheme. Lemma 75.4.2 shows that the category $D_{QCoh}(\mathcal{O}_S)$ can be defined in terms of complexes of $\mathcal{O}_S$-modules on the scheme S or by complexes of $\mathcal{O}$-modules on the small étale site of S. Hence the following definition is compatible with the definition in the case of schemes.

071X **Definition 75.5.1.** Let S be a scheme. Let X be an algebraic space over S. The *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves* is denoted $D_{QCoh}(\mathcal{O}_X)$.

This makes sense by Properties of Spaces, Lemma 66.29.7 and Derived Categories, Section 13.17. Thus we obtain a canonical functor

08F1 (75.5.1.1)
$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

Observe that a flat morphism $f : Y \to X$ of algebraic spaces induces an exact functor $f^* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_Y)$, see Morphisms of Spaces, Lemma 67.30.9 and Modules on Sites, Lemma 18.31.2. In particular $Lf^* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$ is computed on any representative complex (Derived Categories, Lemma 13.16.9). We will write $Lf^* = f^*$ when f is flat and we have $H^i(f^*E) = f^*H^i(E)$ for E in $D(\mathcal{O}_X)$ in this case. We will use this often when f is étale. Of course in the étale case the pullback functor is just the restriction to $Y_{étale}$, see Properties of Spaces, Equation (66.26.1.1).

08F2 **Lemma 75.5.2.** Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent

(1) E is in $D_{QCoh}(\mathcal{O}_X)$,
(2) for every étale morphism $\varphi : U \to X$ where U is an affine scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,
(3) for every étale morphism $\varphi : U \to X$ where U is a scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,
(4) there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$, and
(5) there exists a surjective étale morphism of algebraic spaces $f : Y \to X$ such that Lf^*E is an object of $D_{QCoh}(\mathcal{O}_Y)$.

Proof. This follows immediately from the discussion preceding the lemma and Properties of Spaces, Lemma 66.29.6. □

08F3 **Lemma 75.5.3.** Let S be a scheme. Let X be an algebraic space over S. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.

Proof. By Injectives, Lemma 19.13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 66.29.7. $\square$

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of D_{QCoh}.

0D3E Lemma 75.5.4. Let S be a scheme. Let X be an algebraic space over S. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ with derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$. Assume $H^q(K_{n+1}) \to H^q(K_n)$ is surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then

1. $H^q(K) = \lim H^q(K_n)$,
2. $R\lim H^q(K_n) = \lim H^q(K_n)$, and
3. for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.

Proof. Let $\mathcal{B} \subset \text{Ob}(X_{\acute{e}tale})$ be the set of affine objects. Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in \mathcal{B}$ by the discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \to H^0(U, H^q(K_n))$ are surjective for $U \in \mathcal{B}$ by similar reasoning. Part (1) follows from Cohomology on Sites, Lemma 21.23.12 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology on Sites, Lemma 21.23.5. $\square$

08F4 Lemma 75.5.5. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. Since $a^* \circ Lf^* = Lh^* \circ b^*$ the result follows from Lemma 75.5.2 and the case of schemes which is Derived Categories of Schemes, Lemma 36.3.8. $\square$

08F5 Lemma 75.5.6. Let S be a scheme. Let X be an algebraic space over S. For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let $\varphi : U \to X$ be a surjective étale morphism from a scheme U. Since $\varphi^*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = \varphi^*K \otimes^{\mathbf{L}}_{\mathcal{O}_U} \varphi^*L$ we see from Lemma 75.5.2 that this follows from the case of schemes which is Derived Categories of Schemes, Lemma 36.3.9. $\square$

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

08F6 Lemma 75.5.7. Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism[1].

[1]In particular, E has a K-injective representative as in Cohomology on Sites, Lemma 21.24.1.

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$ as U is an affine scheme. See discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2. Thus the lemma follows from Cohomology on Sites, Lemma 21.23.10 with $d = 0$. $\square$

08F7 Lemma 75.5.8. Let S be a scheme. Let X be an algebraic space over S. Let $F : \text{Mod}(\mathcal{O}_X) \to \text{Ab}$ be a functor and $N \geq 0$ an integer. Assume that

(1) F is left exact,
(2) F commutes with countable direct products,
(3) $R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.

Then for $E \in D_{QCoh}(\mathcal{O}_X)$

(1) $H^i(RF(\tau_{\leq a}E) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,
(2) $H^i(RF(E)) \to H^i(RF(\tau_{\geq b-N+1}E))$ is an isomorphism for $i \geq b$,
(3) if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + N - 1]$.

Proof. Statement (1) is Derived Categories, Lemma 13.16.1.

Proof of statement (2). Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 75.5.7. Thus $RF(E) = R\lim RF(E_n)$ in $D(\text{Ab})$ by Injectives, Lemma 19.13.6. Thus for every $i \in \mathbf{Z}$ we have a short exact sequence

$$0 \to R^1 \lim H^{i-1}(RF(E_n)) \to H^i(RF(E)) \to \lim H^i(RF(E_n)) \to 0$$

see More on Algebra, Remark 15.86.10. To prove (2) we will show that the term on the left is zero and that the term on the right equals $H^i(RF(E_{-b+N-1})$ for any b with $i \geq b$.

For every n we have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 13.12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$H^i(RF(H^{-n}(E)[n])) = R^{i+n}F(H^{-n}(E)) = 0$$

for $i + n \geq N$ and

$$H^i(RF(H^{-n}(E)[n+1])) = R^{i+n+1}F(H^{-n}(E)) = 0$$

for $i + n + 1 \geq N$. We conclude that

$$H^i(RF(E_n)) \to H^i(RF(E_{n-1}))$$

is an isomorphism for $n \geq N - i$. Thus the systems $H^i(RF(E_n))$ all satisfy the ML condition and the $R^1 \lim$ term in our short exact sequence is zero (see discussion in More on Algebra, Section 15.86). Moreover, the system $H^i(RF(E_n))$ is constant starting with $n = N - i - 1$ as desired.

Proof of (3). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a - 1$ from (1). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b + n$ from part (2). $\square$

75.6. Total direct image

08F9 The following lemma is the analogue of Cohomology of Spaces, Lemma 69.8.1.

08FA **Lemma 75.6.1.** Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S.

(1) The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.
(2) If Y is quasi-compact, there exists an integer $N = N(X, Y, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.
(3) In fact, if Y is quasi-compact we can find $N = N(X, Y, f)$ such that for every morphism of algebraic spaces $Y' \to Y$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to Y'$ is the base change of f.

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. This question is local on Y, hence we may assume Y is quasi-compact. Pick $N = N(X, Y, f)$ as in Cohomology of Spaces, Lemma 69.8.1. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$. Moreover $R^pf_*\mathcal{F}$ is quasi-coherent for all p by Cohomology of Spaces, Lemma 69.3.1. These statements remain true after base change.

First, assume E is bounded below. We will show (1) and (2) and (3) hold for such E with our choice of N. In this case we can for example use the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

(Derived Categories, Lemma 13.21.3), the quasi-coherence of $R^pf_*H^q(E)$, and the vanishing of $R^pf_*H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let V be an affine object of $Y_{étale}$. We have $H^p(V \times_Y X, \mathcal{F}) = 0$ for $p \geq N$, see Cohomology of Spaces, Lemma 69.3.2. Hence we may apply Lemma 75.5.8 to the functor $\Gamma(V \times_Y X, -)$ to see that

$$R\Gamma(V, Rf_*E) = R\Gamma(V \times_Y X, E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all V affine in $Y_{étale}$ we conclude that $H^m(Rf_*E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1}E \to E \to \tau_{\geq -n}E \to (\tau_{\leq -n-1}E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 13.12.4. By (2) we see that $Rf_*\tau_{\leq -n-1}E$ has vanishing cohomology sheaves in degrees $\geq -n+N$. Thus, given an integer q we see that R^qf_*E is equal to $R^qf_*\tau_{\geq -n}E$ for some n and the result above applies. $\square$

08FB **Lemma 75.6.2.** Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ commutes with direct sums.

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 75.6.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited.

Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 13.21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Spaces, Lemma 69.5.2. □

08GH Remark 75.6.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of representable algebraic spaces X and Y over S. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then the diagram

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_{X_0}) \arrow[r, equal, "\text{Lemma 75.4.2}"'] & D_{QCoh}(\mathcal{O}_X) \\ D_{QCoh}(\mathcal{O}_{Y_0}) \arrow[r, equal, "\text{Lemma 75.4.2}"] \arrow[u, "Lf_0^*"] & D_{QCoh}(\mathcal{O}_Y) \arrow[u, "Lf^*"'] \end{tikzcd}$$

(Lemma 75.5.5 and Derived Categories of Schemes, Lemma 36.3.8) is commutative. This follows as the equivalences $D_{QCoh}(\mathcal{O}_{X_0}) \to D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_{Y_0}) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 75.4.2 come from pulling back by the (flat) morphisms of ringed sites $\epsilon : X_{étale} \to X_{0,Zar}$ and $\epsilon : Y_{étale} \to Y_{0,Zar}$ and the diagram of ringed sites

$$\begin{tikzcd} X_{0,Zar} \arrow[d, "f_0"'] & X_{étale} \arrow[l, "\epsilon"'] \arrow[d, "f"] \\ Y_{0,Zar} & Y_{étale} \arrow[l, "\epsilon"] \end{tikzcd}$$

is commutative (details omitted). If f is quasi-compact and quasi-separated, equivalently if f_0 is quasi-compact and quasi-separated, then we claim

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_{X_0}) \arrow[r, equal, "\text{Lemma 75.4.2}"'] \arrow[d, "Rf_{0,*}"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D_{QCoh}(\mathcal{O}_{Y_0}) \arrow[r, equal, "\text{Lemma 75.4.2}"] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

(Lemma 75.6.1 and Derived Categories of Schemes, Lemma 36.4.1) is commutative as well. This also follows from the commutative diagram of sites displayed above as the proof of Lemma 75.4.2 shows that the functor $R\epsilon_*$ gives the equivalences $D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{X_0})$ and $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{Y_0})$.

08II Lemma 75.6.4. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms.

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is étale local on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 36.5.2) via Lemma 75.4.2 and Remark 75.6.3. □

08IJ Lemma 75.6.5. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_Y)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{O}_X$.

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Spaces, Lemma 69.8.2). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 36.5.3) via Lemma 75.4.2 and Remark 75.6.3. □

75.7. Being proper over a base

0CZB This section is the analogue of Cohomology of Schemes, Section 30.26. As usual with material having to do with topology on the sets of points, we have to be careful translating the material to algebraic spaces.

0CZC **Lemma 75.7.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T \subset |X|$ be a closed subset. The following are equivalent

(1) the morphism $Z \to Y$ is proper if Z is the reduced induced algebraic space structure on T (Properties of Spaces, Definition 66.12.5),
(2) for some closed subspace $Z \subset X$ with $|Z| = T$ the morphism $Z \to Y$ is proper, and
(3) for any closed subspace $Z \subset X$ with $|Z| = T$ the morphism $Z \to Y$ is proper.

Proof. The implications (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are immediate. Thus it suffices to prove that (2) implies (3). We urge the reader to find their own proof of this fact. Let Z' and Z'' be closed subspaces with $T = |Z'| = |Z''|$ such that $Z' \to Y$ is a proper morphism of algebraic spaces. We have to show that $Z'' \to Y$ is proper too. Let $Z''' = Z' \cup Z''$ be the scheme theoretic union, see Morphisms of Spaces, Definition 67.14.4. Then Z''' is another closed subspace with $|Z'''| = T$. This follows for example from the description of scheme theoretic unions in Morphisms of Spaces, Lemma 67.14.6. Since $Z'' \to Z'''$ is a closed immersion it suffices to prove that $Z''' \to Y$ is proper (see Morphisms of Spaces, Lemmas 67.40.5 and 67.40.4). The morphism $Z' \to Z'''$ is a bijective closed immersion and in particular surjective and universally closed. Then the fact that $Z' \to Y$ is separated implies that $Z''' \to Y$ is separated, see Morphisms of Spaces, Lemma 67.9.8. Moreover $Z''' \to Y$ is locally of finite type as $X \to Y$ is locally of finite type (Morphisms of Spaces, Lemmas 67.23.7 and 67.23.2). Since $Z' \to Y$ is quasi-compact and $Z' \to Z'''$ is a universal homeomorphism we see that $Z''' \to Y$ is quasi-compact. Finally, since $Z' \to Y$ is universally closed, we see that the same thing is true for $Z''' \to Y$ by Morphisms of Spaces, Lemma 67.40.7. This finishes the proof. □

0CZD **Definition 75.7.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T \subset |X|$ be a closed subset. We say T is *proper over* Y if the equivalent conditions of Lemma 75.7.1 are satisfied.

The lemma used in the definition above is false if the morphism $f : X \to Y$ is not locally of finite type. Therefore we urge the reader not to use this terminology if f is not locally of finite type.

0CZE **Lemma 75.7.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T' \subset T \subset |X|$ be closed subsets. If T is proper over Y, then the same is true for T'.

Proof. Omitted. $\square$

0CZF **Lemma 75.7.4.** Let S be a scheme. Consider a cartesian diagram of algebraic spaces over S

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f locally of finite type. If T is a closed subset of $|X|$ proper over Y, then $|g'|^{-1}(T)$ is a closed subset of $|X'|$ proper over Y'.

Proof. Observe that the statement makes sense as f' is locally of finite type by Morphisms of Spaces, Lemma 67.23.3. Let $Z \subset X$ be the reduced induced closed subspace structure on T. Denote $Z' = (g')^{-1}(Z)$ the scheme theoretic inverse image. Then $Z' = X' \times_X Z = (Y' \times_Y X) \times_X Z = Y' \times_Y Z$ is proper over Y' as a base change of Z over Y (Morphisms of Spaces, Lemma 67.40.3). On the other hand, we have $T' = |Z'|$. Hence the lemma holds. $\square$

0CZG **Lemma 75.7.5.** Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which are locally of finite type over B.

(1) If Y is separated over B and $T \subset |X|$ is a closed subset proper over B, then $|f|(T)$ is a closed subset of $|Y|$ proper over B.
(2) If f is universally closed and $T \subset |X|$ is a closed subset proper over B, then $|f|(T)$ is a closed subset of Y proper over B.
(3) If f is proper and $T \subset |Y|$ is a closed subset proper over B, then $|f|^{-1}(T)$ is a closed subset of $|X|$ proper over B.

Proof. Proof of (1). Assume Y is separated over B and $T \subset |X|$ is a closed subset proper over B. Let Z be the reduced induced closed subspace structure on T and apply Morphisms of Spaces, Lemma 67.40.8 to $Z \to Y$ over B to conclude.

Proof of (2). Assume f is universally closed and $T \subset |X|$ is a closed subset proper over B. Let Z be the reduced induced closed subspace structure on T and let Z' be the reduced induced closed subspace structure on $|f|(T)$. We obtain an induced morphism $Z \to Z'$. Denote $Z'' = f^{-1}(Z')$ the scheme theoretic inverse image. Then $Z'' \to Z'$ is universally closed as a base change of f (Morphisms of Spaces, Lemma 67.40.3). Hence $Z \to Z'$ is universally closed as a composition of the closed immersion $Z \to Z''$ and $Z'' \to Z'$ (Morphisms of Spaces, Lemmas 67.40.5 and 67.40.4). We conclude that $Z' \to B$ is separated by Morphisms of Spaces, Lemma 67.9.8. Since $Z \to B$ is quasi-compact and $Z \to Z'$ is surjective we see that $Z' \to B$ is quasi-compact. Since $Z' \to B$ is the composition of $Z' \to Y$ and $Y \to B$ we see that $Z' \to B$ is locally of finite type (Morphisms of Spaces, Lemmas 67.23.7 and 67.23.2). Finally, since $Z \to B$ is universally closed, we see that the same thing is true for $Z' \to B$ by Morphisms of Spaces, Lemma 67.40.7. This finishes the proof.

Proof of (3). Assume f is proper and $T \subset |Y|$ is a closed subset proper over B. Let Z be the reduced induced closed subspace structure on T. Denote $Z' = f^{-1}(Z)$

the scheme theoretic inverse image. Then $Z' \to Z$ is proper as a base change of f (Morphisms of Spaces, Lemma 67.40.3). Whence $Z' \to B$ is proper as the composition of $Z' \to Z$ and $Z \to B$ (Morphisms of Spaces, Lemma 67.40.4). This finishes the proof. $\square$

0CZH Lemma 75.7.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T_i \subset |X|$, $i = 1, \ldots, n$ be closed subsets. If T_i, $i = 1, \ldots, n$ are proper over Y, then the same is true for $T_1 \cup \ldots \cup T_n$.

Proof. Let Z_i be the reduced induced closed subscheme structure on T_i. The morphism

$$Z_1 \amalg \ldots \amalg Z_n \longrightarrow X$$

is finite by Morphisms of Spaces, Lemmas 67.45.10 and 67.45.11. As finite morphisms are universally closed (Morphisms of Spaces, Lemma 67.45.9) and since $Z_1 \amalg \ldots \amalg Z_n$ is proper over S we conclude by Lemma 75.7.5 part (2) that the image $Z_1 \cup \ldots \cup Z_n$ is proper over S. $\square$

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Then the support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of $|X|$, see Morphisms of Spaces, Lemma 67.15.2. Hence it makes sense to say "the support of $\mathcal{F}$ is proper over Y".

0CZI Lemma 75.7.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

(1) the support of $\mathcal{F}$ is proper over Y,
(2) the scheme theoretic support of $\mathcal{F}$ (Morphisms of Spaces, Definition 67.15.4) is proper over Y, and
(3) there exists a closed subspace $Z \subset X$ and a finite type, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that (a) $Z \to Y$ is proper, and (b) $(Z \to X)_*\mathcal{G} = \mathcal{F}$.

Proof. The support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of $|X|$, see Morphisms of Spaces, Lemma 67.15.2. Hence we can apply Definition 75.7.2. Since the scheme theoretic support of $\mathcal{F}$ is a closed subspace whose underlying closed subset is $\text{Supp}(\mathcal{F})$ we see that (1) and (2) are equivalent by Definition 75.7.2. It is clear that (2) implies (3). Conversely, if (3) is true, then $\text{Supp}(\mathcal{F}) \subset |Z|$ and hence $\text{Supp}(\mathcal{F})$ is proper over Y for example by Lemma 75.7.3. $\square$

0CZJ Lemma 75.7.8. Let S be a scheme. Consider a cartesian diagram of algebraic spaces over S

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. If the support of $\mathcal{F}$ is proper over Y, then the support of $(g')^*\mathcal{F}$ is proper over Y'.

Proof. Observe that the statement makes sense because $(g') * \mathcal{F}$ is of finite type by Modules on Sites, Lemma 18.23.4. We have $\text{Supp}((g')^*\mathcal{F}) = |g'|^{-1}(\text{Supp}(\mathcal{F}))$ by Morphisms of Spaces, Lemma 67.15.2. Thus the lemma follows from Lemma 75.7.4. $\square$

0CZK Lemma 75.7.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$, $\mathcal{G}$ be finite type, quasi-coherent $\mathcal{O}_X$-module.

(1) If the supports of $\mathcal{F}$, $\mathcal{G}$ are proper over Y, then the same is true for $\mathcal{F} \oplus \mathcal{G}$, for any extension of $\mathcal{G}$ by $\mathcal{F}$, for $\mathrm{Im}(u)$ and $\mathrm{Coker}(u)$ given any $\mathcal{O}_X$-module map $u : \mathcal{F} \to \mathcal{G}$, and for any quasi-coherent quotient of $\mathcal{F}$ or $\mathcal{G}$.
(2) If Y is locally Noetherian, then the category of coherent $\mathcal{O}_X$-modules with support proper over Y is a Serre subcategory (Homology, Definition 12.10.1) of the abelian category of coherent $\mathcal{O}_X$-modules.

Proof. Proof of (1). Let T, T' be the support of $\mathcal{F}$ and $\mathcal{G}$. Then all the sheaves mentioned in (1) have support contained in $T \cup T'$. Thus the assertion itself is clear from Lemmas 75.7.3 and 75.7.6 provided we check that these sheaves are finite type and quasi-coherent. For quasi-coherence we refer the reader to Properties of Spaces, Section 66.29. For "finite type" we refer the reader to Properties of Spaces, Section 66.30.

Proof of (2). The proof is the same as the proof of (1). Note that the assertions make sense as X is locally Noetherian by Morphisms of Spaces, Lemma 67.23.5 and by the description of the category of coherent modules in Cohomology of Spaces, Section 69.12. □

08GC Lemma 75.7.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with support proper over Y. Then $R^pf_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $p \geq 0$.

Proof. By Lemma 75.7.7 there exists a closed immersion $i : Z \to X$ with $g = f \circ i : Z \to Y$ proper and $\mathcal{F} = i_*\mathcal{G}$ for some coherent module $\mathcal{G}$ on Z. We see that $R^pg_*\mathcal{G}$ is coherent on S by Cohomology of Spaces, Lemma 69.20.2. On the other hand, $R^qi_*\mathcal{G} = 0$ for $q > 0$ (Cohomology of Spaces, Lemma 69.12.9). By Cohomology on Sites, Lemma 21.14.7 we get $R^pf_*\mathcal{F} = R^pg_*\mathcal{G}$ and the lemma follows. □

75.8. Derived category of coherent modules

08GI Let S be a scheme. Let X be a locally Noetherian algebraic space over S. In this case the category $\mathrm{Coh}(\mathcal{O}_X) \subset \mathrm{Mod}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 12.10 and Cohomology of Spaces, Lemma 69.12.3. Denote

$$D_{\mathrm{Coh}}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13.17. Thus we obtain a canonical functor

08GJ (75.8.0.1) $$D(\mathrm{Coh}(\mathcal{O}_X)) \longrightarrow D_{\mathrm{Coh}}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

08GK Lemma 75.8.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is Noetherian. Let E be an object of $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over Y for all i. Then Rf_*E is an object of $D^b_{\mathrm{Coh}}(\mathcal{O}_Y)$.

Proof. Consider the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

see Derived Categories, Lemma 13.21.3. By assumption and Lemma 75.7.10 the sheaves $R^p f_* H^q(E)$ are coherent. Hence $R^{p+q} f_* E$ is coherent, i.e., $E \in D_{\text{Coh}}(\mathcal{O}_Y)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Spaces, Lemma 69.8.1 or from Lemma 75.6.1. $\square$

0D0R Lemma 75.8.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is Noetherian. Let E be an object of $D^+_{\text{Coh}}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^+_{\text{Coh}}(\mathcal{O}_Y)$.

Proof. The proof is the same as the proof of Lemma 75.8.1. You can also deduce it from Lemma 75.8.1 by considering what the exact functor Rf_* does to the distinguished triangles $\tau_{\leq a}E \to E \to \tau_{\geq a+1}E \to \tau_{\leq a}E[1]$. $\square$

0D0S Lemma 75.8.3. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. If L is in $D^+_{\text{Coh}}(\mathcal{O}_X)$ and K in $D^-_{\text{Coh}}(\mathcal{O}_X)$, then $R\mathcal{H}om(K, L)$ is in $D^+_{\text{Coh}}(\mathcal{O}_X)$.

Proof. We can check whether an object of $D(\mathcal{O}_X)$ is in $D_{\text{Coh}}(\mathcal{O}_X)$ étale locally on X, see Cohomology of Spaces, Lemma 69.12.2. Hence this lemma follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.11.5. $\square$

0D0T Lemma 75.8.4. Let A be a Noetherian ring. Let X be a proper algebraic space over A. For L in $D^+_{\text{Coh}}(\mathcal{O}_X)$ and K in $D^-_{\text{Coh}}(\mathcal{O}_X)$, the A-modules $\text{Ext}^n_{\mathcal{O}_X}(K, L)$ are finite.

Proof. Recall that

$$\text{Ext}^n_{\mathcal{O}_X}(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\text{Spec}(A), Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology on Sites, Lemma 21.35.1 and Cohomology on Sites, Section 21.14. Thus the result follows from Lemmas 75.8.3 and 75.8.2. $\square$

75.9. Induction principle

08GL In this section we discuss an induction principle for algebraic spaces analogous to what is Cohomology of Schemes, Lemma 30.4.1 for schemes. To formulate it we introduce the notion of an elementary distinguished square; this terminology is borrowed from [MV99]. The principle as formulated here is implicit in the paper [GR71] by Raynaud and Gruson. A related principle for algebraic stacks is [Ryd10, Theorem D] by David Rydh.

08GM Definition 75.9.1. Let S be a scheme. A commutative diagram

$$\begin{tikzcd} U \times_W V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & W \end{tikzcd}$$

of algebraic spaces over S is called an elementary distinguished square if

(1) U is an open subspace of W and j is the inclusion morphism,
(2) f is étale, and

(3) setting $T = W \setminus U$ (with reduced induced subspace structure) the morphism $f^{-1}(T) \to T$ is an isomorphism.

We will indicate this by saying: "Let $(U \subset W, f : V \to W)$ be an elementary distinguished square."

Note that if $(U \subset W, f : V \to W)$ is an elementary distinguished square, then we have $W = U \cup f(V)$. Thus $\{U \to W, V \to W\}$ is an étale covering of W. It turns out that these étale coverings have nice properties and that in some sense there are "enough" of them.

08GN Lemma 75.9.2. Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square of algebraic spaces over S.

(1) If $V' \subset V$ and $U \subset U' \subset W$ are open subspaces and $W' = U' \cup f(V')$ then $(U' \subset W', f|_{V'} : V' \to W')$ is an elementary distinguished square.
(2) If $p : W' \to W$ is a morphism of algebraic spaces, then $(p^{-1}(U) \subset W', V \times_W W' \to W')$ is an elementary distinguished square.
(3) If $S' \to S$ is a morphism of schemes, then $(S' \times_S U \subset S' \times_S W, S' \times_S V \to S' \times_S W)$ is an elementary distinguished square.

Proof. Omitted. □

08GP Lemma 75.9.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let P be a property of the quasi-compact and quasi-separated objects of $X_{spaces,étale}$. Assume that

(1) P holds for every affine object of $X_{spaces,étale}$,
(2) for every elementary distinguished square $(U \subset W, f : V \to W)$ such that
   (a) W is a quasi-compact and quasi-separated object of $X_{spaces,étale}$,
   (b) U is quasi-compact,
   (c) V is affine, and
   (d) P holds for U, V, and $U \times_W V$,

   then P holds for W.

Then P holds for every quasi-compact and quasi-separated object of $X_{spaces,étale}$ and in particular for X.

Proof. We first claim that P holds for every representable quasi-compact and quasi-separated object of $X_{spaces,étale}$. Namely, suppose that $U \to X$ is étale and U is a quasi-compact and quasi-separated scheme. By assumption (1) property P holds for every affine open of U. Moreover, if $W, V \subset U$ are quasi-compact open with V affine and P holds for W, V, and $W \cap V$, then P holds for $W \cup V$ by (2) (as the pair $(W \subset W \cup V, V \to W \cup V)$ is an elementary distinguished square). Thus P holds for U by the induction principle for schemes, see Cohomology of Schemes, Lemma 30.4.1.

To finish the proof it suffices to prove P holds for X (because we can simply replace X by any quasi-compact and quasi-separated object of $X_{spaces,étale}$ we want to prove the result for). We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Decent Spaces, Lemma 68.8.6. We will prove that P holds for U_p by descending induction on p. Note that P holds for U_{n+1} by (1) as an empty algebraic space is affine. Assume P holds for U_{p+1}. Note that

$(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, but (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = U_{p+1}$ and

$$W_{p,i} = U_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X. Then we have

$$U_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = U_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 75.9.2. Note that P holds for each $V_{p,1}$ (as affine schemes) and for $W_{p,i} \times_{W_{p,i+1}} V_{p,i+1}$ as this is a quasi-compact open of $V_{p,i+1}$ and hence P holds for it by the first paragraph of this proof. Thus (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = U_p$. $\square$

08GQ Lemma 75.9.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{B} \subset \text{Ob}(X_{spaces,étale})$. Let P be a property of the elements of $\mathcal{B}$. Assume that

(1) every $W \in \mathcal{B}$ is quasi-compact and quasi-separated,
(2) if $W \in \mathcal{B}$ and $U \subset W$ is quasi-compact open, then $U \in \mathcal{B}$,
(3) if $V \in \text{Ob}(X_{spaces,étale})$ is affine, then (a) $V \in \mathcal{B}$ and (b) P holds for V,
(4) for every elementary distinguished square $(U \subset W, f : V \to W)$ such that
   (a) $W \in \mathcal{B}$,
   (b) U is quasi-compact,
   (c) V is affine, and
   (d) P holds for U, V, and $U \times_W V$,
   then P holds for W.

Then P holds for every $W \in \mathcal{B}$.

Proof. This is proved in exactly the same manner as the proof of Lemma 75.9.3. (We remark that (4)(d) makes sense as $U \times_W V$ is a quasi-compact open of V hence an element of $\mathcal{B}$ by conditions (2) and (3).) $\square$

08GR Remark 75.9.5. How to choose the collection $\mathcal{B}$ in Lemma 75.9.4? Here are some examples:

(1) If X is quasi-compact and separated, then we can choose $\mathcal{B}$ to be the set of quasi-compact and separated objects of $X_{spaces,étale}$. Then $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a). With this choice of $\mathcal{B}$ Lemma 75.9.4 reproduces Lemma 75.9.3.
(2) If X is quasi-compact with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1), then we can choose $\mathcal{B}$ to be the set of objects of $X_{spaces,étale}$ which are quasi-compact and have affine diagonal over $\mathbf{Z}$. Again $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a).
(3) If X is quasi-compact and quasi-separated, then the smallest subset $\mathcal{B}$ which contains X and satisfies (1), (2), and (3)(a) is given by the rule $W \in \mathcal{B}$ if and only if either W is a quasi-compact open subspace of X, or W is a quasi-compact open of an affine object of $X_{spaces,étale}$.

Here is a variant where we extend the truth from an open to larger opens.

09IT Lemma 75.9.6. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open subspace. Let P be a property of quasi-compact open subspaces of X. Assume that

(1) P holds for W, and
(2) for every elementary distinguished square $(W_1 \subset W_2, f : V \to W_2)$ where such that
  (a) W_1, W_2 are quasi-compact open subspaces of X,
  (b) $W \subset W_1$,
  (c) V is affine, and
  (d) P holds for W_1,
  then P holds for W_2.

Then P holds for X.

Proof. We can deduce this from Lemma 75.9.4, but instead we will give a direct argument by explicitly redoing the proof of Lemma 75.9.3. We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Decent Spaces, Lemma 68.8.6. We will prove that P holds for $W_p = W \cup U_p$ by descending induction on p. This will finish the proof as $W_1 = X$. Note that P holds for $W_{n+1} = W \cap U_{n+1} = W$ by (1). Assume P holds for W_{p+1}. Observe that $W_p \setminus W_{p+1}$ (with reduced induced subspace structure) is a closed subspace of $U_p \setminus U_{p+1}$. Since $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, the same is true for $(W_{p+1} \subset W_p, f_p : V_p \to W_p)$. However (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = W_{p+1}$ and

$$W_{p,i} = W_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X containing W. Then we have

$$W_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = W_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 75.9.2. Now (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = W_p$. □

75.10. Mayer-Vietoris

08GS In this section we prove that an elementary distinguished triangle gives rise to various Mayer-Vietoris sequences.

Let S be a scheme. Let $U \to X$ be an étale morphism of algebraic spaces over S. In Properties of Spaces, Section 66.27 it was shown that $U_{spaces,étale} = X_{spaces,étale}/U$ compatible with structure sheaves. Hence in this situation we often think of the morphism $j_U : U \to X$ as a localization morphism (see Modules on Sites, Definition 18.19.1). In particular we think of pullback j_U^* as restriction to U and we often denote it by $|_U$; this is compatible with Properties of Spaces, Equation (66.26.1.1). In particular we see that

08GT (75.10.0.1) $$(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}}$$

if $\overline{u}$ is a geometric point of U and $\overline{x}$ the image of $\overline{u}$ in X. Moreover, restriction has an exact left adjoint $j_{U!}$, see Modules on Sites, Lemmas 18.19.2 and 18.19.3. Finally, recall that if $\mathcal{G}$ is an $\mathcal{O}_X$-module, then

08GU (75.10.0.2) $$(j_{U!}\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

for any geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ where the direct sum is over those morphisms $\overline{u} : \mathrm{Spec}(k) \to U$ such that $j_U \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 18.38.1 and Properties of Spaces, Lemma 66.19.13.

08GV **Lemma 75.10.1.** Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.

(1) For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence
$$0 \to j_{U \times_X V!}\mathcal{F}|_{U \times_X V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$
(2) For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle
$$j_{U \times_X V!}E|_{U \times_X V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U \times_X V!}E|_{U \times_X V}[1]$$
in $D(\mathcal{O}_X)$.

Proof. To show the sequence of (1) is exact we may check on stalks at geometric points by Properties of Spaces, Theorem 66.19.12. Let $\overline{x}$ be a geometric point of X. By Equations (75.10.0.1) and (75.10.0.2) taking stalks at $\overline{x}$ we obtain the sequence
$$0 \to \bigoplus\nolimits_{(\overline{u},\overline{v})} \mathcal{F}_{\overline{x}} \to \bigoplus\nolimits_{\overline{u}} \mathcal{F}_{\overline{x}} \oplus \bigoplus\nolimits_{\overline{v}} \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to 0$$
This sequence is exact because for every $\overline{x}$ there either is exactly one $\overline{u}$ mapping to $\overline{x}$, or there is no $\overline{u}$ and exactly one $\overline{v}$ mapping to $\overline{x}$.

Proof of (2). We have seen in Cohomology on Sites, Section 21.20 that the restriction functors and the extension by zero functors on derived categories are computed by just applying the functor to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules
$$0 \to j_{U \times_X V!}\mathcal{E}^\bullet|_{U \times_X V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$
which is short exact by (1). $\square$

08GW **Lemma 75.10.2.** Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.

(1) For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence
$$0 \to \mathcal{F} \to j_{U,*}\mathcal{F}|_U \oplus j_{V,*}\mathcal{F}|_V \to j_{U \times_X V,*}\mathcal{F}|_{U \times_X V} \to 0$$
(2) For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle
$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U \times_X V,*}E|_{U \times_X V} \to E[1]$$
in $D(\mathcal{O}_X)$.

Proof. Let W be an object of $X_{\acute{e}tale}$. We claim the sequence
$$0 \to \mathcal{F}(W) \to \mathcal{F}(W \times_X U) \oplus \mathcal{F}(W \times_X V) \to \mathcal{F}(W \times_X U \times_X V)$$
is exact and that an element of the last group can locally on W be lifted to the middle one. By Lemma 75.9.2 the pair $(W \times_X U \subset W, V \times_X W \to W)$ is an

elementary distinguished square. Thus we may assume $W = X$ and it suffices to prove the same thing for

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \times_X V)$$

We have seen that

$$0 \to j_{U \times_X V!}\mathcal{O}_{U \times_X V} \to j_{U!}\mathcal{O}_U \oplus j_{V!}\mathcal{O}_V \to \mathcal{O}_X \to 0$$

is a exact sequence of $\mathcal{O}_X$-modules in Lemma 75.10.1 and applying the right exact functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{F})$ gives the sequence above. This also means that the obstruction to lifting $s \in \mathcal{F}(U \times_X V)$ to an element of $\mathcal{F}(U) \oplus \mathcal{F}(V)$ lies in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$. By locality of cohomology (Cohomology on Sites, Lemma 21.7.3) this obstruction vanishes étale locally on X and the proof of (1) is complete.

Proof of (2). Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $\mathrm{Mod}(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then $\mathcal{I}^\bullet|U$ is a K-injective complex (Cohomology on Sites, Lemma 21.20.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \times_X V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U \times_X V,*}\mathcal{I}^\bullet|_{U \times_X V} \to 0.$$

This sequence is exact by (1). $\square$

08JK Lemma 75.10.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $(U \subset X, V \to X)$ be an elementary distinguished square. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \times_X V} : U \times_X V \to Y$ the restrictions. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U \times_X V}) \to Rf_*E[1]$$

in $D(\mathcal{O}_Y)$. This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $\mathrm{Mod}(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Cohomology on Sites, Lemma 21.20.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \times_X V} \to 0.$$

To see this is a short exact sequence of complexes we argue as follows. Pick an injective object $\mathcal{I}$ of $\mathrm{Mod}(\mathcal{O}_X)$. Apply f_* to the short exact sequence

$$0 \to \mathcal{I} \to j_{U,*}\mathcal{I}|_U \oplus j_{V,*}\mathcal{I}|_V \to j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \to 0$$

of Lemma 75.10.2 and use that $R^1f_*\mathcal{I} = 0$ to get a short exact sequence

$$0 \to f_*\mathcal{I} \to f_*j_{U,*}\mathcal{I}|_U \oplus f_*j_{V,*}\mathcal{I}|_V \to f_*j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \to 0$$

The proof is finished by observing that $a_* = f_*j_{U,*}$ and similarly for b_* and c_*. $\square$

08H9 **Lemma 75.10.4.** Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence

$$\begin{tikzcd} & \ldots \arrow[r] & \mathrm{Ext}^{-1}(E_{U \times_X V}, F_{U \times_X V}) \arrow[dll] \\ \mathrm{Hom}(E, F) \arrow[r] & \mathrm{Hom}(E_U, F_U) \oplus \mathrm{Hom}(E_V, F_V) \arrow[r] & \mathrm{Hom}(E_{U \times_X V}, F_{U \times_X V}) \end{tikzcd}$$

where the subscripts denote restrictions to the relevant opens and the Hom's are taken in the relevant derived categories.

Proof. Use the distinguished triangle of Lemma 75.10.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 13.4.2) and use that $\mathrm{Hom}(j_{U!}E|_U, F) = \mathrm{Hom}(E|_U, F|_U)$ by Cohomology on Sites, Lemma 21.20.8. $\square$

0CRS **Lemma 75.10.5.** Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle

$$R\Gamma(X, E) \to R\Gamma(U, E) \oplus R\Gamma(V, E) \to R\Gamma(U \times_X V, E) \to R\Gamma(X, E)[1]$$

and in particular a long exact cohomology sequence

$$\ldots \to H^n(X, E) \to H^n(U, E) \oplus H^n(V, E) \to H^n(U \times_X V, E) \to H^{n+1}(X, E) \to \ldots$$

The construction of the distinguished triangle and the long exact sequence is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $\mathit{Mod}(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. In the proof of Lemma 75.10.2 we found a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U \times_X V,*}\mathcal{I}^\bullet|_{U \times_X V} \to 0$$

Since $H^1(X, \mathcal{I}^n) = 0$, we see that taking global sections gives an exact sequence of complexes

$$0 \to \Gamma(X, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{I}^\bullet) \oplus \Gamma(V, \mathcal{I}^\bullet) \to \Gamma(U \times_X V, \mathcal{I}^\bullet) \to 0$$

Since these complexes represent $R\Gamma(X, E)$, $R\Gamma(U, E)$, $R\Gamma(V, E)$, and $R\Gamma(U \times_X V, E)$ we get a distinguished triangle by Derived Categories, Section 13.12 and especially Lemma 13.12.1. $\square$

08HA **Lemma 75.10.6.** Let S be a scheme. Let $j : U \to X$ be a étale morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $W = V \times_X U$ and denote $j_W : W \to V$ the projection morphism. Then $(j_!E)|_V = j_{W!}(E|_W)$ for E in $D(\mathcal{O}_U)$.

Proof. This is true because $(j_!\mathcal{F})|_V = j_{W!}(\mathcal{F}|_W)$ for an $\mathcal{O}_X$-module $\mathcal{F}$ as follows immediately from the construction of the functors $j_!$ and $j_{W!}$, see Modules on Sites, Lemma 18.19.2. $\square$

08GG **Lemma 75.10.7.** Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Set $T = |X| \setminus |U|$.

(1) If E is an object of $D(\mathcal{O}_X)$ supported on T, then (a) $E \to Rj_*(E|_V)$ and (b) $j_!(E|_V) \to E$ are isomorphisms.

(2) If F is an object of $D(\mathcal{O}_V)$ supported on $j^{-1}T$, then (a) $F \to (j_!F)|_V$, (b) $(Rj_*F)|_V \to F$, and (c) $j_!F \to Rj_*F$ are isomorphisms.

Proof. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. Then we see that $E|_U = 0$ and $E|_{U \times_X V} = 0$ as T doesn't meet U and $j^{-1}T$ doesn't meet $U \times_X V$. Thus (1)(a) follows from Lemma 75.10.2. In exactly the same way (1)(b) follows from Lemma 75.10.1.

Let F be an object of $D(\mathcal{O}_V)$ whose cohomology sheaves are supported on $j^{-1}T$. By Lemma 75.3.1 we have $(Rj_*F)|_U = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. Similarly $(j_!F)|_U = j_{W!}(F|_W) = 0$ by Lemma 75.10.6. Thus $j_!F$ and Rj_*F are supported on T and $(j_!F)|_V$ and $(Rj_*F)|_V$ are supported on $j^{-1}(T)$. To check that the maps (2)(a), (b), (c) are isomorphisms in the derived category, it suffices to check that these map induce isomorphisms on stalks of cohomology sheaves at geometric points of T and $j^{-1}(T)$ by Properties of Spaces, Theorem 66.19.12. This we may do after replacing X by V, U by $U \times_X V$, V by $V \times_X V$ and F by $F|_{V \times_X V}$ (restriction via first projection), see Lemmas 75.3.1, 75.10.6, and 75.9.2. Since $V \times_X V \to V$ has a section this reduces (2) to the case that $j : V \to X$ has a section.

Assume j has a section $\sigma : X \to V$. Set $V' = \sigma(X)$. This is an open subspace of V. Set $U' = j^{-1}(U)$. This is another open subspace of V. Then $(U' \subset V, V' \to V)$ is an elementary distinguished square. Observe that $F|_{U'} = 0$ and $F|_{V' \cap U'} = 0$ because F is supported on $j^{-1}(T)$. Denote $j' : V' \to V$ the open immersion and $j_{V'} : V' \to X$ the composition $V' \to V \to X$ which is the inverse of σ. Set $F' = \sigma^*F$. The distinguished triangles of Lemmas 75.10.1 and 75.10.2 show that $F = j'_!(F|_{V'})$ and $F = Rj'_*(F|_{V'})$. It follows that $j_!F = j_!j'_!(F|_{V'}) = j_{V'!}F = F'$ because $j_{V'} : V' \to X$ is an isomorphism and the inverse of σ. Similarly, $Rj_*F = Rj_*Rj'_*F = Rj_{V',*}F = F'$. This proves (2)(c). To prove (2)(a) and (2)(b) it suffices to show that $F = F'|_V$. This is clear because both F and $F'|_V$ restrict to zero on U' and $U' \cap V'$ and the same object on V'. $\square$

We can glue complexes!

08HB Lemma 75.10.8. Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. Suppose given

(1) an object A of $D(\mathcal{O}_U)$,
(2) an object B of $D(\mathcal{O}_V)$, and
(3) an isomorphism $c : A|_{U \times_X V} \to B|_{U \times_X V}$.

Then there exists an object F of $D(\mathcal{O}_X)$ and isomorphisms $f : F|_U \to A$, $g : F|_V \to B$ such that $c = g|_{U \times_X V} \circ f^{-1}|_{U \times_X V}$. Moreover, given

(1) an object E of $D(\mathcal{O}_X)$,
(2) a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,
(3) a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,

such that

$$a|_{U \times_X V} = b|_{U \times_X V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is $a \circ f$ and whose restriction to V is $b \circ g$.

Proof. Denote j_U, j_V, $j_{U \times_X V}$ the corresponding morphisms towards X. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U \times_X V,*}(B|_{U \times_X V}) \to F[1]$$

Here the map $Rj_{V,*}B \to Rj_{U \times_X V,*}(B|_{U \times_X V})$ is the obvious one. The map $Rj_{U,*}A \to Rj_{U \times_X V,*}(B|_{U \times_X V})$ is the composition of $Rj_{U,*}A \to Rj_{U \times_X V,*}(A|_{U \times_X V})$ with $Rj_{U \times_X V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U \to F|_U[1]$$

Denote $j : U \times_X V \to U$. Compatibility of restriction and total direct image (Lemma 75.3.1) shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U \times_X V})$. Hence the second arrow of the last displayed equation has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism.

To see that the morphism $F|_V \to B$ is an isomorphism we will use a trick. Namely, choose a distinguished triangle

$$F|_V \to B \to B' \to F[1]|_V$$

in $D(\mathcal{O}_V)$. Since $F|_U \to A$ is an isomorphism, and since we have the isomorphism $c : A|_{U \times_X V} \to B|_{U \times_X V}$ the restriction of $F|_V \to B$ is an isomorphism over $U \times_X V$. Thus B' is supported on $j_V^{-1}(T)$ where $T = |X| \setminus |U|$. On the other hand, there is a morphism of distinguished triangles

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & Rj_{U,*}F|_U \oplus Rj_{V,*}F|_V \arrow[r] \arrow[d] & Rj_{U \times_X V,*}F|_{U \times_X V} \arrow[r] \arrow[d] & F[1] \arrow[d] \\
F \arrow[r] & Rj_{U,*}A \oplus Rj_{V,*}B \arrow[r] & Rj_{U \times_X V,*}(B|_{U \times_X V}) \arrow[r] & F[1]
\end{tikzcd}$$

The all of the vertical maps in this diagram are isomorphisms, except for the map $Rj_{V,*}F|_V \to Rj_{V,*}B$, hence that is an isomorphism too (Derived Categories, Lemma 13.4.3). This implies that $Rj_{V,*}B' = 0$. Hence $B' = 0$ by Lemma 75.10.7.

The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 75.10.4. □

75.11. The coherator

08GX Let S be a scheme. Let X be an algebraic space over S. The coherator is a functor

$$Q_X : \text{Mod}(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$. It exists for any algebraic space X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties of Spaces, Proposition 66.32.2. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(QCoh(\mathcal{O}_X)).$$

Since Q_X is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ we see that RQ_X is right adjoint to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$ by Derived Categories, Lemma 13.30.3.

In this section we will study the functor RQ_X. In Section 75.19 we will study the (closely related) right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ (when it exists).

08GY Lemma 75.11.1. Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$. This functor has the property that

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "f_*"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Spaces, Lemma 69.8.2. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 13.16.9. For any complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ there is a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when $\mathcal{F}^\bullet$ is a complex with each $\mathcal{F}^n$ quasi-coherent. The statement is étale local on Y hence we may assume Y affine. As an affine morphism is representable we reduce to the case of schemes by the compatibility of Remark 75.6.3. The case of schemes is Derived Categories of Schemes, Lemma 36.7.1. □

08GZ Lemma 75.11.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact, quasi-separated, and flat. Then, denoting

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(\mathcal{O}_Y)) \to D(\mathcal{O}_X)$.

Since f is quasi-compact and quasi-separated, we see that f_* preserves quasi-coherence, see Morphisms of Spaces, Lemma 67.11.2. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 66.32.2). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology on Sites, Lemma 21.19.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 13.30.3.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. $\square$

08H0 Lemma 75.11.3. Let S be a scheme. Let X be an affine algebraic space over S. Set $A = \Gamma(X, \mathcal{O}_X)$. Then

(1) $Q_X : \text{Mod}(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,
(2) $RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,
(3) restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (75.5.1.1).

Proof. Let $X_0 = \text{Spec}(A)$ be the affine scheme representing X. Recall that there is a morphism of ringed sites $\epsilon : X_{\acute{e}tale} \to X_{0,Zar}$ which induces equivalences

$$\begin{tikzcd} QCoh(\mathcal{O}_X) \arrow[r, shift left, "\epsilon_*"] & QCoh(\mathcal{O}_{X_0}) \arrow[l, shift left, "\epsilon^*"] \end{tikzcd}$$

see Lemma 75.4.2. Hence we see that $Q_X = \epsilon^* \circ Q_{X_0} \circ \epsilon_*$ by uniqueness of adjoint functors. Hence (1) follows from the description of Q_{X_0} in Derived Categories of Schemes, Lemma 36.7.3 and the fact that $\Gamma(X_0, \epsilon_*\mathcal{F}) = \Gamma(X, \mathcal{F})$. Part (2) follows from (1) and the fact that the functor from A-modules to quasi-coherent $\mathcal{O}_X$-modules is exact. The third assertion now follows from the result for schemes (Derived Categories of Schemes, Lemma 36.7.3) and Lemma 75.4.2. $\square$

Next, we prove a criterion for when the functor $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence.

09TG Lemma 75.11.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Suppose that for every étale morphism $j : V \to W$ with $W \subset X$ quasi-compact open and V affine the right derived functor

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_W))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_V) \to QCoh(\mathcal{O}_W)$ fits into a commutative diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_V)) \arrow[r, "i_V"'] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_V) \arrow[d, "Rj_*"] \\ D(QCoh(\mathcal{O}_W)) \arrow[r, "i_W"] & D_{QCoh}(\mathcal{O}_W) \end{tikzcd}$$

Then the functor (75.5.1.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. We first use the induction principle to prove i_X is fully faithful. More precisely, we will use Lemma 75.9.6. Let $(U \subset W, V \to W)$ be an elementary distinguished square with V affine and U, W quasi-compact open in X. Assume that i_U is fully faithful. We have to show that i_W is fully faithful. We may replace X by W, i.e., we may assume $W = X$ (we do this just to simplify the notation – observe that the condition in the statement of the lemma is preserved under this operation).

Suppose that A, B are objects of $D(QCoh(\mathcal{O}_X))$. We want to show that

$$\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))$$

is bijective. Let $T = |X| \setminus |U|$.

Assume first $i_X(B)$ is supported on T. In this case the map

$$i_X(B) \to Rj_{V,*}(i_X(B)|_V) = Rj_{V,*}(i_V(B|_V))$$

is a quasi-isomorphism (Lemma 75.10.7). By assumption we have an isomorphism $i_X(\Phi(B|_V)) \to Rj_{V,*}(i_V(B|_V))$ in $D(\mathcal{O}_X)$. Moreover, Φ and $-|_V$ are adjoint functors on the derived categories of quasi-coherent modules (by Derived Categories, Lemma 13.30.3). The adjunction map $B \to \Phi(B|_V)$ becomes an isomorphism after applying i_X, whence is an isomorphism in $D(QCoh(\mathcal{O}_X))$. Hence

$$\begin{aligned}
\mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, B) &= \mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, \Phi(B|_V)) \\
&= \mathrm{Mor}_{D(QCoh(\mathcal{O}_V))}(A|_V, B|_V) \\
&= \mathrm{Mor}_{D(\mathcal{O}_V)}(i_V(A|_V), i_V(B|_V)) \\
&= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), Rj_{V,*}(i_V(B|_V))) \\
&= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))
\end{aligned}$$

as desired. Here we have used that i_V is fully faithful (Lemma 75.11.3).

In general, choose any complex $\mathcal{B}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules representing B. Next, choose any quasi-isomorphism $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ of complexes of quasi-coherent modules on U. As $j_U : U \to X$ is quasi-compact and quasi-separated the functor $j_{U,*}$ transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 67.11.2). Thus there is a canonical map $\mathcal{B}^\bullet \to j_{U,*}(\mathcal{B}^\bullet|_U) \to j_{U,*}\mathcal{C}^\bullet$ of complexes of quasi-coherent modules on X. Set $B'' = j_{U,*}\mathcal{C}^\bullet$ in $D(QCoh(\mathcal{O}_X))$ and choose a distinguished triangle

$$B \to B'' \to B' \to B[1]$$

in $D(QCoh(\mathcal{O}_X))$. Since the first arrow of the triangle restricts to an isomorphism over U we see that B' is supported on T. Hence in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1]) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1]) \arrow[d] \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B)) \arrow[d] \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'') \arrow[r] \arrow[d] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'')) \arrow[d] \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B') \arrow[r] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'))
\end{tikzcd}$$

we have exact columns and the top and bottom horizontal arrows are bijective. Finally, choose a complex $\mathcal{A}^\bullet$ of quasi-coherent modules representing A.

Let $\alpha : i_X(A) \to i_X(B)$ be a morphism between in $D(\mathcal{O}_X)$. The restriction $\alpha|_U$ comes from a morphism in $D(QCoh(\mathcal{O}_U))$ as i_U is fully faithful. Hence there exists a choice of $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ as above such that $\alpha|_U$ is represented by an actual map of complexes $\mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$. This corresponds to a map of complexes $\mathcal{A} \to j_{U,*}\mathcal{C}^\bullet$. In other words, the image of α in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B''))$ comes from an element of $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. A diagram chase then shows that α comes from a morphism $A \to B$ in $D(QCoh(\mathcal{O}_X))$. Finally, suppose that $a : A \to B$ is a morphism of $D(QCoh(\mathcal{O}_X))$ which becomes zero in $D(\mathcal{O}_X)$. After choosing $\mathcal{B}^\bullet$ suitably, we may assume a is represented by a morphism of complexes $a^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$. Since i_U is fully faithul the restriction $a^\bullet|_U$ is zero in $D(QCoh(\mathcal{O}_U))$. Thus we can choose s such that $s \circ a^\bullet|_U : \mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$ is homotopic to zero. Applying the functor $j_{U,*}$ we conclude that $\mathcal{A}^\bullet \to j_{U,*}\mathcal{C}^\bullet$ is homotopic to zero. Thus a maps to zero in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. Thus we may assume that a is the image of an element of $b \in \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1])$. The image of b in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1])$ comes from a $\gamma \in \mathrm{Hom}_{D(\mathcal{O}_X)}(A, B''[-1])$ (as a maps to zero in the group on the right). Since we've seen above the horizontal arrows are surjective, we see that γ comes from a c in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B''[-1])$ which implies $a = 0$ as desired.

At this point we know that i_X is fully faithful for our original X. Since RQ_X is its right adjoint, we see that $RQ_X \circ i_X = \mathrm{id}$ (Categories, Lemma 4.24.4). To finish the proof we show that for any E in $D_{QCoh}(\mathcal{O}_X)$ the map $i_X(RQ_X(E)) \to E$ is an isomorphism. Choose a distinguished triangle

$$i_X(RQ_X(E)) \to E \to E' \to i_X(RQ_X(E))[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. A formal argument using the above shows that $i_X(RQ_X(E')) = 0$. Thus it suffices to prove that for $E \in D_{QCoh}(\mathcal{O}_X)$ the condition $i_X(RQ_X(E)) = 0$ implies that $E = 0$. Consider an étale morphism $j : V \to X$ with V affine. By Lemmas 75.11.3 and 75.11.2 and our assumption we have

$$Rj_*(E|_V) = Rj_*(i_V(RQ_V(E|_V))) = i_X(\Phi(RQ_V(E|_V))) = i_X(RQ_X(Rj_*(E|_V)))$$

Choose a distinguished triangle

$$E \to Rj_*(E|_V) \to E' \to E[1]$$

Apply RQ_X to get a distinguished triangle

$$0 \to RQ_X(Rj_*(E|_V)) \to RQ_X(E') \to 0[1]$$

in other words the map in the middle is an isomorphism. Combined with the string of equalities above we find that our first distinguished triangle becomes a distinguished triangle

$$E \to i_X(RQ_X(E')) \to E' \to E[1]$$

where the middle morphism is the adjunction map. However, the composition $E \to E'$ is zero, hence $E \to i_X(RQ_X(E'))$ is zero by adjunction! Since this morphism is isomorphic to the morphism $E \to Rj_*(E|_V)$ adjoint to $\mathrm{id} : E|_V \to E|_V$ we conclude that $E|_V$ is zero. Since this holds for all affine V étale over X we conclude E is zero as desired. □

08H1 Proposition 75.11.5. Let S be a scheme. Let X be a quasi-compact algebraic space over S with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1). Then the functor (75.5.1.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let $V \to W$ be an étale morphism with V affine and W a quasi-compact open subspace of X. Then the morphism $V \to W$ is affine as W has affine diagonal over $\mathbf{Z}$ and V is affine (Morphisms of Spaces, Lemma 67.20.11). Lemma 75.11.1 then guarantees that the assumption of Lemma 75.11.4 holds. Hence we conclude. $\square$

0CSR Lemma 75.11.6. Let S be a scheme and let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y are quasi-compact and have affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1). Then, denoting

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

is commutative.

Proof. Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 75.11.5. Hence we can identify these categories (and similarly for other quasi-compact algebraic spaces with affine diagonal) and then the statement of the lemma is that the canonical map $\Phi(K) \to Rf_*(K)$ is an isomorphism for all K in $D(QCoh(\mathcal{O}_X))$. Note that if $K_1 \to K_2 \to K_3 \to K_1[1]$ is a distinguished triangle in $D(QCoh(\mathcal{O}_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $\mathcal{B} \subset \text{Ob}(X_{spaces,étale})$ be the set of objects which are quasi-compact and have affine diagonal. For $U \in \mathcal{B}$ and any morphism $g : U \to Z$ where Z is a quasi-compact algebraic space over S with affine diagonal, denote

$$\Phi_g : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_Z))$$

the derived extension of g_*. Let $P(U)$ = "for any K in $D(QCoh(\mathcal{O}_U))$ and any $g : U \to Z$ as above the map $\Phi_g(K) \to Rg_*K$ is an isomorphism". By Remark 75.9.5 conditions (1), (2), and (3)(a) of Lemma 75.9.4 hold and we are left with proving (3)(b) and (4).

Checking condition (3)(b). Let U be an affine scheme étale over X. Let $g : U \to Z$ be as above. Since the diagonal of Z is affine the morphism $g : U \to Z$ is affine (Morphisms of Spaces, Lemma 67.20.11). Hence $P(U)$ holds by Lemma 75.11.1.

Checking condition (4). Let $(U \subset W, V \to W)$ be an elementary distinguished square in $X_{spaces,étale}$ with U, W, V in $\mathcal{B}$ and V affine. Assume that P holds for U, V, and $U \times_W V$. We have to show that P holds for W. Let $g : W \to Z$ be a morphism to a quasi-compact algebraic space with affine diagonal. Let K be an object of $D(QCoh(\mathcal{O}_W))$. Consider the distinguished triangle

$$K \to Rj_{U,*}K|_U \oplus Rj_{V,*}K|_V \to Rj_{U \times_W V,*}K|_{U \times_W V} \to K[1]$$

in $D(\mathcal{O}_W)$. By the two-out-of-three property mentioned above, it suffices to show that $\Phi_g(Rj_{U,*}K|_U) \to Rg_*(Rj_{U,*}K|_U)$ is an isomorphism and similarly for V and $U \times_W V$. This is discussed in the next paragraph.

Let $j : U \to W$ be a morphism $X_{spaces,étale}$ with U, W in $\mathcal{B}$ and P holds for U. Let $g : W \to Z$ be a morphism to a quasi-compact algebraic space with affine diagonal. To finish the proof we have to show that $\Phi_g(Rj_*K) \to Rg_*(Rj_*K)$ is an isomorphism for any K in $D(QCoh(\mathcal{O}_U))$. Let $\mathcal{I}^\bullet$ be a K-injective complex in $QCoh(\mathcal{O}_U)$ representing K. From $P(U)$ applied to j we see that $j_*\mathcal{I}^\bullet$ represents Rj_*K. Since $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ has an exact left adjoint $j^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ we see that $j_*\mathcal{I}^\bullet$ is a K-injective complex in $QCoh(\mathcal{O}_W)$, see Derived Categories, Lemma 13.31.9. Hence $\Phi_g(Rj_*K)$ is represented by $g_*j_*\mathcal{I}^\bullet = (g \circ j)_*\mathcal{I}^\bullet$. By $P(U)$ applied to $g \circ j$ we see that this represents $R_{g\circ j,*}(K) = Rg_*(Rj_*K)$. This finishes the proof. □

75.12. The coherator for Noetherian spaces

09TH We need a little bit more about injective modules to treat the case of a Noetherian algebraic space.

09TI **Lemma 75.12.1.** *Let S be a Noetherian affine scheme. Every injective object of $QCoh(\mathcal{O}_S)$ is a filtered colimit $\mathrm{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to S)_*\mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Let $S = \mathrm{Spec}(A)$. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_S)$. Since $QCoh(\mathcal{O}_S)$ is equivalent to the category of A-modules we see that $\mathcal{J}$ is equal to $\widetilde{J}$ for some injective A-module J. By Dualizing Complexes, Proposition 47.5.9 we can write $J = \bigoplus E_\alpha$ with E_α indecomposable and therefore isomorphic to the injective hull of a reside field at a point. Thus (because finite disjoint unions of Artinian schemes are Artinian) we may assume that J is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Then $J = \bigcup J[\mathfrak{p}^n]$ where $J[\mathfrak{p}^n]$ is the injective hull of $\kappa(\mathfrak{p})$ over $A_/\mathfrak{p}^nA_\mathfrak{p}$, see Dualizing Complexes, Lemma 47.7.3. Thus $\widetilde{J}$ is the colimit of the sheaves $(Z_n \to X)_*\mathcal{G}_n$ where $Z_n = \mathrm{Spec}(A_\mathfrak{p}/\mathfrak{p}^nA_\mathfrak{p})$ and $\mathfrak{G}_n$ the coherent sheaf associated to the finite $A_/\mathfrak{p}^nA_\mathfrak{p}$-module $J[\mathfrak{p}^n]$. Finiteness follows from Dualizing Complexes, Lemma 47.6.1. □

09TJ **Lemma 75.12.2.** *Let S be an affine scheme. Let X be a Noetherian algebraic space over S. Every injective object of $QCoh(\mathcal{O}_X)$ is a direct summand of a filtered colimit $\mathrm{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to X)_*\mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Choose an affine scheme U and a surjective étale morphism $j : U \to X$ (Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian affine scheme. Choose an injective object $\mathcal{J}'$ of $QCoh(\mathcal{O}_U)$ such that there exists an injection $\mathcal{J}|_U \to \mathcal{J}'$. Then

$$\mathcal{J} \to j_*\mathcal{J}'$$

is an injective morphism in $QCoh(\mathcal{O}_X)$, hence identifies $\mathcal{J}$ as a direct summand of $j_*\mathcal{J}'$. Thus the result follows from the corresponding result for $\mathcal{J}'$ proved in Lemma 75.12.1. □

09TK **Lemma 75.12.3.** *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, and quasi-separated morphism of algebraic spaces over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then $f_*\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_Y)$.*

Proof. Since f is quasi-compact and quasi-separated, the functor f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces, Lemma 67.11.2). The functor f^* is a left adjoint to f_* which transforms injections into injections. Hence the result follows from Homology, Lemma 12.29.1 $\square$

09TL Lemma 75.12.4. Let S be a scheme. Let X be a Noetherian algebraic space over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then

(1) $H^p(U, \mathcal{J}|_U) = 0$ for $p > 0$ and for every quasi-compact and quasi-separated algebraic space U étale over X,
(2) for any morphism $f : X \to Y$ of algebraic spaces over S with Y quasi-separated we have $R^pf_*\mathcal{J} = 0$ for $p > 0$.

Proof. Proof of (1). Write $\mathcal{J}$ as a direct summand of $\mathrm{colim}\, \mathcal{F}_i$ with $\mathcal{F}_i = (Z_i \to X)_*\mathcal{G}_i$ as in Lemma 75.12.2. It is clear that it suffices to prove the vanishing for $\mathrm{colim}\, \mathcal{F}_i$. Since pullback commutes with colimits and since U is quasi-compact and quasi-separated, it suffices to prove $H^p(U, \mathcal{F}_i|_U) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 69.5.1. Observe that $Z_i \to X$ is an affine morphism, see Morphisms of Spaces, Lemma 67.20.12. Thus

$$\mathcal{F}_i|_U = (Z_i \times_X U \to U)_*\mathcal{G}'_i = R(Z_i \times_X U \to U)_*\mathcal{G}'_i$$

where $\mathcal{G}'_i$ is the pullback of $\mathcal{G}_i$ to $Z_i \times_X U$, see Cohomology of Spaces, Lemma 69.11.1. Since $Z_i \times_X U$ is affine we conclude that $\mathcal{G}'_i$ has no higher cohomology on $Z_i \times_X U$. By the Leray spectral sequence we conclude the same thing is true for $\mathcal{F}_i|_U$ (Cohomology on Sites, Lemma 21.14.6).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \to Y$ be an étale morphism with V affine. Then $V \times_Y X \to X$ is an étale morphism and $V \times_Y X$ is a quasi-compact and quasi-separated algebraic space étale over X (details omitted). Hence $H^p(V \times_Y X, \mathcal{J})$ is zero by part (1). Since $R^pf_*\mathcal{J}$ is the sheaf associated to the presheaf $V \mapsto H^p(V \times_Y X, \mathcal{J})$ the result is proved. $\square$

09TM Lemma 75.12.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over S. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. Since X and Y are Noetherian the morphism is quasi-compact and quasi-separated (see Morphisms of Spaces, Lemma 67.8.10). Thus f_* preserve quasi-coherence, see Morphisms of Spaces, Lemma 67.11.2. Next, let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 66.32.2), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 75.6.1. Consider the short exact sequence

$$0 \to \sigma_{\geq m-N-1}\mathcal{I}^\bullet \to \mathcal{I}^\bullet \to \sigma_{\leq m-N-2}\mathcal{I}^\bullet \to 0$$

of complexes of quasi-coherent sheaves on X. By Lemma 75.6.1 we see that the cohomology sheaves of $Rf_*\sigma_{\leq m-N-2}\mathcal{I}^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $R^mf_*\mathcal{I}^\bullet$ is isomorphic to $R^mf_*\sigma_{\geq m-N-1}\mathcal{I}^\bullet$. In other words, we may assume that $\mathcal{I}^\bullet$ is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This case follows from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) with required vanishing because of Lemma 75.12.4. □

09TN Proposition 75.12.6. Let S be a scheme. Let X be a Noetherian algebraic space over S. Then the functor (75.5.1.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Follows immediately from Lemmas 75.12.5 and 75.11.4. □

75.13. Pseudo-coherent and perfect complexes

08HC In this section we study the general notions defined in Cohomology on Sites, Sections 21.44, 21.45, 21.46, and 21.47 for the étale site of an algebraic space. In particular we match this with what happens for schemes.

First we compare the notion of a pseudo-coherent complex on a scheme and on its associated small étale site.

08HD Lemma 75.13.1. Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is of finite type as an $\mathcal{O}_X$-module, and
(2) $\epsilon^*\mathcal{F}$ is of finite type as an $\mathcal{O}_{étale}$-module on the small étale site of X.

Here ϵ is as in (75.4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Modules on Sites, Lemma 18.23.4. Assume (2). By assumption there exists an étale covering $\{f_i : X_i \to X\}$ such that $\epsilon^*\mathcal{F}|_{(X_i)_{étale}}$ is generated by finitely many sections. Let $x \in X$. We will show that $\mathcal{F}$ is generated by finitely many sections in a neighbourhood of x. Say x is in the image of $X_i \to X$ and denote $X' = X_i$. Let $s_1, \ldots, s_n \in \Gamma(X', \epsilon^*\mathcal{F}|_{X'_{étale}})$ be generating sections. As $\epsilon^*\mathcal{F} = \epsilon^{-1}\mathcal{F} \otimes_{\epsilon^{-1}\mathcal{O}_X} \mathcal{O}_{étale}$ we can find an étale morphism $X'' \to X'$ such that x is in the image of $X' \to X$ and such that $s_i|_{X''} = \sum s_{ij} \otimes a_{ij}$ for some sections $s_{ij} \in \epsilon^{-1}\mathcal{F}(X'')$ and $a_{ij} \in \mathcal{O}_{étale}(X'')$. Denote $U \subset X$ the image of $X'' \to X$. This is an open subscheme as $f'' : X'' \to X$ is étale (Morphisms, Lemma 29.36.13). After possibly shrinking X'' more we may assume s_{ij} come from elements $t_{ij} \in \mathcal{F}(U)$ as follows from the construction of the inverse image functor ϵ^{-1}. Now we claim that t_{ij} generate $\mathcal{F}|_U$ which finishes the proof of the lemma. Namely, the corresponding map $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ has the property that its pullback by f'' to X'' is surjective. Since $f'' : X'' \to U$ is a surjective flat morphism of schemes, this implies that $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ is surjective by looking at stalks and using that $\mathcal{O}_{U,f''(z)} \to \mathcal{O}_{X'',z}$ is faithfully flat for all $z \in X''$. □

In the situation above the morphism of sites ϵ is flat hence defines a pullback on complexes of modules.

08HE Lemma 75.13.2. Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent

(1) E is m-pseudo-coherent, and
(2) ϵ^*E is m-pseudo-coherent on the small étale site of X.

Here ϵ is as in (75.4.0.1).

Proof. The implication (1) ⇒ (2) is a general fact, see Cohomology on Sites, Lemma 21.45.3. Assume ϵ^*E is m-pseudo-coherent. We will use without further mention that ϵ^* is an exact functor and that therefore

$$\epsilon^*H^i(E) = H^i(\epsilon^*E).$$

To show that E is m-pseudo-coherent we may work locally on X, hence we may assume that X is quasi-compact (for example affine). Since X is quasi-compact every étale covering $\{U_i \to X\}$ has a finite refinement. Thus we see that ϵ^*E is an object of $D^-(\mathcal{O}_{étale})$, see comments following Cohomology on Sites, Definition 21.45.1. By Lemma 75.4.1 it follows that E is an object of $D^-(\mathcal{O}_X)$.

Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(E)$ is nonzero; then n is also the largest integer such that $H^n(\epsilon^*E)$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is clearly true. If $n \geq m$, then $H^n(\epsilon^*E)$ is a finite $\mathcal{O}_{étale}$-module, see Cohomology on Sites, Lemma 21.45.7. Hence $H^n(E)$ is a finite $\mathcal{O}_X$-module, see Lemma 75.13.1. After replacing X by the members of an open covering, we may assume there exists a surjection $\mathcal{O}_X^{\oplus t} \to H^n(E)$. We may locally on X lift this to a map of complexes $\alpha : \mathcal{O}_X^{\oplus t}[-n] \to E$ (details omitted). Choose a distinguished triangle

$$\mathcal{O}_X^{\oplus t}[-n] \to E \to C \to \mathcal{O}_X^{\oplus t}[-n+1]$$

Then C has vanishing cohomology in degrees $\geq n$. On the other hand, the complex ϵ^*C is m-pseudo-coherent, see Cohomology on Sites, Lemma 21.45.4. Hence by induction we see that C is m-pseudo-coherent. Applying Cohomology on Sites, Lemma 21.45.4 once more we conclude. □

08HF Lemma 75.13.3. Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then

(1) E has tor amplitude in $[a, b]$ if and only if ϵ^*E has tor amplitude in $[a, b]$.
(2) E has finite tor dimension if and only if ϵ^*E has finite tor dimension.

Here ϵ is as in (75.4.0.1).

Proof. The easy implication follows from Cohomology on Sites, Lemma 21.46.5. For the converse, assume that ϵ^*E has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. As ϵ is a flat morphism of ringed sites (Lemma 75.4.1) we have

$$\epsilon^*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = \epsilon^*E \otimes_{\mathcal{O}_{étale}}^{\mathbf{L}} \epsilon^*\mathcal{F}$$

Thus the (assumed) vanishing of cohomology sheaves on the right hand side implies the desired vanishing of the cohomology sheaves of $E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$ via Lemma 75.4.1. □

0DK7 **Lemma 75.13.4.** Let $f : X \to Y$ be a morphism of schemes. Let E be an object of $D(\mathcal{O}_X)$. Then

1. E as an object of $D(f^{-1}\mathcal{O}_Y)$ has tor amplitude in $[a, b]$ if and only if ϵ^*E has tor amplitude in $[a, b]$ as an object of $D(f_{small}^{-1}\mathcal{O}_{Y_{étale}})$.
2. E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$ if and only if ϵ^*E locally has finite tor dimension as an object of $D(f_{small}^{-1}\mathcal{O}_{Y_{étale}})$.

Here ϵ is as in (75.4.0.1).

Proof. The easy direction in (1) follows from Cohomology on Sites, Lemma 21.46.5. Let $x \in X$ be a point and let $\overline{x}$ be a geometric point lying over x. Let $y = f(x)$ and denote $\overline{y}$ the geometric point of Y coming from $\overline{x}$. Then $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,y}$ (Sheaves, Lemma 6.21.5) and

$$(f_{small}^{-1}\mathcal{O}_{Y_{étale}})_{\overline{x}} = \mathcal{O}_{Y_{étale},\overline{y}} = \mathcal{O}_{Y,y}^{sh}$$

is the strict henselization (by Étale Cohomology, Lemmas 59.36.2 and 59.33.1). Since the stalk of $\mathcal{O}_{X_{étale}}$ at X is $\mathcal{O}_{X,x}^{sh}$ we obtain

$$(\epsilon^*E)_{\overline{x}} = E_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X,x}^{sh}$$

by transitivity of pullbacks. If ϵ^*E has tor amplitude in $[a, b]$ as a complex of $f_{small}^{-1}\mathcal{O}_{Y_{étale}}$-modules, then $(\epsilon^*E)_{\overline{x}}$ has tor amplitude in $[a, b]$ as a complex of $\mathcal{O}_{Y,y}^{sh}$-modules (because taking stalks is a pullback and lemma cited above). By More on Flatness, Lemma 38.2.6 we find the tor amplitude of $(\epsilon^*E)_{\overline{x}}$ as a complex of $\mathcal{O}_{Y,y}$-modules is in $[a, b]$. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{sh}$ is faithfully flat (More on Algebra, Lemma 15.45.1) and since $(\epsilon^*E)_{\overline{x}} = E_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X,x}^{sh}$ we may apply More on Algebra, Lemma 15.66.18 to conclude the tor amplitude of E_x as a complex of $\mathcal{O}_{Y,y}$-modules is in $[a, b]$. By Cohomology, Lemma 20.48.5 we conclude that E as an object of $D(f^{-1}\mathcal{O}_Y)$ has tor amplitude in $[a, b]$. This gives the reverse implication in (1). Part (2) follows formally from (1). □

08HG **Lemma 75.13.5.** Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if ϵ^*E is a perfect object of $D(\mathcal{O}_{étale})$. Here ϵ is as in (75.4.0.1).

Proof. The easy implication follows from the general result contained in Cohomology on Sites, Lemma 21.47.5. For the converse, we can use the equivalence of Cohomology on Sites, Lemma 21.47.4 and the corresponding results for pseudo-coherent and complexes of finite tor dimension, namely Lemmas 75.13.2 and 75.13.3. Some details omitted. □

08JL **Lemma 75.13.6.** Let S be a scheme. Let X be an algebraic space over S. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.

Proof. Locally $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ strictly perfect. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

08IK **Lemma 75.13.7.** Let S be a scheme. Let X be a Noetherian algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent

1. $H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and
2. E is m-pseudo-coherent.

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{\text{Coh}}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we can find an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.3). Observe that U is Noetherian. Note that E is m-pseudo-coherent if and only if $E|_U$ is m-pseudo-coherent (follows from the definition or from Cohomology on Sites, Lemma 21.45.2). Similarly, $H^i(E)$ is coherent if and only if $H^i(E)|_U = H^i(E|_U)$ is coherent (see Cohomology of Spaces, Lemma 69.12.2). Thus we may assume that X is representable.

If X is representable by a scheme X_0 then (Lemma 75.4.2) we can write $E = \epsilon^*E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{étale} \to (X_0)_{Zar}$ is as in (75.4.0.1). In this case E is m-pseudo-coherent if and only if E_0 is by Lemma 75.13.2. Similarly, $H^i(E_0)$ is of finite type (i.e., coherent) if and only if $H^i(E)$ is by Lemma 75.13.1. Finally, $H^i(E_0) = 0$ if and only if $H^i(E) = 0$ by Lemma 75.4.1. Thus we reduce to the case of schemes which is Derived Categories of Schemes, Lemma 36.10.3. □

08IL Lemma 75.13.8. Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent

(1) E has tor amplitude in $[a, b]$, and
(2) for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.

Proof. It is clear that (1) implies (2). Assume (2). Let $j : U \to X$ be an étale morphism with U affine. As X is quasi-separated $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 67.11.2). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Since it suffices to prove that $E|_U$ has tor amplitude in $[a, b]$ we reduce to the case where X is representable.

If X is representable by a scheme X_0 then (Lemma 75.4.2) we can write $E = \epsilon^*E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{étale} \to (X_0)_{Zar}$ is as in (75.4.0.1). For every quasi-coherent module $\mathcal{F}_0$ on X_0 the module $\epsilon^*\mathcal{F}_0$ is quasi-coherent on X and

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \epsilon^*\mathcal{F}_0) = \epsilon^*H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$$

as ϵ is flat (Lemma 75.4.1). Moreover, the vanishing of these sheaves for $i \notin [a, b]$ implies the same thing for $H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$ by the same lemma. Thus we've reduced the problem to the case of schemes which is treated in Derived Categories of Schemes, Lemma 36.10.6. □

08JP Lemma 75.13.9. Let X be a scheme. Let E, F be objects of $D(\mathcal{O}_X)$. Assume either

(1) E is pseudo-coherent and F lies in $D^+(\mathcal{O}_X)$, or
(2) E is perfect and F arbitrary,

then there is a canonical isomorphism

$$\epsilon^*R\mathcal{H}om(E, F) \longrightarrow R\mathcal{H}om(\epsilon^*E, \epsilon^*F)$$

Here ϵ is as in (75.4.0.1).

Proof. Recall that ϵ is flat (Lemma 75.4.1) and hence $\epsilon^* = L\epsilon^*$. There is a canonical map from left to right by Cohomology on Sites, Remark 21.35.11. To see this is an isomorphism we can work locally, i.e., we may assume X is an affine scheme.

In case (1) we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules, see Derived Categories of Schemes, Lemma 36.13.3. We may also represent F by a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules. Applying Cohomology, Lemma 20.46.11 we see that $R\mathcal{H}om(E, F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

Applying Cohomology on Sites, Lemma 21.44.10 we see that $R\mathcal{H}om(\epsilon^*E, \epsilon^*F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\mathcal{E}^p, \epsilon^*\mathcal{F}^q)$$

Thus the statement of the lemma boils down to the true fact that the canonical map

$$\epsilon^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\mathcal{E}, \epsilon^*\mathcal{F})$$

is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$ and finite free $\mathcal{O}_X$-module $\mathcal{E}$.

In case (2) we can represent E by a strictly perfect complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$-modules, use Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7 and the fact that a perfect complex of modules is represented by a finite complex of finite projective modules. Thus we can do the exact same proof as above, replacing the reference to Cohomology, Lemma 20.46.11 by a reference to Cohomology, Lemma 20.46.9. □

0A8A Lemma 75.13.10. Let S be a scheme. Let X be an algebraic space over S. Let L, K be objects of $D(\mathcal{O}_X)$. If either

(1) L in $D^+_{QCoh}(\mathcal{O}_X)$ and K is pseudo-coherent,
(2) L in $D_{QCoh}(\mathcal{O}_X)$ and K is perfect,

then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.

Proof. This follows from the analogue for schemes (Derived Categories of Schemes, Lemma 36.10.8) via the criterion of Lemma 75.5.2, the criterion of Lemmas 75.13.2 and 75.13.5, and the result of Lemma 75.13.9. □

0E4Q Lemma 75.13.11. Let S be a scheme. Let X be an algebraic space over S. Let K, L, M be objects of $D_{QCoh}(\mathcal{O}_X)$. The map

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

of Cohomology on Sites, Lemma 21.35.7 is an isomorphism in the following cases

(1) M perfect, or
(2) K is perfect, or
(3) M is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and K has finite tor dimension.

Proof. Checking whether or not the map is an isomorphism can be done étale locally hence we may assume X is an affine scheme. Then we can write K, L, M as $\epsilon^*K_0, \epsilon^*L_0, \epsilon^*M_0$ for some K_0, L_0, M_0 in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.4.2. Then we see that Lemma 75.13.9 reduces cases (1) and (3) to the case of schemes which is Derived Categories of Schemes, Lemma 36.10.9. If K is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because K will be represented (after localizing further) by a finite complex of finite free modules in which case it is clear. □

75.14. Approximation by perfect complexes

08HH In this section we continue the discussion started in Derived Categories of Schemes, Section 36.14.

08HI Definition 75.14.1. Let S be a scheme. Let X be an algebraic space over S. Consider triples (T, E, m) where

(1) $T \subset |X|$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say approximation holds for the triple (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Please read the remarks following Derived Categories of Schemes, Definition 36.14.1 to see why.

08HJ Definition 75.14.2. Let S be a scheme. Let X be an algebraic space over S. We say approximation by perfect complexes holds on X if for any closed subset $T \subset |X|$ such that the morphism $X \setminus T \to X$ is quasi-compact there exists an integer r such that for every triple (T, E, m) as in Definition 75.14.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

08HK Lemma 75.14.3. Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic space over S. Let E be a perfect object of $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ where $T = |X| \setminus |U|$. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.

Proof. Being perfect is local on $X_{étale}$. Thus it suffices to check that Rj_*E is perfect when restricted to U and V. We have $Rj_*E|_V = E$ by Lemma 75.10.7 which is perfect. We have $Rj_*E|_U = 0$ because $E|_{V \setminus j^{-1}(T)} = 0$ (use Lemma 75.3.1). $\square$

08HL Lemma 75.14.4. Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Let T be a closed subset of $|X| \setminus |U|$ and let (T, E, m) be a triple as in Definition 75.14.1. If

(1) approximation holds for $(j^{-1}T, E|_V, m)$, and
(2) the sheaves $H^i(E)$ for $i \geq m$ are supported on T,

then approximation holds for (T, E, m).

Proof. Let $P \to E|_V$ be an approximation of the triple $(j^{-1}T, E|_V, m)$ over V. Then Rj_*P is a perfect object of $D(\mathcal{O}_X)$ by Lemma 75.14.3. On the other hand, $Rj_*P = j_!P$ by Lemma 75.10.7. We see that $j_!P$ is supported on T for example by (75.10.0.2). Hence we obtain an approximation $Rj_*P = j_!P \to j_!(E|_V) \to E$. $\square$

08HM Lemma 75.14.5. Let S be a scheme. Let X be an algebraic space over S which is representable by an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 75.14.1 such that there exists an integer $r \geq 0$ with

(1) E is m-pseudo-coherent,
(2) $H^i(E)$ is supported on T for $i \geq m - r + 1$,
(3) $X \setminus T$ is the union of r affine opens.

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Let X_0 be an affine scheme representing X. Let $T_0 \subset X_0$ by the closed subset corresponding to T. Let $\epsilon : X_{\acute{e}tale} \to X_{0,Zar}$ be the morphism (75.4.0.1). We may write $E = \epsilon^*E_0$ for some object E_0 of $D_{QCoh}(\mathcal{O}_{X_0})$, see Lemma 75.4.2. Then E_0 is m-pseudo-coherent, see Lemma 75.13.2. Comparing stalks of cohomology sheaves (see proof of Lemma 75.4.1) we see that $H^i(E_0)$ is supported on T_0 for $i \geq m-r+1$. By Derived Categories of Schemes, Lemma 36.14.4 there exists an approximation $P_0 \to E_0$ of (T_0, E_0, m). By Lemma 75.13.5 we see that $P = \epsilon^*P_0$ is a perfect object of $D(\mathcal{O}_X)$. Pulling back we obtain an approximation $P = \epsilon^*P_0 \to \epsilon^*E_0 = E$ as desired. □

08HN Lemma 75.14.6. Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Assume U quasi-compact, V affine, and $U \times_X V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation by perfect complexes holds on X.

Proof. Let $T \subset |X|$ be a closed subset with $X \setminus T \to X$ quasi-compact. Let r_U be the integer of Definition 75.14.2 adapted to the pair $(U, T \cap |U|)$. Set $T' = T \setminus |U|$. Endow T' with the induced reduced subspace structure. Since $|T'|$ is contained in $|X| \setminus |U|$ we see that $j^{-1}(T') \to T'$ is an isomorphism. Moreover, $V \setminus j^{-1}(T')$ is quasi-compact as it is the fibre product of $U \times_X V$ with $X \setminus T$ over X and we've assumed $U \times_X V$ quasi-compact and $X \setminus T \to X$ quasi-compact. Let r' be the number of affines needed to cover $V \setminus j^{-1}(T')$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m-r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m-r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 75.14.5 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 75.14.4 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap |U|, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. In the rest of the proof we will use the equivalence of Lemma 75.4.2 (and the compatibilities of Remark 75.6.3) for the representable algebraic spaces V and $U \times_X V$. We will also use the fact that $(m-r)$-pseudo-coherence, resp. perfectness on the Zariski site and étale site agree, see Lemmas 75.13.2 and 75.13.5. Thus we can use the results of Derived Categories of Schemes, Section 36.13 for the open immersion $U \times_X V \subset V$. In this way Derived Categories of Schemes, Lemma 36.13.9 implies there exists a perfect object Q in $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ and an isomorphism $Q|_{U \times_X V} \to (P \oplus P[1])|_{U \times_X V}$. By Derived Categories of Schemes, Lemma 36.13.6 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \times_X V} \longrightarrow (P \oplus P[1])|_{U \times_X V} \longrightarrow P|_{U \times_X V} \longrightarrow E|_{U \times_X V}$$

lifts to $Q \to E|_V$. By Lemma 75.10.8 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguished triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m-r)$-pseudo-coherent (Cohomology on Sites, Lemma 21.45.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m-r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distinguished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R'' \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R[1] \arrow[d] \\ R \arrow[r] & E \arrow[r] & E' \arrow[r] & R[1] \end{tikzcd}$$

using TR3. Then R'' is a perfect complex (Cohomology on Sites, Lemma 21.47.6) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. $\square$

08HP Theorem 75.14.7. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then approximation by perfect complexes holds on X.

Proof. This follows from the induction principle of Lemma 75.9.3 and Lemmas 75.14.6 and 75.14.5. $\square$

75.15. Generating derived categories

09IU This section is the analogue of Derived Categories of Schemes, Section 36.15. However, we first prove the following lemma which is the analogue of Derived Categories of Schemes, Lemma 36.13.10.

09IV Lemma 75.15.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is a quasi-compact morphism. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_W$ be a map where P is a perfect object of $D(\mathcal{O}_W)$ supported on $T \cap W$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_W$ in $D(\mathcal{O}_W)$ compatible α and $\beta|_W$.

Proof. We will use the induction principle of Lemma 75.9.6 to prove this. Thus we immediately reduce to the case where we have an elementary distinguished square $(W \subset X, f : V \to X)$ with V affine and $P \to E|_W$ as in the statement of the lemma. In the rest of the proof we will use Lemma 75.4.2 (and the compatibilities of Remark 75.6.3) for the representable algebraic spaces V and $W \times_X V$. We will also use the fact that perfectness on the Zariski site and étale site agree, see Lemma 75.13.5.

By Derived Categories of Schemes, Lemma 36.13.9 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $f^{-1}T$ and an isomorphism $Q|_{W \times_X V} \to (P \oplus P[1])|_{W \times_X V}$. By Derived Categories of Schemes, Lemma 36.13.6 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $f^{-1}T$) and assume that the map

$$Q|_{W \times_X V} \to (P \oplus P[1])|_{W \times V} \longrightarrow P|_{W \times_X V} \longrightarrow E|_{W \times_X V}$$

lifts to $Q \to E|_V$. By Lemma 75.10.8 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_W$ is isomorphic to $P \oplus P[1] \to E|_W$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. $\square$

09IW Remark 75.15.2. The proof of Lemma 75.15.1 shows that

$$R|_W = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_W$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_W$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_W$ and P above any given degree.

09IX Lemma 75.15.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let W be a quasi-compact open subspace of X. Let P be a perfect object of $D(\mathcal{O}_W)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.

Proof. Special case of Lemma 75.15.1. $\square$

09IY Theorem 75.15.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent

(1) $E = 0$, and
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ for all $n \in \mathbf{Z}$.

Proof. We will prove this using the induction principle of Lemma 75.9.3.

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square with U quasi-compact such that the theorem holds for U and V is an affine scheme. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 75.15.3 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 75.9.1). This is a retrocompact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rf_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Lemma 75.10.7 we have $K' = f_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 36.15.2 (using also Lemma 75.4.2) the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \times_X V \to V)_*E|_{U \times_X V}$. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

0E4R Remark 75.15.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 75.15.4). Then the following are equivalent

(1) for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
(2) $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
(3) Lf^*E is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that Lf^* is the left adjoint to Rf_*. These conditions hold if f is affine (Lemma 75.6.4) or if f is an open immersion, or if f is a composition of such.

The following result is an strengthening of Theorem 75.15.4 proved using exactly the same methods. Let $T \subset |X|$ be a closed subset where X is an algebraic space. Let's denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T.

0AEC Lemma 75.15.6. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.

Proof. We will prove this using the induction principle of Lemma 75.9.3. The property is true for representable quasi-compact and quasi-separated objects of the site $X_{spaces,étale}$ by Derived Categories of Schemes, Lemma 36.15.4.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square such that the lemma holds for U and V is affine. To finish the proof we have to show that the result holds for X. Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T\cap U}(\mathcal{O}_U)$. Using Lemma 75.15.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then $f^{-1}Z$ is a closed subset of V such that $V \setminus f^{-1}Z$ is quasi-compact. As X is quasi-separated, it follows that $f^{-1}Z \cap f^{-1}T = f^{-1}(Z \cap T)$ is a closed subset of V such that $W = V \setminus f^{-1}(Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $f^{-1}(Z \cap T) = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over B. Note that since K is supported on $f^{-1}(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Lemma 75.10.7 we have $K' = R(V \to X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 36.15.2 we have $E|_V = Rj_*E|_W$ where $j : W \to V$ is the inclusion. Picture

$$\begin{tikzcd}
W \arrow[r, "j"'] & V & Z \cap T \arrow[l] \arrow[d] \\
V \setminus f^{-1}Z \arrow[u, "j'"] \arrow[ur, "j''"'] & & Z \arrow[ul]
\end{tikzcd}$$

Since E is supported on T we see that $E|_W$ is supported on $f^{-1}T \cap W = f^{-1}T \cap (V \setminus f^{-1}Z)$ which is closed in W. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

Here the second equality is part (1) of Cohomology, Lemma 20.33.6 which applies because V is a scheme and E has quasi-coherent cohomology sheaves hence pushforward along the quasi-compact open immersion j' agrees with pushforward on the underlying schemes, see Remark 75.6.3. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

75.16. Compact and perfect objects

09M7 This section is the analogue of Derived Categories of Schemes, Section 36.17.

09M8 **Proposition 75.16.1.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.

Proof. If K is a perfect object of $D(\mathcal{O}_X)$ with dual $K^\vee$ (Cohomology on Sites, Lemma 21.48.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums and since $H^0(X, -)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.6.2 we conclude that K is compact in $D_{QCoh}(\mathcal{O}_X)$.

Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine scheme U étale over X, see Cohomology on Sites, Lemma 21.47.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 75.6.2. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a perfect object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine, in particular a quasi-compact and quasi-separated scheme. Via Lemma 75.4.2 and 75.13.5 we reduce to the case of schemes, i.e., to Derived Categories of Schemes, Proposition 36.17.1. $\square$

0GFC **Remark 75.16.2.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let G be a perfect object of $D(\mathcal{O}_X)$ which is a generator for $D_{QCoh}(\mathcal{O}_X)$. By Theorem 75.15.4 there is at least one of these. Combining Lemma 75.5.3 with Proposition 75.16.1 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf}(\mathcal{O}_X)$.

The following result is a strengthening of Proposition 75.16.1. Let $T \subset |X|$ be a closed subset where X is an algebraic space. As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T. Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

0AED Lemma 75.16.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 75.16.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 75.15.6. Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 13.37.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. $\square$

0GFD Remark 75.16.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. Let G be a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$ which is a generator for $D_{QCoh,T}(\mathcal{O}_X)$. By Lemma 75.15.6 there is at least one of these. Combining the fact that $D_{QCoh,T}(\mathcal{O}_X)$ has direct sums with Lemma 75.16.3 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf,T}(\mathcal{O}_X)$.

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

0AEE Lemma 75.16.5. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that the complement $U \subset X$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either

(1) P is perfect and E supported on T, or
(2) P pseudo-coherent, E supported on T, and E bounded below.

Then there exists a perfect complex of $\mathcal{O}_X$-modules I and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 75.13.10 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 75.15.6. Write

$$K = \text{hocolim} K_n$$

as in Derived Categories, Lemma 13.37.3 using the generator E. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ holds in

$D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to α, see Derived Categories, Lemma 13.33.9. By Derived Categories, Lemma 13.37.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \to Q \to K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction I is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over U, and the composition $I \to K$ is zero as α factors through Q. This proves the lemma. $\square$

75.17. Derived categories as module categories

09M9 The section is the analogue of Derived Categories of Schemes, Section 36.18.

09MA Lemma 75.17.1. Let S be a scheme. Let X be an algebraic space over S. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property P. Let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 22.20. Then we have

$$P \otimes_E K^\bullet = \mathrm{hocolim}\ F_iP \otimes_E K^\bullet$$

Each of the F_iP has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. $\square$

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

09MB Lemma 75.17.2. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let K be a perfect object of $D(\mathcal{O}_X)$. Then

1. there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$ for $L \in D_{QCoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, and
2. if L is bounded, then $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L)$ is zero for all but finitely many n.

Proof. Part (2) follows from (1) as $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n])$. We prove (1). Since K is perfect we have

$$\mathrm{Ext}^i_{D(\mathcal{O}_X)}(K, L) = H^i(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\vee$ is the "dual" perfect complex to K, see Cohomology on Sites, Lemma 21.48.4. Note that $P = K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 75.5.6 and 75.13.6 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say $K^\vee$ has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

shows that $H^j(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ is zero if $H^q(L) = 0$ for $q \in [j-b, j-a]$. Let N be the integer $\max(d_p + p)$ of Cohomology of Spaces, Lemma 69.7.3. Then $H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ H^{-N+1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ \ldots,\ H^0(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

are zero. Namely, by the lemma cited and Lemma 75.5.8, we have

$$H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq 1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$ which is zero by Derived Categories, Lemma 13.16.1. It follows that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b-N, -a]$. □

The following is the analogue of Derived Categories of Schemes, Theorem 36.18.3.

09MC Theorem 75.17.3. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, \mathrm{d})$.

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 75.15.4 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 22.35. Since $K^\bullet$ is K-injective we have

09MD (75.17.3.1) $$H^n(E) = \mathrm{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 75.17.2. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 75.16.1. Combined with (75.17.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 22.35.6. It has a right adjoint

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 22.35.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 75.17.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 13.7.2. □

0DK8 Remark 75.17.4 (Variant with support). Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. The analogue of Theorem 75.17.3 holds for $D_{QCoh,T}(\mathcal{O}_X)$. This follows from the exact same argument as in the proof of the theorem, using Lemmas 75.15.6 and 75.16.3 and a variant of Lemma 75.17.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

0DK9 Remark 75.17.5 (Uniqueness of dga). Let X be a quasi-compact and quasi-separated algebraic space over a ring R. By the construction of the proof of Theorem 75.17.3 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R-linearly equivalent to $D(A, \text{d})$ as a triangulated category. One may ask: how unique is (A, d)? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$-module and Ω' is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \text{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \text{d})$$

is given by the functor $- \otimes_A^{\mathbf{L}} \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 22.37.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

75.18. Characterizing pseudo-coherent complexes, I

0DKA This material will be continued in More on Morphisms of Spaces, Section 76.51. We can characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

0DKB Lemma 75.18.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $K \in D(\mathcal{O}_X)$. The following are equivalent

(1) K is pseudo-coherent, and
(2) $K = \text{hocolim} K_n$ where K_n is perfect and $\tau_{\geq -n} K_n \to \tau_{\geq -n} K$ is an isomorphism for all n.

Proof. The implication (2) $\Rightarrow$ (1) is true on any ringed site. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology on Sites, Lemma 21.47.4. Then it follows from the definitions that $\tau_{\geq -n} K_n$ is $(-n+1)$-pseudo-coherent and hence $\tau_{\geq -n} K$ is $(-n+1)$-pseudo-coherent, hence K is $(-n+1)$-pseudo-coherent. This is true for all n, hence K is pseudo-coherent, see Cohomology on Sites, Definition 21.45.1.

Assume (1). We start by choosing an approximation $K_1 \to K$ of $(X, K, -2)$ by a perfect complex K_1, see Definitions 75.14.1 and 75.14.2 and Theorem 75.14.7. Suppose by induction we have

$$K_1 \to K_2 \to \ldots \to K_n \to K$$

with K_i perfect such that such that $\tau_{\geq -i} K_i \to \tau_{\geq -i} K$ is an isomorphism for all $1 \leq i \leq n$. Then we pick $a \leq b$ as in Lemma 75.17.2 for the perfect object K_n.

Choose an approximation $K_{n+1} \to K$ of $(X, K, \min(a-1, -n-1))$. Choose a distinguished triangle

$$K_{n+1} \to K \to C \to K_{n+1}[1]$$

Then we see that $C \in D_{QCoh}(\mathcal{O}_X)$ has $H^i(C) = 0$ for $i \geq a$. Thus by our choice of a, b we see that $\text{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$. Hence the composition $K_n \to K \to C$ is zero. Hence by Derived Categories, Lemma 13.4.2 we can factor $K_n \to K$ through K_{n+1} proving the induction step.

We still have to prove that $K = \text{hocolim} K_n$. This follows by an application of Derived Categories, Lemma 13.33.8 to the functors $H^i(-) : D(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ and our choice of K_n. □

0DKC Lemma 75.18.2. Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $K \in D(\mathcal{O}_X)$ supported on T. The following are equivalent

(1) K is pseudo-coherent, and
(2) $K = \text{hocolim} K_n$ where K_n is perfect, supported on T, and $\tau_{\geq -n} K_n \to \tau_{\geq -n} K$ is an isomorphism for all n.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 75.18.1 except that in the choice of the approximations we use the triples (T, K, m). □

75.19. The coherator revisited

0CR3 In Section 75.11 we constructed and studied the right adjoint RQ_X to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : \text{Mod}(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{QCoh}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote[2] it

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated algebraic spaces have such a right adjoint.

0CR4 Lemma 75.19.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ has a right adjoint.

First proof. We will use the induction principle in Lemma 75.9.3 to prove this. If $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor RQ_X of Section 75.11 is a right adjoint to the functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. In particular, our lemma is true for affine algebraic spaces, see Lemma 75.11.3. Thus we see that it suffices to show: if $(U \subset X, f : V \to X)$ is an elementary distinguished square with U quasi-compact and V affine and the lemma holds for U, V, and $U \times_X V$, then the lemma holds for X.

The adjoint exists if and only if for every object K of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \to E \to K \to E'[1]$$

[2]This is probably nonstandard notation. However, we have already used Q_X for the coherator and RQ_X for its derived extension.

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 13.40.7. Consider the distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U \times_X V,*}E|_{U \times_X V} \to E[1]$$

in $D(\mathcal{O}_X)$ of Lemma 75.10.2. By Derived Categories, Lemma 13.40.5 it suffices to construct the desired distinguished triangles for $Rj_{U,*}E|_U$, $Rj_{V,*}E|_V$, and $Rj_{U \times_X V,*}E|_{U \times_X V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \to X$ be an étale morphism corresponding with U quasi-compact and quasi-separated and the lemma is true for U. Let L be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \to Rj_*L \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \to L \to Q \to L'[1]$$

in $D(\mathcal{O}_U)$ such that L' is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\mathrm{Hom}(N, Q) = 0$ for all N in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 13.40.7 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \to Rj_*L \to Rj_*Q \to Rj_*L'[1]$$

in $D(\mathcal{O}_X)$. Observe that Rj_*L' is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.6.1. On the other hand, if M in $D_{QCoh}(\mathcal{O}_X)$, then

$$\mathrm{Hom}(M, Rj_*Q) = \mathrm{Hom}(Lj^*M, Q) = 0$$

because Lj^*M is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 75.5.5. This finishes the proof. □

Second proof. The adjoint exists by Derived Categories, Proposition 13.38.2. The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 75.5.3). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 75.15.4 and because perfect objects are compact by Proposition 75.16.1. □

0CR5 Lemma 75.19.2. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. If the right adjoints DQ_X and DQ_Y of the inclusion functors $D_{QCoh} \to D$ exist for X and Y, then

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

Proof. The statement makes sense because Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Lemma 75.6.1. The statement is true because Lf^* similarly maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (Lemma 75.5.5) and hence both $Rf_* \circ DQ_X$ and $DQ_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_X)$. □

0CR6 Remark 75.19.3. Let S be a scheme. Let $(U \subset X, f : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Assume X, U, V are quasi-compact and quasi-separated. By Lemma 75.19.1 the functors DQ_X, DQ_U, DQ_V, $DQ_{U \times_X V}$ exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \to Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \to Rj_{U \times_X V,*}DQ_{U \times_X V}(K|_{U \times_X V}) \to$$

for any $K \in D(\mathcal{O}_X)$. This follows by applying the exact functor DQ_X to the distinguished triangle of Lemma 75.10.2 and using Lemma 75.19.2 three times.

0CSS Lemma 75.19.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The functor DQ_X of Lemma 75.19.1 has the following boundedness property: there exists an integer $N = N(X)$ such that, if K in $D(\mathcal{O}_X)$ with $H^i(U, K) = 0$ for U affine étale over X and $i \not\in [a, b]$, then the cohomology sheaves $H^i(DQ_X(K))$ are zero for $i \not\in [a, b + N]$.

Proof. We will prove this using the induction principle of Lemma 75.9.3.

If X is affine, then the lemma is true with $N = 0$ because then $RQ_X = DQ_X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemma 75.11.3.

Let $(U \subset W, f : V \to W)$ be an elementary distinguished square with W quasi-compact and quasi-separated, $U \subset W$ quasi-compact open, V affine such that the lemma holds for U, V, and $U \times_W V$. Say with integers $N(U)$, $N(V)$, and $N(U \times_W V)$. Now suppose K is in $D(\mathcal{O}_X)$ with $H^i(W, K) = 0$ for all affine W étale over X and all $i \not\in [a, b]$. Then $K|_U$, $K|_V$, $K|_{U \times_W V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U \cap V}(K|_{U \times_W V})$ have vanishing cohomology sheaves outside the inverval $[a, b + \max(N(U), N(V), N(U \times_W V))$. Since the functors $Rj_{U,*}$, $Rj_{V,*}$, $Rj_{U \times_W V,*}$ have finite cohomological dimension on D_{QCoh} by Lemma 75.6.1 we see that there exists an N such that $Rj_{U,*}DQ_U(K|_U)$, $Rj_{V,*}DQ_V(K|_V)$, and $Rj_{U \cap V,*}DQ_{U \times_W V}(K|_{U \times_W V})$ have vanishing cohomology sheaves outside the interval $[a, b + N]$. Then finally we conclude by the distinguished triangle of Remark 75.19.3. $\square$

0CST Example 75.19.5. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $(\mathcal{F}_n)$ be an inverse system of quasi-coherent sheaves on X. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{QCoh}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first $R\lim$ is taken in $D(\mathcal{O}_X)$. In fact, let's write $K = R\lim \mathcal{F}_n$ for this. For any affine U étale over X we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves $\mathcal{F}_n$ have no higher cohomology on affines. By the computation of $R\lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R\lim$ in $D_{QCoh}(\mathcal{O}_X)$, which is $DQ_X(K)$ by the above, is in $D^b_{QCoh}(\mathcal{O}_X)$ and has vanishing cohomology sheaves in negative degrees by Lemma 75.19.4.

75.20. Cohomology and base change, IV

08IM This section is the analogue of Derived Categories of Schemes, Section 36.22.

08IN Lemma 75.20.1. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ we have

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Cohomology on Sites, Lemmas 21.18.4 and 21.19.1. To check it is an isomorphism we may work étale locally on Y. Hence we reduce to the case that Y is an affine scheme.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 75.6.2. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence by Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5. Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 15.59.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. $\square$

08IP Definition 75.20.2. Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. We say X and Y are Tor independent over B if and only if for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "\overline{x}"'] \arrow[d, "\overline{y}"'] \arrow[dr, "\overline{b}"'] & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

of geometric points the rings $\mathcal{O}_{X,\overline{x}}$ and $\mathcal{O}_{Y,\overline{y}}$ are Tor independent over $\mathcal{O}_{B,\overline{b}}$ (see More on Algebra, Definition 15.61.1).

The following lemma shows in particular that this definition agrees with our definition in the case of representable algebraic spaces.

08IQ Lemma 75.20.3. Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. The following are equivalent

(1) X and Y are Tor independent over B,
(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & W \arrow[d] & V \arrow[l] \arrow[d] \\ X \arrow[r] & B & Y \arrow[l] \end{tikzcd}$$

with étale vertical arrows U and V are Tor independent over W,
(3) for some commutative diagram as in (2) with (a) $W \to B$ étale surjective, (b) $U \to X \times_B W$ étale surjective, (c) $V \to Y \times_B W$ étale surjective, the spaces U and V are Tor independent over W, and

(4) for some commutative diagram as in (3) with U, V, W schemes, the schemes U and V are Tor independent over W in the sense of Derived Categories of Schemes, Definition 36.22.2.

Proof. For an étale morphism $\varphi : U \to X$ of algebraic spaces and geometric point $\overline{u}$ the map of local rings $\mathcal{O}_{X,\varphi(\overline{u})} \to \mathcal{O}_{U,\overline{u}}$ is an isomorphism. Hence the equivalence of (1) and (2) follows. So does the implication (1) $\Rightarrow$ (3). Assume (3) and pick a diagram of geometric points as in Definition 75.20.2. The assumptions imply that we can first lift $\overline{b}$ to a geometric point $\overline{w}$ of W, then lift the geometric point $(\overline{x}, \overline{b})$ to a geometric point $\overline{u}$ of U, and finally lift the geometric point $(\overline{y}, \overline{b})$ to a geometric point $\overline{v}$ of V. Use Properties of Spaces, Lemma 66.19.4 to find the lifts. Using the remark on local rings above we conclude that the condition of the definition is satisfied for the given diagram.

Having made these initial points, it is clear that (4) comes down to the statement that Definition 75.20.2 agrees with Derived Categories of Schemes, Definition 36.22.2 when X, Y, and B are schemes.

Let $\overline{x}, \overline{b}, \overline{y}$ be as in Definition 75.20.2 lying over the points x, y, b. Recall that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{X,x}^{sh}$ (Properties of Spaces, Lemma 66.22.1) and similarly for the other two. By Algebra, Lemma 10.155.12 we see that $\mathcal{O}_{X,\overline{x}}$ is a strict henselization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}}$. In particular, the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is flat (More on Algebra, Lemma 15.45.1). By More on Algebra, Lemma 15.61.3 we see that

$$\text{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}} (\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}) = \text{Tor}_i^{\mathcal{O}_{B,\overline{b}}}(\mathcal{O}_{X,\overline{x}}, \mathcal{O}_{Y,\overline{y}})$$

Hence it follows that if X and Y are Tor independent over B as schemes, then X and Y are Tor independent as algebraic spaces over B.

For the converse, we may assume X, Y, and B are affine. Observe that the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}$$

is flat by the observations given above. Moreover, the image of the map on spectra includes all primes $\mathfrak{s} \subset \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}$ lying over $\mathfrak{m}_x$ and $\mathfrak{m}_y$. Hence from this and the displayed formula of Tor's above we see that if X and Y are Tor independent over B as algebraic spaces, then

$$\text{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y})_{\mathfrak{s}} = 0$$

for all $i > 0$ and all $\mathfrak{s}$ as above. By More on Algebra, Lemma 15.61.6 applied to the ring maps $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ and $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ this implies that X and Y are Tor independent over B. □

08IR Lemma 75.20.4. Let S be a scheme. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

If X and Y' are Tor independent over Y, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*L(g')^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology on Sites, Remark 21.19.3 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*L(g')^*E$. To check this is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*L(g')^*E = Rf_*(Rg'_*L(g')^*E)$$

is an isomorphism, see Lemma 75.6.4 (and use Lemmas 75.5.5, 75.5.6, and 75.6.1 to see that the objects $Rf'_*L(g')^*E$ and Lg^*Rf_*E have quasi-coherent cohomology sheaves). Note that g' is affine as well (Morphisms of Spaces, Lemma 67.20.5). By Lemma 75.6.5 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g_*\mathcal{O}_{Y'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{Y'}$. Thus by Lemma 75.20.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. This follows from our assumption that X and Y' are Tor independent over Y. Namely, to check it we may work étale locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$ and $Y' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (see Lemma 75.20.3 and More on Algebra, Lemma 15.61.6), i.e., $\mathrm{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. $\square$

The following lemma will be used in the chapter on dualizing complexes.

0E4S Lemma 75.20.5. Let $g : S' \to S$ be a morphism of affine schemes. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of quasi-compact and quasi-separated algebraic spaces. Assume g and f Tor independent. Write $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\,\mathrm{Hom}_X(M, K) \otimes^{\mathbf{L}}_R R' \longrightarrow R\,\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

(1) $M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or
(2) $M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^+_{QCoh}(X)$, and R' has finite tor dimension over R.

Proof. There is a canonical map $R\,\mathrm{Hom}_X(M, K) \to R\,\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology on Sites, Section 21.36. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness of restriction and $- \otimes^{\mathbf{L}}_R R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\,\mathrm{Hom}_X(M, R(g')_*L(g')^*K) = R\,\mathrm{Hom}_X(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

by Lemma 75.6.5. In both cases the complex $R\mathcal{H}om(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.13.10. There is a natural map

$$R\mathcal{H}om(M, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'} \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases Lemma 75.13.11. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf^*g_*\mathcal{O}_X$ the second equality by Lemma 75.20.4. Using Derived Categories of Schemes, Lemma 36.10.4, Lemma 75.13.3, and Cohomology on Sites, Lemma 21.46.5 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing K by $R\mathcal{H}om(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 75.6.5. Hence the result follows from Lemma 75.20.4. □

0E4T Remark 75.20.6. With notation as in Lemma 75.20.5. The diagram

$$\begin{tikzcd}
R\operatorname{Hom}_X(M, Rg'_*L) \otimes^{\mathbf{L}}_R R' \arrow[r] \arrow[d, "\mu"] & R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*Rg'_*L) \arrow[d, "a"] \\
R\operatorname{Hom}_X(M, R(g')_*L) \arrow[r, equal] & R\operatorname{Hom}_{X'}(L(g')^*M, L)
\end{tikzcd}$$

is commutative where the top horizontal arrow is the map from the lemma, μ is the multiplication map, and a comes from the adjunction map $L(g')^*Rg'_*L \to L$. The multiplication map is the adjunction map $K' \otimes^{\mathbf{L}}_R R' \to K'$ for any $K' \in D(R')$.

0DKD Lemma 75.20.7. Let S be a scheme. Consider a cartesian square of algebraic spaces

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

over S. Assume g and f Tor independent.

(1) If $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules, then $L(g')^*E$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_{Y'}$-modules.
(2) If $\mathcal{G}$ is an $\mathcal{O}_X$-module flat over Y, then $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$.

Proof. We can compute tor dimension at stalks, see Cohomology on Sites, Lemma 21.46.10 and Properties of Spaces, Theorem 66.19.12. If $\overline{x}'$ is a geometric point of X' with image $\overline{x}$ in X, then

$$(L(g')^*E)_{\overline{x}'} = E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'}$$

Let $\overline{y}'$ in Y' and $\overline{y}$ in Y be the image of $\overline{x}'$ and $\overline{x}$. Since X and Y' are tor independent over Y, we can apply More on Algebra, Lemma 15.61.2 to see that the right hand side of the displayed formula is equal to $E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}$ in $D(\mathcal{O}_{Y',\overline{y}'})$. Thus (1) follows from More on Algebra, Lemma 15.66.13. To see (2) observe that flatness of $\mathcal{G}$ is equivalent to the condition that $\mathcal{G}[0]$ has tor amplitude in $[0, 0]$. Applying (1) we conclude. □

75.21. Cohomology and base change, V

0DKE This section is the analogue of Derived Categories of Schemes, Section 36.26. In Section 75.20 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 15.61. In this section we analyze when one can get a base change result "one complex at a time".

To make this work, let S be a base scheme and suppose we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of algebraic spaces over S (usually we will assume it is cartesian). Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. For a geometric point $\overline{x}'$ of X' consider the geometric points $\overline{x} = g'(\overline{x}')$, $\overline{y}' = f'(\overline{x}')$, $\overline{y} = f(\overline{x}) = g(\overline{y}')$ of X, Y', Y. Then we can consider the maps

$$K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'} \to K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'} \to K'_{\overline{x}'}$$

where the first arrow is More on Algebra, Equation (15.61.0.1) and the second comes from $(L(g')^*K)_{\overline{x}'} = K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'}$ and the given map $L(g')^*K \to K'$. For each $i \in \mathbf{Z}$ we obtain a $\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}$-module structure on $H^i(K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})$. Putting everything together we obtain canonical maps

0DKF (75.21.0.1)
$$H^i(K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}) \otimes_{(\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})} \mathcal{O}_{X',\overline{x}'} \longrightarrow H^i(K'_{\overline{x}'})$$

of $\mathcal{O}_{X',\overline{x}'}$-modules.

0DKG **Lemma 75.21.1.** Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. The following are equivalent

(1) for any $x' \in X'$ and $i \in \mathbf{Z}$ the map (75.21.0.1) is an isomorphism,
(2) for any commutative diagram

$$\begin{tikzcd} & U \arrow[d] \arrow[dr, "a"] & \\ V' \arrow[r] \arrow[dr, "c"'] & V \arrow[dr, "b"] & X \arrow[d, "f"] \\ & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with a, b, c étale, U, V, V' schemes, and with $U' = V' \times_V U$ the equivalent conditions of Derived Categories of Schemes, Lemma 75.21.1 hold for $(U \to X)^*K$ and $(U' \to X')^*K'$, and

(3) there is some diagram as in (2) with $U' \to X'$ surjective.

Proof. Observe that (1) is étale local on X'. Working through formal implications of what is known, we see that it suffices to prove condition (1) of this lemma is equivalent to condition (1) of Derived Categories of Schemes, Lemma 36.26.1 if X, Y, Y', X' are representable by schemes X_0, Y_0, Y_0', X_0'. Denote f_0, g_0, g_0', f_0' the morphisms between these schemes corresponding to f, g, g', f'. We may assume $K = \epsilon^* K_0$ and $K' = \epsilon^* K_0'$ for some objects $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ and $K_0' \in D_{QCoh}(\mathcal{O}_{X_0'})$, see Lemma 75.4.2. Moreover, the map $Lg^*K \to K'$ is the pullback of a map $L(g_0)^*K_0 \to K_0'$ with notation as in Remark 75.6.3. Recall that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x}$ (Properties of Spaces, Lemma 66.22.1) and that we have

$$K_{\overline{x}} = K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}} \quad \text{and} \quad K'_{\overline{x}'} = K'_{0,x'} \otimes^{\mathbf{L}}_{\mathcal{O}_{X',x'}} \mathcal{O}_{X',\overline{x}'}$$

(akin to Properties of Spaces, Lemma 66.29.4). Consider the commutative diagram

$$\begin{tikzcd}
H^i(K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}) \otimes_{(\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})} \mathcal{O}_{X',\overline{x}'} \arrow[r] & H^i(K'_{\overline{x}'}) \\
H^i(K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'})} \mathcal{O}_{X',x'} \arrow[r] \arrow[u] & H^i(K'_{0,x'}) \arrow[u]
\end{tikzcd}$$

We have to show that the lower horizontal arrow is an isomorphism if and only if the upper horizontal arrow is an isomorphism. Since $\mathcal{O}_{X',x'} \to \mathcal{O}_{X',\overline{x}'}$ is faithfully flat (More on Algebra, Lemma 15.45.1) it suffices to show that the top arrow is the base change of the bottom arrow by this map. This follows immediately from the relationships between stalks given above for the objects on the right. For the objects on the left it suffices to show that

$$\begin{aligned}
&H^i\left((K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}}) \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}\right) \\
&= H^i(K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'})} (\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})
\end{aligned}$$

This follows from More on Algebra, Lemma 15.61.5. The flatness assumptions of this lemma hold by what was said above as well as Algebra, Lemma 10.155.12 implying that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,\overline{y}}$ and that $\mathcal{O}_{Y',\overline{y}'}$ is the strict henselization of $\mathcal{O}_{Y',y'} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,\overline{y}}$. ☐

0DKH Lemma 75.21.2. Let S be a scheme. Let

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If

(1) the equivalent conditions of Lemma 75.21.1 hold, and
(2) f is quasi-compact and quasi-separated,

then the composition $Lg^*Rf_*K \to Rf'_*L(g')^*K \to Rf'_*K'$ is an isomorphism.

Proof. To check the map is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In this case, we will use the induction principle of Lemma 75.9.3 to prove that for a quasi-compact and quasi-separated algebraic space U étale over X the similarly constructed map $Lg^*R(U \to Y)_*K|_U \to R(U' \to Y')_*K'|_{U'}$ is an isomorphism. Here $U' = X' \times_{g',X} U = Y' \times_{g,Y} U$.

If U is a scheme (for example affine), then the result holds. Namely, then Y, Y', U, U' are schemes, K and K' come from objects of the derived category of the underlying schemes by Lemma 75.4.2 and the condition of Derived Categories of Schemes, Lemma 36.26.1 holds for these complexes by Lemma 75.21.1. Thus (by the compatibilities explained in Remark 75.6.3) we can apply the result in the case of schemes which is Derived Categories of Schemes, Lemma 36.26.2.

The induction step. Let $(U \subset W, V \to W)$ be an elementary distinguished square with W a quasi-compact and quasi-separated algebraic space étale over X, with U quasi-compact, V affine and the result holds for U, V, and $U \times_W V$. To easy notation we replace W by X (this is permissible at this point). Denote $a : U \to Y$, $b : V \to Y$, and $c : U \times_X V \to Y$ the obvious morphisms. Let $a' : U' \to Y'$, $b' : V' \to Y'$ and $c' : U' \times_{X'} V' \to Y'$ be the base changes of a, b, and c. Using the distinguished triangles from relative Mayer-Vietoris (Lemma 75.10.3) we obtain a commutative diagram

$$\begin{tikzcd}
Lg^*Rf_*K \arrow[r] \arrow[d] & Rf'_*K' \arrow[d] \\
Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V \arrow[r] \arrow[d] & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \arrow[d] \\
Lg^*Rc_*K|_{U\times_X V} \arrow[r] \arrow[d] & Rc'_*K'|_{U'\times_{X'}V'} \arrow[d] \\
Lg^*Rf_*K[1] \arrow[r] & Rf'_*K'[1]
\end{tikzcd}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 13.4.3) and the proof of the lemma is finished. □

0DKI Lemma 75.21.3. Let S be a scheme. Let

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If the equivalent conditions of Lemma 75.21.1 hold, then

(1) for $E \in D_{QCoh}(\mathcal{O}_X)$ the equivalent conditions of Lemma 75.21.1 hold for $L(g')^*(E \otimes^{\mathbf{L}} K) \to L(g')^*E \otimes^{\mathbf{L}} K'$,
(2) if E in $D(\mathcal{O}_X)$ is perfect the equivalent conditions of Lemma 75.21.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$, and
(3) if K is bounded below and E in $D(\mathcal{O}_X)$ pseudo-coherent the equivalent conditions of Lemma 75.21.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$.

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 75.5.5, 75.5.6, and 75.13.10 and Cohomology on Sites, Lemmas 21.45.3 and 21.47.5. Having said this, we can check the maps (75.21.0.1) are isomorphisms in case (1) by computing the source and target of (75.21.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 15.59.15. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E,K) \to R\mathcal{H}om(L(g')^*E, L(g')^*K) \to R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology on Sites, Remark 21.35.11 and the second arrow comes from the given map $L(g')^*K \to K'$. To prove the maps (75.21.0.1) are isomorphisms one represents E_x by a bounded complex of finite projective $\mathcal{O}_{X.x}$-modules in case (2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. □

0A1K Lemma 75.21.4. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a bounded above complex of quasi-coherent $\mathcal{O}_X$-modules flat over Y. Then formation of

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in other words $X' = Y' \times_Y X$. The lemma asserts that

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \longrightarrow Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do not use derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 75.21.2 and 75.21.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 75.21.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $\mathcal{G}^\bullet_{\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}$ computes the derived tensor product by our assumptions on the complex $\mathcal{G}^\bullet$. □

08JQ Lemma 75.21.5. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. If

(1) E is perfect, $\mathcal{G}^\bullet$ is a bounded above, and $\mathcal{G}^n$ is flat over Y, or
(2) E is pseudo-coherent, $\mathcal{G}^\bullet$ is bounded, and $\mathcal{G}^n$ is flat over Y,

then formation of

$$Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in other words $X' = Y' \times_Y X$. The lemma asserts that

$$Lg^* Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_* R\mathcal{H}om(L(g')^* E, (g')^* \mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do not use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 75.21.2 and 75.21.3 to see that it suffices to prove the canonical map

$$L(g')^* \mathcal{G}^\bullet \to (g')^* \mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 75.21.1. This was shown in the proof of Lemma 75.21.4. □

75.22. Producing perfect complexes

0A1L The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation.

08IS Lemma 75.22.1. Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ such that

(1) $E \in D^b_{\text{Coh}}(\mathcal{O}_X)$,
(2) the support of $H^i(E)$ is proper over Y for all i,
(3) E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$.

Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.

Proof. By Lemma 75.8.1 we see that Rf_*E is an object of $D^b_{\text{Coh}}(\mathcal{O}_Y)$. Hence Rf_*E is pseudo-coherent (Lemma 75.13.7). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology on Sites, Lemma 21.47.4. By Lemma 75.13.8 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on Y. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset |X|$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over Y by assumptions (1) and (2) and Lemma 75.7.6. In particular there exists a quasi-compact open subspace $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. We have assumed f is quasi-separated. Thus

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F} = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_*\left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}\right)$$

by Lemma 75.20.1 and Cohomology on Sites, Lemma 21.18.5. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. □

0DKJ Lemma 75.22.2. Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over B with support proper over B. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_B)$.

Proof. The object K is perfect by Lemma 75.22.1. We check the lemma applies: Locally E is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ is isomorphic to a finite complex whose terms are of the form

$$\bigoplus\nolimits_{i=a,\ldots,b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers $a, b, r_a, \ldots, r_b$. This immediately implies the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are coherent. The hypothesis on the tor dimension also follows as $\mathcal{G}^i$ is flat over $f^{-1}\mathcal{O}_Y$. $\square$

0DKK Lemma 75.22.3. Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over B with support proper over B. Then $K = Rf_* R\mathcal{H}om(E, \mathcal{G})$ is a perfect object of $D(\mathcal{O}_B)$.

Proof. Since E is a perfect complex there exists a dual perfect complex $E^\vee$, see Cohomology on Sites, Lemma 21.48.4. Observe that $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$. Thus the perfectness of K follows from Lemma 75.22.2. $\square$

75.23. A projection formula for Ext

08JM Lemma 75.23.3 (or similar results in the literature) is sometimes useful to verify properties of an obstruction theory needed to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems. Suppose that $f : X \to Y$ is a proper, flat, finitely presented morphism of algebraic spaces and $E \in D(\mathcal{O}_X)$ is perfect. Here the lemma says

$$\mathrm{Ext}^i_X(E, f^*\mathcal{F}) = \mathrm{Ext}^i_Y((Rf_*E^\vee)^\vee, \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on Y. Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 75.20.1.

0A1M Lemma 75.23.1. Assumptions and notation as in Lemma 75.22.2. Then there are functorial isomorphisms

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology on Sites, Lemma 21.18.5, the second as $\mathcal{G}^n$ is a flat $f^{-1}\mathcal{O}_B$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(B, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(B, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \\ &= H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology on Sites, Remark 21.14.4), and the third equality by Lemma 75.20.1. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \arrow[d, "\delta"] \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) \arrow[r] & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}^n$ is flat over B. We omit the verification of the commutativity of the displayed diagram. □

08JN Lemma 75.23.2. Assumption and notation as in Lemma 75.22.3. Then there are functorial isomorphisms

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow \operatorname{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. As in the proof of Lemma 75.22.3 let $E^\vee$ be the dual perfect complex and recall that $K = Rf_*(E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$. Since we also have

$$\operatorname{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of $E^\vee$, the existence of the isomorphisms follows from Lemma 75.23.1 applied to $E^\vee$ and $\mathcal{G}^\bullet$. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & \operatorname{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \arrow[d, "\delta"] \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) \arrow[r] & \operatorname{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}^n$ is flat over B. We omit the verification of the commutativity of the displayed diagram. □

08JR Lemma 75.23.3. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S, $E \in D(\mathcal{O}_X)$, and $\mathcal{F}^\bullet$ a complex of $\mathcal{O}_X$-modules. Assume

(1) B is Noetherian,
(2) f is locally of finite type and quasi-separated,

(3) $E \in D^-_{\textit{Coh}}(\mathcal{O}_X)$,
(4) $\mathcal{G}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-module flat over B with support proper over B.

Then the following two statements are true

(A) for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_B)$ and functorial maps

$$\alpha^i_\mathcal{F} : \text{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof) such that $\alpha^i_\mathcal{F}$ is an isomorphism for $i \leq m$, and

(B) there exists a pseudo-coherent $L \in D(\mathcal{O}_B)$ and functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_B}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps.

Proof. Proof of (A). Suppose $\mathcal{G}^i$ is nonzero only for $i \in [a, b]$. We may replace X by a quasi-compact open neighbourhood of the union of the supports of $\mathcal{G}^i$. Hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m-1+a)$ (possible by Theorem 75.14.7). Then the induced map

$$\text{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\text{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \text{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 13.27.3. This reduces us to the case that E is a perfect complex which is Lemma 75.23.2. The statement on boundaries is explained in the proof of Lemma 75.23.2.

Proof of (B). As in the proof of (A) we may assume X is Noetherian. Observe that E is pseudo-coherent by Lemma 75.13.7. By Lemma 75.18.1 we can write $E = \text{hocolim} E_n$ with E_n perfect and $E_n \to E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $E_n^\vee$ be the dual perfect complex (Cohomology on Sites, Lemma 21.48.4). We obtain an inverse system $\ldots \to E_3^\vee \to E_2^\vee \to E_1^\vee$ of perfect objects. This in turn gives rise to an inverse system

$$\ldots \to K_3 \to K_2 \to K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

perfect on Y, see Lemma 75.22.2. By Lemma 75.23.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) for any quasi-coherent $\mathcal{F}$ on Y we have functorial canonical maps

$$\begin{tikzcd} & \text{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \arrow[ld] \arrow[rd] & \\ H^i(Y, K_{n+1} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}) \arrow[rr] & & H^i(Y, K_n \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}) \end{tikzcd}$$

which are isomorphisms for $i \leq n + a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \to L_2 \to L_3 \to \ldots$ is a system of perfect objects in $D(\mathcal{O}_Y)$ such that for any quasi-coherent $\mathcal{F}$ on Y the maps

$$\mathrm{Ext}^i_{\mathcal{O}_Y}(L_{n+1}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_Y}(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n + a - 1$. This implies that $L_n \to L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \to L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \mathrm{hocolim} L_n$ is pseudo-coherent, see Lemma 75.18.1. The mapping property of homotopy colimits gives that $\mathrm{Ext}^i_{\mathcal{O}_Y}(L, \mathcal{F}) = \mathrm{Ext}^i_{\mathcal{O}_Y}(L_n, \mathcal{F})$ for $i \leq n + a - 3$ which finishes the proof. □

0DKL Remark 75.23.4. The pseudo-coherent complex L of part (B) of Lemma 75.23.3 is canonically associated to the situation. For example, formation of L as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes we have canonical functorial isomorphisms

$$\mathrm{Ext}^i_{\mathcal{O}_{Y'}}(Lg^*L, \mathcal{F}') \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for $\mathcal{F}'$ quasi-coherent on Y'. Obsere that we do not use derived pullback on $\mathcal{G}^\bullet$ on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

75.24. Limits and derived categories

09RG In this section we collect some results about the derived category of an algebraic space which is the limit of an inverse system of algebraic spaces. More precisely, we will work in the following setting.

09RH Situation 75.24.1. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We denote $f_i : X \to X_i$ the projection. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

09RI Lemma 75.24.2. In Situation 75.24.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{X_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{X_i})}(E_i, K_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(E, K)$$

is an isomorphism if either

(1) E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$, or
(2) E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has finite tor dimension.

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the canonical map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0 by the induction principle of Lemma 75.9.3. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Lemma 75.10.4. Thus it suffices to prove the lemma when X_0 is affine.

If X_0 is affine, then the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.29.2. To see this use the equivalence of Lemma 75.4.2 and use the translation of properties explained in Lemmas 75.13.2, 75.13.3, and 75.13.5. □

09RJ Lemma 75.24.3. In Situation 75.24.1 the category of perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{X_i})$.

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\mathrm{colim}_{i\geq 0}\, D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for every U_0 by the induction principle of Lemma 75.9.3. First, we observe that we already know the functor is fully faithful by Lemma 75.24.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have an elementary distinguished square $(U_0 \subset X_0, V_0 \to X_0)$ and that P holds for U_0, V_0, and $U_0 \times_{X_0} V_0$. Let E be a perfect object of $D(\mathcal{O}_X)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (R(X \to X_i)_*E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (R(X \to X_i)_*E)|_{V_i}$$

the maps adjoint to the isomorphisms $L(U \to U_i)^*E_{U,i} \to E|_U$ and $L(V \to V_i)^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i\times_{X_i}V_i} \to E_{V,i}|_{U_i\times_{X_i}V_i}$ which pulls back to the identifications

$$L(U \to U_i)^*E_{U,i}|_{U\times_X V} \to E|_{U\times_X V} \to L(V \to V_i)^*E_{V,i}|_{U\times_X V}.$$

Apply Lemma 75.10.8 to get an object E_i on X_i and a map $d : E_i \to R(X \to X_i)_*E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $L(X \to X_i)^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when X_0 is affine. This follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.29.3. To see this use the equivalence of Lemma 75.4.2 and use the translation of Lemma 75.13.5. □

75.25. Cohomology and base change, VI

0A1N A final section on cohomology and base change continuing the discussion of Sections 75.20, 75.21, and 75.22. An easy to grok special case is given in Remark 75.25.2.

0A1P Lemma 75.25.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over Y, with support proper over Y. Then

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 75.21.4. Thus it suffices to show that K is a perfect object. If Y is Noetherian, then this follows from Lemma 75.22.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on Y, hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 70.7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemma 70.7.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits of Spaces, Lemma 70.6.12. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits of Spaces, Lemma 70.12.3. Finally, by Lemma 75.24.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. By Lemma 75.22.2 we have that $K_i = Rf_{i,*}(E_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i}} \mathcal{G}_i^\bullet)$ is perfect on $\mathrm{Spec}(R_i)$ where $f_i : X_i \to \mathrm{Spec}(R_i)$ is the structure morphism. By the base change result (Lemma 75.21.4) the pullback of K_i to $Y = \mathrm{Spec}(R)$ is K and we conclude. $\square$

0A1Q Remark 75.25.2. Let R be a ring. Let X be an algebraic space of finite presentation over R. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over R with support proper over R. By Lemma 75.25.1 there exists a finite complex of finite projective R-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R-algebra R'.

0CTL Lemma 75.25.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a pseudo-coherent object. Let $\mathcal{G}^\bullet$ be a bounded above complex of finitely presented $\mathcal{O}_X$-modules, flat over Y, with support proper over Y. Then

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a pseudo-coherent object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 75.21.4. Thus it suffices to show that K is a pseudo-coherent object. This will follow from Lemma 75.25.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is étale local on Y, hence we may assume Y is affine. Then X is quasi-compact and quasi-separated. Moreover, there exists an integer N such that total direct image $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ has cohomological dimension N as explained in Lemma 75.6.1. Choose an integer b such that $\mathcal{G}^i = 0$ for $i > b$. It suffices to show that K is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m-N-1-b)$. This is possible by Theorem 75.14.7. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ are zero in degrees $\geq m - N - 1$. Thus the cohomology sheaves of $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G})$ are zero in degrees $\geq m-1$. Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}) \to Rf_*(E \otimes^{\mathbf{L}} \mathcal{G})$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. Next, suppose that $H^i(P) = 0$ for $i > a$. Then $P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet \longrightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is an isomorphism on cohomology sheaves in degrees $\geq m-N-1$. Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet) \to Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. By Lemma 75.25.1 the source is a perfect complex. We conclude that K is m-pseudo-coherent as desired. □

0CTM Lemma 75.25.4. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of finite presentation of algebraic spaces over S.

(1) Let $E \in D(\mathcal{O}_X)$ be perfect and f flat. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.
(2) Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. Special cases of Lemma 75.25.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. □

0CTN Lemma 75.25.5. Let S be a scheme. Let $f : X \to Y$ be a flat proper morphism of finite presentation of algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then Rf_*E is a pseudo-coherent object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

More generally, if $f : X \to Y$ is proper and E on X is pseudo-coherent relative to Y (More on Morphisms of Spaces, Definition 76.45.3), then Rf_*E is pseudo-coherent (but formation does not commute with base change in this generality). The case of this for schemes is proved in [Kie72].

Proof. Special case of Lemma 75.25.3 applied with $\mathcal{G} = \mathcal{O}_X$. □

0D3F Lemma 75.25.6. Let R be a ring. Let X be an algebraic space and let $f : X \to \mathrm{Spec}(R)$ be proper, flat, and of finite presentation. Let (M_n) be an inverse system of R-modules with surjective transition maps. Then the canonical map

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to DQ_X applied to the target.

Proof. The statement means that for any object E of $D_{QCoh}(\mathcal{O}_X)$ the induced map

$$\mathrm{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \mathrm{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Theorem 75.15.4) it suffices to check this for perfect E. By Lemma 75.5.4 we have $\lim \mathcal{O}_X \otimes_R M_n = R\lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$ of Cohomology on Sites, Section 21.36 commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R\lim R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let $E^\vee$ be the dual perfect complex, see Cohomology on Sites, Lemma 21.48.4. We have

$$R\operatorname{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*M_n) = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n$$

by Lemma 75.20.1. From Lemma 75.25.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of R-modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 15.102.3 we obtain

$$R\lim R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R \lim M_n$$

as desired. $\square$

0CWH Lemma 75.25.7. Let A be a ring. Let X be an algebraic space over A which is quasi-compact and quasi-separated. Let $K \in D^-_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect E in $D(\mathcal{O}_X)$, then $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$.

Proof. There exists an integer N such that $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ has cohomological dimension N as explained in Lemma 75.6.1. Let $b \in \mathbf{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let E be pseudo-coherent on X. It suffices to show that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 75.14.7. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^{\mathbf{L}} K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \to R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent as desired. $\square$

0A1R Lemma 75.25.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over Y, with support proper over Y. Then

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 75.21.5. Thus it suffices to show that K is a perfect object. If Y is Noetherian, then this follows from Lemma 75.22.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on Y, hence we may assume Y is affine. Say $Y = \operatorname{Spec}(R)$. We write $R = \operatorname{colim} R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 70.7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemma 70.7.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-module $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits of Spaces, Lemma 70.6.12. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits of Spaces, Lemma 70.12.3. Finally, by Lemma 75.13.5 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 75.23.2 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. $\square$

75.26. Perfect complexes

0D1X We first talk about jumping loci for betti numbers of perfect complexes. First we have to define betti numbers.

Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. Let $x \in |X|$. We want to define $\beta_i(x) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$. To do this, choose a morphism $f : \mathrm{Spec}(k) \to X$ in the equivalence class of x. Then Lf^*E is an object of $D(\mathrm{Spec}(k)_{\acute{e}tale}, \mathcal{O})$. By Étale Cohomology, Lemma 59.59.4 and Theorem 59.17.4 we find that $D(\mathrm{Spec}(k)_{\acute{e}tale}, \mathcal{O}) = D(k)$ is the derived category of k-vector spaces. Hence Lf^*E is a complex of k-vector spaces and we can take $\beta_i(x) = \dim_k H^i(Lf^*E)$. It is easy to see that this does not depend on the choice of the representative in x. Moreover, if X is a scheme, this is the same as the notion used in Derived Categories of Schemes, Section 36.31.

0D1Y Lemma 75.26.1. Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbf{Z}$ consider the function

$$\beta_i : |X| \longrightarrow \{0, 1, 2, \ldots\}$$

defined above. Then we have

(1) formation of β_i commutes with arbitrary base change,
(2) the functions β_i are upper semi-continuous, and
(3) the level sets of β_i are étale locally constructible.

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Then $L\varphi^*E$ is a pseudo-coherent complex on the scheme U (use Lemma 75.13.2) and we can apply the result for schemes, see Derived Categories of Schemes, Lemma 36.31.1. The meaning of part (3) is that the inverse image of the level sets to U are locally constructible, see Properties of Spaces, Definition 66.8.2. $\square$

0E0R Lemma 75.26.2. Let Y be a scheme and let X be an algebraic space over Y such that the structure morphism $f : X \to Y$ is flat, proper, and of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over Y. For fixed $i \in \mathbf{Z}$ consider the function

$$\beta_i : |Y| \to \{0, 1, 2, \ldots\}, \quad y \longmapsto \dim_{\kappa(y)} H^i(X_y, \mathcal{F}_y)$$

Then we have

(1) formation of β_i commutes with arbitrary base change,
(2) the functions β_i are upper semi-continuous, and
(3) the level sets of β_i are locally constructible in Y.

Proof. By cohomology and base change (more precisely by Lemma 75.25.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of Y whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_y, \mathcal{F}_y) = H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \kappa(y))$$

Thus the lemma follows from Lemma 75.26.1. ☐

0D1Z Lemma 75.26.3. Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be perfect. The function

$$\chi_E : |X| \longrightarrow \mathbf{Z}, \quad x \longmapsto \sum (-1)^i \beta_i(x)$$

is locally constant on X.

Proof. Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 36.31.2. ☐

0D20 Lemma 75.26.4. Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbf{Z}$, there exists an open subspace $U \subset X$ characterized by the following

(1) $E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free $\mathcal{O}_U$-module of rank r,
(2) a morphism $f : Y \to X$ factors through U if and only if Lf^*E is isomorphic to a locally free module of rank r placed in degree i.

Proof. Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 36.31.3. ☐

0E6A Lemma 75.26.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is proper, flat, and of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over Y. Fix $i, r \in \mathbf{Z}$. Then there exists an open subspace $V \subset Y$ with the following property: A morphism $T \to Y$ factors through V if and only if $Rf_{T,*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank r placed in degree i.

Proof. By cohomology and base change (Lemma 75.25.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of Y whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 75.26.4. ☐

0D21 Lemma 75.26.6. Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $r \geq 0$. Then there exists a locally closed subspace $j : Z \to X$ characterized by the following

(1) $H^a(Lj^*E)$ is a locally free $\mathcal{O}_Z$-module of rank r, and
(2) a morphism $f : Y \to X$ factors through Z if and only if for all morphisms $g : Y' \to Y$ the $\mathcal{O}_{Y'}$-module $H^a(L(f \circ g)^*E)$ is locally free of rank r.

Moreover, $j : Z \to X$ is of finite presentation and we have

(3) if $f : Y \to X$ factors as $Y \xrightarrow{g} Z \to X$, then $H^a(Lf^*E) = g^*H^a(Lj^*E)$,
(4) if $\beta_a(x) \leq r$ for all $x \in |X|$, then j is a closed immersion and given $f : Y \to X$ the following are equivalent
  (a) $f : Y \to X$ factors through Z,
  (b) $H^0(Lf^*E)$ is a locally free $\mathcal{O}_Y$-module of rank r,
  and if $r = 1$ these are also equivalent to
  (c) $\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E))$ is injective.

Proof. Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 36.31.4. ☐

0E6B **Lemma 75.26.7.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is proper, flat, and of finite presentation, and
(2) for a morphism $\mathrm{Spec}(k) \to Y$ where k is a field, we have $k = H^0(X_k, \mathcal{O}_{X_k})$.

Then we have

(a) $f_*\mathcal{O}_X = \mathcal{O}_Y$ and this holds after any base change,
(b) étale locally on Y we have
$$Rf_*\mathcal{O}_X = \mathcal{O}_Y \oplus P$$
in $D(\mathcal{O}_Y)$ where P is perfect of tor amplitude in $[1, \infty)$.

Proof. It suffices to prove (a) and (b) étale locally on Y, thus we may and do assume Y is an affine scheme. By cohomology and base change (Lemma 75.25.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. In particular, for $y \in Y$ we see that $H^0(E \otimes^{\mathbf{L}} \kappa(y)) = H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$. Thus $\beta_0(y) \leq 1$ for all $y \in Y$ with notation as in Lemma 75.26.1. Apply Lemma 75.26.6 with $a = 0$ and $r = 1$. We obtain a universal closed subscheme $j : Z \to Y$ with $H^0(Lj^*E)$ invertible characterized by the equivalence of (4)(a), (b), and (c) of the lemma. Since formation of E commutes with base change, we have
$$Lf^*E = R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X}$$
The morphism $\mathrm{pr}_1 : X \times_Y X$ has a section namely the diagonal morphism Δ for X over Y. We obtain maps
$$\mathcal{O}_X \longrightarrow R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X} \longrightarrow \mathcal{O}_X$$
in $D(\mathcal{O}_X)$ whose composition is the identity. Thus $R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X} = \mathcal{O}_X \oplus E'$ in $D(\mathcal{O}_X)$. Thus $\mathcal{O}_X$ is a direct summand of $H^0(Lf^*E)$ and we conclude that $X \to Y$ factors through Z by the equivalence of (4)(c) and (4)(a) of the lemma cited above. Since $\{X \to Y\}$ is an fppf covering, we have $Z = Y$. Thus $f_*\mathcal{O}_X$ is an invertible $\mathcal{O}_Y$-module. We conclude $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism because a ring map $A \to B$ such that B is invertible as an A-module is an isomorphism. Since the assumptions are preserved under base change, we see that (a) is true.

Proof of (b). Above we have seen that for every $y \in Y$ the map $\mathcal{O}_Y \to H^0(E \otimes^{\mathbf{L}} \kappa(y))$ is surjective. Thus we may apply More on Algebra, Lemma 15.76.2 to see that in an open neighbourhood of y we have a decomposition $Rf_*\mathcal{O}_X = \mathcal{O}_Y \oplus P$ □

0E0S **Lemma 75.26.8.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is proper, flat, and of finite presentation, and
(2) the geometric fibres of f are reduced and connected.

Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ and this holds after any base change.

Proof. By Lemma 75.26.7 it suffices to show that $k = H^0(X_k, \mathcal{O}_{X_k})$ for all morphisms $\mathrm{Spec}(k) \to Y$ where k is a field. This follows from Spaces over Fields, Lemma 72.14.3 and the fact that X_k is geometrically connected and geometrically reduced. □

75.27. Other applications

0CRT In this section we state and prove some results that can be deduced from the theory worked out above.

0CRU **Lemma 75.27.1.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine schemes étale over X. Then for any quasi-compact and quasi-separated étale morphism $U \to X$ and any perfect object E in $D(\mathcal{O}_X)$ the sets

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology on Sites, Lemma 21.48.4 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^{\mathbf{L}} E)$. We will use the induction principle to prove the lemma, see Lemma 75.9.3.

When $U = \mathrm{Spec}(A)$ is affine the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.33.2.

To finish the proof it suffices to show: if $(U \subset W, V \to W)$ is an elementary distinguished triangle and the result holds for U, V, and $U \times_W V$, then the result holds for W. This is an immediate consquence of the Mayer-Vietoris sequence, see Lemma 75.10.5. □

0CRV **Lemma 75.27.2.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume the sets of sections of $\mathcal{O}_X$ over affines étale over X are countable. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

(1) $K = \mathrm{hocolim} E_n$ with E_n a perfect object of $D(\mathcal{O}_X)$, and
(2) the cohomology sheaves $H^i(K)$ have countable sets of sections over affines étale over X.

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 13.33.8) and because a perfect complex is locally isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules and therefore satisfies (2) by assumption on X.

Assume (2). Choose a K-injective complex $\mathcal{K}^\bullet$ representing K. Choose a perfect generator E of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a K-injective complex $\mathcal{I}^\bullet$. According to Theorem 75.17.3 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \to D(A, \mathrm{d})$ where (A, d) is the differential graded algebra

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps K to the differential graded module

$$M = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \mathrm{Ext}^i(E, E)$ and $H^i(M) = \mathrm{Ext}^i(E, K)$. Moreover, since F is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write M as a homotopy colimit of compact objects of $D(A, \mathrm{d})$. By Differential Graded Algebra, Lemma 22.38.3 it suffices show that $\mathrm{Ext}^i(E, E)$ and $\mathrm{Ext}^i(E, K)$ are countable for each i. This follows from Lemma 75.27.1. □

0CRW Lemma 75.27.3. Let A be a ring. Let $f : U \to X$ be a flat morphism of algebraic spaces of finite presentation over A. Then

(1) there exists an inverse system of perfect objects L_n of $D(\mathcal{O}_X)$ such that
$$R\Gamma(U, Lf^*K) = \text{hocolim } R\operatorname{Hom}_X(L_n, K)$$
in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$, and

(2) there exists a system of perfect objects E_n of $D(\mathcal{O}_X)$ such that
$$R\Gamma(U, Lf^*K) = \text{hocolim } R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$
in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 75.20.1 we have
$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$
functorially in K. Observe that $R\Gamma(X, -)$ commutes with homotopy colimits because it commutes with direct sums by Lemma 75.6.2. Similarly, $- \otimes^{\mathbf{L}} K$ commutes with derived colimits because $- \otimes^{\mathbf{L}} K$ commutes with direct sums (because direct sums in $D(\mathcal{O}_X)$ are given by direct sums of representing complexes). Hence to prove (2) it suffices to write $Rf_*\mathcal{O}_U = \text{hocolim} E_n$ for a system of perfect objects E_n of $D(\mathcal{O}_X)$. Once this is done we obtain (1) by setting $L_n = E_n^\vee$, see Cohomology on Sites, Lemma 21.48.4.

Write $A = \operatorname{colim} A_i$ with A_i of finite type over $\mathbf{Z}$. By Limits of Spaces, Lemma 70.7.1 we can find an i and morphisms $U_i \to X_i \to \operatorname{Spec}(A_i)$ of finite presentation whose base change to $\operatorname{Spec}(A)$ recovers $U \to X \to \operatorname{Spec}(A)$. After increasing i we may assume that $f_i : U_i \to X_i$ is flat, see Limits of Spaces, Lemma 70.6.12. By Lemma 75.20.4 the derived pullback of $Rf_{i,*}\mathcal{O}_{U_i}$ by $g : X \to X_i$ is equal to $Rf_*\mathcal{O}_U$. Since Lg^* commutes with derived colimits, it suffices to prove what we want for f_i. Hence we may assume that U and X are of finite type over $\mathbf{Z}$.

Assume $f : U \to X$ is a morphism of algebraic spaces of finite type over $\mathbf{Z}$. To finish the proof we will show that $Rf_*\mathcal{O}_U$ is a homotopy colimit of perfect complexes. To see this we apply Lemma 75.27.2. Thus it suffices to show that $R^if_*\mathcal{O}_U$ has countable sets of sections over affines étale over X. This follows from Lemma 75.27.1 applied to the structure sheaf. □

75.28. The resolution property

0GUR This section is the analogue of Derived Categories of Schemes, Section 36.36 for algebraic spaces; please read that section first. It is currently not known if a smooth proper algebraic space over a field always has the resolution property or if this is false. If you know the answer to this question, please email stacks.project@gmail.com.

We can make the following definition although it scarcely makes sense to consider it for general algebraic spaces.

0GUS Definition 75.28.1. Let S be a scheme. Let X be an algebraic space over S. We say X has the resolution property if every quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a finite locally free $\mathcal{O}_X$-module.

If X is a quasi-compact and quasi-separated algebraic space, then it suffices to check every $\mathcal{O}_X$-module module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free $\mathcal{O}_X$-module, see Limits of Spaces, Lemma 70.9.3.

If X is a Noetherian algebraic space, then finite type quasi-coherent modules are exactly the coherent $\mathcal{O}_X$-modules, see Cohomology of Spaces, Lemma 69.12.2.

0GUT Lemma 75.28.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) Y is quasi-compact and quasi-separated and has the resolution property,
(2) there exists an f-ample invertible module on X (Divisors on Spaces, Definition 71.14.1).

Then X has the resolution property.

Proof. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an f-ample invertible module. Choose an affine scheme V and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. Then $\mathcal{L}|_U$ is ample on U. By Properties, Proposition 28.26.13 we know there exists finitely many maps $s_i : \mathcal{L}^{\otimes n_i}|_U \to \mathcal{F}|_U$ which are jointly surjective. Consider the quasi-coherent $\mathcal{O}_Y$-modules

$$\mathcal{H}_n = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

We may think of s_i as a section over V of the sheaf $\mathcal{H}_{-n_i}$. Suppose we can find finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}_i$ and maps $\mathcal{E}_i \to \mathcal{H}_{-n_i}$ such that s_i is in the image. Then the corresponding maps

$$f^*\mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n_i} \longrightarrow \mathcal{F}$$

are going to be jointly surjective and the lemma is proved. By Limits of Spaces, Lemma 70.9.2 for each i we can find a finite type quasi-coherent submodule $\mathcal{H}'_i \subset \mathcal{H}_{-n_i}$ which contains the section s_i over V. Thus the resolution property of Y produces surjections $\mathcal{E}_i \to \mathcal{H}'_i$ and we conclude. □

0GUU Lemma 75.28.3. Let S be a scheme. Let $f : X \to Y$ be an affine or quasi-affine morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. If Y has the resolution property, so does X.

Proof. By Divisors on Spaces, Lemma 71.14.7 this is a special case of Lemma 75.28.2. □

Here is a case where one can prove the resolution property goes down.

0GUV Lemma 75.28.4. Let S be a scheme. Let $f : X \to Y$ be a surjective finite locally free morphism of algebraic spaces over S. If X has the resolution property, so does Y.

Proof. The condition means that f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module of positive rank. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. By assumption there exists a surjection $\mathcal{E} \to f^*\mathcal{G}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Since f_* is exact (Cohomology of Spaces, Section 69.4) we get a surjection

$$f_*\mathcal{E} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$$

Taking duals we get a surjection

$$f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{G}$$

Since $f_*\mathcal{E}$ is finite locally free, we conclude. □

For more on the resolution property of algebraic spaces, please see More on Morphisms of Spaces, Section 76.56.

75.29. Detecting Boundedness

0GFE In this section, we show that compact generators of D_{QCoh} of a quasi-compact, quasi-separated scheme, as constructed in Section 75.15, have a special property. We recommend reading that section first as it is very similar to this one.

0GFF **Lemma 75.29.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

1. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \gg 0$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\geq a}E) = 0$ for $i \gg 0$.*

Proof. Using the triangle $\tau_{<a}E \to E \to \tau_{\geq a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{<a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{<a}E)[i]) = 0$$

for $i \gg 0$. As P is perfect this is true by Lemma 75.17.2. $\square$

0GFG **Lemma 75.29.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

1. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \ll 0$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\leq a}E) = 0$ for $i \ll 0$.*

Proof. Using the triangle $\tau_{\leq a}E \to E \to \tau_{>a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{>a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{>a}E)[i]) = 0$$

for $i \ll 0$. As P is perfect this is true by Lemma 75.17.2. $\square$

0GFH **Proposition 75.29.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$E \in D^-_{QCoh}(\mathcal{O}_X)$,*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \gg 0$,*
3. *$\mathrm{Ext}^i_X(G, E) = 0$ for $i \gg 0$,*
4. *$R\,\mathrm{Hom}_X(G, E)$ is in $D^-(\mathbf{Z})$,*
5. *$H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,*
6. *$R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$,*
7. *for every perfect object P of $D(\mathcal{O}_X)$*
   1. *(a) the assertions (2), (3), (4) hold with G replaced by P, and*
   2. *(b) $H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,*
   3. *(c) $R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$.*

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \gg 0$ by Lemma 75.17.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology on Sites, Section 21.36. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology on Sites, Lemma 21.48.4 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 75.16.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\operatorname{Hom}_X(P, E)$ is in $D^-(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is perserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Lemma 75.9.3.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. This follows from the case of schemes, see Derived Categories of Schemes, Proposition 36.40.5.

Now assume $(U \subset X, j : V \to X)$ is an elementary distinguished square of quasi-compact and quasi-separated algebraic spaces over S and assume the implication (2) – (7) $\Rightarrow$ (1) is known for U, V, and $U \times_X V$. To finish the proof we have to show the implication (2) – (7) $\Rightarrow$ (1) for X. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 75.15.3 and Theorem 75.15.4 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$.

Say $V = \operatorname{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 75.9.1). This is a retro-compact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rj_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). By assumption, we know $R\operatorname{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\operatorname{Hom}_{\mathcal{O}_X}(K', E)$ are bounded above.

By Lemma 75.10.7 we have $K' = j_!K$ and hence

$$\operatorname{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = \operatorname{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = 0$$

for $i \gg 0$. Therefore, we may apply Derived Categories of Schemes, Lemma 36.40.1 to $E|_V$ to obtain an integer a such that $\tau_{\geq a}(E|_V) = \tau_{\geq a}R(U \times_X V \to V)_*(E|_{U \times_X V})$. Then $\tau_{\geq a}E = \tau_{\geq a}R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 75.29.1 twice we see that

$$\operatorname{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \operatorname{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \gg 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^-_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^-_{QCoh} (by Lemma 75.6.1), we get $\tau_{\geq a}E \in D^-_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^-_{QCoh}(\mathcal{O}_X)$. $\square$

0GFI Proposition 75.29.4. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

(1) $E \in D^+_{QCoh}(\mathcal{O}_X)$,
(2) $\operatorname{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \ll 0$,
(3) $\operatorname{Ext}^i_X(G, E) = 0$ for $i \ll 0$,

(4) $R\operatorname{Hom}_X(G, E)$ is in $D^+(\mathbf{Z})$,
(5) $H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,
(6) $R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$,
(7) for every perfect object P of $D(\mathcal{O}_X)$
  (a) the assertions (2), (3), (4) hold with G replaced by P, and
  (b) $H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,
  (c) $R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$.

Proof. Assume (1). Since $\operatorname{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \operatorname{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \ll 0$ by Lemma 75.17.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology on Sites, Section 21.36. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology on Sites, Lemma 21.48.4 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 75.16.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\operatorname{Hom}_X(P, E)$ is in $D^+(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is perserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Lemma 75.9.3.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. This follows from the case of schemes, see Derived Categories of Schemes, Proposition 36.40.6.

Now assume $(U \subset X, j : V \to X)$ is an elementary distinguished square of quasi-compact and quasi-separated algebraic spaces over S and assume the implication (2) – (7) $\Rightarrow$ (1) is known for U, V, and $U \times_X V$. To finish the proof we have to show the implication (2) – (7) $\Rightarrow$ (1) for X. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 75.15.3 and Theorem 75.15.4 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$.

Say $V = \operatorname{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 75.9.1). This is a retro-compact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rj_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). By assumption, we know $R\operatorname{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\operatorname{Hom}_{\mathcal{O}_X}(K', E)$ are bounded below.

By Lemma 75.10.7 we have $K' = j_!K$ and hence

$$\operatorname{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = \operatorname{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = 0$$

for $i \ll 0$. Therefore, we may apply Derived Categories of Schemes, Lemma 36.40.2 to $E|_V$ to obtain an integer a such that $\tau_{\leq a}(E|_V) = \tau_{\leq a}R(U \times_X V \to V)_*(E|_{U \times_X V})$.

Then $\tau_{\leq a}E = \tau_{\leq a}R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 75.29.2 twice we see that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \mathrm{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \ll 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^+_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^+_{QCoh} (by Lemma 75.6.1), we get $\tau_{\leq a}E \in D^+_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^+_{QCoh}(\mathcal{O}_X)$. $\square$

75.30. Quasi-coherent objects in the derived category

0H05 Let S be a scheme. Let X be an algebraic space over S. Recall that $X_{affine,étale}$ denotes the category of affine objects of $X_{étale}$ with topology given by standard étale coverings, see Properties of Spaces, Definition 66.18.5. We remind the reader that the topos of $X_{affine,étale}$ is the small étale topos of X, see Properties of Spaces, Lemma 66.18.6. The site $X_{étale}$ comes with a structure sheaf $\mathcal{O}_X$ whose restriction to $X_{affine,étale}$ we also denote $\mathcal{O}_X$. Then there is an equivalence of ringed topoi

$$(Sh(X_{affine,étale}), \mathcal{O}_X) \longrightarrow (Sh(X_{étale}), \mathcal{O}_X)$$

See Descent on Spaces, Equation (74.5.0.1) and the discussion in Descent on Spaces, Section 74.5.

In this section we denote X_{affine} the underlying category of $X_{affine,étale}$ endowed with the chaotic topology, i.e., such that sheaves agree with presheaves. In particular, the structure sheaf $\mathcal{O}_X$ becomes a sheaf on X_{affine} as well. We obtain a morphisms of ringed sites

$$\epsilon : (X_{affine,étale}, \mathcal{O}_X) \longrightarrow (X_{affine}, \mathcal{O}_X)$$

as in Cohomology on Sites, Section 21.27. In this section we will identify $D_{QCoh}(\mathcal{O}_X)$ with the category $QC(X_{affine}, \mathcal{O}_X)$ introduced in Cohomology on Sites, Section 21.43.

0H06 **Lemma 75.30.1.** *In the sitation above there are canonical exact equivalences between the following triangulated categories*

1. *$D_{QCoh}(\mathcal{O}_X)$,*
2. *$D_{QCoh}(X_{affine,étale}, \mathcal{O}_X)$,*
3. *$D_{QCoh}(X_{affine}, \mathcal{O}_X)$, and*
4. *$QC(X_{affine}, \mathcal{O}_X)$.*

Proof. If $U \to V \to X$ are étale morphisms with U and V affine, then the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is flat. Hence the equivalence between (3) and (4) is a special case of Cohomology on Sites, Lemma 21.43.11 (the proof also clarifies the statement).

The discussion preceding the lemma shows that we have an equivalence of ringed topoi $(Sh(X_{affine,étale}), \mathcal{O}_X) \to (Sh(X_{étale}), \mathcal{O}_X)$ and hence an equivalence between abelian categories of modules. Since the notion of quasi-coherent modules is intrinsic (Modules on Sites, Lemma 18.23.2) we see that this equivalence preserves the subcategories of quasi-coherent modules. Thus we get a canonical exact equivalence between the triangulated categories in (1) and (2).

To get an exact equivalence between the triangulated categories in (2) and (3) we will apply Cohomology on Sites, Lemma 21.29.1 to the morphism $\epsilon : (X_{affine,étale}, \mathcal{O}_X) \to$

$(X_{affine}, \mathcal{O}_X)$ above. We take $\mathcal{B} = \text{Ob}(X_{affine})$ and we take $\mathcal{A} \subset \text{PMod}(X_{affine}, \mathcal{O})$ to be the full subcategory of those presheaves $\mathcal{F}$ such that $\mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \to \mathcal{F}(U)$ is an isomorphism. Observe that by Descent on Spaces, Lemma 74.5.1 objects of $\mathcal{A}$ are exactly those sheaves in the étale topology which are quasi-coherent modules on $(X_{affine,étale}, \mathcal{O}_X)$. On the other hand, by Modules on Sites, Lemma 18.24.2, the objects of $\mathcal{A}$ are exactly the quasi-coherent modules on $(X_{affine}, \mathcal{O}_X)$, i.e., in the chaotic topology. Thus if we show that Cohomology on Sites, Lemma 21.29.1 applies, then we do indeed get the canonical equivalence between the categories of (2) and (3) using ϵ^* and $R\epsilon_*$.

We have to verify 4 conditions:

(1) Every object of $\mathcal{A}$ is a sheaf for the étale topology. This we have seen above.
(2) $\mathcal{A}$ is a weak Serre subcategory of $\text{Mod}(X_{affine,étale}, \mathcal{O}_X)$. Above we have seen that $\mathcal{A} = QCoh(X_{affine,étale}, \mathcal{O}_X)$ and we have seen above that these, via the equivalence $\text{Mod}(X_{affine,étale}, \mathcal{O}) = \text{Mod}(X_{étale}, \mathcal{O}_X)$, correspond to the quasi-coherent modules on X. Thus the result by Properties of Spaces, Lemma 66.29.7 and Homology, Lemma 12.10.3.
(3) Every object of X_{affine} has a covering in the chaotic topology whose members are elements of $\mathcal{B}$. This holds because $\mathcal{B}$ contains all objects.
(4) For every object U of X_{affine} and $\mathcal{F}$ in $\mathcal{A}$ we have $H^p_{Zar}(U, \mathcal{F}) = 0$ for $p > 0$. This holds by the vanishing of cohomology of quasi-coherent modules on affines, see discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2.

This finishes the proof. □

0H07 Remark 75.30.2. Let S be a scheme. Let X be an algebraic space over S. We will later show that also $QC((\text{Aff}/X), \mathcal{O})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_X)$. See Sheaves on Stacks, Proposition 96.26.4.

75.31. Other chapters

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CHAPTER 76

More on Morphisms of Spaces

049F 76.1. Introduction

049G In this chapter we continue our study of properties of morphisms of algebraic spaces. A fundamental reference is [Knu71].

76.2. Conventions

049H The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

76.3. Radicial morphisms

0480 It turns out that a radicial morphism is not the same thing as a universally injective morphism, contrary to what happens with morphisms of schemes. In fact it is a bit stronger.

0481 **Definition 76.3.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *radicial* if for any morphism $\text{Spec}(K) \to Y$ where K is a field the reduction $(\text{Spec}(K) \times_Y X)_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K.

0482 **Lemma 76.3.2.** A radicial morphism of algebraic spaces is universally injective.

Proof. Let S be a scheme. Let $f : X \to Y$ be a radicial morphism of algebraic spaces over S. It is clear from the definition that given a morphism $\text{Spec}(K) \to Y$ there is at most one lift of this morphism to a morphism into X. Hence we conclude that f is universally injective by Morphisms of Spaces, Lemma 67.19.2. $\square$

0483 **Example 76.3.3.** It is no longer true that universally injective is equivalent to radicial. For example the morphism

$$X = [\text{Spec}(\overline{\mathbf{Q}})/\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = \text{Spec}(\mathbf{Q})$$

of Spaces, Example 65.14.7 is universally injective, but is not radicial in the sense above.

Nonetheless it is often the case that the reverse implication holds.

0484 **Lemma 76.3.4.** Let S be a scheme. Let $f : X \to Y$ be a universally injective morphism of algebraic spaces over S.

(1) If f is decent then f is radicial.

(2) If f is quasi-separated then f is radicial.
(3) If f is locally separated then f is radicial.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is stable under base change and composition and holds for closed immersions. Assume $f : X \to Y$ has $\mathcal{P}$ and is universally injective. Then, in the situation of Definition 76.3.1 the morphism $(\mathrm{Spec}(K) \times_Y X)_{red} \to \mathrm{Spec}(K)$ is universally injective and has $\mathcal{P}$. This reduces the problem of proving

$$\mathcal{P} + \text{universally injective} \Rightarrow \text{radicial}$$

to the problem of proving that any nonempty reduced algebraic space X over field whose structure morphism $X \to \mathrm{Spec}(K)$ is universally injective and $\mathcal{P}$ is representable by the spectrum of a field. Namely, then $X \to \mathrm{Spec}(K)$ will be a morphism of schemes and we conclude by the equivalence of radicial and universally injective for morphisms of schemes, see Morphisms, Lemma 29.10.2.

Let us prove (1). Assume f is decent and universally injective. By Decent Spaces, Lemmas 68.17.4, 68.17.6, and 68.17.2 (to see that an immersion is decent) we see that the discussion in the first paragraph applies. Let X be a nonempty decent reduced algebraic space universally injective over a field K. In particular we see that $|X|$ is a singleton. By Decent Spaces, Lemma 68.14.2 we conclude that $X \cong \mathrm{Spec}(L)$ for some extension $K \subset L$ as desired.

A quasi-separated morphism is decent, see Decent Spaces, Lemma 68.17.2. Hence (1) implies (2).

Let us prove (3). Recall that the separation axioms are stable under base change and composition and that closed immersions are separated, see Morphisms of Spaces, Lemmas 67.4.4, 67.4.8, and 67.10.7. Thus the discussion in the first paragraph of the proof applies. Let X be a reduced algebraic space universally injective and locally separated over a field K. In particular $|X|$ is a singleton hence X is quasi-compact, see Properties of Spaces, Lemma 66.5.2. We can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 66.6.3. Consider the morphism of schemes

$$j : U \times_X U \longrightarrow U \times_{\mathrm{Spec}(K)} U$$

As $X \to \mathrm{Spec}(K)$ is universally injective j is surjective, and as $X \to \mathrm{Spec}(K)$ is locally separated j is an immersion. A surjective immersion is a closed immersion, see Schemes, Lemma 26.10.4. Hence $R = U \times_X U$ is affine as a closed subscheme of an affine scheme. In particular R is quasi-compact. It follows that $X = U/R$ is quasi-separated, and the result follows from (2). $\square$

049E Remark 76.3.5. Let $X \to Y$ be a morphism of algebraic spaces. For some applications (of radicial morphisms) it is enough to require that for every $\mathrm{Spec}(K) \to Y$ where K is a field

(1) the space $|\mathrm{Spec}(K) \times_Y X|$ is a singleton,
(2) there exists a monomorphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K) \times_Y X$, and
(3) $K \subset L$ is purely inseparable.

If needed later we will may call such a morphism weakly radicial. For example if $X \to Y$ is a surjective weakly radicial morphism then $X(k) \to Y(k)$ is surjective for every algebraically closed field k. Note that the base change $X_{\overline{\mathbf{Q}}} \to \mathrm{Spec}(\overline{\mathbf{Q}})$ of the morphism in Example 76.3.3 is weakly radicial, but not radicial. The analogue of Lemma 76.3.4 is that if $X \to Y$ has property (β) and is universally injective, then it is weakly radicial (proof omitted).

0AGE Lemma 76.3.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is locally of finite type,
(2) for every étale morphism $V \to Y$ the map $|X \times_Y V| \to |V|$ is injective.

Then f is universally injective.

Proof. The question is étale local on Y by Morphisms of Spaces, Lemma 67.19.6. Hence we may assume that Y is a scheme. Then Y is in particular decent and by Decent Spaces, Lemma 68.18.9 we see that f is locally quasi-finite. Let $y \in Y$ be a point and let X_y be the scheme theoretic fibre. Assume X_y is not empty. By Spaces over Fields, Lemma 72.10.8 we see that X_y is a scheme which is locally quasi-finite over $\kappa(y)$. Since $|X_y| \subset |X|$ is the fibre of $|X| \to |Y|$ over y we see that X_y has a unique point x. The same is true for $X_y \times_{\text{Spec}(\kappa(y))} \text{Spec}(k)$ for any finite separable extension $k/\kappa(y)$ because we can realize k as the residue field at a point lying over y in an étale scheme over Y, see More on Morphisms, Lemma 37.35.2. Thus X_y is geometrically connected, see Varieties, Lemma 33.7.11. This implies that the finite extension $\kappa(x)/\kappa(y)$ is purely inseparable.

We conclude (in the case that Y is a scheme) that for every $y \in Y$ either the fibre X_y is empty, or $(X_y)_{red} = \text{Spec}(\kappa(x))$ with $\kappa(y) \subset \kappa(x)$ purely inseparable. Hence f is radicial (some details omitted), whence universally injective by Lemma 76.3.2. □

76.4. Monomorphisms

0B89 This section is the continuation of Morphisms of Spaces, Section 67.10. We would like to know whether or not every monomorphism of algebraic spaces is representable. If you can prove this is true or have a counterexample, please email stacks.project@gmail.com. For the moment this is known in the following cases

(1) for monomorphisms which are locally of finite type (more generally any separated, locally quasi-finite morphism is representable by Morphisms of Spaces, Lemma 67.51.1 and a monomorphism which is locally of finite type is locally quasi-finite by Morphisms of Spaces, Lemma 67.27.10),
(2) if the target is a disjoint union of spectra of zero dimensional local rings (Decent Spaces, Lemma 68.19.1), and
(3) for flat monomorphisms (see below).

0B8A Lemma 76.4.1 (David Rydh). A flat monomorphism of algebraic spaces is representable by schemes.

Proof. Let $f : X \to Y$ be a flat monomorphism of algebraic spaces. To prove f is representable, we have to show $X \times_Y V$ is a scheme for every scheme V mapping to Y. Since being a scheme is local (Properties of Spaces, Lemma 66.13.1), we may assume V is affine. Thus we may assume $Y = \text{Spec}(B)$ is an affine scheme. Next, we can assume that X is quasi-compact by replacing X by a quasi-compact open. The space X is separated as $X \to X \times_{\text{Spec}(B)} X$ is an isomorphism. Applying Limits of Spaces, Lemma 70.17.3 we reduce to the case where B is local, $X \to \text{Spec}(B)$ is a flat monomorphism, and there exists a point $x \in X$ mapping to the closed point of $\text{Spec}(B)$. Then $X \to \text{Spec}(B)$ is surjective as generalizations lift along flat morphisms of separated algebraic spaces, see Decent Spaces, Lemma 68.7.4. Hence we see that $\{X \to \text{Spec}(B)\}$ is an fpqc cover. Then $X \to \text{Spec}(B)$ is a morphism which becomes an isomorphism after base change by $X \to \text{Spec}(B)$. Hence it is an isomorphism by fpqc descent, see Descent on Spaces, Lemma 74.11.15. $\square$

The following is (in some sense) a variant of the lemma above.

0B8B Lemma 76.4.2. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact monomorphism of algebraic spaces such that for every $T \to Y$ the map

$$\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X \times_Y T}$$

is injective. Then f is an isomorphism (and hence representable by schemes).

Proof. The question is étale local on Y, hence we may assume $Y = \text{Spec}(A)$ is affine. Then X is quasi-compact and we may choose an affine scheme $U = \text{Spec}(B)$ and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.3). Note that $U \times_X U = \text{Spec}(B \otimes_A B)$. Hence the category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category $DD_{B/A}$ of descent data on modules for $A \to B$. See Properties of Spaces, Proposition 66.32.1, Descent, Definition 35.3.1, and Descent, Subsection 35.4.14. On the other hand,

$$A \to B$$

is a universally injective ring map. Namely, given an A-module M we see that $A \oplus M \to B \otimes_A (A \oplus M)$ is injective by the assumption of the lemma. Hence $DD_{B/A}$ is equivalent to the category of A-modules by Descent, Theorem 35.4.22. Thus pullback along $f : X \to \text{Spec}(A)$ determines an equivalence of categories of quasi-coherent modules. In particular f^* is exact on quasi-coherent modules and we see that f is flat (small detail omitted). Moreover, it is clear that f is surjective (for example because $\text{Spec}(B) \to \text{Spec}(A)$ is surjective). Hence we see that $\{X \to \text{Spec}(A)\}$ is an fpqc cover. Then $X \to \text{Spec}(A)$ is a morphism which becomes an isomorphism after base change by $X \to \text{Spec}(A)$. Hence it is an isomorphism by fpqc descent, see Descent on Spaces, Lemma 74.11.15. $\square$

0B8C Lemma 76.4.3. A quasi-compact flat surjective monomorphism of algebraic spaces is an isomorphism.

Proof. Such a morphism satisfies the assumptions of Lemma 76.4.2. $\square$

76.5. Conormal sheaf of an immersion

04CM Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 67.13.1. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Morphisms of Spaces, Lemma 67.14.1. This quasi-coherent $\mathcal{O}_Z$-module is the conormal sheaf of Z in X and is often denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Morphisms of Spaces, Section 67.14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of

Spaces, Remark 67.12.4. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

04CN Definition 76.5.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf* $\mathcal{C}_{Z/X}$ of Z in X or the *conormal sheaf of* i is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [DG67, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

04CO Lemma 76.5.2. Let S be a scheme. Let $i : Z \to X$ be an immersion. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then

$$\mathcal{C}_{Z/X}|_{Z_U} = \mathcal{C}_{Z_U/U}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma just states that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 67.13.1. $\square$

04CP Lemma 76.5.3. Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 67.13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 67.13.1). Thus we obtain a commutative

diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^2 = g^{-1}((\mathcal{I}')^2)$. Hence the diagram induces a map $g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{C}_{Z/X}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives a map $f^{-1}\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$, which induces the desired map. □

04G2 Lemma 76.5.4. Let S be a scheme. The conormal sheaf of Definition 76.5.1, and its functoriality of Lemma 76.5.3 satisfy the following properties:

(1) If $Z \to X$ is an immersion of schemes over S, then the conormal sheaf agrees with the one from Morphisms, Definition 29.31.1.
(2) If in Lemma 76.5.3 all the spaces are schemes, then the map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ is the same as the one constructed in Morphisms, Lemma 29.31.3.
(3) Given a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "g'"] \\ Z'' \arrow[r, "i''"] & X'' \end{tikzcd}$$

then the map $(f' \circ f)^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z/X}$ is the same as the composition of $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ with the pullback by f of $(f')^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z'/X'}$

Proof. Omitted. Note that Part (1) is a special case of Lemma 76.5.2. □

04CQ Lemma 76.5.5. Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S. Assume i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 76.5.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ U' \arrow[r] & X' \end{tikzcd}$$

where U, U' are schemes and the horizontal arrows are surjective and étale, see Spaces, Lemma 65.11.6. Then using Lemmas 76.5.2 and 76.5.4 we see that the question reduces to the case of a morphism of schemes. In the schemes case this is Morphisms, Lemma 29.31.4. ☐

06BD Lemma 76.5.6. Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces. Then there is a canonical exact sequence

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 76.5.3 and $i : Z \to Y$ is the first morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Via Lemmas 76.5.2 and 76.5.4 the exactness of the sequence translates immediately into the exactness of the corresponding sequence for the immersions of schemes $Z \times_X U \to Y \times_X U \to U$. Hence the lemma follows from Morphisms, Lemma 29.31.5. ☐

76.6. The normal cone of an immersion

09RM Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 67.13.1. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms of Spaces, Lemma 67.14.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the conormal algebra of Z in X and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms of Spaces, Section 67.14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal algebra of i as the conormal algebra of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 67.12.4. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

09RN Definition 76.6.1. Let $i : Z \to X$ be an immersion. The conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X or the conormal algebra of i is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

09RP (76.6.1.1)
$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

09RQ (76.6.1.2)
$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

09RR Lemma 76.6.2. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then

$$\mathcal{C}_{Z/X,*}|_{Z_U} = \mathcal{C}_{Z_U/U,*}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma follows from the fact that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 67.13.1. □

09RS **Lemma 76.6.3.** Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 67.13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 67.13.1). Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^n = g^{-1}((\mathcal{I}')^n)$ for all $n \geq 1$. Hence the diagram induces a map $g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{C}_{Z/X,n}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives maps $f^{-1}\mathcal{C}_{Z'/X',n} \to \mathcal{C}_{Z/X,n}$, which induce the desired map. □

09RT **Lemma 76.6.4.** Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a cartesian square of algebraic spaces over S with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 76.6.3 is surjective. If g is flat, then it is an isomorphism.

Proof. We may check the statement after étale localizing X'. In this case we may assume $X' \to X$ is a morphism of schemes, hence Z and Z' are schemes and the result follows from the case of schemes, see Divisors, Lemma 31.19.4. □

We use the same conventions for cones and vector bundles over algebraic spaces as we do for schemes (where we use the conventions of EGA), see Constructions, Sections 27.7 and 27.6. In particular, a vector bundle is a very general gadget (and not locally isomorphic to an affine space bundle).

09RU Definition 76.6.5. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. The normal cone $C_Z X$ of Z in X is

$$C_Z X = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Morphisms of Spaces, Definition 67.20.8. The normal bundle of Z in X is the vector bundle

$$N_Z X = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z. Moreover, the canonical surjection (76.6.1.2) of graded algebras defines a canonical closed immersion

09RV (76.6.5.1)
$$C_Z X \longrightarrow N_Z X$$

of cones over Z.

76.7. Sheaf of differentials of a morphism

04CR We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131), the corresponding section in the chapter on morphism of schemes (Morphisms, Section 29.32) as well as Modules on Sites, Section 18.33. We first show that the notion of sheaf of differentials for a morphism of schemes agrees with the corresponding morphism of small étale (ringed) sites.

To clearly state the following lemma we temporarily go back to denoting $\mathcal{F}^a$ the sheaf of $\mathcal{O}_{X_{étale}}$-modules associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on the scheme X, see Descent, Definition 35.8.2.

04CS Lemma 76.7.1. Let $f : X \to Y$ be a morphism of schemes. Let $f_{small} : X_{étale} \to Y_{étale}$ be the associated morphism of small étale sites, see Descent, Remark 35.8.4. Then there is a canonical isomorphism

$$(\Omega_{X/Y})^a = \Omega_{X_{étale}/Y_{étale}}$$

compatible with universal derivations. Here the first module is the sheaf on $X_{étale}$ associated to the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, see Morphisms, Definition 29.32.1, and the second module is the one from Modules on Sites, Definition 18.33.3.

Proof. Let $h : U \to X$ be an étale morphism. In this case the natural map $h^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an isomorphism, see More on Morphisms, Lemma 37.9.9. This means that there is a natural $\mathcal{O}_{Y_{étale}}$-derivation

$$\mathrm{d}^a : \mathcal{O}_{X_{étale}} \longrightarrow (\Omega_{X/Y})^a$$

since we have just seen that the value of $(\Omega_{X/Y})^a$ on any object U of $X_{étale}$ is canonically identified with $\Gamma(U, \Omega_{U/Y})$. By the universal property of $\mathrm{d}_{X/Y} : \mathcal{O}_{X_{étale}} \to \Omega_{X_{étale}/Y_{étale}}$ there is a unique $\mathcal{O}_{X_{étale}}$-linear map $c : \Omega_{X_{étale}/Y_{étale}} \to (\Omega_{X/Y})^a$ such that $\mathrm{d}^a = c \circ \mathrm{d}_{X/Y}$.

Conversely, suppose that $\mathcal{F}$ is an $\mathcal{O}_{X_{étale}}$-module and $D : \mathcal{O}_{X_{étale}} \to \mathcal{F}$ is a $\mathcal{O}_{Y_{étale}}$-derivation. Then we can simply restrict D to the small Zariski site X_{Zar} of X.

Since sheaves on X_{Zar} agree with sheaves on X, see Descent, Remark 35.8.3, we see that $D|_{X_{Zar}} : \mathcal{O}_X \to \mathcal{F}|_{X_{Zar}}$ is just a "usual" Y-derivation. Hence we obtain a map $\psi : \Omega_{X/Y} \longrightarrow \mathcal{F}|_{X_{Zar}}$ such that $D|_{X_{Zar}} = \psi \circ \text{d}$. In particular, if we apply this with $\mathcal{F} = \Omega_{X_{étale}/Y_{étale}}$ we obtain a map

$$c' : \Omega_{X/Y} \longrightarrow \Omega_{X_{étale}/Y_{étale}}|_{X_{Zar}}$$

Consider the morphism of ringed sites $\text{id}_{small,étale,Zar} : X_{étale} \to X_{Zar}$ discussed in Descent, Remark 35.8.4 and Lemma 35.8.5. Since the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{X_{Zar}}$ is equal to $\text{id}_{small,étale,Zar,*}$, since $\text{id}^*_{small,étale,Zar}$ is left adjoint to $\text{id}_{small,étale,Zar,*}$ and since $(\Omega_{X/Y})^a = \text{id}^*_{small,étale,Zar}\Omega_{X/Y}$ we see that c' is adjoint to a map

$$c'' : (\Omega_{X/Y})^a \longrightarrow \Omega_{X_{étale}/Y_{étale}}.$$

We claim that c'' and c' are mutually inverse. This claim finishes the proof of the lemma. To see this it is enough to show that $c''(\text{d}(f)) = \text{d}_{X/Y}(f)$ and $c(\text{d}_{X/Y}(f)) = \text{d}(f)$ if f is a local section of $\mathcal{O}_X$ over an open of X. We omit the verification. □

This clears the way for the following definition. For an alternative, see Remark 76.7.5.

04CT Definition 76.7.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The sheaf of differentials $\Omega_{X/Y}$ of X over Y is sheaf of differentials (Modules on Sites, Definition 18.33.10) for the morphism of ringed topoi

$$(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$$

of Properties of Spaces, Lemma 66.21.3. The universal Y-derivation will be denoted $\text{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$.

By Lemma 76.7.1 this does not conflict with the already existing notion in case X and Y are representable. From now on, if X and Y are representable, we no longer distinguish between the sheaf of differentials defined above and the one defined in Morphisms, Definition 29.32.1. We want to relate this to the usual modules of differentials for morphisms of schemes. Here is the key lemma.

04CU Lemma 76.7.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider any commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where the vertical arrows are étale morphisms of algebraic spaces. Then

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{U/V}$$

In particular, if U, V are schemes, then this is equal to the usual sheaf of differentials of the morphism of schemes $U \to V$.

Proof. By Properties of Spaces, Lemma 66.18.11 and Equation (66.18.11.1) we may think of the restriction of a sheaf on $X_{étale}$ to $U_{étale}$ as the pullback by a_{small}. Similarly for b. By Modules on Sites, Lemma 18.33.6 we have

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{\mathcal{O}_{U_{étale}}/a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}}$$

Since $a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}} = \psi_{small}^{-1}b_{small}^{-1}\mathcal{O}_{Y_{étale}} = \psi_{small}^{-1}\mathcal{O}_{V_{étale}}$ we see that the lemma holds. □

04CV **Lemma 76.7.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is a quasi-coherent $\mathcal{O}_X$-module.

Proof. Choose a diagram as in Lemma 76.7.3 with a and b surjective and U and V schemes. Then we see that $\Omega_{X/Y}|_U = \Omega_{U/V}$ which is quasi-coherent (for example by Morphisms, Lemma 29.32.7). Hence we conclude that $\Omega_{X/Y}$ is quasi-coherent by Properties of Spaces, Lemma 66.29.6. $\square$

04CW **Remark 76.7.5.** Now that we know that $\Omega_{X/Y}$ is quasi-coherent we can attempt to construct it in another manner. For example we can use the result of Properties of Spaces, Section 66.32 to construct the sheaf of differentials by glueing. For example if Y is a scheme and if $U \to X$ is a surjective étale morphism from a scheme towards X, then we see that $\Omega_{U/Y}$ is a quasi-coherent $\mathcal{O}_U$-module, and since $s, t : R \to U$ are étale we get an isomorphism

$$\alpha : s^*\Omega_{U/Y} \to \Omega_{R/Y} \to t^*\Omega_{U/Y}$$

by using Morphisms, Lemma 29.34.16. You check that this satisfies the cocycle condition and you're done. If Y is not a scheme, then you define $\Omega_{U/Y}$ as the cokernel of the map $(U \to Y)^*\Omega_{Y/S} \to \Omega_{U/S}$, and proceed as before. This two step process is a little bit ugly. Another possibility is to glue the sheaves $\Omega_{U/V}$ for any diagram as in Lemma 76.7.3 but this is not very elegant either. Both approaches will work however, and will give a slightly more elementary construction of the sheaf of differentials.

04CX **Lemma 76.7.6.** Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*\text{d}_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*\text{d}_{X/Y}(t)$ mapsto $\text{d}_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is a special case of Modules on Sites, Lemma 18.33.11. $\square$

05Z7 **Lemma 76.7.7.** Let S be a scheme. Let

$$\begin{tikzcd} X'' \arrow[r, "g"'] \arrow[d] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Then we have

$$c_{f \circ g} = c_g \circ g^*c_f$$

as maps $(f \circ g)^*\Omega_{X/Y} \to \Omega_{X''/Y''}$.

Proof. Omitted. Hint: Use the characterization of $c_f, c_g, c_{f\circ g}$ in terms of the effect these maps have on local sections. $\square$

05Z8 Lemma 76.7.8. Let S be a scheme. Let $f : X \to Y$, $g : Y \to B$ be morphisms of algebraic spaces over S. Then there is a canonical exact sequence

$$f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 76.7.6.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.9, of this result via étale localization, see Lemma 76.7.3. $\square$

05Z9 Lemma 76.7.9. Let S be a scheme. If $X \to Y$ is an immersion of algebraic spaces over S then $\Omega_{X/S}$ is zero.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.14, of this result via étale localization, see Lemma 76.7.3. $\square$

05ZA Lemma 76.7.10. Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B. There is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/B}$ and the second arrow comes from Lemma 76.7.6.

Proof. This is the algebraic spaces version of Morphisms, Lemma 29.32.15 and will be a consequence of that lemma by étale localization, see Lemmas 76.7.3 and 76.5.2. However, we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/B}$" here. Namely, we may assume that i is a closed immersion after replacing X by an open subspace. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Morphisms of Spaces, Lemma 67.14.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. $\square$

05ZB Lemma 76.7.11. Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B, and assume i (étale locally) has a left inverse. Then the canonical sequence

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 76.7.10 is (étale locally) split exact.

Proof. Clarification: we claim that if $g : X \to Z$ is a left inverse of i over B, then i^*c_g is a right inverse of the map $i^*\Omega_{X/B} \to \Omega_{Z/B}$. Having said this, the result follows from the corresponding result for morphisms of schemes by étale localization, see Lemmas 76.7.3 and 76.5.2. $\square$

05ZC Lemma 76.7.12. Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $X' = X_{Y'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map

$$(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

of Lemma 76.7.6 is an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.10 and étale localization, see Lemma 76.7.3. ☐

05ZD Lemma 76.7.13. Let S be a scheme. Let $f : X \to B$ and $g : Y \to B$ be morphisms of algebraic spaces over S with the same target. Let $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$ be the projection morphisms. The maps from Lemma 76.7.6

$$p^*\Omega_{X/B} \oplus q^*\Omega_{Y/B} \longrightarrow \Omega_{X \times_B Y/B}$$

give an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.11 and étale localization, see Lemma 76.7.3. ☐

05ZE Lemma 76.7.14. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is a finite type $\mathcal{O}_X$-module.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.12 and étale localization, see Lemma 76.7.3. ☐

05ZF Lemma 76.7.15. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite presentation, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.13 and étale localization, see Lemma 76.7.3. ☐

0CK5 Lemma 76.7.16. Let S be a scheme. Let $f : X \to Y$ be a smooth morphism of algebraic spaces over S. Then the module of differentials $\Omega_{X/Y}$ is finite locally free.

Proof. The statement is étale local on X and Y by Lemma 76.7.3. Hence this follows from the case of schemes, see Morphisms, Lemma 29.34.12. ☐

76.8. Topological invariance of the étale site

05ZG We show that the site $X_{spaces,étale}$ is a "topological invariant". It then follows that $X_{étale}$, which consists of the representable objects in $X_{spaces,étale}$, is a topological invariant too, see Lemma 76.8.2.

05ZH Theorem 76.8.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral, universally injective and surjective. The functor

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$.

Proof. The morphism f is representable and a universal homeomorphism, see Morphisms of Spaces, Section 67.53.

We first prove that the functor is faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $a, b : V' \to V$ are distinct morphisms over Y. Since V', V are étale over Y the equalizer

$$E = V' \times_{(a,b), V \times_Y V, \Delta_{V/Y}} V$$

of a, b is étale over Y also. Hence $E \to V'$ is an étale monomorphism (i.e., an open immersion) which is an isomorphism if and only if it is surjective. Since $X \to Y$ is a universal homeomorphism we see that this is the case if and only if $E_X = V'_X$, i.e., if and only if $a_X = b_X$.

Next, we prove that the functor is fully faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $c : V'_X \to V_X$ is a morphism over X. We want to construct a morphism $a : V' \to V$ over Y such that $a_X = c$. Let $a' : V'' \to V'$ be a surjective étale morphism such that V'' is a separated algebraic space. If we can construct a morphism $a'' : V'' \to V$ such that $a''_X = c \circ a'_X$, then the two compositions

$$V'' \times_{V'} V'' \xrightarrow{\text{pr}_i} V'' \xrightarrow{a''} V$$

will be equal by the faithfulness of the functor proved in the first paragraph. Hence a'' will factor through a unique morphism $a : V' \to V$ as V' is (as a sheaf) the quotient of V'' by the equivalence relation $V'' \times_{V'} V''$. Hence we may assume that V' is separated. In this case the graph

$$\Gamma_c \subset (V' \times_Y V)_X$$

is open and closed (details omitted). Since $X \to Y$ is a universal homeomorphism, there exists an open and closed subspace $\Gamma \subset V' \times_Y V$ such that $\Gamma_X = \Gamma_c$. The projection $\Gamma \to V'$ is an étale morphism whose base change to X is an isomorphism. Hence $\Gamma \to V'$ is étale, universally injective, and surjective, so an isomorphism by Morphisms of Spaces, Lemma 67.51.2. Thus Γ is the graph of a morphism $a : V' \to V$ as desired.

Finally, we prove that the functor is essentially surjective. Suppose that U is an object of $X_{spaces,étale}$. We have to find an object V of $Y_{spaces,étale}$ such that $V_X \cong U$. Let $U' \to U$ be a surjective étale morphism such that $U' \cong V'_X$ and $U' \times_U U' \cong V''_X$ for some objects V'', V' of $Y_{spaces,étale}$. Then by fully faithfulness of the functor we obtain morphisms $s, t : V'' \to V'$ with $t_X = \text{pr}_0$ and $s_X = \text{pr}_1$ as morphisms $U' \times_U U' \to U'$. Using that $(\text{pr}_0, \text{pr}_1) : U' \times_U U' \to U' \times_S U'$ is an étale equivalence relation, and that $U' \to V'$ and $U' \times_U U' \to V''$ are universally injective and surjective we deduce that $(t, s) : V'' \to V' \times_S V'$ is an étale equivalence relation. Then the quotient $V = V'/V''$ (see Spaces, Theorem 65.10.5) is an algebraic space V over Y. There is a morphism $V' \to V$ such that $V'' = V' \times_V V'$. Thus we obtain a morphism $V \to Y$ (see Descent on Spaces, Lemma 74.7.2). On base change to X we see that we have a morphism $U' \to V_X$ and a compatible isomorphism $U' \times_{V_X} U' = U' \times_U U'$, which implies that $V_X \cong U$ (by the lemma just cited once more).

Pick a scheme W and a surjective étale morphism $W \to Y$. Pick a scheme U' and a surjective étale morphism $U' \to U \times_X W_X$. Note that U' and $U' \times_U U'$ are schemes étale over X whose structure morphism to X factors through the scheme W_X. Hence by Étale Cohomology, Theorem 59.45.2 there exist schemes V', V'' étale over W whose base change to W_X is isomorphic to respectively U' and $U' \times_U U'$. This finishes the proof. □

07VW Lemma 76.8.2. With assumption and notation as in Theorem 76.8.1 the equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$ restricts to equivalences of categories $Y_{étale} \to X_{étale}$ and $Y_{affine,étale} \to X_{affine,étale}$.

Proof. This is just the statement that given an object $V \in Y_{spaces,étale}$ we have V is a(n affine) scheme if and only if $V \times_Y X$ is a(n affine) scheme. Since $V \times_Y X \to V$

is integral, universally injective, and surjective (as a base change of $X \to Y$) this follows from Limits of Spaces, Lemma 70.15.4 and Proposition 70.15.2. ☐

05ZI Remark 76.8.3. A universal homeomorphism of algebraic spaces need not be representable, see Morphisms of Spaces, Example 67.53.3. In fact Theorem 76.8.1 does not hold for universal homeomorphisms. To see this, let k be an algebraically closed field of characteristic 0 and let Email by Lenny Taelman dated May 1, 2016.

$$\mathbf{A}^1 \to X \to \mathbf{A}^1$$

be as in Morphisms of Spaces, Example 67.53.3. Recall that the first morphism is étale and identifies t with $-t$ for $t \in \mathbf{A}^1_k \setminus \{0\}$ and that the second morphism is our universal homeomorphism. Since $\mathbf{A}^1_k$ has no nontrivial connected finite étale coverings (because k is algebraically closed of characteristic zero; details omitted), it suffices to construct a nontrivial connected finite étale covering $Y \to X$. To do this, let Y be the affine line with zero doubled (Schemes, Example 26.14.3). Then $Y = Y_1 \cup Y_2$ with $Y_i = \mathbf{A}^1_k$ glued along $\mathbf{A}^1_k \setminus \{0\}$. To define the morphism $Y \to X$ we use the morphisms

$$Y_1 \xrightarrow{1} \mathbf{A}^1_k \to X \quad \text{and} \quad Y_2 \xrightarrow{-1} \mathbf{A}^1_k \to X.$$

These glue over $Y_1 \cap Y_2$ by the construction of X and hence define a morphism $Y \to X$. In fact, we claim that

$$\begin{tikzcd} Y \arrow[d] & Y_1 \amalg Y_2 \arrow[l] \arrow[d] \\ X & \mathbf{A}^1_k \arrow[l] \end{tikzcd}$$

is a cartesian square. We omit the details; you can use for example Groupoids, Lemma 39.20.7. Since $\mathbf{A}^1_k \to X$ is étale and surjective, this proves that $Y \to X$ is finite étale of degree 2 which gives the desired example.

More simply, you can argue as follows. The scheme Y has a free action of the group $G = \{+1, -1\}$ where -1 acts by swapping Y_1 and Y_2 and changing the sign of the coordinate. Then $X = Y/G$ (see Spaces, Definition 65.14.4) and hence $Y \to X$ is finite étale. You can also show directly that there exists a universal homeomorphism $X \to \mathbf{A}^1_k$ by using $t \mapsto t^2$ on affine spaces. In fact, this X is the same as the X above.

76.9. Thickenings

05ZJ The following terminology may not be completely standard, but it is convenient.

05ZK Definition 76.9.1. Thickenings. Let S be a scheme.

(1) We say an algebraic space X' is a thickening of an algebraic space X if X is a closed subspace of X' and the associated topological spaces are equal.
(2) We say X' is a first order thickening of X if X is a closed subspace of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a morphism of thickenings is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subspace Y. In this situation we set $f = f'|_X :$

$X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.

(4) Let B be an algebraic space. We similarly define thickenings over B, and morphisms of thickenings over B. This means that the spaces X, X', Y, Y' above are algebraic spaces endowed with a structure morphism to B, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over B.

The fundamental equivalence. Note that if $X \subset X'$ is a thickening, then $X \to X'$ is integral and universally bijective. This implies that

05ZL (76.9.1.1)
$$X_{spaces,étale} = X'_{spaces,étale}$$

via the pullback functor, see Theorem 76.8.1. Hence we may think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$. Thus a canonical equivalence of locally ringed topoi

05ZM (76.9.1.2)
$$(Sh(X'_{spaces,étale}), \mathcal{O}_{X'}) \cong (Sh(X_{spaces,étale}), \mathcal{O}_{X'})$$

Below we will frequently combine this with the fully faithfulness result of Properties of Spaces, Theorem 66.28.4. For example the closed immersion $i_X : X \to X'$ corresponds to the surjective map $i_X^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X$.

Let S be a scheme, and let B be an algebraic space over S. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings over B. Note that the diagram of continuous functors

$$\begin{tikzcd} X_{spaces,étale} & Y_{spaces,étale} \arrow[l] \\ X'_{spaces,étale} \arrow[u] & Y'_{spaces,étale} \arrow[l] \arrow[u] \end{tikzcd}$$

is commutative and the vertical arrows are equivalences. Hence $f_{spaces,étale}$, f_{small}, $f'_{spaces,étale}$, and f'_{small} all define the same morphism of topoi. Thus we may think of

$$(f')^\sharp : f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{X'}$$

as a map of sheaves of $\mathcal{O}_B$-algebras fitting into the commutative diagram

$$\begin{tikzcd} f_{spaces,étale}^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\ f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "(f')^\sharp"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"'] \end{tikzcd}$$

Here $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the names of the given closed immersions.

05ZN Lemma 76.9.2. Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be thickenings of algebraic spaces over B. Let $f : X \to Y$ be a morphism of algebraic spaces over B. Given any map of $\mathcal{O}_B$-algebras

$$\alpha : f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

such that

$$\begin{tikzcd} f_{spaces,étale}^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\ f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "\alpha"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"'] \end{tikzcd}$$

commutes, there exists a unique morphism of (f, f') of thickenings over B such that $\alpha = (f')^\sharp$.

Proof. To find f', by Properties of Spaces, Theorem 66.28.4, all we have to do is show that the morphism of ringed topoi

$$(f_{spaces,étale}, \alpha) : (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \longrightarrow (Sh(Y_{spaces,étale}), \mathcal{O}_{Y'})$$

is a morphism of locally ringed topoi. This follows directly from the definition of morphisms of locally ringed topoi (Modules on Sites, Definition 18.40.9), the fact that $(f, f^\sharp)$ is a morphism of locally ringed topoi (Properties of Spaces, Lemma 66.28.1), that α fits into the given commutative diagram, and the fact that the kernels of $i_X^\sharp$ and $i_Y^\sharp$ are locally nilpotent. Finally, the fact that $f' \circ i_X = i_Y \circ f$ follows from the commutativity of the diagram and another application of Properties of Spaces, Theorem 66.28.4. We omit the verification that f' is a morphism over B. $\square$

05ZP Lemma 76.9.3. Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. For any open subspace $U \subset X$ there exists a unique open subspace $U' \subset X'$ such that $U = X \times_{X'} U'$.

Proof. Let $U' \to X'$ be the object of $X'_{spaces,étale}$ corresponding to the object $U \to X$ of $X_{spaces,étale}$ via (76.9.1.1). The morphism $U' \to X'$ is étale and universally injective, hence an open immersion, see Morphisms of Spaces, Lemma 67.51.2. $\square$

Finite order thickenings. Let $i_X : X \to X'$ be a thickening of algebraic spaces. Any local section of the kernel $\mathcal{I} = \mathrm{Ker}(i_X^\sharp) \subset \mathcal{O}_{X'}$ is locally nilpotent. Let us say that $X \subset X'$ is a finite order thickening if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

05ZQ Lemma 76.9.4. Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. Let U be an affine object of $X_{spaces,étale}$. Then

$$\Gamma(U, \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is surjective where we think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$ via (76.9.1.2).

Proof. Let $U' \to X'$ be the étale morphism of algebraic spaces such that $U = X \times_{X'} U'$, see Theorem 76.8.1. By Limits of Spaces, Lemma 70.15.1 we see that U' is an affine scheme. Hence $\Gamma(U, \mathcal{O}_{X'}) = \Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ is surjective as $U \to U'$ is a closed immersion of affine schemes. Below we give a direct proof for finite order thickenings which is the case most used in practice. $\square$

Proof for finite order thickenings. We may assume that $X \subset X'$ is a first order thickening by the principle explained above. Denote $\mathcal{I}$ the kernel of the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. As $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{X'}$-module and since $\mathcal{I}^2 = 0$ by the definition of a first order thickening we may apply Morphisms of Spaces, Lemma 67.14.1 to see that $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. Hence the lemma follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and the fact that $H^1_{\acute{e}tale}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. $\square$

05ZR Lemma 76.9.5. Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. If X is (representable by) a scheme, then so is X'.

Proof. Note that $X'_{red} = X_{red}$. Hence if X is a scheme, then X'_{red} is a scheme. Thus the result follows from Limits of Spaces, Lemma 70.15.3. Below we give a direct proof for finite order thickenings which is the case most often used in practice. $\square$

Proof for finite order thickenings. It suffices to prove this when X' is a first order thickening of X. By Properties of Spaces, Lemma 66.13.1 there is a largest open subspace of X' which is a scheme. Thus we have to show that every point x of $|X'| = |X|$ is contained in an open subspace of X' which is a scheme. Using Lemma 76.9.3 we may replace $X \subset X'$ by $U \subset U'$ with $x \in U$ and U an affine scheme. Hence we may assume that X is affine. Thus we reduce to the case discussed in the next paragraph.

Assume $X \subset X'$ is a first order thickening where X is an affine scheme. Set $A = \Gamma(X, \mathcal{O}_X)$ and $A' = \Gamma(X', \mathcal{O}_{X'})$. By Lemma 76.9.4 the map $A \to A'$ is surjective. The kernel I is an ideal of square zero. By Properties of Spaces, Lemma 66.33.1 we obtain a canonical morphism $f : X' \to \mathrm{Spec}(A')$ which fits into the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X' \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A') \end{tikzcd}$$

Because the horizontal arrows are thickenings it is clear that f is universally injective and surjective. Hence it suffices to show that f is étale, since then Morphisms of Spaces, Lemma 67.51.2 will imply that f is an isomorphism.

To prove that f is étale choose an affine scheme U' and an étale morphism $U' \to X'$. It suffices to show that $U' \to X' \to \mathrm{Spec}(A')$ is étale, see Properties of Spaces, Definition 66.16.2. Write $U' = \mathrm{Spec}(B')$. Set $U = X \times_{X'} U'$. Since U is a closed subspace of U', it is a closed subscheme, hence $U = \mathrm{Spec}(B)$ with $B' \to B$ surjective. Denote $J = \mathrm{Ker}(B' \to B)$ and note that $J = \Gamma(U, \mathcal{I})$ where

$\mathcal{I} = \text{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ on $X_{spaces,étale}$ as in the proof of Lemma 76.9.4. The morphism $U' \to X' \to \text{Spec}(A')$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now, since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module we have $\mathcal{I} = (\widetilde{I})^a$, see Descent, Definition 35.8.2 for notation and Descent, Proposition 35.8.9 for why this is true. Hence we see that $J = I \otimes_A B$. Finally, note that $A \to B$ is étale as $U \to X$ is étale as the base change of the étale morphism $U' \to X'$. We conclude that $A' \to B'$ is étale by Algebra, Lemma 10.143.11. $\square$

05ZS Lemma 76.9.6. Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. The functor

$$V' \longmapsto V = X \times_{X'} V'$$

defines an equivalence of categories $X'_{étale} \to X_{étale}$.

Proof. The functor $V' \mapsto V$ defines an equivalence of categories $X'_{spaces,étale} \to X_{spaces,étale}$, see Theorem 76.8.1. Thus it suffices to show that V is a scheme if and only if V' is a scheme. This is the content of Lemma 76.9.5. $\square$

First order thickening are described as follows.

05ZT Lemma 76.9.7. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Consider a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X_{étale}$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_B$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_B$-algebras, and $\mathcal{I}$ is its kernel. If

(1) $\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and
(2) $\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module

then there exists a first order thickening $X \subset X'$ over B and an isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ of $f^{-1}\mathcal{O}_B$-algebras compatible with the surjections to $\mathcal{O}_X$.

Proof. In this proof we redo some of the arguments used in the proofs of Lemmas 76.9.4 and 76.9.5. We first handle the case $B = S = \text{Spec}(\mathbf{Z})$. Let U be an affine scheme, and let $U \to X$ be étale. Then

$$0 \to \mathcal{I}(U) \to \mathcal{A}(U) \to \mathcal{O}_X(U) \to 0$$

is exact as $H^1(U_{étale}, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. If $V \to U$ is a morphism of affine objects of $X_{spaces,étale}$ then

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$$

since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module, see Descent, Proposition 35.8.9. Hence $\mathcal{A}(U) \to \mathcal{A}(V)$ is an étale ring map, see Algebra, Lemma 10.143.11. Hence we see that

$$U \longmapsto U' = \text{Spec}(\mathcal{A}(U))$$

is a functor from $X_{affine,étale}$ to the category of affine schemes and étale morphisms. In fact, we claim that this functor can be extended to a functor $U \mapsto U'$ on all of $X_{étale}$. To see this, if U is an object of $X_{étale}$, note that

$$0 \to \mathcal{I}|_{U_{Zar}} \to \mathcal{A}|_{U_{Zar}} \to \mathcal{O}_X|_{U_{Zar}} \to 0$$

and $\mathcal{I}|_{U_{Zar}}$ is a quasi-coherent sheaf on U, see Descent, Proposition 35.9.4. Hence by More on Morphisms, Lemma 37.2.2 we obtain a first order thickening $U \subset U'$ of schemes such that $\mathcal{O}_{U'}$ is isomorphic to $\mathcal{A}|_{U_{Zar}}$. It is clear that this construction is compatible with the construction for affines above.

Choose a presentation $X = U/R$, see Spaces, Definition 65.9.3 so that $s, t : R \to U$ define an étale equivalence relation. Applying the functor above we obtain an étale equivalence relation $s', t' : R' \to U'$ in schemes. Consider the algebraic space $X' = U'/R'$ (see Spaces, Theorem 65.10.5). The morphism $X = U/R \to U'/R' = X'$ is a first order thickening. Consider $\mathcal{O}_{X'}$ viewed as a sheaf on $X_{étale}$. By construction we have an isomorphism

$$\gamma : \mathcal{O}_{X'}|_{U_{étale}} \longrightarrow \mathcal{A}|_{U_{étale}}$$

such that $s^{-1}\gamma$ agrees with $t^{-1}\gamma$ on $R_{étale}$. Hence by Properties of Spaces, Lemma 66.18.14 this implies that γ comes from a unique isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ as desired.

To handle the case of a general base algebraic space B, we first construct X' as an algebraic space over $\mathbf{Z}$ as above. Then we use the isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ to define $f^{-1}\mathcal{O}_B \to \mathcal{O}_{X'}$. According to Lemma 76.9.2 this defines a morphism $X' \to B$ compatible with the given morphism $X \to B$ and we are done. $\square$

09ZX Lemma 76.9.8. Let S be a scheme. Let $Y \subset Y'$ be a thickening of algebraic spaces over S. Let $X' \to Y'$ be a morphism and set $X = Y \times_{Y'} X'$. Then $(X \subset X') \to (Y \subset Y')$ is a morphism of thickenings. If $Y \subset Y'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.

Proof. Omitted. $\square$

0BPH Lemma 76.9.9. Let S be a scheme. If $X \subset X'$ and $X' \subset X''$ are thickenings of algebraic spaces over S, then so is $X \subset X''$.

Proof. Omitted. $\square$

0BPI Lemma 76.9.10. The property of being a thickening is fpqc local. Similarly for first order thickenings.

Proof. The statement means the following: Let S be a scheme and let $X \to X'$ be a morphism of algebraic spaces over S. Let $\{g_i : X'_i \to X'\}$ be an fpqc covering of algebraic spaces such that the base change $X_i \to X'_i$ is a thickening for all i. Then $X \to X'$ is a thickening. Since the morphisms g_i are jointly surjective we conclude that $X \to X'$ is surjective. By Descent on Spaces, Lemma 74.11.17 we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. $\square$

76.10. Morphisms of thickenings

0CG4 If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of algebraic spaces, then often properties of the morphism f are inherited by f'. There are several variants.

09ZY **Lemma 76.10.1.** Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S. Then

(1) f is an affine morphism if and only if f' is an affine morphism,
(2) f is a surjective morphism if and only if f' is a surjective morphism,
(3) f is quasi-compact if and only if f' quasi-compact,
(4) f is universally closed if and only if f' is universally closed,
(5) f is integral if and only if f' is integral,
(6) f is (quasi-)separated if and only if f' is (quasi-)separated,
(7) f is universally injective if and only if f' is universally injective,
(8) f is universally open if and only if f' is universally open,
(9) f is representable if and only if f' is representable, and
(10) add more here.

Proof. Observe that $Y \to Y'$ and $X \to X'$ are integral and universal homeomorphisms. This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Limits of Spaces, Proposition 70.15.2 which tells us that there is a 1-to-1 correspondence between affine schemes étale over X and X' and between affine schemes étale over Y and Y'. Part (5) follows from (1) and (4) by Morphisms of Spaces, Lemma 67.45.7. Finally, note that

$$X \times_Y X = X \times_{Y'} X \to X \times_{Y'} X' \to X' \times_{Y'} X'$$

is a thickening (the two arrows are thickenings by Lemma 76.9.8). Hence applying (3) and (4) to the morphism $(X \subset X') \to (X \times_Y X \to X' \times_{Y'} X')$ we obtain (6). Finally, part (9) follows from the fact that an algebraic space thickening of a scheme is again a scheme, see Lemma 76.9.5. $\square$

09ZZ **Lemma 76.10.2.** Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S such that $X = Y \times_{Y'} X'$. If $X \subset X'$ is a finite order thickening, then

(1) f is a closed immersion if and only if f' is a closed immersion,
(2) f is locally of finite type if and only if f' is locally of finite type,
(3) f is locally quasi-finite if and only if f' is locally quasi-finite,
(4) f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,
(5) $\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,
(6) f is unramified if and only if f' is unramified,
(7) f is proper if and only if f' is proper,
(8) f is a finite morphism if and only if f' is an finite morphism,
(9) f is a monomorphism if and only if f' is a monomorphism,
(10) f is an immersion if and only if f' is an immersion, and
(11) add more here.

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 37.3.3. This proves (2), (3), (4), (5), and (6).

The properties of morphisms in (1), (7), (8), (9), (10) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 65.12.3, and Morphisms of Spaces, Lemmas 67.40.3, 67.45.5, and 67.10.5.

The interesting direction in (1), (7), (8), (9), (10) is to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $Y \subset Y'$ is a first order thickening, see discussion on finite order thickenings above.

Proof of (1). Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$ and $U = X \times_Y V$. Then $U \to V$ is a closed immersion, which implies that U is a scheme, which in turn implies that U' is a scheme (Lemma 76.9.5). Thus we can apply the lemma in the case of schemes (More on Morphisms, Lemma 37.3.3) to $(U \subset U') \to (V \subset V')$ to conclude.

Proof of (7). Follows by combining (2) with results of Lemma 76.10.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 76.10.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 29.44.4).

Proof of (9). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $X' \to X' \times_{Y'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{Y'} X'$ is contained in the pullback of the ideal $\mathcal{I} \subset \mathcal{O}_{Y'}$ cutting out Y in Y'. Indeed, $X = X \times_Y X = (X' \times_{Y'} X') \times_{Y'} Y$ is contained in X'. The conormal sheaf of the closed immersion $\Delta : X' \to X' \times_{Y'} X'$ is equal to $\Omega_{X'/Y'}$ (this is the analogue of Morphisms, Lemma 29.32.7 for algebraic spaces and follows either by étale localization or by combining Lemmas 76.7.11 and 76.7.13; some details omitted). Thus it suffices to show that $\Omega_{X'/Y'} = 0$ which follows from (5) and the corresponding statement for X/Y.

Proof of (10). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open subspace and $X \to V$ is a closed immersion, see Morphisms of Spaces, Remark 67.12.4. Let $V' \subset Y'$ be the open subspace whose underlying topological space $|V'|$ is the same as $|V| \subset |Y| = |Y'|$. Then $X' \to Y'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (1). This finishes the proof. $\square$

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from f to f'), we instead take as given that each of f and f' is locally of finite type.

0BPJ Lemma 76.10.3. Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings of algebraic spaces over S. Assume f and f' are locally of finite type and $X = Y \times_{Y'} X'$. Then

(1) f is locally quasi-finite if and only if f' is locally quasi-finite,
(2) f is finite if and only if f' is finite,
(3) f is a closed immersion if and only if f' is a closed immersion,
(4) $\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,
(5) f is unramified if and only if f' is unramified,
(6) f is a monomorphism if and only if f' is a monomorphism,
(7) f is an immersion if and only if f' is an immersion,

(8) f is proper if and only if f' is proper, and
(9) add more here.

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 37.3.4. This proves (1), (4), and (5).

The properties in (2), (3), (6), (7), and (8) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 65.12.3, and Morphisms of Spaces, Lemmas 67.40.3, 67.45.5, and 67.10.5. Hence in each case we need only to prove that if f has the desired property, so does f'.

Case (2) follows from case (5) of Lemma 76.10.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms of Spaces, Lemma 67.45.6).

Case (3). This follows immediately from Limits of Spaces, Lemma 70.15.5.

Proof of (6). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by (3). In fact $\Delta_{X'/Y'}$ induces a bijection $|X'| \to |X' \times_{Y'} X'|$, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms of Spaces, Lemma 67.28.10. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_{Y'} X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_{Y'} X'$ is zero. (The conormal sheaf of the closed immersion $\Delta_{X'/Y'}$ is equal to $\Omega_{X'/Y'}$; this is the analogue of Morphisms, Lemma 29.32.7 for algebraic spaces and follows either by étale localization or by combining Lemmas 76.7.11 and 76.7.13; some details omitted.) In other words, $\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma 10.21.5.

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open subspace and $X \to V$ is a closed immersion, see Morphisms of Spaces, Remark 67.12.4. Let $V' \subset Y'$ be the open subspace whose underlying topological space $|V'|$ is the same as $|V| \subset |Y| = |Y'|$. Then $X' \to Y'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma 76.10.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. ☐

76.11. Picard groups of thickenings

0DNL Some material on Picard groups of thickenings.

0DNM Lemma 76.11.1. Let S be a scheme. Let $X \subset X'$ be a first order thickening of algebraic spaces over S with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{I}) \arrow[r] & H^0(X', \mathcal{O}_{X'}^*) \arrow[r] & H^0(X, \mathcal{O}_X^*) \arrow[dll, rounded corners] \\ & H^1(X, \mathcal{I}) \arrow[r] & \mathrm{Pic}(X') \arrow[r] & \mathrm{Pic}(X) \arrow[dll, rounded corners] \\ & H^2(X, \mathcal{I}) \arrow[r] & \ldots \arrow[r] & \ldots \end{tikzcd}$$

of abelian groups.

Proof. Recall that $X_{\acute{e}tale} = X'_{\acute{e}tale}$, see Lemma 76.9.6 and more generally the discussion in Section 76.9. The sequence of the lemma is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

on $X_{\acute{e}tale}$ where the first map sends a local section f of $\mathcal{I}$ to the invertible section $1+f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed site with the first cohomology group of the sheaf of invertible functions, see Cohomology on Sites, Lemma 21.6.1. $\square$

76.12. First order infinitesimal neighbourhood

05ZU A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of algebraic spaces. Choose an open subspace $U \subset X$ such that i identifies Z with a closed subspace $Z \subset U$ (see Morphisms of Spaces, Remark 67.12.4). Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U, see Morphisms of Spaces, Lemma 67.13.1. Then we can consider the closed subspace $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

05ZV Definition 76.12.1. Let $i : Z \to X$ be an immersion of algebraic spaces. The first order infinitesimal neighbourhood of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

05ZW Lemma 76.12.2. Let $i : Z \to X$ be an immersion of algebraic spaces. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open subspace used in the construction of Z', i.e., an open such that Z is identified with a closed subspace of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $|b|(|T'|) \subset |U|$. Hence we can think of

b as a morphism into U, see Properties of Spaces, Lemma 66.4.9. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the square zero quasi-coherent sheaf of ideals cutting out T. By the commutativity of the diagram we have $b|_T = i \circ a$ where $i : Z \to U$ is the closed immersion. We conclude that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$ by Morphisms of Spaces, Lemma 67.13.1. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Morphisms of Spaces, Lemma 67.13.1 this implies that b factors through Z'. Letting $a' : T' \to Z'$ be this factorization we win. □

05ZX Lemma 76.12.3. Let $i : Z \to X$ be an immersion of algebraic spaces. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Lemma 76.5.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. □

76.13. Formally smooth, étale, unramified transformations

04G3 Recall that a ring map $R \to A$ is called formally smooth, resp. formally étale, resp. formally unramified (see Algebra, Definition 10.138.1, resp. Definition 10.150.1, resp. Definition 10.148.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists a, resp. exists a unique, resp. exists at most one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of algebraic spaces, and more generally functors.

049S Definition 76.13.1. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Consider commutative solid diagrams of the form

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where T and T' are affine schemes and i is a closed immersion defined by an ideal of square zero.

(1) We say a is formally smooth if given any solid diagram as above there exists a dotted arrow making the diagram commute[1].

[1]This is just one possible definition that one can make here. Another slightly weaker condition would be to require that the dotted arrow exists fppf locally on T'. This weaker notion has in some sense better formal properties.

(2) We say a is formally étale if given any solid diagram as above there exists exactly one dotted arrow making the diagram commute.
(3) We say a is formally unramified if given any solid diagram as above there exists at most one dotted arrow making the diagram commute.

04G4 Lemma 76.13.2. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Then a is formally étale if and only if a is both formally smooth and formally unramified.

Proof. Formal from the definition. □

049T Lemma 76.13.3. Composition.

(1) A composition of formally smooth transformations of functors is formally smooth.
(2) A composition of formally étale transformations of functors is formally étale.
(3) A composition of formally unramified transformations of functors is formally unramified.

Proof. This is formal. □

049U Lemma 76.13.4. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

(1) If a is formally smooth, then the base change a' is formally smooth.
(2) If a is formally étale, then the base change a' is formally étale.
(3) If a is formally unramified, then the base change a' is formally unramified.

Proof. This is formal. □

04AL Lemma 76.13.5. Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors.

(1) If a is smooth then a is formally smooth.
(2) If a is étale, then a is formally étale.
(3) If a is unramified, then a is formally unramified.

Proof. Consider a solid commutative diagram

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 76.13.1. Then $F \times_G T'$ is a scheme smooth (resp. étale, resp. unramified) over T'. Hence by More on Morphisms, Lemma 37.11.7 (resp. Lemma 37.8.9, resp. Lemma 37.6.8) we can fill in (resp. uniquely fill in, resp. fill in at most one way) the dotted arrow in the diagram

$$\begin{tikzcd} F \times_G T' \arrow[d] & T \arrow[l] \arrow[d, "i"] \\ T' & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

an hence we also obtain the corresponding assertion in the first diagram. □

04CY Lemma 76.13.6. Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume that a is representable, surjective, and étale.

(1) If b is formally smooth, then $b \circ a$ is formally smooth.
(2) If b is formally étale, then $b \circ a$ is formally étale.
(3) If b is formally unramified, then $b \circ a$ is formally unramified.

Conversely, consider a solid commutative diagram

$$\begin{tikzcd} G \arrow[d, "b"'] & T \arrow[l] \arrow[d, "i"] \\ H & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

with T' an affine scheme over S and $i : T \to T'$ a closed immersion defined by an ideal of square zero.

(4) If $b \circ a$ is formally smooth, then for every $t \in T$ there exists an étale morphism of affines $U' \to T'$ and a morphism $U' \to G$ such that

$$\begin{tikzcd} G \arrow[d, "b"'] & T \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

commutes and t is in the image of $U' \to T'$.
(5) If $b \circ a$ is formally unramified, then there exists at most one dotted arrow in the diagram above, i.e., b is formally unramified.
(6) If $b \circ a$ is formally étale, then there exists exactly one dotted arrow in the diagram above, i.e., b is formally étale.

Proof. Assume b is formally smooth (resp. formally étale, resp. formally unramified). Since an étale morphism is both smooth and unramified we see that a is representable and smooth (resp. étale, resp. unramified). Hence parts (1), (2) and (3) follow from a combination of Lemma 76.13.5 and Lemma 76.13.3.

Assume that $b \circ a$ is formally smooth. Consider a diagram as in the statement of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 41.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Choose an affine open subscheme $U' \subset W'$ such that t is in the image of $U' \to T'$. Because $b \circ a$ is formally smooth we see that the exist morphisms $U' \to F$ such that

$$\begin{tikzcd} F \arrow[d, "b\circ a"'] & W \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

commutes. Taking the composition $U' \to F \to G$ gives a map as in part (5) of the lemma.

Assume that $f, g : T' \to G$ are two dotted arrows fitting into the diagram of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 41.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Since a is formally étale the compositions

$$W' \to T' \xrightarrow{f} G \quad \text{and} \quad W' \to T' \xrightarrow{g} G$$

lift to morphisms $f', g' : W' \to F$ (lift on affine opens and glue by uniqueness). Now if $b \circ a : F \to H$ is formally unramified, then $f' = g'$ and hence $f = g$ as $W' \to T'$ is an étale covering. This proves part (6) of the lemma.

Assume that $b \circ a$ is formally étale. Then by part (4) we can étale locally on T' find a dotted arrow fitting into the diagram and by part (5) this dotted arrow is unique. Hence we may glue the local solutions to get assertion (6). Some details omitted. □

04CZ Remark 76.13.7. It is tempting to think that in the situation of Lemma 76.13.6 we have "b formally smooth" ⇔ "$b \circ a$ formally smooth". However, this is likely not true in general.

04G5 Lemma 76.13.8. Let S be a scheme. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume b is formally unramified.

(1) If $b \circ a$ is formally unramified then a is formally unramified.
(2) If $b \circ a$ is formally étale then a is formally étale.
(3) If $b \circ a$ is formally smooth then a is formally smooth.

Proof. Let $T \subset T'$ be a closed immersion of affine schemes defined by an ideal of square zero. Let $g' : T' \to G$ and $f : T \to F$ be given such that $g'|_T = a \circ f$. Because b is formally unramified, there is a one to one correspondence between

$$\{f' : T' \to F \mid f = f'|_T \text{ and } a \circ f' = g'\}$$

and

$$\{f' : T' \to F \mid f = f'|_T \text{ and } b \circ a \circ f' = b \circ g'\}.$$

From this the lemma follows formally. □

76.14. Formally unramified morphisms

04G6 In this section we work out what it means that a morphism of algebraic spaces is formally unramified.

04G7 Definition 76.14.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be formally unramified if it is formally unramified as a transformation of functors as in Definition 76.13.1.

We will not restate the results proved in the more general setting of formally unramified transformations of functors in Section 76.13. It turns out we can characterize this property in terms of vanishing of the module of relative differentials, see Lemma 76.14.6.

04G8 Lemma 76.14.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is formally unramified,
(2) for every diagram
$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally unramified (as in More on Morphisms, Definition 37.6.1), and
(3) for one such diagram with surjective vertical arrows the morphism ψ is formally unramified.

Proof. Assume f is formally unramified. By Lemma 76.13.5 the morphisms $U \to X$ and $V \to Y$ are formally unramified. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 76.13.8 that $U \to V$ is formally unramified. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 76.13.5 the morphism $V \to Y$ is formally unramified. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 76.13.6 that $X \to Y$ is formally unramified, i.e., (1) holds. $\square$

05ZY Lemma 76.14.3. Let S be a scheme. If $f : X \to Y$ is a formally unramified morphism of algebraic spaces over S, then given any solid commutative diagram
$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$
where $T \subset T'$ is a first order thickening of algebraic spaces over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 76.14.1 the condition that T be an affine scheme may be dropped.

Proof. This is true because there exists a surjective étale morphism $U' \to T'$ where U' is a disjoint union of affine schemes (see Properties of Spaces, Lemma 66.6.1) and a morphism $T' \to X$ is determined by its restriction to U'. $\square$

05ZZ Lemma 76.14.4. A composition of formally unramified morphisms is formally unramified.

Proof. This is formal. $\square$

0600 Lemma 76.14.5. A base change of a formally unramified morphism is formally unramified.

Proof. This is formal. $\square$

04G9 **Lemma 76.14.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is formally unramified, and
(2) $\Omega_{X/Y} = 0$.

Proof. This is a combination of Lemma 76.14.2, More on Morphisms, Lemma 37.6.7, and Lemma 76.7.3. $\square$

04GA **Lemma 76.14.7.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) The morphism f is unramified,
(2) the morphism f is locally of finite type and $\Omega_{X/Y} = 0$, and
(3) the morphism f is locally of finite type and formally unramified.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. Then we see

$$\begin{aligned} f \text{ unramified} &\Leftrightarrow \psi \text{ unramified} \\ &\Leftrightarrow \psi \text{ locally finite type and } \Omega_{U/V} = 0 \\ &\Leftrightarrow f \text{ locally finite type and } \Omega_{X/Y} = 0 \\ &\Leftrightarrow f \text{ locally finite type and formally unramified} \end{aligned}$$

Here we have used Morphisms, Lemma 29.35.2 and Lemma 76.14.6. $\square$

05W6 **Lemma 76.14.8.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is unramified and a monomorphism,
(2) f is unramified and universally injective,
(3) f is locally of finite type and a monomorphism,
(4) f is universally injective, locally of finite type, and formally unramified.

Moreover, in this case f is also representable, separated, and locally quasi-finite.

Proof. We have seen in Lemma 76.14.7 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_Y X$ is both an open immersion (Morphisms of Spaces, Lemma 67.38.9) and surjective (Morphisms of Spaces, Lemma 67.19.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1). Finally, we see that f is representable, separated, and locally quasi-finite by Morphisms of Spaces, Lemmas 67.27.10 and 67.51.1. $\square$

05W8 **Lemma 76.14.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is a closed immersion,
(2) f is universally closed, unramified, and a monomorphism,
(3) f is universally closed, unramified, and universally injective,

(4) f is universally closed, locally of finite type, and a monomorphism,
(5) f is universally closed, universally injective, locally of finite type, and formally unramified.

Proof. The equivalence of (2) – (5) follows immediately from Lemma 76.14.8. Moreover, if (2) – (5) are satisfied then f is representable. Similarly, if (1) is satisfied then f is representable. Hence the result follows from the case of schemes, see Étale Morphisms, Lemma 41.7.2. □

76.15. Universal first order thickenings

0601 Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. A universal first order thickening of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

0602 (76.15.0.1)
$$\begin{tikzcd}
 & Z \arrow[ld] & & T \arrow[ll, "a"'] \arrow[rd] & \\
Z' \arrow[rrd] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[lld, "b"] \\
 & & X & &
\end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does. Before we prove this, let us show that a universal first order thickening in the category of schemes is a universal first order thickening in the category of algebraic spaces.

0603 Lemma 76.15.1. Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. Let $Z \subset Z'$ be a first order thickening over X. The following are equivalent

(1) $Z \subset Z'$ is a universal first order thickening,
(2) for any diagram (76.15.0.1) with T' a scheme a unique dotted arrow exists making the diagram commute, and
(3) for any diagram (76.15.0.1) with T' an affine scheme a unique dotted arrow exists making the diagram commute.

Proof. The implications (1) ⇒ (2) ⇒ (3) are formal. Assume (3) a assume given an arbitrary diagram (76.15.0.1). Choose a presentation $T' = U'/R'$, see Spaces, Definition 65.9.3. We may assume that $U' = \coprod U'_i$ is a disjoint union of affines, so $R' = U' \times_{T'} U' = \coprod_{i,j} U'_i \times'_T U'_j$. For each pair (i, j) choose an affine open covering $U'_i \times'_T U'_j = \bigcup_k R'_{ijk}$. Denote U_i, R_{ijk} the fibre products with T over T'. Then each $U_i \subset U'_i$ and $R_{ijk} \subset R'_{ijk}$ is a first order thickening of affine schemes. Denote $a_i : U_i \to Z$, resp. $a_{ijk} : R_{ijk} \to Z$ the composition of $a : T \to Z$ with the morphism $U_i \to T$, resp. $R_{ijk} \to T$. By (3) applied to $a_i : U_i \to Z$ we obtain unique morphisms $a'_i : U'_i \to Z'$. By (3) applied to a_{ijk} we see that the two compositions $R'_{ijk} \to R'_i \to Z'$ and $R'_{ijk} \to R'_j \to Z'$ are equal. Hence

$a' = \coprod a'_i : U' = \coprod U'_i \to Z'$ descends to the quotient sheaf $T' = U'/R'$ and we win. $\square$

0604 Lemma 76.15.2. Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. If $Z \subset Z'$ is a universal first order thickening of Z over Y and $Y \to X$ is formally étale, then $Z \subset Z'$ is a universal first order thickening of Z over X.

Proof. This is formal. Namely, by Lemma 76.15.1 it suffices to consider solid commutative diagrams (76.15.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y$ as $Y \to X$ is assumed formally étale. Hence the fact that $Z \subset Z'$ is a universal first order thickening over Y produces the desired morphism $a' : T' \to Z'$. $\square$

0605 Lemma 76.15.3. Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. Assume $Z \to Y$ is étale.

(1) If $Y \subset Y'$ is a universal first order thickening of Y over X, then the unique étale morphism $Z' \to Y'$ such that $Z = Y \times_{Y'} Z'$ (see Theorem 76.8.1) is a universal first order thickening of Z over X.
(2) If $Z \to Y$ is surjective and $(Z \subset Z') \to (Y \subset Y')$ is an étale morphism of first order thickenings over X and Z' is a universal first order thickening of Z over X, then Y' is a universal first order thickening of Y over X.

Proof. Proof of (1). By Lemma 76.15.1 it suffices to consider solid commutative diagrams (76.15.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y'$ as Y' is the universal first order thickening. Then the fact that $Z' \to Y'$ is étale implies (see Lemma 76.13.5) that $T' \to Y'$ lifts to the desired morphism $a' : T' \to Z'$.

Proof of (2). Let $T \subset T'$ be a first order thickening over X and let $a : T \to Y$ be a morphism. Set $W = T \times_Y Z$ and denote $c : W \to Z$ the projection Let $W' \to T'$ be the unique étale morphism such that $W = T \times_{T'} W'$, see Theorem 76.8.1. Note that $W' \to T'$ is surjective as $Z \to Y$ is surjective. By assumption we obtain a unique morphism $c' : W' \to Z'$ over X restricting to c on W. By uniqueness the two restrictions of c' to $W' \times_{T'} W'$ are equal (as the two restrictions of c to $W \times_T W$ are equal). Hence c' descends to a unique morphism $a' : T' \to Y'$ and we win. $\square$

0606 Lemma 76.15.4. Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S. There exists a universal first order thickening $Z \subset Z'$ of Z over X.

Proof. Choose any commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where V and U are schemes and the vertical arrows are étale. Note that $V \to U$ is a formally unramified morphism of schemes, see Lemma 76.14.2. Combining Lemma 76.15.1 and More on Morphisms, Lemma 37.7.1 we see that a universal first order thickening $V \subset V'$ of V over U exists. By Lemma 76.15.2 part (1) V' is a universal first order thickening of V over X.

Fix a scheme U and a surjective étale morphism $U \to X$. The argument above shows that for any $V \to Z$ étale with V a scheme such that $V \to Z \to X$ factors through U a universal first order thickening $V \subset V'$ of V over X exists (but does not depend on the chosen factorization of $V \to X$ through U). Now we may choose V such that $V \to Z$ is surjective étale (see Spaces, Lemma 65.11.6). Then $R = V \times_Z V$ a scheme étale over Z such that $R \to X$ factors through U also. Hence we obtain universal first order thickenings $V \subset V'$ and $R \subset R'$ over X. As $V \subset V'$ is a universal first order thickening, the two projections $s, t : R \to V$ lift to morphisms $s', t' : R' \to V'$. By Lemma 76.15.3 as R' is the universal first order thickening of R over X these morphisms are étale. Then $(t', s') : R' \to V'$ is an étale equivalence relation and we can set $Z' = V'/R'$. Since $V' \to Z'$ is surjective étale and v' is the universal first order thickening of V over X we conclude from Lemma 76.15.2 part (2) that Z' is a universal first order thickening of Z over X. □

0607 Definition 76.15.5. Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S.

(1) The universal first order thickening of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 76.15.4.
(2) The conormal sheaf of Z over X is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z_{étale}$ and $\mathcal{C}_{Z/X}$ is a quasi-coherent $\mathcal{O}_Z$-module. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

0608 Lemma 76.15.6. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then

(1) i is formally unramified,
(2) the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 76.12.1,
(3) the conormal sheaf of i in the sense of Definition 76.5.1 agrees with the conormal sheaf of i in the sense of Definition 76.15.5.

Proof. An immersion of algebraic spaces is by definition a representable morphism. Hence by Morphisms, Lemmas 29.35.7 and 29.35.8 an immersion is unramified (via the abstract principle of Spaces, Lemma 65.5.8). Hence it is formally unramified by Lemma 76.14.7. The other assertions follow by combining Lemmas 76.12.2 and 76.12.3 and the definitions. □

0609 Lemma 76.15.7. Let S be a scheme. Let $Z \to X$ be a formally unramified morphism of algebraic spaces over S. Then the universal first order thickening Z' is formally unramified over X.

Proof. Let $T \subset T'$ be a first order thickening of affine schemes over X. Let

$$\begin{tikzcd} Z' & T \arrow[l, "c"'] \arrow[d] \\ X \arrow[from=u] & T' \arrow[l] \arrow[ul, "a,b"] \end{tikzcd}$$

be a commutative diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. ☐

060A **Lemma 76.15.8.** Let S be a scheme Consider a commutative diagram of algebraic spaces over S

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

$$\begin{tikzcd} & & & Z' \arrow[dl] \arrow[d, "f'"] \\ Z \arrow[urrr] \arrow[rr] \arrow[d, "f"'] & & X \arrow[d] & W' \arrow[dl] \\ W \arrow[urrr] \arrow[rr] & & Y & \end{tikzcd}$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Lemma 76.5.3 applied to $(Z \subset Z') \to (W \subset W')$. ☐

060B **Lemma 76.15.9.** Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 76.15.8 is surjective.

Proof. The morphism h is formally unramified by Lemma 76.14.5. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 76.5.5 for why this implies that the map of conormal sheaves is surjective. $\square$

060C Lemma 76.15.10. Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms of Spaces, Lemma 67.30.4. Hence the first statement follows from the description of W' in Lemma 76.15.9. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 76.5.5 for why this implies that the map of conormal sheaves is an isomorphism. $\square$

060D Lemma 76.15.11. Taking the universal first order thickenings commutes with étale localization. More precisely, let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over a base scheme S. Let

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

be a commutative diagram with étale vertical arrows. Let Z' be the universal first order thickening of Z over X. Then $V \to U$ is formally unramified and the universal first order thickening V' of V over U is étale over Z'. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 76.14.2. The compatibility of universal first order thickenings is a consequence of Lemmas 76.15.2 and 76.15.3. $\square$

060E Lemma 76.15.12. Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map

$$\mathrm{d}h' : (h')^*\Omega_{X/B} \to \Omega_{Z'/B}$$

induces an isomorphism $h^*\Omega_{X/B} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Lemma 76.7.6. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the case of schemes by étale localization, see Lemma

76.15.11 and Lemma 76.7.3. In this case the result is More on Morphisms, Lemma 37.7.9. □

060F Lemma 76.15.13. Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. There is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0.$$

The first arrow is induced by $\mathrm{d}_{Z'/B}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Lemma 76.7.10. Hence the result follows on applying Lemma 76.15.12. □

06BE Lemma 76.15.14. Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 76.15.8 and the second from Lemma 76.15.13.

Proof. Since the maps have been defined, checking the sequence is exact reduces to the case of schemes by étale localization, see Lemma 76.15.11 and Lemma 76.7.3. In this case the result is More on Morphisms, Lemma 37.7.11. □

06BF Lemma 76.15.15. Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S.

(1) If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.
(2) There is a canonical exact sequence

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 76.15.8 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 76.15.8. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Lemma 76.5.6 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Lemma 76.5.5 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

76.16. Formally étale morphisms

04GB In this section we work out what it means that a morphism of algebraic spaces is formally étale.

04GC Definition 76.16.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be formally étale if it is formally étale as a transformation of functors as in Definition 76.13.1.

We will not restate the results proved in the more general setting of formally étale transformations of functors in Section 76.13.

04GD Lemma 76.16.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is formally étale,
(2) for every diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally étale (as in More on Morphisms, Definition 37.8.1), and
(3) for one such diagram with surjective vertical arrows the morphism ψ is formally étale.

Proof. Assume f is formally étale. By Lemma 76.13.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 76.13.8 that $U \to V$ is formally étale. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 76.13.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 76.13.6 that $X \to Y$ is formally étale, i.e., (1) holds. $\square$

0611 Lemma 76.16.3. Let S be a scheme. Let $f : X \to Y$ be a formally étale morphism of algebraic spaces over S. Then given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over Y there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 76.16.1 the condition that T be affine may be dropped.

Proof. Let $U' \to T'$ be a surjective étale morphism where $U' = \coprod U'_i$ is a disjoint union of affine schemes. Let $U_i = T \times_{T'} U'_i$. Then we get morphisms $a'_i : U'_i \to X$ such that $a'_i|_{U_i}$ equals the composition $U_i \to T \to X$. By uniqueness (see Lemma 76.14.3) we see that a'_i and a'_j agree on the fibre product $U'_i \times_{T'} U'_j$. Hence $\coprod a'_i : U' \to X$ descends to give a unique morphism $a' : T' \to X$. $\square$

0612 **Lemma 76.16.4.** A composition of formally étale morphisms is formally étale.

Proof. This is formal. $\square$

0613 **Lemma 76.16.5.** A base change of a formally étale morphism is formally étale.

Proof. This is formal. $\square$

0614 **Lemma 76.16.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S The following are equivalent:

(1) f is formally étale,
(2) f is formally unramified and the universal first order thickening of X over Y is equal to X,
(3) f is formally unramified and $\mathcal{C}_{X/Y} = 0$, and
(4) $\Omega_{X/Y} = 0$ and $\mathcal{C}_{X/Y} = 0$.

Proof. Actually, the last assertion only make sense because $\Omega_{X/Y} = 0$ implies that $\mathcal{C}_{X/Y}$ is defined via Lemma 76.14.6 and Definition 76.15.5. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/Y} = 0$ since after all by definition $\mathcal{C}_{X/Y} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. $\square$

0615 **Lemma 76.16.7.** An unramified flat morphism is formally étale.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 37.8.7 and étale localization, see Lemmas 76.14.2 and 76.16.2 and Morphisms of Spaces, Lemma 67.30.5. $\square$

0616 **Lemma 76.16.8.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) The morphism f is étale, and
(2) the morphism f is locally of finite presentation and formally étale.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 37.8.9 and étale localization, see Lemma 76.16.2 and Morphisms of Spaces, Lemmas 67.28.4 and 67.39.2. $\square$

76.17. Infinitesimal deformations of maps

0617 In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X, see Equations (76.9.1.1) and (76.9.1.2).

0618 **Lemma 76.17.1.** Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings of algebraic spaces over B. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over B. Assume that

(1) $a = b$, and
(2) the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Lemma 76.5.3) are equal.

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_B$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_B$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_B$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_B$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. $\square$

Note that in the situation of the lemma above we may write D as

0619 (76.17.1.1) $$D = \mathrm{d}_{Y/B} \circ \theta$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/B} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$.

04D0 Lemma 76.17.2. Let S be a scheme. Let B be an algebraic space over S. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over B. Let

$$\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 76.17.1 hold and the derivation D and θ are related by Equation (76.17.1.1).

Proof. Consider the map

$$\alpha = (a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (76.17.1.1). As D is an $\mathcal{O}_B$-derivation it follows that α is a map of sheaves of $\mathcal{O}_B$-algebras. By construction we have $i_X^\sharp \circ \alpha = a^\sharp \circ i_Y^\sharp$ where $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the given closed immersions. By Lemma 76.9.2 we obtain a unique morphism $(a, b') : (X \subset X') \to (Y \subset Y')$ of thickenings over B such that $\alpha = (b')^\sharp$. Setting $b = a$ we win. $\square$

0CK6 Remark 76.17.3. Assumptions and notation as in Lemma 76.17.2. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (76.17.1.1).

061A Lemma 76.17.4. Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over B. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an object U' of $(X')_{spaces,étale}$ with $U = X \times_{X'} U'$ consider morphisms $a' : U' \to Y'$ such that

(1) a' is a morphism over B,

(2) $a'|_U = a|_U$, and
(3) the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.

Then the rule

061B (76.17.4.1) $$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on $(X')_{spaces,étale}$.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms of algebraic spaces satisfy étale descent, see Descent on Spaces, Lemma 74.7.2. $\square$

061C **Lemma 76.17.5.** Same notation and assumptions as in Lemma 76.17.4. We identify sheaves on X and X' via (76.9.1.1). There is an action of the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$$

on the sheaf (76.17.4.1). Moreover, the action is simply transitive for any object U' of $(X')_{spaces,étale}$ over which the sheaf (76.17.4.1) has a section.

Proof. This is a combination of Lemmas 76.17.1, 76.17.2, and 76.17.4. $\square$

061D **Remark 76.17.6.** A special case of Lemmas 76.17.1, 76.17.2, 76.17.4, and 76.17.5 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 76.17.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } B \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$.

0CK7 **Remark 76.17.7.** Another special case of Lemmas 76.17.1, 76.17.2, 76.17.4, and 76.17.5 is where B itself is a thickening $Z \subset Z' = B$ and $Y = Z \times_{Z'} Y'$. Picture

$$\begin{tikzcd} (X \subset X') \arrow[rr, dotted, "{(a,?)}"] \arrow[dr, "{(g,g')}"'] & & (Y \subset Y') \arrow[dl, "{(h,h')}"] \\ & (Z \subset Z') & \end{tikzcd}$$

In this case the map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is determined by a: the map $h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'}$ is surjective (because we assumed $Y = Z \times_{Z'} Y'$), hence the pullback $g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'}$ is surjective, and the composition $g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ has to be the canonical map induced by g'. Thus the sheaf of Lemma 76.17.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$.

0CK8 **Lemma 76.17.8.** Let S be a scheme. Consider a commutative diagram of first order thickenings

$$\begin{tikzcd} (T_2 \subset T_2') \arrow[r, "{(a_2,a_2')}"'] \arrow[d, "{(h,h')}"'] & (X_2 \subset X_2') \arrow[d, "{(f,f')}"] \\ (T_1 \subset T_1') \arrow[r, "{(a_1,a_1')}"] & (X_1 \subset X_1') \end{tikzcd} \quad \text{and a commutative diagram} \quad \begin{tikzcd} X_2' \arrow[r] \arrow[d] & B_2 \arrow[d] \\ X_1' \arrow[r] & B_1 \end{tikzcd}$$

of algebraic spaces over S with $X_2 \to X_1$ and $B_2 \to B_1$ étale. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/B_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/B_2} = h^*a_1^*\Omega_{X_1/B_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{tikzcd} T_2' \arrow[r, "\theta_2 \cdot a_2'"'] \arrow[d] & X_2' \arrow[d] \\ T_1' \arrow[r, "\theta_1 \cdot a_1'"] & X_1' \end{tikzcd}$$

commutes where the actions $\theta_2 \cdot a_2'$ and $\theta_1 \cdot a_1'$ are as in Remark 76.17.3.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ we get as the construction of the sheaf of differentials is compatible with étale localization (both on source and target), see Lemma 76.7.3. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^*f^*\Omega_{X_1/S_1} = h^*a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on étale locally. Thus we may assume T_i', X_i', B_2, and B_1 are schemes and in this case the lemma follows from More on Morphisms, Lemma 37.9.10. Alternative proof: using Lemma 76.9.2 it suffices to show a certain diagram of sheaves of rings on X_1' is commutative; then argue exactly as in the proof of the aforementioned More on Morphisms, Lemma 37.9.10 to see that this is indeed the case. $\square$

76.18. Infinitesimal deformations of algebraic spaces

06BG The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

06BH **Lemma 76.18.1.** Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings of algebraic spaces over S. Assume that f is flat. Then the following are equivalent

(1) f' is flat and $X = Y \times_{Y'} X'$, and
(2) the canonical map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism.

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $U = X \times_{X'} U'$ and $V = Y \times_{Y'} V'$. According to our definition of a flat morphism of algebraic spaces we see that the induced map $g : U \to V$ is a flat morphism of schemes and that f' is flat if and only if the corresponding morphism $g' : U' \to V'$ is flat. Also, $X = Y \times_{Y'} X'$ if and only if $U = V \times_{V'} V'$. Finally, the map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism if and only if $g^*\mathcal{C}_{V/V'} \to \mathcal{C}_{U/U'}$ is an isomorphism. Hence the lemma follows from its analogue for morphisms of schemes, see More on Morphisms, Lemma 37.10.1. $\square$

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 76.23.

0CG5 **Lemma 76.18.2.** Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "(f,f')"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume

1. X' is flat over B',
2. f is flat,
3. $B \subset B'$ is a finite order thickening, and
4. $X = B \times_{B'} X'$ and $Y = B \times_{B'} Y'$.

Then f' is flat and Y' is flat over B' at all points in the image of f'.

Proof. Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of schemes to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. See More on Morphisms, Lemma 37.10.2. The statement about flatness of Y'/B' at points in the image of f' follows in the same manner. $\square$

Many properties of morphisms of schemes are preserved under flat deformations.

0CG6 **Lemma 76.18.3.** Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "(f,f')"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume $B \subset B'$ is a finite order thickening, X' flat over B', $X = B \times_{B'} X'$, and $Y = B \times_{B'} Y'$. Then

0CG7 (1) f is representable if and only if f' is representable,
0CG8 (2) f is flat if and only if f' is flat,
0CG9 (3) f is an isomorphism if and only if f' is an isomorphism,
0CGA (4) f is an open immersion if and only if f' is an open immersion,
0CGB (5) f is quasi-compact if and only if f' is quasi-compact,
0CGC (6) f is universally closed if and only if f' is universally closed,
0CGD (7) f is (quasi-)separated if and only if f' is (quasi-)separated,
0CGE (8) f is a monomorphism if and only if f' is a monomorphism,
0CGF (9) f is surjective if and only if f' is surjective,
0CGG (10) f is universally injective if and only if f' is universally injective,

0CGH (11) f is affine if and only if f' is affine,
0CGI (12) f is locally of finite type if and only if f' is locally of finite type,
0CGJ (13) f is locally quasi-finite if and only if f' is locally quasi-finite,
0CGK (14) f is locally of finite presentation if and only if f' is locally of finite presentation,
0CGL (15) f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,
0CGM (16) f is universally open if and only if f' is universally open,
0CGN (17) f is syntomic if and only if f' is syntomic,
0CGP (18) f is smooth if and only if f' is smooth,
0CGQ (19) f is unramified if and only if f' is unramified,
0CGR (20) f is étale if and only if f' is étale,
0CGS (21) f is proper if and only if f' is proper,
0CGT (22) f is integral if and only if f' is integral,
0CGU (23) f is finite if and only if f' is finite,
0CGV (24) f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and
(25) add more here.

Proof. Case (1) follows from Lemma 76.10.1.

Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Consider the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. For any of the properties which are étale local on the source-and-target the result follows immediately from the corresponding result for morphisms of thickenings of schemes applied to the diagram above. Thus cases (2), (12), (13), (14), (15), (17), (18), (19), (20) follow from the corresponding cases of More on Morphisms, Lemma 37.10.3.

Since $X \to X'$ and $Y \to Y'$ are universal homeomorphisms we see that any question about the topology of the maps $X \to Y$ and $X' \to Y'$ has the same answer. Thus we see that cases (5), (6), (9), (10), and (16) hold.

In each of the remaining cases we only prove the implication f has $P \Rightarrow f'$ has P since the other implication follows from the fact that P is stable under base change, see Spaces, Lemma 65.12.3 and Morphisms of Spaces, Lemmas 67.4.4, 67.10.5, 67.20.5, 67.40.3, 67.45.5, and 67.46.5.

The case (4). Assume f is an open immersion. Then f' is étale by (20) and universally injective by (10) hence f' is an open immersion, see Morphisms of Spaces, Lemma 67.51.2. You can avoid using this lemma at the cost of first using (1) to reduce to the case of schemes.

The case (3). Follows from cases (4) and (9).

The case (7). See Lemma 76.10.1.

The case (8). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $B \to B'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (3) we conclude that $\Delta_{X'/Y'}$ is an isomorphism and hence f' is a monomorphism.

The case (11). See Lemma 76.10.1.

The case (21). See Lemma 76.10.2.

The case (22). See Lemma 76.10.1.

The case (23). See Lemma 76.10.2.

The case (24). Assume f finite locally free. By (23) we see that f' is finite. By (2) we see that f' is flat. By (14) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms of Spaces, Lemma 67.46.6. □

The following lemma is the "locally nilpotent" version of the "critère de platitude par fibres", see Section 76.23.

0CGW Lemma 76.18.4. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume

(1) $Y' \to B'$ is locally of finite type,
(2) $X' \to B'$ is flat and locally of finite presentation,
(3) f is flat, and
(4) $X = B \times_{B'} X'$ and $Y = B \times_{B'} Y'$.

Then f' is flat and for all $y' \in |Y'|$ in the image of $|f'|$ the morphism $Y' \to B'$ is flat at y'.

Proof. Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of schemes to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. See More on Morphisms, Lemma 37.10.4. The statement about flatness of Y'/B' at points in the image of f' follows in the same manner. □

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

0CGX **Lemma 76.18.5.** Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[rd] & & (Y \subset Y') \arrow[ld] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume $Y' \to B'$ locally of finite type, $X' \to B'$ flat and locally of finite presentation, $X = B \times_{B'} X'$, and $Y = B \times_{B'} Y'$. Then

0CGY (1) f is representable if and only if f' is representable,
0CGZ (2) f is flat if and only if f' is flat,
0CH0 (3) f is an isomorphism if and only if f' is an isomorphism,
0CH1 (4) f is an open immersion if and only if f' is an open immersion,
0CH2 (5) f is quasi-compact if and only if f' is quasi-compact,
0CH3 (6) f is universally closed if and only if f' is universally closed,
0CH4 (7) f is (quasi-)separated if and only if f' is (quasi-)separated,
0CH5 (8) f is a monomorphism if and only if f' is a monomorphism,
0CH6 (9) f is surjective if and only if f' is surjective,
0CH7 (10) f is universally injective if and only if f' is universally injective,
0CH8 (11) f is affine if and only if f' is affine,
0CH9 (12) f is locally quasi-finite if and only if f' is locally quasi-finite,
0CHA (13) f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,
0CHB (14) f is universally open if and only if f' is universally open,
0CHC (15) f is syntomic if and only if f' is syntomic,
0CHD (16) f is smooth if and only if f' is smooth,
0CHE (17) f is unramified if and only if f' is unramified,
0CHF (18) f is étale if and only if f' is étale,
0CHG (19) f is proper if and only if f' is proper,
0CHH (20) f is finite if and only if f' is finite,
0CHI (21) f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and
(22) add more here.

Proof. Case (1) follows from Lemma 76.10.1.

Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Consider the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[rd] & & (V \subset V') \arrow[ld] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. For any of the properties which are étale local on the source-and-target the result follows immediately from the corresponding result for morphisms of thickenings of schemes applied to the diagram above. Thus cases (2), (12), (13), (15), (16), (17), (18) follow from the corresponding cases of More on Morphisms, Lemma 37.10.5.

Since $X \to X'$ and $Y \to Y'$ are universal homeomorphisms we see that any question about the topology of the maps $X \to Y$ and $X' \to Y'$ has the same answer. Thus we see that cases (5), (6), (9), (10), and (14) hold.

In each of the remaining cases we only prove the implication f has $P \Rightarrow f'$ has P since the other implication follows from the fact that P is stable under base change, see Spaces, Lemma 65.12.3 and Morphisms of Spaces, Lemmas 67.4.4, 67.10.5, 67.20.5, 67.40.3, 67.45.5, and 67.46.5.

The case (4). Assume f is an open immersion. Then f' is étale by (18) and universally injective by (10) hence f' is an open immersion, see Morphisms of Spaces, Lemma 67.51.2. You can avoid using this lemma at the cost of first using (1) to reduce to the case of schemes.

The case (3). Follows from cases (4) and (9).

The case (7). See Lemma 76.10.1.

The case (8). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $B \to B'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (3) we conclude that $\Delta_{X'/Y'}$ is an isomorphism and hence f' is a monomorphism.

The case (11). See Lemma 76.10.1.

The case (19). See Lemma 76.10.3.

The case (20). See Lemma 76.10.3.

The case (21). Assume f finite locally free. By (20) we see that f' is finite. By (2) we see that f' is flat. Also f' is locally finite presentation by Morphisms of Spaces, Lemma 67.28.9. Hence f' is finite locally free by Morphisms of Spaces, Lemma 67.46.6. □

76.19. Formally smooth morphisms

049R In this section we introduce the notion of a formally smooth morphism $X \to Y$ of algebraic spaces. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 76.19.6. It turns out that this criterion is often easier to use than the Jacobian criterion.

060G **Definition 76.19.1.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally smooth* if it is formally smooth as a transformation of functors as in Definition 76.13.1.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 76.14.3 and 76.16.3. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow étale locally on T'. In fact, analyzing the proof of Lemma 76.19.6 shows that this would be equivalent to the definition as it currently stands. It is also true that requiring the existence of the dotted arrow fppf locally on T' would be sufficient, but that is slightly more difficult to prove.

We will not restate the results proved in the more general setting of formally smooth transformations of functors in Section 76.13.

061E Lemma 76.19.2. A composition of formally smooth morphisms is formally smooth.

Proof. Omitted. $\square$

061F Lemma 76.19.3. A base change of a formally smooth morphism is formally smooth.

Proof. Omitted, but see Algebra, Lemma 10.138.2 for the algebraic version. $\square$

061G Lemma 76.19.4. Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.

Proof. Omitted. $\square$

Here is a helper lemma which will be superseded by Lemma 76.19.10.

061H Lemma 76.19.5. Let S be a scheme. Let

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. If the vertical arrows are étale and f is formally smooth, then ψ is formally smooth.

Proof. By Lemma 76.13.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. By Lemma 76.13.3 the composition $U \to Y$ is formally smooth. By Lemma 76.13.8 we see $\psi : U \to V$ is formally smooth. $\square$

The following lemma is the main result of this section. It implies, combined with Limits of Spaces, Proposition 70.3.10, that we can recognize whether a morphism of algebraic spaces $f : X \to Y$ is smooth in terms of "simple" properties of the transformation of functors $X \to Y$.

04AM Lemma 76.19.6 (Infinitesimal lifting criterion). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) The morphism f is smooth.
(2) The morphism f is locally of finite presentation, and formally smooth.

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. By Lemma 76.19.5 we see $\psi : U \to V$ is formally smooth. By Morphisms of Spaces, Lemma 67.28.4 the morphism ψ is locally of finite presentation. Hence by the case of schemes the morphism ψ is smooth, see More on Morphisms, Lemma 37.11.7. Hence f is smooth, see Morphisms of Spaces, Lemma 67.37.4.

Conversely, assume that $f : X \to Y$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 76.19.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 76.17.4 as in the special case discussed in Remark 76.17.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{spaces,étale}$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 76.17.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 21.4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 21.4.3.

First we prove (I). To see this choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. As f is assumed smooth we see that ψ is smooth and hence formally smooth by Lemma 76.13.5. By the same lemma the morphism $V \to Y$ is formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 76.13.6 part (4).

Finally we prove (II). By Lemma 76.7.15 we see that $\Omega_{X/S}$ is of finite presentation. Hence $a^*\Omega_{X/S}$ is of finite presentation (see Properties of Spaces, Section 66.30). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$ is quasi-coherent by Properties of Spaces, Lemma 66.29.7. Thus by Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2 we have

$$H^1(T_{spaces,étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}) = H^1(T, \mathcal{H}) = 0$$

as desired. $\square$

Smooth morphisms satisfy strong local lifting property, see Lemma 76.19.7. If in the lemma we assume T' is affine, then we do not know if it is necessary to take an étale covering. More precisely, if we have a commutative diagram

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] \\ Y & T' \arrow[l] \arrow[lu, dotted] \end{tikzcd}$$

of algebraic spaces where $X \to Y$ is smooth and $T \to T'$ is a thickening of affine schemes, the does a dotted arrow making the diagram commute always exist? If you know the answer, or if you have a reference, please email stacks.project@gmail.com.

0CHJ **Lemma 76.19.7.** Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] \\ Y & T' \arrow[l] \end{tikzcd}$$

of algebraic spaces over S where $X \to Y$ is smooth and $T \to T'$ is a thickening. Then there exists an étale covering $\{T_i' \to T'\}$ such that we can find the dotted arrow in

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] & T \times_{T'} T_i' \arrow[l] \arrow[d] \\ Y & T' \arrow[l] & T_i' \arrow[l] \arrow[llu, dotted] \end{tikzcd}$$

making the diagram commute (for all i).

Proof. Choose an étale covering $\{Y_i \to Y\}$ with each Y_i affine. After replacing T' by the induced étale covering we may assume Y is affine.

Assume Y is affine. Choose an étale covering $\{X_i \to X\}$. This gives rise to an étale covering of T. This étale covering of T comes from an étale covering of T' (by Theorem 76.8.1, see discussion in Section 76.9). Hence we may assume X is affine.

Assume X and Y are affine. We can do one more étale covering of T' and assume T' is affine. In this case the lemma follows from Algebra, Lemma 10.138.17. $\square$

We do a bit more work to show that being formally smooth is étale local on the source. To begin we show that a formally smooth morphism has a nice sheaf of differentials. The notion of a locally projective quasi-coherent module is defined in Properties of Spaces, Section 66.31.

061I **Lemma 76.19.8.** Let S be a scheme. Let $f : X \to Y$ be a formally smooth morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is locally projective on X.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are affine(!) schemes and the vertical arrows are étale. By Lemma 76.19.5 we see $\psi : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see More on Morphisms, Lemma 37.11.6. Hence by Algebra, Lemma 10.138.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 28.21. Since $\Omega_{X/Y}|_U = \Omega_{U/V}$ we see that $\Omega_{X/Y}$ is locally projective too. (Because we can find an étale covering of X by the affine U's fitting into diagrams as above – details omitted.) $\square$

061J **Lemma 76.19.9.** Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules on $T_{étale}$. Consider the internal hom sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$ on $T_{étale}$. If $\mathcal{F}$ is locally projective, then $H^1(T_{étale}, \mathcal{H}) = 0$.

Proof. By the definition of a locally projective sheaf on an algebraic space (see Properties of Spaces, Definition 66.31.2) we see that $\mathcal{F}_{Zar} = \mathcal{F}|_{T_{Zar}}$ is a locally projective sheaf on the scheme T. Thus $\mathcal{F}_{Zar}$ is a direct summand of a free $\mathcal{O}_{T_{Zar}}$-module. Whereupon we conclude (as $\mathcal{F} = (\mathcal{F}_{Zar})^a$, see Descent, Proposition 35.8.9) that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module on $T_{étale}$. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology on Sites, Lemma 21.12.5 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. □

061K **Lemma 76.19.10.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is formally smooth,
(2) for every diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally smooth (as in More on Morphisms, Definition 37.6.1), and
(3) for one such diagram with surjective vertical arrows the morphism ψ is formally smooth.

Proof. We have seen that (1) implies (2) and (3) in Lemma 76.19.5. Assume (3). The proof that f is formally smooth is entirely similar to the proof of (1) $\Rightarrow$ (2) of Lemma 76.19.6.

Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 76.19.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 76.17.4 as in the special case discussed in Remark 76.17.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{spaces,étale}$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 76.17.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 21.4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 21.4.3.

First we prove (I). To see this consider a diagram (which exists because we are assuming (3))

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale and surjective, and ψ is formally smooth. By Lemma 76.13.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 76.13.6 part (4).

Finally we prove (II). By Lemma 76.19.8 we see that $\Omega_{U/V}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Descent on Spaces, Lemma 74.6.5. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties of Spaces, Lemma 66.31.3. Hence

$$H^1(T_{étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 76.19.9 as desired. □

06CS Lemma 76.19.11. The property $\mathcal{P}(f) =$"f is formally smooth" is fpqc local on the base.

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. Choose an index set I and diagrams

$$\begin{tikzcd} U_i \arrow[r, "\psi_i"'] \arrow[d] & V_i \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with étale vertical arrows and U_i, V_i affine schemes. Moreover, assume that $\coprod U_i \to X$ and $\coprod V_i \to Y$ are surjective, see Properties of Spaces, Lemma 66.6.1. By Lemma 76.19.10 we see that f is formally smooth if and only if each of the morphisms ψ_i are formally smooth. Hence we reduce to the case of a morphism of affine schemes. In this case the result follows from Algebra, Lemma 10.138.16. Some details omitted. □

06BI Lemma 76.19.12. Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume f is formally smooth. Then

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

Lemma 76.7.8 is short exact.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 37.11.11, by étale localization, see Lemmas 76.19.10 and 76.7.3. □

06BJ Lemma 76.19.13. Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Assume that Z is formally smooth over B. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 76.15.13 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 76.15.13 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z \to \Omega_{Z/B} \to 0.$$

Since $Z \to S$ is formally smooth we can étale locally on Z' find a left inverse $Z' \to Z$ over B to the inclusion map $Z \to Z'$. Thus the sequence is étale locally split, see Lemma 76.7.11. □

06BK Lemma 76.19.14. Let S be a scheme. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 76.15.14 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 76.15.13 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"] \arrow[dr, "j'"'] & Z' \arrow[r, "a"] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

The sequence above is identified with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

via our definitions concerning conormal sheaves of formally unramified morphisms. Let $U'' \to Z''$ be an étale morphism with U'' affine. Denote $U \to Z$ and $U' \to Z'$ the corresponding affine schemes étale over Z and Z'. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ur, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. □

76.20. Smoothness over a Noetherian base

0APM This section is the analogue of More on Morphisms, Section 37.12.

0APN **Lemma 76.20.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume that Y is locally Noetherian and f locally of finite type. The following are equivalent:

(1) f is smooth at x,
(2) for every solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a surjection of local rings with $\mathrm{Ker}(B' \to B)$ of square zero, and α mapping the closed point of $\mathrm{Spec}(B)$ to x there exists a dotted arrow making the diagram commute, and
(3) same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 10.141.1).

Proof. Condition (1) means there is an open subspace $X' \subset X$ such that $X' \to Y$ is smooth. Hence (1) implies conditions (2) and (3) by Lemma 76.19.6. Condition (2) implies condition (3) trivially. Assume (3). Choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \\ Y & V \arrow[l] \end{tikzcd}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Next, consider a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & V \arrow[l] & \mathrm{Spec}(B') \arrow[l, "\beta"] \end{tikzcd}$$

as in (3) but for $u \in U \to V$. Let $\gamma : \mathrm{Spec}(B') \to X$ be the arrow we get from our assumption that (3) holds for X. Because $U \to X$ is étale and hence formally étale (Lemma 76.16.8) the morphism γ has a unique lift to U compatible with α. Then because $V \to Y$ is étale hence formally étale this lift is compatible with β. Hence (3) holds for $u \in U \to V$ and we conclude that $U \to V$ is smooth at u by More on Morphisms, Lemma 37.12.1. This proves that $X \to Y$ is smooth at x, thereby finishing the proof. $\square$

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms of Spaces, Section 67.25).

0APP **Lemma 76.20.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. The following are equivalent:

(1) f is smooth,
(2) for every solid commutative diagram

$$\begin{tikzcd} X & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y \arrow[from=1-1, d, "f"'] & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 76.19.6) says f is formally smooth and (2) holds.

Assume f is not smooth. The set of points $x \in X$ where f is not smooth forms a closed subset T of $|X|$. By Morphisms of Spaces, Lemma 67.25.6, there exists a point $x \in T \subset X$ with $x \in X_{\text{ft-pts}}$. Choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & u \arrow[d, mapsto] \\ Y & V \arrow[l] & v \end{tikzcd}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Then u is a finite type point of U. Since $U \to V$ is not smooth at the point u, by More on Morphisms, Lemma 37.12.1 there is a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & V \arrow[l] & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

with $B' \to B$ a small extension of (Artinian) local rings such that the residue field of B is equal to $\kappa(v)$ and such that the dotted arrow does not exist. Since $U \to V$ is of finite type, we see that v is a finite type point of V. By Morphisms, Lemma 29.16.2 the morphism β is of finite type, hence the composition $\mathrm{Spec}(B) \to Y$ is of finite type also. Arguing exactly as in the proof of Lemma 76.20.1 (using that $U \to X$ and $V \to Y$ are étale hence formally étale) we see that there cannot be an arrow $\mathrm{Spec}(B) \to X$ fitting into the outer rectangle of the last displayed diagram. In other words, (2) doesn't hold and the proof is complete. $\square$

Here is a useful application.

0APQ Lemma 76.20.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Let $Z \subset Y$ be a closed subspace with nth infinitesimal neighbourhood $Z_n \subset Y$. Set $X_n = Z_n \times_Y X$.

(1) If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.
(2) If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 76.20.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. □

76.21. The naive cotangent complex

0D0U This section is the continuation of Modules on Sites, Section 18.35 which in turn continues the discussion in Algebra, Section 10.134.

0D0V Definition 76.21.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The naive cotangent complex of f is the complex defined in Modules on Sites, Definition 18.35.4 for the morphism of ringed topoi f_{small} between the small étale sites of X and Y, see Properties of Spaces, Lemma 66.21.3. Notation: NL_f or $NL_{X/Y}$.

The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $NL_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

0D0W Lemma 76.21.2. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of algebraic spaces over S with p and q étale. Then there is a canonical identification $NL_{X/Y}|_{U_{étale}} = NL_{U/V}$ in $D(\mathcal{O}_U)$.

Proof. Formation of the naive cotangent complex commutes with pullback (Modules on Sites, Lemma 18.35.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{étale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{étale}} = \mathcal{O}_{V_{étale}}$ by Properties of Spaces, Lemma 66.26.1. Tracing through the definitions we conclude that $NL_{X/Y}|_{U_{étale}} = NL_{U/V}$. □

0D0X Lemma 76.21.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y representable by schemes X_0 and Y_0. Then there is a canonical identification $NL_{X/Y} = \epsilon^* NL_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where ϵ is as in Derived Categories of Spaces, Section 75.4 and NL_{X_0/Y_0} is as in More on Morphisms, Definition 37.13.1.

Proof. Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to f. There is a canonical map $\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0} \to f_{small}^{-1}\mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1}\mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$\begin{tikzcd} X_{0,Zar} \arrow[d, "f_0"'] & X_{étale} \arrow[l, "\epsilon"'] \arrow[d, "f"] \\ Y_{0,Zar} & Y_{étale} \arrow[l, "\epsilon"] \end{tikzcd}$$

see Derived Categories of Spaces, Remark 75.6.3. Thus we obtain a canonical map

$$\epsilon^{-1} NL_{X_0/Y_0} = \epsilon^{-1} NL_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = NL_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \to NL_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = NL_{X/Y}$$

by functoriality of the naive cotangent complex. To see that the induced map $\epsilon^* NL_{X_0/Y_0} \to NL_{X/Y}$ is an isomorphism in $D(\mathcal{O}_X)$ we may check on stalks at geometric points (Properties of Spaces, Theorem 66.19.12). Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$NL_{X/Y,\overline{x}} = NL_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

This is true because taking stalks at $\overline{x}$ is the same as taking inverse image via $\overline{x} : \mathrm{Spec}(k) \to X$ and we may apply Modules on Sites, Lemma 18.35.3. On the other hand we have

$$(\epsilon^* NL_{X_0/Y_0})_{\overline{x}} = NL_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = NL_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 7.34.2, as well as the corresponding result for modules, see Modules on Sites, Lemma 18.36.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 66.22.1). Thus the result follows from More on Algebra, Lemma 15.33.8. □

0D0Y Lemma 76.21.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees -1, 0 and equal to $\Omega_{X/Y}$ in degree 0.

Proof. By construction of the naive cotangent complex in Modules on Sites, Section 18.35 we have that $NL_{X/Y}$ is a complex sitting in degrees -1, 0 and that its cohomology in degree 0 is $\Omega_{X/Y}$ (by our construction of $\Omega_{X/Y}$ in Section 76.7). The sheaf of differentials is quasi-coherent (by Lemma 76.7.4). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking étale locally (allowed by Lemma 76.21.2 and Properties of Spaces, Lemma 66.29.6) reducing to the case of schemes (Lemma 76.21.3) and finally using the result in the case of schemes (More on Morphisms, Lemma 37.13.3). □

0D0Z Lemma 76.21.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite presentation, then $NL_{X/Y}$ is étale locally on X quasi-isomorphic to a complex

$$\ldots \to 0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to 0 \to \ldots$$

of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}^0$ of finite presentation and $\mathcal{F}^{-1}$ of finite type.

Proof. Formation of the naive cotangent complex commutes with étale localization by Lemma 76.21.2. This reduces us to the case of schemes by Lemma 76.21.3. The result in the case of schemes is More on Morphisms, Lemma 37.13.4. □

0D10 Lemma 76.21.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is formally smooth,
(2) $H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.

Proof. This follows from Lemma 76.19.10, Lemma 76.21.2, Lemma 76.21.3 and the case of schemes which is More on Morphisms, Lemma 37.13.5. □

0D11 Lemma 76.21.7. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is formally étale,

(2) $H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.

Proof. Assume (1). A formally étale morphism is a formally smooth morphism. Thus $H^{-1}(NL_{X/Y}) = 0$ by Lemma 76.21.6. On the other hand, a formally étale morphism if formally unramified hence we have $\Omega_{X/Y} = 0$ by Lemma 76.14.6. Conversely, if (2) holds, then f is formally smooth by Lemma 76.21.6 and formally unramified by Lemma 76.14.6 and hence formally étale by Lemmas 76.19.4. $\square$

0D12 Lemma 76.21.8. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is smooth, and
(2) f is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.

Proof. This follows from Lemma 76.19.10, Lemma 76.21.2, Lemma 76.21.3 and the case of schemes which is More on Morphisms, Lemma 37.13.7. $\square$

76.22. Openness of the flat locus

05WU This section is analogue of More on Morphisms, Section 37.15. Note that we have defined the notion of flatness for quasi-coherent modules on algebraic spaces in Morphisms of Spaces, Section 67.31.

05WV Theorem 76.22.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume f is locally of finite presentation and that $\mathcal{F}$ is an $\mathcal{O}_X$-module which is locally of finite presentation. Then

$$\{x \in |X| : \mathcal{F} \text{ is flat over } Y \text{ at } x\}$$

is open in $|X|$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "p"] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

with U, V schemes and p, q surjective and étale as in Spaces, Lemma 65.11.6. By More on Morphisms, Theorem 37.15.1 the set $U' = \{u \in |U| : p^*\mathcal{F} \text{ is flat over } V \text{ at } u\}$ is open in U. By Morphisms of Spaces, Definition 67.31.2 the image of U' in $|X|$ is the set of the theorem. Hence we are done because the map $|U| \to |X|$ is open, see Properties of Spaces, Lemma 66.4.6. $\square$

05WW Lemma 76.22.2. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation. Then

$$\{x' \in |X'| : (g')^*\mathcal{F} \text{ is flat over } Y' \text{ at } x'\}$$

is the inverse image of the open subset of Theorem 76.22.1 under the continuous map $|g'| : |X'| \to |X|$.

Proof. This follows from Morphisms of Spaces, Lemma 67.31.3. $\square$

76.23. Critère de platitude par fibres

05WX Let S be a scheme. Consider a commutative diagram of algebraic spaces over S

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in |X|$ we consider the question as to whether $\mathcal{F}$ is flat over Y at x. If $\mathcal{F}$ is flat over Z at x, then the theorem below states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre of $X \to Z$ over $g(x)$ is flat over the fibre of $Y \to Z$ over $g(x)$. To make sense out of this we offer the following preliminary lemma.

05WY **Lemma 76.23.1.** In the situation above the following are equivalent

1. Pick a geometric point $\overline{x}$ of X lying over x. Set $\overline{y} = f \circ \overline{x}$ and $\overline{z} = g \circ \overline{x}$. Then the module $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}$.
2. Pick a morphism $x : \mathrm{Spec}(K) \to X$ in the equivalence class of x. Set $z = g \circ x$, $X_z = \mathrm{Spec}(K) \times_{z,Z} X$, $Y_z = \mathrm{Spec}(K) \times_{z,Z} Y$, and $\mathcal{F}_z$ the pullback of $\mathcal{F}$ to X_z. Then $\mathcal{F}_z$ is flat at x over Y_z (as defined in Morphisms of Spaces, Definition 67.31.2).
3. Pick a commutative diagram

$$\begin{tikzcd} & & & U \arrow[llld, "a"'] \arrow[rr] \arrow[rd] & & V \arrow[llld, "b"'] \arrow[ld] \\ X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] & & W \arrow[lllld, "c"'] & \\ & Z & & & & \end{tikzcd}$$

where U, V, W are schemes, and a, b, c are étale, and a point $u \in U$ mapping to x. Let $w \in W$ be the image of u. Let $\mathcal{F}_w$ be the pullback of $\mathcal{F}$ to the fibre U_w of $U \to W$ at w. Then $\mathcal{F}_w$ is flat over V_w at u.

Proof. Note that in (2) the morphism $x : \mathrm{Spec}(K) \to X$ defines a K-rational point of X_z, hence the statement makes sense. Moreover, the condition in (2) is independent of the choice of $\mathrm{Spec}(K) \to X$ in the equivalence class of x (details omitted; this will also follow from the arguments below because the other conditions do not depend on this choice). Also note that we can always choose a diagram as in (3) by: first choosing a scheme W and a surjective étale morphism $W \to Z$, then choosing a scheme V and a surjective étale morphism $V \to W \times_Z Y$, and finally choosing a scheme U and a surjective étale morphism $U \to V \times_Y X$. Having made these choices we set $U \to W$ equal to the composition $U \to V \to W$ and we can pick a point $u \in U$ mapping to x because the morphism $U \to X$ is surjective.

Suppose given both a diagram as in (3) and a geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ as in (1). By Properties of Spaces, Lemma 66.19.4 we can choose a geometric point

$\overline{u} : \mathrm{Spec}(k) \to U$ lying over u such that $\overline{x} = a \circ \overline{u}$. Denote $\overline{v} : \mathrm{Spec}(k) \to V$ and $\overline{w} : \mathrm{Spec}(k) \to W$ the induced geometric points of V and W. In this setting we know that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ and similarly for Y and Z, see Properties of Spaces, Lemma 66.22.1. In the same vein we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}^{sh}_{U,u}$$

see Properties of Spaces, Lemma 66.29.4. Note that the stalk of $\mathcal{F}_w$ at u is given by

$$(\mathcal{F}_w)_u = (a^*\mathcal{F})_u / \mathfrak{m}_w (a^*\mathcal{F})_u$$

and the local ring of V_w at v is given by

$$\mathcal{O}_{V_w,v} = \mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v}.$$

Since $\mathfrak{m}_{\overline{z}} = \mathfrak{m}_w\mathcal{O}_{Z,\overline{z}} = \mathfrak{m}_w\mathcal{O}^{sh}_{W,w}$ we see that

$$\begin{aligned}
\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}} &= (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U,u}/\mathfrak{m}_w\mathcal{O}^{sh}_{U,u} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U_w,\overline{u}} \\
&= (\mathcal{F}_w)_{\overline{u}}
\end{aligned}$$

the penultimate equality by Algebra, Lemma 10.156.4 and the last equality by Properties of Spaces, Lemma 66.29.4. The same arguments applied to the structure sheaves of V and Y show that

$$\mathcal{O}^{sh}_{V_w,\overline{v}} = \mathcal{O}^{sh}_{V,v}/\mathfrak{m}_w\mathcal{O}^{sh}_{V,v} = \mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}.$$

OK, and now we can use Morphisms of Spaces, Lemma 67.31.1 to see that (1) is equivalent to (3).

Finally we prove the equivalence of (2) and (3). To do this we pick a field extension $\tilde{K}$ of K and a morphism $\tilde{x} : \mathrm{Spec}(\tilde{K}) \to U$ which lies over u (this is possible because $u \times_{X,x} \mathrm{Spec}(K)$ is a nonempty scheme). Set $\tilde{z} : \mathrm{Spec}(\tilde{K}) \to U \to W$ be the composition. We obtain a commutative diagram

$$\begin{tikzcd}
 & & & U_w \times_w \tilde{z} \arrow[rr] \arrow[dlll, "a"'] \arrow[dr] & & V_w \times_w \tilde{z} \arrow[dlll, "b"'] \arrow[dl] \\
X_z \arrow[rr, "f"] \arrow[dr, "g"'] & & Y_z \arrow[dl, "h"] & & \tilde{z} \arrow[dlll, "c"] & \\
 & z & & & &
\end{tikzcd}$$

where $z = \mathrm{Spec}(K)$ and $w = \mathrm{Spec}(\kappa(w))$. Now it is clear that $\mathcal{F}_w$ and $\mathcal{F}_z$ pull back to the same module on $U_w \times_w \tilde{z}$. This leads to a commutative diagram

$$\begin{tikzcd}
X_z \arrow[d] & U_w \times_w \tilde{z} \arrow[l] \arrow[r] \arrow[d] & U_w \arrow[d] \\
Y_z & V_w \times_w \tilde{z} \arrow[l] \arrow[r] & V_w
\end{tikzcd}$$

both of whose squares are cartesian and whose bottom horizontal arrows are flat: the lower left horizontal arrow is the composition of the morphism $Y \times_Z \tilde{z} \to Y \times_Z z = Y_z$ (base change of a flat morphism), the étale morphism $V \times_Z \tilde{z} \to Y \times_Z \tilde{z}$, and the étale morphism $V \times_W \tilde{z} \to V \times_Z \tilde{z}$. Thus it follows from Morphisms of Spaces, Lemma 67.31.3 that

$$\mathcal{F}_z \text{ flat at } x \text{ over } Y_z \Leftrightarrow \mathcal{F}|_{U_w \times_w \tilde{z}} \text{ flat at } \tilde{x} \text{ over } V_w \times_w \tilde{z} \Leftrightarrow \mathcal{F}_w \text{ flat at } u \text{ over } V_w$$

and we win. $\square$

05WZ Definition 76.23.2. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and denote $z \in |Z|$ its image.

(1) We say the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z if the equivalent conditions of Lemma 76.23.1 are satisfied.
(2) We say the fibre of X over z is flat at x over the fibre of Y over z if the equivalent conditions of Lemma 76.23.1 hold with $\mathcal{F} = \mathcal{O}_X$.
(3) We say the fibre of X over z is flat over the fibre of Y over z if for all $x \in |X|$ lying over z the fibre of X over z is flat at x over the fibre of Y over z

With this definition in hand we can state a version of the criterion as follows. The Noetherian version can be found in Section 76.24.

05X0 Theorem 76.23.3. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) X is locally of finite presentation over Z,
(2) $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and
(3) Y is locally of finite type over Z.

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) $\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and
(2) Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.

Moreover, the set of points x where (1) and (2) hold is open in $\text{Supp}(\mathcal{F})$.

Proof. Choose a diagram as in Lemma 76.23.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 37.16.2. $\square$

05X1 Lemma 76.23.4. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume

(1) X is locally of finite presentation over Z,
(2) X is flat over Z,
(3) for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z, and
(4) Y is locally of finite type over Z.

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 76.23.3. $\square$

05X2 Lemma 76.23.5. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) X is locally of finite presentation over Z,
(2) $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,
(3) $\mathcal{F}$ is flat over Z, and
(4) Y is locally of finite type over Z.

Then the set

$$A = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change: If $Z' \to Z$ is a morphism of algebraic spaces, and A' is the set of points of $X' = X \times_Z Z'$ where $\mathcal{F}' = \mathcal{F} \times_Z Z'$ is flat over $Y' = Y \times_Z Z'$, then A' is the inverse image of A under the continuous map $|X'| \to |X|$.

Proof. One way to prove this is to translate the proof as given in More on Morphisms, Lemma 37.16.4 into the category of algebraic spaces. Instead we will prove this by reducing to the case of schemes. Namely, choose a diagram as in Lemma 76.23.1 part (3) such that a, b, and c are surjective. It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. The only minor point to make is that given a morphism of algebraic spaces $Z' \to Z$ we choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Then we set $U' = W' \times_W U$ and $V' = W' \times_W V$. We write a', b', c' for the morphisms from U', V', W' to X', Y', Z'. In this case A, resp. A' are images of the open subsets of U, resp. U' associated to $a^*\mathcal{F}$, resp. $(a')^*\mathcal{F}'$. This indeed does reduce the lemma to More on Morphisms, Lemma 37.16.4. □

05X3 Lemma 76.23.6. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume

(1) X is locally of finite presentation over Z,
(2) X is flat over Z, and
(3) Y is locally of finite type over Z.

Then the set

$$\{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change $Z' \to Z$.

Proof. This is a special case of Lemma 76.23.5. □

0CZS Lemma 76.23.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in |X|$ with image $y \in |Y|$. If $\mathcal{F}$ is flat at x over Y, then the following are equivalent

(1) $(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a flat $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module,
(2) $(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a free $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module,
(3) $\mathcal{F}_{\overline{y}}$ is finite free in an étale neighbourhood of $\overline{x}$ in $X_{\overline{y}}$, and
(4) $\mathcal{F}$ is finite free in an étale neighbourhood of x in X.

Here $\overline{x}$ is a geometric point of X lying over x and $\overline{y} = f \circ \overline{x}$.

Proof. Pick a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale such that there is a point $u \in U$ mapping to x. Let $v \in V$ be the image of u. Applying Lemma 76.23.1 to $\mathrm{id} : X \to X$ over Y we see that (1) translates into the condition "$\mathcal{F}|_{U_v}$ is flat over U_v at u". In other words, (1) is equivalent to $(\mathcal{F}|_{U_v})_u$ being a flat $\mathcal{O}_{U_v,u}$-module. By the case of schemes (More on Morphisms, Lemma 37.16.7), we find that this implies that $\mathcal{F}|_U$ is finite free in an open neighbourhood of u. In this way we see that (1) implies (4). The implications (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are immediate. For the implication (3) $\Rightarrow$ (2) use the description of local rings and stalks in Properties of Spaces, Lemmas 66.22.1 and 66.29.4. $\square$

0CZT Lemma 76.23.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over Y. Then the set

$$\{x \in |X| : \mathcal{F} \text{ free in an étale neighbourhood of } x\}$$

is open in $|X|$ and its formation commutes with arbitrary base change $Y' \to Y$.

Proof. Openness holds trivially. Let $Y' \to Y$ be a morphism of algebraic spaces, set $X' = Y' \times_Y X$, and let $x' \in |X'|$ be a point lying over $x \in |X|$. By Lemma 76.23.7 we see that x is in our set if and only if $(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a flat $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module. Simiarly, x' is in the analogue of our set for the pullback $\mathcal{F}'$ of $\mathcal{F}$ to X' if and only if $(\mathcal{F}'_{\overline{y}'})_{\overline{x}'}$ is a flat $\mathcal{O}_{X'_{\overline{y}'},\overline{x}'}$-module (with obvious notation). These two assertions are equivalent by Lemma 76.23.1 applied to the morphism $\mathrm{id} : X \to X$ over Y. Thus the statement on base change holds. $\square$

76.24. Flatness over a Noetherian base

08VN Here is the "Critère de platitude par fibres" in the Noetherian case.

0APR Theorem 76.24.1. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) X, Y, Z locally Noetherian, and
(2) $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) $\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and
(2) Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.

Proof. Choose a diagram as in Lemma 76.23.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 37.16.1. $\square$

0APS Lemma 76.24.2. Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume

(1) X, Y, Z locally Noetherian,
(2) X is flat over Z,
(3) for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z.

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 76.24.1. □

Just like for checking smoothness, if the base is Noetherian it suffices to check flatness over Artinian rings. Here is a sample statement.

08VP Lemma 76.24.3. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let X be an algebraic space locally of finite presentation over $S = \mathrm{Spec}(A)$. For $n \geq 1$ set $S_n = \mathrm{Spec}(A/I^n)$ and $X_n = S_n \times_S X$. Let $\mathcal{F}$ be coherent $\mathcal{O}_X$-module. If for every $n \geq 1$ the pullback $\mathcal{F}_n$ of $\mathcal{F}$ to X is flat over S_n, then the (open) locus where $\mathcal{F}$ is flat over X contains the inverse image of $V(I)$ under $X \to S$.

Proof. The locus where $\mathcal{F}$ is flat over S is open in $|X|$ by Theorem 76.22.1. The statement is insensitive to replacing X by the members of an étale covering, hence we may assume X is an affine scheme. In this case the result follows immediately from Algebra, Lemma 10.99.11. Some details omitted. □

76.25. Normalization revisited

082D Normalization commutes with smooth base change.

082E Lemma 76.25.1. Let S be a scheme. Let $f : Y \to X$ be a smooth morphism of algebraic spaces over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.

Proof. By our construction of the integral closure, see Morphisms of Spaces, Definition 67.48.2, this reduces immediately to the case where X and Y are affine. In this case the result is Algebra, Lemma 10.147.4. □

082F Lemma 76.25.2 (Normalization commutes with smooth base change). Let S be a scheme. Let

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[d, "f"] \\ X_2 \arrow[r, "\varphi"] & X_1 \end{tikzcd}$$

be a fibre square of algebraic spaces over S. Assume f is quasi-compact and quasi-separated and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_1$ and $X_2 \times_{X_1} X_1' \to X_1$ is integral (Morphisms of Spaces, Lemma 67.45.5). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms of Spaces, Lemma 67.48.5. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$ as the construction of the relative spectrum commutes with arbitrary base change. By Cohomology of Spaces, Lemma 69.11.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 76.25.1. □

76.26. Cohen-Macaulay morphisms

0E0T This is the analogue of More on Morphisms, Section 37.22.

0E0U **Lemma 76.26.1.** The property of morphisms of germs of schemes

$$\mathcal{P}((X,x) \to (S,s)) = \text{the local ring } \mathcal{O}_{X_s,x} \text{ of the fibre is Noetherian and Cohen-Macaulay}$$

is étale local on the source-and-target (Descent, Definition 35.33.1).

Proof. Given a diagram as in Descent, Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Descent, Lemma 35.33.5. Thus the strict henselizations of the local rings $\mathcal{O}_{U'_{v'},u'}$ and $\mathcal{O}_{U_v,u}$ are the same. We conclude by More on Algebra, Lemma 15.45.9. □

0E0V **Definition 76.26.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian (Divisors on Spaces, Definition 71.4.2).

(1) Let $x \in |X|$, and $y = f(x)$. We say that f is *Cohen-Macaulay at x* if f is flat at x and the equivalent conditions of Morphisms of Spaces, Lemma 67.22.5 hold for the property $\mathcal{P}$ described in Lemma 76.26.1.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

0E0W **Lemma 76.26.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. The following are equivalent

(1) f is Cohen-Macaulay,
(2) f is flat and for some surjective étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Cohen-Macaulay algebraic spaces, and
(3) f is flat and for any étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Cohen-Macaulay algebraic spaces.

Given $x \in |X|$ with image $y \in |Y|$ the following are equivalent

(a) f is Cohen-Macaulay at x, and
(b) $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat and $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$ is Cohen-Macaulay.

Proof. Given an étale morphism $V \to Y$ where V is a scheme choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let $u \in U$ with images $x \in |X|$, $y \in |Y|$, and $v \in V$. Then f is Cohen-Macaulay at x if and only if $U \to V$ is Cohen-Macaulay at u (by definition). Moreover the morphism $U_v \to X_v = (X_V)_v$ is surjective étale. Hence the scheme U_v is Cohen-Macaulay if and only if the algebraic space X_v is Cohen-Macaulay. Thus the equivalence of (1), (2), and (3) follows from the corresponding equivalence for morphisms of schemes, see More on Morphisms, Lemma 37.22.2 by a formal argument.

Proof of equivalence of (a) and (b). The corresponding equivalence for flatness is Morphisms of Spaces, Lemma 67.30.8. Thus we may assume f is flat at x when proving the equivalence. Consider a diagram and x, y, u, v as above. Then $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is equal to the map $\mathcal{O}_{V,v}^{sh} \to \mathcal{O}_{U,u}^{sh}$ on strict henselizations of local rings, see Properties of Spaces, Lemma 66.22.1. Thus we have

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})^{sh}$$

by Algebra, Lemma 10.156.4. Thus we have to show that the Noetherian local ring $\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u}$ is Cohen-Macaulay if and only if its strict henselization is. This is More on Algebra, Lemma 15.45.9. □

0E0X Lemma 76.26.4. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in |X|$ with images $y \in |Y|$ and $z \in |Z|$.

(1) If f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at x.
(2) If f and g are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.
(3) If $g \circ f$ is Cohen-Macaulay at x and f is flat at x, then f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$.
(4) If $f \circ g$ is Cohen-Macaulay and f is flat, then f is Cohen-Macaulay and g is Cohen-Macaulay at every point in the image of f.

Proof. Working étale locally this follows from the corresponding result for schemes, see More on Morphisms, Lemma 37.22.4. Alternatively, we can use the equivalence of (a) and (b) in Lemma 76.26.3. Thus we consider the local homomorphism of Noetherian local rings

$$\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}}$$

whose fibre is

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$$

and we use Algebra, Lemma 10.163.3. □

0E0Y Lemma 76.26.5. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of locally Noetherian algebraic spaces over S. If X is Cohen-Macaulay, then f is Cohen-Macaulay and $\mathcal{O}_{Y,f(\overline{x})}$ is Cohen-Macaulay for all $x \in |X|$.

Proof. After translating into algebra using Lemma 76.26.3 (compare with the proof of Lemma 76.26.4) this follows from Algebra, Lemma 10.163.3. □

0E0Z Lemma 76.26.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in |X'|$ be a point with image $x \in |X|$.

(1) If f is Cohen-Macaulay at x, then $f' : X' \to Y'$ is Cohen-Macaulay at x'.
(2) If f is flat at x and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.
(3) If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.

Proof. Denote $y \in |Y|$ and $y' \in |Y'|$ the image of x'. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. Choose a surjectiev étale morphism $V' \to Y' \times_Y V$ where V' is a scheme. Then $U' = U \times_V V'$ is a scheme which comes equipped with a surjective étale morphism $U' \to X'$. Choose $u' \in U'$ mapping to x'. Denote $u \in U$ the image of u'. Then the lemma follows from the lemma for $U \to V$ and its base change $U' \to V'$ and the points u' and u (this follows from the definitions). Thus the lemma follows from the case of schemes, see More on Morphisms, Lemma 37.22.6. $\square$

0E10 **Lemma 76.26.7.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

Then W is open in $|X|$ and the formation of W commutes with arbitrary base change of f: For any morphism $g : Y' \to Y$, consider the base change $f' : X' \to Y'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows and U and V schemes. Let $u \in U$ with image $x \in |X|$. Then f is Cohen-Macaulay at x if and only if $U \to V$ is Cohen-Macaulay at u (by definition). Thus we reduce to the case of the morphism $U \to V$. See More on Morphisms, Lemma 37.22.7. $\square$

0E11 **Lemma 76.26.8.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is locally of finite presentation and Cohen-Macaulay. Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to Y$ has relative dimension d.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows and U and V schemes. Then $U \to V$ is locally of finite presentation and Cohen-Macaulay (immediate from our definitions). Thus we have a decomposition $U = \coprod_{d \geq 0} U_d$ into open and closed subschemes with $f|_{U_d} : U_d \to V$ of relative dimension d, see Morphisms, Lemma 29.29.4. Let $u \in U$ with image $x \in |X|$. Then f has relative dimension d at x if and only if $U \to V$ has relative dimension d at u (this follows from our definitions). In this way we see that U_d is the inverse image of a subset $X_d \subset |X|$ which is necessarily open and closed. Denoting X_d the corresponding open and closed algebraic subspace of X we see that the lemma is true. $\square$

76.27. Gorenstein morphisms

0E12 This is the analogue of Duality for Schemes, Section 48.25.

0E13 **Lemma 76.27.1.** *The property of morphisms of germs of schemes*

$$\mathcal{P}((X, x) \to (S, s)) = \text{the local ring } \mathcal{O}_{X_s, x} \text{ of the fibre is Noetherian and Gorenstein}$$

is étale local on the source-and-target (Descent, Definition 35.33.1).

Proof. Given a diagram as in Descent, Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Descent, Lemma 35.33.5. Thus $\mathcal{O}_{U_v,u} \to \mathcal{O}_{U'_{v'},u'}$ is the localization of an étale ring map. Hence the first is Noetherian if and only if the second is Noetherian, see More on Algebra, Lemma 15.44.1. Then, since $\mathcal{O}_{U'_{v'},u'}/\mathfrak{m}_u\mathcal{O}_{U'_{v'},u'} = \kappa(u')$ (Algebra, Lemma 10.143.5) is a Gorenstein ring, we see that $\mathcal{O}_{U_v,u}$ is Gorenstein if and only if $\mathcal{O}_{U'_{v'},u'}$ is Gorenstein by Dualizing Complexes, Lemma 47.21.8. $\square$

0E14 **Definition 76.27.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian (Divisors on Spaces, Definition 71.4.2).

(1) Let $x \in |X|$, and $y = f(x)$. We say that f is *Gorenstein at x* if f is flat at x and the equivalent conditions of Morphisms of Spaces, Lemma 67.22.5 hold for the property $\mathcal{P}$ described in Lemma 76.27.1.
(2) We say f is a *Gorenstein morphism* if f is Gorenstein at every point of X.

Here is a translation.

0E15 **Lemma 76.27.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Gorenstein,*
(2) *f is flat and for some surjective étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Gorenstein algebraic spaces, and*
(3) *f is flat and for any étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Gorenstein algebraic spaces.*

Given $x \in |X|$ with image $y \in |Y|$ the following are equivalent

(a) *f is Gorenstein at x, and*
(b) *$\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat and $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$ is Gorenstein.*

Proof. Given an étale morphism $V \to Y$ where V is a scheme choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let $u \in U$ with images $x \in |X|$, $y \in |Y|$, and $v \in V$. Then f is Gorenstein at x if and only if $U \to V$ is Gorenstein at u (by definition). Moreover the morphism $U_v \to X_v = (X_V)_v$ is surjective étale. Hence the scheme U_v is Gorenstein if and only if the algebraic space X_v is Gorenstein. Thus the equivalence of (1), (2), and (3) follows from the corresponding equivalence for morphisms of schemes, see Duality for Schemes, Lemma 48.24.4 by a formal argument.

Proof of equivalence of (a) and (b). The corresponding equivalence for flatness is Morphisms of Spaces, Lemma 67.30.8. Thus we may assume f is flat at x when proving the equivalence. Consider a diagram and x, y, u, v as above. Then $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is equal to the map $\mathcal{O}_{V,v}^{sh} \to \mathcal{O}_{U,u}^{sh}$ on strict henselizations of local rings, see Properties of Spaces, Lemma 66.22.1. Thus we have

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})^{sh}$$

by Algebra, Lemma 10.156.4. Thus we have to show that the Noetherian local ring $\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u}$ is Gorenstein if and only if its strict henselization is. This follows immediately from Dualizing Complexes, Lemma 47.22.3 and the definition of a Gorenstein local ring as a Noetherian local ring which is a dualizing complex over itself. □

0E16 Lemma 76.27.4. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in |X|$ with images $y \in |Y|$ and $z \in |Z|$.

(1) If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x.
(2) If f and g are Gorenstein, then $g \circ f$ is Gorenstein.
(3) If $g \circ f$ is Gorenstein at x and f is flat at x, then f is Gorenstein at x and g is Gorenstein at $f(x)$.
(4) If $f \circ g$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f.

Proof. Working étale locally this follows from the corresponding result for schemes, see Duality for Schemes, Lemma 48.25.6. Alternatively, we can use the equivalence of (a) and (b) in Lemma 76.27.3. Thus we consider the local homomorphism of Noetherian local rings

$$\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}}$$

whose fibre is

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$$

and we use Dualizing Complexes, Lemma 47.21.8. □

0E17 Lemma 76.27.5. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of locally Noetherian algebraic spaces over S. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(\overline{x})}$ is Gorenstein for all $x \in |X|$.

Proof. After translating into algebra using Lemma 76.27.3 (compare with the proof of Lemma 76.27.4) this follows from Dualizing Complexes, Lemma 47.21.8. □

0E18 Lemma 76.27.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in |X'|$ be a point with image $x \in |X|$.

(1) If f is Gorenstein at x, then $f' : X' \to Y'$ is Gorenstein at x'.
(2) If f is flat at x and f' is Gorenstein at x', then f is Gorenstein at x.

(3) If $Y' \to Y$ is flat at $f'(x')$ and f' is Gorenstein at x', then f is Gorenstein at x.

Proof. Denote $y \in |Y|$ and $y' \in |Y'|$ the image of x'. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. Choose a surjectiev étale morphism $V' \to Y' \times_Y V$ where V' is a scheme. Then $U' = U \times_V V'$ is a scheme which comes equipped with a surjective étale morphism $U' \to X'$. Choose $u' \in U'$ mapping to x'. Denote $u \in U$ the image of u'. Then the lemma follows from the lemma for $U \to V$ and its base change $U' \to V'$ and the points u' and u (this follows from the definitions). Thus the lemma follows from the case of schemes, see Duality for Schemes, Lemma 48.25.8. ☐

0E19 Lemma 76.27.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

Then W is open in $|X|$ and the formation of W commutes with arbitrary base change of f: For any morphism $g : Y' \to Y$, consider the base change $f' : X' \to Y'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let $u \in U$ with image $x \in |X|$. Then f is Gorenstein at x if and only if $U \to V$ is Gorenstein at u (by definition). Thus we reduce to the case of the morphism $U \to V$ of schemes. Openness is proven in Duality for Schemes, Lemma 48.25.11 and compatibility with base change in Duality for Schemes, Lemma 48.25.9. ☐

76.28. Slicing Cohen-Macaulay morphisms

06LV Let S be a scheme. Let X be an algebraic space over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. In this case we denote $V(f_1, \ldots, f_r)$ the closed subspace of X cut out by $f_1, \ldots, f_r$. More precisely, we can define $V(f_1, \ldots, f_r)$ as the closed subspace of X corresponding to the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_r$, see Morphisms of Spaces, Lemma 67.13.1. Alternatively, we can choose a presentation $X = U/R$ and consider the closed subscheme $Z \subset U$ cut out by $f_1|U, \ldots, f_r|_U$. It is clear that Z is an R-invariant (see Groupoids, Definition 39.19.1) closed subscheme and we may set $V(f_1, \ldots, f_r) = Z/R_Z$.

06LW Lemma 76.28.1. Let S be a scheme. Consider a cartesian diagram

$$\begin{tikzcd} X \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ Y & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is flat and locally of finite presentation, and where k is a field over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ and

$z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow Y$$

is flat and locally of finite presentation.

Proof. Set $Z = V(f_1, \ldots, f_r)$. It is clear that $Z \to X$ is locally of finite presentation, hence the composition $Z \to Y$ is locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.2. Hence it suffices to show that $Z \to Y$ is flat in a neighbourhood of $p(z)$. Let k'/k be an extension field. Then $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is surjective and flat over F, hence we can find a point $z' \in |F'|$ mapping to z and the local ring map $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 67.30.8. Hence the image of $f_1, \ldots, f_r$ in $\mathcal{O}_{F',\overline{z}'}$ is a regular sequence too, see Algebra, Lemma 10.68.5. Thus, during the proof we may replace k by an extension field. In particular, we may assume that $z \in |F|$ comes from a section $z : \mathrm{Spec}(k) \to F$ of the structure morphism $F \to \mathrm{Spec}(k)$.

Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. After possibly enlarging k once more we may assume that $\mathrm{Spec}(k) \to F \to X$ factors through U (as $U \to X$ is surjective). Let $u : \mathrm{Spec}(k) \to U$ be such a factorization and denote $v \in V$ the image of u. Note that the morphisms

$$U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k) = U \times_V \mathrm{Spec}(k) \to U \times_Y \mathrm{Spec}(k) \to F$$

are étale (the first as the base change of $V \to V \times_Y V$ and the second as the base change of $U \to X$). Moreover, by construction the point $u : \mathrm{Spec}(k) \to U$ gives a point of the left most space which maps to z on the right. Hence the elements $f_1, \ldots, f_r$ map to a regular sequence in the local ring on the right of the following map

$$\mathcal{O}_{U_v,u} \longrightarrow \mathcal{O}_{U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k),\overline{u}} = \mathcal{O}_{U \times_V \mathrm{Spec}(k),\overline{u}}.$$

But since the displayed arrow is flat (combine More on Flatness, Lemma 38.2.5 and Morphisms of Spaces, Lemma 67.30.8) we see from Algebra, Lemma 10.68.5 that $f_1, \ldots, f_r$ maps to a regular sequence in $\mathcal{O}_{U_v,u}$. By More on Morphisms, Lemma 37.23.2 we conclude that the morphism of schemes

$$V(f_1, \ldots, f_r) \times_X U = V(f_1|_U, \ldots, f_r|_U) \to V$$

is flat in an open neighbourhood U' of u. Let $X' \subset X$ be the open subspace corresponding to the image of $|U'| \to |X|$ (see Properties of Spaces, Lemmas 66.4.6 and 66.4.8). We conclude that $V(f_1, \ldots, f_r) \cap X' \to Y$ is flat (see Morphisms of Spaces, Definition 67.30.1) as we have the commutative diagram

$$\begin{tikzcd} V(f_1, \ldots, f_r) \times_X U' \arrow[r] \arrow[d, "a"'] & V \arrow[d, "b"] \\ V(f_1, \ldots, f_r) \cap X' \arrow[r] & Y \end{tikzcd}$$

with a, b étale and a surjective. $\square$

76.29. Reduced fibres

0E06 This section is the analogue of More on Morphisms, Section 37.26.

0E07 Lemma 76.29.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. The following are equivalent

(1) for some morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is geometrically reduced over k,
(2) for every morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is geometrically reduced over k,
(3) for every morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is reduced.

Proof. This follows immediately from Spaces over Fields, Lemma 72.11.6 and the definition of the equivalence relation defining $|X|$ given in Properties of Spaces, Section 66.4. □

0E08 Definition 76.29.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. We say the fibre of $f : X \to Y$ at y is geometrically reduced if the equivalent conditions of Lemma 76.29.1 hold.

Here are the obligatory lemmas.

0E09 Lemma 76.29.3. Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of algebraic spaces over S. Denote $f' : X' \to Y'$ the base change of f by g. Then

$$\begin{aligned} &\{y' \in |Y'| : \text{the fibre of } f' : X' \to Y' \text{ at } y' \text{ is geometrically reduced}\} \\ = \; &g^{-1}(\{y \in |Y| : \text{the fibre of } f : X \to Y \text{ at } y \text{ is geometrically reduced}\}). \end{aligned}$$

Proof. For $y' \in |Y'|$ choose a morphism $\mathrm{Spec}(k) \to Y'$ in the equivalence class of y'. Then $g(y')$ is represented by the composition $\mathrm{Spec}(k) \to Y' \to Y$. Hence $X' \times_{Y'} \mathrm{Spec}(k) = X \times_Y \mathrm{Spec}(k)$ and the result follows from the definition. □

0E0A Lemma 76.29.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is quasi-compact and locally of finite presentation. Then the set

$$E = \{y \in |Y| : \text{the fibre of } f : X \to Y \text{ at } y \text{ is geometrically reduced}\}$$

is étale locally constructible.

Proof. Choose an affine scheme V and an étale morphism $V \to Y$. The meaning of the statement is that the inverse image of E in $|V|$ is constructible. By Lemma 76.29.3 we may replace Y by V, i.e., we may assume that Y is an affine scheme. Then X is quasi-compact. Choose an affine scheme U and a surjective étale morphism $U \to X$. For a morphism $\mathrm{Spec}(k) \to Y$ the morphism between fibres $U_k \to X_k$ is surjective étale. Hence U_k is geometrically reduced over k if and only if X_k is geometrically reduced over k, see Spaces over Fields, Lemma 72.11.7. Thus the set E for $X \to Y$ is the same as the set E for $U \to Y$. In this way we see that the lemma follows from the case of schemes, see More on Morphisms, Lemma 37.26.5. □

0E0B Lemma 76.29.5. Let X be an algebraic space over a discrete valuation ring R whose structure morphism $X \to \mathrm{Spec}(R)$ is proper and flat. If the special fibre is reduced, then both X and the generic fibre X_η are reduced.

Proof. Choose an étale morphism $U \to X$ where U is an affine scheme. Then U is of finite type over R. Let $u \in U$ be in the special fibre. The local ring $A = \mathcal{O}_{U,u}$ is essentially of finite type over R, hence Noetherian. Let $\pi \in R$ be a uniformizer. Since X is flat over R, we see that $\pi \in \mathfrak{m}_A$ is a nonzerodivisor on A and since the special fibre of X is reduced, we have that $A/\pi A$ is reduced. If $a \in A$, $a \neq 0$ then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \notin \pi A$. This follows from Krull intersection theorem (Algebra, Lemma 10.51.4). If a is nilpotent, so is a', because π is a nonzerodivisor. But a' maps to a nonzero element of the reduced ring $A/\pi A$ so this is impossible. Hence A is reduced. It follows that there exists an open neighbourhood of u in U which is reduced (small detail omitted; use that U is Noetherian). Thus we can find an étale morphism $U \to X$ with U a reduced scheme, such that every point of the special fibre of X is in the image. Since X is proper over R it follows that $U \to X$ is surjective. Hence X is reduced. Since the generic fibre of $U \to \text{Spec}(R)$ is reduced as well (on affine pieces it is computed by taking localizations), we conclude the same thing is true for the generic fibre. □

0E0C Lemma 76.29.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat, proper, and of finite presentation, then the set

$$E = \{y \in |Y| : \text{the fibre of } f : X \to Y \text{ at } y \text{ is geometrically reduced}\}$$

is open in $|Y|$.

Proof. By Lemma 76.29.3 formation of E commutes with base change. To check a subset of $|Y|$ is open, we may replace Y by the members of an étale covering. Thus we may assume Y is affine. Then Y is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to Y$ is the base change of $X_0 \to Y_0$ where Y_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to Y_0$ is flat and proper. See Limits of Spaces, Lemma 70.7.1, 70.6.12, and 70.6.13. Since the formation of E commutes with base change (see above), we may assume the base is Noetherian.

Assume Y is Noetherian. The set is constructible by Lemma 76.29.4. Hence it suffices to show the set is stable under generalization (Topology, Lemma 5.19.10). By Properties, Lemma 28.5.10 we reduce to the case where $Y = \text{Spec}(R)$, R is a discrete valuation ring, and the closed fibre X_y is geometrically reduced. To show: the generic fibre X_η is geometrically reduced.

If not then there exists a finite extension L of the fraction field of R such that X_L is not reduced, see Spaces over Fields, Lemmas 72.11.4 (characteristic zero) and 72.11.5 (positive characteristic). There exists a discrete valuation ring $R' \subset L$ with fraction field L dominating R, see Algebra, Lemma 10.120.18. After replacing R by R' we reduce to Lemma 76.29.5. □

76.30. Connected components of fibres

0E1A This section is the analogue of More on Morphisms, Section 37.28.

0E1B Lemma 76.30.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let

$$n_{X/Y} : |Y| \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of X_k where $\mathrm{Spec}(k) \to Y$ is in the equivalence class of y with k algebraically closed. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Let $\mathrm{Spec}(k') \to Y'$ be in the equivalence class of y' with k' algebraically closed. Then we can choose a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[rd] & \mathrm{Spec}(k') \arrow[r] & Y' \arrow[d] \\ & \mathrm{Spec}(k) \arrow[r] & Y \end{tikzcd}$$

where K is an algebraically closed field. The result follows as the morphisms of schemes

$$X'_{k'} \longleftarrow (X'_{k'})_K = (X_k)_K \longrightarrow X_k$$

induce bijections between connected components, see Spaces over Fields, Lemma 72.12.4. To use this to prove the function is well defined take $Y' = Y$. □

76.31. Dimension of fibres

0D4L This section is the analogue of More on Morphisms, Section 37.30.

0D4M **Lemma 76.31.1.** Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let $y \in |Y|$. The following quantities are the same

(1) $d = -\infty$ if y is not in the image of $|f|$ and otherwise the minimal integer d such that f has relative dimension $\leq d$ at every $x \in |X|$ mapping to y,
(2) the dimension of the algebraic space $X_k = \mathrm{Spec}(k) \times_Y X$ for any morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class defining y.

Proof. To parse this one has to consult Morphisms of Spaces, Definition 67.33.1, Properties of Spaces, Definition 66.9.2, Properties of Spaces, Definition 66.9.1. We will show that the numbers in (1) and (2) are equal for a fixed morphism $\mathrm{Spec}(k) \to Y$. Choose an étale morphism $V \to Y$ where V is an affine scheme and a point $v \in V$ mapping to y. Since $V \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale (by Properties of Spaces, Lemma 66.4.3) we can find a finite separable extension k'/k (by Morphisms, Lemma 29.36.7) and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & Y \end{tikzcd}$$

We may replace $X \to Y$ by $V \times_Y X \to V$ and X_k by $X_{k'} = \mathrm{Spec}(k') \times_V (V \times_Y X)$ because this does not change the dimensions in question by Properties of Spaces, Lemma 66.22.5 and Morphisms of Spaces, Lemma 67.34.3. Thus we may assume that Y is an affine scheme. In this case we may assume that $k = \kappa(y)$ because the dimension of $X_{\kappa(y)}$ and X_k are the same by the aforementioned Morphisms of Spaces, Lemma 67.34.3 and the fact that for an algebraic space Z over a field K

the relative dimension of Z at a point $z \in |Z|$ is the same as $\dim_z(Z)$ by definition. Assume Y is affine and $k = \kappa(y)$. Then X is quasi-compact we can choose an affine scheme U and an surjective étale morphism $U \to X$. Then $\dim(X_k) = \dim(U_k) = \max \dim_u(U_k)$ is equal to the number given in (1) by definition. □

0D4N Lemma 76.31.2. Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let

$$n_{X/Y} : |Y| \to \{-\infty, 0, 1, 2, 3, \ldots\}$$

be the function which associates to $y \in |Y|$ the integer discussed in Lemma 76.31.1. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ |g|$$

where $X' \to Y'$ is the base change of f.

Proof. This follows immediately from Lemma 76.31.1. □

0D4P Lemma 76.31.3. Let S be a scheme. Let $f : X \to Y$ be a flat morphism of finite presentation of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 76.31.2. Then $n_{X/Y}$ is lower semi-continuous.

Proof. Let $V \to Y$ be a surjective étale morphism where V is a scheme. If we can show that the composition $n_{X/Y} \circ |g|$ is lower semi-continuous, then the lemma follows as $|g|$ is open. Hence we may assume Y is a scheme. Working locally we may assume V is an affine scheme. Then we can choose an affine scheme U and a surjective étale morphism $U \to X$. Then $n_{X/Y} = n_{U/Y}$. Hence we may assume X and Y are both schemes. In this case the lemma follows from More on Morphisms, Lemma 37.30.4. □

0D4Q Lemma 76.31.4. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 76.31.2. Then $n_{X/Y}$ is upper semi-continuous.

Proof. Let $Z_d = \{x \in |X| : \text{the fibre of } f \text{ at } x \text{ has dimension } > d\}$. Then Z_d is a closed subset of $|X|$ by Morphisms of Spaces, Lemma 67.34.4. Since f is proper $f(Z_d)$ is closed in $|Y|$. Since $y \in f(Z_d) \Leftrightarrow n_{X/Y}(y) > d$ we see that the lemma is true. □

0D4R Lemma 76.31.5. Let S be a scheme. Let $f : X \to Y$ be a proper, flat, finitely presented morphism of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 76.31.2. Then $n_{X/Y}$ is locally constant.

Proof. Immediate consequence of Lemmas 76.31.3 and 76.31.4. □

76.32. Catenary algebraic spaces

0EDL This section continues the discussion started in Decent Spaces, Section 68.25. The following lemma will be used in the proof of the next one.

0EDM Lemma 76.32.1. Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Let $y \in |Y|$ be a point which can be represented by a closed immersion $y : \mathrm{Spec}(k) \to Y$. Then there exists a factorization $X \to X' \to Y$ of f such that

(1) $X' \to Y$ is integral,
(2) $X \to X'$ is an isomorphism over $X' \setminus X'_y$,
(3) X'_y has a unique point x' with $\kappa(x') = k$.

Moreover, if f is finite and Y is locally Noetherian, then $X' \to Y$ is finite.

Proof. By Morphisms of Spaces, Lemma 67.11.2 the sheaves $f_*\mathcal{O}_X$, $(X_y \to Y)_*\mathcal{O}_{X_y}$, and $y_*\mathcal{O}_{\mathrm{Spec}(k)}$ are quasi-coherent sheaves of $\mathcal{O}_Y$-algebras. Consider the maps

$$f_*\mathcal{O}_Y \longrightarrow (X_y \to Y)_*\mathcal{O}_{X_y} \longleftarrow y_*\mathcal{O}_{\mathrm{Spec}(k)}$$

The fibre product is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A}'$ and we can define $X' \to Y$ as the relative spectrum of $\mathcal{A}'$ over Y, see Morphisms, Lemma 29.11.5. This construction commutes with arbitrary change of base. In particular, it is clear that over the open subspace $|Y| \setminus \{y\}$ the morphism $X \to X'$ is an isomorphism and over $|Y| \setminus \{y\}$ the morphism $X' \to Y$ is integral. It remains to prove the statements in a small neighbourhood of y. Choose an affine scheme $V = \mathrm{Spec}(R)$ and an étale morphism $\varphi : V \to Y$ such that y is in the image of φ. Then V_y is a closed subscheme of V étale over k, whence consists of finitely many points each with residue field separable over k (see Decent Spaces, Remark 68.4.1). After shrinking V we may assume there is a unique closed point $v = \mathrm{Spec}(l) \to V$ mapping to y with l/k finite separable. We may write $V \times_Y X = \mathrm{Spec}(C)$ with $R \to C$ an integral ring map. The stated compatibility with base change gives us that $U \times_X Y' = \mathrm{Spec}(C')$ where

$$C' = C \times_{C \otimes_R l} l$$

Since $R \to l$ is surjective, also $C \to C \otimes_R l$ is surjective and we see that this is a fibre product of the kind studied in More on Algebra, Situation 15.6.1 (with A, A', B, B' corresponding to $C \otimes_R l, C, l, C'$). Observe that C' is an R-subalgebra of C and hence is integral over R; this proves (1). Finally, More on Algebra, Lemma 15.6.2 shows that $V \times_X Y' = \mathrm{Spec}(C')$ has a unique point y'' lying over v with residue l (this corresponds with the obvious surjective map $C' \to l$). Thus $X_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$ has a unique point with residue field l. Since l/k is finite separable, this implies X'_y has a unique point with residue field k, i.e., (3) holds.

To prove the final statement, observe that if Y is locally Noetherian, then R is a Noetherian ring and if f is finite, then $R \to C$ is finite. Then C' is a finite type R-algebra by More on Algebra, Lemma 15.5.1. This proves that $X' \to Y$ is finite. $\square$

0EDN Lemma 76.32.2. Let S be a scheme. Let B be an algebraic space over S. Let $\delta : |B| \to \mathbf{Z}$ be a function. Assume B is decent, locally Noetherian, and universally catenary and δ is a dimension function. If X is a decent algebraic space over B whose structure morphism $f : X \to B$ is locally of finite type we define $\delta_X : |X| \to \mathbf{Z}$ by the rule

$$\delta_X(x) = \delta(f(x)) + \text{transcendence degreeof } x/f(x)$$

(Morphisms of Spaces, Definition 67.33.1). Then δ_X is a dimension function.

Proof. The problem is local on B. Thus we may assume B is quasi-compact. By Decent Spaces, Lemma 68.14.1 we see B is quasi-separated. By Limits of Spaces, Proposition 70.16.1 we can choose a finite surjective morphism $\pi : Y \to X$ where Y is a scheme. Claim: δ_Y is a dimension function.

The claim implies the lemma. With $X \to B$ as in the lemma set $Z = Y \times_B X$ with projections $p : Z \to Y$ and $q : Z \to X$. Then we have

$$\delta_Z(z) = \delta_Y(p(z)) + \text{transcendence degreeof } z/p(z)$$

and $\delta_Z(z) = \delta_X(q(z))$. This follows from Morphisms of Spaces, Lemma 67.34.2 and the fact that these transcendence degrees are zero for finite morphisms. By Decent Spaces, Lemma 68.25.2 and the claim we find that δ_Z is a dimension function. Then we find that δ_X is a dimension function by Decent Spaces, Lemma 68.25.6.

Proof of the claim. Consider a specialization $y \rightsquigarrow y'$, $y \neq y'$ of points of the Noetherian scheme Y. Then $\delta_Y(y) > \delta_Y(y')$ because there are no specializations between points in fibres of Y (see Decent Spaces, Lemma 68.18.10). Using this for a chain of specializations we find

$$\delta_Y(y) - \delta_Y(y') \geq \text{codim}(\overline{\{y'\}}, \overline{\{y\}})$$

Our task is to show equality. By Properties, Lemma 28.5.9 we can choose a specialization $y' \rightsquigarrow y_0$. It suffices to show $\delta_Y(y) - \delta_Y(y_0) = \text{codim}(\overline{\{y_0\}}, \overline{\{y\}})$ because this will imply the equality for both $y \rightsquigarrow y'$ and $y' \rightsquigarrow y_0$.

Choose a maximal chain $y = y_c \rightsquigarrow y_{c-1} \rightsquigarrow \ldots \rightsquigarrow y_0$ of specializations in Y. Set $b = \pi(y)$ and $b_0 = \pi(y_0)$. Choose a maximal chain $b = b_e \rightsquigarrow b_{e-1} \rightsquigarrow \ldots \rightsquigarrow b_0$ of specializations in $|B|$. We have to show $e = c$. Since π is closed (Morphisms of Spaces, Lemma 67.45.9) we can find a sequence of specializations $y = y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ mapping to $b = b_e \rightsquigarrow b_{e-1} \rightsquigarrow \ldots \rightsquigarrow b_0$. Observe that $y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ is a maximal chain as well. If $y_0 = y'_0$, then because Y is catenary, we conclude that $e = c$ as desired. In the next paragraph we reduce to this case by sleight of hand and we conclude in the same manner.

Since π is closed we see that b_0 is a closed point of $|B|$. By Decent Spaces, Lemma 68.14.6 we can represent b_0 by a closed immersion $b_0 : \text{Spec}(k) \to B$. By Lemma 76.32.1 we can find a factorization

$$Y \to Y' \to X$$

with $\pi' : Y' \to X$ finite and $Y \to Y'$ a morphism which map y_0 and y'_0 to the same point and is an isomorphism away from the inverse image of b_0. (Of course Y' won't be a scheme but this doesn't matter for the argument that follows.) Clearly the maximal chains of specializations $y_c \rightsquigarrow y_{c-1} \rightsquigarrow \ldots \rightsquigarrow y_0$ and $y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ map to maximal chains of specializations in Y' having the same start and end. Since B is universally catenary, we see that $|Y'|$ is catenary and we conclude as before. $\square$

76.33. Étale localization of morphisms

082G The section is the analogue of More on Morphisms, Section 37.41.

082H **Lemma 76.33.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Let $x_1, \ldots, x_n \in |X|$ mapping to y. Assume that

(1) f is locally of finite type,
(2) f is separated,
(3) f is quasi-finite at $x_1, \ldots, x_n$, and
(4) f is quasi-compact or Y is decent.

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite, every point of the fibre of $|V| \to |U|$ over u maps to an x_i, and the fibre of $|W| \to |U|$ over u contains no point mapping to an x_i.

Proof. Let $(U, u) \to (Y, y)$ be an étale morphism of algebraic spaces and consider the set of $w \in |U \times_Y X|$ mapping to $u \in |U|$ and one of the $x_i \in |X|$. By Decent Spaces, Lemma 68.18.4 (if f is of finite type) or Decent Spaces, Lemma 68.18.5 (if Y is decent) this set is finite. It follows that we may replace f by the base change $U \times_Y X \to U$ and $x_1, \ldots, x_n$ by the set of these w. In particular we may and do assume that Y is an affine scheme, whence X is a separated algebraic space.

Choose an affine scheme Z and an étale morphism $Z \to X$ such that $x_1, \ldots, x_n$ are in the image of $|Z| \to |X|$. The fibres of $|Z| \to |X|$ are finite, see Properties of Spaces, Lemma 66.6.7 (or the more general discussion in Decent Spaces, Section 68.6). Let $\{z_1, \ldots, z_m\} \subset |Z|$ be the preimage of $\{x_1, \ldots, x_n\}$. By More on Morphisms, Lemma 37.41.4 there exists an étale morphism $(U, u) \to (Y, y)$ such that $U \times_Y Z = Z_1 \amalg Z_2$ with $Z_1 \to U$ finite and $(Z_1)_y = \{z_1, \ldots, z_m\}$. We may assume that U is affine and hence Z_1 is affine too.

Since f is separated, the image V of $Z_1 \to X$ is both open and closed (Morphisms of Spaces, Lemma 67.40.6). Set $W = X \setminus V$ to get a decomposition as in the lemma. To finish the proof we have to show that $V \to U$ is finite. As $Z_1 \to V$ is surjective and étale, V is the quotient of Z_1 by the étale equivalence relation $R = Z_1 \times_V Z_1$, see Spaces, Lemma 65.9.1. Since f is separated, $V \to U$ is separated and R is closed in $Z_1 \times_U Z_1$. Since $Z_1 \to U$ is finite, the projections $s, t : R \to Z_1$ are finite. Thus V is an affine scheme by Groupoids, Proposition 39.23.9. By Morphisms, Lemma 29.41.9 we conclude that $V \to U$ is proper and by Morphisms, Lemma 29.44.11 we conclude that $V \to U$ is finite, thereby finishing the proof. □

0ADU Lemma 76.33.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Assume that

(1) f is locally of finite type,
(2) f is separated, and
(3) f is quasi-finite at x.

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite and there exists a point $v \in |V|$ which maps to x in $|X|$ and u in $|U|$.

Proof. Pick a scheme U, a point $u \in U$, and an étale morphism $U \to Y$ mapping u to y. There exists a point $x' \in |U \times_Y X|$ mapping to x in $|X|$ and u in $|U|$ (Properties of Spaces, Lemma 66.4.3). To finish, apply Lemma 76.33.1 to the morphism $U \times_Y X \to U$ and the point x'. It applies because U is a scheme and hence u comes from the monomorphism $\mathrm{Spec}(\kappa(u)) \to U$. □

76.34. Zariski's Main Theorem

05W7 In this section we apply the results of the previous section to prove Zariski's main theorem for morphisms of algebraic spaces. This section is the analogue of More on Morphisms, Section 37.43.

082I **Lemma 76.34.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is of finite type and separated. Let Y' be the normalization of Y in X. Picture:

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\nu"] \\ & Y & \end{tikzcd}$$

Then there exists an open subspace $U' \subset Y'$ such that

(1) $(f')^{-1}(U') \to U'$ is an isomorphism, and
(2) $(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.

Proof. By Morphisms of Spaces, Lemma 67.34.7 there is an open subspace $U \subset X$ corresponding to the points of $|X|$ where f is quasi-finite. We have to prove

(a) the image of $|U| \to |Y'|$ is $|U'|$ for some open subspace U' of Y',
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Since formation of U commutes with arbitrary base change (Morphisms of Spaces, Lemma 67.34.7), since formation of the normalization Y' commutes with smooth base change (Lemma 76.25.2), since étale morphisms are open, and since "being an isomorphism" is fpqc local on the base (Descent on Spaces, Lemma 74.11.15), it suffices to prove (a), (b), (c) étale locally on Y (some details omitted). Thus we may assume Y is an affine scheme. This implies that Y' is an (affine) scheme as well.

Let $x \in |U|$. Claim: there exists an open neighbourhood $f'(x) \in V \subset Y'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is a scheme (as an open of Y'), locally of finite type over Y (as an open subspace of X), and for $v \in V$ the residue field extension $\kappa(v)/\kappa(\nu(v))$ is algebraic (as $V \subset Y'$ and Y' is integral over Y). Hence the fibres of $V \to Y$ are discrete (Morphisms, Lemma 29.20.2) and $(f')^{-1}V \to Y$ is locally quasi-finite (Morphisms, Lemma 29.20.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $y = f(x) \in |Y|$. Let $(T, t) \to (Y, y)$ be an étale morphism of pointed schemes. Denote by a subscript $_T$ the base change to T. Let $z \in X_T$ be a point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms of Spaces, Lemma 67.34.7. Note that

$$X_T \xrightarrow{f'_T} Y'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 76.25.2. Suppose that the claim holds for $z \in U_T \subset X_T \to Y'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(z) \in V' \subset Y'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $Y'_T \to Y'$ is étale hence the image $V \subset Y'$ of V' is open. Observe that $f'(x) \in V$

as $f'_T(z) \in V'$. Observe that

$$\begin{tikzcd} (f'_T)^{-1}V' \arrow[r] \arrow[d] & (f')^{-1}(V) \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is a fibre square (as $Y'_T \times_{Y'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent on Spaces, Lemma 74.11.15. In other words, the claim holds for $x \in U \subset X \to Y' \to Y$.

By the result of the previous paragraph to prove the claim for $x \in |U|$, we may replace Y by an étale neighbourhood T of $y = f(x)$ and x by any point lying over x in $T \times_Y X$. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subspaces where $V \to Y$ is finite and $x \in V$, see Lemma 76.33.1. Since X is a disjoint union of V and W over Y and since $V \to Y$ is finite we see that the normalization of Y in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of Y in W, see Morphisms of Spaces, Lemmas 67.48.8, 67.45.6, and 67.48.10. The claim follows and we win. $\square$

The following lemma is a duplicate of Morphisms of Spaces, Lemma 67.52.2. The reason for having two copies of the same lemma is that the proofs are somewhat different. The proof given below rests on Zariski's Main Theorem for nonrepresentable morphisms of algebraic spaces as presented above, whereas the proof of Morphisms of Spaces, Lemma 67.52.2 rests on Morphisms of Spaces, Proposition 67.50.2 to reduce to the case of morphisms of schemes.

082J Lemma 76.34.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:

$$\begin{tikzcd} X \arrow[rr, "f'"'] \arrow[rd, "f"'] & & Y' \arrow[ld, "\nu"] \\ & Y & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 76.34.1. Namely, by that lemma there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Morphisms of Spaces, Lemma 67.8.9). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. $\square$

082K Lemma 76.34.3 (Zariski's Main Theorem). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated and assume that Y is quasi-compact and quasi-separated. Then there exists a factorization

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to Y' \to Y$ be as in the conclusion of Lemma 76.34.2. By Limits of Spaces, Lemma 70.9.7 we can write $\nu_*\mathcal{O}_{Y'} = \mathrm{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{Y'}$. Then $\pi_i : T_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \to Y$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $Y' = \lim T_i$.

By Limits of Spaces, Lemma 70.5.7 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in Y' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits of Spaces, Lemma 70.4.1. By Limits of Spaces, Lemma 70.5.12 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms of Spaces, Lemma 67.12.6 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

0874 Lemma 76.34.4. With notation and hypotheses as in Lemma 76.34.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over Y.

Proof. By Limits of Spaces, Lemma 70.11.3 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits of Spaces, Lemma 70.5.7 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits of Spaces, Lemma 70.5.12 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. □

76.35. Applications of Zariski's Main Theorem, I

0F43 A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

0A4X Lemma 76.35.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:

(1) f is finite,
(2) f is proper and locally quasi-finite,
(3) f is proper and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field,
(4) f is universally closed, separated, locally of finite type and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field.

Proof. We have (1) $\Rightarrow$ (2) by Morphisms of Spaces, Lemmas 67.45.9, 67.45.8. We have (2) $\Rightarrow$ (3) by Morphisms of Spaces, Lemma 67.27.5. By definition (3) implies (4).

Assume (4). Since f is universally closed it is quasi-compact (Morphisms of Spaces, Lemma 67.9.7). Pick a point y of $|Y|$. We represent y by a morphism $\mathrm{Spec}(k) \to Y$. Note that $|X_k|$ is finite discrete as a quasi-compact discrete space. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 66.4.3). By Morphisms of Spaces, Lemma 67.34.8 we see that $X \to Y$ is quasi-finite at all the points of the fibre of $|X| \to |Y|$ over y. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$ and decomposition $X_U = V \amalg W$ as in Lemma 76.33.1 adapted to all the points of $|X|$ lying over y. Note that $W_u = \emptyset$ because we used all the points in the fibre of $|X| \to |Y|$ over y. Since f is universally closed we see that the image of $|W|$ in $|U|$ is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $y \in |Y|$ was arbitrary this means there exists a family $\{U_i \to Y\}$ of étale morphisms whose images cover Y such that the base changes $X_{U_i} \to U_i$ are finite. We conclude that f is finite by Morphisms of Spaces, Lemma 67.45.3. $\square$

As a consequence we have the following useful result.

0A4Y Lemma 76.35.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Assume

(1) f is proper, and
(2) f is quasi-finite at all $x \in |X|$ lying over y (Decent Spaces, Lemma 68.18.10).

Then there exists an open neighbourhood $V \subset Y$ of y such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. By Morphisms of Spaces, Lemma 67.34.7 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $y \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of y with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ is finite by Lemma 76.35.1. $\square$

0AEJ Lemma 76.35.3. Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphism of algebraic spaces over S. Let $b \in B$ and let $\mathrm{Spec}(k) \to B$ be a morphism in the equivalence class of b. Assume

(1) $X \to B$ is a proper morphism,
(2) $Y \to B$ is separated and locally of finite type,
(3) one of the following is true
  (a) the image of $|X_k| \to |Y_k|$ is finite,
  (b) the image of $|f|^{-1}(\{b\})$ in $|Y|$ is finite and B is decent.

Then there is an open subspace $B' \subset B$ containing b such that $X_{B'} \to Y_{B'}$ factors through a closed subspace $Z \subset Y_{B'}$ finite over B'.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms of Spaces, Section 67.16. By Morphisms of Spaces, Lemma 67.40.8 the morphism $X \to Z$ is surjective and $Z \to B$ is proper. Thus

$$\{x \in |X| \text{ lying over } b\} \to \{z \in |Z| \text{ lying over } b\}$$

and $|X_k| \to |Z_k|$ are surjective. We see that either (3)(a) or (3)(b) imply that $Z \to B$ is quasi-finite all points of $|Z|$ lying over b by Decent Spaces, Lemma 68.18.10. Hence $Z \to B$ is finite in an open neighbourhood of b by Lemma 76.35.2. $\square$

76.36. Stein factorization

0A18 Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

0A19 Lemma 76.36.1. Let S be a scheme. Let $f : X \to Y$ be a universally closed and quasi-separated morphism of algebraic spaces over S. There exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

(1) the morphism f' is universally closed, quasi-compact, quasi-separated, and surjective,
(2) the morphism $\pi : Y' \to Y$ is integral,
(3) we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,
(4) we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and
(5) Y' is the normalization of Y in X as defined in Morphisms of Spaces, Definition 67.48.3.

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms of Spaces, Lemma 67.9.7 the morphism f is quasi-compact. We just define Y' as the normalization of Y in X, so (5) and (2) hold automatically. By Morphisms of Spaces, Lemma 67.48.9 we see that (4) holds. The morphism f' is universally closed by Morphisms of Spaces, Lemma 67.40.6. It is quasi-compact by Morphisms of Spaces, Lemma 67.8.9 and quasi-separated by Morphisms of Spaces, Lemma 67.4.10.

To show the remaining statements we may assume the base Y is affine (as taking normalization commutes with étale localization). Say $Y = \mathrm{Spec}(R)$. Then $Y' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{Y'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Morphisms of Spaces, Lemma 67.11.2, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset Y' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the Jacobson radical of A and we conclude that $V(I) = Y'$ as desired. $\square$

0E1C **Lemma 76.36.2.** In Lemma 76.36.1 assume in addition that f is locally of finite type and Y affine. Then for $y \in Y$ the fibre $\pi^{-1}(\{y\}) = \{y_1, \ldots, y_n\}$ is finite and the field extensions $\kappa(y_i)/\kappa(y)$ are finite.

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{y\})$, see Algebra, Lemma 10.36.20. As f' is surjective, we find that $|X_y| \to \pi^{-1}(\{y\})$ is surjective. Observe that X_y is a quasi-separated algebraic space of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus $|X_y|$ is a Noetherian topological space (Morphisms of Spaces, Lemma 67.28.6). A topological argument (omitted) now shows that $\pi^{-1}(\{y\})$ is finite. For each i we can pick a finite type point $x_i \in |X_y|$ mapping to y_i (Morphisms of Spaces, Lemma 67.25.6). We conclude that $\kappa(y_i)/\kappa(y)$ is finite: x_i can be represented by a morphism $\mathrm{Spec}(k_i) \to X_y$ of finite type (by our definition of finite type points) and hence $\mathrm{Spec}(k_i) \to y = \mathrm{Spec}(\kappa(y))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(y)$ is finite (Morphisms, Lemma 29.16.1). $\square$

Let $f : X \to Y$ be a morphism of algebraic spaces and let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Then the fibre of f over $\overline{y}$ is the algebraic space $X_{\overline{y}} = X \times_{Y,\overline{y}} \mathrm{Spec}(k)$ over k. If Y is a scheme and $y \in Y$ is a point, then we denote $X_y = X \times_Y \mathrm{Spec}(\kappa(y))$ the fibre as usual.

0A1A **Lemma 76.36.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Then $X_{\overline{y}}$ is connected, if and only if for every étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ where V is a scheme the base change $X_V \to V$ has connected fibre X_v.

Proof. Since the category of étale neighbourhoods of $\overline{y}$ is cofiltered and contains a cofinal collection of schemes (Properties of Spaces, Lemma 66.19.3) we may replace Y by one of these neighbourhoods and assume that Y is a scheme. Let $y \in Y$ be the point corresponding to $\overline{y}$. Then X_y is geometrically connected over $\kappa(y)$ if and only if $X_{\overline{y}}$ is connected and if and only if $(X_y)_{k'}$ is connected for every finite separable extension k' of $\kappa(y)$. See Spaces over Fields, Section 72.12 and especially Lemma 72.12.8. By More on Morphisms, Lemma 37.35.2 there exists an affine étale neighbourhood $(V, v) \to (Y, y)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k'$ any given finite separable extension. The lemma follows. $\square$

0A1B **Theorem 76.36.4** (Stein factorization; Noetherian case)**.** Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. There exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

(1) the morphism f' is proper with connected geometric fibres,
(2) the morphism $\pi : Y' \to Y$ is finite,
(3) we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,
(4) we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and
(5) Y' is the normalization of Y in X, see Morphisms, Definition 29.53.3.

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 76.36.1. Note that besides the conclusions of Lemma 76.36.1 we also have that f' is separated (Morphisms of Spaces, Lemma 67.4.10) and finite type (Morphisms of Spaces, Lemma 67.23.6). Hence f' is proper. By Cohomology of Spaces, Lemma 69.20.2 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that the geometric fibres of f' are connected. It is clear from the discussion above that we may replace Y by Y'. Then Y is locally Noetherian, $f : X \to Y$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $\overline{y}$ be a geometric point of Y. At this point we apply the theorem on formal functions, more precisely Cohomology of Spaces, Lemma 69.22.7. It tells us that

$$\mathcal{O}^\wedge_{Y,\overline{y}} = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where $X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \times_Y X$. Note that $X_1 = X_{\overline{y}} \to X_n$ is a (finite order) thickening and hence the underlying topological space of X_n is equal to that of $X_{\overline{y}}$. Thus, if $X_{\overline{y}} = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subspaces, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}^\wedge_{Y,\overline{y}}$ is a local ring. Thus the assumption was wrong, i.e., $X_{\overline{y}}$ is connected as desired. □

0A1C **Theorem 76.36.5** (Stein factorization; general case). Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. There exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

(1) the morphism f' is proper with connected geometric fibres,
(2) the morphism $\pi : Y' \to Y$ is integral,
(3) we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,
(4) we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and
(5) Y' is the normalization of Y in X (Morphisms of Spaces, Definition 67.48.3).

Proof. We may apply Lemma 76.36.1 to get the morphism $f' : X \to Y'$. Note that besides the conclusions of Lemma 76.36.1 we also have that f' is separated (Morphisms of Spaces, Lemma 67.4.10) and finite type (Morphisms of Spaces, Lemma 67.23.6). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has connected geometric fibres.

It is clear from the discussion that we may replace Y by Y'. Then $f : X \to Y$ is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Note that these conditions are preserved under flat base change (Morphisms of Spaces, Lemma 67.40.3 and Cohomology of Spaces, Lemma 69.11.2). Let $\overline{y}$ be a geometric point of Y. By Lemma 76.36.3 and the remark just made we reduce to the case where Y is a scheme, $y \in Y$ is a point, $f : X \to Y$ is a proper algebraic space over Y with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and we have to show the fibre X_y

is connected. Replacing Y by an affine neighbourhood of y we may assume that $Y = \mathrm{Spec}(R)$ is affine. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ signifies that the ring map $R \to \Gamma(X, \mathcal{O}_X)$ is bijective.

By Limits of Spaces, Lemma 70.12.2 we can write $(X \to Y) = \lim(X_i \to Y_i)$ with $X_i \to Y_i$ proper and of finite presentation and Y_i Noetherian. For i large enough Y_i is affine (Limits of Spaces, Lemma 70.5.10). Say $Y_i = \mathrm{Spec}(R_i)$. Let $R_i' = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that we have ring maps $R_i \to R_i' \to R$. Namely, we have the first because X_i is an algebraic space over R_i and the second because we have $X \to X_i$ and $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathrm{colim}\, R_i'$ by Limits of Spaces, Lemma 70.5.6. Then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[d] & \\ Y \arrow[r] & Y_i' \arrow[r] & Y_i \end{tikzcd}$$

is commutative with $Y_i' = \mathrm{Spec}(R_i')$. Let $y_i' \in Y_i'$ be the image of y. We have $X_y = \lim X_{i,y_i'}$ because $X = \lim X_i$, $Y = \lim Y_i'$, and $\kappa(y) = \mathrm{colim}\, \kappa(y_i')$. Now let $X_y = U \amalg V$ with U and V open and closed. Then U, V are the inverse images of opens U_i, V_i in $X_{i,y_i'}$ (Limits of Spaces, Lemma 70.5.7). By Theorem 76.36.4 the fibres of $X_i \to Y_i'$ are connected, hence either U or V is empty. This finishes the proof. □

Here is an application.

0AYI Lemma 76.36.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is proper,
(2) Y is integral (Spaces over Fields, Definition 72.4.1) with generic point ξ,
(3) Y is normal,
(4) X is reduced,
(5) every generic point of an irreducible component of $|X|$ maps to ξ,
(6) we have $H^0(X_\xi, \mathcal{O}) = \kappa(\xi)$.

Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ and f has geometrically connected fibres.

Proof. Apply Theorem 76.36.5 to get a factorization $X \to Y' \to Y$. It is enough to show that $Y' = Y$. It suffices to show that $Y' \times_Y V \to V$ is an isomorphism, where $V \to Y$ is an étale morphism and V an affine integral scheme, see Spaces over Fields, Lemma 72.4.5. The formation of Y' commutes with étale base change, see Morphisms of Spaces, Lemma 67.48.4. The generic points of $X \times_Y V$ lie over the generic points of X (Decent Spaces, Lemma 68.20.1) hence map to the generic point of V by assumption (5). Moreover, condition (6) is preserved under the base change by $V \to Y$, for example by flat base change (Cohomology of Spaces, Lemma 69.11.2). Thus it suffices to prove the lemma in case Y is a normal integral affine scheme.

Assume Y is a normal integral affine scheme. We will show $Y' \to Y$ is an isomorphism by an application of Morphisms, Lemma 29.54.8. Namely, Y' is reduced because X is reduced (Morphisms of Spaces, Lemma 67.48.6). The morphism $Y' \to Y$ is integral by the theorem cited above. Since Y is decent and $X \to Y$ is separated, we see that X is decent too; to see this use Decent Spaces, Lemmas

68.17.2 and 68.17.5. By assumption (5), Morphisms of Spaces, Lemma 67.48.7, and Decent Spaces, Lemma 68.20.1 we see that every generic point of an irreducible component of $|Y'|$ maps to ξ. On the other hand, since Y' is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre Y'_ξ is the spectrum of $H^0(X_\xi, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence Y' is an integral scheme with function field equal to the function field of Y. This finishes the proof. $\square$

Here is another application.

0E1D Lemma 76.36.7. Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. If f is proper, flat, and of finite presentation, then the function $n_{X/Y} : |Y| \to \mathbf{Z}$ counting the number of geometric connected components of fibres of f (Lemma 76.30.1) is lower semi-continuous.

Proof. The question is étale local on Y, hence we may and do assume Y is an affine scheme. Let $y \in Y$. Set $n = n_{X/S}(y)$. Note that $n < \infty$ as the geometric fibre of $X \to Y$ at y is a proper algebraic space over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood V of y such that $n_{X/S}|_V \geq n$. Let $X \to Y' \to Y$ be the Stein factorization as in Theorem 76.36.5. By Lemma 76.36.2 there are finitely many points $y'_1, \ldots, y'_m \in Y'$ lying over y and the extensions $\kappa(y'_i)/\kappa(y)$ are finite. More on Morphisms, Lemma 37.42.1 tells us that after replacing Y by an étale neighbourhood of y we may assume $Y' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $y'_i \in V_i$ and $\kappa(y'_i)/\kappa(y)$ purely inseparable. Then the algebraic spaces $X_{y'_i}$ are geometrically connected over $\kappa(y)$, hence $m = n$. The algebraic spaces $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are flat and of finite presentation over Y. Hence the image of $X_i \to Y$ is open (Morphisms of Spaces, Lemma 67.30.6). Thus in a neighbourhood of y we see that $n_{X/Y}$ is at least n. $\square$

0E1E Lemma 76.36.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume

(1) f is proper, flat, and of finite presentation, and
(2) the geometric fibres of f are reduced.

Then the function $n_{X/S} : |Y| \to \mathbf{Z}$ counting the numbers of geometric connected components of fibres of f (Lemma 76.30.1) is locally constant.

Proof. By Lemma 76.36.7 the function $n_{X/Y}$ is lower semincontinuous. Thus it suffices to show it is upper semi-continuous. To do this we may work étale locally on Y, hence we may assume Y is an affine scheme. For $y \in Y$ consider the $\kappa(y)$-algebra

$$A = H^0(X_y, \mathcal{O}_{X_y})$$

By Spaces over Fields, Lemma 72.14.3 and the fact that X_y is geometrically reduced A is finite product of finite separable extensions of $\kappa(y)$. Hence $A \otimes_{\kappa(y)} \kappa(\overline{y})$ is a product of $\beta_0(y) = \dim_{\kappa(y)} A$ copies of $\kappa(\overline{y})$. Thus $X_{\overline{y}}$ has $\beta_0(y)$ connected components. In other words, we have $n_{X/S} = \beta_0$ as functions on Y. Thus $n_{X/Y}$ is upper semi-continuous by Derived Categories of Spaces, Lemma 75.26.2. This finishes the proof. $\square$

0E0D Lemma 76.36.9. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $X \to Y' \to Y$ be the Stein factorization of f (Theorem 76.36.5). If f is of finite presentation, flat, with geometrically reduced fibres (Definition 76.29.2), then $Y' \to Y$ is finite étale.

Proof. Formation of the Stein factorization commutes with flat base change, see Lemma 76.36.1. Thus we may work étale locally on Y and we may assume Y is an affine scheme. Then Y' is an affine scheme and $Y' \to Y$ is integral.

Let $y \in Y$. Set n be the number of connected components of the geometric fibre $X_{\overline{y}}$. Note that $n < \infty$ as the geometric fibre of $X \to Y$ at y is a proper algebraic space over a field, hence Noetherian, hence has a finite number of connected components. By Lemma 76.36.2 there are finitely many points $y'_1, \ldots, y'_m \in Y'$ lying over y and for each i we can pick a finite type point $x_i \in |X_y|$ mapping to y'_i the extension $\kappa(y'_i)/\kappa(y)$ is finite. Thus More on Morphisms, Lemma 37.42.1 tells us that after replacing Y by an étale neighbourhood of y we may assume $Y' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $y'_i \in V_i$ and $\kappa(y'_i)/\kappa(y)$ purely inseparable. In this case the algebraic spaces $X_{y'_i}$ are geometrically connected over $\kappa(y)$, hence $m = n$. The algebraic spaces $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are proper, flat, of finite presentation, with geometrically reduced fibres over Y. It suffices to prove the lemma for each of the morphisms $X_i \to Y$. This reduces us to the case where $X_{\overline{y}}$ is connected.

Assume that $X_{\overline{y}}$ is connected. By Lemma 76.36.8 we see that $X \to Y$ has geometrically connected fibres in a neighbourhood of y. Thus we may assume the fibres of $X \to Y$ are geometrically connected. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ by Derived Categories of Spaces, Lemma 75.26.8 which finishes the proof. □

The proof of the following lemma uses Stein factorization for schemes which is why it ended up in this section.

0CWI **Lemma 76.36.10.** *Let (A, I) be a henselian pair. Let X be an algebraic space separated and of finite type over A. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subspace such that $Y \to \mathrm{Spec}(A/I)$ is proper. Then there exists an open and closed subspace $W \subset X$ which is proper over A with $W \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) = Y$.*

Proof. We will denote $T \mapsto T_0$ the base change by $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. By a weak version of Chow's lemma (in the form of Cohomology of Spaces, Lemma 69.18.1) there exists a surjective proper morphism $\varphi : X' \to X$ such that X' admits an immersion into $\mathbf{P}^n_A$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of X'_0. The lemma holds for (X', Y') by More on Morphisms, Lemma 37.53.9. Let $W' \subset X'$ be the open and closed subscheme proper over A such that $Y' = W'_0$. By Morphisms of Spaces, Lemma 67.40.6 $Q_1 = \varphi(|W'|) \subset |X|$ and $Q_2 = \varphi(|X' \setminus W'|) \subset |X|$ are closed subsets and by Morphisms of Spaces, Lemma 67.40.7 any closed subspace structure on Q_1 is proper over A. The image of $Q_1 \cap Q_2$ in $\mathrm{Spec}(A)$ is closed. Since (A, I) is henselian, if $Q_1 \cap Q_2$ is nonempty, then we find that $Q_1 \cap Q_2$ has a point lying over $\mathrm{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that Q_1 is open and closed in $|X|$. Let $W \subset X$ be the corresponding open and closed subspace. Then W is proper over A with $W_0 = Y$. □

76.37. Extending properties from an open

0875 In this section we collect a number of results of the form: If $f : X \to Y$ is a flat morphism of algebraic spaces and f satisfies some property over a dense open of Y, then f satisfies the same property over all of Y.

0876 **Lemma 76.37.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Assume

(1) f is locally of finite presentation,
(2) $\mathcal{F}$ is of finite type and flat over Y,
(3) $V \to Y$ is quasi-compact and scheme theoretically dense,
(4) $\mathcal{F}|_{f^{-1}V}$ is of finite presentation.

Then $\mathcal{F}$ is of finite presentation.

Proof. It suffices to prove the pullback of $\mathcal{F}$ to a scheme surjective and étale over X is of finite presentation. Hence we may assume X is a scheme. Similarly, we can replace Y by a scheme surjective and étale and over Y (the inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 67.17). Thus we reduce to the case of schemes which is More on Flatness, Lemma 38.11.1. ☐

0877 **Lemma 76.37.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Assume

(1) f is locally of finite type and flat,
(2) $V \to Y$ is quasi-compact and scheme theoretically dense,
(3) $f|_{f^{-1}V} : f^{-1}V \to V$ is locally of finite presentation.

Then f is of locally of finite presentation.

Proof. The proof is identical to the proof of Lemma 76.37.1 except one uses More on Flatness, Lemma 38.11.2. ☐

0878 **Lemma 76.37.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite type. Let $V \subset Y$ be an open subspace such that $|V| \subset |Y|$ is dense and such that $X_V \to V$ has relative dimension $\leq d$. If also either

(1) f is locally of finite presentation, or
(2) $V \to Y$ is quasi-compact,

then $f : X \to Y$ has relative dimension $\leq d$.

Proof. We may replace Y by its reduction, hence we may assume Y is reduced. Then V is scheme theoretically dense in Y, see Morphisms of Spaces, Lemma 67.17.7. By definition the property of having relative dimension $\leq d$ can be checked on an étale covering, see Morphisms of Spaces, Sections 67.33. Thus it suffices to prove f has relative dimension $\leq d$ after replacing X by a scheme surjective and étale over X. Similarly, we can replace Y by a scheme surjective and étale and over Y. The inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 67.17. Since a scheme theoretically dense open of a scheme is in particular dense, we reduce to the case of schemes which is More on Flatness, Lemma 38.11.3. ☐

0B4J **Lemma 76.37.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and proper. Let $V \to Y$ be an open subspace with $|V| \subset |Y|$ dense such that $X_V \to V$ is finite. If also either f is locally of finite presentation or $V \to Y$ is quasi-compact, then f is finite.

Proof. By Lemma 76.37.3 the fibres of f have dimension zero. By Morphisms of Spaces, Lemma 67.34.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 67.51.1 this implies that f is representable. We can check whether f is finite étale locally on Y, hence we may assume Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Flatness, Lemma 38.11.4. □

0879 Lemma 76.37.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. If

(1) f is separated, locally of finite type, and flat,
(2) $f^{-1}(V) \to V$ is an isomorphism, and
(3) $V \to Y$ is quasi-compact and scheme theoretically dense,

then f is an open immersion.

Proof. Applying Lemma 76.37.2 we see that f is locally of finite presentation. Applying Lemma 76.37.3 we see that f has relative dimension ≤ 0. By Morphisms of Spaces, Lemma 67.34.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 67.51.1 this implies that f is representable. By Descent on Spaces, Lemma 74.11.14 we can check whether f is an open immersion étale locally on Y. Hence we may assume that Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Flatness, Lemma 38.11.5. □

76.38. Blowing up and flatness

087A Instead of redoing the work in More on Flatness, Section 38.30 we prove an analogue of More on Flatness, Lemma 38.30.5 which tells us that the problem of finding a suitable blowup is often étale local on the base.

087B Lemma 76.38.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\varphi : W \to X$ be a quasi-compact separated étale morphism. Let $U \subset X$ be a quasi-compact open subspace. Let $\mathcal{I} \subset \mathcal{O}_W$ be a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}) \cap \varphi^{-1}(U) = \emptyset$. Then there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that

(1) $V(\mathcal{J}) \cap U = \emptyset$, and
(2) $\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathcal{I}\mathcal{I}'$ for some finite type quasi-coherent ideal $\mathcal{I}' \subset \mathcal{O}_W$.

Proof. Choose a factorization $W \to Y \to X$ where $j : W \to Y$ is a quasi-compact open immersion and $\pi : Y \to X$ is a finite morphism of finite presentation (Lemma 76.34.4). Let $V = j(W) \cup \pi^{-1}(U) \subset Y$. Note that $\mathcal{I}$ on $W \cong j(W)$ and $\mathcal{O}_{\pi^{-1}(U)}$ glue to a finite type quasi-coherent sheaf of ideals $\mathcal{I}_1 \subset \mathcal{O}_V$. By Limits of Spaces, Lemma 70.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_2 \subset \mathcal{O}_Y$ such that $\mathcal{I}_2|_V = \mathcal{I}_1$. In other words, $\mathcal{I}_2 \subset \mathcal{O}_Y$ is a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_2)$ is disjoint from $\pi^{-1}(U)$ and $j^{-1}\mathcal{I}_2 = \mathcal{I}$. Denote $i : Z \to Y$ the corresponding closed immersion which is of finite presentation (Morphisms of Spaces, Lemma 67.28.12). In particular the composition $\tau = \pi \circ i : Z \to X$ is finite and of finite presentation (Morphisms of Spaces, Lemmas 67.28.2 and 67.45.4).

Let $\mathcal{F} = \tau_*\mathcal{O}_Z$ which we think of as a quasi-coherent $\mathcal{O}_X$-module. By Descent on Spaces, Lemma 74.6.7 we see that $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{J} = \text{Fit}_0(\mathcal{F})$. (Insert reference to fitting modules on ringed topoi here.) This is a finite type quasi-coherent sheaf of ideals on X (as $\mathcal{F}$ is of finite presentation, see More on

Algebra, Lemma 15.8.4). Part (1) of the lemma holds because $|\tau|(|Z|) \cap |U| = \emptyset$ by our choice of $\mathcal{I}_2$ and because the 0th Fitting ideal of the trivial module equals the structure sheaf. To prove (2) note that $\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \text{Fit}_0(\varphi^*\mathcal{F})$ because taking Fitting ideals commutes with base change. On the other hand, as $\varphi : W \to X$ is separated and étale we see that $(1, j) : W \to W \times_X Y$ is an open and closed immersion. Hence $W \times_Y Z = V(\mathcal{I}) \amalg Z'$ for some finite and finitely presented morphism of algebraic spaces $\tau' : Z' \to W$. Thus we see that

$$\begin{aligned} \text{Fit}_0(\varphi^*\mathcal{F}) &= \text{Fit}_0((W \times_Y Z \to W)_*\mathcal{O}_{W \times_Y Z}) \\ &= \text{Fit}_0(\mathcal{O}_W/\mathcal{I}) \cdot \text{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \\ &= \mathcal{I} \cdot \text{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \end{aligned}$$

the second equality by More on Algebra, Lemma 15.8.4 translated in sheaves on ringed topoi. Setting $\mathcal{I}' = \text{Fit}_0(\tau'_*\mathcal{O}_{Z'})$ finishes the proof of the lemma. ☐

087C Theorem 76.38.2. Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset B$ be a quasi-compact open subspace. Assume

(1) X is quasi-compact,
(2) X is locally of finite presentation over B,
(3) $\mathcal{F}$ is a module of finite type,
(4) $\mathcal{F}_U$ is of finite presentation, and
(5) $\mathcal{F}_U$ is flat over U.

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_B B'}$-module of finite presentation and flat over B'.

Proof. Choose an affine scheme V and a surjective étale morphism $V \to X$. Because strict transform commutes with étale localization (Divisors on Spaces, Lemma 71.18.2) it suffices to prove the result with X replaced by V. Hence we may assume that $X \to B$ is representable (in addition to the hypotheses of the lemma).

Assume that $X \to B$ is representable. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to B$. Note that $X \times_B W$ is a scheme. By the case of schemes (More on Flatness, Theorem 38.30.7) we can find a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_W$ such that (a) $|V(\mathcal{I})| \cap |\varphi^{-1}(U)| = \emptyset$ and (b) the strict transform of $\mathcal{F}|_{X \times_B W}$ with respect to the blowing up $W' \to W$ in $\mathcal{I}$ becomes flat over W' and is a module of finite presentation. Choose a finite type sheaf of ideals $\mathcal{J} \subset \mathcal{O}_B$ as in Lemma 76.38.1. Let $B' \to B$ be the blowing up of $\mathcal{J}$. We claim that this blowup works. Namely, it is clear that $B' \to B$ is U-admissible by our choice of ideal $\mathcal{J}$. Moreover, the base change $B' \times_B W \to W$ is the blowup of W in $\varphi^{-1}\mathcal{J} = \mathcal{I}\mathcal{I}'$ (compatibility of blowup with flat base change, see Divisors on Spaces, Lemma 71.17.3). Hence there is a factorization

$$W \times_B B' \to W' \to W$$

where the first morphism is a blowup as well, see Divisors on Spaces, Lemma 71.17.10). The restriction of $\mathcal{F}'$ (which lives on $B' \times_B X$) to $W \times_B B' \times_B X$ is the strict transform of $\mathcal{F}|_{X \times_B W}$ (Divisors on Spaces, Lemma 71.18.2) and hence is the twice repeated strict transform of $\mathcal{F}|_{X \times_B W}$ by the two blowups displayed above (Divisors on Spaces, Lemma 71.18.7). After the first blowup our sheaf is already flat over the base and of finite presentation (by construction). Whence this holds after the second strict transform as well (since this is a pullback by Divisors on Spaces, Lemma 71.18.4). Thus we see that the restriction of $\mathcal{F}'$ to an étale cover of $B' \times_B X$ has the desired properties and the theorem is proved. □

76.39. Applications

087D In this section we apply the result on flattening by blowing up.

087E Lemma 76.39.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume

(1) B is quasi-compact and quasi-separated,
(2) U is quasi-compact,
(3) $f : X \to B$ is of finite type and quasi-separated, and
(4) $f^{-1}(U) \to U$ is flat and locally of finite presentation.

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X is flat and of finite presentation over B'.

Proof. Let $B' \to B$ be a U-admissible blowup. Note that the strict transform of X is quasi-compact and quasi-separated over B' as X is quasi-compact and quasi-separated over B. Hence we only need to worry about finding a U-admissible blowup such that the strict transform becomes flat and locally of finite presentation. We cannot directly apply Theorem 76.38.2 because X is not locally of finite presentation over B.

Choose an affine scheme V and a surjective étale morphism $V \to X$. (This is possible as X is quasi-compact as a finite type space over the quasi-compact space B.) Then it suffices to show the result for the morphism $V \to B$ (as strict transform commutes with étale localization, see Divisors on Spaces, Lemma 71.18.2). Hence we may assume that $X \to B$ is separated as well as finite type. In this case we can find a closed immersion $i : X \to Y$ with $Y \to B$ separated and of finite presentation, see Limits of Spaces, Proposition 70.11.7.

Apply Theorem 76.38.2 to $\mathcal{F} = i_*\mathcal{O}_X$ on Y/B. We find a U-admissible blowup $B' \to B$ such that strict transform of $\mathcal{F}$ is flat over B' and of finite presentation. Let X' be the strict transform of X under the blowup $B' \to B$. Let $i' : X' \to Y \times_B B'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors on Spaces, Lemma 71.18.5), we see that $i'_*\mathcal{O}_{X'}$ is flat over B' and of finite presentation as a $\mathcal{O}_{Y \times_B B'}$-module. Thus $X' \to B'$ is flat and locally of finite presentation. This implies the lemma by our earlier remarks. □

0B4K Lemma 76.39.2. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume

(1) B is quasi-compact and quasi-separated,
(2) U is quasi-compact,
(3) $f : X \to B$ is proper, and
(4) $f^{-1}(U) \to U$ is finite locally free.

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X is finite locally free over B'.

Proof. By Lemma 76.39.1 we may assume that $X \to B$ is flat and of finite presentation. After replacing B by a U-admissible blowup if necessary, we may assume that $U \subset B$ is scheme theoretically dense. Then f is finite by Lemma 76.37.4. Hence f is finite locally free by Morphisms of Spaces, Lemma 67.46.6. □

0GUW Lemma 76.39.3. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume

(1) B is quasi-compact and quasi-separated,
(2) U is quasi-compact,
(3) $f : X \to B$ is proper, and
(4) $f^{-1}(U) \to U$ is an isomorphism.

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X maps isomorphically to B'.

Proof. By Lemma 76.39.1 we may assume that $X \to B$ is flat and of finite presentation. After replacing B by a U-admissible blowup if necessary, we may assume that $U \subset B$ is scheme theoretically dense. Then f is finite by Lemma 76.37.4 and an open immersion by Lemma 76.37.5. Hence f is an open immersion whose image is closed and contains the dense open U, whence f is an isomorphism. □

087F Lemma 76.39.4. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume

(1) B quasi-compact and quasi-separated,
(2) U is quasi-compact,
(3) f is of finite type
(4) $f^{-1}(U) \to U$ is an isomorphism.

Then there exists a U-admissible blowup $B' \to B$ such that U is scheme theoretically dense in B' and such that the strict transform X' of X maps isomorphically to an open subspace of B'.

Proof. This lemma is a generalization of Lemma 76.39.3. As the composition of U-admissible blowups is U-admissible (Divisors on Spaces, Lemma 71.19.2) we can proceed in stages. Pick a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_B$ with $|B| \setminus |U| = |V(\mathcal{I})|$. Replace B by the blowup of B in $\mathcal{I}$ and X by the strict transform of X. After this replacement $B \setminus U$ is the support of an effective Cartier divisor D (Divisors on Spaces, Lemma 71.17.4). In particular U is scheme theoretically dense in B (Divisors on Spaces, Lemma 71.6.4). Next, we do another U-admissible blowup to get to the situation where $X \to B$ is flat and of finite presentation, see Lemma 76.39.1. Note that U is still scheme theoretically dense in B. Hence $X \to B$ is an open immersion by Lemma 76.37.5. □

The following lemma says that a modification can be dominated by a blowup.

087G Lemma 76.39.5. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume

(1) B is quasi-compact and quasi-separated,
(2) U is quasi-compact,
(3) $f : X \to B$ is proper,
(4) $f^{-1}(U) \to U$ us an isomorphism.

Then there exists a U-admissible blowup $B' \to B$ which dominates X, i.e., such that there exists a factorization $B' \to X \to B$ of the blowup morphism.

Proof. By Lemma 76.39.3 we may find a U-admissible blowup $B' \to B$ such that the strict transform X' maps isomorphically to B'. Then we can use $B' = X' \to X$ as the factorization. $\square$

0CPI **Lemma 76.39.6.** Let S be a scheme. Let X, Y be algebraic spaces over S. Let $U \subset W \subset Y$ be open subspaces. Let $f : X \to W$ and let $s : U \to X$ be morphisms such that $f \circ s = \text{id}_U$. Assume

(1) f is proper,
(2) Y is quasi-compact and quasi-separated, and
(3) U and W are quasi-compact.

Then there exists a U-admissible blowup $b : Y' \to Y$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms of Spaces, Lemma 67.16.7. By Lemma 76.39.5 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s. We finish the proof by applying Divisors on Spaces, Lemma 71.19.3 to extend $W' \to W$ to a U-admissible blowup of Y. $\square$

76.40. Chow's lemma

088P In this section we prove Chow's lemma (Lemma 76.40.5). We encourage the reader to take a look at Cohomology of Spaces, Section 69.18 for a weak version of Chow's lemma that is easy to prove and sufficient for many applications.

Since we have yet to define projective morphisms of algebraic spaces, the statements of lemmas (see for example Lemma 76.40.2) will involve representable proper morphisms, rather than projective ones.

088Q **Lemma 76.40.1.** Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $U \to X_1$ and $U \to X_2$ be open immersions of algebraic spaces over Y and assume U, X_1, X_2 of finite type and separated over Y. Then there exists a commutative diagram

$$\begin{tikzcd} X_1' \arrow[r] \arrow[d] & X & X_2' \arrow[l] \arrow[d] \\ X_1 & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & X_2 \end{tikzcd}$$

of algebraic spaces over Y where $X_i' \to X_i$ is a U-admissible blowup, $X_i' \to X$ is an open immersion, and X is separated and finite type over Y.

Proof. Throughout the proof all the algebraic spaces will be separated of finite type over Y. This in particular implies these algebraic spaces are quasi-compact and quasi-separated and that the morphisms between them will be quasi-compact and separated. See Morphisms of Spaces, Sections 67.4 and 67.8. We will use that if $U \to W$ is an immersion of such spaces over Y, then the scheme theoretic image Z of U in W is a closed subspace of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $|U| \subset |Z|$ is dense. See Morphisms of Spaces, Lemma 67.17.7.

Let $X_{12} \subset X_1 \times_Y X_2$ be the scheme theoretic image of $U \to X_1 \times_Y X_2$. The projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$ by Morphisms of

Spaces, Lemma 67.16.7. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subspace of X_i^i, see Lemma 76.39.4. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$, see Divisors on Spaces, Lemma 71.18.3. Let X_{12}' be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$, see Divisors on Spaces, Lemma 71.17.10 for what this entails. We obtain a commutative diagram

$$\begin{tikzcd} X_{12}' \arrow[r] \arrow[d] & X_{12}^2 \arrow[d] \\ X_{12}^1 \arrow[r] & X_{12} \end{tikzcd}$$

where all the morphisms are U-admissible blowing ups. Since $X_{12}^i \subset X_i^i$ is an open we may choose a U-admissible blowup $X_i' \to X_i^i$ restricting to $X_{12}' \to X_{12}^i$, see Divisors on Spaces, Lemma 71.19.3. Then $X_{12}' \subset X_i'$ is an open subspace and the diagram

$$\begin{tikzcd} X_{12}' \arrow[r] \arrow[d] & X_i' \arrow[d] \\ X_{12}^i \arrow[r] & X_i^i \end{tikzcd}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X_{12}' \to X_1' \times_Y X_2'$ is an immersion and proper (use that $X_{12}' \to X_{12}$ is proper and $X_{12} \to X_1 \times_Y X_2$ is closed and $X_1' \times_Y X_2' \to X_1 \times_Y X_2$ is separated and apply Morphisms of Spaces, Lemma 67.40.6). Thus $X_{12}' \to X_1' \times_Y X_2'$ is a closed immersion. If we define X by glueing X_1' and X_2' along the common open subspace X_{12}', then $X \to Y$ is of finite type and separated[2]. As compositions of U-admissible blowups are U-admissible blowups (Divisors on Spaces, Lemma 71.19.2) the lemma is proved. $\square$

088R Lemma 76.40.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \subset X$ be an open subspace. Assume

(1) U is quasi-compact,
(2) Y is quasi-compact and quasi-separated,
(3) there exists an immersion $U \to \mathbf{P}^n_Y$ over Y,
(4) f is of finite type and separated.

Then there exists a commutative diagram

$$\begin{tikzcd} & U \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[dl] & Z \arrow[dl] \\ & Y & \mathbf{P}^n_Y \arrow[l] & \end{tikzcd}$$

[2] Because we may check closedness of the diagonal $X \to X \times_Y X$ over the four open parts $X_i' \times_Y X_j'$ of $X \times_Y X$ where it is clear.

where the arrows with source U are open immersions, $X' \to X$ is a U-admissible blowup, $X' \to Z'$ is an open immersion, $Z' \to Y$ is a proper and representable morphism of algebraic spaces. More precisely, $Z' \to Z$ is a U-admissible blowup and $Z \to \mathbf{P}^n_Y$ is a closed immersion.

Proof. Let $Z \subset \mathbf{P}^n_Y$ be the scheme theoretic image of the immersion $U \to \mathbf{P}^n_Y$. Since $U \to \mathbf{P}^n_Y$ is quasi-compact we see that $U \subset Z$ is a (scheme theoretically) dense open subspace (Morphisms of Spaces, Lemma 67.17.7). Apply Lemma 76.40.1 to find a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X}' & Z' \arrow[l] \arrow[d] \\ X & U \arrow[l] \arrow[r] \arrow[ul] \arrow[u] \arrow[ur] & Z \end{tikzcd}$$

with properties as listed in the statement of that lemma. As $X' \to X$ and $Z' \to Z$ are U-admissible blowups we find that U is a scheme theoretically dense open of both X' and Z' (see Divisors on Spaces, Lemmas 71.17.4 and 71.6.4). Since $Z' \to Z \to Y$ is proper we see that $Z' \subset \overline{X}'$ is a closed subspace (see Morphisms of Spaces, Lemma 67.40.6). It follows that $X' \subset Z'$ (scheme theoretically), hence X' is an open subspace of Z' (small detail omitted) and the lemma is proved. ☐

088S Lemma 76.40.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated, of finite type, and Y Noetherian. Then there exists a dense open subspace $U \subset X$ and a commutative diagram

$$\begin{tikzcd} & U \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[dl] & Z \arrow[dl] \\ & Y & \mathbf{P}^n_Y \arrow[l] & \end{tikzcd}$$

where the arrows with source U are open immersions, $X' \to X$ is a U-admissible blowup, $X' \to Z'$ is an open immersion, $Z' \to Y$ is a proper and representable morphism of algebraic spaces. More precisely, $Z' \to Z$ is a U-admissible blowup and $Z \to \mathbf{P}^n_Y$ is a closed immersion.

Proof. By Limits of Spaces, Lemma 70.13.3 there exists a dense open subspace $U \subset X$ and an immersion $U \to \mathbf{A}^n_Y$ over Y. Composing with the open immersion $\mathbf{A}^n_Y \to \mathbf{P}^n_Y$ we obtain a situation as in Lemma 76.40.2 and the result follows. ☐

088T Remark 76.40.4. In Lemmas 76.40.2 and 76.40.3 the morphism $g : Z' \to Y$ is a composition of projective morphisms. Presumably (by the analogue for algebraic spaces of Morphisms, Lemma 29.37.8) there exists a g-ample invertible sheaf on Z'. If we ever need this, then we will state and prove this here.

The following result is [Knu71, IV Theorem 3.1]. Note that the immersion $X' \to \mathbf{P}^n_Y$ is quasi-compact, hence can be factored as $X' \to Z' \to \mathbf{P}^n_Y$ where the first morphism is an open immersion and the second morphism a closed immersion (Morphisms of Spaces, Lemma 67.17.7).

088U Lemma 76.40.5 (Chow's lemma). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and Noetherian. Then there exists a commutative diagram [Knu71, IV Theorem 3.1]

$$\begin{tikzcd} X \arrow[rd] & X' \arrow[l] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[ld] \\ & Y & \end{tikzcd}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. In this first paragraph of the proof we reduce the lemma to the case where Y is of finite type over $\mathrm{Spec}(\mathbf{Z})$. We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. We can write $Y = \lim Y_i$ as a directed limit of separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 70.8.1 and Lemma 70.5.9. For all i sufficiently large we can find a separated finite type morphism $X_i \to Y_i$ such that $X = Y \times_{Y_i} X_i$, see Limits of Spaces, Lemmas 70.7.1 and 70.6.9. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of $|X|$ (X is Noetherian as a finite type separated algebraic space over the Noetherian algebraic space Y and therefore $|X|$ is a Noetherian topological space). By Limits of Spaces, Lemma 70.5.2 we find that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are distinct for i large enough. We may replace X_i by the scheme theoretic image of the (quasi-compact, in fact affine) morphism $X \to X_i$. After this replacement we see that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are the generic points of the irreducible components of $|X_i|$, see Morphisms of Spaces, Lemma 67.16.3. Having said this, suppose we can find a diagram

$$\begin{tikzcd} X_i \arrow[rd] & X'_i \arrow[l] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_i} \arrow[ld] \\ & Y & \end{tikzcd}$$

where $X'_i \to X_i$ is a U_i-admissible blowup for some dense open $U_i \subset X_i$ and the morphism $X'_i \to \mathbf{P}^n_{Y_i}$ is an immersion. Then the strict transform $X' \to X$ of X relative to $X'_i \to X_i$ is a U-admissible blowing up where $U \subset X$ is the inverse image of U_i in X. Because of our carefuly chosen index i it follows that $\eta_1, \ldots, \eta_n \in |U|$ and $U \subset X$ is dense. Moreover, $X' \to \mathbf{P}^n_Y$ is an immersion as X' is closed in $X'_i \times_{X_i} X = X'_i \times_{Y_i} Y$ which comes with an immersion into $\mathbf{P}^n_Y$. Thus we have reduced to the situation of the following paragraph.

Assume that Y is separated of finite type over $\mathrm{Spec}(\mathbf{Z})$. Then $X \to \mathrm{Spec}(\mathbf{Z})$ is separated of finite type as well. We apply Lemma 76.40.3 to $X \to \mathrm{Spec}(\mathbf{Z})$ to find a dense open subspace $U \subset X$ and a commutative diagram

$$\begin{tikzcd} & U \arrow[ld] \arrow[d] \arrow[rd] \arrow[rrd] & & \\ X \arrow[rd] & X' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[ld] & Z \arrow[ld] \\ & \mathrm{Spec}(\mathbf{Z}) & \mathbf{P}^n_{\mathbf{Z}} \arrow[l] & \end{tikzcd}$$

with all the properties listed in the lemma. Note that Z has an ample invertible sheaf, namely $\mathcal{O}_{\mathbf{P}^n}(1)|_Z$. Hence $Z' \to Z$ is a H-projective morphism by Morphisms, Lemma 29.43.16. It follows that $Z' \to \mathrm{Spec}(\mathbf{Z})$ is H-projective by Morphisms, Lemma 29.43.7. Thus there exists a closed immersion $Z' \to \mathbf{P}^m_{\mathrm{Spec}(\mathbf{Z})}$ for some $m \geq 0$. It follows that the diagonal morphism

$$X' \to Y \times \mathbf{P}^m_{\mathbf{Z}} = \mathbf{P}^m_Y$$

is an immersion (because the composition with the projection to $\mathbf{P}^m_{\mathbf{Z}}$ is an immersion) and we win. □

76.41. Variants of Chow's Lemma

089K In this section we prove a number of variants of Chow's lemma dealing with morphisms between non-Noetherian algebraic spaces. The Noetherian versions are Lemma 76.40.3 and Lemma 76.40.5.

089L **Lemma 76.41.1.** Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a separated morphism of finite type. Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & \overline{X}' \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to X$ is proper surjective, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Proposition 70.11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 70.8.1. By Limits of Spaces, Lemma 70.7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 70.6.9 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 76.40.3. □

089M **Lemma 76.41.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and quasi-compact. Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to X$ is proper surjective morphism and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. By Limits of Spaces, Proposition 70.11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\text{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\text{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 70.8.1. By Limits of Spaces, Lemma 70.5.9 we may assume that Y_i is separated for all i. By Limits of Spaces, Lemma 70.7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 70.6.9 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 76.40.5. □

76.42. Grothendieck's existence theorem

089N In this section we discuss Grothendieck's existence theorem for algebraic spaces. Instead of developing a theory of "formal algebraic spaces" we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal space".

Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ of such inverse systems is simply a compatible system of morphisms $\alpha_n : \mathcal{F}_n \to \mathcal{G}_n$. Let us denote the category of these inverse systems with $\text{Coh}(X, \mathcal{I})$. We will develop some theory regarding these systems that will parallel to the corresponding results in the case of schemes, see Cohomology of Schemes, Sections 30.24, 30.25, 30.27, and 30.28.

Functoriality. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over a scheme S, and let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. In this situation there is a functor

$$f^* : \text{Coh}(Y, \mathcal{J}) \longrightarrow \text{Coh}(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. Compare with Cohomology of Schemes, Lemma 30.23.9. If f is étale, then we may think of this as simply the restriction of the system to X, see Properties of Spaces, Equation 66.26.1.1.

Étale descent. Let S be a scheme. Let $U_0 \to X$ be a surjective étale morphism of Noetherian algebraic spaces. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I}_i = \mathcal{I}|_{U_i}$. In this situation we obtain a diagram of categories

$$\text{Coh}(X, \mathcal{I}) \longrightarrow \text{Coh}(U_0, \mathcal{I}_0) \rightrightarrows \text{Coh}(U_1, \mathcal{I}_1) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \text{Coh}(U_2, \mathcal{I}_2)$$

an the first arrow presents $\text{Coh}(X, \mathcal{I})$ as the homotopy limit of the right part of the diagram. More precisely, given a descent datum, i.e., a pair $((\mathcal{G}_n), \varphi)$ where $(\mathcal{G}_n)$ is an object of $\text{Coh}(U_0, \mathcal{I}_0)$ and $\varphi : \text{pr}_0^*(\mathcal{G}_n) \to \text{pr}_1^*(\mathcal{G}_n)$ is an isomorphism in $\text{Coh}(U_1, \mathcal{I}_1)$ satisfying the cocycle condition in $\text{Coh}(U_2, \mathcal{I}_2)$, then there exists a unique object $(\mathcal{F}_n)$ of $\text{Coh}(X, \mathcal{I})$ whose associated canonical descent datum is isomorphic to $((\mathcal{G}_n), \varphi)$. Compare with Descent on Spaces, Definition 74.3.3. The proof of this statement follows immediately by applying Descent on Spaces, Proposition 74.4.1 to the descent data $(\mathcal{G}_n, \varphi_n)$ for varying n.

089P Lemma 76.42.1. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.

(1) The category $\mathrm{Coh}(X, \mathcal{I})$ is abelian.
(2) Exactness in $\mathrm{Coh}(X, \mathcal{I})$ can be checked étale locally.
(3) For any flat morphism $f : X' \to X$ of Noetherian algebraic spaces the functor $f^* : \mathrm{Coh}(X, \mathcal{I}) \to \mathrm{Coh}(X', f^{-1}\mathcal{I}\mathcal{O}_{X'})$ is exact.

Proof. Proof of (1). Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X$. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$ as in our discussion of étale descent above. The categories $\mathrm{Coh}(U_i, \mathcal{I}_i)$ are abelian (Cohomology of Schemes, Lemma 30.23.2) and the pullback functors are exact functors $\mathrm{Coh}(U_0, \mathcal{I}_0) \to \mathrm{Coh}(U_1, \mathcal{I}_1)$ and $\mathrm{Coh}(U_1, \mathcal{I}_1) \to \mathrm{Coh}(U_2, \mathcal{I}_2)$ (Cohomology of Schemes, Lemma 30.23.9). The lemma then follows formally from the description of $\mathrm{Coh}(X, \mathcal{I})$ as a category of descent data. Some details omitted; compare with the proof of Groupoids, Lemma 39.14.6.

Part (2) follows immediately from the discussion in the previous paragraph. In the situation of (3) choose a commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

where U' and U are affine schemes and the vertical morphisms are surjective étale. Then $U' \to U$ is a flat morphism of Noetherian schemes (Morphisms of Spaces, Lemma 67.30.5) whence the pullback functor $\mathrm{Coh}(U, \mathcal{I}\mathcal{O}_U) \to \mathrm{Coh}(U', \mathcal{I}\mathcal{O}_{U'})$ is exact by Cohomology of Schemes, Lemma 30.23.9. Since we can check exactness in $\mathrm{Coh}(X, \mathcal{O}_X)$ on U and similarly for X', U' the assertion follows. $\square$

08B3 Lemma 76.42.2. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $\mathrm{Coh}(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.

Proof. We can check on an affine étale cover of X by Lemma 76.42.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.3. $\square$

Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

08B4 (76.42.2.1) $$\mathrm{Coh}(\mathcal{O}_X) \longrightarrow \mathrm{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $\mathrm{Coh}(X, \mathcal{I})$.

08B5 Lemma 76.42.3. The functor (76.42.2.1) is exact.

Proof. It suffices to check this étale locally on X, see Lemma 76.42.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.4. $\square$

08B6 **Lemma 76.42.4.** Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{\mathrm{Coh}(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge).$$

Proof. Since $\mathcal{H}$ is a sheaf on $X_{étale}$ and since we have étale descent for objects of $\mathrm{Coh}(X, \mathcal{I})$ it suffices to prove this étale locally. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.5. □

We introduce the setting that we will focus on throughout the rest of this section.

08B7 **Situation 76.42.5.** Here A is a Noetherian ring complete with respect to an ideal I. Also $f : X \to \mathrm{Spec}(A)$ is a finite type separated morphism of algebraic spaces and $\mathcal{I} = I\mathcal{O}_X$.

In this situation we denote

$$\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $\mathrm{Coh}(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose support is proper over $\mathrm{Spec}(A)$, or equivalently whose scheme theoretic support is proper over $\mathrm{Spec}(A)$, see Derived Categories of Spaces, Lemma 75.7.7. Similarly, we let

$$\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $\mathrm{Coh}(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (76.42.2.1) induces a functor

08B8 $$\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow \mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I}) \quad (76.42.5.1)$$

Our first result is that this functor is fully faithful.

08B9 **Lemma 76.42.6.** In Situation 76.42.5. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that the intersection of the supports of $\mathcal{F}$ and $\mathcal{G}$ is proper over $\mathrm{Spec}(A)$. Then the map

$$\mathrm{Mor}_{\mathrm{Coh}(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{\mathrm{Coh}(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge)$$

coming from (76.42.2.1) is a bijection. In particular, (76.42.5.1) is fully faithful.

Proof. Let $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_X$-module because its restriction of schemes étale over X is coherent by Modules, Lemma 17.22.6. By Lemma 76.42.4 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{\mathrm{Coh}(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that Z is a closed subspace such that $|Z|$ is contained in the intersection of the supports of $\mathcal{F}$ and $\mathcal{G}$. Hence $Z \to \mathrm{Spec}(A)$ is proper by assumption (see Derived Categories of Spaces, Section 75.7). Write $\mathcal{H} = i_*\mathcal{H}'$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}'$. We have $i_*(\mathcal{H}'/I^n\mathcal{H}') = \mathcal{H}/I^n\mathcal{H}$. Hence we obtain

$$\begin{aligned}
\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) &= \lim_n H^0(Z, \mathcal{H}'/\mathcal{I}^n\mathcal{H}') \\
&= H^0(Z, \mathcal{H}') \\
&= H^0(X, \mathcal{H}) \\
&= \mathrm{Mor}_{\mathrm{Coh}(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G})
\end{aligned}$$

the second equality by the theorem on formal functions (Cohomology of Spaces, Lemma 69.22.6). This proves the lemma. □

08BA Remark 76.42.7. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $\text{Coh}(X, \mathcal{I})$. Given an affine scheme $U = \text{Spec}(A)$ and a surjective étale morphism $U \to X$ denote $I, K \subset A$ the ideals corresponding to the restrictions $\mathcal{I}|_U, \mathcal{K}|_U$. Denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Cohomology of Schemes, Lemma 30.23.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\text{Ker}(\alpha_n)$ and $\text{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any (or some) affine open $\text{Spec}(A) = U \subset X$ as above the modules $\text{Ker}(\alpha_U)$ and $\text{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$.

If these equivalent conditions hold we will say that α is a map whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. To see the equivalence we refer to Cohomology of Schemes, Remark 30.25.1.

08BB Lemma 76.42.8. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $\text{Coh}(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where

(1) $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,
(2) $a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,
(3) $\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and
(4) $\alpha = a^\wedge \circ \beta$.

Proof. The uniqueness and étale descent for objects of $\text{Coh}(X, \mathcal{I})$ and $\text{Coh}(\mathcal{O}_X)$ implies it suffices to construct $(\mathcal{F}, a, \beta)$ étale locally on X. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.6. □

08BC Lemma 76.42.9. In Situation 76.42.5. Let $\mathcal{K} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $X_e \subset X$ be the closed subspace cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. Assume

(1) the functor $\text{Coh}_{\text{support proper over } A}(\mathcal{O}_{X_e}) \to \text{Coh}_{\text{support proper over } A}(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and
(2) there exists an object $\mathcal{H}$ of $\text{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$ and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Then $(\mathcal{F}_n)$ is in the essential image of (76.42.5.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Cohomology of Spaces, Lemma 69.12.8. In particular we think of

$$\text{Coh}_{\text{support proper over } A}(\mathcal{O}_{X_e}) \subset \text{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$$

as the full subcategory of consisting of modules annihilated by $\mathcal{K}^e$ and

$$\text{Coh}_{\text{support proper over } A}(X_e, \mathcal{I}_e) \subset \text{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

as the full subcategory of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (76.42.5.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $\text{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. We obtain an object $(\mathcal{G}_e)$ of the category $\text{Coh}_{\text{support proper over } A}(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$ (details omitted; see Cohomology of Schemes, Remark 30.25.1). Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

(details omitted; see Cohomology of Schemes, Remark 30.25.1). We apply Lemma 76.42.8 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $\text{Coh}(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $\text{Coh}(X, \mathcal{K})$. Thus the uniqueness in Lemma 76.42.8 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight.

To finish the proof of the lemma we still have to show that the support of $\mathcal{F}$ is proper over A. By construction the kernel of $a : \mathcal{F} \to \mathcal{H}$ is annihilated by a power of $\mathcal{K}$. Hence the support of this kernel is contained in the support of $\mathcal{G}_1$. Since $\mathcal{G}_1$ is an object of $\text{Coh}_{\text{support proper over } A}(\mathcal{O}_{X_1})$ we see this is proper over A. Combined with the fact that the support of $\mathcal{H}$ is proper over A we conclude that the support of $\mathcal{F}$ is proper over A by Derived Categories of Spaces, Lemma 75.7.6. □

08BD Lemma 76.42.10. Let S be a scheme. Let $f : X \to Y$ be a representable proper morphism of Noetherian algebraic spaces over S. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Assume f is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $\text{Coh}(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $\text{Coh}(X, \mathcal{I})$. Then there exists a map

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $\text{Coh}(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Cohomology of Spaces, Lemma 69.20.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness and étale descent for $\text{Coh}(Y, \mathcal{J})$ it suffices to prove this étale locally on Y. Thus we may assume Y is the spectrum of a Noetherian ring. As f is representable we see that X is a scheme as well. Thus we reduce to the case of schemes, see proof of Cohomology of Schemes, Lemma 30.25.3. □

08BE Theorem 76.42.11 (Grothendieck's existence theorem). In Situation 76.42.5 the functor (76.42.5.1) is an equivalence.

Proof. We will use the equivalence of categories of Cohomology of Spaces, Lemma 69.12.8 without further mention in the proof of the theorem. By Lemma 76.42.6 the functor is fully faithful. Thus we need to prove the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $\text{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Cohomology of Spaces, Lemma 69.13.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $\text{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is in the essential image, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 76.40.5) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Note that Y is a scheme and f representable. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 29.43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $|T_n| = |T_1|$, hence T_n is proper over A for all n (Morphisms of Spaces, Lemma 67.40.7). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms of Spaces, Lemma 67.40.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 29.41.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $\text{Coh}(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Cohomology of Schemes, Lemma 30.24.3) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $\text{Coh}(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \text{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\text{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the Jacobson radical of A (Algebra, Lemma 10.96.6). But we've seen above that $Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Cohomology of Spaces, Lemma 69.20.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 76.42.10 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ in the category $\text{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $|Z_0| \subset f(|Z'|)$. Hence $Z_0 \to \text{Spec}(A)$ is proper (Morphisms of Spaces, Lemma

67.40.7). Thus $\mathcal{H}$ is an object of $\text{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$. Since each of the sheaves of ideals $\mathcal{K}^e$ is an element of Ξ we see that the assumptions of Lemma 76.42.9 are satisfied and we conclude. $\square$

08BF Remark 76.42.12 (Unwinding Grothendieck's existence theorem). Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots & X \arrow[d] \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots & S \end{tikzcd}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\text{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 76.42.11 says that the completion functor

$$\begin{matrix} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{matrix} \longrightarrow \begin{matrix} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{matrix}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

76.43. Grothendieck's algebraization theorem

0A00 This section is the analogue of Cohomology of Schemes, Section 30.28. However, this section is missing the result on algebraization of deformations of proper algebraic spaces endowed with ample invertible sheaves, as a proper algebraic space which comes with an ample invertible sheaf is already a scheme. We do have an algebraization result on proper algebraic spaces of relative dimension 1. Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

08BG Lemma 76.43.1. Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. Assume that

(1) $Z_1 \to X_1$ is a closed immersion, and
(2) $Z_1 \to S_1$ is proper.

Then there exists a closed immersion of algebraic spaces $Z \to X$ such that $Z_n = Z \times_S S_n$ for all $n \geq 1$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Limits of Spaces, Lemma 70.15.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 76.42.12, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 76.42.3) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. $\square$

0A01 Lemma 76.43.2. Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. Assume that

(1) $Y_1 \to X_1$ is a finite morphism, and
(2) $Y_1 \to S_1$ is proper.

Then there exists a finite morphism of algebraic spaces $Y \to X$ such that $Y_n = Y \times_S S_n$ for all $n \geq 1$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of f_n to X_1 is f_1. Thus Lemma 76.10.2 shows that f_n is a finite morphism. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 76.42.12, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fully faithful (Theorem 76.42.11) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. $\square$

0A4Z Lemma 76.43.3. Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let X, Y be algebraic spaces over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams

$$\begin{tikzcd} & & X_{n+1} \arrow[rr, "g_{n+1}"] \arrow[dr] & & Y_{n+1} \arrow[dl] \\ X_n \arrow[urr] \arrow[rr, "g_n"] \arrow[dr] & & Y_n \arrow[urr] \arrow[dl] & S_{n+1} & \\ & S_n \arrow[urr] & & & \end{tikzcd}$$

Assume that

(1) $X \to S$ is proper, and
(2) $Y \to S$ is separated of finite type.

Then there exists a unique morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Morphisms of Spaces, Lemma 67.4.7). Thus by Lemma 76.43.1 there exists a closed subspace $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms of Spaces, Lemma 67.40.6) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is quasi-finite on an open subspace of Z containing every point of Z_0 for example by Morphisms of Spaces, Lemma 67.34.7. As Z is proper over S this open neighbourhood is all of Z. We conclude that $p : Z \to X$ is finite by Zariski's main theorem (for example apply Lemma 76.34.3 and use properness of Z over X to see that the immersion is a closed immersion). Applying the equivalence of Theorem 76.42.11 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. $\square$

0GHK Remark 76.43.4. We can ask if in Grothendieck's algebraization theorem (in the form of Lemma 76.43.3), we can get by with weaker separation axioms on the target. Let us be more precise. Let A, I, S, S_n, X, Y, X_n, Y_n, and g_n be as in the statement of Lemma 76.43.3 and assume that

(1) $X \to S$ is proper, and
(2) $Y \to S$ is locally of finite type.

Does there exist a morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n? We don't know the answer in general; if you do please email stacks.project@gmail.com. If $Y \to S$ is separated, then the result holds by the lemma (there is an immediate reduction to the case where X is finite type over S, by choosing a quasi-compact open containing the image of g_1). If we only assume $Y \to S$ is quasi-separated, then the result is true as well. First, as before we may assume Y is quasi-compact as well as quasi-separated. Then we can use either [Bha16] or from [HR19] to algebraize (g_n). Namely, to apply the first reference, we use

$$D_{perf}(X) \to \lim D_{perf}(X_n) \xrightarrow{\lim Lg_n^*} \lim D_{perf}(Y_n) = D_{perf}(Y)$$

where the last step uses a Grothendieck existence result for the derived category of the proper algebraic space Y over R (compare with Flatness on Spaces, Remark 77.13.7). The paper cited shows that this arrow determines a morphism $Y \to X$ as desired. To apply the second reference we use the same argument with coherent modules:

$$\mathrm{Coh}(\mathcal{O}_X) \to \lim \mathrm{Coh}(\mathcal{O}_{X_n}) \xrightarrow{\lim g_n^*} \lim \mathrm{Coh}(\mathcal{O}_{Y_n}) = \mathrm{Coh}(\mathcal{O}_Y)$$

where the final equality is a consequence of Grothendieck's existence theorem (Theorem 76.42.11). The second reference tells us that this functor corresponds to a morphism $Y \to X$ over R. If we ever need this generalization we will precisely state and carefully prove the result here.

0E7R Lemma 76.43.5. Let $(A, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/\mathfrak{m}^n)$. Consider a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. If $\dim(X_1) \leq 1$, then there exists a projective morphism of schemes $X \to S$ and isomorphisms $X_n \cong X \times_S S_n$ compatible with i_n.

Proof. By Spaces over Fields, Lemma 72.9.3 the algebraic space X_1 is a scheme. Hence X_1 is a proper scheme of dimension ≤ 1 over κ. By Varieties, Lemma 33.43.4 we see that X_1 is H-projective over κ. Let $\mathcal{L}_1$ be an ample invertible sheaf on X_1.

We are going to show that $\mathcal{L}_1$ lifts to a compatible system $\{\mathcal{L}_n\}$ of invertible sheaves on $\{X_n\}$. Observe that X_n is a scheme too by Lemma 76.9.5. Recall that $X_1 \to X_n$ induces homeomorphisms of underlying topological spaces. In the rest of the proof we do not distinguish between sheaves on X_n and sheaves on X_1. Suppose, given a lift $\mathcal{L}_n$ to X_n. We consider the exact sequence

$$1 \to (1 + \mathfrak{m}^n\mathcal{O}_{X_{n+1}})^* \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 1$$

of sheaves on X_{n+1}. The class of $\mathcal{L}_n$ in $H^1(X_n, \mathcal{O}^*_{X_n})$ (see Cohomology, Lemma 20.6.1) can be lifted to an element of $H^1(X_{n+1}, \mathcal{O}^*_{X_{n+1}})$ if and only if the obstruction in $H^2(X_{n+1}, (1 + \mathfrak{m}^n\mathcal{O}_{X_{n+1}})^*)$ is zero. As X_1 is a Noetherian scheme of dimension ≤ 1 this cohomology group vanishes (Cohomology, Proposition 20.20.7).

By Grothendieck's algebraization theorem (Cohomology of Schemes, Theorem 30.28.4) we find a projective morphism of schemes $X \to S = \mathrm{Spec}(A)$ and a compatible system of isomorphisms $X_n = S_n \times_S X$. □

0AE7 Lemma 76.43.6. Let $(A, \mathfrak{m}, \kappa)$ be a complete Noetherian local ring. Let X be an algebraic space over $\mathrm{Spec}(A)$. If $X \to \mathrm{Spec}(A)$ is proper and $\dim(X_\kappa) \leq 1$, then X is a scheme projective over A.

Proof. Set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$. By Lemma 76.43.5 there exists a projective morphism $Y \to \mathrm{Spec}(A)$ and compatible isomorphisms $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n) \cong X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$. By Lemma 76.43.3 we see that $X \cong Y$ and the proof is complete. □

76.44. Regular immersions

06BL This section is the analogue of Divisors, Section 31.21 for morphisms of algebraic spaces. The reader is encouraged to read up on regular immersions of schemes in that section first.

In Divisors, Section 31.21 we defined four types of regular immersions for morphisms of schemes. Of these only three are (as far as we know) local on the target for the étale topology; as usual plain old regular immersions aren't. This is why for morphisms of algebraic spaces we cannot actually define regular immersions. (These kinds of annoyances prompted Grothendieck and his school to replace original notion of a regular immersion by a Koszul-regular immersions, see [BGI71, Exposee

VII, Definition 1.4].) But we can define Koszul-regular, H_1-regular, and quasi-regular immersions. Another remark is that since Koszul-regular immersions are not preserved by arbitrary base change, we cannot use the strategy of Morphisms of Spaces, Section 67.3 to define them. Similarly, as Koszul-regular immersions are not étale local on the source, we cannot use Morphisms of Spaces, Lemma 67.22.1 to define them either. We replace this lemma instead by the following.

06BM Lemma 76.44.1. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the target. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Consider commutative diagrams

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where V is a scheme and $V \to Y$ is étale. The following are equivalent

(1) for any diagram as above the projection $X \times_Y V \to V$ has property $\mathcal{P}$, and
(2) for some diagram as above with $V \to Y$ surjective the projection $X \times_Y V \to V$ has property $\mathcal{P}$.

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate. Assume

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Y V' \arrow[r] \arrow[d] & V' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are two diagrams as in the lemma. Assume $V \to Y$ is surjective and $X \times_Y V \to V$ has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that $X \times_Y V' \to V'$ has $\mathcal{P}$. To do this consider the diagram

$$\begin{tikzcd} X \times_Y V \arrow[d] & (X \times_Y V) \times_X (X \times_Y V') \arrow[l] \arrow[r] \arrow[d] & X \times_Y V' \arrow[d] \\ V & V \times_Y V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By our assumption that $\mathcal{P}$ is étale local on the source, we see that $\mathcal{P}$ is preserved under étale base change, see Descent, Lemma 35.22.2. Hence if the left vertical arrow has $\mathcal{P}$ the so does the middle vertical arrow. Since $U \times_X U' \to U'$ is surjective and étale (hence defines an étale covering of U') this implies (as $\mathcal{P}$ is assumed local for the étale topology on the target) that the left vertical arrow has $\mathcal{P}$.

If X and Y are representable, then we can take $\mathrm{id}_Y : Y \to Y$ as our étale covering to see the final statement of the lemma is true. $\square$

Note that "being a Koszul-regular (resp. H_1-regular, resp. quasi-regular) immersion" is a property of morphisms of schemes which is fpqc local on the target, see Descent, Lemma 35.23.32. Hence the following definition now makes sense.

06BN Definition 76.44.2. Let S be a scheme. Let $i : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say i is a Koszul-regular immersion if i is representable and the equivalent conditions of Lemma 76.44.1 hold with $\mathcal{P}(f) =$"f is a Koszul-regular immersion".
(2) We say i is an H_1-regular immersion if i is representable and the equivalent conditions of Lemma 76.44.1 hold with $\mathcal{P}(f) =$"f is an H_1-regular immersion".
(3) We say i is a quasi-regular immersion if i is representable and the equivalent conditions of Lemma 76.44.1 hold with $\mathcal{P}(f) =$"f is a quasi-regular immersion".

06BP Lemma 76.44.3. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. We have the following implications: i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.

Proof. Via the definition this lemma immediately reduces to Divisors, Lemma 31.21.2. □

09RW Lemma 76.44.4. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Assume X is locally Noetherian. Then i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.

Proof. Via Definition 76.44.2 (and the definition of a locally Noetherian algebraic space in Properties of Spaces, Section 66.7) this immediately translates to the case of schemes which is Divisors, Lemma 31.21.3. □

09RX Lemma 76.44.5. Let S be a scheme. Let $i : Z \to X$ be a Koszul-regular, H_1-regular, or quasi-regular immersion of algebraic spaces over S. Let $X' \to X$ be a flat morphism of algebraic spaces over S. Then the base change $i' : Z \times_X X' \to X'$ is a Koszul-regular, H_1-regular, or quasi-regular immersion.

Proof. Via Definition 76.44.2 (and the definition of a flat morphism of algebraic spaces in Morphisms of Spaces, Section 67.30) this lemma reduces to the case of schemes, see Divisors, Lemma 31.21.4. □

09RY Lemma 76.44.6. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then i is a quasi-regular immersion if and only if the following conditions are satisfied

(1) i is locally of finite presentation,
(2) the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and
(3) the map (76.6.1.2) is an isomorphism.

Proof. Follows from the case of schemes (Divisors, Lemma 31.21.5) via étale localization (use Definition 76.44.2 and Lemma 76.6.2). □

09RZ Lemma 76.44.7. Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces over S. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Lemma 76.5.6

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and (étale) locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 76.44.6 and Lemma 76.44.3) it suffices to prove that the sequence is exact. It suffices to show that the first map is injective as the sequence is already right exact in general. After étale localization on X this reduces to the case of schemes, see Divisors, Lemma 31.21.6. $\square$

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 10.69.8. The other types of regular immersions are preserved under composition.

09S0 Lemma 76.44.8. Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S.

(1) If i and j are Koszul-regular immersions, so is $j \circ i$.
(2) If i and j are H_1-regular immersions, so is $j \circ i$.
(3) If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.

Proof. Immediate from the case of schemes, see Divisors, Lemma 31.21.7. $\square$

09S1 Lemma 76.44.9. Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume that the sequence

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 76.5.6 is exact and locally split.

(1) If $j \circ i$ is a quasi-regular immersion, so is i.
(2) If $j \circ i$ is a H_1-regular immersion, so is i.
(3) If both j and $j \circ i$ are Koszul-regular immersions, so is i.

Proof. Immediate from the case of schemes, see Divisors, Lemma 31.21.8. $\square$

09S2 Lemma 76.44.10. Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume X is locally Noetherian. The following are equivalent

(1) i and j are Koszul regular immersions,
(2) i and $j \circ i$ are Koszul regular immersions,
(3) $j \circ i$ is a Koszul regular immersion and the conormal sequence

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Immediate from the case of schemes, see Divisors, Lemma 31.21.9. $\square$

76.45. Relative pseudo-coherence

0CSV This section is the analogue of More on Morphisms, Section 37.59. However, in the treatment of this material for algebraic spaces we have decided to work exclusively with objects in the derived category whose cohomology sheaves are quasi-coherent. There are two reasons for this: (1) it greatly simplifies the exposition and (2) we currently have no use for the more general notion.

0CSW Remark 76.45.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of representable algebraic spaces over S which is locally of finite type. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then f_0 is locally of finite type. If E is an object of $D_{QCoh}(\mathcal{O}_X)$, then E is the pullback of a unique object E_0 in $D_{QCoh}(\mathcal{O}_{X_0})$, see Derived Categories of Spaces, Lemma 75.4.2. In this situation the phrase "E is m-pseudo-coherent relative to Y" will be taken to mean "E_0 is m-pseudo-coherent relative to Y_0" as defined in More on Morphisms, Section 37.59.

0CSX Lemma 76.45.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $m \in \mathbf{Z}$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. With notation as explained in Remark 76.45.1 the following are equivalent:

(1) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale, the complex $E|_U$ is m-pseudo-coherent relative to V,

(2) for some commutative diagram as in (1) with $U \to X$ surjective, the complex $E|_U$ is m-pseudo-coherent relative to V,

(3) for every commutative diagram as in (1) with U and V affine the complex $R\Gamma(U, E)$ of $\mathcal{O}_X(U)$-modules is m-pseudo-coherent relative to $\mathcal{O}_Y(V)$.

Proof. Part (1) implies (3) by More on Morphisms, Lemma 37.59.7.

Assume (3). Pick any commutative diagram as in (1) with $U \to X$ surjective. Choose an affine open covering $V = \bigcup V_j$ and affine open coverings $(U \to V)^{-1}(V_j) = \bigcup U_{ij}$. By (3) and More on Morphisms, Lemma 37.59.7 we see that $E|_U$ is m-pseudo-coherent relative to V. Thus (3) implies (2).

Assume (2). Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, the morphism $U \to X$ is surjective, and $E|_U$ is m-pseudo-coherent relative to V. Next, suppose given a second commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & V' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows and U', V' schemes. We want to show that $E|_{U'}$ is m-pseudo-coherent relative to V'. The morphism $U'' = U \times_X U' \to U'$ is surjective étale and $U'' \to V'$ factors through $V'' = V' \times_Y V$ which is étale over V'. Hence it suffices to show that $E|_{U''}$ is m-pseudo-coherent relative to V'', see More on Morphisms, Lemmas 37.70.1 and 37.70.2. Using the second lemma once more it suffices to show that $E|_{U''}$ is m-pseudo-coherent relative to V. This is true by More on Morphisms, Lemma 37.59.16 and the fact that an étale morphism of schemes is pseudo-coherent by More on Morphisms, Lemma 37.60.6. □

0CSY Definition 76.45.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is m-pseudo-coherent relative to Y if the equivalent conditions of Lemma 76.45.2 are satisfied.
(2) We say E is pseudo-coherent relative to Y if E is m-pseudo-coherent relative to Y for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is m-pseudo-coherent relative to Y if $\mathcal{F}$ viewed as an object of $D_{QCoh}(\mathcal{O}_X)$ is m-pseudo-coherent relative to Y.
(4) We say $\mathcal{F}$ is pseudo-coherent relative to Y if $\mathcal{F}$ viewed as an object of $D_{QCoh}(\mathcal{O}_X)$ is pseudo-coherent relative to Y.

Most of the properties of pseudo-coherent complexes relative to a base will follow immediately from the corresponding properties in the case of schemes. We will add the relevant lemmas here as needed.

0DII Lemma 76.45.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let E in $D_{QCoh}(\mathcal{O}_X)$. If f is flat and locally of finite presentation, then the following are equivalent

(1) E is pseudo-coherent relative to Y, and
(2) E is pseudo-coherent on X.

Proof. By étale localization and the definitions we may assume X and Y are schemes. For the case of schemes this follows from More on Morphisms, Lemma 37.59.18. $\square$

76.46. Pseudo-coherent morphisms

06BQ This section is the analogue of More on Morphisms, Section 37.60 for morphisms of schemes. The reader is encouraged to read up on pseudo-coherent morphisms of schemes in that section first.

The property "pseudo-coherent" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 37.60.10 and 37.60.13 and Descent, Lemma 35.32.6. By Morphisms of Spaces, Lemma 67.22.1 we may define the notion of a pseudo-coherent morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 37.60 when the algebraic spaces in question are representable.

06BR Definition 76.46.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is pseudo-coherent if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}$ ="pseudo-coherent".
(2) Let $x \in |X|$. We say f is pseudo-coherent at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is pseudo-coherent.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

06BS Lemma 76.46.2. A flat base change of a pseudo-coherent morphism is pseudo-coherent.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.3. $\square$

06BT Lemma 76.46.3. A composition of pseudo-coherent morphisms is pseudo-coherent.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.4. □

06BU Lemma 76.46.4. A pseudo-coherent morphism is locally of finite presentation.

Proof. Immediate from the definitions. □

06BV Lemma 76.46.5. A flat morphism which is locally of finite presentation is pseudo-coherent.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.6. □

06BW Lemma 76.46.6. Let $f : X \to Y$ be a morphism of algebraic spaces pseudo-coherent over a base algebraic space B. Then f is pseudo-coherent.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.7. □

06BX Lemma 76.46.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.9. □

76.47. Perfect morphisms

06BY This section is the analogue of More on Morphisms, Section 37.61 for morphisms of schemes. The reader is encouraged to read up on perfect morphisms of schemes in that section first.

The property "perfect" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 37.61.10 and 37.61.14 and Descent, Lemma 35.32.6. By Morphisms of Spaces, Lemma 67.22.1 we may define the notion of a perfect morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 37.61 when the algebraic spaces in question are representable.

06BZ Definition 76.47.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is perfect if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}$ ="perfect".
(2) Let $x \in |X|$. We say f is perfect at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is perfect.

Note that a perfect morphism is pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

06C0 Lemma 76.47.2. A flat base change of a perfect morphism is perfect.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.61.3. □

06C1 Lemma 76.47.3. A composition of perfect morphisms is perfect.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.61.4. □

06C2 Lemma 76.47.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is flat and perfect, and
(2) f is flat and locally of finite presentation.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.61.5. □

0E4U Lemma 76.47.5. Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a perfect proper morphism of algebraic spaces. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.

Proof. We claim that Derived Categories of Spaces, Lemma 75.22.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and Y affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Thus it suffices to show that $\mathcal{O}_X$ has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_Y$-modules on the étale site. By Derived Categories of Spaces, Lemma 75.13.4 it suffices to check this on the Zariski site. This is equivalent to being perfect for finite type morphisms of schemes by More on Morphisms, Lemma 37.61.11. □

76.48. Local complete intersection morphisms

06C3 This section is the analogue of More on Morphisms, Section 37.62 for morphisms of schemes. The reader is encouraged to read up on local complete intersection morphisms of schemes in that section first.

The property "being a local complete intersection morphism" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 37.62.19 and 37.62.20 and Descent, Lemma 35.32.6. By Morphisms of Spaces, Lemma 67.22.1 we may define the notion of a local complete intersection morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 37.62 when the algebraic spaces in question are representable.

06C4 Definition 76.48.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is a Koszul morphism, or that f is a local complete intersection morphism if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}(f) =$"f is a local complete intersection morphism".
(2) Let $x \in |X|$. We say f is Koszul at x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is a local complete intersection morphism.

In some sense the defining property of a local complete intersection morphism is the result of the following lemma.

06C5 Lemma 76.48.2. Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Let P be an algebraic space smooth over Y.

Let $U \to X$ be an étale morphism of algebraic spaces and let $i : U \to P$ an immersion of algebraic spaces over Y. Picture:

$$
\begin{tikzcd}
X \arrow[dr] & U \arrow[l] \arrow[r, "i"'] \arrow[d] & P \arrow[dl] \\
& Y &
\end{tikzcd}
$$

Then i is a Koszul-regular immersion of algebraic spaces.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme W and a surjective étale morphism $W \to P \times_Y V$. Set $U' = U \times_P W$, which is a scheme étale over U. We have to show that $U' \to W$ is a Koszul-regular immersion of schemes, see Definition 76.44.2. By Definition 76.48.1 above the morphism of schemes $U' \to V$ is a local complete intersection morphism. Hence the result follows from More on Morphisms, Lemma 37.62.3. $\square$

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

06C6 Lemma 76.48.3. Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then

(1) f is locally of finite presentation,
(2) f is pseudo-coherent, and
(3) f is perfect.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.4. $\square$

Beware that a base change of a Koszul morphism is not Koszul in general.

06C7 Lemma 76.48.4. A flat base change of a local complete intersection morphism is a local complete intersection morphism.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.6. $\square$

06C8 Lemma 76.48.5. A composition of local complete intersection morphisms is a local complete intersection morphism.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.7. $\square$

06C9 Lemma 76.48.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent

(1) f is flat and a local complete intersection morphism, and
(2) f is syntomic.

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.8. $\square$

0CHK Lemma 76.48.7. Let S be a scheme. A Koszul-regular immersion of algebraic spaces over S is a local complete intersection morphism.

Proof. Let $i : X \to Y$ be a Koszul-regular immersion of algebraic spaces over S. By definition there exists a surjective étale morphism $V \to Y$ where V is a scheme such that $X \times_Y V$ is a scheme and the base change $X \times_Y V \to V$ is a Koszul-regular immersion of schemes. By More on Morphisms, Lemma 37.62.9 we see that $X \times_Y V \to V$ is a local complete intersection morphism. From Definition 76.48.1 we conclude that i is a local complete intersection morphism of algebraic spaces. □

0CHL Lemma 76.48.8. Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume $Y \to Z$ is smooth and $X \to Z$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to W$ is a local complete intersection morphism of schemes and $V \to W$ is a smooth morphism of schemes. By the result for schemes (More on Morphisms, Lemma 37.62.10) we conclude that $U \to V$ is a local complete intersection morphism. By definition this means that f is a local complete intersection morphism. □

0CHM Lemma 76.48.9. The property $\mathcal{P}(f)$ ="f is a local complete intersection morphism" is fpqc local on the base.

Proof. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}$ be an fpqc covering (Topologies on Spaces, Definition 73.9.1). Let $f_i : X_i \to Y_i$ be the base change of f by $Y_i \to Y$. If f is a local complete intersection morphism, then each f_i is a local complete intersection morphism by Lemma 76.48.4.

Conversely, assume each f_i is a local complete intersection morphism. We may replace the covering by a refinement (again because flat base change preserves the property of being a local complete intersection morphism). Hence we may assume Y_i is a scheme for each i, see Topologies on Spaces, Lemma 73.9.5. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. We have to show that $U \to V$ is a local complete intersection morphism of schemes. By Topologies on Spaces, Lemma 73.9.4 we have that $\{Y_i \times_Y V \to V\}$ is an fpqc covering of schemes. By the case of schemes (More on Morphisms, Lemma 37.62.19) it suffices to prove the base change

$$U \times_Y Y_i = U \times_V (V \times_Y Y_i) \longrightarrow V$$

of $U \to V$ by $V \times_Y Y_i \to V$ is a local complete intersection morphism. We can write this as the composition

$$U \times_Y Y_i \longrightarrow (V \times_Y X) \times_Y Y_i = V \times_Y X_i \longrightarrow V \times_Y Y_i$$

The first arrow is an étale morphism of schemes (as a base change of $U \to V \times_Y X$) and the second arrow is a local complete intersection morphism of schemes as a flat base change of f_i. The result follows as being a local complete intersection morphism is syntomic local on the source and since étale morphisms are syntomic (More on Morphisms, Lemma 37.62.20 and Morphisms, Lemma 29.36.10). □

0CHN Lemma 76.48.10. The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is syntomic local on the source.

Proof. This follows from Descent on Spaces, Lemma 74.14.3 and More on Morphisms, Lemma 37.62.20. □

06CA Lemma 76.48.11. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces over S. Assume that both p and q are flat and locally of finite presentation. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is Koszul. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of More on Morphisms, Lemma 37.62.21 and in fact we will deduce the lemma from it. By Definition 76.48.1 the set $\{x \in |X| : f \text{ is Koszul at } x\}$ is open in $|X|$ hence by Properties of Spaces, Lemma 66.4.8 it corresponds to an open subspace $U(f)$ of X. Hence we only need to prove the final statement.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 66.16.7). Note that by definition $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism of schemes $U \to V$ is Koszul. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism of schemes $U' \to V'$ is Koszul. Now, by construction the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is cartesian (in the category of schemes). Hence the aforementioned More on Morphisms, Lemma 37.62.21 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

06CB Lemma 76.48.12. Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.

Proof. This follows from the corresponding result for morphisms of schemes, see More on Morphisms, Lemma 37.62.22, by étale localization, see Lemma 76.15.11. (Note that in the situation of this lemma the morphism $V \to U$ is unramified and a local complete intersection morphism by definition.) $\square$

06CC Lemma 76.48.13. Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 76.5.6 is short exact.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme V and a surjective étale morphism $V \to U \times_X Y$. Choose a scheme W and a surjective étale morphism $W \to V \times_Y Z$. By Lemma 76.15.11 the morphisms $W \to V$ and $V \to U$ are formally unramified. Moreover the sequence $i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$ restricts to the corresponding sequence $i^*\mathcal{C}_{V/U} \to \mathcal{C}_{W/U} \to \mathcal{C}_{W/V} \to 0$ for $W \to V \to U$. Hence the result follows from the result for schemes (More on Morphisms, Lemma 37.62.23) as by definition the morphism $W \to V$ is a local complete intersection morphism. $\square$

76.49. When is a morphism an isomorphism?

05X7 More generally we can ask: "When does a morphism have property $\mathcal{P}$?" A more precise question is the following. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Does there exist a monomorphism of algebraic spaces $W \to Z$ with the following two properties:

(1) the base change $f_W : X_W \to Y_W$ has property $\mathcal{P}$, and
(2) any morphism $Z' \to Z$ of algebraic spaces factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ has property $\mathcal{P}$.

In many cases, if $W \to Z$ exists, then it is an immersion, open immersion, or closed immersion.

The answer to this question may depend on auxiliary properties of the morphisms f, p, and q. An example is $\mathcal{P}(f) =$"f is flat" which we have discussed for morphisms of schemes in the case $Y = S$ in great detail in the chapter "More on Flatness", starting with More on Flatness, Section 38.20.

05X8 Lemma 76.49.1. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that p is locally of finite type and closed. Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified.

Proof. By Morphisms of Spaces, Lemma 67.38.10 there exists an open subspace $U(f) \subset X$ which is the set of points where f is unramified. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| (|X| \setminus |U(f)|)$$

i.e., $z \in |Z|$ is a point of W if and only if f is unramified at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 66.4.9) that W has the desired universal property. □

05X9 Lemma 76.49.2. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) p is locally of finite type,
(2) p is closed, and
(3) $p_2 : X \times_Y X \to Z$ is closed.

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified and universally injective.

Proof. After replacing Z by the open subspace found in Lemma 76.49.1 we may assume that f is already unramified; note that this does not destroy assumption (2) or (3). By Morphisms of Spaces, Lemma 67.38.9 we see that $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion. This remains true after any base change. Hence by Morphisms of Spaces, Lemma 67.19.2 we see that $f_{Z'}$ is universally injective if and only if the base change of the diagonal $X_{Z'} \to (X \times_Y X)_{Z'}$ is an isomorphism. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p_2| \left(|X \times_Y X| \setminus \mathrm{Im}(|\Delta_{X/Y}|)\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the fibre of $|X \times_Y X| \to |Z|$ over z is in the image of $|X| \to |X \times_Y X|$. Then it is clear from the discussion above that the restriction $p^{-1}(W) \to q^{-1}(W)$ of f is unramified and universally injective.

Conversely, suppose that $f_{Z'}$ is unramified and universally injective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 66.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. The discussion above shows that

$$|X_{Z'}| \longrightarrow |(X \times_Y X)_{Z'}|$$

is surjective. By Properties of Spaces, Lemma 66.4.3 in the commutative diagram

$$\begin{tikzcd} |X_{Z'}| \arrow[r] \arrow[d] & |(X \times_Y X)_{Z'}| \arrow[d] \\ |p|^{-1}(\{z\}) \arrow[r] & |p_2|^{-1}(\{z\}) \end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. $\square$

05XA Lemma 76.49.3. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) p is locally of finite type,
(2) p is universally closed, and
(3) $q : Y \to Z$ is separated.

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a closed immersion.

Proof. We will use the characterization of closed immersions as universally closed, unramified, and universally injective morphisms, see Lemma 76.14.9. First, note that since p is universally closed and q is separated, we see that f is universally closed, see Morphisms of Spaces, Lemma 67.40.6. It follows that any base change of f is universally closed, see Morphisms of Spaces, Lemma 67.9.3. Thus to finish the proof of the lemma it suffices to prove that the assumptions of Lemma 76.49.2 are satisfied. The projection $\text{pr}_0 : X \times_Y X \to X$ is universally closed as a base change of f, see Morphisms of Spaces, Lemma 67.9.3. Hence $X \times_Y X \to Z$ is universally closed as a composition of universally closed morphisms (see Morphisms of Spaces, Lemma 67.9.4). This finishes the proof of the lemma. $\square$

05XB Lemma 76.49.4. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) p is locally of finite presentation,
(2) p is flat,
(3) p is closed, and
(4) q is locally of finite type.

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is flat.

Proof. By Lemma 76.23.6 the set

$$A = \{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus A\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the whole fibre of $|X| \to |Z|$ over z is contained in A. This is open because p is closed. Since the formation of A commutes with arbitrary base change it follows that W works. □

05XC Lemma 76.49.5. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) p is locally of finite presentation,
(2) p is flat,
(3) p is closed,
(4) q is locally of finite type, and
(5) q is closed.

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is surjective and flat.

Proof. By Lemma 76.49.4 we may assume that f is flat. Note that f is locally of finite presentation by Morphisms of Spaces, Lemma 67.28.9. Hence f is open, see Morphisms of Spaces, Lemma 67.30.6. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |q| \left(|Y| \setminus |f|(|X|)\right).$$

in other words for $z \in |Z|$ we have $z \in |W|$ if and only if the whole fibre of $|Y| \to |Z|$ over z is in the image of $|X| \to |Y|$. Since q is closed this set is open in $|Z|$. The morphism $X_W \to Y_W$ is surjective by construction. Finally, suppose that $X_{Z'} \to Y_{Z'}$ is surjective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 66.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. By Properties of Spaces, Lemma 66.4.3 in the commutative diagram

$$\begin{tikzcd} |X_{Z'}| \arrow[r] \arrow[d] & |Y_{Z'}| \arrow[d] \\ |p|^{-1}(\{z\}) \arrow[r] & |q|^{-1}(\{z\}) \end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

05XD Lemma 76.49.6. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) p is locally of finite presentation,
(2) p is flat,
(3) p is universally closed,
(4) q is locally of finite type,
(5) q is closed, and
(6) q is separated.

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is an isomorphism.

Proof. By Lemma 76.49.5 there exists an open subspace $W_1 \subset Z$ such that $f_{Z'}$ is surjective and flat if and only if $Z' \to Z$ factors through W_1. By Lemma 76.49.3 there exists an open subspace $W_2 \subset Z$ such that $f_{Z'}$ is a closed immersion if and only if $Z' \to Z$ factors through W_2. We claim that $W = W_1 \cap W_2$ works. Certainly, if $f_{Z'}$ is an isomorphism, then $Z' \to Z$ factors through W. Hence it suffices to show that f_W is an isomorphism. By construction f_W is a surjective flat closed immersion. In particular f_W is representable. Since a surjective flat closed immersion of schemes is an isomorphism (see Morphisms, Lemma 29.26.1) we win. (Note that actually f_W is locally of finite presentation, whence open, so you can avoid the use of this lemma if you like.) □

06CE Lemma 76.49.7. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) p is flat and locally of finite presentation,
(2) p is closed, and
(3) q is flat and locally of finite presentation,

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a local complete intersection morphism.

Proof. By Lemma 76.48.11 there exists an open subspace $U(f) \subset X$ which is the set of points where f is Koszul. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is Koszul at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 66.4.9) that W has the desired universal property. □

76.50. Exact sequences of differentials and conormal sheaves

06CD In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to composable morphisms of algebraic spaces.

In the sequences below each of the maps are as constructed in either Lemma 76.7.6 or Lemma 76.15.8. Let S be a scheme. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of algebraic spaces over S.

(1) There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Lemma 76.7.8. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 76.19.12.

(2) If g is formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$

see Lemma 76.15.13. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 76.19.13.

(3) if g and $f \circ g$ are formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$

see Lemma 76.15.14. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 76.19.14.

(4) if g and f are formally unramified, then there is a canonical exact sequence

$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$

see Lemma 76.15.15. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 76.48.13.

76.51. Characterizing pseudo-coherent complexes, II

0CTP In this section we discuss a characterization of pseudo-coherent complexes in terms of cohomology. Earlier material on pseudo-coherent complexes on algebraic spaces may be found in Derived Categories of Spaces, Section 75.13 and in Derived Categories of Spaces, Section 75.18. The analogue of this section for schemes is More on Morphisms, Section 37.69. A basic tool will be to reduce to the case of projective space using a derived version of Chow's lemma, see Lemma 76.51.2.

0CTQ **Lemma 76.51.1.** Let S be a scheme. Consider a commutative diagram of algebraic spaces

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X' \arrow[r] & B' \end{tikzcd}$$

over S. Let $B \to B'$ be a morphism. Denote by X and Y the base changes of X' and Y' to B. Assume $Y' \to B'$ and $Z' \to X'$ are flat. Then $X \times_B Y$ and Z' are Tor independent over $X' \times_{B'} Y'$.

Proof. By Derived Categories of Spaces, Lemma 75.20.3 we may check tor independence étale locally on $X \times_B Y$ and Z'. This[3] reduces the lemma to the case of schemes which is More on Morphisms, Lemma 37.69.1. $\square$

0CTR Lemma 76.51.2 (Derived Chow's lemma). Let A be a ring. Let X be a separated algebraic space of finite presentation over A. Let $x \in |X|$. Then there exist an $n \geq 0$, a closed subspace $Z \subset X \times_A \mathbf{P}^n_A$, a point $z \in |Z|$, an open $V \subset \mathbf{P}^n_A$, and an object E in $D(\mathcal{O}_{X \times_A \mathbf{P}^n_A})$ such that

(1) $Z \to X \times_A \mathbf{P}^n_A$ is of finite presentation,
(2) $c : Z \to \mathbf{P}^n_A$ is a closed immersion over V, set $W = c^{-1}(V)$,
(3) the restriction of $b : Z \to X$ to W is étale, $z \in W$, and $b(z) = x$,
(4) $E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V}$,
(5) E is pseudo-coherent and supported on Z.

Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and an algebraic space X' separated and of finite presentation over A' whose base change to A is X. See Limits of Spaces, Lemmas 70.7.1 and 70.6.9. Let $x' \in |X'|$ be the image of x. If we can prove the lemma for $(X'/A', x')$, then the lemma follows for $(X/A, x)$. Namely, if n', Z', z', V', E' provide the solution for $(X'/A', x')$, then we can let $n = n'$, let $Z \subset X \times \mathbf{P}^n$ be the inverse image of Z', let $z \in Z$ be the unique point mapping to x, let $V \subset \mathbf{P}^n_A$ be the inverse image of V', and let E be the derived pullback of E'. Observe that E is pseudo-coherent by Cohomology on Sites, Lemma 21.45.3. It only remains to check (5). To see this set $W = c^{-1}(V)$ and $W' = (c')^{-1}(V')$ and consider the cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "{(b,c)}"'] & W' \arrow[d, "{(b',c')}"] \\ X \times_A V \arrow[r] & X' \times_{A'} V' \end{tikzcd}$$

By Lemma 76.51.1 $X \times_A V$ and W' are tor-independent over $X' \times_{A'} V'$. Thus the derived pullback of $(b', c')_* \mathcal{O}_{W'}$ to $X \times_A V$ is $(b, c)_* \mathcal{O}_W$ by Derived Categories of Spaces, Lemma 75.20.4. This also uses that $R(b', c')_* \mathcal{O}_{Z'} = (b', c')_* \mathcal{O}_{Z'}$ because (b', c') is a closed immersion and simiarly for $(b, c)_* \mathcal{O}_Z$. Since $E'|_{U' \times_{A'} V'} = (b', c')_* \mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b, c)_* \mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume A is of finite type over $\mathbf{Z}$. Choose an étale morphism $U \to X$ where U is an affine scheme and a point $u \in U$ mapping to x. Then U is of finite type over A. Choose a closed immersion $U \to \mathbf{A}^n_A$ and denote $j : U \to \mathbf{P}^n_A$ the immersion we get

[3] Here is the argument in more detail. Choose a surjective étale morphism $W' \to B'$ with W' a scheme. Choose a surjective étale morphism $W \to B \times_{B'} W'$ with W a scheme. Choose a surjective étale morphism $U' \to X' \times_{B'} W'$ with U' a scheme. Choose a surjective étale morphism $V' \to Y' \times_{B'} W'$ with V' a scheme. Observe that $U' \times_{W'} V' \to X' \times_{B'} Y'$ is surjective étale. Choose a surjective étale morphism $T' \to Z' \times_{X' \times_{B'} Y'} U' \times_{W'} V'$ with T' a scheme. Denote U and V the base changes of U' and V' to W. Then the lemma says that $X \times_B Y$ and Z' are Tor independent over $X' \times_{B'} Y'$ as algebraic spaces if and only if $U \times_W V$ and T' are Tor independent over $U' \times_{W'} V'$ as schemes. Thus it suffices to prove the lemma for the square with corners T', U', V', W' and base change by $W \to W'$. The flatness of $Y' \to B'$ and $Z' \to X'$ implies flatness of $V' \to W'$ and $T' \to U'$.

by composing with the open immersion $\mathbf{A}^n_A \to \mathbf{P}^n_A$. Let Z be the scheme theoretic closure of

$$(\mathrm{id}_U, j) : U \longrightarrow X \times_A \mathbf{P}^n_A$$

Let $z \in Z$ be the image of u. Let $Y \subset \mathbf{P}^n_A$ be the scheme theoretic closure of j. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\mathrm{id}_U, j) : U \to X \times_A Y$. As X is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by Morphisms of Spaces, Lemma 67.16.7. Choose $V \subset \mathbf{P}^n_A$ open with $V \cap Y = j(U)$. Then we see that (2) holds, that $W = (\mathrm{id}_U, j)(U)$, and hence that (3) holds. Part (1) holds because A is Noetherian.

Because A is Noetherian we see that X and $X \times_A \mathbf{P}^n_A$ are Noetherian algebraic spaces. Hence we can take $E = (b, c)_*\mathcal{O}_Z$ in this case: (4) is clear and for (5) see Derived Categories of Spaces, Lemma 75.13.7. This finishes the proof. □

0CTS Lemma 76.51.3. Let X/A, $x \in |X|$, and n, Z, z, V, E be as in Lemma 76.51.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = R(W \to V)_*K|_W$$

where $p : X \times_A \mathbf{P}^n_A \to X$ and $q : X \times_A \mathbf{P}^n_A \to \mathbf{P}^n_A$ are the projections and where the morphism $W \to V$ is the finitely presented closed immersion $c|_W : W \to V$.

Proof. Since $W = c^{-1}(V)$ and since c is a closed immersion over V, we see that $c|_W$ is a closed immersion. It is of finite presentation because W and V are of finite presentation over A, see Morphisms of Spaces, Lemma 67.28.9. First we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right)$$

where $q' : X \times_A V \to V$ is the projection because formation of total direct image commutes with localization. Denote $i = (b, c)|_W : W \to X \times_A V$ the given closed immersion. Then

$$Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right) = Rq'_*(Lp^*K|_{X \times_A V} \otimes^{\mathbf{L}} i_*\mathcal{O}_W)$$

by property (5). Since i is a closed immersion we have $i_*\mathcal{O}_W = Ri_*\mathcal{O}_W$. Using Derived Categories of Spaces, Lemma 75.20.1 we can rewrite this as

$$Rq'_*Ri_*Li^*Lp^*K|_{X \times_A V} = R(q' \circ i)_*Lb^*K|_W = R(W \to V)_*K|_W$$

which is what we want. (Note that restricting to W and derived pulling back via $W \to X$ is the same thing as W is étale over X.) □

0CTT Lemma 76.51.4. Let A be a ring. Let X be an algebraic space separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A (Definition 76.45.3).

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$. Let $x \in |X|$. We will show that K is pseudo-coherent relative to A in an étale neighbourhood of x. This will prove the lemma by our definition of relative pseudo-coherence.

Choose n, Z, z, V, E as in Lemma 76.51.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$\begin{aligned}
&R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(i)) \\
&= R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i)) \\
&= R\Gamma(X, K \otimes^{\mathbf{L}} Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i)))
\end{aligned}$$

by Derived Categories of Spaces, Lemma 75.20.1. By Derived Categories of Spaces, Lemma 75.25.5 the complex $Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on X. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 36.34.2 we conclude that $Rq_*(Lp^*K \otimes^{\mathbf{L}} E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 76.51.3 we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V} = R(W \to V)_* K|_W$$

Since $W \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_W$ is pseudo-coherent relative to A for example by More on Morphisms, Lemma 37.59.18. □

0GFJ Lemma 76.51.5. Let A be a ring. Let X be an algebraic space separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.

Proof. In view of Lemma 76.51.4, it suffices to show $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E \in D(\mathcal{O}_X)$. By Derived Categories of Spaces, Proposition 75.29.3 it follows that $K \in D^-_{QCoh}(\mathcal{O}_X)$. Now the result follows by Derived Categories of Spaces, Lemma 75.25.7. □

76.52. Relatively perfect objects

0DKM In this section we introduce a notion from [Lie06a]. This notion has been discussed for morphisms of schemes in Derived Categories of Schemes, Section 36.35.

0DKN Definition 76.52.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is perfect relative to Y or Y-perfect if E is pseudo-coherent (Cohomology on Sites, Definition 21.45.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$ (Cohomology on Sites, Definition 21.46.1).

Please see Derived Categories of Schemes, Remark 36.35.14 for a discussion; here we just mention that E being pseudo-coherent is the same thing as E being pseudo-coherent relative to Y by Lemma 76.45.4. Moreover, pseudo-coherence of E implies $E \in D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 75.13.6.

0DKP Example 76.52.2. Let k be a field. Let X be an algebraic space of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{\text{Coh}}(X)$ (by Derived Categories of Spaces, Lemma 75.13.7). Thus being relatively perfect does not mean "perfect on the fibres".

The corresponding algebra concept is studied in More on Algebra, Section 15.83. We can link the notion for algebraic spaces with the algebraic notion as follows.

0DKQ **Lemma 76.52.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent:

(1) E is Y-perfect,
(2) for every commutative diagram
$$\begin{tikzcd} U \arrow[r, "g"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
where U, V are schemes and the vertical arrows are étale, the complex $E|_U$ is V-perfect in the sense of Derived Categories of Schemes, Definition 36.35.1,
(3) for some commutative diagram as in (2) with $U \to X$ surjective, the complex $E|_U$ is V-perfect in the sense of Derived Categories of Schemes, Definition 36.35.1,
(4) for every commutative diagram as in (2) with U and V affine the complex $R\Gamma(U, E)$ is $\mathcal{O}_Y(V)$-perfect.

Proof. To make sense of parts (2), (3), (4) of the lemma, observe that the object $E|_U$ of $D_{QCoh}(\mathcal{O}_U)$ corresponds to an object E_0 of $D_{QCoh}(\mathcal{O}_{U_0})$ where U_0 denotes the scheme underlying U, see Derived Categories of Spaces, Lemma 75.4.2. Moreover, in this case E_0 is pseudo-coherent if and only if $E|_U$ is pseudo-coherent, see Derived Categories of Spaces, Lemma 75.13.2. Also, $E|_U$ locally has finite tor dimension over $f^{-1}\mathcal{O}_Y|_U = g^{-1}\mathcal{O}_V$ if and only if E_0 locally has finite tor dimension over $g_0^{-1}\mathcal{O}_{V_0}$ by Derived Categories of Spaces, Lemma 75.13.4. Here $g_0 : U_0 \to V_0$ is the morphism of schemes representing $g : U \to V$ (notation as in Derived Categories of Spaces, Remark 75.6.3). Finally, observe that "being pseudo-coherent" is étale local and of course "having locally finite tor dimension" is étale local. Thus we see that it suffices to check Y-perfectness étale locally and by the above discussion we see that (1) implies (2) and (3) implies (1). Since part (4) is equivalent to (2) and (3) by Derived Categories of Schemes, Lemma 36.35.3 the proof is complete. □

0DKR **Lemma 76.52.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of Y-perfect objects is a saturated[4] triangulated subcategory.

Proof. This follows from Cohomology on Sites, Lemmas 21.45.4, 21.45.6, 21.46.6, and 21.46.8. □

0DKS **Lemma 76.52.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is Y-perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ is Y-perfect if K is perfect and M is Y-perfect.

Proof. Reduce to the case of schemes using Lemma 76.52.3 and then apply Derived Categories of Schemes, Lemma 36.35.5. □

0DKT **Lemma 76.52.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let $g : Y' \to Y$ be a

[4]Derived Categories, Definition 13.6.1.

morphism of algebraic spaces over S. Set $X' = Y' \times_Y X$ and denote $g' : X' \to X$ the projection. If $K \in D(\mathcal{O}_X)$ is Y-perfect, then $L(g')^*K$ is Y'-perfect.

Proof. Reduce to the case of schemes using Lemma 76.52.3 and then apply Derived Categories of Schemes, Lemma 36.35.6. □

0DKU Situation 76.52.7. Let S be a scheme. Let $Y = \lim_{i \in I} Y_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $g_{i'i} : Y_{i'} \to Y_i$. We assume that Y_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : Y \to Y_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \to Y_0$. We set $X_i = Y_i \times_{Y_0} X_0$ and $X = Y \times_{Y_0} X_0$ and we denote the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $f_i : X \to X_i$ the projections.

0DKV Lemma 76.52.8. In Situation 76.52.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^*K_0$ and $L_i = Lf_{i0}^*L_0$ for $i \geq 0$ and set $K = Lf_0^*K_0$ and $L = Lf_0^*L_0$. Then the map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \to Y_0)^{-1}\mathcal{O}_{Y_0})$

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0.

Suppose that $(U_0 \subset W_0, V_0 \to W_0)$ is an elementary distinguished square in $(X_0)_{spaces,étale}$ and P holds for $U_0, V_0, U_0 \times_{W_0} V_0$. Then P holds for W_0 by Mayer-Vietoris for hom in the derived category, see Derived Categories of Spaces, Lemma 75.10.4.

We first consider $U_0 = W_0 \times_{Y_0} X_0$ with W_0 a quasi-compact and quasi-separated object of $(Y_0)_{spaces,étale}$. By the induction principle of Derived Categories of Spaces, Lemma 75.9.3 applied to these W_0 and the previous paragraph, we find that it is enough to prove P for $U_0 = W_0 \times_{Y_0} X_0$ with W_0 affine. In other words, we have reduced to the case where Y_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0, to reduce to the case where X_0 is affine as well.

If X_0 and Y_0 are affine, then we are back in the case of schemes which is proved in Derived Categories of Schemes, Lemma 36.35.8. The reader may use Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, 75.13.2, and 75.13.4 to accomplish the translation of the statement into a statement involving only schemes and derived categories of modules on schemes. □

0DKW Lemma 76.52.9. In Situation 76.52.7 the category of Y-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of Y_i-perfect objects of $D(\mathcal{O}_{X_i})$.

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\mathrm{colim}_{i \geq 0} D_{Y_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{Y\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We observe that we already know this functor is fully faithful by Lemma 76.52.8. Thus it suffices to prove essential surjectivity.

Suppose that $(U_0 \subset W_0, V_0 \to W_0)$ is an elementary distinguished square in $(X_0)_{spaces,étale}$ and P holds for $U_0, V_0, U_0 \times_{W_0} V_0$. We claim that P holds for W_0. We will use the notation $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0$, and similarly for V_0 and W_0. We will abusively use the symbol f_i for all the morphisms $U \to U_i$, $V \to V_i$, $U \times_W V \to U_i \times_{W_i} V_i$, and $W \to W_i$. Suppose E is an Y-perfect object of $D(\mathcal{O}_W)$. Goal: show E is in the essential image of the functor. By assumption, we can find $i \geq 0$, an Y_i-perfect object $E_{U,i}$ on U_i, an Y_i-perfect object $E_{V,i}$ on V_i, and isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. Let

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \times_{W_i} V_i} \to E_{V,i}|_{U_i \times_{W_i} V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \times_W V} \to E|_{U \times_W V} \to Lf_i^*E_{V,i}|_{U \times_W V}.$$

Apply Derived Categories of Spaces, Lemma 75.10.8 to get an object E_i on W_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is Y_i-perfect (because being relatively perfect is an étale local property) and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

By exactly the same argument as used in the proof of Lemma 76.52.8 using the induction principle (Derived Categories of Spaces, Lemma 75.9.3) we reduce to the case where both X_0 and Y_0 are affine: first work with quasi-compact and quasi-separated objects in $(Y_0)_{spaces,étale}$ to reduce to Y_0 affine, then work with quasi-compact and quasi-separated object in $(X_0)_{spaces,étale}$ to reduce to X_0 affine. In the affine case the result follows from the case of schemes which is Derived Categories of Schemes, Lemma 36.35.9. The translation into the case for schemes is done by Lemma 76.52.3. ☐

0DKX Lemma 76.52.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be Y-perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Derived Categories of Spaces, Lemma 75.21.4 (with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0). Thus it suffices to show that Rf_*E is a perfect object. We will reduce to the case where Y is Noetherian affine by a limit argument.

The question is étale local on Y, hence we may assume Y is affine. Say $Y = \text{Spec}(R)$. We write $R = \text{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 70.7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemmas 70.6.13 and 70.6.12 we may assume X_i is proper and flat over R_i. By Lemma 76.52.9 we may assume there exists a R_i-perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Derived Categories of Spaces, Lemma 75.22.1 to $X_i \to \text{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. ☐

0DKY **Lemma 76.52.11.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $E, K \in D(\mathcal{O}_X)$. Assume

(1) Y is quasi-compact and quasi-separated,
(2) f is proper, flat, and of finite presentation,
(3) E is Y-perfect,
(4) K is pseudo-coherent.

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_Y)$ such that

$$Rf_* R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_Y)$$

and the same is true after arbitrary base change: given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \quad \begin{array}{c} \text{cartesian, then we have} \\ Rf'_* R\mathcal{H}om(L(g')^*K, L(g')^*E) \\ = R\mathcal{H}om(Lg^*L, \mathcal{O}_{Y'}) \end{array}$$

Proof. Since Y is quasi-compact and quasi-separated, the same is true for X. By Derived Categories of Spaces, Lemma 75.18.1 we can write $K = \text{hocolim} K_n$ with K_n perfect and $K_n \to K$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $K_n^\vee$ be the dual perfect complex (Cohomology on Sites, Lemma 21.48.4). We obtain an inverse system $\ldots \to K_3^\vee \to K_2^\vee \to K_1^\vee$ of perfect objects. By Lemma 76.52.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is Y-perfect. Thus we may apply Lemma 76.52.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\ldots \to M_3 \to M_2 \to M_1$$

of perfect complexes on Y with

$$M_n = Rf_*(K_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = Rf_* R\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^*M_n = Rf'_*((L(g')^*K_n)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*E) = Rf'_* R\mathcal{H}om(L(g')^*K_n, L(g')^*E)$.

As $K_n \to K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \to K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \to K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules. Then the same is true after any base change, see Derived Categories of Spaces, Lemma 75.20.7. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \to M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \to M_n \to C_n \to M_{n+1}[1]$$

Pick $y \in |Y|$. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$; this is possible by Decent Spaces, Lemma 68.11.4. Take Y' equal to the spectrum of the residue field at u. Pull back to see that $C_n|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \kappa(u)$ has nonzero cohomology only in degrees $\geq n + a$. By More on Algebra, Lemma 15.75.6 we see that the perfect complex $C_n|_U$ has tor amplitude in $[n + a, m_n]$ for some integer m_n and after possibly shrinking U. Thus C_n has tor amplitude in $[n + a, m_n]$ for some integer m_n (because Y is quasi-compact). In particular, the dual perfect complex $C_n^\vee$ has tor amplitude in $[-m_n, -n - a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Derived Categories of Spaces, Lemma 75.18.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology on Sites, Lemma 21.48.8) and since $R\lim$ commutes with Rf_* we find that

$$Rf_* R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_Y) = R\mathcal{H}om(L, \mathcal{O}_Y)$$

This proves the formula over Y. Since the construction of M_n is compatible with base chance, the formula continues to hold after any base change. $\square$

0DKZ Remark 76.52.12. The reader may have noticed the similarity between Lemma 76.52.11 and Derived Categories of Spaces, Lemma 75.23.3. Indeed, the pseudo-coherent complex L of Lemma 76.52.11 may be characterized as the unique pseudo-coherent complex on Y such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_Y}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})$$

compatible with boundary maps for $\mathcal{F}$ ranging over $QCoh(\mathcal{O}_Y)$. If we ever need this we will formulate a precise result here and give a detailed proof.

0GFK Lemma 76.52.13. Let S be a scheme. Let X be an algebraic space over S such that the structure morphism $f : X \to S$ is flat and locally of finite presentation. Let E be a pseudo-coherent object of $D(\mathcal{O}_X)$. The following are equivalent

1. (1) E is S-perfect, and
2. (2) E is locally bounded below and for every point $s \in S$ the object $L(X_s \to X)^*E$ of $D(\mathcal{O}_{X_s})$ is locally bounded below.

Proof. Since everything is local we immediately reduce to the case that X and S are affine, see Lemma 76.52.3. This case is handled by Derived Categories of Schemes, Lemma 36.35.13. $\square$

0GFL Lemma 76.52.14. Let A be a ring. Let X be an algebraic space separated, of finite presentation, and flat over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is perfect in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is $\text{Spec}(A)$-perfect.

Proof. By Lemma 76.51.5, K is pseudo-coherent relative to A. By Lemma 76.45.4, K is pseudo-coherent in $D(\mathcal{O}_X)$. By Derived Categories of Spaces, Proposition 75.29.4 we see that K is in $D^-(\mathcal{O}_X)$. Let $\mathfrak{p}$ be a prime ideal of A and denote $i : Y \to X$ the inclusion of the scheme theoretic fibre over $\mathfrak{p}$, i.e., Y is a scheme over $\kappa(\mathfrak{p})$. By Lemma 76.52.13, we will be done if we can show $Li^*(K)$ is bounded below. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Theorem 75.15.4. We have

$$\begin{aligned} R\text{Hom}_{\mathcal{O}_Y}(Li^*(G), Li^*(K)) &= R\Gamma(Y, Li^*(G^\vee \otimes^{\mathbf{L}} K)) \\ &= R\Gamma(X, G^\vee \otimes^{\mathbf{L}} K) \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p}) \end{aligned}$$

The first equality uses that Li^* preserves perfect objects and duals and Cohomology on Sites, Lemma 21.48.4; we omit some details. The second equality follows from Derived Categories of Spaces, Lemma 75.20.4 as X is flat over A. It follows from our hypothesis that this is a perfect object of $D(\kappa(\mathfrak{p}))$. The object $Li^*(G) \in D_{perf}(\mathcal{O}_Y)$ generates $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Spaces, Remark 75.15.5.

Hence Derived Categories of Spaces, Proposition 75.29.4 now implies that $Li^*(K)$ is bounded below and we win. $\square$

76.53. Theorem of the cube

0D22 This section is the analogue of More on Morphisms, Section 37.33. The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

0D23 **Lemma 76.53.1.** Let S be a scheme. Let $f : X \to Y$ be a flat, proper morphism of finite presentation of algebraic spaces over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : Y' \to Y$ consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is an isomorphism for all $g : Y' \to Y$. Then there exists an immersion $j : Z \to Y$ of finite presentation such that a morphism $g : Y' \to Y$ factors through Z if and only if there exists a finite locally free $\mathcal{O}_{Y'}$-module $\mathcal{N}$ with $(f')^*\mathcal{N} \cong (g')^*\mathcal{L}$.

Proof. Let $y : \mathrm{Spec}(k) \to Y$ be a field valued point. Then the fibre X_y of f at y is connected by our assumption that $H^0(X_y, \mathcal{O}_{X_y}) = k$. Thus the rank of $\mathcal{E}$ is constant on the fibres. Since f is open (Morphisms of Spaces, Lemma 67.30.6) and closed we conclude that there is a decomposition $Y = \coprod Y_r$ of Y into open and closed subspaces such that $\mathcal{E}$ has constant rank r on the inverse image of Y_r. Thus we may assume $\mathcal{E}$ has constant rank r. We will denote $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ the dual rank r module.

By cohomology and base change (more precisely by Derived Categories of Spaces, Lemma 75.25.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of Y and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}^\vee$. Since there is never any cohomology in degrees < 0, we see that E and E' have (locally) tor-amplitude in $[0, b]$ for some b. Observe that for any $g : Y' \to Y$ we have $f'_*((g')^*\mathcal{E}) = H^0(Lg^*E)$ and $f'_*((g')^*\mathcal{E}^\vee) = H^0(Lg^*E')$. Let $j : Z \to Y$ and $j' : Z' \to Y$ be the locally closed immersions constructed in Derived Categories of Spaces, Lemma 75.26.6 for E and E' with $a = 0$ and $r = r$; these are characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are locally free modules of rank r compatible with pullback.

Let $g : Y' \to Y$ be a morphism. If there exists an $\mathcal{N}$ as in the lemma, then, using the projection formula Cohomology on Sites, Lemma 21.50.1, we see that the modules

$$f'_*((g')^*\mathcal{E}) \cong f'_*((f')^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_{Y'}} f'_*\mathcal{O}_{X'} \cong \mathcal{N} \quad \text{and similarly} \quad f'_*((g')^*\mathcal{E}^\vee) \cong \mathcal{N}^\vee$$

are locally free of rank r and remain locally free of rank r after any further base change $Y'' \to Y'$. Hence in this case $g : Y' \to Y$ factors through j and through j'. Thus we may replace Y by $Z \times_Y Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}^\vee$ are locally free $\mathcal{O}_Y$-modules of rank r whose formation commutes with arbitrary change of base.

In this sitation if $g : Y' \to Y$ is a morphism and there exists an $\mathcal{N}$ as in the lemma, then the map (cup product in degree 0)

$$f'_*((g')^*\mathcal{E}) \otimes_{\mathcal{O}_{Y'}} f'_*((g')^*\mathcal{E}^\vee) \longrightarrow \mathcal{O}_{Y'}$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on Y' we have a basis of sections $\sigma_1, \ldots, \sigma_r$ in $f'_*((g')^*\mathcal{L})$ and $\tau_1, \ldots, \tau_r$ in $f'_*((g')^*\mathcal{E}^\vee)$ whose products satisfy $\sigma_i\tau_j = \delta_{ij}$. Thinking of σ_i as a section of $(g')^*\mathcal{L}$ on X' and τ_j as a section of $(g')^*\mathcal{L}^\vee$ on X', we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X'}^{\oplus r} \longrightarrow (g')^*\mathcal{E}$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : (g')^*\mathcal{E} \longrightarrow \mathcal{O}_{X'}^{\oplus r}$$

In other words, we see that $(f')^*f'_*(g')^*\mathcal{E} \cong (g')^*\mathcal{E}$. But the condition that the cupproduct is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. □

76.54. Descent of finiteness properties of complexes

0DL0 This section is the analogue of More on Morphisms, Section 37.70 and Derived Categories of Schemes, Section 36.12.

0DL1 Lemma 76.54.1. Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an fpqc covering of algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology on Sites, Lemma 21.45.3. Thus it suffices to assume Lf_i^*E is m-pseudo-coherent and to prove that E is m-pseudo-coherent. Then first we may assume X_i is a scheme for all i, see Topologies on Spaces, Lemma 73.9.5. Next, choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_i = U \times_X X_i$ is a scheme and we obtain an fpqc covering $\{U_i \to U\}$ of schemes, see Topologies on Spaces, Lemma 73.9.4. We know the result is true for $\{U_i \to U\}_{i \in I}$ by the case for schemes, see Derived Categories of Schemes, Lemma 36.12.2. On the other hand, the restriction $E|_U$ comes from an object of $D_{QCoh}(\mathcal{O}_U)$ (defined using the Zariski topology and the "usual" structure sheaf of U), see Derived Categories of Spaces, Lemma 75.4.2. The lemma follows as the two notions of pseudo-coherent (étale and Zariski) agree by Derived Categories of Spaces, Lemma 75.13.2. □

0DL2 Lemma 76.54.2. Let S be a scheme. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces and set $X_i = Y_i \times_Y X$ with projections $f_i : X_i \to Y_i$ and $g'_i : X_i \to X$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$. Then the following are equivalent

(1) E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_Y)$, and
(2) $L(g'_i)^*E$ has tor amplitude in $[a, b]$ as a object of $D(f_i^{-1}\mathcal{O}_{Y_i})$ for all i.

Also true if "tor amplitude in $[a, b]$" is replaced by "locally finite tor dimension".

Proof. Pullback preserves "tor amplitude in $[a, b]$" by Derived Categories of Spaces, Lemma 75.20.7 Observe that Y_i and X are tor independent over Y as $Y_i \to Y$ is flat. Let us assume (2) and prove (1). We can compute tor dimension at stalks, see Cohomology on Sites, Lemma 21.46.10 and Properties of Spaces, Theorem 66.19.12.

Let $\overline{x}$ be a geometric point of X. Choose an i and a geometric point $\overline{x}_i$ in X_i with image $\overline{x}$ in X. Then

$$(L(g_i')^*E)_{\overline{x}_i} = E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X,\overline{x}_i}$$

Let $\overline{y}_i$ in Y_i and $\overline{y}$ in Y be the image of $\overline{x}_i$ and $\overline{x}$. Since X and Y_i are tor independent over Y, we can apply More on Algebra, Lemma 15.61.2 to see that the right hand side of the displayed formula is equal to $E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y_i,\overline{y}_i}$ in $D(\mathcal{O}_{Y_i,\overline{y}_i})$. Since we have assume the tor amplitude of this is in $[a, b]$, we conclude that the tor amplitude of $E_{\overline{x}}$ in $D(\mathcal{O}_{Y,\overline{y}})$ is in $[a, b]$ by More on Algebra, Lemma 15.66.17. Thus (1) follows.

Using some elementary topology the case "locally finite tor dimension" follows too. □

The following lemmas do not really belong in this section.

0DL3 Lemma 76.54.3. Let S be a scheme. Let $i : X \to X'$ be a finite order thickening of algebraic spaces. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then K' is pseudo-coherent.

Proof. We first prove K' has quasi-coherent cohomology sheaves; we urge the reader to skip this part. To do this, we may reduce to the case of a first order thickening, see Section 76.9. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out X. Tensoring the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to i_*\mathcal{O}_X \to 0$$

with K' we obtain a distinguished triangle

$$K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I} \to K' \to K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \to (K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = Ri_*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}) \to K' \to i_*K \to i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology of Spaces, Lemma 69.4.4 to identify the terms. Since K is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that K' is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Spaces, Lemmas 75.13.6, 75.5.6, and 75.6.1.

Assume K' is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is étale local on X' hence we may assume X' is affine. In this case the result follows from the case of schemes (More on Morphisms, Lemma 37.71.1). The translation into the language of schemes uses Derived Categories of Spaces, Lemmas 75.4.2 and 75.13.2 and Remark 75.6.3. □

0DL4 Lemma 76.54.4. Let S be a scheme. Consider a cartesian diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of algebraic spaces over S. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is Y-perfect, then E' is Y'-perfect.

Proof. Recall that being Y-perfect for E means E is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Definition 76.52.1). By Lemma 76.54.3 we find that E' is pseudo-coherent. In particular, E' is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Spaces, Lemma 75.13.6. By Lemma 76.52.3 this reduces us to the case of schemes. The case of schemes is More on Morphisms, Lemma 37.71.2. $\square$

0DL5 Lemma 76.54.5. Let (R, I) be a pair consisting of a ring and an ideal I contained in the Jacobson radical. Set $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$. Let X be an algebraic space over R whose structure morphism $f : X \to S$ is proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. If the derived restriction E_0 of E to X_0 is S_0-perfect, then E is S-perfect.

Proof. Choose a surjective étale morphism $U \to X$ with U affine. Choose a closed immersion $U \to \mathbf{A}^d_S$. Set $U_0 = S_0 \times_S U$. The complex $E_0|_{U_0}$ has tor amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $\overline{x}$ be a geometric point of X. We will show that the tor amplitude of $E_{\overline{x}}$ over R is in $[a - d, b]$. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology on Sites, Lemma 21.46.10 and Properties of Spaces, Theorem 66.19.12.

Let $x \in |X|$ be the point determined by $\overline{x}$. Recall that $|X| \to |S|$ is closed (by definition of proper morphisms). Since I is contained in the Jacobson radical, any nonempty closed subset of S contains a point of the closed subscheme S_0. Hence we can find a specialization $x \rightsquigarrow x_0$ in $|X|$ with $x_0 \in |X_0|$. Choose $u_0 \in U_0$ mapping to x_0. By Decent Spaces, Lemma 68.7.4 (or by Decent Spaces, Lemma 68.7.3 which applies directly to étale morphisms) we find a specialization $u \rightsquigarrow u_0$ in U such that u maps to x. We may lift $\overline{x}$ to a geometric point $\overline{u}$ of U lying over u. Then we have $E_{\overline{x}} = (E|_U)_{\overline{u}}$.

Write $U = \mathrm{Spec}(A)$. Then A is a flat, finitely presented R-algebra which is a quotient of a polynomial R-algebra in d-variables. The restriction $E|_U$ corresponds (by Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, and 75.13.2 and Derived Categories of Schemes, Lemma 36.3.5 and 36.10.2) to a pseudo-coherent object K of $D(A)$. Observe that E_0 corresponds to $K \otimes_A^{\mathbf{L}} A/IA$. Let $\mathfrak{q} \subset \mathfrak{q}_0 \subset A$ be the prime ideals corresponding to $u \rightsquigarrow u_0$. Then

$$E_{\overline{x}} = (E|_U)_{\overline{u}} = E_u \otimes_{\mathcal{O}_{U,u}}^{\mathbf{L}} \mathcal{O}_{U,\overline{u}} = K_\mathfrak{q} \otimes_{A_\mathfrak{q}}^{\mathbf{L}} A_\mathfrak{q}^{sh}$$

(some details omitted). Since $A_\mathfrak{q} \to A_\mathfrak{q}^{sh}$ is flat, the tor amplitude of this as an R-module is the same as the tor amplitude of $K_\mathfrak{q}$ as an R-module (More on Algebra, Lemma 15.66.18). Also, $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}_0}$. Hence it suffices to show that $K_{\mathfrak{q}_0}$ has tor amplitude in $[a - d, b]$ as a complex of R-modules.

Let $I \subset \mathfrak{p}_0 \subset R$ be the prime ideal corresponding to $f(x_0)$. Then we have

$$\begin{aligned} K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}_0) &= (K \otimes_R^{\mathbf{L}} R/I) \otimes_{R/I}^{\mathbf{L}} \kappa(\mathfrak{p}_0) \\ &= (K \otimes_A^{\mathbf{L}} A/IA) \otimes_{R/I}^{\mathbf{L}} \kappa(\mathfrak{p}_0) \end{aligned}$$

the second equality because $R \to A$ is flat. By our choice of a, b this complex has cohomology only in degrees in the interval $[a, b]$. Thus we may finally apply More on Algebra, Lemma 15.83.9 to $R \to A$, $\mathfrak{q}_0$, $\mathfrak{p}_0$ and K to conclude. $\square$

76.55. Families of nodal curves

0DSD This section is the continuation of Algebraic Curves, Section 53.20. Please also see that section for our choice of terminology.

The property "at-worst-nodal of relative dimension 1" of morphisms of schemes is étale local on the source-and-target, see Descent, Lemma 35.32.6 and Algebraic Curves, Lemmas 53.20.8, 53.20.9, and 53.20.7. It is also stable under base change and fpqc local on the target, see Algebraic Curves, Lemmas 53.20.4 and 53.20.9. Hence, by Morphisms of Spaces, Lemma 67.22.1 we may define the notion of an at-worst-nodal morphism of relative dimension 1 for algebraic spaces as follows and it agrees with the already existing notion defined in Morphisms of Spaces, Section 67.3 when the morphism is representable.

0DSE **Definition 76.55.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *at-worst-nodal of relative dimension 1* if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}$ ="at-worst-nodal of relative dimension 1".

0DSF **Lemma 76.55.2.** *The property of being at-worst-nodal of relative dimension 1 is preserved under base change.*

Proof. See Morphisms of Spaces, Remark 67.22.4 and Algebraic Curves, Lemma 53.20.4. ☐

0DSG **Lemma 76.55.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is at-worst-nodal of relative dimension 1,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is at-worst-nodal of relative dimension 1,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is at-worst-nodal of relative dimension 1,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is at-worst-nodal of relative dimension 1,*
(5) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is at-worst-nodal of relative dimension 1,*
(6) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is at-worst-nodal of relative dimension 1,
(7) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is at-worst-nodal of relative dimension 1, and

(8) there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is at-worst-nodal of relative dimension 1.

Proof. Omitted. $\square$

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

0DSH Lemma 76.55.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Then there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with arbitrary base change.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective. By the lemma for the case of schemes (Algebraic Curves, Lemma 53.20.5) we find a maximal open subscheme $U' \subset U$ such that $h|_{U'} : U' \to V$ is at-worst-nodal of relative dimension 1 and such that formation of U' commutes with base change. Let $X' \subset X$ be the open subspace whose points correspond to the open subset $\text{Im}(|U'| \to |X|)$. By Lemma 76.55.3 we see that $X' \to Y$ is at-worst-nodal of relative dimension 1 and that X' is the largest open subspace with this property (this also implies that U' is the inverse image of X' in U, but we do not need this). Since the same is true after base change the proof is complete. $\square$

76.56. The resolution property

0GUX We continue the discussion in Derived Categories of Spaces, Section 75.28.

0GUY Situation 76.56.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $V \to X$ be a surjective étale morphism where V is an affine scheme (such a thing exists by Properties of Spaces, Lemma 66.6.3). Choose a commutative diagram

$$\begin{tikzcd} V \arrow[rr, "j"'] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

where j is an open immersion and π is a finite morphism of algebraic spaces (such a diagram exists by Lemma 76.34.3). Let $\mathcal{I} \subset \mathcal{O}_Y$ be a finite type quasi-coherent sheaf of ideals on Y with $V(\mathcal{I}) = Y \setminus j(V)$ (such a sheaf of ideals exists by Limits of Spaces, Lemma 70.14.1).

0GUZ **Lemma 76.56.2.** In Situation 76.56.1, assume X is Noetherian. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exist $r \geq 0$, integers $n_1, \ldots, n_r \geq 0$, and a surjection

$$\bigoplus\nolimits_{i=1,\ldots,r} \pi_*(\mathcal{I}^{n_i}) \longrightarrow \mathcal{F}$$

of $\mathcal{O}_X$-modules.

Proof. Denote $\omega_{Y/X}$ the coherent $\mathcal{O}_Y$-module such that there is an isomorphism

$$\pi_*\omega_{Y/X} \cong \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X)$$

of $\pi_*\mathcal{O}_Y$-modules, see Morphisms of Spaces, Lemma 67.20.10 and Descent on Spaces, Lemma 74.6.6. The canonical map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ produces a canonical map

$$\mathrm{Tr}_\pi : \pi_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

Since V is Noetherian affine we may choose sections

$$s_1, \ldots, s_r \in \Gamma(V, \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X})$$

generating the coherent module $\pi^*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_{Y/X}$ over V. By Cohomology of Spaces, Lemma 69.13.4 we can choose integers $n_i \geq 0$ such that s_i extends to a map $s_i' : \mathcal{I}^{n_i} \to \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$. Pushing to X we obtain maps

$$\sigma_i : \pi_*\mathcal{I}^{n_i} \xrightarrow{\pi_* s_i'} \pi_*(\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{F} \otimes_{\mathcal{O}_X} \pi_*\omega_{Y/X} \xrightarrow{\mathrm{Tr}_\pi} \mathcal{F}$$

where the equality sign is Cohomology of Spaces, Lemma 69.4.3. To finish the proof we will show that the sum of these maps is surjective.

Let $x \in |X|$ be a point of X. Let $v \in |V|$ be a point mapping to x. We may choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$U \times_X Y = W \coprod W'$$

(disjoint union of algebraic spaces) such that $W \to U$ is an isomorphism and such that the unique point $w \in W$ lying over u maps to v in $V \subset Y$. To see this is true use Lemma 76.33.2 and Étale Morphisms, Lemma 41.18.1. After shrinking U further if necessary we may assume W maps into $V \subset Y$ by the projection. Since the formation of $\omega_{Y/X}$ commutes with étale localization we see that

$$\pi_*\omega_{Y/X}|_U = (\pi|_W)_*\omega_{W/U} \oplus (\pi|_{W'})_*\omega_{W'/U}$$

We have $(\pi|_W)_*\omega_{W/U} = \mathcal{O}_U$ and this isomorphism is given by the trace map $\mathrm{Tr}_\pi|_U$ restricted to the first summand in the decomposition above. Since W maps into V we see that $\mathcal{I}^{n_i}|_W = \mathcal{O}_W$. Hence

$$\pi_*(\mathcal{I}^{n_i})|_U = \mathcal{O}_U \oplus (W' \to U)_*(\mathcal{I}^{n_i}|_{W'})$$

Chasing diagrams the reader sees (details omitted) that $\sigma_i|_U$ on the summand $\mathcal{O}_U$ is the map $\mathcal{O}_U \to \mathcal{F}$ corresponding to the section

$$s_i|_W \in \Gamma(W, \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \Gamma(W, \mathcal{F}|_W \otimes_{\mathcal{O}_W} \omega_{W/U}) = \Gamma(U, \mathcal{F})$$

Since the sections s_i generate the module $\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$ over V and since W maps into V we conclude that the restriction of $\bigoplus \sigma_i$ to U is surjective. Since x was an arbitrary point the proof is complete. $\square$

0GV0 **Lemma 76.56.3.** In Situation 76.56.1, assume X is Noetherian. Then X has the resolution property if and only if $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module.

Proof. The module $\pi_*\mathcal{I}$ is coherent by Cohomology of Spaces, Lemma 69.12.9. Hence if X has the resolution property then $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module. Conversely, assume given a surjection $\mathcal{E} \to \pi_*\mathcal{I}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Observe that for all $n \geq 1$ there is a surjection

$$\pi_*\mathcal{I} \otimes_{\mathcal{O}_X} \pi_*\mathcal{I}^n \longrightarrow \pi_*\mathcal{I}^{n+1}$$

Hence $\mathcal{E}^{\otimes n}$ surjects onto $\pi_*\mathcal{I}^n$ for all $n \geq 1$. We conclude that X has the resolution property if we combine this with the result of Lemma 76.56.2. $\square$

0GV1 Lemma 76.56.4. In Situation 76.56.1, the algebraic space X has the resolution property if and only if $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module.

Proof. The pushforward $\pi_*\mathcal{G}$ of a finite type quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_X$-module by Descent on Spaces, Lemma 74.6.6. In particular, if X has the resolution property, then $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module by Derived Categories of Spaces, Definition 75.28.1.

Assume that we have a surjection $\mathcal{E} \to \pi_*\mathcal{I}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. In the rest of the proof we show that X has the resolution property by reducing to the Noetherian case handled in Lemma 76.56.3. We suggest the reader skip the rest of the proof.

A first reduction is that we may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2. (This doesn't affect the conditions nor the conclusion of the lemma.)

By Limits of Spaces, Lemma 70.11.3 we can write $Y = \lim Y_i$ with Y_i finite and of finite presentation over X and where the transition maps are closed immersions. Consider the closed subspace $Z = V(\mathcal{I})$ of Y. Since $\mathcal{I}$ is of finite type, the morphism $Z \to Y$ is of finite presentation. Hence we can find an i and a morphism $Z_i \to Y_i$ of finite presentation whose base change to Y is $Z \to Y$, see Limits of Spaces, Lemma 70.7.1. For $i' \geq i$ denote $Z_{i'} = Z_i \times_{Y_i} Y_{i'}$. After increasing i we may assume $Z_i \to Y_i$ is a closed immersion (of finite presentation), see Limits of Spaces, Lemma 70.6.8. Denote $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$ the ideal sheaf of Z_i and denote $\pi_i : Y_i \to X$ the structure morphism. Similarly for $i' \geq i$. Since $Z = \lim_{i' \geq i} Z_{i'}$ we have

$$\pi_*\mathcal{I} = \text{colim}\, \pi_{i',*}\mathcal{I}_{i'}$$

The transition maps in the system are all surjective as follows from the surjectivity of the maps $\pi_{i,*}\mathcal{O}_{Y_i} \to \pi_{i',*}\mathcal{O}_{Y_{i'}}$ and the fact that $Z_{i'} = Z_i \times_{Y_i} Y_{i'}$. By Cohomology of Spaces, Lemma 69.5.3 for some $i' \geq i$ the map $\mathcal{E} \to \pi_*\mathcal{I}$ lifts to a map $\mathcal{E} \to \pi_{i',*}\mathcal{I}_{i'}$. After increasing i' this map $\mathcal{E} \to \pi_{i',*}\mathcal{I}_{i'}$ becomes surjective (since if not the colimit of the cokernels, having surjective transition maps, is nonzero). This reduces us to the case discussed in the next paragraph.

Assume X is an algebraic space over $\mathbf{Z}$ and that $Y \to X$ is of finite presentation. By absolute Noetherian approximation we can write $X = \lim X_i$ as a directed limit, where each X_i is a quasi-separated algebraic space of finite type over $\mathbf{Z}$ and the transition morphisms are affine, see Limits of Spaces, Proposition 70.8.1. Since $\pi : Y \to X$ is of finite presentation we can find an i and a morphism $\pi_i : Y_i \to X_i$ of finite presentation whose base change to X is π, see Limits of Spaces, Lemma 70.7.1. After increasing i we may assume π_i is finite, see Limits of Spaces, Lemma 70.6.7. Next, we may assume there exists a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ whose pullback to X is $\mathcal{E}$, see Limits of Spaces, Lemma 70.7.3. We may also assume there is a map $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ whose pullback to X is the composition $\mathcal{E} \to \pi_*\mathcal{I} \to \pi_*\mathcal{O}_Y$, see Limits of Spaces, Lemma 70.7.2. The cokernel

$$\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i} \to \mathcal{Q}_i \to 0$$

is a coherent $\mathcal{O}_{Y_i}$-module whose pullback to X is the (finitely presented) cokernel $\mathcal{Q}$ of the map $\mathcal{E} \to \pi_*\mathcal{O}_Y$. In other words, we have $\mathcal{Q} = \pi_*(\mathcal{O}_Y/\mathcal{I})$. Consider the map

$$\mathcal{E}_i \otimes_{\mathcal{O}_{X_i}} \pi_{i,*}\mathcal{O}_{Y_i} \longrightarrow \pi_{i,*}\mathcal{O}_{Y_i} \otimes_{\mathcal{O}_{X_i}} \pi_{i,*}\mathcal{O}_{Y_i} \to \pi_{i,*}\mathcal{O}_{Y_i} \to \mathcal{Q}_i$$

where the second arrow is given by the algebra structure on $\pi_{i,*}\mathcal{O}_{Y_i}$. The pullback of this map to Y is zero because the image of $\mathcal{E} \to \pi_*\mathcal{O}_Y$ is the ideal $\pi_*\mathcal{I}$. Hence by Limits of Spaces, Lemma 70.7.2 after increasing i we may assume the displayed composition is zero. This exactly means that the imag of $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ is of the form $\pi_{i,*}\mathcal{I}_i$ for some coherent ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$. Since $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ pulls back to $\mathcal{E} \to \pi_*\mathcal{O}_Y$ we see that the pullback of $\mathcal{I}_i$ to Y generates $\mathcal{I}$. Denote $V_i \subset Y_i$ the open subspace whose complement is $V(\mathcal{I}_i) \subset Y_i$. Then V is the inverse image of V_i by the comments above. After increasing i we may assume that V_i is affine and that $\pi_i|_{V_i} : V_i \to X_i$ is étale, see Limits of Spaces, Lemmas 70.5.10 and 70.6.2. Having said all of this, we may apply Lemma 76.56.3 to conclude that X_i has the resolution property. Since $X \to X_i$ is affine we conclude that X has the resolution property too by Derived Categories of Spaces, Lemma 75.28.3. □

0GV2 Lemma 76.56.5. Let S be a scheme. Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated algebraic spaces over S with affine transition morphisms. Then X has the resolution property if and only if X_i has the resolution properties for some i.

Proof. If X_i has the resolution property, then X does by Derived Categories of Spaces, Lemma 75.28.3. Assume X has the resolution property. Choose $i \in I$. We may choose an affine scheme V_i and a surjective étale morphism $V_i \to X_i$ (Properties of Spaces, Lemma 66.6.3). We may choose an embedding $j : V_i \to Y_i$ with Y_i finite and finitely presented over X_i (Lemma 76.34.4). We may choose a finite type quasi-coherent ideal $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$ such that $V_i = Y_i \setminus V(\mathcal{I}_i)$ (Limits of Spaces, Lemma 70.14.1). Denote $V \to Y \to X$ the base changes of $V_i \to Y_i \to X_i$ to X. Denote $\mathcal{I} \subset \mathcal{O}_Y$ the pullback of the ideal $\mathcal{I}_i$. By the easy direction of Lemma 76.56.4 there exists a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \pi_*\mathcal{I}$. Note that since $\pi_i : Y_i \to X_i$ is finite and of finite presentation we also have that $\pi : Y \to X$ is finite and of finite presentation and that the $\mathcal{O}_{X_i}$-modules $\pi_{i,*}\mathcal{O}_{Y_i}$ and $\pi_{i,*}(\mathcal{O}_{Y_i}/\mathcal{I}_i)$ are of finite presentation and pullback to X to give $\pi_*\mathcal{O}_Y$ and $\pi_*(\mathcal{O}_Y/\mathcal{I})$. Thus by Limits of Spaces, Lemma 70.7.2 after increasing i we can find a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ and a map $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ whose base change to X recovers the composition $\mathcal{E} \to \pi_*\mathcal{I} \to \pi_*\mathcal{O}_Y$. The pullbacks of the finitely presented $\mathcal{O}_{X_i}$-modules $\text{Coker}(\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i})$ and $\pi_{i,*}(\mathcal{O}_{Y_i}/\mathcal{I}_i)$ to X agree as quotients of $\pi_*\mathcal{O}_Y$. Hence by Limits of Spaces, Lemma 70.7.2 we may assume that these agree, in other words that the image of $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{X_i}$ is equal to $\pi_{i,*}\mathcal{I}_i$. Then we conclude that X_i has the resolution property by Lemma 76.56.4. □

0GV3 Lemma 76.56.6. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space with the resolution property. Then X has affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1).

Proof. We could prove this as in the case of schemes, but instead we will deduce the lemma from the case of schemes. First, we may and do assume $S = \mathrm{Spec}(\mathbf{Z})$. Next, we choose a scheme Y and a surjective integral morphism $f : Y \to X$, see Decent Spaces, Lemma 68.9.2. Then f is affine, hence Y has the resolution property by Derived Categories of Spaces, Lemma 75.28.3. Hence by the case of schemes, the scheme Y has affine diagonal, see Derived Categories of Schemes, Lemma 36.36.10. Next, we consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\Delta_Y"'] \arrow[d] & Y \times_{\mathbf{Z}} Y \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times_{\mathbf{Z}} X \end{tikzcd}$$

Observe that the right vertical arrow is integral, in particular affine. Let $W \to X \times_{\mathbf{Z}} X$ be a morphism with W affine. Then we see that

$$Y \times_{X \times_{\mathbf{Z}} X} W = Y \times_{\Delta_Y, Y \times_{\mathbf{Z}} Y} (Y \times_{\mathbf{Z}} Y) \times_{X \times_{\mathbf{Z}} X} W$$

is affine. On the other hand, $Y \to X$ is integral and surjective hence

$$Y \times_{X \times_{\mathbf{Z}} X} W \longrightarrow X \times_{X \times_{\mathbf{Z}} X} W$$

is integral surjective as the base change of $Y \to X$ to W. We conclude that the target of this arrow is affine by Limits of Spaces, Proposition 70.15.2. It follows that Δ_X is affine as desired. □

76.57. Blowing up and the resolution property

0GV4 We prove that the resolution property is satisfied after a blowing up.

0GV5 Lemma 76.57.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume that $|X|$ has finitely many irreducible components. There exists a dense quasi-compact open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that the algebraic space X' has the resolution property.

Proof. By Limits of Spaces, Lemma 70.16.3 there exists a surjective, finite, and finitely presented morphism $f : Y \to X$ where Y is a scheme and a quasi-compact dense open $U \subset X$ such that $f^{-1}(U) \to U$ is finite étale. By More on Morphisms, Lemma 37.80.2 there is a quasi-compact dense open $V \subset Y$ and a V-admissible blowing up $Y' \to Y$ such that Y' has an ample family of invertible modules. After shrinking U we may assume that $f^{-1}(U) \subset V$ (details omitted). Hence $f' : Y' \to X$ is finite étale over U and in particular, the morphism $(f')^{-1}(U) \to U$ is finite locally free. By Lemma 76.39.2 there is a U-admissible blowing up $X' \to X$ such that the strict transform Y'' of Y' is finite locally free over X'. Picture

$$\begin{tikzcd} Y'' \arrow[r, "g"'] \arrow[d] & Y' \arrow[r] & Y \arrow[d] \\ X' \arrow[rr] & & X \end{tikzcd}$$

Since $g : Y'' \to Y'$ is a blowing up (Divisors on Spaces, Lemma 71.18.3) in the inverse image of the center of $X' \to X$, we see that $g : Y'' \to Y'$ is projective and that there exists some g-ample invertible module on Y''. Hence by More on Morphisms, Lemma 37.79.1 we see that Y'' has an ample family of invertible modules. Hence Y'' has the resolution property, see Derived Categories of Schemes, Lemma 36.36.7. We conclude that X' has the resolution property by Derived Categories of Spaces, Lemma 75.28.4. $\square$

0GV6 Lemma 76.57.2. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. There exists a $t \geq 0$ and closed subspaces

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is of finite presentation, $Z_0 \subset X$ is a thickening, and for each $i = 0, \ldots t-1$ there exists a $(Z_i \setminus Z_{i-1})$-admissible blowing up $Z_i' \to Z_i$ such that Z_i' has the resolution property.

Proof. In this paragraph we use absolute Noetherian approximation to reduce to the case of algebraic spaces of finite presentation over $\text{Spec}(\mathbf{Z})$. We may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2 and Properties of Spaces, Definition 66.3.1. Thus we may apply Limits of Spaces, Proposition 70.8.1. It follows that we can find an affine morphism $X \to X_0$ with X_0 of finite presentation over $\mathbf{Z}$. If we can prove the lemma for X_0, then we can pull back the stratification and the centers of the blowing ups to X and get the result for X; this uses that the resolution property goes up along affine morphisms (Derived Categories of Spaces, Lemma 75.28.3) and that the strict transform of an affine morphism is affine – details omitted. This reduces us to the case discussed in the next paragraph.

Assume X is of finite presentation over $\mathbf{Z}$. Then X is Noetherian and $|X|$ is a Noetherian topological space (with finitely many irreducible components) of finite dimension. Hence we may use induction on $\dim(|X|)$. By Lemma 76.57.1 there exists a dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that X' has the resolution property. Set $Z_0 = X$ and let $Z_1 \subset X$ be the reduced closed subspace with $|Z_1| = |X| \setminus |U|$. By induction we find an integer $t \geq 0$ and a filtration

$$Z_1 \supset Z_{1,0} \supset Z_{1,1} \supset \ldots \supset Z_{1,t} = \emptyset$$

by closed subspaces, where $Z_{1,0} \to Z_1$ is a thickening and there exist $(Z_{1,i} \setminus Z_{1,i+1})$-admissible blowing ups $Z_{1,i}' \to Z_{1,i}$ such that $Z_{1,i}'$ has the resolution property. Since Z_1 is reduced, we have $Z_1 = Z_{1,0}$. Hence we can set $Z_i = Z_{1,i-1}$ and $Z_i' = Z_{1,i-1}'$ for $i \geq 1$ and the lemma is proved. $\square$

76.58. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
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(10) Commutative Algebra
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(12) Homological Algebra
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(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
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(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

CHAPTER 77

Flatness on Algebraic Spaces

0CU3 77.1. Introduction

0CU4 In this chapter, we discuss some advanced results on flat modules and flat morphisms in the setting of algebraic spaces. We strongly encourage the reader to take a look at the corresponding chapter in the setting of schemes first, see More on Flatness, Section 38.1. A reference is the paper [GR71] by Raynaud and Gruson.

77.2. Impurities

0CV5 The section is the analogue of More on Flatness, Section 38.15.

0CV6 Situation 77.2.1. Let S be a scheme. Let $f : X \to Y$ be a finite type, decent[1] morphism of algebraic spaces over S. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally $y \in |Y|$ is a point of Y.

In this situation consider a scheme T, a morphism $g : T \to Y$, a point $t \in T$ with $g(t) = y$, a specialization $t' \rightsquigarrow t$ in T, and a point $\xi \in |X_T|$ lying over t'. Here $X_T = T \times_Y X$. Picture

0CV7 (77.2.1.1)
$$
\begin{tikzcd}
\xi \arrow[d, mapsto] & & \\
t' \arrow[r, rightsquigarrow] & t \arrow[r, mapsto] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
X_T \arrow[r] \arrow[d, "f_T"'] & X \arrow[d, "f"] \\
T \arrow[r, "g"] & Y
\end{tikzcd}
$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

0CV8 Definition 77.2.2. In Situation 77.2.1 we say a diagram (77.2.1.1) defines an impurity of $\mathcal{F}$ above y if $\xi \in \text{Ass}_{X_T/T}(\mathcal{F}_T)$ and $t \notin f_T(\overline{\{\xi\}})$. We will indicate this by saying "let $(g : T \to Y, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above y".

Another way to say this is: $(g : T \to Y, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above y if there exists no specialization $\xi \rightsquigarrow \theta$ in the topological space $|X_T|$ with $f_T(\theta) = t$. Specializations in non-decent algebraic spaces do not behave well. If the morphism f is decent, then X_T is a decent algebraic space for all morphisms $g : T \to Y$ as above, see Decent Spaces, Definition 68.17.1.

0CV9 Lemma 77.2.3. In Situation 77.2.1. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above y. Assume $T = \lim_{i \in I} T_i$ is a directed limit of affine schemes over Y. Then for some i the triple $(T_i \to Y, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above y.

[1] Quasi-separated morphisms are decent, see Decent Spaces, Lemma 68.17.2. For any morphism $\text{Spec}(k) \to Y$ where k is a field, the algebraic space X_k is of finite presentation over k because it is of finite type over k and quasi-separated by Decent Spaces, Lemma 68.14.1.

Proof. The notation in the statement means this: Let $p_i : T \to T_i$ be the projection morphisms, let $t_i = p_i(t)$ and $t'_i = p_i(t')$. Finally $\xi_i \in |X_{T_i}|$ is the image of ξ. By Divisors on Spaces, Lemma 71.4.7 we have $\xi_i \in \text{Ass}_{X_{T_i}/T_i}(\mathcal{F}_{T_i})$. Thus the only point is to show that $t_i \notin f_{T_i}(\overline{\{\xi_i\}})$ for some i.

Let $Z_i \subset X_{T_i}$ be the reduced induced scheme structure on $\overline{\{\xi_i\}} \subset |X_{T_i}|$ and let $Z \subset X_T$ be the reduced induced scheme structure on $\overline{\{\xi\}} \subset |X_T|$. Then $Z = \lim Z_i$ by Limits of Spaces, Lemma 70.5.4 (the lemma applies because each X_{T_i} is decent). Choose a field k and a morphism $\text{Spec}(k) \to T$ whose image is t. Then

$$\emptyset = Z \times_T \text{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \text{Spec}(k) = \lim Z_i \times_{T_i} \text{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each $Z_i \times_{T_i} \text{Spec}(k)$ is quasi-compact because $X_{T_i} \to T_i$ is of finite type and hence $Z_i \to T_i$ is of finite type. Hence $Z_i \times_{T_i} \text{Spec}(k)$ is empty for some i by Limits of Spaces, Lemma 70.5.3. Since the image of the composition $\text{Spec}(k) \to T \to T_i$ is t_i we obtain what we want. $\square$

Impurities go up along flat base change.

0CVA **Lemma 77.2.4.** In Situation 77.2.1. Let $(Y_1, y_1) \to (Y, y)$ be a morphism of pointed algebraic spaces over S. Assume $Y_1 \to Y$ is flat at y_1. If $(T \to Y, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above y, then there exists an impurity $(T_1 \to Y_1, t'_1 \rightsquigarrow t_1, \xi_1)$ of the pullback $\mathcal{F}_1$ of $\mathcal{F}$ to $X_1 = Y_1 \times_Y X$ over y_1 such that T_1 is étale over $Y_1 \times_Y T$.

Proof. Choose an étale morphism $T_1 \to Y_1 \times_Y T$ where T_1 is a scheme and let $t_1 \in T_1$ be a point mapping to y_1 and t. It is possible to find a pair (T_1, t_1) like this by Properties of Spaces, Lemma 66.4.3. The morphism of schemes $T_1 \to T$ is flat at t_1 (use Morphisms of Spaces, Lemma 67.30.4 and the definition of flat morphisms of algebraic spaces) there exists a specialization $t'_1 \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Morphisms, Lemma 29.25.9. Choose a point $\xi_1 \in |X_{T_1}|$ mapping to t'_1 and ξ with $\xi_1 \in \text{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. point of $\text{Spec}(\kappa(t'_1) \otimes_{\kappa(t')} \kappa(\xi))$. This is possible by Divisors on Spaces, Lemma 71.4.7. As the closure Z_1 of $\{\xi_1\}$ in $|X_{T_1}|$ maps into the closure of $\{\xi\}$ in $|X_T|$ we conclude that the image of Z_1 in $|T_1|$ cannot contain t_1. Hence $(T_1 \to Y_1, t'_1 \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_1$ above Y_1. $\square$

0CVB **Lemma 77.2.5.** In Situation 77.2.1. Let $\overline{y}$ be a geometric point lying over y. Let $\mathcal{O} = \mathcal{O}_{Y,\overline{y}}$ be the étale local ring of Y at $\overline{y}$. Denote $Y^{sh} = \text{Spec}(\mathcal{O})$, $X^{sh} = X \times_Y Y^{sh}$, and $\mathcal{F}^{sh}$ the pullback of $\mathcal{F}$ to X^{sh}. The following are equivalent

(1) there exists an impurity $(Y^{sh} \to Y, y' \rightsquigarrow \overline{y}, \xi)$ of $\mathcal{F}$ above y,
(2) every point of $\text{Ass}_{X^{sh}/Y^{sh}}(\mathcal{F}^{sh})$ specializes to a point of the closed fibre $X_{\overline{y}}$,
(3) there exists an impurity $(T \to Y, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above y such that $(T, t) \to (Y, y)$ is an étale neighbourhood, and
(4) there exists an impurity $(T \to Y, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above y such that $T \to Y$ is quasi-finite at t.

Proof. That parts (1) and (2) are equivalent is immediate from the definition.

Recall that $\mathcal{O} = \mathcal{O}_{Y,\overline{y}}$ is the filtered colimit of $\mathcal{O}(V)$ over the category of étale neighbourhoods $(V, \overline{v}) \to (Y, \overline{y})$ (Properties of Spaces, Lemma 66.19.3). Moreover, it suffices to consider affine étale neighbourhoods V. Hence $Y^{sh} = \text{Spec}(\mathcal{O}) = \lim \text{Spec}(\mathcal{O}(V)) = \lim V$. Thus we see that (1) implies (3) by Lemma 77.2.3.

Since an étale morphism is locally quasi-finite (Morphisms of Spaces, Lemma 67.39.5) we see that (3) implies (4).

Finally, assume (4). After replacing T by an open neighbourhood of t we may assume $T \to Y$ is locally quasi-finite. By Lemma 77.2.4 we find an impurity $(T_1 \to Y^{sh}, t'_1 \leadsto t_1, \xi_1)$ with $T_1 \to T \times_Y Y^{sh}$ étale. Since an étale morphism is locally quasi-finite and using Morphisms of Spaces, Lemma 67.27.4 and Morphisms, Lemma 29.20.12 we see that $T_1 \to Y^{sh}$ is locally quasi-finite. As $\mathcal{O}$ is strictly henselian, we can apply More on Morphisms, Lemma 37.41.1 to see that after replacing T_1 by an open and closed neighbourhood of t_1 we may assume that $T_1 \to Y^{sh} = \mathrm{Spec}(\mathcal{O})$ is finite. Let $\theta \in |X^{sh}|$ be the image of ξ_1 and let $y' \in \mathrm{Spec}(\mathcal{O})$ be the image of t'_1. By Divisors on Spaces, Lemma 71.4.7 we see that $\theta \in \mathrm{Ass}_{X^{sh}/Y^{sh}}(\mathcal{F}^{sh})$. Since $\pi : X_{T_1} \to X^{sh}$ is finite, it induces a closed map $|X_{T_1}| \to |X^{sh}|$. Hence the image of $\overline{\{\xi_1\}}$ is $\overline{\{\theta\}}$. It follows that $(Y^{sh} \to Y, y' \leadsto \overline{y}, \theta)$ is an impurity of $\mathcal{F}$ above y and the proof is complete. $\square$

77.3. Relatively pure modules

0CVC This section is the analogue of More on Flatness, Section 38.16.

0CVD Definition 77.3.1. In Situation 77.2.1.

(1) We say $\mathcal{F}$ is pure above y if none of the equivalent conditions of Lemma 77.2.5 hold.
(2) We say $\mathcal{F}$ is universally pure above y if there does not exist any impurity of $\mathcal{F}$ above y.
(3) We say that X is pure above y if $\mathcal{O}_X$ is pure above y.
(4) We say $\mathcal{F}$ is universally Y-pure, or universally pure relative to Y if $\mathcal{F}$ is universally pure above y for every $y \in |Y|$.
(5) We say $\mathcal{F}$ is Y-pure, or pure relative to Y if $\mathcal{F}$ is pure above y for every $y \in |Y|$.
(6) We say that X is Y-pure or pure relative to Y if $\mathcal{O}_X$ is pure relative to Y.

The obligatory lemmas follow.

0CVE Lemma 77.3.2. In Situation 77.2.1.

(1) $\mathcal{F}$ is universally pure above y, and
(2) for every morphism $(Y', y') \to (Y, y)$ of pointed algebraic spaces the pullback $\mathcal{F}_{Y'}$ is pure above y'.

In particular, $\mathcal{F}$ is universally pure relative to Y if and only if every base change $\mathcal{F}_{Y'}$ of $\mathcal{F}$ is pure relative to Y'.

Proof. This is formal. $\square$

0CVF Lemma 77.3.3. In Situation 77.2.1. Let $(Y', y') \to (Y, y)$ be a morphism of pointed algebraic spaces. If $Y' \to Y$ is quasi-finite at y' and $\mathcal{F}$ is pure above y, then $\mathcal{F}_{Y'}$ is pure above y'.

Proof. It $(T \to Y', t' \leadsto t, \xi)$ is an impurity of $\mathcal{F}_{Y'}$ above y' with $T \to Y'$ quasi-finite at t, then $(T \to Y, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above y with $T \to Y$ quasi-finite at t, see Morphisms of Spaces, Lemma 67.27.3. Hence the lemma follows immediately from the definition of purity. $\square$

Purity satisfies flat descent.

0CVG Lemma 77.3.4. In Situation 77.2.1. Let $(Y_1, y_1) \to (Y, y)$ be a morphism of pointed algebraic spaces. Assume $Y_1 \to Y$ is flat at y_1.

(1) If $\mathcal{F}_{Y_1}$ is pure above y_1, then $\mathcal{F}$ is pure above y.
(2) If $\mathcal{F}_{Y_1}$ is universally pure above y_1, then $\mathcal{F}$ is universally pure above y.

Proof. This is true because impurities go up along a flat base change, see Lemma 77.2.4. For example part (1) follows because by any impurity $(T \to Y, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above y with $T \to Y$ quasi-finite at t by the lemma leads to an impurity $(T_1 \to Y_1, t'_1 \rightsquigarrow t_1, \xi_1)$ of the pullback $\mathcal{F}_1$ of $\mathcal{F}$ to $X_1 = Y_1 \times_Y X$ over y_1 such that T_1 is étale over $Y_1 \times_Y T$. Hence $T_1 \to Y_1$ is quasi-finite at t_1 because étale morphisms are locally quasi-finite and compositions of locally quasi-finite morphisms are locally quasi-finite (Morphisms of Spaces, Lemmas 67.39.5 and 67.27.3). Similarly for part (2). $\square$

0CVH Lemma 77.3.5. In Situation 77.2.1. Let $i : Z \to X$ be a closed immersion and assume that $\mathcal{F} = i_*\mathcal{G}$ for some finite type, quasi-coherent sheaf $\mathcal{G}$ on Z. Then $\mathcal{G}$ is (universally) pure above y if and only if $\mathcal{F}$ is (universally) pure above y.

Proof. This follows from Divisors on Spaces, Lemma 71.4.9. $\square$

0CVI Lemma 77.3.6. In Situation 77.2.1.

(1) If the support of $\mathcal{F}$ is proper over Y, then $\mathcal{F}$ is universally pure relative to Y.
(2) If f is proper, then $\mathcal{F}$ is universally pure relative to Y.
(3) If f is proper, then X is universally pure relative to Y.

Proof. First we reduce (1) to (2). Namely, let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$ (Morphisms of Spaces, Definition 67.15.4). Let $i : Z \to X$ be the corresponding closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some finite type quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$. In case (1) $Z \to Y$ is proper by assumption. Thus by Lemma 77.3.5 case (1) reduces to case (2).

Assume f is proper. Let $(g : T \to Y, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above y. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. $\square$

77.4. Flat finite type modules

0CVJ Please compare with More on Flatness, Sections 38.10, 38.13, and 38.26. Most of these results have immediate consequences of algebraic spaces by étale localization.

0CWJ Lemma 77.4.1. Let S be a scheme. Let $X \to Y$ be a finite type morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$ be a point. There exists an étale morphism $(Y', y') \to (Y, y)$ with Y' an affine scheme and étale morphisms $h_i : W_i \to X_{Y'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/W_i/Y'$ over y', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to W_i and such that $|(X_{Y'})_{y'}| \subset \bigcup h_i(W_i)$.

Proof. The question is étale local on Y hence we may assume Y is an affine scheme. Then X is quasi-compact, hence we can choose an affine scheme X' and a surjective étale morphism $X' \to X$. Then we may apply More on Flatness, Lemma 38.5.8 to $X' \to Y$, $(X' \to Y)^*\mathcal{F}$, and y to get what we want. $\square$

0CWK Lemma 77.4.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $y \in |Y|$ and $F = f^{-1}(\{y\}) \subset |X|$. Then the set

$$\{x \in F \mid \mathcal{F} \text{ flat over } Y \text{ at } x\}$$

is open in F.

Proof. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Y$ mapping v to y. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $|U_v| \to F$ is an open continuous map of topological spaces as $|U| \to |X|$ is continuous and open. Hence the result follows from the case of schemes which is More on Flatness, Lemma 38.10.4. $\square$

0CVK Lemma 77.4.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. If $\mathcal{F}$ is flat at x over Y, then

$$x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow y \in \text{WeakAss}_Y(\mathcal{G}) \text{ and } x \in \text{Ass}_{X/Y}(\mathcal{F}).$$

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective étale. Choose $u \in U$ mapping to x. Let $\mathcal{E} = \mathcal{F}|_U$ and $\mathcal{H} = \mathcal{G}|_V$. Let $v \in V$ be the image of u. Then $x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $u \in \text{WeakAss}_X(\mathcal{E} \otimes_{\mathcal{O}_X} g^*\mathcal{H})$ by Divisors on Spaces, Definition 71.2.2. Similarly, $y \in \text{WeakAss}_Y(\mathcal{G})$ if and only if $v \in \text{WeakAss}_V(\mathcal{H})$. Finally, we have $x \in \text{Ass}_{X/Y}(\mathcal{F})$ if and only if $u \in \text{Ass}_{U_v}(\mathcal{E}|_{U_v})$ by Divisors on Spaces, Definition 71.4.5. Observe that flatness of $\mathcal{F}$ at x is equivalent to flatness of $\mathcal{E}$ at u, see Morphisms of Spaces, Definition 67.31.2. The equivalence for $g : U \to V$, $\mathcal{E}$, $\mathcal{H}$, u, and v is More on Flatness, Lemma 38.13.3. $\square$

0CVL Lemma 77.4.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over Y. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then we have

$$\text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \text{Ass}_{X/Y}(\mathcal{F}) \cap |f|^{-1}(\text{WeakAss}_Y(\mathcal{G}))$$

Proof. Immediate consequence of Lemma 77.4.3. $\square$

0DLR Theorem 77.4.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) $X \to Y$ is locally of finite presentation,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and
(3) the set of weakly associated points of Y is locally finite in Y.

Then $U = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over Y.

Proof. Condition (3) means that if $V \to Y$ is a surjective étale morphism where V is a scheme, then the weakly associated points of V are locally finite on the scheme V. (Recall that the weakly associated points of V are exactly the inverse image of the weakly associated points of Y by Divisors on Spaces, Definition 71.2.2.) Having said this the question is étale local on X and Y, hence we may assume X and Y are schemes. Thus the result follows from More on Flatness, Theorem 38.13.6. $\square$

0CVW Lemma 77.4.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$. Assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat over Y at all $x \in F$.

Then there exists an étale morphism $(Y', y') \to (Y, y)$ where Y' is a scheme and a commutative diagram of algebraic spaces

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ Y & \mathrm{Spec}(\mathcal{O}_{Y',y'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_Y \mathrm{Spec}(\mathcal{O}_{Y',y'})$ is étale, $|X'_{y'}| \to F$ is surjective, X' is affine, and $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{Y',y'}$-module.

Proof. Choose an étale morphism $(Y', y') \to (Y, y)$ where Y' is an affine scheme. Then $X \times_Y Y'$ is quasi-compact. Choose an affine scheme X' and a surjective étale morphism $X' \to X \times_Y Y'$. Picture

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ Y & Y' \arrow[l] \end{tikzcd}$$

Then $\mathcal{F}' = g^*\mathcal{F}$ is flat over Y' at all points of $X'_{y'}$, see Morphisms of Spaces, Lemma 67.31.3. Hence we can apply the lemma in the case of schemes (More on Flatness, Lemma 38.12.11) to the morphism $X' \to Y'$, the quasi-coherent sheaf $g^*\mathcal{F}$, and the point y'. This gives an étale morphism $(Y'', y'') \to (Y', y')$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] & X'' \arrow[l, "g'"'] \arrow[d] \\ Y & Y' \arrow[l] & \mathrm{Spec}(\mathcal{O}_{Y'',y''}) \arrow[l] \end{tikzcd}$$

To get what we want we take $(Y'', y'') \to (Y, y)$ and $g \circ g' : X'' \to X$. $\square$

0CWL Theorem 77.4.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in |X|$ with image $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$. Consider the conditions

(1) $\mathcal{F}$ is flat at x over Y, and

(2) for every $x' \in F \cap \text{Ass}_{X/Y}(\mathcal{F})$ which specializes to x we have that $\mathcal{F}$ is flat at x' over Y.

Then we always have (2) $\Rightarrow$ (1). If X and Y are decent, then (1) $\Rightarrow$ (2).

Proof. Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a point $u \in U$ mapping to x. Let $v \in V$ be the image of u. We will deduce the result from the corresponding result for $\mathcal{F}|_U = (U \to X)^*\mathcal{F}$ and the point u. U_v. This works because $\text{Ass}_{U/V}(\mathcal{F}|_U) \cap |U_v|$ is equal to $\text{Ass}_{U_v}(\mathcal{F}|_{U_v})$ and equal to the inverse image of $F \cap \text{Ass}_{X/Y}(\mathcal{F})$. Since the map $|U_v| \to F$ is continuous we see that specializations in $|U_v|$ map to specializations in F, hence condition (2) is inherited by $U \to V$, $\mathcal{F}|_U$, and the point u. Thus More on Flatness, Theorem 38.26.1 applies and we conclude that (1) holds.

If Y is decent, then we can represent y by a quasi-compact monomorphism $\text{Spec}(k) \to Y$ (by definition of decent spaces, see Decent Spaces, Definition 68.6.1). Then $F = |X_k|$, see Decent Spaces, Lemma 68.18.6. If in addition X is decent (or more generally if f is decent, see Decent Spaces, Definition 68.17.1 and Decent Spaces, Lemma 68.17.3), then X_y is a decent space too. Furthermore, specializations in F can be lifted to specializations in $U_v \to X_y$, see Decent Spaces, Lemma 68.12.2. Having said this it is clear that the reverse implication holds, because it holds in the case of schemes. $\square$

0CWM Lemma 77.4.8. Let S be a local scheme with closed point s. Let $f : X \to S$ be a morphism from an algebraic space X to S which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume that

(1) every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s[2],
(2) $\mathcal{F}$ is flat over S at every point of X_s.

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 77.4.7. $\square$

77.5. Flat finitely presented modules

0CVX This is the analogue of More on Flatness, Section 38.12.

0CVY Proposition 77.5.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$. Assume that

(1) f is locally of finite presentation,
(2) $\mathcal{F}$ is of finite presentation, and
(3) $\mathcal{F}$ is flat at x over Y.

Then there exists a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (Y, y) & (Y', y') \arrow[l] \end{tikzcd}$$

[2]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

whose horizontal arrows are étale such that X', Y' are affine and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(Y', \mathcal{O}_{Y'})$-module.

Proof. As formulated this proposition immmediately reduces to the case of schemes, which is More on Flatness, Proposition 38.12.4. $\square$

0CVZ Lemma 77.5.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$. Assume that

(1) f is of finite presentation,
(2) $\mathcal{F}$ is of finite presentation, and
(3) $\mathcal{F}$ is flat over Y at all $x \in F$.

Then there exists a commutative diagram of algebraic spaces

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ Y & Y' \arrow[l, "h"'] \end{tikzcd}$$

such that h and g are étale, there is a point $y' \in |Y'|$ mapping to y, we have $F \subset g(|X'|)$, the algebraic spaces X', Y' are affine, and $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(Y', \mathcal{O}_{Y'})$-module.

Proof. As formulated this lemma immmediately reduces to the case of schemes, which is More on Flatness, Lemma 38.12.5. $\square$

77.6. A criterion for purity

0CW0 This section is the analogue of More on Flatness, Section 38.18.

0CW1 Lemma 77.6.1. Let S be a scheme. Let X be a decent algebraic space locally of finite type over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ such that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in \text{Ass}_{X/S}(\mathcal{F})$. If the closure of $\{x'\}$ in $|X|$ meets $|X_s|$, then the closure meets $\text{Ass}_{X/S}(\mathcal{F}) \cap |X_s|$.

Proof. Observe that $|X_s| \subset |X|$ is the set of points of $|X|$ lying over $s \in S$, see Decent Spaces, Lemma 68.18.6. Let $t \in |X_s|$ be a specialization of x' in $|X|$. Choose an affine scheme U and a point $u \in U$ and an étale morphism $\varphi : U \to X$ mapping u to t. By Decent Spaces, Lemma 68.12.2 we can choose a specialization $u' \rightsquigarrow u$ with u' mapping to x'. Set $g = f \circ \varphi$. Observe that $s' = g(u') = f(x')$ specializes to s. By our definition of $\text{Ass}_{X/S}(\mathcal{F})$ we have $u' \in \text{Ass}_{U/S}(\varphi^*\mathcal{F})$. By the schemes version of this lemma (More on Flatness, Lemma 38.18.1) we see that there is a specialization $u' \rightsquigarrow u$ with $u \in \text{Ass}_{U_s}(\varphi^*\mathcal{F}_s) = \text{Ass}_{U/S}(\varphi^*\mathcal{F}) \cap U_s$. Hence $x = \varphi(u) \in \text{Ass}_{X/S}(\mathcal{F})$ lies over s and the lemma is proved. $\square$

0CW2 Lemma 77.6.2. Let Y be an algebraic space over a scheme S. Let $g : X' \to X$ be a morphism of algebraic spaces over Y with X locally of finite type over Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$, then for any morphism $Z \to Y$ we have $\text{Ass}_{X_Z/Z}(\mathcal{F}_Z) \subset g_Z(|X'_Z|)$.

Proof. By Properties of Spaces, Lemma 66.4.3 the map $|X'_Z| \to |X_Z| \times_{|X|} |X'|$ is surjective as X'_Z is equal to $X_Z \times_X X'$. By Divisors on Spaces, Lemma 71.4.7 the map $|X_Z| \to |X|$ sends $\text{Ass}_{X_Z/Z}(\mathcal{F}_Z)$ into $\text{Ass}_{X/Y}(\mathcal{F})$. The lemma follows. $\square$

0CW3 Lemma 77.6.3. Let Y be an algebraic space over a scheme S. Let $g : X' \to X$ be an étale morphism of algebraic spaces over Y. Assume the structure morphisms $X' \to Y$ and $X \to Y$ are decent and of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$.

(1) If $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$ and $g^*\mathcal{F}$ is (universally) pure above y, then $\mathcal{F}$ is (universally) pure above y.
(2) If $\mathcal{F}$ is pure above y, $g(|X'|)$ contains F, and Y is affine local with closed point y, then $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$.
(3) If $\mathcal{F}$ is pure above y, $\mathcal{F}$ is flat at all points of F, $g(|X'|)$ contains $\text{Ass}_{X/Y}(\mathcal{F}) \cap F$, and Y is affine local with closed point y, then $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$.
(4) Add more here.

Proof. The assumptions on $X \to Y$ and $X' \to Y$ guarantee that we may apply the material in Sections 77.2 and 77.3 to these morphisms and the sheaves $\mathcal{F}$ and $g^*\mathcal{F}$. Since g is étale we see that $\text{Ass}_{X'/Y}(g^*\mathcal{F})$ is the inverse image of $\text{Ass}_{X/Y}(\mathcal{F})$ and the same remains true after base change.

Proof of (1). Assume $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$. Suppose that $(T \to Y, t' \leadsto t, \xi)$ is an impurity of $\mathcal{F}$ above y. Since $\text{Ass}_{X_T/T}(\mathcal{F}_T) \subset g_T(|X'_T|)$ by Lemma 77.6.2 we can choose a point $\xi' \in |X'_T|$ mapping to ξ. By the above we see that $(T \to Y, t' \leadsto t, \xi')$ is an impurity of $g^*\mathcal{F}$ above y'. This implies (1) is true.

Proof of (2). This follows from the fact that $g(|X'|)$ is open in $|X|$ and the fact that by purity every point of $\text{Ass}_{X/Y}(\mathcal{F})$ specializes to a point of F.

Proof of (3). This follows from the fact that $g(|X'|)$ is open in $|X|$ and the fact that by purity combined with Lemma 77.6.1 every point of $\text{Ass}_{X/Y}(\mathcal{F})$ specializes to a point of $\text{Ass}_{X/Y}(\mathcal{F}) \cap F$. $\square$

0CW4 Lemma 77.6.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$. Assume

(1) f is decent and of finite type,
(2) $\mathcal{F}$ is of finite type,
(3) $\mathcal{F}$ is flat over Y at all points lying over y, and
(4) $\mathcal{F}$ is pure above y.

Then $\mathcal{F}$ is universally pure above y.

Proof. Consider the morphism $\text{Spec}(\mathcal{O}_{Y,\overline{y}}) \to Y$. This is a flat morphism from the spectrum of a stricly henselian local ring which maps the closed point to y. By Lemma 77.3.4 we reduce to the case described in the next paragraph.

Assume Y is the spectrum of a strictly henselian local ring R with closed point y. By Lemma 77.4.6 there exists an étale morphism $g : X' \to X$ with $g(|X'|) \supset |X_y|$, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free R-module. Then $g^*\mathcal{F}$ is universally pure relative to Y, see More on Flatness, Lemma 38.17.4. Hence it suffices to prove that $g(|X'|)$ contains $\text{Ass}_{X/Y}(\mathcal{F})$ by Lemma 77.6.3 part (1). This in turn follows from Lemma 77.6.3 part (2). $\square$

0CW5 Lemma 77.6.5. Let S be a scheme. Let $f : X \to Y$ be a decent, finite type morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over Y. In this case $\mathcal{F}$ is pure relative to Y if and only if $\mathcal{F}$ is universally pure relative to Y.

Proof. Immediate consequence of Lemma 77.6.4 and the definitions. □

0CW6 Lemma 77.6.6. Let Y be an algebraic space over a scheme S. Let $g : X' \to X$ be a flat morphism of algebraic spaces over Y with X locally of finite type over Y. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module which is flat over Y. If $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$ then the canonical map

$$\mathcal{F} \longrightarrow g_*g^*\mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_Z \to (g_Z)_*g_Z^*\mathcal{F}_Z$ is injective for any morphism $Z \to Y$. Since the assumption on the relative assassin is preserved by base change (Lemma 77.6.2) it suffices to prove the injectivity of the displayed arrow.

Let $\mathcal{K} = \text{Ker}(\mathcal{F} \to g_*g^*\mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors on Spaces, Lemma 71.2.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors on Spaces, Lemma 71.2.4. As $\mathcal{F}$ is flat we see from Lemma 77.4.4 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/Y}(\mathcal{F})$. By assumption any point x of $\text{Ass}_{X/Y}(\mathcal{F})$ is the image of some $x' \in |X'|$. Since g is flat the local ring map $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X',\overline{x}'}$ is faithfully flat, hence the map

$$\mathcal{F}_{\overline{x}} \longrightarrow (g^*\mathcal{F})_{\overline{x}'} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'}$$

is injective (see Algebra, Lemma 10.82.11). Since the displayed arrow factors through $\mathcal{F}_{\overline{x}} \to (g_*g^*\mathcal{F})_{\overline{x}}$, we conclude that $\mathcal{K}_{\overline{x}} = 0$. Hence x cannot be a weakly associated point of $\mathcal{K}$ and we win. □

77.7. Flattening functors

083E This section is the analogue of More on Flatness, Section 38.20. We urge the reader to skip this section on a first reading.

083F Situation 77.7.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_B T \to X$. In this situation we can consider the functor

083G (77.7.1.1)
$$F_{iso} : (Sch/B)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

In Situation 77.7.1 we sometimes think of the functors F_{iso}, F_{inj}, F_{surj}, and F_{zero} as functors $(Sch/S)^{opp} \to \text{Sets}$ endowed with a morphism $F_{iso} \to B$, $F_{inj} \to B$, $F_{surj} \to B$, and $F_{zero} \to B$. Namely, if T is a scheme over S, then an element $h \in F_{iso}(T)$ is a morphism $h : T \to B$ such that the base change of u via h is an isomorphism. In particular, when we say that F_{iso} is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to \text{Sets}$ is an algebraic space.

083H Lemma 77.7.2. In Situation 77.7.1. Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 73.9.3. In particular, for every $x \in |X_T|$ there exists an $i \in I$ and an $x_i \in |X_i|$ mapping to x. Since $\mathcal{O}_{X_T,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is flat, hence faithfully flat (see Morphisms of Spaces, Section 67.30). we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. The lemma follows. □

083I Lemma 77.7.3. In Situation 77.7.1 let $X' \to X$ be a flat morphism of algebraic spaces. Denote $u' : \mathcal{F}' \to \mathcal{G}'$ the pullback of u to X'. Denote F'_{iso}, F'_{inj}, F'_{surj}, F'_{zero} the functors on Sch/B associated to u'.

(1) If $\mathcal{G}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{G}$, then $F_{surj} = F'_{surj}$ and $F_{zero} = F'_{zero}$.
(2) If $\mathcal{F}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{F}$, then $F_{inj} = F'_{inj}$ and $F_{zero} = F'_{zero}$.
(3) If $\mathcal{F}$ and $\mathcal{G}$ are of finite type and the image of $|X'| \to |X|$ contains the supports of $\mathcal{F}$ and $\mathcal{G}$, then $F_{iso} = F'_{iso}$.

Proof. let $v : \mathcal{H} \to \mathcal{E}$ be a map of quasi-coherent modules on an algebraic space Y and let $\varphi : Y' \to Y$ be a surjective flat morphism of algebraic spaces, then v is an isomorphism, injective, surjective, or zero if and only if φ^*v is an isomorphism, injective, surjective, or zero. Namely, for every $y \in |Y|$ there exists a $y' \in |Y'|$ and the map of local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{Y',\overline{y'}}$ is faithfully flat (see Morphisms of Spaces, Section 67.30). Of course, to check for injectivity or being zero it suffices to look at the points in the support of $\mathcal{H}$, and to check for surjectivity it suffices to look at points in the support of $\mathcal{E}$. Moreover, under the finite type assumptions as in the statement of the lemma, taking the supports commutes with base change, see Morphisms of Spaces, Lemma 67.15.2. Thus the lemma is clear. □

Recall that we've defined the scheme theoretic support of a finite type quasi-coherent module in Morphisms of Spaces, Definition 67.15.4.

083J Lemma 77.7.4. In Situation 77.7.1.

(1) If $\mathcal{G}$ is of finite type and the scheme theoretic support of $\mathcal{G}$ is quasi-compact over B, then F_{surj} is limit preserving.
(2) If $\mathcal{F}$ of finite type and the scheme theoretic support of $\mathcal{F}$ is quasi-compact over B, then F_{zero} is limit preserving.
(3) If $\mathcal{F}$ is of finite type, $\mathcal{G}$ is of finite presentation, and the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ are quasi-compact over B, then F_{iso} is limit preserving.

Proof. Proof of (1). Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{G}$ and think of $\mathcal{G}$ as a finite type quasi-coherent module on Z. We may replace X by Z and u by the map $i^*\mathcal{F} \to \mathcal{G}$ (details omitted). Hence we may assume f is quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove (1) we have to show that u_i is surjective for some i. Pick $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Since f is quasi-compact we see X_0 is quasi-compact. Hence we may choose a surjective étale morphism $\varphi_0 : W_0 \to X_0$ where W_0 is an affine scheme. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with a surjective étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$. Hence $\varphi^* u_T$ is surjective and it suffices to prove that $\varphi_i^* u_i$ is surjective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 10.127.5 part (2).

Proof of (2). Assume $\mathcal{F}$ is of finite type with scheme theoretic support $Z \subset B$ quasi-compact over B. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is zero. Set $X_i = T_i \times_B X$ and denote $u_i : \mathcal{F}_i \to \mathcal{G}_i$ the pullback. Choose $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Set $Z_0 = Z \times_X X_0$. By Morphisms of Spaces, Lemma 67.15.2 the support of $\mathcal{F}_i$ is $|Z_0|$. Since $|Z_0|$ is quasi-compact we can find an affine scheme W_0 and an étale morphism $W_0 \to X_0$ such that $|Z_0| \subset \mathrm{Im}(|W_0| \to |X_0|)$. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$ and that the support of $\mathcal{F}_T$ and $\mathcal{F}_i$ is contained in the image of $|W| \to |X_T|$ and $|W_i| \to |X_i|$. Now $\varphi^* u_T$ is injective and it suffices to prove that $\varphi_i^* u_i$ is injective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 10.127.5 part (1).

Proof of (3). This can be proven in exactly the same manner as in the previous two paragraphs using Algebra, Lemma 10.127.5 part (3). We can also deduce it from (1) and (2) as follows. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is an isomorphism. By part (1) there exists an $0 \in I$ such that u_{T_0} is surjective. Set $\mathcal{K} = \mathrm{Ker}(u_{T_0})$ and consider the map of quasi-coherent modules $v : \mathcal{K} \to \mathcal{F}_{T_0}$. For $i \geq 0$ the base change v_{T_i} is zero if and only if u_i is an isomorphism. Moreover, v_T is zero. Since $\mathcal{G}_{T_0}$ is of finite presentation, $\mathcal{F}_{T_0}$ is of finite type, and u_{T_0} is surjective we conclude that $\mathcal{K}$ is of finite type (Modules on Sites, Lemma 18.24.1). It is clear that the support of $\mathcal{K}$ is contained in the support of $\mathcal{F}_{T_0}$ which is quasi-compact over T_0. Hence we can apply part (2) to see that v_{T_i} is zero for some i. □

0CVM Lemma 77.7.5. In Situation 77.7.1 suppose given an exact sequence

$$\mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{v} \mathcal{H} \to 0$$

Then we have $F_{v,iso} = F_{u,zero}$ with obvious notation.

Proof. Since pullback is right exact we see that $\mathcal{F}_T \to \mathcal{G}_T \to \mathcal{H}_T \to 0$ is exact for every scheme T over B. Hence u_T is surjective if and only if v_T is an isomorphism. □

0CW7 Lemma 77.7.6. In Situation 77.7.1 suppose given an affine morphism $i : Z \to X$ and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{H}$ such that $\mathcal{G} = i_*\mathcal{H}$. Let $v : i^*\mathcal{F} \to \mathcal{H}$ be the map adjoint to u. Then

(1) $F_{v,zero} = F_{u,zero}$, and
(2) if i is a closed immersion, then $F_{v,surj} = F_{u,surj}$.

Proof. Let T be a scheme over B. Denote $i_T : Z_T \to X_T$ the base change of i and $\mathcal{H}_T$ the pullback of $\mathcal{H}$ to Z_T. Observe that $(i^*\mathcal{F})_T = i_T^*\mathcal{F}_T$ and $i_{T,*}\mathcal{H}_T = (i_*\mathcal{H})_T$. The first statement follows from commutativity of pullbacks and the second from Cohomology of Spaces, Lemma 69.11.1. Hence we see that u_T and v_T are adjoint maps as well. Thus $u_T = 0$ if and only if $v_T = 0$. This proves (1). In case (2) we see that u_T is surjective if and only if v_T is surjective because u_T factors as

$$\mathcal{F}_T \to i_{T,*}i_T^*\mathcal{F}_T \xrightarrow{i_{T,*}v_T} i_{T,*}\mathcal{H}_T$$

and the fact that $i_{T,*}$ is an exact functor fully faithfully embedding the category of quasi-coherent modules on Z_T into the category of quasi-coherent $\mathcal{O}_{X_T}$-modules. See Morphisms of Spaces, Lemma 67.14.1. □

0CW8 Lemma 77.7.7. In Situation 77.7.1 suppose given an affine morphism $g : X \to X'$. Set $u' = f_*u : f_*\mathcal{F} \to f_*\mathcal{G}$. Then $F_{u,iso} = F_{u',iso}$, $F_{u,inj} = F_{u',inj}$, $F_{u,surj} = F_{u',surj}$, and $F_{u,zero} = F_{u',zero}$.

Proof. By Cohomology of Spaces, Lemma 69.11.1 we have $g_{T,*}u_T = u'_T$. Moreover, $g_{T,*} : QCoh(\mathcal{O}_{X_T}) \to QCoh(\mathcal{O}_X)$ is a faithful, exact functor reflecting isomorphisms, injective maps, and surjective maps. □

0CWX Situation 77.7.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over Y we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_Y T \to X$. Since the base change of a flat module is flat we obtain a functor

0CWY (77.7.8.1) $$F_{flat} : (Sch/Y)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

In Situation 77.7.8 we sometimes think of F_{flat} as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $F_{flat} \to Y$. Namely, if T is a scheme over S, then an element $h \in F_{flat}(T)$ is a morphism $h : T \to Y$ such that the base change of $\mathcal{F}$ via h is flat over T. In particular, when we say that F_{flat} is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

0CWZ Lemma 77.7.9. In Situation 77.7.8.

(1) The functor F_{flat} satisfies the sheaf property for the fpqc topology.
(2) If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of algebraic spaces over Y, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see Morphisms of Spaces, Lemma 67.31.3. Part (2) follows from Limits of Spaces, Lemma 70.6.12 if f is also quasi-separated (i.e., f is of finite presentation). For the general case, first reduce to the case where the base is affine and then cover X by finitely many affines to reduce to the quasi-separated case. Details omitted. □

77.8. Making a map zero

0CW9 This section has no analogue in the corresponding chapter on schemes.

0CWA Situation 77.8.1. Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{G}$ flat over S.

083K Lemma 77.8.2. In Situation 77.8.1. Let $T \to S$ be a quasi-compact morphism of schemes such that the base change u_T is zero. Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) the base change u_Z is zero. If $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module and the scheme theoretic support of $\mathcal{F}$ is quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Let $U \to X$ be a surjective étale morphism of algebraic spaces where $U = \coprod U_i$ is a disjoint union of affine schemes (see Properties of Spaces, Lemma 66.6.1). By Lemma 77.7.3 we see that we may replace X by U. In other words, we may assume that $X = \coprod X_i$ is a disjoint union of affine schemes X_i. Suppose that we can prove the lemma for $u_i = u|_{X_i}$. Then we find a closed subscheme $Z_i \subset S$ such that $T \to S$ factors through Z_i and u_{i,Z_i} is zero. If $Z_i = \mathrm{Spec}(R/I_i) \subset \mathrm{Spec}(R) = S$, then taking $Z = \mathrm{Spec}(R/\sum I_i)$ works. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine.

Choose a finite affine open covering $T = T_1 \cup \ldots \cup T_m$. It is clear that we may replace T by $\coprod_{j=1,\ldots,m} T_j$. Hence we may assume T is affine. Say $T = \mathrm{Spec}(R')$. Let $u : M \to N$ be the homomorphisms of A-modules corresponding to $u : \mathcal{F} \to \mathcal{G}$. Then N is a flat R-module as $\mathcal{G}$ is flat over S. The assumption of the lemma means that the composition

$$M \otimes_R R' \to N \otimes_R R'$$

is zero. Let $z \in M$. By Lazard's theorem (Algebra, Theorem 10.81.4) and the fact that $\otimes$ commutes with colimits we can find free R-module F_z, an element $\tilde{z} \in F_z$, and a map $F_z \to N$ such that $u(z)$ is the image of $\tilde{z}$ and $\tilde{z}$ maps to zero in $F_z \otimes_R R'$. Choose a basis $\{e_{z,\alpha}\}$ of F_z and write $\tilde{z} = \sum f_{z,\alpha} e_{z,\alpha}$ with $f_{z,\alpha} \in R$. Let $I \subset R$ be the ideal generated by the elements $f_{z,\alpha}$ with z ranging over all elements of M. By construction I maps to zero in R' and the elements $\tilde{z}$ map to zero in F_z/IF_z whence in N/IN. Thus $Z = \mathrm{Spec}(R/I)$ is a solution to the problem in this case.

Assume $\mathcal{F}$ is of finite type with quasi-compact scheme theoretic support. Write $Z = \mathrm{Spec}(R/I)$. Write $I = \bigcup I_\lambda$ as a filtered union of finitely generated ideals. Set $Z_\lambda = \mathrm{Spec}(R/I_\lambda)$, so $Z = \mathrm{colim}\, Z_\lambda$. Since u_Z is zero, we see that u_{Z_λ} is zero for some λ by Lemma 77.7.4. This finishes the proof of the lemma. $\square$

083L Lemma 77.8.3. Let A be a ring. Let $u : M \to N$ be a map of A-modules. If N is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent

(1) $u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is zero, and
(2) $\varphi(I) = 0$.

Proof. As N is projective we can find a projective A-module C such that $F = N \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume N is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\mathrm{Im}(u)$ with respect to some (fixed) basis of N. $\square$

0CWB Lemma 77.8.4. In Situation 77.8.1. Let $T \subset S$ be a subset. Let $s \in S$ be in the closure of T. For $t \in T$, let u_t be the pullback of u to X_t and let u_s be the pullback of u to X_s. If X is locally of finite presentation over S, $\mathcal{G}$ is of finite presentation[3], and $u_t = 0$ for all $t \in T$, then $u_s = 0$.

[3]It would suffice if X is locally of finite type over S and $\mathcal{G}$ is finitely presented relative to S, but this notion hasn't yet been defined in the setting of algebraic spaces. The definition for schemes is given in More on Morphisms, Section 37.58.

Proof. To check whether u_s is zero, is étale local on the fibre X_s. Hence we may pick a point $x \in |X_s| \subset |X|$ and check in an étale neighbourhood. Choose

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

as in Proposition 77.5.1. Let $T' \subset S'$ be the inverse image of T. Observe that s' is in the closure of T' because $S' \to S$ is open. Hence we reduce to the algebra problem described in the next paragraph.

We have an R-module map $u : M \to N$ such that N is projective as an R-module and such that $u_t : M \otimes_R \kappa(t) \to N \otimes_R \kappa(t)$ is zero for each $t \in T$. Problem: show that $u_s = 0$. Let $I \subset R$ be the ideal defined in Lemma 77.8.3. Then I maps to zero in $\kappa(t)$ for all $t \in T$. Hence $T \subset V(I)$. Since s is in the closure of T we see that $s \in V(I)$. Hence $u_s = 0$. □

It would be interesting to find a "simple" direct proof of either Lemma 77.8.5 or Lemma 77.8.6 using arguments like those used in Lemmas 77.8.2 and 77.8.4. A "classical" proof of this lemma when $f : X \to B$ is a projective morphism and B a Noetherian scheme would be: (a) choose a relatively ample invertible sheaf $\mathcal{O}_X(1)$, (b) set $u_n : f_*\mathcal{F}(n) \to f_*\mathcal{G}(n)$, (c) observe that $f_*\mathcal{G}(n)$ is a finite locally free sheaf for all $n \gg 0$, and (d) F_{zero} is represented by the vanishing locus of u_n for some $n \gg 0$.

0CWC Lemma 77.8.5. In Situation 77.7.1. Assume

(1) f is of finite presentation, and
(2) $\mathcal{G}$ is of finite presentation, flat over B, and pure relative to B.

Then F_{zero} is an algebraic space and $F_{zero} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then $F_{zero} \to B$ is of finite presentation.

Proof. By Lemma 77.6.5 the module $\mathcal{G}$ is universally pure relative to B. In order to prove that F_{zero} is an algebraic space, it suffices to show that $F_{zero} \to B$ is representable, see Spaces, Lemma 65.11.3. Let $B' \to B$ be a morphism where B' is a scheme and let $u' : \mathcal{F}' \to \mathcal{G}'$ be the pullback of u to $X' = X_{B'}$. Then the associated functor F'_{zero} equals $F_{zero} \times_B B'$. This reduces us to the case that B is a scheme.

Assume B is a scheme. We will show that F_{zero} is representable by a closed subscheme of B. By Lemma 77.7.2 and Descent, Lemmas 35.37.2 and 35.39.1 the question is local for the étale topology on B. Let $b \in B$. We first replace B by an affine neighbourhood of b. Choose a diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ B & B' \arrow[l] \end{tikzcd}$$

and $b' \in B'$ mapping to $b \in B$ as in Lemma 77.5.2. As we are working étale locally, we may replace B by B' and assume that we have a diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & B & \end{tikzcd}$$

with B and X' affine such that $\Gamma(X', g^*\mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module and $g(|X'|) \supset |X_b|$. Let $U \subset X$ be the open subspace with $|U| = g(|X'|)$. By Divisors on Spaces, Lemma 71.4.10 the set

$$E = \{t \in B : \text{Ass}_{X_t}(\mathcal{G}_t) \subset |U_t|\} = \{t \in B : \text{Ass}_{X/B}(\mathcal{G}) \cap |X_t| \subset |U_t|\}$$

is constructible in B. By Lemma 77.6.3 part (2) we see that E contains $\text{Spec}(\mathcal{O}_{B,b})$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of b. Hence after replacing B by a smaller affine neighbourhood of b we may assume that $\text{Ass}_{X/B}(\mathcal{G}) \subset g(|X'|)$.

From Lemma 77.6.6 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to B$ is a morphism of affine schemes and $\Gamma(X, \mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module. This case follows immediately from Lemma 77.8.3.

We still have to show that $F_{zero} \to B$ is of finite presentation if $\mathcal{F}$ is of finite type. This follows from Lemma 77.7.4 combined with Limits of Spaces, Proposition 70.3.10. □

083M Lemma 77.8.6. In Situation 77.7.1. Assume

(1) f is locally of finite presentation,
(2) $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation flat over B,
(3) the support of $\mathcal{G}$ is proper over B.

Then the functor F_{zero} is an algebraic space and $F_{zero} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then $F_{zero} \to B$ is of finite presentation.

Proof. If f is of finite presentation, then this follows immediately from Lemmas 77.8.5 and 77.3.6. This is the only case of interest and we urge the reader to skip the rest of the proof, which deals with the possibility (allowed by the assumptions in this lemma) that f is not quasi-separated or quasi-compact.

Let $i : Z \to X$ be the closed subspace cut out by the zeroth fitting ideal of $\mathcal{G}$ (Divisors on Spaces, Section 71.5). Then $Z \to B$ is proper by assumption (see Derived Categories of Spaces, Section 75.7). On the other hand i is of finite presentation (Divisors on Spaces, Lemma 71.5.2 and Morphisms of Spaces, Lemma 67.28.12). There exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{H}$ of finite type with $i_*\mathcal{H} = \mathcal{G}$ (Divisors on Spaces, Lemma 71.5.3). In fact $\mathcal{H}$ is of finite presentation as an $\mathcal{O}_Z$-module by Algebra, Lemma 10.6.4 (details omitted). Then F_{zero} is the same as the functor F_{zero} for the map $i^*\mathcal{F} \to \mathcal{H}$ adjoint to u, see Lemma 77.7.6. The sheaf $\mathcal{H}$ is flat relative to B because the same is true for $\mathcal{G}$ (check on stalks; details omitted). Moreover, note that if $\mathcal{F}$ is of finite type, then $i^*\mathcal{F}$ is of finite type. Hence we have reduced the lemma to the case discussed in the first paragraph of the proof. □

77.9. Flattening a map

0CVN This section is the analogue of More on Flatness, Section 38.23. In particular the following result is a variant of More on Flatness, Theorem 38.23.3.

0CWD **Theorem 77.9.1.** *In Situation 77.7.1 assume*

1. *f is of finite presentation,*
2. *$\mathcal{F}$ is of finite presentation, flat over B, and pure relative to B, and*
3. *u is surjective.*

Then F_{iso} is representable by a closed immersion $Z \to B$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. Let $\mathcal{K} = \mathrm{Ker}(u)$ and denote $v : \mathcal{K} \to \mathcal{F}$ the inclusion. By Lemma 77.7.5 we see that $F_{u,iso} = F_{v,zero}$. By Lemma 77.8.5 applied to v we see that $F_{u,iso} = F_{v,zero}$ is representable by a closed subspace of B. Note that $\mathcal{K}$ is of finite type if $\mathcal{G}$ is of finite presentation, see Modules on Sites, Lemma 18.24.1. Hence we also obtain the final statement of the lemma. □

083N **Lemma 77.9.2.** *In Situation 77.7.1. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is locally of finite presentation and flat over B,*
3. *the support of $\mathcal{F}$ is proper over B, and*
4. *u is surjective.*

Then the functor F_{iso} is an algebraic space and $F_{iso} \to B$ is a closed immersion. If $\mathcal{G}$ is of finite presentation, then $F_{iso} \to B$ is of finite presentation.

Proof. Let $\mathcal{K} = \mathrm{Ker}(u)$ and denote $v : \mathcal{K} \to \mathcal{F}$ the inclusion. By Lemma 77.7.5 we see that $F_{u,iso} = F_{v,zero}$. By Lemma 77.8.6 applied to v we see that $F_{u,iso} = F_{v,zero}$ is representable by a closed subspace of B. Note that $\mathcal{K}$ is of finite type if $\mathcal{G}$ is of finite presentation, see Modules on Sites, Lemma 18.24.1. Hence we also obtain the final statement of the lemma. □

We will use the following (easy) result when discussing the Quot functor.

09TP **Lemma 77.9.3.** *In Situation 77.7.1. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{G}$ is of finite type,*
3. *the support of $\mathcal{G}$ is proper over B.*

Then F_{surj} is an algebraic space and $F_{surj} \to B$ is an open immersion.

Proof. Consider $\mathrm{Coker}(u)$. Observe that $\mathrm{Coker}(u_T) = \mathrm{Coker}(u)_T$ for any T/B. Note that formation of the support of a finite type quasi-coherent module commutes with pullback (Morphisms of Spaces, Lemma 67.15.1). Hence F_{surj} is representable by the open subspace of B corresponding to the open set

$$|B| \setminus |f|(\mathrm{Supp}(\mathrm{Coker}(u)))$$

see Properties of Spaces, Lemma 66.4.8. This is an open because $|f|$ is closed on $\mathrm{Supp}(\mathcal{G})$ and $\mathrm{Supp}(\mathrm{Coker}(u))$ is a closed subset of $\mathrm{Supp}(\mathcal{G})$. □

77.10. Flattening in the local case

0CWN This section is the analogue of More on Flatness, Section 38.24.

0CWP **Lemma 77.10.1.** Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of algebraic spaces which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset |X_s|$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

(1) $\mathcal{F}_T$ is flat over T at all points of $|X_t|$ which map to a point of $E \subset |X_s|$, and
(2) $\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset |X_s|$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. Choose a scheme U and an étale morphism $\varphi : U \to X$. Let $E' \subset |U_s|$ be the inverse image of E. If $E' \to E$ is surjective, then condition (1) is equivalent to: $(\varphi^*\mathcal{F})_T$ is flat over T at all points of $|U_t|$ which map to a point of $E' \subset |U_t|$. Choosing φ to be surjective, we reduced to the case of schemes which is More on Flatness, Lemma 38.24.3. If E is closed and quasi-compact, then we may choose U to be affine such that $E' \to E$ is surjective. Then E' is closed and quasi-compact and the final statement follows from the final statement of More on Flatness, Lemma 38.24.3. $\square$

77.11. Universal flattening

0CWQ This section is the analogue of More on Flatness, Section 38.27. Our main aim is to prove Lemma 77.11.8. However, we do not see a way to deduce this result from the corresponding result for schemes directly. Hence we have to redevelop some of the material here. But first a definition.

0CWR **Definition 77.11.1.** Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 77.7.8 is an algebraic space. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

This is a bit unsatisfactory, because here the definition of universal flattening does not agree with the one used in the case of schemes, as we don't know whether every monomorphism of algebraic spaces is representable (More on Morphisms of Spaces, Section 76.4). Hopefully no confusion will ever result from this.

0CWS **Lemma 77.11.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. The following are equivalent

(1) for some commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\varphi"'] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with surjective, étale vertical arrows where U and V are schemes, the sheaf $\varphi^*\mathcal{F}$ is flat over V in dimensions $\geq n$ (More on Flatness, Definition 38.20.10),

(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\varphi"'] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows where U and V are schemes, the sheaf $\varphi^*\mathcal{F}$ is flat over V in dimensions $\geq n$, and

(3) for $x \in |X|$ such that $\mathcal{F}$ is not flat at x over Y the transcendence degree of $x/f(x)$ is $< n$ (Morphisms of Spaces, Definition 67.33.1).

If this is true, then it remains true after any base change $Y' \to Y$.

Proof. Suppose that we have a diagram as in (1). Then the equivalence of the conditions in More on Flatness, Lemma 38.20.9 shows that (1) and (3) are equivalent. But condition (3) is inherited by $\varphi^*\mathcal{F}$ for any $U \to V$ as in (2). Whence we see that (3) implies (2) by the result for schemes again. The result for schemes also implies the statement on base change. $\square$

0CWT Definition 77.11.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. We say $\mathcal{F}$ is flat over Y in dimensions $\geq n$ if the equivalent conditions of Lemma 77.11.2 are satisfied.

0CWU Situation 77.11.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over Y we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_Y T \to X$. Note that $f_T : X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type (Morphisms of Spaces, Lemma 67.23.3 and Modules on Sites, Lemma 18.23.4). Let $n \geq 0$. By Definition 77.11.3 and Lemma 77.11.2 we obtain a functor

(77.11.4.1)

0CWV $$F_n : (Sch/Y)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

In Situation 77.11.4 we sometimes think of F_n as a functor $(Sch/S)^{opp} \to \text{Sets}$ endowed with a morphism $F_n \to Y$. Namely, if T is a scheme over S, then an element $h \in F_n(T)$ is a morphism $h : T \to Y$ such that the base change of $\mathcal{F}$ via h is flat over T in dim $\geq n$. In particular, when we say that F_n is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to \text{Sets}$ is an algebraic space.

0CWW Lemma 77.11.5. In Situation 77.11.4.

(1) The functor F_n satisfies the sheaf property for the fpqc topology.
(2) If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.

Proof. Proof of (1). Suppose that $\{T_i \to T\}$ is an fpqc covering of a scheme T over Y. We have to show that if $F_n(T_i)$ is nonempty for all i, then $F_n(T)$ is nonempty. Choose a diagram as in part (1) of Lemma 77.11.2. Denote F'_n the corresponding functor for $\varphi^*\mathcal{F}$ and the morphism $U \to V$. By More on Flatness, Lemma 38.20.12 we have the sheaf property for F'_n. Thus we get the sheaf property for F_n because for $T \to Y$ we have $F_n(T) = F'_n(V \times_Y T)$ by Lemma 77.11.2 and because $\{V \times_Y T_i \to V \times_Y T\}$ is an fpqc covering.

Proof of (2). Suppose that $T = \lim_{i \in I} T_i$ is a filtered limit of affine schemes T_i over Y and assume that $F_n(T)$ is nonempty. We have to show that $F_n(T_i)$ is nonempty for some i. Choose a diagram as in part (1) of Lemma 77.11.2. Fix $i \in I$ and choose an affine open $W_i \subset V \times_Y T_i$ mapping surjectively onto T_i. For $i' \geq i$ let $W_{i'}$ be the inverse image of W_i in $V \times_Y T_{i'}$ and let $W \subset V \times_Y T$ be the inverse image of W_i. Then $W = \lim_{i' \geq i} W_i$ is a filtered limit of affine schemes over V. By Lemma 77.11.2 again it suffices to show that $F'_n(W_{i'})$ is nonempty for some $i' \geq i$. But we know that $F'_n(W)$ is nonempty because of our assumption that $F_n(T) = F'_n(V \times_Y T)$ is nonempty. Thus we can apply More on Flatness, Lemma 38.20.12 to conclude. $\square$

0CX0 Lemma 77.11.6. In Situation 77.11.4. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (77.11.4.1) associated to $(f' : X' \to Y, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset \text{Ass}_{X/Y}(\mathcal{F})$, then $F_n = F'_n$.

Proof. Choose $U \to X$, $V \to Y$, $U \to V$ as in part (1) of Lemma 77.11.2. Choose a surjective étale morphism $U' \to U \times_X X'$ where U' is a scheme. Then we have the lemma for the two functors $F_{U,n}$ and $F_{U',n}$ determined by $U' \to U$ and $\mathcal{F}|_U$ over V, see More on Flatness, Lemma 38.27.2. On the other hand, Lemma 77.11.2 tells us that given $T \to Y$ we have $F_n(T) = F_{U,n}(V \times_Y T)$ and $F'_n(T) = F_{U',n}(V \times_Y T)$. This proves the lemma. $\square$

0CX1 Theorem 77.11.7. In Situation 77.11.4. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to Y. Then F_n is an algebraic space and $F_n \to Y$ is a monomorphism of finite presentation.

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 77.11.5. Since $F_n \to Y$ is a monomorphism of sheaves on $(Sch/S)_{fppf}$ we see that $\Delta : F_n \to F_n \times F_n$ is the pullback of the diagonal $\Delta_Y : Y \to Y \times_S Y$. Hence the representability of Δ_Y implies the same thing for F_n. Therefore it suffices to prove that there exists a scheme W over S and a surjective étale morphism $W \to F_n$.

To construct $W \to F_n$ choose an étale covering $\{Y_i \to Y\}$ with Y_i a scheme. Let $X_i = X \times_Y Y_i$ and let $\mathcal{F}_i$ be the pullback of $\mathcal{F}$ to X_i. Then $\mathcal{F}_i$ is pure relative to Y_i either by definition or by Lemma 77.3.3. The other assumptions of the theorem are preserved as well. Finally, the restriction of F_n to Y_i is the functor F_n corresponding to $X_i \to Y_i$ and $\mathcal{F}_i$. Hence it suffices to show: Given $\mathcal{F}$ and $f : X \to Y$ as in the statement of the theorem where Y is a scheme, the functor F_n is representable by a scheme Z_n and $Z_n \to Y$ is a monomorphism of finite presentation.

Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 29.20.15). Hence combining Descent, Lemma 35.39.1, More on Morphisms, Lemma 37.57.1 , and Descent, Lemmas 35.23.31 and 35.23.13 we see that the question is local for the étale topology on Y.

In particular the situation is local for the Zariski topology on Y and we may assume that Y is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to Y$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_Y X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to Y$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 77.3.3 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $Y_{n+1} = Y$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over Y.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over the affine scheme Y. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on Y it suffices to show that for every $y \in Y$ there exists an étale neighbourhood $(Y', y') \to (Y, y)$ such that the result holds after base change to Y'. Thus by Lemma 77.4.1 we may assume there exist étale morphisms $h_j : W_j \to X$, $j = 1, \ldots, m$ such that for each j there exists a complete dévissage of $\mathcal{F}_j/W_j/Y$ over y, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to W_j and such that $|X_y| \subset \bigcup h_j(W_j)$. Since h_j is étale, by Lemma 77.11.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over Y. Set $W = \bigcup h_j(W_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_y we see that

$$E = \{t \in |Y| : \text{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of y. By Divisors on Spaces, Lemma 71.4.10 E is a constructible subset of Y. We have seen that $\text{Spec}(\mathcal{O}_{Y,y}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of y. Hence after shrinking Y we may assume that $E = Y$. It follows from Lemma 77.11.6 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod W_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $W_j \to Y$ and the sheaf $\mathcal{F}_j$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to Y$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_Y \ldots \times_Y Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in More on Flatness, Lemma 38.27.4. □

Thus we finally obtain the desired result.

0CX2 Lemma 77.11.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to Y, then there exists a universal flattening $Y' \to Y$ of $\mathcal{F}$. Moreover $Y' \to Y$ is a monomorphism of finite presentation.
(2) If f is of finite presentation and X is pure relative to Y, then there exists a universal flattening $Y' \to Y$ of X. Moreover $Y' \to Y$ is a monomorphism of finite presentation.
(3) If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $Y' \to Y$ of $\mathcal{F}$. Moreover $Y' \to Y$ is a monomorphism of finite presentation.
(4) If f is proper and of finite presentation then there exists a universal flattening $Y' \to Y$ of X.

Proof. These statements follow immediately from Theorem 77.11.7 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 77.3.6. □

77.12. Grothendieck's Existence Theorem

0CX3 This section is the analogue of More on Flatness, Section 38.28 and continues the discussion in More on Morphisms of Spaces, Section 76.42. We will work in the following situation.

0CX4 Situation 77.12.1. Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have an algebraic space X separated and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subspace of X. We assume further given a system $(\mathcal{F}_n, \varphi_n)$ where $\mathcal{F}_n$ is a finitely presented $\mathcal{O}_{X_n}$-module, flat over A_n, with support proper over A_n, and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms of Spaces, Lemma 67.14.1).

Our goal is to see if we can find a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n.

0CX5 Lemma 77.12.2. In Situation 77.12.1 consider

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(\mathcal{F}_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

Then K is in $D^b_{QCoh}(\mathcal{O}_X)$ and in fact K has nonzero cohomology sheaves only in degrees ≥ 0.

Proof. Special case of Derived Categories of Spaces, Example 75.19.5. □

0CX6 Lemma 77.12.3. In Situation 77.12.1 let K be as in Lemma 77.12.2. For any perfect object E of $D(\mathcal{O}_X)$ we have

(1) $M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$ in $D(A_n)$,
(2) $N = R\mathrm{Hom}_X(E, K)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\mathrm{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes_A^{\mathbf{L}} A_n$ in $D(A_n)$.

In both statements $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Proof of (2). Write $E_n = E|_{X_n}$ and $N_n = R\mathrm{Hom}_{X_n}(E_n, \mathcal{F}_n)$. Recall that $R\mathrm{Hom}_{X_n}(-,-)$ is equal to $R\Gamma(X_n, R\mathcal{H}om(-,-))$, see Cohomology on Sites, Section 21.36. Hence by Derived Categories of Spaces, Lemma 75.25.8 we see that N_n is a perfect object of $D(A_n)$ whose formation commutes with base change. Thus the maps $N_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \to N_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.97.3 we find that $R\lim N_n$ is perfect and that its base change back to A_n recovers N_n. On the other hand, the exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, K) = R\lim R\mathrm{Hom}_X(E, \mathcal{F}_n) = R\lim R\mathrm{Hom}_X(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology on Sites, Lemma 21.48.4. □

0CX7 **Lemma 77.12.4.** In Situation 77.12.1 let K be as in Lemma 77.12.2. Then K is pseudo-coherent relative to A.

Proof. Combinging Lemma 77.12.3 and Derived Categories of Spaces, Lemma 75.25.7 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus the lemma follows from More on Morphisms of Spaces, Lemma 76.51.4. $\square$

0CX8 **Lemma 77.12.5.** In Situation 77.12.1 let K be as in Lemma 77.12.2. For any étale morphism $U \to X$ with U quasi-compact and quasi-separated we have

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in $D(A_n)$ where $U_n = U \times_X X_n$.

Proof. Fix n. By Derived Categories of Spaces, Lemma 75.27.3 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to $\mathcal{F}_n$ we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \text{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 77.12.3 and the fact that $- \otimes_A^{\mathbf{L}} A_n$ commutes with homotopy colimits we obtain the result. $\square$

0CX9 **Lemma 77.12.6.** In Situation 77.12.1 let K be as in Lemma 77.12.2. Denote $X_0 \subset |X|$ the closed subset consisting of points lying over the closed subset $\text{Spec}(A_1) = \text{Spec}(A_2) = \ldots$ of $\text{Spec}(A)$. There exists an open subspace $W \subset X$ containing X_0 such that

(1) $H^i(K)|_W$ is zero unless $i = 0$,
(2) $\mathcal{F} = H^0(K)|_W$ is of finite presentation, and
(3) $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.

Proof. Fix $n \geq 1$. By construction there is a canonical map $K \to \mathcal{F}_n$ in $D_{QCoh}(\mathcal{O}_X)$ and hence a canonical map $H^0(K) \to \mathcal{F}_n$ of quasi-coherent sheaves. This explains the meaning of part (3).

Let $x \in X_0$ be a point. We will find an open neighbourhood W of x such that (1), (2), and (3) are true. Since X_0 is quasi-compact this will prove the lemma. Let $U \to X$ be an étale morphism with U affine and $u \in U$ a point mapping to x. Since $|U| \to |X|$ is open it suffices to find an open neighbourhood of u in U where (1), (2), and (3) are true. Say $U = \text{Spec}(B)$. Choose a surjection $P \to B$ with P smooth over A. By Lemma 77.12.4 and the definition of relative pseudo-coherence there exists a bounded above complex $F^\bullet$ of finite free P-modules representing Ri_*K where $i : U \to \text{Spec}(P)$ is the closed immersion induced by the presentation. Let M_n be the B-module corresponding to $\mathcal{F}_n|_U$. By Lemma 77.12.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ M_n & \text{if} \quad i = 0 \end{cases}$$

Let i be the maximal index such that F^i is nonzero. If $i \leq 0$, then (1), (2), and (3) are true. If not, then $i > 0$ and we see that the rank of the map

$$F^{i-1} \to F^i$$

in the point u is maximal. Hence in an open neighbourhood of u inside $\text{Spec}(P)$ the rank is maximal. Thus after replacing P by a principal localization we may assume that the displayed map is surjective. Since F^i is finite free we may choose a splitting $F^{i-1} = F' \oplus F^i$. Then we may replace $F^\bullet$ by the complex

$$\ldots \to F^{i-2} \to F' \to 0 \to \ldots$$

and we win by induction on i. $\square$

0CXA Lemma 77.12.7. In Situation 77.12.1 let K be as in Lemma 77.12.2. Let $W \subset X$ be as in Lemma 77.12.6. Set $\mathcal{F} = H^0(K)|_W$. Then, after possibly shrinking the open W, the support of $\mathcal{F}$ is proper over A.

Proof. Fix $n \geq 1$. Let $I_n = \text{Ker}(A \to A_n)$. By More on Algebra, Lemma 15.11.3 the pair (A, I_n) is henselian. Let $Z \subset W$ be the scheme theoretic support of $\mathcal{F}$. This is a closed subspace as $\mathcal{F}$ is of finite presentation. By part (3) of Lemma 77.12.6 we see that $Z \times_{\text{Spec}(A)} \text{Spec}(A_n)$ is equal to the support of $\mathcal{F}_n$ and hence proper over $\text{Spec}(A/I)$. By More on Morphisms of Spaces, Lemma 76.36.10 we can write $Z = Z_1 \amalg Z_2$ with Z_1, Z_2 open and closed in Z, with Z_1 proper over A, and with $Z_1 \times_{\text{Spec}(A)} \text{Spec}(A/I_n)$ equal to the support of $\mathcal{F}_n$. In other words, $|Z_2|$ does not meet X_0. Hence after replacing W by $W \setminus Z_2$ we obtain the lemma. $\square$

0CXB Theorem 77.12.8 (Grothendieck Existence Theorem). In Situation 77.12.1 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$, flat over A, with support proper over A, such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.

Proof. Apply Lemmas 77.12.2, 77.12.3, 77.12.4, 77.12.5, 77.12.6, and 77.12.7 to get an open subspace $W \subset X$ containing all points lying over $\text{Spec}(A_n)$ and a finitely presented $\mathcal{O}_W$-module $\mathcal{F}$ whose support is proper over A with $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$ for all $n \geq 1$. (This makes sense as $X_n \subset W$.) By Lemma 77.3.6 we see that $\mathcal{F}$ is universally pure relative to $\text{Spec}(A)$. By Theorem 77.11.7 (for explanation, see Lemma 77.11.8) there exists a universal flattening $S' \to \text{Spec}(A)$ of $\mathcal{F}$ and moreover the morphism $S' \to \text{Spec}(A)$ is a monomorphism of finite presentation. In particular S' is a scheme (this follows from the proof of the theorem but it also follows a postoriori by Morphisms of Spaces, Proposition 67.50.2). Since the base change of $\mathcal{F}$ to $\text{Spec}(A_n)$ is $\mathcal{F}_n$ we find that $\text{Spec}(A_n) \to \text{Spec}(A)$ factors (uniquely) through S' for each n. By More on Flatness, Lemma 38.28.8 we see that $S' = \text{Spec}(A)$. This means that $\mathcal{F}$ is flat over A. Finally, since the scheme theoretic support Z of $\mathcal{F}$ is proper over $\text{Spec}(A)$, the morphism $Z \to X$ is closed. Hence the pushforward $(W \to X)_*\mathcal{F}$ is supported on W and has all the desired properties. $\square$

77.13. Grothendieck's Existence Theorem, bis

0DIJ In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 77.12 for quasi-coherent modules. This section is the analogue of More on Flatness, Section 38.29 for algebraic spaces. The classical case (for algebraic spaces) is discussed in More on Morphisms of Spaces, Section 76.42. We will work in the following situation.

0DIK Situation 77.13.1. Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have an algebraic space X proper, flat, and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subspace of X. We assume further given a system (K_n, φ_n) where K_n is a pseudo-coherent object of $D(\mathcal{O}_{X_n})$ and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in $D(\mathcal{O}_{X_n})$ which induces an isomorphism $K_n \otimes^{\mathbf{L}}_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \to K_{n-1}$ in $D(\mathcal{O}_{X_{n-1}})$.

More precisely, we should write $\varphi_n : K_n \to Ri_{n-1,*}K_{n-1}$ where $i_{n-1} : X_{n-1} \to X_n$ is the inclusion morphism and in this notation the condition is that the adjoint map $Li^*_{n-1}K_n \to K_{n-1}$ is an isomorphism. Our goal is to find a pseudo-coherent $K \in D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n (with the same abuse of notation).

0DIL Lemma 77.13.2. In Situation 77.13.1 consider

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} K_n)$$

Then K is in $D^-_{QCoh}(\mathcal{O}_X)$.

Proof. The functor DQ_X exists because X is quasi-compact and quasi-separated, see Derived Categories of Spaces, Lemma 75.19.1. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Spaces, Lemma 75.19.4 the functor DQ_X has bounded cohomological dimension. Hence it suffices to show that $R\lim K_n \in D^-(\mathcal{O}_X)$. To see this, let $U \to X$ be étale with U affine. Then there is a canonical exact sequence

$$0 \to R^1\lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

by Cohomology on Sites, Lemma 21.23.2. Since U is affine and K_n is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Spaces, Lemma 75.13.6) we see that $H^m(U, K_n) = H^m(K_n)(U)$ by Derived Categories of Schemes, Lemma 36.3.5. Thus we conclude that it suffices to show that K_n is bounded above independent of n.

Since K_n is pseudo-coherent we have $K_n \in D^-(\mathcal{O}_{X_n})$. Suppose that a_n is maximal such that $H^{a_n}(K_n)$ is nonzero. Of course $a_1 \leq a_2 \leq a_3 \leq \ldots$. Note that $H^{a_n}(K_n)$ is an $\mathcal{O}_{X_n}$-module of finite presentation (Cohomology on Sites, Lemma 21.45.7). We have $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Since $X_{n-1} \to X_n$ is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that if $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ is zero, then $H^{a_n}(K_n)$ is zero too (argue by checking on stalks for example; small detail omitted). Thus $a_{n-1} = a_n$ for all n and we conclude. □

0DIM Lemma 77.13.3. In Situation 77.13.1 let K be as in Lemma 77.13.2. For any perfect object E of $D(\mathcal{O}_X)$ the cohomology

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

is a pseudo-coherent object of $D(A)$ and there is a canonical isomorphism

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$$

in $D(A_n)$. Here $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Write $E_n = E|_{X_n}$ and $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$. By Derived Categories of Spaces, Lemma 75.25.5 we see that M_n is a pseudo-coherent object of $D(A_n)$ whose formation commutes with base change. Thus the maps $M_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to M_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.97.1 we find that $R\lim M_n$ is pseudo-coherent and that its base change back to A_n recovers M_n. On the other hand, the exact functor $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired. $\square$

0DIN Lemma 77.13.4. In Situation 77.13.1 let K be as in Lemma 77.13.2. Then K is pseudo-coherent on X.

Proof. Combinging Lemma 77.13.3 and Derived Categories of Spaces, Lemma 75.25.7 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus it follows from More on Morphisms of Spaces, Lemma 76.51.4 that K is pseudo-coherent relative to A. Since X is of flat and of finite presentation over A, this is the same as being pseudo-coherent on X, see More on Morphisms of Spaces, Lemma 76.45.4. $\square$

0DIP Lemma 77.13.5. In Situation 77.13.1 let K be as in Lemma 77.13.2. For any étale morphism $U \to X$ with U quasi-compact and quasi-separated we have

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, K_n)$$

in $D(A_n)$ where $U_n = U \times_X X_n$.

Proof. Fix n. By Derived Categories of Spaces, Lemma 75.27.3 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to K_n we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \text{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 77.13.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. $\square$

0DIQ Theorem 77.13.6 (Derived Grothendieck Existence Theorem). In Situation 77.13.1 there exists a pseudo-coherent K in $D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.

Proof. Apply Lemmas 77.13.2, 77.13.3, 77.13.4 to get a pseudo-coherent object K of $D(\mathcal{O}_X)$. Choosing affine U in Lemma 77.13.5 it follows immediately that K restricts to K_n over X_n. $\square$

0DIR Remark 77.13.7. The result in this section can be generalized. It is probably correct if we only assume $X \to \text{Spec}(A)$ to be separated, of finite presentation, and K_n pseudo-coherent relative to A_n supported on a closed subset of X_n proper over A_n. The outcome will be a K which is pseudo-coherent relative to A supported on a closed subset proper over A. If we ever need this, we will formulate a precise statement and prove it here.

77.14. Other chapters

CHAPTER 78

Groupoids in Algebraic Spaces

0437 78.1. Introduction

0438 This chapter is devoted to generalities concerning groupoids in algebraic spaces. We recommend reading the beautiful paper [KM97] by Keel and Mori.

A lot of what we say here is a repeat of what we said in the chapter on groupoid schemes, see Groupoids, Section 39.1. The discussion of quotient stacks is new here.

78.2. Conventions

0439 The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

We continue our convention to label projection maps starting with index 0, so we have $\text{pr}_0 : X \times_S Y \to X$ and $\text{pr}_1 : X \times_S Y \to Y$.

78.3. Notation

043A Let S be a scheme; this will be our base scheme and all algebraic spaces will be over S. Let B be an algebraic space over S; this will be our base algebraic space, and often other algebraic spaces, and schemes will be over B. If we say that X is an algebraic space over B, then we mean that X is an algebraic space over S which comes equipped with structure morphism $X \to B$. Moreover, we try to reserve the letter T to denote a "test" scheme over B. In other words T is a scheme which comes equipped with a structure morphism $T \to B$. In this situation we denote $X(T)$ for the set of T-valued points of X over B. In a formula:

$$X(T) = \text{Mor}_B(T, X).$$

Similarly, given a second algebraic space Y over B we set

$$X(Y) = \text{Mor}_B(Y, X).$$

Suppose we are given algebraic spaces X, Y over B as above and a morphism $f : X \to Y$ over B. For any scheme T over B we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which is functorial in the scheme T over B. As f is a map of sheaves on $(Sch/S)_{fppf}$ over the sheaf B it is clear that f determines and is determined by this rule. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are algebraic spaces over B, and if $m : X \times_B Y \to Z \times_B Z$ is a morphism of algebraic spaces over B, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

Finally, given two maps $f, g : X \to Y$ of algebraic spaces over B, if the induced maps $f, g : X(T) \to Y(T)$ are equal for every scheme T over B, then $f = g$, and hence also $f, g : X(Z) \to Y(Z)$ are equal for every third algebraic space Z over B. Hence, for example, to check the axioms for an group algebraic space G over B, it suffices to check commutativity of diagram on T-valued points where T is a scheme over B as we do in Definition 78.5.1 below.

78.4. Equivalence relations

043B Please refer to Groupoids, Section 39.3 for notation.

043C Definition 78.4.1. Let $B \to S$ as in Section 78.3. Let U be an algebraic space over B.

(1) A pre-relation on U over B is any morphism $j : R \to U \times_B U$ of algebraic spaces over B. In this case we set $t = \text{pr}_0 \circ j$ and $s = \text{pr}_1 \circ j$, so that $j = (t, s)$.
(2) A relation on U over B is a monomorphism $j : R \to U \times_B U$ of algebraic spaces over B.
(3) A pre-equivalence relation is a pre-relation $j : R \to U \times_B U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all schemes T over B.
(4) We say a morphism $R \to U \times_B U$ of algebraic spaces over B is an equivalence relation on U over B if and only if for every T over B the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

043D Lemma 78.4.2. Let $B \to S$ as in Section 78.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Finally, set

$$R' = (U' \times_B U') \times_{U \times_B U} R \xrightarrow{j'} U' \times_B U'$$

Then j' is a pre-relation on U' over B. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. $\square$

043E Definition 78.4.3. Let $B \to S$ as in Section 78.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The pre-relation $j' : R' \to U' \times_B U'$ of Lemma 78.4.2 is called the restriction, or pullback of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

043F Lemma 78.4.4. Let $B \to S$ as in Section 78.3. Let $j : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Consider the relation on $|U|$ defined by the rule

$$x \sim y \Leftrightarrow \exists\, r \in |R| : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in |R|$ with $t(r) = x$, $s(r) = y$ and pick $r' \in |R|$ with $t(r') = y$, $s(r') = z$. We may pick a field K such that r and r' can be represented by morphisms $r, r' : \mathrm{Spec}(K) \to R$ with $s \circ r = t \circ r'$. Denote $x = t \circ r$, $y = s \circ r = t \circ r'$, and $z = s \circ r'$, so $x, y, z : \mathrm{Spec}(K) \to U$. By construction $(x, y) \in j(R(K))$ and $(y, z) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x, z) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. $\square$

78.5. Group algebraic spaces

043G Please refer to Groupoids, Section 39.4 for notation.

043H Definition 78.5.1. Let $B \to S$ as in Section 78.3.

(1) A group algebraic space over B is a pair (G, m), where G is an algebraic space over B and $m : G \times_B G \to G$ is a morphism of algebraic spaces over B with the following property: For every scheme T over B the pair $(G(T), m)$ is a group.
(2) A morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B is a morphism $\psi : G \to G'$ of algebraic spaces over B such that for every T/B the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group algebraic space over the algebraic space B. By the discussion in Groupoids, Section 39.4 we obtain morphisms of algebraic spaces over B (identity) $e : B \to G$ and (inverse) $i : G \to G$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group.

Let (G, m), (G', m') be group algebraic spaces over B. Let $f : G \to G'$ be a morphism of algebraic spaces over B. It follows from the definition that f is a morphism of group algebraic spaces over B if and only if the following diagram is commutative:

$$\begin{tikzcd} G \times_B G \arrow[r, "f \times f"'] \arrow[d, "m"'] & G' \times_B G' \arrow[d, "m"] \\ G \arrow[r, "f"] & G' \end{tikzcd}$$

043I Lemma 78.5.2. Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B. Let $B' \to B$ be a morphism of algebraic spaces. The pullback $(G_{B'}, m_{B'})$ is a group algebraic space over B'.

Proof. Omitted. $\square$

78.6. Properties of group algebraic spaces

06P5 In this section we collect some simple properties of group algebraic spaces which hold over any base.

06P6 **Lemma 78.6.1.** Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Then $G \to B$ is separated (resp. quasi-separated, resp. locally separated) if and only if the identity morphism $e : B \to G$ is a closed immersion (resp. quasi-compact, resp. an immersion).

Proof. We recall that by Morphisms of Spaces, Lemma 67.4.7 we have that e is a closed immersion (resp. quasi-compact, resp. an immersion) if $G \to B$ is separated (resp. quasi-separated, resp. locally separated). For the converse, consider the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta_{G/B}"'] \arrow[d] & G \times_B G \arrow[d, "(g,g') \mapsto m(i(g),g')"] \\ B \arrow[r, "e"] & G \end{tikzcd}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/B}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact, resp. an immersion) so is $\Delta_{G/B}$, see Spaces, Lemma 65.12.3 (resp. Morphisms of Spaces, Lemma 67.8.4, resp. Spaces, Lemma 65.12.3). ☐

0DSI **Lemma 78.6.2.** Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Assume $G \to B$ is locally of finite type. Then $G \to B$ is unramified (resp. locally quasi-finite) if and only if $G \to B$ is unramified (resp. quasi-finite) at $e(b)$ for all $b \in |B|$.

Proof. By Morphisms of Spaces, Lemma 67.38.10 (resp. Morphisms of Spaces, Lemma 67.27.2) there is a maximal open subspace $U \subset G$ such that $U \to B$ is unramified (resp. locally quasi-finite) and formation of U commutes with base change. Thus we reduce to the case where $B = \mathrm{Spec}(k)$ is the spectrum of a field. Let $g \in G(K)$ be a point with values in an extension K/k. Then to check whether or not g is in U, we may base change to K. Hence it suffices to show

$$G \to \mathrm{Spec}(k) \text{ is unramified at } e \Leftrightarrow G \to \mathrm{Spec}(k) \text{ is unramified at } g$$

for a k-rational point g (resp. similarly for quasi-finite at g and e). Since translation by g is an automorphism of G over k this is clear. ☐

0DSJ **Lemma 78.6.3.** Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Assume $G \to B$ is locally of finite type.

(1) There exists a maximal open subspace $U \subset B$ such that $G_U \to U$ is unramified and formation of U commutes with base change.
(2) There exists a maximal open subspace $U \subset B$ such that $G_U \to U$ is locally quasi-finite and formation of U commutes with base change.

Proof. By Morphisms of Spaces, Lemma 67.38.10 (resp. Morphisms of Spaces, Lemma 67.27.2) there is a maximal open subspace $W \subset G$ such that $W \to B$ is unramified (resp. locally quasi-finite). Moreover formation of W commutes with base change. By Lemma 78.6.2 we see that $U = e^{-1}(W)$ in either case. ☐

78.7. Examples of group algebraic spaces

06P7 If $G \to S$ is a group scheme over the base scheme S, then the base change G_B to any algebraic space B over S is an group algebraic space over B by Lemma 78.5.2. We will frequently use this in the examples below.

043J Example 78.7.1 (Multiplicative group algebraic space). Let $B \to S$ as in Section 78.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{m,B} = B \times_S \mathbf{G}_{m,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the multiplicative group scheme over S, see Groupoids, Example 39.5.1.

043K Example 78.7.2 (Roots of unity as a group algebraic space). Let $B \to S$ as in Section 78.3. Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T over B the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the group algebraic space

$$\mu_{n,B} = B \times_S \mu_{n,S}$$

over B. Here $\mu_{n,S}$ is the group scheme of nth roots of unity over S, see Groupoids, Example 39.5.2.

043L Example 78.7.3 (Additive group algebraic space). Let $B \to S$ as in Section 78.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{a,B} = B \times_S \mathbf{G}_{a,S}$$

over B. Here $\mathbf{G}_{a,S}$ is the additive group scheme over S, see Groupoids, Example 39.5.3.

043M Example 78.7.4 (General linear group algebraic space). Let $B \to S$ as in Section 78.3. Let $n \geq 1$. Consider the functor which associates to any scheme T over B the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathrm{GL}_{n,B} = B \times_S \mathrm{GL}_{n,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the general linear group scheme over S, see Groupoids, Example 39.5.4.

043N Example 78.7.5. Let $B \to S$ as in Section 78.3. Let $n \geq 1$. The determinant defines a morphism of group algebraic spaces

$$\det : \mathrm{GL}_{n,B} \longrightarrow \mathbf{G}_{m,B}$$

over B. It is the base change of the determinant morphism over S from Groupoids, Example 39.5.5.

043O Example 78.7.6 (Constant group algebraic space). Let $B \to S$ as in Section 78.3. Let G be an abstract group. Consider the functor which associates to any scheme T over B the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the group algebraic space

$$G_B = B \times_S G_S$$

over B. Here G_S is the constant group scheme introduced in Groupoids, Example 39.5.6.

78.8. Actions of group algebraic spaces

043P Please refer to Groupoids, Section 39.10 for notation.

043Q Definition 78.8.1. Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B.

(1) An action of G on the algebraic space X/B is a morphism $a : G \times_B X \to X$ over B such that for every scheme T over B the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.
(2) Suppose that X, Y are algebraic spaces over B each endowed with an action of G. An equivariant or more precisely a G-equivariant morphism $\psi : X \to Y$ is a morphism of algebraic spaces over B such that for every T over B the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

043R (78.8.1.1)
$$\begin{tikzcd} G \times_B G \times_B X \arrow[r, "1_G \times a"'] \arrow[d, "m \times 1_X"'] & G \times_B X \arrow[d, "a"] \\ G \times_B X \arrow[r, "a"] & X \end{tikzcd} \qquad \begin{tikzcd} G \times_B X \arrow[r, "a"] & X \\ X \arrow[u, "e \times 1_X"] \arrow[ur, "1_X"'] & \end{tikzcd}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{tikzcd} G \times_B X \arrow[r, "\mathrm{id} \times f"'] \arrow[d, "a"'] & G \times_B Y \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

06P8 Definition 78.8.2. Let $B \to S$, $G \to B$, and $X \to B$ as in Definition 78.8.1. Let $a : G \times_B X \to X$ be an action of G on X/B. We say the action is free if for every scheme T over B the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

06P9 Lemma 78.8.3. Situation as in Definition 78.8.2, The action a is free if and only if

$$G \times_B X \to X \times_B X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism of algebraic spaces.

Proof. Immediate from the definitions. □

78.9. Principal homogeneous spaces

04TV This section is the analogue of Groupoids, Section 39.11. We suggest reading that section first.

04TW Definition 78.9.1. Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on X.

(1) We say X is a pseudo G-torsor or that X is formally principally homogeneous under G if the induced morphism $G \times_B X \to X \times_B X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism.
(2) A pseudo G-torsor X is called trivial if there exists an G-equivariant isomorphism $G \to X$ over B where G acts on G by left multiplication.

It is clear that if $B' \to B$ is a morphism of algebraic spaces then the pullback $X_{B'}$ of a pseudo G-torsor over B is a pseudo $G_{B'}$-torsor over B'.

04TX Lemma 78.9.2. In the situation of Definition 78.9.1.

(1) The algebraic space X is a pseudo G-torsor if and only if for every scheme T over B the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.
(2) A pseudo G-torsor X is trivial if and only if the morphism $X \to B$ has a section.

Proof. Omitted. □

04TY Definition 78.9.3. Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be a pseudo G-torsor over B.

(1) We say X is a principal homogeneous space, or more precisely a principal homogeneous G-space over B if there exists a fpqc covering[1] $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section (i.e., is a trivial pseudo G_{B_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a G-torsor in the τ topology, or a τ G-torsor, or simply a τ torsor if there exists a τ covering $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section.
(3) If X is a principal homogeneous G-space over B, then we say that it is quasi-isotrivial if it is a torsor for the étale topology.
(4) If X is a principal homogeneous G-space over B, then we say that it is locally trivial if it is a torsor for the Zariski topology.

We sometimes say "let X be a G-principal homogeneous space over B" to indicate that X is an algebraic space over B equipped with an action of G which turns it into a principal homogeneous space over B. Next we show that this agrees with the notation introduced earlier when both apply.

04TZ Lemma 78.9.4. Let S be a scheme. Let (G, m) be a group algebraic space over S. Let X be an algebraic space over S, and let $a : G \times_S X \to X$ be an action of G on X. Then X is a G-torsor in the $fppf$-topology in the sense of Definition 78.9.3 if and only if X is a G-torsor on $(Sch/S)_{fppf}$ in the sense of Cohomology on Sites, Definition 21.4.1.

Proof. Omitted. □

0DSK Lemma 78.9.5. Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Let X be a pseudo G-torsor over B. Assume G and X locally of finite type over B.

(1) If $G \to B$ is unramified, then $X \to B$ is unramified.
(2) If $G \to B$ is locally quasi-finite, then $X \to B$ is locally quasi-finite.

Proof. Proof of (1). By Morphisms of Spaces, Lemma 67.38.10 we reduce to the case where B is the spectrum of a field. If X is empty, then the result holds. If X is nonempty, then after increasing the field, we may assume X has a point. Then $G \cong X$ and the result holds.

[1] The default type of torsor in Groupoids, Definition 39.11.3 is a pseudo torsor which is trivial on an fpqc covering. Since G, as an algebraic space, can be seen a sheaf of groups there already is a notion of a G-torsor which corresponds to fppf-torsor, see Lemma 78.9.4. Hence we use "principal homogeneous space" for a pseudo torsor which is fpqc locally trivial, and we try to avoid using the word torsor in this situation.

The proof of (2) works in exactly the same way using Morphisms of Spaces, Lemma 67.27.2. ☐

78.10. Equivariant quasi-coherent sheaves

043S Please compare with Groupoids, Section 39.12.

043T **Definition 78.10.1.** Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on the algebraic space X over B. An G-equivariant quasi-coherent $\mathcal{O}_X$-module, or simply a equivariant quasi-coherent $\mathcal{O}_X$-module, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G\times_B X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_B X \to X$ is the projection such that

(1) the diagram

$$\begin{tikzcd}
(1_G \times a)^*\mathrm{pr}_2^*\mathcal{F} \arrow[r, "\mathrm{pr}_{12}^*\alpha"'] & \mathrm{pr}_2^*\mathcal{F} \\
(1_G \times a)^*a^*\mathcal{F} \arrow[u, "(1_G\times a)^*\alpha"] \arrow[r, equal] & (m \times 1_X)^*a^*\mathcal{F} \arrow[u, "(m\times 1_X)^*\alpha"']
\end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{G\times_B G\times_B X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (78.8.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

043U **Lemma 78.10.2.** Let $B \to S$ as in Section 78.3. Let G be a group algebraic space over B. Let $f : X \to Y$ be a G-equivariant morphism between algebraic spaces over B endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of quasi-coherent G-equivariant sheaves on Y to the category of quasi-coherent G-equivariant sheaves on X.

Proof. Omitted. ☐

78.11. Groupoids in algebraic spaces

043V Please refer to Groupoids, Section 39.13 for notation.

043W **Definition 78.11.1.** Let $B \to S$ as in Section 78.3.

(1) A groupoid in algebraic spaces over B is a quintuple (U, R, s, t, c) where U and R are algebraic spaces over B, and $s, t : R \to U$ and $c : R\times_{s,U,t}R \to R$ are morphisms of algebraic spaces over B with the following property: For any scheme T over B the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

(2) A morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B is given by morphisms of algebraic spaces $f : U \to U'$ and $f : R \to R'$ over B with the following property: For any scheme T over B the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Note that there are unique morphisms of algebraic spaces $e : U \to R$ and $i : R \to R$ over B such that for every scheme T over B the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) of Groupoids, Section 39.13. Conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over B. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid in algebraic spaces over B we denote

$$j = (t, s) : R \longrightarrow U \times_B U$$

which is compatible with our conventions in Section 78.4 above. We sometimes say "let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B" to stress the existence of identity and inverse.

043X Lemma 78.11.2. Let $B \to S$ as in Section 78.3. Given a groupoid in algebraic spaces (U, R, s, t, c) over B the morphism $j : R \to U \times_B U$ is a pre-equivalence relation.

Proof. Omitted. This is a nice exercise in the definitions. $\square$

043Y Lemma 78.11.3. Let $B \to S$ as in Section 78.3. Given an equivalence relation $j : R \to U \times_B U$ over B there is a unique way to extend it to a groupoid in algebraic spaces (U, R, s, t, c) over B.

Proof. Omitted. This is a nice exercise in the definitions. $\square$

043Z Lemma 78.11.4. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram

$$\begin{tikzcd} & & U & & \\ R \arrow[urr, "t"] \arrow[d, "s"] & & R \times_{s,U,t} R \arrow[ll, "\mathrm{pr}_0"'] \arrow[rr, "c"'] \arrow[d, "\mathrm{pr}_1"] & & R \arrow[ull, "t"'] \arrow[d, "s"] \\ U & & R \arrow[ll, "t"] \arrow[rr, "s"] & & U \end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. $\square$

0450 Lemma 78.11.5. Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. The diagram

04P3 (78.11.5.1)
$$\begin{tikzcd}
R \times_{t,U,t} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_0 \times c \circ (i,1)"'] & R \arrow[r, "t"] \arrow[d, "\mathrm{id}_R"] & U \arrow[d, "\mathrm{id}_U"] \\
R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between $\text{Arrows} \times_{t,\text{Ob},t} \text{Arrows}$ and $\text{Arrows} \times_{s,\text{Ob},t} \text{Arrows}$. Hence the second assertion of the lemma. The last assertion follows from Lemma 78.11.4. $\square$

0DTA Lemma 78.11.6. Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $B' \to B$ be a morphism of algebraic spaces. Then the base changes $U' = B' \times_B U$, $R' = B' \times_B R$ endowed with the base changes s', t', c' of the morphisms s, t, c form a groupoid in algebraic spaces (U', R', s', t', c') over B' and the projections determine a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B.

Proof. Omitted. Hint: $R' \times_{s',U',t'} R' = B' \times_B (R \times_{s,U,t} R)$. $\square$

78.12. Quasi-coherent sheaves on groupoids

0440 Please compare with Groupoids, Section 39.14.

0441 Definition 78.12.1. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. A quasi-coherent module on (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

$$\begin{tikzcd}
& \mathrm{pr}_1^* t^*\mathcal{F} \arrow[r, "\mathrm{pr}_1^*\alpha"'] \arrow[dl, equal] & \mathrm{pr}_1^* s^*\mathcal{F} \arrow[dr, equal] & \\
\mathrm{pr}_0^* s^*\mathcal{F} & & & c^* s^*\mathcal{F} \\
& \mathrm{pr}_0^* t^*\mathcal{F} \arrow[ul, "\mathrm{pr}_0^*\alpha"] \arrow[r, equal] & c^* t^*\mathcal{F} \arrow[ur, "c^*\alpha"'] &
\end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{R \times_{s,U,t} R}$-modules, and

(2) the pullback
$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$
is the identity map.

Compare with the commutative diagrams of Lemma 78.11.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

077W Lemma 78.12.2. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.

Proof. Pull back the commutative diagram of Definition 78.12.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. $\square$

0442 Lemma 78.12.3. Let $B \to S$ as in Section 78.3. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid in algebraic spaces over B. Then pullback f^* given by
$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$
defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. $\square$

0GPM Lemma 78.12.4. Let $B \to S$ as in Section 78.3. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B. Assume that

(1) $f : U \to U'$ is quasi-compact and quasi-separated,
(2) the square
$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "t"] & R' \arrow[d, "t'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$
is cartesian, and
(3) s' and t' are flat.

Then pushforward f_* given by
$$(\mathcal{F}, \alpha) \mapsto (f_*\mathcal{F}, f_*\alpha)$$
defines a functor from the category of quasi-coherent sheaves on (U, R, s, t, c) to the category of quasi-coherent sheaves on (U', R', s', t', c') which is right adjoint to pullback as defined in Lemma 78.12.3.

Proof. Since $U \to U'$ is quasi-compact and quasi-separated we see that f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces, Lemma 67.11.2). Moreover, since the squares

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "t"'] & R' \arrow[d, "t'"] \\ U \arrow[r, "f"] & U' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "f"] \arrow[d, "s"'] & R' \arrow[d, "s'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

are cartesian we find that $(t')^*f_*\mathcal{F} = f_*t^*\mathcal{F}$ and $(s')^*f_*\mathcal{F} = f_*s^*\mathcal{F}$, see Cohomology of Spaces, Lemma 69.11.2. Thus it makes sense to think of $f_*\alpha$ as a map $(t')^*f_*\mathcal{F} \to (s')^*f_*\mathcal{F}$. A similar argument shows that $f_*\alpha$ satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and pushforward on modules on ringed spaces are adjoint. Some details omitted. □

077X Lemma 78.12.5. Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Properties of Spaces, Lemma 66.29.7. Since colimits commute with pullback we see that $s^*\mathcal{F} = \operatorname{colim} s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = \operatorname{colim} t^*\mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

06VZ Lemma 78.12.6. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^* \operatorname{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^* \operatorname{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^* \operatorname{Ker}(\varphi) \to s^* \operatorname{Ker}(\varphi)$ and $\lambda : t^* \operatorname{Coker}(\varphi) \to s^* \operatorname{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\operatorname{Ker}(\varphi), \kappa)$ and $(\operatorname{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\operatorname{Coim}(\varphi) = \operatorname{Im}(\varphi)$ follows because it holds over U. □

78.13. Colimits of quasi-coherent modules

0GPN This section is the analogue of Groupoids, Section 39.15.

0GPP Lemma 78.13.1. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Assume s, t are flat, quasi-compact, and quasi-separated. For any quasi-coherent module $\mathcal{G}$ on U, there exists a canonical isomorphism $\alpha : t^*s_*t^*\mathcal{G} \to s^*s_*t^*\mathcal{G}$ which turns $(s_*t^*\mathcal{G}, \alpha)$ into a quasi-coherent module on (U, R, s, t, c). This construction defines a functor

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$.

Proof. The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Morphisms of Spaces, Lemma 67.11.2. Hence $s_*t^*\mathcal{G}$ is quasi-coherent. With notation as in Lemma 78.11.4 we have

$$t^*s_*t^*\mathcal{G} = \text{pr}_{1,*}\text{pr}_0^*t^*\mathcal{G} = \text{pr}_{1,*}c^*t^*\mathcal{G} = s^*s_*t^*\mathcal{G}$$

The middle equality because $t \circ c = t \circ \text{pr}_0$ as morphisms $R \times_{s,U,t} R \to U$, and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Spaces, Lemma 69.11.2.

To verify the cocycle condition of Definition 78.12.1 for α and the adjointness property we describe the construction $\mathcal{G} \mapsto (s_*t^*\mathcal{G}, \alpha)$ in another way. Consider the groupoid scheme $(R, R \times_{t,U,t} R, \text{pr}_0, \text{pr}_1, \text{pr}_{02})$ associated to the equivalence relation $R \times_{t,U,t} R$ on R, see Lemma 78.11.3. There is a morphism

$$f : (R, R \times_{t,U,t} R, \text{pr}_1, \text{pr}_0, \text{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by $s : R \to U$ and $R \times_{t,U,t} R \to R$ given by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$; we omit the verification of the commutativity of the required diagrams. Since $t, s : R \to U$ are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{tikzcd} R \times_{t,U,t} R \arrow[r, "(r_0,r_1)\mapsto r_0^{-1}\circ r_1"'] \arrow[d, "\text{pr}_0"'] & R \arrow[d, "t"] \\ R \arrow[r, "s"] & U \end{tikzcd}$$

by Lemma 78.11.5 it follows that Lemma 78.12.4 applies to f. Thus pushforward and pullback of quasi-coherent modules along f are adjoint functors. To finish the proof we will identify these functors with the functors described above. To do this, note that

$$t^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(R, R \times_{t,U,t} R, \text{pr}_1, \text{pr}_0, \text{pr}_{02})$$

is an equivalence by the theory of descent of quasi-coherent sheaves as $\{t : R \to U\}$ is an fpqc covering, see Descent on Spaces, Proposition 74.4.1.

Pushforward along f precomposed with the equivalence t^* sends $\mathcal{G}$ to $(s_*t^*\mathcal{G}, \alpha)$; we omit the verification that the isomorphism α obtained in this fashion is the same as the one constructed above.

Pullback along f postcomposed with the inverse of the equivalence t^* sends $(\mathcal{F}, \beta)$ to the descent relative to $\{t : R \to U\}$ of the module $s^*\mathcal{F}$ endowed with the descent datum γ on $R \times_{t,U,t} R$ which is the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. Consider the isomorphism $\beta : t^*\mathcal{F} \to s^*\mathcal{F}$. The canonical descent datum (Descent on Spaces, Definition 74.3.3) on $t^*\mathcal{F}$ relative to $\{t : R \to U\}$ translates via β into the map

$$\text{pr}_0^*s^*\mathcal{F} \xrightarrow{\text{pr}_0^*\beta^{-1}} \text{pr}_0^*t^*\mathcal{F} \xrightarrow{can} \text{pr}_1^*t^*\mathcal{F} \xrightarrow{\text{pr}_1^*\beta} \text{pr}_1^*s^*\mathcal{F}$$

Since β satisfies the cocycle condition, this is equal to the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. To see this take the actual cocycle relation in Definition 78.12.1 and pull it back by the morphism $(\text{pr}_0, c \circ (i, 1)) : R \times_{t,U,t} R \to R \times_{s,U,t} R$ which also plays a role in the commutative diagram of Lemma 78.11.5. It follows that $(s^*\mathcal{F}, \gamma)$ is isomorphic to $(t^*\mathcal{F}, can)$. All in all, we conclude that pullback by f postcomposed with the inverse of the equivalence t^* is isomorphic to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$. $\square$

0GPQ Remark 78.13.2. In the situation of Lemma 78.13.1 denote

$$F : QCoh(U, R, s, t, c) \to QCoh(\mathcal{O}_U), \quad (\mathcal{F}, \beta) \mapsto \mathcal{F}$$

the forgetful functor and denote

$$G : QCoh(\mathcal{O}_U) \to QCoh(U, R, s, t, c), \quad \mathcal{G} \mapsto (s_*t^*\mathcal{G}, \alpha)$$

the right adjoint constructed in the lemma. Then the unit $\eta : \text{id} \to G \circ F$ of the adjunction evaluated on $(\mathcal{F}, \beta)$ is given by the map

$$\mathcal{F} \to s_*s^*\mathcal{F} \xrightarrow{\beta^{-1}} s_*t^*\mathcal{F}$$

We omit the verification.

0GPR Lemma 78.13.3. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module, and let $\varphi : \mathcal{G} \to f^*\mathcal{F}$ be a module map. Assume

(1) φ is injective,
(2) f is quasi-compact, quasi-separated, flat, and surjective,
(3) X, Y are locally Noetherian, and
(4) $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module.

Then $\mathcal{F} \cap f_*\mathcal{G}$ defined as the pullback

$$\begin{tikzcd} \mathcal{F} \arrow[r] & f_*f^*\mathcal{F} \\ \mathcal{F} \cap f_*\mathcal{G} \arrow[u] \arrow[r] & f_*\mathcal{G} \arrow[u] \end{tikzcd}$$

is a coherent $\mathcal{O}_X$-module.

Proof. We will freely use the characterization of coherent modules of Cohomology of Spaces, Lemma 69.12.2 as well as the fact that coherent modules form a Serre subcategory of $QCoh(\mathcal{O}_X)$, see Cohomology of Spaces, Lemma 69.12.4. If f has a section σ, then we see that $\mathcal{F} \cap f_*\mathcal{G}$ is contained in the image of $\sigma^*\mathcal{G} \to \sigma^*f^*\mathcal{F} = \mathcal{F}$, hence coherent. In general, to show that $\mathcal{F} \cap f_*\mathcal{G}$ is coherent, it suffices the show that $f^*(\mathcal{F} \cap f_*\mathcal{G})$ is coherent (see Descent on Spaces, Lemma 74.6.1). Since f is flat this is equal to $f^*\mathcal{F} \cap f^*f_*\mathcal{G}$. Since f is flat, quasi-compact, and quasi-separated we see $f^*f_*\mathcal{G} = p_*q^*\mathcal{G}$ where $p, q : Y \times_X Y \to Y$ are the projections, see Cohomology of Spaces, Lemma 69.11.2. Since p has a section we win. $\square$

Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Assume that U is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is coherent if $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module.

0GPS Lemma 78.13.4. Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Assume that

(1) U, R are Noetherian,
(2) s, t are flat, quasi-compact, and quasi-separated.

Then every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is a filtered colimit of coherent modules.

Proof. We will use the characterization of Cohomology of Spaces, Lemma 69.12.2 of coherent modules on locally Noetherian algebraic spaces without further mention. We can write $\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ as the filtered colimit of coherent submodules $\mathcal{H}_i \subset \mathcal{F}$, see Cohomology of Spaces, Lemma 69.15.1. Given a quasi-coherent sheaf $\mathcal{H}$ on U we denote $(s_*t^*\mathcal{H}, \alpha)$ the quasi-coherent sheaf on (U, R, s, t, c) of Lemma 78.13.1. Consider the adjunction map $(\mathcal{F}, \beta) \to (s_*t^*\mathcal{F}, \alpha)$ in $QCoh(U, R, s, t, c)$, see Remark 78.13.2. Set

$$(\mathcal{F}_i, \beta_i) = (\mathcal{F}, \beta) \times_{(s_*t^*\mathcal{F}, \alpha)} (s_*t^*\mathcal{H}_i, \alpha)$$

in $QCoh(U, R, s, t, c)$. Since restriction to U is an exact functor on $QCoh(U, R, s, t, c)$ by the proof of Lemma 78.12.6 we obtain a pullback diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & s_*t^*\mathcal{F} \\ \mathcal{F}_i \arrow[r] \arrow[u] & s_*t^*\mathcal{H}_i \arrow[u] \end{tikzcd}$$

in other words $\mathcal{F}_i = \mathcal{F} \cap s_*t^*\mathcal{H}_i$. By the description of the adjunction map in Remark 78.13.2 this diagram is isomorphic to the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & s_*s^*\mathcal{F} \\ \mathcal{F}_i \arrow[r] \arrow[u] & s_*t^*\mathcal{H}_i \arrow[u] \end{tikzcd}$$

where the right vertical arrow is the result of appplying s_* to the map

$$t^*\mathcal{H}_i \to t^*\mathcal{F} \xrightarrow{\beta} s^*\mathcal{F}$$

This arrow is injective as t is a flat morphism. It follows that $\mathcal{F}_i$ is coherent by Lemma 78.13.3. Finally, because s is quasi-compact and quasi-separated we see that s_* commutes with colimits (see Cohomology of Schemes, Lemma 30.6.1). Hence $s_*t^*\mathcal{F} = \operatorname{colim} s_*t^*\mathcal{H}_i$ and hence $(\mathcal{F}, \beta) = \operatorname{colim}(\mathcal{F}_i, \beta_i)$ as desired. □

78.14. Crystals in quasi-coherent sheaves

077Y Let (I, Φ, j) be a pair consisting of a set I and a pre-relation $j : \Phi \to I \times I$. Assume given for every $i \in I$ a scheme X_i and for every $\phi \in \Phi$ a morphism of schemes $f_\phi : X_{i'} \to X_i$ where $j(\phi) = (i, i')$. Set $X = (\{X_i\}_{i \in I}, \{f_\phi\}_{\phi \in \Phi})$. Define a crystal in quasi-coherent modules on X as a rule which associates to every $i \in \mathrm{Ob}(\mathcal{I})$ a quasi-coherent sheaf $\mathcal{F}_i$ on X_i and for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ an isomorphism

$$\alpha_\phi : f_\phi^*\mathcal{F}_i \longrightarrow \mathcal{F}_{i'}$$

of quasi-coherent sheaves on $X_{i'}$. These crystals in quasi-coherent modules form an additive category $\mathrm{CQC}(X)$[2]. This category has colimits (proof is the same as the proof of Lemma 78.12.5). If all the morphisms f_ϕ are flat, then $\mathrm{CQC}(X)$ is abelian (proof is the same as the proof of Lemma 78.12.6). Let κ be a cardinal.

[2]We could single out a set of triples $\phi, \phi', \phi'' \in \Phi$ with $j(\phi) = (i, i')$, $j(\phi') = (i', i'')$, and $j(\phi'') = (i, i'')$ such that $f_{\phi''} = f_\phi \circ f_{\phi'}$ and require that $\alpha_{\phi'} \circ f_{\phi'}^*\alpha_\phi = \alpha_{\phi''}$ for these triples. This would define an additive subcategory. For example the data (I, Φ) could be the set of objects and arrows of an index category and X could be a diagram of schemes over this index category. The result of Lemma 78.14.1 immediately gives the corresponding result in the subcategory.

We say that a crystal in quasi-coherent modules $\mathcal{F}$ on X is κ-generated if each $\mathcal{F}_i$ is κ-generated (see Properties, Definition 28.23.1).

077Z Lemma 78.14.1. In the situation above, if all the morphisms f_ϕ are flat, then there exists a cardinal κ such that every object $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ of $\mathrm{CQC}(X)$ is the directed colimit of its κ-generated submodules.

Proof. In the lemma and in this proof a submodule of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ means the data of a quasi-coherent submodule $\mathcal{G}_i \subset \mathcal{F}_i$ for all i such that $\alpha_\phi(f_\phi^*\mathcal{G}_i) = \mathcal{G}_{i'}$ as subsheaves of $\mathcal{F}_{i'}$ for all $\phi \in \Phi$. This makes sense because since f_ϕ is flat the pullback f_ϕ^* is exact, i.e., preserves subsheaves. The proof will be a variant to the proof of Properties, Lemma 28.23.3. We urge the reader to read that proof first.

We claim that it suffices to prove the lemma in case all the schemes X_i are affine. To see this let

$$J = \coprod\nolimits_{i\in I}\{U \subset X_i \text{ affine open}\}$$

and let

$$\begin{aligned}\Psi = & \coprod\nolimits_{\phi\in\Phi}\{(U,V) \mid U \subset X_i, V \subset X_{i'} \text{ affine open with } f_\phi(U) \subset V\} \\ & \amalg \coprod\nolimits_{i\in I}\{(U,U') \mid U,U' \subset X_i \text{ affine open with } U \subset U'\}\end{aligned}$$

endowed with the obvious map $\Psi \to J \times J$. Then our $(\mathcal{F}, \alpha)$ induces a crystal in quasi-coherent sheaves $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ on $Y = (J, \Psi)$ by setting $\mathcal{H}_{(i,U)} = \mathcal{F}_i|_U$ for $(i,U) \in J$ and setting β_ψ for $\psi \in \Psi$ equal to the restriction of α_ϕ to U if $\psi = (\phi, U, V)$ and equal to $\mathrm{id} : (\mathcal{F}_i|_{U'})|_U \to \mathcal{F}_i|_U$ when $\psi = (i, U, U')$. Moreover, submodules of $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ correspond 1-to-1 with submodules of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$. We omit the proof (hint: use Sheaves, Section 6.30). Moreover, it is clear that if κ works for Y, then the same κ works for X (by the definition of κ-generated modules). Hence it suffices to proof the lemma for crystals in quasi-coherent sheaves on Y.

Assume that all the schemes X_i are affine. Let κ be an infinite cardinal larger than the cardinality of I or Φ. Let $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ be an object of $\mathrm{CQC}(X)$. For each i write $X_i = \mathrm{Spec}(A_i)$ and $M_i = \Gamma(X_i, \mathcal{F}_i)$. For every $\phi \in \Phi$ with $j(\phi) = (i, i')$ the map α_ϕ translates into an $A_{i'}$-module isomorphism

$$\alpha_\phi : M_i \otimes_{A_i} A_{i'} \longrightarrow M_{i'}$$

Using the axiom of choice choose a rule

$$(\phi, m) \longmapsto S(\phi, m')$$

where the source is the collection of pairs (ϕ, m') such that $\phi \in \Phi$ with $j(\phi) = (i, i')$ and $m' \in M_{i'}$ and where the output is a finite subset $S(\phi, m') \subset M_i$ so that

$$m' = \alpha_\phi\left(\sum\nolimits_{m\in S(\phi,m')} m \otimes a'_m\right)$$

for some $a'_m \in A_{i'}$.

Having made these choices we claim that any section of any $\mathcal{F}_i$ over any X_i is in a κ-generated submodule. To see this suppose that we are given a collection

$\mathcal{S} = \{S_i\}_{i \in I}$ of subsets $S_i \subset M_i$ each with cardinality at most κ. Then we define a new collection $\mathcal{S}' = \{S'_i\}_{i \in I}$ with

$$S'_i = S_i \cup \bigcup\nolimits_{(\phi, m'),\ j(\phi) = (i, i'),\ m' \in S_{i'}} S(\phi, m')$$

Note that each S'_i still has cardinality at most κ. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $S_i^{(\infty)} = \bigcup_{n \geq 0} S_i^{(n)}$ and $\mathcal{S}^{(\infty)} = \{S_i^{(\infty)}\}_{i \in I}$. By construction, for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ and every $m' \in S_{i'}^{(\infty)}$ we can write m' as a finite linear combination of images $\alpha_\phi(m \otimes 1)$ with $m \in S_i^{(\infty)}$. Thus we see that setting N_i equal to the A_i-submodule of M_i generated by $S_i^{(\infty)}$ the corresponding quasi-coherent submodules $\widetilde{N_i} \subset \mathcal{F}_i$ form a κ-generated submodule. This finishes the proof. $\square$

0780 Lemma 78.14.2. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then there exists a set T and a family of objects $(\mathcal{F}_t, \alpha_t)_{t \in T}$ of $QCoh(U, R, s, t, c)$ such that every object $(\mathcal{F}, \alpha)$ is the directed colimit of its submodules isomorphic to one of the objects $(\mathcal{F}_t, \alpha_t)$.

Proof. This lemma is a generalization of Groupoids, Lemma 39.15.7 which deals with the case of a groupoid in schemes. We can't quite use the same argument, so we use the material on "crystals of quasi-coherent sheaves" we developed above.

Choose a scheme W and a surjective étale morphism $W \to U$. Choose a scheme V and a surjective étale morphism $V \to W \times_{U,s} R$. Choose a scheme V' and a surjective étale morphism $V' \to R \times_{t,U} W$. Consider the collection of schemes

$$I = \{W, W \times_U W, V, V', V \times_R V'\}$$

and the set of morphisms of schemes

$$\Phi = \{\mathrm{pr}_i : W \times_U W \to W, V \to W, V' \to W, V \times_R V' \to V, V \times_R V' \to V'\}$$

Set $X = (I, \Phi)$. Recall that we have defined a category $\mathrm{CQC}(X)$ of crystals of quasi-coherent sheaves on X. There is a functor

$$QCoh(U, R, s, t, c) \longrightarrow \mathrm{CQC}(X)$$

which assigns to $(\mathcal{F}, \alpha)$ the sheaf $\mathcal{F}|_W$ on W, the sheaf $\mathcal{F}|_{W \times_U W}$ on $W \times_U W$, the pullback of $\mathcal{F}$ via $V \to W \times_{U,s} R \to W \to U$ on V, the pullback of $\mathcal{F}$ via $V' \to R \times_{t,U} W \to W \to U$ on V', and finally the pullback of $\mathcal{F}$ via $V \times_R V' \to V \to W \times_{U,s} R \to W \to U$ on $V \times_R V'$. As comparison maps $\{\alpha_\phi\}_{\phi \in \Phi}$ we use the obvious ones (coming from associativity of pullbacks) except for the map $\phi = \mathrm{pr}_{V'} : V \times_R V' \to V'$ we use the pullback of $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ to $V \times_R V'$. This makes sense because of the following commutative diagram

$$\begin{tikzcd}
 & V \times_R V' \arrow[dl] \arrow[dr] & \\
V \arrow[dr] \arrow[dd] & & V' \arrow[dl] \arrow[dd] \\
 & R \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] & \\
W \arrow[dr] & & W \arrow[dl] \\
 & U &
\end{tikzcd}$$

The functor displayed above isn't an equivalence of categories. However, since $W \to U$ is surjective étale it is faithful[3]. Since all the morphisms in the diagram above are flat we see that it is an exact functor of abelian categories. Moreover, we claim that given $(\mathcal{F}, \alpha)$ with image $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ there is a 1-to-1 correspondence between quasi-coherent submodules of $(\mathcal{F}, \alpha)$ and $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$. Namely, given a submodule of $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ compatibility of the submodule over W with the projection maps $W \times_U W \to W$ will guarantee the submodule comes from a quasi-coherent submodule of $\mathcal{F}$ (by Properties of Spaces, Proposition 66.32.1) and compatibility with $\alpha_{\mathrm{pr}_{V'}}$ will insure this subsheaf is compatible with α (details omitted).

Choose a cardinal κ as in Lemma 78.14.1 for the system $X = (I, \Phi)$. It is clear from Properties, Lemma 28.23.2 that there is a set of isomorphism classes of κ-generated crystals in quasi-coherent sheaves on X. Hence the result is clear. $\square$

78.15. Groupoids and group spaces

0443 Please compare with Groupoids, Section 39.16.

0444 Lemma 78.15.1. Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B with identity e_G and inverse i_G. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Then we get a groupoid in algebraic spaces (U, R, s, t, c, e, i) over B in the following manner:

(1) We set $U = X$, and $R = G \times_B X$.
(2) We set $s : R \to U$ equal to $(g, x) \mapsto x$.
(3) We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.
(4) We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.
(5) We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.
(6) We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. $\square$

[3]In fact the functor is fully faithful, but we won't need this.

0445 Lemma 78.15.2. Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Let (U, R, s, t, c) be the groupoid in algebraic spaces constructed in Lemma 78.15.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_B X \to X$, see Definitions 78.10.1 and 78.12.1. Using the translation in Lemma 78.15.1 the commutativity requirements of the two definitions match up exactly. $\square$

78.16. The stabilizer group algebraic space

0446 Please compare with Groupoids, Section 39.17. Given a groupoid in algebraic spaces we get a group algebraic space as follows.

0447 Lemma 78.16.1. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The algebraic space G defined by the cartesian square

$$\begin{tikzcd} G \arrow[r] \arrow[d] & R \arrow[d, "j=(t,s)"] \\ U \arrow[r, "\Delta"] & U \times_B U \end{tikzcd}$$

is a group algebraic space over U with composition law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. $\square$

Since Δ is a monomorphism we see that $G = j^{-1}(\Delta_{U/B})$ is a subsheaf of R. Thinking of it in this way, the structure morphism $G = j^{-1}(\Delta_{U/B}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

0448 Definition 78.16.2. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The group algebraic space $j^{-1}(\Delta_{U/B}) \to U$ is called the stabilizer of the groupoid in algebraic spaces (U, R, s, t, c).

In the literature the stabilizer group algebraic space is often denoted S (because the word stabilizer starts with an "s" presumably); we cannot do this since we have already used S for the base scheme.

0449 Lemma 78.16.3. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B, and let G/U be its stabilizer. Denote R_t/U the algebraic space R seen as an algebraic space over U via the morphism $t : R \to U$. There is a canonical left action

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/B we define $a(g, r) = c(g, r)$. $\square$

78.17. Restricting groupoids

044A Please refer to Groupoids, Section 39.18 for notation.

044B **Lemma 78.17.1.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces. Consider the following diagram*

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid in algebraic spaces over B and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, for any scheme T over B the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see Groupoids, Section 39.18) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. $\square$

044C **Definition 78.17.2.** Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The morphism of groupoids in algebraic spaces $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 78.17.1 is called the *restriction of (U, R, s, t, c) to U'*. We sometime use the notation $R' = R|_{U'}$ in this case.

044D **Lemma 78.17.3.** *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 78.17.2 and 78.4.3 agree via the constructions of Lemmas 78.11.2 and 78.11.3.*

Proof. What we are saying here is that R' of Lemma 78.17.1 is also equal to

$$R' = (U' \times_B U') \times_{U \times_B U} R \longrightarrow U' \times_B U'$$

In fact this might have been a clearer way to state that lemma. $\square$

78.18. Invariant subspaces

044E In this section we discuss briefly the notion of an invariant subspace.

044F **Definition 78.18.1.** Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over the base B.

(1) We say an open subspace $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
(2) A locally closed subspace $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$ as locally closed subspaces of R.

(3) A monomorphism of algebraic spaces $T \to U$ is R-invariant if $T \times_{U,t} R = R \times_{s,U} T$ as algebraic spaces over R.

For an open subspace $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-invariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant locally closed subspace, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

044G Lemma 78.18.2. Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B.

(1) If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R-invariant.
(2) If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subspaces.

Proof. Assume s and t open and $W \subset U$ open. Since s is open we see that $W' = s(t^{-1}(W))$ is an open subspace of U. Now it is quite easy to using the functorial point of view that this is an R-invariant open subset of U, but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)}, U, t} R \xrightarrow{\text{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s, U, s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\text{pr}_0} R.$$

The algebraic spaces A, B on the left of the arrows above are open subspaces of $R \times_{s,U,t} R$ and $R \times_{s,U,s} R$ respectively. By Lemma 78.11.4 the diagram

$$\begin{tikzcd} R \times_{s,U,t} R \arrow[rr, "(\text{pr}_1, c)"'] \arrow[dr, "\text{pr}_1"'] & & R \times_{s,U,s} R \arrow[dl, "\text{pr}_0"] \\ & R & \end{tikzcd}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\text{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R-invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over images of affines étale over U we see (2). $\square$

78.19. Quotient sheaves

044H Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. For each scheme S' over S we can take the equivalence relation $\sim_{S'}$ generated by the image of $j(S') : R(S') \to U(S') \times U(S')$. Hence we get a presheaf

044I (78.19.0.1)
$$\begin{matrix} (Sch/S)_{fppf}^{opp} & \longrightarrow & \text{Sets}, \\ S' & \longmapsto & U(S')/\sim_{S'} \end{matrix}$$

Note that since j is a morphism of algebraic spaces over B and into $U \times_B U$ there is a canonical transformation of presheaves from the presheaf (78.19.0.1) to B.

044J Definition 78.19.1. Let $B \to S$ and the pre-relation $j : R \to U \times_B U$ be as above. In this setting the quotient sheaf U/R associated to j is the sheafification of the presheaf (78.19.0.1) on $(Sch/S)_{fppf}$. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G over B on U as in Lemma 78.15.1 then we denote the quotient sheaf U/G.

This means exactly that the diagram

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_{fppf}$. Again there is a canonical map of sheaves $U/R \to B$ as j is a morphism of algebraic spaces over B into $U \times_B U$.

044K Remark 78.19.2. A variant of the construction above would have been to sheafify the functor

$$\begin{matrix} (\text{Spaces}/B)_{fppf}^{opp} & \longrightarrow & \text{Sets}, \\ X & \longmapsto & U(X)/\sim_X \end{matrix}$$

where now $\sim_X \subset U(X) \times U(X)$ is the equivalence relation generated by the image of $j : R(X) \to U(X) \times U(X)$. Here of course $U(X) = \text{Mor}_B(X, U)$ and $R(X) = \text{Mor}_B(X, R)$. In fact, the result would have been the same, via the identifications of (insert future reference in Topologies of Spaces here).

044L Definition 78.19.3. In the situation of Definition 78.19.1. We say that the pre-relation j has a quotient representable by an algebraic space if the sheaf U/R is an algebraic space. We say that the pre-relation j has a representable quotient if the sheaf U/R is representable by a scheme. We will say a groupoid in algebraic spaces (U, R, s, t, c) over B has a representable quotient (resp. quotient representable by an algebraic space if the quotient U/R with $j = (t, s)$ is representable (resp. an algebraic space).

If the quotient U/R is representable by M (either a scheme or an algebraic space over S), then it comes equipped with a canonical structure morphism $M \to B$ as we've seen above.

The following lemma characterizes M representing the quotient. It applies for example if $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

044M Lemma 78.19.4. In the situation of Definition 78.19.1. Assume there is an algebraic space M over S, and a morphism $U \to M$ such that

(1) the morphism $U \to M$ equalizes s, t,
(2) the map $U \to M$ is a surjection of sheaves, and
(3) the induced map $(t, s) : R \to U \times_M U$ is a surjection of sheaves.

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $U \to M$ factors through U/R. Condition (2) says that $U/R \to M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to M$ is injective as a map of sheaves. Hence the lemma follows. □

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

046O Lemma 78.19.5. Let S be a scheme. Let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. For a scheme S' over S and $a, b \in U(S')$ the following are equivalent:

(1) a and b map to the same element of $(U/R)(S')$, and
(2) there exists an fppf covering $\{f_i : S_i \to S'\}$ of S' and morphisms $r_i : S_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.

In other words, in this case the map of sheaves

$$R \longrightarrow U \times_{U/R} U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (78.19.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 78.4.1. □

046P Lemma 78.19.6. Let S be a scheme. Let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B and $g : U' \to U$ a morphism of algebraic spaces over B. Let $j' : R' \to U' \times_B U'$ be the restriction of j to U'. The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If $U' \to U$ is surjective as a map of sheaves, for example if $\{g : U' \to U\}$ is an fppf covering (see Topologies on Spaces, Definition 73.7.1), then $U'/R' \to U/R$ is an isomorphism of sheaves.

Proof. Suppose $\xi, \xi' \in (U'/R')(S')$ are sections which map to the same section of U/R. Then we can find an fppf covering $\mathcal{S} = \{S_i \to S'\}$ of S' such that $\xi|_{S_i}, \xi'|_{S_i}$ are given by $a_i, a_i' \in U'(S_i)$. By Lemma 78.19.5 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : S_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(S_i)$ and this shows that a_i and a_i' define the same section of U'/R' over S_i. By the sheaf condition this implies $\xi = \xi'$.

If $U' \to U$ is a surjective map of sheaves, then $U'/R' \to U/R$ is surjective also. Finally, if $\{g : U' \to U\}$ is a fppf covering, then the map of sheaves $U' \to U$ is surjective, see Topologies on Spaces, Lemma 73.7.5. □

044N Lemma 78.19.7. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$\begin{tikzcd} U' \times_{g,U,t} R \arrow[r, "\mathrm{pr}_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

is a surjection of fppf sheaves then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is an *fppf*-covering, or if $U' \to U$ is a surjection of sheaves, or if $\{g : U' \to U\}$ is a covering in the fppf topology.

Proof. Injectivity follows on combining Lemmas 78.11.2 and 78.19.6. To see surjectivity (see Sites, Section 7.11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ if the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f_i' = \text{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f_i' \in U'(T_i)$ maps to $g \circ f_i' \in U(T_i)$ and that $g \circ f_i' \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (78.19.0.1). Namely, the element of $R(T_i)$ giving the relation is $\text{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 73.7.5. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). $\square$

78.20. Quotient stacks

044O In this section and the next few sections we describe a kind of generalization of Section 78.19 above and Groupoids, Section 39.20. It is different in the following way: We are going to take quotient stacks instead of quotient sheaves.

Let us assume we have a scheme S, and algebraic space B over S and a groupoid in algebraic spaces (U, R, s, t, c) over B. Given these data we consider the functor

044P (78.20.0.1)
$$\begin{array}{ccc} (Sch/S)_{fppf}^{opp} & \longrightarrow & \text{Groupoids} \\ S' & \longmapsto & (U(S'), R(S'), s, t, c) \end{array}$$

By Categories, Example 4.37.1 this "presheaf in groupoids" corresponds to a category fibred in groupoids over $(Sch/S)_{fppf}$. In this chapter we will denote this

$$[U/_pR] \to (Sch/S)_{fppf}$$

where the subscript $_p$ is there to distinguish from the quotient stack.

044Q **Definition 78.20.1.** Quotient stacks. Let $B \to S$ be as above.

(1) Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The quotient stack
$$p : [U/R] \longrightarrow (Sch/S)_{fppf}$$
of (U, R, s, t, c) is the stackification (see Stacks, Lemma 8.9.1) of the category fibred in groupoids $[U/_pR]$ over $(Sch/S)_{fppf}$ associated to (78.20.0.1).

(2) Let (G, m) be a group algebraic space over B. Let $a : G \times_B X \to X$ be an action of G on an algebraic space over B. The quotient stack
$$p : [X/G] \longrightarrow (Sch/S)_{fppf}$$
is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ over B of Lemma 78.15.1.

Thus $[U/R]$ and $[X/G]$ are stacks in groupoids over $(Sch/S)_{fppf}$. These stacks will be very important later on and hence it makes sense to give a detailed description. Recall that given an algebraic space X over S we use the notation $\mathcal{S}_X \to (Sch/S)_{fppf}$ to denote the stack in sets associated to the sheaf X, see Categories, Lemma 4.38.6 and Stacks, Lemma 8.6.2.

044R **Lemma 78.20.2.** Assume $B \to S$ and (U, R, s, t, c) as in Definition 78.20.1 (1). There are canonical 1-morphisms $\pi : \mathcal{S}_U \to [U/R]$, and $[U/R] \to \mathcal{S}_B$ of stacks in groupoids over $(Sch/S)_{fppf}$. The composition $\mathcal{S}_U \to \mathcal{S}_B$ is the 1-morphism associated to the structure morphism $U \to B$.

Proof. During this proof let us denote $[U/_pR]$ the category fibred in groupoids associated to the presheaf in groupoids (78.20.0.1). By construction of the stackification there is a 1-morphism $[U/_pR] \to [U/R]$. The 1-morphism $\mathcal{S}_U \to [U/R]$ is simply the composition $\mathcal{S}_U \to [U/_pR] \to [U/R]$, where the first arrow associates to the scheme S'/S and morphism $x : S' \to U$ over S the object $x \in U(S')$ of the fibre category of $[U/_pR]$ over S'.

To construct the 1-morphism $[U/R] \to \mathcal{S}_B$ it is enough to construct the 1-morphism $[U/_pR] \to \mathcal{S}_B$, see Stacks, Lemma 8.9.2. On objects over S'/S we just use the map

$$U(S') \longrightarrow B(S')$$

coming from the structure morphism $U \to B$. And clearly, if $a \in R(S')$ is an "arrow" with source $s(a) \in U(S')$ and target $t(a) \in U(S')$, then since s and t are morphisms over B these both map to the same element $\overline{a}$ of $B(S')$. Hence we can map an arrow $a \in R(S')$ to the identity morphism of $\overline{a}$. (This is good because the fibre category $(\mathcal{S}_B)_{S'}$ only contains identities.) We omit the verification that this rule is compatible with pullback on these split fibred categories, and hence defines a 1-morphism $[U/_pR] \to \mathcal{S}_B$ as desired.

We omit the verification of the last statement. $\square$

044S **Lemma 78.20.3.** Assumptions and notation as in Lemma 78.20.2. There exists a canonical 2-morphism $\alpha : \pi \circ s \to \pi \circ t$ making the diagram

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"'] & \mathcal{S}_U \arrow[d, "\pi"] \\ \mathcal{S}_U \arrow[r, "\pi"] & [U/R] \end{tikzcd}$$

2-commutative.

Proof. Let S' be a scheme over S. Let $r : S' \to R$ be a morphism over S. Then $r \in R(S')$ is an isomorphism between the objects $s \circ r, t \circ r \in U(S')$. Moreover, this construction is compatible with pullbacks. This gives a canonical 2-morphism $\alpha_p : \pi_p \circ s \to \pi_p \circ t$ where $\pi_p : \mathcal{S}_U \to [U/_pR]$ is as in the proof of Lemma 78.20.2. Thus even the diagram

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"] & \mathcal{S}_U \arrow[d, "\pi_p"] \\ \mathcal{S}_U \arrow[r, "\pi_p"] & [U/_pR] \end{tikzcd}$$

is 2-commutative. Thus a fortiori the diagram of the lemma is 2-commutative. $\square$

04M7 **Remark 78.20.4.** In future chapters we will use the ambiguous notation where instead of writing $\mathcal{S}_X$ for the stack in sets associated to X we simply write X. Using this notation the diagram of Lemma 78.20.3 becomes the familiar diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d, "\pi"] \\ U \arrow[r, "\pi"] & [U/R] \end{tikzcd}$$

In the following sections we will show that this diagram has many good properties. In particular we will show that it is a 2-fibre product (Section 78.22) and that it is close to being a 2-coequalizer of s and t (Section 78.23).

78.21. Functoriality of quotient stacks

04Y3 A morphism of groupoids in algebraic spaces gives an associated morphism of quotient stacks.

046Q **Lemma 78.21.1.** Let S be a scheme. Let B be an algebraic space over S. Let $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be a morphism of groupoids in algebraic spaces over B. Then f induces a canonical 1-morphism of quotient stacks

$$[f] : [U/R] \longrightarrow [U'/R'].$$

Proof. Denote $[U/_pR]$ and $[U'/_pR']$ the categories fibred in groupoids over the base site $(Sch/S)_{fppf}$ associated to the functors (78.20.0.1). It is clear that f defines a 1-morphism $[U/_pR] \to [U'/_pR']$ which we can compose with the stackyfication functor for $[U'/R']$ to get $[U/_pR] \to [U'/R']$. Then, by the universal property of the stackyfication functor $[U/_pR] \to [U/R]$, see Stacks, Lemma 8.9.2 we get $[U/R] \to [U'/R']$. $\square$

Let $B \to S$ and $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be as in Lemma 78.21.1. In this situation, we define a third groupoid in algebraic spaces over B as follows, using the language of T-valued points where T is a (varying) scheme over B:

1. $U'' = U \times_{f,U',t'} R'$ so that a T-valued point is a pair (u, r') with $f(u) = t'(r')$,
2. $R'' = R \times_{f\circ s,U',t'} R'$ so that a T-valued point is a pair (r, r') with $f(s(r)) = t'(r')$,
3. $s'' : R'' \to U''$ is given by $s''(r, r') = (s(r), r')$,
4. $t'' : R'' \to U''$ is given by $t''(r, r') = (t(r), c'(f(r), r'))$,
5. $c'' : R'' \times_{s'',U'',t''} R'' \to R''$ is given by $c''((r_1, r'_1), (r_2, r'_2)) = (c(r_1, r_2), r'_2)$.

The formula for c'' makes sense as $s''(r_1, r'_1) = t''(r_2, r'_2)$. It is clear that c'' is associative. The identity e'' is given by $e''(u, r) = (e(u), r)$. The inverse of (r, r') is given by $(i(r), c'(f(r), r'))$. Thus we do indeed get a groupoid in algebraic spaces over B.

Clearly the maps $U'' \to U$ and $R'' \to R$ define a morphism $g : (U'', R'', s'', t'', c'') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, the maps $U'' \to U'$, $(u, r') \mapsto s'(r')$ and $R'' \to U'$, $(r, r') \mapsto s'(r')$ show that in fact $(U'', R'', s'', t'', c'')$ is a groupoid in algebraic spaces over U'.

04Y4 **Lemma 78.21.2.** Notation and assumption as in Lemma 78.21.1. Let $(U'', R'', s'', t'', c'')$ be the groupoid in algebraic spaces over B constructed above. There is a 2-commutative square

$$\begin{tikzcd} {[U''/R'']} \arrow[r, "{[g]}"'] \arrow[d] & {[U/R]} \arrow[d, "{[f]}"] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/R']} \end{tikzcd}$$

which identifies $[U''/R'']$ with the 2-fibre product.

Proof. The maps $[f]$ and $[g]$ come from an application of Lemma 78.21.1 and the other two maps come from Lemma 78.20.2 (and the fact that $(U'', R'', s'', t'', c'')$ lives over U'). To show the 2-fibre product property, it suffices to prove the lemma for the diagram

$$\begin{tikzcd} {[U''/_p R'']} \arrow[r, "{[g]}"'] \arrow[d] & {[U/_p R]} \arrow[d, "{[f]}"] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/_p R']} \end{tikzcd}$$

of categories fibred in groupoids, see Stacks, Lemma 8.9.3. In other words, it suffices to show that an object of the 2-fibre product $\mathcal{S}_U \times_{[U'/_p R']} [U/_p R]$ over T corresponds to a T-valued point of U'' and similarly for morphisms. And of course this is exactly how we constructed U'' and R'' in the first place.

In detail, an object of $\mathcal{S}_U \times_{[U'/_p R']} [U/_p R]$ over T is a triple (u', u, r') where u' is a T-valued point of U', u is a T-valued point of U, and r' is a morphism from u' to $f(u)$ in $[U'/R']_T$, i.e., r' is a T-valued point of R with $s'(r') = u'$ and $t'(r') = f(u)$. Clearly we can forget about u' without losing information and we see that these objects are in one-to-one correspondence with T-valued points of R''.

Similarly for morphisms: Let (u'_1, u_1, r'_1) and (u'_2, u_2, r'_2) be two objects of the fibre product over T. Then a morphism from (u'_2, u_2, r'_2) to (u'_1, u_1, r'_1) is given by $(1, r)$ where $1 : u'_1 \to u'_2$ means simply $u'_1 = u'_2$ (this is so because $\mathcal{S}_U$ is fibred in sets), and r is a T-valued point of R with $s(r) = u_2$, $t(r) = u_1$ and moreover $c'(f(r), r'_2) = r'_1$. Hence the arrow

$$(1, r) : (u'_2, u_2, r'_2) \to (u'_1, u_1, r'_1)$$

is completely determined by knowing the pair (r, r'_2). Thus the functor of arrows is represented by R'', and moreover the morphisms s'', t'', and c'' clearly correspond to source, target and composition in the 2-fibre product $\mathcal{S}_U \times_{[U'/_p R']} [U/_p R]$. $\square$

78.22. The 2-cartesian square of a quotient stack

04M8 In this section we compute the *Isom*-sheaves for a quotient stack and we deduce that the defining diagram of a quotient stack is a 2-fibre product.

044V Lemma 78.22.1. Assume $B \to S$, (U, R, s, t, c) and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 78.20.2. Let S' be a scheme over S. Let $x, y \in \mathrm{Ob}([U/R]_{S'})$ be objects of the quotient stack over S'. If $x = \pi(x')$ and $y = \pi(y')$ for some morphisms $x', y' : S' \to U$, then

$$\mathit{Isom}(x, y) = S' \times_{(y', x'), U \times_S U} R$$

as sheaves over S'.

Proof. Let $[U/_p R]$ be the category fibred in groupoids associated to the presheaf in groupoids (78.20.0.1) as in the proof of Lemma 78.20.2. By construction the sheaf $\mathit{Isom}(x, y)$ is the sheaf associated to the presheaf $\mathit{Isom}(x', y')$. On the other hand, by definition of morphisms in $[U/_p R]$ we have

$$\mathit{Isom}(x', y') = S' \times_{(y', x'), U \times_S U} R$$

and the right hand side is an algebraic space, therefore a sheaf. $\square$

04M9 **Lemma 78.22.2.** Assume $B \to S$, (U, R, s, t, c), and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 78.20.2. The 2-commutative square

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"'] & \mathcal{S}_U \arrow[d, "\pi"] \\ \mathcal{S}_U \arrow[r, "\pi"] & {[U/R]} \end{tikzcd}$$

of Lemma 78.20.3 is a 2-fibre product of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. According to Stacks, Lemma 8.5.6 the lemma makes sense. It also tells us that we have to show that the functor

$$\mathcal{S}_R \longrightarrow \mathcal{S}_U \times_{[U/R]} \mathcal{S}_U$$

which maps $r : T \to R$ to $(T, t(r), s(r), \alpha(r))$ is an equivalence, where the right hand side is the 2-fibre product as described in Categories, Lemma 4.32.3. This is, after spelling out the definitions, exactly the content of Lemma 78.22.1. (Alternative proof: Work out the meaning of Lemma 78.21.2 in this situation will give you the result also.) $\square$

044W **Lemma 78.22.3.** Assume $B \to S$ and (U, R, s, t, c) are as in Definition 78.20.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ has the following property: There exists a fppf covering $\{T_i \to T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is representable by an algebraic space.

Proof. Follows immediately from Lemma 78.22.1 and the fact that both x and y locally in the fppf topology come from objects of $\mathcal{S}_U$ by construction of the quotient stack. $\square$

78.23. The 2-coequalizer property of a quotient stack

04MA On a groupoid we have the composition, which leads to a cocycle condition for the canonical 2-morphism of the lemma above. To give the precise formulation we will use the notation introduced in Categories, Sections 4.28 and 4.29.

044T **Lemma 78.23.1.** Assumptions and notation as in Lemmas 78.20.2 and 78.20.3. The vertical composition of

$$\begin{tikzcd} \mathcal{S}_{R \times_{s,U,t} R} \arrow[r, bend left=30, "\pi \circ s \circ \mathrm{pr}_1 = \pi \circ s \circ c"] \arrow[r, "\pi \circ t \circ \mathrm{pr}_1 = \pi \circ s \circ \mathrm{pr}_0"'] \arrow[r, bend right=30, "\pi \circ t \circ \mathrm{pr}_0 = \pi \circ t \circ c"'] & {[U/R]} \end{tikzcd}$$

with 2-morphisms $\Downarrow \alpha \star \mathrm{id}_{\mathrm{pr}_1}$ (upper) and $\Downarrow \alpha \star \mathrm{id}_{\mathrm{pr}_0}$ (lower)

is the 2-morphism $\alpha \star \mathrm{id}_c$. In a formula $\alpha \star \mathrm{id}_c = (\alpha \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\alpha \star \mathrm{id}_{\mathrm{pr}_1})$.

Proof. We make two remarks:

(1) The formula $\alpha \star \mathrm{id}_c = (\alpha \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\alpha \star \mathrm{id}_{\mathrm{pr}_1})$ only makes sense if you realize the equalities $\pi \circ s \circ \mathrm{pr}_1 = \pi \circ s \circ c$, $\pi \circ t \circ \mathrm{pr}_1 = \pi \circ s \circ \mathrm{pr}_0$, and $\pi \circ t \circ \mathrm{pr}_0 = \pi \circ t \circ c$. Namely, the second one implies the vertical composition $\circ$ makes sense, and the other two guarantee the two sides of the formula are 2-morphisms with the same source and target.

(2) The reason the lemma holds is that composition in the category fibred in groupoids $[U/_pR]$ associated to the presheaf in groupoids (78.20.0.1) comes from the composition law $c : R \times_{s,U,t} R \to R$.

We omit the proof of the lemma. $\square$

Note that, in the situation of the lemma, we actually have the equalities $s \circ \mathrm{pr}_1 = s \circ c$, $t \circ \mathrm{pr}_1 = s \circ \mathrm{pr}_0$, and $t \circ \mathrm{pr}_0 = t \circ c$ before composing with π. Hence the formula in the lemma below makes sense in exactly the same way that the formula in the lemma above makes sense.

044U Lemma 78.23.2. Assumptions and notation as in Lemmas 78.20.2 and 78.20.3. The 2-commutative diagram of Lemma 78.20.3 is a 2-coequalizer in the following sense: Given

(1) a stack in groupoids $\mathcal{X}$ over $(Sch/S)_{fppf}$,
(2) a 1-morphism $f : \mathcal{S}_U \to \mathcal{X}$, and
(3) a 2-arrow $\beta : f \circ s \to f \circ t$

such that

$$\beta \star \mathrm{id}_c = (\beta \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\beta \star \mathrm{id}_{\mathrm{pr}_1})$$

then there exists a 1-morphism $[U/R] \to \mathcal{X}$ which makes the diagram

$$\begin{tikzcd}
\mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"] & \mathcal{S}_U \arrow[d] \arrow[ddr, "f"] & \\
\mathcal{S}_U \arrow[r] \arrow[drr, "f"'] & {[U/R]} \arrow[dr] & \\
 & & \mathcal{X}
\end{tikzcd}$$

2-commute.

Proof. Suppose given $\mathcal{X}$, f and β as in the lemma. By Stacks, Lemma 8.9.2 it suffices to construct a 1-morphism $g : [U/_pR] \to \mathcal{X}$. First we note that the 1-morphism $\mathcal{S}_U \to [U/_pR]$ is bijective on objects. Hence on objects we can set $g(x) = f(x)$ for $x \in \mathrm{Ob}(\mathcal{S}_U) = \mathrm{Ob}([U/_pR])$. A morphism $\varphi : x \to y$ of $[U/_pR]$ arises from a commutative diagram

$$\begin{tikzcd}
S_2 \arrow[r, "x"'] \arrow[dr, "\varphi"] \arrow[dd, "h"'] & U \\
 & R \arrow[u, "s"'] \arrow[d, "t"] \\
S_1 \arrow[r, "y"] & U.
\end{tikzcd}$$

Thus we can set $g(\varphi)$ equal to the composition

$$\begin{tikzcd}
f(x) \arrow[r, equal] \arrow[drrrrr] & f(s \circ \varphi) \arrow[r, equal] & (f \circ s)(\varphi) \arrow[r, "\beta"] & (f \circ t)(\varphi) \arrow[r, equal] & f(t \circ \varphi) \arrow[r, equal] & f(y \circ h) \arrow[d] \\
 & & & & & f(y).
\end{tikzcd}$$

The vertical arrow is the result of applying the functor f to the canonical morphism $y \circ h \to y$ in $\mathcal{S}_U$ (namely, the strongly cartesian morphism lifting h with target y). Let us verify that f so defined is compatible with composition, at least on fibre categories. So let S' be a scheme over S, and let $a : S' \to R \times_{s,U,t} R$ be a morphism. In this situation we set $x = s \circ \mathrm{pr}_1 \circ a = s \circ c \circ a$, $y = t \circ \mathrm{pr}_1 \circ a = s \circ \mathrm{pr}_0 \circ a$, and $z = t \circ \mathrm{pr}_0 \circ a = t \circ \mathrm{pr}_0 \circ c$ to get a commutative diagram

$$\begin{tikzcd} x \arrow[rr, "c\circ a"'] \arrow[dr, "\mathrm{pr}_1\circ a"'] & & z \\ & y \arrow[ur, "\mathrm{pr}_0\circ a"'] & \end{tikzcd}$$

in the fibre category $[U/_pR]_{S'}$. Moreover, any commutative triangle in this fibre category has this form. Then we see by our definitions above that f maps this to a commutative diagram if and only if the diagram

$$\begin{tikzcd} & (f \circ s)(c \circ a) \arrow[r, "\beta"'] & (f \circ t)(c \circ a) \arrow[dr, equal] & \\ (f \circ s)(\mathrm{pr}_1 \circ a) \arrow[ur, equal] \arrow[dr, "\beta"] & & & (f \circ t)(\mathrm{pr}_0 \circ a) \\ & (f \circ t)(\mathrm{pr}_1 \circ a) \arrow[r, equal] & (f \circ s)(\mathrm{pr}_0 \circ a) \arrow[ur, "\beta"] & \end{tikzcd}$$

is commutative which is exactly the condition expressed by the formula in the lemma. We omit the verification that f maps identities to identities and is compatible with composition for arbitrary morphisms. □

78.24. Explicit description of quotient stacks

04MB In order to formulate the result we need to introduce some notation. Assume $B \to S$ and (U, R, s, t, c) are as in Definition 78.20.1 (1). Let T be a scheme over S. Let $\mathcal{T} = \{T_i \to T\}_{i \in I}$ be an fppf covering. A $[U/R]$-descent datum relative to $\mathcal{T}$ is given by a system (u_i, r_{ij}) where

(1) for each i a morphism $u_i : T_i \to U$, and
(2) for each i, j a morphism $r_{ij} : T_i \times_T T_j \to R$

such that

(a) as morphisms $T_i \times_T T_j \to U$ we have
$$s \circ r_{ij} = u_i \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{ij} = u_j \circ \mathrm{pr}_1,$$
(b) as morphisms $T_i \times_T T_j \times_T T_k \to R$ we have
$$c \circ (r_{jk} \circ \mathrm{pr}_{12}, r_{ij} \circ \mathrm{pr}_{01}) = r_{ik} \circ \mathrm{pr}_{02}.$$

A morphism $(u_i, r_{ij}) \to (u'_i, r'_{ij})$ between two $[U/R]$-descent data over the same covering $\mathcal{T}$ is a collection $(r_i : T_i \to R)$ such that

(α) as morphisms $T_i \to U$ we have
$$u_i = s \circ r_i \quad \text{and} \quad u'_i = t \circ r_i$$
(β) as morphisms $T_i \times_T T_j \to R$ we have
$$c \circ (r'_{ij}, r_i \circ \mathrm{pr}_0) = c \circ (r_j \circ \mathrm{pr}_1, r_{ij}).$$

There is a natural composition law on morphisms of descent data relative to a fixed covering and we obtain a category of descent data. This category is a groupoid. Finally, if $\mathcal{T}' = \{T'_j \to T\}_{j \in J}$ is a second fppf covering which refines $\mathcal{T}$ then there is a notion of pullback of descent data. This is particularly easy to describe explicitly in this case. Namely, if $\alpha : J \to I$ and $\varphi_j : T'_j \to T_{\alpha(i)}$ is the morphism of coverings, then the pullback of the descent datum $(u_i, r_{ii'})$ is simply

$$(u_{\alpha(i)} \circ \varphi_j, r_{\alpha(j)\alpha(j')} \circ \varphi_j \times \varphi_{j'}).$$

Pullback defined in this manner defines a functor from the category of descent data over $\mathcal{T}$ to the category of descend data over $\mathcal{T}'$.

044X Lemma 78.24.1. Assume $B \to S$ and (U, R, s, t, c) are as in Definition 78.20.1 (1). Let $\pi : \mathcal{S}_U \to [U/R]$ be as in Lemma 78.20.2. Let T be a scheme over S.

(1) for every object x of the fibre category $[U/R]_T$ there exists an fppf covering $\{f_i : T_i \to T\}_{i \in I}$ such that $f_i^*x \cong \pi(u_i)$ for some $u_i \in U(T_i)$,
(2) the composition of the isomorphisms
$$\pi(u_i \circ \mathrm{pr}_0) = \mathrm{pr}_0^*\pi(u_i) \cong \mathrm{pr}_0^*f_i^*x \cong \mathrm{pr}_1^*f_j^*x \cong \mathrm{pr}_1^*\pi(u_j) = \pi(u_j \circ \mathrm{pr}_1)$$
are of the form $\pi(r_{ij})$ for certain morphisms $r_{ij} : T_i \times_T T_j \to R$,
(3) the system (u_i, r_{ij}) forms a $[U/R]$-descent datum as defined above,
(4) any $[U/R]$-descent datum (u_i, r_{ij}) arises in this manner,
(5) if x corresponds to (u_i, r_{ij}) as above, and $y \in \mathrm{Ob}([U/R]_T)$ corresponds to (u'_i, r'_{ij}) then there is a canonical bijection
$$\mathrm{Mor}_{[U/R]_T}(x, y) \longleftrightarrow \left\{ \begin{matrix} \text{morphisms } (u_i, r_{ij}) \to (u'_i, r'_{ij}) \\ \text{of } [U/R]\text{-descent data} \end{matrix} \right\}$$
(6) this correspondence is compatible with refinements of fppf coverings.

Proof. Statement (1) is part of the construction of the stackyfication. Part (2) follows from Lemma 78.22.1. We omit the verification of (3). Part (4) is a translation of the fact that in a stack all descent data are effective. We omit the verifications of (5) and (6). $\square$

78.25. Restriction and quotient stacks

046R In this section we study what happens to the quotient stack when taking a restriction.

046S Lemma 78.25.1. Notation and assumption as in Lemma 78.21.1. The morphism of quotient stacks

$$[f] : [U/R] \longrightarrow [U'/R']$$

is fully faithful if and only if R is the restriction of R' via the morphism $f : U \to U'$.

Proof. Let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/R']_T$. The functor $[f]$ is fully faithful if and only if the map of sheaves

$$\mathit{Isom}(x, y) \longrightarrow \mathit{Isom}(x', y')$$

is an isomorphism for every T, x, y. We may test this locally on T (in the fppf topology). Hence, by Lemma 78.24.1 we may assume that x, y come from $a, b \in$

$U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 78.22.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R'.$$

This is an isomorphism if R is the restriction, because in that case $R = (U \times_B U) \times_{U' \times_B U'} R'$, see Lemma 78.17.3 and its proof. Conversely, if the last displayed map is an isomorphism for all T, a, b, then it follows that $R = (U \times_B U) \times_{U' \times_B U'} R'$, i.e., R is the restriction of R'. □

046T Lemma 78.25.2. Notation and assumption as in Lemma 78.21.1. The morphism of quotient stacks

$$[f] : [U/R] \longrightarrow [U'/R']$$

is an equivalence if and only if

(1) (U, R, s, t, c) is the restriction of (U', R', s', t', c') via $f : U \to U'$, and
(2) the map

$$\begin{tikzcd} U \times_{f,U',t'} R' \arrow[r, "\mathrm{pr}_1"'] \arrow[rr, bend left, "h"] & R' \arrow[r, "s'"'] & U' \end{tikzcd}$$

is a surjection of sheaves.

Part (2) holds for example if $\{h : U \times_{f,U',t'} R' \to U'\}$ is an fppf covering, or if $f : U \to U'$ is a surjection of sheaves, or if $\{f : U \to U'\}$ is an fppf covering.

Proof. We already know that part (1) is equivalent to fully faithfulness by Lemma 78.25.1. Hence we may assume that (1) holds and that $[f]$ is fully faithful. Our goal is to show, under these assumptions, that $[f]$ is an equivalence if and only if (2) holds. We may use Stacks, Lemma 8.4.8 which characterizes equivalences.

Assume (2). We will use Stacks, Lemma 8.4.8 to prove $[f]$ is an equivalence. Suppose that T is a scheme and $x' \in \mathrm{Ob}([U'/R']_T)$. There exists a covering $\{g_i : T_i \to T\}$ such that $g_i^* x'$ is the image of some element $a'_i \in U'(T_i)$, see Lemma 78.24.1. Hence we may assume that x' is the image of $a' \in U'(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $b_i : T_i \to U \times_{g,U',t'} R'$ such that $a' \circ \varphi_i = h \circ b_i$. Denote $a_i = \mathrm{pr}_0 \circ b_i : T_i \to U$. Then we see that $a_i \in U(T_i)$ maps to $f \circ a_i \in U'(T_i)$ and that $f \circ a_i \cong_{T_i} h \circ b_i = a' \circ \varphi_i$, where $\cong_{T_i}$ denotes isomorphism in the fibre category $[U'/R']_{T_i}$. Namely, the element of $R'(T_i)$ giving the isomorphism is $\mathrm{pr}_1 \circ b_i$. This means that the restriction of x to T_i is in the essential image of the functor $[U/R]_{T_i} \to [U'/R']_{T_i}$ as desired.

Assume $[f]$ is an equivalence. Let $\xi' \in [U'/R']_{U'}$ denote the object corresponding to the identity morphism of U'. Applying Stacks, Lemma 8.4.8 we see there exists an fppf covering $\mathcal{U}' = \{g'_i : U'_i \to U'\}$ such that $(g'_i)^* \xi' \cong [f](\xi_i)$ for some ξ_i in $[U/R]_{U'_i}$. After refining the covering $\mathcal{U}'$ (using Lemma 78.24.1) we may assume ξ_i comes from a morphism $a_i : U'_i \to U$. The fact that $[f](\xi_i) \cong (g'_i)^* \xi'$ means that, after possibly refining the covering $\mathcal{U}'$ once more, there exist morphisms $r'_i : U'_i \to R'$ with $t' \circ r'_i = f \circ a_i$ and $s' \circ r'_i = \mathrm{id}_{U'} \circ g'_i$. Picture

$$\begin{tikzcd} U \arrow[d, "f"] & & U'_i \arrow[ll, "a_i"'] \arrow[dl, "r'_i"'] \arrow[d, "g'_i"] \\ U' & R' \arrow[l, "t'"'] \arrow[r, "s'"] & U' \end{tikzcd}$$

Thus $(a_i, r_i') : U_i' \to U \times_{g, U', t'} R'$ are morphisms such that $h \circ (a_i, r_i') = g_i'$ and we conclude that $\{h : U \times_{g, U', t'} R' \to U'\}$ can be refined by the fppf covering $\mathcal{U}'$ which means that h induces a surjection of sheaves, see Topologies on Spaces, Lemma 73.7.5.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 73.7.5. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid in algebraic spaces). $\square$

04ZN Lemma 78.25.3. Notation and assumption as in Lemma 78.21.1. Assume that

$$\begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "s"'] & R' \arrow[d, "s'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

is cartesian. Then

$$\begin{tikzcd} \mathcal{S}_U \arrow[r] \arrow[d] & {[U/R]} \arrow[d, "{[f]}"] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/R']} \end{tikzcd}$$

is a 2-fibre product square.

Proof. Applying the inverse isomorphisms $i : R \to R$ and $i' : R' \to R'$ to the (first) cartesian diagram of the statement of the lemma we see that

$$\begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "t"'] & R' \arrow[d, "t'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

is cartesian as well. By Lemma 78.21.2 we have a 2-fibre square

$$\begin{tikzcd} {[U''/R'']} \arrow[r] \arrow[d] & {[U/R]} \arrow[d] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/R']} \end{tikzcd}$$

where $U'' = U \times_{f, U', t'} R'$ and $R'' = R \times_{f \circ s, U', t'} R'$. By the above we see that $(t, f) : R \to U''$ is an isomorphism, and that

$$R'' = R \times_{f \circ s, U', t'} R' = R \times_{s, U} U \times_{f, U', t'} R' = R \times_{s, U, t} \times R.$$

Explicitly the isomorphism $R \times_{s, U, t} R \to R''$ is given by the rule $(r_0, r_1) \mapsto (r_0, f(r_1))$. Moreover, s'', t'', c'' translate into the maps

$$R \times_{s, U, t} R \to R, \quad s''(r_0, r_1) = r_1, \quad t''(r_0, r_1) = c(r_0, r_1)$$

and

$$\begin{array}{rccc} c'' : & (R \times_{s, U, t} R) \times_{s'', R, t''} (R \times_{s, U, t} R) & \longrightarrow & R \times_{s, U, t} R, \\ & ((r_0, r_1), (r_2, r_3)) & \longmapsto & (c(r_0, r_2), r_3). \end{array}$$

Precomposing with the isomorphism

$$R \times_{s, U, s} R \longrightarrow R \times_{s, U, t} R, \quad (r_0, r_1) \longmapsto (c(r_0, i(r_1)), r_1)$$

we see that t'' and s'' turn into pr_0 and pr_1 and that c'' turns into $\text{pr}_{02} : R \times_{s,U,s} R \times_{s,U,s} R \to R \times_{s,U,s} R$. Hence we see that there is an isomorphism $[U''/R''] \cong [R/R \times_{s,U,s} R]$ where as a groupoid in algebraic spaces $(R, R \times_{s,U,s} R, s'', t'', c'')$ is the restriction of the trivial groupoid $(U, U, \text{id}, \text{id}, \text{id})$ via $s : R \to U$. Since $s : R \to U$ is a surjection of fppf sheaves (as it has a right inverse) the morphism

$$[U''/R''] \cong [R/R \times_{s,U,s} R] \longrightarrow [U/U] = \mathcal{S}_U$$

is an equivalence by Lemma 78.25.2. This proves the lemma. $\square$

78.26. Inertia and quotient stacks

06PA The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 8.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.34.

06PB **Lemma 78.26.1.** *Assume $B \to S$ and (U, R, s, t, c) as in Definition 78.20.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Definition 78.16.2. Set $R' = R \times_{s,U} G$ and set*

1. *$s' : R' \to G$, $(r, g) \mapsto g$,*
2. *$t' : R' \to G$, $(r, g) \mapsto c(r, c(g, i(r)))$,*
3. *$c' : R' \times_{s',G,t'} R' \to R'$, $((r_1, g_1), (r_2, g_2) \mapsto (c(r_1, r_2), g_1)$.*

Then (G, R', s', t', c') is a groupoid in algebraic spaces over B and

$$\mathcal{I}_{[U/R]} = [G/R'].$$

i.e., the associated quotient stack is the inertia stack of $[U/R]$.

Proof. By Stacks, Lemma 8.8.5 it suffices to prove that $\mathcal{I}_{[U/_pR]} = [G/_pR']$. Let T be a scheme over S. Recall that an object of the inertia fibred category of $[U/_pR]$ over T is given by a pair (x, g) where x is an object of $[U/_pR]$ over T and g is an automorphism of x in its fibre category over T. In other words, $x : T \to U$ and $g : T \to R$ such that $x = s \circ g = t \circ g$. This means exactly that $g : T \to G$. A morphism in the inertia fibred category from $(x, g) \to (y, h)$ over T is given by $r : T \to R$ such that $s(r) = x$, $t(r) = y$ and $c(r, g) = c(h, r)$, see the commutative diagram in Categories, Lemma 4.34.1. In a formula

$$h = c(r, c(g, i(r))) = c(c(r, g), i(r)).$$

The notation $s(r)$, etc is a short hand for $s \circ r$, etc. The composition of $r_1 : (x_2, g_2) \to (x_1, g_1)$ and $r_2 : (x_1, g_1) \to (x_2, g_2)$ is $c(r_1, r_2) : (x_1, g_1) \to (x_3, g_3)$.

Note that in the above we could have written g in stead of (x, g) for an object of $\mathcal{I}_{[U/_pR]}$ over T as x is the image of g under the structure morphism $G \to U$. Then the morphisms $g \to h$ in $\mathcal{I}_{[U/_pR]}$ over T correspond exactly to morphisms $r' : T \to R'$ with $s'(r') = g$ and $t'(r') = h$. Moreover, the composition corresponds to the rule explained in (3). Thus the lemma is proved. $\square$

06PC **Lemma 78.26.2.** *Assume $B \to S$ and (U, R, s, t, c) as in Definition 78.20.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see*

Definition 78.16.2. There is a canonical 2-cartesian diagram

$$\begin{tikzcd} \mathcal{S}_G \arrow[r] \arrow[d] & \mathcal{S}_U \arrow[d] \\ \mathcal{I}_{[U/R]} \arrow[r] & [U/R] \end{tikzcd}$$

of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. By Lemma 78.25.3 it suffices to prove that the morphism $s' : R' \to G$ of Lemma 78.26.1 isomorphic to the base change of s by the structure morphism $G \to U$. This base change property is clear from the construction of s'. $\square$

78.27. Gerbes and quotient stacks

06PD In this section we relate quotient stacks to the discussion Stacks, Section 8.11 and especially gerbes as defined in Stacks, Definition 8.11.4. The stacks in groupoids occurring in this section are generally speaking not algebraic stacks!

06PE Lemma 78.27.1. Notation and assumption as in Lemma 78.21.1. The morphism of quotient stacks

$$[f] : [U/R] \longrightarrow [U'/R']$$

turns $[U/R]$ into a gerbe over $[U'/R']$ if $f : U \to U'$ and $R \to R'|_U$ are surjective maps of fppf sheaves. Here $R'|_U$ is the restriction of R' to U via $f : U \to U'$.

Proof. We will verify that Stacks, Lemma 8.11.3 properties (2) (a) and (2) (b) hold. Property (2)(a) holds because $U \to U'$ is a surjective map of sheaves (use Lemma 78.24.1 to see that objects in $[U'/R']$ locally come from U'). To prove (2)(b) let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/'R]_T$. Condition (2)(b) requires us to check the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

on $(Sch/T)_{fppf}$ is surjective. To see this we may work fppf locally on T and assume that come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 78.22.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U\times_B U} R \longrightarrow T \times_{f\circ a,f\circ b,U'\times_B U'} R' = T \times_{(a,b),U\times_B U} R'|_U.$$

Hence the assumption that $R \to R'|_U$ is a surjective map of fppf sheaves on $(Sch/S)_{fppf}$ implies the desired surjectivity. $\square$

06PF Lemma 78.27.2. Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. The morphism

$$[B/G] \longrightarrow \mathcal{S}_B$$

(Lemma 78.20.2) turns $[B/G]$ into a gerbe over B.

Proof. Immediate from Lemma 78.27.1 as the morphisms $B \to B$ and $B \times_B G \to B$ are surjective as morphisms of sheaves. $\square$

78.28. Quotient stacks and change of big site

04WW We suggest skipping this section on a first reading. Pullbacks of stacks are defined in Stacks, Section 8.12.

04WX **Lemma 78.28.1.** Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 34.12. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let $B, U, R \in Sh((Sch/S)_{fppf})$ be algebraic spaces, and let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : Sch_{fppf} \to Sch'_{fppf}$. Then we have a canonical equivalence

$$[f^{-1}U/f^{-1}R] \longrightarrow f^{-1}[U/R]$$

of stacks in groupoids over $(Sch'/S)_{fppf}$.

Proof. Note that $f^{-1}B, f^{-1}U, f^{-1}R \in Sh((Sch'/S)_{fppf})$ are algebraic spaces by Spaces, Lemma 65.15.1 and hence $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Thus the statement makes sense.

The category $u_p[U/_pR]$ is the localization of the category $u_{pp}[U/_pR]$ at right multiplicative system I of morphisms. An object of $u_{pp}[U/_pR]$ is a triple

$$(T', \phi : T' \to T, x)$$

where $T' \in \mathrm{Ob}((Sch'/S)_{fppf})$, $T \in \mathrm{Ob}((Sch/S)_{fppf})$, ϕ is a morphism of schemes over S, and $x : T \to U$ is a morphism of sheaves on $(Sch/S)_{fppf}$. Note that the morphism of schemes $\phi : T' \to T$ is the same thing as a morphism $\phi : T' \to u(T)$, and since $u(T)$ represents $f^{-1}T$ it is the same thing as a morphism $T' \to f^{-1}T$. Moreover, as f^{-1} on algebraic spaces is fully faithful, see Spaces, Lemma 65.15.2, we may think of x as a morphism $x : f^{-1}T \to f^{-1}U$ as well. From now on we will make such identifications without further mention. A morphism

$$(a, a', \alpha) : (T_1', \phi_1 : T_1' \to T_1, x_1) \longrightarrow (T_2', \phi_2 : T_2' \to T_2, x_2)$$

of $u_{pp}[U/_pR]$ is a commutative diagram

$$\begin{tikzcd}
& & U \\
T_1' \arrow[r, "\phi_1"'] \arrow[d, "a'"'] & T_1 \arrow[r, "\alpha"'] \arrow[d, "a"'] \arrow[ru, "x_1"] & R \arrow[u, "s"'] \arrow[d, "t"] \\
T_2' \arrow[r, "\phi_2"] & T_2 \arrow[r, "x_2"] & U
\end{tikzcd}$$

and such a morphism is an element of I if and only if $T_1' = T_2'$ and $a' = \mathrm{id}$. We define a functor

$$u_{pp}[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

by the rules

$$(T', \phi : T' \to T, x) \longmapsto (x \circ \phi : T' \to f^{-1}U)$$

on objects and

$$(a, a', \alpha) \longmapsto (\alpha \circ \phi_1 : T_1' \to f^{-1}R)$$

on morphisms as above. It is clear that elements of I are transformed into isomorphisms as $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Hence this functor factors in a canonical way through a functor

$$u_p[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

Applying stackification we obtain a functor of stacks

$$f^{-1}[U/R] \longrightarrow [f^{-1}U/f^{-1}R]$$

over $(Sch'/S)_{fppf}$, as by Stacks, Lemma 8.12.11 the stack $f^{-1}[U/R]$ is the stackification of $u_p[U/_pR]$.

At this point we have a morphism of stacks, and to verify that it is an equivalence it suffices to show that it is fully faithful and that objects are locally in the essential image, see Stacks, Lemmas 8.4.7 and 8.4.8. The statement on objects holds as $f^{-1}R$ admits a surjective étale morphism $f^{-1}W \to f^{-1}R$ for some object W of $(Sch/S)_{fppf}$. To show that the functor is "full", it suffices to show that morphisms are locally in the image of the functor which holds as $f^{-1}U$ admits a surjective étale morphism $f^{-1}W \to f^{-1}U$ for some object W of $(Sch/S)_{fppf}$. We omit the proof that the functor is faithful. □

78.29. Separation conditions

0453 This really means conditions on the morphism $j : R \to U \times_B U$ when given a groupoid in algebraic spaces (U, R, s, t, c) over B. As in the previous section we first formulate the corresponding diagram.

0454 **Lemma 78.29.1.** Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space. The commutative diagram

$$\begin{tikzcd} R \arrow[rr, "f\mapsto(f,s(f))"'] \arrow[d, "\Delta_{R/U\times_B U}"] & & R \times_{s,U} U \arrow[r] \arrow[d] & U \arrow[d] \\ R \times_{(U\times_B U)} R \arrow[rr, "(f,g)\mapsto(f,f^{-1}\circ g)"] & & R \times_{s,U} G \arrow[r] & G \end{tikzcd}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

0455 **Lemma 78.29.2.** Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space.

(1) The following are equivalent
   (a) $j : R \to U \times_B U$ is separated,
   (b) $G \to U$ is separated, and
   (c) $e : U \to G$ is a closed immersion.
(2) The following are equivalent
   (a) $j : R \to U \times_B U$ is locally separated,
   (b) $G \to U$ is locally separated, and
   (c) $e : U \to G$ is an immersion.
(3) The following are equivalent
   (a) $j : R \to U \times_B U$ is quasi-separated,

(b) $G \to U$ is quasi-separated, and
(c) $e : U \to G$ is quasi-compact.

Proof. The group algebraic space $G \to U$ is the base change of $R \to U \times_B U$ by the diagonal morphism $U \to U \times_B U$, see Lemma 78.16.1. Hence if j is separated (resp. locally separated, resp. quasi-separated), then $G \to U$ is separated (resp. locally separated, resp. quasi-separated). See Morphisms of Spaces, Lemma 67.4.4. Thus (a) $\Rightarrow$ (b) in (1), (2), and (3).

Conversely, if $G \to U$ is separated (resp. locally separated, resp. quasi-separated), then the morphism $e : U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. an immersion, resp. quasi-compact), see Morphisms of Spaces, Lemma 67.4.7. Thus (b) $\Rightarrow$ (c) in (1), (2), and (3).

If e is a closed immersion (resp. an immersion, resp. quasi-compact) then by the result of Lemma 78.29.1 (and Spaces, Lemma 65.12.3, and Morphisms of Spaces, Lemma 67.8.4) we see that $\Delta_{R/U \times_B U}$ is a closed immersion (resp. an immersion, resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in (1), (2), and (3). $\square$

78.30. Other chapters

CHAPTER 79

More on Groupoids in Spaces

04P4

79.1. Introduction

04P5 This chapter is devoted to advanced topics on groupoids in algebraic spaces. Even though the results are stated in terms of groupoids in algebraic spaces, the reader should keep in mind the 2-cartesian diagram

04P6 (79.1.0.1)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & {[U/R]} \end{tikzcd}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 78.20.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [KM97] by Keel and Mori.

79.2. Notation

04P7 We continue to abide by the conventions and notation introduced in Groupoids in Spaces, Section 78.3.

79.3. Useful diagrams

04P8 We briefly restate the results of Groupoids in Spaces, Lemmas 78.11.4 and 78.11.5 for easy reference in this chapter. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram

04P9 (79.3.0.1)
$$\begin{tikzcd} & & U & & \\ R \arrow[urr, "t"] \arrow[d, "s"'] & & R \times_{s,U,t} R \arrow[ll, "\mathrm{pr}_0"'] \arrow[rr, "c"] \arrow[d, "\mathrm{pr}_1"] & & R \arrow[ull, "t"'] \arrow[d, "s"] \\ U & & R \arrow[ll, "t"'] \arrow[rr, "s"] & & U \end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

0451 (79.3.0.2)
$$\begin{tikzcd}
R \times_{t, U, t} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_0 \times c \circ (i, 1)"'] & R \arrow[r, "t"] \arrow[d, "\mathrm{id}_R"] & U \arrow[d, "\mathrm{id}_U"] \\
R \times_{s, U, t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

79.4. Local structure

0CK9 Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over S. Let $\overline{u}$ be a geometric point of U. In this section we explain what kind of structure we obtain on the local rings (Properties of Spaces, Definition 66.22.2)

$$A = \mathcal{O}_{U, \overline{u}} \quad \text{and} \quad B = \mathcal{O}_{R, e(\overline{u})}$$

The convention we will use is to denote the local ring homomorphisms induced by the morphisms s, t, c, e, i by the corresponding letters. In particular we have a commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "t"'] \arrow[drr, "1"] & & \\
& B \arrow[r, "e"] & A \\
A \arrow[ur, "s"] \arrow[urr, "1"'] & &
\end{tikzcd}$$

of local rings. Thus if $I \subset B$ denotes the kernel of $e : B \to A$, then $B = s(A) \oplus I = t(A) \oplus I$. Let us denote

$$C = \mathcal{O}_{R \times_{s, U, t} R, (e, e)(\overline{u})}$$

Then we have

$$C = (B \otimes_{s, A, t} B)^h_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$$

because the localization $(B \otimes_{s, A, t} B)_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$ has separably closed residue field. Let $J \subset C$ be the ideal of C generated by $I \otimes B + B \otimes I$. Then J is also the kernel of the local ring homomorphism

$$(e, e) : C \longrightarrow A$$

The composition law $c : R \times_{s, U, t} R \to R$ corresponds to a ring map

$$c : B \longrightarrow C$$

sending I into J.

0CKA **Lemma 79.4.1.** The map $I/I^2 \to J/J^2$ induced by c is the composition

$$I/I^2 \xrightarrow{(1,1)} I/I^2 \oplus I/I^2 \to J/J^2$$

where the second arrow comes from the equality $J = (I \otimes B + B \otimes I)C$. The map $i : B \to B$ induces the map $-1 : I/I^2 \to I/I^2$.

Proof. To describe a local homomorphism from C to another henselian local ring it is enough to say what happens to elements of the form $b_1 \otimes b_2$ by Algebra, Lemma 10.155.6 for example. Keeping this in mind we have the two canonical maps

$$e_2 : C \to B,\ b_1 \otimes b_2 \mapsto b_1 s(e(b_2)), \quad e_1 : C \to B,\ b_1 \otimes b_2 \mapsto t(e(b_1))b_2$$

corresponding to the embeddings $R \to R \times_{s,U,t} R$ given by $r \mapsto (r, e(s(r)))$ and $r \mapsto (e(t(r)), r)$. These maps define maps $J/J^2 \to I/I^2$ which jointly give an inverse to the map $I/I^2 \oplus I/I^2 \to J/J^2$ of the lemma. Thus to prove statement we only have to show that $e_1 \circ c : B \to B$ and $e_2 \circ c : B \to B$ are the identity maps. This follows from the fact that both compositions $R \to R \times_{s,U,t} R \to R$ are identities.

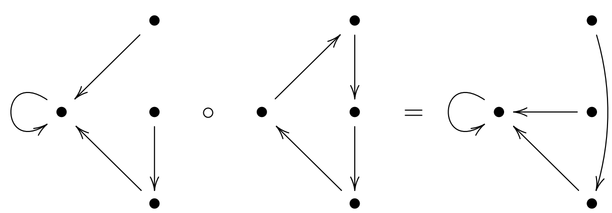
The statement on i follows from the statement on c and the fact that $c \circ (1, i) = e \circ t$. Some details omitted. $\square$

79.5. Groupoid of sections

0CKB Suppose we have a groupoid $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$. Then we can construct a monoid Γ whose elements are maps $\delta : \mathrm{Ob} \to \mathrm{Arrows}$ with $s \circ \delta = \mathrm{id}_{\mathrm{Ob}}$ and composition given by

$$\delta_1 \circ \delta_2 = c(\delta_1 \circ t \circ \delta_2, \delta_2)$$

In other words, an element of Γ is a rule δ which prescribes an arrow emanating from every object and composition is the natural thing. For example



with obvious notation

The same procedure can be applied to a groupoid in algebraic spaces (U, R, s, t, c, e, i) over a scheme S. Namely, as elements of Γ we take the set

$$\Gamma = \{\delta : U \to R \mid s \circ \delta = \mathrm{id}_U\}$$

and composition $\circ : \Gamma \times \Gamma \to \Gamma$ is given by the rule above

0CKC (79.5.0.1) $$\delta_1 \circ \delta_2 = c(\delta_1 \circ t \circ \delta_2, \delta_2)$$

The identity is given by $e \in \Gamma$. The groupoid Γ is not a group in general because there may be elements $\delta \in \Gamma$ which do not have an inverse. Namely, it is clear that $\delta \in \Gamma$ will have an inverse if and only if $t \circ \delta$ is an automorphism of U and in this case $\delta^{-1} = i \circ \delta \circ (t \circ \delta)^{-1}$.

For later use we discuss what happens with the subgroupoid Γ_0 of Γ of sections which are infinitesimally close to the identity e. More precisely, suppose given an R-invariant closed subspace $U_0 \subset U$ such that U is a first order thickening of U_0. Denote $R_0 = s^{-1}(U_0) = t^{-1}(U_0)$ and let $(U_0, R_0, s_0, t_0, c_0, e_0, i_0)$ be the corresponding groupoid in algebraic spaces. Set

$$\Gamma_0 = \{\delta \in \Gamma \mid \delta|_{U_0} = e_0\}$$

If s and t are flat, then every element in Γ_0 is invertible. This follows because $t \circ \delta$ will be a morphism $U \to U$ inducing the identity on $\mathcal{O}_{U_0}$ and on $\mathcal{C}_{U_0/U}$ (Lemma 79.5.1) and we conclude because we have a short exact sequence $0 \to \mathcal{C}_{U_0/U} \to \mathcal{O}_U \to \mathcal{O}_{U_0} \to 0$.

0CKD Lemma 79.5.1. In the situation discussed in this section, let $\delta \in \Gamma_0$ and $f = t \circ \delta : U \to U$. If s, t are flat, then the canonical map $\mathcal{C}_{U_0/U} \to \mathcal{C}_{U_0/U}$ induced by f (More on Morphisms of Spaces, Lemma 76.5.3) is the identity map.

Proof. To see this we extend the bottom of the diagram (79.3.0.2) as follows

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
U \arrow[r, "\delta"'] & R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}$$

where the left square is cartesian and this is our definition of Y; we will not need to know more about Y. There is a similar diagram with similar properties obtained by base change to U_0 everywhere. We are trying to show that $\mathrm{id}_U = s \circ \delta$ and $f = t \circ \delta$ induce the same maps on conormal sheaves. Since s is flat and surjective, it suffices to prove the same thing for the two compositions $a, b : Y \to R$ along the top row. Observe that $a_0 = b_0$ and that one of a and b is an isomorphism as we know that $s \circ \delta$ is an isomorphism. Therefore the two morphisms $a, b : Y \to R$ are morphisms between algebraic spaces flat over U (via the morphism $t : R \to U$ and the morphism $t \circ a = t \circ b : Y \to U$). This implies what we want. Namely, by the compatibility with compositions in More on Morphisms of Spaces, Lemma 76.5.4 we conclude that both maps $a_0^*\mathcal{C}_{R_0/R} \to \mathcal{C}_{Y_0/Y}$ fit into a commutative diagram

$$\begin{tikzcd}
a_0^*\mathcal{C}_{R_0/R} \arrow[rr] & & \mathcal{C}_{Y_0/Y} \\
a_0^*t_0^*\mathcal{C}_{U_0/U} \arrow[u] \arrow[rr, equal] & & (t_0 \circ a_0)^*\mathcal{C}_{U_0/U} \arrow[u]
\end{tikzcd}$$

whose vertical arrows are isomorphisms by More on Morphisms of Spaces, Lemma 76.18.1. Thus the lemma holds. $\square$

Let us identify the group Γ_0. Applying the discussion in More on Morphisms of Spaces, Remarks 76.17.3 and 76.17.7 to the diagram

$$\begin{tikzcd}
(U_0 \subset U) \arrow[rr, dotted, "{(e_0, \delta)}"'] \arrow[dr, "{(\mathrm{id}_{U_0}, \mathrm{id}_U)}"'] & & (R_0 \subset R) \arrow[dl, "{(s_0, s)}"] \\
& (U_0 \subset U) &
\end{tikzcd}$$

we see that $\delta = \theta \cdot e$ for a unique $\mathcal{O}_{U_0}$-linear map $\theta : e_0^*\Omega_{R_0/U_0} \to \mathcal{C}_{U_0/U}$. Thus we get a bijection

0CKE (79.5.1.1) $$\mathrm{Hom}_{\mathcal{O}_{U_0}}(e_0^*\Omega_{R_0/U_0}, \mathcal{C}_{U_0/U}) \longrightarrow \Gamma_0$$

by applying More on Morphisms of Spaces, Lemma 76.17.5.

0CKF Lemma 79.5.2. The bijection (79.5.1.1) is an isomorphism of groups.

Proof. Let $\delta_1, \delta_2 \in \Gamma_0$ correspond to θ_1, θ_2 as above and the composition $\delta = \delta_1 \circ \delta_2$ in Γ_0 correspond to θ. We have to show that $\theta = \theta_1 + \theta_2$. Recall (More on Morphisms of Spaces, Lemma 76.17.2) that $\theta_1, \theta_2, \theta$ correspond to derivations $D_1, D_2, D : e_0^{-1}\mathcal{O}_{R_0} \to \mathcal{C}_{U_0/U}$ given by $D_1 = \theta_1 \circ \text{d}_{R_0/U_0}$ and so on. It suffices to check that $D = D_1 + D_2$.

We may check equality on stalks. Let $\overline{u}$ be a geometric point of U and let us use the local rings A, B, C introduced in Section 79.4. The morphisms δ_i correspond to ring maps $\delta_i : B \to A$. Let $K \subset A$ be the ideal of square zero such that $A/K = \mathcal{O}_{U_0,\overline{u}}$. In other words, K is the stalk of $\mathcal{C}_{U_0/U}$ at $\overline{u}$. The fact that $\delta_i \in \Gamma_0$ means exactly that $\delta_i(I) \subset K$. The derivation D_i is just the map $\delta_i - e : B \to A$. Since $B = s(A) \oplus I$ we see that D_i is determined by its restriction to I and that this is just given by $\delta_i|_I$. Moreover D_i and hence δ_i annihilates I^2 because $I = \text{Ker}(I)$.

To finish the proof we observe that δ corresponds to the composition

$$B \to C = (B \otimes_{s,A,t} B)^h_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B} \to A$$

where the first arrow is c and the second arrow is determined by the rule $b_1 \otimes b_2 \mapsto \delta_2(t(\delta_1(b_1)))\delta_2(b_2)$ as follows from (79.5.0.1). By Lemma 79.4.1 we see that an element ζ of I maps to $\zeta \otimes 1 + 1 \otimes \zeta$ plus higher order terms. Hence we conclude that

$$D(\zeta) = (\delta_2 \circ t)(D_1(\zeta)) + D_2(\zeta)$$

However, by Lemma 79.5.1 the action of $\delta_2 \circ t$ on $K = \mathcal{C}_{U_0/U,\overline{u}}$ is the identity and we win. □

79.6. Properties of groupoids

044Y This section is the analogue of More on Groupoids, Section 40.6. The reader is strongly encouraged to read that section first.

The following lemma is the analogue of More on Groupoids, Lemma 40.6.4.

044Z Lemma 79.6.1. Let $B \to S$ be as in Section 79.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $\tau \in \{fppf, \textit{étale}, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target (Descent on Spaces, Definition 74.10.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant (Groupoids in Spaces, Definition 78.18.1).

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 74.10.3. In Diagram (79.3.0.1) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\text{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent on Spaces, Lemma 74.10.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. □

06R4 Lemma 79.6.2. Let $B \to S$ be as in Section 79.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be its stabilizer group algebraic space. Let $\tau \in \{fppf, \textit{étale}, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids in Spaces, Definition 78.18.1).

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 74.10.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism of algebraic spaces over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent on Spaces, Lemma 74.10.3. $\square$

79.7. Comparing fibres

04PA This section is the analogue of More on Groupoids, Section 40.7. The reader is strongly encouraged to read that section first.

0452 Lemma 79.7.1. Let $B \to S$ be as in Section 79.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let K be a field and let $r, r' : \mathrm{Spec}(K) \to R$ be morphisms such that $t \circ r = t \circ r' : \mathrm{Spec}(K) \to U$. Set $u = s \circ r$, $u' = s \circ r'$ and denote $F_u = \mathrm{Spec}(K) \times_{u,U,s} R$ and $F_{u'} = \mathrm{Spec}(K) \times_{u',U,s} R$ the fibre products. Then $F_u \cong F_{u'}$ as algebraic spaces over K.

Proof. We use the properties and the existence of Diagram (79.3.0.1). There exists a morphism $\xi : \mathrm{Spec}(K) \to R \times_{s,U,t} R$ with $\mathrm{pr}_0 \circ \xi = r$ and $c \circ \xi = r'$. Let $\tilde{r} = \mathrm{pr}_1 \circ \xi : \mathrm{Spec}(K) \to R$. Then looking at the bottom two squares of Diagram (79.3.0.1) we see that both F_u and $F_{u'}$ are identified with the algebraic space $\mathrm{Spec}(K) \times_{\tilde{r},R,\mathrm{pr}_1} (R \times_{s,U,t} R)$. $\square$

Actually, in the situation of the lemma the morphisms of pairs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally isomorphic in the τ-topology, provided $\{s : R \to U\}$ is a τ-covering. We will insert a precise statement here if needed.

79.8. Restricting groupoids

04RM In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

04RN (79.8.0.1)

$$\begin{tikzcd} R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\ U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\ U' \arrow[r, "g"] & U & \end{tikzcd}$$

of a restriction. See Groupoids in Spaces, Lemma 78.17.1.

04RP Lemma 79.8.1. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.

(1) If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.
(2) If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.
(3) If s, t are flat and g is flat, then s', t' are flat.
(4) Add more here.

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms of Spaces, Lemmas 67.23.2 and 67.23.3. Hence (1) is clear from Diagram (79.8.0.1). For the other cases, see Morphisms of Spaces, Lemmas 67.28.2, 67.28.3, 67.30.3, and 67.30.4. $\square$

79.9. Properties of groups over fields and groupoids on fields

06DW The reader is advised to first look at the corresponding sections for groupoid schemes, see Groupoids, Section 39.7 and More on Groupoids, Section 40.10.

06DX Situation 79.9.1. Here S is a scheme, k is a field over S, and (G, m) is a group algebraic space over $\mathrm{Spec}(k)$.

06DY Situation 79.9.2. Here S is a scheme, B is an algebraic space, and (U, R, s, t, c) is a groupoid in algebraic spaces over B with $U = \mathrm{Spec}(k)$ for some field k.

Note that in Situation 79.9.1 we obtain a groupoid in algebraic spaces

06DZ (79.9.2.1)
$$(\mathrm{Spec}(k), G, p, p, m)$$

where $p : G \to \mathrm{Spec}(k)$ is the structure morphism of G, see Groupoids in Spaces, Lemma 78.15.1. This is a situation as in Situation 79.9.2. We will use this without further mention in the rest of this section.

06E0 Lemma 79.9.3. In Situation 79.9.2 the composition morphism $c : R \times_{s,U,t} R \to R$ is flat and universally open. In Situation 79.9.1 the group law $m : G \times_k G \to G$ is flat and universally open.

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (79.3.0.2). The projection is flat as a base change of the flat morphism t and open by Morphisms of Spaces, Lemma 67.6.6. The second assertion follows immediately from the first because m matches c in (79.9.2.1). $\square$

Note that the following lemma applies in particular when working with either quasi-separated or locally separated algebraic spaces (Decent Spaces, Lemma 68.15.2).

08BH Lemma 79.9.4. In Situation 79.9.2 assume R is a decent space. Then R is a separated algebraic space. In Situation 79.9.1 assume that G is a decent algebraic space. Then G is separated algebraic space.

Proof. We first prove the second assertion. By Groupoids in Spaces, Lemma 78.6.1 we have to show that $e : S \to G$ is a closed immersion. This follows from Decent Spaces, Lemma 68.14.5.

Next, we prove the first assertion. To do this we may replace B by S. By the paragraph above the stabilizer group scheme $G \to U$ is separated. By Groupoids in Spaces, Lemma 78.29.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 26.17). Hence R is separated, see Morphisms of Spaces, Lemma 67.4.9. $\square$

06E1 **Lemma 79.9.5.** In Situation 79.9.2. Let k'/k be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[dr, dotted] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r] & U &
\end{tikzcd}$$

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 29.23.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms of Spaces, Lemmas 67.5.5, 67.5.4, 67.6.4, 67.20.5, 67.20.4, 67.30.4, and 67.30.3. $\square$

06E2 **Lemma 79.9.6.** In Situation 79.9.2. For any point $r \in |R|$ there exist

(1) a field extension k'/k with k' algebraically closed,
(2) a point $r' : \mathrm{Spec}(k') \to R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$

such that

(1) the point r' maps to r under the morphism $R' \to R$, and
(2) the maps $s' \circ r', t' \circ r' : \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ are automorphisms.

Proof. Let's represent r by a morphism $r : \mathrm{Spec}(K) \to R$ for some field K. To prove the lemma we have to find an algebraically closed field k' and a commutative diagram

$$\begin{tikzcd}
k' & k' \arrow[l, "1"'] & \\
k' \arrow[u, "\tau"] & K \arrow[ul, "\sigma"] & k \arrow[l, "s"] \arrow[ul, "i"'] \\
& k \arrow[u, "t"'] \arrow[ul, "i"] &
\end{tikzcd}$$

where $s, t : k \to K$ are the field maps coming from $s \circ r$ and $t \circ r$. In the proof of More on Groupoids, Lemma 40.10.5 it is shown how to construct such a diagram. $\square$

06E3 **Lemma 79.9.7.** In Situation 79.9.2. If $r : \mathrm{Spec}(k) \to R$ is a morphism such that $s \circ r, t \circ r$ are automorphisms of $\mathrm{Spec}(k)$, then the map

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

is an automorphism $R \to R$ which maps e to r.

Proof. Proof is identical to the proof of More on Groupoids, Lemma 40.10.6. $\square$

06E4 Lemma 79.9.8. In Situation 79.9.2 the algebraic space R is geometrically unibranch. In Situation 79.9.1 the algebraic space G is geometrically unibranch.

Proof. Let $r \in |R|$. We have to show that R is geometrically unibranch at r. Combining Lemma 79.9.5 with Descent on Spaces, Lemma 74.9.1 we see that it suffices to prove this in case k is algebraically closed and r comes from a morphism $r : \mathrm{Spec}(k) \to R$ such that $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. By Lemma 79.9.7 we reduce to the case that $r = e$ is the identity of R and k is algebraically closed.

Assume $r = e$ and k is algebraically closed. Let $A = \mathcal{O}_{R,e}$ be the étale local ring of R at e and let $C = \mathcal{O}_{R \times_{s,U,t} R,(e,e)}$ be the étale local ring of $R \times_{s,U,t} R$ at (e, e). By More on Algebra, Lemma 15.107.4 the minimal prime ideals $\mathfrak{q}$ of C correspond 1-to-1 to pairs of minimal primes $\mathfrak{p}, \mathfrak{p}' \subset A$. On the other hand, the composition law induces a flat ring map

$$\begin{tikzcd} A \arrow[r, "c^\sharp"'] & C \\ & A \otimes_{s^\sharp,k,t^\sharp} A \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{q} \\ \mathfrak{p} \otimes A + A \otimes \mathfrak{p}' \arrow[u, mapsto] \end{tikzcd}$$

Note that $(c^\sharp)^{-1}(\mathfrak{q})$ contains both $\mathfrak{p}$ and $\mathfrak{p}'$ as the diagrams

$$\begin{tikzcd} A \arrow[r, "c^\sharp"'] & C \\ A \otimes_{s^\sharp,k} k \arrow[u] & A \otimes_{s^\sharp,k,t^\sharp} A \arrow[l, "1\otimes e^\sharp"'] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "c^\sharp"'] & C \\ k \otimes_{k,t^\sharp} A \arrow[u] & A \otimes_{s^\sharp,k,t^\sharp} A \arrow[l, "e^\sharp\otimes 1"'] \arrow[u] \end{tikzcd}$$

commute by (79.3.0.1). Since $c^\sharp$ is flat (as c is a flat morphism by Lemma 79.9.3), we see that $(c^\sharp)^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence $\mathfrak{p} = (c^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}'$. $\square$

In the following lemma we use dimension of algebraic spaces (at a point) as defined in Properties of Spaces, Section 66.9. We also use the dimension of the local ring defined in Properties of Spaces, Section 66.10 and transcendence degree of points, see Morphisms of Spaces, Section 67.33.

06FD Lemma 79.9.9. In Situation 79.9.2 assume s, t are locally of finite type. For all $r \in |R|$

(1) $\dim(R) = \dim_r(R)$,
(2) the transcendence degree of r over $\mathrm{Spec}(k)$ via s equals the transcendence degree of r over $\mathrm{Spec}(k)$ via t, and
(3) if the transcendence degree mentioned in (2) is 0, then $\dim(R) = \dim(\mathcal{O}_{R,\overline{r}})$.

Proof. Let $r \in |R|$. Denote $\mathrm{trdeg}(r/_s k)$ the transcendence degree of r over $\mathrm{Spec}(k)$ via s. Choose an étale morphism $\varphi : V \to R$ where V is a scheme and $v \in V$ mapping to r. Using the definitions mentioned above the lemma we see that

$$\dim_r(R) = \dim_v(V) = \dim(\mathcal{O}_{V,v}) + \mathrm{trdeg}_{s(k)}(\kappa(v)) = \dim(\mathcal{O}_{R,\overline{r}}) + \mathrm{trdeg}(r/_s k)$$

and similarly for t (the second equality by Morphisms, Lemma 29.28.1). Hence we see that $\mathrm{trdeg}(r/_s k) = \mathrm{trdeg}(r/_t k)$, i.e., (2) holds.

Let k'/k be a field extension. Note that the restriction R' of R to $\mathrm{Spec}(k')$ (see Lemma 79.9.5) is obtained from R by two base changes by morphisms of fields.

Thus Morphisms of Spaces, Lemma 67.34.3 shows the dimension of R at a point is unchanged by this operation. Hence in order to prove (1) we may assume, by Lemma 79.9.6, that r is represented by a morphism $r : \mathrm{Spec}(k) \to R$ such that both $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. In this case there exists an automorphism $R \to R$ which maps r to e (Lemma 79.9.7). Hence we see that $\dim_r(R) = \dim_e(R)$ for any r. By definition this means that $\dim_r(R) = \dim(R)$.

Part (3) is a formal consequence of the results obtained in the discussion above. $\square$

06FE Lemma 79.9.10. In Situation 79.9.1 assume G locally of finite type. For all $g \in |G|$

(1) $\dim(G) = \dim_g(G)$,
(2) if the transcendence degree of g over k is 0, then $\dim(G) = \dim(\mathcal{O}_{G,\overline{g}})$.

Proof. Immediate from Lemma 79.9.9 via (79.9.2.1). $\square$

06FF Lemma 79.9.11. In Situation 79.9.2 assume s, t are locally of finite type. Let $G = \mathrm{Spec}(k) \times_{\Delta, \mathrm{Spec}(k) \times_B \mathrm{Spec}(k), t \times s} R$ be the stabilizer group algebraic space. Then we have $\dim(R) = \dim(G)$.

Proof. Since G and R are equidimensional (see Lemmas 79.9.9 and 79.9.10) it suffices to prove that $\dim_e(R) = \dim_e(G)$. Let V be an affine scheme, $v \in V$, and let $\varphi : V \to R$ be an étale morphism of schemes such that $\varphi(v) = e$. Note that V is a Noetherian scheme as $s \circ \varphi$ is locally of finite type as a composition of morphisms locally of finite type and as V is quasi-compact (use Morphisms of Spaces, Lemmas 67.23.2, 67.39.8, and 67.28.5 and Morphisms, Lemma 29.15.6). Hence V is locally connected (see Properties, Lemma 28.5.5 and Topology, Lemma 5.9.6). Thus we may replace V by the connected component containing v (it is still affine as it is an open and closed subscheme of V). Set $T = V_{red}$ equal to the reduction of V. Consider the two morphisms $a, b : T \to \mathrm{Spec}(k)$ given by $a = s \circ \varphi|_T$ and $b = t \circ \varphi|_T$. Note that a, b induce the same field map $k \to \kappa(v)$ because $\varphi(v) = e$! Let $k_a \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $a^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. Similarly, let $k_b \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $b^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. By Varieties, Proposition 33.31.1 we see that $k_a = k_b$. Thus we obtain the following commutative diagram

$$\begin{tikzcd}
k \arrow[dr, "a"] \arrow[drrr, bend left] & & & \\
& k_a = k_b \arrow[r] & \Gamma(T, \mathcal{O}_T) \arrow[r, shift left] \arrow[r, shift right] & \kappa(v) \\
k \arrow[ur, "b"'] \arrow[urrr, bend right] & & &
\end{tikzcd}$$

As discussed above the long arrows are equal. Since $k_a = k_b \to \kappa(v)$ is injective we conclude that the two morphisms a and b agree. Hence $T \to R$ factors through G. It follows that $R_{red} = G_{red}$ in an open neighbourhood of e which certainly implies that $\dim_e(R) = \dim_e(G)$. $\square$

79.10. Group algebraic spaces over fields

0B8D There exists a nonseparated group algebraic space over a field, namely $\mathbf{G}_a/\mathbf{Z}$ over a field of characteristic zero, see Examples, Section 110.49. In fact any group scheme over a field is separated (Lemma 79.9.4) hence every nonseparated group algebraic space over a field is nonrepresentable. On the other hand, a group algebraic space over a field is separated as soon as it is decent, see Lemma 79.9.4. In this section we will show that a separated group algebraic space over a field is representable, i.e., a scheme.

0B8E Lemma 79.10.1. Let k be a field with algebraic closure $\overline{k}$. Let G be a group algebraic space over k which is separated[1]. Then $G_{\overline{k}}$ is a scheme.

Proof. By Spaces over Fields, Lemma 72.10.2 it suffices to show that G_K is a scheme for some field extension K/k. Denote $G'_K \subset G_K$ the schematic locus of G_K as in Properties of Spaces, Lemma 66.13.1. By Properties of Spaces, Proposition 66.13.3 we see that $G'_K \subset G_K$ is dense open, in particular not empty. Choose a scheme U and a surjective étale morphism $U \to G$. By Varieties, Lemma 33.14.2 if K is an algebraically closed field of large enough transcendence degree, then U_K is a Jacobson scheme and every closed point of U_K is K-rational. Hence G'_K has a K-rational point and it suffices to show that every K-rational point of G_K is in G'_K. If $g \in G_K(K)$ is a K-rational point and $g' \in G'_K(K)$ a K-rational point in the schematic locus, then we see that g is in the image of G'_K under the automorphism

$$G_K \longrightarrow G_K, \quad h \longmapsto g(g')^{-1}h$$

of G_K. Since automorphisms of G_K as an algebraic space preserve G'_K, we conclude that $g \in G'_K$ as desired. □

0B8F Lemma 79.10.2. Let k be a field. Let G be a group algebraic space over k. If G is separated and locally of finite type over k, then G is a scheme.

Proof. This follows from Lemma 79.10.1, Groupoids, Lemma 39.8.6, and Spaces over Fields, Lemma 72.10.7. □

0B8G Proposition 79.10.3. Let k be a field. Let G be a group algebraic space over k. If G is separated, then G is a scheme.

Proof. This lemma generalizes Lemma 79.10.2 (which covers all cases one cares about in practice). The proof is very similar to the proof of Spaces over Fields, Lemma 72.10.7 used in the proof of Lemma 79.10.2 and we encourage the reader to read that proof first.

By Lemma 79.10.1 the base change $G_{\overline{k}}$ is a scheme. Let K/k be a purely transcendental extension of very large transcendence degree. By Spaces over Fields, Lemma 72.10.5 it suffices to show that G_K is a scheme. Let K^{perf} be the perfect closure of K. By Spaces over Fields, Lemma 72.10.1 it suffices to show that $G_{K^{perf}}$ is a scheme. Let $K \subset K^{perf} \subset \overline{K}$ be the algebraic closure of K. We may choose an embedding $\overline{k} \to \overline{K}$ over k, so that $G_{\overline{K}}$ is the base change of the scheme $G_{\overline{k}}$ by $\overline{k} \to \overline{K}$. By Varieties, Lemma 33.14.2 we see that $G_{\overline{K}}$ is a Jacobson scheme all of whose closed points have residue field $\overline{K}$.

Since $G_{\overline{K}} \to G_{K^{perf}}$ is surjective, it suffices to show that the image $g \in |G_{K^{perf}}|$ of an arbitrary closed point of $G_{\overline{K}}$ is in the schematic locus of G_K. In particular, we

[1]It is enough to assume G is decent, e.g., locally separated or quasi-separated by Lemma 79.9.4.

may represent g by a morphism $g : \mathrm{Spec}(L) \to G_{K^{perf}}$ where L/K^{perf} is separable algebraic (for example we can take $L = \overline{K}$). Thus the scheme

$$\begin{aligned} T &= \mathrm{Spec}(L) \times_{G_{K^{perf}}} G_{\overline{K}} \\ &= \mathrm{Spec}(L) \times_{\mathrm{Spec}(K^{perf})} \mathrm{Spec}(\overline{K}) \\ &= \mathrm{Spec}(L \otimes_{K^{perf}} \overline{K}) \end{aligned}$$

is the spectrum of a $\overline{K}$-algebra which is a filtered colimit of algebras which are finite products of copies of $\overline{K}$. Thus by Groupoids, Lemma 39.7.13 we can find an affine open $W \subset G_{\overline{K}}$ containing the image of $g_{\overline{K}} : T \to G_{\overline{K}}$.

Choose a quasi-compact open $V \subset G_{K^{perf}}$ containing the image of W. By Spaces over Fields, Lemma 72.10.2 we see that $V_{K'}$ is a scheme for some finite extension K'/K^{perf}. After enlarging K' we may assume that there exists an affine open $U' \subset V_{K'} \subset G_{K'}$ whose base change to $\overline{K}$ recovers W (use that $V_{\overline{K}}$ is the limit of the schemes $V_{K''}$ for $K' \subset K'' \subset \overline{K}$ finite and use Limits, Lemmas 32.4.11 and 32.4.13). We may assume that K'/K^{perf} is a Galois extension (take the normal closure Fields, Lemma 9.16.3 and use that K^{perf} is perfect). Set $H = \mathrm{Gal}(K'/K^{perf})$. By construction the H-invariant closed subscheme $\mathrm{Spec}(L) \times_{G_{K^{perf}}} G_{K'}$ is contained in U'. By Spaces over Fields, Lemmas 72.10.3 and 72.10.4 we conclude. □

79.11. No rational curves on groups

0AEK In this section we prove that there are no nonconstant morphisms from $\mathbf{P}^1$ to a group algebraic space locally of finite type over a field.

0AEL Lemma 79.11.1. Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ and $g : X \to Z$ be morphisms of algebraic spaces over B. Assume

1. $Y \to B$ is separated,
2. g is surjective, flat, and locally of finite presentation,
3. there is a scheme theoretically dense open $V \subset Z$ such that $f|_{g^{-1}(V)} : g^{-1}(V) \to Y$ factors through V.

Then f factors through g.

Proof. Set $R = X \times_Z X$. By (2) we see that $Z = X/R$ as sheaves. Also (2) implies that the inverse image of V in R is scheme theoretically dense in R (Morphisms of Spaces, Lemma 67.30.11). The we see that the two compositions $R \to X \to Y$ are equal by Morphisms of Spaces, Lemma 67.17.8. The lemma follows. □

0AEM Lemma 79.11.2. Let k be a field. Let $n \geq 1$ and let $(\mathbf{P}^1_k)^n$ be the n-fold self product over $\mathrm{Spec}(k)$. Let $f : (\mathbf{P}^1_k)^n \to Z$ be a morphism of algebraic spaces over k. If Z is separated of finite type over k, then f factors as

$$(\mathbf{P}^1_k)^n \xrightarrow{projection} (\mathbf{P}^1_k)^m \xrightarrow{finite} Z.$$

Proof. We may assume k is algebraically closed (details omitted); we only do this so we may argue using rational points, but the reader can work around this if she/he so desires. In the proof products are over k. The automorphism group algebraic space of $(\mathbf{P}^1_k)^n$ contains $G = (\mathrm{GL}_{2,k})^n$. If $C \subset (\mathbf{P}^1_k)^n$ is a closed subvariety (in particular irreducible over k) which is mapped to a point, then we can apply More on Morphisms of Spaces, Lemma 76.35.3 to the morphism

$$G \times C \to G \times Z, \quad (g, c) \mapsto (g, f(g \cdot c))$$

over G. Hence $g(C)$ is mapped to a point for $g \in G(k)$ lying in a Zariski open $U \subset G$. Suppose $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are k-valued points of $(\mathbf{P}^1_k)^n$. Let $I \subset \{1, \ldots, n\}$ be the set of indices i such that $x_i = y_i$. Then

$$\{g(x) \mid g(y) = y,\ g \in U(k)\}$$

is Zariski dense in the fibre of the projection $\pi_I : (\mathbf{P}^1_k)^n \to \prod_{i \in I} \mathbf{P}^1_k$ (exercise). Hence if $x, y \in C(k)$ are distinct, we conclude that f maps the whole fibre of π_I containing x, y to a single point. Moreover, the $U(k)$-orbit of C meets a Zariski open set of fibres of π_I. By Lemma 79.11.1 the morphism f factors through π_I. After repeating this process finitely many times we reach the stage where all fibres of f over k points are finite. In this case f is finite by More on Morphisms of Spaces, Lemma 76.35.2 and the fact that k points are dense in Z (Spaces over Fields, Lemma 72.16.2). □

0AEN Lemma 79.11.3. Let k be a field. Let G be a separated group algebraic space locally of finite type over k. There does not exist a nonconstant morphism $f : \mathbf{P}^1_k \to G$ over $\mathrm{Spec}(k)$.

Proof. Assume f is nonconstant. Consider the morphisms

$$\mathbf{P}^1_k \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathbf{P}^1_k \longrightarrow G, \quad (t_1, \ldots, t_n) \longmapsto f(g_1) \ldots f(g_n)$$

where on the right hand side we use multiplication in the group. By Lemma 79.11.2 and the assumption that f is nonconstant this morphism is finite onto its image. Hence $\dim(G) \geq n$ for all n, which is impossible by Lemma 79.9.10 and the fact that G is locally of finite type over k. □

79.12. The finite part of a morphism

04PB Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. For an algebraic space or a scheme T over S consider pairs (a, Z) where

04PC (79.12.0.1)
$$\begin{array}{l} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace} \\ \text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite.} \end{array}$$

Suppose $h : T' \to T$ is a morphism of algebraic spaces over S and (a, Z) is a pair as in (79.12.0.1) over T. Set $a' = a \circ h$ and $Z' = (h \times \mathrm{id}_X)^{-1}(Z) = T' \times_T Z$. Then (a', Z') is a pair as in (79.12.0.1) over T'. This follows as finite morphisms are preserved under base change, see Morphisms of Spaces, Lemma 67.45.5. Thus we obtain a functor

04PD (79.12.0.2)
$$\begin{array}{rccc} (X/Y)_{fin} : & (Sch/S)^{opp} & \longrightarrow & \text{Sets} \\ & T & \longmapsto & \{(a, Z) \text{ as above}\} \end{array}$$

For applications we are mainly interested in this functor $(X/Y)_{fin}$ when f is separated and locally of finite type. To get an idea of what this is all about, take a look at Remark 79.12.6.

04PE Lemma 79.12.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then we have

(1) The presheaf $(X/Y)_{fin}$ satisfies the sheaf condition for the fppf topology.
(2) If T is an algebraic space over S, then there is a canonical bijection

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, (X/Y)_{fin}) = \{(a, Z) \text{ satisfying 79.12.0.1}\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering (by algebraic spaces). Let $s_i = (a_i, Z_i)$ be pairs over T_i satisfying 79.12.0.1 such that we have $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. First, this implies in particular that a_i and a_j define the same morphism $T_i \times_T T_j \to Y$. By Descent on Spaces, Lemma 74.7.2 we deduce that there exists a unique morphism $a : T \to Y$ such that a_i equals the composition $T_i \to T \to Y$. Second, this implies that $Z_i \subset T_i \times_Y X$ are open subspaces whose inverse images in $(T_i \times_T T_j) \times_Y X$ are equal. Since $\{T_i \times_Y X \to T \times_Y X\}$ is an fppf covering we deduce that there exists a unique open subspace $Z \subset T \times_Y X$ which restricts back to Z_i over T_i, see Descent on Spaces, Lemma 74.7.1. We claim that the projection $Z \to T$ is finite. This follows as being finite is local for the fpqc topology, see Descent on Spaces, Lemma 74.11.23.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair satisfying conditions 79.12.0.1.

Let $v : T \to (X/Y)_{fin}$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in (X/Y)_{fin}(U)$ corresponds to a pair (a_U, Z_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, Z_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, Z) over T.

Conversely, let (a, Z) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, Z)|_U$ gives rise to a transformation of functors $v : h_U \to (X/Y)_{fin}$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, Z)|_U$ and $t^*(a, Z)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 65.9.1 and since $(X/Y)_{fin}$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to (X/Y)_{fin}$.

We omit the verification that the two constructions above are mutually inverse. $\square$

04PF Lemma 79.12.2. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[rr, "j"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

of algebraic spaces over S. If j is an open immersion, then there is a canonical injective map of sheaves $j : (X'/Y)_{fin} \to (X/Y)_{fin}$.

Proof. If (a, Z) is a pair over T for X'/Y, then $(a, j(Z))$ is a pair over T for X/Y. $\square$

04PG Lemma 79.12.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $X' \subset X$ be the maximal open subspace over which f is locally quasi-finite, see Morphisms of Spaces, Lemma 67.34.7. Then $(X/Y)_{fin} = (X'/Y)_{fin}$.

Proof. Lemma 79.12.2 gives us an injective map $(X'/Y)_{fin} \to (X/Y)_{fin}$. Morphisms of Spaces, Lemma 67.34.7 assures us that formation of X' commutes with base change. Hence everything comes down to proving that if $Z \subset X$ is an open subspace such that $f|_Z : Z \to Y$ is finite, then $Z \subset X'$. This is true because a finite morphism is locally quasi-finite, see Morphisms of Spaces, Lemma 67.45.8. □

04PH Lemma 79.12.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let T be an algebraic space over S, and let (a, Z) be a pair as in 79.12.0.1. If f is separated, then Z is closed in $T \times_Y X$.

Proof. A finite morphism of algebraic spaces is universally closed by Morphisms of Spaces, Lemma 67.45.9. Since f is separated so is the morphism $T \times_Y X \to T$, see Morphisms of Spaces, Lemma 67.4.4. Thus the closedness of Z follows from Morphisms of Spaces, Lemma 67.40.6. □

04PI Remark 79.12.5. Let $f : X \to Y$ be a separated morphism of algebraic spaces. The sheaf $(X/Y)_{fin}$ comes with a natural map $(X/Y)_{fin} \to Y$ by mapping the pair $(a, Z) \in (X/Y)_{fin}(T)$ to the element $a \in Y(T)$. We can use Lemma 79.12.4 to define operations

$$\star_i : (X/Y)_{fin} \times_Y (X/Y)_{fin} \longrightarrow (X/Y)_{fin}$$

by the rules

$$\begin{aligned}
\star_1 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \cup Z_2) \\
\star_2 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \cap Z_2) \\
\star_3 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \setminus Z_2) \\
\star_4 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_2 \setminus Z_1).
\end{aligned}$$

The reason this works is that $Z_1 \cap Z_2$ is both open and closed inside Z_1 and Z_2 (which also implies that $Z_1 \cup Z_2$ is the disjoint union of the other three pieces). Thus we can think of $(X/Y)_{fin}$ as an $\mathbf{F}_2$-algebra (without unit) over Y with multiplication given by $ss' = \star_2(s, s')$, and addition given by

$$s + s' = \star_1(\star_3(s, s'), \star_4(s, s'))$$

which boils down to taking the symmetric difference. Note that in this sheaf of algebras $0 = (1_Y, \emptyset)$ and that indeed $s + s = 0$ for any local section s. If $f : X \to Y$ is finite, then this algebra has a unit namely $1 = (1_Y, X)$ and $\star_3(s, s') = s(1 + s')$, and $\star_4(s, s') = (1 + s)s'$.

04PJ Remark 79.12.6. Let $f : X \to Y$ be a separated, locally quasi-finite morphism of schemes. In this case the sheaf $(X/Y)_{fin}$ is closely related to the sheaf $f_!\mathbf{F}_2$ (insert future reference here) on $Y_{étale}$. Namely, if $V \to Y$ is étale, and $s \in \Gamma(V, f_!\mathbf{F}_2)$, then $s \in \Gamma(V \times_Y X, \mathbf{F}_2)$ is a section with proper support $Z = \text{Supp}(s)$ over V. Since f is also locally quasi-finite we see that the projection $Z \to V$ is actually finite. Since the support of a section of a constant abelian sheaf is open we see that the pair $(V \to Y, \text{Supp}(s))$ satisfies 79.12.0.1. In fact, $f_!\mathbf{F}_2 \cong (X/Y)_{fin}|_{Y_{étale}}$ in this case which also explains the $\mathbf{F}_2$-algebra structure introduced in Remark 79.12.5.

04PK Lemma 79.12.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The diagonal of $(X/Y)_{fin} \to Y$

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times_Y (X/Y)_{fin}$$

is representable (by schemes) and an open immersion and the "absolute" diagonal

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times (X/Y)_{fin}$$

is representable (by schemes).

Proof. The second statement follows from the first as the absolute diagonal is the composition of the relative diagonal and a base change of the diagonal of Y (which is representable by schemes), see Spaces, Section 65.3. To prove the first assertion we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 79.12.0.1 there is an open subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h(T') \subset V \Leftrightarrow \Big(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\Big)$$

Let us construct V. Note that $Z_1 \cap Z_2$ is open in Z_1 and in Z_2. Since $\text{pr}_0|_{Z_i} : Z_i \to T$ is finite, hence proper (see Morphisms of Spaces, Lemma 67.45.9) we see that

$$E = \text{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2)) \cup \text{pr}_0|_{Z_2}(Z_2 \setminus Z_1 \cap Z_2))$$

is closed in T. Now it is clear that $V = T \setminus E$ works. $\square$

04QE **Lemma 79.12.8.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Suppose that U is a scheme, $U \to Y$ is an étale morphism and $Z \subset U \times_Y X$ is an open subspace finite over U. Then the induced morphism $U \to (X/Y)_{fin}$ is étale.

Proof. This is formal from the description of the diagonal in Lemma 79.12.7 but we write it out since it is an important step in the development of the theory. We have to check that for any scheme T over S and a morphism $T \to (X/Y)_{fin}$ the projection map

$$T \times_{(X/Y)_{fin}} U \longrightarrow T$$

is étale. Note that

$$T \times_{(X/Y)_{fin}} U = (X/Y)_{fin} \times_{((X/Y)_{fin} \times_Y (X/Y)_{fin})} (T \times_Y U)$$

Applying the result of Lemma 79.12.7 we see that $T \times_{(X/Y)_{fin}} U$ is represented by an open subscheme of $T \times_Y U$. As the projection $T \times_Y U \to T$ is étale by Morphisms of Spaces, Lemma 67.39.4 we conclude. $\square$

04QF **Lemma 79.12.9.** Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a fibre product square of algebraic spaces over S. Then

$$\begin{tikzcd} (X'/Y')_{fin} \arrow[r] \arrow[d] & (X/Y)_{fin} \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a fibre product square of sheaves on $(Sch/S)_{fppf}$.

Proof. It follows immediately from the definitions that the sheaf $(X'/Y')_{fin}$ is equal to the sheaf $Y' \times_Y (X/Y)_{fin}$. $\square$

04QG **Lemma 79.12.10.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated and locally quasi-finite, then there exists a scheme U étale over Y and a surjective étale morphism $U \to (X/Y)_{fin}$ over Y.

Proof. Note that the assertion makes sense by the result of Lemma 79.12.7 on the diagonal of $(X/Y)_{fin}$, see Spaces, Lemma 65.5.10. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. By Lemma 79.12.9 the morphism $(V \times_Y X/V)_{fin} \to (X/Y)_{fin}$ is a base change of the map $V \to Y$ and hence is surjective and étale, see Spaces, Lemma 65.5.5. Hence it suffices to prove the lemma for $(V \times_Y X/V)_{fin}$. (Here we implicitly use that the composition of representable, surjective, and étale transformations of functors is again representable, surjective, and étale, see Spaces, Lemmas 65.3.2 and 65.5.4, and Morphisms, Lemmas 29.9.2 and 29.36.3.) Note that the properties of being separated and locally quasi-finite are preserved under base change, see Morphisms of Spaces, Lemmas 67.4.4 and 67.27.4. Hence $V \times_Y X \to V$ is separated and locally quasi-finite as well, and by Morphisms of Spaces, Proposition 67.50.2 we see that $V \times_Y X$ is a scheme as well. Thus we may assume that $f : X \to Y$ is a separated and locally quasi-finite morphism of schemes.

Pick a point $y \in Y$. Pick $x_1, \ldots, x_n \in X$ points lying over y. Pick an étale neighbourhood $a : (U, u) \to (Y, y)$ and a decomposition

$$U \times_S X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_j} V_{i,j}$$

as in More on Morphisms, Lemma 37.41.5. Pick any subset

$$I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}.$$

Given these choices we obtain a pair (a, Z) with $Z = \bigcup_{(i,j) \in I} V_{i,j}$ which satisfies conditions 79.12.0.1. In other words we obtain a morphism $U \to (X/Y)_{fin}$. The construction of this morphism depends on all the things we picked above, so we should really write

$$U(y, n, x_1, \ldots, x_n, a, I) \longrightarrow (X/Y)_{fin}$$

This morphism is étale by Lemma 79.12.8.

Claim: The disjoint union of all of these is surjective onto $(X/Y)_{fin}$. It is clear that if the claim holds, then the lemma is true.

To show surjectivity we have to show the following (see Spaces, Remark 65.5.2): Given a scheme T over S, a point $t \in T$, and a map $T \to (X/Y)_{fin}$ we can find a datum $(y, n, x_1, \ldots, x_n, a, I)$ as above such that t is in the image of the projection map

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T \longrightarrow T.$$

To prove this we may clearly replace T by $\mathrm{Spec}(\overline{\kappa(t)})$ and $T \to (X/Y)_{fin}$ by the composition $\mathrm{Spec}(\overline{\kappa(t)}) \to T \to (X/Y)_{fin}$. In other words, we may assume that T is the spectrum of an algebraically closed field.

Let $T = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field k. The morphism $T \to (X/Y)_{fin}$ is given by a pair $(T \to Y, Z)$ satisfying conditions 79.12.0.1. Here is a picture:

$$\begin{tikzcd} & Z \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, equal] & T \arrow[r] & Y \end{tikzcd}$$

Let $y \in Y$ be the image point of $T \to Y$. Since Z is finite over k it has finitely many points. Thus there exist finitely many points $x_1, \ldots, x_n \in X$ such that the image of Z in X is contained in $\{x_1, \ldots, x_n\}$. Choose $a : (U, u) \to (Y, y)$ adapted to y and $x_1, \ldots, x_n$ as above, which gives the diagram

$$\begin{tikzcd} W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} \arrow[r] \arrow[d] & X \arrow[d] \\ U \arrow[r] & Y. \end{tikzcd}$$

Since k is algebraically closed and $\kappa(y) \subset \kappa(u)$ is finite separable we may factor the morphism $T = \mathrm{Spec}(k) \to Y$ through the morphism $u = \mathrm{Spec}(\kappa(u)) \to \mathrm{Spec}(\kappa(y)) = y \subset Y$. With this choice we obtain the commutative diagram:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & U \arrow[r] & Y \end{tikzcd}$$

We know that the image of the left upper arrow ends up in $\coprod V_{i,j}$. Recall also that Z is an open subscheme of $\mathrm{Spec}(k) \times_Y X$ by definition of $(X/Y)_{fin}$ and that the right hand square is a fibre product square. Thus we see that

$$Z \subset \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} \mathrm{Spec}(k) \times_U V_{i,j}$$

is an open subscheme. By construction (see More on Morphisms, Lemma 37.41.5) each $V_{i,j}$ has a unique point $v_{i,j}$ lying over u with purely inseparable residue field extension $\kappa(v_{i,j})/\kappa(u)$. Hence each scheme $\mathrm{Spec}(k) \times_U V_{i,j}$ has exactly one point. Thus we see that

$$Z = \coprod_{(i,j) \in I} \mathrm{Spec}(k) \times_U V_{i,j}$$

for a unique subset $I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$. Unwinding the definitions this shows that

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T$$

with I as found above is nonempty as desired. $\square$

04QH Proposition 79.12.11. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated and locally of finite type. Then $(X/Y)_{fin}$ is an algebraic space. Moreover, the morphism $(X/Y)_{fin} \to Y$ is étale.

Proof. By Lemma 79.12.3 we may replace X by the open subscheme which is locally quasi-finite over Y. Hence we may assume that f is separated and locally quasi-finite. We will check the three conditions of Spaces, Definition 65.6.1. Condition (1) follows from Lemma 79.12.1. Condition (2) follows from Lemma 79.12.7. Finally, condition (3) follows from Lemma 79.12.10. Thus $(X/Y)_{fin}$ is an algebraic space. Moreover, that lemma shows that there exists a commutative diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & (X/Y)_{fin} \arrow[dl] \\ & Y & \end{tikzcd}$$

with horizontal arrow surjective and étale and south-east arrow étale. By Properties of Spaces, Lemma 66.16.3 this implies that the south-west arrow is étale as well. $\square$

04QI Remark 79.12.12. The condition that f be separated cannot be dropped from Proposition 79.12.11. An example is to take X the affine line with zero doubled, see Schemes, Example 26.14.3, $Y = \mathbf{A}^1_k$ the affine line, and $X \to Y$ the obvious map. Recall that over $0 \in Y$ there are two points 0_1 and 0_2 in X. Thus $(X/Y)_{fin}$ has four points over 0, namely $\emptyset, \{0_1\}, \{0_2\}, \{0_1, 0_2\}$. Of these four points only three can be lifted to an open subscheme of $U \times_Y X$ finite over U for $U \to Y$ étale, namely $\emptyset, \{0_1\}, \{0_2\}$. This shows that $(X/Y)_{fin}$ if representable by an algebraic space is not étale over Y. Similar arguments show that $(X/Y)_{fin}$ is really not an algebraic space. Details omitted.

04QJ Remark 79.12.13. Let $Y = \mathbf{A}^1_{\mathbf{R}}$ be the affine line over the real numbers, and let $X = \text{Spec}(\mathbf{C})$ mapping to the $\mathbf{R}$-rational point 0 in Y. In this case the morphism $f : X \to Y$ is finite, but it is not the case that $(X/Y)_{fin}$ is a scheme. Namely, one can show that in this case the algebraic space $(X/Y)_{fin}$ is isomorphic to the algebraic space of Spaces, Example 65.14.2 associated to the extension $\mathbf{R} \subset \mathbf{C}$. Thus it is really necessary to leave the category of schemes in order to represent the sheaf $(X/Y)_{fin}$, even when f is a finite morphism.

04RI Lemma 79.12.14. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated, flat, and locally of finite presentation. In this case

(1) $(X/Y)_{fin} \to Y$ is separated, representable, and étale, and
(2) if Y is a scheme, then $(X/Y)_{fin}$ is (representable by) a scheme.

Proof. Since f is in particular separated and locally of finite type (see Morphisms of Spaces, Lemma 67.28.5) we see that $(X/Y)_{fin}$ is an algebraic space by Proposition 79.12.11. To prove that $(X/Y)_{fin} \to Y$ is separated we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 79.12.0.1 there is a closed subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h \text{ factors through } V \Leftrightarrow \Big(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\Big)$$

In the proof of Lemma 79.12.7 we have seen that $V = T' \setminus E$ is an open subscheme of T' with closed complement

$$E = \text{pr}_0|_{Z_1}\left(Z_1 \setminus Z_1 \cap Z_2)\right) \cup \text{pr}_0|_{Z_2}\left(Z_2 \setminus Z_1 \cap Z_2)\right).$$

Thus everything comes down to showing that E is also open. By Lemma 79.12.4 we see that Z_1 and Z_2 are closed in $T' \times_Y X$. Hence $Z_1 \setminus Z_1 \cap Z_2$ is open in Z_1. As f is flat and locally of finite presentation, so is $\text{pr}_0|_{Z_1}$. This is true as Z_1 is an open subspace of the base change $T' \times_Y X$, and Morphisms of Spaces, Lemmas 67.28.3 and Lemmas 67.30.4. Hence $\text{pr}_0|_{Z_1}$ is open, see Morphisms of Spaces, Lemma 67.30.6. Thus $\text{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2))$ is open and it follows that E is open as desired.

We have already seen that $(X/Y)_{fin} \to Y$ is étale, see Proposition 79.12.11. Hence now we know it is locally quasi-finite (see Morphisms of Spaces, Lemma 67.39.5) and separated, hence representable by Morphisms of Spaces, Lemma 67.51.1. The final assertion is clear (if you like you can use Morphisms of Spaces, Proposition 67.50.2). □

Variant: Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. For an algebraic space or a scheme T over S consider pairs (a, Z) where

04RQ (79.12.14.1)
$$\begin{matrix} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace} \\ \text{such that } \text{pr}_0|_Z : Z \to T \text{ is finite and} \\ (1_T, \sigma \circ a) : T \to T \times_Y X \text{ factors through } Z. \end{matrix}$$

We will denote $(X/Y, \sigma)_{fin}$ the subfunctor of $(X/Y)_{fin}$ parametrizing these pairs.

04RR Lemma 79.12.15. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. Consider the transformation of functors

$$t : (X/Y, \sigma)_{fin} \longrightarrow (X/Y)_{fin}.$$

defined above. Then

(1) t is representable by open immersions,
(2) if f is separated, then t is representable by open and closed immersions,
(3) if $(X/Y)_{fin}$ is an algebraic space, then $(X/Y, \sigma)_{fin}$ is an algebraic space and an open subspace of $(X/Y)_{fin}$, and
(4) if $(X/Y)_{fin}$ is a scheme, then $(X/Y, \sigma)_{fin}$ is an open subscheme of it.

Proof. Omitted. Hint: Given a pair (a, Z) over T as in (79.12.0.1) the inverse image of Z by $(1_T, \sigma \circ a) : T \to T \times_Y X$ is the open subscheme of T we are looking for. □

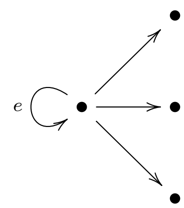
79.13. Finite collections of arrows

04RS Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 39.13 this corresponds to a septuple $(\text{Ob}, \text{Arrows}, s, t, c, e, i)$.

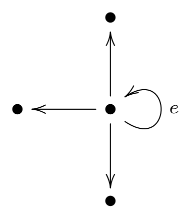
Using this data we can make another groupoid $\mathcal{C}_{fin}$ as follows:

(1) An object of $\mathcal{C}_{fin}$ consists of a finite subset $Z \subset \text{Arrows}$ with the following properties:
   (a) $s(Z) = \{u\}$ is a singleton, and
   (b) $e(u) \in Z$.
(2) A morphism of $\mathcal{C}_{fin}$ consists of a pair (Z, z), where Z is an object of $\mathcal{C}_{fin}$ and $z \in Z$.
(3) The source of (Z, z) is Z.
(4) The target of (Z, z) is $t(Z, z) = \{z' \circ z^{-1}; z' \in Z\}$.
(5) Given (Z_1, z_1), (Z_2, z_2) such that $s(Z_1, z_1) = t(Z_2, z_2)$ the composition $(Z_1, z_1) \circ (Z_2, z_2)$ is $(Z_2, z_1 \circ z_2)$.

We omit the verification that this defines a groupoid. Pictorially an object of $\mathcal{C}_{fin}$ can be viewed as a diagram



To make a morphism of $\mathcal{C}_{fin}$ you pick one of the arrows and you precompose the other arrows by its inverse. For example if we pick the middle horizontal arrow then the target is the picture



Note that the cardinalities of $s(Z, z)$ and $t(Z, z)$ are equal. So $\mathcal{C}_{fin}$ is really a countable disjoint union of groupoids.

79.14. The finite part of a groupoid

04RT In this section we are going to use the idea explained in Section 79.13 to take the finite part of a groupoid in algebraic spaces.

Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assumption: The morphisms s, t are separated and locally of finite type. This notation and assumption will we be fixed throughout this section.

Denote R_s the algebraic space R seen as an algebraic space over U via s. Let $U' = (R_s/U, e)_{fin}$. Since s is separated and locally of finite type, by Proposition 79.12.11 and Lemma 79.12.15, we see that U' is an algebraic space endowed with an étale morphism $g : U' \to U$. Moreover, by Lemma 79.12.1 there exists a universal open subspace $Z_{univ} \subset R \times_{s,U,g} U'$ which is finite over U' and such that $(1_{U'}, e \circ g) : U' \to R \times_{s,U,g} U'$ factors through Z_{univ}. Moreover, by Lemma 79.12.4 the open subspace Z_{univ} is also closed in $R \times_{s,U',g} U$. Picture so far:

$$\begin{tikzcd} Z_{univ} \arrow[d] \arrow[dr] & \\ R \times_{s,U,g} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\ R \arrow[r, "s"] & U \end{tikzcd}$$

Let T be a scheme over B. We see that a T-valued point of Z_{univ} may be viewed as a triple (u, Z, z) where

(1) $u : T \to U$ is a T-valued point of U,

(2) $Z \subset R \times_{s,U,u} T$ is an open and closed subspace finite over T such that $(e \circ u, 1_T)$ factors through it, and
(3) $z : T \to R$ is a T-valued point of R with $s \circ z = u$ and such that $(z, 1_T)$ factors through Z.

Having said this, it is morally clear from the discussion in Section 79.13 that we can turn (Z_{univ}, U') into a groupoid in algebraic spaces over B. To make sure will define the morphisms s', t', c', e', i' one by one using the functorial point of view. (Please don't read this before reading and understanding the simple construction in Section 79.13.)

The morphism $s' : Z_{univ} \to U'$ corresponds to the rule

$$s' : (u, Z, z) \mapsto (u, Z).$$

The morphism $t' : Z_{univ} \to U'$ is given by the rule

$$t' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z)).$$

The entry $c(Z, i \circ z)$ makes sense as the map $c(-, i \circ z) : R \times_{s,U,u} T \to R \times_{s,U,t\circ z} T$ is an isomorphism with inverse $c(-, z)$. The morphism $e' : U' \to Z_{univ}$ is given by the rule

$$e' : (u, Z) \mapsto (u, Z, (e \circ u, 1_T)).$$

Note that this makes sense by the requirement that $(e \circ u, 1_T)$ factors through Z. The morphism $i' : Z_{univ} \to Z_{univ}$ is given by the rule

$$i' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z), i \circ z).$$

Finally, composition is defined by the rule

$$c' : ((u_1, Z_1, z_1), (u_2, Z_2, z_2)) \mapsto (u_2, Z_2, z_1 \circ z_2).$$

We omit the verification that the axioms of a groupoid in algebraic spaces hold for $(U', Z_{univ}, s', t', c', e', i')$.

A final piece of information is that there is a canonical morphism of groupoids

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

Namely, the morphism $U' \to U$ is the morphism $g : U' \to U$ which is defined by the rule $(u, Z) \mapsto u$. The morphism $Z_{univ} \to R$ is defined by the rule $(u, Z, z) \mapsto z$. This finishes the construction. Let us summarize our findings as follows.

04RU Lemma 79.14.1. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assume the morphisms s, t are separated and locally of finite type. There exists a canonical morphism

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

of groupoids in algebraic spaces over B where

(1) $g : U' \to U$ is identified with $(R_s/U, e)_{fin} \to U$, and
(2) $Z_{univ} \subset R \times_{s,U,g} U'$ is the universal open (and closed) subspace finite over U' which contains the base change of the unit e.

Proof. See discussion above. $\square$

79.15. Étale localization of groupoid schemes

04RJ In this section we prove results similar to [KM97, Proposition 4.2]. We try to be a bit more general, and we try to avoid using Hilbert schemes by using the finite part of a morphism instead. The goal is to "split" a groupoid in algebraic spaces over a point after étale localization. Here is the definition (very similar to [KM97, Definition 4.1]).

04RK Definition 79.15.1. Let S be a scheme. Let B be an algebraic space over S Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $u \in |U|$ be a point.

(1) We say R is strongly split over u if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $\{r \in |R| : s(r) = u, t(r) = u\} \subset |P|$.
   The choice of such a P will be called a strong splitting of R over u.
(2) We say R is split over u if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $\{g \in |G| : g \text{ maps to } u\} \subset |P|$ where $G \to U$ is the stabilizer.
   The choice of such a P will be called a splitting of R over u.
(3) We say R is quasi-split over u if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $e(u) \in |P|$[2].
   The choice of such a P will be called a quasi-splitting of R over u.

Note the similarity of the conditions on P to the conditions on pairs in (79.12.0.1). In particular, if s, t are separated, then P is also closed in R (see Lemma 79.12.4).

Suppose we start with a groupoid in algebraic spaces (U, R, s, t, c) over B and a point $u \in |U|$. Since the goal is to split the groupoid after étale localization we may as well replace U by an affine scheme (what we mean is that this is harmless for any possible application). Moreover, the additional hypotheses we are going to have to impose will force R to be a scheme at least in a neighbourhood of $\{r \in |R| : s(r) = u, t(r) = u\}$ or $e(u)$. This is why we start with a groupoid scheme as described below. However, our technique of proof leads us outside of the category of schemes, which is why we have formulated a splitting for the case of groupoids in algebraic spaces above. On the other hand, we know of no applications but the case where the morphisms s, t are also flat and of finite presentation, in which case we end up back in the category of schemes.

04RL Situation 79.15.2 (Strong splitting). Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{r \in R : s(r) = u, t(r) = u\}$ is finite, and
(4) s is quasi-finite at each point of the set in (3).

[2] This condition is implied by (a).

Note that assumptions (3) and (4) are implied by the assumption that the fibre $s^{-1}(\{u\})$ is finite, see Morphisms, Lemma 29.20.7.

0DTB Situation 79.15.3 (Splitting). Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{g \in G : g \text{ maps to } u\}$ is finite where $G \to U$ is the stabilizer, and
(4) s is quasi-finite at each point of the set in (3).

04RV Situation 79.15.4 (Quasi-splitting). Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type, and
(3) s is quasi-finite at $e(u)$.

For our application to the existence theorems for algebraic spaces the case of quasi-splittings is sufficient. Moreover, the quasi-splitting case will allow us to prove an étale local structure theorem for quasi-DM stacks. The splitting case will be used to prove a version of the Keel-Mori theorem. The strong splitting case applies to give an étale local structure theorem for quasi-DM algebraic stacks with quasi-compact diagonal.

03FM Lemma 79.15.5 (Existence of strong splitting). In Situation 79.15.2 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 79.14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The set $\{r \in R : s(r) = u, t(r) = u\}$ is finite by assumption and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each of its elements by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $\{r \in R : s(r) = u, t(r) = u\}$. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 79.14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{r' \in |R'| : s'(r') = u', t'(r') = u'\}$ is contained in $|Z_{univ}|$. Pick any point r' in this set and represent it by a morphism $z' : \mathrm{Spec}(k) \to R'$. Denote $z : \mathrm{Spec}(k) \to R$ the composition of z' with the map $R' \to R$. Clearly, z defines an element of the set $\{r \in R : s(r) = u, t(r) = u\}$. Also, the compositions

$s \circ z, t \circ z : \mathrm{Spec}(k) \to U$ factor through u, so we may think of $s \circ z, t \circ z$ as a morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(u))$. Then $z' = (z, u' \circ t \circ z, u' \circ s \circ u)$ as morphisms into $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Consider the triple

$$(s \circ z, Z_u \times_{\mathrm{Spec}(\kappa(u)), s\circ z} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\mathrm{Spec}(k)$-valued point of Z_{univ} whose image via s', t' in U' is u' and whose image via $Z_{univ} \to R'$ is the point r' by the relationship between z and z' mentioned above. This finishes the proof. $\square$

0DTC Lemma 79.15.6 (Existence of splitting). In Situation 79.15.3 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 79.14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. Let $G_u \subset F_u$ be the scheme theoretic fibre of $G \to U$ over u. By assumption G_u is finite and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each point of G_u by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is G_u. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 79.14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{g' \in |G'| : g' \text{ maps to } u'\}$ is contained in $|Z_{univ}|$. Pick any point g' in this set and represent it by a morphism $z' : \mathrm{Spec}(k) \to G'$. Denote $z : \mathrm{Spec}(k) \to G$ the composition of z' with the map $G' \to G$. Clearly, z defines a point of G_u. In fact, let us write $\tilde{u} : \mathrm{Spec}(k) \to u \to U$ for the corresponding map to u or U. Consider the triple

$$(\tilde{u}, Z_u \times_{u, \tilde{u}} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\mathrm{Spec}(k)$-valued point of Z_{univ} whose image via s', t' in U' is u' and whose image via $Z_{univ} \to R'$ is the point z' (because the image in R is z). This finishes the proof. $\square$

04RW Lemma 79.15.7 (Existence of quasi-splitting). In Situation 79.15.4 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 79.14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The morphism $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at $e(u)$ by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $e(u)$. Hence by the construction of U' in Section 79.14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'. To finish the proof we have to show that $e'(u') \in Z_{univ}$ which is clear. □

Finally, when we add additional assumptions we obtain schemes.

04RX Lemma 79.15.8. In Situation 79.15.2 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.

Proof. This follows from the construction of U' in the proof of Lemma 79.15.5 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 79.12.14 and 79.12.15. □

0DTD Lemma 79.15.9. In Situation 79.15.3 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.

Proof. This follows from the construction of U' in the proof of Lemma 79.15.6 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 79.12.14 and 79.12.15. □

04RY Lemma 79.15.10. In Situation 79.15.4 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.

Proof. This follows from the construction of U' in the proof of Lemma 79.15.7 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 79.12.14 and 79.12.15. □

In fact we can obtain affine schemes by applying an earlier result on finite locally free groupoids.

04RZ Lemma 79.15.11. In Situation 79.15.2 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.

Proof. Let $U' \to U$ and $u' \in U'$ be the separated étale morphism of schemes we found in Lemma 79.15.8. Let $P \subset R'$ be the strong splitting of R' over u'. By More on Groupoids, Lemma 40.9.1 the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation. They are finite by assumption. Hence s', t' are finite locally free, see Morphisms, Lemma 29.48.2. In particular $t(s^{-1}(u'))$ is a finite set of points $\{u'_1, u'_2, \ldots, u'_n\}$ of U'. Choose a quasi-compact open $W \subset U'$ containing each u'_i. As U is affine the morphism $W \to U$ is quasi-compact (see Schemes, Lemma 26.19.2). The morphism $W \to U$ is also locally quasi-finite (see Morphisms, Lemma 29.36.6) and separated. Hence by More on Morphisms, Lemma 37.43.2 (a version of Zariski's Main Theorem) we conclude that W is quasi-affine. By Properties, Lemma 28.29.5 we see that $\{u'_1, \ldots, u'_n\}$ are contained in an affine open of U'. Thus we may apply Groupoids, Lemma 39.24.1 to conclude that there exists an affine P-invariant open $U'' \subset U'$ which contains u'.

To finish the proof denote $R'' = R|_{U''}$ the restriction of R to U''. This is the same as the restriction of R' to U''. As $P \subset R'$ is an open and closed subscheme, so is $P|_{U''} \subset R''$. By construction the open subscheme $U'' \subset U'$ is P-invariant which means that $P|_{U''} = (s'|_P)^{-1}(U'') = (t'|_P)^{-1}(U'')$ (see discussion in Groupoids, Section 39.19) so the restrictions of s'' and t'' to $P|_{U''}$ are still finite. The sub groupoid scheme $P|_{U''}$ is still a strong splitting of R'' over u''; above we verified (a), (b) and (c) holds as $\{r' \in R' : t'(r') = u', s'(r') = u'\} = \{r'' \in R'' : t''(r'') = u', s''(r'') = u'\}$ trivially. The lemma is proved. □

0DTE Lemma 79.15.12. In Situation 79.15.3 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.

Proof. The proof of this lemma is literally the same as the proof of Lemma 79.15.11 except that "strong splitting" needs to be replaced by "splitting" (2 times) and that the reference to Lemma 79.15.8 needs to be replaced by a reference to Lemma 79.15.9. □

04S0 Lemma 79.15.13. In Situation 79.15.4 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.

Proof. The proof of this lemma is literally the same as the proof of Lemma 79.15.11 except that "strong splitting" needs to be replaced by "quasi-splitting" (2 times) and that the reference to Lemma 79.15.8 needs to be replaced by a reference to Lemma 79.15.10. □

79.16. Other chapters

Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
(93) Deformation Problems

Algebraic Stacks

(94) Algebraic Stacks
(95) Examples of Stacks
(96) Sheaves on Algebraic Stacks
(97) Criteria for Representability
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(100) Properties of Algebraic Stacks
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(103) Cohomology of Algebraic Stacks
(104) Derived Categories of Stacks
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CHAPTER 80

Bootstrap

046A
80.1. Introduction

046B In this chapter we use the material from the preceding sections to give criteria under which a presheaf of sets on the category of schemes is an algebraic space. Some of this material comes from the work of Artin, see [Art69b], [Art70], [Art73], [Art71b], [Art71a], [Art69a], [Art69c], and [Art74]. However, our method will be to use as much as possible arguments similar to those of the paper by Keel and Mori, see [KM97].

80.2. Conventions

046C The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

80.3. Morphisms representable by algebraic spaces

02YP Here we define the notion of one presheaf being relatively representable by algebraic spaces over another, and we prove some properties of this notion.

02YQ Definition 80.3.1. Let S be a scheme contained in Sch_{fppf}. Let F, G be presheaves on Sch_{fppf}/S. We say a morphism $a : F \to G$ is representable by algebraic spaces if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi : U \to G$ the fiber product $U \times_{\xi, G} F$ is an algebraic space.

Here is a sanity check.

03BN Lemma 80.3.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is representable by algebraic spaces.

Proof. This is formal. It relies on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 65.7.3. $\square$

03Y0 Lemma 80.3.3. Let S be a scheme. Let

$$\begin{tikzcd} G' \times_G F \arrow[r] \arrow[d, "a'"] & F \arrow[d, "a"] \\ G' \arrow[r] & G \end{tikzcd}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces so is a'.

Proof. Omitted. Hint: This is formal. ☐

02YR Lemma 80.3.4. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is a sheaf, then so is F.

Proof. (Same as the proof of Spaces, Lemma 65.3.5.) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma, G, a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. ☐

05LA Lemma 80.3.5. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. Then $\Delta_{F/G} : F \to F \times_G F$ is representable by algebraic spaces.

Proof. (Same as the proof of Spaces, Lemma 65.3.6.) Let U be a scheme. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist an algebraic space V and a morphism $V \to U$ representing the fibre product $U \times_{\xi', G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $U \times_{\xi', G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is an algebraic space. ☐

The proof of Lemma 80.3.6 below is actually slightly tricky. Namely, we cannot use the argument of the proof of Spaces, Lemma 65.11.3 because we do not yet know that a composition of transformations representable by algebraic spaces is representable by algebraic spaces. In fact, we will use this lemma to prove that statement.

02YS Lemma 80.3.6. Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is an algebraic space, then so is F.

Proof. We have seen in Lemma 80.3.4 that F is a sheaf.

Let U be a scheme and let $U \to G$ be a surjective étale morphism. In this case $U \times_G F$ is an algebraic space. Let W be a scheme and let $W \to U \times_G F$ be a surjective étale morphism.

First we claim that $W \to F$ is representable. To see this let X be a scheme and let $X \to F$ be a morphism. Then

$$W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X)$$

Since both $U \times_G F$ and G are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to F$ is surjective and étale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are étale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and étale, and the composition of surjective étale morphisms is surjective and étale.

Set $R = W \times_F W$. By the above R is a scheme and the projections $t, s : R \to W$ are étale. It is clear that R is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence R is an étale equivalence relation and $F = W/R$. Hence F is an algebraic space by Spaces, Theorem 65.10.5. $\square$

03XY Lemma 80.3.7. Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Suppose $a : F \to G$ is representable by algebraic spaces. If X is an algebraic space over S, and $X \to G$ is a map of presheaves then $X \times_G F$ is an algebraic space.

Proof. By Lemma 80.3.3 the transformation $X \times_G F \to X$ is representable by algebraic spaces. Hence it is an algebraic space by Lemma 80.3.6. $\square$

03Y1 Lemma 80.3.8. Let S be a scheme. Let

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a and b are representable by algebraic spaces, so is $b \circ a$.

Proof. Let T be a scheme over S, and let $T \to H$ be a morphism. By assumption $T \times_H G$ is an algebraic space. Hence by Lemma 80.3.7 we see that $T \times_H F = (T \times_H G) \times_G F$ is an algebraic space as well. $\square$

046D Lemma 80.3.9. Let S be a scheme. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Then

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable by algebraic spaces.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 80.3.8 and 80.3.3. $\square$

0AMN Lemma 80.3.10. Let S be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(Sch/S)_{fppf}^{opp} \to Sets$. Assume

(1) $\Delta : G \to G \times_H G$ is representable by algebraic spaces, and
(2) $b \circ a : F \to H$ is representable by algebraic spaces.

Then a is representable by algebraic spaces.

Proof. Let U be a scheme over S and let $\xi \in G(U)$. Then

$$U \times_{\xi, G, a} F = (U \times_{b(\xi), H, b \circ a} F) \times_{(\xi, a), (G \times_H G), \Delta} G$$

Hence the result using Lemma 80.3.7. $\square$

07WE Lemma 80.3.11. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume

(1) F is a sheaf for the Zariski topology on $(Sch/S)_{fppf}$,
(2) there exists an index set I and subfunctors $F_i \subset F$ such that
    (a) each F_i is an fppf sheaf,
    (b) each $F_i \to F$ is representable by algebraic spaces,
    (c) $\coprod F_i \to F$ becomes surjective after fppf sheafification.

Then F is an fppf sheaf.

Proof. Let $T \in \mathrm{Ob}((Sch/S)_{fppf})$ and let $s \in F(T)$. By (2)(c) there exists an fppf covering $\{T_j \to T\}$ such that $s|_{T_j}$ is a section of $F_{\alpha(j)}$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 29.25.10. By (2)(b) we see $F_{\alpha(j)} \times_{F,s|_{W_j}} W_j \to W_j$ is a monomorphism of algebraic spaces through which T_j factors. Since $\{T_j \to W_j\}$ is an fppf covering, we conclude that $F_{\alpha(j)} \times_{F,s|_{W_j}} W_j = W_j$, in other words $s|_{W_j} \in F_{\alpha(j)}(W_j)$. Hence we conclude that $\coprod F_i \to F$ is surjective for the Zariski topology.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$. Let $s, s' \in F(T)$ with $s|_{T_j} = s'|_{T_j}$ for all j. We want to show that s, s' are equal. As F is a Zariski sheaf by (1) we may work Zariski locally on T. By the result of the previous paragraph we may assume there exist i such that $s \in F_i(T)$. Then we see that $s'|_{T_j}$ is a section of F_i. By (2)(b) we see $F_i \times_{F,s'} T \to T$ is a monomorphism of algebraic spaces through which all of the T_j factor. Hence we conclude that $s' \in F_i(T)$. Since F_i is a sheaf for the fppf topology we conclude that $s = s'$.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$ and let $s_j \in F(T_j)$ such that $s_j|_{T_j \times_T T_{j'}} = s_{j'}|_{T_j \times_T T_{j'}}$. By assumption (2)(b) we may refine the covering and assume that $s_j \in F_{\alpha(j)}(T_j)$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 29.25.10. Then $\{T_j \to W_j\}$ is an fppf covering. Since $F_{\alpha(j)}$ is a sub presheaf of F we see that the two restrictions of s_j to $T_j \times_{W_j} T_j$ agree as elements of $F_{\alpha(j)}(T_j \times_{W_j} T_j)$. Hence, the sheaf condition for $F_{\alpha(j)}$ implies there exists a $s'_j \in F_{\alpha(j)}(W_j)$ whose restriction to T_j is s_j. For a pair of indices j and j' the sections $s'_j|_{W_j \cap W_{j'}}$ and $s'_{j'}|_{W_j \cap W_{j'}}$ of F agree by the result of the previous paragraph. This finishes the proof by the fact that F is a Zariski sheaf. □

80.4. Properties of maps of presheaves representable by algebraic spaces

046E Here is the definition that makes this work.

03XZ Definition 80.4.1. Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 74.10.1.

In this case we say that a has property $\mathcal{P}$ if for every scheme U and $\xi : U \to G$ the resulting morphism of algebraic spaces $U \times_G F \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

The definition above applies[1] for example to the properties of being "surjective", "quasi-compact", "étale", "smooth", "flat", "separated", "(locally) of finite type",

[1]Being preserved under base change holds by Morphisms of Spaces, Lemmas 67.5.5, 67.8.4, 67.39.4, 67.37.3, 67.30.4, 67.4.4, 67.23.3, 67.27.4, 67.28.3, 67.20.5, 67.40.3, and Spaces, Lemma 65.12.3. Being fppf local on the base holds by Descent on Spaces, Lemmas 74.11.6, 74.11.1, 74.11.28, 74.11.26, 74.11.13, 74.11.18, 74.11.11, 74.11.24, 74.11.10, 74.11.16, 74.11.19, and 74.11.17.

"(locally) quasi-finite", "(locally) of finite presentation", "affine", "proper", and "a closed immersion". In other words, a is surjective (resp. quasi-compact, étale, smooth, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion) if for every scheme T and map $\xi : T \to G$ the morphism of algebraic spaces $T \times_{\xi, G} F \to T$ is surjective (resp. quasi-compact, étale, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion).

Next, we check consistency with the already existing notions. By Lemma 80.3.2 any morphism between algebraic spaces over S is representable by algebraic spaces. And by Morphisms of Spaces, Lemma 67.5.3 (resp. 67.8.8, 67.39.2, 67.37.4, 67.30.5, 67.4.12, 67.23.4, 67.27.6, 67.28.4, 67.20.3, 67.40.2, 67.12.1) the definition of surjective (resp. quasi-compact, étale, smooth, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, affine, proper, closed immersion) above agrees with the already existing definition of morphisms of algebraic spaces.

Some formal lemmas follow.

046F Lemma 80.4.2. Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 80.4.1. Let

$$\begin{tikzcd} G' \times_G F \arrow[r] \arrow[d, "a'"] & F \arrow[d, "a"] \\ G' \arrow[r] & G \end{tikzcd}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces and has $\mathcal{P}$ so does a'.

Proof. Omitted. Hint: This is formal. $\square$

046G Lemma 80.4.3. Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 80.4.1, and assume $\mathcal{P}$ is stable under composition. Let

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a, b are representable by algebraic spaces and has $\mathcal{P}$ so does $b \circ a$.

Proof. Omitted. Hint: See Lemma 80.3.8 and use stability under composition. $\square$

046H Lemma 80.4.4. Let S be a scheme. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Let $\mathcal{P}$ be a property as in Definition 80.4.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.

Proof. Note that the lemma makes sense by Lemma 80.3.9. Proof omitted. $\square$

0AM1 Lemma 80.4.5. Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$. Let $a : F \to G$ be a transformation of functors representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 80.4.1. Suppose that for any morphism $f : X \to Y$ of algebraic spaces over S we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$, then a has property $\mathcal{P}'$.

Proof. Formal. $\square$

04S1 **Lemma 80.4.6.** Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be sheaves. Let $a : F \to G$ be representable by algebraic spaces, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is an algebraic space and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective morphism of algebraic spaces. Let $U \to T'$ be a surjective étale morphism, where U is a scheme. Then by the definition of flat morphisms of algebraic spaces the morphism of schemes $U \to T$ is flat. Similarly for "locally of finite presentation". The morphism $U \to T$ is surjective also, see Morphisms of Spaces, Lemma 67.5.3. Hence we see that $\{U \to T\}$ is an fppf covering such that $g|_U \in G(U)$ comes from an element of $F(U)$, namely the map $U \to T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 7.11.1. $\square$

80.5. Bootstrapping the diagonal

046I In this section we prove that the diagonal of a sheaf F on $(Sch/S)_{fppf}$ is representable as soon as there exists an "fppf cover" of F by a scheme or by an algebraic space, see Lemma 80.5.3.

03Y2 **Lemma 80.5.1.** Let S be a scheme. If F is a presheaf on $(Sch/S)_{fppf}$. The following are equivalent:

(1) $\Delta_F : F \to F \times F$ is representable by algebraic spaces,
(2) for every scheme T any map $T \to F$ is representable by algebraic spaces, and
(3) for every algebraic space X any map $X \to F$ is representable by algebraic spaces.

Proof. Assume (1). Let $X \to F$ be as in (3). Let T be a scheme, and let $T \to F$ be a morphism. Then we have

$$T \times_F X = (T \times_S X) \times_{F \times F, \Delta} F$$

which is an algebraic space by Lemma 80.3.7 and (1). Hence $X \to F$ is representable, i.e., (3) holds. The implication (3) $\Rightarrow$ (2) is trivial. Assume (2). Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Then

$$F \times_{\Delta_F, F \times F} T = (T \times_{a,F,b} T) \times_{T \times T, \Delta_T} T$$

which is an algebraic space by assumption. Hence Δ_F is representable by algebraic spaces, i.e., (1) holds. $\square$

In particular if F is a presheaf satisfying the equivalent conditions of the lemma, then for any morphism $X \to F$ where X is an algebraic space it makes sense to say that $X \to F$ is surjective (resp. étale, flat, locally of finite presentation) by using Definition 80.4.1.

Before we actually do the bootstrap we prove a fun lemma.

046J **Lemma 80.5.2.** Let S be a scheme. Let

$$\begin{tikzcd} E \arrow[r, "a"'] \arrow[d, "f"'] & F \arrow[d, "g"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

be a cartesian diagram of sheaves on $(Sch/S)_{fppf}$, so $E = H \times_G F$. If

(1) g is representable by algebraic spaces, surjective, flat, and locally of finite presentation, and
(2) a is representable by algebraic spaces, separated, and locally quasi-finite

then b is representable (by schemes) as well as separated and locally quasi-finite.

Proof. Let T be a scheme, and let $T \to G$ be a morphism. We have to show that $T \times_G H$ is a scheme, and that the morphism $T \times_G H \to T$ is separated and locally quasi-finite. Thus we may base change the whole diagram to T and assume that G is a scheme. In this case F is an algebraic space. Let U be a scheme, and let $U \to F$ be a surjective étale morphism. Then $U \to F$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 67.39.7 and 67.39.8. By Lemma 80.3.8 $U \to G$ is surjective, flat and locally of finite presentation also. Note that the base change $E \times_F U \to U$ of a is still separated and locally quasi-finite (by Lemma 80.4.2). Hence we may replace the upper part of the diagram of the lemma by $E \times_F U \to U$. In other words, we may assume that $F \to G$ is a surjective, flat morphism of schemes which is locally of finite presentation. In particular, $\{F \to G\}$ is an fppf covering of schemes. By Morphisms of Spaces, Proposition 67.50.2 we conclude that E is a scheme also. By Descent, Lemma 35.39.1 the fact that $E = H \times_G F$ means that we get a descent datum on E relative to the fppf covering $\{F \to G\}$. By More on Morphisms, Lemma 37.57.1 this descent datum is effective. By Descent, Lemma 35.39.1 again this implies that H is a scheme. By Descent, Lemmas 35.23.6 and 35.23.24 it now follows that b is separated and locally quasi-finite. □

Here is the result that the section title refers to.

046K Lemma 80.5.3. Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that

(1) the presheaf F is a sheaf,
(2) there exists an algebraic space X and a map $X \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation.

Then Δ_F is representable (by schemes).

Proof. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $U \to X$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 67.39.7 and 67.39.8. By Lemma 80.4.3 the composition $U \to F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation also. Thus we see that $R = U \times_F U$ is an algebraic space, see Lemma 80.3.7. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism, see Morphisms of Spaces, Lemma 67.10.2). Since $U \to F$ is locally of finite presentation, both morphisms $R \to U$ are locally of finite presentation, see Lemma 80.4.2. Hence $R \to U \times_S U$ is locally of finite type (use Morphisms of Spaces, Lemmas 67.28.5 and 67.23.6). Altogether this means that $R \to U \times_S U$ is a monomorphism which is locally of finite type, hence a separated and locally quasi-finite morphism, see Morphisms of Spaces, Lemma 67.27.10.

Now we are ready to prove that Δ_F is representable. Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Set

$$T' = (U \times_S U) \times_{F \times F} T.$$

Note that $U \times_S U \to F \times F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation by Lemma 80.4.4. Hence T' is an algebraic space, and the projection morphism $T' \to T$ is surjective, flat, and locally of finite presentation. Consider $Z = T \times_{F \times F} F$ (this is a sheaf) and

$$Z' = T' \times_{U \times_S U} R = T' \times_T Z.$$

We see that Z' is an algebraic space, and $Z' \to T'$ is separated and locally quasi-finite by the discussion in the first paragraph of the proof which showed that R is an algebraic space and that the morphism $R \to U \times_S U$ has those properties. Hence we may apply Lemma 80.5.2 to the diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & T' \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

and we conclude. $\square$

Here is a variant of the result above.

0AHV Lemma 80.5.4. Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ be a functor. Let X be a scheme and let $X \to F$ be representable by algebraic spaces and locally quasi-finite. Then $X \to F$ is representable (by schemes).

Proof. Let T be a scheme and let $T \to F$ be a morphism. We have to show that the algebraic space $X \times_F T$ is representable by a scheme. Consider the morphism

$$X \times_F T \longrightarrow X \times_{\mathrm{Spec}(\mathbf{Z})} T$$

Since $X \times_F T \to T$ is locally quasi-finite, so is the displayed arrow (Morphisms of Spaces, Lemma 67.27.8). On the other hand, the displayed arrow is a monomorphism and hence separated (Morphisms of Spaces, Lemma 67.10.3). Thus $X \times_F T$ is a scheme by Morphisms of Spaces, Proposition 67.50.2. $\square$

80.6. Bootstrap

03XV We warn the reader right away that the result of this section will be superseded by the stronger Theorem 80.10.1. On the other hand, the theorem in this section is quite a bit easier to prove and still provides quite a bit of insight into how things work, especially for those readers mainly interested in Deligne-Mumford stacks.

In Spaces, Section 65.6 we defined an algebraic space as a sheaf in the fppf topology whose diagonal is representable, and such that there exist a surjective étale morphism from a scheme towards it. In this section we show that a sheaf in the fppf topology whose diagonal is representable by algebraic spaces and which has an étale surjective covering by an algebraic space is also an algebraic space. In other words, the category of algebraic spaces is an enlargement of the category of schemes by those fppf sheaves F which have a representable diagonal and an étale covering by a scheme. The result of this section says that doing the same process again starting with the category of algebraic spaces, does not lead to yet another category.

Another motivation for the material in this section is that it will guarantee later that a Deligne-Mumford stack whose inertia stack is trivial is equivalent to an algebraic space, see Algebraic Stacks, Lemma 94.13.2.

Here is the main result of this section (as we mentioned above this will be superseded by the stronger Theorem 80.10.1).

03Y3 **Theorem 80.6.1.** Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that

(1) the presheaf F is a sheaf,
(2) the diagonal morphism $F \to F \times F$ is representable by algebraic spaces, and
(3) there exists an algebraic space X and a map $X \to F$ which is surjective, and étale.

or assume that

(a) the presheaf F is a sheaf, and
(b) there exists an algebraic space X and a map $X \to F$ which is representable by algebraic paces, surjective, and étale.

Then F is an algebraic space.

Proof. We will use the remarks directly below Definition 80.4.1 without further mention.

Assume (1), (2), and (3) and let $X \to F$ be as in (3). By Lemma 80.5.1 the morphism $X \to F$ is representable by algebraic spaces. Thus we see that (a) and (b) hold.

Assume (a) and (b) and let $X \to F$ be as in (b). Let $U \to X$ be a surjective étale morphism from a scheme towards X. By Lemma 80.3.8 the transformation $U \to F$ is representable by algebraic spaces, surjective, and étale. Hence to prove that F is an algebraic space boils down to proving that Δ_F is representable (Spaces, Definition 65.6.1). This follows immediately from Lemma 80.5.3. On the other hand we can circumvent this lemma and show directly F is an algebraic space as in the next paragraph.

Namely, let U be a scheme and let $U \to F$ be representable by algebraic spaces, surjective, and étale. Consider the fibre product $R = U \times_F U$. Both projections $R \to U$ are representable by algebraic spaces, surjective, and étale (Lemma 80.4.2). In particular R is an algebraic space by Lemma 80.3.6. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism). Since $R \to U$ is étale, we see that $R \to U$ is locally quasi-finite, see Morphisms of Spaces, Lemma 67.39.5. We conclude that also $R \to U \times_S U$ is locally quasi-finite by Morphisms of Spaces, Lemma 67.27.8. Hence Morphisms of Spaces, Proposition 67.50.2 applies and R is a scheme. By Lemma 80.4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$. We conclude that F is an algebraic space by Spaces, Theorem 65.10.5. $\square$

80.7. Finding opens

04S2 First we prove a lemma which is a slight improvement and generalization of Spaces, Lemma 65.10.2 to quotient sheaves associated to groupoids.

046M **Lemma 80.7.1.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism. Assume

(1) the composition

$$\begin{tikzcd} U' \times_{g,U,t} R \arrow[r, "\mathrm{pr}_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

has an open image $W \subset U$, and

(2) the resulting map $h : U' \times_{g,U,t} R \to W$ defines a surjection of sheaves in the fppf topology.

Let $R' = R|_{U'}$ be the restriction of R to U'. Then the map of quotient sheaves

$$U'/R' \to U/R$$

in the fppf topology is representable, and is an open immersion.

Proof. Note that W is an R-invariant open subscheme of U. This is true because the set of points of W is the set of points of U which are equivalent in the sense of Groupoids, Lemma 39.3.4 to a point of $g(U') \subset U$ (the lemma applies as $j : R \to U \times_S U$ is a pre-equivalence relation by Groupoids, Lemma 39.13.2). Also $g : U' \to U$ factors through W. Let $R|_W$ be the restriction of R to W. Then it follows that R' is also the restriction of $R|_W$ to U'. Hence we can factor the map of sheaves of the lemma as

$$U'/R' \longrightarrow W/R|_W \longrightarrow U/R$$

By Groupoids, Lemma 39.20.6 we see that the first arrow is an isomorphism of sheaves. Hence it suffices to show the lemma in case g is the immersion of an R-invariant open into U.

Assume $U' \subset U$ is an R-invariant open and g is the inclusion morphism. Set $F = U/R$ and $F' = U'/R'$. By Groupoids, Lemma 39.20.5 or 39.20.6 the map $F' \to F$ is injective. Let $\xi \in F(T)$. We have to show that $T \times_{\xi,F} F'$ is representable by an open subscheme of T. There exists an fppf covering $\{f_i : T_i \to T\}$ such that $\xi|_{T_i}$ is the image via $U \to U/R$ of a morphism $a_i : T_i \to U$. Set $V_i = a_i^{-1}(U')$. We claim that $V_i \times_T T_j = T_i \times_T V_j$ as open subschemes of $T_i \times_T T_j$.

As $a_i \circ \mathrm{pr}_0$ and $a_j \circ \mathrm{pr}_1$ are morphisms $T_i \times_T T_j \to U$ which both map to the section $\xi|_{T_i \times_T T_j} \in F(T_i \times_T T_j)$ we can find an fppf covering $\{f_{ijk} : T_{ijk} \to T_i \times_T T_j\}$ and morphisms $r_{ijk} : T_{ijk} \to R$ such that

$$a_i \circ \mathrm{pr}_0 \circ f_{ijk} = s \circ r_{ijk}, \quad a_j \circ \mathrm{pr}_1 \circ f_{ijk} = t \circ r_{ijk},$$

see Groupoids, Lemma 39.20.4. Since U' is R-invariant we have $s^{-1}(U') = t^{-1}(U')$ and hence $f_{ijk}^{-1}(V_i \times_T T_j) = f_{ijk}^{-1}(T_i \times_T V_j)$. As $\{f_{ijk}\}$ is surjective this implies the claim above. Hence by Descent, Lemma 35.13.6 there exists an open subscheme $V \subset T$ such that $f_i^{-1}(V) = V_i$. We claim that V represents $T \times_{\xi,F} F'$.

As a first step, we will show that $\xi|_V$ lies in $F'(V) \subset F(V)$. Namely, the family of morphisms $\{V_i \to V\}$ is an fppf covering, and by construction we have $\xi|_{V_i} \in F'(V_i)$. Hence by the sheaf property of F' we get $\xi|_V \in F'(V)$. Finally, let $T' \to T$ be a morphism of schemes and that $\xi|_{T'} \in F'(T')$. To finish the proof we have to show that $T' \to T$ factors through V. We can find a fppf covering $\{T'_j \to T'\}_{j \in J}$ and morphisms $b_j : T'_j \to U'$ such that $\xi|_{T'_j}$ is the image via $U' \to U/R$ of b_j. Clearly, it is enough to show that the compositions $T'_j \to T$ factor through V. Hence we may assume that $\xi|_{T'}$ is the image of a morphism $b : T' \to U'$. Now, it is enough to show that $T' \times_T T_i \to T_i$ factors through V_i. Over the scheme $T' \times_T T_i$ the restriction of ξ is the image of two elements of $(U/R)(T' \times_T T_i)$, namely $a_i \circ \mathrm{pr}_1$, and $b \circ \mathrm{pr}_0$, the second of which factors through the R-invariant open U'. Hence by Groupoids, Lemma 39.20.4 there exists a covering $\{h_k : Z_k \to T' \times_T T_i\}$ and morphisms $r_k : Z_k \to R$ such that $a_i \circ \mathrm{pr}_1 \circ h_k = s \circ r_k$ and $b \circ \mathrm{pr}_0 \circ h_k = t \circ r_k$. As U' is an R-invariant open the fact that b has image in U' then implies that each $a_i \circ \mathrm{pr}_1 \circ h_k$ has image in U'. It follows from this that $T' \times_T T_i \to T_i$ has image in V_i by definition of V_i which concludes the proof. $\square$

80.8. Slicing equivalence relations

046L In this section we explain how to "improve" a given equivalence relation by slicing. This is not a kind of "étale slicing" that you may be used to but a much coarser kind of slicing.

0489 Lemma 80.8.1. Let S be a scheme. Let $j : R \to U \times_S U$ be an equivalence relation on schemes over S. Assume $s, t : R \to U$ are flat and locally of finite presentation. Then there exists an equivalence relation $j' : R' \to U' \times_S U'$ on schemes over S, and an isomorphism

$$U'/R' \longrightarrow U/R$$

induced by a morphism $U' \to U$ which maps R' into R such that $s', t' : R \to U$ are flat, locally of finite presentation and locally quasi-finite.

Proof. We will prove this lemma in several steps. We will use without further mention that an equivalence relation gives rise to a groupoid scheme and that the restriction of an equivalence relation is an equivalence relation, see Groupoids, Lemmas 39.3.2, 39.13.3, and 39.18.3.

Step 1: We may assume that $s, t : R \to U$ are locally of finite presentation and Cohen-Macaulay morphisms. Namely, as in More on Groupoids, Lemma 40.8.1 let $g : U' \to U$ be the open subscheme such that $t^{-1}(U') \subset R$ is the maximal open over which $s : R \to U$ is Cohen-Macaulay, and denote R' the restriction of R to U'. By the lemma cited above we see that

$$\begin{tikzcd} t^{-1}(U') \arrow[r, equal] & U' \times_{g,U,t} R \arrow[r, "\mathrm{pr}_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

is surjective. Since h is flat and locally of finite presentation, we see that $\{h\}$ is a fppf covering. Hence by Groupoids, Lemma 39.20.6 we see that $U'/R' \to U/R$ is an isomorphism. By the construction of U' we see that s', t' are Cohen-Macaulay and locally of finite presentation.

Step 2. Assume s, t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 40.12.4 there exists an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) h is flat, locally of finite presentation and locally quasi-finite, and
(5) the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.

Here we have used the notation introduced in More on Groupoids, Situation 40.12.1.

Step 3. For each point $u \in U$ which is of finite type choose a $g_u : U'_u \to U$ as in Step 2 and denote R'_u the restriction of R to U'_u. Denote $h_u = s \circ \text{pr}_1 : U'_u \times_{g_u, U, t} R \to U$. Set $U' = \coprod_{u \in U} U'_u$, and $g = \coprod g_u$. Let R' be the restriction of R to U' as above. We claim that the pair (U', g) works[2]. Note that

$$\begin{aligned} R' &= \coprod\nolimits_{u_1, u_2 \in U} (U'_{u_1} \times_{g_{u_1}, U, t} R) \times_R (R \times_{s, U, g_{u_2}} U'_{u_2}) \\ &= \coprod\nolimits_{u_1, u_2 \in U} (U'_{u_1} \times_{g_{u_1}, U, t} R) \times_{h_{u_1}, U, g_{u_2}} U'_{u_2} \end{aligned}$$

Hence the projection $s' : R' \to U' = \coprod U'_{u_2}$ is flat, locally of finite presentation and locally quasi-finite as a base change of $\coprod h_{u_1}$. Finally, by construction the morphism $h : U' \times_{g, U, t} R \to U$ is equal to $\coprod h_u$ hence its image contains all points of finite type of U. Since each h_u is flat and locally of finite presentation we conclude that h is flat and locally of finite presentation. In particular, the image of h is open (see Morphisms, Lemma 29.25.10) and since the set of points of finite type is dense (see Morphisms, Lemma 29.16.7) we conclude that the image of h is U. This implies that $\{h\}$ is an fppf covering. By Groupoids, Lemma 39.20.6 this means that $U'/R' \to U/R$ is an isomorphism. This finishes the proof of the lemma. $\square$

80.9. Quotient by a subgroupoid

04S3 We need one more lemma before we can do our final bootstrap. Let us discuss what is going on in terms of "plain" groupoids before embarking on the scheme theoretic version.

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 39.13 this corresponds to a quintuple $(\text{Ob}, \text{Arrows}, s, t, c)$. Suppose we are given a subset $P \subset \text{Arrows}$ such that $(\text{Ob}, P, s|_P, t|_P, c|_P)$ is also a groupoid and such that there are no nontrivial automorphisms in P. Then we can construct the quotient groupoid $(\overline{\text{Ob}}, \overline{\text{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ as follows:

(1) $\overline{\text{Ob}} = \text{Ob}/P$ is the set of P-isomorphism classes,
(2) $\overline{\text{Arrows}} = P\backslash\text{Arrows}/P$ is the set of arrows in $\mathcal{C}$ up to pre-composing and post-composing by arrows of P,
(3) the source and target maps $\overline{s}, \overline{t} : P\backslash\text{Arrows}/P \to \text{Ob}/P$ are induced by s, t,
(4) composition is defined by the rule $\overline{c}(\overline{a}, \overline{b}) = \overline{c(a, b)}$ which is well defined.

In fact, it turns out that the original groupoid $(\text{Ob}, \text{Arrows}, s, t, c)$ is canonically isomorphic to the restriction (see discussion in Groupoids, Section 39.18) of the groupoid $(\overline{\text{Ob}}, \overline{\text{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ via the quotient map $g : \text{Ob} \to \overline{\text{Ob}}$. Recall that this means that

$$\text{Arrows} = \text{Ob} \times_{g, \overline{\text{Ob}}, \overline{t}} \overline{\text{Arrows}} \times_{\overline{s}, \overline{\text{Ob}}, g} \text{Ob}$$

which holds as P has no nontrivial automorphisms. We omit the details.

[2]Here we should check that U' is not too large, i.e., that it is isomorphic to an object of the category Sch_{fppf}, see Section 80.2. This is a purely set theoretical matter; let us use the notion of size of a scheme introduced in Sets, Section 3.9. Note that each U'_u has size at most the size of U and that the cardinality of the index set is at most the cardinality of $|U|$ which is bounded by the size of U. Hence U' is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9 part (6).

The following lemma holds in much greater generality, but this is the version we use in the proof of the final bootstrap (after which we can more easily prove the more general versions of this lemma).

04S4 **Lemma 80.9.1.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $P \to R$ be monomorphism of schemes. Assume that

(1) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid scheme,
(2) $s|_P, t|_P : P \to U$ are finite locally free,
(3) $j|_P : P \to U \times_S U$ is a monomorphism.
(4) U is affine, and
(5) $j : R \to U \times_S U$ is separated and locally quasi-finite,

Then U/P is representable by an affine scheme $\overline{U}$, the quotient morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. Moreover, R is the restriction of a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ on $\overline{U}$ via the quotient morphism $U \to \overline{U}$.

Proof. Conditions (1), (2), (3), and (4) and Groupoids, Proposition 39.23.9 imply the affine scheme $\overline{U}$ representing U/P exists, the morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. The identification $P = U \times_{\overline{U}} U$ is such that $t|_P = \mathrm{pr}_0$ and $s|_P = \mathrm{pr}_1$, and such that composition is equal to $\mathrm{pr}_{02} : U \times_{\overline{U}} U \times_{\overline{U}} U \to U \times_{\overline{U}} U$. A product of finite locally free morphisms is finite locally free (see Spaces, Lemma 65.5.7 and Morphisms, Lemmas 29.48.4 and 29.48.3). To get $\overline{R}$ we are going to descend the scheme R via the finite locally free morphism $U \times_S U \to \overline{U} \times_S \overline{U}$. Namely, note that

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = P \times_S P$$

by the above. Thus giving a descent datum (see Descent, Definition 35.34.1) for $R/U \times_S U/\overline{U} \times_S \overline{U}$ consists of an isomorphism

$$\varphi : R \times_{(U \times_S U), t \times t} (P \times_S P) \longrightarrow (P \times_S P) \times_{s \times s, (U \times_S U)} R$$

over $P \times_S P$ satisfying a cocycle condition. We define φ on T-valued points by the rule

$$\varphi : (r, (p, p')) \longmapsto ((p, p'), p^{-1} \circ r \circ p')$$

where the composition is taken in the groupoid category $(U(T), R(T), s, t, c)$. This makes sense because for $(r, (p, p'))$ to be a T-valued point of the source of φ it needs to be the case that $t(r) = t(p)$ and $s(r) = t(p')$. Note that this map is an isomorphism with inverse given by $((p, p'), r') \mapsto (p \circ r' \circ (p')^{-1}, (p, p'))$. To check the cocycle condition we have to verify that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$ as maps over

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = (P \times_S P) \times_{s \times s, (U \times_S U), t \times t} (P \times_S P)$$

By explicit calculation we see that

$$\begin{array}{llll} \varphi_{02} & (r, (p_1, p'_1), (p_2, p'_2)) & \mapsto & ((p_1, p'_1), (p_2, p'_2), (p_1 \circ p_2)^{-1} \circ r \circ (p'_1 \circ p'_2)) \\ \varphi_{01} & (r, (p_1, p'_1), (p_2, p'_2)) & \mapsto & ((p_1, p'_1), p_1^{-1} \circ r \circ p'_1, (p_2, p'_2)) \\ \varphi_{12} & ((p_1, p'_1), r, (p_2, p'_2)) & \mapsto & ((p_1, p'_1), (p_2, p'_2), p_2^{-1} \circ r \circ p'_2) \end{array}$$

(with obvious notation) which implies what we want. As j is separated and locally quasi-finite by (5) we may apply More on Morphisms, Lemma 37.57.1 to get a scheme $\overline{R} \to \overline{U} \times_S \overline{U}$ and an isomorphism

$$R \to \overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$$

which identifies the descent datum φ with the canonical descent datum on $\overline{R}\times_{(\overline{U}\times_S\overline{U})}(U \times_S U)$, see Descent, Definition 35.34.10.

Since $U \times_S U \to \overline{U} \times_S \overline{U}$ is finite locally free we conclude that $R \to \overline{R}$ is finite locally free as a base change. Hence $R \to \overline{R}$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$. Our choice of φ implies that given T-valued points $r, r' \in R(T)$ these have the same image in $\overline{R}$ if and only if $p^{-1} \circ r \circ p'$ for some $p, p' \in P(T)$. Thus $\overline{R}$ represents the sheaf

$$T \longmapsto \overline{R(T)} = P(T)\backslash R(T)/P(T)$$

with notation as in the discussion preceding the lemma. Hence we can define the groupoid structure on $(\overline{U} = U/P, \overline{R} = P\backslash R/P)$ exactly as in the discussion of the "plain" groupoid case. It follows from this that (U, R, s, t, c) is the pullback of this groupoid structure via the morphism $U \to \overline{U}$. This concludes the proof. $\square$

80.10. Final bootstrap

04S5 The following result goes quite a bit beyond the earlier results.

04S6 Theorem 80.10.1. Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Any one of the following conditions implies that F is an algebraic space:

(1) $F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,
(2) $F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,
(3) F is a sheaf and there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation,
(4) F is a sheaf and there exists a scheme U and a morphism $U \to F$ which is representable by algebraic spaces or schemes, surjective, flat and locally of finite presentation,
(5) F is a sheaf, Δ_F is representable by algebraic spaces, and there exists an algebraic space U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation, or
(6) F is a sheaf, Δ_F is representable, and there exists a scheme U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation.

Proof. Trivial observations: (6) is a special case of (5) and (4) is a special case of (3). We first prove that cases (5) and (3) reduce to case (1). Namely, by bootstrapping the diagonal Lemma 80.5.3 we see that (3) implies (5). In case (5) we set $R = U \times_F U$ which is an algebraic space by assumption. Moreover, by assumption both projections $s, t : R \to U$ are surjective, flat and locally of finite presentation. The map $j : R \to U \times_S U$ is clearly an equivalence relation. By Lemma 80.4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$ which reduces us to case (1).

Next, we show that (1) reduces to (2). Namely, let (U, R, s, t, c) be a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Choose a scheme U' and a surjective étale morphism $U' \to U$. Let $R' = R|_{U'}$ be the restriction of R

to U'. By Groupoids in Spaces, Lemma 78.19.6 we see that $U/R = U'/R'$. Since $s', t' : R' \to U'$ are also flat and locally of finite presentation (see More on Groupoids in Spaces, Lemma 79.8.1) this reduces us to the case where U is a scheme. As j is an equivalence relation we see that j is a monomorphism. As $s : R \to U$ is locally of finite presentation we see that $j : R \to U \times_S U$ is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. By Morphisms of Spaces, Lemma 67.27.10 we see that j is locally quasi-finite and separated. Hence if U is a scheme, then R is a scheme by Morphisms of Spaces, Proposition 67.50.2. Thus we reduce to proving the theorem in case (2).

Assume $F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. By Lemma 80.8.1 we reduce to that case where s, t are flat, locally of finite presentation, and locally quasi-finite. Let $U = \bigcup_{i \in I} U_i$ be an affine open covering (with index set I of cardinality $\leq$ than the size of U to avoid set theoretic problems later – most readers can safely ignore this remark). Let $(U_i, R_i, s_i, t_i, c_i)$ be the restriction of R to U_i. It is clear that s_i, t_i are still flat, locally of finite presentation, and locally quasi-finite as R_i is the open subscheme $s^{-1}(U_i) \cap t^{-1}(U_i)$ of R and s_i, t_i are the restrictions of s, t to this open. By Lemma 80.7.1 (or the simpler Spaces, Lemma 65.10.2) the map $U_i/R_i \to U/R$ is representable by open immersions. Hence if we can show that $F_i = U_i/R_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 65.8.4. As $U = \bigcup U_i$ is an open covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 65.8.5. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite and j is an equivalence.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, such that s, t are flat, locally of finite presentation, and locally quasi-finite, and j is an equivalence relation. Choose $u \in U$. We apply More on Groupoids in Spaces, Lemma 79.15.13 to $u \in U, R, s, t, c$. We obtain an affine scheme U', an étale morphism $g : U' \to U$, a point $u' \in U'$ with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ is quasi-split over u'. Note that the image $g(U')$ is open as g is étale and contains u. Hence, repeatedly applying the lemma, we can find finitely many points $u_i \in U$, $i = 1, \ldots, n$, affine schemes U_i', étale morphisms $g_i : U_i' \to U$, points $u_i' \in U_i'$ with $g(u_i') = u_i$ such that (a) each restriction R_i' is quasi-split over some point in U_i' and (b) $U = \bigcup_{i = 1, \ldots, n} g_i(U_i')$. Now we rerun the last part of the argument in the preceding paragraph: Using Lemma 80.7.1 (or the simpler Spaces, Lemma 65.10.2) the map $U_i'/R_i' \to U/R$ is representable by open immersions. If we can show that $F_i = U_i'/R_i'$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 65.8.4. As $\{g_i : U_i' \to U\}$ is an étale covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 65.8.5. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite, j is an equivalence, and R is quasi-split over u for some $u \in U$.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and R is quasi-split over u. Let $P \subset R$ be a quasi-splitting of R over u. By Lemma 80.9.1 we see that (U, R, s, t, c) is the restriction of a groupoid $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ by a surjective finite locally free morphism $U \to \overline{U}$ such that $P = U \times_{\overline{U}} U$. Note that s admits a factorization

$$R = U \times_{\overline{U}, \overline{t}} \overline{R} \times_{\overline{s}, \overline{U}} U \xrightarrow{\text{pr}_{23}} \overline{R} \times_{\overline{s}, \overline{U}} U \xrightarrow{\text{pr}_2} U$$

The map pr_2 is the base change of $\overline{s}$, and the map pr_{23} is a base change of the surjective finite locally free map $U \to \overline{U}$. Since s is flat, locally of finite presentation, and locally quasi-finite and since pr_{23} is surjective finite locally free (as a base change of such), we conclude that pr_2 is flat, locally of finite presentation, and locally quasi-finite by Descent, Lemmas 35.27.1 and 35.28.1 and Morphisms, Lemma 29.20.18. Since pr_2 is the base change of the morphism $\overline{s}$ by $U \to \overline{U}$ and $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}$ is flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 35.23.15, 35.23.11, and 35.23.24. The same goes for $\overline{t}$. Consider the commutative diagram

$$\begin{tikzcd} U \times_{\overline{U}} U \arrow[r, equal] \arrow[dr] & P \arrow[r] \arrow[d] & R \arrow[d] \\ & \overline{U} \arrow[r, "\overline{e}"] & \overline{R} \end{tikzcd}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R (by definition of a quasi-splitting) we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 35.23.16. An application of Groupoids, Lemma 39.20.5 shows that $U/R = \overline{U}/\overline{R}$. Hence we have reduced to the case where (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and $e : U \to R$ is an open immersion!

But of course, if e is an open immersion and s, t are flat and locally of finite presentation then the morphisms t, s are étale. For example you can see this by applying More on Groupoids, Lemma 40.4.1 which shows that $\Omega_{R/U} = 0$ which in turn implies that $s, t : R \to U$ is G-unramified (see Morphisms, Lemma 29.35.2), which in turn implies that s, t are étale (see Morphisms, Lemma 29.36.16). And if s, t are étale then finally U/R is an algebraic space by Spaces, Theorem 65.10.5. $\square$

80.11. Applications

04SJ As a first application we obtain the following fundamental fact:

$\boxed{\text{A sheaf which is fppf locally an algebraic space is an algebraic space.}}$

This is the content of the following lemma. Note that assumption (2) is equivalent to the condition that $F|_{(Sch/S_i)_{fppf}}$ is an algebraic space, see Spaces, Lemma 65.16.4. Assumption (3) is a set theoretic condition which may be ignored by those not worried about set theoretic questions.

04SK Lemma 80.11.1. Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that

(1) F is a sheaf,
(2) each $F_i = h_{S_i} \times F$ is an algebraic space, and
(3) $\coprod_{i \in I} F_i$ is an algebraic space (see Spaces, Lemma 65.8.4).

Then F is an algebraic space.

Proof. Consider the morphism $\coprod F_i \to F$. This is the base change of $\coprod S_i \to S$ via $F \to S$. Hence it is representable, locally of finite presentation, flat and surjective by our definition of an fppf covering and Lemma 80.4.2. Thus Theorem 80.10.1 applies to show that F is an algebraic space. $\square$

Here is a special case of Lemma 80.11.1 where we do not need to worry about set theoretical issues.

04U0 **Lemma 80.11.2.** Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that

(1) F is a sheaf,
(2) each $F_i = h_{S_i} \times F$ is an algebraic space, and
(3) the morphisms $F_i \to S_i$ are of finite type.

Then F is an algebraic space.

Proof. We will use Lemma 80.11.1 above. To do this we will show that the assumption that F_i is of finite type over S_i to prove that the set theoretic condition in the lemma is satisfied (after perhaps refining the given covering of S a bit). We suggest the reader skip the rest of the proof.

If $S'_i \to S_i$ is a morphism of schemes then

$$h_{S'_i} \times F = h_{S'_i} \times_{h_{S_i}} h_{S_i} \times F = h_{S'_i} \times_{h_{S_i}} F_i$$

is an algebraic space of finite type over S'_i, see Spaces, Lemma 65.7.3 and Morphisms of Spaces, Lemma 67.23.3. Thus we may refine the given covering. After doing this we may assume: (a) each S_i is affine, and (b) the cardinality of I is at most the cardinality of the set of points of S. (Since to cover all of S it is enough that each point is in the image of $S_i \to S$ for some i.)

Since each S_i is affine and each F_i of finite type over S_i we conclude that F_i is quasi-compact. Hence by Properties of Spaces, Lemma 66.6.3 we can find an affine $U_i \in \text{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U_i \to F_i$. The fact that $F_i \to S_i$ is locally of finite type then implies that $U_i \to S_i$ is locally of finite type, and in particular $U_i \to S$ is locally of finite type. By Sets, Lemma 3.9.7 we conclude that $\text{size}(U_i) \leq \text{size}(S)$. Since also $|I| \leq \text{size}(S)$ we conclude that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 3.9.5 and the construction of Sch. This implies that $\coprod F_i$ is an algebraic space by Spaces, Lemma 65.8.4 and we win. $\square$

As a second application we obtain

$$\boxed{\text{Any fppf descent datum for algebraic spaces is effective.}}$$

This holds modulo set theoretical difficulties; as an example result we offer the following lemma.

0ADV **Lemma 80.11.3.** Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S.

(1) If I is countable[3], then any descent datum for algebraic spaces relative to $\{X_i \to X\}$ is effective.

[3] The restriction on countablility can be ignored by those who do not care about set theoretical issues. We can allow larger index sets here if we can bound the size of the algebraic spaces which we are descending. See for example Lemma 80.11.2.

(2) Any descent datum (Y_i, φ_{ij}) relative to $\{X_i \to X\}_{i \in I}$ (Descent on Spaces, Definition 74.22.3) with $Y_i \to X_i$ of finite type is effective.

Proof. Proof of (1). By Descent on Spaces, Lemma 74.23.1 this translates into the statement that an fppf sheaf F endowed with a map $F \to X$ is an algebraic space provided that each $F \times_X X_i$ is an algebraic space. The restriction on the cardinality of I implies that coproducts of algebraic spaces indexed by I are algebraic spaces, see Spaces, Lemma 65.8.4 and Sets, Lemma 3.9.9. The morphism

$$\coprod F \times_X X_i \longrightarrow F$$

is representable by algebraic spaces (as the base change of $\coprod X_i \to X$, see Lemma 80.3.3), and surjective, flat, and locally of finite presentation (as the base change of $\coprod X_i \to X$, see Lemma 80.4.2). Hence part (1) follows from Theorem 80.10.1.

Proof of (2). First we apply Descent on Spaces, Lemma 74.23.1 to obtain an fppf sheaf F endowed with a map $F \to X$ such that $F \times_X X_i = Y_i$ for all $i \in I$. Our goal is to show that F is an algebraic space. Choose a scheme U and a surjective étale morphism $U \to X$. Then $F' = U \times_X F \to F$ is representable, surjective, and étale as the base change of $U \to X$. By Theorem 80.10.1 it suffices to show that $F' = U \times_X F$ is an algebraic space. We may choose an fppf covering $\{U_j \to U\}_{j \in J}$ where U_j is a scheme refining the fppf covering $\{X_i \times_X U \to U\}_{i \in I}$, see Topologies on Spaces, Lemma 73.7.4. Thus we get a map $a : J \to I$ and for each j a morphism $U_j \to X_{a(j)}$ over X. Then we see that $U_j \times_U F' = U_j \times_{X_{a(j)}} Y_{a(j)}$ is of finite type over U_j. Hence F' is an algebraic space by Lemma 80.11.2. □

Here is a different type of application.

0AMP Lemma 80.11.4. Let S be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(Sch/S)_{fppf}^{opp} \to Sets$. Assume

(1) F, G, H are sheaves,
(2) $a : F \to G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective, and
(3) $b \circ a : F \to H$ is representable by algebraic spaces.

Then b is representable by algebraic spaces.

Proof. Let U be a scheme over S and let $\xi \in H(U)$. We have to show that $U \times_{\xi, H} G$ is an algebraic space. On the other hand, we know that $U \times_{\xi, H} F$ is an algebraic space and that $U \times_{\xi, H} F \to U \times_{\xi, H} G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective as a base change of the morphism a (see Lemma 80.4.2). Thus the result follows from Theorem 80.10.1. □

04TB Lemma 80.11.5. Assume $B \to S$ and (U, R, s, t, c) are as in Groupoids in Spaces, Definition 78.20.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ is an algebraic space.

Proof. By Groupoids in Spaces, Lemma 78.22.3 there exists an fppf covering $\{T_i \to T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is an algebraic space for each i. By Spaces, Lemma 65.16.4 this means that each $F_i = h_{S_i} \times Isom(x, y)$ is an algebraic space. Thus to prove the lemma we only have to verify the set theoretic condition that $\coprod F_i$ is an algebraic space of Lemma 80.11.1 above to conclude. To do this we use Spaces, Lemma 65.8.4 which requires showing that I and the F_i are not "too large". We suggest the reader skip the rest of the proof.

Choose $U' \in \text{Ob}(Sch/S)_{fppf}$ and a surjective étale morphism $U' \to U$. Let R' be the restriction of R to U'. Since $[U/R] = [U'/R']$ we may, after replacing U by U', assume that U is a scheme. (This step is here so that the fibre products below are over a scheme.)

Note that if we refine the covering $\{T_i \to T\}$ then it remains true that each F_i is an algebraic space. Hence we may assume that each T_i is affine. Since $T_i \to T$ is locally of finite presentation, this then implies that $\text{size}(T_i) \leq \text{size}(T)$, see Sets, Lemma 3.9.7. We may also assume that the cardinality of the index set I is at most the cardinality of the set of points of T since to get a covering it suffices to check that each point of T is in the image. Hence $|I| \leq \text{size}(T)$. Choose $W \in \text{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to R$. Note that in the proof of Groupoids in Spaces, Lemma 78.22.3 we showed that F_i is representable by $T_i \times_{(y_i, x_i), U \times_B U} R$ for some $x_i, y_i : T_i \to U$. Hence now we see that $V_i = T_i \times_{(y_i, x_i), U \times_B U} W$ is a scheme which comes with an étale surjection $V_i \to F_i$. By Sets, Lemma 3.9.6 we see that

$$\text{size}(V_i) \leq \max\{\text{size}(T_i), \text{size}(W)\} \leq \max\{\text{size}(T), \text{size}(W)\}$$

Hence, by Sets, Lemma 3.9.5 we conclude that

$$\text{size}(\coprod\nolimits_{i \in I} V_i) \leq \max\{|I|, \text{size}(T), \text{size}(W)\}.$$

Hence we conclude by our construction of Sch that $\coprod_{i \in I} V_i$ is isomorphic to an object V of $(Sch/S)_{fppf}$. This verifies the hypothesis of Spaces, Lemma 65.8.4 and we win. $\square$

06PG Lemma 80.11.6. Let S be a scheme. Consider an algebraic space F of the form $F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Then $U \to F$ is surjective, flat, and locally of finite presentation.

Proof. This is almost but not quite a triviality. Namely, by Groupoids in Spaces, Lemma 78.19.5 and the fact that j is a monomorphism we see that $R = U \times_F U$. Choose a scheme W and a surjective étale morphism $W \to F$. As $U \to F$ is a surjection of sheaves we can find an fppf covering $\{W_i \to W\}$ and maps $W_i \to U$ lifting the morphisms $W_i \to F$. Then we see that

$$W_i \times_F U = W_i \times_U U \times_F U = W_i \times_{U,t} R$$

and the projection $W_i \times_F U \to W_i$ is the base change of $t : R \to U$ hence flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 67.30.4 and 67.28.3. Hence by Descent on Spaces, Lemmas 74.11.13 and 74.11.10 we see that $U \to F$ is flat and locally of finite presentation. It is surjective by Spaces, Remark 65.5.2. $\square$

06PH Lemma 80.11.7. Let S be a scheme. Let $X \to B$ be a morphism of algebraic spaces over S. Let G be a group algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. If

(1) a is a free action, and
(2) $G \to B$ is flat and locally of finite presentation,

then X/G (see Groupoids in Spaces, Definition 78.19.1) is an algebraic space, the morphism $X \to X/G$ is surjective, flat, and locally of finite presentation, and X is an fppf G-torsor over X/G.

Proof. The fact that X/G is an algebraic space is immediate from Theorem 80.10.1 and the definitions. Namely, $X/G = X/R$ where $R = G \times_B X$. The morphisms $s, t : G \times_B X \to X$ are flat and locally of finite presentation (clear for s as a base change of $G \to B$ and by symmetry using the inverse it follows for t) and the morphism $j : G \times_B X \to X \times_B X$ is a monomorphism by Groupoids in Spaces, Lemma 78.8.3 as the action is free. The morphism $X \to X/G$ is surjective, flat, and locally of finite presentation by Lemma 80.11.6. To see that $X \to X/G$ is an fppf G-torsor (Groupoids in Spaces, Definition 78.9.3) we have to show that $G \times_S X \to X \times_{X/G} X$ is an isomorphism and that $X \to X/G$ fppf locally has sections. The second part is clear from the properties of $X \to X/G$ already shown. The map $G \times_S X \to X \times_{X/G} X$ is injective (as a map of fppf sheaves) as the action is free. Finally, the map is also surjective as a map of sheaves by Groupoids in Spaces, Lemma 78.19.5. This finishes the proof. □

04U1 Lemma 80.11.8. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Let G be a group algebraic space over S, and denote $G_i = G_{S_i}$ the base changes. Suppose given

(1) for each $i \in I$ an fppf G_i-torsor X_i over S_i, and
(2) for each $i, j \in I$ a $G_{S_i \times_S S_j}$-equivariant isomorphism $\varphi_{ij} : X_i \times_S S_j \to S_i \times_S X_j$ satisfying the cocycle condition over every $S_i \times_S S_j \times_S S_j$.

Then there exists an fppf G-torsor X over S whose base change to S_i is isomorphic to X_i such that we recover the descent datum φ_{ij}.

Proof. We may think of X_i as a sheaf on $(Sch/S_i)_{fppf}$, see Spaces, Section 65.16. By Sites, Section 7.26 the descent datum (X_i, φ_{ij}) is effective in the sense that there exists a unique sheaf X on $(Sch/S)_{fppf}$ which recovers the algebraic spaces X_i after restricting back to $(Sch/S_i)_{fppf}$. Hence we see that $X_i = h_{S_i} \times X$. By Lemma 80.11.1 we see that X is an algebraic space, modulo verifying that $\coprod X_i$ is an algebraic space which we do at the end of the proof. By the equivalence of categories in Sites, Lemma 7.26.5 the action maps $G_i \times_{S_i} X_i \to X_i$ glue to give a map $a : G \times_S X \to X$. Now we have to show that a is an action and that X is a pseudo-torsor, and fppf locally trivial (see Groupoids in Spaces, Definition 78.9.3). These may be checked fppf locally, and hence follow from the corresponding properties of the actions $G_i \times_{S_i} X_i \to X_i$. Hence the lemma is true.

We suggest the reader skip the rest of the proof, which is purely set theoretical. Pick coverings $\{S_{ij} \to S_j\}_{j \in J_i}$ of $(Sch/S)_{fppf}$ which trivialize the G_i torsors X_i (possible by assumption, and Topologies, Lemma 34.7.7 part (1)). Then $\{S_{ij} \to S\}_{i \in I, j \in J_i}$ is a covering of $(Sch/S)_{fppf}$ and hence we may assume that each X_i is the trivial torsor! Of course we may also refine the covering further, hence we may assume that each S_i is affine and that the index set I has cardinality bounded by the cardinality of the set of points of S. Choose $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to G$. Then we see that $U_i = U \times_S S_i$ comes with an étale surjective morphism to $X_i \cong G_i$. By Sets, Lemma 3.9.6 we see $\text{size}(U_i) \leq \max\{\text{size}(U), \text{size}(S_i)\}$. By Sets, Lemma 3.9.7 we have $\text{size}(S_i) \leq \text{size}(S)$. Hence we see that $\text{size}(U_i) \leq \max\{\text{size}(U), \text{size}(S)\}$ for all $i \in I$. Together with the bound on $|I|$ we found above we conclude from Sets, Lemma 3.9.5 that $\text{size}(\coprod U_i) \leq \max\{\text{size}(U), \text{size}(S)\}$. Hence Spaces, Lemma 65.8.4 applies to show that $\coprod X_i$ is an algebraic space which is what we had to prove. □

80.12. Algebraic spaces in the étale topology

076L Let S be a scheme. Instead of working with sheaves over the big fppf site $(Sch/S)_{fppf}$ we could work with sheaves over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Spaces, Sections 65.3 and 65.5 makes sense for sheaves over $(Sch/S)_{étale}$. Thus we get a second notion of algebraic spaces by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Spaces, Definition 65.6.1 since a sheaf in the fppf topology is certainly a sheaf in the étale topology. However, the notions are equivalent as is shown by the following lemma.

076M **Lemma 80.12.1.** Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let

$$F : (Sch_\alpha/S)^{opp} \longrightarrow \text{Sets}$$

be a presheaf with the following properties:

(1) F is a sheaf for the étale topology,
(2) the diagonal $\Delta : F \to F \times F$ is representable, and
(3) there exists $U \in \text{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and étale.

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Spaces, Definition 65.6.1 are independent of the topology. This is true because these properties involve only the notion of a fibre product of presheaves, maps of presheaves, the notion of a representable transformation of functors, and what it means for such a transformation to be surjective and étale. Thus all we have to prove is that an étale sheaf F with properties (2) and (3) is also an fppf sheaf.

To do this, let $R = U \times_F U$. By (2) the presheaf R is representable by a scheme and by (3) the projections $R \to U$ are étale. Thus $j : R \to U \times_S U$ is an étale equivalence relation. Moreover $U \to F$ identifies F as the quotient of U by R for the étale topology: (a) if $T \to F$ is a morphism, then $\{T \times_F U \to T\}$ is an étale covering, hence $U \to F$ is a surjection of sheaves for the étale topology, (b) if $a, b : T \to U$ map to the same section of F, then $(a, b) : T \to R$ hence a and b have the same image in the quotient of U by R for the étale topology. Next, let U/R denote the quotient sheaf in the fppf topology which is an algebraic space by Spaces, Theorem 65.10.5. Thus we have morphisms (transformations of functors)

$$U \to F \to U/R.$$

By the aforementioned Spaces, Theorem 65.10.5 the composition is representable, surjective, and étale. Hence for any scheme T and morphism $T \to U/R$ the fibre product $V = T \times_{U/R} U$ is a scheme surjective and étale over T. In other words, $\{V \to U\}$ is an étale covering. This proves that $U \to U/R$ is surjective as a map of sheaves in the étale topology. It follows that $F \to U/R$ is surjective as a map of sheaves in the étale topology. On the other hand, the map $F \to U/R$ is injective (as a map of presheaves) since $R = U \times_{U/R} U$ again by Spaces, Theorem 65.10.5. It follows that $F \to U/R$ is an isomorphism of étale sheaves, see Sites, Lemma 7.11.2 which concludes the proof. $\square$

There is also an analogue of Spaces, Lemma 65.11.1.

0BH4 **Lemma 80.12.2.** Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let

$$F : (Sch_\alpha/S)^{opp} \longrightarrow \text{Sets}$$

be a presheaf with the following properties:

(1) F is a sheaf for the étale topology,
(2) there exists an algebraic space U over S and a map $U \to F$ which is representable by algebraic spaces, surjective, and étale.

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. Set $R = U \times_F U$. This is an algebraic space as $U \to F$ is assumed representable by algebraic spaces. The projections $s, t : R \to U$ are étale morphisms of algebraic spaces as $U \to F$ is assumed étale. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 80.10.1 the fppf quotient sheaf $F' = U/R$ is an algebraic space. The morphism $U \to F'$ is surjective, flat, and locally of finite presentation by Lemma 80.11.6. The map $R \to U \times_{F'} U$ is surjective as a map of fppf sheaves by Groupoids in Spaces, Lemma 78.19.5 and since j is a monomorphism it is an isomorphism. Hence the base change of $U \to F'$ by $U \to F'$ is étale, and we conclude that $U \to F'$ is étale by Descent on Spaces, Lemma 74.11.28. Thus $U \to F'$ is surjective as a map of étale sheaves. This means that F' is equal to the quotient sheaf U/R in the étale topology (small check omitted). Hence we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of étale sheaves and hence so is $F' \to F$. This means that $F' = F$ and the proof is complete. □

In fact, it suffices to have a smooth cover by a scheme and it suffices to assume the diagonal is representable by algebraic spaces.

07WF **Lemma 80.12.3.** Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α.

$$F : (Sch_\alpha/S)^{opp} \longrightarrow \text{Sets}$$

be a presheaf with the following properties:

(1) F is a sheaf for the étale topology,
(2) the diagonal $\Delta : F \to F \times F$ is representable by algebraic spaces, and
(3) there exists $U \in \text{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and smooth.

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. The proof mirrors the proof of Lemma 80.12.1. Let $R = U \times_F U$. By (2) the presheaf R is an algebraic space and by (3) the projections $R \to U$ are smooth and surjective. Denote (U, R, s, t, c) the groupoid associated to the equivalence relation $j : R \to U \times_S U$ (see Groupoids in Spaces, Lemma 78.11.3). By Theorem 80.10.1 we see that $X = U/R$ (quotient in the fppf-topology) is an algebraic space. Using that the smooth topology and the étale topology have the same sheaves (by More on Morphisms, Lemma 37.38.7) we see the map $U \to F$ identifies F as the quotient of U by R for the smooth topology (details omitted). Thus we have morphisms (transformations of functors)

$$U \to F \to X.$$

By Lemma 80.11.6 we see that $U \to X$ is surjective, flat and locally of finite presentation. By Groupoids in Spaces, Lemma 78.19.5 (and the fact that j is a monomorphism) we have $R = U \times_X U$. By Descent on Spaces, Lemma 74.11.26 we conclude that $U \to X$ is smooth and surjective (as the projections $R \to U$ are smooth and surjective and $\{U \to X\}$ is an fppf covering). Hence for any scheme T and morphism $T \to X$ the fibre product $T \times_X U$ is an algebraic space surjective and smooth over T. Choose a scheme V and a surjective étale morphism $V \to T \times_X U$. Then $\{V \to T\}$ is a smooth covering such that $V \to T \to X$ lifts to a morphism $V \to U$. This proves that $U \to X$ is surjective as a map of sheaves in the smooth topology. It follows that $F \to X$ is surjective as a map of sheaves in the smooth topology. On the other hand, the map $F \to X$ is injective (as a map of presheaves) since $R = U \times_X U$. It follows that $F \to X$ is an isomorphism of smooth (= étale) sheaves, see Sites, Lemma 7.11.2 which concludes the proof. □

Finally, here is the analogue of Spaces, Lemma 65.11.1 with a smooth morphism covering the space.

0GE0 Lemma 80.12.4. Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let

$$F : (Sch_\alpha/S)^{opp} \longrightarrow \text{Sets}$$

be a presheaf with the following properties:

(1) F is a sheaf for the étale topology,
(2) there exists an algebraic space U over S and a map $U \to F$ which is representable by algebraic spaces, surjective, and smooth.

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. The proof is identical to the proof of Lemma 80.12.2. Set $R = U \times_F U$. This is an algebraic space as $U \to F$ is assumed representable by algebraic spaces. The projections $s, t : R \to U$ are smooth morphisms of algebraic spaces as $U \to F$ is assumed smooth. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 80.10.1 the fppf quotient sheaf $F' = U/R$ is an algebraic space. The morphism $U \to F'$ is surjective, flat, and locally of finite presentation by Lemma 80.11.6. The map $R \to U \times_{F'} U$ is surjective as a map of fppf sheaves by Groupoids in Spaces, Lemma 78.19.5 and since j is a monomorphism it is an isomorphism. Hence the base change of $U \to F'$ by $U \to F'$ is smooth, and we conclude that $U \to F'$ is smooth by Descent on Spaces, Lemma 74.11.26. Thus $U \to F'$ is surjective as a map of étale sheaves (as the smooth topology is equal to the étale topology by More on Morphisms, Lemma 37.38.7). This means that F' is equal to the quotient sheaf U/R in the étale topology (small check omitted). Hence we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of étale sheaves (as the smooth topology is the same as the étale topology) and hence so is $F' \to F$. This means that $F' = F$ and the proof is complete. □

80.13. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory

CHAPTER 81

Pushouts of Algebraic Spaces

0AHT 81.1. Introduction

0AHU The goal of this chapter is to discuss pushouts in the category of algebraic spaces. This can be done with varying assumptions. A fairly general pushout construction is given in [TT13]: one of the morphisms is affine and the other is a closed immersion. We discuss a particular case of this in Section 81.6 where we assume one of the morphisms is affine and the other is a thickening, a situation that often comes up in deformation theory.

In Sections 81.10 and 81.11 we discuss diagrams

$$\begin{tikzcd} f^{-1}(X \setminus Z) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X \setminus Z \arrow[r] & X \end{tikzcd}$$

where f is a quasi-compact and quasi-separated morphism of algebraic spaces, $Z \to X$ is a closed immersion of finite presentation, the map $f^{-1}(Z) \to Z$ is an isomorphism, and f is flat along $f^{-1}(Z)$. In this situation we glue quasi-coherent modules on $X \setminus Z$ and Y (in Section 81.10) to quasi-coherent modules on X and we glue algebraic spaces over $X \setminus Z$ and Y (in Section 81.11) to algebraic spaces over X.

In Section 81.13 we discuss how proper birational morphisms of Noetherian algebraic spaces give rise to coequalizer diagrams in algebraic spaces in some sense.

In Section 81.14 we use the construction of elementary distinguished squares in Section 81.9 to prove Nagata's theorem on compactifications in the setting of algebraic spaces.

81.2. Conventions

0GFM The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

81.3. Colimits of algebraic spaces

0GFN We briefly discuss colimits of algebraic spaces. Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram (see Categories, Section 4.14). For each i we may consider the small étale site $X_{i,\acute{e}tale}$ whose objects are schemes étale over X_i, see Properties of Spaces, Section 66.18. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,\acute{e}tale} \to X_{i,\acute{e}tale}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,\acute{e}tale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a system of étale morphisms $U_i \to X_i$ over $\mathcal{I}$ such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & U_j \arrow[d] \\ X_i \arrow[r] & X_j \end{tikzcd}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to T$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to T$ is equal to f_i. Then we get a functor $T_{\acute{e}tale} \to \lim X_{i,\acute{e}tale}$. With this notation in hand we can formulate our lemma.

07SX Lemma 81.3.1. Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of schemes over S as above. Assume that

(1) $X = \operatorname{colim} X_i$ exists in the category of schemes,
(2) $\coprod X_i \to X$ is surjective,
(3) if $U \to X$ is étale and $U_i = X_i \times_X U$, then $U = \operatorname{colim} U_i$ in the category of schemes, and
(4) every object $(U_i \to X_i)$ of $\lim X_{i,\acute{e}tale}$ with $U_i \to X_i$ separated is in the essential image of the functor $X_{\acute{e}tale} \to \lim X_{i,\acute{e}tale}$.

Then $X = \operatorname{colim} X_i$ in the category of algebraic spaces over S also.

Proof. Let Z be an algebraic space over S. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to Z. We may assume that W is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,\acute{e}tale}$ with $U_i \to X_i$ separated. By assumption (4) we can find an étale morphism $U \to X$ and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (3) there exists a morphism $h : U \to W$ such that the compositions $U_i \to U \to W$ are h_i. Let $g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_S W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the schemes $U_i \times_{X_i} U_i$ by (3) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. □

We can check that a cocone is a colimit (fpqc) locally on the cocone.

0GFQ Lemma 81.3.2. Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Let $(X, X_i \to X)$ be a cocone for the diagram in the category of algebraic spaces over B (Categories, Remark 4.14.5). If there exists a fpqc covering $\{U_a \to X\}_{a \in A}$ such that

(1) for all $a \in A$ we have $U_a = \mathrm{colim}\, X_i \times_X U_a$ in the category of algebraic spaces over B, and
(2) for all $a, b \in A$ we have $U_a \times_X U_b = \mathrm{colim}\, X_i \times_X U_a \times_X U_b$ in the category of algebraic spaces over B,

then $X = \mathrm{colim}\, X_i$ in the category of algebraic spaces over B.

Proof. Namely, for an algebraic space Y over B a morphism $X \to Y$ over B is the same thing as a collection of morphism $U_a \to Y$ which agree on the overlaps $U_a \times_X U_b$ for all $a, b \in A$, see Descent on Spaces, Lemma 74.7.2. □

We are going to find a common partial generalization of Lemmas 81.3.1 and 81.3.2 which can in particular be used to reduce a colimit construction to a subcategory of the category of all algebraic spaces.

Let S be a scheme and let B be an algebraic space over S. Let $\mathcal{I}$ be an index category and let $i \mapsto X_i$ be a diagram in the category of algebraic spaces over B, see Categories, Section 4.14. For each i we may consider the small étale site $X_{i,spaces,étale}$ whose objects are algebraic spaces étale over X_i, see Properties of Spaces, Section 66.18. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,spaces,étale} \to X_{i,spaces,étale}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,spaces,étale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a diagram $i \mapsto (U_i \to X_i)$ in the category of arrows of algebraic spaces over B such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & U_j \arrow[d] \\ X_i \arrow[r] & X_j \end{tikzcd}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to Z$ is a family of morphisms of algebraic spaces over B such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. Then we get a functor $Z_{spaces,étale} \to \lim X_{i,spaces,étale}$. With this notation in hand we can formulate our next lemma.

0GHL Lemma 81.3.3. Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Let $(X, X_i \to X)$ be a cocone for the diagram in the category of algebraic spaces over B (Categories, Remark 4.14.5). Assume that

(1) the base change functor $X_{spaces,étale} \to \lim X_{i,spaces,étale}$, sending U to $U_i = X_i \times_X U$ is an equivalence,
(2) given
    (a) B' affine and étale over B,
    (b) Z an affine scheme over B',

(c) $U \to X \times_B B'$ an étale morphism of algebraic spaces with U affine,
(d) $f_i : U_i \to Z$ a cocone over B' of the diagram $i \mapsto U_i = U \times_X X_i$,

there exists a unique morphism $f : U \to Z$ over B' such that f_i equals the composition $U_i \to U \to Z$.

Then $X = \operatorname{colim} X_i$ in the category of all algebraic spaces over B.

Proof. In this paragraph we reduce to the case where B is an affine scheme. Let $B' \to B$ be an étale morphism of algebraic spaces. Observe that conditions (1) and (2) are preserved if we replace B, X_i, X by B', $X_i \times_B B'$, $X \times_B B'$. Let $\{B_a \to B\}_{a \in A}$ be an étale covering with B_a affine, see Properties of Spaces, Lemma 66.6.1. For $a \in A$ denote X_a, $X_{a,i}$ the base changes of X and the diagram to B_a. For $a, b \in A$ denote $X_{a,b}$ and $X_{a,b,i}$ the base changes of X and the diagram to $B_a \times_B B_b$. By Lemma 81.3.2 it suffices to prove that $X_a = \operatorname{colim} X_{a,i}$ and $X_{a,b} = \operatorname{colim} X_{a,b,i}$. This reduces us to the case where $B = B_a$ (an affine scheme) or $B = B_a \times_B B_b$ (a separated scheme). Repeating the argument once more, we conclude that we may assume B is an affine scheme (this uses that the intersection of affine opens in a separated scheme is affine).

Assume B is an affine scheme. Let Z be an algebraic space over B. We have to show

$$\operatorname{Mor}_B(X, Z) \longrightarrow \lim \operatorname{Mor}_B(X_i, Z)$$

is a bijection.

Proof of injectivity. Let $f, g : X \to Z$ be morphisms such that the compositions $f_i, g_i : X_i \to Z$ are the same for all i. Choose an affine scheme Z' and an étale morphism $Z' \to Z$. By Properties of Spaces, Lemma 66.6.1 we know we can cover Z by such affines. Set $U = X \times_{f,Z} Z'$ and $U' = X \times_{g,Z} Z'$ and denote $p : U \to X$ and $p' : U' \to X$ the projections. Since $f_i = g_i$ for all i, we see that

$$U_i = X_i \times_{f_i,Z} Z' = X_i \times_{g_i,Z} Z' = U'_i$$

compatible with transition morphisms. By (1) there is a unique isomorphism $\epsilon : U \to U'$ as algebraic spaces over X, i.e., with $p = p' \circ \epsilon$ which is compatible with the displayed identifications. Choose an étale covering $\{h_a : U_a \to U\}$ with U_a affine. By (2) we see that $f \circ p \circ h_a = g \circ p' \circ \epsilon \circ h_a = g \circ p \circ h_a$. Since $\{h_a : U_a \to U\}$ is an étale covering we conclude $f \circ p = g \circ p$. Since the collection of morphisms $p : U \to X$ we obtain in this manner is an étale covering, we conclude that $f = g$.

Proof of surjectivity. Let $f_i : X_i \to Z$ be an element of the right hand side of the displayed arrow in the first paragraph of the proof. It suffices to find an étale covering $\{U_c \to X\}_{c \in C}$ such that the families $f_{c,i} \in \lim_i \operatorname{Mor}_B(X_i \times_X U_c, Z)$ come from morphisms $f_c : U_c \to Z$. Namely, by the uniqueness proved above the morphisms f_c will agree on $U_c \times_X U_b$ and hence will descend to give the desired morphism $f : X \to Z$. To find our covering, we first choose an étale covering $\{g_a : Z_a \to Z\}_{a \in A}$ where each Z_a is affine. Then we let $U_{a,i} = X_i \times_{f_i,Z} Z_a$. By (1) we find $U_{a,i} = X_i \times_X U_a$ for some algebraic spaces U_a étale over X. Then we choose étale coverings $\{U_{a,b} \to U_a\}_{b \in B_a}$ with $U_{a,b}$ affine and we consider the morphisms

$$U_{a,b,i} = X_i \times_X U_{a,b} \to X_i \times_X U_a = X_i \times_{f_i,Z} Z_a \to Z_a$$

By (2) we obtain morphisms $f_{a,b} : U_{a,b} \to Z_a$ compatible with these morphisms. Setting $C = \coprod_{a \in A} B_a$ and for $c \in C$ corresponding to $b \in B_a$ setting $U_c = U_{a,b}$ and $f_c = g_a \circ f_{a,b} : U_c \to Z$ we conclude. $\square$

Here is an application of these ideas to reduce the general case to the case of separated algebraic spaces.

0GFP **Lemma 81.3.4.** Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Assume that

(1) each X_i is separated over B,
(2) $X = \mathrm{colim}\, X_i$ exists in the category of algebraic spaces separated over B,
(3) $\coprod X_i \to X$ is surjective,
(4) if $U \to X$ is an étale separated morphism of algebraic spaces and $U_i = X_i \times_X U$, then $U = \mathrm{colim}\, U_i$ in the category of algebraic spaces separated over B, and
(5) every object $(U_i \to X_i)$ of $\lim X_{i,spaces,étale}$ with $U_i \to X_i$ separated is of the form $U_i = X_i \times_X U$ for some étale separated morphism of algebraic spaces $U \to X$.

Then $X = \mathrm{colim}\, X_i$ in the category of all algebraic spaces over B.

Proof. We encourage the reader to look instead at Lemma 81.3.3 and its proof.

Let Z be an algebraic space over B. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ over B such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to Z. We may assume that W is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated and that W is separated over B. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,spaces,étale}$ with $U_i \to X_i$ separated. By assumption (5) we can find a separated étale morphism $U \to X$ of algebraic spaces and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (4) there exists a morphism $h : U \to W$ over B such that the compositions $U_i \to U \to W$ are h_i. Let $g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_S W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the algebraic spaces $U_i \times_{X_i} U_i$ in the category of algebraic spaces separated over B by (4) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. $\square$

81.4. Descending étale sheaves

0GFR This section is the analogue for algebraic spaces of Étale Cohomology, Section 59.104.

In order to conveniently express our results we need some notation. Let S be a scheme. Let $\mathcal{U} = \{f_i : X_i \to X\}$ be a family of morphisms of algebraic spaces over S with fixed target. A descent datum for étale sheaves with respect to $\mathcal{U}$ is a family $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ where

(1) $\mathcal{F}_i$ is in $Sh(X_{i,étale})$, and
(2) $\varphi_{ij} : \mathrm{pr}_{0,small}^{-1}\mathcal{F}_i \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}_j$ is an isomorphism in $Sh((X_i \times_X X_j)_{étale})$

such that the cocycle condition holds: the diagrams

$$\begin{tikzcd} \mathrm{pr}_{0,small}^{-1}\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{01,small}^{-1}\varphi_{ij}"] \arrow[dr, "\mathrm{pr}_{02,small}^{-1}\varphi_{ik}"'] & & \mathrm{pr}_{1,small}^{-1}\mathcal{F}_j \arrow[dl, "\mathrm{pr}_{12,small}^{-1}\varphi_{jk}"] \\ & \mathrm{pr}_{2,small}^{-1}\mathcal{F}_k & \end{tikzcd}$$

commute in $Sh((X_i \times_X X_j \times_X X_k)_{étale})$. There is an obvious notion of morphisms of descent data and we obtain a category of descent data. A descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is called effective if there exist a $\mathcal{F}$ in $Sh(X_{étale})$ and isomorphisms $\varphi_i : f_{i,small}^{-1}\mathcal{F} \to \mathcal{F}_i$ in $Sh(X_{i,étale})$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \mathrm{pr}_{1,small}^{-1}(\varphi_j) \circ \mathrm{pr}_{0,small}^{-1}(\varphi_i^{-1})$$

Another way to say this is the following. Given an object $\mathcal{F}$ of $Sh(X_{étale})$ we obtain the canonical descent datum $(f_{i,small}^{-1}\mathcal{F}_i, c_{ij})$ where c_{ij} is the canonical isomorphism

$$c_{ij} : \mathrm{pr}_{0,small}^{-1} f_{i,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1} f_{j,small}^{-1}\mathcal{F}$$

The descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is effective if and only if it is isomorphic to the canonical descent datum associated to some $\mathcal{F}$ in $Sh(X_{étale})$.

If the family consists of a single morphism $\{X \to Y\}$, then we think of a descent datum as a pair $(\mathcal{F}, \varphi)$ where $\mathcal{F}$ is an object of $Sh(X_{étale})$ and φ is an isomorphism

$$\mathrm{pr}_{0,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}$$

in $Sh((X \times_Y X)_{étale})$ such that the cocycle condition holds:

$$\begin{tikzcd} \mathrm{pr}_{0,small}^{-1}\mathcal{F} \arrow[rr, "\mathrm{pr}_{01,small}^{-1}\varphi"] \arrow[dr, "\mathrm{pr}_{02,small}^{-1}\varphi"'] & & \mathrm{pr}_{1,small}^{-1}\mathcal{F} \arrow[dl, "\mathrm{pr}_{12,small}^{-1}\varphi"] \\ & \mathrm{pr}_{2,small}^{-1}\mathcal{F} & \end{tikzcd}$$

commutes in $Sh((X \times_Y X \times_Y X)_{étale})$. There is a notion of morphisms of descent data and effectivity exactly as before.

0GFS Lemma 81.4.1. Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an étale covering of algebraic spaces. The functor

$$Sh(X_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. In Properties of Spaces, Section 66.18 we have defined a site $X_{spaces,étale}$ whose objects are algebraic spaces étale over X with étale coverings. Moreover, we have a identifications $Sh(X_{étale}) = Sh(X_{spaces,étale})$ compatible with morphisms of algebraic spaces, i.e., compatible with pushforward and pullback. Hence the statement of the lemma follows from the much more general discussion in Sites, Section 7.26. $\square$

0GFT Lemma 81.4.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of algebraic spaces. If for each $i \in I$ the functor

$$Sh(Y_{i,\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Y_i \to Y_i\}$$

is an equivalence of categories and for each $i, j \in I$ the functor

$$Sh((Y_i \times_Y Y_j)_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Y_i \times_Y Y_j \to Y_i \times_Y Y_j\}$$

is an equivalence of categories, then

$$Sh(Y_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Formal consequence of Lemma 81.4.1 and the definitions. □

0GFU Lemma 81.4.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable (by schemes) and f has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation Then

$$Sh(Y_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Each of the properties of morphisms of algebraic spaces mentioned in the statement of the lemma is preserved by arbitrary base change, see the lists in Spaces, Section 65.4. Thus we can apply Lemma 81.4.2 to see that we can work étale locally on Y. In this way we reduce to the case where Y is a scheme; some details omitted. In this case X is also a scheme and the result follows from Étale Cohomology, Lemma 59.104.2, 59.104.3, or 59.104.5. □

0GFV Lemma 81.4.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\pi : X' \to X$ be a morphism of algebraic spaces. Assume

(1) $f \circ \pi$ is representable (by schemes),
(2) $f \circ \pi$ has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation.

Then

$$Sh(Y_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Formal consequence of Lemma 81.4.3 and Stacks, Lemma 8.3.7. □

0GFW Lemma 81.4.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation. Then the functor

$$Sh(Y_{\acute{e}tale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Observe that the base change of a proper surjective morphism is proper and surjective, see Morphisms of Spaces, Lemmas 67.40.3 and 67.5.5. Hence by Lemma 81.4.2 we may work étale locally on Y. Hence we reduce to Y being an affine scheme; some details omitted.

Assume Y is affine. By Lemma 81.4.4 it suffices to find a morphism $X' \to X$ where X' is a scheme such that $X' \to Y$ is surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation.

In case $X \to Y$ is integral and surjective, we can take $X = X'$ as an integral morphism is representable.

If f is proper and surjective, then the algebraic space X is quasi-compact and separated, see Morphisms of Spaces, Section 67.8 and Lemma 67.4.9. Choose a scheme X' and a surjective finite morphism $X' \to X$, see Limits of Spaces, Proposition 70.16.1. Then $X' \to Y$ is surjective and proper.

Finally, if $X \to Y$ is surjective and flat and locally of finite presentation then we can take an affine étale covering $\{U_i \to X\}$ and set X' equal to the disjoint $\coprod U_i$. $\square$

0GFX Lemma 81.4.6. Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an fppf covering of algebraic spaces over S. The functor

$$Sh(X_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. We have Lemma 81.4.5 for the morphism $f : \coprod X_i \to X$. Then a formal argument shows that descent data for f are the same thing as descent data for the covering, compare with Descent, Lemma 35.34.5. Details omitted. $\square$

0GFY Lemma 81.4.7. Let S be a scheme. Let $f : Y' \to Y$ be a proper morphism of algebraic spaces over S. Let $i : Z \to Y$ be a closed immersion. Set $E = Z \times_Y Y'$. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

If f is an isomorphism over $Y \setminus Z$, then the functor

$$Sh(Y_{étale}) \longrightarrow Sh(Y'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale})$$

is an equivalence of categories.

Proof. Observe that $X = Y' \coprod Z \to Y$ is a proper surjective morphism. Thus it suffice to construct an equivalence of categories

$$Sh(Y'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

compatible with pullback functors from Y because then we can use Lemma 81.4.5 to conclude. Thus let $(\mathcal{G}', \mathcal{G}, \alpha)$ be an object of $Sh(Y'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale})$ with notation as in Categories, Example 4.31.3. Then we can consider the sheaf $\mathcal{F}$ on X defined by taking $\mathcal{G}'$ on the summand Y' and $\mathcal{G}$ on the summand Z. We have

$$X \times_Y X = Y' \times_Y Y' \amalg Y' \times_Y Z \amalg Z \times_Y Y' \amalg Z \times_Y Z = Y' \times_Y Y' \amalg E \amalg E \amalg Z$$

The isomorphisms of the two pullbacks of $\mathcal{F}$ to this algebraic space are obvious over the summands E, E, Z. The interesting part of the proof is to find an isomorphism

$\mathrm{pr}_{0,small}^{-1}\mathcal{G}' \to \mathrm{pr}_{1,small}^{-1}\mathcal{G}'$ over $Y' \times_Y Y'$ satisfying the cocycle condition. However, our assumption that $Y' \to Y$ is an isomorphism over $Y \setminus Z$ implies that

$$h : Y \coprod E \times_Z E \longrightarrow Y' \times_Y Y'$$

is a surjective proper morphism. (It is in fact a finite morphism as it is the disjoint union of two closed immersions.) Hence it suffices to construct an isomorphism of the pullbacks of $\mathrm{pr}_{0,small}^{-1}\mathcal{G}'$and $\mathrm{pr}_{1,small}^{-1}\mathcal{G}'$ by h_{small} satisfying a certain cocycle condition. For the diagonal, it is clear how to do this. And for the pullback to $E \times_Z E$ we use that both sheaves pull back to the pullback of $\mathcal{G}$ by the morphism $E \times_Z E \to Z$. We omit the details. □

81.5. Descending étale morphisms of algebraic spaces

0GFZ In this section we combine the glueing results for étale sheaves given in Section 81.4 with the flexibility of algebraic spaces to get some descent statements for étale morphisms of algebraic spaces.

0GG0 Lemma 81.5.1. Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of algebraic spaces over S. Any descent datum $(U/X, \varphi)$ relative to f (Descent on Spaces, Definition 74.22.1) with U étale over X is effective (Descent on Spaces, Definition 74.22.10). More precisely, there exists an étale morphism $V \to Y$ of algebraic spaces whose corresponding canonical descent datum is isomorphic to $(U/X, \varphi)$.

Proof. Recall that U gives rise to a representable sheaf $\mathcal{F} = h_U$ in $Sh(X_{spaces,étale}) = Sh(X_{étale})$, see Properties of Spaces, Section 66.18. The descent datum on U relative to f exactly gives a descent datum $(\mathcal{F}, \varphi)$ for étale sheaves with respect to $\{X \to Y\}$. By Lemma 81.4.5 this descent datum is effective. Let $\mathcal{G}$ be the corresponding sheaf on $Y_{étale}$. By Properties of Spaces, Lemma 66.27.3 we obtain an étale morphism $V \to Y$ of algebraic spaces corresponding to $\mathcal{G}$; we omit the verification of the set theoretic condition[1]. The given isomorphism $\mathcal{F} \to f_{small}^{-1}\mathcal{G}$ corresponds to an isomorphism $U \to V \times_Y X$ compatible with the descent datum. □

0GG1 Lemma 81.5.2. Let S be a scheme. Let $f : Y' \to Y$ be a proper morphism of algebraic spaces over S. Let $i : Z \to Y$ be a closed immersion. Set $E = Z \times_Y Y'$. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

If f is an isomorphism over $Y \setminus Z$, then the functor

$$Y_{spaces,étale} \longrightarrow Y'_{spaces,étale} \times_{E_{spaces,étale}} Z_{spaces,étale}$$

is an equivalence of categories.

Proof. Let $(V' \to Y', W \to Z, \alpha)$ be an object of the right hand side. Recall that V', resp. W gives rise to a representable sheaf $\mathcal{G}' = h_{V'}$ in $Sh(Y'_{spaces,étale}) = Sh(Y'_{étale})$, resp. $\mathcal{G} = h_W$ in $Sh(Z_{spaces,étale}) = Sh(Z_{étale})$, see Properties of Spaces, Section 66.18. The isomorphism $\alpha : V' \times_{Y'} E \to W \times_Z E$ determines an isomorphism $j_{small}^{-1}\mathcal{G}' \to g_{small}^{-1}\mathcal{G}$ of sheaves on E. By Lemma 81.4.7 we obtain a unique sheaf $\mathcal{F}$

[1]It follows from the fact that $\mathcal{F}$ satisfies the corresponding condition.

on Y pulling pack to $\mathcal{G}'$ and $\mathcal{G}$ compatibly with the isomorphism. By Properties of Spaces, Lemma 66.27.3 we obtain an étale morphism $V \to Y$ of algebraic spaces corresponding to $\mathcal{F}$; we omit the verification of the set theoretic condition[2]. The given isomorphism $\mathcal{G}' \to f_{small}^{-1}\mathcal{F}$ and $\mathcal{G} \to i_{small}^{-1}\mathcal{F}$ corresponds to isomorphisms $V' \to V \times_Y Y'$ and $W \to V \times_Y Z$ compatible with α as desired. $\square$

81.6. Pushouts along thickenings and affine morphisms

07SW This section is analogue of More on Morphisms, Section 37.14.

07SY **Lemma 81.6.1.** Let S be a scheme. Let $X \to X'$ be a thickening of schemes over S and let $X \to Y$ be an affine morphism of schemes over S. Let $Y' = Y \amalg_X X'$ be the pushout in the category of schemes (see More on Morphisms, Lemma 37.14.3). Then Y' is also a pushout in the category of algebraic spaces over S.

Proof. This is an immediate consequence of Lemma 81.3.1 and More on Morphisms, Lemmas 37.14.3, 37.14.4, and 37.14.6. $\square$

07VX **Lemma 81.6.2.** Let S be a scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Then there exists a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y \amalg_X X' \end{tikzcd}$$

in the category of algebraic spaces over S. Moreover $Y' = Y \amalg_X X'$ is a thickening of Y and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $Y_{\acute{e}tale} = (Y')_{\acute{e}tale}$.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. This is a scheme affine over V with a surjective étale morphism $U \to X$. By More on Morphisms of Spaces, Lemma 76.9.6 there exists a $U' \to X'$ surjective étale with $U = U' \times_{X'} X$. In particular the morphism of schemes $U \to U'$ is a thickening too. Apply More on Morphisms, Lemma 37.14.3 to obtain a pushout $V' = V \amalg_U U'$ in the category of schemes.

We repeat this procedure to construct a pushout

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d] & U' \times_{X'} U' \arrow[d] \\ V \times_Y V \arrow[r] & R' \end{tikzcd}$$

in the category of schemes. Consider the morphisms

$$U \times_X U \to U \to V', \quad U' \times_{X'} U' \to U' \to V', \quad V \times_Y V \to V \to V'$$

where we use the first projection in each case. Clearly these glue to give a morphism $t' : R' \to V'$ which is étale by More on Morphisms, Lemma 37.14.6. Similarly, we obtain $s' : R' \to V'$ étale. The morphism $j' = (t', s') : R' \to V' \times_S V'$ is unramified (as t' is étale) and a monomorphism when restricted to the closed

[2]It follows from the fact that $\mathcal{G}$ and $\mathcal{G}'$ satisfies the corresponding condition.

subscheme $V \times_Y V \subset R'$. As $V \times_Y V \subset R'$ is a thickening it follows that j' is a monomorphism too. Finally, j' is an equivalence relation as we can use the functoriality of pushouts of schemes to construct a morphism $c' : R' \times_{s', V', t'} R' \to R'$ (details omitted). At this point we set $Y' = U'/R'$, see Spaces, Theorem 65.10.5.

We have morphisms $X' = U'/U' \times_{X'} U' \to V'/R' = Y'$ and $Y = V/V \times_Y V \to V'/R' = Y'$. By construction these fit into the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

Since $Y \to Y'$ is a thickening we have $Y_{\acute{e}tale} = (Y')_{\acute{e}tale}$, see More on Morphisms of Spaces, Lemma 76.9.6. The commutativity of the diagram gives a map of sheaves

$$\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

on this set. By More on Morphisms, Lemma 37.14.3 this map is an isomorphism when we restrict to the scheme V', hence it is an isomorphism.

To finish the proof we show that the diagram above is a pushout in the category of algebraic spaces. To see this, let Z be an algebraic space and let $a' : X' \to Z$ and $b : Y \to Z$ be morphisms of algebraic spaces. By Lemma 81.6.1 we obtain a unique morphism $h : V' \to Z$ fitting into the commutative diagrams

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & V' \arrow[d, "h"] \\ X' \arrow[r, "a'"] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V \arrow[r] \arrow[d] & V' \arrow[d, "h"] \\ Y \arrow[r, "b"] & Z \end{tikzcd}$$

The uniqueness shows that $h \circ t' = h \circ s'$. Hence h factors uniquely as $V' \to Y' \to Z$ and we win. $\square$

In the following lemma we use the fibre product of categories as defined in Categories, Example 4.31.3.

07VY **Lemma 81.6.3.** Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 81.6.2). Base change gives a functor

$$F : (\text{Spaces}/Y') \longrightarrow (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$. The functor F has a left adjoint

$$G : (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X') \longrightarrow (\text{Spaces}/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$ in the category of algebraic spaces over S. The functor G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Proof. The proof is completely formal. Since the morphisms $X \to X'$ and $X \to Y$ are representable it is clear that F sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$.

Let us construct G. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies

U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms of Spaces, Lemma 67.20.5. Hence we can apply Lemma 81.6.2 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

If (V, U', φ) is an object of $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ then $U = U' \times_{X'} X$ is a scheme too and we can form the pushout $V' = V \amalg_U U'$ in the category of schemes by More on Morphisms, Lemma 37.14.3. By Lemma 81.6.1 this is also a pushout in the category of schemes, hence G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Let us prove that G is a left adjoint to F. Let Z be an algebraic space over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse. □

07VZ Lemma 81.6.4. Let S be a scheme. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r] & D \arrow[r] & F \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume that A, B, C, D and A, B, E, F form cartesian squares and that $B \to D$ is surjective étale. Then C, D, E, F is a cartesian square.

Proof. This is formal. □

07W0 Lemma 81.6.5. In the situation of Lemma 81.6.3 the functor $F \circ G$ is isomorphic to the identity functor.

Proof. We will prove that $F \circ G$ is isomorphic to the identity by reducing this to the corresponding statement of More on Morphisms, Lemma 37.14.4.

Choose a scheme Y_1 and a surjective étale morphism $Y_1 \to Y$. Set $X_1 = Y_1 \times_Y X$. This is a scheme affine over Y_1 with a surjective étale morphism $X_1 \to X$. By More on Morphisms of Spaces, Lemma 76.9.6 there exists a $X_1' \to X'$ surjective étale with $X_1 = X_1' \times_{X'} X$. In particular the morphism of schemes $X_1 \to X_1'$ is a thickening too. Apply More on Morphisms, Lemma 37.14.3 to obtain a pushout $Y_1' = Y_1 \amalg_{X_1} X_1'$ in the category of schemes. In the proof of Lemma 81.6.2 we constructed Y' as a quotient of an étale equivalence relation on Y_1' such that we get a commutative diagram

07W1 (81.6.5.1)
$$
\begin{tikzcd}
 & X \arrow[rr] \arrow[dd] & & X' \arrow[dd] \\
X_1 \arrow[ur] \arrow[rr] \arrow[dd] & & X_1' \arrow[ur] \arrow[dd] & \\
 & Y \arrow[rr] & & Y' \\
Y_1 \arrow[ur] \arrow[rr] & & Y_1' \arrow[ur] &
\end{tikzcd}
$$

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. Denote F_1, G_1 the functors constructed in More on Morphisms, Lemma 37.14.4 for the front square. Then the diagram of categories

$$
\begin{tikzcd}
(Sch/Y_1') \arrow[r, shift right, "F_1"'] \arrow[d] & (Sch/Y_1) \times_{(Sch/Y_1')} (Sch/X_1') \arrow[l, shift right, "G_1"'] \arrow[d] \\
(\text{Spaces}/Y') \arrow[r, shift right, "F"'] & (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X') \arrow[l, shift right, "G"']
\end{tikzcd}
$$

is commutative by simple considerations regarding base change functors and the agreement of pushouts in schemes with pushouts in spaces of Lemma 81.6.1.

Let (V, U', φ) be an object of $(\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X')$. Denote $U = U' \times_{X'} X$ so that $G(V, U', \varphi) = V \amalg_U U'$. Choose a scheme V_1 and a surjective étale morphism $V_1 \to Y_1 \times_Y V$. Set $U_1 = V_1 \times_Y X$. Then

$$U_1 = V_1 \times_Y X \longrightarrow (Y_1 \times_Y V) \times_Y X = X_1 \times_Y V = X_1 \times_X X \times_Y V = X_1 \times_X U$$

is surjective étale too. By More on Morphisms of Spaces, Lemma 76.9.6 there exists a thickening $U_1 \to U_1'$ and a surjective étale morphism $U_1' \to X_1' \times_{X'} U'$ whose base change to $X_1 \times_X U$ is the displayed morphism. At this point (V_1, U_1', φ_1) is an object of $(Sch/Y_1) \times_{(Sch/Y_1')} (Sch/X_1')$. In the proof of Lemma 81.6.2 we constructed $G(V, U', \varphi) = V \amalg_U U'$ as a quotient of an étale equivalence relation on

$G_1(V_1, U'_1, \varphi_1) = V_1 \amalg_{U_1} U'_1$ such that we get a commutative diagram

07W2 (81.6.5.2)

$$\begin{tikzcd} & U \arrow[rr] \arrow[dd] & & U' \arrow[dd] \\ U_1 \arrow[ur] \arrow[rr] \arrow[dd] & & U'_1 \arrow[ur] \arrow[dd] & \\ & V \arrow[rr] & & G(V, U', \varphi) \\ V_1 \arrow[ur] \arrow[rr] & & G_1(V_1, U'_1, \varphi_1) \arrow[ur] & \end{tikzcd}$$

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. In particular

$$G_1(V_1, U'_1, \varphi_1) \to G(V, U', \varphi)$$

is surjective étale.

Finally, we come to the proof of the lemma. We have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. We know $(V_1, U'_1, \varphi_1) \to F_1(G_1(V_1, U'_1, \varphi_1))$ is an isomorphism by More on Morphisms, Lemma 37.14.4. Recall that F and F_1 are given by base change. Using the properties of (81.6.5.2) and Lemma 81.6.4 we see that $V \to G(V, U', \varphi) \times_{Y'} Y$ and $U' \to G(V, U', \varphi) \times_{Y'} X'$ are isomorphisms, i.e., $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. $\square$

08KV Lemma 81.6.6. Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 81.6.2). Let $V' \to Y'$ be a morphism of algebraic spaces over S. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between

(1) quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and
(2) the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where
    (a) $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_V$-module flat over Y,
    (b) $\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{U'}$-module flat over X, and
    (c) $\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ is an isomorphism of $\mathcal{O}_U$-modules.

The equivalence maps $\mathcal{G}'$ to $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$. Suppose $\mathcal{G}'$ corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

(a) $\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.
(b) if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines étale over V' and Y' we recover the equivalence of More on Algebra, Lemma 15.7.5. Details omitted.

Parts (a) and (b) reduce by étale localization (Properties of Spaces, Section 66.30) to the case where V' and Y' are affine in which case the result follows from More on Algebra, Lemmas 15.7.4 and 15.7.6. $\square$

07W3 **Lemma 81.6.7.** In the situation of Lemma 81.6.5. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then

(1) $V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,
(2) $V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,
(3) $V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,
(4) $V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,
(5) $V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and
(6) add more here as needed.

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of spaces flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Choose a diagram (81.6.5.1) as in the proof of Lemma 81.6.5.

Proof of (1) – (5). Let (V, U', φ) be an object of $(\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X')$. Construct a diagram (81.6.5.2) as in the proof of Lemma 81.6.5. Then the base change of $G(V, U', \varphi) \to Y'$ to Y'_1 is $G_1(V_1, U'_1, \varphi_1) \to Y'_1$. Hence (1) – (5) follow immediately from the corresponding statements of More on Morphisms, Lemma 37.14.6 for schemes.

Suppose that $W' \to Y'$ is flat. Choose a scheme W'_1 and a surjective étale morphism $W'_1 \to Y'_1 \times_{Y'} W'$. Observe that $W'_1 \to W'$ is surjective étale as a composition of surjective étale morphisms. We know that $G_1(F_1(W'_1)) \to W'_1$ is an isomorphism by More on Morphisms, Lemma 37.14.6 applied to W'_1 over Y'_1 and the front of the diagram (with functors G_1 and F_1 as in the proof of Lemma 81.6.5). Then the construction of $G(F(W'))$ (as a pushout, i.e., as constructed in Lemma 81.6.2) shows that $G_1(F_1(W'_1)) \to G(F(W))$ is surjective étale. Whereupon we conclude that $G(F(W)) \to W$ is étale, see for example Properties of Spaces, Lemma 66.16.3. But $G(F(W)) \to W$ is an isomorphism on underlying reduced algebraic spaces (by construction), hence it is an isomorphism. $\square$

81.7. Pushouts along closed immersions and integral morphisms

0GG2 This section is analogue of More on Morphisms, Section 37.67.

0EDP **Lemma 81.7.1.** In More on Morphisms, Situation 37.67.1 let $Y \amalg_Z X$ be the pushout in the category of schemes (More on Morphisms, Proposition 37.67.3). Then $Y \amalg_Z X$ is also a pushout in the category of algebraic spaces over S.

Proof. This is a consequence of Lemma 81.3.1, the proposition mentioned in the lemma and More on Morphisms, Lemmas 37.67.6 and 37.67.7. Conditions (1) and (2) of Lemma 81.3.1 follow immediately. To see (3) and (4) note that an étale morphism is locally quasi-finite and use that the equivalence of categories of More on Morphisms, Lemma 37.67.7 is constructed using the pushout construction of More on Morphisms, Lemmas 37.67.6. Minor details omitted. $\square$

81.8. Pushouts and derived categories

0DL6 In this section we discuss the behaviour of the derived category of modules under pushouts.

0DL7 **Lemma 81.8.1.** Let S be a scheme. Consider a pushout

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

in the category of algebraic spaces over S as in Lemma 81.6.2. Assume i is a thickening. Then the essential image of the functor[3]

$$D(\mathcal{O}_{Y'}) \longrightarrow D(\mathcal{O}_Y) \times_{D(\mathcal{O}_X)} D(\mathcal{O}_{X'})$$

contains every triple (M, K', α) where $M \in D(\mathcal{O}_Y)$ and $K' \in D(\mathcal{O}_{X'})$ are pseudo-coherent.

Proof. Let (M, K', α) be an object of the target of the functor of the lemma. Here $\alpha : Lf^*M \to Li^*K'$ is an isomorphism which is adjoint to a map $\beta : M \to Rf_*Li^*K'$. Thus we obtain maps

$$Rj_*M \xrightarrow{Rj_*\beta} Rj_*Rf_*Li^*K' = Rf'_*Ri_*Li^*K' \leftarrow Rf'_*K'$$

where the arrow pointing left comes from $K' \to Ri_*Li^*K'$. Choose a distinguished triangle

$$M' \to Rj_*M \oplus Rf'_*K' \to Rj_*Rf_*Li^*K' \to M'[1]$$

in $D(\mathcal{O}_{Y'})$. The first arrow defines canonical maps $Lj^*M' \to M$ and $L(f')^*M' \to K'$ compatible with α. Thus it suffices to show that the maps $Lj^*M' \to M$ and $L(f')^*M' \to K$ are isomorphisms. This we may check étale locally on Y', hence we may assume Y' is étale.

Assume Y' affine and $M \in D(\mathcal{O}_Y)$ and $K' \in D(\mathcal{O}_{X'})$ are pseudo-coherent. Say our pushout corresponds to the fibre product

$$\begin{tikzcd} B & B' \arrow[l] \\ A \arrow[u] & A' \arrow[l] \arrow[u] \end{tikzcd}$$

of rings where $B' \to B$ is surjective with locally nilpotent kernel I (and hence $A' \to A$ is surjective with locally nilpotent kernel I as well). The assumption on M and K' imply that M comes from a pseudo-coherent object of $D(A)$ and K' comes from a pseudo-coherent object of $D(B')$, see Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, and 75.13.2 and Derived Categories of Schemes, Lemma 36.3.5 and 36.10.2. Moreover, pushforward and derived pullback agree with the corresponding operations on derived categories of modules, see Derived Categories of Spaces, Remark 75.6.3 and Derived Categories of Schemes, Lemmas 36.3.7 and 36.3.8. This reduces us to the statement formulated in the next paragraph. (To be sure these references show the object M' lies $D_{QCoh}(\mathcal{O}_{Y'})$ as this is a triangulated subcategory of $D(\mathcal{O}_{Y'})$.)

[3]All functors given by derived pullback.

Given a diagram of rings as above and a triple (M, K', α) where $M \in D(A)$, $K' \in D(B')$ are pseudo-coherent and $\alpha : M \otimes_A^{\mathbf{L}} B \to K' \otimes_{B'}^{\mathbf{L}} B$ is an isomorphism suppose we have distinguished triangle

$$M' \to M \oplus K' \to K' \otimes_{B'}^{\mathbf{L}} B \to M'[1]$$

in $D(A')$. Goal: show that the induced maps $M' \otimes_{A'}^{\mathbf{L}} A \to M$ and $M' \otimes_{A'}^{\mathbf{L}} B' \to K'$ are isomorphisms. To do this, choose a bounded above complex $E^\bullet$ of finite free A-modules representing M. Since (B', I) is a henselian pair (More on Algebra, Lemma 15.11.2) with $B = B'/I$ we may apply More on Algebra, Lemma 15.75.8 to see that there exists a bounded above complex $P^\bullet$ of free B'-modules such that α is represented by an isomorphism $E^\bullet \otimes_A B \cong P^\bullet \otimes_{B'} B$. Then we can consider the short exact sequence

$$0 \to L^\bullet \to E^\bullet \oplus P^\bullet \to P^\bullet \otimes_{B'} B \to 0$$

of complexes of B'-modules. More on Algebra, Lemma 15.6.9 implies $L^\bullet$ is a bounded above complex of finite projective A'-modules (in fact it is rather easy to show directly that L^n is finite free in our case) and that we have $L^\bullet \otimes_{A'} A = E^\bullet$ and $L^\bullet \otimes_{A'} B' = P^\bullet$. The short exact sequence gives a distinguished triangle

$$L^\bullet \to M \oplus K' \to K' \otimes_{B'}^{\mathbf{L}} B \to (L^\bullet)[1]$$

in $D(A')$ (Derived Categories, Section 13.12) which is isomorphic to the given distinguished triangle by general properties of triangulated categories (Derived Categories, Section 13.4). In other words, $L^\bullet$ represents M' compatibly with the given maps. Thus the maps $M' \otimes_{A'}^{\mathbf{L}} A \to M$ and $M' \otimes_{A'}^{\mathbf{L}} B' \to K'$ are isomorphisms because we just saw that the corresponding thing is true for $L^\bullet$. □

81.9. Constructing elementary distinguished squares

0DVH Elementary distinguished squares were defined in Derived Categories of Spaces, Section 75.9.

0DVI **Lemma 81.9.1.** Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square. Then

$$\begin{tikzcd} U \times_W V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r] & W \end{tikzcd}$$

is a pushout in the category of algebraic spaces over S.

Proof. Observe that $U \amalg V \to W$ is a surjective étale morphism. The fibre product

$$(U \amalg V) \times_W (U \amalg V)$$

is the disjoint union of four pieces, namely $U = U \times_W U$, $U \times_W V$, $V \times_W U$, and $V \times_W V$. There is a surjective étale morphism

$$V \amalg (U \times_W V) \times_U (U \times_W V) \longrightarrow V \times_W V$$

because f induces an isomorphism over $W \setminus U$ (part of the definition of being an elementary distinguished square). Let B be an algebraic space over S and let $g : V \to B$ and $h : U \to B$ be morphisms over S which agree after restricting to $U \times_W V$. Then the description of $(U \amalg V) \times_W (U \amalg V)$ given above shows that

$h \amalg g : U \amalg V \to B$ equalizes the two projections. Since B is a sheaf for the étale topology we obtain a unique factorization of $h \amalg g$ through W as desired. □

0DVJ **Lemma 81.9.2.** Let S be a scheme. Let V, U be algebraic spaces over S. Let $V' \subset V$ be an open subspace and let $f' : V' \to U$ be a separated étale morphism of algebraic spaces over S. Then there exists a pushout

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r] & W \end{tikzcd}$$

in the category of algebraic spaces over S and moreover $(U \subset W, f : V \to W)$ is an elementary distinguished square.

Proof. We are going to construct W as the quotient of an étale equivalence relation R on $U \amalg V$. Such a quotient is an algebraic space for example by Bootstrap, Theorem 80.10.1. Moreover, the proof of Lemma 81.9.1 tells us to take

$$R = U \amalg V' \amalg V' \amalg V \amalg (V' \times_U V' \setminus \Delta_{V'/U}(V'))$$

Since we assumed $V' \to U$ is separated, the image of $\Delta_{V'/U}$ is closed and hence the complement is an open subspace. The morphism $j : R \to (U \amalg V) \times_S (U \amalg V)$ is given by

$$u,\ v',\ v',\ v,\ (v'_1, v'_2) \mapsto (u, u),\ (f'(v'), v'),\ (v', f'(v')),\ (v, v),\ (v'_1, v'_2)$$

with obvious notation. It is immediately verified that this is a monomorphism, an equivalence relation, and that the induced morphisms $s, t : R \to U \amalg V$ are étale. Let $W = (U \amalg V)/R$ be the quotient algebraic space. We obtain a commutative diagram as in the statement of the lemma. To finish the proof it suffices to show that this diagram is an elementary distinguished square, since then Lemma 81.9.1 implies that it is a pushout. Thus we have to show that $U \to W$ is open and that f is étale and is an isomorphism over $W \setminus U$. This follows from the choice of R; we omit the details. □

81.10. Formal glueing of quasi-coherent modules

0AEP This section is the analogue of More on Algebra, Section 15.89. In the case of morphisms of schemes, the result can be found in the paper by Joyet [Joy96]; this is a good place to start reading. For a discussion of applications to descent problems for stacks, see the paper by Moret-Bailly [MB96]. In the case of an affine morphism of schemes there is a statement in the appendix of the paper [FR70] but one needs to add the hypothesis that the closed subscheme is cut out by a finitely generated ideal (as in the paper by Joyet) since otherwise the result does not hold. A generalization of this material to (higher) derived categories with potential applications to nonflat situations can be found in [Bha16, Section 5].

We start with a lemma on abelian sheaves supported on closed subsets.

0AEQ **Lemma 81.10.1.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = f(\overline{y})$. We have

$$(Rf_*Q)_{\overline{x}} = Q_{\overline{y}}$$

in $D(\textit{Ab})$ for any object Q of $D(Y_{\acute{e}tale})$ supported on $|f^{-1}Z|$.

Proof. Consider the commutative diagram of algebraic spaces

$$\begin{array}{ccc} f^{-1}Z & \xrightarrow{i'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

By Cohomology of Spaces, Lemma 69.9.4 we can write $Q = Ri'_*K'$ for some object K' of $D(f^{-1}Z_{étale})$. By Morphisms of Spaces, Lemma 67.53.7 we have $K' = (f')^{-1}K$ with $K = Rf'_*K'$. Then we have $Rf_*Q = Rf_*Ri'_*K' = Ri_*Rf'_*K' = Ri_*K$. Let $\overline{z}$ be the geometric point of Z corresponding to $\overline{x}$ and let $\overline{z}'$ be the geometric point of $f^{-1}Z$ corresponding to $\overline{y}$. We obtain the result of the lemma as follows

$$Q_{\overline{y}} = (Ri'_*K')_{\overline{y}} = K'_{\overline{z}'} = (f')^{-1}K_{\overline{z}'} = K_{\overline{z}} = Ri_*K_{\overline{x}} = Rf_*Q_{\overline{x}}$$

The middle equality holds because of the description of the stalk of a pullback given in Properties of Spaces, Lemma 66.19.9. □

0AER Lemma 81.10.2. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = f(\overline{y})$. Let $\mathcal{G}$ be an abelian sheaf on Y. Then the map of two term complexes

$$(f_*\mathcal{G}_{\overline{x}} \to (f \circ j')_*(\mathcal{G}|_V)_{\overline{x}}) \longrightarrow (\mathcal{G}_{\overline{y}} \to j'_*(\mathcal{G}|_V)_{\overline{y}})$$

induces an isomorphism on kernels and an injection on cokernels. Here $V = Y \setminus f^{-1}Z$ and $j' : V \to Y$ is the inclusion.

Proof. Choose a distinguished triangle

$$\mathcal{G} \to Rj'_*\mathcal{G}|_V \to Q \to \mathcal{G}[1]$$

n $D(Y_{étale})$. The cohomology sheaves of Q are supported on $|f^{-1}Z|$. We apply Rf_* and we obtain

$$Rf_*\mathcal{G} \to Rf_*Rj'_*\mathcal{G}|_V \to Rf_*Q \to Rf_*\mathcal{G}[1]$$

Taking stalks at $\overline{x}$ we obtain an exact sequence

$$0 \to (R^{-1}f_*Q)_{\overline{x}} \to f_*\mathcal{G}_{\overline{x}} \to (f \circ j')_*(\mathcal{G}|_V)_{\overline{x}} \to (R^0f_*Q)_{\overline{x}}$$

We can compare this with the exact sequence

$$0 \to H^{-1}(Q)_{\overline{y}} \to \mathcal{G}_{\overline{y}} \to j'_*(\mathcal{G}|_V)_{\overline{y}} \to H^0(Q)_{\overline{y}}$$

Thus we see that the lemma follows because $Q_{\overline{y}} = Rf_*Q_{\overline{x}}$ by Lemma 81.10.1. □

0AES Lemma 81.10.3. Let S be a scheme. Let X be an algebraic space over S. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism. Let $\overline{x}$ be a geometric point of X and let $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to X$ be the canonical morphism. For a quasi-coherent module $\mathcal{G}$ on Y we have

$$f_*\mathcal{G}_{\overline{x}} = \Gamma(Y \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), p^*\mathcal{F})$$

where $p : Y \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to Y$ is the projection.

Proof. Observe that $f_*\mathcal{G}_{\overline{x}} = \Gamma(\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), h^*f_*\mathcal{G})$ where $h : \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to X$. Hence the result is true because h is flat so that Cohomology of Spaces, Lemma 69.11.2 applies. □

0AET Lemma 81.10.4. Let S be a scheme. Let X be an algebraic space over S. Let $i : Z \to X$ be a closed immersion of finite presentation. Let $Q \in D_{QCoh}(\mathcal{O}_X)$ be supported on $|Z|$. Let $\overline{x}$ be a geometric point of X and let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$ be the stalk of the ideal sheaf of Z. Then the cohomology modules $H^n(Q_{\overline{x}})$ are $I_{\overline{x}}$-power torsion (see More on Algebra, Definition 15.88.1).

Proof. Choose an affine scheme U and an étale morphism $U \to X$ such that $\overline{x}$ lifts to a geometric point $\overline{u}$ of U. Then we can replace X by U, Z by $U \times_X Z$, Q by the restriction $Q|_U$, and $\overline{x}$ by $\overline{u}$. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the ideal defining Z. Since $i : Z \to X$ is of finite presentation, the ideal $I = (f_1, \ldots, f_r)$ is finitely generated. The object Q comes from a complex of A-modules $M^\bullet$, see Derived Categories of Spaces, Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5. Since the cohomology sheaves of Q are supported on Z we see that the localization $M^\bullet_f$ is acyclic for each $f \in I$. Take $x \in H^p(M^\bullet)$. By the above we can find n_i such that $f_i^{n_i}x = 0$ in $H^p(M^\bullet)$ for each i. Then with $n = \sum n_i$ we see that I^n annihilates x. Thus $H^p(M^\bullet)$ is I-power torsion. Since the ring map $A \to \mathcal{O}_{X,\overline{x}}$ is flat and since $I_{\overline{x}} = I\mathcal{O}_{X,\overline{x}}$ we conclude. □

0AEU Lemma 81.10.5. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume $f^{-1}Z \to Z$ is an isomorphism and that f is flat in every point of $f^{-1}Z$. For any Q in $D_{QCoh}(\mathcal{O}_Y)$ supported on $|f^{-1}Z|$ we have $Lf^*Rf_*Q = Q$.

Proof. We show the canonical map $Lf^*Rf_*Q \to Q$ is an isomorphism by checking on stalks at $\overline{y}$. If $\overline{y}$ is not in $f^{-1}Z$, then both sides are zero and the result is true. Assume the image $\overline{x}$ of $\overline{y}$ is in Z. By Lemma 81.10.1 we have $Rf_*Q_{\overline{x}} = Q_{\overline{y}}$ and since f is flat at $\overline{y}$ we see that

$$(Lf^*Rf_*Q)_{\overline{y}} = (Rf_*Q)_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} = Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}}$$

Thus we have to check that the canonical map

$$Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} \longrightarrow Q_{\overline{y}}$$

is an isomorphism in the derived category. Let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$ be the stalk of the ideal sheaf defining Z. Since $Z \to X$ is locally of finite presentation this ideal is finitely generated and the cohomology groups of $Q_{\overline{y}}$ are $I_{\overline{y}} = I_{\overline{x}}\mathcal{O}_{Y,\overline{y}}$-power torsion by Lemma 81.10.4 applied to Q on Y. It follows that they are also $I_{\overline{x}}$-power torsion. The ring map $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}$ is flat and induces an isomorphism after dividing by $I_{\overline{x}}$ and $I_{\overline{y}}$ because we assumed that $f^{-1}Z \to Z$ is an isomorphism. Hence we see that the cohomology modules of $Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}}$ are equal to the cohomology modules of $Q_{\overline{y}}$ by More on Algebra, Lemma 15.89.2 which finishes the proof. □

0AEV Situation 81.10.6. Here S is a base scheme, $f : Y \to X$ is a quasi-compact and quasi-separated morphism of algebraic spaces over S, and $Z \to X$ is a closed immersion of finite presentation. We assume that $f^{-1}(Z) \to Z$ is an isomorphism and that f is flat in every point $x \in |f^{-1}Z|$. We set $U = X \setminus Z$ and $V = Y \setminus f^{-1}(Z)$. Picture

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f|_V"'] & Y \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

In Situation 81.10.6 we define $\mathrm{QCoh}(Y \to X, Z)$ as the category of triples $(\mathcal{H}, \mathcal{G}, \varphi)$ where $\mathcal{H}$ is a quasi-coherent sheaf of $\mathcal{O}_U$-modules, $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, and $\varphi : f^*\mathcal{H} \to \mathcal{G}|_V$ is an isomorphism of $\mathcal{O}_V$-modules. There is a canonical functor

0AEW (81.10.6.1)
$$QCoh(\mathcal{O}_X) \longrightarrow \mathrm{QCoh}(Y \to X, Z)$$

which maps $\mathcal{F}$ to the system $(\mathcal{F}|_U, f^*\mathcal{F}, can)$. By analogy with the proof given in the affine case, we construct a functor in the opposite direction. To an object $(\mathcal{H}, \mathcal{G}, \varphi)$ we assign the $\mathcal{O}_X$-module

0AEX (81.10.6.2)
$$\mathrm{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)$$

Observe that j and j' are quasi-compact morphisms as $Z \to X$ is of finite presentation. Hence f_*, j_*, and $(f \circ j')_*$ transform quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 67.11.2). Thus the module (81.10.6.2) is quasi-coherent.

0AEY **Lemma 81.10.7.** In Situation 81.10.6. The functor (81.10.6.2) is right adjoint to the functor (81.10.6.1).

Proof. This follows easily from the adjointness of f^* to f_* and j^* to j_*. Details omitted. □

0AEZ **Lemma 81.10.8.** In Situation 81.10.6. Let $X' \to X$ be a flat morphism of algebraic spaces. Set $Z' = X' \times_X Z$ and $Y' = X' \times_X Y$. The pullbacks $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{X'})$ and $QCoh(Y \to X, Z) \to QCoh(Y' \to X', Z')$ are compatible with the functors (81.10.6.2) and 81.10.6.1).

Proof. This is true because pullback commutes with pullback and because flat pullback commutes with pushforward along quasi-compact and quasi-separated morphisms, see Cohomology of Spaces, Lemma 69.11.2. □

0AF0 **Proposition 81.10.9.** In Situation 81.10.6 the functor (81.10.6.1) is an equivalence with quasi-inverse given by (81.10.6.2).

Proof. We first treat the special case where X and Y are affine schemes and where the morphism f is flat. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Then f corresponds to a flat ring map $R \to S$. Moreover, $Z \subset X$ is cut out by a finitely generated ideal $I \subset R$. Choose generators $f_1, \ldots, f_t \in I$. By the description of quasi-coherent modules in terms of modules (Schemes, Section 26.7), we see that the category $\mathrm{QCoh}(Y \to X, Z)$ is canonically equivalent to the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ of More on Algebra, Remark 15.89.10 such that the functors (81.10.6.1) and (81.10.6.2) correspond to the functors Can and H^0. Hence the result follows from More on Algebra, Proposition 15.89.15 in this case.

We return to the general case. Let $\mathcal{F}$ be a quasi-coherent module on X. We will show that
$$\alpha : \mathcal{F} \longrightarrow \mathrm{Ker}\left(j_*\mathcal{F}|_U \oplus f_*f^*\mathcal{F} \to (f \circ j')_*f^*\mathcal{F}|_V\right)$$
is an isomorphism. Let $(\mathcal{H}, \mathcal{G}, \varphi)$ be an object of $QCoh(Y \to X, Z)$. We will show that
$$\beta : f^* \mathrm{Ker}\left(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V\right) \longrightarrow \mathcal{G}$$
and
$$\gamma : j^* \mathrm{Ker}\left(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V\right) \longrightarrow \mathcal{H}$$

are isomorphisms. To see these statements are true it suffices to look at stalks. Let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ of X.

Fix an object $(\mathcal{H}, \mathcal{G}, \varphi)$ of $QCoh(Y \to X, Z)$. By Lemma 81.10.2 and a diagram chase (omitted) the canonical map

$$\mathrm{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)_{\overline{x}} \longrightarrow \mathrm{Ker}(j_*\mathcal{H}_{\overline{x}} \oplus \mathcal{G}_{\overline{y}} \to j'_*\mathcal{G}_{\overline{y}})$$

is an isomorphism.

In particular, if $\overline{y}$ is a geometric point of V, then we see that $j'_*\mathcal{G}_{\overline{y}} = \mathcal{G}_{\overline{y}}$ and hence that this kernel is equal to $\mathcal{H}_{\overline{x}}$. This easily implies that $\alpha_{\overline{x}}$, $\beta_{\overline{x}}$, and $\beta_{\overline{y}}$ are isomorphisms in this case.

Next, assume that $\overline{y}$ is a point of $f^{-1}Z$. Let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$, resp. $I_{\overline{y}} \subset \mathcal{O}_{Y,\overline{y}}$ be the stalk of the ideal cutting out Z, resp. $f^{-1}Z$. Then $I_{\overline{x}}$ is a finitely generated ideal, $I_{\overline{y}} = I_{\overline{x}}\mathcal{O}_{Y,\overline{y}}$, and $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}$ is a flat local homomorphism inducing an isomorphism $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}} = \mathcal{O}_{Y,\overline{y}}/I_{\overline{y}}$. At this point we can bootstrap using the diagram of categories

$$\begin{tikzcd}
QCoh(\mathcal{O}_X) \arrow[r, "(81.10.6.1)"'] \arrow[d] & QCoh(Y \to X, Z) \arrow[l, bend right, "(81.10.6.2)"'] \arrow[d] \\
\mathrm{Mod}_{\mathcal{O}_{X,\overline{x}}} \arrow[r, "\mathrm{Can}"] & \mathrm{Glue}(\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}, f_1, \ldots, f_t) \arrow[l, bend left, "H^0"]
\end{tikzcd}$$

Namely, as in the first paragraph of the proof we identify

$$\mathrm{Glue}(\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}, f_1, \ldots, f_t) = QCoh(\mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), V(I_{\overline{x}}))$$

The right vertical functor is given by pullback, and it is clear that the inner square is commutative. Our computation of the stalk of the kernel in the third paragraph of the proof combined with Lemma 81.10.3 implies that the outer square (using the curved arrows) commutes. Thus we conclude using the case of a flat morphism of affine schemes which we handled in the first paragraph of the proof. □

0AFJ Lemma 81.10.10. In Situation 81.10.6 the functor Rf_* induces an equivalence between $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ and $D_{QCoh,|Z|}(\mathcal{O}_X)$ with quasi-inverse given by Lf^*.

Proof. Since f is quasi-compact and quasi-separated we see that Rf_* defines a functor from $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ to $D_{QCoh,|Z|}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 75.6.1. By Derived Categories of Spaces, Lemma 75.5.5 we see that Lf^* maps $D_{QCoh,|Z|}(\mathcal{O}_X)$ into $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. In Lemma 81.10.5 we have seen that $Lf^*Rf_*Q = Q$ for Q in $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. By the dual of Derived Categories, Lemma 13.7.2 to finish the proof it suffices to show that $Lf^*K = 0$ implies $K = 0$ for K in $D_{QCoh,|Z|}(\mathcal{O}_X)$. This follows from the fact that f is flat at all points of $f^{-1}Z$ and the fact that $f^{-1}Z \to Z$ is surjective. □

0AF1 Lemma 81.10.11. In Situation 81.10.6 there exists an fpqc covering $\{X_i \to X\}_{i \in I}$ refining the family $\{U \to X, Y \to X\}$.

Proof. For the definition and general properties of fpqc coverings we refer to Topologies, Section 34.9. In particular, we can first choose an étale covering $\{X_i \to X\}$ with X_i affine and by base changing Y, Z, and U to each X_i we reduce to the case where X is affine. In this case U is quasi-compact and hence a finite union $U = U_1 \cup \ldots \cup U_n$ of affine opens. Then Z is quasi-compact hence also $f^{-1}Z$ is quasi-compact. Thus we can choose an affine scheme W and an étale morphism $h : W \to Y$ such that $h^{-1}f^{-1}Z \to f^{-1}Z$ is surjective. Say $W = \mathrm{Spec}(B)$ and $h^{-1}f^{-1}Z = V(J)$ where $J \subset B$ is an ideal of finite type. By Pro-étale Cohomology, Lemma 61.5.1 there exists a localization $B \to B'$ such that points of $\mathrm{Spec}(B')$ correspond exactly to points of $W = \mathrm{Spec}(B)$ specializing to $h^{-1}f^{-1}Z = V(J)$. It follows that the composition $\mathrm{Spec}(B') \to \mathrm{Spec}(B) = W \to Y \to X$ is flat as by assumption $f : Y \to X$ is flat at all the points of $f^{-1}Z$. Then $\{\mathrm{Spec}(B') \to X, U_1 \to X, \ldots, U_n \to X\}$ is an fpqc covering by Topologies, Lemma 34.9.2. □

81.11. Formal glueing of algebraic spaces

0AF2 In Situation 81.10.6 we consider the category $\mathrm{Spaces}(Y \to X, Z)$ of commutative diagrams of algebraic spaces over S of the form

$$\begin{tikzcd} U' \arrow[d] & V' \arrow[l] \arrow[r] \arrow[d] & Y' \arrow[d] \\ U & V \arrow[l] \arrow[r] & Y \end{tikzcd}$$

where both squares are cartesian. There is a canonical functor

0AF3 (81.11.0.1) $$\mathrm{Spaces}/X \longrightarrow \mathrm{Spaces}(Y \to X, Z)$$

which maps $X' \to X$ to the morphisms $U \times_X X' \leftarrow V \times_X X' \to Y \times_X X'$.

0AF4 **Lemma 81.11.1.** In Situation 81.10.6 the functor (81.11.0.1) restricts to an equivalence

(1) from the category of algebraic spaces affine over X to the full subcategory of $\mathrm{Spaces}(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ affine,
(2) from the category of closed immersions $X' \to X$ to the full subcategory of $\mathrm{Spaces}(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ closed immersions, and
(3) same statement as in (2) for finite morphisms.

Proof. The category of algebraic spaces affine over X is equivalent to the category of quasi-coherent sheaves $\mathcal{A}$ of $\mathcal{O}_X$-algebras. The full subcategory of $\mathrm{Spaces}(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ affine is equivalent to the category of algebra objects of $QCoh(Y \to X, Z)$. In both cases this follows from Morphisms of Spaces, Lemma 67.20.7 with quasi-inverse given by the relative spectrum construction (Morphisms of Spaces, Definition 67.20.8) which commutes with arbitrary base change. Thus part (1) of the lemma follows from Proposition 81.10.9.

Fully faithfulness in part (2) follows from part (1). For essential surjectivity, we reduce by part (1) to proving that $X' \to X$ is a closed immersion if and only if both $U \times_X X' \to U$ and $Y \times_X X' \to Y$ are closed immersions. By Lemma 81.10.11 $\{U \to X, Y \to X\}$ can be refined by an fpqc covering. Hence the result follows from Descent on Spaces, Lemma 74.11.17.

For (3) use the argument proving (2) and Descent on Spaces, Lemma 74.11.23. □

0AF5 **Lemma 81.11.2.** In Situation 81.10.6 the functor (81.11.0.1) reflects isomorphisms.

Proof. By a formal argument with base change, this reduces to the following question: A morphism $a : X' \to X$ of algebraic spaces such that $U \times_X X' \to U$ and $Y \times_X X' \to Y$ are isomorphisms, is an isomorphism. The family $\{U \to X, Y \to X\}$ can be refined by an fpqc covering by Lemma 81.10.11. Hence the result follows from Descent on Spaces, Lemma 74.11.15. □

0AF6 **Lemma 81.11.3.** In Situation 81.10.6 the functor (81.11.0.1) is fully faithful on algebraic spaces separated over X. More precisely, it induces a bijection

$$\mathrm{Mor}_X(X_1', X_2') \longrightarrow \mathrm{Mor}_{\mathrm{Spaces}(Y \to X, Z)}(F(X_1'), F(X_2'))$$

whenever $X_2' \to X$ is separated.

Proof. Since $X_2' \to X$ is separated, the graph $i : X_1' \to X_1' \times_X X_2'$ of a morphism $X_1' \to X_2'$ over X is a closed immersion, see Morphisms of Spaces, Lemma 67.4.6. Moreover a closed immersion $i : T \to X_1' \times_X X_2'$ is the graph of a morphism if and only if $\mathrm{pr}_1 \circ i$ is an isomorphism. The same is true for

(1) the graph of a morphism $U \times_X X_1' \to U \times_X X_2'$ over U,
(2) the graph of a morphism $V \times_X X_1' \to V \times_X X_2'$ over V, and
(3) the graph of a morphism $Y \times_X X_1' \to Y \times_X X_2'$ over Y.

Moreover, if morphisms as in (1), (2), (3) fit together to form a morphism in the category $\mathrm{Spaces}(Y \to X, Z)$, then these graphs fit together to give an object of $\mathrm{Spaces}(Y \times_X (X_1' \times_X X_2') \to X_1' \times_X X_2', Z \times_X (X_1' \times_X X_2'))$ whose triple of morphisms are closed immersions. The proof is finished by applying Lemmas 81.11.1 and 81.11.2. □

81.12. Glueing and the Beauville-Laszlo theorem

0F9M Let $R \to R'$ be a ring homomorphism and let $f \in R$ be an element such that

$$0 \to R \to R_f \oplus R' \to R'_f \to 0$$

is a short exact sequence. This implies that $R/f^nR \cong R'/f^nR'$ for all n and $(R \to R', f)$ is a glueing pair in the sense of More on Algebra, Section 15.90. Set $X = \mathrm{Spec}(R)$, $U = \mathrm{Spec}(R_f)$, $X' = \mathrm{Spec}(R')$ and $U' = \mathrm{Spec}(R'_f)$. Picture

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

In this situation we can consider the category $\mathrm{Spaces}(U \leftarrow U' \to X')$ whose objects are commutative diagrams

$$\begin{tikzcd} V \arrow[d] & V' \arrow[l] \arrow[r] \arrow[d] & Y' \arrow[d] \\ U & U' \arrow[l] \arrow[r] & X' \end{tikzcd}$$

of algebraic spaces with both squares cartesian and whose morphism are defined in the obvious manner. An object of this category will be denoted (V, V', Y') with arrows surpressed from the notation. There is a functor

0F9N (81.12.0.1) $$\mathrm{Spaces}/X \longrightarrow \mathrm{Spaces}(U \leftarrow U' \to X')$$

given by base change: $Y \mapsto (U \times_X Y, U' \times_X Y, X' \times_X Y)$.

We have seen in More on Algebra, Section 15.90 that not every R-module M can be recovered from its gluing data. Similarly, the functor (81.12.0.1) won't be fully faithful on the category of all spaces over X. In order to single out a suitable subcategory of algebraic spaces over X we need a lemma.

0F9P Lemma 81.12.1. Let $(R \to R', f)$ be a glueing pair, see above. Let Y be an algebraic space over X. The following are equivalent

(1) there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ with Y_i affine and $\Gamma(Y_i, \mathcal{O}_{Y_i})$ glueable as an R-module,
(2) for every étale morphism $W \to Y$ with W affine $\Gamma(W, \mathcal{O}_W)$ is a glueable R-module.

Proof. It is immediate that (2) implies (1). Assume $\{Y_i \to Y\}$ is as in (1) and let $W \to Y$ be as in (2). Then $\{Y_i \times_Y W \to W\}_{i \in I}$ is an étale covering, which we may refine by an étale covering $\{W_j \to W\}_{j=1,\ldots,m}$ with W_j affine (Topologies, Lemma 34.4.4). Thus to finish the proof it suffices to show the following three algebraic statements:

(1) if $R \to A \to B$ are ring maps with $A \to B$ étale and A glueable as an R-module, then B is glueable as an R-module,
(2) finite products of glueable R-modules are glueable,
(3) if $R \to A \to B$ are ring maps with $A \to B$ faithfully étale and B glueable as an R-module, then A is glueable as an R-module.

Namely, the first of these will imply that $\Gamma(W_j, \mathcal{O}_{W_j})$ is a glueable R-module, the second will imply that $\prod \Gamma(W_j, \mathcal{O}_{W_j})$ is a glueable R-module, and the third will imply that $\Gamma(W, \mathcal{O}_W)$ is a glueable R-module.

Consider an étale R-algebra homomorphism $A \to B$. Set $A' = A \otimes_R R'$ and $B' = B \otimes_R R' = A' \otimes_A B$. Statements (1) and (3) then follow from the following facts: (a) A, resp. B is glueable if and only if the sequence

$$0 \to A \to A_f \oplus A' \to A'_f \to 0, \quad \text{resp.} \quad 0 \to B \to B_f \oplus B' \to B'_f \to 0,$$

is exact, (b) the second sequence is equal to the functor $- \otimes_A B$ applied to the first and (c) (faithful) flatness of $A \to B$. We omit the proof of (2). $\square$

Let $(R \to R', f)$ be a glueing pair, see above. We will say an algebraic space Y over $X = \text{Spec}(R)$ is glueable for $(R \to R', f)$ if the equivalent conditions of Lemma 81.12.1 are satisfied.

0F9Q Lemma 81.12.2. Let $(R \to R', f)$ be a glueing pair, see above. The functor (81.12.0.1) restricts to an equivalence between the category of affine Y/X which are glueable for $(R \to R', f)$ and the full subcategory of objects (V, V', Y') of $\text{Spaces}(U \leftarrow U' \to X')$ with V, V', Y' affine.

Proof. Let (V, V', Y') be an object of $\text{Spaces}(U \leftarrow U' \to X')$ with V, V', Y' affine. Write $V = \text{Spec}(A_1)$ and $Y' = \text{Spec}(A')$. By our definition of the category $\text{Spaces}(U \leftarrow U' \to X')$ we find that V' is the spectrum of $A_1 \otimes_{R_f} R'_f = A_1 \otimes_R R'$ and the spectrum of A'_f. Hence we get an isomorphism $\varphi : A'_f \to A_1 \otimes_R R'$ of R'_f-algebras. By More on Algebra, Theorem 15.90.17 there exists a unique glueable

R-module A and isomorphisms $A_f \to A_1$ and $A \otimes_R R' \to A'$ of modules compatible with φ. Since the sequence

$$0 \to A \to A_1 \oplus A' \to A'_f \to 0$$

is short exact, the multiplications on A_1 and A' define a unique R-algebra structure on A such that the maps $A \to A_1$ and $A \to A'$ are ring homomorphisms. We omit the verification that this construction defines a quasi-inverse to the functor (81.12.0.1) restricted to the subcategories mentioned in the statement of the lemma. □

0F9R Lemma 81.12.3. Let P be one of the following properties of morphisms: "finite", "closed immersion", "flat", "finite type", "flat and finite presentation", "étale". Under the equivalence of Lemma 81.12.2 the morphisms having P correspond to morphisms of triples whose components have P.

Proof. Let P' be one of the following properties of homomorphisms of rings: "finite", "surjective", "flat", "finite type", "flat and of finite presentation", "étale". Translated into algebra, the statement means the following: If $A \to B$ is an R-algebra homomorphism and A and B are glueable for $(R \to R', f)$, then $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ have P' if and only if $A \to B$ has P'.

By More on Algebra, Lemmas 15.90.5 and 15.90.19 the algebraic statement is true for P' equal to "finite" or "flat".

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are surjective, then $N = B/A$ is an R-module with $N_f = 0$ and $N \otimes_R R' = 0$ and hence vanishes by More on Algebra, Lemma 15.90.3. Thus $A \to B$ is surjective.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are finite type, then we can choose an A-algebra homomorphism $A[x_1, \ldots, x_n] \to B$ such that $A_f[x_1, \ldots, x_n] \to B_f$ and $(A \otimes_R R')[x_1, \ldots, x_n] \to B \otimes_R R'$ are surjective (small detail omitted). We conclude that $A[x_1, \ldots, x_n] \to B$ is surjective by the previous result. Thus $A \to B$ is of finite type.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are flat and of finite presentation, then we know that $A \to B$ is flat and of finite type by what we have already shown. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and denote I the kernel. By flatness of B over A we see that I_f is the kernel of $A_f[x_1, \ldots, x_n] \to B_f$ and $I \otimes_R R'$ is the kernel of $A \otimes_R R'[x_1, \ldots, x_n] \to B \otimes_R R'$. Thus I_f is a finite $A_f[x_1, \ldots, x_n]$-module and $I \otimes_R R'$ is a finite $(A \otimes_R R')[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 15.90.5 applied to I viewed as a module over $A[x_1, \ldots, x_n]$ we conclude that I is a finitely generated ideal and we conclude $A \to B$ is flat and of finite presentation.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are étale, then we know that $A \to B$ is flat and of finite presentation by what we have already shown. Since the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are isomorphic to fibres of $\mathrm{Spec}(B_f) \to \mathrm{Spec}(A_f)$ or $\mathrm{Spec}(B/fB) \to \mathrm{Spec}(A/fA)$, we conclude that $A \to B$ is unramified, see Morphisms, Lemmas 29.35.11 and 29.35.12. We conclude that $A \to B$ is étale by Morphisms, Lemma 29.36.16 for example. □

0F9S Lemma 81.12.4. Let $(R \to R', f)$ be a glueing pair, see above. The functor (81.12.0.1) is faithful on the full subcategory of algebraic spaces Y/X glueable for $(R \to R', f)$.

Proof. Let $f, g : Y \to Z$ be two morphisms of algebraic spaces over X with Y and Z glueable for $(R \to R', f)$ such that f and g are mapped to the same morphism in the category $\text{Spaces}(U \leftarrow U' \to X')$. We have to show the equalizer $E \to Y$ of f and g is an isomorphism. Working étale locally on Y we may assume Y is an affine scheme. Then E is a scheme and the morphism $E \to Y$ is a monomorphism and locally quasi-finite, see Morphisms of Spaces, Lemma 67.4.1. Moreover, the base change of $E \to Y$ to U and to X' is an isomorphism. As Y is the disjoint union of the affine open $V = U \times_X Y$ and the affine closed $V(f) \times_X Y$, we conclude E is the disjoint union of their isomorphic inverse images. It follows in particular that E is quasi-compact. By Zariski's main theorem (More on Morphisms, Lemma 37.43.3) we conclude that E is quasi-affine. Set $B = \Gamma(E, \mathcal{O}_E)$ and $A = \Gamma(Y, \mathcal{O}_Y)$ so that we have an R-algebra homomorphism $A \to B$. Since $E \to Y$ becomes an isomorphism after base change to U and X' we obtain ring maps $B \to A_f$ and $B \to A \otimes_R R'$ agreeing as maps into $A \otimes_R R'_f$. Since A is glueable for $(R \to R', f)$ we get a ring map $B \to A$ which is left inverse to the map $A \to B$. The corresponding morphism $Y = \text{Spec}(A) \to \text{Spec}(B)$ maps into the open subscheme $E \subset \text{Spec}(B)$ pointwise because this is true after base change to U and X'. Hence we get a morphism $Y \to E$ over Y. Since $E \to Y$ is a monomorhism we conclude $Y \to E$ is an isomorphism as desired. $\square$

0F9T Lemma 81.12.5. Let $(R \to R', f)$ be a glueing pair, see above. The functor (81.12.0.1) is fully faithful on the full subcategory of algebraic spaces Y/X which are (a) glueable for $(R \to R', f)$ and (b) have affine diagonal $Y \to Y \times_X Y$.

Proof. Let Y, Z be two algebraic spaces over X which are both glueable for $(R \to R', f)$ and assume the diagonal of Z is affine. Let $a : U \times_X Y \to U \times_X Z$ over U and $b : X' \times_X Y \to X' \times_X Z$ over X' be two morphisms of algebraic spaces which induce the same morphism $c : U' \times_X Y \to U' \times_X Z$ over U'. We want to construct a morphism $f : Y \to Z$ over X which produces the morphisms a, b on base change to U, X'. By the faithfulness of Lemma 81.12.4, it suffices to construct the morphism f étale locally on Y (details omitted). Thus we may and do assume Y is affine.

Let $y \in |Y|$ be a point. If y maps into the open $U \subset X$, then $U \times_X Y$ is an open of Y on which the morphism f is defined (we can just take a). Thus we may assume y maps into the closed subset $V(f)$ of X. Since $R/fR = R'/fR'$ there is a unique point $y' \in |X' \times_X Y|$ mapping to y. Denote $z' = b(y') \in |X' \times_X Z|$ and $z \in |Z|$ the images of y'. Choose an étale neighbourhood $(W, w) \to (Z, z)$ with W affine. Observe that

$$(U \times_X W) \times_{U \times_X Z, a} (U \times_X Y), \quad (U' \times_X W) \times_{U' \times_X Z, c} (U' \times_X Y),$$

and

$$(X' \times_X W) \times_{X' \times_X Z, b} (X' \times_X Y)$$

form an object of $\text{Spaces}(U \leftarrow U' \to X')$ with affine parts (this is where we use that Z has affine diagonal). Hence by Lemma 81.12.2 there exists a unique affine scheme V glueable for $(R \to R', f)$ such that

$$(U \times_X V, U' \times_X V, X' \times_X V)$$

is the triple displayed above. By fully faithfulness for the affine case (Lemma 81.12.2) we get a unique morphisms $V \to W$ and $V \to Y$ agreeing with the first and second projection morphisms over U and X' in the construction above. By

Lemma 81.12.3 the morphism $V \to Y$ is étale. To finish the proof, it suffices to show that there is a point $v \in |V|$ mapping to y (because then f is defined on an étale neighbourhood of y, namely V). There is a unique point $w' \in |X' \times_X W|$ mapping to w. By uniqueness w' is mapped to z' under the map $|X' \times_X W| \to |X' \times_X Z|$. Then we consider the cartesian diagram

$$\begin{tikzcd} X' \times_X V \arrow[r] \arrow[d] & X' \times_X W \arrow[d] \\ X' \times_X Y \arrow[r] & X' \times_X Z \end{tikzcd}$$

to see that there is a point $v' \in |X' \times_X V|$ mapping to y' and w', see Properties of Spaces, Lemma 66.4.3. Of course the image v of v' in $|V|$ maps to y and the proof is complete. $\square$

0F9U Lemma 81.12.6. Let $(R \to R', f)$ be a glueing pair, see above. Any object (V, V', Y') of $\text{Spaces}(U \leftarrow U' \to X')$ with V, V', Y' quasi-affine is isomorphic to the image under the functor (81.12.0.1) of a separated algebraic space Y over X.

Proof. Choose n', $T' \to Y'$ and n_1, $T_1 \to V$ as in Properties, Lemma 28.18.6. Picture

$$\begin{tikzcd} & & T_1 \times_V V' \times_Y T' \arrow[dl] \arrow[dr] & & \\ T_1 \arrow[d] & T_1 \times_V V' \arrow[l] \arrow[dr] & & V' \times_{Y'} T' \arrow[r] \arrow[dl] & T' \arrow[d] \\ V & & V' \arrow[ll] \arrow[rr] & & Y' \end{tikzcd}$$

Observe that $T_1 \times_V V'$ and $V' \times_{Y'} T'$ are affine (namely the morphisms $V' \to V$ and $V' \to Y'$ are affine as base changes of the affine morphisms $U' \to U$ and $U' \to X'$). By construction we see that

$$\mathbf{A}^{n'}_{T_1 \times_V V'} \cong T_1 \times_V V' \times_{Y'} T' \cong \mathbf{A}^{n_1}_{V' \times_{Y'} T'}$$

In other words, the affine schemes $\mathbf{A}^{n'}_{T_1}$ and $\mathbf{A}^{n_1}_{T'}$ are part of a triple making an affine object of $\text{Spaces}(U \leftarrow U' \to X')$. By Lemma 81.12.2 there exists a morphism of affine schemes $T \to X$ and isomorphisms $U \times_X T \cong \mathbf{A}^{n'}_{T_1}$ and $X' \times_X T \cong \mathbf{A}^{n_1}_{T'}$ compatible with the isomorphisms displayed above. These isomorphisms produce morphisms

$$U \times_X T \longrightarrow V \quad \text{and} \quad X' \times_X T \longrightarrow Y'$$

satisfying the property of Properties, Lemma 28.18.6 with $n = n' + n_1$ and moreover define a morphism from the triple $(U \times_X T, U' \times_X T, X' \times_X T)$ to our triple (V, V', Y') in the category $\text{Spaces}(U \leftarrow U' \to X')$.

By Lemma 81.12.2 there is an affine scheme W whose image in $\text{Spaces}(U \leftarrow U' \to X')$ is isomorphic to the triple

$$((U \times_X T) \times_V (U \times_X T), (U' \times_X T) \times_{V'} (U' \times_X T), (X' \times_X T) \times_{Y'} (X' \times_X T))$$

By fully faithfulness of this construction, we obtain two maps $p_0, p_1 : W \to T$ whose base changes to U, U', X' are the projection morphisms. By Lemma 81.12.3 the morphisms p_0, p_1 are flat and of finite presentation and the morphism $(p_0, p_1) :$

$W \to T \times_X T$ is a closed immersion. In fact, $W \to T \times_X T$ is an equivalence relation: by the lemmas used above we may check symmetry, reflexivity, and transitivity after base change to U and X', where these are obvious (details omitted). Thus the quotient sheaf

$$Y = T/W$$

is an algebraic space for example by Bootstrap, Theorem 80.10.1. Since it is clear that Y/X is sent to the triple (V, V', Y'). The base change of the diagonal $\Delta : Y \to Y \times_X Y$ by the quasi-compact surjective flat morphism $T \times_X T \to Y \times_X Y$ is the closed immersion $W \to T \times_X T$. Thus Δ is a closed immersion by Descent on Spaces, Lemma 74.11.17. Thus the algebraic space Y is separated and the proof is complete. □

81.13. Coequalizers and glueing

0AGF Let X be a Noetherian algebraic space and $Z \to X$ a closed subspace. Let $X' \to X$ be the blowing up in Z. In this section we show that X can be recovered from X', Z_n and glueing data where Z_n is the nth infinitesimal neighbourhood of Z in X.

0AGG **Lemma 81.13.1.** *Let S be a scheme. Let*

$$g : Y \longrightarrow X$$

be a morphism of algebraic spaces over S. Assume X is locally Noetherian, and g is proper. Let $R = Y \times_X Y$ with projection morphisms $t, s : R \to Y$. There exists a coequalizer X' of $s, t : R \to Y$ in the category of algebraic spaces over S. Moreover

1. *The morphism $X' \to X$ is finite.*
2. *The morphism $Y \to X'$ is proper.*
3. *The morphism $Y \to X'$ is surjective.*
4. *The morphism $X' \to X$ is universally injective.*
5. *If g is surjective, the morphism $X' \to X$ is a universal homeomorphism.*

Proof. Denote $h : R \to X$ denote the composition of either s or t with g. Then h is proper by Morphisms of Spaces, Lemmas 67.40.3 and 67.40.4. The sheaves

$$g_*\mathcal{O}_Y \quad \text{and} \quad h_*\mathcal{O}_R$$

are coherent $\mathcal{O}_X$-algebras by Cohomology of Spaces, Lemma 69.20.2. The X-morphisms s, t induce $\mathcal{O}_X$-algebra maps $s^\sharp, t^\sharp$ from the first to the second. Set

$$\mathcal{A} = \text{Equalizer}\left(s^\sharp, t^\sharp : g_*\mathcal{O}_Y \longrightarrow h_*\mathcal{O}_R\right)$$

Then $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra and we can define

$$X' = \underline{\text{Spec}}_X(\mathcal{A})$$

as in Morphisms of Spaces, Definition 67.20.8. By Morphisms of Spaces, Remark 67.20.9 and functoriality of the $\underline{\text{Spec}}$ construction there is a factorization

$$Y \longrightarrow X' \longrightarrow X$$

and the morphism $g' : Y \to X'$ equalizes s and t.

Before we show that X' is the coequalizer of s and t, we show that $Y \to X'$ and $X' \to X$ have the desired properties. Since $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module it is clear that $X' \to X$ is a finite morphism of algebraic spaces. This proves (1). The morphism $Y \to X'$ is proper by Morphisms of Spaces, Lemma 67.40.6. This proves (2). Denote $Y \to Y' \to X$ with $Y' = \underline{\text{Spec}}_X(g_*\mathcal{O}_Y)$ the Stein factorization of g, see More on Morphisms of Spaces, Theorem 76.36.4. Of course we obtain morphisms $Y \to Y' \to X' \to X$ fitting with the morphisms studied above. Since $\mathcal{O}_{X'} \subset g_*\mathcal{O}_Y$ is a finite extension we see that $Y' \to X'$ is finite and surjective. Some details omitted; hint: use Algebra, Lemma 10.36.17 and reduce to the affine case by étale localization. Since $Y \to Y'$ is surjective (with geometrically connected fibres) we conclude that $Y \to X'$ is surjective. This proves (3). To show that $X' \to X$ is universally injective, we have to show that $X' \to X' \times_X X'$ is surjective, see Morphisms of Spaces, Definition 67.19.3 and Lemma 67.19.2. Since $Y \to X'$ is surjective (see above) and since base changes and compositions of surjective morphisms are surjective by Morphisms of Spaces, Lemmas 67.5.5 and 67.5.4 we see that $Y \times_X Y \to X' \times_X X'$ is surjective. However, since $Y \to X'$ equalizes s and t, we see that $Y \times_X Y \to X' \times_X X'$ factors through $X' \to X' \times_X X'$ and we conclude this latter map is surjective. This proves (4). Finally, if g is surjective, then since g factors through $X' \to X$ we see that $X' \to X$ is surjective. Since a surjective, universally injective, finite morphism is a universal homeomorphism (because it is universally bijective and universally closed), this proves (5).

In the rest of the proof we show that $Y \to X'$ is the coequalizer of s and t in the category of algebraic spaces over S. Observe that X' is locally Noetherian (Morphisms of Spaces, Lemma 67.23.5). Moreover, observe that $Y \times_{X'} Y \to Y \times_X Y$ is an isomorphism as $Y \to X'$ equalizes s and t (this is a categorical statement). Hence in order to prove the statement that $Y \to X'$ is the coequalizer of s and t, we may and do assume $X = X'$. In other words, $\mathcal{O}_X$ is the equalizer of the maps $s^\sharp, t^\sharp : g_*\mathcal{O}_Y \to h_*\mathcal{O}_R$.

Let $X_1 \to X$ be a flat morphism of algebraic spaces over S with X_1 locally Noetherian. Denote $g_1 : Y_1 \to X_1$, $h_1 : R_1 \to X_1$ and $s_1, t_1 : R_1 \to Y_1$ the base changes of g, h, s, t to X_1. Of course g_1 is proper and $R_1 = Y_1 \times_{X_1} Y_1$. Since we have flat base change for pushforward of quasi-coherent modules, Cohomology of Spaces, Lemma 69.11.2, we see that $\mathcal{O}_{X_1}$ is the equalizer of the maps $s_1^\sharp, t_1^\sharp : g_{1,*}\mathcal{O}_{Y_1} \to h_{1,*}\mathcal{O}_{R_1}$. Hence all the assumptions we have are preserved by this base change.

At this point we are going to check conditions (1) and (2) of Lemma 81.3.3. Condition (1) follows from Lemma 81.5.1 and the fact that g is proper and surjective (because $X = X'$). To check condition (2), by the remarks on base change above, we reduce to the statement discussed and proved in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ is an affine scheme, $X = X'$ is an affine scheme, and Z is an affine scheme over S. We have to show that

$$\mathrm{Mor}_S(X, Z) \longrightarrow \mathrm{Equalizer}(s, t : \mathrm{Mor}_S(Y, Z) \to \mathrm{Mor}_S(R, Z))$$

is bijective. However, this is clear from the fact that $X = X'$ which implies $\mathcal{O}_X$ is the equalizer of the maps $s^\sharp, t^\sharp : g_*\mathcal{O}_Y \to h_*\mathcal{O}_R$ which in turn implies

$$\Gamma(X, \mathcal{O}_X) = \mathrm{Equalizer}\left(s^\sharp, t^\sharp : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(R, \mathcal{O}_R)\right)$$

Namely, we have

$$\mathrm{Mor}_S(X, Z) = \mathrm{Hom}_A(\Gamma(Z, \mathcal{O}_Z), \Gamma(X, \mathcal{O}_X))$$

and similarly for Y and R, see Properties of Spaces, Lemma 66.33.1. $\square$

We will work in the following situation.

0AGH Situation 81.13.2. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $Z \to X$ be a closed immersion and let $U \subset X$ be the complementary open subspace. Finally, let $f : X' \to X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \to U$ is an isomorphism.

0AGI Lemma 81.13.3. In Situation 81.13.2 let $Y = X' \amalg Z$ and $R = Y \times_X Y$ with projections $t, s : R \to Y$. There exists a coequalizer X_1 of $s, t : R \to Y$ in the category of algebraic spaces over S. The morphism $X_1 \to X$ is a finite universal homeomorphism, an isomorphism over U, and $Z \to X$ lifts to X_1.

Proof. Existence of X_1 and the fact that $X_1 \to X$ is a finite universal homeomorphism is a special case of Lemma 81.13.1. The formation of X_1 commutes with étale localization on X (see proof of Lemma 81.13.1). Thus the morphism $X_1 \to X$ is an isomorphism over U. It is immediate from the construction that $Z \to X$ lifts to X_1. $\square$

In Situation 81.13.2 for $n \geq 1$ let $Z_n \subset X$ be the nth order infinitesimal neighbourhood of Z in X, i.e., the closed subscheme defined by the nth power of the sheaf of ideals cutting out Z. Consider $Y_n = X' \amalg Z_n$ and $R_n = Y_n \times_X Y_n$ and the coequalizer

$$R_n \rightrightarrows Y_n \longrightarrow X_n \longrightarrow X$$

as in Lemma 81.13.3. The maps $Y_n \to Y_{n+1}$ and $R_n \to R_{n+1}$ induce morphisms

0AGJ (81.13.3.1)
$$X_1 \to X_2 \to X_3 \to \ldots \to X$$

Each of these morphisms is a universal homeomorphism as the morphisms $X_n \to X$ are universal homeomorphisms.

0AGK Lemma 81.13.4. In Situation 81.13.2 assume X quasi-compact. In (81.13.3.1) for all n large enough, there exists an m such that $X_n \to X_{n+m}$ factors through a closed immersion $X \to X_{n+m}$.

Proof. Let's look a bit more closely at the construction of X_n and how it changes as we increase n. We have $X_n = \underline{\mathrm{Spec}}(\mathcal{A}_n)$ where $\mathcal{A}_n$ is the equalizer of $s_n^\sharp$ and $t_n^\sharp$ going from $g_{n,*}\mathcal{O}_{Y_n}$ to $h_{n,*}\mathcal{O}_{R_n}$. Here $g_n : Y_n = X' \amalg Z_n \to X$ and $h_n : R_n = Y_n \times_X Y_n \to X$ are the given morphisms. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals corresponding to Z. Then

$$g_{n,*}\mathcal{O}_{Y_n} = f_*\mathcal{O}_{X'} \times \mathcal{O}_X/\mathcal{I}^n$$

Similarly, we have a decomposition

$$R_n = X' \times_X X' \amalg X' \times_X Z_n \amalg Z_n \times_X X' \amalg Z_n \times_X Z_n$$

As $Z_n \to X$ is a monomorphism, we see that $X' \times_X Z_n = Z_n \times_X X'$ and that this identification is compatible with the two morphisms to X, with the two morphisms to X', and with the two morphisms to Z_n. Denote $f_n : X' \times_X Z_n \to X$ the morphism to X. Denote

$$\mathcal{A} = \text{Equalizer}(\ f_*\mathcal{O}_{X'} \rightrightarrows (f \times f)_*\mathcal{O}_{X' \times_X X'}\ )$$

By the remarks above we find that

$$\mathcal{A}_n = \text{Equalizer}(\ \mathcal{A} \times \mathcal{O}_X/\mathcal{I}^n \rightrightarrows f_{n,*}\mathcal{O}_{X' \times_X Z_n}\ )$$

We have canonical maps

$$\mathcal{O}_X \to \ldots \to \mathcal{A}_3 \to \mathcal{A}_2 \to \mathcal{A}_1$$

of coherent $\mathcal{O}_X$-algebras. The statement of the lemma means that for n large enough there exists an $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \to \mathcal{A}_n$ is isomorphic to $\mathcal{O}_X$. This we may check étale locally on X. Hence by Properties of Spaces, Lemma 66.6.3 we may assume X is an affine Noetherian scheme.

Since $X_n \to X$ is an isomorphism over U we see that the kernel of $\mathcal{O}_X \to \mathcal{A}_n$ is supported on $|Z|$. Since X is Noetherian, the sequence of kernels $\mathcal{J}_n = \text{Ker}(\mathcal{O}_X \to \mathcal{A}_n)$ stabilizes (Cohomology of Spaces, Lemma 69.13.1). Say $\mathcal{J}_{n_0} = \mathcal{J}_{n_0+1} = \ldots = \mathcal{J}$. By Cohomology of Spaces, Lemma 69.13.2 we find that $\mathcal{I}^t\mathcal{J} = 0$ for some $t \geq 0$. On the other hand, there is an $\mathcal{O}_X$-algebra map $\mathcal{A}_n \to \mathcal{O}_X/\mathcal{I}^n$ and hence $\mathcal{J} \subset \mathcal{I}^n$ for all n. By Artin-Rees (Cohomology of Spaces, Lemma 69.13.3) we find that $\mathcal{J} \cap \mathcal{I}^n \subset \mathcal{I}^{n-c}\mathcal{J}$ for some $c \geq 0$ and all $n \gg 0$. We conclude that $\mathcal{J} = 0$.

Pick $n \geq n_0$ as in the previous paragraph. Then $\mathcal{O}_X \to \mathcal{A}_n$ is injective. Hence it now suffices to find $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \to \mathcal{A}_n$ is equal to the image of $\mathcal{O}_X$. Observe that $\mathcal{A}_n$ sits in a short exact sequence

$$0 \to \text{Ker}(\mathcal{A} \to f_{n,*}\mathcal{O}_{X' \times_X Z_n}) \to \mathcal{A}_n \to \mathcal{O}_X/\mathcal{I}^n \to 0$$

and similarly for $\mathcal{A}_{n+m}$. Hence it suffices to show

$$\text{Ker}(\mathcal{A} \to f_{n+m,*}\mathcal{O}_{X' \times_X Z_{n+m}}) \subset \text{Im}(\mathcal{I}^n \to \mathcal{A})$$

for some $m \geq 0$. To do this we may work étale locally on X and since X is Noetherian we may assume that X is a Noetherian affine scheme. Say $X = \text{Spec}(R)$ and $\mathcal{I}$ corresponds to the ideal $I \subset R$. Let $\mathcal{A} = \widetilde{A}$ for a finite R-algebra A. Let $f_*\mathcal{O}_{X'} = \widetilde{B}$ for a finite R-algebra B. Then $R \to A \subset B$ and these maps become isomorphisms on inverting any element of I.

Note that $f_{n,*}\mathcal{O}_{X' \times_X Z_n}$ is equal to $f_*(\mathcal{O}_{X'}/I^n\mathcal{O}_{X'})$ in the notation used in Cohomology of Spaces, Section 69.22. By Cohomology of Spaces, Lemma 69.22.4 we see that there exists a $c \geq 0$ such that

$$\text{Ker}(B \to \Gamma(X, f_*(\mathcal{O}_{X'}/I^{n+m+c}\mathcal{O}_{X'}))$$

is contained in $I^{n+m}B$. On the other hand, as $R \to B$ is finite and an isomorphism after inverting any element of I we see that $I^{n+m}B \subset \text{Im}(I^n \to B)$ for m large enough (can be chosen independent of n). This finishes the proof as $A \subset B$. $\square$

0AGL Remark 81.13.5. The meaning of Lemma 81.13.4 is the system $X_1 \to X_2 \to X_3 \to \ldots$ is essentially constant with value X. See Categories, Definition 4.22.1.

81.14. Compactifications

0F44 This section is the analogue of More on Flatness, Section 38.33. The theorem in this section is the main theorem in [CLO12].

Let B be a quasi-compact and quasi-separated algebraic space over some base scheme S. We will say an algebraic space X over B has a compactification over B or is compactifyable over B if there exists a quasi-compact open immersion $X \to \overline{X}$ into an algebraic space $\overline{X}$ proper over B. If X has a compactification over B, then $X \to B$ is separated and of finite type. The main theorem of this section is that the converse is true as well.

0F45 **Lemma 81.14.1.** Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. If $(U \subset X, f : V \to X)$ is an elementary distinguished square such that $U \to Y$ and $V \to Y$ are separated and $U \times_X V \to U \times_Y V$ is closed, then $X \to Y$ is separated.

Proof. We have to check that $\Delta : X \to X \times_Y X$ is a closed immersion. There is an étale covering of $X \times_Y X$ given by the four parts $U \times_Y U$, $U \times_Y V$, $V \times_Y U$, and $V \times_Y V$. Observe that $(U \times_Y U) \times_{(X \times_Y X), \Delta} X = U$, $(U \times_Y V) \times_{(X \times_Y X), \Delta} X = U \times_X V$, $(V \times_Y U) \times_{(X \times_Y X), \Delta} X = V \times_X U$, and $(V \times_Y V) \times_{(X \times_Y X), \Delta} X = V$. Thus the assumptions of the lemma exactly tell us that Δ is a closed immersion. $\square$

0F46 **Lemma 81.14.2.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open.

(1) If $Z_1, Z_2 \subset X$ are closed subspaces of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transforms of Z_1 and Z_2 are disjoint.
(2) If $T_1, T_2 \subset |U|$ are disjoint constructible closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $\mathcal{I}_i$ of Z_i is of finite type, see Morphisms of Spaces, Lemma 67.28.12. Denote $Z \subset X$ the closed subspace cut out by the product $\mathcal{I}_1\mathcal{I}_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors on Spaces, Lemma 71.19.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of Z_1 and Z_2 are disjoint. Denote $Y \subset Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors on Spaces, Definition 71.19.1 and Morphisms of Spaces, Lemma 67.28.2). Thus the blowing up $X' \to X$ of Y is a U-admissible blowing up. By general properties of strict transforms, the strict transform of Z_1, Z_2 with respect to $X' \to X$ is the same as the strict transform of Z_1, Z_2 with respect to $Z' \to Z$, see Divisors on Spaces, Lemma 71.18.3. Thus (1) is proved.

Proof of (2). By Limits of Spaces, Lemma 70.14.1 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_U$ such that $T_i = V(\mathcal{J}_i)$ (set theoretically). By Limits of Spaces, Lemma 70.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_X$ whose restriction to U is $\mathcal{J}_i$. Apply the result of part (1) to the closed subspaces $Z_i = V(\mathcal{I}_i)$ to conclude. $\square$

0F47 **Lemma 81.14.3.** Let S be a scheme. Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $V \subset Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subset |V|$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of T in V. Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of f is an isomorphism over an open neighbourhood of the closure of T in $|Y'|$.

Proof. Let $T' \subset |V|$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then T', T are closed in $|V|$ and $T \cap T' = \emptyset$. Since $|V|$ is a spectral topological space (Properties of Spaces, Lemma 66.15.2) we can find constructible closed subsets T_c, T'_c of $|V|$ with $T \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open W of $|V|$ containing T' not meeting T and set $T_c = |V| \setminus W$, then choose a quasi-compact open W' of $|V|$ containing T_c not meeting T' and set $T'_c = |V| \setminus W'$). By Lemma 81.14.2 we may, after replacing Y by a V-admissible blowing up, assume that T_c and T'_c have disjoint closures in $|Y|$. Let Y_0 be the open subspace of Y corresponding to the open $|Y| \setminus \overline{T}'_c$ and set $V_0 = V \cap Y_0$, $U_0 = U \times_V V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to V_0$ is an isomorphism, we can find a V_0-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform X'_0 of X_0 maps isomorphically to Y'_0, see More on Morphisms of Spaces, Lemma 76.39.4. By Divisors on Spaces, Lemma 71.19.3 there exists a V-admissible blow up $Y' \to Y$ whose restriction to Y_0 is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of f, then we see what we want is true because f' restricts to an isomorphism over Y'_0. ☐

0F48 **Lemma 81.14.4.** Let S be a scheme. Consider a diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l] \arrow[d, "f|_U"'] & A \arrow[l] \arrow[d] \\ Y & V \arrow[l] & B \arrow[l] \end{tikzcd}$$

of quasi-compact and quasi-separated algebraic spaces over S. Assume

(1) f is proper,
(2) V is a quasi-compact open of Y, $U = f^{-1}(V)$,
(3) $B \subset V$ and $A \subset U$ are closed subspaces,
(4) $f|_A : A \to B$ is an isomorphism, and f is étale at every point of A.

Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ satisfies: for every geometric point $\overline{a}$ of the closure of $|A|$ in $|X'|$ there exists a quotient $\mathcal{O}_{X',\overline{a}} \to \mathcal{O}$ such that $\mathcal{O}_{Y',f'(\overline{a})} \to \mathcal{O}$ is finite flat.

As you can see from the proof, more is true, but the statement is already long enough and this will be sufficient later on.

Proof. Let $T' \subset |U|$ be the complement of the maximal open on which $f|_U$ is étale. Then T' is closed in $|U|$ and disjoint from $|A|$. Since $|U|$ is a spectral topological space (Properties of Spaces, Lemma 66.15.2) we can find constructible closed subsets T_c, T'_c of $|U|$ with $|A| \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (see proof of Lemma 81.14.3). By Lemma 81.14.2 there is a U-admissible blowing up $X_1 \to X$ such that T_c and T'_c have disjoint closures in $|X_1|$. Let $X_{1,0}$ be the open subspace of X_1 corresponding to the open $|X_1| \setminus \overline{T}'_c$ and set $U_0 = U \cap X_{1,0}$. Observe that the scheme theoretic image $\overline{A}_1 \subset X_1$ of A is contained in $X_{1,0}$ by construction.

After replacing Y by a V-admissible blowing up and taking strict transforms, we may assume $X_{1,0} \to Y$ is flat, quasi-finite, and of finite presentation, see More on Morphisms of Spaces, Lemmas 76.39.1 and 76.37.3. Consider the commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd} \quad \text{and the diagram} \quad \begin{tikzcd} \overline{A}_1 \arrow[rr] \arrow[dr] & & \overline{A} \arrow[dl] \\ & \overline{B} & \end{tikzcd}$$

of scheme theoretic images. The morphism $\overline{A}_1 \to \overline{A}$ is surjective because it is proper and hence the scheme theoretic image of $\overline{A}_1 \to \overline{A}$ must be equal to $\overline{A}$ and then we can use Morphisms of Spaces, Lemma 67.40.8. The statement on étale local rings follows by choosing a lift of the geometric point $\overline{a}$ to a geometric point $\overline{a}_1$ of $\overline{A}_1$ and setting $\mathcal{O} = \mathcal{O}_{X_1,\overline{a}_1}$. Namely, since $X_1 \to Y$ is flat and quasi-finite on $X_{1,0} \supset \overline{A}_1$, the map $\mathcal{O}_{Y',f'(\overline{a})} \to \mathcal{O}_{X_1,\overline{a}_1}$ is finite flat, see Algebra, Lemmas 10.156.3 and 10.153.3. □

0F49 Lemma 81.14.5. Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. Let $U \subset X$ be an open subspace. Let $V \to X \times_B Y$ be a quasi-compact morphism whose composition with the first projection maps into U. Let $Z \subset X \times_B Y$ be the scheme theoretic image of $V \to X \times_B Y$. Let $X' \to X$ be a U-admissible blowup. Then the scheme theoretic image of $V \to X' \times_B Y$ is the strict transform of Z with respect to the blowing up.

Proof. Denote $Z' \to Z$ the strict transform. The morphism $Z' \to X'$ induces a morphism $Z' \to X' \times_B Y$ which is a closed immersion (as Z' is a closed subspace of $X' \times_X Z$ by definition). Thus to finish the proof it suffices to show that the scheme theoretic image Z'' of $V \to Z'$ is Z'. Observe that $Z'' \subset Z'$ is a closed subspace such that $V \to Z'$ factors through Z''. Since both $V \to X \times_B Y$ and $V \to X' \times_B Y$ are quasi-compact (for the latter this follows from Morphisms of Spaces, Lemma 67.8.9 and the fact that $X' \times_B Y \to X \times_B Y$ is separated as a base change of a proper morphism), by Morphisms of Spaces, Lemma 67.16.3 we see that $Z \cap (U \times_B Y) = Z'' \cap (U \times_B Y)$. Thus the inclusion morphism $Z'' \to Z'$ is an isomorphism away from the exceptional divisor E of $Z' \to Z$. However, the structure sheaf of Z' does not have any nonzero sections supported on E (by definition of strict transforms) and we conclude that the surjection $\mathcal{O}_{Z'} \to \mathcal{O}_{Z''}$ must be an isomorphism. □

0F4A Lemma 81.14.6. Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let U be an algebraic space of finite type and separated over B. Let $V \to U$ be an étale morphism. If V has a compactification $V \subset Y$ over B, then there exists a V-admissible blowing up $Y' \to Y$ and an open $V \subset V' \subset Y'$ such that $V \to U$ extends to a proper morphism $V' \to U$.

Proof. Consider the scheme theoretic image $Z \subset Y \times_B U$ of the "diagonal" morphism $V \to Y \times_B U$. If we replace Y by a V-admissible blowing up, then Z is replaced by the strict transform with respect to this blowing up, see Lemma 81.14.5. Hence by More on Morphisms of Spaces, Lemma 76.39.4 we may assume $Z \to Y$ is an open immersion. If $V' \subset Y$ denotes the image, then we see that the induced morphism $V' \to U$ is proper because the projection $Y \times_B U \to U$ is proper and $V' \cong Z$ is a closed subspace of $Y \times_B U$. □

The following lemma is formulated for finite type separated algebraic spaces over a finite type algebraic space over $\mathbf{Z}$. The version for quasi-compact and quasi-separated algebraic spaces is true as well (with essentially the same proof), but will be trivially implied by the main theorem in this section. We strongly urge the reader to read the proof of this lemma in the case of schemes first.

0F4B Lemma 81.14.7. Let B be an algebraic space of finite type over $\mathbf{Z}$. Let U be an algebraic space of finite type and separated over B. Let $(U_2 \subset U, f : U_1 \to U)$ be an elementary distinguished square. Assume U_1 and U_2 have compactifications over B and $U_1 \times_U U_2 \to U$ has dense image. Then U has a compactification over B.

Proof. Choose a compactification $U_i \subset X_i$ over B for $i = 1, 2$. We may assume U_i is scheme theoretically dense in X_i. We may assume there is an open $V_i \subset X_i$ and a proper morphism $\psi_i : V_i \to U$ extending $U_i \to U$, see Lemma 81.14.6. Picture

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[r] \arrow[dl, "\psi_i"] & X_i \\ U \end{tikzcd}$$

Denote $Z_1 \subset U$ the reduced closed subspace corresponding to the closed subset $|U| \setminus |U_2|$. Recall that $f^{-1}Z_1$ is a closed subspace of U_1 mapping isomorphically to Z_1. Denote $Z_2 \subset U$ the reduced closed subspace corresponding to the closed subset $|U| \setminus \mathrm{Im}(|f|) = |U_2| \setminus \mathrm{Im}(|U_1 \times_U U_2| \to |U_2|)$. Thus we have

$$U = U_2 \amalg Z_1 = Z_2 \amalg \mathrm{Im}(f) = Z_2 \amalg \mathrm{Im}(U_1 \times_U U_2 \to U_2) \amalg Z_1$$

set theoretically. Denote $Z_{i,i} \subset V_i$ the inverse image of Z_i under ψ_i. Observe that ψ_2 is an isomorphism over an open neighbourhood of Z_2. Observe that $Z_{1,1} = \psi_1^{-1}Z_1 = f^{-1}Z_1 \amalg T$ for some closed subspace $T \subset V_1$ disjoint from $f^{-1}Z_1$ and furthermore ψ_1 is étale along $f^{-1}Z_1$. Denote $Z_{i,j} \subset V_i$ the inverse image of Z_j under ψ_i. Observe that $\psi_i : Z_{i,j} \to Z_j$ is a proper morphism. Since Z_i and Z_j are disjoint closed subspaces of U, we see that $Z_{i,i}$ and $Z_{i,j}$ are disjoint closed subspaces of V_i.

Denote $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ the scheme theoretic images of $Z_{i,i}$ and $Z_{i,j}$ in X_i. We recall that $|Z_{i,j}|$ is dense in $|\overline{Z}_{i,j}|$, see Morphisms of Spaces, Lemma 67.17.7. After replacing X_i by a V_i-admissible blowup we may assume that $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ are disjoint, see Lemma 81.14.2. We assume this holds for both X_1 and X_2. Observe that this property is preserved if we replace X_i by a further V_i-admissible blowup. Hence we may replace X_1 by another V_1-admissible blowup and assume $|\overline{Z}_{1,1}|$ is the disjoint union of the closures of $|T|$ and $|f^{-1}Z_1|$ in $|X_1|$.

Set $V_{12} = V_1 \times_U V_2$. We have an immersion $V_{12} \to X_1 \times_B X_2$ which is the composition of the closed immersion $V_{12} = V_1 \times_U V_2 \to V_1 \times_B V_2$ (Morphisms of Spaces, Lemma 67.4.5) and the open immersion $V_1 \times_B V_2 \to X_1 \times_B X_2$. Let $X_{12} \subset X_1 \times_B X_2$ be the scheme theoretic image of $V_{12} \to X_1 \times_B X_2$. The projection morphisms

$$p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2$$

are proper as X_1 and X_2 are proper over B. If we replace X_1 by a V_1-admissible blowing up, then X_{12} is replaced by the strict transform with respect to this blowing up, see Lemma 81.14.5.

Denote $\psi : V_{12} \to U$ the compositions $\psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}}$. Consider the closed subspace

$$Z_{12,2} = (p_1|_{V_{12}})^{-1}Z_{1,2} = (p_2|_{V_{12}})^{-1}Z_{2,2} = \psi^{-1}Z_2 \subset V_{12}$$

The morphism $p_1|_{V_{12}} : V_{12} \to V_1$ is an isomorphism over an open neighbourhood of $Z_{1,2}$ because $\psi_2 : V_2 \to U$ is an isomorphism over an open neighbourhood of Z_2 and $V_{12} = V_1 \times_U V_2$. By Lemma 81.14.3 there exists a V_1-admissible blowing up $X_1' \to X_1$ such that the strict tranform $p_1' : X_{12}' \to X_1'$ of p_1 is an isomorphism over an open neighbourhood of the closure of $|Z_{1,2}|$ in $|X_1'|$. After replacing X_1 by X_1' and X_{12} by X_{12}' we may assume that p_1 is an isomorphism over an open neighbourhood of $|\overline{Z}_{1,2}|$.

The result of the previous paragraph tells us that

$$X_{12} \cap (\overline{Z}_{1,2} \times_B \overline{Z}_{2,1}) = \emptyset$$

where the intersection taken in $X_1 \times_B X_2$. Namely, the inverse image $p_1^{-1}\overline{Z}_{1,2}$ in X_{12} maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $|Z_{12,2}|$ is dense in $|p_1^{-1}\overline{Z}_{1,2}|$. Thus p_2 maps $|p_1^{-1}\overline{Z}_{1,2}|$ into $|\overline{Z}_{2,2}|$. Since $|\overline{Z}_{2,2}| \cap |\overline{Z}_{2,1}| = \emptyset$ we conclude.

It turns out that we need to do one additional blowing up before we can conclude the argument. Namely, let $V_2 \subset W_2 \subset X_2$ be the open subspace with underlying topological space

$$|W_2| = |V_2| \cup (|X_2| \setminus |\overline{Z}_{2,1}|) = |X_2| \setminus \left(|\overline{Z}_{2,1}| \setminus |Z_{2,1}|\right)$$

Since $p_2(p_1^{-1}\overline{Z}_{1,2})$ is contained in W_2 (see above) we see that replacing X_2 by a W_2-admissible blowup and X_{21} by the corresponding strict transform will preserve the property of p_1 being an isomorphism over an open neighbourhood of $\overline{Z}_{1,2}$. Since $\overline{Z}_{2,1} \cap W_2 = \overline{Z}_{2,1} \cap V_2 = Z_{2,1}$ we see that $Z_{2,1}$ is a closed subspace of W_2 and V_2. Observe that $V_{12} = V_1 \times_U V_2 = p_1^{-1}(V_1) = p_2^{-1}(V_2)$ as open subspaces of X_{12} as it is the largest open subspace of X_{12} over which the morphism $\psi : V_{12} \to U$ extends; details omitted[4]. We have the following equalities of closed subspaces of V_{12}:

$$p_2^{-1}Z_{2,1} = p_2^{-1}\psi_2^{-1}Z_1 = p_1^{-1}\psi_1^{-1}Z_1 = p_1^{-1}Z_{1,1} = p_1^{-1}f^{-1}Z_1 \amalg p_1^{-1}T$$

Here and below we use the slight abuse of notation of writing p_2 in stead of the restriction of p_2 to V_{12}, etc. Since $p_2^{-1}(Z_{2,1})$ is a closed subspace of $p_2^{-1}(W_2)$ as $Z_{2,1}$ is a closed subspace of W_2 we conclude that also $p_1^{-1}f^{-1}Z_1$ is a closed subspace of $p_2^{-1}(W_2)$. Finally, the morphism $p_2 : X_{12} \to X_2$ is étale at points of $p_1^{-1}f^{-1}Z_1$ as ψ_1 is étale along $f^{-1}Z_1$ and $V_{12} = V_1 \times_U V_2$. Thus we may apply Lemma 81.14.4 to the morphism $p_2 : X_{12} \to X_2$, the open W_2, the closed subspace $Z_{2,1} \subset W_2$, and the closed subspace $p_1^{-1}f^{-1}Z_1 \subset p_2^{-1}(W_2)$. Hence after replacing X_2 by a W_2-admissible blowup and X_{12} by the corresponding strict transform, we obtain for every geometric point $\overline{y}$ of the closure of $|p_1^{-1}f^{-1}Z_1|$ a local ring map $\mathcal{O}_{X_{12},\overline{y}} \to \mathcal{O}$ such that $\mathcal{O}_{X_2,p_2(\overline{y})} \to \mathcal{O}$ is finite flat.

Consider the algebraic space

$$W_2 = U \coprod\nolimits_{U_2} (X_2 \setminus \overline{Z}_{2,1}),$$

and with $T \subset V_1$ as in the first paragraph the algebraic space

$$W_1 = U \coprod\nolimits_{U_1} (X_1 \setminus \overline{Z}_{1,2} \cup \overline{T}),$$

obtained by pushout, see Lemma 81.9.2. Let us apply Lemma 81.14.1 to see that $W_i \to B$ is separated. First, $U \to B$ and $X_i \to B$ are separated. Let us check the quasi-compact immersion $U_i \to U \times_B (X_i \setminus \overline{Z}_{i,j})$ is closed using the valuative criterion, see Morphisms of Spaces, Lemma 67.42.1. Choose a valuation ring A over B with fraction field K and compatible morphisms $(u, x_i) : \mathrm{Spec}(A) \to U \times_B X_i$ and $u_i : \mathrm{Spec}(K) \to U_i$. Since ψ_i is proper, we can find a unique $v_i : \mathrm{Spec}(A) \to V_i$ compatible with u and u_i. Since X_i is proper over B we see that $x_i = v_i$. If v_i does not factor through $U_i \subset V_i$, then we conclude that x_i maps the closed point

[4]Namely, $V_1 \times_U V_2$ is proper over U so if ψ extends to a larger open of X_{12}, then $V_1 \times_U V_2$ would be closed in this open by Morphisms of Spaces, Lemma 67.40.6. Then we get equality as $V_{12} \subset X_{12}$ is dense.

of $\mathrm{Spec}(A)$ into $Z_{i,j}$ or T when $i = 1$. This finishes the proof because we removed $\overline{Z}_{i,j}$ and $\overline{T}$ in the construction of W_i.

On the other hand, for any valuation ring A over B with fraction field K and any morphism

$$\gamma : \mathrm{Spec}(K) \to \mathrm{Im}(U_1 \times_U U_2 \to U)$$

over B, we claim that after replacing A by an extension of valuation rings, there is an i and an extension of γ to a morphism $h_i : \mathrm{Spec}(A) \to W_i$. Namely, we first extend γ to a morphism $g_2 : \mathrm{Spec}(A) \to X_2$ using the valuative criterion of properness. If the image of g_2 does not meet $\overline{Z}_{2,1}$, then we obtain our morphism into W_2. Otherwise, denote $\overline{z} \in \overline{Z}_{2,1}$ a geometric point lying over the image of the closed point under g_2. We may lift this to a geometric point $\overline{y}$ of X_{12} in the closure of $|p_1^{-1}f^{-1}Z_1|$ because the map of spaces $|p_1^{-1}f^{-1}Z_1| \to |\overline{Z}_{2,1}|$ is closed with image containing the dense open $|Z_{2,1}|$. After replacing A by its strict henselization (More on Algebra, Lemma 15.123.6) we get the following diagram

$$\begin{tikzcd}
A \arrow[rr, dotted] & & A' \\
\mathcal{O}_{X_2,\overline{z}} \arrow[u] \arrow[r] & \mathcal{O}_{X_{12},\overline{y}} \arrow[r] & \mathcal{O} \arrow[u, dotted]
\end{tikzcd}$$

where $\mathcal{O}_{X_{12},\overline{y}} \to \mathcal{O}$ is the map we found in the 5th paragraph of the proof. Since the horizontal composition is finite and flat we can find an extension of valuation rings A'/A and dotted arrow making the diagram commute. After replacing A by A' this means that we obtain a lift $g_{12} : \mathrm{Spec}(A) \to X_{12}$ whose closed point maps into the closure of $|p_1^{-1}f^{-1}Z_1|$. Then $g_1 = p_1 \circ g_{12} : \mathrm{Spec}(A) \to X_1$ is a morphism whose closed point maps into the closure of $|f^{-1}Z_1|$. Since the closure of $|f^{-1}Z_1|$ is disjoint from the closure of $|T|$ and contained in $|\overline{Z}_{1,1}|$ which is disjoint from $|\overline{Z}_{1,2}|$ we conclude that g_1 defines a morphism $h_1 : \mathrm{Spec}(A) \to W_1$ as desired.

Consider a diagram

$$\begin{tikzcd}
W_1' \arrow[r] \arrow[d] & W & W_2' \arrow[l] \arrow[d] \\
W_1 & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & W_2
\end{tikzcd}$$

as in More on Morphisms of Spaces, Lemma 76.40.1. By the previous paragraph for every solid diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "\gamma"'] \arrow[d] & W \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}$$

where $\mathrm{Im}(\gamma) \subset \mathrm{Im}(U_1 \times_U U_2 \to U)$ there is an i and an extension $h_i : \mathrm{Spec}(A) \to W_i$ of γ after possibly replacing A by an extension of valuation rings. Using the valuative criterion of properness for $W_i' \to W_i$, we can then lift h_i to $h_i' : \mathrm{Spec}(A) \to W_i'$. Hence the dotted arrow in the diagram exists after possibly extending A. Since W is separated over B, we see that the choice of extension isn't needed and the arrow is unique as well, see Morphisms of Spaces, Lemmas 67.41.5 and 67.43.1. Then finally the existence of the dotted arrow implies that $W \to B$ is universally closed by Morphisms of Spaces, Lemma 67.42.5. As $W \to B$ is already of finite type and separated, we win. $\square$

0F4C Lemma 81.14.8. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $U \subset X$ be a proper dense open subspace. Then there exists an affine scheme V and an étale morphism $V \to X$ such that

(1) the open subspace $W = U \cup \mathrm{Im}(V \to X)$ is strictly larger than U,
(2) $(U \subset W, V \to W)$ is a distinguished square, and
(3) $U \times_W V \to U$ has dense image.

Proof. Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and morphisms $f_p : V_p \to U_p$ as in Decent Spaces, Lemma 68.8.6. Let p be the smallest integer such that $U_p \not\subset U$ (this is possible as $U \neq X$). Choose an affine open $V \subset V_p$ such that the étale morphism $f_p|_V : V \to X$ does not factor through U. Consider the open $W = U \cup \mathrm{Im}(V \to X)$ and the reduced closed subspace $Z \subset W$ with $|Z| = |W| \setminus |U|$. Then $f^{-1}Z \to Z$ is an isomorphism because we have the corresponding property for the morphism f_p, see the lemma cited above. Thus $(U \subset W, f : V \to W)$ is a distinguished square. It may not be true that the open $I = \mathrm{Im}(U \times_W V \to U)$ is dense in U. The algebraic space $U' \subset U$ whose underlying set is $|U| \setminus \overline{|I|}$ is Noetherian and hence we can find a dense open subscheme $U'' \subset U'$, see for example Properties of Spaces, Proposition 66.13.3. Then we can find a dense open affine $U''' \subset U''$, see Properties, Lemmas 28.5.7 and 28.29.1. After we replace f by $V \amalg U''' \to X$ everything is clear. $\square$

0F4D Theorem 81.14.9. Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let $X \to B$ be a separated, finite type morphism. Then X has a compactification over B. [CLO12]

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. First, we may replace S by $\mathrm{Spec}(\mathbf{Z})$. See Spaces, Section 65.16 and Properties of Spaces, Definition 66.3.1. There exists a closed immersion $X \to X'$ with $X' \to B$ of finite presentation and separated. See Limits of Spaces, Proposition 70.11.7. If we find a compactification of X' over B, then taking the scheme theoretic closure of X in this will give a compactification of X over B. Thus we may assume $X \to B$ is separated and of finite presentation. We may write $B = \lim B_i$ as a directed limit of a system of Noetherian algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$ with affine transition morphisms. See Limits of Spaces, Proposition 70.8.1. We can choose an i and a morphism $X_i \to B_i$ of finite presentation whose base change to B is $X \to B$, see Limits of Spaces, Lemma 70.7.1. After increasing i we may assume $X_i \to B_i$ is separated, see Limits of Spaces, Lemma 70.6.9. If we can find a compactification of X_i over B_i, then the base change of this to B will be a compactification of X over B. This reduces us to the case discussed in the next paragraph.

Assume B is of finite type over $\mathbf{Z}$ in addition to being quasi-compact and quasi-separated. Let $U \to X$ be an étale morphism of algebraic spaces such that U has a compactification Y over $\mathrm{Spec}(\mathbf{Z})$. The morphism

$$U \longrightarrow B \times_{\mathrm{Spec}(\mathbf{Z})} Y$$

is separated and quasi-finite by Morphisms of Spaces, Lemma 67.27.10 (the displayed morphism factors into an immersion hence is a monomorphism). Hence by Zariski's main theorem (More on Morphisms of Spaces, Lemma 76.34.3) there is an open immersion of U into an algebraic space Y' finite over $B \times_{\mathrm{Spec}(\mathbf{Z})} Y$. Then $Y' \to B$ is proper as the composition $Y' \to B \times_{\mathrm{Spec}(\mathbf{Z})} Y \to B$ of two proper morphisms (use Morphisms of Spaces, Lemmas 67.45.9, 67.40.4, and 67.40.3). We conclude that U has a compactification over B.

There is a dense open subspace $U \subset X$ which is a scheme. (Properties of Spaces, Proposition 66.13.3). In fact, we may choose U to be an affine scheme (Properties, Lemmas 28.5.7 and 28.29.1). Thus U has a compactification over $\mathrm{Spec}(\mathbf{Z})$; this is easily shown directly but also follows from the theorem for schemes, see More on Flatness, Theorem 38.33.8. By the previous paragraph U has a compactification over B. By Noetherian induction we can find a maximal dense open subspace $U \subset X$ which has a compactification over B. We will show that the assumption that $U \neq X$ leads to a contradiction. Namely, by Lemma 81.14.8 we can find a strictly larger open $U \subset W \subset X$ and a distinguished square $(U \subset W, f : V \to W)$ with V affine and $U \times_W V$ dense image in U. Since V is affine, as before it has a compactification over B. Hence Lemma 81.14.7 applies to show that W has a compactification over B which is the desired contradiction. ☐

81.15. Other chapters

Part 5

Topics in Geometry

CHAPTER 82

Chow Groups of Spaces

0EDQ 82.1. Introduction

0EDR In this chapter we first discuss Chow groups of algebraic spaces. Having defined these, we define Chern classes of vector bundles as operators on these chow groups. The strategy will be entirely the same as the strategy in the case of schemes. Therefore we urge the reader to take a look at the introduction (Chow Homology, Section 42.1) of the corresponding chapter for schemes.

Some related papers: [EG98] and [Kre99].

82.2. Setup

0EDS We first fix the category of algebraic spaces we will be working with. Please keep in mind throughout this chapter that "decent + locally Noetherian" is the same as "quasi-separated + locally Noetherian" according to Decent Spaces, Lemma 68.14.1.

0EDT Situation 82.2.1. Here S is a scheme and B is an algebraic space over S. We assume B is quasi-separated, locally Noetherian, and universally catenary (Decent Spaces, Definition 68.25.4). Moreover, we assume given a dimension function $\delta : |B| \longrightarrow \mathbf{Z}$. We say X/B is good if X is an algebraic space over B whose structure morphism $f : X \to B$ is quasi-separated and locally of finite type. In this case we define

$$\delta = \delta_{X/B} : |X| \longrightarrow \mathbf{Z}$$

as the map sending x to $\delta(f(x))$ plus the transcendence degree of $x/f(x)$ (Morphisms of Spaces, Definition 67.33.1). This is a dimension function by More on Morphisms of Spaces, Lemma 76.32.2.

A special case is when $S = B$ is a scheme and (S, δ) is as in Chow Homology, Situation 42.7.1. Thus B might be the spectrum of a field (Chow Homology, Example 42.7.2) or $B = \mathrm{Spec}(\mathbf{Z})$ (Chow Homology, Example 42.7.3).

Many lemma, proposition, theorems, definitions on algebraic spaces are easier in the setting of Situation 82.2.1 because the algebraic spaces we are working with are quasi-separated (and thus a fortiori decent) and locally Noetherian. We will sprinkle this chapter with remarks such as the following to point this out.

0EDU Remark 82.2.2. In Situation 82.2.1 if X/B is good, then $|X|$ is a sober topological space. See Properties of Spaces, Lemma 66.15.1 or Decent Spaces, Proposition 68.12.4. We will use this without further mention to choose generic points of irreducible closed subsets of $|X|$.

0EDV Remark 82.2.3. In Situation 82.2.1 if X/B is good, then X is integral (Spaces over Fields, Definition 72.4.1) if and only if X is reduced and $|X|$ is irreducible.

Moreover, for any point $\xi \in |X|$ there is a unique integral closed subspace $Z \subset X$ such that ξ is the generic point of the closed subset $|Z| \subset |X|$, see Spaces over Fields, Lemma 72.4.7.

If B is Jacobson and δ sends closed points to zero, then δ is the function sending a point to the dimension of its closure.

0EDW Lemma 82.2.4. In Situation 82.2.1 assume B is Jacobson and that $\delta(b) = 0$ for every closed point b of $|B|$. Let X/B be good. If $Z \subset X$ is an integral closed subspace with generic point $\xi \in |Z|$, then the following integers are the same:

(1) $\delta(\xi) = \delta_{X/B}(\xi)$,
(2) $\dim(|Z|)$,
(3) $\text{codim}(\{z\}, |Z|)$ for $z \in |Z|$ closed,
(4) the dimension of the local ring of Z at z for $z \in |Z|$ closed, and
(5) $\dim(\mathcal{O}_{Z,\overline{z}})$ for $z \in |Z|$ closed.

Proof. Let X, Z, ξ be as in the lemma. Since X is locally of finite type over B we see that X is Jacobson, see Decent Spaces, Lemma 68.23.1. Hence $X_{\text{ft-pts}} \subset |X|$ is the set of closed points by Decent Spaces, Lemma 68.23.3. Given a chain $T_0 \supset \ldots \supset T_e$ of irreducible closed subsets of $|Z|$ we have $T_e \cap X_{\text{ft-pts}}$ nonempty by Morphisms of Spaces, Lemma 67.25.6. Thus we can always assume such a chain ends with $T_e = \{z\}$ for some $z \in |Z|$ closed. It follows that $\dim(Z) = \sup_z \text{codim}(\{z\}, |Z|)$ where z runs over the closed points of $|Z|$. We have $\text{codim}(\{z\}, Z) = \delta(\xi) - \delta(z)$ by Topology, Lemma 5.20.2. By Morphisms of Spaces, Lemma 67.25.4 the image of z is a finite type point of B, i.e., a closed point of $|B|$. By Morphisms of Spaces, Lemma 67.33.4 the transcendence degree of z/b is 0. We conclude that $\delta(z) = \delta(b) = 0$ by assumption. Thus we obtain equality

$$\dim(|Z|) = \text{codim}(\{z\}, Z) = \delta(\xi)$$

for all $z \in |Z|$ closed. Finally, we have that $\text{codim}(\{z\}, Z)$ is equal to the dimension of the local ring of Z at z by Decent Spaces, Lemma 68.20.2 which in turn is equal to $\dim(\mathcal{O}_{Z,\overline{z}})$ by Properties of Spaces, Lemma 66.22.4. □

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. This motivates the following definition.

0EDX Definition 82.2.5. In Situation 82.2.1 for any good X/B and any irreducible closed subset $T \subset |X|$ we define

$$\dim_\delta(T) = \delta(\xi)$$

where $\xi \in T$ is the generic point of T. We will call this the δ-dimension of T. If $T \subset |X|$ is any closed subset, then we define $\dim_\delta(T)$ as the supremum of the δ-dimensions of the irreducible components of T. If Z is a closed subspace of X, then we set $\dim_\delta(Z) = \dim_\delta(|Z|)$.

Of course this just means that $\dim_\delta(T) = \sup\{\delta(t) \mid t \in T\}$.

82.3. Cycles

0EDY This is the analogue of Chow Homology, Section 42.8

Since we are not assuming our spaces are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

0EDZ Definition 82.3.1. In Situation 82.2.1 let X/B be good. Let $k \in \mathbf{Z}$.

(1) A cycle on X is a formal sum
$$\alpha = \sum n_Z[Z]$$
where the sum is over integral closed subspaces $Z \subset X$, each $n_Z \in \mathbf{Z}$, and $\{|Z|; n_Z \neq 0\}$ is a locally finite collection of subsets of $|X|$ (Topology, Definition 5.28.4).

(2) A k-cycle on X is a cycle
$$\alpha = \sum n_Z[Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(3) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subspaces (Remark 82.2.3) of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by
$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$
i.e., by adding the coefficients.

82.4. Multiplicities

0EE0 A section with a few simple results on lengths and multiplicities.

0EE1 Lemma 82.4.1. Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent

(1) $\text{length}_{\mathcal{O}_{X,\overline{x}}} \mathcal{F}_{\overline{x}} = d$

(2) for some étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have $\text{length}_{\mathcal{O}_{U,u}} (\mathcal{F}|_U)_u = d$

(3) for any étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have $\text{length}_{\mathcal{O}_{U,u}} (\mathcal{F}|_U)_u = d$

Proof. Let $U \to X$ and $u \in U$ be as in (2) or (3). Then we know that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ and that
$$\mathcal{F}_{\overline{x}} = (\mathcal{F}|_U)_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$
See Properties of Spaces, Lemmas 66.22.1 and 66.29.4. Thus the equality of the lengths follows from Algebra, Lemma 10.52.13 the fact that $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is flat and the fact that $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_u\mathcal{O}_{X,\overline{x}}$ is equal to the residue field of $\mathcal{O}_{X,\overline{x}}$. These facts about strict henselizations can be found in More on Algebra, Lemma 15.45.1. $\square$

0EE2 Definition 82.4.2. Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. Let $d \in \{0, 1, 2, \ldots, \infty\}$. We say $\mathcal{F}$ has length d at x if the equivalent conditions of Lemma 82.4.1 are satisfied.

0EE3 Lemma 82.4.3. Let S be a scheme. Let $i : Y \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $y \in |Y|$ with image $x \in |X|$. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent

(1) $\mathcal{G}$ has length d at y, and
(2) $i_*\mathcal{G}$ has length d at x.

Proof. Choose an étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x. Set $V = Y \times_X U$. Denote $g : V \to Y$ and $j : V \to U$ the projections. Then $j : V \to U$ is a closed immersion and there is a unique point $v \in V$ mapping to $y \in |Y|$ and $u \in U$ (use Properties of Spaces, Lemma 66.4.3 and Spaces, Lemma 65.12.3). We have $j_*(\mathcal{G}|_V) = (i_*\mathcal{G})|_U$ as modules on the scheme V and j_* the "usual" pushforward of modules for the morphism of schemes j, see discussion surrounding Cohomology of Spaces, Equation (69.3.0.1). In this way we reduce to the case of schemes: if $i : Y \to X$ is a closed immersion of schemes, then

$$(i_*\mathcal{G})_x = \mathcal{G}_y$$

as modules over $\mathcal{O}_{X,x}$ where the module structure on the right hand side is given by the surjection $i_y^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. Thus equality by Algebra, Lemma 10.52.5. $\square$

0EE4 Lemma 82.4.4. Let S be a scheme and let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $x \in |X|$. The following are equivalent

(1) for some étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have u is a generic point of an irreducible component of $\text{Supp}(\mathcal{F}|_U)$,
(2) for any étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have u is a generic point of an irreducible component of $\text{Supp}(\mathcal{F}|_U)$,
(3) the length of $\mathcal{F}$ at x is finite and nonzero.

If X is decent (equivalently quasi-separated) then these are also equivalent to

(4) x is a generic point of an irreducible component of $\text{Supp}(\mathcal{F})$.

Proof. Assume $f : U \to X$ is an étale morphism with U a scheme and $u \in U$ maps to x. Then $\mathcal{F}|_U = f^*\mathcal{F}$ is a coherent $\mathcal{O}_U$-module on the locally Noetherian scheme U and in particular $(\mathcal{F}|_U)_u$ is a finite $\mathcal{O}_{U,u}$-module, see Cohomology of Spaces, Lemma 69.12.2 and Cohomology of Schemes, Lemma 30.9.1. Recall that the support of $\mathcal{F}|_U$ is a closed subset of U (Morphisms, Lemma 29.5.3) and that the support of $(\mathcal{F}|_U)_u$ is the pullback of the support of $\mathcal{F}|_U$ by the morphism $\text{Spec}(\mathcal{O}_{U,u}) \to U$. Thus u is a generic point of an irreducible component of $\text{Supp}(\mathcal{F}|_U)$ if and only if the support of $(\mathcal{F}|_U)_u$ is equal to the maximal ideal of $\mathcal{O}_{U,u}$. Now the equivalence of (1), (2), (3) follows from by Algebra, Lemma 10.62.3.

If X is decent we choose an étale morphism $f : U \to X$ and a point $u \in U$ mapping to x. The support of $\mathcal{F}$ pulls back to the support of $\mathcal{F}|_U$, see Morphisms of Spaces, Lemma 67.15.2. Also, specializations $x' \rightsquigarrow x$ in $|X|$ lift to specializations $u' \rightsquigarrow u$ in U and any nontrivial specialization $u' \rightsquigarrow u$ in U maps to a nontrivial specialization $f(u') \rightsquigarrow f(u)$ in $|X|$, see Decent Spaces, Lemmas 68.12.2 and 68.12.1. Using that $|X|$ and U are sober topological spaces (Decent Spaces, Proposition 68.12.4 and Schemes, Lemma 26.11.1) we conclude x is a generic point of the support of $\mathcal{F}$ if and only if u is a generic point of the support of $\mathcal{F}|_U$. We conclude (4) is equivalent to (1).

The parenthetical statement follows from Decent Spaces, Lemma 68.14.1. $\square$

0EE6 Lemma 82.4.5. In Situation 82.2.1 let X/B be good. Let $T \subset |X|$ be a closed subset and $t \in T$. If $\dim_\delta(T) \leq k$ and $\delta(t) = k$, then t is a generic point of an irreducible component of T.

Proof. We know t is contained in an irreducible component $T' \subset T$. Let $t' \in T'$ be the generic point. Then $k \geq \delta(t') \geq \delta(t)$. Since δ is a dimension function we see that $t = t'$. □

82.5. Cycle associated to a closed subspace

0EE7 This section is the analogue of Chow Homology, Section 42.9.

0EE8 Remark 82.5.1. In Situation 82.2.1 let X/B be good. Let $Y \subset X$ be a closed subspace. By Remarks 82.2.2 and 82.2.3 there are 1-to-1 correspondences between

(1) irreducible components T of $|Y|$,
(2) generic points of irreducible components of $|Y|$, and
(3) integral closed subspaces $Z \subset Y$ with the property that $|Z|$ is an irreducible component of $|Y|$.

In this chapter we will call Z as in (3) an irreducible component of Y and we will call $\xi \in |Z|$ its generic point.

0EE9 Definition 82.5.2. In Situation 82.2.1 let X/B be good. Let $Y \subset X$ be a closed subspace.

(1) For an irreducible component $Z \subset Y$ with generic point ξ the length of $\mathcal{O}_Y$ at ξ (Definition 82.4.2) is called the multiplicity of Z in Y. By Lemma 82.4.4 applied to $\mathcal{O}_Y$ on Y this is a positive integer.
(2) Assume $\dim_\delta(Y) \leq k$. The k-cycle associated to Y is
$$[Y]_k = \sum m_{Z,Y}[Z]$$
where the sum is over the irreducible components Z of Y of δ-dimension k and $m_{Z,Y}$ is the multiplicity of Z in Y. This is a k-cycle by Spaces over Fields, Lemma 72.6.1.

It is important to note that we only define $[Y]_k$ if the δ-dimension of Y does not exceed k. In other words, by convention, if we write $[Y]_k$ then this implies that $\dim_\delta(Y) \leq k$.

82.6. Cycle associated to a coherent sheaf

0EEA This is the analogue of Chow Homology, Section 42.10.

0EEB Definition 82.6.1. In Situation 82.2.1 let X/B be good. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) For an integral closed subspace $Z \subset X$ with generic point ξ such that $|Z|$ is an irreducible component of $\text{Supp}(\mathcal{F})$ the length of $\mathcal{F}$ at ξ (Definition 82.4.2) is called the multiplicity of Z in $\mathcal{F}$. By Lemma 82.4.4 this is a positive integer.
(2) Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The k-cycle associated to $\mathcal{F}$ is
$$[\mathcal{F}]_k = \sum m_{Z,\mathcal{F}}[Z]$$
where the sum is over the integral closed subspaces $Z \subset X$ corresponding to irreducible components of $\text{Supp}(\mathcal{F})$ of δ-dimension k and $m_{Z,\mathcal{F}}$ is the multiplicity of Z in $\mathcal{F}$. This is a k-cycle by Spaces over Fields, Lemma 72.6.1.

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\text{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

0EEC Lemma 82.6.2. In Situation 82.2.1 let X/B be good. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. Let $\xi \in |Z|$ be the generic point. Then the coefficient of Z in $[\mathcal{F}]_k$ is the length of $\mathcal{F}$ at ξ.

Proof. Observe that $|Z|$ is an irreducible component of $\text{Supp}(\mathcal{F})$ if and only if $\xi \in \text{Supp}(\mathcal{F})$, see Lemma 82.4.5. Moreover, the length of $\mathcal{F}$ at ξ is zero if $\xi \notin \text{Supp}(\mathcal{F})$. Combining this with Definition 82.6.1 we conclude. □

0EED Lemma 82.6.3. In Situation 82.2.1 let X/B be good. Let $Y \subset X$ be a closed subspace. If $\dim_\delta(Y) \leq k$, then $[Y]_k = [i_*\mathcal{O}_Y]_k$ where $i : Y \to X$ is the inclusion morphism.

Proof. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. If $Z \not\subset Y$ the Z has coefficient zero in both $[Y]_k$ and $[i_*\mathcal{O}_Y]_k$. If $Z \subset Y$, then the generic point of Z may be viewed as a point $y \in |Y|$ whose image $x \in |X|$. Then the coefficient of Z in $[Y]_k$ is the length of $\mathcal{O}_Y$ at y and the coefficient of Z in $[i_*\mathcal{O}_Y]_k$ is the length of $i_*\mathcal{O}_Y$ at x. Thus the equality of the coefficients follows from Lemma 82.4.3. □

0EEE Lemma 82.6.4. In Situation 82.2.1 let X/B be good. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.

Proof. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. It suffices to show that the coefficients of Z in $[\mathcal{G}]_k$, $[\mathcal{F}]_k$, and $[\mathcal{H}]_k$ satisfy the corresponding additivity. By Lemma 82.6.2 it suffices to show

$$\text{the length of } \mathcal{G} \text{ at } x = \text{the length of } \mathcal{F} \text{ at } x + \text{the length of } \mathcal{H} \text{ at } x$$

for any $x \in |X|$. Looking at Definition 82.4.2 this follows immediately from additivity of lengths, see Algebra, Lemma 10.52.3. □

82.7. Preparation for proper pushforward

0EEF This section is the analogue of Chow Homology, Section 42.11.

0EEG Lemma 82.7.1. In Situation 82.2.1 let $X, Y/B$ be good and let $f : X \to Y$ be a morphism over B. If $Z \subset X$ is an integral closed subspace, then there exists a unique integral closed subspace $Z' \subset Y$ such that there is a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z' \arrow[r] & Y \end{tikzcd}$$

with $Z \to Z'$ dominant. If f is proper, then $Z \to Z'$ is proper and surjective.

Proof. Let $\xi \in |Z|$ be the generic point. Let $Z' \subset Y$ be the integral closed subspace whose generic point is $\xi' = f(\xi)$, see Remark 82.2.3. Since $\xi \in |f^{-1}(Z')| = |f|^{-1}(|Z'|)$ by Properties of Spaces, Lemma 66.4.3 and since Z is the reduced with $|Z| = \overline{\{\xi\}}$ we see that $Z \subset f^{-1}(Z')$ as closed subspaces of X (see Properties of Spaces, Lemma 66.12.4). Thus we obtain our morphism $Z \to Z'$. This morphism is dominant as the generic point of Z maps to the generic point of Z'. Uniqueness of Z' is clear. If f is proper, then $Z \to Y$ is proper as a composition of proper morphisms (Morphisms of Spaces, Lemmas 67.40.3 and 67.40.5). Then we conclude that $Z \to Z'$ is proper by Morphisms of Spaces, Lemma 67.40.6. Surjectivity then follows as the image of a proper morphism is closed. □

0ENW Remark 82.7.2. In Situation 82.2.1 let X/B be good. Every $x \in |X|$ can be represented by a (unique) monomorphism $\mathrm{Spec}(k) \to X$ where k is a field, see Decent Spaces, Lemma 68.11.1. Then k is the residue field of x and is denoted $\kappa(x)$. Recall that X has a dense open subscheme $U \subset X$ (Properties of Spaces, Proposition 66.13.3). If $x \in U$, then $\kappa(x)$ agrees with the residue field of x on U as a scheme. See Decent Spaces, Section 68.11.

0ENX Remark 82.7.3. In Situation 82.2.1 let X/B be good. Assume X is integral. In this case the function field $R(X)$ of X is defined and is equal to the residue field of X at its generic point. See Spaces over Fields, Definition 72.4.3. Combining this with Remark 82.2.3 we find that for any $x \in X$ the residue field $\kappa(x)$ is the function field of the unique integral closed subspace $Z \subset X$ whose generic point is x.

0ENY Lemma 82.7.4. In Situation 82.2.1 let $X, Y/B$ be good and let $f : X \to Y$ be a morphism over B. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either f factors through a proper closed subspace of Y, or f is dominant and the extension of function fields $R(X)/R(Y)$ is finite.

Proof. By Lemma 82.7.1 there is a unique integral closed subspace $Z \subset Y$ such that f factors through a dominant morphism $X \to Z$. Then $Z = Y$ if and only if $\dim_\delta(Z) = \dim_\delta(Y)$. On the other hand, by our construction of dimension functions (see Situation 82.2.1) we have $\dim_\delta(X) = \dim_\delta(Z) + r$ where r the transcendence degree of the extension $R(X)/R(Z)$. Combining this with Spaces over Fields, Lemma 72.5.1 the lemma follows. □

0ENZ Lemma 82.7.5. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is quasi-compact, and $\{T_i\}_{i\in I}$ is a locally finite collection of closed subsets of $|X|$. Then $\{\overline{|f|(T_i)}\}_{i\in I}$ is a locally finite collection of closed subsets of $|Y|$.

Proof. Let $V \subset |Y|$ be a quasi-compact open subset. Then $|f|^{-1}(V) \subset |X|$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.3. Hence the set $\{i \in I : T_i \cap |f|^{-1}(V) \neq \emptyset\}$ is finite by a simple topological argument which we omit. Since this is the same as the set

$$\{i \in I : |f|(T_i) \cap V \neq \emptyset\} = \{i \in I : \overline{|f|(T_i)} \cap V \neq \emptyset\}$$

the lemma is proved. □

82.8. Proper pushforward

0EP0 This section is the analogue of Chow Homology, Section 42.12.

0EP1 Definition 82.8.1. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subspace with $\dim_\delta(Z) = k$. Let $Z' \subset Y$ be the image of Z as in Lemma 82.7.1. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(Z') < k, \\ \deg(Z/Z')[Z'] & \text{if} \quad \dim_\delta(Z') = k. \end{cases}$$

The degree of Z over Z' is defined and finite if $\dim_\delta(Z') = \dim_\delta(Z)$ by Lemma 82.7.4 and Spaces over Fields, Definition 72.5.2.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The pushforward of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 82.7.5 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category whose object are good algebraic spaces over B and whose morphisms are proper morphisms over B.

0EP2 Lemma 82.8.2. In Situation 82.2.1 let $X, Y, Z/B$ be good. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms over B. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.

Proof. Let $W \subset X$ be an integral closed subspace of dimension k. Consider the integral closed subspaces $W' \subset Y$ and $W'' \subset Z$ we get by applying Lemma 82.7.1 to f and W and then to g and W'. Then $W \to W'$ and $W' \to W''$ are surjective and proper. We have to show that $g_*(f_*[W]) = (f \circ g)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see $W \to W'$ and $W' \to W''$ both satisfy the hypotheses of Lemma 82.7.4. Hence

$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Spaces over Fields, Lemma 72.5.3 to conclude. $\square$

0EP3 Lemma 82.8.3. In Situation 82.2.1 let $f : X \to Y$ be a proper morphism of good algebraic spaces over B.

(1) Let $Z \subset X$ be a closed subspace with $\dim_\delta(Z) \leq k$. Then

$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Then

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Spaces, Lemma 69.20.2.

Proof. Part (1) follows from (2) and Lemma 82.6.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. By Cohomology of Spaces, Lemma 69.12.7 there exists a closed immersion $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$, see Morphisms of Spaces, Definition 67.16.2. Consider the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f|_Z"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i'"] & Y \end{tikzcd}$$

of algebraic spaces over B. Observe that $f|_Z$ is surjective (follows from Morphisms of Spaces, Lemma 67.16.3 and the fact that $|f|$ is closed) and proper (follows from Morphisms of Spaces, Lemmas 67.40.3, 67.40.5, and 67.40.6). We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion follows from Lemma 82.4.3 and the definitions. Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and surjective.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and surjective. For every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subspace Z is actually an irreducible component of Y of δ-dimension k (see Lemma 82.4.5). Pick such an integral closed subspace $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 82.4.5). By Spaces over Fields, Lemma 72.3.2 there exists an open subspace $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Since η is a generic point of an irreducible component of $|Y|$ we may assume V is an affine scheme, see Properties of Spaces, Proposition 66.13.3. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite. In particular X and Y are schemes and we reduce to the corresponding result for schemes, see Chow Homology, Lemma 42.12.4 (applied with $S = Y$). ☐

82.9. Preparation for flat pullback

0EP4 This section is the analogue of Chow Homology, Section 42.13.

Recall that a morphism of algebraic spaces is said to have relative dimension r if étale locally on the source and the target we get a morphism of schemes which has relative dimension r. The precise definition is equivalent, but in fact slightly different, see Morphisms of Spaces, Definition 67.33.2.

0EP5 **Lemma 82.9.1.** In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is flat of relative dimension r. For any closed subset $T \subset |Y|$ we have

$$\dim_\delta(|f|^{-1}(T)) = \dim_\delta(T) + r.$$

provided $|f|^{-1}(T)$ is nonempty. If $Z \subset Y$ is an integral closed subscheme and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. Since the δ-dimension of a closed subset is the supremum of the δ-dimensions of the irreducible components, it suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms of Spaces, Lemma 67.30.6 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Thus Z' dominates $Z = Y$. Finally, we see that ξ and η are in the schematic locus of X and Y by Properties of Spaces, Proposition 66.13.3. Since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0 (we may use usual local rings as ξ and η are in the schematic loci of X and Y). Thus by Morphisms of Spaces, Lemma 67.34.1 (and the definition of morphisms of given relative dimension) we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

0EP6 Lemma 82.9.2. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume $\{T_i\}_{i\in I}$ is a locally finite collection of closed subsets of $|Y|$. Then $\{|f|^{-1}(T_i)\}_{i\in I}$ is a locally finite collection of closed subsets of X.

Proof. Let $U \subset |X|$ be a quasi-compact open subset. Since the image $|f|(U) \subset |Y|$ is a quasi-compact subset there exists a quasi-compact open $V \subset |Y|$ such that $|f|(U) \subset V$. Note that

$$\{i \in I : |f|^{-1}(T_i) \cap U \neq \emptyset\} \subset \{i \in I : T_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

82.10. Flat pullback

0EP7 This section is the analogue of Chow Homology, Section 42.14.

Let S be a scheme and let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $Z \subset Y$ be a closed subspace. In this chapter we will sometimes use the terminology scheme theoretic inverse image for the inverse image $f^{-1}(Z)$ of Z constructed in Morphisms of Spaces, Definition 67.13.2. The scheme theoretic inverse image is the fibre product

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

If $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y, then $f^{-1}(\mathcal{I})\mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to $f^{-1}(Z)$ in X.

0EP8 Definition 82.10.1. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subspace of δ-dimension k. We define $f^*[Z]$ to be the $(k + r)$-cycle on X associated to the scheme theoretic inverse image

$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$

This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 82.9.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The flat pullback of α by f is the sum

$$f^*\alpha = \sum n_i f^*[Z_i]$$

where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 82.9.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the restriction of the cycle to U. Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all surjective. The reason is that given any integral closed subspace $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subspace of X (see Properties of Spaces, Definition 66.12.5). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

0EP9 Lemma 82.10.2. In Situation 82.2.1 let X/B be good. Let $U \subset X$ be an open subspace. Let Y be the reduced closed subspace of X with $|Y| = |X| \setminus |U|$ and denote $i : Y \to X$ the inclusion morphism. For every $k \in \mathbf{Z}$ the sequence

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Surjectivity of j^* we saw above. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. Such a basis element maps either to the basis element $[Z \cap U]$ of $Z_k(U)$ or to zero if $Z \subset Y$. Hence the lemma is clear in this case. The general case is similar and the proof is omitted. □

0EPY Lemma 82.10.3. In Situation 82.2.1 let $f : X \to Y$ be an étale morphism of good algebraic spaces over B. If $Z \subset Y$ is an integral closed subspace, then $f^*[Z] = \sum[Z']$ where the sum is over the irreducible components (Remark 82.5.1) of $f^{-1}(Z)$.

Proof. The meaning of the lemma is that the coefficient of $[Z']$ is 1. This follows from the fact that $f^{-1}(Z)$ is a reduced algebraic space because it is étale over the integral algebraic space Z. □

0EPA Lemma 82.10.4. In Situation 82.2.1 let $X, Y, Z/B$ be good. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s over B. Then $g \circ f$ is flat of relative dimension $r + s$ and

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms of Spaces, Lemmas 67.34.2 and 67.30.3. Suppose that

(1) $A \subset Z$ is a closed integral subspace of δ-dimension k,
(2) $A' \subset Y$ is a closed integral subspace of δ-dimension $k+s$ with $A' \subset g^{-1}(A)$, and

(3) $A'' \subset Y$ is a closed integral subspace of δ-dimension $k + s + r$ with $A'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[A'']$ in $(g \circ f)^*[A]$ agrees with the coefficient m of $[A'']$ in $f^*(g^*[A])$. We may choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

where U, V, W are schemes, the vertical arrows are étale, and there exist points $u \in U$, $v \in V$, $w \in W$ such that $u \mapsto v \mapsto w$ and such that u, v, w map to the generic points of A'', A', A. (Details omitted.) Then we have flat local ring homorphisms $\mathcal{O}_{W,w} \to \mathcal{O}_{V,v}$, $\mathcal{O}_{V,v} \to \mathcal{O}_{U,u}$, and repeatedly using Lemma 82.4.1 we find

$$n = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_w\mathcal{O}_{U,u})$$

and

$$m = \text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v})\text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

Hence the equality follows from Algebra, Lemma 10.52.14. □

0EPB Lemma 82.10.5. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r.

(1) Let $Z \subset Y$ be a closed subspace with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.
(2) Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Then we have $\dim_\delta(\text{Supp}(f^*\mathcal{F})) \leq k + r$ and
$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$
in $Z_{k+r}(X)$.

Proof. Part (1) follows from part (2) by Lemma 82.6.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Spaces, Lemma 69.12.2 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules on Sites, Lemma 18.23.4), hence $f^*\mathcal{F}$ is coherent (Cohomology of Spaces, Lemma 69.12.2 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subspace of δ-dimension k, and let $W' \subset X$ be an integral closed subspace of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W']$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[f^*\mathcal{F}]_{k+r}$. We may choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and there exist points $u \in U$, $v \in V$ such that $u \mapsto v$ and such that u, v map to the generic points of W', W. (Details omitted.) Consider the stalk $M = (\mathcal{F}|_V)_v$ as an $\mathcal{O}_{V,v}$-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 82.4.4.) We have $(f^*\mathcal{F}|_U)_u = \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{V,v}} M$. Thus we see that

$$n = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u} \otimes_{\mathcal{O}_{V,v}} M) \quad \text{and} \quad m = \text{length}_{\mathcal{O}_{V,v}}(M)\text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

Thus the equality follows from Algebra, Lemma 10.52.13. □

82.11. Push and pull

0EPC This section is the analogue of Chow Homology, Section 42.14.

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

0EPD **Lemma 82.11.1.** In Situation 82.2.1 let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of good algebraic spaces over B. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^*f_*\alpha = f'_*(g')^*\alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms of Spaces, Lemma 67.40.3. The assertion that g' is flat of relative dimension r follows from Morphisms of Spaces, Lemmas 67.34.3 and 67.30.4. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subspace $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 82.6.3. By Lemmas 82.8.3 and 82.10.5 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^*f_*\mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Spaces, Lemma 69.11.2. □

0EPE **Lemma 82.11.2.** In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms of Spaces, Definition 67.46.2). Then f is both proper and flat of relative dimension 0, and

$$f_*f^*\alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms of Spaces, Lemma 67.46.6, and a finite morphism is proper by Morphisms of Spaces, Lemma 67.45.9. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms of Spaces, Lemma 67.46.5) we see that $f_*f^*\mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Thus clearly $f_*f^*\mathcal{O}_Z$ has length d at the generic point of Z. Hence

$$f_*f^*[Z] = f_*f^*[\mathcal{O}_Z]_k = [f_*f^*\mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 82.10.5 and 82.8.3. □

82.12. Preparation for principal divisors

0EPF This section is the analogue of Chow Homology, Section 42.16. Some of the material in this section partially overlaps with the discussion in Spaces over Fields, Section 72.6.

0EPZ **Lemma 82.12.1.** *In Situation 82.2.1 let X/B be good. Assume X is integral.*

1. *If $Z \subset X$ is an integral closed subspace, then the following are equivalent:*
   (a) *Z is a prime divisor,*
   (b) *$|Z|$ has codimension 1 in $|X|$, and*
   (c) *$\dim_\delta(Z) = \dim_\delta(X) - 1$.*
2. *If Z is an irreducible component of an effective Cartier divisor on X, then $\dim_\delta(Z) = \dim_\delta(X) - 1$.*

Proof. Part (1) follows from the definition of a prime divisor (Spaces over Fields, Definition 72.6.2), Decent Spaces, Lemma 68.20.2, and the definition of a dimension function (Topology, Definition 5.20.1).

Let $D \subset X$ be an effective Cartier divisor. Let $Z \subset D$ be an irreducible component and let $\xi \in |Z|$ be the generic point. Choose an étale neighbourhood $(U, u) \to (X, \xi)$ where $U = \mathrm{Spec}(A)$ and $D \times_X U$ is cut out by a nonzerodivisor $f \in A$, see Divisors on Spaces, Lemma 71.6.2. Then u is a generic point of $V(f)$ by Decent Spaces, Lemma 68.20.1. Hence $\mathcal{O}_{U,u}$ has dimension 1 by Krull's Hauptidealsatz (Algebra, Lemma 10.60.11). Thus ξ is a codimension 1 point on X and Z is a prime divisor as desired. □

82.13. Principal divisors

0EQ0 This section is the analogue of Chow Homology, Section 42.17. The following definition is the analogue of Spaces over Fields, Definition 72.6.7 in our current setup.

0EQ1 **Definition 82.13.1.** In Situation 82.2.1 let X/B be good. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to f* is the $(n-1)$-cycle

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

defined in Spaces over Fields, Definition 72.6.7. This makes sense because prime divisors have δ-dimension $n-1$ by Lemma 82.12.1.

In the situation of the definition for $f, g \in R(X)^*$ we have

$$\mathrm{div}_X(fg) = \mathrm{div}_X(f) + \mathrm{div}_X(g)$$

in $Z_{n-1}(X)$. See Spaces over Fields, Lemma 72.6.8. The following lemma will allow us to reduce statements about principal divisors to the case of schemes.

0EQ2 **Lemma 82.13.2.** *In Situation 82.2.1 let $f : X \to Y$ be an étale morphism of good algebraic spaces over B. Assume Y is integral. Let $g \in R(Y)^*$. As cycles on X we have*

$$f^*(\mathrm{div}_Y(g)) = \sum\nolimits_{X'} (X' \to X)_* \mathrm{div}_{X'}(g \circ f|_{X'})$$

where the sum is over the irreducible components of X (Remark 82.5.1).

Proof. The map $|X| \to |Y|$ is open. The set of irreducible components of $|X|$ is locally finite in $|X|$. We conclude that $f|_{X'} : X' \to Y$ is dominant for every irreducible component $X' \subset X$. Thus $g \circ f|_{X'}$ is defined (Morphisms of Spaces, Section 67.47), hence $\text{div}_{X'}(g \circ f|_{X'})$ is defined. Moreover, the sum is locally finite and we find that the right hand side indeed is a cycle on X. The left hand side is defined by Definition 82.10.1 and the fact that an étale morphism is flat of relative dimension 0.

Since f is étale we see that $\delta_X(x) = \delta_y(f(x))$ for all $x \in |X|$. Thus if $\dim_\delta(Y) = n$, then $\dim_\delta(X') = n$ for every irreducible component X' of X (since generic points of X map to the generic point of Y, see above). Thus both left and right hand side are $(n-1)$-cycles.

Let $Z \subset X$ be an integral closed subspace with $\dim_\delta(Z) = n - 1$. To prove the equality, we need to show that the coefficients of Z are the same. Let $Z' \subset Y$ be the integral closed subspace constructed in Lemma 82.7.1. Then $\dim_\delta(Z') = n - 1$ too. Let $\xi \in |Z|$ be the generic point. Then $\xi' = f(\xi) \in |Z'|$ is the generic point. Consider the commutative diagram

$$\begin{tikzcd} \text{Spec}(\mathcal{O}^h_{X,\xi}) \arrow[r] \arrow[d] & X \arrow[d] \\ \text{Spec}(\mathcal{O}^h_{Y,\xi'}) \arrow[r] & Y \end{tikzcd}$$

of Decent Spaces, Remark 68.11.11. We have to be slightly careful as the reduced Noetherian local rings $\mathcal{O}^h_{X,\xi}$ and $\mathcal{O}^h_{Y,\xi'}$ need not be domains. Thus we work with total rings of fractions $Q(-)$ rather than fraction fields. By definition, to get the coefficient of Z' in $\text{div}_Y(g)$ we write the image of g in $Q(\mathcal{O}^h_{Y,\xi'})$ as a/b with $a, b \in \mathcal{O}^h_{Y,\xi'}$ nonzerodivisors and we take

$$\text{ord}_{Z'}(g) = \text{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/a\mathcal{O}^h_{Y,\xi'}) - \text{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/b\mathcal{O}^h_{Y,\xi'})$$

Observe that the coefficient of Z in $f^*\text{div}_Y(G)$ is the same integer, see Lemma 82.10.3. Suppose that $\xi \in X'$. Then we can consider the maps

$$\mathcal{O}^h_{Y,\xi'} \to \mathcal{O}^h_{X,\xi} \to \mathcal{O}^h_{X',\xi}$$

The first arrow is flat and the second arrow is a surjective map of reduced local Noetherian rings of dimension 1. Therefore both these maps send nonzerodivisors to nonzerodivisors and we conclude the coefficient of Z' in $\text{div}_{X'}(g \circ f|_{X'})$ is

$$\text{ord}_Z(g \circ f|_{X'}) = \text{length}_{\mathcal{O}^h_{X',\xi}}(\mathcal{O}^h_{X',\xi}/a\mathcal{O}^h_{X',\xi}) - \text{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{X',\xi}/b\mathcal{O}^h_{X',\xi})$$

by the same prescription as above. Thus it suffices to show

$$\text{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/a\mathcal{O}^h_{Y,\xi'}) = \sum\nolimits_{\xi \in |X'|} \text{length}_{\mathcal{O}^h_{X',\xi}}(\mathcal{O}^h_{X',\xi}/a\mathcal{O}^h_{X',\xi})$$

First, since the ring map $\mathcal{O}^h_{Y,\xi'} \to \mathcal{O}^h_{X,\xi}$ is flat and unramified, we have

$$\text{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/a\mathcal{O}^h_{Y,\xi'}) = \text{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi})$$

by Algebra, Lemma 10.52.13. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the nonmaximal primes of $\mathcal{O}^h_{X,\xi}$ and set $R_j = \mathcal{O}^h_{X,\xi}/\mathfrak{q}_j$. For X' as above, denote $J(X') \subset \{1, \ldots, t\}$ the set of indices such that $\mathfrak{q}_j$ corresponds to a point of X', i.e., such that under the surjection

$\mathcal{O}^h_{X,\xi} \to \mathcal{O}_{X',\xi}$ the prime $\mathfrak{q}_j$ corresponds to a prime of $\mathcal{O}_{X',\xi}$. By Chow Homology, Lemma 42.3.2 we get

$$\text{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi}) = \sum\nolimits_j \text{length}_{R_j}(R_j/aR_j)$$

and

$$\text{length}_{\mathcal{O}^h_{X',\xi}}(\mathcal{O}^h_{X',\xi}/a\mathcal{O}^h_{X',\xi}) = \sum\nolimits_{j \in J(X')} \text{length}_{R_j}(R_j/aR_j)$$

Thus the result of the lemma holds because $\{1, \ldots, t\}$ is the disjoint union of the sets $J(X')$: each point of codimension 0 on X lies on a unique X'. □

82.14. Principal divisors and pushforward

0EQ3 This section is the analogue of Chow Homology, Section 42.18.

0EQ4 Lemma 82.14.1. In Situation 82.2.1 let $X, Y/B$ be good. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set

$$g = \text{Nm}_{R(X)/R(Y)}(f).$$

Then we have $p_*\text{div}(f) = \text{div}(g)$.

Proof. We are going to deduce this from the case of schemes by étale localization. Let $Z \subset Y$ be an integral closed subspace of δ-dimension $n - 1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. Apply Spaces over Fields, Lemma 72.3.2 to the morphism $p : X \to Y$ and the generic point $\xi \in |Z|$. We find that we may replace Y by an open subspace containing ξ and assume that $p : X \to Y$ is finite. Pick an étale neighbourhood $(V, v) \to (Y, \xi)$ where V is an affine scheme. By Lemma 82.10.3 it suffices to prove the equality of cycles after pulling back to V. Set $U = V \times_Y X$ and consider the commutative diagram

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ V \arrow[r, "b"] & Y \end{tikzcd}$$

Let $V_j \subset V$, $j = 1, \ldots, m$ be the irreducible components of V. For each i, let $U_{j,i}$, $i = 1, \ldots, n_j$ be the irreducible components of U dominating V_j. Denote $p'_{j,i} : U_{j,i} \to V_j$ the restriction of $p' : U \to V$. By the case of schemes (Chow Homology, Lemma 42.18.1) we see that

$$p'_{j,i,*}\text{div}_{U_{j,i}}(f_{j,i}) = \text{div}_{V_j}(g_{j,i})$$

where $f_{j,i}$ is the restriction of f to $U_{j,i}$ and $g_{j,i}$ is the norm of $f_{j,i}$ along the finite extension $R(U_{j,i})/R(V_j)$. We have

$$\begin{aligned} b^*p_*\text{div}_X(f) &= p'_*a^*\text{div}_X(f) \\ &= p'_*\left(\sum\nolimits_{j,i}(U_{j,i} \to U)_*\text{div}_{U_{j,i}}(f_{j,i})\right) \\ &= \sum\nolimits_{j,i}(V_j \to V)_*p'_{j,i,*}\text{div}_{U_{j,i}}(f_{j,i}) \\ &= \sum\nolimits_j (V_j \to V)_*\left(\sum\nolimits_i \text{div}_{V_j}(g_{j,i})\right) \\ &= \sum\nolimits_j (V_j \to V)_*\text{div}_{V_j}(\prod\nolimits_i g_{j,i}) \end{aligned}$$

by Lemmas 82.11.1, 82.13.2, and 82.8.2. To finish the proof, using Lemma 82.13.2 again, it suffices to show that

$$g \circ b|_{V_j} = \prod\nolimits_i g_{j,i}$$

as elements of the function field of V_j. In terms of fields this is the following statement: let L/K be a finite extension. Let M/K be a finite separable extension. Write $M \otimes_K L = \prod M_i$. Then for $t \in L$ with images $t_i \in M_i$ the image of $\text{Norm}_{L/K}(t)$ in M is $\prod \text{Norm}_{M_i/M}(t_i)$. We omit the proof. □

82.15. Rational equivalence

0EQ5 This section is the analogue of Chow Homology, Section 42.19. In this section we define rational equivalence on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena (see examples in the chapter on schemes). However, if we do not allow these then we do not know how to prove that capping with Chern classes of line bundles factors through rational equivalence.

0EQ6 **Definition 82.15.1.** In Situation 82.2.1 let X/B be good. Let $k \in \mathbf{Z}$.

1. Given any locally finite collection $\{W_j \subset X\}$ of integral closed subspaces with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider
$$\sum (i_j)_* \text{div}(f_j) \in Z_k(X)$$
where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.
2. We say that $\alpha \in Z_k(X)$ is rationally equivalent to zero if α is a cycle of the form displayed above.
3. We say $\alpha, \beta \in Z_k(X)$ are rationally equivalent and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
4. We define
$$\text{CH}_k(X) = Z_k(X)/\sim_{rat}$$
to be the Chow group of k-cycles on X. This is sometimes called the Chow group of k-cycles modulo rational equivalence on X.

There are many other interesting equivalence relations. Rational equivalence is the coarsest of them all. A very simple but important lemma is the following.

0EQ7 **Lemma 82.15.2.** In Situation 82.2.1 let X/B be good. Let $U \subset X$ be an open subspace. Let Y be the reduced closed subspace of X with $|Y| = |X| \setminus |U|$ and denote $i : Y \to X$ the inclusion morphism. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that

$$\alpha \sim_{rat} \beta + i_*\gamma.$$

In other words, the sequence

$$\text{CH}_k(Y) \xrightarrow{i_*} \text{CH}_k(X) \xrightarrow{j^*} \text{CH}_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j \in J}$ be a locally finite collection of integral closed subspaces of U of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_* \text{div}(f_j)$ as in the definition. Let $W_j' \subset X$ be the corresponding integral closed subspace of X, i.e., having the same generic point as W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that

$$\alpha - \beta - \sum (i_j')_* \text{div}(f_j)$$

is a cycle on X whose restriction to U is zero. The lemma follows by applying Lemma 82.10.2. $\square$

0EQ8 Remark 82.15.3. In Situation 82.2.1 let X/B be good. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ of $|X|$ such that α_i, β_i are supported on T_i and such that $\alpha_i \sim_{rat} \beta_i$ "on" T_i. More precisely, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of integral closed subspaces $W_{i,j}$ with $|W_{i,j}| \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

82.16. Rational equivalence and push and pull

0EQ9 This section is the analogue of Chow Homology, Section 42.20. In this section we show that flat pullback and proper pushforward commute with rational equivalence.

0EQA Lemma 82.16.1. In Situation 82.2.1 let $X, Y/B$ be good. Assume Y integral with $\dim_\delta(Y) = k$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then for $g \in R(Y)^*$ we have

$$f^* \text{div}_Y(g) = \sum m_{X',X} (X' \to X)_* \text{div}_{X'}(g \circ f|_{X'})$$

as $(k+r-1)$-cycles on X where the sum is over the irreducible components X' of X and $m_{X',X}$ is the multiplicity of X' in X.

Proof. Observe that any irreducible component of X dominates Y (Lemma 82.9.1) and hence the composition $g \circ f|_{X'}$ is defined (Morphisms of Spaces, Section 67.47). We will reduce this to the case of schemes. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Picture

$$\begin{tikzcd} U \arrow[r, "a"'] \arrow[d, "h"'] & X \arrow[d, "f"] \\ V \arrow[r, "b"] & Y \end{tikzcd}$$

Since a is surjective and étale it follows from Lemma 82.10.3 that it suffices to prove the equality of cycles after pulling back by a. We can use Lemma 82.13.2 to write

$$b^*\text{div}_Y(g) = \sum (V' \to V)_*\text{div}_{V'}(g \circ b|_{V'})$$

where the sum is over the irreducible components V' of V. Using Lemma 82.11.1 we find

$$h^*b^*\text{div}_Y(g) = \sum (V' \times_V U \to U)_*(h')^*\text{div}_{V'}(g \circ b|_{V'})$$

where $h' : V' \times_V U \to V'$ is the projection. We may apply the lemma in the case of schemes (Chow Homology, Lemma 42.20.1) to the morphism $h' : V' \times_V U \to V'$ to see that we have

$$(h')^*\text{div}_{V'}(g \circ b|_{V'}) = \sum m_{U', V' \times_V U}(U' \to V' \times_V U)_*\text{div}_{U'}(g \circ b|_{V'} \circ h'|_{U'})$$

where the sum is over the irreducible components U' of $V' \times_V U$. Each U' occurring in this sum is an irreducible component of U and conversely every irreducible component U' of U is an irreducible component of $V' \times_V U$ for a unique irreducible component $V' \subset V$. Given an irreducible component $U' \subset U$, denote $\overline{a(U')} \subset X$ the "image" in X (Lemma 82.7.1); this is an irreducible component of X for example by Lemma 82.9.1. The mupltiaity $m_{U', V' \times_V U}$ is equal to the multiplicity $m_{\overline{a(U')}, X}$. This follows from the equality $h^*a^*[Y] = b^*f^*[Y]$ (Lemma 82.10.4), the definitions, and Lemma 82.10.3. Combining all of what we just said we obtain

$$a^*f^*\text{div}_Y(g) = h^*b^*\text{div}_Y(g) = \sum m_{\overline{a(U')}, X}(U' \to U)_*\text{div}_{U'}(g \circ (f \circ a)|_{U'})$$

Next, we analyze what happens with the right hand side of the formula in the statement of the lemma if we pullback by a. First, we use Lemma 82.11.1 to get

$$a^* \sum m_{X', X}(X' \to X)_*\text{div}_{X'}(g \circ f|_{X'}) = \sum m_{X', X}(X' \times_X U \to U)_*(a')^*\text{div}_{X'}(g \circ f|_{X'})$$

where $a' : X' \times_X U \to X'$ is the projection. By Lemma 82.13.2 we get

$$(a')^*\text{div}_{X'}(g \circ f|_{X'}) = \sum (U' \to X' \times_X U)_*\text{div}_{U'}(g \circ (f \circ a)|_{U'})$$

where the sum is over the irreducible components U' of $X' \times_X U$. These U' are irreducible components of U and in fact are exactly the irreducible components of U such that $\overline{a(U')} = X'$. Comparing with what we obtained above we conclude. $\square$

0EQB Lemma 82.16.2. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k+r)$-cycles on X.

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $g_j \in R(W_j)^*$. Moreover, assume that the collection $\{|i_j|(|W_j|)\}_{j \in J}$ is locally finite in $|Y|$. Then we have to show that

$$f^*(\sum i_{j,*}\text{div}(g_j)) = \sum f^*i_{j,*}\text{div}(g_j)$$

is rationally equivalent to zero on X. The sum on the right makes sense by Lemma 82.9.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote $f_j : W'_j \to W_j$ the first projection. By Lemma 82.11.1 we can write the sum above as

$$\sum i'_{j,*}(f_j^*\mathrm{div}(g_j))$$

By Lemma 82.16.1 we see that each $f_j^*\mathrm{div}(g_j)$ is rationally equivalent to zero on W'_j. Hence each $i'_{j,*}(f_j^*\mathrm{div}(g_j))$ is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 82.15.3. $\square$

0EQC Lemma 82.16.3. In Situation 82.2.1 let $X, Y/B$ be good. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j\in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\mathrm{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_*i_{j,*}\mathrm{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subspace which is the image of $p \circ i_j$, see Lemma 82.7.1. The collection $\{W'_j\}$ is locally finite in Y by Lemma 82.7.5. Hence it suffices to show, for a given j, that either $p_*i_{j,*}\mathrm{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\mathrm{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a since integral closed subspace $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{tikzcd} W \arrow[r, "i"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "i'"] & Y \end{tikzcd}$$

Note that $p_*i_*\mathrm{div}(f) = i'_*(p')_*\mathrm{div}(f)$ by Lemma 82.8.2. As explained above we have to show that $(p')_*\mathrm{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_*\mathrm{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_*\mathrm{div}(f) = 0$ in this case. Since $(p')_*\mathrm{div}(f)$ is a k-cycle, we see that $(p')_*\mathrm{div}(f) = n[W']$ for some $n \in \mathbf{Z}$. In order to prove that $n = 0$ we may replace W' by a nonempty open subspace. In particular, we may and do assume that W' is a scheme. Let $\eta \in W'$ be the generic point. Let $K = \kappa(\eta) = R(W')$ be the function field. Consider the base change diagram

$$\begin{tikzcd} W_\eta \arrow[r] \arrow[d, "c"'] & W \arrow[d, "p'"] \\ \mathrm{Spec}(K) \arrow[r, "\eta"] & W' \end{tikzcd}$$

Observe that c is proper. Also $|W_\eta|$ has dimension 1: use Decent Spaces, Lemma 68.18.6 to identify $|W_\eta|$ as the subspace of $|W|$ of points mapping to η and note that since $\dim_\delta(W) = k+1$ and $\delta(\eta) = k$ points of W_η must have δ-value either k or $k+1$. Thus the local rings have dimension ≤ 1 by Decent Spaces, Lemma 68.20.2. By Spaces over Fields, Lemma 72.9.3 we find that W_η is a scheme. Since $\mathrm{Spec}(K)$ is the limit of the nonempty affine open subschemes of W' we conclude that we may assume that W is a scheme by Limits of Spaces, Lemma 70.5.11. Then finally by the case of schemes (Chow Homology, Lemma 42.20.3) we find that $n = 0$.

The case $\dim_\delta(W') = k+1$. In this case Lemma 82.14.1 applies, and we see that indeed $p'_*\mathrm{div}(f) = \mathrm{div}(g)$ for some $g \in R(W')^*$ as desired. □

82.17. The divisor associated to an invertible sheaf

0EQD This section is the analogue of Chow Homology, Section 42.24. The following definition is the analogue of Spaces over Fields, Definition 72.7.4 in our current setup.

0EQE Definition 82.17.1. In Situation 82.2.1 let X/B be good. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the Weil divisor associated to s is the $(n-1)$-cycle

$$\mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z]$$

defined in Spaces over Fields, Definition 72.7.4. This makes sense because Weil divisors have δ-dimension $n-1$ by Lemma 82.12.1.

(2) We define Weil divisor associated to $\mathcal{L}$ as

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \mathrm{div}_\mathcal{L}(s) \in \mathrm{CH}_{n-1}(X)$$

where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Spaces over Fields, Lemma 72.7.3.

The zero scheme of a nonzero section is an effective Cartier divisor whose Weil divisor class computes the Weil divisor associated to the invertible module.

0EQF Lemma 82.17.2. In Situation 82.2.1 let X/B be good. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then

$$\mathrm{div}_\mathcal{L}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $\mathrm{CH}_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subspace of δ-dimension $n-1$. Let $\xi \in |Z|$ be its generic point. To prove the first equality we compare the coefficients of Z on both sides. Choose an elementary étale neighbourhood $(U, u) \to (X, \xi)$, see Decent Spaces, Section 68.11 and recall that $\mathcal{O}^h_{X,\xi} = \mathcal{O}^h_{U,u}$ in this case. After replacing U by an open neighbourhood of u we may assume there is a trivializing section s_U of $\mathcal{L}|_U$. Write $s|_U = fs_U$ for some $f \in \Gamma(U, \mathcal{O}_U)$. Then $Z \times_X U$ is equal to $V(f)$ as a closed subscheme of U, see Divisors on Spaces, Definition 71.7.6. As in Spaces over Fields, Section 72.7 denote $\mathcal{L}_\xi$ the pullback of $\mathcal{L}$ under the canonical morphism $c_\xi : \mathrm{Spec}(\mathcal{O}^h_{X,\xi}) \to X$. Denote s_ξ the pullback of s_U; it is a trivialization of $\mathcal{L}_\xi$. Then we see that $c_\xi^*(s) = fs_\xi$. The coefficient of Z in $[Z(s)]_{n-1}$ is by definition

$$\text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/f\mathcal{O}_{U,u})$$

Since $\mathcal{O}_{U,u} \to \mathcal{O}^h_{X,\xi}$ is flat and identifies residue fields this is equal to

$$\text{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/f\mathcal{O}^h_{X,\xi})$$

by Algebra, Lemma 10.52.13. This final quantity is equal to $\text{ord}_{Z,\mathcal{L}}(s)$ by Spaces over Fields, Definition 72.7.1, i.e., to the coefficient of Z in $\text{div}_\mathcal{L}(s)$ as desired. $\square$

0EQG Lemma 82.17.3. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The morphism

$$q : T = \underline{\mathrm{Spec}}\left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}\right) \longrightarrow X$$

has the following properties:

(1) q is surjective, smooth, affine, of relative dimension 1,
(2) there is an isomorphism $\alpha : q^*\mathcal{L} \cong \mathcal{O}_T$,
(3) formation of $(q : T \to X, \alpha)$ commutes with base change,
(4) $q^* : Z_k(X) \to Z_{k+1}(T)$ is injective,
(5) if $Z \subset X$ is an integral closed subspace, then $q^{-1}(Z) \subset T$ is an integral closed subspace,
(6) if $Z \subset X$ is a closed subspace of X of δ-dimension $\leq k$, then $q^{-1}(Z)$ is a closed subspace of T of δ-dimension $\leq k+1$ and $q^*[Z]_k = [q^{-1}(Z)]_{k+1}$,
(7) if $\xi' \in |T|$ is the generic point of the fibre of $|T| \to |X|$ over ξ, then the ring map $\mathcal{O}^h_{X,\xi} \to \mathcal{O}^h_{T,\xi'}$ is flat, we have $\mathfrak{m}^h_{\xi'} = \mathfrak{m}^h_\xi \mathcal{O}^h_{T,\xi'}$, and the residue field extension is purely transcendental of transcendence degree 1, and
(8) add more here as needed.

Proof. Let $U \to X$ be an étale morphism such that $\mathcal{L}|_U$ is trivial. Then $T \times_X U \to U$ is isomorphic to the projection morphism $\mathbf{G}_m \times U \to U$, where $\mathbf{G}_m$ is the multipliciative group scheme, see Groupoids, Example 39.5.1. Thus (1) is clear.

To see (2) observe that $q_*q^*\mathcal{L} = \bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n+1}$. Thus there is an obvious isomorphism $q_*q^*\mathcal{L} \to q_*\mathcal{O}_T$ of $q_*\mathcal{O}_T$-modules. By Morphisms of Spaces, Lemma 67.20.10 this determines an isomorphism $q^*\mathcal{L} \to \mathcal{O}_T$.

Part (3) holds because forming the relative spectrum commutes with arbitrary base change and the same thing is clearly true for the isomorphism α.

Part (4) follows immediately from (1) and the definitions.

Part (5) follows from the fact that if Z is an integral algebraic space, then $\mathbf{G}_m \times Z$ is an integral algebraic space.

Part (6) follows from the fact that lengths are preserved: if $(A, \mathfrak{m})$ is a local ring and $B = A[x]_{\mathfrak{m}A[x]}$ and if M is an A-module, then $\text{length}_A(M) = \text{length}_B(M \otimes_A B)$. This implies that if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module and $\xi \in |X|$ with $\xi' \in |T|$ the generic point of the fibre over ξ, then the length of $\mathcal{F}$ at ξ is the same as the length of $q^*\mathcal{F}$ at ξ'. Tracing through the definitions this gives (6) and more.

The map in part (7) comes from Decent Spaces, Remark 68.11.11. However, in our case we have

$$\text{Spec}(\mathcal{O}_{X,\xi}^h) \times_X T = \mathbf{G}_m \times \text{Spec}(\mathcal{O}_{X,\xi}^h) = \text{Spec}(\mathcal{O}_{X,\xi}^h[t, t^{-1}])$$

and ξ' corresponds to the generic point of the special fibre of this over $\text{Spec}(\mathcal{O}_{X,\xi}^h)$. Thus $\mathcal{O}_{T,\xi'}^h$ is the henselization of the localization of $\mathcal{O}_{X,\xi}^h[t, t^{-1}]$ at the corresponding prime. Part (7) follows from this and some commutative algebra; details omitted. $\square$

0EQH Lemma 82.17.4. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume X is integral. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $q : T \to X$ be the morphism of Lemma 82.17.3. Then

$$q^*\text{div}_\mathcal{L}(s) = \text{div}_T(q^*(s))$$

where we view the pullback $q^*(s)$ as a nonzero meromorphic function on T using the isomorphism $q^*\mathcal{L} \to \mathcal{O}_T$

Proof. Observe that $\text{div}_T(q^*(s)) = \text{div}_{\mathcal{O}_T}(q^*(s))$ by the compatibility between the constructions given in Spaces over Fields, Sections 72.6 and 72.7. We will show the agreement with $\text{div}_{\mathcal{O}_T}(q^*(s))$ in this proof. We will use all the properties of $q : T \to X$ stated in Lemma 82.17.3 without further mention. Let $Z \subset T$ be a prime divisor. Then either $Z \to X$ is dominant or $Z = q^{-1}(Z')$ for some prime divisor $Z' \subset X$. If $Z \to X$ is dominant, then the coefficient of Z in either side of the equality of the lemma is zero. Thus we may assume $Z = q^{-1}(Z')$ where $Z' \subset X$ is a prime divisor. Let $\xi' \in |Z'|$ and $\xi \in |Z|$ be the generic points. Then we obtain a commutative diagram

$$\begin{tikzcd} \text{Spec}(\mathcal{O}_{T,\xi}^h) \arrow[r, "c_\xi"'] \arrow[d, "h"'] & T \arrow[d, "q"] \\ \text{Spec}(\mathcal{O}_{X,\xi'}^h) \arrow[r, "c_{\xi'}"] & X \end{tikzcd}$$

see Decent Spaces, Remark 68.11.11. Choose a trivialization $s_{\xi'}$ of $\mathcal{L}_{\xi'} = c_{\xi'}^*\mathcal{L}$. Then we can use the pullback s_ξ of $s_{\xi'}$ via h as our trivialization of $\mathcal{L}_\xi = c_\xi^* q^*\mathcal{L}$. Write $s/s_{\xi'} = a/b$ for $a, b \in \mathcal{O}_{X,\xi'}$ nonzerodivisors. By definition the coefficient of Z' in $\text{div}_\mathcal{L}(s)$ is

$$\text{length}_{\mathcal{O}_{X,\xi'}^h}(\mathcal{O}_{X,\xi'}^h/a\mathcal{O}_{X,\xi'}^h) - \text{length}_{\mathcal{O}_{X,\xi'}^h}(\mathcal{O}_{X,\xi'}^h/b\mathcal{O}_{X,\xi'}^h)$$

Since $Z = q^{-1}(Z')$, this is also the coefficient of Z in $q^*\text{div}_\mathcal{L}(s)$. Since $\mathcal{O}_{X,\xi'}^h \to \mathcal{O}_{T,\xi}^h$ is flat the elements a, b map to nonzerodivisors in $\mathcal{O}_{T,\xi}^h$. Thus $q^*(s)/s_\xi = a/b$ in the total quotient ring of $\mathcal{O}_{T,\xi}^h$. By definition the coefficient of Z in $\text{div}_T(q^*(s))$ is

$$\text{length}_{\mathcal{O}_{T,\xi}^h}(\mathcal{O}_{T,\xi}^h/a\mathcal{O}_{T,\xi}^h) - \text{length}_{\mathcal{O}_{T,\xi}^h}(\mathcal{O}_{T,\xi}^h/b\mathcal{O}_{T,\xi}^h)$$

The proof is finished because these lengths are the same as before by Algebra, Lemma 10.52.13 and the fact that $\mathfrak{m}_\xi^h = \mathfrak{m}_{\xi'}^h\mathcal{O}_{T,\xi}^h$ shown in Lemma 82.17.3. $\square$

82.18. Intersecting with an invertible sheaf

0EQI This section is the analogue of Chow Homology, Section 42.25. In this section we study the following construction.

0EQJ **Definition 82.18.1.** In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to \mathrm{CH}_k(X)$$

called *intersection with the first Chern class of $\mathcal{L}$*.

(1) Given an integral closed subspace $i : W \to X$ with $\dim_\delta(W) = k+1$ we define

$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$

where the right hand side is defined in Definition 82.17.1.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set

$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subspaces of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subspaces on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subspaces of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, often more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $\mathrm{CH}_k(\coprod W_i) = \prod \mathrm{CH}_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence. This is not a triviality.

0EQK **Lemma 82.18.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $\mathrm{CH}_k(X)$ for every $\alpha \in Z_{k-1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Spaces over Fields, Lemma 72.7.5 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subspace $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. □

Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on an algebraic space X, see Divisors on Spaces, Definition 71.7.6.

0EQL **Lemma 82.18.3.** *In Situation 82.2.1 let Y/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $s \in \Gamma(Y, \mathcal{L})$ be a regular section and assume $\dim_\delta(Y) \leq k+1$. Write $[Y]_{k+1} = \sum n_i[Y_i]$ where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$. Set $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$. Then*

0EQM (82.18.3.1)
$$[Z(s)]_k = \sum n_i[Z(s_i)]_k$$

as k-cycles on Y.

Proof. Let $\varphi : V \to Y$ be a surjective étale morphism where V is a scheme. It suffices to prove the equality after pulling back by φ, see Lemma 82.10.3. That same lemma tells us that $\varphi^*[Y_i] = [\varphi^{-1}(Y_i)] = \sum [V_{i,j}]$ where $V_{i,j}$ are the irreducible components of V lying over Y_i. Hence if we first apply the case of schemes (Chow Homology, Lemma 42.25.3) to $\varphi^* s_i$ on $Y_i \times_Y V$ we find that $\varphi^*[Z(s_i)]_k = [Z(\varphi^* s_i)] = \sum [Z(s_{i,j})]_k$ where $s_{i,j}$ is the pullback of s to $V_{i,j}$. Applying the case of schemes to $\varphi^* s$ we get

$$\varphi^*[Z(s)]_k = [Z(\varphi^* s)]_k = \sum n_i [Z(s_{i,j})]_k$$

by our remark on multiplicities above. Combining all of the above the proof is complete. □

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 31.14.8.

0EQN Lemma 82.18.4. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y \subset X$ be a closed subscheme with $\dim_\delta(Y) \leq k+1$ and let $s \in \Gamma(Y, \mathcal{L}|_Y)$ be a regular section. Then

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in $\mathrm{CH}_k(X)$.

Proof. Write

$$[Y]_{k+1} = \sum n_i [Y_i]$$

where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ is not zero, and hence is a regular section. By Lemma 82.17.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{Y_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i [Z(s_i)]_k$$

Thus the result follows from Lemma 82.18.3. □

82.19. Intersecting with an invertible sheaf and push and pull

0EQP This section is the analogue of Chow Homology, Section 42.26. In this section we prove that the operation $c_1(\mathcal{L}) \cap -$ commutes with flat pullback and proper pushforward.

0EQQ Lemma 82.19.1. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume Y is integral and $n = \dim_\delta(Y)$. Let s be a nonzero meromorphic section of $\mathcal{L}$. Then we have

$$f^* \mathrm{div}_{\mathcal{L}}(s) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

in $Z_{n+r-1}(X)$. Here the sum is over the irreducible components $X_i \subset X$ of δ-dimension $n+r$, the section $s_i = f|_{X_i}^*(s)$ is the pullback of s, and $n_i = m_{X_i,X}$ is the multiplicity of X_i in X.

Proof. Using sleight of hand we will deduce this from Lemma 82.16.1. (An alternative is to redo the proof of that lemma in the setting of meromorphic sections of invertible modules.) Namely, let $q : T \to Y$ be the morphism of Lemma 82.17.3 constructed using $\mathcal{L}$ on Y. We will use all the properties of T stated in this lemma. Consider the fibre product diagram

$$\begin{tikzcd} T' \arrow[r, "q'"] \arrow[d, "h"] & X \arrow[d, "f"] \\ T \arrow[r, "q"] & Y \end{tikzcd}$$

Then $q' : T' \to X$ is the morphism constructed using $f^*\mathcal{L}$ on X. Then it suffices to prove

$$(q')^* f^* \mathrm{div}_{\mathcal{L}}(s) = \sum n_i (q')^* \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

Observe that $T'_i = q^{-1}(X_i)$ are the irreducible components of T' and that n_i is the multiplicity of T'_i in T'. The left hand side is equal to

$$h^* q^* \mathrm{div}_{\mathcal{L}}(s) = h^* \mathrm{div}_T(q^*(s))$$

by Lemma 82.17.4 (and Lemma 82.10.4). On the other hand, denoting $q'_i : T'_i \to X_i$ the restriction of q' we find that Lemma 82.17.4 also tells us the right hand side is equal to

$$\sum n_i \mathrm{div}_{T_i}((q'_i)^*(s_i))$$

In these two formulas the expressions $q^*(s)$ and $(q'_i)^*(s_i)$ represent the rational functions corresponding to the pulled back meromorphic sections of $q^*\mathcal{L}$ and $(q'_i)^* f^*\mathcal{L}|_{X_i}$ via the isomorphism $\alpha : q^*\mathcal{L} \to \mathcal{O}_T$ and its pullbacks to spaces over T. With this convention it is clear that $(q'_i)^*(s_i)$ is the composition of the rational function $q^*(s)$ on T and the morphism $h|_{T'_i} : T'_i \to T$. Thus Lemma 82.16.1 exactly says that

$$h^* \mathrm{div}_T(q^*(s)) = \sum n_i \mathrm{div}_{T_i}((q'_i)^*(s_i))$$

as desired. □

0EQR Lemma 82.19.2. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $\mathrm{CH}_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i [W_i]$. We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$$

in $\mathrm{CH}_{k+r-1}(X)$ by producing a rational equivalence on the closed subspace $f^{-1}(W_i)$ of X. By the discussion in Remark 82.15.3 this will prove the equality of the lemma is true.

Let $W \subset Y$ be an integral closed subspace of δ-dimension k. Consider the closed subspace $W' = f^{-1}(W) = W \times_Y X$ so that we have the fibre product diagram

$$\begin{tikzcd} W' \arrow[r] \arrow[d, "h"] & X \arrow[d, "f"] \\ W \arrow[r] & Y \end{tikzcd}$$

We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Let $W'_i \subset W'$ be the irreducible components of δ-dimension $k + r$. Write $[W']_{k+r} = \sum n_i [W'_i]$ with n_i the multiplicity of W'_i in W' as per definition. So $f^*[W] = \sum n_i [W'_i]$ in $Z_{k+r}(X)$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. By Lemma 82.19.1 we have the following equality of cycles

$$h^*\mathrm{div}_{\mathcal{L}|_W}(s) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{W'_i}}(s_i)$$

in $Z_{k+r-1}(W')$. This finishes the proof since the left hand side is a cycle on W' which pushes to $f^*(c_1(\mathcal{L}) \cap [W])$ in $\mathrm{CH}_{k+r-1}(X)$ and the right hand side is a cycle on W' which pushes to $c_1(f^*\mathcal{L}) \cap f^*[W]$ in $\mathrm{CH}_{k+r-1}(X)$. □

0EQS Lemma 82.19.3. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Then

$$f_*\left(\mathrm{div}_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)]\mathrm{div}_{\mathcal{L}}(s).$$

as cycles on Y. In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. Proof of the first equaltion. Let $q : T \to Y$ be the morphism of Lemma 82.17.3 constructed using $\mathcal{L}$ on Y. We will use all the properties of T stated in this lemma. Consider the fibre product diagram

$$\begin{tikzcd} T' \arrow[r, "q'"] \arrow[d, "h"] & X \arrow[d, "f"] \\ T \arrow[r, "q"] & Y \end{tikzcd}$$

Then $q' : T' \to X$ is the morphism constructed using $f^*\mathcal{L}$ on X. It suffices to prove the equality after pulling back to T'. The left hand side pulls back to

$$\begin{aligned} q^*f_*\left(\mathrm{div}_{f^*\mathcal{L}}(f^*s)\right) &= h_*(q')^*\mathrm{div}_{f^*\mathcal{L}}(f^*s) \\ &= h_*\mathrm{div}_{(q')^*f^*\mathcal{L}}((q')^*f^*s) \\ &= h_*\mathrm{div}_{h^*q^*\mathcal{L}}(h^*q^*s) \end{aligned}$$

The first equality by Lemma 82.11.1. The second by Lemma 82.19.1 using that T' is integral. The third because the displayed diagram commutes. The right hand side pulls back to

$$[R(X) : R(Y)]q^*\mathrm{div}_{\mathcal{L}}(s) = [R(T') : R(T)]\mathrm{div}_{q^*\mathcal{L}}(q^*s)$$

This follows from Lemma 82.19.1, the fact that T is integral, and the equality $[R(T') : R(T)] = [R(X) : R(Y)]$ whose proof we omit (it follows from Lemma 82.11.1 but that would be a silly way to prove the equality). Thus it suffices to prove the lemma for $h : T' \to T$, the invertible module $q^{\mathcal{L}}$ and the section q^*s. Since $q^*\mathcal{L} = \mathcal{O}_T$ we reduce to the case where $\mathcal{L} \cong \mathcal{O}$ discussed in the next paragraph.

Assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$. Using the embedding $R(Y) \subset R(X)$ we may think of g as a rational on X and we are simply trying to prove

$$f_* (\mathrm{div}_X(g)) = [R(X) : R(Y)]\mathrm{div}_Y(g).$$

Comparing with the result of Lemma 82.14.1 we see this true since $\mathrm{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. □

0EQT Lemma 82.19.4. In Situation 82.2.1 let $X, Y/B$ be good. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $\mathrm{CH}_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subspace $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the "image" of W_i as in Lemma 82.7.1. Consider the diagram

$$\begin{tikzcd} X' = \coprod W_i \arrow[r, "q"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' = \coprod W_i' \arrow[r, "q'"] & Y. \end{tikzcd}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 82.19.3. □

82.20. The key formula

0EQU This section is the analogue of Chow Homology, Section 42.27. We strongly urge the reader to read the proof in that case first.

In Situation 82.2.1 let X/B be good. Assume X is integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s be a nonzero meromorphic section of $\mathcal{L}$ and let t be a nonzero meromorphic section of $\mathcal{N}$. Let $Z \subset X$ be a prime divisor with generic point $\xi \in |Z|$. Consider the morphism

$$c_\xi : \mathrm{Spec}(\mathcal{O}_{X,\xi}^h) \longrightarrow X$$

used in Spaces over Fields, Section 72.7. We denote $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$ the pullbacks of $\mathcal{L}$ and $\mathcal{N}$ by c_ξ; we often think of $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$ as the rank 1 free $\mathcal{O}_{X,\xi}^h$-modules they give rise to. Note that the pullback of s, resp. t is a regular meromorphic section of $\mathcal{L}_\xi$, resp. $\mathcal{N}_\xi$.

Let $Z_i \subset X$, $i \in I$ be a locally finite set of prime divisors with the following property: If $Z \notin \{Z_i\}$, then s is a generator for $\mathcal{L}_\xi$ and t is a generator for $\mathcal{N}_\xi$. Such a set exists by Spaces over Fields, Lemma 72.7.2. Then

$$\mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{Z_i,\mathcal{L}}(s)[Z_i]$$

and similarly

$$\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each i generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ where ξ_i is the generic point of Z_i. Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

with f_i, g_i invertible elements of the total ring of fractions $Q(\mathcal{O}^h_{X,\xi_i})$. We abbreviate $B_i = \mathcal{O}^h_{X,\xi_i}$. Let us denote

$$\mathrm{ord}_{B_i} : Q(B_i)^* \longrightarrow \mathbf{Z}, \quad a/b \longmapsto \mathrm{length}_{B_i}(B_i/aB_i) - \mathrm{length}_{B_i}(B_i/bB_i)$$

In other words, we temporarily extend Algebra, Definition 10.121.2 to these reduced Noetherian local rings of dimension 1. Then by definition

$$\mathrm{ord}_{Z_i,\mathcal{L}}(s) = \mathrm{ord}_{B_i}(f_i) \quad \text{and} \quad \mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$$

Since ξ_i is the generic point of Z_i we see that the residue field $\kappa(\xi_i)$ is the function field of Z_i. Moreover $\kappa(\xi_i)$ is the residue field of B_i, see Decent Spaces, Lemma 68.11.10. Since t_i is a generator of $\mathcal{N}_{\xi_i}$ we see that its image in the fibre $\mathcal{N}_{\xi_i} \otimes_{B_i} \kappa(\xi_i)$ is a nonzero meromorphic section of $\mathcal{N}|_{Z_i}$. We will denote this image $t_i|_{Z_i}$. From our definitions it follows that

$$c_1(\mathcal{N}) \cap \mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{B_i}(f_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in $\mathrm{CH}_{n-2}(X)$. We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^* = R(Z_i)^*$$

see Chow Homology, Section 42.5.

0EQV **Lemma 82.20.1** (Key formula). *In the situation above the cycle*

$$\sum (Z_i \to X)_* \left(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \mathrm{ord}_{B_i}(g_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})\right)$$

is equal to the cycle

$$\sum (Z_i \to X)_*\mathrm{div}(\partial_{B_i}(f_i, g_i))$$

Proof. The strategy of the proof will be: first reduce to the case where $\mathcal{L}$ and $\mathcal{N}$ are trivial invertible modules, then change our choices of local trivializations, and then finally use étale localization to reduce to the case of schemes[1].

First step. Let $q : T \to X$ be the morphism constructed in Lemma 82.17.3. We will use all properties stated in that lemma without further mention. In particular, it suffices to show that the cycles are equal after pulling back by q. Denote s' and t' the pullbacks of s and t to meromorphic sections of $q^*\mathcal{L}$ and $q^*\mathcal{N}$. Denote $Z'_i = q^{-1}(Z_i)$, denote $\xi'_i \in |Z'_i|$ the generic point, denote $B'_i = \mathcal{O}^h_{T,\xi'_i}$, denote $\mathcal{L}_{\xi'_i}$

[1] It is possible that a shorter proof can be given by immediately applying étale localization.

and $\mathcal{N}_{\xi'_i}$ the pullbacks of $\mathcal{L}$ and $\mathcal{N}$ to $\text{Spec}(B'_i)$. Recall that we have commutative diagrams

$$\begin{tikzcd} \text{Spec}(B'_i) \arrow[r, "c_{\xi'_i}"'] \arrow[d] & T \arrow[d, "q"] \\ \text{Spec}(B_i) \arrow[r, "c_{\xi_i}"] & X \end{tikzcd}$$

see Decent Spaces, Remark 68.11.11. Denote s'_i and t'_i the pullbacks of s_i and t_i which are generators of $\mathcal{L}_{\xi'_i}$ and $\mathcal{N}_{\xi'_i}$. Then we have

$$s' = f'_i s'_i \quad \text{and} \quad t' = g'_i t'_i$$

where f'_i and g'_i are the images of f_i, g_i under the map $Q(B_i) \to Q(B'_i)$ induced by $B_i \to B'_i$. By Algebra, Lemma 10.52.13 we have

$$\text{ord}_{B_i}(f_i) = \text{ord}_{B'_i}(f'_i) \quad \text{and} \quad \text{ord}_{B_i}(g_i) = \text{ord}_{B'_i}(g'_i)$$

By Lemma 82.19.1 applied to $q : Z'_i \to Z_i$ we have

$$q^*\text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) = \text{div}_{q^*\mathcal{N}|_{Z'_i}}(t'_i|_{Z'_i}) \quad \text{and} \quad q^*\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = \text{div}_{q^*\mathcal{L}|_{Z'_i}}(s'_i|_{Z'_i})$$

This already shows that the first cycle in the statement of the lemma pulls back to the corresponding cycle for s', t', Z'_i, s'_i, t'_i. To see the same is true for the second, note that by Chow Homology, Lemma 42.5.4 we have

$$\partial_{B_i}(f_i, g_i) \mapsto \partial_{B'_i}(f'_i, g'_i) \quad \text{via} \quad \kappa(\xi_i) \to \kappa(\xi'_i)$$

Hence the same lemma as before shows that

$$q^*\text{div}(\partial_{B_i}(f_i, g_i)) = \text{div}(\partial_{B'_i}(f'_i, g'_i))$$

Since $q^*\mathcal{L} \cong \mathcal{O}_T$ we find that it suffices to prove the equality in case $\mathcal{L}$ is trivial. Exchanging the roles of $\mathcal{L}$ and $\mathcal{N}$ we see that we may similarly assume $\mathcal{N}$ is trivial. This finishes the proof of the first step.

Second step. Assume $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$. Denote 1 the trivializing section of $\mathcal{L}$. Then $s_i = u \cdot 1$ for some unit $u \in B_i$. Let us examine what happens if we replace s_i by 1. Then f_i gets replaced by uf_i. Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$\text{ord}_{B_i}(g_i)\text{div}(u|_{Z_i})$$

where $u|_{Z_i}$ is the image of u in the residue field by Spaces over Fields, Lemma 72.7.3 and in the second expression we add

$$\text{div}(\partial_{B_i}(u, g_i))$$

by bi-linearity of the tame symbol. These terms agree by the property of the tame symbol given in Chow Homology, Equation (6).

Let $Y \subset X$ be an integral closed subspace with $\dim_\delta(Y) = n - 2$. To show that the coefficients of Y of the two cycles of the lemma is the same, we may do a replacement of s_i by 1 as in the previous paragraph. In exactly the same way one shows that we may do a replacement of t_i by 1. Since there are only a finite number of Z_i such that $Y \subset Z_i$ we may assume $s_i = 1$ and $t_i = 1$ for all these Z_i.

Third step. Here we prove the coefficients of Y in the cycles of the lemma agree for an integral closed subspace Y with $\dim_\delta(Y) = n - 2$ such that moreover $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$ and $s_i = 1$ and $t_i = 1$ for all Z_i such that $Y \subset Z_i$. After replacing X

by a smaller open subspace we may in fact assume that s_i and t_i are equal to 1 for all i. In this case the first cycle is zero. Our task is to show that the coefficient of Y in the second cycle is zero as well.

First, since $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$ we may and do think of s, t as rational functions f, g on X. Since s_i and t_i are equal to 1 we find that f_i, resp. g_i is the image of f, resp. g in $Q(B_i)$ for all i. Let $\zeta \in |Y|$ be the generic point. Choose an étale neighbourhood

$$(U, u) \longrightarrow (X, \zeta)$$

and denote $Y' = \overline{\{u\}} \subset U$. Since an étale morphism is flat, we can pullback f and g to regular meromorphic functions on U which we will also denote f and g. For every prime divisor $Y \subset Z \subset X$ the scheme $Z \times_X U$ is a union of prime divisors of U. Conversely, given a prime divisor $Y' \subset Z' \subset U$, there is a prime divisor $Y \subset Z \subset X$ such that Z' is a component of $Z \times_X U$. Given such a pair (Z, Z') the ring map

$$\mathcal{O}^h_{X,\xi} \to \mathcal{O}^h_{U,\xi'}$$

is étale (in fact it is finite étale). Hence we find that

$$\partial_{\mathcal{O}^h_{X,\xi}}(f, g) \mapsto \partial_{\mathcal{O}^h_{U,\xi'}}(f, g) \quad \text{via} \quad \kappa(\xi) \to \kappa(\xi')$$

by Chow Homology, Lemma 42.5.4. Thus Lemma 82.13.2 applies to show

$$(Z \times_X U \to Z)^* \text{div}_Z(\partial_{\mathcal{O}^h_{X,\xi}}(f, g)) = \sum\nolimits_{Z' \subset Z \times_X U} \text{div}_{Z'}(\partial_{\mathcal{O}^h_{U,\xi'}}(f, g))$$

Since flat pullback commutes with pushforward along closed immersions (Lemma 82.11.1) we see that it suffices to prove that the coefficient of Y' in

$$\sum\nolimits_{Z' \subset U} (Z' \to U)_* \text{div}_{Z'}(\partial_{\mathcal{O}^h_{U,\xi'}}(f, g))$$

is zero.

Let $A = \mathcal{O}_{U,u}$. Then $f, g \in Q(A)^*$. Thus we can write $f = a/b$ and $g = c/d$ with $a, b, c, d \in A$ nonzerodivisors. The coefficient of Y' in the expression above is

$$\sum\nolimits_{\mathfrak{q} \subset A \text{ height } 1} \text{ord}_{A/\mathfrak{q}}(\partial_{A_\mathfrak{q}}(f, g))$$

By bilinearity of ∂_A it suffices to prove

$$\sum\nolimits_{\mathfrak{q} \subset A \text{ height } 1} \text{ord}_{A/\mathfrak{q}}(\partial_{A_\mathfrak{q}}(a, c))$$

is zero and similarly for the other pairs (a, d), (b, c), and (b, d). This is true by Chow Homology, Lemma 42.6.2. $\square$

82.21. Intersecting with an invertible sheaf and rational equivalence

0EQW This section is the analogue of Chow Homology, Section 42.28. Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that $c_1(\mathcal{L}) \cap -$ factors through rational equivalence and that these operations for different invertible sheaves commute.

0EQX **Lemma 82.21.1.** In Situation 82.2.1 let X/B be good. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = \text{div}_\mathcal{L}(s)$ and $\beta = \text{div}_\mathcal{N}(t)$. Then

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $\mathrm{CH}_{n-2}(X)$.

Proof. Immediate from the key Lemma 82.20.1 and the discussion preceding it. □

0EQY Lemma 82.21.2. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 82.18.1 is zero. By Definition 82.15.1 there exists a locally finite family $\{W_j\}$ of integral closed subspaces with $\dim_\delta(W_j) = k + 2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \mathrm{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\mathrm{div}_{W_j}(f_j)$. By Lemma 82.19.4 we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \mathrm{div}_{\mathcal{L}}(s)$. By Lemma 82.21.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 82.18.2 we see that the right hand side is zero in $\mathrm{CH}_k(X)$ as desired. □

In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be invertible on X. We will denote

$$c_1(\mathcal{L})^s \cap - : \mathrm{CH}_{k+s}(X) \to \mathrm{CH}_k(X)$$

the operation $c_1(\mathcal{L}) \cap -$. This makes sense by Lemma 82.21.2. We will denote $c_1(\mathcal{L}^s \cap -$ the s-fold iterate of this operation for all $s \geq 0$.

0EQZ Lemma 82.21.3. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in \mathrm{CH}_{k+2}(X)$ we have

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $\mathrm{CH}_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subspaces $Z_j \subset X$ with $\dim_\delta(Z_j) = k + 2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k + 2)$-cycle on $\coprod Z_j$. By several applications of Lemma 82.19.4 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 82.21.1 and the definitions. □

82.22. Intersecting with effective Cartier divisors

0ER0 This section is the analogue of Chow Homology, Section 42.29. Please read the introduction of that section we motivation.

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and s is a global section, see Divisors on Spaces, Lemma 71.7.8. If D corresponds to $(\mathcal{L}, s)$, then $\mathcal{L} = \mathcal{O}_X(D)$. Please keep this in mind while reading this section.

0ER1 Definition 82.22.1. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s)$ be a pair consisting of an invertible sheaf and a global section $s \in \Gamma(X, \mathcal{L})$. Let $D = Z(s)$ be the vanishing locus of s, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a (refined) Gysin homomorphism

$$i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D).$$

by the following rules:

(1) Given a integral closed subspace $W \subset X$ with $\dim_\delta(W) = k + 1$ we define
 (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
 (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.

(2) For a general $(k + 1)$-cycle $\alpha = \sum n_j[W_j]$ we set

$$i^*\alpha = \sum n_j i^*[W_j]$$

(3) If D is an effective Cartier divisor, then we denote $D \cdot \alpha = i_*i^*\alpha$ the pushforward of the class to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the pullback of the class α.

0ER2 Remark 82.22.2. Let S, B, X, $\mathcal{L}$, s, $i : D \to X$ be as in Definition 82.22.1 and assume that $\mathcal{L}|_D \cong \mathcal{O}_D$. In this case we can define a canonical map $i^* : Z_{k+1}(X) \to Z_k(D)$ on cycles, by requiring that $i^*[W] = 0$ whenever $W \subset D$. The possibility to do this will be useful later on.

0ER3 Remark 82.22.3. Let $f : X' \to X$ be a morphism of good algebraic spaces over B as in Situation 82.2.1. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 82.22.1. Then we can set $\mathcal{L}' = f^*\mathcal{L}$, $s' = f^*s$, and $D' = X' \times_X D = Z(s')$. This gives a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

and we can ask for various compatibilities between i^* and $(i')^*$.

0ER4 Lemma 82.22.4. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Let α be a $(k+1)$-cycle on X. Then $i_*i^*\alpha = c_1(\mathcal{L}) \cap \alpha$ in $\mathrm{CH}_k(X)$. In particular, if D is an effective Cartier divisor, then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.

Proof. Write $\alpha = \sum n_j[W_j]$ where $i_j : W_j \to X$ are integral closed subspaces with $\dim_\delta(W_j) = k$. Since D is the vanishing locus of s we see that $D \cap W_j$ is the vanishing locus of the restriction $s|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$ by Lemma 82.18.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{L})|_{W_j}) \cap [W_j])$$

in $\mathrm{CH}_k(X)$ by Definition 82.18.1. The right hand side matches (termwise) the pushforward of the class $i^*\alpha$ on D from Definition 82.22.1. Hence we win. $\square$

0ER5 Lemma 82.22.5. In Situation 82.2.1. Let $f : X' \to X$ be a proper morphism of good algebraic spaces over B. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Form the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Remark 82.22.3. For any $(k+1)$-cycle α' on X' we have $i^* f_* \alpha' = g_*(i')^*\alpha'$ in $\mathrm{CH}_k(D)$ (this makes sense as f_* is defined on the level of cycles).

Proof. Suppose $\alpha = [W']$ for some integral closed subspace $W' \subset X'$. Let $W \subset X$ be the "image" of W' as in Lemma 82.7.1. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \mathrm{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \mathrm{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Lemma 82.19.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$ is equal to $c_1(\mathcal{L}|_W) \cap [W]$ in $\mathrm{CH}_k(W)$ by the second assertion of Lemma 82.19.3. By Remark 82.15.3 the result follows for general α'. $\square$

0ER6 Lemma 82.22.6. In Situation 82.2.1. Let $f : X' \to X$ be a flat morphism of relative dimension r of good algebraic spaces over B. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Form the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Remark 82.22.3. For any $(k+1)$-cycle α on X we have $(i')^* f^*\alpha = g^* i^* \alpha'$ in $\mathrm{CH}_{k+r}(D)$ (this makes sense as f^* is defined on the level of cycles).

Proof. Suppose $\alpha = [W]$ for some integral closed subspace $W \subset X$. Let $W' = f^{-1}(W) \subset X'$. In case $W \not\subset D$, then $W' \not\subset D'$ and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subspaces of D'. Hence the equality holds as cycles, see Lemma 82.10.5. In case $W \subset D$, then $W' \subset D'$ and $W' = g^{-1}(W)$ with $[W']_{k+1+r} = g^*[W]$ and equality holds in $\mathrm{CH}_{k+r}(D')$ by Lemma 82.19.2. By Remark 82.15.3 the result follows for general α'. $\square$

0ER7 Lemma 82.22.7. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*([Z]_{k+1}) = [D \cap Z]_k$.

Proof. The assumption means that $s|_Z$ is a regular section of $\mathcal{L}|_Z$. Thus $D \cap Z = Z(s)$ and we get

$$[D \cap Z]_k = \sum n_i [Z(s_i)]_k$$

as cycles where $s_i = s|_{Z_i}$, the Z_i are the irreducible components of δ-dimension $k+1$, and $[Z]_{k+1} = \sum n_i [Z_i]$. See Lemma 82.18.3. We have $D \cap Z_i = Z(s_i)$. Comparing with the definition of the gysin map we conclude. □

82.23. Gysin homomorphisms

0ER8 This section is the analogue of Chow Homology, Section 42.30. In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence.

0ER9 Lemma 82.23.1. In Situation 82.2.1 let X/B be good. Assume X integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module and let t be a nonzero meromorphic section of $\mathcal{N}$. Then $i^*\text{div}_\mathcal{N}(t) = c_1(\mathcal{N}) \cap [D]_{n-1}$ in $\text{CH}_{n-2}(D)$.

Proof. Write $\text{div}_\mathcal{N}(t) = \sum \text{ord}_{Z_i,\mathcal{N}}(t)[Z_i]$ for some integral closed subspaces $Z_i \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_i\}$ is locally finite, that $t \in \Gamma(U, \mathcal{N}|_U)$ is a generator where $U = X \setminus \bigcup Z_i$, and that every irreducible component of D is one of the Z_i, see Spaces over Fields, Lemmas 72.6.1, 72.6.6, and 72.7.2.

Set $\mathcal{L} = \mathcal{O}_X(D)$. Denote $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$ the canonical section. We will apply the discussion of Section 82.20 to our current situation. For each i let $\xi_i \in |Z_i|$ be its generic point. Let $B_i = \mathcal{O}^h_{X,\xi_i}$. For each i we pick generators s_i of $\mathcal{L}_{\xi_i}$ and t_i of $\mathcal{N}_{\xi_i}$ over B_i but we insist that we pick $s_i = s$ if $Z_i \not\subset D$. Write $s = f_i s_i$ and $t = g_i t_i$ with $f_i, g_i \in B_i$. Then $\text{ord}_{Z_i,\mathcal{N}}(t) = \text{ord}_{B_i}(g_i)$. On the other hand, we have $f_i \in B_i$ and

$$[D]_{n-1} = \sum \text{ord}_{B_i}(f_i)[Z_i]$$

because of our choices of s_i. We claim that

$$i^*\text{div}_\mathcal{N}(t) = \sum \text{ord}_{B_i}(g_i)\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for $\text{div}_\mathcal{N}(t)$ and the fact that $\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$ when $Z_i \not\subset D$ because in that case $s_i|_{Z_i} = s|_{Z_i}$ is a regular section, see Lemma 82.17.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \text{ord}_{B_i}(f_i)\text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 82.20.1) gives the equality

$$\sum \left(\text{ord}_{B_i}(f_i)\text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \text{ord}_{B_i}(g_i)\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})\right) = \sum \text{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If $Z_i \not\subset D$, then $f_i = 1$ and hence $\text{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$. Thus we get a rational equivalence between our specific cycles representing $i^*\text{div}_\mathcal{N}(t)$ and $c_1(\mathcal{N}) \cap [D]_{n-1}$ on D. This finishes the proof. □

0ERA Lemma 82.23.2. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. The Gysin homomorphism factors through rational equivalence to give a map $i^* : \text{CH}_{k+1}(X) \to \text{CH}_k(D)$.

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subspaces $W_j \subset X$ of δ-dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\text{div}_{W_j}(f_j)$. Set $X' = \coprod W_i$ and consider the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "q"] & X' \arrow[d, "p"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

of Remark 82.22.3. Since $X' \to X$ is proper we see that $i^*p_* = q_*(i')^*$ by Lemma 82.22.5. As we know that q_* factors through rational equivalence (Lemma 82.16.3), it suffices to prove the result for $\alpha' = \sum \text{div}_{W_j}(f_j)$ on X'. Clearly this reduces us to the case where X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$. If $X = D$, then we see that $i^*\alpha$ is equal to $c_1(\mathcal{L}) \cap \alpha$. This is rationally equivalent to zero by Lemma 82.21.2. If $D \neq X$, then we see that $i^*\text{div}_X(f)$ is equal to $c_1(\mathcal{O}_D) \cap [D]_{n-1}$ in $\text{CH}_k(D)$ by Lemma 82.23.1. Of course capping with $c_1(\mathcal{O}_D)$ is the zero map. $\square$

0ERB Lemma 82.23.3. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 82.22.1. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module. Then $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^*\mathcal{N}) \cap i^*\alpha$ in $\text{CH}_{k-2}(D)$ for all $\alpha \in \text{CH}_k(Z)$.

Proof. With exactly the same proof as in Lemma 82.23.2 this follows from Lemmas 82.19.4, 82.21.3, and 82.23.1. $\square$

0ERC Lemma 82.23.4. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ and $(\mathcal{L}', s', i' : D' \to X)$ be two triples as in Definition 82.22.1. Then the diagram

$$\begin{tikzcd} \text{CH}_k(X) \arrow[r, "i^*"] \arrow[d, "(i')^*"] & \text{CH}_{k-1}(D) \arrow[d] \\ \text{CH}_{k-1}(D') \arrow[r] & \text{CH}_{k-2}(D \cap D') \end{tikzcd}$$

commutes where each of the maps is a gysin map.

Proof. Denote $j : D \cap D' \to D$ and $j' : D \cap D' \to D'$ the closed immersions corresponding to $(\mathcal{L}|_{D'}, s|_{D'}$ and $(\mathcal{L}'_D, s|_D)$. We have to show that $(j')^*i^*\alpha = j^*(i')^*\alpha$ for all $\alpha \in \text{CH}_k(X)$. Let $W \subset X$ be an integral closed subscheme of dimension k. We will prove the equality in case $\alpha = [W]$. The general case will then follow from the observation in Remark 82.15.3 (and the specific shape of our rational equivalence produced below). We will deduce the equality for $\alpha = [W]$ from the key formula.

We let σ be a nonzero meromorphic section of $\mathcal{L}|_W$ which we require to be equal to $s|_W$ if $W \not\subset D$. We let σ' be a nonzero meromorphic section of $\mathcal{L}'|_W$ which we require to be equal to $s'|_W$ if $W \not\subset D'$. Write

$$\text{div}_{\mathcal{L}|_W}(\sigma) = \sum \text{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\text{div}_{\mathcal{L}'|_W}(\sigma') = \sum \text{ord}_{Z_i, \mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 82.20. Then we see that $Z_i \subset D$ if $n_i \neq 0$ and $Z'_i \subset D'$ if $n'_i \neq 0$. For each i, let $\xi_i \in |Z_i|$ be the generic point. As in Section

82.20 we choose for each i an element $\sigma_i \in \mathcal{L}_{\xi_i}$, resp. $\sigma'_i \in \mathcal{L}'_{\xi_i}$ which generates over $B_i = \mathcal{O}^h_{W,\xi_i}$ and which is equal to the image of s, resp. s' if $Z_i \not\subset D$, resp. $Z_i \not\subset D'$. Write $\sigma = f_i\sigma_i$ and $\sigma' = f'_i\sigma'_i$ so that $n_i = \text{ord}_{B_i}(f_i)$ and $n'_i = \text{ord}_{B_i}(f'_i)$. From our definitions it follows that

$$(j')^*i^*[W] = \sum \text{ord}_{B_i}(f_i)\text{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^*[W] = \sum \text{ord}_{B_i}(f'_i)\text{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 82.20.1) now gives the equality

$$\sum \left(\text{ord}_{B_i}(f_i)\text{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \text{ord}_{B_i}(f'_i)\text{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})\right) = \sum \text{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that $\text{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$ if $Z_i \not\subset D \cap D'$ because in this case either $f_i = 1$ or $f'_i = 1$. Thus we get a rational equivalence between our specific cycles representing $(j')^*i^*[W]$ and $j^*(i')^*[W]$ on $D \cap D' \cap W$. □

82.24. Relative effective Cartier divisors

0ERD This section is the analogue of Chow Homology, Section 42.31. Relative effective Cartier divisors are defined in Divisors on Spaces, Section 71.9. To develop the basic results on Chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

0ERE Lemma 82.24.1. In Situation 82.2.1. Let $X, Y/B$ be good. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be a relative effective Cartier divisor (Divisors on Spaces, Definition 71.9.2). Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in \text{CH}_{k+1}(Y)$ we have

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $\text{CH}_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $\text{CH}_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subspace of δ-dimension $k+1$. By Divisors on Spaces, Lemma 71.9.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 82.22.7 we get the first equality in

$$i^*[p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$ as algebraic spaces. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ we see that $i^*p^*[W] = (p|_D)^*[W]$ as cycles. If $\alpha = \sum m_j[W_j]$ is a general $k+1$ cycle, then we get $i^*\alpha = \sum m_j i^*p^*[W_j] = \sum m_j(p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 82.22.4. □

82.25. Affine bundles

0ERF This section is the analogue of Chow Homology, Section 42.32. For an affine bundle the pullback map is surjective on Chow groups.

0ERG Lemma 82.25.1. In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a quasi-compact flat morphism over B of relative dimension r. Assume that for every $y \in Y$ we have $X_y \cong \mathbf{A}^r_{\kappa(y)}$. Then $f^* : \text{CH}_k(Y) \to \text{CH}_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.

Proof. Let $\alpha \in \mathrm{CH}_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subspaces of δ-dimension $k+r$. Then the family $\{W_j\}$ is locally finite in X. Let $Z_j \subset Y$ be the integral closed subspace such that we obtain a dominant morphism $W_j \to Z_j$ as in Lemma 82.7.1. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection Z_j of closures of images is a locally finite collection of integral closed subspaces of Y.

Consider the fibre product diagrams

$$\begin{tikzcd} f^{-1}(Z_j) \arrow[r] \arrow[d, "f_j"] & X \arrow[d, "f"] \\ Z_j \arrow[r] & Y \end{tikzcd}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in \mathrm{CH}_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$ will be rationally equivalent to α (see Remark 82.15.3). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $\mathrm{CH}_{k+r}(X) = 0$ and the lemma holds; this is the base case of the induction. Consider a nonempty open $V \subset Y$. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 82.10.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $\mathrm{CH}_k(f^{-1}(Y \setminus V)) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(f^{-1}(V))$ of Lemma 82.15.2 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in \mathrm{CH}_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

In particular, by replacing Y by a suitable open we may assume Y is a scheme with generic point η. The isomorphism $Y_\eta \cong \mathbf{A}^r_\eta$ extends to an isomorphism over a nonempty open $V \subset Y$, see Limits of Spaces, Lemma 70.7.1. This reduces us to the case of schemes which is Chow Homology, Lemma 42.32.1. □

0ERH Lemma 82.25.2. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let

$$p : L = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{L})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+1}(L)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 82.25.1. Let $o : X \to L$ be the zero section of $L \to X$, i.e., the morphism corresponding to the surjection $\mathrm{Sym}^*(\mathcal{L}) \to \mathcal{O}_X$ which maps $\mathcal{L}^{\otimes n}$ to zero for all $n > 0$. Then $p \circ o = \mathrm{id}_X$ and $o(X)$ is an effective Cartier divisor on L. Hence by Lemma 82.24.1 we see that $o^* \circ p^* = \mathrm{id}$ and we conclude that p^* is injective too. □

82.26. Bivariant intersection theory

0ERI This section is the analogue of Chow Homology, Section 42.33. In order to intelligently talk about higher Chern classes of vector bundles we introduce the following notion, following [FM81]. It follows from [Ful98, Theorem 17.1] that our definition agrees with that of [Ful98] modulo the caveat that we are working in different settings.

0ERJ Definition 82.26.1. In Situation 82.2.1 let $f : X \to Y$ be a morphism of good algebraic spaces over B. Let $p \in \mathbf{Z}$. A bivariant class c of degree p for f is given by a rule which assigns to every morphism $Y' \to Y$ of good algebraic spaces over B and every k a map Similar to [Ful98, Definition 17.1]

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $X' = Y' \times_Y X$, satisfying the following conditions

(1) if $Y'' \to Y'$ is a proper morphism, then $c \cap (Y'' \to Y')_*\alpha'' = (X'' \to X')_*(c \cap \alpha'')$ for all α'' on Y'',
(2) if $Y'' \to Y'$ a morphism of good algebraic spaces over B which is flat of relative dimension r, then $c \cap (Y'' \to Y')^*\alpha' = (X'' \to X')^*(c \cap \alpha')$ for all α' on Y',
(3) if $(\mathcal{L}', s', i' : D' \to Y')$ is as in Definition 82.22.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X', then we have $c \cap (i')^*\alpha' = (j')^*(c \cap \alpha')$ for all α' on Y'.

The collection of all bivariant classes of degree p for f is denoted $A^p(X \to Y)$.

In Situation 82.2.1 let $X \to Y$ and $Y \to Z$ be morphisms of good algebraic spaces over B. Let $p \in \mathbf{Z}$. It is clear that $A^p(X \to Y)$ is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \to Y) \times A^q(Y \to Z) \to A^{p+q}(X \to Z)$$

which is associative. We will be most interested in $A^p(X) = A^p(X \to X)$, which will always mean the bivariant cohomology classes for id_X. Namely, that is where Chern classes will live.

0ERK Definition 82.26.2. In Situation 82.2.1 let X/B be good. The Chow cohomology of X is the graded $\mathbf{Z}$-algebra $A^*(X)$ whose degree p component is $A^p(X \to X)$.

Warning: It is not clear that the $\mathbf{Z}$-algebra structure on $A^*(X)$ is commutative, but we will see that Chern classes live in its center.

0ERL Remark 82.26.3. In Situation 82.2.1 let $f : X \to Y$ be a morphism of good algebraic spaces over B. Then there is a canonical $\mathbf{Z}$-algebra map $A^*(Y) \to A^*(X)$. Namely, given $c \in A^p(Y)$ and $X' \to X$, then we can let f^*c be defined by the map $c \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-p}(X')$ which is given by thinking of X' as an algebraic space over Y.

0ERM Lemma 82.26.4. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the rule that to $f : X' \to X$ assigns $c_1(f^*\mathcal{L}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(X')$ is a bivariant class of degree 1.

Proof. This follows from Lemmas 82.21.2, 82.19.4, 82.19.2, and 82.23.3. □

0ERN Lemma 82.26.5. In Situation 82.2.1 let $f : X \to Y$ be a morphism of good algebraic spaces over B which is flat of relative dimension r. Then the rule that to $Y' \to Y$ assigns $(f')^* : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ where $X' = X \times_Y Y'$ is a bivariant class of degree $-r$.

Proof. This follows from Lemmas 82.16.2, 82.10.4, 82.11.1, and 82.22.6. □

0ERP Lemma 82.26.6. In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 82.22.1. Then the rule that to $f : X' \to X$ assigns $(i')^* : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(D')$ where $D' = D \times_X X'$ is a bivariant class of degree 1.

Proof. This follows from Lemmas 82.23.2, 82.22.5, 82.22.6, and 82.23.4. □

0ERQ Lemma 82.26.7. In Situation 82.2.1 let $f : X \to Y$ and $g : Y \to Z$ be morphisms of good algebraic spaces over B. Let $c \in A^p(X \to Z)$ and assume f is proper. Then the rule that to $X' \to X$ assigns $\alpha \longmapsto f_*(c \cap \alpha)$ is a bivariant class of degree p.

Proof. This follows from Lemmas 82.8.2, 82.11.1, and 82.22.5. □

Here we see that $c_1(\mathcal{L})$ is in the center of $A^*(X)$.

0ERR Lemma 82.26.8. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $c_1(\mathcal{L}) \in A^1(X)$ commutes with every element $c \in A^p(X)$.

Proof. Let $p : L \to X$ be as in Lemma 82.25.2 and let $o : X \to L$ be the zero section. Observe that $p^*\mathcal{L}^{\otimes -1}$ has a canonical section whose vanishing locus is exactly the effective Cartier divisor $o(X)$. Let $\alpha \in \mathrm{CH}_k(X)$. Then we see that

$$p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) = c_1(p^*\mathcal{L}^{\otimes -1}) \cap p^*\alpha = o_*o^*p^*\alpha$$

by Lemmas 82.19.2 and 82.24.1. Since c is a bivariant class we have

$$\begin{aligned} p^*(c \cap c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) \\ &= c \cap o_*o^*p^*\alpha \\ &= o_*o^*p^*(c \cap \alpha) \\ &= p^*(c_1(\mathcal{L}^{\otimes -1}) \cap c \cap \alpha) \end{aligned}$$

(last equality by the above applied to $c \cap \alpha$). Since p^* is injective by a lemma cited above we get that $c_1(\mathcal{L}^{\otimes -1})$ is in the center of $A^*(X)$. This proves the lemma. □

Here a criterion for when a bivariant class is zero.

0ERS Lemma 82.26.9. In Situation 82.2.1 let X/B be good. Let $c \in A^p(X)$. Then c is zero if and only if $c \cap [Y] = 0$ in $\mathrm{CH}_*(Y)$ for every integral algebraic space Y locally of finite type over X.

Proof. The if direction is clear. For the converse, assume that $c \cap [Y] = 0$ in $\mathrm{CH}_*(Y)$ for every integral algebraic space Y locally of finite type over X. Let $X' \to X$ be locally of finite type. Let $\alpha \in \mathrm{CH}_k(X')$. Write $\alpha = \sum n_i[Y_i]$ with $Y_i \subset X'$ a locally finite collection of integral closed subschemes of δ-dimension k. Then we see that α is pushforward of the cycle $\alpha' = \sum n_i[Y_i]$ on $X'' = \coprod Y_i$ under the proper morphism $X'' \to X'$. By the properties of bivariant classes it suffices to prove that $c \cap \alpha' = 0$ in $\mathrm{CH}_{k-p}(X'')$. We have $\mathrm{CH}_{k-p}(X'') = \prod \mathrm{CH}_{k-p}(Y_i)$ as follows immediately from the definitions. The projection maps $\mathrm{CH}_{k-p}(X'') \to \mathrm{CH}_{k-p}(Y_i)$ are given by flat pullback. Since capping with c commutes with flat pullback, we see that it suffices to show that $c \cap [Y_i]$ is zero in $\mathrm{CH}_{k-p}(Y_i)$ which is true by assumption. □

82.27. Projective space bundle formula

0ERT In Situation 82.2.1 let X/B be good. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the projective bundle associated to $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

0ERU Lemma 82.27.1. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in \mathrm{CH}_k(X)$ the element

$$\pi_* \left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha\right) \in \mathrm{CH}_{k+r-1-s}(X)$$

is 0 if $s < r-1$ and is equal to α when $s = r-1$.

Proof. Let $Z \subset X$ be an integral closed subspace of δ-dimension k. We will prove the lemma for $\alpha = [Z]$. We omit the argument deducing the general case from this special case; hint: argue as in Remark 82.15.3.

Let $P_Z = P \times_X Z$ be the base change; of course $\pi_Z : P_Z \to Z$ is the projective bundle associated to $\mathcal{E}|_Z$ and $\mathcal{O}_P(1)$ pulls back to the corresponding invertible module on P_Z. Since $c_1(\mathcal{O}_P(1)\cap -$, and π^* are bivariant classes by Lemmas 82.26.4 and 82.26.5 we see that

$$\pi_* \left(c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]\right) = (Z \to X)_* \pi_{Z,*} \left(c_1(\mathcal{O}_{P_Z}(1))^s \cap \pi_Z^*[Z]\right)$$

Hence it suffices to prove the lemma in case X is integral and $\alpha = [X]$.

Assume X is integral, $\dim_\delta(X) = k$, and $\alpha = [X]$. Note that $\pi^*[X] = [P]$ as P is integral of δ-dimension $r-1$. If $s < r-1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap [P]$ a $(k+r-1-s)$-cycle. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap [P]) = n[X]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by a dense open. Thus we may assume X is a scheme and the result follows from Chow Homology, Lemma 42.36.1. $\square$

0ERV Lemma 82.27.2 (Projective space bundle formula). Let (S, δ) be as in Situation 82.2.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map

$$\bigoplus\nolimits_{i=0}^{r-1} \mathrm{CH}_{k+i}(X) \longrightarrow \mathrm{CH}_{k+r-1}(P),$$

$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 82.27.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

To prove the map is surjective, we will argue exactly as in the proof of Lemma 82.25.1 to reduce to the case of schemes. We urge the reader to skip the proof.

Let $\beta \in \mathrm{CH}_{k+r-1}(P)$. Write $\beta = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subspaces of δ-dimension $k + r$. Then the family $\{W_j\}$ is locally finite in P. Let $Z_j \subset X$ be the "image" of W_j as in Lemma 82.7.1. For any quasi-compact open $U \subset X$ we see that $\pi^{-1}(U) \cap W_j$ is nonempty only for finitely many j. Hence the collection Z_j of images is a locally finite collection of integral closed subspaces of X.

Consider the fibre product diagrams

$$\begin{tikzcd} P_j \arrow[r] \arrow[d, "\pi_j"'] & P \arrow[d, "\pi"] \\ Z_j \arrow[r] & X \end{tikzcd}$$

Suppose that $[W_j] \in Z_{k+r-1}(P_j)$ is rationally equivalent to

$$\pi_j^*\alpha_{j,0} + c_1(\mathcal{O}(1)) \cap \pi_j^*\alpha_{j,1} + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi_j^*\alpha_{j,r-1}$$

for some $(k+i)$-cycle $\alpha_{j,i} \in \mathrm{CH}_{k+i}(Z_j)$. Then $\alpha_i = \sum m_j\beta_{j,i}$ will be a $(k+i)$-cycle on X and

$$\pi^*\alpha_0 + c_1(\mathcal{O}(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

will be rationally equivalent to β (see Remark 82.15.3). This reduces us to the case X integral, and $\alpha = [W]$ for some integral closed subscheme of P dominating X. In particular we may assume that $d = \dim_\delta(X) < \infty$.

Hence we can use induction on $d = \dim_\delta(X)$. If $d < k$, then $\mathrm{CH}_{k+r-1}(X) = 0$ and the lemma holds; this is the base case of the induction. Consider a nonempty open $U \subset X$. Suppose that we can show that

$$\beta|_{\pi^{-1}(U)} = \pi^*\alpha_0 + c_1(\mathcal{O}(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

for some $\alpha_i \in Z_{k+i}(U)$. By Lemma 82.10.2 we see that $\alpha_i = \alpha_i'|_U$ for some $\alpha_i' \in Z_{k+i}(X)$. By the exact sequences $\mathrm{CH}_{k+i}(\pi^{-1}(X \setminus U)) \to \mathrm{CH}_{k+i}(P) \to \mathrm{CH}_{k+i}(\pi^{-1}(U))$ of Lemma 82.15.2 we see that

$$\beta - \left(\pi^*\alpha_0' + c_1(\mathcal{O}(1)) \cap \pi^*\alpha_1' + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^*\alpha_{r-1}'\right)$$

comes from a cycle $\beta' \in \mathrm{CH}_{k+r}(\pi^{-1}(X \setminus U))$. Since $\dim_\delta(X \setminus U) < d$ we win by induction on d.

In particular, by replacing X by a suitable open we may assume X is a scheme and we have reduced our problem to Chow Homology, Lemma 42.36.2. $\square$

0ERW Lemma 82.27.3. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let

$$p : E = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$ is an isomorphism for all k.

Proof. (For the case of linebundles, see Lemma 82.25.2.) For surjectivity see Lemma 82.25.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors on Spaces, Lemma 71.7.8). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : \mathrm{CH}_{k+r}(P) \longrightarrow \mathrm{CH}_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 82.15.2. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in \mathrm{CH}_{k+r}(D)$. By Lemma 82.27.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in \mathrm{CH}_{k+i}(X)$. By Lemmas 82.24.1 and 82.19.4 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 82.19.4 unless all α and all β_i are zero. □

82.28. The Chern classes of a vector bundle

0ERX This section is the analogue of Chow Homology, Sections 42.37 and 42.38. However, contrary to what is done there, we directly define the Chern classes of a vector bundle as bivariant classes. This saves a considerable amount of work.

0ERY Lemma 82.28.1. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$. For every morphism $X' \to X$ of good algebraic spaces over B there are unique maps

$$c_i(\mathcal{E}) \cap - : \mathrm{CH}_k(X') \longrightarrow \mathrm{CH}_{k-i}(X'), \quad i = 0, \ldots, r$$

such that for $\alpha \in \mathrm{CH}_k(X')$ we have $c_0(\mathcal{E}) \cap \alpha = \alpha$ and

$$\sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^* (c_{r-i}(\mathcal{E}) \cap \alpha) = 0$$

where $\pi' : P' \to X'$ is the base change of π. Moreover, these maps define a bivariant class $c_i(\mathcal{E})$ of degree i on X.

Proof. Uniqueness and existence of the maps $c_i(\mathcal{E}) \cap -$ follows immediately from Lemma 82.27.2 and the given description of $c_0(\mathcal{E})$. For every $i \in \mathbf{Z}$ the rule which to every morphism $X' \to X$ of good algebraic spaces over B assigns the map

$$t_i(\mathcal{E}) \cap - : \mathrm{CH}_k(X') \longrightarrow \mathrm{CH}_{k-i}(X'), \quad \alpha \longmapsto \pi'_*(c_1(\mathcal{O}_{P'}(1))^{r-1+i} \cap (\pi')^*\alpha)$$

is a bivariant class[2] by Lemmas 82.26.4, 82.26.5, and 82.26.7. By Lemma 82.27.1 we have $t_i(\mathcal{E}) = 0$ for $i < 0$ and $t_0(\mathcal{E}) = 1$. Applying pushforward to the equation in the statement of the lemma we find from Lemma 82.27.1 that

$$(-1)^r t_1(\mathcal{E}) + (-1)^{r-1} c_1(\mathcal{E}) = 0$$

In particular we find that $c_1(\mathcal{E})$ is a bivariant class. If we multiply the equation in the statement of the lemma by $c_1(\mathcal{O}_{P'}(1))$ and push the result forward to X' we find

$$(-1)^r t_2(\mathcal{E}) + (-1)^{r-1} t_1(\mathcal{E}) \cap c_1(\mathcal{E}) + (-1)^{r-2} c_2(\mathcal{E}) = 0$$

As before we conclude that $c_2(\mathcal{E})$ is a bivariant class. And so on. $\square$

0ERZ Definition 82.28.2. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. For $i = 0, \ldots, r$ the ith Chern class of $\mathcal{E}$ is the bivariant class $c_i(\mathcal{E}) \in A^i(X)$ of degree i constructed in Lemma 82.28.1. The total Chern class of $\mathcal{E}$ is the formal sum

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots + c_r(\mathcal{E})$$

which is viewed as a nonhomogeneous bivariant class on X.

For convenience we often set $c_i(\mathcal{E}) = 0$ for $i > r$ and $i < 0$. By definition we have $c_0(\mathcal{E}) = 1 \in A^0(X)$. Here is a sanity check.

0ES0 Lemma 82.28.3. In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first Chern class of $\mathcal{L}$ on X of Definition 82.28.2 is equal to the bivariant class of Lemma 82.26.4.

Proof. Namely, in this case $P = \mathbf{P}(\mathcal{L}) = X$ with $\mathcal{O}_P(1) = \mathcal{L}$ by our normalization of the projective bundle, see Section 82.27. Hence the equation in Lemma 82.28.1 reads

$$(-1)^0 c_1(\mathcal{L})^0 \cap c_1^{new}(\mathcal{L}) \cap \alpha + (-1)^1 c_1(\mathcal{L})^1 \cap c_0^{new}(\mathcal{L}) \cap \alpha = 0$$

where $c_i^{new}(\mathcal{L})$ is as in Definition 82.28.2. Since $c_0^{new}(\mathcal{L}) = 1$ and $c_1(\mathcal{L})^0 = 1$ we conclude. $\square$

Next we see that Chern classes are in the center of the bivariant Chow cohomology ring $A^*(X)$.

0ES1 Lemma 82.28.4. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then $c_j(\mathcal{L}) \in A^j(X)$ commutes with every element $c \in A^p(X)$. In particular, if $\mathcal{F}$ is a second locally free $\mathcal{O}_X$-module on X of rank s, then

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $\mathrm{CH}_{k-i-j}(X)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. Let $X' \to X$ be a morphism of good algebraic spaces over B. Let $\alpha \in \mathrm{CH}_k(X')$. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 82.28.1 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi' : P' \to X', \mathcal{O}_{P'}(1))$ associated to $(X' \to X)^*\mathcal{E}$. Applying $c \cap -$ and using Lemma 82.26.8 and the properties of bivariant classes we obtain

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap \pi^*(c \cap \alpha_{r-i}) = 0$$

[2]Up to signs these are the Segre classes of $\mathcal{E}$.

in the Chow group of P'. Hence we see that $c \cap \alpha_j$ is equal to $c_j(\mathcal{E}) \cap (c \cap \alpha)$ by the uniqueness in Lemma 82.28.1. This proves the lemma. $\square$

0ES2 Remark 82.28.5. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. If the rank of $\mathcal{E}$ is not constant then we can still define the Chern classes of $\mathcal{E}$. Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

where $X_r \subset X$ is the open and closed subspace where the rank of $\mathcal{E}$ is r. If $X' \to X$ is a morphism of good algebraic spaces over B, then we obtain by pullback a corresponding decomposition of X' and we find that

$$\mathrm{CH}_*(X') = \prod\nolimits_{r \geq 0} \mathrm{CH}_*(X'_r)$$

by our definitions. Then we simply define $c_i(\mathcal{E})$ to be the bivariant class which preserves these direct product decompositions and acts by the already defined operations $c_i(\mathcal{E}|_{X_r}) \cap -$ on the factors. Observe that in this setting it may happen that $c_i(\mathcal{E})$ is nonzero for infinitely many i.

82.29. Polynomial relations among Chern classes

0ES3 In Situation 82.2.1 let X/B be good. Let $\mathcal{E}_i$ be a finite collection of finite locally free $\mathcal{O}_X$-modules. By Lemma 82.28.4 we see that the Chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central) $\mathbf{Z}$-subalgebra of the Chow cohomology $A^*(X)$. Thus we can say what it means for a polynomial in these Chern classes to be zero, or for two polynomials to be the same. As an example, saying that $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$ means that the operations

$$\mathrm{CH}_k(Y) \longrightarrow \mathrm{CH}_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms $f : Y \to X$ of good algebraic spaces over B. By Lemma 82.26.9 this is equivalent to the requirement that given any morphism $f : Y \to X$ where Y is an integral algebraic space locally of finite type over X the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in $\mathrm{CH}_{\dim(Y)-5}(Y)$.

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 82.18.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

0ES4 Lemma 82.29.1. In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have

0ES5 (82.29.1.1)
$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.39.1 replacing the lemmas used there by Lemmas 82.26.9 and 82.28.1. $\square$

82.30. Additivity of Chern classes

0ES6 This section is the analogue of Chow Homology, Section 42.40.

0ES7 **Lemma 82.30.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.40.1 replacing the lemmas used there by Lemmas 82.26.9, 82.24.1, 82.19.4, and 82.28.1. □

0ES8 **Lemma 82.30.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.40.2 replacing the lemmas used there by Lemmas 82.30.1 and 82.29.1. □

0ES9 **Lemma 82.30.3.** *In Situation 82.2.1 let X/B be good. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. The total Chern classes satisfy

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.40.3 replacing the lemmas used there by Lemmas 82.26.9, 82.30.2, and 82.28.1. □

0ESA **Lemma 82.30.4.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules. Let $\mathcal{E}$ be a locally free rank $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r}(1 + x_i)$$

in $A^*(X)$.

Proof. Apply Lemma 82.30.2 and induction. □

82.31. The splitting principle

0ESB This section is the analogue of Chow Homology, Section 42.40.

0ESC **Lemma 82.31.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a projective flat morphism $\pi : P \to X$ of relative dimension d such that*

1. *for any morphism $f : Y \to X$ of good algebraic spaces over B the map $\pi_Y^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(Y \times_X P)$ is injective, and*
2. *each $\pi^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.*

Proof. We prove this by induction on the integer $r = \sum r_i$. If $r = 0$ we can take $\pi = \mathrm{id}_X$. If $r_i = 1$ for all i, then we can also take $\pi = \mathrm{id}_X$. Assume that $r_{i_0} > 1$ for some i_0. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}_{i_0}$. The canonical map $\pi^*\mathcal{E}_{i_0} \to \mathcal{O}_P(1)$ is surjective and hence its kernel $\mathcal{E}_{i_0}'$ is finite locally free of rank $r_{i_0} - 1$. Observe that π_Y^* is injective for any morphism $f : Y \to X$ of good algebraic spaces over B, see Lemma 82.27.2. Thus it suffices to prove the lemma for P and the locally free sheaves $\pi^*\mathcal{E}_i$. However, because we have the subbundle $\mathcal{E}_{i_0} \subset \pi^*\mathcal{E}_{i_0}$ with invertible quotient, it now suffices to prove the lemma for the collection $\{\mathcal{E}_i\}_{i \neq i_0} \cup \{\mathcal{E}_{i_0}'\}$. This decreases r by 1 and we win by induction hypothesis. □

Rather than explaining what the splitting principle says, let us use it in the proof of some lemmas.

0ESD **Lemma 82.31.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$$

in $A^i(X)$.

Proof. Choose a morphism $\pi : P \to X$ as in Lemma 82.31.1. By the injectivity of π^* (after any base change) it suffices to prove the relation between the Chern classes of $\mathcal{E}$ and $\mathcal{E}^\vee$ after pulling back to P. Thus we may assume there exist invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ and a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Then we obtain the dual filtration

$$0 = \mathcal{E}_r^\perp \subset \mathcal{E}_1^\perp \subset \mathcal{E}_2^\perp \subset \ldots \subset \mathcal{E}_0^\perp = \mathcal{E}^\vee$$

such that $\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp \cong \mathcal{L}_i^{\otimes -1}$. Set $x_i = c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{L}_i^{\otimes -1}) = -x_i$ by Lemma 82.18.2. By Lemma 82.30.4 we have

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^r (1 + x_i) \quad \text{and} \quad c(\mathcal{E}^\vee) = \prod\nolimits_{i=1}^r (1 - x_i)$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

0ESE **Lemma 82.31.3.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ and $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-modules of ranks r and s. Then we have*

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$
$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2c_2(\mathcal{F}) + rsc_1(\mathcal{F})c_1(\mathcal{E}) + s^2c_2(\mathcal{E})$$

and so on (see proof).

Proof. Arguing exactly as in the proof of Lemma 82.31.2 we may assume we have invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ $\mathcal{N}_i$, $i = 1, \ldots, s$ filtrations

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E} \quad \text{and} \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ and such that $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{N}_j$. Ordering pairs (i, j) lexicographically we obtain a filtration

$$0 \subset \ldots \subset \mathcal{E}_i \otimes \mathcal{F}_j + \mathcal{E}_{i-1} \otimes \mathcal{F} \subset \ldots \subset \mathcal{E} \otimes \mathcal{F}$$

with successive quotients

$$\mathcal{L}_1 \otimes \mathcal{N}_1, \mathcal{L}_1 \otimes \mathcal{N}_2, \ldots, \mathcal{L}_1 \otimes \mathcal{N}_s, \mathcal{L}_2 \otimes \mathcal{N}_1, \ldots, \mathcal{L}_r \otimes \mathcal{N}_s$$

By Lemma 82.30.4 we have

$$c(\mathcal{E}) = \prod (1 + x_i), \quad c(\mathcal{F}) = \prod (1 + y_j), \quad \text{and} \quad c(\mathcal{F}) = \prod (1 + x_i + y_j),$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

82.32. Degrees of zero cycles

0ESF This section is the analogue of Chow Homology, Section 42.41. We start with defining the degree of a zero cycle on a proper algebraic space over a field.

0ESG **Definition 82.32.1.** Let k be a field. Let $p : X \to \mathrm{Spec}(k)$ be a proper morphism of algebraic spaces. The *degree of a zero cycle* on X is given by proper pushforward

$$p_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(\mathrm{Spec}(k)) \longrightarrow \mathbf{Z}$$

(Lemma 82.16.3) composed with the natural isomorphism $\mathrm{CH}_0(\mathrm{Spec}(k)) \to \mathbf{Z}$ which maps $[\mathrm{Spec}(k)]$ to 1. Notation: $\deg(\alpha)$.

Let us spell this out further.

0ESH **Lemma 82.32.2.** Let k be a field. Let X be a proper algebraic space over k. Let $\alpha = \sum n_i[Z_i]$ be in $Z_0(X)$. Then

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where $\deg(Z_i)$ is the degree of $Z_i \to \mathrm{Spec}(k)$, i.e., $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$.

Proof. This is the definition of proper pushforward (Definition 82.8.1). □

0ESI **Lemma 82.32.3.** Let k be a field. Let X be a proper algebraic space over k. Let $Z \subset X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_1) \cap [Z]_d)$$

where the left hand side is defined in Spaces over Fields, Definition 72.18.3.

Proof. Let $Z_i \subset Z$, $i = 1, \ldots, t$ be the irreducible components of dimension d. Let m_i be the multiplicity of Z_i in Z. Then $[Z]_d = \sum m_i[Z_i]$ and $c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d$ is the sum of the cycles $m_i c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z_i]$. Since we have a similar decomposition for $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ by Spaces over Fields, Lemma 72.18.2 it suffices to prove the lemma in case $Z = X$ is a proper integral algebraic space over k.

By Chow's lemma there exists a proper morphism $f : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that X' is a scheme. See More on Morphisms of Spaces, Lemma 76.40.5. Then X' is a proper scheme over k. After replacing

X' by the scheme theoretic closure of $f^{-1}(U)$ we may assume that X' is integral. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot X') = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Spaces over Fields, Lemma 72.18.7 and we have

$$f_*(c_1(f^*\mathcal{L}_1) \cap \ldots \cap c_1(f^*\mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 82.19.4. Thus we may replace X by X' and assume that X is a proper scheme over k. This case was proven in Chow Homology, Lemma 42.41.4. □

82.33. Other chapters

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Algebraic Spaces

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CHAPTER 83

Quotients of Groupoids

048A 83.1. Introduction

048B This chapter is devoted to generalities concerning groupoids and their quotients (as far as they exist). There is a lot of literature on this subject, see for example [MFK94], [Ses72], [Kol97], [KM97], [Kol08] and many more.

83.2. Conventions and notation

048C In this chapter the conventions and notation are those introduced in Groupoids in Spaces, Sections 78.2 and 78.3.

83.3. Invariant morphisms

048D

048E Definition 83.3.1. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. We say a morphism $\phi : U \to X$ of algebraic spaces over B is R-invariant if the diagram

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"] & U \arrow[d, "\phi"] \\ U \arrow[r, "\phi"] & X \end{tikzcd}$$

is commutative. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G on U over B as in Groupoids in Spaces, Lemma 78.15.1, then we say that ϕ is G-invariant.

In other words, a morphism $U \to X$ is R-invariant if it equalizes s and t. We can reformulate this in terms of associated quotient sheaves as follows.

048F Lemma 83.3.2. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. A morphism of algebraic spaces $\phi : U \to X$ is R-invariant if and only if it factors as $U \to U/R \to X$.

Proof. This is clear from the definition of the quotient sheaf in Groupoids in Spaces, Section 78.19. □

048G Lemma 83.3.3. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $X' \to X$ be any morphism of algebraic spaces.

(1) Setting $U' = X' \times_X U$, $R' = X' \times_X R$ we obtain a pre-relation $j' : R' \to U' \times_B U'$.
(2) If j is a relation, then j' is a relation.

(3) If j is a pre-equivalence relation, then j' is a pre-equivalence relation.
(4) If j is an equivalence relation, then j' is an equivalence relation.
(5) If j comes from a groupoid in algebraic spaces (U, R, s, t, c) over B, then
   (a) (U, R, s, t, c) is a groupoid in algebraic spaces over X, and
   (b) j' comes from the base change (U', R', s', t', c') of this groupoid to X', see Groupoids in Spaces, Lemma 78.11.6.
(6) If j comes from the action of a group algebraic space G/B on U as in Groupoids in Spaces, Lemma 78.15.1 then j' comes from the induced action of G on U'.

Proof. Omitted. Hint: Functorial point of view combined with the picture:

$$\begin{tikzcd}
R' = X' \times_X R \arrow[rr] \arrow[dr] \arrow[dd] & & X' \times_X U = U' \arrow[dr] \arrow[dd] & \\
& R \arrow[rr] \arrow[dd] & & U \arrow[dd] \\
U' = X' \times_X U \arrow[rr] \arrow[dr] & & X' \arrow[dr] & \\
& U \arrow[rr] & & X
\end{tikzcd}$$

$\square$

048H Definition 83.3.4. In the situation of Lemma 83.3.3 we call $j' : R' \to U' \times_B U'$ the base change of the pre-relation j to X'. We say it is a flat base change if $X' \to X$ is a flat morphism of algebraic spaces.

This kind of base change interacts well with taking quotient sheaves and quotient stacks.

0DTF Lemma 83.3.5. In the situation of Lemma 83.3.3 there is an isomorphism of sheaves

$$U'/R' = X' \times_X U/R$$

For the construction of quotient sheaves, see Groupoids in Spaces, Section 78.19.

Proof. Since $U \to X$ is R-invariant, it is clear that the map $U \to X$ factors through the quotient sheaf U/R. Recall that by definition

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & U/R \end{tikzcd}$$

is a coequalizer diagram in the category Sh of sheaves of sets on $(Sch/S)_{fppf}$. In fact, this is a coequalizer diagram in the comma category Sh/X. Since the base change functor $X' \times_X - : Sh/X \to Sh/X'$ is exact (true in any topos), we conclude. $\square$

0DTG Lemma 83.3.6. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $g : X' \to X$ be a morphism of algebraic spaces over B and let (U', R', s', t', c') be the base change as in Lemma

83.3.3. Then

$$\begin{tikzcd} [U'/R'] \arrow[r] \arrow[d] & [U/R] \arrow[d] \\ \mathcal{S}_{X'} \arrow[r] & \mathcal{S}_X \end{tikzcd}$$

is a 2-fibre product of stacks in groupoids over $(Sch/S)_{fppf}$. For the construction of quotient stacks and the morphisms in this diagram, see Groupoids in Spaces, Section 78.20.

Proof. We will prove this by using the explicit description of the quotient stacks given in Groupoids in Spaces, Lemma 78.24.1. However, we strongly urge the reader to find their own proof. First, we may view (U, R, s, t, c) as a groupoid in algebraic spaces over X, hence we obtain a map $f : [U/R] \to \mathcal{S}_X$, see Groupoids in Spaces, Lemma 78.20.2. Similarly, we have $f' : [U'/R'] \to X'$.

An object of the 2-fibre product $\mathcal{S}_{X'} \times_{\mathcal{S}_X} [U/R]$ over a scheme T over S is the same as a morphism $x' : T \to X'$ and an object y of $[U/R]$ over T such that such that the composition $g \circ x'$ is equal to $f(y)$. This makes sense because objects of $\mathcal{S}_X$ over T are morphisms $T \to X$. By Groupoids in Spaces, Lemma 78.24.1 we may assume y is given by a $[U/R]$-descent datum (u_i, r_{ij}) relative to an fppf covering $\{T_i \to T\}$. The agreement of $g \circ x' = f(y)$ means that the diagrams

$$\begin{tikzcd} T_i \arrow[rr, "u_i"'] \arrow[d] & & U \arrow[d] \\ T \arrow[r, "x'"] & X' \arrow[r, "g"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T_i \times_T T_j \arrow[rr, "r_{ij}"'] \arrow[d] & & R \arrow[d] \\ T \arrow[r, "x'"] & X' \arrow[r, "g"] & X \end{tikzcd}$$

are commutative.

On the other hand, an object y' of $[U'/R']$ over a scheme T over S by Groupoids in Spaces, Lemma 78.24.1 is given by a $[U'/R']$-descent datum (u'_i, r'_{ij}) relative to an fppf covering $\{T_i \to T\}$. Setting $f'(y') = x' : T \to X'$ we see that the diagrams

$$\begin{tikzcd} T_i \arrow[r, "u'_i"'] \arrow[d] & U' \arrow[d] \\ T \arrow[r, "x'"] & X' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T_i \times_T T_j \arrow[r, "r'_{ij}"'] \arrow[d] & U' \arrow[d] \\ T \arrow[r, "x'"] & X' \end{tikzcd}$$

are commutative.

With this notation in place, we define a functor

$$[U'/R'] \longrightarrow \mathcal{S}_{X'} \times_{\mathcal{S}_X} [U/R]$$

by sending $y' = (u'_i, r'_{ij})$ as above to the object $(x', (u_i, r_{ij}))$ where $x' = f'(y')$, where u_i is the composition $T_i \to U' \to U$, and where r_{ij} is the composition $T_i \times_T T_j \to R' \to R$. Conversely, given an object $(x', (u_i, r_{ij})$ of the right hand side we can send this to the object $((x', u_i), (x', r_{ij}))$ of the left hand side. We omit the discussion of what to do with morphisms (works in exactly the same manner). $\square$

83.4. Categorical quotients

048I This is the most basic kind of quotient one can consider.

048J Definition 83.4.1. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in algebraic spaces over B.

(1) We say a morphism $\phi : U \to X$ of algebraic spaces over B is a categorical quotient if it is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ of algebraic spaces over B there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(2) Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B. Assume U, R are objects of $\mathcal{C}$. In this situation we say a morphism $\phi : U \to X$ of algebraic spaces over B is a categorical quotient in $\mathcal{C}$ if $X \in \mathrm{Ob}(\mathcal{C})$, and ϕ is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ with $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(3) If $B = S$ and $\mathcal{C}$ is the category of schemes over S, then we say $U \to X$ is a categorical quotient in the category of schemes, or simply a categorical quotient in schemes.

We often single out a category $\mathcal{C}$ of algebraic spaces over B by some separation axiom, see Example 83.4.3 for some standard cases. Note that $\phi : U \to X$ is a categorical quotient if and only if $U \to X$ is a coequalizer for the morphisms $t, s : R \to U$ in the category. Hence we immediately deduce the following lemma.

048K Lemma 83.4.2. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation in algebraic spaces over B. If a categorical quotient in the category of algebraic spaces over B exists, then it is unique up to unique isomorphism. Similarly for categorical quotients in full subcategories of Spaces/B.

Proof. See Categories, Section 4.11. $\square$

049V Example 83.4.3. Let S be a scheme, and let B be an algebraic space over S. Here are some standard examples of categories $\mathcal{C}$ that we often come up when applying Definition 83.4.1:

(1) $\mathcal{C}$ is the category of all algebraic spaces over B,
(2) B is separated and $\mathcal{C}$ is the category of all separated algebraic spaces over B,
(3) B is quasi-separated and $\mathcal{C}$ is the category of all quasi-separated algebraic spaces over B,
(4) B is locally separated and $\mathcal{C}$ is the category of all locally separated algebraic spaces over B,
(5) B is decent and $\mathcal{C}$ is the category of all decent algebraic spaces over B, and
(6) $S = B$ and $\mathcal{C}$ is the category of schemes over S.

In this case, if $\phi : U \to X$ is a categorical quotient then we say $U \to X$ is (1) a categorical quotient, (2) a categorical quotient in separated algebraic spaces, (3) a categorical quotient in quasi-separated algebraic spaces, (4) a categorical quotient in locally separated algebraic spaces, (5) a categorical quotient in decent algebraic spaces, (6) a categorical quotient in schemes.

048L Definition 83.4.4. Let S be a scheme, and let B be an algebraic space over S. Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B closed under fibre products. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in $\mathcal{C}$, and let $U \to X$ be an R-invariant morphism with $X \in \mathrm{Ob}(\mathcal{C})$.

(1) We say $U \to X$ is a universal categorical quotient in $\mathcal{C}$ if for every morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the base change $j' : R' \to U'$ of j.
(2) We say $U \to X$ is a uniform categorical quotient in $\mathcal{C}$ if for every flat morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the base change $j' : R' \to U'$ of j.

049W Lemma 83.4.5. In the situation of Definition 83.4.1. If $\phi : U \to X$ is a categorical quotient and U is reduced, then X is reduced. The same holds for categorical quotients in a category of spaces $\mathcal{C}$ listed in Example 83.4.3.

Proof. Let X_{red} be the reduction of the algebraic space X. Since U is reduced the morphism $\phi : U \to X$ factors through $i : X_{red} \to X$ (Properties of Spaces, Lemma 66.12.4). Denote this morphism by $\phi_{red} : U \to X_{red}$. Since $\phi \circ s = \phi \circ t$ we see that also $\phi_{red} \circ s = \phi_{red} \circ t$ (as $i : X_{red} \to X$ is a monomorphism). Hence by the universal property of ϕ there exists a morphism $\chi : X \to X_{red}$ such that $\phi_{red} = \phi \circ \chi$. By uniqueness we see that $i \circ \chi = \mathrm{id}_X$ and $\chi \circ i = \mathrm{id}_{X_{red}}$. Hence i is an isomorphism and X is reduced.

To show that this argument works in a category $\mathcal{C}$ one just needs to show that the reduction of an object of $\mathcal{C}$ is an object of $\mathcal{C}$. We omit the verification that this holds for each of the standard examples. $\square$

83.5. Quotients as orbit spaces

048M Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. If j is a pre-equivalence relation, then loosely speaking the "orbits" of R on U are the subsets $t(s^{-1}(\{u\}))$ of U. However, if j is just a pre-relation, then we need to take the equivalence relation generated by R.

048N Definition 83.5.1. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. If $u \in |U|$, then the orbit, or more precisely the R-orbit of u is

$$O_u = \left\{ u' \in |U| \ : \ \begin{matrix} \exists n \geq 1,\ \exists u_0, \ldots, u_n \in |U| \text{ such that } u_0 = u \text{ and } u_n = u' \\ \text{and for all } i \in \{0, \ldots, n-1\} \text{ either } u_i = u_{i+1} \text{ or} \\ \exists r \in |R|,\ s(r) = u_i, t(r) = u_{i+1} \text{ or} \\ \exists r \in |R|,\ t(r) = u_i, s(r) = u_{i+1} \end{matrix} \right\}$$

It is clear that these are the equivalence classes of an equivalence relation, i.e., we have $u' \in O_u$ if and only if $u \in O_{u'}$. The following lemma is a reformulation of Groupoids in Spaces, Lemma 78.4.4.

048O Lemma 83.5.2. Let $B \to S$ as in Section 83.2. Let $j : R \to U \times_B U$ be a pre-equivalence relation of algebraic spaces over B. Then

$$O_u = \{u' \in |U| \text{ such that } \exists r \in |R|,\ s(r) = u,\ t(r) = u'\}.$$

Proof. By the aforementioned Groupoids in Spaces, Lemma 78.4.4 we see that the orbits O_u as defined in the lemma give a disjoint union decomposition of $|U|$. Thus we see they are equal to the orbits as defined in Definition 83.5.1. $\square$

048P Lemma 83.5.3. In the situation of Definition 83.5.1. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then $|\phi| : |U| \to |X|$ is constant on the orbits.

Proof. To see this we just have to show that $\phi(u) = \phi(u')$ for all $u, u' \in |U|$ such that there exists an $r \in |R|$ such that $s(r) = u$ and $t(r) = u'$. And this is clear since ϕ equalizes s and t. $\square$

There are several problems with considering the orbits $O_u \subset |U|$ as a tool for singling out properties of quotient maps. One issue is the following. Suppose that $\mathrm{Spec}(k) \to B$ is a geometric point of B. Consider the canonical map

$$U(k) \longrightarrow |U|.$$

Then it is usually not the case that the equivalence classes of the equivalence relation generated by $j(R(k)) \subset U(k) \times U(k)$ are the inverse images of the orbits $O_u \subset |U|$. A silly example is to take $S = B = \mathrm{Spec}(\mathbf{Z})$, $U = R = \mathrm{Spec}(k)$ with $s = t = \mathrm{id}_k$. Then $|U| = |R|$ is a single point but $U(k)/R(k)$ is enormous. A more interesting example is to take $S = B = \mathrm{Spec}(\mathbf{Q})$, choose some of number fields $K \subset L$, and set $U = \mathrm{Spec}(L)$ and $R = \mathrm{Spec}(L \otimes_K L)$ with obvious maps $s, t : R \to U$. In this case $|U|$ still has just one point, but the quotient

$$U(k)/R(k) = \mathrm{Hom}(K, k)$$

consists of more than one element. We conclude from both examples that if $U \to X$ is an R-invariant map and if we want it to "separate orbits" we get a much stronger and interesting notion by considering the induced maps $U(k) \to X(k)$ and ask that those maps separate orbits.

There is an issue with this too. Namely, suppose that $S = B = \mathrm{Spec}(\mathbf{R})$, $U = \mathrm{Spec}(\mathbf{C})$, and $R = \mathrm{Spec}(\mathbf{C}) \amalg \mathrm{Spec}(K)$ for some field extension $\sigma : \mathbf{C} \to K$. Let the maps s, t be given by the identity on the component $\mathrm{Spec}(\mathbf{C})$, but by $\sigma, \sigma \circ \tau$ on the second component where τ is complex conjugation. If K is a nontrivial extension of $\mathbf{C}$, then the two points $1, \tau \in U(\mathbf{C})$ are not equivalent under $j(R(\mathbf{C}))$. But after choosing an extension $\mathbf{C} \subset \Omega$ of sufficiently large cardinality (for example larger than the cardinality of K) then the images of $1, \tau \in U(\mathbf{C})$ in $U(\Omega)$ do become equivalent! It seems intuitively clear that this happens either because $s, t : R \to U$ are not locally of finite type or because the cardinality of the field k is not large enough.

Keeping this in mind we make the following definition.

048Q Definition 83.5.4. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.

(1) We say $\overline{u}, \overline{u}' \in U(k)$ are weakly R-equivalent if they are in the same equivalence class for the equivalence relation generated by the relation $j(R(k)) \subset U(k) \times U(k)$.
(2) We say $\overline{u}, \overline{u}' \in U(k)$ are R-equivalent if for some overfield $k \subset \Omega$ the images in $U(\Omega)$ are weakly R-equivalent.
(3) The weak orbit, or more precisely the weak R-orbit of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are weakly R-equivalent to $\overline{u}$.
(4) The orbit, or more precisely the R-orbit of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are R-equivalent to $\overline{u}$.

It turns out that in good cases orbits and weak orbits agree, see Lemma 83.5.7. The following lemma illustrates the difference in the special case of a pre-equivalence relation.

048R Lemma 83.5.5. Let S be a scheme, and let B be an algebraic space over S. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. In this case the weak orbit of $\overline{u} \in U(k)$ is simply

$$\{\overline{u}' \in U(k) \text{ such that } \exists \overline{r} \in R(k),\ s(\overline{r}) = \overline{u},\ t(\overline{r}) = \overline{u}'\}$$

and the orbit of $\overline{u} \in U(k)$ is

$$\{\overline{u}' \in U(k) : \exists \text{ field extension } K/k,\ \exists\ r \in R(K),\ s(r) = \overline{u},\ t(r) = \overline{u}'\}$$

Proof. This is true because by definition of a pre-equivalence relation the image $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation. $\square$

Let us describe the recipe for turning any pre-relation into a pre-equivalence relation. We will use the morphisms

048S (83.5.5.1)
$$\begin{array}{lcccccc} j_{diag} & : & U & \longrightarrow & U \times_B U, & u & \longmapsto & (u, u) \\ j_{flip} & : & R & \longrightarrow & U \times_B U, & r & \longmapsto & (s(r), t(r)) \\ j_{comp} & : & R \times_{s,U,t} R & \longrightarrow & U \times_B U, & (r, r') & \longmapsto & (t(r), s(r')) \end{array}$$

We define $j_1 = (t_1, s_1) : R_1 \to U \times_B U$ to be the morphism

$$j \amalg j_{diag} \amalg j_{flip} : R \amalg U \amalg R \longrightarrow U \times_B U$$

with notation as in Equation (83.5.5.1). For $n > 1$ we set

$$j_n = (t_n, s_n) : R_n = R_1 \times_{s_1,U,t_{n-1}} R_{n-1} \longrightarrow U \times_B U$$

where t_n comes from t_1 precomposed with projection onto R_1 and s_n comes from s_{n-1} precomposed with projection onto R_{n-1}. Finally, we denote

$$j_\infty = (t_\infty, s_\infty) : R_\infty = \coprod\nolimits_{n \geq 1} R_n \longrightarrow U \times_B U.$$

048T Lemma 83.5.6. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Then $j_\infty : R_\infty \to U \times_B U$ is a pre-equivalence relation over B. Moreover

(1) $\phi : U \to X$ is R-invariant if and only if it is R_∞-invariant,
(2) the canonical map of quotient sheaves $U/R \to U/R_\infty$ (see Groupoids in Spaces, Section 78.19) is an isomorphism,
(3) weak R-orbits agree with weak R_∞-orbits,
(4) R-orbits agree with R_∞-orbits,
(5) if s, t are locally of finite type, then s_∞, t_∞ are locally of finite type,
(6) add more here as needed.

Proof. Omitted. Hint for (5): Any property of s, t which is stable under composition and stable under base change, and Zariski local on the source will be inherited by s_∞, t_∞. $\square$

048U Lemma 83.5.7. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.

(1) If $s, t : R \to U$ are locally of finite type then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.
(2) If k has sufficiently large cardinality then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.

Proof. We first prove (1). Assume s, t locally of finite type. By Lemma 83.5.6 we may assume that R is a pre-equivalence relation. Let k be an algebraically closed field over B. Suppose $\overline{u}, \overline{u}' \in U(k)$ are R-equivalent. Then for some extension field Ω/k there exists a point $\overline{r} \in R(\Omega)$ mapping to $(\overline{u}, \overline{u}') \in (U \times_B U)(\Omega)$, see Lemma 83.5.5. Hence

$$Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$$

is nonempty. As s is locally of finite type we see that also j is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. This implies Z is a nonempty algebraic space locally of finite type over the algebraically closed field k (use Morphisms of Spaces, Lemma 67.23.3). Thus Z has a k-valued point, see Morphisms of Spaces, Lemma 67.24.1. Hence we conclude there exists a $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$, and we conclude that $\overline{u}, \overline{u}'$ are R-equivalent as desired.

The proof of part (2) is the same, except that it uses Morphisms of Spaces, Lemma 67.24.2 instead of Morphisms of Spaces, Lemma 67.24.1. This shows that the assertion holds as soon as $|k| > \lambda(R)$ with $\lambda(R)$ as introduced just above Morphisms of Spaces, Lemma 67.24.1. ☐

In the following definition we use the terminology "k is a field over B" to mean that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to B$.

048V Definition 83.5.8. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say $\phi : U \to X$ is set-theoretically R-invariant if and only if the map $U(k) \to X(k)$ equalizes the two maps $s, t : R(k) \to U(k)$ for every algebraically closed field k over B.
(2) We say $\phi : U \to X$ separates orbits, or separates R-orbits if it is set-theoretically R-invariant and $\phi(\overline{u}) = \phi(\overline{u}')$ in $X(k)$ implies that $\overline{u}, \overline{u}' \in U(k)$ are in the same orbit for every algebraically closed field k over B.

In Example 83.5.12 we show that being set-theoretically invariant is "too weak" a notion in the category of algebraic spaces. A more geometric reformulation of what it means to be set-theoretically invariant or to separate orbits is in Lemma 83.5.17.

048W Lemma 83.5.9. In the situation of Definition 83.5.8. A morphism $\phi : U \to X$ is set-theoretically R-invariant if and only if for any algebraically closed field k over B the map $U(k) \to X(k)$ is constant on orbits.

Proof. This is true because the condition is supposed to hold for all algebraically closed fields over B. ☐

048X Lemma 83.5.10. In the situation of Definition 83.5.8. An invariant morphism is set-theoretically invariant.

Proof. This is immediate from the definitions. ☐

048Y Lemma 83.5.11. In the situation of Definition 83.5.8. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Assume

(1) ϕ is set-theoretically R-invariant,
(2) R is reduced, and
(3) X is locally separated over B.

Then ϕ is R-invariant.

Proof. Consider the equalizer

$$Z = R \times_{(\phi,\phi)\circ j, X \times_B X, \Delta_{X/B}} X$$

algebraic space. Then $Z \to R$ is an immersion by assumption (3). By assumption (1) $|Z| \to |R|$ is surjective. This implies that $Z \to R$ is a bijective closed immersion (use Schemes, Lemma 26.10.4) and by assumption (2) we conclude that $Z = R$. □

048Z Example 83.5.12. There exist reduced quasi-separated algebraic spaces X, Y and a pair of morphisms $a, b : Y \to X$ which agree on all k-valued points but are not equal. To get an example take $Y = \mathrm{Spec}(k[[x]])$ and

$$X = \mathbf{A}^1_k \Big/ \big(\Delta \amalg \{(x, -x) \mid x \neq 0\}\big)$$

the algebraic space of Spaces, Example 65.14.1. The two morphisms $a, b : Y \to X$ come from the two maps $x \mapsto x$ and $x \mapsto -x$ from Y to $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$. On the generic point the two maps are the same because on the open part $x \neq 0$ of the space X the functions x and $-x$ are equal. On the closed point the maps are obviously the same. It is also true that $a \neq b$. This implies that Lemma 83.5.11 does not hold with assumption (3) replaced by the assumption that X be quasi-separated. Namely, consider the diagram

$$\begin{tikzcd} Y \arrow[r, "1"'] \arrow[d, "-1"'] & Y \arrow[d, "a"] \\ Y \arrow[r, "a"] & X \end{tikzcd}$$

then the composition $a \circ (-1) = b$. Hence we can set $R = Y$, $U = Y$, $s = 1$, $t = -1$, $\phi = a$ to get an example of a set-theoretically invariant morphism which is not invariant.

The example above is instructive because the map $Y \to X$ even separates orbits. It shows that in the category of algebraic spaces there are simply too many set-theoretically invariant morphisms lying around. Next, let us define what it means for R to be a set-theoretic equivalence relation, while remembering that we need to allow for field extensions to make this work correctly.

0490 Definition 83.5.13. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say j is a set-theoretic pre-equivalence relation if for all algebraically closed fields k over B the relation $\sim_R$ on $U(k)$ defined by

$$\overline{u} \sim_R \overline{u}' \Leftrightarrow \begin{array}{c} \exists \text{ field extension } K/k,\ \exists\, r \in R(K), \\ s(r) = \overline{u},\ t(r) = \overline{u}' \end{array}$$

is an equivalence relation.
(2) We say j is a set-theoretic equivalence relation if j is universally injective and a set-theoretic pre-equivalence relation.

Let us reformulate this in more geometric terms.

0491 **Lemma 83.5.14.** In the situation of Definition 83.5.13. The following are equivalent:

(1) The morphism j is a set-theoretic pre-equivalence relation.
(2) The subset $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (83.5.5.1).
(3) For every algebraically closed field k over B of sufficiently large cardinality the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.

If s, t are locally of finite type these are also equivalent to

(4) For every algebraically closed field k over B the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.

Proof. Assume (2). Let k be an algebraically closed field over B. We are going to show that $\sim_R$ is an equivalence relation. Suppose that $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2$ are k-valued points of U. Suppose that $(\overline{u}_1, \overline{u}_2)$ is the image of a K-valued point $r \in R(K)$. Consider the solid commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Spec}(k) \arrow[d, "{(\overline{u}_2,\overline{u}_1)}"'] & \mathrm{Spec}(K) \arrow[l] \arrow[d] \\
R \arrow[r, "j"] & U \times_B U & R \arrow[l, "j_{flip}"']
\end{tikzcd}$$

We also denote $r \in |R|$ the image of r. By assumption the image of $|j_{flip}|$ is contained in the image of $|j|$, in other words there exists a $r' \in |R|$ such that $|j|(r') = |j_{flip}|(r)$. But note that $(\overline{u}_2, \overline{u}_1)$ is in the equivalence class that defines $|j|(r')$ (by the commutativity of the solid part of the diagram). This means there exists a field extension K'/k and a morphism $r' : \mathrm{Spec}(K) \to R$ (abusively denoted r' as well) with $j \circ r' = (\overline{u}_2, \overline{u}_1) \circ i$ where $i : \mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is the obvious map. In other words the dotted part of the diagram commutes. This proves that $\sim_R$ is a symmetric relation on $U(k)$. In the similar way, using that the image of $|j_{diag}|$ is contained in the image of $|j|$ we see that $\sim_R$ is reflexive (details omitted).

To show that $\sim_R$ is transitive assume given $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2, 3$ and field extensions K_i/k and points $r_i : \mathrm{Spec}(K_i) \to R$, $i = 1, 2$ such that $j(r_1) = (\overline{u}_1, \overline{u}_2)$ and $j(r_1) = (\overline{u}_2, \overline{u}_3)$. Then we may choose a commutative diagram of fields

$$\begin{tikzcd}
K & K_2 \arrow[l] \\
K_1 \arrow[u] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

and we may think of $r_1, r_2 \in R(K)$. We consider the commutative solid diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Spec}(k) \arrow[d, "{(\overline{u}_1,\overline{u}_3)}"'] & \mathrm{Spec}(K) \arrow[l] \arrow[d, "{(r_1,r_2)}"] \\
R \arrow[r, "j"] & U \times_B U & R \times_{s,U,t} R \arrow[l, "j_{comp}"']
\end{tikzcd}$$

By exactly the same reasoning as in the first part of the proof, but this time using that $|j_{comp}|((r_1, r_2))$ is in the image of $|j|$, we conclude that a field K' and dotted arrows exist making the diagram commute. This proves that $\sim_R$ is transitive and concludes the proof that (2) implies (1).

Assume (1) and let k be an algebraically closed field over B whose cardinality is larger than $\lambda(R)$, see Morphisms of Spaces, Lemma 67.24.2. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption there exists a point in $|R|$ mapping to $(\overline{u}, \overline{u}') \in |U \times_B U|$. Hence by Morphisms of Spaces, Lemma 67.24.2 we conclude there exists an $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$. In this way we see that (1) implies (3).

Assume (3). Let us show that $\mathrm{Im}(|j_{comp}|) \subset \mathrm{Im}(|j|)$. Pick any point $c \in |R \times_{s,U,t} R|$. We may represent this by a morphism $\overline{c} : \mathrm{Spec}(k) \to R \times_{s,U,t} R$, with k over B having sufficiently large cardinality. By assumption we see that $j_{comp}(\overline{c}) \in U(k) \times U(k) = (U \times_B U)(k)$ is also the image $j(\overline{r})$ for some $\overline{r} \in R(k)$. Hence $j_{comp}(c) = j(r)$ in $|U \times_B U|$ as desired (with $r \in |R|$ the equivalence class of $\overline{r}$). The same argument shows also that $\mathrm{Im}(|j_{diag}|) \subset \mathrm{Im}(|j|)$ and $\mathrm{Im}(|j_{flip}|) \subset \mathrm{Im}(|j|)$ (details omitted). In this way we see that (3) implies (2). At this point we have shown that (1), (2) and (3) are all equivalent.

It is clear that (4) implies (3) (without any assumptions on s, t). To finish the proof of the lemma we show that (1) implies (4) if s, t are locally of finite type. Namely, let k be an algebraically closed field over B. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption the algebraic space $Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$ is nonempty. On the other hand, since $j = (t, s)$ is locally of finite type the morphism $Z \to \mathrm{Spec}(k)$ is locally of finite type as well (use Morphisms of Spaces, Lemmas 67.23.6 and 67.23.3). Hence Z has a k point by Morphisms of Spaces, Lemma 67.24.1 and we conclude that $(\overline{u}, \overline{u}') \in j(R(k))$ as desired. This finishes the proof of the lemma. □

049X Lemma 83.5.15. In the situation of Definition 83.5.13. The following are equivalent:

1. The morphism j is a set-theoretic equivalence relation.
2. The morphism j is universally injective and $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (83.5.5.1).
3. For every algebraically closed field k over B of sufficiently large cardinality the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.

If j is decent, or locally separated, or quasi-separated these are also equivalent to

4. For every algebraically closed field k over B the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) follow from Lemma 83.5.14 and the definitions. The same lemma shows that (3) implies j is a set-theoretic pre-equivalence relation. But of course condition (3) also implies that j is universally injective, see Morphisms of Spaces, Lemma 67.19.2, so that j is indeed a set-theoretic equivalence relation. At this point we know that (1), (2), (3) are all equivalent.

Condition (4) implies (3) without any further hypotheses on j. Assume j is decent, or locally separated, or quasi-separated and the equivalent conditions (1), (2), (3) hold. By More on Morphisms of Spaces, Lemma 76.3.4 we see that j is radicial. Let k be any algebraically closed field over B. Let $\overline{u}, \overline{u}' \in U(k)$ with $\overline{u} \sim_R \overline{u}'$. We see that $R \times_{U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$ is nonempty. Hence, as j is radicial, its reduction is the spectrum of a field purely inseparable over k. As $k = \overline{k}$ we see that it is the spectrum of k. Whence a point $\overline{r} \in R(k)$ with $t(\overline{r}) = \overline{u}$ and $s(\overline{r}) = \overline{u}'$ as desired. □

0492 Lemma 83.5.16. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) If j is a pre-equivalence relation, then j is a set-theoretic pre-equivalence relation. This holds in particular when j comes from a groupoid in algebraic spaces, or from an action of a group algebraic space on U.
(2) If j is an equivalence relation, then j is a set-theoretic equivalence relation.

Proof. Omitted. □

049Y Lemma 83.5.17. Let $B \to S$ be as in Section 83.2. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Consider the diagram

$$\begin{tikzcd} (U \times_X U) \times_{(U \times_B U)} R \arrow[r, "p"'] \arrow[d, "q"] & R \arrow[d, "j"] \\ U \times_X U \arrow[r, "c"] & U \times_B U \end{tikzcd}$$

Then we have:

(1) The morphism ϕ is set-theoretically invariant if and only if p is surjective.
(2) If j is a set-theoretic pre-equivalence relation then ϕ separates orbits if and only if p and q are surjective.
(3) If p and q are surjective, then j is a set-theoretic pre-equivalence relation (and ϕ separates orbits).
(4) If ϕ is R-invariant and j is a set-theoretic pre-equivalence relation, then ϕ separates orbits if and only if the induced morphism $R \to U \times_X U$ is surjective.

Proof. Assume ϕ is set-theoretically invariant. This means that for any algebraically closed field k over B and any $\overline{r} \in R(k)$ we have $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. Hence $((\phi(t(\overline{r})), \phi(s(\overline{r}))), \overline{r})$ defines a point in the fibre product mapping to $\overline{r}$ via p. This shows that p is surjective. Conversely, assume p is surjective. Pick $\overline{r} \in R(k)$. As p is surjective, we can find a field extension K/k and a K-valued point $\tilde{r}$ of the fibre product with $p(\tilde{r}) = \overline{r}$. Then $q(\tilde{r}) \in U \times_X U$ maps to $(t(\overline{r}), s(\overline{r}))$ in $U \times_B U$ and we conclude that $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. This proves that ϕ is set-theoretically invariant.

The proofs of (2), (3), and (4) are omitted. Hint: Assume k is an algebraically closed field over B of large cardinality. Consider the associated diagram of sets

$$\begin{tikzcd} (U(k) \times_{X(k)} U(k)) \times_{U(k) \times U(k)} R(k) \arrow[r, "p"'] \arrow[d, "q"] & R(k) \arrow[d, "j"] \\ U(k) \times_{X(k)} U(k) \arrow[r, "c"] & U(k) \times U(k) \end{tikzcd}$$

By the lemmas above the equivalences posed in (2), (3), and (4) become set-theoretic questions related to the diagram we just displayed, using that surjectivity translates into surjectivity on k-valued points by Morphisms of Spaces, Lemma 67.24.2. □

Because we have seen above that the notion of a set-theoretically invariant morphism is a rather weak one in the category of algebraic spaces, we define an orbit space for a pre-relation as follows.

0493 Definition 83.5.18. Let $B \to S$ as in Section 83.2. Let $j : R \to U \times_B U$ be a pre-relation. We say $\phi : U \to X$ is an orbit space for R if

(1) ϕ is R-invariant,
(2) ϕ separates R-orbits, and
(3) ϕ is surjective.

The definition of separating R-orbits involves a discussion of points with values in algebraically closed fields. But as we've seen in many cases this just corresponds to the surjectivity of certain canonically associated morphisms of algebraic spaces. We summarize some of the discussion above in the following characterization of orbit spaces.

049Z Lemma 83.5.19. Let $B \to S$ as in Section 83.2. Let $j : R \to U \times_B U$ be a set-theoretic pre-equivalence relation. A morphism $\phi : U \to X$ is an orbit space for R if and only if

(1) $\phi \circ s = \phi \circ t$, i.e., ϕ is invariant,
(2) the induced morphism $(t, s) : R \to U \times_X U$ is surjective, and
(3) the morphism $\phi : U \to X$ is surjective.

This characterization applies for example if j is a pre-equivalence relation, or comes from a groupoid in algebraic spaces over B, or comes from the action of a group algebraic space over B on U.

Proof. Follows immediately from Lemma 83.5.17 part (4). $\square$

In the following lemma it is (probably) not good enough to assume just that the morphisms s, t are locally of finite type. The reason is that it may happen that some map $\phi : U \to X$ is an orbit space, yet is not locally of finite type. In that case $U(k) \to X(k)$ may not be surjective for all algebraically closed fields k over B.

04A0 Lemma 83.5.20. Let $B \to S$ as in Section 83.2. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. Assume R, U are locally of finite type over B. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then ϕ is an orbit space for R if and only if the natural map

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \longrightarrow X(k)$$

is bijective for all algebraically closed fields k over B.

Proof. Note that since U, R are locally of finite type over B all of the morphisms s, t, j, ϕ are locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. We will also use without further mention Morphisms of Spaces, Lemma 67.24.1. Assume ϕ is an orbit space. Let k be any algebraically closed field over B. Let $\overline{x} \in X(k)$. Consider $U \times_{\phi, X, \overline{x}} \mathrm{Spec}(k)$. This is a nonempty algebraic space which is locally of finite type over k. Hence it has a k-valued point. This shows the displayed map of the lemma is surjective. Suppose that $\overline{u}, \overline{u}' \in U(k)$ map to the same element of $X(k)$. By Definition 83.5.8 this means that $\overline{u}, \overline{u}'$ are in the same R-orbit. By Lemma 83.5.7 this means that they are equivalent under the equivalence relation generated by $j(R(k))$. Thus the displayed morphism is injective.

Conversely, assume the displayed map is bijective for all algebraically closed fields k over B. This condition clearly implies that ϕ is surjective. We have already assumed that ϕ is R-invariant. Finally, the injectivity of all the displayed maps implies that ϕ separates orbits. Hence ϕ is an orbit space. $\square$

83.6. Coarse quotients

04A1 We only add this here so that we can later say that coarse quotients correspond to coarse moduli spaces (or moduli schemes).

04A2 Definition 83.6.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a coarse quotient if

(1) ϕ is a categorical quotient, and
(2) ϕ is an orbit space.

If $S = B$, U, R are all schemes, then we say a morphism of schemes $\phi : U \to X$ is a coarse quotient in schemes if

(1) ϕ is a categorical quotient in schemes, and
(2) ϕ is an orbit space.

In many situations the algebraic spaces R and U are locally of finite type over B and the orbit space condition simply means that

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \cong X(k)$$

for all algebraically closed fields k. See Lemma 83.5.20. If j is also a (set-theoretic) pre-equivalence relation, then the condition is simply equivalent to $U(k)/j(R(k)) \to X(k)$ being bijective for all algebraically closed fields k.

83.7. Topological properties

04A3 Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. We say a subset $T \subset |U|$ is R-invariant if $s^{-1}(T) = t^{-1}(T)$ as subsets of $|R|$. Note that if T is closed, then it may not be the case that the corresponding reduced closed subspace of U is R-invariant (as in Groupoids in Spaces, Definition 78.18.1) because the pullbacks $s^{-1}(T)$, $t^{-1}(T)$ may not be reduced. Here are some conditions that we can consider for an invariant morphism $\phi : U \to X$.

04A4 Definition 83.7.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B.

04A5 (1) The morphism ϕ is submersive.
04A6 (2) For any R-invariant closed subset $Z \subset |U|$ the image $\phi(Z)$ is closed in $|X|$.
04A7 (3) Condition (2) holds and for any pair of R-invariant closed subsets $Z_1, Z_2 \subset |U|$ we have
$$\phi(Z_1 \cap Z_2) = \phi(Z_1) \cap \phi(Z_2)$$
04A8 (4) The morphism $(t, s) : R \to U \times_X U$ is universally submersive.

For each of these properties we can also require them to hold after any flat base change, or after any base change, see Definition 83.3.4. In this case we say condition (1), (2), (3), or (4) holds uniformly or universally.

83.8. Invariant functions

04A9 In some cases it is convenient to pin down the structure sheaf of a quotient by requiring any invariant function to be a local section of the structure sheaf of the quotient.

04AA Definition 83.8.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism. Denote $\phi' = \phi \circ s = \phi \circ t : R \to X$.

(1) We denote $(\phi_*\mathcal{O}_U)^R$ the $\mathcal{O}_X$-sub-algebra of $\phi_*\mathcal{O}_U$ which is the equalizer of the two maps

$$\begin{tikzcd} \phi_*\mathcal{O}_U \arrow[r, shift left, "\phi_* s^\sharp"] \arrow[r, shift right, "\phi_* t^\sharp"'] & \phi'_*\mathcal{O}_R \end{tikzcd}$$

on $X_{\acute{e}tale}$. We sometimes call this the sheaf of R-invariant functions on X.

(2) We say the functions on X are the R-invariant functions on U if the natural map $\mathcal{O}_X \to (\phi_*\mathcal{O}_U)^R$ is an isomorphism.

Of course we can require this property holds after any (flat or any) base change, leading to a (uniform or) universal notion. This condition is often thrown in with other conditions in order to obtain a (more) unique quotient. And of course a good deal of motivation for the whole subject comes from the following special case: $U = \mathrm{Spec}(A)$ is an affine scheme over a field $S = B = \mathrm{Spec}(k)$ and where $R = G \times U$, with G an affine group scheme over k. In this case you have the option of taking for the quotient:

$$X = \mathrm{Spec}(A^G)$$

so that at least the condition of the definition above is satisfied. Even though this is a nice thing you can do it is often not the right quotient; for example if $U = \mathrm{GL}_{n,k}$ and G is the group of upper triangular matrices, then the above gives $X = \mathrm{Spec}(k)$, whereas a much better quotient (namely the flag variety) exists.

83.9. Good quotients

04AB Especially when taking quotients by group actions the following definition is useful.

04AC Definition 83.9.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a good quotient if

(1) ϕ is invariant,
(2) ϕ is affine,
(3) ϕ is surjective,
(4) condition (3) holds universally, and
(5) the functions on X are the R-invariant functions on U.

In [Ses72] Seshadri gives almost the same definition, except that instead of (4) he simply requires the condition (3) to hold – he does not require it to hold universally.

83.10. Geometric quotients

04AD This is Mumford's definition of a geometric quotient (at least the definition from the first edition of GIT; as far as we can tell later editions changed "universally submersive" to "submersive").

04AE Definition 83.10.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a geometric quotient if

(1) ϕ is an orbit space,
(2) condition (1) holds universally, i.e., ϕ is universally submersive, and
(3) the functions on X are the R-invariant functions on U.

83.11. Other chapters

CHAPTER 84

More on Cohomology of Spaces

0DFR

84.1. Introduction

0DFS In this chapter continues the discussion started in Cohomology of Spaces, Section 69.1. One can also view this chapter as the analogue for algebraic spaces of the chapter on étale cohomology for schemes, see Étale Cohomology, Section 59.1.

In fact, we intend this chapter to be mainly a translation of the results already proved for schemes into the language of algebraic spaces. Some of our results can be found in [Knu71].

84.2. Conventions

0DFT The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

84.3. Transporting results from schemes

0DFU In this section we explain briefly how results for schemes imply results for (representable) algebraic spaces and (representable) morphisms of algebraic spaces. For quasi-coherent modules more is true (because étale cohomology of a quasi-coherent module over a scheme agrees with Zariski cohomology) and this has already been discussed in Cohomology of Spaces, Section 69.3.

Let S be a scheme. Let X be an algebraic space over S. Now suppose that X is representable by the scheme X_0 (awkward but temporary notation; we usually just say "X is a scheme"). In this case X and X_0 have the same small étale sites:

$$X_{\acute{e}tale} = (X_0)_{\acute{e}tale}$$

This is pointed out in Properties of Spaces, Section 66.18. Moreover, if $f : X \to Y$ is a morphism of representable algebraic spaces over S and if $f_0 : X_0 \to Y_0$ is a morphism of schemes representing f, then the induced morphisms of small étale topoi agree:

$$\begin{tikzcd} Sh(X_{\acute{e}tale}) \arrow[r, "f_{small}"'] \arrow[d, equal] & Sh(Y_{\acute{e}tale}) \arrow[d, equal] \\ Sh((X_0)_{\acute{e}tale}) \arrow[r, "(f_0)_{small}"] & Sh((Y_0)_{\acute{e}tale}) \end{tikzcd}$$

See Properties of Spaces, Lemma 66.18.8 and Topologies, Lemma 34.4.17.

Thus there is absolutely no difference between étale cohomology of a scheme and the étale cohomology of the corresponding algebraic space. Similarly for higher direct images along morphisms of schemes. In fact, if $f : X \to Y$ is a morphism of algebraic spaces over S which is representable (by schemes), then the higher direct images $R^i f_* \mathcal{F}$ of a sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ can be computed étale locally on Y (Cohomology on Sites, Lemma 21.7.4) hence this often reduces computations and proofs to the case where Y and X are schemes.

We will use the above without further mention in this chapter. For other topologies the same thing is true; we state it explicitly as a lemma for cohomology here.

0DFV Lemma 84.3.1. Let S be a scheme. Let $\tau \in \{\acute{e}tale, fppf, ph\}$ (add more here). The inclusion functor

$$(Sch/S)_\tau \longrightarrow (\text{Spaces}/S)_\tau$$

is a special cocontinuous functor (Sites, Definition 7.29.2) and hence identifies topoi.

Proof. The conditions of Sites, Lemma 7.29.1 are immediately verified as our functor is fully faithful and as every algebraic space has an étale covering by schemes. $\square$

84.4. Proper base change

0DFW The proper base change theorem for algebraic spaces follows from the proper base change theorem for schemes and Chow's lemma with a little bit of work.

0DFX Lemma 84.4.1. Let S be a scheme. Let $f : Y \to X$ be a surjective proper morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}tale}$. Then $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is injective with image the equalizer of the two maps $f_* f^{-1}\mathcal{F} \to g_* g^{-1}\mathcal{F}$ where g is the structure morphism $g : Y \times_X Y \to X$.

Proof. For any surjective morphism $f : Y \to X$ of algebraic spaces over S, the map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is injective. Namely, if $\overline{x}$ is a geometric point of X, then we choose a geometric point $\overline{y}$ of Y lying over $\overline{x}$ and we consider

$$\mathcal{F}_{\overline{x}} \to (f_* f^{-1}\mathcal{F})_{\overline{x}} \to (f^{-1}\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}}$$

See Properties of Spaces, Lemma 66.19.9 for the last equality.

The second statement is local on X in the étale topology, hence we may and do assume Y is an affine scheme.

Choose a surjective proper morphism $Z \to Y$ where Z is a scheme, see Cohomology of Spaces, Lemma 69.18.1. The result for $Z \to X$ implies the result for $Y \to X$. Since $Z \to X$ is a surjective proper morphism of schemes and hence a ph covering (Topologies, Lemma 34.8.6) the result for $Z \to X$ follows from Étale Cohomology, Lemma 59.102.1 (in fact it is in some sense equivalent to this lemma). $\square$

0DFY Lemma 84.4.2. Let (A, I) be a henselian pair. Let X be an algebraic space over A such that the structure morphism $f : X \to \text{Spec}(A)$ is proper. Let $i : X_0 \to X$ be the inclusion of $X \times_{\text{Spec}(A)} \text{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, i^{-1}\mathcal{F})$.

Proof. Choose a surjective proper morphism $Y \to X$ where Y is a scheme, see Cohomology of Spaces, Lemma 69.18.1. Consider the diagram

$$\begin{tikzcd} \Gamma(X_0, \mathcal{F}_0) \arrow[r] & \Gamma(Y_0, \mathcal{G}_0) \arrow[r, shift left] \arrow[r, shift right] & \Gamma((Y \times_X Y)_0, \mathcal{H}_0) \\ \Gamma(X, \mathcal{F}) \arrow[r] \arrow[u] & \Gamma(Y, \mathcal{G}) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \Gamma(Y \times_X Y, \mathcal{H}) \arrow[u] \end{tikzcd}$$

Here $\mathcal{G}$, resp. $\mathcal{H}$ is the pullbackf or $\mathcal{F}$ to Y, resp. $Y \times_X Y$ and the index 0 indicates base change to $\mathrm{Spec}(A/I)$. By the case of schemes (Étale Cohomology, Lemma 59.91.2) we see that the middle and right vertical arrows are bijective. By Lemma 84.4.1 it follows that the left one is too. □

0DFZ Lemma 84.4.3. Let A be a henselian local ring. Let X be an algebraic space over A such that $f : X \to \mathrm{Spec}(A)$ is a proper morphism. Let $X_0 \subset X$ be the fibre of f over the closed point. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})$.

Proof. This is a special case of Lemma 84.4.2. □

0DG0 Lemma 84.4.4. Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be a morphisms of algebraic spaces over S. Assume f is proper. Set $X' = Y' \times_Y X$ with projections $f' : X' \to Y'$ and $g' : X' \to X$. Let $\mathcal{F}$ be any sheaf on $X_{\acute{e}tale}$. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$.

Proof. The question is étale local on Y'. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then we may replace Y' by V' and Y by V. Hence we may assume Y and Y' are schemes. Then we may work Zariski locally on Y and Y' and hence we may assume Y and Y' are affine schemes.

Assume Y and Y' are affine schemes. Choose a surjective proper morphism $h_1 : X_1 \to X$ where X_1 is a scheme, see Cohomology of Spaces, Lemma 69.18.1. Set $X_2 = X_1 \times_X X_1$ and denote $h_2 : X_2 \to X$ the structure morphism. Observe this is a scheme. By the case of schemes (Étale Cohomology, Lemma 59.91.5) we know the lemma is true for the cartesian diagrams

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X'_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and the sheaves $\mathcal{F}_i = (X_i \to X)^{-1}\mathcal{F}$. By Lemma 84.4.1 we have an exact sequence $0 \to \mathcal{F} \to h_{1,*}\mathcal{F}_1 \to h_{2,*}\mathcal{F}_2$ and similarly for $(g')^{-1}\mathcal{F}$ because $X'_2 = X'_1 \times_{X'} X'_1$. Hence we conlude that the lemma is true (some details omitted). □

Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\overline{x} : \mathrm{Spec}(k) \to S$ be a geometric point. The fibre of f at $\overline{x}$ is the algebraic space $Y_{\overline{x}} = \mathrm{Spec}(k) \times_{\overline{x},X} Y$ over $\mathrm{Spec}(k)$. If $\mathcal{F}$ is a sheaf on $Y_{\acute{e}tale}$, then denote $\mathcal{F}_{\overline{x}} = p^{-1}\mathcal{F}$ the pullback of $\mathcal{F}$ to $(Y_{\overline{x}})_{\acute{e}tale}$. Here $p : Y_{\overline{x}} \to Y$ is the projection. In the following we will consider the set $\Gamma(Y_{\overline{x}}, \mathcal{F}_{\overline{x}})$.

0DG1 **Lemma 84.4.5.** Let S be a scheme. Let $f : Y \to X$ be a proper morphism of algebraic spaces over S. Let $\overline{x} \to X$ be a geometric point. For any sheaf $\mathcal{F}$ on $Y_{étale}$ the canonical map

$$(f_*\mathcal{F})_{\overline{x}} \longrightarrow \Gamma(Y_{\overline{x}}, \mathcal{F}_{\overline{x}})$$

is bijective.

Proof. This is a special case of Lemma 84.4.4. □

0DG2 **Theorem 84.4.6.** Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces over S. Assume f is proper. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Then the base change map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism.

Proof. This proof repeats a few of the arguments given in the proof of the proper base change theorem for schemes. See Étale Cohomology, Section 59.91 for more details.

The statement is étale local on Y' and Y, hence we may assume both Y and Y' are affine schemes. Observe that this in particular proves the theorem in case f is representable (we will use this below).

For every $n \geq 1$ let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$. By Cohomology of Spaces, Lemma 69.5.2 the functors $g^{-1}R^pf_*$ and $R^pf'_*(g')^{-1}$ commute with filtered colimits. Hence it suffices to prove the theorem if $\mathcal{F}$ is killed by n.

Let $\mathcal{F} \to \mathcal{I}^\bullet$ be a resolution by injective sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Observe that $g^{-1}f_*\mathcal{I}^\bullet = f'_*(g')^{-1}\mathcal{I}^\bullet$ by Lemma 84.4.4. Applying Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we conclude it suffices to prove $R^pf'_*(g')^{-1}\mathcal{I}^m = 0$ for $p > 0$ and $m \in \mathbf{Z}$.

Choose a surjective proper morphism $h : Z \to X$ where Z is a scheme, see Cohomology of Spaces, Lemma 69.18.1. Choose an injective map $h^{-1}\mathcal{I}^m \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $Z_{étale}$. Since h is surjective the map $\mathcal{I}^m \to h_*\mathcal{J}$ is injective (see Lemma 84.4.1). Since $\mathcal{I}^m$ is injective we see that $\mathcal{I}^m$ is a direct summand of $h_*\mathcal{J}$. Thus it suffices to prove the desired vanishing for $h_*\mathcal{J}$.

Denote h' the base change by g and denote $g'' : Z' \to Z$ the projection. There is a spectral sequence

$$E_2^{p,q} = R^pf'_*R^qh'_*(g'')^{-1}\mathcal{J}$$

converging to $R^{p+q}(f' \circ h')_*(g'')^{-1}\mathcal{J}$. Since h and $f \circ h$ are representable (by schemes) we know the result we want holds for them. Thus in the spectral sequence we see that $E_2^{p,q} = 0$ for $q > 0$ and $R^{p+q}(f' \circ h')_*(g'')^{-1}\mathcal{J} = 0$ for $p + q > 0$. It follows that $E_2^{p,0} = 0$ for $p > 0$. Now

$$E_2^{p,0} = R^pf'_*h'_*(g'')^{-1}\mathcal{J} = R^pf'_*(g')^{-1}h_*\mathcal{J}$$

by Lemma 84.4.4. This finishes the proof. □

0DG3 Lemma 84.4.7. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces over S. Assume f is proper. Let $E \in D^+(X_{\acute{e}tale})$ have torsion cohomology sheaves. Then the base change map $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.

Proof. This is a simple consequence of the proper base change theorem (Theorem 84.4.6) using the spectral sequences

$$E_2^{p,q} = R^pf_*H^q(E) \quad \text{and} \quad {E'}_2^{p,q} = R^pf'_*(g')^{-1}H^q(E)$$

converging to R^nf_*E and $R^nf'_*(g')^{-1}E$. The spectral sequences are constructed in Derived Categories, Lemma 13.21.3. Some details omitted. □

0DG4 Lemma 84.4.8. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces. Let $\overline{y} \to Y$ be a geometric point.

(1) For a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $(R^nf_*\mathcal{F})_{\overline{y}} = H^n_{\acute{e}tale}(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.
(2) For $E \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves we have $(R^nf_*E)_{\overline{y}} = H^n_{\acute{e}tale}(X_{\overline{y}}, E_{\overline{y}})$.

Proof. In the statement, $\mathcal{F}_{\overline{y}}$ denotes the pullback of $\mathcal{F}$ to $X_{\overline{y}} = \overline{y} \times_Y X$. Since pulling back by $\overline{y} \to Y$ produces the stalk of $\mathcal{F}$, the first statement of the lemma is a special case of Theorem 84.4.6. The second one is a special case of Lemma 84.4.7. □

0DG5 Lemma 84.4.9. Let k'/k be an extension of separably closed fields. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a torsion abelian sheaf on X. Then the map $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.

Proof. This is a special case of Theorem 84.4.6. □

84.5. Comparing big and small topoi

0DG6 Let S be a scheme and let X be an algebraic space over S. In Topologies on Spaces, Lemma 73.4.8 we have introduced comparison morphisms $\pi_X : (\mathit{Spaces}/X)_{\acute{e}tale} \to X_{spaces,\acute{e}tale}$ and $i_X : Sh(X_{\acute{e}tale}) \to Sh((\mathit{Spaces}/X)_{\acute{e}tale})$ with $\pi_X \circ i_X = \text{id}$ as morphisms of topoi and $\pi_{X,*} = i_X^{-1}$. More generally, if $f : Y \to X$ is an object of $(\mathit{Spaces}/X)_{\acute{e}tale}$, then there is a morphism $i_f : Sh(Y_{\acute{e}tale}) \to Sh((\mathit{Spaces}/X)_{\acute{e}tale})$ such that $f_{small} = \pi_X \circ i_f$, see Topologies on Spaces, Lemmas 73.4.7 and 73.4.11. In Topologies on Spaces, Remark 73.4.14 we have extended these to a morphism of ringed sites

$$\pi_X : ((\mathit{Spaces}/X)_{\acute{e}tale}, \mathcal{O}) \to (X_{spaces,\acute{e}tale}, \mathcal{O}_X)$$

and morphisms of ringed topoi

$$i_X : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \to (Sh((\mathit{Spaces}/X)_{\acute{e}tale}), \mathcal{O})$$

and

$$i_f : (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y) \to (Sh((\mathit{Spaces}/X)_{\acute{e}tale}, \mathcal{O}))$$

Note that the restriction $i_X^{-1} = \pi_{X,*}$ (see Topologies, Definition 34.4.15) transforms $\mathcal{O}$ into $\mathcal{O}_X$. Similarly, i_f^{-1} transforms $\mathcal{O}$ into $\mathcal{O}_Y$. See Topologies on Spaces, Remark 73.4.14. Hence $i_X^*\mathcal{F} = i_X^{-1}\mathcal{F}$ and $i_f^*\mathcal{F} = i_f^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(\textit{Spaces}/X)_{\acute{e}tale}$. In particular i_X^* and i_f^* are exact functors. The functor i_X^* is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{X_{\acute{e}tale}}$ (and this does not conflict with the notation in Topologies on Spaces, Definition 73.4.9).

0DG7 **Lemma 84.5.1.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}tale}$. Then $\pi_X^{-1}\mathcal{F}$ is given by the rule

$$(\pi_X^{-1}\mathcal{F})(Y) = \Gamma(Y_{\acute{e}tale}, f_{small}^{-1}\mathcal{F})$$

for $f : Y \to X$ in $(\textit{Spaces}/X)_{\acute{e}tale}$. Moreover, $\pi_Y^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to smooth, syntomic, fppf, fpqc, and ph coverings.

Proof. Since pullback is transitive and $f_{small} = \pi_X \circ i_f$ (see above) we see that $i_f^{-1}\pi_X^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$. This shows that π_X^{-1} has the description given in the lemma.

To prove that $\pi_X^{-1}\mathcal{F}$ is a sheaf for the ph topology it suffices by Topologies on Spaces, Lemma 73.8.7 to show that for a surjective proper morphism $V \to U$ of algebraic spaces over X we have $(\pi_X^{-1}\mathcal{F})(U)$ is the equalizer of the two maps $(\pi_X^{-1}\mathcal{F})(V) \to (\pi_X^{-1}\mathcal{F})(V \times_U V)$. This we have seen in Lemma 84.4.1.

The case of smooth, syntomic, fppf coverings follows from the case of ph coverings by Topologies on Spaces, Lemma 73.8.2.

Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an fpqc covering of algebraic spaces over X. Let $s_i \in (\pi_X^{-1}\mathcal{F})(U_i)$ be sections which agree over $U_i \times_U U_j$. We have to prove there exists a unique $s \in (\pi_X^{-1}\mathcal{F})(U)$ restricting to s_i over U_i. Case I: U and U_i are schemes. This case follows from Étale Cohomology, Lemma 59.39.2. Case II: U is a scheme. Here we choose surjective étale morphisms $T_i \to U_i$ where T_i is a scheme. Then $\mathcal{T} = \{T_i \to U\}$ is an fpqc covering by schemes and by case I the result holds for $\mathcal{T}$. We omit the verification that this implies the result for $\mathcal{U}$. Case III: general case. Let $W \to U$ be a surjective étale morphism, where W is a scheme. Then $\mathcal{W} = \{U_i \times_U W \to W\}$ is an fpqc covering (by algebraic spaces) of the scheme W. By case II the result hold for $\mathcal{W}$. We omit the verification that this implies the result for $\mathcal{U}$. $\square$

0DG8 **Lemma 84.5.2.** Let S be a scheme. Let $Y \to X$ be a morphism of $(\textit{Spaces}/S)_{\acute{e}tale}$.

(1) If $\mathcal{I}$ is injective in $\textit{Ab}((\textit{Spaces}/X)_{\acute{e}tale})$, then
   (a) $i_f^{-1}\mathcal{I}$ is injective in $\textit{Ab}(Y_{\acute{e}tale})$,
   (b) $\mathcal{I}|_{X_{\acute{e}tale}}$ is injective in $\textit{Ab}(X_{\acute{e}tale})$,
(2) If $\mathcal{I}^\bullet$ is a K-injective complex in $\textit{Ab}((\textit{Spaces}/X)_{\acute{e}tale})$, then
   (a) $i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $\textit{Ab}(Y_{\acute{e}tale})$,
   (b) $\mathcal{I}^\bullet|_{X_{\acute{e}tale}}$ is a K-injective complex in $\textit{Ab}(X_{\acute{e}tale})$,

The corresponding statements for modules do not hold.

Proof. Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor $\pi_{X,*} = i_X^{-1}$ is a right adjoint of the exact functor π_X^{-1}, see Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that i_f^{-1} is a right adjoint of the exact functor $i_{f,!}$. This functor is constructed in Topologies, Lemma 34.4.13 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 18.16.2. It is shown in Modules on Sites, Lemma 18.16.3 that it is exact. Second proof. We can use that $i_f = i_Y \circ f_{big}$ as is shown in Topologies, Lemma 34.4.17. Since f_{big} is a localization, we see that pullback by it preserves injectives and K-injectives, see Cohomology on Sites, Lemmas 21.7.1 and 21.20.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor i_Y^{-1} to conclude.

To see a counter example for the case of modules we refer to Étale Cohomology, Lemma 59.99.1. □

Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The commutative diagram of Topologies on Spaces, Lemma 73.4.11 (3) leads to a commutative diagram of ringed sites

$$\begin{tikzcd} (Y_{spaces,étale}, \mathcal{O}_Y) \arrow[d, "f_{spaces,étale}"'] & ((\mathrm{Spaces}/Y)_{étale}, \mathcal{O}) \arrow[l, "\pi_Y"'] \arrow[d, "f_{big}"] \\ (X_{spaces,étale}, \mathcal{O}_X) & ((\mathrm{Spaces}/X)_{étale}, \mathcal{O}) \arrow[l, "\pi_X"] \end{tikzcd}$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_X^\sharp$, and $\pi_Y^\sharp$. In particular this means that

0DG9 (84.5.2.1) $$(f_{big,*}\mathcal{F})|_{X_{étale}} = f_{small,*}(\mathcal{F}|_{Y_{étale}})$$

for any sheaf $\mathcal{F}$ on $(\mathrm{Spaces}/Y)_{étale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (84.5.2.1) is an isomorphism of $\mathcal{O}_X$-modules on $X_{étale}$.

0DGA **Lemma 84.5.3.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) For K in $D((\mathrm{Spaces}/Y)_{étale})$ we have $(Rf_{big,*}K)|_{X_{étale}} = Rf_{small,*}(K|_{Y_{étale}})$ in $D(X_{étale})$.
(2) For K in $D((\mathrm{Spaces}/Y)_{étale}, \mathcal{O})$ we have $(Rf_{big,*}K)|_{X_{étale}} = Rf_{small,*}(K|_{Y_{étale}})$ in $D(\mathrm{Mod}(X_{étale}, \mathcal{O}_X))$.

More generally, let $g : X' \to X$ be an object of $(\mathrm{Spaces}/X)_{étale}$. Consider the fibre product

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

Then

(3) For K in $D((\mathrm{Spaces}/Y)_{étale})$ we have $i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$ in $D(X'_{étale})$.
(4) For K in $D((\mathrm{Spaces}/Y)_{étale}, \mathcal{O})$ we have $i_g^*(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^*K)$ in $D(\mathrm{Mod}(X'_{étale}, \mathcal{O}_{X'}))$.
(5) For K in $D((\mathrm{Spaces}/Y)_{étale})$ we have $g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$ in $D((\mathrm{Spaces}/X')_{étale})$.
(6) For K in $D((\mathrm{Spaces}/Y)_{étale}, \mathcal{O})$ we have $g_{big}^*(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^*K)$ in $D(\mathrm{Mod}(X'_{étale}, \mathcal{O}_{X'}))$.

Proof. Part (1) follows from Lemma 84.5.2 and (84.5.2.1) on choosing a K-injective complex of abelian sheaves representing K.

Part (3) follows from Lemma 84.5.2 and Topologies, Lemma 34.4.19 on choosing a K-injective complex of abelian sheaves representing K.

Part (5) is Cohomology on Sites, Lemma 21.21.1.

Part (6) is Cohomology on Sites, Lemma 21.21.2.

Part (2) can be proved as follows. Above we have seen that $\pi_X \circ f_{big} = f_{small} \circ \pi_Y$ as morphisms of ringed sites. Hence we obtain $R\pi_{X,*} \circ Rf_{big,*} = Rf_{small,*} \circ R\pi_{Y,*}$ by Cohomology on Sites, Lemma 21.19.2. Since the restriction functors $\pi_{X,*}$ and $\pi_{Y,*}$ are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to $f' : Y' \to X'$. □

Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{H}$ be an abelian sheaf on $(Spaces/X)_{étale}$. Recall that $H^n_{étale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{étale}$.

0DGB Lemma 84.5.4. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then

(1) For K in $D(X_{étale})$ we have $H^n_{étale}(X, \pi_X^{-1}K) = H^n(X_{étale}, K)$.
(2) For K in $D(X_{étale}, \mathcal{O}_X)$ we have $H^n_{étale}(X, L\pi_X^*K) = H^n(X_{étale}, K)$.
(3) For K in $D(X_{étale})$ we have $H^n_{étale}(Y, \pi_X^{-1}K) = H^n(Y_{étale}, f_{small}^{-1}K)$.
(4) For K in $D(X_{étale}, \mathcal{O}_X)$ we have $H^n_{étale}(Y, L\pi_X^*K) = H^n(Y_{étale}, Lf_{small}^*K)$.
(5) For M in $D((Spaces/X)_{étale})$ we have $H^n_{étale}(Y, M) = H^n(Y_{étale}, i_f^{-1}M)$.
(6) For M in $D((Spaces/X)_{étale}, \mathcal{O})$ we have $H^n_{étale}(Y, M) = H^n(Y_{étale}, i_f^*M)$.

Proof. To prove (5) represent M by a K-injective complex of abelian sheaves and apply Lemma 84.5.2 and work out the definitions. Part (3) follows from this as $i_f^{-1}\pi_X^{-1} = f_{small}^{-1}$. Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 21.37.5. Then part (4) follows because $Lf_{small}^* = i_f^* \circ L\pi_X^*$. Part (2) is a special case of (4). □

0DGC Lemma 84.5.5. Let S be a scheme. Let X be an algebraic space over S. For $K \in D(X_{étale})$ the map

$$K \longrightarrow R\pi_{X,*}\pi_X^{-1}K$$

is an isomorphism where $\pi_X : Sh((Spaces/X)_{étale}) \to Sh(X_{étale})$ is as above.

Proof. This is true because both π_X^{-1} and $\pi_{X,*} = i_X^{-1}$ are exact functors and the composition $\pi_{X,*} \circ \pi_X^{-1}$ is the identity functor. □

0DGD Lemma 84.5.6. Let S be a scheme. Let $f : Y \to X$ be a proper morphism of algebraic spaces over S. Then we have

(1) $\pi_X^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_Y^{-1}$ as functors $Sh(Y_{étale}) \to Sh((Spaces/X)_{étale})$,
(2) $\pi_X^{-1}Rf_{small,*}K = Rf_{big,*}\pi_Y^{-1}K$ for K in $D^+(Y_{étale})$ whose cohomology sheaves are torsion, and
(3) $\pi_X^{-1}Rf_{small,*}K = Rf_{big,*}\pi_Y^{-1}K$ for all K in $D(Y_{étale})$ if f is finite.

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf on $Y_{étale}$. Let $g : X' \to X$ be an object of $(\text{Spaces}/X)_{étale}$. Consider the fibre product

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

Then we have

$$(f_{big,*}\pi_Y^{-1}\mathcal{F})(X') = (\pi_Y^{-1}\mathcal{F})(Y') = ((g'_{small})^{-1}\mathcal{F})(Y') = (f'_{small,*}(g'_{small})^{-1}\mathcal{F})(X')$$

the second equality by Lemma 84.5.1. On the other hand

$$(\pi_X^{-1}f_{small,*}\mathcal{F})(X') = (g_{small}^{-1}f_{small,*}\mathcal{F})(X')$$

again by Lemma 84.5.1. Hence by proper base change for sheaves of sets (Lemma 84.4.4) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_X^{-1}f_{small,*}\mathcal{F} = f_{big,*}\pi_Y^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_X^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_Y^{-1}$.

Proof of (2). There is a canonical base change map $\pi_X^{-1}Rf_{small,*}K \to Rf_{big,*}\pi_Y^{-1}K$ for any K in $D(Y_{étale})$, see Cohomology on Sites, Remark 21.19.3. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by $i_g : Sh(X'_{étale}) \to Sh((Sch/X)_{étale})$ is an isomorphism for any object $g : X' \to X$ of $(Sch/X)_{étale}$. Let T', g', f' be as in the previous paragraph. The pullback of the base change map is

$$\begin{aligned} g_{small}^{-1}Rf_{small,*}K &= i_g^{-1}\pi_X^{-1}Rf_{small,*}K \\ &\to i_g^{-1}Rf_{big,*}\pi_Y^{-1}K \\ &= Rf'_{small,*}(i_{g'}^{-1}\pi_Y^{-1}K) \\ &= Rf'_{small,*}((g'_{small})^{-1}K) \end{aligned}$$

where we have used $\pi_X \circ i_g = g_{small}$, $\pi_Y \circ i_{g'} = g'_{small}$, and Lemma 84.5.3. This map is an isomorphism by the proper base change theorem (Lemma 84.4.7) provided K is bounded below and the cohomology sheaves of K are torsion.

Proof of (3). If f is finite, then the functors $f_{small,*}$ and $f_{big,*}$ are exact. This follows from Cohomology of Spaces, Lemma 69.4.1 for f_{small}. Since any base change f' of f is finite too, we conclude from Lemma 84.5.3 part (3) that $f_{big,*}$ is exact too (as the higher derived functors are zero). Thus this case follows from part (1). □

84.6. Comparing fppf and étale topologies

0DGE This section is the analogue of Étale Cohomology, Section 59.100.

Let S be a scheme. Let X be an algebraic space over S. On the category Spaces$/X$ we consider the fppf and étale topologies. The identity functor $(\text{Spaces}/X)_{étale} \to (\text{Spaces}/X)_{fppf}$ is continuous and defines a morphism of sites

$$\epsilon_X : (\text{Spaces}/X)_{fppf} \longrightarrow (\text{Spaces}/X)_{étale}$$

by an application of Sites, Proposition 7.14.7. Please note that $\epsilon_{X,*}$ is the identity functor on underlying presheaves and that ϵ_X^{-1} associates to an étale sheaf the fppf sheafification. Consider the morphism of sites

$$\pi_X : (\text{Spaces}/X)_{étale} \longrightarrow X_{spaces,étale}$$

comparing big and small étale sites, see Section 84.5. The composition determines a morphism of sites

$$a_X = \pi_X \circ \epsilon_X : (\text{Spaces}/X)_{fppf} \longrightarrow X_{spaces,étale}$$

If $\mathcal{H}$ is an abelian sheaf on $(\text{Spaces}/X)_{fppf}$, then we will write $H^n_{fppf}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of $(\text{Spaces}/X)_{fppf}$.

0DGF Lemma 84.6.1. Let S be a scheme. Let X be an algebraic space over S.

(1) For $\mathcal{F} \in Sh(X_{étale})$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$.
(2) For $\mathcal{F} \in \text{Ab}(X_{étale})$ we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.

Proof. We have $a_X^{-1}\mathcal{F} = \epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$. By Lemma 84.5.1 the étale sheaf $\pi_X^{-1}\mathcal{F}$ is a sheaf for the fppf topology and therefore is equal to $a_X^{-1}\mathcal{F}$ (as pulling back by ϵ_X is given by fppf sheafification). Recall moreover that $\epsilon_{X,*}$ is the identity on underlying presheaves. Now part (1) is immediate from the explicit description of π_X^{-1} in Lemma 84.5.1.

We will prove part (2) by reducing it to the case of schemes – see part (1) of Étale Cohomology, Lemma 59.100.6. This will "clearly work" as every algebraic space is étale locally a scheme. The details are given below but we urge the reader to skip the proof.

For an abelian sheaf $\mathcal{H}$ on $(\text{Spaces}/X)_{fppf}$ the higher direct image $R^p\epsilon_{X,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, \mathcal{H})$ on $(\text{Spaces}/X)_{étale}$. See Cohomology on Sites, Lemma 21.7.4. Since every object of $(\text{Spaces}/X)_{étale}$ has a covering by schemes, it suffices to prove that given U/X a scheme and $\xi \in H^p_{fppf}(U, a_X^{-1}\mathcal{F})$ we can find an étale covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. We have

$$\begin{aligned} H^p_{fppf}(U, a_X^{-1}\mathcal{F}) &= H^p((\text{Spaces}/U)_{fppf}, (a_X^{-1}\mathcal{F})|_{\text{Spaces}/U}) \\ &= H^p((Sch/U)_{fppf}, (a_X^{-1}\mathcal{F})|_{Sch/U}) \end{aligned}$$

where the second identification is Lemma 84.3.1 and the first is a general fact about restriction (Cohomology on Sites, Lemma 21.7.1). Looking at the first paragraph and the corresponding result in the case of schemes (Étale Cohomology, Lemma 59.100.1) we conclude that the sheaf $(a_X^{-1}\mathcal{F})|_{Sch/U}$ matches the pullback by the "schemes version of a_U". Therefore we can find an étale covering $\{U_i \to U\}$ such that our class dies in $H^p((Sch/U_i)_{fppf}, (a_X^{-1}\mathcal{F})|_{Sch/U_i})$ for each i, see Étale Cohomology, Lemma 59.100.6 (the precise statement one should use here is that V_n holds for all n which is the statement of part (2) for the case of schemes). Transporting back (using the same formulas as above but now for U_i) we conclude ξ restricts to zero over U_i as desired. $\square$

The hard work done in the case of schemes now tells us that étale and fppf cohomology agree for sheaves coming from the small étale site.

0DGG Lemma 84.6.2. Let S be a scheme. Let X be an algebraic space over S. For $K \in D^+(X_{étale})$ the maps

$$\pi_X^{-1}K \longrightarrow R\epsilon_{X,*}a_X^{-1}K \quad \text{and} \quad K \longrightarrow Ra_{X,*}a_X^{-1}K$$

are isomorphisms with $a_X : Sh((\text{Spaces}/X)_{fppf}) \to Sh(X_{étale})$ as above.

Proof. We only prove the second statement; the first is easier and proved in exactly the same manner. There is an immediate reduction to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 84.6.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 84.6.1 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$. We have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and by Lemma 84.5.5 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. $\square$

0DGH Lemma 84.6.3. Let S be a scheme and let X be an algebraic space over S. With $a_X : Sh((\text{Spaces}/X)_{fppf}) \to Sh(X_{\acute{e}tale})$ as above:

(1) $H^q(X_{\acute{e}tale}, \mathcal{F}) = H^q_{fppf}(X, a_X^{-1}\mathcal{F})$ for an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,
(2) $H^q(X_{\acute{e}tale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ for $K \in D^+(X_{\acute{e}tale})$.

Example: if A is an abelian group, then $H^q_{\acute{e}tale}(X, \underline{A}) = H^q_{fppf}(X, \underline{A})$.

Proof. This follows from Lemma 84.6.2 by Cohomology on Sites, Remark 21.14.4. $\square$

0DGI Lemma 84.6.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there are commutative diagrams of topoi

$$\begin{tikzcd} Sh((\text{Spaces}/X)_{fppf}) \arrow[r, "f_{big,fppf}"'] \arrow[d, "\epsilon_X"'] & Sh((\text{Spaces}/Y)_{fppf}) \arrow[d, "\epsilon_Y"] \\ Sh((\text{Spaces}/X)_{\acute{e}tale}) \arrow[r, "f_{big,\acute{e}tale}"] & Sh((\text{Spaces}/Y)_{\acute{e}tale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((\text{Spaces}/X)_{fppf}) \arrow[r, "f_{big,fppf}"'] \arrow[d, "a_X"'] & Sh((\text{Spaces}/Y)_{fppf}) \arrow[d, "a_Y"] \\ Sh(X_{\acute{e}tale}) \arrow[r, "f_{small}"] & Sh(Y_{\acute{e}tale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. This follows immediately from working out the definitions of the morphisms involved, see Topologies on Spaces, Section 73.7 and Section 84.5. $\square$

0DGJ Lemma 84.6.5. In Lemma 84.6.4 if f is proper, then we have

(1) $a_Y^{-1} \circ f_{small,*} = f_{big,fppf,*} \circ a_X^{-1}$, and
(2) $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$ for K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves.

Proof. Proof of (1). You can prove this by repeating the proof of Lemma 84.5.6 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. Lemma 84.6.1 shows that $\epsilon_{Y,*} \circ$

$a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,fppf,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned}\pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*}a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*}f_{big,fppf,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\pi_X^{-1}\mathcal{F}\end{aligned}$$

is an isomorphism. This is part (1) of Lemma 84.5.6.

To see (2) we use that

$$\begin{aligned}R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K &= Rf_{big,étale,*}R\epsilon_{X,*}a_X^{-1}K \\ &= Rf_{big,étale,*}\pi_X^{-1}K \\ &= \pi_Y^{-1}Rf_{small,*}K \\ &= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K\end{aligned}$$

The first equality by the commutative diagram in Lemma 84.6.4 and Cohomology on Sites, Lemma 21.19.2. Then second equality is Lemma 84.6.2. The third is Lemma 84.5.6 part (2). The fourth is Lemma 84.6.2 again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,fppf,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{étale})$ and M in $D^+((\text{Spaces}/Y)_{fppf})$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by Lemma 84.6.1 because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

0DGK Lemma 84.6.6. In Lemma 84.6.4 if f is finite, then $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$ for K in $D^+(X_{étale})$.

Proof. Let $V \to Y$ be a surjective étale morphism where V is a scheme. It suffices to prove the base change map is an isomorphism after restricting to V. Hence we may assume that Y is a scheme. As the morphism is finite, hence representable, we conclude that we may assume both X and Y are schemes. In this case the result follows from the case of schemes (Étale Cohomology, Lemma 59.100.6 part (2)) using the comparison of topoi discussed in Section 84.3 and in particular given in Lemma 84.3.1. Some details omitted. □

0DGL Lemma 84.6.7. In Lemma 84.6.4 assume f is flat, locally of finite presentation, and surjective. Then the functor

$$Sh(Y_{étale}) \longrightarrow \left\{(\mathcal{G}, \mathcal{H}, \alpha)\middle| \begin{array}{l}\mathcal{G} \in Sh(X_{étale}),\ \mathcal{H} \in Sh((Sch/Y)_{fppf}), \\ \alpha : a_X^{-1}\mathcal{G} \to f_{big,fppf}^{-1}\mathcal{H} \text{ an isomorphism}\end{array}\right\}$$

sending $\mathcal{F}$ to $(f_{small}^{-1}\mathcal{F}, a_Y^{-1}\mathcal{F}, can)$ is an equivalence.

Proof. The functor a_X^{-1} is fully faithful (as $a_{X,*}a_X^{-1} = \text{id}$ by Lemma 84.6.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor a_Y^{-1} is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of f. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is true for f_i and f_{ij} for all i, j, then the lemma is true for f. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{étale}, X_{étale}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 7.26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on Y. In particular, we may assume Y is a scheme.

Assume Y is a scheme. Choose a scheme X' and a surjective étale morphism $s : X' \to X$. Set $f' = f \circ s : X' \to Y$ and observe that f' is surjective, locally of finite presentation, and flat. Claim: if the lemma is true for f', then it is true for f. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for f, we can pullback by s to get a triple $(s_{small}^{-1}\mathcal{G}, \mathcal{H}, s_{big,fppf}^{-1}\alpha)$ for f'. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{étale}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case where both X and Y are schemes. This case follows from Étale Cohomology, Lemma 59.100.4 via the discussion in Section 84.3 and in particular Lemma 84.3.1. $\square$

84.7. Comparing fppf and étale topologies: modules

0DGM We continue the discussion in Section 84.6 but in this section we briefly discuss what happens for sheaves of modules.

Let S be a scheme. Let X be an algebraic space over S. The morphisms of sites ϵ_X, π_X, and their composition a_X introduced in Section 84.6 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_X : ((\text{Spaces}/X)_{fppf}, \mathcal{O}) \longrightarrow ((\text{Spaces}/X)_{étale}, \mathcal{O})$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_X : ((\text{Spaces}/X)_{étale}, \mathcal{O}) \longrightarrow (X_{étale}, \mathcal{O}_X)$$

The third is the morphism

$$a_X : ((\text{Spaces}/X)_{fppf}, \mathcal{O}) \longrightarrow (X_{étale}, \mathcal{O}_X)$$

Let us review what we already know about quasi-coherent modules on these sites.

0DGN Lemma 84.7.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) The rule

$$\mathcal{F}^a : (\text{Spaces}/X)_{étale} \longrightarrow \text{Ab}, \quad (f : Y \to X) \longmapsto \Gamma(Y, f^*\mathcal{F})$$

satisfies the sheaf condition for fpqc and a fortiori fppf and étale coverings,

(2) $\mathcal{F}^a = \pi_X^*\mathcal{F}$ on $(\text{Spaces}/X)_{étale}$,

(3) $\mathcal{F}^a = a_X^*\mathcal{F}$ on $(\text{Spaces}/X)_{fppf}$,

(4) the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_X$-modules and quasi-coherent modules on $((\text{Spaces}/X)_{étale}, \mathcal{O})$,
(5) the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_X$-modules and quasi-coherent modules on $((\text{Spaces}/X)_{fppf}, \mathcal{O})$,
(6) we have $\epsilon_{X,*}a_X^*\mathcal{F} = \pi_X^*\mathcal{F}$ and $a_{X,*}a_X^*\mathcal{F} = \mathcal{F}$,
(7) we have $R^i\epsilon_{X,*}(a_X^*\mathcal{F}) = 0$ and $R^ia_{X,*}(a_X^*\mathcal{F}) = 0$ for $i > 0$.

Proof. Part (1) is a consequence of fppf descent of quasi-coherent modules. Namely, suppose that $\{f_i : U_i \to U\}$ is an fpqc covering in $(\text{Spaces}/X)_{étale}$. Denote $g : U \to X$ the structure morphism. Suppose that we have a family of sections $s_i \in \Gamma(U_i, f_i^*g^*\mathcal{F})$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$. We have to find the correspond section $s \in \Gamma(U, g^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_U = \mathcal{O}_{U_i} \to f_i^*g^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_U$ and $g^*\mathcal{F}$ on U. Hence by Descent on Spaces, Proposition 74.4.1 we see that we may (uniquely) descend these to a map $\mathcal{O}_U \to g^*\mathcal{F}$ which gives us our section s.

We will deduce (2) – (7) from the corresponding statement for schemes. Choose an étale covering $\{X_i \to X\}_{i \in I}$ where each X_i is a scheme. Observe that $X_i \times_X X_j$ is a scheme too. This covering induces a covering of the final object in each of the three sites $(\text{Spaces}/X)_{fppf}$, $(\text{Spaces}/X)_{étale}$, and $X_{étale}$. Hence we see that the category of sheaves on these sites are equivalent to descent data for these coverings, see Sites, Lemma 7.26.5. Parts (2), (3) are local (because we have the glueing statement). Being quasi-coherent is a local property, hence parts (4), (5) are local. Clearly (6) and (7) are local. It follows that it suffices to prove parts (2) – (7) of the lemma when X is a scheme.

Assume X is a scheme. The embeddings $(Sch/X)_{étale} \subset (\text{Spaces}/X)_{étale}$ and $(Sch/X)_{fppf} \subset (\text{Spaces}/X)_{fppf}$ determine equivalences of ringed topoi by Lemma 84.3.1. We conclude that (2) – (7) follows from the case of schemes. Étale Cohomology, Lemma 59.101.1. To transport the property of being quasi-coherent via this equivalence use that being quasi-coherent is an intrinsic property of modules as explained in Modules on Sites, Section 18.23. Some minor details omitted. ☐

0DGP Lemma 84.7.2. Let S be a scheme. Let X be an algebraic space over S. For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module the maps

$$\pi_X^*\mathcal{F} \longrightarrow R\epsilon_{X,*}(a_X^*\mathcal{F}) \quad \text{and} \quad \mathcal{F} \longrightarrow Ra_{X,*}(a_X^*\mathcal{F})$$

are isomorphisms.

Proof. This is an immediate consequence of parts (6) and (7) of Lemma 84.7.1. ☐

0DGQ Lemma 84.7.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Set

$$\mathcal{H}_{étale} = \text{Ker}(\pi_X^*\mathcal{F}_2 \to \pi_X^*\mathcal{F}_3)/\operatorname{Im}(\pi_X^*\mathcal{F}_1 \to \pi_X^*\mathcal{F}_2)$$

on $(\text{Spaces}/X)_{étale}$ and set

$$\mathcal{H}_{fppf} = \text{Ker}(a_X^*\mathcal{F}_2 \to a_X^*\mathcal{F}_3)/\operatorname{Im}(a_X^*\mathcal{F}_1 \to a_X^*\mathcal{F}_2)$$

on $(\text{Spaces}/X)_{fppf}$. Then $\mathcal{H}_{étale} = \epsilon_{X,*}\mathcal{H}_{fppf}$ and

$$H^p_{étale}(U, \mathcal{H}_{étale}) = H^p_{fppf}(U, \mathcal{H}_{fppf}) = 0$$

for $p > 0$ and any affine object U of $(\text{Spaces}/X)_{étale}$.

More is true, namely the collection of modules on $(\text{Spaces}/X)_{fppf}$ which fppf locally look like those in the lemma are called adquate modules. They form a weak Serre subcategory of the category of all $\mathcal{O}$-modules and their cohomology is studied in Adequate Modules, Section 46.5.

Proof. For any object $f : U \to X$ of $(\text{Spaces}/X)_{étale}$ consider the restriction $\mathcal{H}_{étale}|_{U_{étale}}$ of $\mathcal{H}_{étale}$ to $U_{étale}$ via the functor $i_f^* = i_f^{-1}$ discussed in Section 84.5. The sheaf $\mathcal{H}_{étale}|_{U_{étale}}$ is equal to the homology of complex $f^*\mathcal{F}_\bullet$ in degree 1. This is true because $i_f \circ \pi_X = f$ as morphisms of ringed sites $U_{étale} \to X_{étale}$. In particular we see that $\mathcal{H}_{étale}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. Next, let $g : V \to U$ be a flat morphism in $(\text{Spaces}/X)_{étale}$. Since

$$i_{f\circ g}^* \circ \pi_X^* = (f \circ g)^* = g^* \circ f^*$$

as morphisms of sites $V_{étale} \to X_{étale}$ and since g is flat hence g^* is exact, we obtain

$$\mathcal{H}_{étale}|_{V_{étale}} = g^* \left(\mathcal{H}_{étale}|_{U_{étale}}\right)$$

With these preparations we are ready to prove the lemma.

Let $\mathcal{U} = \{g_i : U_i \to U\}_{i\in I}$ be an fppf covering with $f : U \to X$ as above. The sheaf propery holds for $\mathcal{H}_{étale}$ and the covering $\mathcal{U}$ by (1) of Lemma 84.7.1 applied to $\mathcal{H}_{étale}|_{U_{étale}}$ and the above. Therefore we see that $\mathcal{H}_{étale}$ is already an fppf sheaf and this means that $\mathcal{H}_{fppf}$ is equal to $\mathcal{H}_{étale}$ as a presheaf. In particular $\mathcal{H}_{étale} = \epsilon_{X,*}\mathcal{H}_{fppf}$.

Finally, to prove the vanishing, we use Cohomology on Sites, Lemma 21.10.9. We let $\mathcal{B}$ be the affine objects of $(\text{Spaces}/X)_{fppf}$ and we let Cov be the set of finite fppf coverings $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ with U, U_i affine. We have

$$\check{H}^p(\mathcal{U}, \mathcal{H}_{étale}) = \check{H}^p(\mathcal{U}, (\mathcal{H}_{étale}|_{U_{étale}})^a)$$

because the values of $\mathcal{H}_{étale}$ on the affine schemes $U_{i_0} \times_U \ldots \times_U U_{i_p}$ flat over U agree with the values of the pullback of the quasi-coherent module $\mathcal{H}_{étale}|_{U_{étale}}$ by the first paragraph. Hence we obtain vanishing by Descent, Lemma 35.9.2. This finishes the proof. □

0DGR Lemma 84.7.4. Let S be a scheme. Let X be an algebraic space over S. For $K \in D_{QCoh}(\mathcal{O}_X)$ the maps

$$L\pi_X^*K \longrightarrow R\epsilon_{X,*}(La_X^*K) \quad \text{and} \quad K \longrightarrow Ra_{X,*}(La_X^*K)$$

are isomorphisms. Here $a_X : Sh((\text{Spaces}/X)_{fppf}) \to Sh(X_{étale})$ is as above.

Proof. The question is étale local on X hence we may assume X is affine. Say $X = \text{Spec}(A)$. Then we have $D_{QCoh}(\mathcal{O}_X) = D(A)$ by Derived Categories of Spaces, Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5. Hence we can choose an K-flat complex of A-modules $K^\bullet$ whose corresponding complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules represents K. We claim that $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}_X$-modules.

Proof of the claim. By Derived Categories of Schemes, Lemma 36.3.6 we see that $\widetilde{K}^\bullet$ is K-flat on the scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$. Next, note that $\mathcal{K}^\bullet = \epsilon^*\widetilde{K}^\bullet$ where ϵ is as in Derived Categories of Spaces, Lemma 75.4.2 whence $\mathcal{K}^\bullet$ is K-flat by Cohomology on Sites, Lemma 21.18.7 and the fact that the étale site of a scheme has enough points (Étale Cohomology, Remarks 59.29.11).

By the claim we see that $La_X^*K = a_X^*\mathcal{K}^\bullet$ and $L\pi_X^*K = \pi_X^*\mathcal{K}^\bullet$. Since the first part of the proof shows that the pullback $a_X^*\mathcal{K}^n$ of the quasi-coherent module is acyclic for $\epsilon_{X,*}$, resp. $a_{X,*}$, surely the proof is done by Leray's acyclicity lemma? Actually..., no because Leray's acyclicity lemma only applies to bounded below complexes. However, in the next paragraph we will show the result does follow from the bounded below case because our complex is the derived limit of bounded below complexes of quasi-coherent modules.

The cohomology sheaves of $\pi_X^*\mathcal{K}^\bullet$ and $a_X^*\mathcal{K}^\bullet$ have vanishing higher cohomology groups over affine objects of $(\text{Spaces}/X)_{étale}$ by Lemma 84.7.3. Therefore we have

$$L\pi_X^*K = R\lim \tau_{\geq -n}(L\pi_X^*K) \quad \text{and} \quad La_X^*K = R\lim \tau_{\geq -n}(La_X^*K)$$

by Cohomology on Sites, Lemma 21.23.10.

Proof of $L\pi_X^*K = R\epsilon_{X,*}(La_X^*\mathcal{F})$. By the above we have

$$R\epsilon_{X,*}La_X^*K = R\lim R\epsilon_{X,*}(\tau_{\geq -n}(La_X^*K))$$

by Cohomology on Sites, Lemma 21.23.3. Note that $\tau_{\geq -n}(La_X^*K)$ is represented by $\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)$ which may not be the same as $a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)$. But clearly the systems

$$\{\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)\}_{n\geq 1} \quad \text{and} \quad \{a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)\}_{n\geq 1}$$

are isomorphic as pro-systems. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and the first part of the lemma we see that

$$R\epsilon_{X,*}(a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)) = \pi_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)$$

Then we can use that the systems

$$\{\tau_{\geq -n}(\pi_X^*\mathcal{K}^\bullet)\}_{n\geq 1} \quad \text{and} \quad \{\pi_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)\}_{n\geq 1}$$

are isomorphic as pro-systems. Finally, we put everything together as follows

$$\begin{aligned} R\epsilon_{X,*}La_X^*K &= R\epsilon_{X,*}(R\lim \tau_{\geq -n}(La_X^*K)) \\ &= R\lim R\epsilon_{X,*}(\tau_{\geq -n}(La_X^*K)) \\ &= R\lim R\epsilon_{X,*}(\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)) \\ &= R\lim R\epsilon_{X,*}(a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)) \\ &= R\lim \pi_X^*(\tau_{\geq -n}\mathcal{K}^\bullet) \\ &= R\lim \tau_{\geq -n}(\pi_X^*\mathcal{K}^\bullet) \\ &= R\lim \tau_{\geq -n}(L\pi_X^*K) \\ &= L\pi_X^*K \end{aligned}$$

Here in equalities four and six we have used that isomorphic pro-systems have the same $R\lim$ (small detail omitted). You can avoid this step by using more about cohomology of the terms of the complex $\tau_{\geq -n}a_X^*\mathcal{K}^\bullet$ proved in Lemma 84.7.3 as this will prove directly that $R\epsilon_{X,*}(\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)) = \tau_{\geq -n}(\pi_X^*\mathcal{K}^\bullet)$.

The equality $K = Ra_{X,*}(La_X^*\mathcal{F})$ is proved in exactly the same way using in the final step that $K = R\lim \tau_{\geq -n}K$ by Derived Categories of Spaces, Lemma 75.5.7. $\square$

84.8. Comparing ph and étale topologies

0DGS This section is the analogue of Étale Cohomology, Section 59.102.

Let S be a scheme. Let X be an algebraic space over S. On the category Spaces$/X$ we consider the ph and étale topologies. The identity functor $(\text{Spaces}/X)_{étale} \to (\text{Spaces}/X)_{ph}$ is continuous as every étale covering is a ph covering by Topologies on Spaces, Lemma 73.8.2. Hence it defines a morphism of sites

$$\epsilon_X : (\text{Spaces}/X)_{ph} \longrightarrow (\text{Spaces}/X)_{étale}$$

by an application of Sites, Proposition 7.14.7. Please note that $\epsilon_{X,*}$ is the identity functor on underlying presheaves and that ϵ_X^{-1} associates to an étale sheaf the ph sheafification. Consider the morphism of sites

$$\pi_X : (\text{Spaces}/X)_{étale} \longrightarrow X_{spaces,étale}$$

comparing big and small étale sites, see Section 84.5. The composition determines a morphism of sites

$$a_X = \pi_X \circ \epsilon_X : (\text{Spaces}/X)_{ph} \longrightarrow X_{spaces,étale}$$

If $\mathcal{H}$ is an abelian sheaf on $(\text{Spaces}/X)_{ph}$, then we will write $H^n_{ph}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of $(\text{Spaces}/X)_{ph}$.

0DGT **Lemma 84.8.1.** Let S be a scheme. Let X be an algebraic space over S.

(1) For $\mathcal{F} \in Sh(X_{étale})$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$.
(2) For $\mathcal{F} \in \text{Ab}(X_{étale})$ torsion we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.

Proof. We have $a_X^{-1}\mathcal{F} = \epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$. By Lemma 84.5.1 the étale sheaf $\pi_X^{-1}\mathcal{F}$ is a sheaf for the ph topology and therefore is equal to $a_X^{-1}\mathcal{F}$ (as pulling back by ϵ_X is given by ph sheafification). Recall moreover that $\epsilon_{X,*}$ is the identity on underlying presheaves. Now part (1) is immediate from the explicit description of π_X^{-1} in Lemma 84.5.1.

We will prove part (2) by reducing it to the case of schemes – see part (1) of Étale Cohomology, Lemma 59.102.5. This will "clearly work" as every algebraic space is étale locally a scheme. The details are given below but we urge the reader to skip the proof.

For an abelian sheaf $\mathcal{H}$ on $(\text{Spaces}/X)_{ph}$ the higher direct image $R^p\epsilon_{X,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{ph}(U, \mathcal{H})$ on $(\text{Spaces}/X)_{étale}$. See Cohomology on Sites, Lemma 21.7.4. Since every object of $(\text{Spaces}/X)_{étale}$ has a covering by schemes, it suffices to prove that given U/X a scheme and $\xi \in H^p_{ph}(U, a_X^{-1}\mathcal{F})$ we can find an étale covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. We have

$$\begin{aligned} H^p_{ph}(U, a_X^{-1}\mathcal{F}) &= H^p((\text{Spaces}/U)_{ph}, (a_X^{-1}\mathcal{F})|_{\text{Spaces}/U}) \\ &= H^p((Sch/U)_{ph}, (a_X^{-1}\mathcal{F})|_{Sch/U}) \end{aligned}$$

where the second identification is Lemma 84.3.1 and the first is a general fact about restriction (Cohomology on Sites, Lemma 21.7.1). Looking at the first paragraph and the corresponding result in the case of schemes (Étale Cohomology, Lemma 59.102.1) we conclude that the sheaf $(a_X^{-1}\mathcal{F})|_{Sch/U}$ matches the pullback by the "schemes version of a_U". Therefore we can find an étale covering $\{U_i \to U\}$ such that our class dies in $H^p((Sch/U_i)_{ph}, (a_X^{-1}\mathcal{F})|_{Sch/U_i})$ for each i, see Étale Cohomology, Lemma 59.102.5 (the precise statement one should use here is that V_n holds for all n which is the statement of part (2) for the case of schemes). Transporting back (using the same formulas as above but now for U_i) we conclude ξ restricts to zero over U_i as desired. $\square$

The hard work done in the case of schemes now tells us that étale and ph cohomology agree for torsion abelian sheaves coming from the small étale site.

0DGU Lemma 84.8.2. Let S be a scheme. Let X be an algebraic space over S. For $K \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves the maps

$$\pi_X^{-1}K \longrightarrow R\epsilon_{X,*}a_X^{-1}K \quad \text{and} \quad K \longrightarrow Ra_{X,*}a_X^{-1}K$$

are isomorphisms with $a_X : Sh((\text{Spaces}/X)_{ph}) \to Sh(X_{\acute{e}tale})$ as above.

Proof. We only prove the second statement; the first is easier and proved in exactly the same manner. There is a reduction to the case where K is given by a single torsion abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 21.19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^qa_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 84.8.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^qa_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 84.8.1 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$. We have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and by Lemma 84.5.5 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. $\square$

0DGV Lemma 84.8.3. Let S be a scheme and let X be an algebraic space over S. With $a_X : Sh((\text{Spaces}/X)_{ph}) \to Sh(X_{\acute{e}tale})$ as above:

(1) $H^q(X_{\acute{e}tale}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,
(2) $H^q(X_{\acute{e}tale}, K) = H^q_{ph}(X, a_X^{-1}K)$ for $K \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves

Example: if A is a torsion abelian group, then $H^q_{\acute{e}tale}(X, \underline{A}) = H^q_{ph}(X, \underline{A})$.

Proof. This follows from Lemma 84.8.2 by Cohomology on Sites, Remark 21.14.4. $\square$

0DGW Lemma 84.8.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there are commutative diagrams of topoi

$$\begin{tikzcd} Sh((\text{Spaces}/X)_{ph}) \arrow[r, "f_{big,ph}"'] \arrow[d, "\epsilon_X"'] & Sh((\text{Spaces}/Y)_{ph}) \arrow[d, "\epsilon_Y"] \\ Sh((\text{Spaces}/X)_{\acute{e}tale}) \arrow[r, "f_{big,\acute{e}tale}"] & Sh((\text{Spaces}/Y)_{\acute{e}tale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((\text{Spaces}/X)_{ph}) \arrow[r, "f_{big,ph}"'] \arrow[d, "a_X"'] & Sh((\text{Spaces}/Y)_{ph}) \arrow[d, "a_Y"] \\ Sh(X_{\acute{e}tale}) \arrow[r, "f_{small}"] & Sh(Y_{\acute{e}tale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. This follows immediately from working out the definitions of the morphisms involved, see Topologies on Spaces, Section 73.8 and Section 84.5. □

0DGX Lemma 84.8.5. In Lemma 84.8.4 if f is proper, then we have

(1) $a_Y^{-1} \circ f_{small,*} = f_{big,ph,*} \circ a_X^{-1}$, and
(2) $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$ for K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves.

Proof. Proof of (1). You can prove this by repeating the proof of Lemma 84.5.6 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. Lemma 84.8.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1}f_{small,*}\mathcal{F} \to f_{big,ph,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1}f_{small,*}\mathcal{F} &= \epsilon_{Y,*}a_Y^{-1}f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*}f_{big,ph,*}a_X^{-1}\mathcal{F} \\ &= f_{big,\acute{e}tale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} \\ &= f_{big,\acute{e}tale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 84.5.6.

To see (2) we use that

$$\begin{aligned} R\epsilon_{Y,*}Rf_{big,ph,*}a_X^{-1}K &= Rf_{big,\acute{e}tale,*}R\epsilon_{X,*}a_X^{-1}K \\ &= Rf_{big,\acute{e}tale,*}\pi_X^{-1}K \\ &= \pi_Y^{-1}Rf_{small,*}K \\ &= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \end{aligned}$$

The first equality by the commutative diagram in Lemma 84.8.4 and Cohomology on Sites, Lemma 21.19.2. Then second equality is Lemma 84.8.2. The third is Lemma 84.5.6 part (2). The fourth is Lemma 84.8.2 again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,ph,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,ph,*}a_X^{-1}K$$

The proof is finished by the following remark: consider a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{\acute{e}tale})$ having torsion cohomology sheaves and M in $D^+((\text{Spaces}/Y)_{ph})$. If $R\epsilon_{Y,*}\alpha$ is an isomorphism, then α is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by Lemma 84.8.1 because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

84.9. Other chapters

Preliminaries

CHAPTER 85

Simplicial Spaces

09VI 85.1. Introduction

09VJ This chapter develops some theory concerning simplicial topological spaces, simplicial ringed spaces, simplicial schemes, and simplicial algebraic spaces. The theory of simplicial spaces sometimes allows one to prove local to global principles which appear difficult to prove in other ways. Some example applications can be found in the papers [Fal03], [Kie72], and [Del74].

We assume throughout that the reader is familiar with the basic concepts and results of the chapter Simplicial Methods, see Simplicial, Section 14.1. In particular, we continue to write X and not $X_\bullet$ for a simplicial object.

85.2. Simplicial topological spaces

09VK A simplicial space is a simplicial object in the category of topological spaces where morphisms are continuous maps of topological spaces. (We will use "simplicial algebraic space" to refer to simplicial objects in the category of algebraic spaces.) We may picture a simplicial space X as follows

$$X_2 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} X_1 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} X_0$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. It is important to keep in mind that $d^n_i : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s^n_j : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

Let X be a simplicial space. We associate a site X_{Zar}[1] to X as follows.

(1) An object of X_{Zar} is an open U of X_n for some n,
(2) a morphism $U \to V$ of X_{Zar} is given by a $\varphi : [m] \to [n]$ where n, m are such that $U \subset X_n$, $V \subset X_m$ and φ is such that $X(\varphi)(U) \subset V$, and
(3) a covering $\{U_i \to U\}$ in X_{Zar} means that $U, U_i \subset X_n$ are open, the maps $U_i \to U$ are given by $\text{id} : [n] \to [n]$, and $U = \bigcup U_i$.

Note that in particular, if $U \to V$ is a morphism of X_{Zar} given by φ, then $X(\varphi) : X_n \to X_m$ does in fact induce a continuous map $U \to V$ of topological spaces.
It is clear that the above is a special case of a construction that associates to any diagram of topological spaces a site. We formulate the obligatory lemma.

09VL **Lemma 85.2.1.** Let X be a simplicial space. Then X_{Zar} as defined above is a site.

Proof. Omitted. □

[1]This notation is similar to the notation in Sites, Example 7.6.4 and Topologies, Definition 34.3.7.

Let X be a simplicial space. Let $\mathcal{F}$ be a sheaf on X_{Zar}. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the opens of X_n defines a sheaf $\mathcal{F}_n$ on the topological space X_n. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ for pairs $U \subset X_n$, $V \subset X_m$ with $X(\varphi)(U) \subset V$, define an $X(\varphi)$-map $\mathcal{F}(\varphi) : \mathcal{F}_m \to \mathcal{F}_n$, see Sheaves, Definition 6.21.7. Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$\mathcal{F}(\varphi) \circ \mathcal{F}(\psi) = \mathcal{F}(\varphi \circ \psi)$$

(LHS uses composition of f-maps, see Sheaves, Definition 6.21.9). Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n \geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \text{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on X_{Zar}.

09VM Lemma 85.2.2. Let X be a simplicial space. There is an equivalence of categories between

(1) $Sh(X_{Zar})$, and
(2) category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.

Proof. See discussion above. $\square$

09VN Lemma 85.2.3. Let $f : Y \to X$ be a morphism of simplicial spaces. Then the functor $u : X_{Zar} \to Y_{Zar}$ which associates to the open $U \subset X_n$ the open $f_n^{-1}(U) \subset Y_n$ defines a morphism of sites $f_{Zar} : Y_{Zar} \to X_{Zar}$.

Proof. It is clear that u is a continuous functor. Hence we obtain functors $f_{Zar,*} = u^s$ and $f_{Zar}^{-1} = u_s$, see Sites, Section 7.14. To see that we obtain a morphism of sites we have to show that u_s is exact. We will use Sites, Lemma 7.14.6 to see this. Let $V \subset Y_n$ be an open subset. The category $\mathcal{I}_V^u$ (see Sites, Section 7.5) consists of pairs (U, φ) where $\varphi : [m] \to [n]$ and $U \subset X_m$ open such that $Y(\varphi)(V) \subset f_m^{-1}(U)$. Moreover, a morphism $(U, \varphi) \to (U', \varphi')$ is given by a $\psi : [m'] \to [m]$ such that $X(\psi)(U) \subset U'$ and $\varphi \circ \psi = \varphi'$. It is our task to show that $\mathcal{I}_V^u$ is cofiltered.

We verify the conditions of Categories, Definition 4.20.1. Condition (1) holds because $(X_n, \text{id}_{[n]})$ is an object. Let (U, φ) be an object. The condition $Y(\varphi)(V) \subset f_m^{-1}(U)$ is equivalent to $V \subset f_n^{-1}(X(\varphi)^{-1}(U))$. Hence we obtain a morphism $(X(\varphi)^{-1}(U), \text{id}_{[n]}) \to (U, \varphi)$ given by setting $\psi = \varphi$. Moreover, given a pair of objects of the form $(U, \text{id}_{[n]})$ and $(U', \text{id}_{[n]})$ we see there exists an object, namely $(U \cap U', \text{id}_{[n]})$, which maps to both of them. Thus condition (2) holds. To verify condition (3) suppose given two morphisms $a, a' : (U, \varphi) \to (U', \varphi')$ given by $\psi, \psi' : [m'] \to [m]$. Then precomposing with the morphism $(X(\varphi)^{-1}(U), \text{id}_{[n]}) \to (U, \varphi)$ given by φ equalizes a, a' because $\varphi \circ \psi = \varphi' = \varphi \circ \psi'$. This finishes the proof. $\square$

09VP Lemma 85.2.4. Let $f : Y \to X$ be a morphism of simplicial spaces. In terms of the description of sheaves in Lemma 85.2.2 the morphism f_{Zar} of Lemma 85.2.3 can be described as follows.

(1) If $\mathcal{G}$ is a sheaf on Y, then $(f_{Zar,*}\mathcal{G})_n = f_{n,*}\mathcal{G}_n$.
(2) If $\mathcal{F}$ is a sheaf on X, then $(f_{Zar}^{-1}\mathcal{F})_n = f_n^{-1}\mathcal{F}_n$.

Proof. The first part is immediate from the definitions. For the second part, note that in the proof of Lemma 85.2.3 we have shown that for a $V \subset Y_n$ open the category $(\mathcal{I}_V^u)^{opp}$ contains as a cofinal subcategory the category of opens $U \subset X_n$ with $f_n^{-1}(U) \supset V$ and morphisms given by inclusions. Hence we see that the restriction of $u_p\mathcal{F}$ to opens of Y_n is the presheaf $f_{n,p}\mathcal{F}_n$ as defined in Sheaves, Lemma 6.21.3. Since $f_{Zar}^{-1}\mathcal{F} = u_s\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$ and since sheafification uses only coverings and since coverings in Y_{Zar} use only inclusions between opens on the same Y_n, the result follows from the fact that $f_n^{-1}\mathcal{F}_n$ is (correspondingly) the sheafification of $f_{n,p}\mathcal{F}_n$, see Sheaves, Section 6.21. □

Let X be a topological space. In Sites, Example 7.6.4 we denoted X_{Zar} the site consisting of opens of X with inclusions as morphisms and coverings given by open coverings. We identify the topos $Sh(X_{Zar})$ with the category of sheaves on X.

09W0 Lemma 85.2.5. Let X be a simplicial space. The functor $X_{n,Zar} \to X_{Zar}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g_n : Sh(X_n) \to Sh(X_{Zar})$ satisfies

(1) g_n^{-1} associates to the sheaf $\mathcal{F}$ on X the sheaf $\mathcal{F}_n$ on X_n,
(2) $g_n^{-1} : Sh(X_{Zar}) \to Sh(X_n)$ has a left adjoint $g_{n!}^{Sh}$,
(3) $g_{n!}^{Sh}$ commutes with finite connected limits,
(4) $g_n^{-1} : \mathrm{Ab}(X_{Zar}) \to \mathrm{Ab}(X_n)$ has a left adjoint $g_{n!}$, and
(5) $g_{n!}$ is exact.

Proof. Besides the properties of our functor mentioned in the statement, the category $X_{n,Zar}$ has fibre products and equalizers and the functor commutes with them (beware that X_{Zar} does not have all fibre products). Hence the lemma follows from the discussion in Sites, Sections 7.20 and 7.21 and Modules on Sites, Section 18.16. More precisely, Sites, Lemmas 7.21.1, 7.21.5, and 7.21.6 and Modules on Sites, Lemmas 18.16.2 and 18.16.3. □

09W1 Lemma 85.2.6. Let X be a simplicial space. If $\mathcal{I}$ is an injective abelian sheaf on X_{Zar}, then $\mathcal{I}_n$ is an injective abelian sheaf on X_n.

Proof. This follows from Homology, Lemma 12.29.1 and Lemma 85.2.5. □

09W2 Lemma 85.2.7. Let $f : Y \to X$ be a morphism of simplicial spaces. Then

$$\begin{tikzcd} Sh(Y_n) \arrow[r, "f_n"'] \arrow[d] & Sh(X_n) \arrow[d] \\ Sh(Y_{Zar}) \arrow[r, "f_{Zar}"] & Sh(X_{Zar}) \end{tikzcd}$$

is a commutative diagram of topoi.

Proof. Direct from the description of pullback functors in Lemmas 85.2.4 and 85.2.5. □

09W4 Lemma 85.2.8. Let Y be a simplicial space and let $a : Y \to X$ be an augmentation (Simplicial, Definition 14.20.1). Let $a_n : Y_n \to X$ be the corresponding morphisms of topological spaces. There is a canonical morphism of topoi

$$a : Sh(Y_{Zar}) \to Sh(X)$$

with the following properties:

(1) $a^{-1}\mathcal{F}$ is the sheaf restricting to $a_n^{-1}\mathcal{F}$ on Y_n,
(2) $a_m \circ Y(\varphi) = a_n$ for all $\varphi : [m] \to [n]$,
(3) $a \circ g_n = a_n$ as morphisms of topoi with g_n as in Lemma 85.2.5,
(4) $a_*\mathcal{G}$ for $\mathcal{G} \in Sh(Y_{Zar})$ is the equalizer of the two maps $a_{0,*}\mathcal{G}_0 \to a_{1,*}\mathcal{G}_1$.

Proof. Part (2) holds for augmentations of simplicial objects in any category. Thus $Y(\varphi)^{-1}a_m^{-1}\mathcal{F} = a_n^{-1}\mathcal{F}$ which defines an $Y(\varphi)$-map from $a_m^{-1}\mathcal{F}$ to $a_n^{-1}\mathcal{F}$. Thus we can use (1) as the definition of $a^{-1}\mathcal{F}$ (using Lemma 85.2.2) and (4) as the definition of a_*. If this defines a morphism of topoi then part (3) follows because we'll have $g_n^{-1} \circ a^{-1} = a_n^{-1}$ by construction. To check a is a morphism of topoi we have to show that a^{-1} is left adjoint to a_* and we have to show that a^{-1} is exact. The last fact is immediate from the exactness of the functors a_n^{-1}.

Let $\mathcal{F}$ be an object of $Sh(X)$ and let $\mathcal{G}$ be an object of $Sh(Y_{Zar})$. Given $\beta : a^{-1}\mathcal{F} \to \mathcal{G}$ we can look at the components $\beta_n : a_n^{-1}\mathcal{F} \to \mathcal{G}_n$. These maps are adjoint to maps $\beta_n : \mathcal{F} \to a_{n,*}\mathcal{G}_n$. Compatibility with the simplicial structure shows that β_0 maps into $a_*\mathcal{G}$. Conversely, suppose given a map $\alpha : \mathcal{F} \to a_*\mathcal{G}$. For any n choose a $\varphi : [0] \to [n]$. Then we can look at the composition

$$\mathcal{F} \xrightarrow{\alpha} a_*\mathcal{G} \to a_{0,*}\mathcal{G}_0 \xrightarrow{\mathcal{G}(\varphi)} a_{n,*}\mathcal{G}_n$$

These are adjoint to maps $a_n^{-1}\mathcal{F} \to \mathcal{G}_n$ which define a morphism of sheaves $a^{-1}\mathcal{F} \to \mathcal{G}$. We omit the proof that the constructions given above define mutually inverse bijections

$$\mathrm{Mor}_{Sh(Y_{Zar})}(a^{-1}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, a_*\mathcal{G})$$

This finishes the proof. An interesting observation is here that this morphism of topoi does not correspond to any obvious geometric functor between the sites defining the topoi. $\square$

09W5 Lemma 85.2.9. Let X be a simplicial topological space. The complex of abelian presheaves on X_{Zar}

$$\ldots \to \mathbf{Z}_{X_2} \to \mathbf{Z}_{X_1} \to \mathbf{Z}_{X_0}$$

with boundary $\sum (-1)^i d^n_i$ is a resolution of the constant presheaf $\mathbf{Z}$.

Proof. Let $U \subset X_m$ be an object of X_{Zar}. Then the value of the complex above on U is the complex of abelian groups

$$\ldots \to \mathbf{Z}[\mathrm{Mor}_\Delta([2],[m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([1],[m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([0],[m])]$$

In other words, this is the complex associated to the free abelian group on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.26.7, and since "taking free abelian groups" is a functor, we see that the complex above is homotopy equivalent to the free abelian group on $\Delta[0]$ (Simplicial, Remark 14.26.4 and Lemma 14.27.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. $\square$

09W6 Lemma 85.2.10. Let X be a simplicial topological space. Let $\mathcal{F}$ be an abelian sheaf on X. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X_{Zar}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \mathcal{I}^q(X_p)$$

and first differential given by the alternating sum along the maps d^{p+1}_i-maps $\mathcal{I}^q_p \to \mathcal{I}^q_{p+1}$, see Lemma 85.2.2. Note that

$$A^{p,q} = \Gamma(X_p, \mathcal{I}^q_p) = \mathrm{Mor}_{\mathrm{PSh}}(h_{X_p}, \mathcal{I}^q) = \mathrm{Mor}_{\mathrm{PAb}}(\mathbf{Z}_{X_p}, \mathcal{I}^q)$$

Hence it follows from Lemma 85.2.9 and Cohomology on Sites, Lemma 21.10.1 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(X_{Zar}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 85.2.5) the map

$$\mathcal{F}_p \to \mathcal{I}_p^\bullet$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on X_p (Lemma 85.2.6). Hence the cohomology of the columns computes the groups $H^q(X_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 12.25.3 and 12.25.4. $\square$

0D84 Lemma 85.2.11. Let X be a simplicial space and let $a : X \to Y$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on X_{Zar}. Then $R^n a_* \mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto H^n((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$$

Proof. This is the analogue of Cohomology, Lemma 20.7.3 or of Cohomology on Sites, Lemma 21.7.4 and we strongly encourge the reader to skip the proof. Choosing an injective resolution of $\mathcal{F}$ on X_{Zar} and using the definitions we see that it suffices to show: (1) the restriction of an injective abelian sheaf on X_{Zar} to $(X \times_Y V)_{Zar}$ is an injective abelian sheaf and (2) $a_*\mathcal{F}$ is equal to the rule

$$V \longmapsto H^0((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$$

Part (2) follows from the following facts

(2a) $a_*\mathcal{F}$ is the equalizer of the two maps $a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1$ by Lemma 85.2.8,
(2b) $a_{0,*}\mathcal{F}_0(V) = H^0(a_0^{-1}(V), \mathcal{F}_0)$ and $a_{1,*}\mathcal{F}_1(V) = H^0(a_1^{-1}(V), \mathcal{F}_1)$,
(2c) $X_0 \times_Y V = a_0^{-1}(V)$ and $X_1 \times_Y V = a_1^{-1}(V)$,
(2d) $H^0((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$ is the equalizer of the two maps $H^0(X_0 \times_Y V, \mathcal{F}_0) \to H^0(X_1 \times_Y V, \mathcal{F}_1)$ for example by Lemma 85.2.10.

Part (1) follows after one defines an exact left adjoint $j_! : \text{Ab}((X \times_Y V)_{Zar}) \to \text{Ab}(X_{Zar})$ (extension by zero) to restriction $\text{Ab}(X_{Zar}) \to \text{Ab}((X \times_Y V)_{Zar})$ and using Homology, Lemma 12.29.1. $\square$

Let X be a topological space. Denote $X_\bullet$ the constant simplicial topological space with value X. By Lemma 85.2.2 a sheaf on $X_{\bullet,Zar}$ is the same thing as a cosimplicial object in the category of sheaves on X.

09W3 Lemma 85.2.12. Let X be a topological space. Let $X_\bullet$ be the constant simplicial topological space with value X. The functor

$$X_{\bullet,Zar} \longrightarrow X_{Zar}, \quad U \longmapsto U$$

is continuous and cocontinuous and defines a morphism of topoi $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ as well as a left adjoint $g_!$ to g^{-1}. We have

(1) g^{-1} associates to a sheaf on X the constant cosimplicial sheaf on X,
(2) $g_!$ associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the sheaf $\mathcal{F}_0$, and
(3) g_* associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the equalizer of the two maps $\mathcal{F}_0 \to \mathcal{F}_1$.

Proof. The statements about the functor are straightforward to verify. The existence of g and $g_!$ follow from Sites, Lemmas 7.21.1 and 7.21.5. The description of g^{-1} is immediate from Sites, Lemma 7.21.5. The description of g_* and $g_!$ follows as the functors given are right and left adjoint to g^{-1}. $\square$

85.3. Simplicial sites and topoi

09WB It seems natural to define a simplicial site as a simplicial object in the (big) category whose objects are sites and whose morphisms are morphisms of sites. See Sites, Definitions 7.6.2 and 7.14.1 with composition of morphisms as in Sites, Lemma 7.14.4. But here are some variants one might want to consider: (a) we could work with cocontinuous functors (see Sites, Sections 7.20 and 7.21) between sites instead, (b) we could work in a suitable 2-category of sites where one introduces the notion of a 2-morphism between morphisms of sites, (c) we could work in a 2-category constructed out of cocontinuous functors. Instead of picking one of these variants as a definition we will simply develop theory as needed.

Certainly a simplicial topos should probably be defined as a pseudo-functor from Δ^{opp} into the 2-category of topoi. See Categories, Definition 4.29.5 and Sites, Section 7.15 and 7.36. We will try to avoid working with such a beast if possible.

Case A. Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. This morphism is given by a continuous functor in the opposite direction which we will denote $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

09WC **Lemma 85.3.1.** *Let $\mathcal{C}$ be a simplicial object in the category of sites. With notation as above we construct a site $\mathcal{C}_{total}$ as follows.*

1. *An object of $\mathcal{C}_{total}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{total}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : U \to u_\varphi(V)$ of $\mathcal{C}_n$, and*
3. *a covering $\{(\mathrm{id}, f_i) : U_i \to U\}$ in $\mathcal{C}_{total}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, u_\varphi(g) \circ f)$. This uses that $u_\varphi \circ u_\psi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,u_\varphi(U),u_\varphi(f_i)} u_\varphi(U_i)$$

in the category $\mathcal{C}_{total}$. This makes sense as by our definition of morphisms of sites, the required fibre products in $\mathcal{C}_m$ exist since u_φ transforms coverings into coverings. The same reasoning implies the claim (details omitted). Thus we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. $\square$

Case B. Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor denoted $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. The associated morphism of topoi is denoted $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

09WD **Lemma 85.3.2.** *Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. With notation as above, assume the functors $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ have property P of Sites, Remark 7.20.5. Then we can construct a site $\mathcal{C}_{total}$ as follows.*

1. *An object of $\mathcal{C}_{total}$ is an object U of $\mathcal{C}_n$ for some n,*

(2) a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{total}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : u_\varphi(U) \to V$ of $\mathcal{C}_m$, and
(3) a covering $\{(\mathrm{id}, f_i) : U_i \to U\}$ in $\mathcal{C}_{total}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, g \circ u_\psi(f))$. This uses that $u_\psi \circ u_\varphi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,U,f_i} U_i$$

in the category $\mathcal{C}_{total}$ where the right hand side is the object of $\mathcal{C}_m$ defined in Sites, Remark 7.20.5 which exists by property P. Compatibility of this type of fibre product with compositions of functors implies the claim (details omitted). Since the family $\{W \times_{g,U,f_i} U_i \to W\}$ is a covering of $\mathcal{C}_m$ by property P we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. $\square$

09WE Situation 85.3.3. Here we have one of the following two cases:

(A) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. For every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ given by a continuous functor $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.
(B) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors having property P of Sites, Remark 7.20.5. For every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ which induces a morphism of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

As usual we will denote f_φ^{-1} and $f_{\varphi,*}$ the pullback and pushforward. We let $\mathcal{C}_{total}$ denote the site defined in Lemma 85.3.1 (case A) or Lemma 85.3.2 (case B).

Let $\mathcal{C}$ be as in Situation 85.3.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}_{total}$. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the objects of $\mathcal{C}_n$ defines a sheaf $\mathcal{F}_n$ on the site $\mathcal{C}_n$. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ along the morphisms $(\varphi, f) : U \to V$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$ and $V \in \mathrm{Ob}(\mathcal{C}_m)$ define an element $\mathcal{F}(\varphi)$ of

$$\mathrm{Mor}_{Sh(\mathcal{C}_m)}(\mathcal{F}_m, f_{\varphi,*}\mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_n)}(f_\varphi^{-1}\mathcal{F}_m, \mathcal{F}_n)$$

Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ the diagrams

$$\begin{tikzcd} \mathcal{F}_l \arrow[rr, "\mathcal{F}(\varphi\circ\psi)"] \arrow[dr, "\mathcal{F}(\psi)"'] & & f_{\varphi\circ\psi,*}\mathcal{F}_n \\ & f_{\psi,*}\mathcal{F}_m \arrow[ur, "f_{\psi,*}\mathcal{F}(\varphi)"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f_{\varphi\circ\psi}^{-1}\mathcal{F}_l \arrow[rr, "\mathcal{F}(\varphi\circ\psi)"] \arrow[dr, "f_\varphi^{-1}\mathcal{F}(\psi)"'] & & \mathcal{F}_n \\ & f_\varphi^{-1}\mathcal{F}_m \arrow[ur, "\mathcal{F}(\varphi)"'] & \end{tikzcd}$$

commute. Clearly, the converse statement is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi\in\mathrm{Arrows}(\Delta)})$ satisfying the commutativity constraints above, then we obtain a sheaf on $\mathcal{C}_{total}$.

09WF Lemma 85.3.4. In Situation 85.3.3 there is an equivalence of categories between

(1) $Sh(\mathcal{C}_{total})$, and

(2) the category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.

In particular, the topos $Sh(\mathcal{C}_{total})$ only depends on the topoi $Sh(\mathcal{C}_n)$ and the morphisms of topoi f_φ.

Proof. See discussion above. □

09WG Lemma 85.3.5. In Situation 85.3.3 the functor $\mathcal{C}_n \to \mathcal{C}_{total}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ satisfies

(1) g_n^{-1} associates to the sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ the sheaf $\mathcal{F}_n$ on $\mathcal{C}_n$,
(2) $g_n^{-1} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}_n)$ has a left adjoint $g_{n!}^{Sh}$,
(3) for $\mathcal{G}$ in $Sh(\mathcal{C}_n)$ the restriction of $g_{n!}^{Sh}\mathcal{G}$ to $\mathcal{C}_m$ is $\coprod_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$,
(4) $g_{n!}^{Sh}$ commutes with finite connected limits,
(5) $g_n^{-1} : \text{Ab}(\mathcal{C}_{total}) \to \text{Ab}(\mathcal{C}_n)$ has a left adjoint $g_{n!}$,
(6) for $\mathcal{G}$ in $\text{Ab}(\mathcal{C}_n)$ the restriction of $g_{n!}\mathcal{G}$ to $\mathcal{C}_m$ is $\bigoplus_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$, and
(7) $g_{n!}$ is exact.

Proof. Case A. If $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_n$ then the image $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_{total}$ by definition (Lemma 85.3.1). For a morphism $V \to U$ of $\mathcal{C}_n$, the fibre product $V \times_U U_i$ in $\mathcal{C}_n$ is also the fibre product in $\mathcal{C}_{total}$ (by the claim in the proof of Lemma 85.3.1). Therefore our functor is continuous. On the other hand, our functor defines a bijection between coverings of U in $\mathcal{C}_n$ and coverings of U in $\mathcal{C}_{total}$. Therefore it is certainly the case that our functor is cocontinuous.

Case B. If $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_n$ then the image $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_{total}$ by definition (Lemma 85.3.2). For a morphism $V \to U$ of $\mathcal{C}_n$, the fibre product $V \times_U U_i$ in $\mathcal{C}_n$ is also the fibre product in $\mathcal{C}_{total}$ (by the claim in the proof of Lemma 85.3.2). Therefore our functor is continuous. On the other hand, our functor defines a bijection between coverings of U in $\mathcal{C}_n$ and coverings of U in $\mathcal{C}_{total}$. Therefore it is certainly the case that our functor is cocontinuous.

At this point part (1) and the existence of $g_{n!}^{Sh}$ and $g_{n!}$ in cases A and B follows from Sites, Lemmas 7.21.1 and 7.21.5 and Modules on Sites, Lemma 18.16.2.

Proof of (3). Let $\mathcal{G}$ be a sheaf on $\mathcal{C}_n$. Consider the sheaf $\mathcal{H}$ on $\mathcal{C}_{total}$ whose degree m part is the sheaf

$$\mathcal{H}_m = \coprod\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

given in part (3) of the statement of the lemma. Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by the identifications

$$f_\psi^{-1} f_\varphi^{-1}\mathcal{G} \to f_{\psi\circ\varphi}^{-1}\mathcal{G}$$

Observe that given a map $\alpha : \mathcal{H} \to \mathcal{F}$ of sheaves on $\mathcal{C}_{total}$ we obtain a map $\mathcal{G} \to \mathcal{F}_n$ corresponding to the restriction of α_n to the component $\mathcal{G}$ in $\mathcal{H}_n$. Conversely, given a map $\beta : \mathcal{G} \to \mathcal{F}_n$ of sheaves on $\mathcal{C}_n$ we can define $\alpha : \mathcal{H} \to \mathcal{F}$ by letting α_m be the map which on components

$$f_\varphi^{-1}\mathcal{G} \to \mathcal{F}_m$$

uses the maps adjoint to $\mathcal{F}(\varphi) \circ f_\varphi^{-1}\beta$. We omit the arguments showing these two constructions give mutually inverse maps

$$\text{Mor}_{Sh(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \text{Mor}_{Sh(\mathcal{C}_{total})}(\mathcal{H}, \mathcal{F})$$

Thus $\mathcal{H} = g_{n!}^{Sh}\mathcal{G}$ as desired.

Proof of (4). If $\mathcal{G}$ is an abelian sheaf on $\mathcal{C}_n$, then we proceed in exactly the same ammner as above, except that we define $\mathcal{H}$ is the abelian sheaf on $\mathcal{C}_{total}$ whose degree m part is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

with transition maps defined exactly as above. The bijection

$$\mathrm{Mor}_{\mathrm{Ab}(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{\mathrm{Ab}(\mathcal{C}_{total})}(\mathcal{H}, \mathcal{F})$$

is proved exactly as above. Thus $\mathcal{H} = g_{n!}\mathcal{G}$ as desired.

The exactness properties of $g_{n!}^{Sh}$ and $g_{n!}$ follow from formulas given for these functors. $\square$

09WH Lemma 85.3.6. In Situation 85.3.3. If $\mathcal{I}$ is injective in $\mathrm{Ab}(\mathcal{C}_{total})$, then $\mathcal{I}_n$ is injective in $\mathrm{Ab}(\mathcal{C}_n)$. If $\mathcal{I}^\bullet$ is a K-injective complex in $\mathrm{Ab}(\mathcal{C}_{total})$, then $\mathcal{I}_n^\bullet$ is K-injective in $\mathrm{Ab}(\mathcal{C}_n)$.

Proof. The first statement follows from Homology, Lemma 12.29.1 and Lemma 85.3.5. The second statement from Derived Categories, Lemma 13.31.9 and Lemma 85.3.5. $\square$

85.4. Augmentations of simplicial sites

0D93 We continue in the fashion described in Section 85.3 working out the meaning of augmentations in cases A and B treated in that section.

0D6Z Remark 85.4.1. In Situation 85.3.3 an augmentation a_0 towards a site $\mathcal{D}$ will mean

(A) $a_0 : \mathcal{C}_0 \to \mathcal{D}$ is a morphism of sites given by a continuous functor $u_0 : \mathcal{D} \to \mathcal{C}_0$ such that for all $\varphi, \psi : [0] \to [n]$ we have $u_\varphi \circ u_0 = u_\psi \circ u_0$.
(B) $a_0 : Sh(\mathcal{C}_0) \to Sh(\mathcal{D})$ is a morphism of topoi given by a cocontinuous functor $u_0 : \mathcal{C}_0 \to \mathcal{D}$ such that for all $\varphi, \psi : [0] \to [n]$ we have $u_0 \circ u_\varphi = u_0 \circ u_\psi$.

0D70 Lemma 85.4.2. In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. Then a_0 induces

(1) a morphism of topoi $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$ for all $n \geq 0$,
(2) a morphism of topoi $a : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{D})$

such that

(1) for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$,
(2) if $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ is as in Lemma 85.3.5, then $a \circ g_n = a_n$, and
(3) $a_*\mathcal{F}$ for $\mathcal{F} \in Sh(\mathcal{C}_{total})$ is the equalizer of the two maps $a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1$.

Proof. Case A. Let $u_n : \mathcal{D} \to \mathcal{C}_n$ be the common value of the functors $u_\varphi \circ u_0$ for $\varphi : [0] \to [n]$. Then u_n corresponds to a morphism of sites $a_n : \mathcal{C}_n \to \mathcal{D}$, see Sites, Lemma 7.14.4. The same lemma shows that for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$.

Case B. Let $u_n : \mathcal{C}_n \to \mathcal{D}$ be the common value of the functors $u_0 \circ u_\varphi$ for $\varphi : [0] \to [n]$. Then u_n is cocontinuous and hence defines a morphism of topoi $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$, see Sites, Lemma 7.21.2. The same lemma shows that for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$.

Consider the functor $a^{-1} : Sh(\mathcal{D}) \to Sh(\mathcal{C}_{total})$ which to a sheaf of sets $\mathcal{G}$ associates the sheaf $\mathcal{F} = a^{-1}\mathcal{G}$ whose components are $a_n^{-1}\mathcal{G}$ and whose transition maps $\mathcal{F}(\varphi)$ are the identifications

$$f_\varphi^{-1}\mathcal{F}_m = f_\varphi^{-1}a_m^{-1}\mathcal{G} = a_n^{-1}\mathcal{G} = \mathcal{F}_n$$

for $\varphi : [m] \to [n]$, see the description of $Sh(\mathcal{C}_{total})$ in Lemma 85.3.4. Since the functors a_n^{-1} are exact, a^{-1} is an exact functor. Finally, for $a_* : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{D})$ we take the functor which to a sheaf $\mathcal{F}$ on $Sh(\mathcal{D})$ associates

$$a_*\mathcal{F} = \text{Equalizer}(a_{0,*}\mathcal{F}_0 \rightrightarrows a_{1,*}\mathcal{F}_1)$$

Here the two maps come from the two maps $\varphi : [0] \to [1]$ via

$$a_{0,*}\mathcal{F}_0 \to a_{0,*}f_{\varphi,*}f_\varphi^{-1}\mathcal{F}_0 \xrightarrow{\mathcal{F}(\varphi)} a_{0,*}f_{\varphi,*}\mathcal{F}_1 = a_{1,*}\mathcal{F}_1$$

where the first arrow comes from $1 \to f_{\varphi,*}f_\varphi^{-1}$. Let $\mathcal{G}_\bullet$ denote the constant simplicial sheaf with value $\mathcal{G}$ and let $a_{\bullet,*}\mathcal{F}$ denote the simplicial sheaf having $a_{n,*}\mathcal{F}_n$ in degree n. By the usual adjuntion for the morphisms of topoi a_n we see that a map $a^{-1}\mathcal{G} \to \mathcal{F}$ is the same thing as a map

$$\mathcal{G}_\bullet \longrightarrow a_{\bullet,*}\mathcal{F}$$

of simplicial sheaves. By Simplicial, Lemma 14.20.2 this is the same thing as a map $\mathcal{G} \to a_*\mathcal{F}$. Thus a^{-1} and a_* are adjoint functors and we obtain our morphism of topoi a[2]. The equalities $a \circ g_n = f_n$ follow immediately from the definitions. □

85.5. Morphisms of simplicial sites

0D94 We continue in the fashion described in Section 85.3 working out the meaning of morphisms of simplicial sites in cases A and B treated in that section.

0D95 Remark 85.5.1. Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. A morphism h between simplicial sites will mean

(A) Morphisms of sites $h_n : \mathcal{C}_n \to \mathcal{C}'_n$ such that $f'_\varphi \circ h_n = h_m \circ f_\varphi$ as morphisms of sites for all $\varphi : [m] \to [n]$.

(B) Cocontinuous functors $v_n : \mathcal{C}_n \to \mathcal{C}'_n$ inducing morphisms of topoi $h_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}'_n)$ such that $u'_\varphi \circ v_n = v_m \circ u_\varphi$ as functors for all $\varphi : [m] \to [n]$.

In both cases we have $f'_\varphi \circ h_n = h_m \circ f_\varphi$ as morphisms of topoi, see Sites, Lemma 7.21.2 for case B and Sites, Definition 7.14.5 for case A.

0D96 Lemma 85.5.2. Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let h be a morphism between simplicial sites as in Remark 85.5.1. Then we obtain a morphism of topoi

$$h_{total} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}'_{total})$$

and commutative diagrams

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[r, "h_n"'] \arrow[d, "g_n"'] & Sh(\mathcal{C}'_n) \arrow[d, "g'_n"] \\ Sh(\mathcal{C}_{total}) \arrow[r, "h_{total}"] & Sh(\mathcal{C}'_{total}) \end{tikzcd}$$

[2] In case B the morphism a corresponds to the cocontinuous functor $\mathcal{C}_{total} \to \mathcal{D}$ sending U in $\mathcal{C}_n$ to $u_n(U)$.

Moreover, we have $(g'_n)^{-1} \circ h_{total,*} = h_{n,*} \circ g_n^{-1}$.

Proof. Case A. Say h_n corresponds to the continuous functor $v_n : \mathcal{C}'_n \to \mathcal{C}_n$. Then we can define a functor $v_{total} : \mathcal{C}'_{total} \to \mathcal{C}_{total}$ by using v_n in degree n. This is clearly a continuous functor (see definition of coverings in Lemma 85.3.1). Let $h_{total}^{-1} = v_{total,s} : Sh(\mathcal{C}'_{total}) \to Sh(\mathcal{C}_{total})$ and $h_{total,*} = v_{total}^s = v_{total}^p : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}'_{total})$ be the adjoint pair of functors constructed and studied in Sites, Sections 7.13 and 7.14. To see that h_{total} is a morphism of topoi we still have to verify that h_{total}^{-1} is exact. We first observe that $(g'_n)^{-1} \circ h_{total,*} = h_{n,*} \circ g_n^{-1}$; this is immediate by computing sections over an object U of $\mathcal{C}'_n$. Thus, if we think of a sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ as a system $(\mathcal{F}_n, \mathcal{F}(\varphi))$ as in Lemma 85.3.4, then $h_{total,*}\mathcal{F}$ corresponds to the system $(h_{n,*}\mathcal{F}_n, h_{n,*}\mathcal{F}(\varphi))$. Clearly, the functor $(\mathcal{F}'_n, \mathcal{F}'(\varphi)) \to (h_n^{-1}\mathcal{F}'_n, h_n^{-1}\mathcal{F}'(\varphi))$ is its left adjoint. By uniqueness of adjoints, we conclude that h_{total}^{-1} is given by this rule on systems. In particular, h_{total}^{-1} is exact (by the description of sheaves on $\mathcal{C}_{total}$ given in the lemma and the exactness of the functors h_n^{-1}) and we have our morphism of topoi. Finally, we obtain $g_n^{-1} \circ h_{total}^{-1} = h_n^{-1} \circ (g'_n)^{-1}$ as well, which proves that the displayed diagram of the lemma commutes.

Case B. Here we have a functor $v_{total} : \mathcal{C}_{total} \to \mathcal{C}'_{total}$ by using v_n in degree n. This is clearly a cocontinuous functor (see definition of coverings in Lemma 85.3.2). Let h_{total} be the morphism of topoi associated to v_{total}. The commutativity of the displayed diagram of the lemma follows immediately from Sites, Lemma 7.21.2. Taking left adjoints the final equality of the lemma becomes

$$h_{total}^{-1} \circ (g'_n)_!^{Sh} = g_{n!}^{Sh} \circ h_n^{-1}$$

This follows immediately from the explicit description of the functors $(g'_n)_!^{Sh}$ and $g_{n!}^{Sh}$ in Lemma 85.3.5, the fact that $h_n^{-1} \circ (f'_\varphi)^{-1} = f_\varphi^{-1} \circ h_m^{-1}$ for $\varphi : [m] \to [n]$, and the fact that we already know h_{total}^{-1} commutes with restrictions to the degree n parts of the simplicial sites. $\square$

0D97 Lemma 85.5.3. With notation and hypotheses as in Lemma 85.5.2. For $K \in D(\mathcal{C}_{total})$ we have $(g'_n)^{-1} Rh_{total,*}K = Rh_{n,*}g_n^{-1}K$.

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex on $\mathcal{C}_{total}$ representing K. Then $g_n^{-1}K$ is represented by $g_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ which is K-injective by Lemma 85.3.6. We have $(g'_n)^{-1}h_{total,*}\mathcal{I}^\bullet = h_{n,*}g_n^{-1}\mathcal{I}_n^\bullet$ by Lemma 85.5.2 which gives the desired equality. $\square$

0D98 Remark 85.5.4. Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let a_0, resp. a'_0 be an augmentation towards a site $\mathcal{D}$, resp. $\mathcal{D}'$ as in Remark 85.4.1. Let h be a morphism between simplicial sites as in Remark 85.5.1. We say a morphism of topoi $h_{-1} : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ is compatible with h, a_0, a'_0 if

(A) h_{-1} comes from a morphism of sites $h_{-1} : \mathcal{D} \to \mathcal{D}'$ such that $a'_0 \circ h_0 = h_{-1} \circ a_0$ as morphisms of sites.
(B) h_{-1} comes from a cocontinuous functor $v_{-1} : \mathcal{D} \to \mathcal{D}'$ such that $u'_0 \circ v_0 = v_{-1} \circ u_0$ as functors.

In both cases we have $a'_0 \circ h_0 = h_{-1} \circ a_0$ as morphisms of topoi, see Sites, Lemma 7.21.2 for case B and Sites, Definition 7.14.5 for case A.

0D99 Lemma 85.5.5. Let $\mathcal{C}_n, f_\varphi, u_\varphi, \mathcal{D}, a_0$, $\mathcal{C}'_n, f'_\varphi, u'_\varphi, \mathcal{D}', a'_0$, and h_n, $n \geq -1$ be as in Remark 85.5.4. Then we obtain a commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C}_{total}) \arrow[r, "h_{total}"'] \arrow[d, "a"'] & Sh(\mathcal{C}'_{total}) \arrow[d, "a'"] \\ Sh(\mathcal{D}) \arrow[r, "h_{-1}"] & Sh(\mathcal{D}') \end{tikzcd}$$

Proof. The morphism h is defined in Lemma 85.5.2. The morphisms a and a' are defined in Lemma 85.4.2. Thus the only thing is to prove the commutativity of the diagram. To do this, we prove that $a^{-1} \circ h_{-1}^{-1} = h_{total}^{-1} \circ (a')^{-1}$. By the commutative diagrams of Lemma 85.5.2 and the description of $Sh(\mathcal{C}_{total})$ and $Sh(\mathcal{C}'_{total})$ in terms of components in Lemma 85.3.4, it suffices to show that

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[r, "h_n"'] \arrow[d, "a_n"'] & Sh(\mathcal{C}'_n) \arrow[d, "a'_n"] \\ Sh(\mathcal{D}) \arrow[r, "h_{-1}"] & Sh(\mathcal{D}') \end{tikzcd}$$

commutes for all n. This follows from the case for $n = 0$ (which is an assumption in Remark 85.5.4) and for $n > 0$ we pick $\varphi : [0] \to [n]$ and then the required commutativity follows from the case $n = 0$ and the relations $a_n = a_0 \circ f_\varphi$ and $a'_n = a'_0 \circ f'_\varphi$ as well as the commutation relations $f'_\varphi \circ h_n = h_0 \circ f_\varphi$. $\square$

85.6. Ringed simplicial sites

0D71 Let us endow our simplicial topos with a sheaf of rings.

0D72 Lemma 85.6.1. In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. There is a canonical morphism of ringed topoi $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ agreeing with the morphism g_n of Lemma 85.3.5 on underlying topoi. The functor $g_n^* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_n)$ has a left adjoint $g_{n!}$. For $\mathcal{G}$ in $\text{Mod}(\mathcal{O}_n)$-modules the restriction of $g_{n!}\mathcal{G}$ to $\mathcal{C}_m$ is

$$\bigoplus\nolimits_{\varphi : [n] \to [m]} f_\varphi^* \mathcal{G}$$

where $f_\varphi : (Sh(\mathcal{C}_m), \mathcal{O}_m) \to (Sh(\mathcal{C}_n), \mathcal{O}_n)$ is the morphism of ringed topoi agreeing with the previously defined f_φ on topoi and using the map $\mathcal{O}(\varphi) : f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ on sheaves of rings.

Proof. By Lemma 85.3.5 we have $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ and hence we obtain our morphism of ringed topoi. By Modules on Sites, Lemma 18.41.1 we obtain the adjoint $g_{n!}$. To prove the formula for $g_{n!}$ we first define a sheaf of $\mathcal{O}$-modules $\mathcal{H}$ on $\mathcal{C}_{total}$ with degree m component the $\mathcal{O}_m$-module

$$\mathcal{H}_m = \bigoplus\nolimits_{\varphi : [n] \to [m]} f_\varphi^* \mathcal{G}$$

Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by

$$f_\psi^{-1} f_\varphi^* \mathcal{G} \to f_\psi^* f_\varphi^* \mathcal{G} \to f_{\psi \circ \varphi}^* \mathcal{G}$$

Since this map $f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is $\mathcal{O}(\psi) : f_\psi^{-1}\mathcal{O}_m \to \mathcal{O}_{m'}$-semi-linear, this indeed does define an $\mathcal{O}$-module (use Lemma 85.3.4). Then one proves directly that

$$\text{Mor}_{\mathcal{O}_n}(\mathcal{G}, \mathcal{F}_n) = \text{Mor}_{\mathcal{O}}(\mathcal{H}, \mathcal{F})$$

proceeding as in the proof of Lemma 85.3.5. Thus $\mathcal{H} = g_{n!}\mathcal{G}$ as desired. $\square$

0D73 Lemma 85.6.2. In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $\mathcal{I}$ is injective in $\text{Mod}(\mathcal{O})$, then $\mathcal{I}_n$ is a totally acyclic sheaf on $\mathcal{C}_n$.

Proof. This follows from Cohomology on Sites, Lemma 21.37.4 applied to the inclusion functor $\mathcal{C}_n \to \mathcal{C}_{total}$ and its properties proven in Lemma 85.3.5. $\square$

0D74 Lemma 85.6.3. With assumptions as in Lemma 85.6.1 the functor $g_{n!} : \text{Mod}(\mathcal{O}_n) \to \text{Mod}(\mathcal{O})$ is exact if the maps $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ are flat for all $\varphi : [n] \to [m]$.

Proof. Recall that $g_{n!}\mathcal{G}$ is the $\mathcal{O}$-module whose degree m part is the $\mathcal{O}_m$-module

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{G}$$

Here the morphism of ringed topoi $f_\varphi : (Sh(\mathcal{C}_m), \mathcal{O}_m) \to (Sh(\mathcal{C}_n), \mathcal{O}_n)$ uses the map $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ of the statement of the lemma. If these maps are flat, then f_φ^* is exact (Modules on Sites, Lemma 18.31.2). By definition of the site $\mathcal{C}_{total}$ we see that these functors have the desired exactness properties and we conclude. $\square$

0D75 Lemma 85.6.4. In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for all $\varphi : [n] \to [m]$. If $\mathcal{I}$ is injective in $\text{Mod}(\mathcal{O})$, then $\mathcal{I}_n$ is injective in $\text{Mod}(\mathcal{O}_n)$.

Proof. This follows from Homology, Lemma 12.29.1 and Lemma 85.6.3. $\square$

85.7. Morphisms of ringed simplicial sites

0DGY We continue the discussion of Section 85.5.

0DGZ Remark 85.7.1. Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{C}'_{total}$. We will say that $(h, h^\sharp)$ is a morphism between ringed simplicial sites if h is a morphism between simplicial sites as in Remark 85.5.1 and $h^\sharp : h_{total}^{-1}\mathcal{O}' \to \mathcal{O}$ or equivalently $h^\sharp : \mathcal{O}' \to h_{total,*}\mathcal{O}$ is a homomorphism of sheaves of rings.

0DH0 Lemma 85.7.2. Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{C}'_{total}$. Let $(h, h^\sharp)$ be a morphism between simplicial sites as in Remark 85.7.1. Then we obtain a morphism of ringed topoi

$$h_{total} : (Sh(\mathcal{C}_{total}, \mathcal{O}) \to (Sh(\mathcal{C}'_{total}), \mathcal{O}')$$

and commutative diagrams

$$\begin{tikzcd} (Sh(\mathcal{C}_n), \mathcal{O}_n) \arrow[r, "h_n"'] \arrow[d, "g_n"'] & (Sh(\mathcal{C}'_n), \mathcal{O}'_n) \arrow[d, "g'_n"] \\ (Sh(\mathcal{C}_{total}), \mathcal{O}) \arrow[r, "h_{total}"] & (Sh(\mathcal{C}'_{total}), \mathcal{O}') \end{tikzcd}$$

of ringed topoi where g_n and g'_n are as in Lemma 85.6.1. Moreover, we have $(g'_n)^* \circ h_{total,*} = h_{n,*} \circ g_n^*$ as functor $\text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}'_n)$.

Proof. Follows from Lemma 85.5.2 and 85.6.1 by keeping track of the sheaves of rings. A small point is that in order to define h_n as a morphism of ringed topoi we set $h_n^\sharp = g_n^{-1}h^\sharp : g_n^{-1}h_{total}^{-1}\mathcal{O}' \to g_n^{-1}\mathcal{O}$ which makes sense because $g_n^{-1}h_{total}^{-1}\mathcal{O}' =$

$h_n^{-1}(g'_n)^{-1}\mathcal{O}' = h_n^{-1}\mathcal{O}'_n$ and $g_n^{-1}\mathcal{O} = \mathcal{O}_n$. Note that $g_n^*\mathcal{F} = g_n^{-1}\mathcal{F}$ for a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ and similarly for g'_n and this helps explain why $(g'_n)^* \circ h_{total,*} = h_{n,*} \circ g_n^*$ follows from the corresponding statement of Lemma 85.5.2. □

0DH1 Lemma 85.7.3. With notation and hypotheses as in Lemma 85.7.2. For $K \in D(\mathcal{O})$ we have $(g'_n)^* Rh_{total,*}K = Rh_{n,*}g_n^*K$.

Proof. Recall that $g_n^* = g_n^{-1}$ because $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ by the construction in Lemma 85.6.1. In particular g_n^* is exact and Lg_n^* is given by applying g_n^* to any representative complex of modules. Similarly for g'_n. There is a canonical base change map $(g'_n)^* Rh_{total,*}K \to Rh_{n,*}g_n^*K$, see Cohomology on Sites, Remark 21.19.3. By Cohomology on Sites, Lemma 21.20.7 the image of this in $D(\mathcal{C}'_n)$ is the map $(g'_n)^{-1}Rh_{total,*}K_{ab} \to Rh_{n,*}g_n^{-1}K_{ab}$ where K_{ab} is the image of K in $D(\mathcal{C}_{total})$. This we proved to be an isomorphism in Lemma 85.5.3 and the result follows. □

85.8. Cohomology on simplicial sites

0D76 Let $\mathcal{C}$ be as in Situation 85.3.3. In statement of the following lemmas we will let $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ be the morphism of topoi of Lemma 85.3.5. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[rr, "f_\varphi"'] \arrow[dr, "g_n"'] & & Sh(\mathcal{C}_m) \arrow[dl, "g_m"] \\ & Sh(\mathcal{C}_{total}) & \end{tikzcd}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 7.36.

09WI Lemma 85.8.1. In Situation 85.3.3 and with notation as above there is a complex

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

of abelian sheaves on $\mathcal{C}_{total}$ which forms a resolution of the constant sheaf with value $\mathbf{Z}$ on $\mathcal{C}_{total}$.

Proof. We will use the description of the functors $g_{n!}$ in Lemma 85.3.5 without further mention. As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}\mathbf{Z} \to g_{n-1!}\mathbf{Z}$ is the adjoint to the map $\mathbf{Z} \to \bigoplus_{[n-1]\to[n]} \mathbf{Z} = g_n^{-1}g_{n-1!}\mathbf{Z}$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha \in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha \in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha \in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free abelian sheaf on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.26.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 14.26.4 and Lemma 14.27.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

0D77 Lemma 85.8.2. In Situation 85.3.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{total}$ there is a canonical complex

$$0 \to \Gamma(\mathcal{C}_{total}, \mathcal{F}) \to \Gamma(\mathcal{C}_0, \mathcal{F}_0) \to \Gamma(\mathcal{C}_1, \mathcal{F}_1) \to \Gamma(\mathcal{C}_2, \mathcal{F}_2) \to \ldots$$

which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is injective.

Proof. Observe that $\text{Hom}(\mathbf{Z}, \mathcal{F}) = \Gamma(\mathcal{C}_{total}, \mathcal{F})$ and $\text{Hom}(g_{n!}\mathbf{Z}, \mathcal{F}) = \Gamma(\mathcal{C}_n, \mathcal{F}_n)$. Hence this lemma is an immediate consequence of Lemma 85.8.1 and the fact that $\text{Hom}(-, \mathcal{F})$ is exact if $\mathcal{F}$ is injective. $\square$

09WJ Lemma 85.8.3. In Situation 85.3.3. For K in $D^+(\mathcal{C}_{total})$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_1^{p,q} = H^q(\mathcal{C}_p, K_p), \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $H^{p+q}(\mathcal{C}_{total}, K)$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives representing K. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$$

where the horizontal arrows come from Lemma 85.8.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{total}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 85.3.6). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, K_p)$. We conclude by applying Homology, Lemmas 12.25.3 and 12.25.4. $\square$

0H0V Remark 85.8.4. Assumptions and notation as in Lemma 85.8.3 except we do not require K in $D(\mathcal{C}_{total})$ to be bounded below. We claim there is a natural spectral sequence in this case also. Namely, suppose that $\mathcal{I}^\bullet$ is a K-injective complex of sheaves on $\mathcal{C}_{total}$ with injective terms representing K. We have

$$\begin{aligned} R\Gamma(\mathcal{C}_{total}, K) &= R\,\text{Hom}(\mathbf{Z}, K) \\ &= R\,\text{Hom}(\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}, K) \\ &= \Gamma(\mathcal{C}_{total}, \mathcal{H}om^\bullet(\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}, \mathcal{I}^\bullet)) \\ &= \text{Tot}_\pi(A^{\bullet,\bullet}) \end{aligned}$$

where $A^{\bullet,\bullet}$ is the double complex with terms $A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$ and Tot_π denotes the product totalization of this double complex. Namely, the first equality holds in any site. The second equality holds by Lemma 85.8.1. The third equality holds because $\mathcal{I}^\bullet$ is K-injective, see Cohomology on Sites, Sections 21.34 and 21.35. The final equality holds by the construction of $\mathcal{H}om^\bullet$ and the fact that $\text{Hom}(g_{p!}\mathbf{Z}, \mathcal{I}^q) = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$. Then we get our spectral sequence by viewing $\text{Tot}_\pi(A^{\bullet,\bullet})$ as a filtered complex with $F^i\text{Tot}_\pi^n(A^{\bullet,\bullet}) = \prod_{p+q=n,\ p\geq i} A^{p,q}$. The spectral sequence we obtain behaves like the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ in Homology, Section 12.25 (where the case of the direct sum totalization is discussed) except for regularity, boundedness, convergence, and abutment issues. In particular we obtain $E_1^{p,q} = H^q(\mathcal{C}_p, K_p)$ as in Lemma 85.8.3.

0H0W Lemma 85.8.5. In Situation 85.3.3. Let K be an object of $D(\mathcal{C}_{total})$.

(1) If $H^{-p}(\mathcal{C}_p, K_p) = 0$ for all $p \geq 0$, then $H^0(\mathcal{C}_{total}, K) = 0$.
(2) If $R\Gamma(\mathcal{C}_p, K_p) = 0$ for all $p \geq 0$, then $R\Gamma(\mathcal{C}_{total}, K) = 0$.

Proof. With notation as in Remark 85.8.4 we see that $R\Gamma(\mathcal{C}_{total}, K)$ is represented by $\text{Tot}_\pi(A^{\bullet,\bullet})$. The assumption in (1) tells us that $H^{-p}(A^{p,\bullet}) = 0$. Thus the vanishing in (1) follows from More on Algebra, Lemma 15.103.1. Part (2) follows from part (1) and taking shifts. $\square$

0DBZ Lemma 85.8.6. Let $\mathcal{C}$ be as in Situation 85.3.3. Let $U \in \text{Ob}(\mathcal{C}_n)$. Let $\mathcal{F} \in \text{Ab}(\mathcal{C}_{total})$. Then $H^p(U, \mathcal{F}) = H^p(U, g_n^{-1}\mathcal{F})$ where on the left hand side U is viewed as an object of $\mathcal{C}_{total}$.

Proof. Observe that "U viewed as object of $\mathcal{C}_{total}$" is explained by the construction of $\mathcal{C}_{total}$ in Lemma 85.3.1 in case (A) and Lemma 85.3.2 in case (B). The equality then follows from Lemma 85.3.6 and the definition of cohomology. $\square$

85.9. Cohomology and augmentations of simplicial sites

0D9A Consider a simplicial site $\mathcal{C}$ as in Situation 85.3.3. Let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. By Lemma 85.4.2 we obtain a morphism of topoi

$$a : Sh(\mathcal{C}_{total}) \longrightarrow Sh(\mathcal{D})$$

and morphisms of topoi $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ as in Lemma 85.3.5. The compositions $a \circ g_n$ are denoted $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$. Furthermore, the simplicial structure gives morphisms of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$ such that $a_n \circ f_\varphi = a_m$ for all $\varphi : [m] \to [n]$.

0D78 Lemma 85.9.1. In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. For any abelian sheaf $\mathcal{G}$ on $\mathcal{D}$ there is an exact complex

$$\ldots \to g_{2!}(a_2^{-1}\mathcal{G}) \to g_{1!}(a_1^{-1}\mathcal{G}) \to g_{0!}(a_0^{-1}\mathcal{G}) \to a^{-1}\mathcal{G} \to 0$$

of abelian sheaves on $\mathcal{C}_{total}$.

Proof. We encourage the reader to read the proof of Lemma 85.8.1 first. We will use Lemma 85.4.2 and the description of the functors $g_{n!}$ in Lemma 85.3.5 without further mention. In particular $g_{n!}(a_n^{-1}\mathcal{G})$ is the sheaf on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}a_n^{-1}\mathcal{G} = \bigoplus\nolimits_{\varphi:[n]\to[m]} a_m^{-1}\mathcal{G}$$

As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}(a_n^{-1}\mathcal{G}) \to g_{n-1!}(a_{n-1}^{-1}\mathcal{G})$ is the adjoint to the map $a_n^{-1}\mathcal{G} \to \bigoplus_{[n-1]\to[n]} a_n^{-1}\mathcal{G} = g_n^{-1}g_{n-1!}(a_{n-1}^{-1}\mathcal{G})$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. The map $g_{0!}(a_0^{-1}\mathcal{G}) \to a^{-1}\mathcal{G}$ is adjoint to the identity map of $a_0^{-1}\mathcal{G}$. Then g_m^{-1} applied to the chain complex in degrees $\ldots, 2, 1, 0$ gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in\text{Mor}_\Delta([2],[m])} a_m^{-1}\mathcal{G} \to \bigoplus\nolimits_{\alpha\in\text{Mor}_\Delta([1],[m])} a_m^{-1}\mathcal{G} \to \bigoplus\nolimits_{\alpha\in\text{Mor}_\Delta([0],[m])} a_m^{-1}\mathcal{G}$$

on $\mathcal{C}_m$. This is equal to $a_m^{-1}\mathcal{G}$ tensored over the constant sheaf $\mathbf{Z}$ with the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in\text{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in\text{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in\text{Mor}_\Delta([0],[m])} \mathbf{Z}$$

discussed in the proof of Lemma 85.8.1. There we have seen that this complex is homotopy equivalent to $\mathbf{Z}$ placed in degree 0 which finishes the proof. $\square$

0D79 Lemma 85.9.2. In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ there is a canonical complex

$$0 \to a_*\mathcal{F} \to a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1 \to a_{2,*}\mathcal{F}_2 \to \ldots$$

on $\mathcal{D}$ which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is injective.

Proof. To construct the complex, by the Yoneda lemma, it suffices for any abelian sheaf $\mathcal{G}$ on $\mathcal{D}$ to construct a complex

$$0 \to \mathrm{Hom}(\mathcal{G}, a_*\mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, a_{0,*}\mathcal{F}_0) \to \mathrm{Hom}(\mathcal{G}, a_{1,*}\mathcal{F}_1) \to \ldots$$

functorially in $\mathcal{G}$. To do this apply $\mathrm{Hom}(-, \mathcal{F})$ to the exact complex of Lemma 85.9.1 and use adjointness of pullback and pushforward. The exactness properties in degrees $-1, 0$ follow from the construction as $\mathrm{Hom}(-, \mathcal{F})$ is left exact. If $\mathcal{F}$ is an injective abelian sheaf, then the complex is exact because $\mathrm{Hom}(-, \mathcal{F})$ is exact. $\square$

0D7A Lemma 85.9.3. In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. For any K in $D^+(\mathcal{C}_{total})$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_1^{p,q} = R^q a_{p,*} K_p, \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $R^{p+q} a_* K$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives representing K. Consider the double complex with terms

$$A^{p,q} = a_{p,*}\mathcal{I}_p^q$$

where the horizontal arrows come from Lemma 85.9.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The rows of the double complex are exact in positive degrees and evaluate to $a_*\mathcal{I}^q$ in degree 0. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 85.3.6). Hence the cohomology of the columns computes $R^q a_{p,*} K_p$. We conclude by applying Homology, Lemmas 12.25.3 and 12.25.4. $\square$

85.10. Cohomology on ringed simplicial sites

0D7B This section is the analogue of Section 85.8 for sheaves of modules.

In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. In statement of the following lemmas we will let $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ be the morphism of ringed topoi of Lemma 85.6.1. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C}_n), \mathcal{O}_n) \arrow[rr, "f_\varphi"'] \arrow[dr, "g_n"'] & & (Sh(\mathcal{C}_m), \mathcal{O}_m) \arrow[dl, "g_m"] \\
& (Sh(\mathcal{C}_{total}), \mathcal{O}) &
\end{tikzcd}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 7.36.

0D9B Lemma 85.10.1. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. There is a complex

$$\ldots \to g_{2!}\mathcal{O}_2 \to g_{1!}\mathcal{O}_1 \to g_{0!}\mathcal{O}_0$$

of $\mathcal{O}$-modules which forms a resolution of $\mathcal{O}$. Here $g_{n!}$ is as in Lemma 85.6.1.

Proof. We will use the description of $g_{n!}$ given in Lemma 85.3.5. As maps of the complex we take $\sum (-1)^i d^n_i$ where $d^n_i : g_{n!}\mathcal{O}_n \to g_{n-1!}\mathcal{O}_{n-1}$ is the adjoint to the map $\mathcal{O}_n \to \bigoplus_{[n-1]\to[n]} \mathcal{O}_n = g_n^*g_{n-1!}\mathcal{O}_{n-1}$ corresponding to the factor labeled with $\delta^n_i : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathcal{O}_m \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathcal{O}_m \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathcal{O}_m$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free $\mathcal{O}_m$-module on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.26.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 14.26.4 and Lemma 14.27.2). This complex is acyclic in positive degrees and equal to $\mathcal{O}_m$ in degree 0. □

0D9C Lemma 85.10.2. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a canonical complex

$$0 \to \Gamma(\mathcal{C}_{total}, \mathcal{F}) \to \Gamma(\mathcal{C}_0, \mathcal{F}_0) \to \Gamma(\mathcal{C}_1, \mathcal{F}_1) \to \Gamma(\mathcal{C}_2, \mathcal{F}_2) \to \ldots$$

which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is an injective $\mathcal{O}$-module.

Proof. Observe that $\mathrm{Hom}(\mathcal{O}, \mathcal{F}) = \Gamma(\mathcal{C}_{total}, \mathcal{F})$ and $\mathrm{Hom}(g_{n!}\mathcal{O}_n, \mathcal{F}) = \Gamma(\mathcal{C}_n, \mathcal{F}_n)$. Hence this lemma is an immediate consequence of Lemma 85.10.1 and the fact that $\mathrm{Hom}(-, \mathcal{F})$ is exact if $\mathcal{F}$ is injective. □

0D7E Lemma 85.10.3. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings. For K in $D^+(\mathcal{O})$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with

$$E_1^{p,q} = H^q(\mathcal{C}_p, K_p), \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $H^{p+q}(\mathcal{C}_{total}, K)$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective $\mathcal{O}$-modules representing K. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}^q_p)$$

where the horizontal arrows come from Lemma 85.10.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. Observe that $\Gamma(\mathcal{D}, -) = \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{O}_\mathcal{D}, -)$ on $\mathrm{Mod}(\mathcal{O}_\mathcal{D})$. Hence the lemma says rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{total}, \mathcal{I}^q)$ in degree 0. Thus the total complex associated to the double complex computes $R\Gamma(\mathcal{C}_{total}, K)$ by Homology, Lemma 12.25.4. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}^\bullet_p$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}^q_p$ are totally acyclic on $\mathcal{C}_p$ (Lemma 85.6.2). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, K_p)$ by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Cohomology on Sites, Lemma 21.14.3. We conclude by applying Homology, Lemma 12.25.3. □

0DH2 Lemma 85.10.4. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings. Let $U \in \mathrm{Ob}(\mathcal{C}_n)$. Let $\mathcal{F} \in \mathrm{Mod}(\mathcal{O})$. Then $H^p(U, \mathcal{F}) = H^p(U, g_n^*\mathcal{F})$ where on the left hand side U is viewed as an object of $\mathcal{C}_{total}$.

Proof. Observe that "U viewed as object of $\mathcal{C}_{total}$" is explained by the construction of $\mathcal{C}_{total}$ in Lemma 85.3.1 in case (A) and Lemma 85.3.2 in case (B). In both cases the functor $\mathcal{C}_n \to \mathcal{C}$ is continuous and cocontinuous, see Lemma 85.3.5, and $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ by definition. Hence the result is a special case of Cohomology on Sites, Lemma 21.37.5. □

85.11. Cohomology and augmentations of ringed simplicial sites

0D9D This section is the analogue of Section 85.9 for sheaves of modules.

Consider a simplicial site $\mathcal{C}$ as in Situation 85.3.3. Let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Suppose we are given a morphism

$$a^\sharp : \mathcal{O}_\mathcal{D} \longrightarrow a_*\mathcal{O}$$

where a is as in Lemma 85.4.2. Consequently, we obtain a morphism of ringed topoi

$$a : (Sh(\mathcal{C}_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

We will think of $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ as a morphism of ringed topoi as in Lemma 85.6.1, then taking the composition $a_n = a \circ g_n$ (Lemma 85.4.2) as morphisms of ringed topoi we obtain

$$a_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

Using the transition maps $f_\varphi^{-1}\mathcal{O}_m \to \mathcal{O}_n$ we obtain morphisms of ringed topoi

$$f_\varphi : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_m), \mathcal{O}_m)$$

such that $a_n \circ f_\varphi = a_m$ as morphisms of ringed topoi for all $\varphi : [m] \to [n]$.

0DH3 Lemma 85.11.1. With notation as above. The morphism $a : (Sh(\mathcal{C}_{total}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is flat if and only if $a_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is flat for $n \geq 0$.

Proof. Since $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ is flat, we see that if a is flat, then $a_n = a \circ g_n$ is flat as a composition. Conversely, suppose that a_n is flat for all n. We have to check that $\mathcal{O}$ is flat as a sheaf of $a^{-1}\mathcal{O}_\mathcal{D}$-modules. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of $a^{-1}\mathcal{O}_\mathcal{D}$-modules. We have to show that

$$\mathcal{F} \otimes_{a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O} \to \mathcal{G} \otimes_{a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}$$

is injective. We can check this on $\mathcal{C}_n$, i.e., after applying g_n^{-1}. Since $g_n^* = g_n^{-1}$ because $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ we obtain

$$g_n^{-1}\mathcal{F} \otimes_{g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_n \to g_n^{-1}\mathcal{G} \otimes_{g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_n$$

which is injective because $g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D} = a_n^{-1}\mathcal{O}_\mathcal{D}$ and we assume a_n was flat. □

0D7C Lemma 85.11.2. With notation as above. For a $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ there is an exact complex

$$\ldots \to g_{2!}(a_2^*\mathcal{G}) \to g_{1!}(a_1^*\mathcal{G}) \to g_{0!}(a_0^*\mathcal{G}) \to a^*\mathcal{G} \to 0$$

of sheaves of $\mathcal{O}$-modules on $\mathcal{C}_{total}$. Here $g_{n!}$ is as in Lemma 85.6.1.

Proof. Observe that $a^*\mathcal{G}$ is the $\mathcal{O}$-module on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the $\mathcal{O}_m$-module $a_m^*\mathcal{G}$. The description of the functors $g_{n!}$ on modules in Lemma 85.6.1 shows that $g_{n!}(a_n^*\mathcal{G})$ is the $\mathcal{O}$-module on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the $\mathcal{O}_m$-module

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^* a_n^*\mathcal{G} = \bigoplus\nolimits_{\varphi:[n]\to[m]} a_m^*\mathcal{G}$$

The rest of the proof is exactly the same as the proof of Lemma 85.9.1, replacing $a_m^{-1}\mathcal{G}$ by $a_m^*\mathcal{G}$. □

0D7D Lemma 85.11.3. With notation as above. For an $\mathcal{O}$-module $\mathcal{F}$ on $\mathcal{C}_{total}$ there is a canonical complex

$$0 \to a_*\mathcal{F} \to a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1 \to a_{2,*}\mathcal{F}_2 \to \ldots$$

of $\mathcal{O}_\mathcal{D}$-modules which is exact in degrees $-1, 0$. If $\mathcal{F}$ is an injective $\mathcal{O}$-module, then the complex is exact in all degrees and remains exact on applying the functor $\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, -)$ for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$.

Proof. To construct the complex, by the Yoneda lemma, it suffices for any $\mathcal{O}_\mathcal{D}$-modules $\mathcal{G}$ on $\mathcal{D}$ to construct a complex

$$0 \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_{0,*}\mathcal{F}_0) \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_{1,*}\mathcal{F}_1) \to \ldots$$

functorially in $\mathcal{G}$. To do this apply $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ to the exact complex of Lemma 85.11.2 and use adjointness of pullback and pushforward. The exactness properties in degrees $-1, 0$ follow from the construction as $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ is left exact. If $\mathcal{F}$ is an injective $\mathcal{O}$-module, then the complex is exact because $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ is exact. □

0D7F Lemma 85.11.4. With notation as above for any K in $D^+(\mathcal{O})$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ in $\mathrm{Mod}(\mathcal{O}_\mathcal{D})$ with

$$E_1^{p,q} = R^q a_{p,*} K_p$$

converging to $R^{p+q} a_* K$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective $\mathcal{O}$-modules representing K. Consider the double complex with terms

$$A^{p,q} = a_{p,*}\mathcal{I}_p^q$$

where the horizontal arrows come from Lemma 85.11.3 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The lemma says rows of the double complex are exact in positive degrees and evaluate to $a_*\mathcal{I}^q$ in degree 0. Thus the total complex associated to the double complex computes Ra_*K by Homology, Lemma 12.25.4. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are totally acyclic on $\mathcal{C}_p$ (Lemma 85.6.2). Hence the cohomology of the columns are the sheaves $R^q a_{p,*} K_p$ by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Cohomology on Sites, Lemma 21.14.3. We conclude by applying Homology, Lemma 12.25.3. □

85.12. Cartesian sheaves and modules

0D7G Here is the definition.

07TF Definition 85.12.1. In Situation 85.3.3.

(1) A sheaf $\mathcal{F}$ of sets or of abelian groups on $\mathcal{C}_{total}$ is cartesian if the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(2) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$, then a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is cartesian if the maps $f_\varphi^*\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(3) An object K of $D(\mathcal{C}_{total})$ is cartesian if the maps $f_\varphi^{-1}K_m \to K_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(4) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$, then an object K of $D(\mathcal{O})$ is cartesian if the maps $Lf_\varphi^*K_m \to K_n$ are isomorphisms for all $\varphi : [m] \to [n]$.

Of course there is a general notion of a cartesian section of a fibred category and the above are merely examples of this. The property on pullbacks needs only be checked for the degeneracies.

07TG Lemma 85.12.2. In Situation 85.3.3.

(1) A sheaf $\mathcal{F}$ of sets or abelian groups is cartesian if and only if the maps $(f_{\delta^n_j})^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.
(2) An object K of $D(\mathcal{C}_{total})$ is cartesian if and only if the maps $(f_{\delta^n_j})^{-1}K_{n-1} \to K_n$ are isomorphisms.
(3) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$ a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is cartesian if and only if the maps $(f_{\delta^n_j})^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.
(4) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$ an object K of $D(\mathcal{O})$ is cartesian if and only if the maps $L(f_{\delta^n_j})^*K_{n-1} \to K_n$ are isomorphisms.
(5) Add more here.

Proof. In each case the key is that the pullback functors compose to pullback functor; for part (4) see Cohomology on Sites, Lemma 21.18.3. We show how the argument works in case (1) and omit the proof in the other cases. The category Δ is generated by the morphisms the morphisms δ^n_j and σ^n_j, see Simplicial, Lemma 14.2.2. Hence we only need to check the maps $(f_{\delta^n_j})^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ and $(f_{\sigma^n_j})^{-1}\mathcal{F}_{n+1} \to \mathcal{F}_n$ are isomorphisms, see Simplicial, Lemma 14.3.2 for notation. Since $\sigma^n_j \circ \delta^{n+1}_j = \mathrm{id}_{[n]}$ the composition

$$\mathcal{F}_n = (f_{\sigma^n_j})^{-1}(f_{\delta^{n+1}_j})^{-1}\mathcal{F}_n \to (f_{\sigma^n_j})^{-1}\mathcal{F}_{n+1} \to \mathcal{F}_n$$

is the identity. Thus the result for δ^{n+1}_j implies the result for σ^n_j. □

0D7H Lemma 85.12.3. In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1.

(1) The pullback $a^{-1}\mathcal{G}$ of a sheaf of sets or abelian groups on $\mathcal{D}$ is cartesian.
(2) The pullback $a^{-1}K$ of an object K of $D(\mathcal{D})$ is cartesian.

Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{O}_\mathcal{D}$ a sheaf of rings on $\mathcal{D}$ and $a^\sharp : \mathcal{O}_\mathcal{D} \to a_*\mathcal{O}$ a morphism as in Section 85.11.

(3) The pullback $a^*\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{D}$-modules is cartesian.
(4) The derived pullback La^*K of an object K of $D(\mathcal{O}_\mathcal{D})$ is cartesian.

Proof. This follows immediately from the identities $a_m \circ f_\varphi = a_n$ for all $\varphi : [m] \to [n]$. See Lemma 85.4.2 and the discussion in Section 85.11. □

0D7I Lemma 85.12.4. In Situation 85.3.3. The category of cartesian sheaves of sets (resp. abelian groups) is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a sheaf of sets (resp. abelian groups) on $\mathcal{C}_0$ and

$$\alpha : (f_{\delta^1_1})^{-1}\mathcal{F} \longrightarrow (f_{\delta^1_0})^{-1}\mathcal{F}$$

is an isomorphism of sheaves of sets (resp. abelian groups) on $\mathcal{C}_1$ such that $(f_{\delta^2_1})^{-1}\alpha = (f_{\delta^2_0})^{-1}\alpha \circ (f_{\delta^2_2})^{-1}\alpha$ as maps of sheaves on $\mathcal{C}_2$.

Proof. We abbreviate $d^n_j = f_{\delta^n_j} : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{n-1})$. The condition on α in the statement of the lemma makes sense because

$$d^1_1 \circ d^2_2 = d^1_1 \circ d^2_1, \quad d^1_1 \circ d^2_0 = d^1_0 \circ d^2_2, \quad d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$$

as morphisms of topoi $Sh(\mathcal{C}_2) \to Sh(\mathcal{C}_0)$, see Simplicial, Remark 14.3.3. Hence we can picture these maps as follows

$$
\begin{tikzcd}
 & (d^2_0)^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[r, "(d^2_0)^{-1}\alpha"'] & (d^2_0)^{-1}(d^1_0)^{-1}\mathcal{F} \arrow[rd, equal] & \\
(d^2_2)^{-1}(d^1_0)^{-1}\mathcal{F} \arrow[ru, equal] & & & (d^2_1)^{-1}(d^1_0)^{-1}\mathcal{F} \\
 & (d^2_2)^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[lu, "(d^2_2)^{-1}\alpha"] \arrow[r, equal] & (d^2_1)^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[ru, "(d^2_1)^{-1}\alpha"'] &
\end{tikzcd}
$$

and the condition signifies the diagram is commutative. It is clear that given a cartesian sheaf $\mathcal{G}$ of sets (resp. abelian groups) on $\mathcal{C}_{total}$ we can set $\mathcal{F} = \mathcal{G}_0$ and α equal to the composition

$$(d^1_1)^{-1}\mathcal{G}_0 \to \mathcal{G}_1 \leftarrow (d^0_1)^{-1}\mathcal{G}_0$$

where the arrows are invertible as $\mathcal{G}$ is cartesian. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair $(\mathcal{F}, \alpha)$ as in the lemma we set $\mathcal{G}_n = (f_{\tau^n_n})^{-1}\mathcal{F}$. Given $\varphi : [n] \to [m]$ we define $\mathcal{G}(\varphi) : (f_\varphi)^{-1}\mathcal{G}_n \to \mathcal{G}_m$ using

$$
\begin{tikzcd}
(f_\varphi)^{-1}\mathcal{G}_n \arrow[r, equal] & (f_\varphi)^{-1}(f_{\tau^n_n})^{-1}\mathcal{F} \arrow[r, equal] & (f_{\tau^m_{\varphi(n)}})^{-1}\mathcal{F} \arrow[r, equal] & (f_{\tau^m_{\varphi(n)m}})^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[d, "(f_{\tau^m_{\varphi(n)m}})^{-1}\alpha"] \\
 & \mathcal{G}_m \arrow[r, equal] & (f_{\tau^m_m})^{-1}\mathcal{F} \arrow[r, equal] & (f_{\tau^m_{\varphi(n)m}})^{-1}(d^1_0)^{-1}\mathcal{F}
\end{tikzcd}
$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly and hence give rise to a sheaf on $\mathcal{C}_{total}$, see Lemma 85.3.4. We also omit the verification that the two functors are quasi-inverse to each other. ☐

07TH Lemma 85.12.5. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. The category of cartesian $\mathcal{O}$-modules is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a $\mathcal{O}_0$-module and

$$\alpha : (f_{\delta^1_1})^*\mathcal{F} \longrightarrow (f_{\delta^1_0})^*\mathcal{F}$$

is an isomorphism of $\mathcal{O}_1$-modules such that $(f_{\delta^2_1})^*\alpha = (f_{\delta^2_0})^*\alpha \circ (f_{\delta^2_2})^*\alpha$ as $\mathcal{O}_2$-module maps.

Proof. The proof is identical to the proof of Lemma 85.12.4 with pullback of sheaves of abelian groups replaced by pullback of modules. ☐

0D7J Lemma 85.12.6. In Situation 85.3.3.

(1) The full subcategory of cartesian abelian sheaves forms a weak Serre subcategory of $\mathrm{Ab}(\mathcal{C}_{total})$. Colimits of systems of cartesian abelian sheaves are cartesian.

(2) Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms

$$f_{\delta^n_j} : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{n-1}), \mathcal{O}_{n-1})$$

are flat. The full subcategory of cartesian $\mathcal{O}$-modules forms a weak Serre subcategory of $\text{Mod}(\mathcal{O})$. Colimits of systems of cartesian $\mathcal{O}$-modules are cartesian.

Proof. To see we obtain a weak Serre subcategory in (1) we check the conditions listed in Homology, Lemma 12.10.3. First, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a map between cartesian abelian sheaves, then $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are cartesian too because the restriction functors $Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}_n)$ and the functors f_φ^{-1} are exact. Similarly, if

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$$

is a short exact sequence of abelian sheaves on $\mathcal{C}_{total}$ with $\mathcal{F}$ and $\mathcal{G}$ cartesian, then it follows that $\mathcal{H}$ is cartesian from the 5-lemma. To see the property of colimits, use that colimits commute with pullback as pullback is a left adjoint. In the case of modules we argue in the same manner, using the exactness of flat pullback (Modules on Sites, Lemma 18.31.2) and the fact that it suffices to check the condition for $f_{\delta^n_j}$, see Lemma 85.12.2. $\square$

0D7K Remark 85.12.7 (Warning). Lemma 85.12.6 notwithstanding, it can happen that the category of cartesian $\mathcal{O}$-modules is abelian without being a Serre subcategory of $\text{Mod}(\mathcal{O})$. Namely, suppose that we only know that $f_{\delta^1_1}$ and $f_{\delta^1_0}$ are flat. Then it follows easily from Lemma 85.12.5 that the category of cartesian $\mathcal{O}$-modules is abelian. But if $f_{\delta^2_0}$ is not flat (for example), there is no reason for the inclusion functor from the category of cartesian $\mathcal{O}$-modules to all $\mathcal{O}$-modules to be exact.

0D7L Lemma 85.12.8. In Situation 85.3.3.

(1) An object K of $D(\mathcal{C}_{total})$ is cartesian if and only if $H^q(K)$ is a cartesian abelian sheaf for all q.
(2) Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms $f_{\delta^n_j} : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{n-1}), \mathcal{O}_{n-1})$ are flat. Then an object K of $D(\mathcal{O})$ is cartesian if and only if $H^q(K)$ is a cartesian $\mathcal{O}$-module for all q.

Proof. Part (1) is true because the pullback functors $(f_\varphi)^{-1}$ are exact. Part (2) follows from the characterization in Lemma 85.12.2 and the fact that $L(f_{\delta^n_j})^* = (f_{\delta^n_j})^*$ by flatness. $\square$

0D9E Lemma 85.12.9. In Situation 85.3.3.

(1) An object K of $D(\mathcal{C}_{total})$ is cartesian if and only the canonical map

$$g_{n!}K_n \longrightarrow g_{n!}\mathbf{Z} \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

is an isomorphism for all n.
(2) Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ are flat for all $\varphi : [n] \to [m]$. Then an object K of $D(\mathcal{O})$ is cartesian if and only if the canonical map

$$g_{n!}K_n \longrightarrow g_{n!}\mathcal{O}_n \otimes^{\mathbf{L}}_{\mathcal{O}} K$$

is an isomorphism for all n.

Proof. Proof of (1). Since $g_{n!}$ is exact, it induces a functor on derived categories adjoint to g_n^{-1}. The map is the adjoint of the map $K_n \to (g_n^{-1}g_{n!}\mathbf{Z}) \otimes_{\mathbf{Z}}^{\mathbf{L}} K_n$ corresponding to $\mathbf{Z} \to g_n^{-1}g_{n!}\mathbf{Z}$ which in turn is adjoint to $\text{id} : g_{n!}\mathbf{Z} \to g_{n!}\mathbf{Z}$. Using the description of $g_{n!}$ given in Lemma 85.3.5 we see that the restriction to $\mathcal{C}_m$ of this map is

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}K_n \longrightarrow \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$

Thus the statement is clear.

Proof of (2). Since $g_{n!}$ is exact (Lemma 85.6.3), it induces a functor on derived categories adjoint to g_n^* (also exact). The map is the adjoint of the map $K_n \to (g_n^*g_{n!}\mathcal{O}_n) \otimes_{\mathcal{O}_n}^{\mathbf{L}} K_n$ corresponding to $\mathcal{O}_n \to g_n^*g_{n!}\mathcal{O}_n$ which in turn is adjoint to $\text{id} : g_{n!}\mathcal{O}_n \to g_{n!}\mathcal{O}_n$. Using the description of $g_{n!}$ given in Lemma 85.6.1 we see that the restriction to $\mathcal{C}_m$ of this map is

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*K_n \longrightarrow \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{O}_n \otimes_{\mathcal{O}_m} K_m = \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$

Thus the statement is clear. □

0D7M Lemma 85.12.10. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is quasi-coherent in the sense of Modules on Sites, Definition 18.23.1 if and only if $\mathcal{F}$ is cartesian and $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_n$-module for all n.

Proof. Assume $\mathcal{F}$ is quasi-coherent. Since pullbacks of quasi-coherent modules are quasi-coherent (Modules on Sites, Lemma 18.23.4) we see that $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_n$-module for all n. To show that $\mathcal{F}$ is cartesian, let U be an object of $\mathcal{C}_n$ for some n. Let us view U as an object of $\mathcal{C}_{total}$. Because $\mathcal{F}$ is quasi-coherent there exists a covering $\{U_i \to U\}$ and for each i a presentation

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \mathcal{F}|_{\mathcal{C}_{total}/U_i} \to 0$$

Observe that $\{U_i \to U\}$ is a covering of $\mathcal{C}_n$ by the construction of the site $\mathcal{C}_{total}$. Next, let V be an object of $\mathcal{C}_m$ for some m and let $V \to U$ be a morphism of $\mathcal{C}_{total}$ lying over $\varphi : [n] \to [m]$. The fibre products $V_i = V \times_U U_i$ exist and we get an induced covering $\{V_i \to V\}$ in $\mathcal{C}_m$. Restricting the presentation above to the sites $\mathcal{C}_n/U_i$ and $\mathcal{C}_m/V_i$ we obtain presentations

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \mathcal{F}_n|_{\mathcal{C}_n/U_i} \to 0$$

and

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_m/V_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_m/V_i} \to \mathcal{F}_m|_{\mathcal{C}_m/V_i} \to 0$$

These presentations are compatible with the map $\mathcal{F}(\varphi) : f_\varphi^*\mathcal{F}_n \to \mathcal{F}_m$ (as this map is defined using the restriction maps of $\mathcal{F}$ along morphisms of $\mathcal{C}_{total}$ lying over φ). We conclude that $\mathcal{F}(\varphi)|_{\mathcal{C}_m/V_i}$ is an isomorphism. As $\{V_i \to V\}$ is a covering we conclude $\mathcal{F}(\varphi)|_{\mathcal{C}_m/V}$ is an isomorphism. Since V and U were arbitrary this proves that $\mathcal{F}$ is cartesian. (In case A use Sites, Lemma 7.14.10.)

Conversely, assume $\mathcal{F}_n$ is quasi-coherent for all n and that $\mathcal{F}$ is cartesian. Then for any n and object U of $\mathcal{C}_n$ we can choose a covering $\{U_i \to U\}$ of $\mathcal{C}_n$ and for each i a presentation

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \mathcal{F}_n|_{\mathcal{C}_n/U_i} \to 0$$

Pulling back to $\mathcal{C}_{total}/U_i$ we obtain complexes

$$\bigoplus\nolimits_{j \in J_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \bigoplus\nolimits_{k \in K_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \mathcal{F}|_{\mathcal{C}_{total}/U_i} \to 0$$

of modules on $\mathcal{C}_{total}/U_i$. Then the property that $\mathcal{F}$ is cartesian implies that this is exact. We omit the details. ☐

85.13. Simplicial systems of the derived category

0D9F In this section we are going to prove a special case of [BBD82, Proposition 3.2.9] in the setting of derived categories of abelian sheaves. The case of modules is discussed in Section 85.14.

0D9G Definition 85.13.1. In Situation 85.3.3. A simplicial system of the derived category consists of the following data

(1) for every n an object K_n of $D(\mathcal{C}_n)$,
(2) for every $\varphi : [m] \to [n]$ a map $K_\varphi : f_\varphi^{-1}K_m \to K_n$ in $D(\mathcal{C}_n)$

subject to the condition that

$$K_{\varphi \circ \psi} = K_\varphi \circ f_\varphi^{-1}K_\psi : f_{\varphi\circ\psi}^{-1}K_l = f_\varphi^{-1}f_\psi^{-1}K_l \longrightarrow K_n$$

for any morphisms $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ of Δ. We say the simplicial system is cartesian if the maps K_φ are isomorphisms for all φ. Given two simplicial systems of the derived category there is an obvious notion of a morphism of simplicial systems of the derived category.

We have given this notion a ridiculously long name intentionally. The goal is to show that a simplicial system of the derived category comes from an object of $D(\mathcal{C}_{total})$ under certain hypotheses.

0D9H Lemma 85.13.2. In Situation 85.3.3. If $K \in D(\mathcal{C}_{total})$ is an object, then $(K_n, K(\varphi))$ is a simplicial system of the derived category. If K is cartesian, so is the system.

Proof. This is obvious. ☐

0GME Lemma 85.13.3. In Situation 85.3.3 suppose given $K_0 \in D(\mathcal{C}_0)$ and an isomorphism

$$\alpha : f_{\delta^1_1}^{-1}K_0 \longrightarrow f_{\delta^1_0}^{-1}K_0$$

satisfying the cocycle condition. Set $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and set $K_n = f^{-1}_{\tau^n_n}K_0$. Then the K_n form a cartesian simplicial system of the derived category.

Proof. Please compare with Lemma 85.12.4 and its proof (also to see the cocycle condition spelled out). The construction is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Given $\varphi : [n] \to [m]$ we define $K_\varphi : f_\varphi^{-1}K_n \to K_m$ using

$$\begin{tikzcd}
f_\varphi^{-1}K_n \arrow[r, equal] & f_\varphi^{-1}f^{-1}_{\tau^n_n}K_0 \arrow[r, equal] & f^{-1}_{\tau^m_{\varphi(n)}}K_0 \arrow[r, equal] & f^{-1}_{\tau^m_{\varphi(n)m}}f^{-1}_{\delta^1_1}K_0 \arrow[d, "f^{-1}_{\tau^m_{\varphi(n)m}}\alpha"'] \\
& K_m \arrow[r, equal] & f^{-1}_{\tau^m_m}K_0 \arrow[r, equal] & f^{-1}_{\tau^m_{\varphi(n)m}}f^{-1}_{\delta^1_0}K_0
\end{tikzcd}$$

We omit the verification that the cocycle condition implies the maps compose correctly (in their respective derived categories) and hence give rise to a simplicial system in the derived category. ☐

0D9I Lemma 85.13.4. In Situation 85.3.3. Let K be an object of $D(\mathcal{C}_{total})$. Set

$$X_n = (g_{n!}\mathbf{Z}) \otimes^{\mathbf{L}}_{\mathbf{Z}} K \quad \text{and} \quad Y_n = (g_{n!}\mathbf{Z} \to \ldots \to g_{0!}\mathbf{Z})[-n] \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

as objects of $D(\mathcal{C}_{total})$ where the maps are as in Lemma 85.8.1. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ we have

(1) the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system (Derived Categories, Definition 13.41.1) for $\ldots \to X_2 \to X_1 \to X_0$,
(2) $K = \text{hocolim} Y_n[n]$ in $D(\mathcal{C}_{total})$.

Proof. First, if $K = \mathbf{Z}$, then this is the construction of Derived Categories, Example 13.41.2 applied to the complex

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

in $\text{Ab}(\mathcal{C}_{total})$ combined with the fact that this complex represents $K = \mathbf{Z}$ in $D(\mathcal{C}_{total})$ by Lemma 85.8.1. The general case follows from this, the fact that the exact functor $- \otimes^{\mathbf{L}}_{\mathbf{Z}} K$ sends Postnikov systems to Postnikov systems, and that $- \otimes^{\mathbf{L}}_{\mathbf{Z}} K$ commutes with homotopy colimits. □

0D9J Lemma 85.13.5. In Situation 85.3.3. If $K, K' \in D(\mathcal{C}_{total})$. Assume

(1) K is cartesian,
(2) $\text{Hom}(K_i[i], K'_i) = 0$ for $i > 0$, and
(3) $\text{Hom}(K_i[i+1], K'_i) = 0$ for $i \geq 0$.

Then any map $K \to K'$ which induces the zero map $K_0 \to K'_0$ is zero.

Proof. Consider the objects X_n and the Postnikov system Y_n associated to K in Lemma 85.13.4. As $K = \text{hocolim} Y_n[n]$ the map $K \to K'$ induces a compatible family of morphisms $Y_n[n] \to K'$. By (1) and Lemma 85.12.9 we have $X_n = g_{n!}K_n$. Since $Y_0 = X_0$ we find that $K_0 \to K'_0$ being zero implies $Y_0 \to K'$ is zero. Suppose we've shown that the map $Y_n[n] \to K'$ is zero for some $n \geq 0$. From the distinguished triangle

$$Y_n[n] \to Y_{n+1}[n+1] \to X_{n+1}[n+1] \to Y_n[n+1]$$

we get an exact sequence

$$\text{Hom}(X_{n+1}[n+1], K') \to \text{Hom}(Y_{n+1}[n+1], K') \to \text{Hom}(Y_n[n], K')$$

As $X_{n+1}[n+1] = g_{n+1!}K_{n+1}[n+1]$ the first group is equal to

$$\text{Hom}(K_{n+1}[n+1], K'_{n+1})$$

which is zero by assumption (2). By induction we conclude all the maps $Y_n[n] \to K'$ are zero. Consider the defining distinguished triangle

$$\bigoplus Y_n[n] \to \bigoplus Y_n[n] \to K \to (\bigoplus Y_n[n])[1]$$

for the homotopy colimit. Arguing as above, we find that it suffices to show that

$$\text{Hom}((\bigoplus Y_n[n])[1], K') = \prod \text{Hom}(Y_n[n+1], K')$$

is zero for all $n \geq 0$. To see this, arguing as above, it suffices to show that

$$\text{Hom}(K_n[n+1], K'_n) = 0$$

for all $n \geq 0$ which follows from condition (3). □

0D9K Lemma 85.13.6. In Situation 85.3.3. If $K, K' \in D(\mathcal{C}_{total})$. Assume

(1) K is cartesian,
(2) $\text{Hom}(K_i[i-1], K'_i) = 0$ for $i > 1$.

Then any map $\{K_n \to K'_n\}$ between the associated simplicial systems of K and K' comes from a map $K \to K'$ in $D(\mathcal{C}_{total})$.

Proof. Let $\{K_n \to K'_n\}_{n \geq 0}$ be a morphism of simplicial systems of the derived category. Consider the objects X_n and Postnikov system Y_n associated to K of Lemma 85.13.4. By (1) and Lemma 85.12.9 we have $X_n = g_{n!}K_n$. In particular, the map $K_0 \to K'_0$ induces a morphism $X_0 \to K'$. Since $\{K_n \to K'_n\}$ is a morphism of systems, a computation (omitted) shows that the composition

$$X_1 \to X_0 \to K'$$

is zero. As $Y_0 = X_0$ and as Y_1 fits into a distinguished triangle

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

we conclude that there exists a morphism $Y_1[1] \to K'$ whose composition with $X_0 = Y_0 \to Y_1[1]$ is the morphism $X_0 \to K'$ given above. Suppose given a map $Y_n[n] \to K'$ for $n \geq 1$. From the distinguished triangle

$$X_{n+1}[n] \to Y_n[n] \to Y_{n+1}[n+1] \to X_{n+1}[n+1]$$

we get an exact sequence

$$\text{Hom}(Y_{n+1}[n+1], K') \to \text{Hom}(Y_n[n], K') \to \text{Hom}(X_{n+1}[n], K')$$

As $X_{n+1}[n] = g_{n+1!}K_{n+1}[n]$ the last group is equal to

$$\text{Hom}(K_{n+1}[n], K'_{n+1})$$

which is zero by assumption (2). By induction we get a system of maps $Y_n[n] \to K'$ compatible with transition maps and reducing to the given map on Y_0. This produces a map

$$\gamma : K = \text{hocolim} Y_n[n] \longrightarrow K'$$

This map in any case has the property that the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[dr] & K \arrow[d, "\gamma"] \\ & K' \end{tikzcd}$$

is commutative. Restricting to $\mathcal{C}_0$ we deduce that the map $\gamma_0 : K_0 \to K'_0$ is the same as the first map $K_0 \to K'_0$ of the morphism of simplicial systems. Since K is cartesian, this easily gives that $\{\gamma_n\}$ is the map of simplicial systems we started out with. $\square$

0D9L Lemma 85.13.7. In Situation 85.3.3. Let (K_n, K_φ) be a simplicial system of the derived category. Assume

(1) (K_n, K_φ) is cartesian,
(2) $\text{Hom}(K_i[t], K_i) = 0$ for $i \geq 0$ and $t > 0$.

Then there exists a cartesian object K of $D(\mathcal{C}_{total})$ whose associated simplicial system is isomorphic to (K_n, K_φ).

Proof. Set $X_n = g_{n!}K_n$ in $D(\mathcal{C}_{total})$. For each $n \geq 1$ we have

$$\mathrm{Hom}(X_n, X_{n-1}) = \mathrm{Hom}(K_n, g_n^{-1}g_{n-1!}K_{n-1}) = \bigoplus\nolimits_{\varphi : [n-1] \to [n]} \mathrm{Hom}(K_n, f_\varphi^{-1}K_{n-1})$$

Thus we get a map $X_n \to X_{n-1}$ corresponding to the alternating sum of the maps $K_\varphi^{-1} : K_n \to f_\varphi^{-1}K_{n-1}$ where φ runs over $\delta_0^n, \ldots, \delta_n^n$. We can do this because K_φ is invertible by assumption (1). Please observe the similarity with the definition of the maps in the proof of Lemma 85.8.1. We obtain a complex

$$\ldots \to X_2 \to X_1 \to X_0$$

in $D(\mathcal{C}_{total})$. We omit the computation which shows that the compositions are zero. By Derived Categories, Lemma 13.41.6 if we have

$$\mathrm{Hom}(X_i[i-j-2], X_j) = 0 \text{ for } i > j+2$$

then we can extend this complex to a Postnikov system. The group is equal to

$$\mathrm{Hom}(K_i[i-j-2], g_i^{-1}g_{j!}K_j)$$

Again using that (K_n, K_φ) is cartesian we see that $g_i^{-1}g_{j!}K_j$ is isomorphic to a finite direct sum of copies of K_i. Hence the group vanishes by assumption (2). Let the Postnikov system be given by $Y_0 = X_0$ and distinguished sequences $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ for $n \geq 1$. We set

$$K = \mathrm{hocolim} Y_n[n]$$

To finish the proof we have to show that $g_m^{-1}K$ is isomorphic to K_m for all m compatible with the maps K_φ. Observe that

$$g_m^{-1}K = \mathrm{hocolim} g_m^{-1}Y_n[n]$$

and that $g_m^{-1}Y_n[n]$ is a Postnikov system for $g_m^{-1}X_n$. Consider the isomorphisms

$$g_m^{-1}X_n = \bigoplus\nolimits_{\varphi : [n] \to [m]} f_\varphi^{-1}K_n \xrightarrow{\bigoplus K_\varphi} \bigoplus\nolimits_{\varphi : [n] \to [m]} K_m$$

These maps define an isomorphism of complexes

$$\begin{tikzcd}
\ldots \arrow[r] & g_m^{-1}X_2 \arrow[r] \arrow[d] & g_m^{-1}X_1 \arrow[r] \arrow[d] & g_m^{-1}X_0 \arrow[d] \\
\ldots \arrow[r] & \bigoplus_{\varphi : [2] \to [m]} K_m \arrow[r] & \bigoplus_{\varphi : [1] \to [m]} K_m \arrow[r] & \bigoplus_{\varphi : [0] \to [m]} K_m
\end{tikzcd}$$

in $D(\mathcal{C}_m)$ where the arrows in the bottom row are as in the proof of Lemma 85.8.1. The squares commute by our choice of the arrows of the complex $\ldots \to X_2 \to X_1 \to X_0$; we omit the computation. The bottom row complex has a postnikov tower given by

$$Y'_{m,n} = \left(\bigoplus\nolimits_{\varphi : [n] \to [m]} \mathbf{Z} \to \ldots \to \bigoplus\nolimits_{\varphi : [0] \to [m]} \mathbf{Z} \right) [-n] \otimes_{\mathbf{Z}}^{\mathbf{L}} K_m$$

and $\mathrm{hocolim} Y'_{m,n} = K_m$ (please compare with the proof of Lemma 85.13.4 and Derived Categories, Example 13.41.2). Applying the second part of Derived Categories, Lemma 13.41.6 the vertical maps in the big diagram extend to an isomorphism of Postnikov systems provided we have

$$\mathrm{Hom}(g_m^{-1}X_i[i-j-1], \bigoplus\nolimits_{\varphi : [j] \to [m]} K_m) = 0 \text{ for } i > j+1$$

The is true if $\mathrm{Hom}(K_m[i-j-1], K_m) = 0$ for $i > j+1$ which holds by assumption (2). Choose an isomorphism given by $\gamma_{m,n} : g_m^{-1}Y_n \to Y'_{m,n}$ of Postnikov systems in $D(\mathcal{C}_m)$. By uniqueness of homotopy colimits, we can find an isomorphism

$$g_m^{-1}K = \mathrm{hocolim} g_m^{-1}Y_n[n] \xrightarrow{\gamma_m} \mathrm{hocolim} Y'_{m,n} = K_m$$

compatible with $\gamma_{m,n}$.

We still have to prove that the maps γ_m fit into commutative diagrams

$$\begin{tikzcd} f_\varphi^{-1}g_m^{-1}K \arrow[r, "K(\varphi)"'] \arrow[d, "f_\varphi^{-1}\gamma_m"'] & g_n^{-1}K \arrow[d, "\gamma_n"] \\ f_\varphi^{-1}K_m \arrow[r, "K_\varphi"] & K_n \end{tikzcd}$$

for every $\varphi : [m] \to [n]$. Consider the diagram

$$\begin{tikzcd} f_\varphi^{-1}(\bigoplus_{\psi:[0]\to[m]} f_\psi^{-1}K_0) \arrow[r, equal] \arrow[d, "f_\varphi^{-1}(\bigoplus K_\psi)"'] & f_\varphi^{-1}g_m^{-1}X_0 \arrow[r, "X_0(\varphi)"'] \arrow[d] & g_n^{-1}X_0 \arrow[r, equal] \arrow[d] & \bigoplus_{\chi:[0]\to[n]} f_\chi^{-1}K_0 \arrow[d, "\bigoplus K_\chi"] \\ f_\varphi^{-1}(\bigoplus_{\psi:[0]\to[m]} K_m) \arrow[d, equal] & f_\varphi^{-1}g_m^{-1}K \arrow[r, "K(\varphi)"'] \arrow[d, "f_\varphi^{-1}\gamma_m"'] & g_n^{-1}K \arrow[d, "\gamma_n"] & \bigoplus_{\chi:[0]\to[n]} K_n \arrow[d, equal] \\ f_\varphi^{-1}Y'_{0,m} \arrow[r] & f_\varphi^{-1}K_m \arrow[r, "K_\varphi"] & K_n & Y'_{0,n} \arrow[l] \end{tikzcd}$$

The top middle square is commutative as $X_0 \to K$ is a morphism of simplicial objects. The left, resp. the right rectangles are commutative as γ_m, resp. γ_n is compatible with $\gamma_{0,m}$, resp. $\gamma_{0,n}$ which are the arrows $\bigoplus K_\psi$ and $\bigoplus K_\chi$ in the diagram. Going around the outer rectangle of the diagram is commutative as (K_n, K_φ) is a simplical system and the map $X_0(\varphi)$ is given by the obvious identifications $f_\varphi^{-1}f_\psi^{-1}K_0 = f_{\varphi\circ\psi}^{-1}K_0$. Note that the arrow $\bigoplus_\psi K_m \to Y'_{0,m} \to K_m$ induces an isomorphism on any of the direct summands (because of our explicit construction of the Postnikov systems $Y'_{i,j}$ above). Hence, if we take a direct summand of the upper left and corner, then this maps isomorphically to $f_\varphi^{-1}g_m^{-1}K$ as γ_m is an isomorphism. Working out what the above says, but looking only at this direct summand we conclude the lower middle square commutes as we well. This concludes the proof. □

85.14. Simplicial systems of the derived category: modules

0D9M In this section we are going to prove a special case of [BBD82, Proposition 3.2.9] in the setting of derived categories of $\mathcal{O}$-modules. The (slightly) easier case of abelian sheaves is discussed in Section 85.13.

0D9N Definition 85.14.1. In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. A simplicial system of the derived category of modules consists of the following data

(1) for every n an object K_n of $D(\mathcal{O}_n)$,
(2) for every $\varphi : [m] \to [n]$ a map $K_\varphi : Lf_\varphi^*K_m \to K_n$ in $D(\mathcal{O}_n)$

subject to the condition that

$$K_{\varphi\circ\psi} = K_\varphi \circ Lf_\varphi^*K_\psi : Lf_{\varphi\circ\psi}^*K_l = Lf_\varphi^*Lf_\psi^*K_l \longrightarrow K_n$$

for any morphisms $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ of Δ. We say the simplicial system is cartesian if the maps K_φ are isomorphisms for all φ. Given two simplicial systems of the derived category there is an obvious notion of a morphism of simplicial systems of the derived category of modules.

We have given this notion a ridiculously long name intentionally. The goal is to show that a simplicial system of the derived category of modules comes from an object of $D(\mathcal{O})$ under certain hypotheses.

0D9P Lemma 85.14.2. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K \in D(\mathcal{O})$ is an object, then $(K_n, K(\varphi))$ is a simplicial system of the derived category of modules. If K is cartesian, so is the system.

Proof. This is immediate from the definitions. $\square$

0GMF Lemma 85.14.3. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Suppose given $K_0 \in D(\mathcal{O}_0)$ and an isomorphism

$$\alpha : L(f_{\delta^1_1})^*K_0 \longrightarrow L(f_{\delta^1_0})^*K_0$$

satisfying the cocycle condition. Set $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and set $K_n = Lf^*_{\tau^n_n}K_0$. The objects K_n form the members of a cartesian simplicial system of the derived category of modules.

Proof. Please compare with Lemmas 85.13.3 and 85.12.4 and its proof (also to see the cocycle condition spelled out). The construction is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Given $\varphi : [n] \to [m]$ we define $K_\varphi : L(f_\varphi)^*K_n \to K_m$ using

$$\begin{tikzcd}
L(f_\varphi)^*K_n \arrow[r, equal] & L(f_\varphi)^*L(f_{\tau^n_n})^*K_0 \arrow[r, equal] & L(f_{\tau^m_{\varphi(n)}})^*K_0 \arrow[r, equal] & L(f_{\tau^m_{\varphi(n)m}})^*L(f_{\delta^1_1})^*K_0 \arrow[d, "L(f_{\tau^m_{\varphi(n)m}})^*\alpha"] \\
& K_m \arrow[r, equal] & L(f_{\tau^m_m})^*K_0 \arrow[r, equal] & L(f_{\tau^m_{\varphi(n)m}})^*L(f_{\delta^1_0})^*K_0
\end{tikzcd}$$

We omit the verification that the cocycle condition implies the maps compose correctly (in their respective derived categories) and hence give rise to a simplicial systems of the derived category of modules. $\square$

0D9Q Lemma 85.14.4. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let K be an object of $D(\mathcal{C}_{total})$. Set

$$X_n = (g_{n!}\mathcal{O}_n) \otimes^{\mathbf{L}}_{\mathcal{O}} K \quad \text{and} \quad Y_n = (g_{n!}\mathcal{O}_n \to \ldots \to g_{0!}\mathcal{O}_0)[-n] \otimes^{\mathbf{L}}_{\mathcal{O}} K$$

as objects of $D(\mathcal{O})$ where the maps are as in Lemma 85.8.1. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ we have

(1) the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system (Derived Categories, Definition 13.41.1) for $\ldots \to X_2 \to X_1 \to X_0$,
(2) $K = \text{hocolim} Y_n[n]$ in $D(\mathcal{O})$.

Proof. First, if $K = \mathcal{O}$, then this is the construction of Derived Categories, Example 13.41.2 applied to the complex

$$\ldots \to g_{2!}\mathcal{O}_2 \to g_{1!}\mathcal{O}_1 \to g_{0!}\mathcal{O}_0$$

in $\text{Ab}(\mathcal{C}_{total})$ combined with the fact that this complex represents $K = \mathcal{O}$ in $D(\mathcal{C}_{total})$ by Lemma 85.10.1. The general case follows from this, the fact that the exact functor $- \otimes_{\mathcal{O}}^{\mathbf{L}} K$ sends Postnikov systems to Postnikov systems, and that $- \otimes_{\mathcal{O}}^{\mathbf{L}} K$ commutes with homotopy colimits. □

0D9R Lemma 85.14.5. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K, K' \in D(\mathcal{O})$. Assume

(1) $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for $\varphi : [m] \to [n]$,
(2) K is cartesian,
(3) $\text{Hom}(K_i[i], K'_i) = 0$ for $i > 0$, and
(4) $\text{Hom}(K_i[i+1], K'_i) = 0$ for $i \geq 0$.

Then any map $K \to K'$ which induces the zero map $K_0 \to K'_0$ is zero.

Proof. The proof is exactly the same as the proof of Lemma 85.13.5 except using Lemma 85.14.4 instead of Lemma 85.13.4. □

0D9S Lemma 85.14.6. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K, K' \in D(\mathcal{O})$. Assume

(1) $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for $\varphi : [m] \to [n]$,
(2) K is cartesian,
(3) $\text{Hom}(K_i[i-1], K'_i) = 0$ for $i > 1$.

Then any map $\{K_n \to K'_n\}$ between the associated simplicial systems of K and K' comes from a map $K \to K'$ in $D(\mathcal{O})$.

Proof. The proof is exactly the same as the proof of Lemma 85.13.6 except using Lemma 85.14.4 instead of Lemma 85.13.4. □

0D9T Lemma 85.14.7. In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let (K_n, K_φ) be a simplicial system of the derived category of modules. Assume

(1) $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for $\varphi : [m] \to [n]$,
(2) (K_n, K_φ) is cartesian,
(3) $\text{Hom}(K_i[t], K_i) = 0$ for $i \geq 0$ and $t > 0$.

Then there exists a cartesian object K of $D(\mathcal{O})$ whose associated simplicial system is isomorphic to (K_n, K_φ).

Proof. The proof is exactly the same as the proof of Lemma 85.13.7 with the following changes

(1) use $g_n^* = Lg_n^*$ everywhere instead of g_n^{-1},
(2) use $f_\varphi^* = Lf_\varphi^*$ everywhere instead of f_φ^{-1},
(3) refer to Lemma 85.10.1 instead of Lemma 85.8.1,
(4) in the construction of $Y'_{m,n}$ use $\mathcal{O}_m$ instead of $\mathbf{Z}$,
(5) compare with the proof of Lemma 85.14.4 rather than the proof of Lemma 85.13.4.

This ends the proof. □

85.15. The site associated to a semi-representable object

09WK Let $\mathcal{C}$ be a site. Recall that a semi-representable object of $\mathcal{C}$ is simply a family $\{U_i\}_{i \in I}$ of objects of $\mathcal{C}$. A morphism $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ of semi-representable objects is given by a map $\alpha : I \to J$ and for every $i \in I$ a morphism $f_i : U_i \to$

$V_{\alpha(i)}$ of $\mathcal{C}$. The category of semi-representable objects of $\mathcal{C}$ is denoted $\mathrm{SR}(\mathcal{C})$. See Hypercoverings, Definition 25.2.1 and the enclosing section for more information.

For a semi-representable object $K = \{U_i\}_{i \in I}$ of $\mathcal{C}$ we let

$$\mathcal{C}/K = \coprod\nolimits_{i \in I} \mathcal{C}/U_i$$

be the disjoint union of the localizations of $\mathcal{C}$ at U_i. There is a natural structure of a site on this category, with coverings inherited from the localizations $\mathcal{C}/U_i$. The site $\mathcal{C}/K$ is called the localization of $\mathcal{C}$ at K. Observe that a sheaf on $\mathcal{C}/K$ is the same thing as a family of sheaves $\mathcal{F}_i$ on $\mathcal{C}/U_i$, i.e.,

$$Sh(\mathcal{C}/K) = \prod\nolimits_{i \in I} Sh(\mathcal{C}/U_i)$$

This is occasionally useful to understand what is going on.

Let $\mathcal{C}$ be a site. Let $K = \{U_i\}_{i \in I}$ be an object of $\mathrm{SR}(\mathcal{C})$. There is a continuous and cocontinuous localization functor $j : \mathcal{C}/K \to \mathcal{C}$ which is the product of the localization functors $j_i : \mathcal{C}/V_i \to \mathcal{C}$. We obtain functors $j_!$, j^{-1}, j_* exactly as in Sites, Section 7.25. In terms of the product decomposition $Sh(\mathcal{C}/K) = \prod_{i \in I} Sh(\mathcal{C}/U_i)$ we have

$$\begin{matrix} j_! & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & \coprod j_{i,!}\mathcal{F}_i \\ j^{-1} & : & \mathcal{G} & \longmapsto & (j_i^{-1}\mathcal{G})_{i \in I} \\ j_* & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & \prod j_{i,*}\mathcal{F}_i \end{matrix}$$

as the reader easily verifies.

Let $f : K \to L$ be a morphism of $\mathrm{SR}(\mathcal{C})$. Then we obtain a continuous and cocontinuous functor

$$v : \mathcal{C}/K \longrightarrow \mathcal{C}/L$$

by applying the construction of Sites, Lemma 7.25.8 to the components. More precisely, suppose $f = (\alpha, f_i)$ where $K = \{U_i\}_{i \in I}$, $L = \{V_j\}_{j \in J}$, $\alpha : I \to J$, and $f_i : U_i \to V_{\alpha(i)}$. Then the functor v maps the component $\mathcal{C}/U_i$ into the component $\mathcal{C}/V_{\alpha(i)}$ via the construction of the aforementioned lemma. In particular we obtain a morphism

$$f : Sh(\mathcal{C}/K) \to Sh(\mathcal{C}/L)$$

of topoi. In terms of the product decompositions $Sh(\mathcal{C}/K) = \prod_{i \in I} Sh(\mathcal{C}/U_i)$ and $Sh(\mathcal{C}/L) = \prod_{j \in J} Sh(\mathcal{C}/V_j)$ the reader verifies that

$$\begin{matrix} f_! & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & (\coprod_{i \in I, \alpha(i) = j} f_{i,!}\mathcal{F}_i)_{j \in J} \\ f^{-1} & : & (\mathcal{G}_j)_{j \in J} & \longmapsto & (f_i^{-1}\mathcal{G}_{\alpha(i)})_{i \in I} \\ f_* & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & (\prod_{i \in I, \alpha(i) = j} f_{i,*}\mathcal{F}_i)_{j \in J} \end{matrix}$$

where $f_i : Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C}/V_{\alpha(i)})$ is the morphism associated to the localization functor $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ corresponding to $f_i : U_i \to V_{\alpha(i)}$.

0D85 **Lemma 85.15.1.** Let $\mathcal{C}$ be a site.

(1) For K in $\mathrm{SR}(\mathcal{C})$ the functor $j : \mathcal{C}/K \to \mathcal{C}$ is continuous, cocontinuous, and has property P of Sites, Remark 7.20.5.
(2) For $f : K \to L$ in $\mathrm{SR}(\mathcal{C})$ the functor $v : \mathcal{C}/K \to \mathcal{C}/L$ (see above) is continuous, cocontinuous, and has property P of Sites, Remark 7.20.5.

Proof. Proof of (2). In the notation of the discussion preceding the lemma, the localization functors $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ are continuous and cocontinuous by Sites, Section 7.25 and satisfy P by Sites, Remark 7.25.11. It is formal to deduce v is continuous and cocontinuous and has P. We omit the details. We also omit the proof of (1). $\square$

0D86 Lemma 85.15.2. Let $\mathcal{C}$ be a site and K in $\text{SR}(\mathcal{C})$. For $\mathcal{F}$ in $\mathit{Sh}(\mathcal{C})$ we have

$$j_*j^{-1}\mathcal{F} = \mathcal{H}om(F(K)^\#, \mathcal{F})$$

where F is as in Hypercoverings, Definition 25.2.2.

Proof. Say $K = \{U_i\}_{i \in I}$. Using the description of the functors j^{-1} and j_* given above we see that

$$j_*j^{-1}\mathcal{F} = \prod\nolimits_{i \in I} j_{i,*}(\mathcal{F}|_{\mathcal{C}/U_i}) = \prod\nolimits_{i \in I} \mathcal{H}om(h_{U_i}^\#, \mathcal{F})$$

The second equality by Sites, Lemma 7.26.3. Since $F(K) = \coprod h_{U_i}$ in $\text{PSh}(\mathcal{C}$, we have $F(K)^\# = \coprod h_{U_i}^\#$ in $\mathit{Sh}(\mathcal{C})$ and since $\mathcal{H}om(-, \mathcal{F})$ turns coproducts into products (immediate from the construction in Sites, Section 7.26), we conclude. $\square$

0D87 Lemma 85.15.3. Let $\mathcal{C}$ be a site.

(1) For K in $\text{SR}(\mathcal{C})$ the functor $j_!$ gives an equivalence $\mathit{Sh}(\mathcal{C}/K) \to \mathit{Sh}(\mathcal{C})/F(K)^\#$ where F is as in Hypercoverings, Definition 25.2.2.
(2) The functor $j^{-1} : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{C}/K)$ corresponds via the identification of (1) with $\mathcal{F} \mapsto (\mathcal{F} \times F(K)^\# \to F(K)^\#)$.
(3) For $f : K \to L$ in $\text{SR}(\mathcal{C})$ the functor f^{-1} corresponds via the identifications of (1) to the functor $\mathit{Sh}(\mathcal{C})/F(L)^\# \to \mathit{Sh}(\mathcal{C})/F(K)^\#$, $(\mathcal{G} \to F(L)^\#) \mapsto (\mathcal{G} \times_{F(L)^\#} F(K)^\# \to F(K)^\#)$.

Proof. Observe that if $K = \{U_i\}_{i \in I}$ then the category $\mathit{Sh}(\mathcal{C}/K)$ decomposes as the product of the categories $\mathit{Sh}(\mathcal{C}/U_i)$. Observe that $F(K)^\# = \coprod_{i \in I} h_{U_i}^\#$ (coproduct in sheaves). Hence $\mathit{Sh}(\mathcal{C})/F(K)^\#$ is the product of the categories $\mathit{Sh}(\mathcal{C})/h_{U_i}^\#$. Thus (1) and (2) follow from the corresponding statements for each i, see Sites, Lemmas 7.25.4 and 7.25.7. Similarly, if $L = \{V_j\}_{j \in J}$ and f is given by $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$, then we can apply Sites, Lemma 7.25.9 to each of the re-localization morphisms $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ to get (3). $\square$

0D88 Lemma 85.15.4. Let $\mathcal{C}$ be a site. For K in $\text{SR}(\mathcal{C})$ the functor j^{-1} sends injective abelian sheaves to injective abelian sheaves. Similarly, the functor j^{-1} sends K-injective complexes of abelian sheaves to K-injective complexes of abelian sheaves.

Proof. The first statement is the natural generalization of Cohomology on Sites, Lemma 21.7.1 to semi-representable objects. In fact, it follows from this lemma by the product decomposition of $\mathit{Sh}(\mathcal{C}/K)$ and the description of the functor j^{-1} given above. The second statement is the natural generalization of Cohomology on Sites, Lemma 21.20.1 and follows from it by the product decomposition of the topos.

Alternative: since j induces a localization of topoi by Lemma 85.15.3 part (1) it also follows immediately from Cohomology on Sites, Lemmas 21.7.1 and 21.20.1 by enlarging the site; compare with the proof of Cohomology on Sites, Lemma 21.13.3 in the case of injective sheaves. $\square$

0D89 Remark 85.15.5 (Variant for over an object). Let $\mathcal{C}$ be a site. Let $X \in \mathrm{Ob}(\mathcal{C})$. The category $\mathrm{SR}(\mathcal{C}, X)$ of semi-representable objects over X is defined by the formula $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$. See Hypercoverings, Definition 25.2.1. Thus we may apply the above discussion to the site $\mathcal{C}/X$. Briefly, the constructions above give

(1) a site $\mathcal{C}/K$ for K in $\mathrm{SR}(\mathcal{C}, X)$,
(2) a decomposition $Sh(\mathcal{C}/K) = \prod Sh(\mathcal{C}/U_i)$ if $K = \{U_i/X\}$,
(3) a localization functor $j : \mathcal{C}/K \to \mathcal{C}/X$,
(4) a morphism $f : Sh(\mathcal{C}/K) \to Sh(\mathcal{C}/L)$ for $f : K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$.

All results of this section hold in this situation by replacing $\mathcal{C}$ everywhere by $\mathcal{C}/X$.

0D9U Remark 85.15.6 (Ringed variant). Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. In this case, for any semi-representable object K of $\mathcal{C}$ the site $\mathcal{C}/K$ is a ringed site with sheaf of rings $\mathcal{O}_K = j^{-1}\mathcal{O}_\mathcal{C}$. The constructions above give

(1) a ringed site $(\mathcal{C}/K, \mathcal{O}_K)$ for K in $\mathrm{SR}(\mathcal{C})$,
(2) a decomposition $\mathrm{Mod}(\mathcal{O}_K) = \prod \mathrm{Mod}(\mathcal{O}_{U_i})$ if $K = \{U_i\}$,
(3) a localization morphism $j : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ of ringed topoi,
(4) a morphism $f : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/L), \mathcal{O}_L)$ of ringed topoi for $f : K \to L$ in $\mathrm{SR}(\mathcal{C})$.

Many of the results above hold in this setting. For example, the functor j^* has an exact left adjoint

$$j_! : \mathrm{Mod}(\mathcal{O}_K) \to \mathrm{Mod}(\mathcal{O}_\mathcal{C}),$$

which in terms of the product decomposition given in (2) sends $(\mathcal{F}_i)_{i \in I}$ to $\bigoplus j_{i,!}\mathcal{F}_i$. Similarly, given $f : K \to L$ as above, the functor f^* has an exact left adjoint $f_! : \mathrm{Mod}(\mathcal{O}_K) \to \mathrm{Mod}(\mathcal{O}_L)$. Thus the functors j^* and f^* are exact, i.e., j and f are flat morphisms of ringed topoi (also follows from the equalities $\mathcal{O}_K = j^{-1}\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_K = f^{-1}\mathcal{O}_L$).

0D9V Remark 85.15.7 (Ringed variant over an object). Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Let $X \in \mathrm{Ob}(\mathcal{C})$ and denote $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/U}$. Then we can combine the constructions given in Remarks 85.15.5 and 85.15.6 to get

(1) a ringed site $(\mathcal{C}/K, \mathcal{O}_K)$ for K in $\mathrm{SR}(\mathcal{C}, X)$,
(2) a decomposition $\mathrm{Mod}(\mathcal{O}_K) = \prod \mathrm{Mod}(\mathcal{O}_{U_i})$ if $K = \{U_i\}$,
(3) a localization morphism $j : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$ of ringed topoi,
(4) a morphism $f : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/L), \mathcal{O}_L)$ of ringed topoi for $f : K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$.

Of course all of the results mentioned in Remark 85.15.6 hold in this setting as well.

85.16. The site associate to a simplicial semi-representable object

0D8A Let $\mathcal{C}$ be a site. Let K be a simplicial object of $\mathrm{SR}(\mathcal{C})$. As usual, set $K_n = K([n])$ and denote $K(\varphi) : K_n \to K_m$ the morphism associated to $\varphi : [m] \to [n]$. By the construction in Section 85.15 we obtain a simplicial object $n \mapsto \mathcal{C}/K_n$ in the category whose objects are sites and whose morphisms are cocontinuous functors. In other words, we get a gadget as in Case B of Section 85.3. The functors satisfy property P by Lemma 85.15.1. Hence we may apply Lemma 85.3.2 to obtain a site $(\mathcal{C}/K)_{total}$.

We can describe the site $(\mathcal{C}/K)_{total}$ explicitly as follows. Say $K_n = \{U_{n,i}\}_{i \in I_n}$. For $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by a map $\alpha(\varphi) : I_n \to I_m$ and morphisms $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ for $i \in I_n$. Then we have

(1) an object of $(\mathcal{C}/K)_{total}$ corresponds to an object $(U/U_{n,i})$ of $\mathcal{C}/U_{n,i}$ for some n and some $i \in I_n$,
(2) a morphism between $U/U_{n,i}$ and $V/U_{m,j}$ is a pair (φ, f) where $\varphi : [m] \to [n]$, $j = \alpha(\varphi)(i)$, and $f : U \to V$ is a morphism of $\mathcal{C}$ such that
$$\begin{tikzcd} U \arrow[r, "f"'] \arrow[d] & V \arrow[d] \\ U_{n,i} \arrow[r, "f_{\varphi,i}"] & U_{m,j} \end{tikzcd}$$
is commutative, and
(3) coverings of the object $U/U_{n,i}$ are constructed by starting with a covering $\{f_j : U_j \to U\}$ in $\mathcal{C}$ and letting $\{(\mathrm{id}, f_j) : U_j/U_{n,i} \to U/U_{n,i}\}$ be a covering in $(\mathcal{C}/K)_{total}$.

All of our general theory developed for simplicial sites applies to $(\mathcal{C}/K)_{total}$. Observe that the obvious forgetful functor
$$j_{total} : (\mathcal{C}/K)_{total} \longrightarrow \mathcal{C}$$
is continuous and cocontinuous. It turns out that the associated morphism of topoi comes from an (obvious) augmentation.

0D8B Lemma 85.16.1. Let $\mathcal{C}$ be a site. Let K be a simplicial object of $\mathrm{SR}(\mathcal{C})$. The localization functor $j_0 : \mathcal{C}/K_0 \to \mathcal{C}$ defines an augmentation $a_0 : Sh(\mathcal{C}/K_0) \to Sh(\mathcal{C})$, as in case (B) of Remark 85.4.1. The corresponding morphisms of topoi
$$a_n : Sh(\mathcal{C}/K_n) \longrightarrow Sh(\mathcal{C}), \quad a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C})$$
of Lemma 85.4.2 are equal to the morphisms of topoi associated to the continuous and cocontinuous localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}$ and $j_{total} : (\mathcal{C}/K)_{total} \to \mathcal{C}$.

Proof. This is immediate from working through the definitions. See in particular the footnote in the proof of Lemma 85.4.2 for the relationship between a and j_{total}. $\square$

09WM Lemma 85.16.2. With assumption and notation as in Lemma 85.16.1 we have the following properties:

(1) there is a functor $a_!^{Sh} : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C})$ left adjoint to $a^{-1} : Sh(\mathcal{C}) \to Sh((\mathcal{C}/K)_{total})$,
(2) there is a functor $a_! : \mathrm{Ab}((\mathcal{C}/K)_{total}) \to \mathrm{Ab}(\mathcal{C})$ left adjoint to $a^{-1} : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}((\mathcal{C}/K)_{total})$,
(3) the functor a^{-1} associates to $\mathcal{F}$ in $Sh(\mathcal{C})$ the sheaf on $(\mathcal{C}/K)_{total}$ wich in degree n is equal to $a_n^{-1}\mathcal{F}$,
(4) the functor a_* associates to $\mathcal{G}$ in $\mathrm{Ab}((\mathcal{C}/K)_{total})$ the equalizer of the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$,

Proof. Parts (3) and (4) hold for any augmentation of a simplicial site, see Lemma 85.4.2. Parts (1) and (2) follow as j_{total} is continuous and cocontinuous. The functor $a_!^{Sh}$ is constructed in Sites, Lemma 7.21.5 and the functor $a_!$ is constructed in Modules on Sites, Lemma 18.16.2. $\square$

0DC0 Lemma 85.16.3. Let $\mathcal{C}$ be a site. Let K be a simplicial object of $\text{SR}(\mathcal{C})$. Let $U/U_{n,i}$ be an object of $\mathcal{C}/K_n$. Let $\mathcal{F} \in \text{Ab}((\mathcal{C}/K)_{total})$. Then

$$H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}_{n,i})$$

where

(1) on the left hand side U is viewed as an object of $\mathcal{C}_{total}$, and
(2) on the right hand side $\mathcal{F}_{n,i}$ is the ith component of the sheaf $\mathcal{F}_n$ on $\mathcal{C}/K_n$ in the decomposition $Sh(\mathcal{C}/K_n) = \prod Sh(\mathcal{C}/U_{n,i})$ of Section 85.15.

Proof. This follows immediately from Lemma 85.8.6 and the product decompositions of Section 85.15. $\square$

0D8C Remark 85.16.4 (Variant for over an object). Let $\mathcal{C}$ be a site. Let $X \in \text{Ob}(\mathcal{C})$. Recall that we have a category $\text{SR}(\mathcal{C}, X) = \text{SR}(\mathcal{C}/X)$ of semi-representable objects over X, see Remark 85.15.5. We may apply the above discussion to the site $\mathcal{C}/X$. Briefly, the constructions above give

(1) a site $(\mathcal{C}/K)_{total}$ for a simplicial K object of $\text{SR}(\mathcal{C}, X)$,
(2) a localization functor $j_{total} : (\mathcal{C}/K)_{total} \to \mathcal{C}/X$,
(3) localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}/X$,
(4) a morphism of topoi $a : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/X)$,
(5) morphisms of topoi $a_n : Sh(\mathcal{C}/K_n) \to Sh(\mathcal{C}/X)$,
(6) a functor $a_!^{Sh} : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/X)$ left adjoint to a^{-1}, and
(7) a functor $a_! : \text{Ab}((\mathcal{C}/K)_{total}) \to \text{Ab}(\mathcal{C}/X)$ left adjoint to a^{-1}.

All of the results of this section hold in this setting. To prove this one replaces the site $\mathcal{C}$ everywhere by $\mathcal{C}/X$.

0D9W Remark 85.16.5 (Ringed variant). Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Given a simplicial semi-representable object K of $\mathcal{C}$ we set $\mathcal{O} = a^{-1}\mathcal{O}_\mathcal{C}$, where a is as in Lemmas 85.16.1 and 85.16.2. The constructions above, keeping track of the sheaves of rings as in Remark 85.15.6, give

(1) a ringed site $((\mathcal{C}/K)_{total}, \mathcal{O})$ for a simplicial K object of $\text{SR}(\mathcal{C})$,
(2) a morphism of ringed topoi $a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$,
(3) morphisms of ringed topoi $a_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$,
(4) a functor $a_! : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_\mathcal{C})$ left adjoint to a^*.

The functor $a_!$ exists (but in general is not exact) because $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ and we can replace the use of Modules on Sites, Lemma 18.16.2 in the proof of Lemma 85.16.2 by Modules on Sites, Lemma 18.41.1. As discussed in Remark 85.15.6 there are exact functors $a_{n!} : \text{Mod}(\mathcal{O}_n) \to \text{Mod}(\mathcal{O}_\mathcal{C})$ left adjoint to a_n^*. Consequently, the morphisms a and a_n are flat. Remark 85.15.6 implies the morphism of ringed topoi $f_\varphi : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}/K_m), \mathcal{O}_m)$ for $\varphi : [m] \to [n]$ is flat and there exists an exact functor $f_{\varphi!} : \text{Mod}(\mathcal{O}_n) \to \text{Mod}(\mathcal{O}_m)$ left adjoint to f_φ^*. This in turn implies that for the flat morphism of ringed topoi $g_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh((\mathcal{C}/K)_{total}), \mathcal{O})$ the functor $g_{n!} : \text{Mod}(\mathcal{O}_n) \to \text{Mod}(\mathcal{O})$ left adjoint to g_n^* is exact, see Lemma 85.6.3.

0D9X Remark 85.16.6 (Ringed variant over an object). Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let $X \in \text{Ob}(\mathcal{C})$ and denote $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/X}$. Then we can combine the constructions given in Remarks 85.16.4 and 85.16.5 to get

(1) a ringed site $((\mathcal{C}/K)_{total}, \mathcal{O})$ for a simplicial K object of $\text{SR}(\mathcal{C}, X)$,
(2) a morphism of ringed topoi $a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$,

(3) morphisms of ringed topoi $a_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$,
(4) a functor $a_! : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_X)$ left adjoint to a^*.

Of course, all the results mentioned in Remark 85.16.5 hold in this setting as well.

85.17. Cohomological descent for hypercoverings

0D8D Let $\mathcal{C}$ be a site. In this section we assume $\mathcal{C}$ has equalizers and fibre products. We let K be a hypercovering as defined in Hypercoverings, Definition 25.6.1. We will study the augmentation

$$a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C})$$

of Section 85.16.

0D8E **Lemma 85.17.1.** Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Then

(1) $a^{-1} : Sh(\mathcal{C}) \to Sh((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,
(2) $a^{-1} : \text{Ab}(\mathcal{C}) \to \text{Ab}((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.

In both cases a_* provides the quasi-inverse functor.

Proof. The case of abelian sheaves follows immediately from the case of sheaves of sets as the functor a^{-1} commutes with products. In the rest of the proof we work with sheaves of sets. Observe that $a^{-1}\mathcal{F}$ is cartesian for $\mathcal{F}$ in $Sh(\mathcal{C})$ by Lemma 85.12.3. It suffices to show that the adjunction map $\mathcal{F} \to a_*a^{-1}\mathcal{F}$ is an isomorphism $\mathcal{F}$ in $Sh(\mathcal{C})$ and that for a cartesian sheaf $\mathcal{G}$ on $(\mathcal{C}/K)_{total}$ the adjunction map $a^{-1}a_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.

Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Recall that $a_*a^{-1}\mathcal{F}$ is the equalizer of the two maps $a_{0,*}a_0^{-1}\mathcal{F} \to a_{1,*}a_1^{-1}\mathcal{F}$, see Lemma 85.16.2. By Lemma 85.15.2

$$a_{0,*}a_0^{-1}\mathcal{F} = \mathcal{H}om(F(K_0)^\#, \mathcal{F}) \quad \text{and} \quad a_{1,*}a_1^{-1}\mathcal{F} = \mathcal{H}om(F(K_1)^\#, \mathcal{F})$$

On the other hand, we know that

$$F(K_1)^\# \rightrightarrows F(K_0)^\# \longrightarrow \text{final object } * \text{ of } Sh(\mathcal{C})$$

is a coequalizer diagram in sheaves of sets by definition of a hypercovering. Thus it suffices to prove that $\mathcal{H}om(-, \mathcal{F})$ transforms coequalizers into equalizers which is immediate from the construction in Sites, Section 7.26.

Let $\mathcal{G}$ be a cartesian sheaf on $(\mathcal{C}/K)_{total}$. We will show that $\mathcal{G} = a^{-1}\mathcal{F}$ for some sheaf $\mathcal{F}$ on $\mathcal{C}$. This will finish the proof because then $a^{-1}a_*\mathcal{G} = a^{-1}a_*a^{-1}\mathcal{F} = a^{-1}\mathcal{F} = \mathcal{G}$ by the result of the previous paragraph. Set $\mathcal{K}_n = F(K_n)^\#$ for $n \geq 0$. Then we have maps of sheaves

$$\begin{tikzcd} \mathcal{K}_2 \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] & \mathcal{K}_1 \arrow[r, shift left=0.75] \arrow[r, shift right=0.75] & \mathcal{K}_0 \end{tikzcd}$$

coming from the fact that K is a simplicial semi-representable object. The fact that K is a hypercovering means that

$$\mathcal{K}_1 \to \mathcal{K}_0 \times \mathcal{K}_0 \quad \text{and} \quad \mathcal{K}_2 \to \Big(\text{cosk}_1(\ \mathcal{K}_1 \substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow} \mathcal{K}_0\ )\Big)_2$$

are surjective maps of sheaves. Using the description of cartesian sheaves on $(\mathcal{C}/K)_{total}$ given in Lemma 85.12.4 and using the description of $Sh(\mathcal{C}/K_n)$ in Lemma 85.15.3 we find that our problem can be entirely formulated[3] in terms of

(1) the topos $Sh(\mathcal{C})$, and
(2) the simplicial object $\mathcal{K}$ in $Sh(\mathcal{C})$ whose terms are $\mathcal{K}_n$.

Thus, after replacing $\mathcal{C}$ by a different site $\mathcal{C}'$ as in Sites, Lemma 7.29.5, we may assume $\mathcal{C}$ has all finite limits, the topology on $\mathcal{C}$ is subcanonical, a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}$ is a covering if and only if $\coprod h_{V_j} \to V$ is surjective, and there exists a simplicial object U of $\mathcal{C}$ such that $\mathcal{K}_n = h_{U_n}$ as simplicial sheaves. Working backwards through the equivalences we may assume $K_n = \{U_n\}$ for all n.

Let X be the final object of $\mathcal{C}$. Then $\{U_0 \to X\}$ is a covering, $\{U_1 \to U_0 \times U_0\}$ is a covering, and $\{U_2 \to (\text{cosk}_1\text{sk}_1 U)_2\}$ is a covering. Let us use $d^n_i : U_n \to U_{n-1}$ and $s^n_j : U_n \to U_{n+1}$ the morphisms corresponding to δ^n_i and σ^n_j as in Simplicial, Definition 14.2.1. By abuse of notation, given a morphism $c : V \to W$ of $\mathcal{C}$ we denote the morphism of topoi $c : Sh(\mathcal{C}/V) \to Sh(\mathcal{C}/W)$ by the same letter. Now $\mathcal{G}$ is given by a sheaf $\mathcal{G}_0$ on $\mathcal{C}/U_0$ and an isomorphism $\alpha : (d^1_1)^{-1}\mathcal{G}_0 \to (d^1_0)^{-1}\mathcal{G}_0$ satisfying the cocycle condition on $\mathcal{C}/U_2$ formulated in Lemma 85.12.4. Since $\{U_2 \to (\text{cosk}_1\text{sk}_1 U)_2\}$ is a covering, the corresponding pullback functor on sheaves is faithful (small detail omitted). Hence we may replace U by $\text{cosk}_1\text{sk}_1 U$, because this replaces U_2 by $(\text{cosk}_1\text{sk}_1 U)_2$ and leaves U_1 and U_0 unchanged. Then

$$(d^2_0, d^2_1, d^2_2) : U_2 \to U_1 \times U_1 \times U_1$$

is a monomorphism whose its image on T-valued points is described in Simplicial, Lemma 14.19.6. In particular, there is a morphism c fitting into a commutative diagram

$$\begin{tikzcd}
U_1 \times_{(d^1_1, d^1_0), U_0 \times U_0, (d^1_1, d^1_0)} U_1 \arrow[r, "c"'] \arrow[d] & U_2 \arrow[d] \\
U_1 \times U_1 \arrow[r, "(\text{pr}_1, \text{pr}_2, s^0_0 \circ d^1_1 \circ \text{pr}_1)"] & U_1 \times U_1 \times U_1
\end{tikzcd}$$

as going around the other way defines a point of U_2. Pulling back the cocycle condition for α on U_2 translates into the condition that the pullbacks of α via the projections to $U_1 \times_{(d^1_1, d^1_0), U_0 \times U_0, (d^1_1, d^1_0)} U_1$ are the same as the pullback of α via $s^0_0 \circ d^1_1 \circ \text{pr}_1$ is the identity map (namely, the pullback of α by s^0_0 is the identity). By Sites, Lemma 7.26.1 this means that α comes from an isomorphism

$$\alpha' : \text{pr}_1^{-1}\mathcal{G}_0 \to \text{pr}_2^{-1}\mathcal{G}_0$$

of sheaves on $\mathcal{C}/U_0 \times U_0$. Then finally, the morphism $U_2 \to U_0 \times U_0 \times U_0$ is surjective on associated sheaves as is easily seen using the surjectivity of $U_1 \to U_0 \times U_0$ and the description of U_2 given above. Therefore α' satisfies the cocycle condition on $U_0 \times U_0 \times U_0$. The proof is finished by an application of Sites, Lemma 7.26.5 to the covering $\{U_0 \to X\}$. $\square$

[3]Even though it does not matter what the precise formulation is, we spell it out: the problem is to show that given an object $\mathcal{G}_0/\mathcal{K}_0$ of $Sh(\mathcal{C})/\mathcal{K}_0$ and an isomorphism

$$\alpha : \mathcal{G}_0 \times_{\mathcal{K}_0, \mathcal{K}(\delta^1_1)} \mathcal{K}_1 \to \mathcal{G}_0 \times_{\mathcal{K}_0, \mathcal{K}(\delta^1_0)} \mathcal{K}_1$$

over $\mathcal{K}_1$ satisfying a cocycle condtion in $Sh(\mathcal{C})/\mathcal{K}_2$, there exists $\mathcal{F}$ in $Sh(\mathcal{C})$ and an isomorphism $\mathcal{F} \times \mathcal{K}_0 \to \mathcal{G}_0$ over $\mathcal{K}_0$ compatible with α.

0D8F Lemma 85.17.2. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. The Čech complex of Lemma 85.9.2 associated to $a^{-1}\mathcal{F}$

$$a_{0,*}a_0^{-1}\mathcal{F} \to a_{1,*}a_1^{-1}\mathcal{F} \to a_{2,*}a_2^{-1}\mathcal{F} \to \ldots$$

is equal to the complex $\mathcal{H}om(s(\mathbf{Z}_{F(K)}^\#), \mathcal{F})$. Here $s(\mathbf{Z}_{F(K)}^\#)$ is as in Hypercoverings, Definition 25.4.1.

Proof. By Lemma 85.15.2 we have

$$a_{n,*}a_n^{-1}\mathcal{F} = \mathcal{H}om'(F(K_n)^\#, \mathcal{F})$$

where $\mathcal{H}om'$ is as in Sites, Section 7.26. The boundary maps in the complex of Lemma 85.9.2 come from the simplicial structure. Thus the equality of complexes comes from the canonical identifications $\mathcal{H}om'(\mathcal{G}, \mathcal{F}) = \mathcal{H}om(\mathbf{Z}_\mathcal{G}, \mathcal{F})$ for $\mathcal{G}$ in $Sh(\mathcal{C})$. □

0D8G Lemma 85.17.3. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. For $E \in D(\mathcal{C})$ the map

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. First, let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the spectral sequence of Lemma 85.9.3 for the sheaf $a^{-1}\mathcal{I}$ degenerates as $(a^{-1}\mathcal{I})_p = a_p^{-1}\mathcal{I}$ is injective by Lemma 85.15.4. Thus the complex

$$a_{0,*}a_0^{-1}\mathcal{I} \to a_{1,*}a_1^{-1}\mathcal{I} \to a_{2,*}a_2^{-1}\mathcal{I} \to \ldots$$

computes $Ra_*a^{-1}\mathcal{I}$. By Lemma 85.17.2 this is equal to the complex $\mathcal{H}om(s(\mathbf{Z}_{F(K)}^\#), \mathcal{I})$. Because K is a hypercovering, we see that $s(\mathbf{Z}_{F(K)}^\#)$ is exact in degrees > 0 by Hypercoverings, Lemma 25.4.4 applied to the simplicial presheaf $F(K)$. Since $\mathcal{I}$ is injective, the functor $\mathcal{H}om(-, \mathcal{I})$ is exact and we conclude that $\mathcal{H}om(s(\mathbf{Z}_{F(K)}^\#), \mathcal{I})$ is exact in positive degrees. We conclude that $R^pa_*a^{-1}\mathcal{I} = 0$ for $p > 0$. On the other hand, we have $\mathcal{I} = a_*a^{-1}\mathcal{I}$ by Lemma 85.17.1.

Bounded case. Let $E \in D^+(\mathcal{C})$. Choose a bounded below complex $\mathcal{I}^\bullet$ of injectives representing E. By the result of the first paragraph and Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) $Ra_*a^{-1}\mathcal{I}^\bullet$ is computed by the complex $a_*a^{-1}\mathcal{I}^\bullet = \mathcal{I}^\bullet$ and we conclude the lemma is true in this case.

Unbounded case. We urge the reader to skip this, since the argument is the same as above, except that we use explicit representation by double complexes to get around convergence issues. Let $E \in D(\mathcal{C})$. To show the map $E \to Ra_*a^{-1}E$ is an isomorphism, it suffices to show for every object U of $\mathcal{C}$ that

$$R\Gamma(U, E) = R\Gamma(U, Ra_*a^{-1}E)$$

We will compute both sides and show the map $E \to Ra_*a^{-1}E$ induces an isomorphism. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. Choose a quasi-isomorphism $a^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ for some K-injective complex $\mathcal{J}^\bullet$ on $(\mathcal{C}/K)_{total}$. We have

$$R\Gamma(U, E) = R\operatorname{Hom}(\mathbf{Z}_U^\#, E)$$

and

$$R\Gamma(U, Ra_*a^{-1}E) = R\operatorname{Hom}(\mathbf{Z}_U^\#, Ra_*a^{-1}E) = R\operatorname{Hom}(a^{-1}\mathbf{Z}_U^\#, a^{-1}E)$$

By Lemma 85.9.1 we have a quasi-isomorphism

$$\left(\ldots \to g_{2!}(a_2^{-1}\mathbf{Z}_U^\#) \to g_{1!}(a_1^{-1}\mathbf{Z}_U^\#) \to g_{0!}(a_0^{-1}\mathbf{Z}_U^\#)\right) \longrightarrow a^{-1}\mathbf{Z}_U^\#$$

Hence $R\operatorname{Hom}(a^{-1}\mathbf{Z}_U^\#, a^{-1}E)$ is equal to

$$R\Gamma((\mathcal{C}/K)_{total}, R\mathcal{H}om(\ldots \to g_{2!}(a_2^{-1}\mathbf{Z}_U^\#) \to g_{1!}(a_1^{-1}\mathbf{Z}_U^\#) \to g_{0!}(a_0^{-1}\mathbf{Z}_U^\#), \mathcal{J}^\bullet))$$

By the construction in Cohomology on Sites, Section 21.35 and since $\mathcal{J}^\bullet$ is K-injective, we see that this is represented by the complex of abelian groups with terms

$$\prod\nolimits_{p+q=n} \operatorname{Hom}(g_{p!}(a_p^{-1}\mathbf{Z}_U^\#), \mathcal{J}^q) = \prod\nolimits_{p+q=n} \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, g_p^{-1}\mathcal{J}^q)$$

See Cohomology on Sites, Lemmas 21.34.6 and 21.35.1 for more information. Thus we find that $R\Gamma(U, Ra_*a^{-1}E)$ is computed by the product total complex $\operatorname{Tot}_\pi(B^{\bullet,\bullet})$ with $B^{p,q} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, g_p^{-1}\mathcal{J}^q)$. For the other side we argue similarly. First we note that

$$s(\mathbf{Z}_{F(K)}^\#) \longrightarrow \mathbf{Z}$$

is a quasi-isomorphism of complexes on $\mathcal{C}$ by Hypercoverings, Lemma 25.4.4. Since $\mathbf{Z}_U^\#$ is a flat sheaf of $\mathbf{Z}$-modules we see that

$$s(\mathbf{Z}_{F(K)}^\#) \otimes_{\mathbf{Z}} \mathbf{Z}_U^\# \longrightarrow \mathbf{Z}_U^\#$$

is a quasi-isomorphism. Therefore $R\operatorname{Hom}(\mathbf{Z}_U^\#, E)$ is equal to

$$R\Gamma(\mathcal{C}, R\mathcal{H}om(s(\mathbf{Z}_{F(K)}^\#) \otimes_{\mathbf{Z}} \mathbf{Z}_U^\#, \mathcal{I}^\bullet))$$

By the construction of $R\mathcal{H}om$ and since $\mathcal{I}^\bullet$ is K-injective, this is represented by the complex of abelian groups with terms

$$\prod\nolimits_{p+q=n} \operatorname{Hom}(\mathbf{Z}_{K_p}^\# \otimes_{\mathbf{Z}} \mathbf{Z}_U^\#, \mathcal{I}^q) = \prod\nolimits_{p+q=n} \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}\mathcal{I}^q)$$

The equality of terms follows from the fact that $\mathbf{Z}_{K_p}^\# \otimes_{\mathbf{Z}} \mathbf{Z}_U^\# = a_{p!}a_p^{-1}\mathbf{Z}_U^\#$ by Modules on Sites, Remark 18.27.10. Thus we find that $R\Gamma(U, E)$ is computed by the product total complex $\operatorname{Tot}_\pi(A^{\bullet,\bullet})$ with $A^{p,q} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}\mathcal{I}^q)$.

Since $\mathcal{I}^\bullet$ is K-injective we see that $a_p^{-1}\mathcal{I}^\bullet$ is K-injective, see Lemma 85.15.4. Since $\mathcal{J}^\bullet$ is K-injective we see that $g_p^{-1}\mathcal{J}^\bullet$ is K-injective, see Lemma 85.3.6. Both represent the object $a_p^{-1}E$. Hence for every $p \geq 0$ the map of complexes

$$A^{p,\bullet} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}\mathcal{I}^\bullet) \longrightarrow \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, g_p^{-1}\mathcal{J}^\bullet) = B^{p,\bullet}$$

induced by g_p^{-1} applied to the given map $a^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ is a quasi-isomorphisms as these complexes both compute

$$R\operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}E)$$

By More on Algebra, Lemma 15.103.2 we conclude that the right vertical arrow in the commutative diagram

$$\begin{tikzcd} R\Gamma(U, E) \arrow[r] \arrow[d] & \operatorname{Tot}_\pi(A^{\bullet,\bullet}) \arrow[d] \\ R\Gamma(U, Ra_*a^{-1}E) \arrow[r] & \operatorname{Tot}_\pi(B^{\bullet,\bullet}) \end{tikzcd}$$

is a quasi-isomorphism. Since we saw above that the horizontal arrows are quasi-isomorphisms, so is the left vertical arrow. □

0D8H Lemma 85.17.4. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Then we have a canonical isomorphism

$$R\Gamma(\mathcal{C}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C})$.

Proof. This follows from Lemma 85.17.3 because $R\Gamma((\mathcal{C}/K)_{total}, -) = R\Gamma(\mathcal{C}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0D8I Lemma 85.17.5. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{A} \subset \text{Ab}((\mathcal{C}/K)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence

$$D^+(\mathcal{C}) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/K)_{total})$$

with quasi-inverse Ra_*.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.17.1 and 85.17.3 and Cohomology on Sites, Lemma 21.28.5 □

We urge the reader to skip the following remark.

09X6 Remark 85.17.6. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets on $\mathcal{C}$. If $\mathcal{C}$ has equalizers and fibre products, then we've defined the notion of a hypercovering of $\mathcal{G}$ in Hypercoverings, Definition 25.6.1. We claim that all the results in this section have a valid counterpart in this setting. To see this, define the localization $\mathcal{C}/\mathcal{G}$ of $\mathcal{C}$ at $\mathcal{G}$ exactly as in Sites, Lemma 7.30.3 (which is stated only for sheaves; the topos $Sh(\mathcal{C}/\mathcal{G})$ is equal to the localization of the topos $Sh(\mathcal{C})$ at the sheaf $\mathcal{G}^\#$). Then the reader easily shows that the site $\mathcal{C}/\mathcal{G}$ has fibre products and equalizers and that a hypercovering of $\mathcal{G}$ in $\mathcal{C}$ is the same thing as a hypercovering for the site $\mathcal{C}/\mathcal{G}$. Hence replacing the site $\mathcal{C}$ by $\mathcal{C}/\mathcal{G}$ in the lemmas on hypercoverings above we obtain proofs of the corresponding results for hypercoverings of $\mathcal{G}$. Example: for a hypercovering K of $\mathcal{G}$ we have

$$R\Gamma(\mathcal{C}/\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D^+(\mathcal{C}/\mathcal{G})$ where $a : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/\mathcal{G})$ is the canonical augmentation. This is Lemma 85.17.4. Let $R\Gamma(\mathcal{G}, -) : D(\mathcal{C}) \to D(\text{Ab})$ be defined as the derived functor of the functor $H^0(\mathcal{G}, -) = H^0(\mathcal{G}^\#, -)$ discussed in Hypercoverings, Section 25.6 and Cohomology on Sites, Section 21.13. We have

$$R\Gamma(\mathcal{G}, E) = R\Gamma(\mathcal{C}/\mathcal{G}, j^{-1}E)$$

by the analogue of Cohomology on Sites, Lemma 21.7.1 for the localization fuctor $j : \mathcal{C}/\mathcal{G} \to \mathcal{C}$. Putting everything together we obtain

$$R\Gamma(\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}j^{-1}E) = R\Gamma((\mathcal{C}/K)_{total}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$ where $g : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C})$ is the composition of a and j.

85.18. Cohomological descent for hypercoverings: modules

0D9Y Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Assume $\mathcal{C}$ has equalizers and fibre products and let K be a hypercovering as defined in Hypercoverings, Definition 25.6.1. We will study cohomological descent for the augmentation

$$a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$$

of Remark 85.16.5.

0D9Z **Lemma 85.18.1.** Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. With notation as above

$$a^* : \text{Mod}(\mathcal{O}_\mathcal{C}) \to \text{Mod}(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. Since $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ we have $a^* = a^{-1}$. Hence the lemma follows immediately from Lemma 85.17.1. $\square$

0DA0 **Lemma 85.18.2.** Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. For $E \in D(\mathcal{O}_\mathcal{C})$ the map

$$E \longrightarrow Ra_* La^* E$$

is an isomorphism.

Proof. Since $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ we have $La^* = a^* = a^{-1}$. Moreover Ra_* agrees with Ra_* on abelian sheaves, see Cohomology on Sites, Lemma 21.20.7. Hence the lemma follows immediately from Lemma 85.17.3. $\square$

0DA1 **Lemma 85.18.3.** Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. Then we have a canonical isomorphism

$$R\Gamma(\mathcal{C}, E) = R\Gamma((\mathcal{C}/K)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. This follows from Lemma 85.18.2 because $R\Gamma((\mathcal{C}/K)_{total}, -) = R\Gamma(\mathcal{C}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4 or by Cohomology on Sites, Lemma 21.20.5. $\square$

0DA2 **Lemma 85.18.4.** Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence

$$D^+(\mathcal{O}_\mathcal{C}) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6 (the required hypotheses hold by the discussion in Remark 85.16.5). The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.18.1 and 85.18.2 and Cohomology on Sites, Lemma 21.28.5. $\square$

85.19. Cohomological descent for hypercoverings of an object

0D8J In this section we assume $\mathcal{C}$ has fibre products and $X \in \text{Ob}(\mathcal{C})$. We let K be a hypercovering of X as defined in Hypercoverings, Definition 25.3.3. We will study the augmentation

$$a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C}/X)$$

of Remark 85.16.4. Observe that $\mathcal{C}/X$ is a site which has equalizers and fibre products and that K is a hypercovering for the site $\mathcal{C}/X$[4] by Hypercoverings, Lemma 25.3.9. This means that every single result proved for hypercoverings in Section 85.17 has an immediate analogue in the situation in this section.

0D8K **Lemma 85.19.1.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Then

(1) $a^{-1} : Sh(\mathcal{C}/X) \to Sh((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,
(2) $a^{-1} : \text{Ab}(\mathcal{C}/X) \to \text{Ab}((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.

In both cases a_* provides the quasi-inverse functor.

Proof. Via Remarks 85.15.5 and 85.16.4 and the discussion in the introduction to this section this follows from Lemma 85.17.1. □

0D8L **Lemma 85.19.2.** Let $\mathcal{C}$ be a site with fibre product and $X \in \text{Ob}(\mathcal{C})$. Let K be a hypercovering of X. For $E \in D(\mathcal{C}/X)$ the map

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. Via Remarks 85.15.5 and 85.16.4 and the discussion in the introduction to this section this follows from Lemma 85.17.3. □

09X7 **Lemma 85.19.3.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Then we have a canonical isomorphism

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C}/X)$.

Proof. Via Remarks 85.15.5 and 85.16.4 this follows from Lemma 85.17.4. □

0D8M **Lemma 85.19.4.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Let $\mathcal{A} \subset \text{Ab}((\mathcal{C}/K)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence

$$D^+(\mathcal{C}/X) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/K)_{total})$$

with quasi-inverse Ra_*.

Proof. Via Remarks 85.15.5 and 85.16.4 this follows from Lemma 85.17.5. □

[4] The converse may not be the case, i.e., if K is a simplicial object of $\text{SR}(\mathcal{C}, X) = \text{SR}(\mathcal{C}/X)$ which defines a hypercovering for the site $\mathcal{C}/X$ as in Hypercoverings, Definition 25.6.1, then it may not be true that K defines a hypercovering of X. For example, if $K_0 = \{U_{0,i}\}_{i \in I_0}$ then the latter condition guarantees $\{U_{0,i} \to X\}$ is a covering of $\mathcal{C}$ whereas the former condition only requires $\coprod h_{U_{0,i}}^\# \to h_X^\#$ to be a surjective map of sheaves.

85.20. Cohomological descent for hypercoverings of an object: modules

0DA3 In this section we assume $\mathcal{C}$ has fibre products and $X \in \text{Ob}(\mathcal{C})$. We let K be a hypercovering of X as defined in Hypercoverings, Definition 25.3.3. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Set $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/X}$. We will study the augmentation

$$a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}/X), \mathcal{O}_X)$$

of Remark 85.16.6. Observe that $\mathcal{C}/X$ is a site which has equalizers and fibre products and that K is a hypercovering for the site $\mathcal{C}/X$. Therefore the results in this section are immediate consequences of the corresponding results in Section 85.18.

0DA4 **Lemma 85.20.1.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. With notation as above

$$a^* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.1. $\square$

0DA5 **Lemma 85.20.2.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. For $E \in D(\mathcal{O}_X)$ the map

$$E \longrightarrow Ra_* La^* E$$

is an isomorphism.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.2. $\square$

0DA6 **Lemma 85.20.3.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. Then we have a canonical isomorphism

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{total}, La^* E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.3. $\square$

0DA7 **Lemma 85.20.4.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence

$$D^+(\mathcal{O}_X) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.4. $\square$

85.21. Hypercovering by a simplicial object of the site

09X8 Let $\mathcal{C}$ be a site with fibre products and let $X \in \mathrm{Ob}(\mathcal{C})$. In this section we elucidate the results of Section 85.19 in the case that our hypercovering is given by a simplicial object of the site. Let U be a simplicial object of $\mathcal{C}$. As usual we denote $U_n = U([n])$ and $f_\varphi : U_n \to U_m$ the morphism $f_\varphi = U(\varphi)$ corresponding to $\varphi : [m] \to [n]$. Assume we have an augmentation

$$a : U \to X$$

From this we obtain a simplicial site $(\mathcal{C}/U)_{total}$ and an augmentation morphism

$$a : Sh((\mathcal{C}/U)_{total}) \longrightarrow Sh(\mathcal{C}/X)$$

Namely, from U we obtain a simiplical object K of $\mathrm{SR}(\mathcal{C}, X)$ with degree n part $K_n = \{U_n \to X\}$ and we can apply the constructions in Remark 85.16.4. More precisely, an object of the site $(\mathcal{C}/U)_{total}$ is given by a V/U_n and a morphism $(\varphi, f) : V/U_n \to W/U_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "f"'] \arrow[d] & W \arrow[d] \\ U_n \arrow[r, "f_\varphi"] & U_m \end{tikzcd}$$

is commutative. The morphism of topoi a is given by the cocontinuous functor $V/U_n \mapsto V/X$. That's all folks!

In this section we will say the augmentation $a : U \to X$ is a hypercovering of X in $\mathcal{C}$ if the following hold

(1) $\{U_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{U_1 \to U_0 \times_X U_0\}$ is a covering of $\mathcal{C}$,
(3) $\{U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

This is equivalent to the condition that K (as above) is a hypercovering of X, see Hypercoverings, Example 25.3.5.

0DA8 **Lemma 85.21.1.** Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Then

(1) $a^{-1} : Sh(\mathcal{C}/X) \to Sh((\mathcal{C}/U)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,
(2) $a^{-1} : \mathrm{Ab}(\mathcal{C}/X) \to \mathrm{Ab}((\mathcal{C}/U)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.

In both cases a_* provides the quasi-inverse functor.

Proof. This is a special case of Lemma 85.19.1. □

0D8N **Lemma 85.21.2.** Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. For $E \in D(\mathcal{C}/X)$ the map

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. This is a special case of Lemma 85.19.2. □

09X9 **Lemma 85.21.3.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Then we have a canonical isomorphism

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/U)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C}/X)$.

Proof. This is a special case of Lemma 85.19.3. □

0DA9 **Lemma 85.21.4.** Let $\mathcal{C}$ be a site with fibre product and $X \in \text{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Let $\mathcal{A} \subset \text{Ab}((\mathcal{C}/U)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence

$$D^+(\mathcal{C}/X) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/U)_{total})$$

with quasi-inverse Ra_*.

Proof. This is a special case of Lemma 85.19.4 □

09WL **Lemma 85.21.5.** Let U be a simplicial object of a site $\mathcal{C}$ with fibre products.

(1) $\mathcal{C}/U$ has the structure of a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites,
(2) the construction of Lemma 85.3.1 applied to the structure in (1) reproduces the site $(\mathcal{C}/U)_{total}$ above,
(3) if $a : U \to X$ is an augmentation, then $a_0 : \mathcal{C}/U_0 \to \mathcal{C}/X$ is an augmentation as in Remark 85.4.1 part (A) and gives the same morphism of topoi $a : Sh((\mathcal{C}/U)_{total}) \to Sh(\mathcal{C}/X)$ as the one above.

Proof. Given a morphism of objects $V \to W$ of $\mathcal{C}$ the localization morphism $j : \mathcal{C}/V \to \mathcal{C}/W$ is a left adjoint to the base change functor $\mathcal{C}/W \to \mathcal{C}/V$. The base change functor is continuous and induces the same morphism of topoi as j. See Sites, Lemma 7.27.3. This proves (1).

Part (2) holds because a morphism $V/U_n \to W/U_m$ of the category constructed in Lemma 85.3.1 is a morphism $V \to W \times_{U_m, f_\varphi} U_n$ over U_n which is the same thing as a morphism $f : V \to W$ over the morphism $f_\varphi : U_n \to U_m$, i.e., the same thing as a morphism in the category $(\mathcal{C}/U)_{total}$ defined above. Equality of sets of coverings is immediate from the definition.

We omit the proof of (3). □

85.22. Hypercovering by a simplicial object of the site: modules

0DAA Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Let $U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined in Section 85.21. In this section we study the augmentation

$$a : (Sh((\mathcal{C}/U)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}/X), \mathcal{O}_X)$$

we obtain by thinking of U as a simiplical semi-representable object of $\mathcal{C}/X$ whose degree n part is the singleton element $\{U_n/X\}$ and applying the constructions in Remark 85.16.6. Thus all the results in this section are immediate consequences of the corresponding results in Section 85.20.

0DAB **Lemma 85.22.1.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. With notation as above

$$a^* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. This is a special case of Lemma 85.20.1. □

0DAC **Lemma 85.22.2.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. For $E \in D(\mathcal{O}_X)$ the map

$$E \longrightarrow Ra_* La^* E$$

is an isomorphism.

Proof. This is a special case of Lemma 85.20.2. □

0DAD **Lemma 85.22.3.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. Then we have a canonical isomorphism

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/U)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. This is a special case of Lemma 85.20.3. □

0DAE **Lemma 85.22.4.** Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence

$$D^+(\mathcal{O}_X) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. This is a special case of Lemma 85.20.4. □

85.23. Unbounded cohomological descent for hypercoverings

0DC1 In this section we discuss unbounded cohomological descent. The results themselves will be immediate consequences of our results on bounded cohomological descent in the previous sections and Cohomology on Sites, Lemmas 21.28.6 and/or 21.28.7; the real work lies in setting up notation and choosing appropriate assumptions. Our discussion is motivated by the discussion in [LO08a] although the details are a good bit different.

Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given for every object U of $\mathcal{C}$ a weak Serre subcategory $\mathcal{A}_U \subset \text{Mod}(\mathcal{O}_U)$ satisfying the following properties

0DC2 (1) given a morphism $U \to V$ of $\mathcal{C}$ the restriction functor $\text{Mod}(\mathcal{O}_V) \to \text{Mod}(\mathcal{O}_U)$ sends $\mathcal{A}_V$ into $\mathcal{A}_U$,

0DC3 (2) given a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ an object $\mathcal{F}$ of $\text{Mod}(\mathcal{O}_U)$ is in $\mathcal{A}_U$ if and only if the restriction of $\mathcal{F}$ to $\mathcal{C}/U_i$ is in $\mathcal{A}_{U_i}$ for all $i \in I$.

0DC4 (3) there exists a subset $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that
   (a) every object of $\mathcal{C}$ has a covering whose members are in $\mathcal{B}$, and

(b) for every $V \in \mathcal{B}$ there exists an integer d_V and a cofinal system Cov_V of coverings of V such that

$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \to V\} \in \text{Cov}_V,\ p > d_V,\ \text{and } \mathcal{F} \in \text{Ob}(\mathcal{A}_V)$$

Note that we require this to be true for $\mathcal{F}$ in $\mathcal{A}_V$ and not just for "global" objects (and thus it is stronger than the condition imposed in Cohomology on Sites, Situation 21.25.1). In this situation, there is a weak Serre subcategory $\mathcal{A} \subset \text{Mod}(\mathcal{O}_\mathcal{C})$ consisting of objects whose restriction to $\mathcal{C}/U$ is in $\mathcal{A}_U$ for all $U \in \text{Ob}(\mathcal{C})$. Moreover, there are derived categories $D_\mathcal{A}(\mathcal{O}_\mathcal{C})$ and $D_{\mathcal{A}_U}(\mathcal{O}_U)$ and the restriction functors send these into each other.

0DC5 Example 85.23.1. Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{C} = X_{spaces,étale}$ be the étale site on the category of algebraic spaces étale over X, see Properties of Spaces, Definition 66.18.2. Denote $\mathcal{O}_\mathcal{C}$ the structure sheaf, i.e., the sheaf given by the rule $U \mapsto \Gamma(U, \mathcal{O}_U)$. Denote $\mathcal{A}_U$ the category of quasi-coherent $\mathcal{O}_U$-modules. Let $\mathcal{B} = \text{Ob}(\mathcal{C})$ and for $V \in \mathcal{B}$ set $d_V = 0$ and let Cov_V denote the coverings $\{V_i \to V\}$ with V_i affine for all i. Then the assumptions (1), (2), (3) are satisfied. See Properties of Spaces, Lemmas 66.29.2 and 66.29.7 for properties (1) and (2) and the vanishing in (3) follows from Cohomology of Schemes, Lemma 30.2.2 and the discussion in Cohomology of Spaces, Section 69.3.

0DC6 Example 85.23.2. Let S be one of the following types of schemes

(1) the spectrum of a finite field,
(2) the spectrum of a separably closed field,
(3) the spectrum of a strictly henselian Noetherian local ring,
(4) the spectrum of a henselian Noetherian local ring with finite residue field,
(5) add more here.

Let Λ be a finite ring whose order is invertible on S. Let $\mathcal{C} \subset (Sch/S)_{étale}$ be the full subcategory consisting of schemes locally of finite type over S endowed with the étale topology. Let $\mathcal{O}_\mathcal{C} = \underline{\Lambda}$ be the constant sheaf. Set $\mathcal{A}_U = \text{Mod}(\mathcal{O}_U)$, in other words, we consider all étale sheaves of Λ-modules. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be the set of quasi-compact objects. For $V \in \mathcal{B}$ set

$$d_V = 1 + 2\dim(S) + \text{sup}_{v \in V}(\text{trdeg}_{\kappa(s)}(\kappa(v)) + 2\dim \mathcal{O}_{V,v})$$

and let Cov_V denote the étale coverings $\{V_i \to V\}$ with V_i quasi-compact for all i. Our choice of bound d_V comes from Gabber's theorem on cohomological dimension. To see that condition (3) holds with this choice, use [ILO14, Exposé VIII-A, Corollary 1.2 and Lemma 2.2] plus elementary arguments on cohomological dimensions of fields. We add 1 to the formula because our list contains cases where we allow S to have finite residue field. We will come back to this example later (insert future reference).

Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given weak Serre subcategories $\mathcal{A}_U \subset \text{Mod}(\mathcal{O}_U)$ satisfying condition (1). Then

(1) given a semi-representable object $K = \{U_i\}_{i \in I}$ we get a weak Serre subcategory $\mathcal{A}_K \subset \text{Mod}(\mathcal{O}_K)$ by taking $\prod \mathcal{A}_{U_i} \subset \prod \text{Mod}(\mathcal{O}_{U_i}) = \text{Mod}(\mathcal{O}_K)$, and
(2) given a morphism of semi-representable objects $f : K \to L$ the pullback map $f^* : \text{Mod}(\mathcal{O}_L) \to \text{Mod}(\mathcal{O}_L)$ sends $\mathcal{A}_L$ into $\mathcal{A}_K$.

See Remark 85.15.6 for notation and explanation. In particular, given a simplicial semi-representable object K it is unambiguous to say what it means for an object $\mathcal{F}$ of $\text{Mod}(\mathcal{O})$ as in Remark 85.16.5 to have restrictions $\mathcal{F}_n$ in $\mathcal{A}_{K_n}$ for all n.

0DC7 Lemma 85.23.3. Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given weak Serre subcategories $\mathcal{A}_U \subset \text{Mod}(\mathcal{O}_U)$ satisfying conditions (1), (2), and (3) above. Assume $\mathcal{C}$ has equalizers and fibre products and let K be a hypercovering. Let $((\mathcal{C}/K)_{total}, \mathcal{O})$ be as in Remark 85.16.5. Let $\mathcal{A}_{total} \subset \text{Mod}(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules $\mathcal{F}$ whose restriction $\mathcal{F}_n$ is in $\mathcal{A}_{K_n}$ for all n (as defined above). Then the functor La^* defines an equivalence

$$D_\mathcal{A}(\mathcal{O}_\mathcal{C}) \longrightarrow D_{\mathcal{A}_{total}}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. The cartesian $\mathcal{O}$-modules form a weak Serre subcategory by Lemma 85.12.6 (the required hypotheses hold by the discussion in Remark 85.16.5). Since the restriction functor $g_n^* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_n)$ are exact, it follows that $\mathcal{A}_{total}$ is a weak Serre subcategory.

Let us show that $a^* : \mathcal{A} \to \mathcal{A}_{total}$ is an equivalence of categories with inverse given by La_*. We already know that $La_*a^*\mathcal{F} = \mathcal{F}$ by the bounded version (Lemma 85.18.4). It is clear that $a^*\mathcal{F}$ is in $\mathcal{A}_{total}$ for $\mathcal{F}$ in $\mathcal{A}$. Conversely, assume that $\mathcal{G} \in \mathcal{A}_{total}$. Because $\mathcal{G}$ is cartesian we see that $\mathcal{G} = a^*\mathcal{F}$ for some $\mathcal{O}_\mathcal{C}$-module $\mathcal{F}$ by Lemma 85.18.1. We want to show that $\mathcal{F}$ is in $\mathcal{A}$. Take $U \in \text{Ob}(\mathcal{C})$. We have to show that the restriction of $\mathcal{F}$ to $\mathcal{C}/U$ is in $\mathcal{A}_U$. As usual, write $K_0 = \{U_{0,i}\}_{i \in I_0}$. Since K is a hypercovering, the map $\coprod_{i \in I_0} h_{U_{0,i}} \to *$ becomes surjective after sheafification. This implies there is a covering $\{U_j \to U\}_{j \in J}$ and a map $\tau : J \to I_0$ and for each $j \in J$ a morphism $\varphi_j : U_j \to U_{0,\tau(j)}$. Since $\mathcal{G}_0 = a_0^*\mathcal{F}$ we find that the restriction of $\mathcal{F}$ to $\mathcal{C}/U_j$ is equal to the restriction of the $\tau(j)$th component of $\mathcal{G}_0$ to $\mathcal{C}/U_j$ via the morphism $\varphi_j : U_j \to U_{0,\tau(i)}$. Hence by (1) we find that $\mathcal{F}|_{\mathcal{C}/U_j}$ is in $\mathcal{A}_{U_j}$ and in turn by (2) we find that $\mathcal{F}|_{\mathcal{C}/U}$ is in $\mathcal{A}_U$.

In particular the statement of the lemma makes sense. The lemma now follows from Cohomology on Sites, Lemma 21.28.6. Assumption (1) is clear (see Remark 85.16.5). Assumptions (2) and (3) we proved in the preceding paragraph. Assumption (4) is immediate from (3). For assumption (5) let $\mathcal{B}_{total}$ be the set of objects $U/U_{n,i}$ of the site $(\mathcal{C}/K)_{total}$ such that $U \in \mathcal{B}$ where $\mathcal{B}$ is as in (3). Here we use the description of the site $(\mathcal{C}/K)_{total}$ given in Section 85.16. Moreover, we set $\text{Cov}_{U/U_{n,i}}$ equal to Cov_U and $d_{U/U_{n,i}}$ equal d_U where Cov_U and d_U are given to us by (3). Then we claim that condition (5) holds with these choices. This follows immediately from Lemma 85.16.3 and the fact that $\mathcal{F} \in \mathcal{A}_{total}$ implies $\mathcal{F}_n \in \mathcal{A}_{K_n}$ and hence $\mathcal{F}_{n,i} \in \mathcal{A}_{U_{n,i}}$. (The reader who worries about the difference between cohomology of abelian sheaves versus cohomology of sheaves of modules may consult Cohomology on Sites, Lemma 21.12.4.) □

85.24. Glueing complexes

0DC8 This section is the continuation of Cohomology, Section 20.45. The goal is to prove a slight generalization of [BBD82, Theorem 3.2.4]. Our method will be a tiny bit different in that we use the material from Sections 85.13 and 85.14. We will also reprove the unbounded version as it is proved in [LO08a].

Advice to the reader: We suggest the reader first look at the statement of Lemma 85.24.5 as well as the second proof of this lemma.

Here is the situation we are interested in.

0DC9 Situation 85.24.1. Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. We are given

(1) a category $\mathcal{B}$ and a functor $u : \mathcal{B} \to \mathcal{C}$,
(2) an object E_U in $D(\mathcal{O}_{u(U)})$ for $U \in \text{Ob}(\mathcal{B})$,
(3) an isomorphism $\rho_a : E_U|_{\mathcal{C}/u(V)} \to E_V$ in $D(\mathcal{O}_{u(V)})$ for $a : V \to U$ in $\mathcal{B}$

such that whenever we have composable arrows $b : W \to V$ and $a : V \to U$ of $\mathcal{B}$, then $\rho_{a \circ b} = \rho_b \circ \rho_a|_{\mathcal{C}/u(W)}$.

We won't be able to prove anything about this without making more assumptions. An interesting case is where $\mathcal{B}$ is a full subcategory such that every object of $\mathcal{C}$ has a covering whose members are objects of $\mathcal{B}$ (this is the case considered in [BBD82]). For us it is important to allow cases where this is not the case; the main alternative case is where we have a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ and $\mathcal{B}$ is a full subcategory of $\mathcal{D}$ such that every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$.

In Situation 85.24.1 a solution will be a pair (E, ρ_U) where E is an object of $D(\mathcal{O}_\mathcal{C})$ and $\rho_U : E|_{\mathcal{C}/u(U)} \to E_U$ for $U \in \text{Ob}(\mathcal{B})$ are isomorphisms such that we have $\rho_a \circ \rho_U|_{\mathcal{C}/u(V)} = \rho_V$ for $a : V \to U$ in $\mathcal{B}$.

0DCA Lemma 85.24.2. In Situation 85.24.1. Assume negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero. Let L be a simplicial object of $\text{SR}(\mathcal{B})$. Consider the simplicial object $K = u(L)$ of $\text{SR}(\mathcal{C})$ and let $((\mathcal{C}/K)_{total}, \mathcal{O})$ be as in Remark 85.16.5. There exists a cartesian object E of $D(\mathcal{O})$ such that writing $L_n = \{U_{n,i}\}_{i \in I_n}$ the restriction of E to $D(\mathcal{O}_{\mathcal{C}/u(U_{n,i})})$ is $E_{U_{n,i}}$ compatibly (see proof for details). Moreover, E is unique up to unique isomorphism.

Proof. Recall that $Sh(\mathcal{C}/K_n) = \prod_{i \in I_n} Sh(\mathcal{C}/u(U_{n,i}))$ and similarly for the categories of modules. This product decomposition is also inherited by the derived categories of sheaves of modules. Moreover, this product decomposition is compatible with the morphisms in the simplicial semi-representable object K. See Section 85.15. Hence we can set $E_n = \prod_{i \in I_n} E_{U_{n,i}}$ ("formal" product) in $D(\mathcal{O}_n)$. Taking (formal) products of the maps ρ_a of Situation 85.24.1 we obtain isomorphisms $E_\varphi : f_\varphi^* E_n \to E_m$. The assumption about compostions of the maps ρ_a immediately implies that (E_n, E_φ) defines a simplicial system of the derived category of modules as in Definition 85.14.1. The vanishing of negative exts assumed in the lemma implies that $\text{Hom}(E_n[t], E_n) = 0$ for $n \geq 0$ and $t > 0$. Thus by Lemma 85.14.7 we obtain E. Uniqueness up to unique isomorphism follows from Lemmas 85.14.5 and 85.14.6. $\square$

0DCB Lemma 85.24.3 (BBD glueing lemma). In Situation 85.24.1. Assume

(1) $\mathcal{C}$ has equalizers and fibre products,
(2) there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that
    (a) $\mathcal{D}$ has equalizers and fibre products and u commutes with them,
    (b) $\mathcal{B}$ is a full subcategory of $\mathcal{D}$ and $u : \mathcal{B} \to \mathcal{C}$ is the restriction of u,
    (c) every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$,
(3) all negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero, and
(4) there exists a $t \in \mathbf{Z}$ such that $H^i(E_U) = 0$ for $i < t$ and $U \in \text{Ob}(\mathcal{B})$.

Then there exists a solution unique up to unique isomorphism.

Proof. By Hypercoverings, Lemma 25.12.3 there exists a hypercovering L for the site $\mathcal{D}$ such that $L_n = \{U_{n,i}\}_{i \in I_n}$ with $U_{i,n} \in \text{Ob}(\mathcal{B})$. Set $K = u(L)$. Apply Lemma 85.24.2 to get a cartesian object E of $D(\mathcal{O})$ on the site $(\mathcal{C}/K)_{total}$ restricting to $E_{U_{n,i}}$ on $\mathcal{C}/u(U_{n,i})$ compatibly. The assumption on t implies that $E \in D^+(\mathcal{O})$. By Hypercoverings, Lemma 25.12.4 we see that K is a hypercovering too. By Lemma 85.18.4 we find that $E = a^*F$ for some F in $D^+(\mathcal{O}_\mathcal{C})$.

To prove that F is a solution we will use the construction of L_0 and L_1 given in the proof of Hypercoverings, Lemma 25.12.3. (This is a bit inelegant but there does not seem to be a completely straightforward way around it.)

Namely, we have $I_0 = \text{Ob}(\mathcal{B})$ and so $L_0 = \{U\}_{U \in \text{Ob}(\mathcal{B})}$. Hence the isomorphism $a^*F \to E$ restricted to the components $\mathcal{C}/u(U)$ of $\mathcal{C}/K_0$ defines isomorphisms $\rho_U : F|_{\mathcal{C}/u(U)} \to E_U$ for $U \in \text{Ob}(\mathcal{B})$ by our choice of E.

To prove that ρ_U satisfy the requirement of compatibility with the maps ρ_a of Situation 85.24.1 we use that I_1 contains the set

$$\Omega = \{(U, V, W, a, b) \mid U, V, W \in \mathcal{B}, a : U \to V, b : U \to W\}$$

and that for $i = (U, V, W, a, b)$ in Ω we have $U_{1,i} = U$. Moreover, the component maps $f_{\delta^1_0, i}$ and $f_{\delta^1_1, i}$ of the two morphisms $K_1 \to K_0$ are the morphisms

$$a : U \to V \quad \text{and} \quad b : U \to V$$

Hence the compatibility mentioned in Lemma 85.24.2 gives that

$$\rho_a \circ \rho_V|_{\mathcal{C}/u(U)} = \rho_U \quad \text{and} \quad \rho_b \circ \rho_W|_{\mathcal{C}/u(U)} = \rho_U$$

Taking $i = (U, V, U, a, \text{id}_U) \in \Omega$ for example, we find that we have the desired compatibility. The uniqueness of F follows from the uniqueness of E in the previous lemma (small detail omitted). □

0DCC Lemma 85.24.4 (Unbounded BBD glueing lemma). In Situation 85.24.1. Assume

(1) $\mathcal{C}$ has equalizers and fibre products,
(2) there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that
   (a) $\mathcal{D}$ has equalizers and fibre products and u commutes with them,
   (b) $\mathcal{B}$ is a full subcategory of $\mathcal{D}$ and $u : \mathcal{B} \to \mathcal{C}$ is the restriction of u,
   (c) every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$,
(3) all negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero, and
(4) there exist weak Serre subcategories $\mathcal{A}_U \subset \text{Mod}(\mathcal{O}_U)$ for all $U \in \text{Ob}(\mathcal{C})$ satisfying conditions (1), (2), and (3),
(5) $E_U \in D_{\mathcal{A}_U}(\mathcal{O}_U)$.

Then there exists a solution unique up to unique isomorphism.

Proof. The proof is exactly the same as the proof of Lemma 85.24.3. The only change is that E is an object of $D_{\mathcal{A}_{total}}(\mathcal{O})$ and hence we use Lemma 85.23.3 to obtain F with $E = a^*F$ instead of Lemma 85.18.4. □

Here is an example application of the general theory above.

0GMG **Lemma 85.24.5.** Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume $\mathcal{C}$ has fibre products. Let $\{U_i \to X\}_{i \in I}$ be a covering in $\mathcal{C}$. For $i \in I$ let E_i be an object of $D(\mathcal{O}_{U_i})$ and for $i, j \in I$ let Email of Martin Olsson dated Sep 9, 2021.

$$\rho_{ij} : E_i|_{\mathcal{C}/U_{ij}} \longrightarrow E_j|_{\mathcal{C}/U_{ij}}$$

be an isomorphism in $D(\mathcal{O}_{U_{ij}})$ where $U_{ij} = U_i \times_X U_j$. Assume

(1) the ρ_{ij} satisfy the cocycle condition on $U_i \times_X U_j \times_X U_k$ for all $i, j, k \in I$,
(2) $\mathcal{E}xt^p_{\mathcal{O}_{U_i}}(E_i, E_i) = 0$ for all $p < 0$ and $i \in I$, and
(3) there exists a $t \in \mathbf{Z}$ such that $H^p(E_i) = 0$ for $p < t$ and all $i \in I$.

Then there exists a unique pair (E, ρ_i) where E is an object of $D(\mathcal{O}_X)$ and $\rho_i : E|_{U_i} \to E_i$ are isomorphisms in $D(\mathcal{O}_{U_i})$ compatible with the ρ_{ij}.

First proof. In this proof we deduce the lemma from the very general Lemma 85.24.3. We urge the reader to look at the second proof in stead.

We may replace $\mathcal{C}$ with $\mathcal{C}/X$. Thus we may and do assume X is the final object of $\mathcal{C}$ and that $\mathcal{C}$ has all finite limits.

Let $\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of $U \in \mathrm{Ob}(\mathcal{C})$ such that there exists an $i(U) \in I$ and a morphism $a_U : U \to U_{i(U)}$. We denote $E_U = a_U^* E_{i(U)}$ in $D(\mathcal{O}_U)$ the pullback (restriction) of E_i via a_U. Given a morphism $a : U \to U'$ of $\mathcal{B}$ we obtain a morphism $(a_{U'} \circ a, a_U) : U \to U_{i(U')} \times_X U_{i(U)} = U_{i(U')i(U)}$ and hence an isomorphism

$$\rho_a : a^* E_{U'} = a^* a_{U'}^* E_{i(U')} \xrightarrow{(a_{U'} \circ a, a_U)^* \rho_{i(U')i(U)}} a_U^* E_{i(U)} = E_U$$

in $D(\mathcal{O}_U)$. The data $\mathcal{B}, E_U, \rho_a$ are as in Situation 85.24.1; the isomorphisms ρ_a satisfy the cocycle condition exactly because of condition (1) in the statement of the lemma (details omitted).

We are going to apply Lemma 85.24.3 with $\mathcal{B}$, E_U, ρ_a as above and with $\mathcal{D} = \mathcal{C}$ and $f : \mathcal{C} \to \mathcal{D}$ the identity morphism. Assumptions (1) and (2)(a) of Lemma 85.24.3 we have seen above. Assumption (2)(b) of Lemma 85.24.3 is clear. Assumption (2)(c) of Lemma 85.24.3 holds because $\{U_i \to X\}$ is a covering[5]. Assumption (3) of Lemma 85.24.3 holds because we have assumed the vanishing of all negative Ext sheaves of E_i which certainly implies that for any object U lying over U_i the negative self-Exts of $E_i|_U$ are zero. Assumption (4) of Lemma 85.24.3 holds because we have assumed the cohomology sheaves of each E_i are zero to the left of t.

We obtain a unique solution (E, ρ_U). Setting $\rho_i = \rho_{U_i}$ the lemma follows. $\square$

Second proof. We sketch a more direct proof. Denote K the Čech hypercovering of X associated to the covering $\{U_i \to X\}_{i \in I}$, see Hypercoverings, Example 25.3.4. Thus for example $K_0 = \{U_i \to X\}_{i \in I}$ and $K_1 = \{U_i \times_X U_j \to X\}_{i,j \in I}$ and so on. Let $((\mathcal{C}/K)_{total}, \mathcal{O})$, a, a_n be as in Remark 85.16.6. The objects E_i determine an object M_0 in $D(\mathcal{O}_0) = \prod D(\mathcal{O}_{U_i})$. Similarly, the isomorphisms ρ_{ij} determine an isomorphism

$$\alpha : L(f_{\delta^1_1})^* M_0 \longrightarrow L(f_{\delta^1_0})^* M_0$$

in $D(\mathcal{O}_1)$ satisfying the cocycle condition. By Lemma 85.14.3 we obtain a cartesian simplicial system (M_n) of the derived category. By the assumed vanishing of the

[5]In fact, it would suffice if the map $\coprod_{i \in I} h_{U_i} \to h_X$ becomes surjective on sheafification and the lemma holds in this case with the same proof.

negative Ext sheaves we see that the objects M_n have vanishing negative self-exts. Thus we find a cartesian object M of $D(\mathcal{O})$ whose associated simplicial system is isomorphic to (M_n) by Lemma 85.14.7. Since the cohomology sheaves of M are zero in degrees $< t$ we see that by Lemma 85.20.4 we have $M = La^*E$ for some E in $D(\mathcal{O}_X)$. The isomorphism $La^*E \to M$ restricted to $\mathcal{C}/U_i$ produces the isomorphisms ρ_i. We omit the verification of the compatibility with the isomorphisms ρ_{ij}. □

85.25. Proper hypercoverings in topology

09XA Let's work in the category LC of Hausdorff and locally quasi-compact topological spaces and continuous maps, see Cohomology on Sites, Section 21.31. Let X be an object of LC and let U be a simplicial object of LC. Assume we have an augmentation

$$a : U \to X$$

We say that U is a proper hypercovering of X if

(1) $U_0 \to X$ is a proper surjective map,
(2) $U_1 \to U_0 \times_X U_0$ is a proper surjective map,
(3) $U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}$ is a proper surjective map for $n \geq 1$.

The category LC has all finite limits, hence the coskeleta used in the formulation above exist.

$$\boxed{\text{Principle: Proper hypercoverings can be used to compute cohomology.}}$$

A key idea behind the proof of the principle is to find a topology on LC which is stronger than the usual one such that (a) a surjective proper map defines a covering, and (b) cohomology of usual sheaves with respect to this stronger topology agrees with the usual cohomology. Properties (a) and (b) hold for the qc topology, see Cohomology on Sites, Section 21.31. Once we have (a) and (b) we deduce the principle via the earlier work done in this chapter.

0DAF **Lemma 85.25.1.** Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. There is a commutative diagram

$$\begin{tikzcd} Sh((\text{LC}_{qc}/U)_{total}) \arrow[r, "h"'] \arrow[d, "a_{qc}"'] & Sh(U_{Zar}) \arrow[d, "a"] \\ Sh(\text{LC}_{qc}/X) \arrow[r, "h_{-1}"] & Sh(X) \end{tikzcd}$$

where the left vertical arrow is defined in Section 85.21 and the right vertical arrow is defined in Lemma 85.2.8.

Proof. Write $Sh(X) = Sh(X_{Zar})$. Observe that both $(\text{LC}_{qc}/U)_{total}$ and U_{Zar} fall into case A of Situation 85.3.3. This is immediate from the construction of U_{Zar} in Section 85.2 and it follows from Lemma 85.21.5 for $(\text{LC}_{qc}/U)_{total}$. Next, consider the functors $U_{n,Zar} \to \text{LC}_{qc}/U_n$, $U \mapsto U/U_n$ and $X_{Zar} \to \text{LC}_{qc}/X$, $U \mapsto U/X$. We have seen that these define morphisms of sites in Cohomology on Sites, Section 21.31. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 85.5.4 and we may apply Lemma 85.5.5 to conclude. □

0DAG Lemma 85.25.2. Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If $a : U \to X$ gives a proper hypercovering of X, then

$$a^{-1} : Sh(X) \to Sh(U_{Zar}) \quad \text{and} \quad a^{-1} : \text{Ab}(X) \to \text{Ab}(U_{Zar})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 85.25.1. By Cohomology on Sites, Lemma 21.31.6 the functor $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 85.5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in LC_{qc} (as defined in Section 85.21) by Cohomology on Sites, Lemma 21.31.4. By Lemma 85.21.1 we see that a_{qc}^{-1} is fully faithful with quasi-inverse $a_{qc,*}$ and with essential image the cartesian sheaves on $(\text{LC}_{qc}/U)_{total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on U_{Zar}. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on LC_{qc}/U. Hence $h^{-1}\mathcal{G} = a_{qc}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on LC_{qc}/X. We compute

$$\begin{aligned}
(h_{-1})^{-1}(a_*\mathcal{G}) &= (h_{-1})^{-1}\text{Eq}(\ a_{0,*}\mathcal{G}_0 \rightrightarrows a_{1,*}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ (h_{-1})^{-1}a_{0,*}\mathcal{G}_0 \rightrightarrows (h_{-1})^{-1}a_{1,*}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ a_{qc,0,*}h_0^{-1}\mathcal{G}_0 \rightrightarrows a_{qc,1,*}h_1^{-1}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ a_{qc,0,*}a_{qc,0}^{-1}\mathcal{H} \rightrightarrows a_{qc,1,*}a_{qc,1}^{-1}\mathcal{H}\ ) \\
&= a_{qc,*}a_{qc}^{-1}\mathcal{H} \\
&= \mathcal{H}
\end{aligned}$$

Here the first equality follows from Lemma 85.2.8, the second equality follows as $(h_{-1})^{-1}$ is an exact functor, the third equality follows from Cohomology on Sites, Lemma 21.31.8 (here we use that $a_0 : U_0 \to X$ and $a_1 : U_1 \to X$ are proper), the fourth follows from $a_{qc}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$, the fifth from Lemma 85.4.2, and the sixth we've seen above. Since $a_{qc}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}a_*\mathcal{G}$ which ends the proof by fully faithfulness of h^{-1}. □

09XS Lemma 85.25.3. Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If $a : U \to X$ gives a proper hypercovering of X, then for $K \in D^+(X)$

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. Consider the diagram of Lemma 85.25.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_n)$ by Cohomology on Sites, Lemma 21.31.11. Hence

Rh_*h^{-1} is the identity functor on $D^+(U_{Zar})$ by Lemma 85.5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{qc,*}a_{qc}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.21.2 (U is a hypercovering of X in LC_{qc} by Cohomology on Sites, Lemma 21.31.4), and the last equality by the already used Cohomology on Sites, Lemma 21.31.11. □

09XC Lemma 85.25.4. Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If U is a proper hypercovering of X, then

$$R\Gamma(X, K) = R\Gamma(U_{Zar}, a^{-1}K)$$

for $K \in D^+(X)$ where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. This follows from Lemma 85.25.3 because $R\Gamma(U_{Zar}, -) = R\Gamma(X, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0DAH Lemma 85.25.5. Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset \mathrm{Ab}(U_{Zar})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is a proper hypercovering of X, then the functor a^{-1} defines an equivalence

$$D^+(X) \longrightarrow D^+_{\mathcal{A}}(U_{Zar})$$

with quasi-inverse Ra_* where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.25.2 and 85.25.3 and Cohomology on Sites, Lemma 21.28.5. □

09XB Lemma 85.25.6. Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_n$ be the pullback to U_n. If U is a proper hypercovering of X, then there exists a canonical spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.25.4 and 85.2.10. □

85.26. Simplicial schemes

09XT A simplicial scheme is a simplicial object in the category of schemes, see Simplicial, Definition 14.3.1. Recall that a simplicial scheme looks like

$$\begin{tikzcd} X_2 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & X_0 \arrow[l] \end{tikzcd}$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. These morphisms satisfy some required relations such as $d^1_0 \circ s^0_0 = \mathrm{id}_{X_0} = d^1_1 \circ s^0_0$, see Simplicial, Lemma 14.3.2. It is useful to think of $d^n_i : X_n \to X_{n-1}$ as the “projection forgetting the ith coordinate” and to think of $s^n_j : X_n \to X_{n+1}$ as the “diagonal map repeating the jth coordinate”.

A morphism of simplicial schemes $h : X \to Y$ is the same thing as a morphism of simplicial objects in the category of schemes, see Simplicial, Definition 14.3.1. Thus h consists of morphisms of schemes $h_n : X_n \to Y_n$ such that $h_{n-1} \circ d^n_j = d^n_j \circ h_n$ and $h_{n+1} \circ s^n_j = s^n_j \circ h_n$ whenever this makes sense.

An augmentation of a simplicial scheme X is a morphism of schemes $a_0 : X_0 \to S$ such that $a_0 \circ d^1_0 = a_0 \circ d^1_1$. See Simplicial, Section 14.20.

Let X be a simplicial scheme. The construction of Section 85.2 applied to the underlying simplicial topological space gives a site X_{Zar}. On the other hand, for every n we have the small Zariski site $X_{n,Zar}$ (Topologies, Definition 34.3.7) and for every morphism $\varphi : [m] \to [n]$ we have a morphism of sites $f_\varphi = X(\varphi)_{small} : X_{n,Zar} \to X_{m,Zar}$, associated to the morphism of schemes $X(\varphi) : X_n \to X_m$ (Topologies, Lemma 34.3.17). This gives a simplicial object $\mathcal{C}$ in the category of sites. In Lemma 85.3.1 we constructed an associated site $\mathcal{C}_{total}$. Assigning to an open immersion its image defines an equivalence $\mathcal{C}_{total} \to X_{Zar}$ which identifies sheaves, i.e., $Sh(\mathcal{C}_{total}) = Sh(X_{Zar})$. The difference between $\mathcal{C}_{total}$ and X_{Zar} is similar to the difference between the small Zariski site S_{Zar} and the underlying topological space of S. We will silently identify these sites in what follows.

Let X_{Zar} be the site associated to a simplicial scheme X. There is a sheaf of rings $\mathcal{O}$ on X_{Zar} whose restriction to X_n is the structure sheaf $\mathcal{O}_{X_n}$. This follows from Lemma 85.2.2 or from Lemma 85.3.4. We will say $\mathcal{O}$ is the structure sheaf of the simplicial scheme X. At this point all the material developed for simplicial (ringed) sites applies, see Sections 85.3, 85.4, 85.5, 85.6, 85.8, 85.9, 85.10, 85.11, 85.12, 85.13, and 85.14.

Let X be a simplicial scheme with structure sheaf $\mathcal{O}$. As on any ringed topos, there is a notion of a quasi-coherent $\mathcal{O}$-module on X_{Zar}, see Modules on Sites, Definition 18.23.1. However, a quasi-coherent $\mathcal{O}$-module on X_{Zar} is just a cartesian $\mathcal{O}$-module $\mathcal{F}$ whose restrictions $\mathcal{F}_n$ are quasi-coherent on X_n, see Lemma 85.12.10.

Let $h : X \to Y$ be a morphism of simplicial schemes. Either by Lemma 85.2.3 or by (the proof of) Lemma 85.5.2 we obtain a morphism of sites $h_{Zar} : X_{Zar} \to Y_{Zar}$. Recall that h_{Zar}^{-1} and $h_{Zar,*}$ have a simple description in terms of the components, see Lemma 85.2.4 or Lemma 85.5.2. Let $\mathcal{O}_X$, resp. $\mathcal{O}_Y$ denote the structure sheaf of X, resp. Y. We define $h_{Zar}^\sharp : h_{Zar,*}\mathcal{O}_X \to \mathcal{O}_Y$ to be the map of sheaves of rings on Y_{Zar} given by $h_n^\sharp : h_{n,*}\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ on Y_n. We obtain a morphism of ringed sites

$$h_{Zar} : (X_{Zar}, \mathcal{O}_X) \longrightarrow (Y_{Zar}, \mathcal{O}_Y)$$

Let X be a simplicial scheme with structure sheaf $\mathcal{O}$. Let S be a scheme and let $a_0 : X_0 \to S$ be an augmentation of X. Either by Lemma 85.2.8 or by Lemma 85.4.2 we obtain a corresponding morphism of topoi $a : Sh(X_{Zar}) \to Sh(S)$. Observe that $a^{-1}\mathcal{G}$ is the sheaf on X_{Zar} with components $a_n^{-1}\mathcal{G}$. Hence we can use the maps $a_n^\sharp : a_n^{-1}\mathcal{O}_S \to \mathcal{O}_{X_n}$ to define a map $a^\sharp : a^{-1}\mathcal{O}_S \to \mathcal{O}$, or equivalently by adjunction a map $a^\sharp : \mathcal{O}_S \to a_*\mathcal{O}$ (which as usual has the same name). This puts us in the situation discussed in Section 85.11. Therefore we obtain a morphism of ringed topoi

$$a : (Sh(X_{Zar}), \mathcal{O}) \longrightarrow (Sh(S), \mathcal{O}_S)$$

A final observation is the following. Suppose we are given a morphism $h : X \to Y$ of simplicial schemes X and Y with structure sheaves $\mathcal{O}_X$, $\mathcal{O}_Y$, augmentations $a_0 : X_0 \to X_{-1}$, $b_0 : Y_0 \to Y_{-1}$ and a morphism $h_{-1} : X_{-1} \to Y_{-1}$ such that

$$\begin{tikzcd} X_0 \arrow[r, "h_0"'] \arrow[d, "a_0"'] & Y_0 \arrow[d, "b_0"] \\ X_{-1} \arrow[r, "h_{-1}"] & Y_{-1} \end{tikzcd}$$

commutes. Then from the constructions elucidated above we obtain a commutative diagram of morphisms of ringed topoi as follows

$$\begin{tikzcd} (Sh(X_{Zar}), \mathcal{O}_X) \arrow[r, "h_{Zar}"'] \arrow[d, "a"'] & (Sh(Y_{Zar}), \mathcal{O}_Y) \arrow[d, "b"] \\ (Sh(X_{-1}), \mathcal{O}_{X_{-1}}) \arrow[r, "h_{-1}"] & (Sh(Y_{-1}), \mathcal{O}_{Y_{-1}}) \end{tikzcd}$$

85.27. Descent in terms of simplicial schemes

0248 Cartesian morphisms are defined as follows.

0249 **Definition 85.27.1.** Let $a : Y \to X$ be a morphism of simplicial schemes. We say a is *cartesian*, or that Y is *cartesian over* X, if for every morphism $\varphi : [n] \to [m]$ of Δ the corresponding diagram

$$\begin{tikzcd} Y_m \arrow[r, "a"'] \arrow[d, "Y(\varphi)"'] & X_m \arrow[d, "X(\varphi)"] \\ Y_n \arrow[r, "a"] & X_n \end{tikzcd}$$

is a fibre square in the category of schemes.

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian simplicial schemes over a fixed simplicial scheme. In this lemma we denote $f^* : Sch/X \to Sch/Y$ the base change functor associated to a morphism of schemes $f : Y \to X$.

07TC **Lemma 85.27.2.** Let X be a simplicial scheme. The category of simplicial schemes cartesian over X is equivalent to the category of pairs (V, φ) where V is a scheme over X_0 and

$$\varphi : V \times_{X_0, d^1_1} X_1 \longrightarrow X_1 \times_{d^1_0, X_0} V$$

is an isomorphism over X_1 such that $(s^0_0)^*\varphi = \text{id}_V$ and such that

$$(d^2_1)^*\varphi = (d^2_0)^*\varphi \circ (d^2_2)^*\varphi$$

as morphisms of schemes over X_2.

Proof. The statement of the displayed equality makes sense because $d^1_1 \circ d^2_2 = d^1_1 \circ d^2_1$, $d^1_1 \circ d^2_0 = d^1_0 \circ d^2_2$, and $d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$ as morphisms $X_2 \to X_0$, see Simplicial, Remark 14.3.3 hence we can picture these maps as follows

$$\begin{tikzcd}
 & X_2 \times_{d_1^1 \circ d_0^2, X_0} V \arrow[r, "(d_0^2)^*\varphi"'] & X_2 \times_{d_0^1 \circ d_0^2, X_0} V \arrow[dr, equal] & \\
X_2 \times_{d_0^1 \circ d_2^2, X_0} V \arrow[ur, equal] & & & X_2 \times_{d_0^1 \circ d_1^2, X_0} V \\
 & X_2 \times_{d_1^1 \circ d_2^2, X_0} V \arrow[ul, "(d_2^2)^*\varphi"] \arrow[r, equal] & X_2 \times_{d_1^1 \circ d_1^2, X_0} V \arrow[ur, "(d_1^2)^*\varphi"'] &
\end{tikzcd}$$

and the condition signifies the diagram is commutative. It is clear that given a simplicial scheme Y cartesian over X we can set $V = Y_0$ and φ equal to the composition

$$V \times_{X_0, d_1^1} X_1 = Y_0 \times_{X_0, d_1^1} X_1 = Y_1 = X_1 \times_{X_0, d_0^1} Y_0 = X_1 \times_{X_0, d_0^1} V$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair (V, φ) as in the lemma we set $Y_n = X_n \times_{X(\tau_n^n), X_0} V$. Then given $\beta : [n] \to [m]$ we define $V(\beta) : Y_m \to Y_n$ as the pullback by $X(\tau_{\beta(n)m}^m)$ of the map φ postcomposed by the projection $X_m \times_{X(\beta), X_n} Y_n \to Y_n$. This makes sense because

$$X_m \times_{X(\tau_{\beta(n)m}^m), X_1} X_1 \times_{d_1^1, X_0} V = X_m \times_{X(\tau_m^m), X_0} V = Y_m$$

and

$$X_m \times_{X(\tau_{\beta(n)m}^m), X_1} X_1 \times_{d_0^1, X_0} V = X_m \times_{X(\tau_{\beta(n)}^m), X_0} V = X_m \times_{X(\beta), X_n} Y_n.$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. □

024A Definition 85.27.3. Let $f : X \to S$ be a morphism of schemes. The simplicial scheme associated to f, denoted $(X/S)_\bullet$, is the functor $\Delta^{opp} \to Sch$, $[n] \mapsto X \times_S \ldots \times_S X$ described in Simplicial, Example 14.3.5.

Thus $(X/S)_n$ is the $(n+1)$-fold fibre product of X over S. The morphism $d_0^1 : X \times_S X \to X$ is the map $(x_0, x_1) \mapsto x_1$ and the morphism d_1^1 is the other projection. The morphism s_0^0 is the diagonal morphism $X \to X \times_S X$.

024B Lemma 85.27.4. Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes. Set $V = Y_0$ considered as a scheme over X. The morphisms $d_0^1, d_1^1 : Y_1 \to Y_0$ and the morphism $\pi_1 : Y_1 \to X \times_S X$ induce isomorphisms

$$V \times_S X \xleftarrow{(d_1^1, \mathrm{pr}_1 \circ \pi_1)} Y_1 \xrightarrow{(\mathrm{pr}_0 \circ \pi_1, d_0^1)} X \times_S V.$$

Denote $\varphi : V \times_S X \to X \times_S V$ the resulting isomorphism. Then the pair (V, φ) is a descent datum relative to $X \to S$.

Proof. This is a special case of (part of) Lemma 85.27.2 as the displayed equation of that lemma is equivalent to the cocycle condition of Descent, Definition 35.34.1. □

024C **Lemma 85.27.5.** Let $f : X \to S$ be a morphism of schemes. The construction

$$\begin{matrix}\text{category of cartesian} \\ \text{schemes over } (X/S)_\bullet\end{matrix} \longrightarrow \begin{matrix}\text{category of descent data} \\ \text{relative to } X/S\end{matrix}$$

of Lemma 85.27.4 is an equivalence of categories.

Proof. The functor from left to right is given in Lemma 85.27.4. Hence this is a special case of Lemma 85.27.2. □

We may reinterpret the pullback of Descent, Lemma 35.34.6 as follows. Suppose given a morphism of simplicial schemes $f : X' \to X$ and a cartesian morphism of simplicial schemes $Y \to X$. Then the fibre product (viewed as a "pullback")

$$f^*Y = Y \times_X X'$$

of simplicial schemes is a simplicial scheme cartesian over X'. Suppose given a commutative diagram of morphisms of schemes

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

This gives rise to a morphism of simplicial schemes

$$f_\bullet : (X'/S')_\bullet \longrightarrow (X/S)_\bullet.$$

We claim that the "pullback" $f_\bullet^*$ along the morphism $f_\bullet : (X'/S')_\bullet \to (X/S)_\bullet$ corresponds via Lemma 85.27.5 with the pullback defined in terms of descent data in the aforementioned Descent, Lemma 35.34.6.

85.28. Quasi-coherent modules on simplicial schemes

07TE

07TI **Lemma 85.28.1.** Let $f : V \to U$ be a morphism of simplicial schemes. Given a quasi-coherent module $\mathcal{F}$ on U_{Zar} the pullback $f^*\mathcal{F}$ is a quasi-coherent module on V_{Zar}.

Proof. Recall that $\mathcal{F}$ is cartesian with $\mathcal{F}_n$ quasi-coherent, see Lemma 85.12.10. By Lemma 85.2.4 we see that $(f^*\mathcal{F})_n = f_n^*\mathcal{F}_n$ (some details omitted). Hence $(f^*\mathcal{F})_n$ is quasi-coherent. The same fact and the cartesian property for $\mathcal{F}$ imply the cartesian property for $f^*\mathcal{F}$. Thus $\mathcal{F}$ is quasi-coherent by Lemma 85.12.10 again. □

07TJ **Lemma 85.28.2.** Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. For a quasi-coherent module $\mathcal{G}$ on V_{Zar} the pushforward $f_*\mathcal{G}$ is a quasi-coherent module on U_{Zar}.

Proof. If $\mathcal{F} = f_*\mathcal{G}$, then $\mathcal{F}_n = f_{n,*}\mathcal{G}_n$ by Lemma 85.2.4. The maps $\mathcal{F}(\varphi)$ are defined using the base change maps, see Cohomology, Section 20.17. The sheaves $\mathcal{F}_n$ are quasi-coherent by Schemes, Lemma 26.24.1 and the fact that $\mathcal{G}_n$ is quasi-coherent by Lemma 85.12.10. The base change maps along the degeneracies d^n_j are isomorphisms by Cohomology of Schemes, Lemma 30.5.2 and the fact that $\mathcal{G}$ is cartesian by Lemma 85.12.10. Hence $\mathcal{F}$ is cartesian by Lemma 85.12.2. Thus $\mathcal{F}$ is quasi-coherent by Lemma 85.12.10. □

07TK Lemma 85.28.3. Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. Then f^* and f_* form an adjoint pair of functors between the categories of quasi-coherent modules on U_{Zar} and V_{Zar}.

Proof. We have seen in Lemmas 85.28.1 and 85.28.2 that the statement makes sense. The adjointness property follows immediately from the fact that each f_n^* is adjoint to $f_{n,*}$. □

07TL Lemma 85.28.4. Let $f : X \to S$ be a morphism of schemes which has a section[6]. Let $(X/S)_\bullet$ be the simplicial scheme associated to $X \to S$, see Definition 85.27.3. Then pullback defines an equivalence between the category of quasi-coherent $\mathcal{O}_S$-modules and the category of quasi-coherent modules on $((X/S)_\bullet)_{Zar}$.

Proof. Let $\sigma : S \to X$ be a section of f. Let $(\mathcal{F}, \alpha)$ be a pair as in Lemma 85.12.5. Set $\mathcal{G} = \sigma^*\mathcal{F}$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "(\sigma \circ f, 1)"'] \arrow[d, "f"'] & X \times_S X \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_0"] & X \\ S \arrow[r, "\sigma"] & X & \end{tikzcd}$$

Note that $\mathrm{pr}_0 = d^1_1$ and $\mathrm{pr}_1 = d^1_0$. Hence we see that $(\sigma \circ f, 1)^*\alpha$ defines an isomorphism

$$f^*\mathcal{G} = (\sigma \circ f, 1)^*\mathrm{pr}_0^*\mathcal{F} \longrightarrow (\sigma \circ f, 1)^*\mathrm{pr}_1^*\mathcal{F} = \mathcal{F}$$

We omit the verification that this isomorphism is compatible with α and the canonical isomorphism $\mathrm{pr}_0^* f^*\mathcal{G} \to \mathrm{pr}_1^* f^*\mathcal{G}$. □

85.29. Groupoids and simplicial schemes

07TM Given a groupoid in schemes we can build a simplicial scheme. It will turn out that the category of quasi-coherent sheaves on a groupoid is equivalent to the category of cartesian quasi-coherent sheaves on the associated simplicial scheme.

07TN Lemma 85.29.1. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. There exists a simplicial scheme X over S with the following properties

(1) $X_0 = U$, $X_1 = R$, $X_2 = R \times_{s,U,t} R$,
(2) $s^0_0 = e : X_0 \to X_1$,
(3) $d^1_0 = s : X_1 \to X_0$, $d^1_1 = t : X_1 \to X_0$,
(4) $s^1_0 = (e \circ t, 1) : X_1 \to X_2$, $s^1_1 = (1, e \circ t) : X_1 \to X_2$,
(5) $d^2_0 = \mathrm{pr}_1 : X_2 \to X_1$, $d^2_1 = c : X_2 \to X_1$, $d^2_2 = \mathrm{pr}_0$, and
(6) $X = \mathrm{cosk}_2\mathrm{sk}_2 X$.

For all n we have $X_n = R \times_{s,U,t} \ldots \times_{s,U,t} R$ with n factors. The map $d^n_j : X_n \to X_{n-1}$ is given on functors of points by

$$(r_1, \ldots, r_n) \longmapsto (r_1, \ldots, c(r_j, r_{j+1}), \ldots, r_n)$$

for $1 \leq j \leq n-1$ whereas $d^n_0(r_1, \ldots, r_n) = (r_2, \ldots, r_n)$ and $d^n_n(r_1, \ldots, r_n) = (r_1, \ldots, r_{n-1})$.

[6]In fact, it would be enough to assume that f has fpqc locally on S a section, since we have descent of quasi-coherent modules by Descent, Section 35.5.

Proof. We only have to verify that the rules prescribed in (1), (2), (3), (4), (5) define a 2-truncated simplicial scheme U' over S, since then (6) allows us to set $X = \mathrm{cosk}_2 U'$, see Simplicial, Lemma 14.19.2. Using the functor of points approach, all we have to verify is that if $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$ is a groupoid, then

$$\begin{tikzcd}
\mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \arrow[d, shift right=8, "\mathrm{pr}_1"'] \arrow[d, shift left=4, "c"] \arrow[d, shift left=12, "\mathrm{pr}_0"] \\
\mathrm{Arrows} \arrow[u, shift right=4, "{1,e}"'] \arrow[u, shift left=8, "{e,1}"'] \arrow[d, shift right=4, "s"'] \arrow[d, shift left=4, "t"] \\
\mathrm{Ob} \arrow[u, "e"']
\end{tikzcd}$$

is a 2-truncated simplicial set. We omit the details.

Finally, the description of X_n for $n > 2$ follows by induction from the description of X_0, X_1, X_2, and Simplicial, Remark 14.19.9 and Lemma 14.19.6. Alternately, one shows that cosk_2 applied to the 2-truncated simplicial set displayed above gives a simplicial set whose nth term equals $\mathrm{Arrows} \times_{s,\mathrm{Ob},t} \ldots \times_{s,\mathrm{Ob},t} \mathrm{Arrows}$ with n factors and degeneracy maps as given in the lemma. Some details omitted. $\square$

07TP **Lemma 85.29.2.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let X be the simplicial scheme over S constructed in Lemma 85.29.1. Then the category of quasi-coherent modules on (U, R, s, t, c) is equivalent to the category of quasi-coherent modules on X_{Zar}.

Proof. This is clear from Lemmas 85.12.10 and 85.12.5 and Groupoids, Definition 39.14.1. $\square$

In the following lemma we will use the concept of a cartesian morphism $V \to U$ of simplicial schemes as defined in Definition 85.27.1.

07TQ **Lemma 85.29.3.** Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Let X be the simplicial scheme over S constructed in Lemma 85.29.1. Let $(R/U)_\bullet$ be the simplicial scheme associated to $s : R \to U$, see Definition 85.27.3. There exists a cartesian morphism $t_\bullet : (R/U)_\bullet \to X$ of simplicial schemes with low degree morphisms given by

$$\begin{tikzcd}
R \times_{s,U,s} R \times_{s,U,s} R \arrow[r, shift left=2, "\mathrm{pr}_{12}"] \arrow[r, "\mathrm{pr}_{02}"] \arrow[r, shift right=2, "\mathrm{pr}_{01}"'] \arrow[d, "{(r_0,r_1,r_2) \mapsto (r_0 \circ r_1^{-1}, r_1 \circ r_2^{-1})}"'] & R \times_{s,U,s} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "{(r_0,r_1) \mapsto r_0 \circ r_1^{-1}}"'] & R \arrow[d, "t"] \\
R \times_{s,U,t} R \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "c"] \arrow[r, shift right=2, "\mathrm{pr}_0"'] & R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U
\end{tikzcd}$$

Proof. For arbitrary n we define $(R/U)_\bullet \to X_n$ by the rule

$$(r_0, \ldots, r_n) \longrightarrow (r_0 \circ r_1^{-1}, \ldots, r_{n-1} \circ r_n^{-1})$$

Compatibility with degeneracy maps is clear from the description of the degeneracies in Lemma 85.29.1. We omit the verification that the maps respect the morphisms s^n_j. Groupoids, Lemma 39.13.5 (with the roles of s and t reversed) shows that the two right squares are cartesian. In exactly the same manner one shows all the other squares are cartesian too. Hence the morphism is cartesian. □

85.30. Descent data give equivalence relations

024D In Section 85.27 we saw how descent data relative to $X \to S$ can be formulated in terms of cartesian simplicial schemes over $(X/S)_\bullet$. Here we link this to equivalence relations as follows.

024E **Lemma 85.30.1.** Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes, see Definitions 85.27.1 and 85.27.3. Then the morphism

$$j = (d^1_1, d^1_0) : Y_1 \to Y_0 \times_S Y_0$$

defines an equivalence relation on Y_0 over S, see Groupoids, Definition 39.3.1.

Proof. Note that j is a monomorphism. Namely the composition $Y_1 \to Y_0 \times_S Y_0 \to Y_0 \times_S X$ is an isomorphism as π is cartesian.

Consider the morphism

$$(d^2_2, d^2_0) : Y_2 \to Y_1 \times_{d^1_0, Y_0, d^1_1} Y_1.$$

This works because $d_0 \circ d_2 = d_1 \circ d_0$, see Simplicial, Remark 14.3.3. Also, it is a morphism over $(X/S)_2$. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_1} (X \times_S X \times_S X) = X \times_S Y_0 \times_S X$ because π is cartesian. Details omitted.

As in Groupoids, Definition 39.3.1 we denote $t = \mathrm{pr}_0 \circ j = d^1_1$ and $s = \mathrm{pr}_1 \circ j = d^1_0$. The isomorphism above, combined with the morphism $d^2_1 : Y_2 \to Y_1$ give us a composition morphism

$$c : Y_1 \times_{s, Y_0, t} Y_1 \longrightarrow Y_1$$

over $Y_0 \times_S Y_0$. This immediately implies that for any scheme T/S the relation $Y_1(T) \subset Y_0(T) \times Y_0(T)$ is transitive.

Reflexivity follows from the fact that the restriction of the morphism j to the diagonal $\Delta : X \to X \times_S X$ is an isomorphism (again use the cartesian property of π).

To see symmetry we consider the morphism

$$(d^2_2, d^2_1) : Y_2 \to Y_1 \times_{d^1_1, Y_0, d^1_1} Y_1.$$

This works because $d_1 \circ d_2 = d_1 \circ d_1$, see Simplicial, Remark 14.3.3. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_0} (X \times_S X \times_S X) = Y_0 \times_S X \times_S X$ because π is cartesian. Details omitted.

Let T/S be a scheme. Let $a \sim b$ for $a, b \in Y_0(T)$ be synonymous with $(a, b) \in Y_1(T)$. The isomorphism (d^2_2, d^2_1) above implies that if $a \sim b$ and $a \sim c$, then $b \sim c$. Combined with reflexivity this shows that $\sim$ is an equivalence relation. □

85.31. An example case

024F In this section we show that disjoint unions of spectra of Artinian rings can be descended along a quasi-compact surjective flat morphism of schemes.

024G **Lemma 85.31.1.** *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. For $y \in Y_0$ a point define*

$$T_y = \{y' \in Y_0 \mid \exists\, y_1 \in Y_1 : d^1_1(y_1) = y, d^1_0(y_1) = y'\}$$

as a subset of Y_0. Then $y \in T_y$ and $T_y \cap T_{y'} \neq \emptyset \Rightarrow T_y = T_{y'}$.

Proof. Combine Lemma 85.30.1 and Groupoids, Lemma 39.3.4. □

024H **Lemma 85.31.2.** *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. Let $y \in Y_0$ be a point. If $X \to S$ is quasi-compact, then*

$$T_y = \{y' \in Y_0 \mid \exists\, y_1 \in Y_1 : d^1_1(y_1) = y, d^1_0(y_1) = y'\}$$

is a quasi-compact subset of Y_0.

Proof. Let F_y be the scheme theoretic fibre of $d^1_1 : Y_1 \to Y_0$ at y. Then we see that T_y is the image of the morphism

$$\begin{tikzcd} F_y \arrow[r] \arrow[d] & Y_1 \arrow[r, "d^1_0"] \arrow[d, "d^1_1"] & Y_0 \\ y \arrow[r] & Y_0 & \end{tikzcd}$$

Note that F_y is quasi-compact. This proves the lemma. □

024I **Lemma 85.31.3.** *Let $X \to S$ be a quasi-compact flat surjective morphism. Let (V, φ) be a descent datum relative to $X \to S$. If V is a disjoint union of spectra of Artinian rings, then (V, φ) is effective.*

Proof. Let $Y \to (X/S)_\bullet$ be the cartesian morphism of simplicial schemes corresponding to (V, φ) by Lemma 85.27.5. Observe that $Y_0 = V$. Write $V = \coprod_{i \in I} \mathrm{Spec}(A_i)$ with each A_i local Artinian. Moreover, let $v_i \in V$ be the unique closed point of $\mathrm{Spec}(A_i)$ for all $i \in I$. Write $i \sim j$ if and only if $v_i \in T_{v_j}$ with notation as in Lemma 85.31.1 above. By Lemmas 85.31.1 and 85.31.2 this is an equivalence relation with finite equivalence classes. Let $\overline{I} = I/\sim$. Then we can write $V = \coprod_{\overline{i} \in \overline{I}} V_{\overline{i}}$ with $V_{\overline{i}} = \coprod_{i \in \overline{i}} \mathrm{Spec}(A_i)$. By construction we see that $\varphi : V \times_S X \to X \times_S V$ maps the open and closed subspaces $V_{\overline{i}} \times_S X$ into the open and closed subspaces $X \times_S V_{\overline{i}}$. In other words, we get descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$, and (V, φ) is the coproduct of them in the category of descent data. Since each of the $V_{\overline{i}}$ is a finite union of spectra of Artinian local rings the morphism $V_{\overline{i}} \to X$ is affine, see Morphisms, Lemma 29.11.13. Since $\{X \to S\}$ is an fpqc covering we see that all the descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$ are effective by Descent, Lemma 35.37.1. □

To be sure, the lemma above has very limited applicability!

85.32. Simplicial algebraic spaces

0DE7 Let S be a scheme. A simplicial algebraic space is a simplicial object in the category of algebraic spaces over S, see Simplicial, Definition 14.3.1. Recall that a simplicial algebraic space looks like

$$X_2 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_1 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} X_0$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. These morphisms satisfy some required relations such as $d^1_0 \circ s^0_0 = \text{id}_{X_0} = d^1_1 \circ s^0_0$, see Simplicial, Lemma 14.3.2. It is useful to think of $d^n_i : X_n \to X_{n-1}$ as the "projection forgetting the ith coordinate" and to think of $s^n_j : X_n \to X_{n+1}$ as the "diagonal map repeating the jth coordinate".

A morphism of simplicial algebraic spaces $h : X \to Y$ is the same thing as a morphism of simplicial objects in the category of algebraic spaces over S, see Simplicial, Definition 14.3.1. Thus h consists of morphisms of algebraic spaces $h_n : X_n \to Y_n$ such that $h_{n-1} \circ d^n_j = d^n_j \circ h_n$ and $h_{n+1} \circ s^n_j = s^n_j \circ h_n$ whenever this makes sense.

An augmentation $a : X \to X_{-1}$ of a simplicial algebraic space X is given by a morphism of algebraic spaces $a_0 : X_0 \to X_{-1}$ such that $a_0 \circ d^1_0 = a_0 \circ d^1_1$. See Simplicial, Section 14.20. In this situation we always indicate $a_n : X_n \to X_{-1}$ the induced morphisms for $n \geq 0$.

Let X be a simplicial algebraic space. For every n we have the site $X_{n,spaces,étale}$ (Properties of Spaces, Definition 66.18.2) and for every morphism $\varphi : [m] \to [n]$ we have a morphism of sites

$$f_\varphi = X(\varphi)_{spaces,étale} : X_{n,spaces,étale} \to X_{m,spaces,étale},$$

associated to the morphism of algebraic spaces $X(\varphi) : X_n \to X_m$ (Properties of Spaces, Lemma 66.18.8). This gives a simplicial object in the category of sites. In Lemma 85.3.1 we constructed an associated site which we denote $X_{spaces,étale}$. An object of the site $X_{spaces,étale}$ is a an algebraic space U étale over X_n for some n and a morphism $(\varphi, f) : U/X_n \to V/X_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : U \to V$ of algebraic spaces such that the diagram

$$\begin{tikzcd} U \arrow[r, "f"'] \arrow[d] & V \arrow[d] \\ X_n \arrow[r, "f_\varphi"] & X_m \end{tikzcd}$$

is commutative. Consider the full subcategories

$$X_{affine,étale} \subset X_{étale} \subset X_{spaces,étale}$$

whose objects are U/X_n with U affine, respectively a scheme. Endowing these categories with their natural topologies (see Properties of Spaces, Lemma 66.18.6, Definition 66.18.1, and Lemma 66.18.3) these inclusion functors define equivalences of topoi

$$Sh(X_{affine,étale}) = Sh(X_{étale}) = Sh(X_{spaces,étale})$$

In the following we will silently identify these topoi. We will say that $X_{étale}$ is the small étale site of X and its topos is the small étale topos of X.

Let $X_{étale}$ be the small étale site of a simplicial algebraic space X. There is a sheaf of rings $\mathcal{O}$ on $X_{étale}$ whose restriction to X_n is the structure sheaf $\mathcal{O}_{X_n}$. This follows from Lemma 85.3.4. We will say $\mathcal{O}$ is the structure sheaf of the simplicial algebraic space X. At this point all the material developed for simplicial (ringed) sites applies, see Sections 85.3, 85.4, 85.5, 85.6, 85.8, 85.9, 85.10, 85.11, 85.12, 85.13, and 85.14.

Let X be a simplicial algebraic space with structure sheaf $\mathcal{O}$. As on any ringed topos, there is a notion of a quasi-coherent $\mathcal{O}$-module on $X_{étale}$, see Modules on Sites, Definition 18.23.1. However, a quasi-coherent $\mathcal{O}$-module on $X_{étale}$ is just a cartesian $\mathcal{O}$-module $\mathcal{F}$ whose restrictions $\mathcal{F}_n$ are quasi-coherent on X_n, see Lemma 85.12.10.

Let $h : X \to Y$ be a morphism of simplicial algebraic spaces over S. By Lemma 85.5.2 applied to the morphisms of sites $(h_n)_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ (Properties of Spaces, Lemma 66.18.8) we obtain a morphism of small étale topoi $h_{étale} : Sh(X_{étale}) \to Sh(Y_{étale})$. Recall that $h_{étale}^{-1}$ and $h_{étale,*}$ have a simple description in terms of the components, see Lemma 85.5.2. Let $\mathcal{O}_X$, resp. $\mathcal{O}_Y$ denote the structure sheaf of X, resp. Y. We define $h_{étale}^\sharp : h_{étale,*}\mathcal{O}_X \to \mathcal{O}_Y$ to be the map of sheaves of rings on $Y_{étale}$ given by $h_n^\sharp : h_{n,*}\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ on Y_n. We obtain a morphism of ringed topoi

$$h_{étale} : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

Let X be a simplicial algebraic space with structure sheaf $\mathcal{O}$. Let X_{-1} be an algebraic space over S and let $a_0 : X_0 \to X_{-1}$ be an augmentation of X. By Lemma 85.4.2 applied to the morphism of sites $(a_0)_{spaces,étale} : X_{0,spaces,étale} \to X_{-1,spaces,étale}$ we obtain a corresponding morphism of topoi $a : Sh(X_{étale}) \to Sh(X_{-1,étale})$. Observe that $a^{-1}\mathcal{G}$ is the sheaf on $X_{étale}$ with components $a_n^{-1}\mathcal{G}$. Hence we can use the maps $a_n^\sharp : a_n^{-1}\mathcal{O}_{X_{-1}} \to \mathcal{O}_{X_n}$ to define a map $a^\sharp : a^{-1}\mathcal{O}_{X_{-1}} \to \mathcal{O}$, or equivalently by adjunction a map $a^\sharp : \mathcal{O}_{X_{-1}} \to a_*\mathcal{O}$ (which as usual has the same name). This puts us in the situation discussed in Section 85.11. Therefore we obtain a morphism of ringed topoi

$$a : (Sh(X_{étale}), \mathcal{O}) \longrightarrow (Sh(X_{-1}), \mathcal{O}_{X_{-1}})$$

A final observation is the following. Suppose we are given a morphism $h : X \to Y$ of simplicial algebraic spaces X and Y with structure sheaves $\mathcal{O}_X$, $\mathcal{O}_Y$, augmentations $a_0 : X_0 \to X_{-1}$, $b_0 : Y_0 \to Y_{-1}$ and a morphism $h_{-1} : X_{-1} \to Y_{-1}$ such that

$$\begin{tikzcd} X_0 \arrow[r, "h_0"'] \arrow[d, "a_0"'] & Y_0 \arrow[d, "b_0"] \\ X_{-1} \arrow[r, "h_{-1}"] & Y_{-1} \end{tikzcd}$$

commutes. Then from the constructions elucidated above we obtain a commutative diagram of morphisms of ringed topoi as follows

$$\begin{tikzcd} (Sh(X_{étale}), \mathcal{O}_X) \arrow[r, "h_{étale}"'] \arrow[d, "a"'] & (Sh(Y_{étale}), \mathcal{O}_Y) \arrow[d, "b"] \\ (Sh(X_{-1}), \mathcal{O}_{X_{-1}}) \arrow[r, "h_{-1}"] & (Sh(Y_{-1}), \mathcal{O}_{Y_{-1}}) \end{tikzcd}$$

85.33. Fppf hypercoverings of algebraic spaces

0DH4 This section is the analogue of Section 85.25 for the case of algebraic spaces and fppf hypercoverings. The reader who wishes to do so, can replace "algebraic space" everywhere with "scheme" and get equally valid results. This has the advantage of replacing the references to More on Cohomology of Spaces, Section 84.6 with references to Étale Cohomology, Section 59.100.

We fix a base scheme S. Let X be an algebraic space over S and let U be a simplicial algebraic space over S. Assume we have an augmentation

$$a : U \to X$$

See Section 85.32. We say that U is an fppf hypercovering of X if

(1) $U_0 \to X$ is flat, locally of finite presentation, and surjective,
(2) $U_1 \to U_0 \times_X U_0$ is flat, locally of finite presentation, and surjective,
(3) $U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}$ is flat, locally of finite presentation, and surjective for $n \geq 1$.

The category of algebraic spaces over S has all finite limits, hence the coskeleta used in the formulation above exist.

$$\boxed{\text{Principle: Fppf hypercoverings can be used to compute étale cohomology.}}$$

The key idea behind the proof of the principle is to compare the fppf and étale topologies on the category Spaces/S. Namely, the fppf topology is stronger than the étale topology and we have (a) a flat, locally finitely presented, surjective map defines an fppf covering, and (b) fppf cohomology of sheaves pulled back from the small étale site agrees with étale cohomology as we have seen in More on Cohomology of Spaces, Section 84.6.

0DH5 **Lemma 85.33.1.** Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram

$$\begin{tikzcd} Sh((\text{Spaces}/U)_{fppf,total}) \arrow[r, "h"'] \arrow[d, "a_{fppf}"'] & Sh(U_{\acute{e}tale}) \arrow[d, "a"] \\ Sh((\text{Spaces}/X)_{fppf}) \arrow[r, "h_{-1}"] & Sh(X_{\acute{e}tale}) \end{tikzcd}$$

where the left vertical arrow is defined in Section 85.21 and the right vertical arrow is defined in Section 85.32.

Proof. The notation $(\text{Spaces}/U)_{fppf,total}$ indicates that we are using the construction of Section 85.21 for the site $(\text{Spaces}/S)_{fppf}$ and the simplicial object U of this site[7]. We will use the sites $X_{spaces,\acute{e}tale}$ and $U_{spaces,\acute{e}tale}$ for the topoi on the right hand side; this is permissible see discussion in Section 85.32.

Observe that both $(\text{Spaces}/U)_{fppf,total}$ and $U_{spaces,\acute{e}tale}$ fall into case A of Situation 85.3.3. This is immediate from the construction of $U_{\acute{e}tale}$ in Section 85.32 and it follows from Lemma 85.21.5 for $(\text{Spaces}/U)_{fppf,total}$. Next, consider the functors $U_{n,spaces,\acute{e}tale} \to (\text{Spaces}/U_n)_{fppf}$, $U \mapsto U/U_n$ and $X_{spaces,\acute{e}tale} \to (\text{Spaces}/X)_{fppf}$, $U \mapsto U/X$. We have seen that these define morphisms of sites in More on Cohomology of Spaces, Section 84.6 where these were denoted $a_{U_n} = \epsilon_{U_n} \circ \pi_{u_n}$ and

[7] We could also use the étale topology and this would be denoted $(\text{Spaces}/U)_{\acute{e}tale,total}$.

$a_X = \epsilon_X \circ \pi_X$. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 85.5.4 and we may apply Lemma 85.5.5 to conclude. $\square$

0DH6 Lemma 85.33.2. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then

$$a^{-1} : Sh(X_{\acute{e}tale}) \to Sh(U_{\acute{e}tale}) \quad \text{and} \quad a^{-1} : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(U_{\acute{e}tale})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 85.32.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 85.33.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 84.6. Thus it follows from More on Cohomology of Spaces, Lemma 84.6.1 that $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 85.5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in $(\text{Spaces}/S)_{fppf}$ as defined in Section 85.21. By Lemma 85.21.1 we see that a_{fppf}^{-1} is fully faithful with quasi-inverse $a_{fppf,*}$ and with essential image the cartesian sheaves on $(\text{Spaces}/U)_{fppf,total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on $U_{\acute{e}tale}$. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on $(\text{Spaces}/U)_{fppf,total}$. Hence $h^{-1}\mathcal{G} = a_{fppf}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on $(\text{Spaces}/X)_{fppf}$. In particular we find that $h_0^{-1}\mathcal{G}_0 = (a_{0,big,fppf})^{-1}\mathcal{H}$. Recalling that $h_0 = a_{U_0}$ and that $U_0 \to X$ is flat, locally of finite presentation, and surjective, we find from More on Cohomology of Spaces, Lemma 84.6.7 that there exists a sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ and isomorphism $\mathcal{H} = (h_{-1})^{-1}\mathcal{F}$. Since $a_{fppf}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}\mathcal{F}$. By fully faithfulness of h^{-1} we conclude that $a^{-1}\mathcal{F} \cong \mathcal{G}$.

Fix an isomorphism $\theta : a^{-1}\mathcal{F} \to \mathcal{G}$. To finish the proof we have to show $\mathcal{G} = a^{-1}a_*\mathcal{G}$ (in order to show that the quasi-inverse is given by a_*; everything else has been proven above). Because a^{-1} is fully faithful we have $\text{id} \cong a_*a^{-1}$ by Categories, Lemma 4.24.4. Thus $\mathcal{F} \cong a_*a^{-1}\mathcal{F}$ and $a_*\theta : a_*a^{-1}\mathcal{F} \to a_*\mathcal{G}$ combine to an isomorphism $\mathcal{F} \to a_*\mathcal{G}$. Pulling back by a and precomposing by θ^{-1} we find the desired isomorphism. $\square$

0DH7 Lemma 85.33.3. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then for $K \in D^+(X_{\acute{e}tale})$

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.33.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_{n,\acute{e}tale})$ by More on Cohomology of Spaces, Lemma 84.6.2.

Hence Rh_*h^{-1} is the identity functor on $D^+(U_{étale})$ by Lemma 85.5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{fppf,*}a_{fppf}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.21.2 as U is a hypercovering of X in $(\text{Spaces}/S)_{fppf}$, and the last equality by the already used More on Cohomology of Spaces, Lemma 84.6.2. □

0DH8 Lemma 85.33.4. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then

$$R\Gamma(X_{étale}, K) = R\Gamma(U_{étale}, a^{-1}K)$$

for $K \in D^+(X_{étale})$. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. This follows from Lemma 85.33.3 because $R\Gamma(U_{étale}, -) = R\Gamma(X_{étale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0DH9 Lemma 85.33.5. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset \text{Ab}(U_{étale})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is an fppf hypercovering of X, then the functor a^{-1} defines an equivalence

$$D^+(X_{étale}) \longrightarrow D^+_{\mathcal{A}}(U_{étale})$$

with quasi-inverse Ra_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.33.2 and 85.33.3 and Cohomology on Sites, Lemma 21.28.5. □

0DHA Lemma 85.33.6. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $\mathcal{F}_n$ be the pullback to $U_{n,étale}$. If U is an fppf hypercovering of X, then there exists a canonical spectral sequence

$$E_1^{p,q} = H^q_{étale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{étale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.33.4 and 85.8.3. □

85.34. Fppf hypercoverings of algebraic spaces: modules

0DHB We continue the discussion of (cohomological) descent for fppf hypercoverings started in Section 85.33 but in this section we discuss what happens for sheaves of modules. We mainly discuss quasi-coherent modules and it turns out that we can do unbounded cohomological descent for those.

0DHC Lemma 85.34.1. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram

$$\begin{tikzcd} (Sh((\text{Spaces}/U)_{fppf,total}), \mathcal{O}_{big,total}) \arrow[r, "h"'] \arrow[d, "a_{fppf}"'] & (Sh(U_{étale}), \mathcal{O}_U) \arrow[d, "a"] \\ (Sh((\text{Spaces}/X)_{fppf}), \mathcal{O}_{big}) \arrow[r, "h_{-1}"] & (Sh(X_{étale}), \mathcal{O}_X) \end{tikzcd}$$

of ringed topoi where the left vertical arrow is defined in Section 85.22 and the right vertical arrow is defined in Section 85.32.

Proof. For the underlying diagram of topoi we refer to the discussion in the proof of Lemma 85.33.1. The sheaf $\mathcal{O}_U$ is the structure sheaf of the simplicial algebraic space U as defined in Section 85.32. The sheaf $\mathcal{O}_X$ is the usual structure sheaf of the algebraic space X. The sheaves of rings $\mathcal{O}_{big,total}$ and $\mathcal{O}_{big}$ come from the structure sheaf on $(\text{Spaces}/S)_{fppf}$ in the manner explained in Section 85.22 which also constructs a_{fppf} as a morphism of ringed topoi. The component morphisms $h_n = a_{U_n}$ and $h_{-1} = a_X$ are morphisms of ringed topoi by More on Cohomology of Spaces, Section 84.7. Finally, since the continuous functor $u : U_{spaces,étale} \to (\text{Spaces}/U)_{fppf,total}$ used to define h[8] is given by $V/U_n \mapsto V/U_n$ we see that $h_*\mathcal{O}_{big,total} = \mathcal{O}_U$ which is how we endow h with the structure of a morphism of ringed simplicial sites as in Remark 85.7.1. Then we obtain h as a morphism of ringed topoi by Lemma 85.7.2. Please observe that the morphisms h_n indeed agree with the morphisms a_{U_n} described above. We omit the verification that the diagram is commutative (as a diagram of ringed topoi – we already know it is commutative as a diagram of topoi). $\square$

0DHD Lemma 85.34.2. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then

$$a^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$$

is an equivalence fully faithful with quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.34.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 84.7. Thus it follows from More on Cohomology of Spaces, Lemma 84.7.1 that

$$(h_{-1})^* : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_{big})$$

is an equivalence with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. Recall that $QCoh(\mathcal{O}_U)$ and $QCoh(\mathcal{O}_{big,total})$ consist of cartesian modules whose components are quasi-coherent, see Lemma 85.12.10. Since the functors h^* and h_* of Lemma 85.7.2 agree with the functors h_n^* and $h_{n,*}$ on components we conclude that

$$h^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(\mathcal{O}_{big,total})$$

is an equivalence with quasi-inverse h_*. Observe that U is a hypercovering of X in $(\text{Spaces}/S)_{fppf}$ as defined in Section 85.21. By Lemma 85.22.1 we see that

[8] This happened in the proof of Lemma 85.33.1 via an application of Lemma 85.5.5.

a_{fppf}^* is fully faithful with quasi-inverse $a_{fppf,*}$ and with essential image the cartesian sheaves of $\mathcal{O}_{fppf,total}$-modules. Thus, by the description of $QCoh(\mathcal{O}_{big})$ and $QCoh(\mathcal{O}_{big,total})$ of Lemma 85.12.10, we get an equivalence

$$a_{fppf}^* : QCoh(\mathcal{O}_{big}) \longrightarrow QCoh(\mathcal{O}_{big,total})$$

with quasi-inverse given by $a_{fppf,*}$. A formal argument (chasing around the diagram) now shows that a^* is fully faithful on $QCoh(\mathcal{O}_X)$ and has image contained in $QCoh(\mathcal{O}_U)$.

Finally, suppose that $\mathcal{G}$ is in $QCoh(\mathcal{O}_U)$. Then $h^*\mathcal{G}$ is in $QCoh(\mathcal{O}_{big,total})$. Hence $h^*\mathcal{G} = a_{fppf}^*\mathcal{H}$ with $\mathcal{H} = a_{fppf,*}h^*\mathcal{G}$ in $QCoh(\mathcal{O}_{big})$ (see above). In turn we see that $\mathcal{H} = (h_{-1})^*\mathcal{F}$ with $\mathcal{F} = h_{-1,*}\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. Going around the diagram we deduce that $h^*\mathcal{G} \cong h^*a^*\mathcal{F}$. By fully faithfulness of h^* we conclude that $a^*\mathcal{F} \cong \mathcal{G}$. Since $\mathcal{F} = h_{-1,*}a_{fppf,*}h^*\mathcal{G} = a_*h_*h^*\mathcal{G} = a_*\mathcal{G}$ we also obtain the statement that the quasi-inverse is given by a_*. □

0DHE Lemma 85.34.3. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then for $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module the map

$$\mathcal{F} \to Ra_*(a^*\mathcal{F})$$

is an isomorphism. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.33.1. Let $\mathcal{F}_n = a_n^*\mathcal{F}$ be the nth component of $a^*\mathcal{F}$. This is a quasi-coherent $\mathcal{O}_{U_n}$-module. Then $\mathcal{F}_n = Rh_{n,*}h_n^*\mathcal{F}_n$ by More on Cohomology of Spaces, Lemma 84.7.2. Hence $a^*\mathcal{F} = Rh_*h^*a^*\mathcal{F}$ by Lemma 85.7.3. We have

$$\begin{aligned} Ra_*(a^*\mathcal{F}) &= Ra_*Rh_*h^*a^*\mathcal{F} \\ &= Rh_{-1,*}Ra_{fppf,*}a_{fppf}^*(h_{-1})^*\mathcal{F} \\ &= Rh_{-1,*}(h_{-1})^*\mathcal{F} \\ &= \mathcal{F} \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.22.2 as U is a hypercovering of X in $(\text{Spaces}/S)_{fppf}$ and $La_{fppf}^* = a_{fppf}^*$ as a_{fppf} is flat (namely $a_{fppf}^{-1}\mathcal{O}_{big} = \mathcal{O}_{big,total}$, see Remark 85.16.5), and the last equality by the already used More on Cohomology of Spaces, Lemma 84.7.2. □

0DHF Lemma 85.34.4. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Assume $a : U \to X$ is an fppf hypercovering of X. Then $QCoh(\mathcal{O}_U)$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_U)$ and

$$a^* : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_U)$$

is an equivalence of categories with quasi-inverse given by Ra_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. First observe that the maps $a_n : U_n \to X$ and $d_i^n : U_n \to U_{n-1}$ are flat, locally of finite presentation, and surjective by Hypercoverings, Remark 25.8.4.

Recall that an $\mathcal{O}_U$-module $\mathcal{F}$ is quasi-coherent if and only if it is cartesian and $\mathcal{F}_n$ is quasi-coherent for all n. See Lemma 85.12.10. By Lemma 85.12.6 (and flatness of the maps $d^n_i : U_n \to U_{n-1}$ shown above) the cartesian modules for a weak Serre subcategory of $\text{Mod}(\mathcal{O}_U)$. On the other hand $QCoh(\mathcal{O}_{U_n}) \subset \text{Mod}(\mathcal{O}_{U_n})$ is a weak Serre subcategory for each n (Properties of Spaces, Lemma 66.29.7). Combined we see that $QCoh(\mathcal{O}_U) \subset \text{Mod}(\mathcal{O}_U)$ is a weak Serre subcategory.

To finish the proof we check the conditions (1) – (5) of Cohomology on Sites, Lemma 21.28.6 one by one.

Ad (1). This holds since a_n flat (seen above) implies a is flat by Lemma 85.11.1.

Ad (2). This is the content of Lemma 85.34.2.

Ad (3). This is the content of Lemma 85.34.3.

Ad (4). Recall that we can use either the site $U_{\acute{e}tale}$ or $U_{spaces,\acute{e}tale}$ to define the small étale topos $Sh(U_{\acute{e}tale})$, see Section 85.32. The assumption of Cohomology on Sites, Situation 21.25.1 holds for the triple $(U_{spaces,\acute{e}tale}, \mathcal{O}_U, QCoh(\mathcal{O}_U))$ and by the same reasoning for the triple $(U_{\acute{e}tale}, \mathcal{O}_U, QCoh(\mathcal{O}_U))$. Namely, take

$$\mathcal{B} \subset \text{Ob}(U_{\acute{e}tale}) \subset \text{Ob}(U_{spaces,\acute{e}tale})$$

to be the set of affine objects. For $V/U_n \in \mathcal{B}$ take $d_{V/U_n} = 0$ and take Cov_{V/U_n} to be the set of étale coverings $\{V_i \to V\}$ with V_i affine. Then we get the desired vanishing because for $\mathcal{F} \in QCoh(\mathcal{O}_U)$ and any $V/U_n \in \mathcal{B}$ we have

$$H^p(V/U_n, \mathcal{F}) = H^p(V, \mathcal{F}_n)$$

by Lemma 85.10.4. Here on the right hand side we have the cohomology of the quasi-coherent sheaf $\mathcal{F}_n$ on U_n over the affine obect V of $U_{n,\acute{e}tale}$. This vanishes for $p > 0$ by the discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2.

Ad (5). Follows by taking $\mathcal{B} \subset \text{Ob}(X_{spaces,\acute{e}tale})$ the set of affine objects and the references given above. $\square$

0DHG Lemma 85.34.5. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then

$$R\Gamma(X_{\acute{e}tale}, K) = R\Gamma(U_{\acute{e}tale}, a^*K)$$

for $K \in D_{QCoh}(\mathcal{O}_X)$. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 85.32.

Proof. This follows from Lemma 85.34.4 because $R\Gamma(U_{\acute{e}tale}, -) = R\Gamma(X_{\acute{e}tale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. $\square$

0DHH Lemma 85.34.6. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}_n$ be the pullback to $U_{n,\acute{e}tale}$. If U is an fppf hypercovering of X, then there exists a canonical spectral sequence

$$E_1^{p,q} = H^q_{\acute{e}tale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{\acute{e}tale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.34.5 and 85.10.3. $\square$

85.35. Fppf descent of complexes

0DL8 In this section we pull some of the previously shown results together for fppf coverings of algebraic spaces and derived categories of quasi-coherent modules.

0DL9 Lemma 85.35.1. Let X be an algebraic space over a scheme S. Let $K, E \in D_{QCoh}(\mathcal{O}_X)$. Let $a : U \to X$ be an fppf hypercovering. Assume that for all $n \geq 0$ we have

$$\mathrm{Ext}^i_{\mathcal{O}_{U_n}}(La_n^*K, La_n^*E) = 0 \text{ for } i < 0$$

Then we have

(1) $\mathrm{Ext}^i_{\mathcal{O}_X}(K, E) = 0$ for $i < 0$, and
(2) there is an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(K, E) \to \mathrm{Hom}_{\mathcal{O}_{U_0}}(La_0^*K, La_0^*E) \to \mathrm{Hom}_{\mathcal{O}_{U_1}}(La_1^*K, La_1^*E)$$

Proof. Write $K_n = La_n^*K$ and $E_n = La_n^*E$. Then these are the simplicial systems of the derived category of modules (Definition 85.14.1) associated to La^*K and La^*E (Lemma 85.14.2) where $a : U_{\acute{e}tale} \to X_{\acute{e}tale}$ is as in Section 85.32. Let us prove (2) first. By Lemma 85.34.4 we have

$$\mathrm{Hom}_{\mathcal{O}_X}(K, E) = \mathrm{Hom}_{\mathcal{O}_U}(La^*K, La^*E)$$

Thus the sequence looks like this:

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(La^*K, La^*E) \to \mathrm{Hom}_{\mathcal{O}_{U_0}}(K_0, E_0) \to \mathrm{Hom}_{\mathcal{O}_{U_1}}(K_1, E_1)$$

The first arrow is injective by Lemma 85.14.5. The image of this arrow is the kernel of the second by Lemma 85.14.6. This finishes the proof of (2). Part (1) follows by applying part (2) with $K[i]$ and E for $i > 0$. □

0DLA Lemma 85.35.2. Let X be an algebraic space over a scheme S. Let $a : U \to X$ be an fppf hypercovering. Suppose given $K_0 \in D_{QCoh}(U_0)$ and an isomorphism

$$\alpha : L(f_{\delta_1^1})^*K_0 \longrightarrow L(f_{\delta_0^1})^*K_0$$

satisfying the cocycle condition on U_1. Set $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and set $K_n = Lf^*_{\tau_n^n}K_0$. Assume $\mathrm{Ext}^i_{\mathcal{O}_{U_n}}(K_n, K_n) = 0$ for $i < 0$. Then there exists an object $K \in D_{QCoh}(\mathcal{O}_X)$ and an isomorphism $La_0^*K \to K$ compatible with α.

Proof. The objects K_n form the members of a simplicial system of the derived category of modules by Lemma 85.14.3. Then we obtain an object $K' \in D_{QCoh}(\mathcal{O}_{U_{\acute{e}tale}})$ such that (K_n, K_φ) is the system deduced from K', see Lemma 85.14.7. Finally, we apply Lemma 85.34.4 to see that $K' = La^*K$ for some $K \in D_{QCoh}(\mathcal{O}_X)$ as desired. □

85.36. Proper hypercoverings of algebraic spaces

0DHI This section is the analogue of Section 85.25 for the case of algebraic spaces. The reader who wishes to do so, can replace "algebraic space" everywhere with "scheme" and get equally valid results. This has the advantage of replacing the references to More on Cohomology of Spaces, Section 84.8 with references to Étale Cohomology, Section 59.102.

We fix a base scheme S. Let X be an algebraic space over S and let U be a simplicial algebraic space over S. Assume we have an augmentation

$$a : U \to X$$

See Section 85.32. We say that U is a proper hypercovering of X if

(1) $U_0 \to X$ is proper and surjective,
(2) $U_1 \to U_0 \times_X U_0$ is proper and surjective,
(3) $U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}$ is proper and surjective for $n \geq 1$.

The category of algebraic spaces over S has all finite limits, hence the coskeleta used in the formulation above exist.

Principle: Proper hypercoverings can be used to compute étale cohomology.

The key idea behind the proof of the principle is to compare the ph and étale topologies on the category Spaces$/S$. Namely, the ph topology is stronger than the étale topology and we have (a) a proper surjective map defines a ph covering, and (b) ph cohomology of sheaves pulled back from the small étale site agrees with étale cohomology as we have seen in More on Cohomology of Spaces, Section 84.8.

All results in this section generalize to the case where $U \to X$ is merely a "ph hypercovering", meaning a hypercovering of X in the site $(\text{Spaces}/S)_{ph}$ as defined in Section 85.21. If we ever need this, we will precisely formulate and prove this here.

0DHJ Lemma 85.36.1. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram

$$\begin{tikzcd} Sh((\text{Spaces}/U)_{ph,total}) \arrow[r, "h"'] \arrow[d, "a_{ph}"'] & Sh(U_{\acute{e}tale}) \arrow[d, "a"] \\ Sh((\text{Spaces}/X)_{ph}) \arrow[r, "h_{-1}"] & Sh(X_{\acute{e}tale}) \end{tikzcd}$$

where the left vertical arrow is defined in Section 85.21 and the right vertical arrow is defined in Section 85.32.

Proof. The notation $(\text{Spaces}/U)_{ph,total}$ indicates that we are using the construction of Section 85.21 for the site $(\text{Spaces}/S)_{ph}$ and the simplicial object U of this site[9]. We will use the sites $X_{spaces,\acute{e}tale}$ and $U_{spaces,\acute{e}tale}$ for the topoi on the right hand side; this is permissible see discussion in Section 85.32.

Observe that both $(\text{Spaces}/U)_{ph,total}$ and $U_{spaces,\acute{e}tale}$ fall into case A of Situation 85.3.3. This is immediate from the construction of $U_{\acute{e}tale}$ in Section 85.32 and it follows from Lemma 85.21.5 for $(\text{Spaces}/U)_{ph,total}$. Next, consider the functors $U_{n,spaces,\acute{e}tale} \to (\text{Spaces}/U_n)_{ph}$, $U \mapsto U/U_n$ and $X_{spaces,\acute{e}tale} \to (\text{Spaces}/X)_{ph}$, $U \mapsto U/X$. We have seen that these define morphisms of sites in More on Cohomology of Spaces, Section 84.8 where these were denoted $a_{U_n} = \epsilon_{U_n} \circ \pi_{u_n}$ and $a_X = \epsilon_X \circ \pi_X$. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 85.5.4 and we may apply Lemma 85.5.5 to conclude. $\square$

0DHK Lemma 85.36.2. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then

$$a^{-1} : Sh(X_{\acute{e}tale}) \to Sh(U_{\acute{e}tale}) \quad \text{and} \quad a^{-1} : \text{Ab}(X_{\acute{e}tale}) \to \text{Ab}(U_{\acute{e}tale})$$

[9] To distinguish from $(\text{Spaces}/U)_{fppf,total}$ defined using the fppf topology in Section 85.33.

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 85.36.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 84.8. Thus it follows from More on Cohomology of Spaces, Lemma 84.8.1 that $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 85.5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in $(\text{Spaces}/S)_{ph}$ as defined in Section 85.21 since a surjective proper morphism gives a ph covering by Topologies on Spaces, Lemma 73.8.3. By Lemma 85.21.1 we see that a_{ph}^{-1} is fully faithful with quasi-inverse $a_{ph,*}$ and with essential image the cartesian sheaves on $(\text{Spaces}/U)_{ph,total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on $U_{étale}$. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on $(\text{Spaces}/U)_{ph,total}$. Hence $h^{-1}\mathcal{G} = a_{ph}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on $(\text{Spaces}/X)_{ph}$. We compute using somewhat pedantic notation

$$
\begin{aligned}
(h_{-1})^{-1}(a_*\mathcal{G}) &= (h_{-1})^{-1}\text{Eq}(\ a_{0,small,*}\mathcal{G}_0 \rightrightarrows a_{1,small,*}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ (h_{-1})^{-1}a_{0,small,*}\mathcal{G}_0 \rightrightarrows (h_{-1})^{-1}a_{1,small,*}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ a_{0,big,ph,*}h_0^{-1}\mathcal{G}_0 \rightrightarrows a_{1,big,ph,*}h_1^{-1}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ a_{0,big,ph,*}(a_{0,big,ph})^{-1}\mathcal{H} \rightrightarrows a_{1,big,ph,*}(a_{1,big,ph})^{-1}\mathcal{H}\ ) \\
&= a_{ph,*}a_{ph}^{-1}\mathcal{H} \\
&= \mathcal{H}
\end{aligned}
$$

Here the first equality follows from Lemma 85.4.2, the second equality follows as $(h_{-1})^{-1}$ is an exact functor, the third equality follows from More on Cohomology of Spaces, Lemma 84.8.5 (here we use that $a_0 : U_0 \to X$ and $a_1 : U_1 \to X$ are proper), the fourth follows from $a_{ph}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$, the fifth from Lemma 85.4.2, and the sixth we've seen above. Since $a_{ph}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}a_*\mathcal{G}$ which ends the proof by fully faithfulness of h^{-1}. □

0DHL Lemma 85.36.3. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then for $K \in D^+(X_{étale})$

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.36.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_{n,étale})$ by More on Cohomology of Spaces, Lemma 84.8.2.

Hence Rh_*h^{-1} is the identity functor on $D^+(U_{\acute{e}tale})$ by Lemma 85.5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{ph,*}a_{ph}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.21.2 as U is a hypercovering of X in $(\text{Spaces}/S)_{ph}$ by Topologies on Spaces, Lemma 73.8.3, and the last equality by the already used More on Cohomology of Spaces, Lemma 84.8.2. $\square$

0DHM Lemma 85.36.4. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then

$$R\Gamma(X_{\acute{e}tale}, K) = R\Gamma(U_{\acute{e}tale}, a^{-1}K)$$

for $K \in D^+(X_{\acute{e}tale})$. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 85.32.

Proof. This follows from Lemma 85.36.3 because $R\Gamma(U_{\acute{e}tale}, -) = R\Gamma(X_{\acute{e}tale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. $\square$

0DHN Lemma 85.36.5. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset \text{Ab}(U_{\acute{e}tale})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is a proper hypercovering of X, then the functor a^{-1} defines an equivalence

$$D^+(X_{\acute{e}tale}) \longrightarrow D^+_{\mathcal{A}}(U_{\acute{e}tale})$$

with quasi-inverse Ra_*. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 85.32.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.36.2 and 85.36.3 and Cohomology on Sites, Lemma 21.28.5. $\square$

0DHP Lemma 85.36.6. Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $\mathcal{F}_n$ be the pullback to $U_{n,\acute{e}tale}$. If U is a ph hypercovering of X, then there exists a canonical spectral sequence

$$E_1^{p,q} = H^q_{\acute{e}tale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{\acute{e}tale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.36.4 and 85.8.3. $\square$

85.37. Other chapters

Preliminaries

(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
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Schemes

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(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
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(40) More on Groupoid Schemes
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Topics in Scheme Theory

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(45) Weil Cohomology Theories
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(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
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(57) Derived Categories of Varieties
(58) Fundamental Groups of Schemes
(59) Étale Cohomology
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(61) Pro-étale Cohomology
(62) Relative Cycles
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Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
(67) Morphisms of Algebraic Spaces
(68) Decent Algebraic Spaces
(69) Cohomology of Algebraic Spaces
(70) Limits of Algebraic Spaces
(71) Divisors on Algebraic Spaces
(72) Algebraic Spaces over Fields
(73) Topologies on Algebraic Spaces
(74) Descent and Algebraic Spaces
(75) Derived Categories of Spaces
(76) More on Morphisms of Spaces
(77) Flatness on Algebraic Spaces
(78) Groupoids in Algebraic Spaces
(79) More on Groupoids in Spaces
(80) Bootstrap
(81) Pushouts of Algebraic Spaces

Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
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Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
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Algebraic Stacks

(94) Algebraic Stacks

CHAPTER 86

Duality for Spaces

0E4V 86.1. Introduction

0E4W This chapter is the analogue of the corresponding chapter for schemes, see Duality for Schemes, Section 48.1. The development is similar to the development in the papers [Nee96], [LN07], [Lip09], and [Nee14].

86.2. Dualizing complexes on algebraic spaces

0E4X Let U be a locally Noetherian scheme. Let $\mathcal{O}_{étale}$ be the structure sheaf of U on the small étale site of U. We will say an object $K \in D_{QCoh}(\mathcal{O}_{étale})$ is a dualizing complex on U if $K = \epsilon^*(\omega_U^\bullet)$ for some dualizing complex $\omega_U^\bullet$ in the sense of Duality for Schemes, Section 48.2. Here $\epsilon^* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_{étale})$ is the equivalence of Derived Categories of Spaces, Lemma 75.4.2. Most of the properties of $\omega_U^\bullet$ studied in Duality for Schemes, Section 48.2 are inherited by K via the discussion in Derived Categories of Spaces, Sections 75.4 and 75.13.

We define a dualizing complex on a locally Noetherian algebraic space to be a complex which étale locally comes from a dualizing complex on the corresponding scheme.

0E4Y Lemma 86.2.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

(1) For every étale morphism $U \to X$ where U is a scheme the restriction $K|_U$ is a dualizing complex for U (as discussed above).
(2) There exists a surjective étale morphism $U \to X$ where U is a scheme such that $K|_U$ is a dualizing complex for U.

Proof. Assume $U \to X$ is surjective étale where U is a scheme. Let $V \to X$ be an étale morphism where V is a scheme. Then

$$U \leftarrow U \times_X V \to V$$

are étale morphisms of schemes with the arrow to V surjective. Hence we can use Duality for Schemes, Lemma 48.26.1 to see that if $K|_U$ is a dualizing complex for U, then $K|_V$ is a dualizing complex for V. $\square$

0E4Z Definition 86.2.2. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. An object K of $D_{QCoh}(\mathcal{O}_X)$ is called a dualizing complex if K satisfies the equivalent conditions of Lemma 86.2.1.

0E50 Lemma 86.2.3. Let A be a Noetherian ring and let $X = \mathrm{Spec}(A)$. Let $\mathcal{O}_{étale}$ be the structure sheaf of X on the small étale site of X. Let K, L be objects of $D(A)$. If $K \in D_{\mathrm{Coh}}(A)$ and L has finite injective dimension, then

$$\epsilon^* \widetilde{R\,\mathrm{Hom}_A(K, L)} = R\,\mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\widetilde{K}, \epsilon^*\widetilde{L})$$

in $D(\mathcal{O}_{étale})$ where $\epsilon : (X_{étale}, \mathcal{O}_{étale}) \to (X, \mathcal{O}_X)$ is as in Derived Categories of Spaces, Section 75.4.

Proof. By Duality for Schemes, Lemma 48.2.3 we have a canonical isomorphism

$$\widetilde{R\operatorname{Hom}_A(K, L)} = R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

in $D(\mathcal{O}_X)$. There is a canonical map

$$\epsilon^* R\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) \longrightarrow R\mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\widetilde{K}, \epsilon^*\widetilde{L})$$

in $D(\mathcal{O}_{étale})$, see Cohomology on Sites, Remark 21.35.11. We will show the left and right hand side of this arrow have isomorphic cohomology sheaves, but we will omit the verification that the isomorphism is given by this arrow.

We may assume that L is given by a finite complex $I^\bullet$ of injective A-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A-module. Recall that the cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\widetilde{K}, \epsilon^*\widetilde{L}))$ are the sheafifications of the presheaf sending U étale over X to the ith ext group between the restrictions of $\epsilon^*\widetilde{K}$ and $\epsilon^*\widetilde{L}$ to $U_{étale}$. See Cohomology on Sites, Lemma 21.35.1. If $U = \operatorname{Spec}(B)$ is affine, then this ext group is equal to $\operatorname{Ext}^i_B(K \otimes_A B, L \otimes_A B)$ by the equivalence of Derived Categories of Spaces, Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5 (this also uses the compatibilities detailed in Derived Categories of Spaces, Remark 75.6.3). Since $A \to B$ is étale, we see that $I \otimes_A B$ is an injective B-module by Dualizing Complexes, Lemma 47.26.4. Hence we see that

$$\begin{aligned}
\operatorname{Ext}^n_B(K \otimes_A B, I \otimes_A B) &= \operatorname{Hom}_B(H^{-n}(K \otimes_A B), I \otimes_A B) \\
&= \operatorname{Hom}_{A_f}(H^{-n}(K) \otimes_A B, I \otimes_A B) \\
&= \operatorname{Hom}_A(H^{-n}(K), I) \otimes_A B \\
&= \operatorname{Ext}^n_A(K, I) \otimes_A B
\end{aligned}$$

The penultimate equality because $H^{-n}(K)$ is a finite A-module, see More on Algebra, Lemma 15.65.4. Therefore the cohomology sheaves of the left and right hand side of the equality in the lemma are the same. □

0E51 Lemma 86.2.4. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let K be a dualizing complex on X. Then K is an object of $D_{\mathrm{Coh}}(\mathcal{O}_X)$ and $D = R\mathcal{H}om_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence

$$D : D_{\mathrm{Coh}}(\mathcal{O}_X) \longrightarrow D_{\mathrm{Coh}}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $\mathrm{id} \to D \circ D$. If X is quasi-compact, then D exchanges $D^+_{\mathrm{Coh}}(\mathcal{O}_X)$ and $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$ and induces an equivalence $D^b_{\mathrm{Coh}}(\mathcal{O}_X) \to D^b_{\mathrm{Coh}}(\mathcal{O}_X)$.

Proof. Let $U \to X$ be an étale morphism with U affine. Say $U = \operatorname{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 86.2.1 and

Duality for Schemes, Lemma 48.2.1. By Lemma 86.2.3 the diagram

$$\begin{tikzcd} D_{\mathrm{Coh}}(A) \arrow[r] \arrow[d, "R\mathrm{Hom}_A(-,\omega_A^\bullet)"'] & D_{\mathrm{Coh}}(\mathcal{O}_{\acute{e}tale}) \arrow[d, "R\mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(-,K|_U)"] \\ D_{\mathrm{Coh}}(A) \arrow[r] & D(\mathcal{O}_{\acute{e}tale}) \end{tikzcd}$$

commutes where $\mathcal{O}_{\acute{e}tale}$ is the structure sheaf of the small étale site of U. Since formation of $R\mathcal{H}om$ commutes with restriction, we conclude that D sends $D_{\mathrm{Coh}}(\mathcal{O}_X)$ into $D_{\mathrm{Coh}}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(Cohomology on Sites, Lemma 21.35.5) is an isomorphism for all L in $D_{\mathrm{Coh}}(\mathcal{O}_X)$ because this is true over all U as above by Dualizing Complexes, Lemma 47.15.3. The statement on boundedness properties of the functor D in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 47.15.3. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Recall that an object L of $D(\mathcal{O})$ is invertible if it is an invertible object for the symmetric monoidal structure on $D(\mathcal{O}_X)$ given by derived tensor product. In Cohomology on Sites, Lemma 21.49.2 we we have seen this means L is perfect and if $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}$ of U in $\mathcal{C}$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers n_i.

Let S be a scheme and let X be an algebraic space over S. If L in $D(\mathcal{O}_X)$ is invertible, then there is a disjoint union decomposition $X = \coprod_{n \in \mathbf{Z}} X_n$ such that $L|_{X_n}$ is an invertible module sitting in degree n. In particular, it follows that $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$-modules (with zero differentials) representing L.

0E52 Lemma 86.2.5. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. If K and K' are dualizing complexes on X, then K' is isomorphic to $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for some invertible object L of $D(\mathcal{O}_X)$.

Proof. Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of $D(\mathcal{O}_X)$, because affine locally this is true. Use Lemma 86.2.3 and Dualizing Complexes, Lemma 47.15.5 and its proof. The evaluation map $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K'$ is an isomorphism for the same reason. □

0E53 Lemma 86.2.6. Let S be a scheme. Let X be a locally Noetherian quasi-separated algebraic space over S. Let $\omega_X^\bullet$ be a dualizing complex on X. Then X the function $|X| \to \mathbf{Z}$ defined by

$$x \longmapsto \delta(x) \text{ such that } \omega_{X,\overline{x}}^\bullet[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,\overline{x}}$$

is a dimension function on $|X|$.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Let $\omega_U^\bullet$ be the dualizing complex on U associated to $\omega_X^\bullet|_U$. If $u \in U$ maps to $x \in |X|$, then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$. By Dualizing Complexes, Lemma 47.22.1 we see that if $\omega^\bullet$ is a normalized dualizing complex for $\mathcal{O}_{U,u}$, then $\omega^\bullet \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$ is a normalized dualizing complex for $\mathcal{O}_{X,\overline{x}}$. Hence we see that the dimension function

$U \to \mathbf{Z}$ of Duality for Schemes, Lemma 48.2.7 for the scheme U and the complex $\omega_U^\bullet$ is equal to the composition of $U \to |X|$ with δ. Using the specializations in $|X|$ lift to specializations in U and that nontrivial specializations in U map to nontrivial specializations in X (Decent Spaces, Lemmas 68.12.2 and 68.12.1) an easy topological argument shows that δ is a dimension function on $|X|$. $\square$

86.3. Right adjoint of pushforward

0E54 This is the analogue of Duality for Schemes, Section 48.3.

0E55 **Lemma 86.3.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact algebraic spaces over S. The functor $Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y)$ has a right adjoint.

This is almost the same as [Nee96, Example 4.2].

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 13.38.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Spaces, Lemma 75.5.3. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Spaces, Theorem 75.15.4. Since X and Y are quasi-compact and quasi-separated, so is f, see Morphisms of Spaces, Lemmas 67.4.10 and 67.8.9. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Spaces, Lemma 75.6.2. This finishes the proof. $\square$

0E56 **Lemma 86.3.2.** Notation and assumptions as in Lemma 86.3.1. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_*. Then a maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(\mathcal{O}_X)$. In fact, there exists an integer N such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.

Proof. By Derived Categories of Spaces, Lemma 75.6.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that $H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D^+_{QCoh}(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. $\square$

Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact algebraic spaces over S. Let a denote the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. For every $K \in D_{QCoh}(\mathcal{O}_Y)$ and $L \in D_{QCoh}(\mathcal{O}_X)$ we obtain a canonical map

0E57 (86.3.2.1)
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology on Sites, Remark 21.35.10 and the second arrow is the counit $Rf_*a(K) \to K$ of the adjunction.

0E58 **Lemma 86.3.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$. Then the map (86.3.2.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

becomes an isomorphism after applying the functor $DQ_Y : D(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Y)$ discussed in Derived Categories of Spaces, Section 75.19.

Proof. The statement makes sense as DQ_Y exists by Derived Categories of Spaces, Lemma 75.19.1. Since DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$ to prove the lemma we have to show that for any $M \in D_{QCoh}(\mathcal{O}_Y)$ the map (86.3.2.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

To see this we use the following string of equalities

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\ &= \mathrm{Hom}_X(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, a(K)) \\ &= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L), K) \\ &= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L, K) \\ &= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)) \end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.19.1. The second equality by Cohomology on Sites, Lemma 21.35.2. The third equality by construction of a. The fourth equality by Derived Categories of Spaces, Lemma 75.20.1 (this is the important step). The fifth by Cohomology on Sites, Lemma 21.35.2. $\square$

0GG3 Example 86.3.4. The statement of Lemma 86.3.3 is not true without applying the "coherator" DQ_Y. See Duality for Schemes, Example 48.3.7.

0GG4 Remark 86.3.5. In the situation of Lemma 86.3.3 we have

$$DQ_Y(Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = Rf_*DQ_X(R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

by Derived Categories of Spaces, Lemma 75.19.2. Thus if $R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \in D_{QCoh}(\mathcal{O}_X)$, then we can "erase" the DQ_Y on the left hand side of the arrow. On the other hand, if we know that $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \in D_{QCoh}(\mathcal{O}_Y)$, then we can "erase" the DQ_Y from the right hand side of the arrow. If both are true then we see that (86.3.2.1) is an isomorphism. Combining this with Derived Categories of Spaces, Lemma 75.13.10 we see that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism if

(1) L and Rf_*L are perfect, or
(2) K is bounded below and L and Rf_*L are pseudo-coherent.

For (2) we use that $a(K)$ is bounded below if K is bounded below, see Lemma 86.3.2.

0GG5 Example 86.3.6. Let S be a scheme. Let $f : X \to Y$ be a proper morphism of Noetherian algebraic spaces over S, $L \in D^-_{\mathrm{Coh}}(X)$ and $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Then the map $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism. Namely, the complexes L and Rf_*L are pseudo-coherent by Derived Categories of Spaces, Lemmas 75.13.7 and 75.8.1 and the discussion in Remark 86.3.5 applies.

0E59 Lemma 86.3.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact algebraic spaces over S. For all $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ (86.3.2.1) induces an isomorphism $R\mathrm{Hom}_X(L, a(K)) \to R\mathrm{Hom}_Y(Rf_*L, K)$ of global derived homs.

Proof. By construction (Cohomology on Sites, Section 21.36) the complexes

$$R\operatorname{Hom}_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\operatorname{Hom}_Y(Rf_*L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_X}(Rf_*L, a(K)))$$

Thus the lemma is a consequence of Lemma 86.3.3. Namely, a map $E \to E'$ in $D(\mathcal{O}_Y)$ which induces an isomorphism $DQ_Y(E) \to DQ_Y(E')$ induces a quasi-isomorphism $R\Gamma(Y, E) \to R\Gamma(Y, E')$. Indeed we have $H^i(Y, E) = \operatorname{Ext}^i_Y(\mathcal{O}_Y, E) = \operatorname{Hom}(\mathcal{O}_Y[-i], E) = \operatorname{Hom}(\mathcal{O}_Y[-i], DQ_Y(E))$ because $\mathcal{O}_Y[-i]$ is in $D_{QCoh}(\mathcal{O}_Y)$ and DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$. $\square$

86.4. Right adjoint of pushforward and base change, I

0E5A Let us define the base change map between right adjoints of pushforward. Let S be a scheme. Consider a cartesian diagram

0E5B (86.4.0.1)
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where Y' and X are Tor independent over Y. Denote

$$a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{QCoh}(\mathcal{O}_{Y'}) \to D_{QCoh}(\mathcal{O}_{X'})$$

the right adjoints to Rf_* and Rf'_* (Lemma 86.3.1). The base change map of Cohomology on Sites, Remark 21.19.3 gives a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0E5C Lemma 86.4.1. In diagram (86.4.0.1) the map $a \circ Rg_* \leftarrow Rg'_* \circ a'$ is an isomorphism.

Proof. The base change map $Lg^* \circ Rf_*K \to Rf'_* \circ L(g')^*K$ is an isomorphism for every K in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Spaces, Lemma 75.20.4 (this uses the assumption of Tor independence). Thus the corresponding transformation between adjoint functors is an isomorphism as well. $\square$

Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

0E5D (86.4.1.1) $$L(g')^* \circ a \to L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^*$$

The first arrow comes from the adjunction map $\text{id} \to Rg_*Lg^*$ and the last arrow from the adjunction map $L(g')^*Rg'_* \to \text{id}$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 86.4.1. Alternatively, we can think of (86.4.1.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation

$$a \to a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$\begin{aligned} \mathrm{Hom}_X(M, a(K)) &= \mathrm{Hom}_Y(Rf_*M, K) \\ &\to \mathrm{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\ &= \mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K) \\ &\leftarrow \mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K) \\ &= \mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) \\ &= \mathrm{Hom}_X(M, Rg'_*a'(Lg^*K)) \end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Spaces, Lemma 75.20.4) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology on Sites, Remarks 21.19.4 and 21.19.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0E5E Lemma 86.4.2. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "k"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "l"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ Z' \arrow[r, "m"] & Z \end{tikzcd}$$

of quasi-compact and quasi-separated algebraic spaces over S where both diagrams are cartesian and where f and l as well as g and m are Tor independent. Then the maps (86.4.1.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that $g \circ f$ and m are Tor independent (details omitted), hence the statement makes sense. In this proof we write k^* in place of Lk^* and f_* instead of Rf_*. Let a, b, and c be the right adjoints of Lemma 86.3.1 for f, g, and $g \circ f$ and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \to k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \to a' \circ l^*$$

where $\xi_{top} : k_* \circ a' \to a \circ l_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $l^* \circ f_* \to f'_* \circ k^*$. The outer arrows come from the canonical maps $1 \to l_* \circ l^*$ and $k^* \circ k_* \to 1$. Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \to l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \to b' \circ m^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \to k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \to c' \circ m^*$$

We have $(g \circ f)_* = g_* \circ f_*$ and hence $c = a \circ b$ and similarly $c' = a' \circ b'$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

$$\begin{tikzcd}
 & & k^* \circ a \circ b \arrow[ddll] \arrow[d] \\
 & & k^* \circ a \circ l_* \circ l^* \circ b \arrow[dl] \\
k^* \circ a \circ b \circ m_* \circ m^* \arrow[r] & k^* \circ a \circ l_* \circ l^* \circ b \circ m_* \circ m^* & k^* \circ k_* \circ a' \circ l^* \circ b \arrow[u, "\xi_{top}"'] \arrow[d] \arrow[dl] \\
 & k^* \circ k_* \circ a' \circ l^* \circ b \circ m_* \circ m^* \arrow[u, "\xi_{top}"'] \arrow[dr] & a' \circ l^* \circ b \arrow[d] \\
k^* \circ k_* \circ a' \circ b' \circ m^* \arrow[uu, "\xi_{rect}"'] \arrow[ddrr] & k^* \circ k_* \circ a' \circ l^* \circ l_* \circ b' \circ m^* \arrow[l] \arrow[u, "\xi_{bot}"'] \arrow[dr] & a' \circ l^* \circ b \circ m_* \circ m^* \\
 & & a' \circ l^* \circ l_* \circ b' \circ m^* \arrow[u, "\xi_{bot}"'] \arrow[d] \\
 & & a' \circ b' \circ m^*
\end{tikzcd}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show the diagram

$$\begin{tikzcd}
a \circ l_* \circ l^* \circ b \circ m_* & a \circ b \circ m_* \arrow[l] \\
k_* \circ a' \circ l^* \circ b \circ m_* \arrow[u, "\xi_{top}"'] & \\
k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[u, "\xi_{bot}"'] \arrow[r] & k_* \circ a' \circ b' \arrow[uu, "\xi_{rect}"']
\end{tikzcd}$$

becomes commutative if we invert the arrows ξ_{top}, ξ_{bot}, and ξ_{rect} (note that this is different from asking the diagram to be commutative). However, the diagram

$$\begin{tikzcd}
 & a \circ l_* \circ l^* \circ b \circ m_* & \\
a \circ l_* \circ l^* \circ l_* \circ b' \arrow[ur, "\xi_{bot}"] & & k_* \circ a' \circ l^* \circ b \circ m_* \arrow[ul, "\xi_{top}"'] \\
 & k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[ul, "\xi_{top}"] \arrow[ur, "\xi_{bot}"'] &
\end{tikzcd}$$

commutes by Categories, Lemma 4.28.2. Since the diagrams

$$\begin{tikzcd} a \circ l_* \circ l^* \circ b \circ m_* & a \circ b \circ m \arrow[l] \\ a \circ l_* \circ l^* \circ l_* \circ b' \arrow[u] & a \circ l_* \circ b' \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \circ l_* \circ l^* \circ l_* \circ b' \arrow[r] & a \circ l_* \circ b' \\ k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[r] \arrow[u] & k_* \circ a' \circ b' \arrow[u] \end{tikzcd}$$

commute (see references cited) and since the composition of $l_* \to l_* \circ l^* \circ l_* \to l_*$ is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to ξ_{rect} (via the identifications $a \circ b = c$ and $a' \circ b' = c'$). This is the statement dual to Cohomology on Sites, Remark 21.19.4 and the proof is complete. □

0E5F Lemma 86.4.3. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d, "f''"] & X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated algebraic spaces over S where both diagrams are cartesian and where f and h as well as f' and h' are Tor independent. Then the maps (86.4.1.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that f and $h \circ h'$ are Tor independent (details omitted), hence the statement makes sense. In this proof we write g^* in place of Lg^* and f_* instead of Rf_*. Let a, a', and a'' be the right adjoints of Lemma 86.3.1 for f, f', and f''. The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \to g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \to a' \circ h^*$$

where $\xi_{right} : g_* \circ a' \to a \circ h_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $h^* \circ f_* \to f'_* \circ g^*$. The outer arrows come from the canonical maps $1 \to h_* \circ h^*$ and $g^* \circ g_* \to 1$. Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \to a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \to k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \to a'' \circ m^*$$

where $k = g \circ g'$ and $m = h \circ h'$. We have $k^* = (g')^* \circ g^*$ and $m^* = (h')^* \circ h^*$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

$$\begin{tikzcd}
 & (g')^* \circ g^* \circ a \arrow[d] \arrow[ddl] \\
 & (g')^* \circ g^* \circ a \circ h_* \circ h^* \arrow[dl] \\
(g')^* \circ g^* \circ a \circ h_* \circ (h')_* \circ (h')^* \circ h^* & (g')^* \circ g^* \circ g_* \circ a' \circ h^* \arrow[u, "\xi_{right}"'] \arrow[d] \arrow[dl] \\
(g')^* \circ g^* \circ g_* \circ a' \circ (h')_* \circ (h')^* \circ h^* \arrow[u, "\xi_{right}"'] \arrow[dr] & (g')^* \circ a' \circ h^* \arrow[d] \\
(g')^* \circ g^* \circ g_* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \arrow[u, "\xi_{left}"'] \arrow[dr] \arrow[ddr] & (g')^* \circ a' \circ (h')_* \circ (h')^* \circ h^* \\
 & (g')^* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \arrow[u, "\xi_{left}"'] \arrow[d] \\
 & a'' \circ (h')^* \circ h^*
\end{tikzcd}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to ξ_{rect}. This is the statement dual to Cohomology, Remark 20.28.5 and the proof is complete. □

0E5G Remark 86.4.4. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd}
X'' \arrow[r, "k'"'] \arrow[d, "f''"'] & X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y'' \arrow[r, "l'"] \arrow[d, "g''"'] & Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z'' \arrow[r, "m'"] & Z' \arrow[r, "m"] & Z
\end{tikzcd}$$

of quasi-compact and quasi-separated algebraic spaces over S where all squares are cartesian and where (f, l), (g, m), (f', l'), (g', m') are Tor independent pairs of maps. Let a, a', a'', b, b', b'' be the right adjoints of Lemma 86.3.1 for f, f', f'', g, g', g''. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

Then the maps (86.4.1.1) for the squares are (where we use $k^* = Lk^*$, etc)

$$\gamma_A : (k')^* \circ a' \to a'' \circ (l')^* \quad \gamma_B : k^* \circ a \to a' \circ l^*$$
$$\gamma_C : (l')^* \circ b' \to b'' \circ (m')^* \quad \gamma_D : l^* \circ b \to b' \circ m^*$$

For the 2×1 and 1×2 rectangles we have four further base change maps

$$\gamma_{A+B} : (k \circ k')^* \circ a \to a'' \circ (l \circ l')^*$$
$$\gamma_{C+D} : (l \circ l')^* \circ b \to b'' \circ (m \circ m')^*$$
$$\gamma_{A+C} : (k')^* \circ (a' \circ b') \to (a'' \circ b'') \circ (m')^*$$
$$\gamma_{A+C} : k^* \circ (a \circ b) \to (a' \circ b') \circ m^*$$

By Lemma 86.4.3 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 86.4.2 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \mathrm{id}_{l^*}) \circ (\mathrm{id}_{(k')^*} \star \gamma_B)$ with notation as in Categories, Section 4.28 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 86.4.2 and 86.4.3 of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{tikzcd}
(k')^* \circ a' \circ l^* \circ b \arrow[r, "\gamma_D"'] \arrow[d, "\gamma_A"'] & (k')^* \circ a' \circ b' \circ m^* \arrow[d, "\gamma_A"] \\
a'' \circ (l')^* \circ l^* \circ b \arrow[r, "\gamma_D"] & a'' \circ (l')^* \circ b' \circ m^*
\end{tikzcd}$$

commutes. This is true by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1.

86.5. Right adjoint of pushforward and base change, II

0E5H In this section we prove that the base change map of Section 86.4 is an isomorphism in some cases.

0E5I **Lemma 86.5.1.** In diagram (86.4.0.1) assume in addition $g : Y' \to Y$ is a morphism of affine schemes and $f : X \to Y$ is proper. Then the base change map (86.4.1.1) induces an isomorphism

$$L(g')^* a(K) \longrightarrow a'(Lg^* K)$$

in the following cases

(1) for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is flat of finite presentation,
(2) for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is perfect and Y Noetherian,

(3) for $K \in D^+_{QCoh}(\mathcal{O}_X)$ if g has finite Tor dimension and Y Noetherian.

Proof. Write $Y = \text{Spec}(A)$ and $Y' = \text{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Let M be a perfect generator for $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Theorem 75.15.4. Then $L(g')^*M$ is a generator for $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Spaces, Remark 75.15.5. Hence it suffices to show that (86.4.1.1) induces an isomorphism

0E5J (86.5.1.1) $$R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\operatorname{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology on Sites, Section 21.36, as this will imply the cone of $L(g')^*a(K) \to a'(Lg^*K)$ is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (86.5.1.1).

The left hand side. Because M is perfect, the canonical map

$$R\operatorname{Hom}_X(M, a(K)) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Spaces, Lemma 75.20.5. We can combine this with the isomorphism $R\operatorname{Hom}_Y(Rf_*M, K) = R\operatorname{Hom}_X(M, a(K))$ of Lemma 86.3.7 to get that the left hand side equals $R\operatorname{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A'$.

The right hand side. Here we first use the isomorphism

$$R\operatorname{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\operatorname{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 86.3.7. Since f and g are Tor independent the base change map $Lg^*Rf_*M \to Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Spaces, Lemma 75.20.4. Hence we may rewrite this as $R\operatorname{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since Y, Y' are affine and K, Rf_*M are in $D_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Spaces, Lemma 75.6.1) we have a canonical map

$$\beta : R\operatorname{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (15.99.1.1) where we have used Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.8 to translate back and forth into algebra.

(1) If f is flat and of finite presentation, the complex Rf_*M is perfect on Y by Derived Categories of Spaces, Lemma 75.25.4 and β is an isomorphism by More on Algebra, Lemma 15.99.2 part (1).
(2) If f is perfect and Y Noetherian, the complex Rf_*M is perfect on Y by More on Morphisms of Spaces, Lemma 76.47.5 and β is an isomorphism as before.
(3) If g has finite tor dimension and Y is Noetherian, the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Spaces, Lemmas 75.8.1 and 75.13.7) and β is an isomorphism by More on Algebra, Lemma 15.99.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (86.5.1.1) given in the second and third paragraph are in fact given by (86.5.1.1). To make our formulas manageable we will use $(-,-)_X = R\operatorname{Hom}_X(-,-)$, use $-\otimes A'$

in stead of $- \otimes_A^{\mathbf{L}} A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

$$\begin{tikzcd}
((g')^*M, (g')^*a(K))_{X'} \arrow[d] & (M, a(K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[d] \arrow[r, equal] & (f_*M, K)_Y \otimes A' \arrow[d] \\
((g')^*M, (g')^*a(g_*g^*K))_{X'} & (M, a(g_*g^*K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[r, equal] & (f_*M, g_*g^*K)_Y \otimes A' \arrow[dd, bend right, "\mu'"'] \\
((g')^*M, (g')^*g'_*a'(g^*K))_{X'} \arrow[u] \arrow[d] & (M, g'_*a'(g^*K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[u] \arrow[dl, "\mu"] & (f_*M, K) \otimes A' \arrow[d, "\beta"] \\
((g')^*M, a'(g^*K))_{X'} \arrow[r, equal] & (f'_*(g')^*M, g^*K)_{Y'} \arrow[r] & (g^*f_*M, g^*K)_{Y'}
\end{tikzcd}$$

The arrows labeled α are the maps from Derived Categories of Spaces, Lemma 75.20.5 for the diagram with corners X', X, Y', Y. The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ a' \to a \circ g_*$ of Lemma 86.4.1 and therefore the middle square is commutative. Going down the left hand side is (86.5.1.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including β provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \to E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \to E'$ the map induced by c and the adjointness of restriction and base change; if c is clear we write $\mu = \mu_c$, i.e., we drop c from the notation. The map μ in the diagram is of this form with c given by the identification $(M, g'_*a(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$; the triangle involving μ is commutative by Derived Categories of Spaces, Remark 75.20.6.

Observe that

$$\begin{tikzcd}
(M, a(g_*g^*K))_X \arrow[r, equal] & (f_*M, g_*g^*K)_Y \arrow[r, equal] & (g^*f_*M, g^*K)_{Y'} \\
(M, g'_*a'(g^*K))_X \arrow[u] \arrow[r, equal] & ((g')^*M, a'(g^*K))_{X'} \arrow[r, equal] & (f'_*(g')^*M, g^*K)_{Y'} \arrow[u]
\end{tikzcd}$$

is commutative by the very definition of the transformation $g'_* \circ a' \to a \circ g_*$. Letting μ' be as above corresponding to the identification $(f_*M, g_*g^*K)_X = (g^*f_*M, g^*K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that β is equal to the composition of $(f_*M, K)_Y \otimes A' \to (f_*M, g_*g^*K)_X \otimes A'$ and μ'. To do this, it suffices to prove the two induced maps $(f_*M, K)_Y \to (g^*f_*M, g^*K)_{Y'}$ are the same. In other words, it suffices to show the diagram

$$\begin{tikzcd}
R\operatorname{Hom}_A(E, K) \arrow[rr, "\text{induced by } \beta"'] \arrow[dr] & & R\operatorname{Hom}_{A'}(E \otimes_A^{\mathbf{L}} A', K \otimes_A^{\mathbf{L}} A') \\
& R\operatorname{Hom}_A(E, K \otimes_A^{\mathbf{L}} A') \arrow[ur] &
\end{tikzcd}$$

commutes for all $E, K \in D(A)$. Since this is how β is constructed in More on Algebra, Section 15.99 the proof is complete. $\square$

86.6. Right adjoint of pushforward and trace maps

0E5K Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 86.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\mathrm{Tr}_f : Rf_* \circ a \longrightarrow \mathrm{id}$$

The corresponding map $\mathrm{Tr}_{f,K} : Rf_*a(K) \longrightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the trace map. This is the map which has the property that the bijection

$$\mathrm{Hom}_X(L, a(K)) \longrightarrow \mathrm{Hom}_Y(Rf_*L, K)$$

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

$$\varphi \longmapsto \mathrm{Tr}_{f,K} \circ Rf_*\varphi$$

The canonical map (86.3.2.1)

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

comes about by composition with $\mathrm{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0E5L **Lemma 86.6.1** (Trace map and base change). Suppose we have a diagram (86.4.0.1). Then the maps $1 \star \mathrm{Tr}_f : Lg^* \circ Rf_* \circ a \to Lg^*$ and $\mathrm{Tr}_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \to Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology on Sites, Remark 21.19.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (86.4.1.1). More precisely, the diagram

$$\begin{tikzcd}
Lg^* \circ Rf_* \circ a \arrow[rr, "1\star\mathrm{Tr}_f"'] \arrow[d, "\beta\star1"'] & & Lg^* \\
Rf'_* \circ L(g')^* \circ a \arrow[rr, "1\star\alpha"] & & Rf'_* \circ a' \circ Lg^* \arrow[u, "\mathrm{Tr}_{f'}\star1"']
\end{tikzcd}$$

of transformations of functors commutes.

Proof. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 86.4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \to g^* \circ f_* \circ a \circ g_* \circ g^* \to g^* \circ g_* \circ g^* \to g^*$$

where the first arrow is the unit for (g^*, g_*), the second arrow is Tr_f, and the third arrow is the counit for (g^*, g_*). This is a simple consequence of the fact that the composition $g^* \to g^* \circ g_* \circ g^* \to g^*$ of unit and counit is the identity. Consider the

diagram

$$\begin{tikzcd}
 & g^* \circ f_* \circ a \arrow[rr, "\text{Tr}_f"] \arrow[ld, "\beta"'] \arrow[d] & & g^* & \\
f'_* \circ (g')^* \circ a \arrow[rd] & g^* \circ f_* \circ a \circ g_* \circ g^* \arrow[rru] \arrow[d, "\beta"] & & g^* \circ f_* \circ g'_* \circ a' \circ g^* \arrow[ll, "\beta^\vee"'] \arrow[d, "\beta"] & f'_* \circ a' \circ g^* \arrow[lu, "\text{Tr}_{f'}"'] \\
 & f'_* \circ (g')^* \circ a \circ g_* \circ g^* & & f'_* \circ (g')^* \circ g'_* \circ a' \circ g^* \arrow[ll, "\beta^\vee"'] \arrow[ru] &
\end{tikzcd}$$

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By Categories, Lemma 4.24.8 the square

$$\begin{tikzcd}
g^* \circ f_* \circ g'_* \circ a' \arrow[r, "\beta"'] \arrow[d, "\beta^\vee"'] & f'_* \circ (g')^* \circ g'_* \circ a' \arrow[d] \\
g^* \circ f_* \circ a \circ g_* \arrow[r] & \text{id}
\end{tikzcd}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\text{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma. $\square$

Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* as in Lemma 86.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\eta_f : \text{id} \to a \circ Rf_*$$

which is called the unit of the adjunction.

0E5M **Lemma 86.6.2.** Suppose we have a diagram (86.4.0.1). Then the maps $1 \star \eta_f : L(g')^* \to L(g')^* \circ a \circ Rf_*$ and $\eta_{f'} \star 1 : L(g')^* \to a' \circ Rf'_* \circ L(g')^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology on Sites, Remark 21.19.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (86.4.1.1). More precisely, the diagram

$$\begin{tikzcd}
L(g')^* \arrow[r, "1 \star \eta_f"'] \arrow[d, "\eta_{f'} \star 1"'] & L(g')^* \circ a \circ Rf_* \arrow[d, "\alpha"] \\
a' \circ Rf'_* \circ L(g')^* & a' \circ Lg^* \circ Rf_* \arrow[l, "\beta"']
\end{tikzcd}$$

of transformations of functors commutes.

Proof. This proof is dual to the proof of Lemma 86.6.1. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 86.4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \to (g')^* \circ g'_* \circ (g')^* \to (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \to a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_{f'}$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \to (g')^* \circ (g')_* \circ (g')^* \to (g')^*$ of unit and counit is the identity. Consider the diagram

$$\begin{tikzcd}
 & (g')^* \circ a \circ f_* \arrow[r] & (g')^* \circ a \circ g_* \circ g^* \circ f_* \arrow[dl, "\beta"'] \\
(g')^* \arrow[ur, "\eta_f"] \arrow[dr] \arrow[ddd, "\eta_{f'}"'] & (g')^* \circ a \circ g_* \circ f'_* \circ (g')^* & (g')^* \circ g'_* \circ a' \circ g^* \circ f_* \arrow[u, "\beta^\vee"'] \arrow[dl, "\beta"'] \arrow[d] \\
 & (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \arrow[u, "\beta^\vee"'] \arrow[ddl] & a' \circ g^* \circ f_* \arrow[ddll, "\beta"] \\
 & & \\
a' \circ f'_* \circ (g')^* & &
\end{tikzcd}$$

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 4.24.8 the square

$$\begin{tikzcd}
\mathrm{id} \arrow[r] \arrow[d] & g'_* \circ a' \circ g^* \circ f_* \arrow[d, "\beta"] \\
g'_* \circ a' \circ g^* \circ f_* \arrow[r, "\beta^\vee"] & a \circ g_* \circ f'_* \circ (g')^*
\end{tikzcd}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\beta \circ \alpha \circ \eta_f$ by definition this proves the lemma. □

86.7. Right adjoint of pushforward and pullback

0E5N Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let a be the right adjoint of pushforward as in Lemma 86.3.1. For $K, L \in D_{QCoh}(\mathcal{O}_Y)$ there is a canonical map

$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \longrightarrow a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$$

Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(a(L)) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L$$

(equality by Derived Categories of Spaces, Lemma 75.20.1) for which we use the trace map $Rf_*a(L) \to L$. When $L = \mathcal{O}_Y$ we obtain a map

0E5P (86.7.0.1) $$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in K and compatible with distinguished triangles.

0E5Q **Lemma 86.7.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. The map $Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \to a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$ defined above for $K, L \in D_{QCoh}(\mathcal{O}_Y)$ is an isomorphism if K is perfect. In particular, (86.7.0.1) is an isomorphism if K is perfect.

Proof. Let $K^\vee$ be the "dual" to K, see Cohomology on Sites, Lemma 21.48.4. For $M \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K^\vee, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K^\vee, a(L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) \end{aligned}$$

Second equality by the definition of a and the projection formula (Cohomology on Sites, Lemma 21.50.1) or the more general Derived Categories of Spaces, Lemma 75.20.1. Hence the result by the Yoneda lemma. □

0E5R Lemma 86.7.2. Suppose we have a diagram (86.4.0.1). Let $K \in D_{QCoh}(\mathcal{O}_Y)$. The diagram

$$\begin{tikzcd} L(g')^*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)) \arrow[r] \arrow[d] & L(g')^*a(K) \arrow[d] \\ L(f')^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} a'(\mathcal{O}_{Y'}) \arrow[r] & a'(Lg^*K) \end{tikzcd}$$

commutes where the horizontal arrows are the maps (86.7.0.1) for K and Lg^*K and the vertical maps are constructed using Cohomology on Sites, Remark 21.19.3 and (86.4.1.1).

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. Let us write (86.7.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) &\to a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ &\leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ &\to a(K \otimes \mathcal{O}_Y) \\ &\to a(K) \end{aligned}$$

Here the first arrow is the unit η_f, the second arrow is a applied to Cohomology on Sites, Equation (21.50.0.1) which is an isomorphism by Derived Categories of Spaces, Lemma 75.20.1, the third arrow is a applied to $\mathrm{id}_K \otimes \mathrm{Tr}_f$, and the fourth arrow is a applied to the isomorphism $K \otimes \mathcal{O}_Y = K$. The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For η_f and Tr_f this is Lemmas 86.6.2 and 86.6.1. For the arrow using Cohomology on Sites, Equation (21.50.0.1) this is Cohomology on Sites, Remark 21.50.2. For the multiplication map it is clear. This finishes the proof. □

86.8. Right adjoint of pushforward for proper flat morphisms

0E5S For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated algebraic spaces the right adjoint of pushforward enjoys some remarkable properties.

0E5T Lemma 86.8.1. Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 86.3.1. Then a commutes with direct sums.

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Spaces, Lemma 75.25.4 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\
&= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\
&= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i))
\end{aligned}$$

because a perfect object is compact (Derived Categories of Spaces, Proposition 75.16.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Spaces, Theorem 75.15.4) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. $\square$

0E5U Lemma 86.8.2. Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. The map (86.7.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.

Proof. By Lemma 86.8.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (86.7.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Spaces, Theorem 75.17.3) generated by a single perfect object (Derived Categories of Spaces, Theorem 75.15.4) we can argue as in More on Algebra, Remark 15.59.11 to see that it suffices to prove (86.7.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 86.7.1. $\square$

0E5V Lemma 86.8.3. Let Y be an affine scheme. Let $f : X \to Y$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 86.3.1. Then

(1) $a(\mathcal{O}_Y)$ is a Y-perfect object of $D(\mathcal{O}_X)$,
(2) $Rf_*a(\mathcal{O}_Y)$ has vanishing cohomology sheaves in positive degrees,
(3) $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$ is an isomorphism.

Proof. For a perfect object E of $D(\mathcal{O}_X)$ we have

$$\begin{aligned}
Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) &= Rf_* R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) \\
&= R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y) \\
&= (Rf_*E^\vee)^\vee
\end{aligned}$$

For the first equality, see Cohomology on Sites, Lemma 21.48.4. For the second equality, see Lemma 86.3.3, Remark 86.3.5, and Derived Categories of Spaces, Lemma 75.25.4. The third equality is the definition of the dual. In particular these references also show that the outcome is a perfect object of $D(\mathcal{O}_Y)$. We conclude that $\omega^\bullet_{X/Y}$ is Y-perfect by More on Morphisms of Spaces, Lemma 76.52.14. This proves (1).

Let M be an object of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_Y(M, Rf_*a(\mathcal{O}_Y)) &= \mathrm{Hom}_X(Lf^*M, a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\mathcal{O}_Y, \mathcal{O}_Y)\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.19.1. The second equality by construction of a. The third equality by Derived Categories of Spaces, Lemma 75.20.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some N, see Derived Categories of Spaces, Lemma 75.25.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 15.74.2 and the fact that Y is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since Y is affine we conclude that $H^i(Rf_*a(\mathcal{O}_Y)) = 0$ for $i > 0$. This proves (2).

Let E be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have

$$\begin{aligned}\mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y)) &= \mathrm{Hom}_X(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y), a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_Y(Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, a(\mathcal{O}_Y))), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\ &= R\Gamma(Y, Rf_*E^\vee) \\ &= \mathrm{Hom}_X(E, \mathcal{O}_X)\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.35.2. The second equality is the definition of a. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology on Sites, Lemma 21.48.4. The fourth equality follows from the equality $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$ shown in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology on Sites, Lemma 21.48.4) and the fact that Rf_*E is perfect by Derived Categories of Spaces, Lemma 75.25.4 The last equality is Leray for f. This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\mathcal{H}om(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Spaces, Lemma 75.13.10. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$ corresponding to $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in E we see that the cone on α is an object C of $D_{QCoh}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E, C) = 0$ for all perfect E. Since the perfect objects generate (Derived Categories of Spaces, Theorem 75.15.4) we conclude that α is an isomorphism. □

86.9. Relative dualizing complexes for proper flat morphisms

0E5W Motivated by Duality for Schemes, Sections 48.12 and 48.28 and the material in Section 86.8 we make the following definition.

0E5X Definition 86.9.1. Let S be a scheme. Let $f : X \to Y$ be a proper, flat morphism of algebraic spaces over S which is of finite presentation. A relative dualizing complex for X/Y is a pair $(\omega^\bullet_{X/Y}, \tau)$ consisting of a Y-perfect object $\omega^\bullet_{X/Y}$ of $D(\mathcal{O}_X)$ and a map

$$\tau : Rf_*\omega^\bullet_{X/Y} \longrightarrow \mathcal{O}_Y$$

such that for any cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where Y' is an affine scheme the pair $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau)$ is isomorphic to the pair $(a'(\mathcal{O}_{Y'}), \text{Tr}_{f',\mathcal{O}_{Y'}})$ studied in Sections 86.3, 86.4, 86.5, 86.6, 86.7, and 86.8.

There are several remarks we should make here.

(1) In Definition 86.9.1 one may drop the assumption that $\omega_{X/Y}^\bullet$ is Y-perfect. Namely, running Y' through the members of an étale covering of Y by affines, we see from Lemma 86.8.3 that the restrictions of $\omega_{X/Y}^\bullet$ to the members of an étale covering of X are Y-perfect, which implies $\omega_{X/Y}^\bullet$ is Y-perfect, see More on Morphisms of Spaces, Section 76.52.

(2) Consider a relative dualizing complex $(\omega_{X/Y}^\bullet, \tau)$ and a cartesian square as in Definition 86.9.1. We are going to think of the existence of the isomorphism $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau) \cong (a'(\mathcal{O}_{Y'}), \text{Tr}_{f',\mathcal{O}_{Y'}})$ as follows: it says that for any $M' \in D_{QCoh}(\mathcal{O}_{X'})$ the map

$$\text{Hom}_{X'}(M', L(g')^*\omega_{X/Y}^\bullet) \longrightarrow \text{Hom}_{Y'}(Rf'_*M', \mathcal{O}_{Y'}), \quad \varphi' \longmapsto Lg^*\tau \circ Rf'_*\varphi'$$

is an isomorphism. This follows from the definition of a' and the discussion in Section 86.6. In particular, the Yoneda lemma guarantees that the isomorphism is unique.

(3) If Y is affine itself, then a relative dualizing complex $(\omega_{X/Y}^\bullet, \tau)$ exists and is canonically isomorphic to $(a(\mathcal{O}_Y), \text{Tr}_{f,\mathcal{O}_Y})$ where a is the right adjoint for Rf_* as in Lemma 86.3.1 and Tr_f is as in Section 86.6. Namely, given a diagram as in the definition we get an isomorphism $L(g')^*a(\mathcal{O}_Y) \to a'(\mathcal{O}_{Y'})$ by Lemma 86.5.1 which is compatible with trace maps by Lemma 86.6.1.

This produces exactly enough information to glue the locally given relative dualizing complexes to global ones. We suggest the reader skip the proofs of the following lemmas.

0E5Y Lemma 86.9.2. Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. If $(\omega_{X/Y}^\bullet, \tau)$ is a relative dualizing complex, then $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is an isomorphism and $Rf_*\omega_{X/Y}^\bullet$ has vanishing cohomology sheaves in positive degrees.

Proof. It suffices to prove this after base change to an affine scheme étale over Y in which case it follows from Lemma 86.8.3. $\square$

0E5Z Lemma 86.9.3. Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. If $(\omega_j^\bullet, \tau_j)$, $j = 1, 2$ are two relative dualizing complexes on X/Y, then there is a unique isomorphism $(\omega_1^\bullet, \tau_1) \to (\omega_2^\bullet, \tau_2)$.

Proof. Consider $g : Y' \to Y$ étale with Y' an affine scheme and denote $X' = Y' \times_Y X$ the base change. By Definition 86.9.1 and the discussion following, there is a unique isomorphism $\iota : (\omega_1^\bullet|_{X'}, \tau_1|_{Y'}) \to (\omega_2^\bullet|_{X'}, \tau_2|_{Y'})$. If $Y'' \to Y'$ is a further étale morphism of affines and $X'' = Y'' \times_Y X$, then $\iota|_{X''}$ is the unique isomorphism $(\omega_1^\bullet|_{X''}, \tau_1|_{Y''}) \to (\omega_2^\bullet|_{X''}, \tau_2|_{Y''})$ (by uniqueness). Also we have

$$\mathrm{Ext}^p_{X'}(\omega_1^\bullet|_{X'}, \omega_2^\bullet|_{X'}) = 0, \quad p < 0$$

because $\mathcal{O}_{X'} \cong R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_1^\bullet|_{X'}, \omega_1^\bullet|_{X'}) \cong R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_1^\bullet|_{X'}, \omega_2^\bullet|_{X'})$ by Lemma 86.9.2.

Choose a étale hypercovering $b : V \to Y$ such that each $V_n = \coprod_{i \in I_n} Y_{n,i}$ with $Y_{n,i}$ affine. This is possible by Hypercoverings, Lemma 25.12.2 and Remark 25.12.9 (to replace the hypercovering produced in the lemma by the one having disjoint unions in each degree). Denote $X_{n,i} = Y_{n,i} \times_Y X$ and $U_n = V_n \times_Y X$ so that we obtain an étale hypercovering $a : U \to X$ (Hypercoverings, Lemma 25.12.4) with $U_n = \coprod X_{n,i}$. The assumptions of Simplicial Spaces, Lemma 85.35.1 are satisfied for $a : U \to X$ and the complexes $\omega_1^\bullet$ and $\omega_2^\bullet$. Hence we obtain a unique morphism $\iota : \omega_1^\bullet \to \omega_2^\bullet$ whose restriction to $X_{0,i}$ is the unique isomorphism $(\omega_1^\bullet|_{X_{0,i}}, \tau_1|_{Y_{0,i}}) \to (\omega_2^\bullet|_{X_{0,i}}, \tau_2|_{Y_{0,i}})$ We still have to see that the diagram

$$\begin{tikzcd} Rf_*\omega_1^\bullet \arrow[rr, "Rf_*\iota"'] \arrow[dr, "\tau_1"'] & & Rf_*\omega_1^\bullet \arrow[dl, "\tau_2"] \\ & \mathcal{O}_Y & \end{tikzcd}$$

is commutative. However, we know that $Rf_*\omega_1^\bullet$ and $Rf_*\omega_2^\bullet$ have vanishing cohomology sheaves in positive degrees (Lemma 86.9.2) thus this commutativity may be proved after restricting to the affines $Y_{0,i}$ where it holds by construction. $\square$

0E60 **Lemma 86.9.4.** Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. Let $(\omega^\bullet, \tau)$ be a pair consisting of a Y-perfect object of $D(\mathcal{O}_X)$ and a map $\tau : Rf_*\omega^\bullet \to \mathcal{O}_Y$. Assume we have cartesian diagrams

$$\begin{tikzcd} X_i \arrow[r, "g_i'"'] \arrow[d, "f_i"'] & X \arrow[d, "f"] \\ Y_i \arrow[r, "g_i"] & Y \end{tikzcd}$$

with Y_i affine such that $\{g_i : Y_i \to Y\}$ is an étale covering and isomorphisms of pairs $(\omega^\bullet|_{X_i}, \tau|_{Y_i}) \to (a_i(\mathcal{O}_{Y_i}), \mathrm{Tr}_{f_i, \mathcal{O}_{Y_i}})$ as in Definition 86.9.1. Then $(\omega^\bullet, \tau)$ is a relative dualizing complex for X over Y.

Proof. Let $g : Y' \to Y$ and X', f', g', a' be as in Definition 86.9.1. Set $((\omega')^\bullet, \tau') = (L(g')^*\omega^\bullet, Lg^*\tau)$. We can find a finite étale covering $\{Y_j' \to Y'\}$ by affines which refines $\{Y_i \times_Y Y' \to Y'\}$ (Topologies, Lemma 34.4.4). Thus for each j there is an i_j and a morphism $k_j : Y_j' \to Y_{i_j}$ over Y. Consider the fibre products

$$\begin{tikzcd} X_j' \arrow[r, "h_j'"'] \arrow[d, "f_j'"'] & X' \arrow[d, "f'"] \\ Y_j' \arrow[r, "h_j"] & Y' \end{tikzcd}$$

Denote $k_j' : X_j' \to X_{i_j}$ the induced morphism (base change of k_j by f_{i_j}). Restricting the given isomorphisms to Y_j' via the morphism k_j' we get isomorphisms of pairs

$((\omega')^\bullet|_{X'_j}, \tau'|_{Y'_j}) \to (a_j(\mathcal{O}_{Y'_j}), \text{Tr}_{f'_j, \mathcal{O}_{Y'_j}})$. After replacing $f : X \to Y$ by $f' : X' \to Y'$ we reduce to the problem solved in the next paragraph.

Assume Y is affine. Problem: show $(\omega^\bullet, \tau)$ is isomorphic to $(\omega^\bullet_{X/Y}, \text{Tr}) = (a(\mathcal{O}_Y), \text{Tr}_{f, \mathcal{O}_Y})$. We may assume our covering $\{Y_i \to Y\}$ is given by a single surjective étale morphism $\{g : Y' \to Y\}$ of affines. Namely, we can first replace $\{g_i : Y_i \to Y\}$ by a finite subcovering, and then we can set $g = \coprod g_i : Y' = \coprod Y_i \to Y$; some details omitted. Set $X' = Y' \times_Y X$ with maps f', g' as in Definition 86.9.1. Then all we're given is that we have an isomorphism

$$(\omega^\bullet|_{X'}, \tau|_{Y'}) \to (a'(\mathcal{O}_{Y'}), \text{Tr}_{f', \mathcal{O}_{Y'}})$$

Since $(\omega^\bullet_{X/Y}, \text{Tr})$ is a relative dualizing complex (see discussion following Definition 86.9.1) there is a unique isomorphism

$$(\omega^\bullet_{X/Y}|_{X'}, \text{Tr}|_{Y'}) \to (a'(\mathcal{O}_{Y'}), \text{Tr}_{f', \mathcal{O}_{Y'}})$$

Uniqueness by Lemma 86.9.3 for example. Combining the displayed isomorphisms we find an isomorphism

$$\alpha : (\omega^\bullet|_{X'}, \tau|_{Y'}) \to (\omega^\bullet_{X/Y}|_{X'}, \text{Tr}|_{Y'})$$

Set $Y'' = Y' \times_Y Y'$ and $X'' = Y'' \times_Y X$ the two pullbacks of α to X'' have to be the same by uniqueness again. Since we have vanishing negative self exts for $\omega^\bullet_{X'/Y'}$ over X' (Lemma 86.9.2) and since this remains true after pulling back by any projection $Y' \times_Y \ldots \times_Y Y' \to Y'$ (small detail omitted – compare with the proof of Lemma 86.9.3), we find that α descends to an isomorphism $\omega^\bullet \to \omega^\bullet_{X/Y}$ over X by Simplicial Spaces, Lemma 85.35.1. □

0E61 **Lemma 86.9.5.** Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. There exists a relative dualizing complex $(\omega^\bullet_{X/Y}, \tau)$.

Proof. Choose a étale hypercovering $b : V \to Y$ such that each $V_n = \coprod_{i \in I_n} Y_{n,i}$ with $Y_{n,i}$ affine. This is possible by Hypercoverings, Lemma 25.12.2 and Remark 25.12.9 (to replace the hypercovering produced in the lemma by the one having disjoint unions in each degree). Denote $X_{n,i} = Y_{n,i} \times_Y X$ and $U_n = V_n \times_Y X$ so that we obtain an étale hypercovering $a : U \to X$ (Hypercoverings, Lemma 25.12.4) with $U_n = \coprod X_{n,i}$. For each n, i there exists a relative dualizing complex $(\omega^\bullet_{n,i}, \tau_{n,i})$ on $X_{n,i}/Y_{n,i}$. See discussion following Definition 86.9.1. For $\varphi : [m] \to [n]$ and $i \in I_n$ consider the morphisms $g_{\varphi,i} : Y_{n,i} \to Y_{m,\alpha(\varphi)}$ and $g'_{\varphi,i} : X_{n,i} \to X_{m,\alpha(\varphi)}$ which are part of the structure of the given hypercoverings (Hypercoverings, Section 25.12). Then we have a unique isomorphisms

$$\iota_{n,i,\varphi} : (L(g'_{n,i})^*\omega^\bullet_{n,i}, Lg^*_{n,i}\tau_{n,i}) \longrightarrow (\omega^\bullet_{m,\alpha(\varphi)(i)}, \tau_{m,\alpha(\varphi)(i)})$$

of pairs, see discussion following Definition 86.9.1. Observe that $\omega^\bullet_{n,i}$ has vanishing negative self exts on $X_{n,i}$ by Lemma 86.9.2. Denote $(\omega^\bullet_n, \tau_n)$ the pair on U_n/V_n constructed using the pairs $(\omega^\bullet_{n,i}, \tau_{n,i})$ for $i \in I_n$. For $\varphi : [m] \to [n]$ and $i \in I_n$ consider the morphisms $g_\varphi : V_n \to V_m$ and $g'_\varphi : U_n \to U_m$ which are part of the structure of the simplicial algebraic spaces V and U. Then we have unique isomorphisms

$$\iota_\varphi : (L(g'_\varphi)^*\omega^\bullet_n, Lg^*_\varphi\tau_n) \longrightarrow (\omega^\bullet_m, \tau_m)$$

of pairs constructed from the isomorphisms on the pieces. The uniqueness guarantees that these isomorphisms satisfy the transitivity condition as formulated in Simplicial Spaces, Definition 85.14.1. The assumptions of Simplicial Spaces, Lemma 85.35.2 are satisfied for $a : U \to X$, the complexes $\omega_n^\bullet$ and the isomorphisms ι_φ[1]. Thus we obtain an object $\omega^\bullet$ of $D_{QCoh}(\mathcal{O}_X)$ together with an isomorphism $\iota_0 : \omega^\bullet|_{U_0} \to \omega_0^\bullet$ compatible with the two isomorphisms $\iota_{\delta_0^1}$ and $\iota_{\delta_1^1}$. Finally, we apply Simplicial Spaces, Lemma 85.35.1 to find a unique morphism

$$\tau : Rf_*\omega^\bullet \longrightarrow \mathcal{O}_Y$$

whose restriction to V_0 agrees with τ_0; some details omitted – compare with the end of the proof of Lemma 86.9.3 for example to see why we have the required vanishing of negative exts. By Lemma 86.9.4 the pair $(\omega^\bullet, \tau)$ is a relative dualizing complex and the proof is complete. □

0E6C Lemma 86.9.6. Let S be a scheme. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of algebraic spaces over S. Assume $X \to Y$ is proper, flat, and of finite presentation. Let $(\omega_{X/Y}^\bullet, \tau)$ be a relative dualizing complex for f. Then $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau)$ is a relative dualizing complex for f'.

Proof. Observe that $L(g')^*\omega_{X/Y}^\bullet$ is Y'-perfect by More on Morphisms of Spaces, Lemma 76.52.6. The other condition of Definition 86.9.1 holds by transitivity of fibre products. □

86.10. Comparison with the case of schemes

0E6D We should add a lot more in this section.

0E6E Lemma 86.10.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Assume X and Y are representable and let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* from Lemma 86.3.1. Let $a_0 : D_{QCoh}(\mathcal{O}_{Y_0}) \to D_{QCoh}(\mathcal{O}_{X_0})$ be the right adjoint of Rf_* from Duality for Schemes, Lemma 48.3.1. Then

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_{X_0}) \arrow[r, equal, "\text{Derived Categories of Spaces, Lemma 75.4.2}"'] & D_{QCoh}(\mathcal{O}_X) \\ D_{QCoh}(\mathcal{O}_{Y_0}) \arrow[u, "a_0"] \arrow[r, equal, "\text{Derived Categories of Spaces, Lemma 75.4.2}"] & D_{QCoh}(\mathcal{O}_Y) \arrow[u, "a"'] \end{tikzcd}$$

is commutative.

Proof. Follows from uniqueness of adjoints and the compatibilities of Derived Categories of Spaces, Remark 75.6.3. □

[1]This lemma uses only $\omega_0^\bullet$ and the two maps $\delta_1^1, \delta_0^1 : [1] \to [0]$. The reader can skip the first few lines of the proof of the referenced lemma because here we actually are already given a simplicial system of the derived category of modules.

86.11. Other chapters

CHAPTER 87

Formal Algebraic Spaces

0AHW 87.1. Introduction

0AHX Formal schemes were introduced in [DG67]. A more general version of formal schemes was introduced in [McQ02] and another in [Yas09]. Formal algebraic spaces were introduced in [Knu71]. Related material and much besides can be found in [Abb10] and [FK]. This chapter introduces the notion of formal algebraic spaces we will work with. Our definition is general enough to allow most classes of formal schemes/spaces in the literature as full subcategories.

Although we do discuss the comparison of some of these alternative theories with ours, we do not always give full details when it is not necessary for the logical development of the theory.

Besides introducing formal algebraic spaces, we also prove a few very basic properties and we discuss a few types of morphisms.

87.2. Formal schemes à la EGA

0AHY In this section we review the construction of formal schemes in [DG67]. This notion, although very useful in algebraic geometry, may not always be the correct one to consider. Perhaps it is better to say that in the setup of the theory a number of choices are made, where for different purposes others might work better. And indeed in the literature one can find many different closely related theories adapted to the problem the authors may want to consider. Still, one of the major advantages of the theory as sketched here is that one gets to work with definite geometric objects.

Before we start we should point out an issue with the sheaf condition for sheaves of topological rings or more generally sheaves of topological spaces. Namely, the big categories

(1) category of topological spaces,
(2) category of topological groups,
(3) category of topological rings,
(4) category of topological modules over a given topological ring,

endowed with their natural forgetful functors to Sets are not examples of types of algebraic structures as defined in Sheaves, Section 6.15. Thus we cannot blithely apply to them the machinery developed in that chapter. On the other hand, each of the categories listed above has limits and equalizers and the forgetful functor to sets, groups, rings, modules commutes with them (see Topology, Lemmas 5.14.1, 5.30.3, 5.30.8, and 5.30.11). Thus we can define the notion of a sheaf as in Sheaves, Definition 6.9.1 and the underlying presheaf of sets, groups, rings, or modules is a sheaf. The key difference is that for an open covering $U = \bigcup_{i \in I} U_i$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

has to be an equalizer diagram in the category of topological spaces, topological groups, topological rings, topological modules, i.e., that the first map identifies $\mathcal{F}(U)$ with a subspace of $\prod_{i \in I} \mathcal{F}(U_i)$ which is endowed with the product topology.

The stalk $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ of topological spaces, topological groups, topological rings, or topological modules at a point $x \in X$ is defined as the colimit over open neighbourhoods

$$\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$$

in the corresponding category. This is the same as taking the colimit on the level of sets, groups, rings, or modules (see Topology, Lemmas 5.29.1, 5.30.6, 5.30.9, and 5.30.12) but comes equipped with a topology. Warning: the topology one gets depends on which category one is working with, see Examples, Section 110.77. One can sheafify presheaves of topological spaces, topological groups, topological rings, or topological modules and taking stalks commutes with this operation, see Remark 87.2.4.

Let $f : X \to Y$ be a continuous map of topological spaces. There is a functor f_* from the category of sheaves of topological spaces, topological groups, topological rings, topological modules, to the corresponding category of sheaves on Y which is defined by setting $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$ as usual. (We delay discussing the pullback in this setting till later.) We define the notion of an f-map $\xi : \mathcal{G} \to \mathcal{F}$ between a sheaf of topological spaces $\mathcal{G}$ on Y and a sheaf of topological spaces $\mathcal{F}$ on X in exactly the same manner as in Sheaves, Definition 6.21.7 with the additional constraint that $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ be continuous for every open $V \subset Y$. We have

$$\{f\text{-maps from } \mathcal{G} \text{ to } \mathcal{F}\} = \mathrm{Mor}_{Sh(Y, \mathrm{Top})}(\mathcal{G}, f_*\mathcal{F})$$

as in Sheaves, Lemma 6.21.8. Similarly for sheaves of topological groups, topological rings, topological modules. Finally, let $\xi : \mathcal{G} \to \mathcal{F}$ be an f-map as above. Then given $x \in X$ with image $y = f(x)$ there is a continuous map

$$\xi_x : \mathcal{G}_y \longrightarrow \mathcal{F}_x$$

of stalks defined in exactly the same manner as in the discussion following Sheaves, Definition 6.21.9.

Using the discussion above, we can define a category $LTRS$ of "locally topologically ringed spaces". An object is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of topological rings $\mathcal{O}_X$ whose stalks $\mathcal{O}_{X,x}$ are local rings (if one forgets about the topology). A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $LTRS$ is a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continuous map of topological spaces and $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ is an f-map such that for every $x \in X$ the induced map

$$f^\sharp_x : \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism of local rings (forgetting about the topologies). The composition works in exactly the same manner as composition of morphisms of locally ringed spaces.

Assume now that the topological space X has a basis consisting of quasi-compact opens. Given a sheaf $\mathcal{F}$ of sets, groups, rings, modules over a ring, one can endow $\mathcal{F}$

with the structure of a sheaf of topological spaces, topological groups, topological rings, topological modules. Namely, if $U \subset X$ is quasi-compact open, we endow $\mathcal{F}(U)$ with the discrete topology. If $U \subset X$ is arbitrary, then we choose an open covering $U = \bigcup_{i \in I} U_i$ by quasi-compact opens and we endow $\mathcal{F}(U)$ with the induced topology from $\prod_{i \in I} \mathcal{F}(U_i)$ (as we should do according to our discussion above). The reader may verify (omitted) that we obtain a sheaf of topological spaces, topological groups, topological rings, topological modules in this fashion. Let us say that a sheaf of topological spaces, topological groups, topological rings, topological modules is pseudo-discrete if the topology on $\mathcal{F}(U)$ is discrete for every quasi-compact open $U \subset X$. Then the construction given above is an adjoint to the forgetful functor and induces an equivalence between the category of sheaves of sets and the category of pseudo-discrete sheaves of topological spaces (similarly for groups, rings, modules).

Grothendieck and Dieudonné first define formal affine schemes. These correspond to admissible topological rings A, see More on Algebra, Definition 15.36.1. Namely, given A one considers a fundamental system I_λ of ideals of definition for the ring A. (In any admissible topological ring the family of all ideals of definition forms a fundamental system.) For each λ we can consider the scheme $\mathrm{Spec}(A/I_\lambda)$. For $I_\lambda \subset I_\mu$ the induced morphism

$$\mathrm{Spec}(A/I_\mu) \to \mathrm{Spec}(A/I_\lambda)$$

is a thickening because $I_\mu^n \subset I_\lambda$ for some n. Another way to see this, is to notice that the image of each of the maps

$$\mathrm{Spec}(A/I_\lambda) \to \mathrm{Spec}(A)$$

is a homeomorphism onto the set of open prime ideals of A. This motivates the definition

$$\mathrm{Spf}(A) = \{\text{open prime ideals } \mathfrak{p} \subset A\}$$

endowed with the topology coming from $\mathrm{Spec}(A)$. For each λ we can consider the structure sheaf $\mathcal{O}_{\mathrm{Spec}(A/I_\lambda)}$ as a sheaf on $\mathrm{Spf}(A)$. Let $\mathcal{O}_\lambda$ be the corresponding pseudo-discrete sheaf of topological rings, see above. Then we set

$$\mathcal{O}_{\mathrm{Spf}(A)} = \lim \mathcal{O}_\lambda$$

where the limit is taken in the category of sheaves of topological rings. The pair $(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$ is called the formal spectrum of A.

At this point one should check several things. The first is that the stalks $\mathcal{O}_{\mathrm{Spf}(A),x}$ are local rings (forgetting about the topology). The second is that given $f \in A$, for the corresponding open $D(f) \cap \mathrm{Spf}(A)$ we have

$$\Gamma(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}) = A_{\{f\}} = \lim (A/I_\lambda)_f$$

as topological rings where I_λ is a fundamental system of ideals of definition as above. Moreover, the ring $A_{\{f\}}$ is admissible too and $(\mathrm{Spf}(A_f), \mathcal{O}_{\mathrm{Spf}(A_{\{f\}})})$ is isomorphic to $(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}|_{D(f)\cap \mathrm{Spf}(A)})$. Finally, given a pair of admissible topological rings A, B we have

0AHZ (87.2.0.1) $$\mathrm{Mor}_{LTRS}((\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)}), (\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})) = \mathrm{Hom}_{cont}(A, B)$$

where $LTRS$ is the category of "locally topologically ringed spaces" as defined above.

Having said this, in [DG67] a formal scheme is defined as a pair $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ where $\mathfrak{X}$ is a topological space and $\mathcal{O}_\mathfrak{X}$ is a sheaf of topological rings such that every point has an open neighbourhood isomorphic (in $LTRS$) to an affine formal scheme. A morphism of formal schemes $f : (\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \to (\mathfrak{Y}, \mathcal{O}_\mathfrak{Y})$ is a morphism in the category $LTRS$.

Let A be a ring endowed with the discrete topology. Then A is admissible and the formal scheme $\text{Spf}(A)$ is equal to $\text{Spec}(A)$. The structure sheaf $\mathcal{O}_{\text{Spf}(A)}$ is the pseudo-discrete sheaf of topological rings associated to $\mathcal{O}_{\text{Spec}(A)}$, in other words, its underlying sheaf of rings is equal to $\mathcal{O}_{\text{Spec}(A)}$ and the ring $\mathcal{O}_{\text{Spf}(A)}(U) = \mathcal{O}_{\text{Spec}(A)}(U)$ over a quasi-compact open U has the discrete topology, but not in general. Thus we can associate to every affine scheme a formal affine scheme. In exactly the same manner we can start with a general scheme $(X, \mathcal{O}_X)$ and associate to it $(X, \mathcal{O}'_X)$ where $\mathcal{O}'_X$ is the pseudo-discrete sheaf of topological rings whose underlying sheaf of rings is $\mathcal{O}_X$. This construction is compatible with morphisms and defines a functor

0AI0 (87.2.0.2)
$$\text{Schemes} \longrightarrow \text{Formal Schemes}$$

It follows in a straightforward manner from (87.2.0.1) that this functor is fully faithful.

Let $\mathfrak{X}$ be a formal scheme. Let us define the size of the formal scheme by the formula $\text{size}(\mathfrak{X}) = \max(\aleph_0, \kappa_1, \kappa_2)$ where κ_1 is the cardinality of the formal affine opens of $\mathfrak{X}$ and κ_2 is the supremum of the cardinalities of $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$ where $\mathfrak{U} \subset \mathfrak{X}$ is such a formal affine open.

0AI1 **Lemma 87.2.1.** Choose a category of schemes Sch_α as in Sets, Lemma 3.9.2. Given a formal scheme $\mathfrak{X}$ let

$$h_\mathfrak{X} : (Sch_\alpha)^{opp} \longrightarrow \text{Sets}, \quad h_\mathfrak{X}(S) = \text{Mor}_{\text{Formal Schemes}}(S, \mathfrak{X})$$

be its functor of points. Then we have

$$\text{Mor}_{\text{Formal Schemes}}(\mathfrak{X}, \mathfrak{Y}) = \text{Mor}_{\text{PSh}(Sch_\alpha)}(h_\mathfrak{X}, h_\mathfrak{Y})$$

provided the size of $\mathfrak{X}$ is not too large.

Proof. First we observe that $h_\mathfrak{X}$ satisfies the sheaf property for the Zariski topology for any formal scheme $\mathfrak{X}$ (see Schemes, Definition 26.15.3). This follows from the local nature of morphisms in the category of formal schemes. Also, for an open immersion $\mathfrak{V} \to \mathfrak{W}$ of formal schemes, the corresponding transformation of functors $h_\mathfrak{V} \to h_\mathfrak{W}$ is injective and representable by open immersions (see Schemes, Definition 26.15.3). Choose an open covering $\mathfrak{X} = \bigcup \mathfrak{U}_i$ of a formal scheme by affine formal schemes $\mathfrak{U}_i$. Then the collection of functors $h_{\mathfrak{U}_i}$ covers $h_\mathfrak{X}$ (see Schemes, Definition 26.15.3). Finally, note that

$$h_{\mathfrak{U}_i} \times_{h_\mathfrak{X}} h_{\mathfrak{U}_j} = h_{\mathfrak{U}_i \cap \mathfrak{U}_j}$$

Hence in order to give a map $h_\mathfrak{X} \to h_\mathfrak{Y}$ is equivalent to giving a family of maps $h_{\mathfrak{U}_i} \to h_\mathfrak{Y}$ which agree on overlaps. Thus we can reduce the bijectivity (resp. injectivity) of the map of the lemma to bijectivity (resp. injectivity) for the pairs $(\mathfrak{U}_i, \mathfrak{Y})$ and injectivity (resp. nothing) for $(\mathfrak{U}_i \cap \mathfrak{U}_j, \mathfrak{Y})$. In this way we reduce to the case where $\mathfrak{X}$ is an affine formal scheme. Say $\mathfrak{X} = \text{Spf}(A)$ for some admissible topological ring A. Also, choose a fundamental system of ideals of definition $I_\lambda \subset A$.

We can also localize on $\mathfrak{Y}$. Namely, suppose that $\mathfrak{V} \subset \mathfrak{Y}$ is an open formal subscheme and $\varphi : h_\mathfrak{X} \to h_\mathfrak{Y}$. Then

$$h_\mathfrak{V} \times_{h_\mathfrak{Y}, \varphi} h_\mathfrak{X} \to h_\mathfrak{X}$$

is representable by open immersions. Pulling back to $\mathrm{Spec}(A/I_\lambda)$ for all λ we find an open subscheme $U_\lambda \subset \mathrm{Spec}(A/I_\lambda)$. However, for $I_\lambda \subset I_\mu$ the morphism $\mathrm{Spec}(A/I_\lambda) \to \mathrm{Spec}(A/I_\mu)$ pulls back U_μ to U_λ. Thus these glue to give an open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$. A straightforward argument (omitted) shows that

$$h_{\mathfrak{U}} = h_{\mathfrak{V}} \times_{h_{\mathfrak{Y}}} h_{\mathfrak{X}}$$

In this way we see that given an open covering $\mathfrak{Y} = \bigcup \mathfrak{V}_j$ and a transformation of functors $\varphi : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ we obtain a corresponding open covering of $\mathfrak{X}$. Since $\mathfrak{X}$ is affine, we can refine this covering by a finite open covering $\mathfrak{X} = \mathfrak{U}_1 \cup \ldots \cup \mathfrak{U}_n$ by affine formal subschemes. In other words, for each i there is a j and a map $\varphi_i : h_{\mathfrak{U}_i} \to h_{\mathfrak{V}_j}$ such that

$$\begin{tikzcd} h_{\mathfrak{U}_i} \arrow[r, "\varphi_i"'] \arrow[d] & h_{\mathfrak{V}_j} \arrow[d] \\ h_{\mathfrak{X}} \arrow[r, "\varphi"] & h_{\mathfrak{Y}} \end{tikzcd}$$

commutes. With a few additional arguments (which we omit) this implies that it suffices to prove the bijectivity of the lemma in case both $\mathfrak{X}$ and $\mathfrak{Y}$ are affine formal schemes.

Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are affine formal schemes. Say $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. Let $\varphi : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ be a transformation of functors. Let $I_\lambda \subset A$ be a fundamental system of ideals of definition. The canonical inclusion morphism $i_\lambda : \mathrm{Spec}(A/I_\lambda) \to \mathfrak{X}$ maps to a morphism $\varphi(i_\lambda) : \mathrm{Spec}(A/I_\lambda) \to \mathfrak{Y}$. By (87.2.0.1) this corresponds to a continuous map $\chi_\lambda : B \to A/I_\lambda$. Since φ is a transformation of functors it follows that for $I_\lambda \subset I_\mu$ the composition $B \to A/I_\lambda \to A/I_\mu$ is equal to χ_μ. In other words we obtain a ring map

$$\chi = \lim \chi_\lambda : B \longrightarrow \lim A/I_\lambda = A$$

This is a continuous homomorphism because the inverse image of I_λ is open for all λ (as A/I_λ has the discrete topology and χ_λ is continuous). Thus we obtain a morphism $\mathrm{Spf}(\chi) : \mathfrak{X} \to \mathfrak{Y}$ by (87.2.0.1). We omit the verification that this construction is the inverse to the map of the lemma in this case.

Set theoretic remarks. To make this work on the given category of schemes Sch_α we just have to make sure all the schemes used in the proof above are isomorphic to objects of Sch_α. In fact, a careful analysis shows that it suffices if the schemes $\mathrm{Spec}(A/I_\lambda)$ occurring above are isomorphic to objects of Sch_α. For this it certainly suffices to assume the size of $\mathfrak{X}$ is at most the size of a scheme contained in Sch_α. $\square$

0AI2 Lemma 87.2.2. Let $\mathfrak{X}$ be a formal scheme. The functor of points $h_{\mathfrak{X}}$ (see Lemma 87.2.1) satisfies the sheaf condition for fpqc coverings.

Proof. Topologies, Lemma 34.9.13 reduces us to the case of a Zariski covering and a covering $\{\mathrm{Spec}(S) \to \mathrm{Spec}(R)\}$ with $R \to S$ faithfully flat. We observed in the proof of Lemma 87.2.1 that $h_{\mathfrak{X}}$ satisfies the sheaf condition for Zariski coverings.

Suppose that $R \to S$ is a faithfully flat ring map. Denote $\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the corresponding morphism of schemes. It is surjective and flat. Let $f : \mathrm{Spec}(S) \to \mathfrak{X}$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as maps $\mathrm{Spec}(S \otimes_R S) \to \mathfrak{X}$. By Descent, Lemma 35.13.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$

for some (set theoretic) map $g : \mathrm{Spec}(R) \to \mathfrak{X}$. By Morphisms, Lemma 29.25.12 and the fact that f is continuous we see that g is continuous.

Pick $y \in \mathrm{Spec}(R)$. Choose $\mathfrak{U} \subset \mathfrak{X}$ an affine formal open subscheme containing $g(y)$. Say $\mathfrak{U} = \mathrm{Spf}(A)$ for some admissible topological ring A. By the above we may choose an $r \in R$ such that $y \in D(r) \subset g^{-1}(\mathfrak{U})$. The restriction of f to $\pi^{-1}(D(r))$ into $\mathfrak{U}$ corresponds to a continuous ring map $A \to S_r$ by (87.2.0.1). The two induced ring maps $A \to S_r \otimes_{R_r} S_r = (S \otimes_R S)_r$ are equal by assumption on f. Note that $R_r \to S_r$ is faithfully flat. By Descent, Lemma 35.3.6 the equalizer of the two arrows $S_r \to S_r \otimes_{R_r} S_r$ is R_r. We conclude that $A \to S_r$ factors uniquely through a map $A \to R_r$ which is also continuous as it has the same (open) kernel as the map $A \to S_r$. This map in turn gives a morphism $D(r) \to \mathfrak{U}$ by (87.2.0.1).

What have we proved so far? We have shown that for any $y \in \mathrm{Spec}(R)$ there exists a standard affine open $y \in D(r) \subset \mathrm{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to \mathfrak{X}$ factors uniquely though some morphism $D(r) \to \mathfrak{X}$. We omit the verification that these morphisms glue to the desired morphism $\mathrm{Spec}(R) \to \mathfrak{X}$. $\square$

0AI3 Remark 87.2.3 (McQuillan's variant). There is a variant of the construction of formal schemes due to McQuillan, see [McQ02]. He suggests a slight weakening of the condition of admissibility. Namely, recall that an admissible topological ring is a complete (and separated by our conventions) topological ring A which is linearly topologized such that there exists an ideal of definition: an open ideal I such that any neighbourhood of 0 contains I^n for some $n \geq 1$. McQuillan works with what we will call weakly admissible topological rings. A weakly admissible topological ring A is a complete (and separated by our conventions) topological ring which is linearly topologized such that there exists an weak ideal of definition: an open ideal I such that for all $f \in I$ we have $f^n \to 0$ for $n \to \infty$. Similarly to the admissible case, if I is a weak ideal of definition and $J \subset A$ is an open ideal, then $I \cap J$ is a weak ideal of definition. Thus the weak ideals of definition form a fundamental system of open neighbourhoods of 0 and one can proceed along much the same route as above to define a larger category of formal schemes based on this notion. The analogues of Lemmas 87.2.1 and 87.2.2 still hold in this setting (with the same proof).

0AI4 Remark 87.2.4 (Sheafification of presheaves of topological spaces). In this remark [Gra65]
we briefly discuss sheafification of presheaves of topological spaces. The exact same arguments work for presheaves of topological abelian groups, topological rings, and topological modules (over a given topological ring). In order to do this in the correct generality let us work over a site $\mathcal{C}$. The reader who is interested in the case of (pre)sheaves over a topological space X should think of objects of $\mathcal{C}$ as the opens of X, of morphisms of $\mathcal{C}$ as inclusions of opens, and of coverings in $\mathcal{C}$ as coverings in X, see Sites, Example 7.6.4. Denote $Sh(\mathcal{C}, \mathrm{Top})$ the category of sheaves of topological spaces on $\mathcal{C}$ and denote $\mathrm{PSh}(\mathcal{C}, \mathrm{Top})$ the category of presheaves of topological spaces on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of topological spaces on $\mathcal{C}$. The sheafification $\mathcal{F}^\#$ should satisfy the formula

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C}, \mathrm{Top})}(\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C}, \mathrm{Top})}(\mathcal{F}^\#, \mathcal{G})$$

functorially in $\mathcal{G}$ from $Sh(\mathcal{C}, \mathrm{Top})$. In other words, we are trying to construct the left adjoint to the inclusion functor $Sh(\mathcal{C}, \mathrm{Top}) \to \mathrm{PSh}(\mathcal{C}, \mathrm{Top})$. We first claim that

$\mathit{Sh}(\mathcal{C}, \mathrm{Top})$ has limits and that the inclusion functor commutes with them. Namely, given a category $\mathcal{I}$ and a functor $i \mapsto \mathcal{G}_i$ into $\mathit{Sh}(\mathcal{C}, \mathrm{Top})$ we simply define

$$(\lim \mathcal{G}_i)(U) = \lim \mathcal{G}_i(U)$$

where we take the limit in the category of topological spaces (Topology, Lemma 5.14.1). This defines a sheaf because limits commute with limits (Categories, Lemma 4.14.10) and in particular products and equalizers (which are the operations used in the sheaf axiom). Finally, a morphism of presheaves from $\mathcal{F} \to \lim \mathcal{G}_i$ is clearly the same thing as a compatible system of morphisms $\mathcal{F} \to \mathcal{G}_i$. In other words, the object $\lim \mathcal{G}_i$ is the limit in the category of presheaves of topological spaces and a fortiori in the category of sheaves of topological spaces. Our second claim is that any morphism of presheaves $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ an object of $\mathit{Sh}(\mathcal{C}, \mathrm{Top})$ factors through a subsheaf $\mathcal{G}' \subset \mathcal{G}$ whose size is bounded. Here we define the size $|\mathcal{H}|$ of a sheaf of topological spaces $\mathcal{H}$ to be the cardinal $\sup_{U \in \mathrm{Ob}(\mathcal{C})} |\mathcal{H}(U)|$. To prove our claim we let

$$\mathcal{G}'(U) = \left\{ \quad s \in \mathcal{G}(U) \quad \middle| \quad \begin{array}{c} \text{there exists a covering } \{U_i \to U\}_{i \in I} \\ \text{such that } s|_{U_i} \in \mathrm{Im}(\mathcal{F}(U_i) \to \mathcal{G}(U_i)) \end{array} \right\}$$

We endow $\mathcal{G}'(U)$ with the induced topology. Then $\mathcal{G}'$ is a sheaf of topological spaces (details omitted) and $\mathcal{G}' \to \mathcal{G}$ is a morphism through which the given map $\mathcal{F} \to \mathcal{G}$ factors. Moreover, the size of $\mathcal{G}'$ is bounded by some cardinal κ depending only on $\mathcal{C}$ and the presheaf $\mathcal{F}$ (hint: use that coverings in $\mathcal{C}$ form a set by our conventions). Putting everything together we see that the assumptions of Categories, Theorem 4.25.3 are satisfied and we obtain sheafification as the left adjoint of the inclusion functor from sheaves to presheaves. Finally, let p be a point of the site $\mathcal{C}$ given by a functor $u : \mathcal{C} \to \mathrm{Sets}$, see Sites, Definition 7.32.2. For a topological space M the presheaf defined by the rule

$$U \mapsto \mathrm{Map}(u(U), M) = \prod\nolimits_{x \in u(U)} M$$

endowed with the product topology is a sheaf of topological spaces. Hence the exact same argument as given in the proof of Sites, Lemma 7.32.5 shows that $\mathcal{F}_p = \mathcal{F}_p^\#$, in other words, sheafification commutes with taking stalks at a point.

87.3. Conventions and notation

0AI5 The conventions from now on will be similar to the conventions in Properties of Spaces, Section 66.2. Thus from now on the standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site. For topological rings A we assume only that all discrete quotients have this property (but usually we assume more, compare with Remark 87.11.5).

Let S be a scheme and let X be a "space" over S, i.e., a sheaf on $(\mathit{Sch}/S)_{fppf}$. In this chapter we will write $X \times_S X$ for the product of X with itself in the category of sheaves on $(\mathit{Sch}/S)_{fppf}$ instead of $X \times X$. Moreover, if X and Y are "spaces" then we say "let $f : X \to Y$ be a morphism" to indicate that f is a natural transformation of functors, i.e., a map of sheaves on $(\mathit{Sch}/S)_{fppf}$. Similarly, if U is a scheme over S and X is a "space" over S, then we say "let $f : U \to X$ be a morphism" or "let $g : X \to U$ be a morphism" to indicate that f or g is a map of sheaves $h_U \to X$ or $X \to h_U$ where h_U is as in Categories, Example 4.3.4.

87.4. Topological rings and modules

0AMQ This section is a continuation of More on Algebra, Section 15.36. Let R be a topological ring and let M be a linearly topologized R-module. When we say "let M_λ be a fundamental system of open submodules" we will mean that each M_λ is an open submodule and that any neighbourhood of 0 contains one of the M_λ. In other words, this means that M_λ is a fundamental system of neighbourhoods of 0 in M consisting of submodules. Similarly, if R is a linearly topologized ring, then we say "let I_λ be a fundamental system of open ideals" to mean that I_λ is a fundamental system of neighbourhoods of 0 in R consisting of ideals.

0AMR Example 87.4.1. Let R be a linearly topologized ring and let M be a linearly topologized R-module. Let I_λ be a fundamental system of open ideals in R and let M_μ be a fundamental system of open submodules of M. The continuity of $+ : M \times M \to M$ is automatic and the continuity of $R \times M \to M$ signifies

$$\forall f, x, \mu\ \exists \lambda, \nu,\ (f + I_\lambda)(x + M_\nu) \subset fx + M_\mu$$

Since $fM_\nu + I_\lambda M_\nu \subset M_\mu$ if $M_\nu \subset M_\mu$ we see that the condition is equivalent to

$$\forall x, \mu\ \exists \lambda\ I_\lambda x \subset M_\mu$$

However, it need not be the case that given μ there is a λ such that $I_\lambda M \subset M_\mu$. For example, consider $R = k[[t]]$ with the t-adic topology and $M = \bigoplus_{n \in \mathbf{N}} R$ with fundamental system of open submodules given by

$$M_m = \bigoplus\nolimits_{n \in \mathbf{N}} t^{nm} R$$

Since every $x \in M$ has finitely many nonzero coordinates we see that, given m and x there exists a k such that $t^k x \in M_m$. Thus M is a linearly topologized R-module, but it isn't true that given m there is a k such that $t^k M \subset M_m$. On the other hand, if $R \to S$ is a continuous map of linearly topologized rings, then the corresponding statement does hold, i.e., for every open ideal $J \subset S$ there exists an open ideal $I \subset R$ such that $IS \subset J$ (as the reader can easily deduce from continuity of the map $R \to S$).

0AMS Lemma 87.4.2. Let R be a topological ring. Let M be a linearly topologized R-module and let M_λ, $\lambda \in \Lambda$ be a fundamental system of open submodules. Let $N \subset M$ be a submodule. The closure of N is $\bigcap_{\lambda \in \Lambda}(N + M_\lambda)$.

Proof. Since each $N + M_\lambda$ is open, it is also closed. Hence the intersection is closed. If $x \in M$ is not in the closure of N, then $(x + M_\lambda) \cap N = 0$ for some λ. Hence $x \notin N + M_\lambda$. This proves the lemma. □

Unless otherwise mentioned we endow submodules and quotient modules with the induced topology. Let M be a linearly topologized module over a topological ring R, and let $0 \to N \to M \to Q \to 0$ be a short exact sequence of R-modules. If M_λ is a fundamental system of open submodules of M, then $N \cap M_\lambda$ is a fundamental system of open submodules of N. If $\pi : M \to Q$ is the quotient map, then $\pi(M_\lambda)$ is a fundamental system of open submodules of Q. In particular these induced topologies are linear topologies.

0ARZ Lemma 87.4.3. Let R be a topological ring. Let M be a linearly topologized R-module. Let $N \subset M$ be a submodule. Then

(1) $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge$ is exact, and

(2) $N^\wedge$ is the closure of the image of $N \to M^\wedge$.

Proof. Let M_λ, $\lambda \in \Lambda$ be a fundamental system of open submodules. Then $N \cap M_\lambda$ is a fundamental system of open submodules of N and $M_\lambda + N/N$ is a fundamental system of open submodules of M/N. Thus we see that (1) follows from the exactness of the sequences

$$0 \to N/N \cap M_\lambda \to M/M_\lambda \to M/(M_\lambda + N) \to 0$$

and the fact that taking limits commutes with limits. The second statement follows from this and the fact that $N \to N^\wedge$ has dense image and that the kernel of $M^\wedge \to (M/N)^\wedge$ is closed. □

0AMT Lemma 87.4.4. Let R be a topological ring. Let M be a complete, linearly topologized R-module. Let $N \subset M$ be a closed submodule. If M has a countable fundamental system of neighbourhoods of 0, then M/N is complete and the map $M \to M/N$ is open.

Proof. Let M_n, $n \in \mathbf{N}$ be a fundamental system of open submodules of M. We may assume $M_{n+1} \subset M_n$ for all n. The system $(M_n + N)/N$ is a fundamental system in M/N. Hence we have to show that $M/N = \lim M/(M_n + N)$. Consider the short exact sequences

$$0 \to N/N \cap M_n \to M/M_n \to M/(M_n + N) \to 0$$

Since the transition maps of the system $\{N/N \cap M_n\}$ are surjective we see that $M = \lim M/M_n$ (by completeness of M) surjects onto $\lim M/(M_n+N)$ by Algebra, Lemma 10.86.4. As N is closed we see that the kernel of $M \to \lim M/(M_n + N)$ is N (see Lemma 87.4.2). Finally, $M \to M/N$ is open by definition of the quotient topology. □

0AS0 Lemma 87.4.5. Let R be a topological ring. Let M be a linearly topologized R-module. Let $N \subset M$ be a submodule. Assume M has a countable fundamental system of neighbourhoods of 0. Then [Mat86, Theorem 8.1]

(1) $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge \to 0$ is exact,
(2) $N^\wedge$ is the closure of the image of $N \to M^\wedge$,
(3) $M^\wedge \to (M/N)^\wedge$ is open.

Proof. We have $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge$ is exact and statement (2) by Lemma 87.4.3. This produces a canonical map $c : M^\wedge/N^\wedge \to (M/N)^\wedge$. The module $M^\wedge/N^\wedge$ is complete and $M^\wedge \to M^\wedge/N^\wedge$ is open by Lemma 87.4.4. By the universal property of completion we obtain a canonical map $b : (M/N)^\wedge \to M^\wedge/N^\wedge$. Then b and c are mutually inverse as they are on a dense subset. □

0F1S Lemma 87.4.6. Let R be a topological ring. Let M be a topological R-module. Let $I \subset R$ be a finitely generated ideal. Assume M has an open submodule whose topology is I-adic. Then $M^\wedge$ has an open submodule whose topology is I-adic and we have $M^\wedge/I^nM^\wedge = M/I^nM$ for all $n \geq 1$.

Proof. Let $M' \subset M$ be an open submodule whose topology is I-adic. Then $\{I^nM'\}_{n\geq 1}$ is a fundamental system of open submodules of M. Thus $M^\wedge = \lim M/I^nM'$ contains $(M')^\wedge = \lim M'/I^nM'$ as an open submodule and the topology on $(M')^\wedge$ is I-adic by Algebra, Lemma 10.96.3. Since I is finitely generated, I^n is finitely generated, say by $f_1, \ldots, f_r$. Observe that the surjection $(f_1, \ldots, f_r) :$

$M^{\oplus r} \to I^n M$ is continuous and open by our description of the topology on M above. By Lemma 87.4.5 applied to this surjection and to the short exact sequence $0 \to I^n M \to M \to M/I^n M \to 0$ we conclude that

$$(f_1, \ldots, f_r) : (M^\wedge)^{\oplus r} \longrightarrow M^\wedge$$

surjects onto the kernel of the surjection $M^\wedge \to M/I^n M$. Since $f_1, \ldots, f_r$ generate I^n we conclude. □

0AMU Definition 87.4.7. Let R be a topological ring. Let M and N be linearly topologized R-modules. The tensor product of M and N is the (usual) tensor product $M \otimes_R N$ endowed with the linear topology defined by declaring

$$\mathrm{Im}(M_\mu \otimes_R N + M \otimes_R N_\nu \longrightarrow M \otimes_R N)$$

to be a fundamental system of open submodules, where $M_\mu \subset M$ and $N_\nu \subset N$ run through fundamental systems of open submodules in M and N. The completed tensor product

$$M \widehat{\otimes}_R N = \lim M \otimes_R N/(M_\mu \otimes_R N + M \otimes_R N_\nu) = \lim M/M_\mu \otimes_R N/N_\nu$$

is the completion of the tensor product.

Observe that the topology on R is immaterial for the construction of the tensor product or the completed tensor product. If $R \to A$ and $R \to B$ are continuous maps of linearly topologized rings, then the construction above gives a tensor product $A \otimes_R B$ and a completed tensor product $A \widehat{\otimes}_R B$.

We record here the notions introduced in Remark 87.2.3.

0AMV Definition 87.4.8. Let A be a linearly topologized ring.

(1) An element $f \in A$ is called topologically nilpotent if $f^n \to 0$ as $n \to \infty$.
(2) A weak ideal of definition for A is an open ideal $I \subset A$ consisting entirely of topologically nilpotent elements.
(3) We say A is weakly pre-admissible if A has a weak ideal of definition.
(4) We say A is weakly admissible if A is weakly pre-admissible and complete[1].

Given a weak ideal of definition I in a linearly topologized ring A and an open ideal J the intersection $I \cap J$ is a weak ideal of definition. Hence if there is one weak ideal of definition, then there is a fundamental system of open ideals consisting of weak ideals of definition. In particular, given a weakly admissible topological ring A then $A = \lim A/I_\lambda$ where $\{I_\lambda\}$ is a fundamental system of weak ideals of definition.

0DCZ Lemma 87.4.9. Let A be a weakly admissible topological ring. Let $I \subset A$ be a weak ideal of definition. Then (A, I) is a henselian pair.

Proof. Let $A \to A'$ be an étale ring map and let $\sigma : A' \to A/I$ be an A-algebra map. By More on Algebra, Lemma 15.11.6 it suffices to lift σ to an A-algebra map $A' \to A$. To do this, as A is complete, it suffices to find, for every open ideal $J \subset I$, a unique A-algebra map $A' \to A/J$ lifting σ. Since I is a weak ideal of definition, the ideal I/J is locally nilpotent. We conclude by More on Algebra, Lemma 15.11.2. □

[1]By our conventions this includes separated.

0AMW Lemma 87.4.10. Let B be a linearly topologized ring. The set of topologically nilpotent elements of B is a closed, radical ideal of B. Let $\varphi : A \to B$ be a continuous map of linearly topologized rings.

(1) If $f \in A$ is topologically nilpotent, then $\varphi(f)$ is topologically nilpotent.
(2) If $I \subset A$ consists of topologically nilpotent elements, then the closure of $\varphi(I)B$ consists of topologically nilpotent elements.

Proof. Let $\mathfrak b \subset B$ be the set of topologically nilpotent elements. We omit the proof of the fact that $\mathfrak b$ is a radical ideal (good exercise in the definitions). Let g be an element of the closure of $\mathfrak b$. Our goal is to show that g is topologically nilpotent. Let $J \subset B$ be an open ideal. We have to show $g^e \in J$ for some $e \geq 1$. We have $g \in \mathfrak b + J$ by Lemma 87.4.2. Hence $g = f + h$ for some $f \in \mathfrak b$ and $h \in J$. Pick $m \geq 1$ such that $f^m \in J$. Then $g^{m+1} \in J$ as desired.

Let $\varphi : A \to B$ be as in the statement of the lemma. Assertion (1) is clear and assertion (2) follows from this and the fact that $\mathfrak b$ is a closed ideal. □

0AMZ Lemma 87.4.11. Let $A \to B$ be a continuous map of linearly topologized rings. Let $I \subset A$ be an ideal. The closure of IB is the kernel of $B \to B\widehat{\otimes}_A A/I$.

Proof. Let J_μ be a fundamental system of open ideals of B. The closure of IB is $\bigcap(IB + J_\lambda)$ by Lemma 87.4.2. Let I_μ be a fundamental system of open ideals in A. Then

$$B\widehat{\otimes}_A A/I = \lim (B/J_\lambda \otimes_A A/(I_\mu + I)) = \lim B/(J_\lambda + I_\mu B + IB)$$

Since $A \to B$ is continuous, for every λ there is a μ such that $I_\mu B \subset J_\lambda$, see discussion in Example 87.4.1. Hence the limit can be written as $\lim B/(J_\lambda + IB)$ and the result is clear. □

0GB4 Lemma 87.4.12. Let $B \to A$ and $B \to C$ be continuous homomorphisms of linearly topologized rings.

(1) If A and C are weakly pre-admissible, then $A\widehat{\otimes}_B C$ is weakly admissible.
(2) If A and C are pre-admissible, then $A\widehat{\otimes}_B C$ is admissible.
(3) If A and C have a countable fundamental system of open ideals, then $A\widehat{\otimes}_B C$ has a countable fundamental system of open ideals.
(4) If A and C are pre-adic and have finitely generated ideals of definition, then $A\widehat{\otimes}_B C$ is adic and has a finitely generated ideal of definition.
(5) If A and C are pre-adic Noetherian rings and $B/\mathfrak b \to A/\mathfrak a$ is of finite type where $\mathfrak a \subset A$ and $\mathfrak b \subset B$ are the ideals of topologically nilpotent elements, then $A\widehat{\otimes}_B C$ is adic Noetherian.

Proof. Let $I_\lambda \subset A$, $\lambda \in \Lambda$ and $J_\mu \subset C$, $\mu \in M$ be fundamental systems of open ideals, then by definition

$$A\widehat{\otimes}_B C = \lim_{\lambda,\mu} A/I_\lambda \otimes_B C/J_\mu$$

with the limit topology. Thus a fundamental system of open ideals is given by the kernels $K_{\lambda,\mu}$ of the maps $A\widehat{\otimes}_B C \to A/I_\lambda \otimes_B C/J_\mu$. Note that $K_{\lambda,\mu}$ is the closure of the ideal $I_\lambda(A\widehat{\otimes}_B C)+J_\mu(A\widehat{\otimes}_B C)$. Finally, we have a ring homomorphism $\tau : A \otimes_B C \to A\widehat{\otimes}_B C$ with dense image.

Proof of (1). If I_λ and J_μ consist of topologically nilpotent elements, then so does $K_{\lambda,\mu}$ by Lemma 87.4.10. Hence $A\widehat{\otimes}_B C$ is weakly admissible by definition.

Proof of (2). Assume for some λ_0 and μ_0 the ideals $I = I_{\lambda_0} \subset A$ and $J_{\mu_0} \subset C$ are ideals of definition. Thus for every λ there exists an n such that $I^n \subset I_\lambda$. For every μ there exists an m such that $J^m \subset J_\mu$. Then

$$\left(I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)\right)^{n+m} \subset I_\lambda(A\widehat{\otimes}_B C) + J_\mu(A\widehat{\otimes}_B C)$$

It follows that the open ideal $K = K_{\lambda_0,\mu_0}$ satisfies $K^{n+m} \subset K_{\lambda,\mu}$. Hence K is an ideal of definition of $A\widehat{\otimes}_B C$ and $A\widehat{\otimes}_B C$ is admissible by definition.

Proof of (3). If Λ and M are countable, so is $\Lambda \times M$.

Proof of (4). Assume $\Lambda = \mathbf{N}$ and $M = \mathbf{N}$ and we have finitely generated ideals $I \subset A$ and $J \subset C$ such that $I_n = I^n$ and $J_n = J^n$. Then

$$I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)$$

is a finitely generated ideal and it is easily seen that $A\widehat{\otimes}_B C$ is the completion of $A \otimes_B C$ with respect to this ideal. Hence (4) follows from Algebra, Lemma 10.96.3.

Proof of (5). Let $\mathfrak c \subset C$ be the ideal of topologically nilpotent elements. Since A and C are adic Noetherian, we see that $\mathfrak a$ and $\mathfrak c$ are ideals of definition (details omitted). From part (4) we already know that $A\widehat{\otimes}_B C$ is adic and that $\mathfrak a(A\widehat{\otimes}_B C) + \mathfrak c(A\widehat{\otimes}_B C)$ is a finitely generated ideal of definition. Since

$$A\widehat{\otimes}_B C/\left(\mathfrak a(A\widehat{\otimes}_B C) + \mathfrak c(A\widehat{\otimes}_B C)\right) = A/\mathfrak a \otimes_{B/\mathfrak b} C/\mathfrak c$$

is Noetherian as a finite type algebra over the Noetherian ring $C/\mathfrak c$ we conclude by Algebra, Lemma 10.97.5. $\square$

87.5. Taut ring maps

0GX1 It turns out to be convenient to have a name for the following property of continuous maps between linearly topologized rings.

0AMX Definition 87.5.1. Let $\varphi : A \to B$ be a continuous map of linearly topologized rings. We say φ is taut[2] if for every open ideal $I \subset A$ the closure of the ideal $\varphi(I)B$ is open and these closures form a fundamental system of open ideals.

If $\varphi : A \to B$ is a continuous map of linearly topologized rings and I_λ a fundamental system of open ideals of A, then φ is taut if and only if the closures of $I_\lambda B$ are open and form a fundamental system of open ideals in A.

0AMY Lemma 87.5.2. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings. The following are equivalent

(1) φ is taut,
(2) for every weak ideal of definition $I \subset A$ the closure of $\varphi(I)B$ is a weak ideal of definition of B and these form a fundamental system of weak ideals of definition of B.

Proof. The remarks following Definition 87.5.1 show that (2) implies (1). Conversely, assume φ is taut. If $I \subset A$ is a weak ideal of definition, then the closure of $\varphi(I)B$ is open by definition of tautness and consists of topologically nilpotent elements by Lemma 87.4.10. Hence the closure of $\varphi(I)B$ is a weak ideal of definition.

[2]This is nonstandard notation. The definition generalizes to modules, by saying a linearly topologized A-module M is A-taut if for every open ideal $I \subset A$ the closure of IM in M is open and these closures form a fundamental system of neighbourhoods of 0 in M.

Furthermore, by definition of tautness these ideals form a fundamental system of open ideals and we see that (2) is true. □

0GX2 Lemma 87.5.3. Let A be a linearly topologized ring. The map $A \to A^\wedge$ from A to its completion is taut.

Proof. Let I_λ be a fundamental system of open ideals of A. Recall that $A^\wedge = \lim A/I_\lambda$ with the limit topology, which means that the kernels $J_\lambda = \mathrm{Ker}(A^\wedge \to A/I_\lambda)$ form a fundamental system of open ideals of $A^\wedge$. Since J_λ is the closure of $I_\lambda A^\wedge$ (compare with Lemma 87.4.11) we conclude. □

0GX3 Lemma 87.5.4. Let $A \to B$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to B$ and $B \to C$ are taut, then $A \to C$ is taut.

Proof. Omitted. Hint: if $I \subset A$ is an ideal and J is the closure of IB, then the closure of JC is equal to the closure of IC. □

0GX4 Lemma 87.5.5. Let $A \to B$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to C$ is taut, then $B \to C$ is taut.

Proof. Let $J \subset B$ be an open ideal with inverse image $I \subset A$. Then the closure of JC contains the closure of IC. Hence this closure is open as $A \to C$ is taut. Let I_λ be a fundamental system of open ideals of A. Let K_λ be the closure of $I_\lambda C$. Since $A \to C$ is taut, these form a fundamental system of open ideals of C. Denote $J_\lambda \subset B$ the inverse image of K_λ. Then the closure of $J_\lambda C$ is K_λ. Hence we see that the closures of the ideals JC, where J runs over the open ideals of B form a fundamental system of open ideals of C. □

0GX5 Lemma 87.5.6. Let $A \to B$ and $A \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to B$ is taut, then $C \to B \widehat{\otimes}_A C$ is taut.

Proof. Let $K \subset C$ be an open ideal. Choose any open ideal $I \subset A$ whose image in C is contained in J. By assumption the closure J of IB is open. Since $A \to B$ is taut we see that $B \widehat{\otimes}_A C$ is the limit of the rings $B/J \otimes_{A/I} C/K$ over all choices of K and I, i.e, the ideals $J(B \widehat{\otimes}_A C) + K(B \widehat{\otimes}_A C)$ form a fundamental system of open ideals. Now, since $B \to B \widehat{\otimes}_A C$ is continuous we see that J maps into the closure of $K(B \widehat{\otimes}_A C)$ (as I maps into K). Hence this closure is equal to $J(B \widehat{\otimes}_A C) + K(B \widehat{\otimes}_A C)$ and the proof is complete. □

0GX6 Lemma 87.5.7. Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A has a countable fundamental system of open ideals, then B has a countable fundamental system of open ideals.

Proof. Immediate from the definitions. □

0GX7 Lemma 87.5.8. Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is weakly pre-admissible, then B is weakly pre-admissible.

Proof. Let $I \subset A$ be a weak ideal of definition. Then the closure J of IB is open and consists of topologically nilpotent elements by Lemma 87.4.10. Hence J is a weak ideal of definition of B. □

0GX8 Lemma 87.5.9. Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is pre-admissible, then B is pre-admissible.

Proof. Let $I \subset A$ be an ideal of definition. Let $I_\lambda \subset A$ be a fundamental system of open ideals. Then the closure J of IB is open and the closures J_λ of $I_\lambda B$ are open and form a fundamental system of open ideals of B. For every λ there is an n such that $I^n \subset I_\lambda$. Observe that J^n is contained in the closure of $I^n B$. Thus $J^n \subset J_\lambda$ and we conclude J is an ideal of definition. $\square$

0APT Lemma 87.5.10. Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. Assume

(1) φ is taut and has dense image,
(2) A is complete and has a countable fundamental system of open ideals, and
(3) B is separated.

Then φ is surjective and open, B is complete, and $B = A/K$ for some closed ideal $K \subset A$.

Proof. By the open mapping lemma (More on Algebra, Lemma 15.36.5) combined with tautness of φ, we see the map φ is open. Since the image of φ is dense, we see that φ is surjective. The kernel K of φ is closed as φ is continuous. It follows that $B = A/K$ is complete, see for example Lemma 87.4.4. $\square$

87.6. Adic ring maps

0GX9 Let us make the following definition.

0GBR Definition 87.6.1. Let A and B be pre-adic topological rings. A ring homomorphism $\varphi : A \to B$ is adic[3] if there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology.

If $\varphi : A \to B$ is an adic homomorphism of pre-adic rings, then φ is continuous and the topology on B is the I-adic topology for every ideal of definition I of A.

0GXA Lemma 87.6.2. Let $A \to B$ and $B \to C$ be continuous homomorphisms of pre-adic rings. If $A \to B$ and $B \to C$ are adic, then $A \to C$ is adic.

Proof. Omitted. $\square$

0GXB Lemma 87.6.3. Let $A \to B$ and $B \to C$ be continuous homomorphisms of pre-adic rings. If $A \to C$ is adic, then $B \to C$ is adic.

Proof. Choose an ideal of definition I of A. As $A \to C$ is adic, we see that IC is an ideal of definition of C. As $B \to C$ is continuous, we can find an ideal of definition $J \subset B$ mapping into IC. As $A \to B$ is continuous the inverse image $I' \subset I$ of J in I is an ideal of definition of A too. Hence $I'C \subset JC \subset IC$ is sandwiched between two ideals of definition, hence is an ideal of definition itself. $\square$

0GXC Lemma 87.6.4. Let $\varphi : A \to B$ be a continuous homomorphism between pre-adic topological rings. If φ is adic, then φ is taut.

Proof. Immediate from the definitions. $\square$

The next lemma says two things

(1) the property of being adic ascents along taut maps of complete linearly topologized rings, and

[3] This may be nonstandard terminology.

(2) the properties "φ is taut" and "φ is adic" are equivalent for continuous maps $\varphi : A \to B$ between adic rings if A has a finitely generated ideal of definition.

Because of (2) we can say that "tautness" generalizes "adicness" to continuous ring maps between arbitrary linearly topologized rings. See also Section 87.23.

0APU Lemma 87.6.5. Let $\varphi : A \to B$ be a continuous map of linearly topologized rings. If φ is taut, A is pre-adic and has a finitely generated ideal of definition, and B is complete, then B is adic and has a finitely generated ideal of definition and the ring map φ is adic.

Proof. Choose a finitely generated ideal of definition I of A. Let J_n be the closure of $\varphi(I^n)B$ in B. Since B is complete we have $B = \lim B/J_n$. Let $B' = \lim B/I^nB$ be the I-adic completion of B. By Algebra, Lemma 10.96.3, the I-adic topology on B' is complete and $B'/I^nB' = B/I^nB$. Thus the ring map $B' \to B$ is continuous and has dense image as $B' \to B/I^nB \to B/J_n$ is surjective for all n. Finally, the map $B' \to B$ is taut because $(I^nB')B = I^nB$ and $A \to B$ is taut. By Lemma 87.5.10 we see that $B' \to B$ is open and surjective. Thus the topology on B is the I-adic topology and the proof is complete. $\square$

87.7. Weakly adic rings

0GXD We suggest the reader skip this section. The following is a natural generalization of adic rings.

0GXE Definition 87.7.1. Let A be a linearly topologized ring. [GR04, Definition 8.3.8]

(1) We say A is weakly pre-adic[4] if there exists an ideal $I \subset A$ such that the closure of I^n is open for all $n \geq 0$ and these closures form a fundamental system of open ideals.
(2) We say A is weakly adic if A is weakly pre-adic and complete[5].

For complete linearly topologized rings we have the following implications

$$
\begin{tikzcd}
\text{adic + Noetherian} \arrow[d, Rightarrow] & \\
\text{adic + finitely generated ideal of definition} \arrow[d, Rightarrow] & \\
\text{adic} \arrow[d, Rightarrow] & \\
\text{weakly adic} \arrow[d, Rightarrow] & \\
\text{admissible + first countable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{admissible} \arrow[d, Rightarrow] \\
\text{weakly admissible + first countable} \arrow[r, Rightarrow] & \text{weakly admissible}
\end{tikzcd}
$$

[4] In [GR04] the authors say A is c-adic.

[5] By our conventions this includes separated.

where "first countable" means that our topological ring has a countable fundamental system of open ideals. There is a similar diagram of implications for noncomplete linearly topologized rings (i.e., using the notions of pre-adic, weakly pre-adic, pre-admissible, and weakly pre-admissible). Contrary to what happens with pre-adic rings the completion of a weakly pre-adic ring is weakly adic as the following lemma characterizing weakly pre-adic rings shows.

0GXF Lemma 87.7.2. Let A be a linearly topologized ring. The following are equivalent

(1) A is weakly pre-adic,
(2) there exists a taut continuous ring map $A' \to A$ where A' is a pre-adic topological ring, and
(3) A is pre-admissible and there exists an ideal of definition I such that the closure of I^n is open for all $n \geq 1$, and
(4) A is pre-admissible and for every ideal of definition I the closure of I^n is open for all $n \geq 1$.

The completion of a weakly pre-adic ring is weakly adic. If A is weakly adic, then A is admissible and has a countable fundamental system of open ideals.

Proof. Assume (1). Choose an ideal I such that the closure of I^n is open for all n and such that these closures form a fundamental system of open ideals. Denote $A' = A$ endowed with the I-adic topology. Then $A' \to A$ is taut by definition and we see that (2) holds.

Assume (2). Let $I' \subset A'$ be an ideal of definition. Denote I the closure of $I'A$. Tautness of $A' \to A$ means that the closures I_n of $(I')^nA$ are open and form a fundamental system of open ideals. Thus $I = I_1$ is open and the closures of I^n are equal to I_n and hence open and form a fundamental system of open ideals. Thus certainly I is an ideal of definition such that the closure of I^n is open for all n. Hence (3) holds.

If $I \subset A$ is as in (3), then I is an ideal as in Definition 87.7.1 and we see that (1) holds. Also, if $I' \subset A$ is any other ideal of definition, then I' is open (see More on Algebra, Definition 15.36.1) and hence contains I^n for some $n \geq 1$. Thus $(I')^m$ contains I^{nm} for all $m \geq 1$ and we conclude that the closures of $(I')^m$ are open for all m. In this way we see that (3) implies (4). The implication (4) $\Rightarrow$ (3) is trivial.

Let A be weakly pre-adic. Choose $A' \to A$ as in (2). By Lemmas 87.5.3 and 87.5.4 the composition $A' \to A^\wedge$ is taut. Hence $A^\wedge$ is weakly pre-adic by the equivalence of (2) and (1). Since the completion of a linearly topologized ring A is complete (More on Algebra, Section 15.36) we see that $A^\wedge$ is weakly adic.

Let A be weakly adic. Then A is complete and and pre-admissible by (1) $\Rightarrow$ (3) and hence A is admissible. Of course by definition A has a countable fundamental system of open ideals. □

We give two criteria that guarantee that a weakly adic ring is adic and has a finitely generated ideal of definition.

0GXG Lemma 87.7.3. Let A be a complete linearly topologized ring. Let $I \subset A$ be a finitely generated ideal such that the closure of I^n is open for all $n \geq 0$ and these closures form a fundamental system of open ideals. Then A is adic and has a finitely generated ideal of definition.

Proof. Denote A' the ring A endowed with the I-adic topology. The assumptions tells us that $A' \to A$ is taut. We conclude by Lemma 87.6.5 (to be sure, this lemma also tells us that I is an ideal of definition). □

0GXH Lemma 87.7.4. Let A be a weakly adic topological ring. Let I be an ideal of definition such that I/I_2 is a finitely generated module where I_2 is the closure of I^2. Then A is adic and has a finitely generated ideal of definition.

Proof. We use the characterization of Lemma 87.7.2 without further mention. Choose $f_1, \ldots, f_r \in I$ which map to generators of I/I_2. Set $I' = (f_1, \ldots, f_r)$. We have $I' + I_2 = I$. Then I_2 is the closure of $I^2 = (I' + I_2)^2 \subset I' + I_3$ where I_3 is the closure of I^3. Hence $I' + I_3 = I$. Continuing in this fashion we see that $I' + I_n = I$ for all $n \geq 2$ where I_n is the closure of I^n. In other words, the closure of I' in A is I. Hence the closure of $(I')^n$ is I_n. Thus the closures of $(I')^n$ are a fundamental system of open ideals of A. We conclude by Lemma 87.7.3. □

A key feature of the property "weakly pre-adic" is that it ascents along taut ring homomorphisms of linearly topologized rings.

0GXI Lemma 87.7.5. Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is weakly pre-adic, then B is weakly pre-adic.

Proof. Let $I \subset A$ be an ideal such that the closure I_n of I^n is open and these closures define a fundamental system of open ideals. Then the closure of I^nB is equal to the closure of I_nB. Since φ is taut, these closures are open and form a fundamental system of open ideals of B. Hence B is weakly pre-adic. □

0GXJ Lemma 87.7.6. Let $B \to A$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If A and C are weakly pre-adic, then $A \widehat{\otimes}_B C$ is weakly adic.

Proof. We will use the characterization of Lemma 87.7.2 without further mention. By Lemma 87.4.12 we know that $A \widehat{\otimes}_B C$ is admissible. Moreover, the proof of that lemma shows that the closure $K \subset A \widehat{\otimes}_B C$ is an ideal of definition, when $I \subset A$ and $J \subset C$ of $I(A \widehat{\otimes}_B C) + J(A \widehat{\otimes}_B C)$ are ideals of definition. Then it suffices to show that the closure of K^n is open for all $n \geq 1$. Since the ideal K^n contains $I^n(A \widehat{\otimes}_B C) + J^n(A \widehat{\otimes}_B C)$, since the closure of I^n in A is open, and since the closure of J^n in C is open, we see that the closure of K^n is open in $A \widehat{\otimes}_B C$. □

87.8. Descending properties

0GXK In this section we consider the following situation

(1) $\varphi : A \to B$ is a continuous map of linearly topologized topological rings,
(2) φ is taut, and
(3) for every open ideal $I \subset A$ if $J \subset B$ denotes the closure of IB, then the map $A/I \to B/J$ is faithfully flat.

We are going to show that properties of B are inherited by A in this situation.

0GXL Lemma 87.8.1. In the situation above, if B has a countable fundamental system of open ideals, then A has a countable fundamental system of open ideals.

Proof. Choose a fundamental system $B \supset J_1 \supset J_2 \supset \ldots$ of open ideals. By tautness of φ, for every n we can find an open ideal I_n such that $J_n \supset I_nB$. We claim that I_n is a fundamental system of open ideals of A. Namely, suppose that

$I \subset A$ is open. As φ is taut, the closure of IB is open and hence contains J_n for some n large enough. Hence $I_nB \subset IB$. Let J be the closure of IB in B. Since $A/I \to B/J$ is faithfully flat, it is injective. Hence, since $I_n \to A/I \to B/J$ is zero as $I_nB \subset IB \subset J$, we conclude that $I_n \to A/I$ is zero. Hence $I_n \subset I$ and we win. ☐

0GXM Lemma 87.8.2. In the situation above, if B is weakly pre-admissible, then A is weakly pre-admissible.

Proof. Let $J \subset B$ be a weak ideal of definition. Let $I \subset A$ be an open ideal such that $IB \subset J$. To show that I is a weak ideal of definition we have to show that any $f \in I$ is topologically nilpotent. Let $I' \subset A$ be an open ideal. Denote $J' \subset B$ the closure of $I'B$. Then $A/I' \to B/J'$ is faithfully flat, hence injective. Thus in order to show that $f^n \in I'$ it suffices to show that $\varphi(f)^n \in J'$. This holds for $n \gg 0$ since $\varphi(f) \in J$, the ideal J is a weak ideal of defintion of B, and J' is open in B. ☐

0GXN Lemma 87.8.3. In the situation above, if B is pre-admissible, then A is pre-admissible.

Proof. Let $J \subset B$ be a weak ideal of definition. Let $I \subset A$ be an open ideal such that $IB \subset J$. Let $I' \subset A$ be an open ideal. To show that I is an ideal of definition we have to show that $I^n \subset I'$ for $n \gg 0$. Denote $J' \subset B$ the closure of $I'B$. Then $A/I' \to B/J'$ is faithfully flat, hence injective. Thus in order to show that $I^n \subset I'$ it suffices to show that $\varphi(I)^n \subset J'$. This holds for $n \gg 0$ since $\varphi(I) \subset J$, the ideal J is an ideal of defintion of B, and J' is open in B. ☐

0GXP Lemma 87.8.4. In the situation above, if B is weakly pre-adic, then A is weakly pre-adic.

Proof. We will use the characterization of weakly pre-adic rings given in Lemma 87.7.2 without further mention. By Lemma 87.8.3 the topological ring A is pre-admissible. Let $I \subset A$ be an ideal of definition. Fix $n \geq 1$. To prove the lemma we have to show that the closure of I^n is open. Let $I_\lambda \subset A$ be a fundamental system of open ideals. Denote $J \subset B$, resp. $J_\lambda \subset B$ the closure of IB, resp. $I_\lambda B$. Since B is weakly pre-adic, the closure of J^n is open. Hence there exists a λ such that

$$J_\lambda \subset \bigcap\nolimits_\mu (J^n + J_\mu)$$

because the right hand side is the closure of J^n by Lemma 87.4.2. This means that the image of J_λ in B/J_μ is contained in the image of J^n in B/J_μ. Observe that the image of J^n in B/J_μ is equal to the image of I^nB in B/J_μ (since every element of J is congruent to an element of IB modulo J_μ). Since $A/I_\mu \to B/J_\mu$ is faithfully flat and since $I_\lambda B \subset J_\lambda$, we conclude that the image of I_λ in A/I_μ is contained in the image of I^n. We conclude that I_λ is contained in the closure of I^n and the proof is complete. ☐

0GXQ Lemma 87.8.5. In the situation above, if B is adic and has a finitely generated ideal of definition and A is complete, then A is adic and has a finitely generated ideal of definition.

Proof. We already know that A is weakly adic and a fortiori admissible by Lemma 87.8.4 (and Lemma 87.7.2 to see that adic rings are weakly adic). Let $I \subset A$ be an ideal of definition. Let $J \subset B$ be a finitely generated ideal of definition. Since the closure of IB is open, we can find an $n > 0$ such that J^n is contained in the closure of IB. Thus after replacing J by J^n we may assume J is a finitely generated ideal of definition contained in the closure of IB. By Lemma 87.4.2 this certainly implies that

$$J \subset IB + J^2$$

Consider the finitely generated A-module $M = (J + IB)/IB$. The displayed equation shows that $JM = M$. By Lemma 87.4.9 (for example) we see that J is contained in the Jacobson radical of B. Hence by Nakayama's lemma, more precisely part (2) of Algebra, Lemma 10.20.1, we conclude $M = 0$. Thus $J \subset IB$.

Since J is finitely generated, we can find a finitely generated ideal $I' \subset I$ such that $J \subset I'B$. Since $A \to B$ is continuous, $J \subset B$ is open, and I is an ideal of definition, we can find an $n > 0$ such that $I^nB \subset J$. Let $J_{n+1} \subset B$ be the closure of $I^{n+1}B$. We have

$$I^n \cdot (B/J_{n+1}) \subset J \cdot (B/J_{n+1}) \subset I' \cdot (B/J_{n+1})$$

Since $A/I^{n+1} \to B/J_{n+1}$ is faithfully flat, this implies $I^n \cdot (A/I^{n+1}) \subset I' \cdot (A/I^{n+1})$ which in turn means

$$I^n \subset I' + I^{n+1}$$

This implies $I^n \subset I' + I^{n+k}$ for all $k \geq 1$ which in turn implies that $I^{nm} \subset (I')^m + I^{nm+k}$ for all $k, m \geq 1$. This implies that the closure of $(I')^m$ contains I^{nm}. Since the closure of I^{nm} is open as A is weakly adic, we conclude that the closure $(I')^m$ is open for all m. Since these closures form a fundamental system of open ideals of A (as the same thing is true for the closures of I^n) we conclude by Lemma 87.7.3. □

87.9. Affine formal algebraic spaces

0AI6 In this section we introduce affine formal algebraic spaces. These will in fact be the same as what are called affine formal schemes in [BD]. However, we will call them affine formal algebraic spaces, in order to prevent confusion with the notion of an affine formal scheme as defined in [DG67].

Recall that a thickening of schemes is a closed immersion which induces a surjection on underlying topological spaces, see More on Morphisms, Definition 37.2.1.

0AI7 Definition 87.9.1. Let S be a scheme. We say a sheaf X on $(Sch/S)_{fppf}$ is an affine formal algebraic space if there exist

(1) a directed set Λ,
(2) a system $(X_\lambda, f_{\lambda\mu})$ over Λ in $(Sch/S)_{fppf}$ where
    (a) each X_λ is affine,
    (b) each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening,

such that

$$X \cong \operatorname{colim}_{\lambda \in \Lambda} X_\lambda$$

as fppf sheaves and X satisfies a set theoretic condition (see Remark 87.11.5). A morphism of affine formal algebraic spaces over S is a map of sheaves.

Observe that the system $(X_\lambda, f_{\lambda\mu})$ is not part of the data. Suppose that U is a quasi-compact scheme over S. Since the transition maps are monomorphisms, we see that

$$X(U) = \operatorname{colim} X_\lambda(U)$$

by Sites, Lemma 7.17.7. Thus the fppf sheafification inherent in the colimit of the definition is a Zariski sheafification which does not do anything for quasi-compact schemes.

0AI8 Lemma 87.9.2. Let S be a scheme. If X is an affine formal algebraic space over S, then the diagonal morphism $\Delta : X \to X \times_S X$ is representable and a closed immersion.

Proof. Suppose given $U \to X$ and $V \to X$ where U, V are schemes over S. Let us show that $U \times_X V$ is representable. Write $X = \text{colim}\, X_\lambda$ as in Definition 87.9.1. The discussion above shows that Zariski locally on U and V the morphisms factors through some X_λ. In this case $U \times_X V = U \times_{X_\lambda} V$ which is a scheme. Thus the diagonal is representable, see Spaces, Lemma 65.5.10. Given $(a, b) : W \to X \times_S X$ where W is a scheme over S consider the map $X \times_{\Delta, X \times_S X, (a,b)} W \to W$. As before locally on W the morphisms a and b map into the affine scheme X_λ for some λ and then we get the morphism $X_\lambda \times_{\Delta_\lambda, X_\lambda \times_S X_\lambda, (a,b)} W \to W$. This is the base change of $\Delta_\lambda : X_\lambda \to X_\lambda \times_S X_\lambda$ which is a closed immersion as $X_\lambda \to S$ is separated (because X_λ is affine). Thus $X \to X \times_S X$ is a closed immersion. □

A morphism of schemes $X \to X'$ is a thickening if it is a closed immersion and induces a surjection on underlying sets of points, see (More on Morphisms, Definition 37.2.1). Hence the property of being a thickening is preserved under arbitrary base change and fpqc local on the target, see Spaces, Section 65.4. Thus Spaces, Definition 65.5.1 applies to "thickening" and we know what it means for a representable transformation $F \to G$ of presheaves on $(Sch/S)_{fppf}$ to be a thickening. We observe that this does not clash with our definition (More on Morphisms of Spaces, Definition 76.9.1) of thickenings in case F and G are algebraic spaces.

0AI9 Lemma 87.9.3. Let $X_\lambda, \lambda \in \Lambda$ and $X = \text{colim}\, X_\lambda$ be as in Definition 87.9.1. Then $X_\lambda \to X$ is representable and a thickening.

Proof. The statement makes sense by the discussion in Spaces, Section 65.3 and 65.5. By Lemma 87.9.2 the morphisms $X_\lambda \to X$ are representable. Given $U \to X$ where U is a scheme, then the discussion following Definition 87.9.1 shows that Zariski locally on U the morphism factors through some X_μ with $\lambda \leq \mu$. In this case $U \times_X X_\lambda = U \times_{X_\mu} X_\lambda$ so that $U \times_X X_\lambda \to U$ is a base change of the thickening $X_\lambda \to X_\mu$. □

0AIA Lemma 87.9.4. Let $X_\lambda, \lambda \in \Lambda$ and $X = \text{colim}\, X_\lambda$ be as in Definition 87.9.1. If Y is a quasi-compact algebraic space over S, then any morphism $Y \to X$ factors through an X_λ.

Proof. Choose an affine scheme V and a surjective étale morphism $V \to Y$. The composition $V \to Y \to X$ factors through X_λ for some λ by the discussion following Definition 87.9.1. Since $V \to Y$ is a surjection of sheaves, we conclude. □

0AIB Lemma 87.9.5. Let S be a scheme. Let X be a sheaf on $(Sch/S)_{fppf}$. Then X is an affine formal algebraic space if and only if the following hold

(1) any morphism $U \to X$ where U is an affine scheme over S factors through a morphism $T \to X$ which is representable and a thickening with T an affine scheme over S, and
(2) a set theoretic condition as in Remark 87.11.5.

Proof. It follows from Lemmas 87.9.3 and 87.9.4 that an affine formal algebraic space satisfies (1) and (2). In order to prove the converse we may assume X is not empty. Let Λ be the category of representable morphisms $T \to X$ which are thickenings where T is an affine scheme over S. This category is directed. Since X is not empty, Λ contains at least one object. If $T \to X$ and $T' \to X$ are in Λ, then we can factor $T \amalg T' \to X$ through $T'' \to X$ in Λ. Between any two objects of Λ there is a unique arrow or none. Thus Λ is a directed set and by assumption $X = \mathrm{colim}_{T \to X \text{ in } \Lambda} T$. To finish the proof we need to show that any arrow $T \to T'$ in Λ is a thickening. This is true because $T' \to X$ is a monomorphism of sheaves, so that $T = T \times_{T'} T' = T \times_X T'$ and hence the morphism $T \to T'$ equals the projection $T \times_X T' \to T'$ which is a thickening because $T \to X$ is a thickening. $\square$

For a general affine formal algebraic space X there is no guarantee that X has enough functions to separate points (for example). See Examples, Section 110.74. To characterize those that do we offer the following lemma.

0AIC Lemma 87.9.6. Let S be a scheme. Let X be an fppf sheaf on $(Sch/S)_{fppf}$ which satisfies the set theoretic condition of Remark 87.11.5. The following are equivalent:

(1) there exists a weakly admissible topological ring A over S (see Remark 87.2.3) such that $X = \mathrm{colim}_{I \subset A \text{ weak ideal of definition}} \mathrm{Spec}(A/I)$,
(2) X is an affine formal algebraic space and there exists an S-algebra A and a map $X \to \mathrm{Spec}(A)$ such that for a closed immersion $T \to X$ with T an affine scheme the composition $T \to \mathrm{Spec}(A)$ is a closed immersion,
(3) X is an affine formal algebraic space and there exists an S-algebra A and a map $X \to \mathrm{Spec}(A)$ such that for a closed immersion $T \to X$ with T a scheme the composition $T \to \mathrm{Spec}(A)$ is a closed immersion,
(4) X is an affine formal algebraic space and for some choice of $X = \mathrm{colim}\, X_\lambda$ as in Definition 87.9.1 the projections $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \to \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ are surjective,
(5) X is an affine formal algebraic space and for any choice of $X = \mathrm{colim}\, X_\lambda$ as in Definition 87.9.1 the projections $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \to \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ are surjective.

Moreover, the weakly admissible topological ring is $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ endowed with its limit topology and the weak ideals of definition classify exactly the morphisms $T \to X$ which are representable and thickenings.

Proof. It is clear that (5) implies (4).

Assume (4) for $X = \mathrm{colim}\, X_\lambda$ as in Definition 87.9.1. Set $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$. Let $T \to X$ be a closed immersion with T a scheme (note that $T \to X$ is representable by Lemma 87.9.2). Since $X_\lambda \to X$ is a thickening, so is $X_\lambda \times_X T \to T$. On the other hand, $X_\lambda \times_X T \to X_\lambda$ is a closed immersion, hence $X_\lambda \times_X T$ is affine. Hence T is affine by Limits, Proposition 32.11.2. Then $T \to X$ factors through X_λ for some λ by Lemma 87.9.4. Thus $A \to \Gamma(X_\lambda, \mathcal{O}) \to \Gamma(T, \mathcal{O})$ is surjective. In this way we see that (3) holds.

It is clear that (3) implies (2).

Assume (2) for A and $X \to \mathrm{Spec}(A)$. Write $X = \mathrm{colim}\, X_\lambda$ as in Definition 87.9.1. Then $A_\lambda = \Gamma(X_\lambda, \mathcal{O})$ is a quotient of A by assumption (2). Hence $A^\wedge = \lim A_\lambda$ is a complete topological ring, see discussion in More on Algebra, Section 15.36. The maps $A^\wedge \to A_\lambda$ are surjective as $A \to A_\lambda$ is. We claim that for any λ the kernel

$I_\lambda \subset A^\wedge$ of $A^\wedge \to A_\lambda$ is a weak ideal of definition. Namely, it is open by definition of the limit topology. If $f \in I_\lambda$, then for any $\mu \in \Lambda$ the image of f in A_μ is zero in all the residue fields of the points of X_μ. Hence it is a nilpotent element of A_μ. Hence some power $f^n \in I_\mu$. Thus $f^n \to 0$ as $n \to 0$. Thus $A^\wedge$ is weakly admissible. Finally, suppose that $I \subset A^\wedge$ is a weak ideal of definition. Then $I \subset A^\wedge$ is open and hence there exists some λ such that $I \supset I_\lambda$. Thus we obtain a morphism $\text{Spec}(A^\wedge/I) \to \text{Spec}(A_\lambda) \to X$. Then it follows that $X = \text{colim}\,\text{Spec}(A^\wedge/I)$ where now the colimit is over all weak ideals of definition. Thus (1) holds.

Assume (1). In this case it is clear that X is an affine formal algebraic space. Let $X = \text{colim}\, X_\lambda$ be any presentation as in Definition 87.9.1. For each λ we can find a weak ideal of definition $I \subset A$ such that $X_\lambda \to X$ factors through $\text{Spec}(A/I) \to X$, see Lemma 87.9.4. Then $X_\lambda = \text{Spec}(A/I_\lambda)$ with $I \subset I_\lambda$. Conversely, for any weak ideal of definition $I \subset A$ the morphism $\text{Spec}(A/I) \to X$ factors through X_λ for some λ, i.e., $I_\lambda \subset I$. It follows that each I_λ is a weak ideal of definition and that they form a cofinal subset of the set of weak ideals of definition. Hence $A = \lim A/I = \lim A/I_\lambda$ and we see that (5) is true and moreover that $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$. ☐

With this lemma in hand we can make the following definition.

0AID **Definition 87.9.7.** Let S be a scheme. Let X be an affine formal algebraic space over S. We say X is *McQuillan* if X satisfies the equivalent conditions of Lemma 87.9.6. Let A be the weakly admissible topological ring associated to X. We say

1. X is *classical* if X is McQuillan and A is admissible (More on Algebra, Definition 15.36.1),
2. X is *weakly adic* if X is McQuillan and A is weakly adic (Definition 87.7.1),
3. X is *adic* if X is McQuillan and A is adic (More on Algebra, Definition 15.36.1),
4. X is *adic** if X is McQuillan, A is adic, and A has a finitely generated ideal of definition, and
5. X is *Noetherian* if X is McQuillan and A is both Noetherian and adic.

In [FK] they use the terminology "of finite ideal type" for the property that an adic topological ring A contains a finitely generated ideal of definition. Given an affine formal algebraic space X here are the implications among the notions introduced in the definition:

$$\begin{tikzcd}
X \text{ Noetherian} \arrow[r, Rightarrow] & X \text{ adic*} \arrow[r, Rightarrow] & X \text{ adic} \arrow[dll, Rightarrow] \\
X \text{ weakly adic} \arrow[r, Rightarrow] & X \text{ classical} \arrow[r, Rightarrow] & X \text{ McQuillan}
\end{tikzcd}$$

See discussion in Section 87.7 and for a precise statement see Lemma 87.10.3.

0AIE **Remark 87.9.8.** The classical affine formal algebraic spaces correspond to the affine formal schemes considered in EGA ([DG67]). To explain this we assume our base scheme is $\text{Spec}(\mathbf{Z})$. Let $\mathfrak{X} = \text{Spf}(A)$ be an affine formal scheme. Let $h_\mathfrak{X}$ be its functor of points as in Lemma 87.2.1. Then $h_\mathfrak{X} = \text{colim}\, h_{\text{Spec}(A/I)}$ where the colimit is over the collection of ideals of definition of the admissible topological ring A. This follows from (87.2.0.1) when evaluating on affine schemes and it suffices to check on affine schemes as both sides are fppf sheaves, see Lemma 87.2.2. Thus $h_\mathfrak{X}$ is an affine formal algebraic space. In fact, it is a classical affine formal algebraic space by Definition 87.9.7. Thus Lemma 87.2.1 tells us the category of affine formal schemes is equivalent to the category of classical affine formal algebraic spaces.

Having made the connection with affine formal schemes above, it seems natural to make the following definition.

0AIF Definition 87.9.9. Let S be a scheme. Let A be a weakly admissible topological ring over S, see Definition 87.4.8[6]. The formal spectrum of A is the affine formal algebraic space

$$\mathrm{Spf}(A) = \mathrm{colim}\,\mathrm{Spec}(A/I)$$

where the colimit is over the set of weak ideals of definition of A and taken in the category $Sh((Sch/S)_{fppf})$.

Such a formal spectrum is McQuillan by construction and conversely every McQuillan affine formal algebraic space is isomorphic to a formal spectrum. To be sure, in our theory there exist affine formal algebraic spaces which are not the formal spectrum of any weakly admissible topological ring. Following [Yas09] we could introduce S-pro-rings to be pro-objects in the category of S-algebras, see Categories, Remark 4.22.5. Then every affine formal algebraic space over S would be the formal spectrum of such an S-pro-ring. We will not do this and instead we will work directly with the corresponding affine formal algebraic spaces.

The construction of the formal spectrum is functorial. To explain this let $\varphi : B \to A$ be a continuous map of weakly admissible topological rings over S. Then

$$\mathrm{Spf}(\varphi) : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$$

is the unique morphism of affine formal algebraic spaces such that the diagrams

$$\begin{tikzcd}
\mathrm{Spec}(B/J) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[d] \\
\mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A)
\end{tikzcd}$$

commute for all weak ideals of definition $I \subset A$ and $J \subset B$ with $\varphi(I) \subset J$. Since continuity of φ implies that for every weak ideal of definition $J \subset B$ there is a weak ideal of definition $I \subset A$ with the required property, we see that the required commutativities uniquely determine and define $\mathrm{Spf}(\varphi)$.

0AN0 Lemma 87.9.10. Let S be a scheme. Let A, B be weakly admissible topological rings over S. Any morphism $f : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ of affine formal algebraic spaces over S is equal to $\mathrm{Spf}(f^\sharp)$ for a unique continuous S-algebra map $f^\sharp : A \to B$.

Proof. Let $f : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ be as in the lemma. Let $J \subset B$ be a weak ideal of definition. By Lemma 87.9.4 there exists a weak ideal of definition $I \subset A$ such that $\mathrm{Spec}(B/J) \to \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ factors through $\mathrm{Spec}(A/I)$. By Schemes, Lemma 26.6.4 we obtain an S-algebra map $A/I \to B/J$. These maps are compatible for varying J and define the map $f^\sharp : A \to B$. This map is continuous because for every weak ideal of definition $J \subset B$ there is a weak ideal of definition $I \subset A$ such that $f^\sharp(I) \subset J$. The equality $f = \mathrm{Spf}(f^\sharp)$ holds by our choice of the ring maps $A/I \to B/J$ which make up $f^\sharp$. $\square$

[6]See More on Algebra, Definition 15.36.1 for the classical case and see Remark 87.2.3 for a discussion of differences.

0AIG Lemma 87.9.11. Let S be a scheme. Let $f : X \to Y$ be a map of presheaves on $(Sch/S)_{fppf}$. If X is an affine formal algebraic space and f is representable by algebraic spaces and locally quasi-finite, then f is representable (by schemes).

Proof. Let T be a scheme over S and $T \to Y$ a map. We have to show that the algebraic space $X \times_Y T$ is a scheme. Write $X = \text{colim } X_\lambda$ as in Definition 87.9.1. Let $W \subset X \times_Y T$ be a quasi-compact open subspace. The restriction of the projection $X \times_Y T \to X$ to W factors through X_λ for some λ. Then

$$W \to X_\lambda \times_S T$$

is a monomorphism (hence separated) and locally quasi-finite (because $W \to X \times_Y T \to T$ is locally quasi-finite by our assumption on $X \to Y$, see Morphisms of Spaces, Lemma 67.27.8). Hence W is a scheme by Morphisms of Spaces, Proposition 67.50.2. Thus $X \times_Y T$ is a scheme by Properties of Spaces, Lemma 66.13.1. □

87.10. Countably indexed affine formal algebraic spaces

0AIH These are the affine formal algebraic spaces as in the following lemma.

0AII Lemma 87.10.1. Let S be a scheme. Let X be an affine formal algebraic space over S. The following are equivalent

(1) there exists a system $X_1 \to X_2 \to X_3 \to \ldots$ of thickenings of affine schemes over S such that $X = \text{colim } X_n$,
(2) there exists a choice $X = \text{colim } X_\lambda$ as in Definition 87.9.1 such that Λ is countable.

Proof. This follows from the observation that a countable directed set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. See proof of Algebra, Lemma 10.86.3. □

0AIJ Definition 87.10.2. Let S be a scheme. Let X be an affine formal algebraic space over S. We say X is countably indexed if the equivalent conditions of Lemma 87.10.1 are satisfied.

In the language of [BD] this is expressed by saying that X is an $\aleph_0$-ind scheme.

0AIK Lemma 87.10.3. Let X be an affine formal algebraic space over a scheme S.

(1) If X is Noetherian, then X is adic*.
(2) If X is adic*, then X is adic.
(3) If X is adic, then X is weakly adic.
(4) If X is weakly adic, then X is classical.
(5) If X is weakly adic, then X is countably indexed.
(6) If X is countably indexed, then X is McQuillan.

Proof. Statements (1), (2), (3), and (4) follow by writing $X = \text{Spf}(A)$ and where A is a weakly admissible (hence complete) linearly topologized ring and using the implications between the various types of such rings discussed in Section 87.7.

Proof of (5). By definition there exists a weakly adic topological ring A such that $X = \text{colim } \text{Spec}(A/I)$ where the colimit is over the ideals of definition of A. As A is weakly adic, there exits in particular a countable fundamental system I_λ of open ideals, see Definition 87.7.1. Then $X = \text{colim } \text{Spec}(A/I_n)$ by definition of $\text{Spf}(A)$. Thus X is countably indexed.

Proof of (6). Write $X = \operatorname{colim} X_n$ for some system $X_1 \to X_2 \to X_3 \to \ldots$ of thickenings of affine schemes over S. Then

$$A = \lim \Gamma(X_n, \mathcal{O}_{X_n})$$

surjects onto each $\Gamma(X_n, \mathcal{O}_{X_n})$ because the transition maps are surjections as the morphisms $X_n \to X_{n+1}$ are closed immersions. Hence X is McQuillan. $\square$

0AN1 Lemma 87.10.4. Let S be a scheme. Let X be a presheaf on $(Sch/S)_{fppf}$. The following are equivalent

(1) X is a countably indexed affine formal algebraic space,
(2) $X = \text{Spf}(A)$ where A is a weakly admissible topological S-algebra which has a countable fundamental system of neighbourhoods of 0,
(3) $X = \text{Spf}(A)$ where A is a weakly admissible topological S-algebra which has a fundamental system $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of weak ideals of definition,
(4) $X = \text{Spf}(A)$ where A is a complete topological S-algebra with a fundamental system of open neighbourhoods of 0 given by a countable sequence $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of ideals such that I_n/I_{n+1} is locally nilpotent, and
(5) $X = \text{Spf}(A)$ where $A = \lim B/J_n$ with the limit topology where $B \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ is a sequence of ideals in an S-algebra B with J_n/J_{n+1} locally nilpotent.

Proof. Assume (1). By Lemma 87.10.3 we can write $X = \text{Spf}(A)$ where A is a weakly admissible topological S-algebra. For any presentation $X = \operatorname{colim} X_n$ as in Lemma 87.10.1 part (1) we see that $A = \lim A_n$ with $X_n = \text{Spec}(A_n)$ and $A_n = A/I_n$ for some weak ideal of definition $I_n \subset A$. This follows from the final statement of Lemma 87.9.6 which moreover implies that $\{I_n\}$ is a fundamental system of open neighbourhoods of 0. Thus we have a sequence

$$A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$$

of weak ideals of definition with $A = \lim A/I_n$. In this way we see that condition (1) implies each of the conditions (2) – (5).

Assume (5). First note that the limit topology on $A = \lim B/J_n$ is a linearly topologized, complete topology, see More on Algebra, Section 15.36. If $f \in A$ maps to zero in B/J_1, then some power maps to zero in B/J_2 as its image in J_1/J_2 is nilpotent, then a further power maps to zero in J_2/J_3, etc, etc. In this way we see the open ideal $\text{Ker}(A \to B/J_1)$ is a weak ideal of definition. Thus A is weakly admissible. In this way we see that (5) implies (2).

It is clear that (4) is a special case of (5) by taking $B = A$. It is clear that (3) is a special case of (2).

Assume A is as in (2). Let E_n be a countable fundamental system of neighbourhoods of 0 in A. Since A is a weakly admissible topological ring we can find open ideals $I_n \subset E_n$. We can also choose a weak ideal of definition $J \subset A$. Then $J \cap I_n$ is a fundamental system of weak ideals of definition of A and we get $X = \text{Spf}(A) = \operatorname{colim} \text{Spec}(A/(J \cap I_n))$ which shows that X is a countably indexed affine formal algebraic space. $\square$

0AKM Lemma 87.10.5. Let S be a scheme. Let X be an affine formal algebraic space. The following are equivalent

(1) X is Noetherian,
(2) X is adic* and for every closed immersion $T \to X$ with T a scheme, T is Noetherian,
(3) X is adic* and for some choice of $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 the schemes X_λ are Noetherian, and
(4) X is weakly adic and for some choice $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 the schemes X_λ are Noetherian.

Proof. Assume X is Noetherian. Then $X = \text{Spf}(A)$ where A is a Noetherian adic ring. Let $T \to X$ be a closed immersion where T is a scheme. By Lemma 87.9.6 we see that T is affine and that $T \to \text{Spec}(A)$ is a closed immersion. Since A is Noetherian, we see that T is Noetherian. In this way we see that (1) $\Rightarrow$ (2).

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are immediate (see Lemma 87.10.3).

To prove (3) $\Rightarrow$ (1) write $X = \text{Spf}(A)$ for some adic ring A with finitely generated ideal of definition I. We are also given that the rings A/I_λ are Noetherian for some fundamental system of open ideals I_λ. Since I is open, we can find a λ such that $I_\lambda \subset I$. Then A/I is Noetherian and we conclude that A is Noetherian by Algebra, Lemma 10.97.5.

To prove (4) $\Rightarrow$ (3) write $X = \text{Spf}(A)$ for some weakly adic ring A. Then A is admissible and has an ideal of definition I and the closure I_2 of I^2 is open, see Lemma 87.7.2. We are also given that the rings A/I_λ are Noetherian for some fundamental system of open ideals I_λ. Choose a λ such that $I_\lambda \subset I_2$. Then A/I_2 is Noetherian as a quotient of A/I_λ. Hence I/I_2 is a finite A-module. Hence A is an adic ring with a finitely generated ideal of definition by Lemma 87.7.4. Thus X is adic* and (3) holds. $\square$

87.11. Formal algebraic spaces

0AIL We take a break from our habit of introducing new concepts first for rings, then for schemes, and then for algebraic spaces, by introducing formal algebraic spaces without first introducing formal schemes. The general idea will be that a formal algebraic space is a sheaf in the fppf topology which étale locally is an affine formal scheme in the sense of [BD]. Related material can be found in [Yas09].

In the definition of a formal algebraic space we are going to borrow some terminology from Bootstrap, Sections 80.3 and 80.4.

0AIM Definition 87.11.1. Let S be a scheme. We say a sheaf X on $(Sch/S)_{fppf}$ is a formal algebraic space if there exist a family of maps $\{X_i \to X\}_{i \in I}$ of sheaves such that

(1) X_i is an affine formal algebraic space,
(2) $X_i \to X$ is representable by algebraic spaces and étale,
(3) $\coprod X_i \to X$ is surjective as a map of sheaves

and X satisfies a set theoretic condition (see Remark 87.11.5). A morphism of formal algebraic spaces over S is a map of sheaves.

Discussion. Sanity check: an affine formal algebraic space is a formal algebraic space. In the situation of the definition the morphisms $X_i \to X$ are representable (by schemes), see Lemma 87.9.11. By Bootstrap, Lemma 80.4.6 we could instead of asking $\coprod X_i \to X$ to be surjective as a map of sheaves, require that it be surjective (which makes sense because it is representable).

Our notion of a formal algebraic space is very general. In fact, even affine formal algebraic spaces as defined above are very nasty objects.

0AIP Lemma 87.11.2. Let S be a scheme. If X is a formal algebraic space over S, then the diagonal morphism $\Delta : X \to X \times_S X$ is representable, a monomorphism, locally quasi-finite, locally of finite type, and separated.

Proof. Suppose given $U \to X$ and $V \to X$ with U, V schemes over S. Then $U \times_X V$ is a sheaf. Choose $\{X_i \to X\}$ as in Definition 87.11.1. For every i the morphism

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) = (U \times_X V) \times_X X_i \to U \times_X V$$

is representable and étale as a base change of $X_i \to X$ and its source is a scheme (use Lemmas 87.9.2 and 87.9.11). These maps are jointly surjective hence $U \times_X V$ is an algebraic space by Bootstrap, Theorem 80.10.1. The morphism $U \times_X V \to U \times_S V$ is a monomorphism. It is also locally quasi-finite, because on precomposing with the morphism displayed above we obtain the composition

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) \to (U \times_X X_i) \times_S (V \times_X X_i) \to U \times_S V$$

which is locally quasi-finite as a composition of a closed immersion (Lemma 87.9.2) and an étale morphism, see Descent on Spaces, Lemma 74.19.2. Hence we conclude that $U \times_X V$ is a scheme by Morphisms of Spaces, Proposition 67.50.2. Thus Δ is representable, see Spaces, Lemma 65.5.10.

In fact, since we've shown above that the morphisms of schemes $U \times_X V \to U \times_S V$ are aways monomorphisms and locally quasi-finite we conclude that $\Delta : X \to X \times_S X$ is a monomorphism and locally quasi-finite, see Spaces, Lemma 65.5.11. Then we can use the principle of Spaces, Lemma 65.5.8 to see that Δ is separated and locally of finite type. Namely, a monomorphism of schemes is separated (Schemes, Lemma 26.23.3) and a locally quasi-finite morphism of schemes is locally of finite type (follows from the definition in Morphisms, Section 29.20). $\square$

0AIQ Lemma 87.11.3. Let S be a scheme. Let $f : X \to Y$ be a morphism from an algebraic space over S to a formal algebraic space over S. Then f is representable by algebraic spaces.

Proof. Let $Z \to Y$ be a morphism where Z is a scheme over S. We have to show that $X \times_Y Z$ is an algebraic space. Choose a scheme U and a surjective étale morphism $U \to X$. Then $U \times_Y Z \to X \times_Y Z$ is representable surjective étale (Spaces, Lemma 65.5.5) and $U \times_Y Z$ is a scheme by Lemma 87.11.2. Hence the result by Bootstrap, Theorem 80.10.1. $\square$

0AIR Remark 87.11.4. Modulo set theoretic issues the category of formal schemes à la EGA (see Section 87.2) is equivalent to a full subcategory of the category of formal algebraic spaces. To explain this we assume our base scheme is $\text{Spec}(\mathbf{Z})$. By Lemma 87.2.2 the functor of points $h_{\mathfrak X}$ associated to a formal scheme $\mathfrak X$ is a sheaf in the fppf topology. By Lemma 87.2.1 the assignment $\mathfrak X \mapsto h_{\mathfrak X}$ is a fully faithful embedding of the category of formal schemes into the category of fppf sheaves. Given a formal scheme $\mathfrak X$ we choose an open covering $\mathfrak X = \bigcup \mathfrak X_i$ with $\mathfrak X_i$ affine formal schemes. Then $h_{\mathfrak X_i}$ is an affine formal algebraic space by Remark 87.9.8. The morphisms

$h_{\mathfrak{X}_i} \to h_{\mathfrak{X}}$ are representable and open immersions. Thus $\{h_{\mathfrak{X}_i} \to h_{\mathfrak{X}}\}$ is a family as in Definition 87.11.1 and we see that $h_{\mathfrak{X}}$ is a formal algebraic space.

0AIS Remark 87.11.5. Let S be a scheme and let $(Sch/S)_{fppf}$ be a big fppf site as in Topologies, Definition 34.7.8. As our set theoretic condition on X in Definitions 87.9.1 and 87.11.1 we take: there exist objects U, R of $(Sch/S)_{fppf}$, a morphism $U \to X$ which is a surjection of fppf sheaves, and a morphism $R \to U \times_X U$ which is a surjection of fppf sheaves. In other words, we require our sheaf to be a coequalizer of two maps between representable sheaves. Here are some observations which imply this notion behaves reasonably well:

(1) Suppose $X = \mathrm{colim}_{\lambda \in \Lambda} X_\lambda$ and the system satisfies conditions (1) and (2) of Definition 87.9.1. Then $U = \coprod_{\lambda \in \Lambda} X_\lambda \to X$ is a surjection of fppf sheaves. Moreover, $U \times_X U$ is a closed subscheme of $U \times_S U$ by Lemma 87.9.2. Hence if U is representable by an object of $(Sch/S)_{fppf}$ then $U \times_S U$ is too (see Sets, Lemma 3.9.9) and the set theoretic condition is satisfied. This is always the case if Λ is countable, see Sets, Lemma 3.9.9.
(2) Sanity check. Let $\{X_i \to X\}_{i \in I}$ be as in Definition 87.11.1 (with the set theoretic condition as formulated above) and assume that each X_i is actually an affine scheme. Then X is an algebraic space. Namely, if we choose a larger big fppf site $(Sch'/S)_{fppf}$ such that $U' = \coprod X_i$ and $R' = \coprod X_i \times_X X_j$ are representable by objects in it, then $X' = U'/R'$ will be an object of the category of algebraic spaces for this choice. Then an application of Spaces, Lemma 65.15.2 shows that X is an algebraic space for $(Sch/S)_{fppf}$.
(3) Let $\{X_i \to X\}_{i \in I}$ be a family of maps of sheaves satisfying conditions (1), (2), (3) of Definition 87.11.1. For each i we can pick $U_i \in \mathrm{Ob}((Sch/S)_{fppf})$ and $U_i \to X_i$ which is a surjection of sheaves. Thus if I is not too large (for example countable) then $U = \coprod U_i \to X$ is a surjection of sheaves and U is representable by an object of $(Sch/S)_{fppf}$. To get $R \in \mathrm{Ob}((Sch/S)_{fppf})$ surjecting onto $U \times_X U$ it suffices to assume the diagonal $\Delta : X \to X \times_S X$ is not too wild, for example this always works if the diagonal of X is quasi-compact, i.e., X is quasi-separated.

87.12. The reduction

0GB5 All formal algebraic spaces have an underlying reduced algebraic space as the following lemma demonstrates.

0AIN Lemma 87.12.1. Let S be a scheme. Let X be a formal algebraic space over S. There exists a reduced algebraic space X_{red} and a representable morphism $X_{red} \to X$ which is a thickening. A morphism $U \to X$ with U a reduced algebraic space factors uniquely through X_{red}.

Proof. First assume that X is an affine formal algebraic space. Say $X = \mathrm{colim}\, X_\lambda$ as in Definition 87.9.1. Since the transition morphisms are thickenings, the affine schemes X_λ all have isomorphic reductions X_{red}. The morphism $X_{red} \to X$ is representable and a thickening by Lemma 87.9.3 and the fact that compositions of thickenings are thickenings. We omit the verification of the universal property (use Schemes, Definition 26.12.5, Schemes, Lemma 26.12.7, Properties of Spaces, Definition 66.12.5, and Properties of Spaces, Lemma 66.12.4).

Let X and $\{X_i \to X\}_{i \in I}$ be as in Definition 87.11.1. For each i let $X_{i,red} \to X_i$ be the reduction as constructed above. For $i, j \in I$ the projection $X_{i,red} \times_X X_j \to X_{i,red}$ is an étale (by assumption) morphism of schemes (by Lemma 87.9.11). Hence $X_{i,red} \times_X X_j$ is reduced (see Descent, Lemma 35.18.1). Thus the projection $X_{i,red} \times_X X_j \to X_j$ factors through $X_{j,red}$ by the universal property. We conclude that

$$R_{ij} = X_{i,red} \times_X X_j = X_{i,red} \times_X X_{j,red} = X_i \times_X X_{j,red}$$

because the morphisms $X_{i,red} \to X_i$ are injections of sheaves. Set $U = \coprod X_{i,red}$, set $R = \coprod R_{ij}$, and denote $s, t : R \to U$ the two projections. As a sheaf $R = U \times_X U$ and s and t are étale. Then $(t, s) : R \to U$ defines an étale equivalence relation by our observations above. Thus $X_{red} = U/R$ is an algebraic space by Spaces, Theorem 65.10.5. By construction the diagram

$$\begin{tikzcd} \coprod X_{i,red} \arrow[r] \arrow[d] & \coprod X_i \arrow[d] \\ X_{red} \arrow[r] & X \end{tikzcd}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a thickening we conclude that $X_{red} \to X$ is representable by Bootstrap, Lemma 80.5.2 (to verify the assumptions of the lemma use that a surjective étale morphism is surjective, flat, and locally of finite presentation and use that thickenings are separated and locally quasi-finite). Then we can use Spaces, Lemma 65.5.6 to conclude that $X_{red} \to X$ is a thickening (use that being a thickening is equivalent to being a surjective closed immersion).

Finally, suppose that $U \to X$ is a morphism with U a reduced algebraic space over S. Then each $X_i \times_X U$ is étale over U and therefore reduced (by our definition of reduced algebraic spaces in Properties of Spaces, Section 66.7). Then $X_i \times_X U \to X_i$ factors through $X_{i,red}$. Hence $U \to X$ factors through X_{red} because $\{X_i \times_X U \to U\}$ is an étale covering. □

0GB6 Example 87.12.2. Let A be a weakly admissible topological ring. In this case we have

$$\mathrm{Spf}(A)_{red} = \mathrm{Spec}(A/\mathfrak{a})$$

where $\mathfrak{a} \subset A$ is the ideal of topologically nilpotent elements. Namely, $\mathfrak{a}$ is a radical ideal (Lemma 87.4.10) which is open because A is weakly admissible.

0GB7 Lemma 87.12.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces and smooth (for example étale). Then $X_{red} = X \times_Y Y_{red}$.

Proof. (The étale case follows directly from the construction of the underlying reduced algebraic space in the proof of Lemma 87.12.1.) Assume f is smooth. Observe that $X \times_Y Y_{red} \to Y_{red}$ is a smooth morphism of algebraic spaces. Hence $X \times_Y Y_{red}$ is a reduced algebraic space by Descent on Spaces, Lemma 74.9.5. Then the univeral property of reduction shows that the canonical morphism $X_{red} \to X \times_Y Y_{red}$ is an isomorphism. □

0GB8 Lemma 87.12.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces. Then f is surjective in the sense of Bootstrap, Definition 80.4.1 if and only if $f_{red} : X_{red} \to Y_{red}$ is a surjective morphism of algebraic spaces.

Proof. Omitted. $\square$

87.13. Colimits of algebraic spaces along thickenings

0AIT A special type of formal algebraic space is one which can globally be written as a cofiltered colimit of algebraic spaces along thickenings as in the following lemma. We will see later (in Section 87.18) that any quasi-compact and quasi-separated formal algebraic space is such a global colimit.

0AIU **Lemma 87.13.1.** Let S be a scheme. Suppose given a directed set Λ and a system of algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening. Then $X = \mathrm{colim}_{\lambda \in \Lambda} X_\lambda$ is a formal algebraic space over S.

Proof. Since we take the colimit in the category of fppf sheaves, we see that X is a sheaf. Choose and fix $\lambda \in \Lambda$. Choose an étale covering $\{X_{i,\lambda} \to X_\lambda\}$ where X_i is an affine scheme over S, see Properties of Spaces, Lemma 66.6.1. For each $\mu \geq \lambda$ there exists a cartesian diagram

$$\begin{tikzcd} X_{i,\lambda} \arrow[r] \arrow[d] & X_{i,\mu} \arrow[d] \\ X_\lambda \arrow[r] & X_\mu \end{tikzcd}$$

with étale vertical arrows, see More on Morphisms of Spaces, Theorem 76.8.1 (this also uses that a thickening is a surjective closed immersion which satisfies the conditions of the theorem). Moreover, these diagrams are unique up to unique isomorphism and hence $X_{i,\mu} = X_\mu \times_{X_{\mu'}} X_{i,\mu'}$ for $\mu' \geq \mu$. The morphisms $X_{i,\mu} \to X_{i,\mu'}$ is a thickening as a base change of a thickening. Each $X_{i,\mu}$ is an affine scheme by Limits of Spaces, Proposition 70.15.2 and the fact that $X_{i,\lambda}$ is affine. Set $X_i = \mathrm{colim}_{\mu \geq \lambda} X_{i,\mu}$. Then X_i is an affine formal algebraic space. The morphism $X_i \to X$ is étale because given an affine scheme U any $U \to X$ factors through X_μ for some $\mu \geq \lambda$ (details omitted). In this way we see that X is a formal algebraic space. $\square$

Let S be a scheme. Let X be a formal algebraic space over S. How does one prove or check that X is a global colimit as in Lemma 87.13.1? To do this we look for maps $i : Z \to X$ where Z is an algebraic space over S and i is surjective and a closed immersion, in other words, i is a thickening. This makes sense as i is representable by algebraic spaces (Lemma 87.11.3) and we can use Bootstrap, Definition 80.4.1 as before.

0CB8 **Example 87.13.2.** Let $(A, \mathfrak{m}, \kappa)$ be a valuation ring, which is (π)-adically complete for some nonzero $\pi \in \mathfrak{m}$. Assume also that $\mathfrak{m}$ is not finitely generated. An example is $A = \mathcal{O}_{\mathbf{C}_p}$ and $\pi = p$ where $\mathcal{O}_{\mathbf{C}_p}$ is the ring of integers of the field of p-adic complex numbers $\mathbf{C}_p$ (this is the completion of the algebraic closure of $\mathbf{Q}_p$). Another example is

$$A = \left\{ \sum\nolimits_{\alpha \in \mathbf{Q},\ \alpha \geq 0} a_\alpha t^\alpha \middle| \begin{array}{c} a_\alpha \in \kappa \text{ and for all } n \text{ there are only a} \\ \text{finite number of nonzero } a_\alpha \text{ with } \alpha \leq n \end{array} \right\}$$

and $\pi = t$. Then $X = \text{Spf}(A)$ is an affine formal algebraic space and $\text{Spec}(\kappa) \to X$ is a thickening which corresponds to the weak ideal of definition $\mathfrak{m} \subset A$ which is however not an ideal of definition.

0AIV Remark 87.13.3 (Weak ideals of definition). Let $\mathfrak{X}$ be a formal scheme in the sense of McQuillan, see Remark 87.2.3. An weak ideal of definition for $\mathfrak{X}$ is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_\mathfrak{X}$ such that for all $\mathfrak{U} \subset \mathfrak{X}$ affine formal open subscheme the ideal $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_\mathfrak{X}(\mathfrak{U})$ is a weak ideal of definition of the weakly admissible topological ring $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$. It suffices to check the condition on the members of an affine open covering. There is a one-to-one correspondence

$$\{\text{weak ideals of definition for } \mathfrak{X}\} \leftrightarrow \{\text{thickenings } i : Z \to h_\mathfrak{X} \text{ as above}\}$$

This correspondence associates to $\mathcal{I}$ the scheme $Z = (\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathcal{I})$ together with the obvious morphism to $\mathfrak{X}$. A fundamental system of weak ideals of definition is a collection of weak ideals of definition $\mathcal{I}_\lambda$ such that on every affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ the ideals

$$I_\lambda = \mathcal{I}_\lambda(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_\mathfrak{X})$$

form a fundamental system of weak ideals of definition of the weakly admissible topological ring A. It suffices to check on the members of an affine open covering. We conclude that the formal algebraic space $h_\mathfrak{X}$ associated to the McQuillan formal scheme $\mathfrak{X}$ is a colimit of schemes as in Lemma 87.13.1 if and only if there exists a fundamental system of weak ideals of definition for $\mathfrak{X}$.

0AIW Remark 87.13.4 (Ideals of definition). Let $\mathfrak{X}$ be a formal scheme à la EGA. An ideal of definition for $\mathfrak{X}$ is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_\mathfrak{X}$ such that for all $\mathfrak{U} \subset \mathfrak{X}$ affine formal open subscheme the ideal $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_\mathfrak{X}(\mathfrak{U})$ is an ideal of definition of the admissible topological ring $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$. It suffices to check the condition on the members of an affine open covering. We do not get the same correspondence between ideals of definition and thickenings $Z \to h_\mathfrak{X}$ as in Remark 87.13.3; an example is given in Example 87.13.2. A fundamental system of ideals of definition is a collection of ideals of definition $\mathcal{I}_\lambda$ such that on every affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ the ideals

$$I_\lambda = \mathcal{I}_\lambda(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_\mathfrak{X})$$

form a fundamental system of ideals of definition of the admissible topological ring A. It suffices to check on the members of an affine open covering. Suppose that $\mathfrak{X}$ is quasi-compact and that $\{\mathcal{I}_\lambda\}_{\lambda \in \Lambda}$ is a fundamental system of weak ideals of definition. If A is an admissible topological ring then all sufficiently small open ideals are ideals of definition (namely any open ideal contained in an ideal of definition is an ideal of definition). Thus since we only need to check on the finitely many members of an affine open covering we see that $\mathcal{I}_\lambda$ is an ideal of definition for λ sufficiently large. Using the discussion in Remark 87.13.3 we conclude that the formal algebraic space $h_\mathfrak{X}$ associated to the quasi-compact formal scheme $\mathfrak{X}$ à la EGA is a colimit of schemes as in Lemma 87.13.1 if and only if there exists a fundamental system of ideals of definition for $\mathfrak{X}$.

87.14. Completion along a closed subset

0AIX Our notion of a formal algebraic space is well adapted to taking the completion along a closed subset.

0AIY **Lemma 87.14.1.** Let S be a scheme. Let X be an affine scheme over S. Let $T \subset |X|$ be a closed subset. Then the functor

$$(Sch/S)_{fppf} \longrightarrow \text{Sets}, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a McQuillan affine formal algebraic space.

Proof. Say $X = \text{Spec}(A)$ and T corresponds to the radical ideal $I \subset A$. Let $U = \text{Spec}(B)$ be an affine scheme over S and let $f : U \to X$ be an element of $F(U)$. Then f corresponds to a ring map $\varphi : A \to B$ such that every prime of B contains $\varphi(I)B$. Thus every element of $\varphi(I)$ is nilpotent in B, see Algebra, Lemma 10.17.2. Setting $J = \text{Ker}(\varphi)$ we conclude that I/J is a locally nilpotent ideal in A/J. Equivalently, $V(J) = V(I) = T$. In other words, the functor of the lemma equals $\text{colim}\,\text{Spec}(A/J)$ where the colimit is over the collection of ideals J with $V(J) = T$. Thus our functor is an affine formal algebraic space. It is McQuillan (Definition 87.9.7) because the maps $A \to A/J$ are surjective and hence $A^\wedge = \lim A/J \to A/J$ is surjective, see Lemma 87.9.6. $\square$

0AIZ **Lemma 87.14.2.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Then the functor

$$(Sch/S)_{fppf} \longrightarrow \text{Sets}, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a formal algebraic space.

Proof. Denote F the functor. Let $\{U_i \to U\}$ be an fppf covering. Then $\coprod |U_i| \to |U|$ is surjective. Since X is an fppf sheaf, it follows that F is an fppf sheaf.

Let $\{g_i : X_i \to X\}$ be an étale covering such that X_i is affine for all i, see Properties of Spaces, Lemma 66.6.1. The morphisms $F \times_X X_i \to F$ are étale (see Spaces, Lemma 65.5.5) and the map $\coprod F \times_X X_i \to F$ is a surjection of sheaves. Thus it suffices to prove that $F \times_X X_i$ is an affine formal algebraic space. A U-valued point of $F \times_X X_i$ is a morphism $U \to X_i$ whose image is contained in the closed subset $g_i^{-1}(T) \subset |X_i|$. Thus this follows from Lemma 87.14.1. $\square$

0AMC **Definition 87.14.3.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. The formal algebraic space of Lemma 87.14.2 is called the *completion of X along T*.

In [DG67, Chapter I, Section 10.8] the notation $X_{/T}$ is used to denote the completion and we will occasionally use this notation as well. Let $f : X \to X'$ be a morphism of algebraic spaces over a scheme S. Suppose that $T \subset |X|$ and $T' \subset |X'|$ are closed subsets such that $|f|(T) \subset T'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

0APV **Lemma 87.14.4.** Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $T \subset |X|$ be a closed subset and let $T' = |f|^{-1}(T) \subset |X'|$. Then

$$\begin{tikzcd} X'_{/T'} \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ X_{/T} \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of sheaves. In particular, the morphism $X'_{/T'} \to X_{/T}$ is representable by algebraic spaces.

Proof. Namely, suppose that $Y \to X$ is a morphism from a scheme into X such that $|Y|$ maps into T. Then $Y \times_X X' \to X$ is a morphism of algebraic spaces such that $|Y \times_X X'|$ maps into T'. Hence the functor $Y \times_{X_{/T}} X'_{/T'}$ is represented by $Y \times_X X'$ and we see that the lemma holds. $\square$

0GB9 Lemma 87.14.5. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. The reduction $(X_{/T})_{red}$ of the completion $X_{/T}$ of X along T is the reduced induced closed subspace Z of X corresponding to T.

Proof. It follows from Lemma 87.12.1, Properties of Spaces, Definition 66.12.5 (which uses Properties of Spaces, Lemma 66.12.3 to construct Z), and the definition of $X_{/T}$ that Z and $(X_{/T})_{red}$ are reduced algebraic spaces characterized the same mapping property: a morphism $g : Y \to X$ whose source is a reduced algebraic space factors through them if and only if $|Y|$ maps into $T \subset |X|$. $\square$

0GBA Lemma 87.14.6. Let S be a scheme. Let $X = \mathrm{Spec}(A)$ be an affine scheme over S. Let $T \subset X$ be a closed subset. Let $X_{/T}$ be the formal completion of X along T.

(1) If $X \setminus T$ is quasi-compact, i.e., T is constructible, then $X_{/T}$ is adic*.
(2) If $T = V(I)$ for some finitely generated ideal $I \subset A$, then $X_{/T} = \mathrm{Spf}(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A.
(3) If X is Noetherian, then $X_{/T}$ is Noetherian.

Proof. By Algebra, Lemma 10.29.1 if (1) holds, then we can find an ideal $I \subset A$ as in (2). If (3) holds then we can find an ideal $I \subset A$ as in (2). Moreover, completions of Noetherian rings are Noetherian by Algebra, Lemma 10.97.6. All in all we see that it suffices to prove (2).

Proof of (2). Let $I = (f_1, \ldots, f_r) \subset A$ cut out T. If $Z = \mathrm{Spec}(B)$ is an affine scheme and $g : Z \to X$ is a morphism with $g(Z) \subset T$ (set theoretically), then $g^\sharp(f_i)$ is nilpotent in B for each i. Thus I^n maps to zero in B for some n. Hence we see that $X_{/T} = \mathrm{colim}\,\mathrm{Spec}(A/I^n) = \mathrm{Spf}(A^\wedge)$. $\square$

The following lemma is due to Ofer Gabber.

0APW Lemma 87.14.7. Let S be a scheme. Let $X = \mathrm{Spec}(A)$ be an affine scheme over S. Let $T \subset X$ be a closed subscheme. Email by Ofer Gabber of September 11, 2014.

(1) If the formal completion $X_{/T}$ is countably indexed and there exist countably many $f_1, f_2, f_3, \ldots \in A$ such that $T = V(f_1, f_2, f_3, \ldots)$, then $X_{/T}$ is adic*.
(2) The conclusion of (1) is wrong if we omit the assumption that T can be cut out by countably many functions in X.

Proof. The assumption that $X_{/T}$ is countably indexed means that there exists a sequence of ideals

$$A \supset J_1 \supset J_2 \supset J_3 \supset \ldots$$

with $V(J_n) = T$ such that every ideal $J \subset A$ with $V(J) = T$ there exists an n such that $J \supset J_n$.

To construct an example for (2) let ω_1 be the first uncountable ordinal. Let k be a field and let A be the k-algebra generated by x_α, $\alpha \in \omega_1$ and $y_{\alpha\beta}$ with $\alpha \in \beta \in \omega_1$

subject to the relations $x_\alpha = y_{\alpha\beta} x_\beta$. Let $T = V(x_\alpha)$. Let $J_n = (x_\alpha^n)$. If $J \subset A$ is an ideal such that $V(J) = T$, then $x_\alpha^{n_\alpha} \in J$ for some $n_\alpha \geq 1$. One of the sets $\{\alpha \mid n_\alpha = n\}$ must be unbounded in ω_1. Then the relations imply that $J_n \subset J$.

To see that (2) holds it now suffices to show that $A^\wedge = \lim A/J_n$ is not a ring complete with respect to a finitely generated ideal. For $\gamma \in \omega_1$ let A_γ be the quotient of A by the ideal generated by x_α, $\alpha \in \gamma$ and $y_{\alpha\beta}$, $\alpha \in \gamma$. As A/J_1 is reduced, every topologically nilpotent element f of $\lim A/J_n$ is in $J_1^\wedge = \lim J_1/J_n$. This means f is an infinite series involving only a countable number of generators. Hence f dies in $A_\gamma^\wedge = \lim A_\gamma/J_n A_\gamma$ for some γ. Note that $A^\wedge \to A_\gamma^\wedge$ is continuous and open by Lemma 87.4.5. If the topology on $A^\wedge$ was I-adic for some finitely generated ideal $I \subset A^\wedge$, then I would go to zero in some $A_\gamma^\wedge$. This would mean that $A_\gamma^\wedge$ is discrete, which is not the case as there is a surjective continuous and open (by Lemma 87.4.5) map $A_\gamma^\wedge \to k[[t]]$ given by $x_\alpha \mapsto t$, $y_{\alpha\beta} \mapsto 1$ for $\gamma = \alpha$ or $\gamma \in \alpha$.

Before we prove (1) we first prove the following: If $I \subset A^\wedge$ is a finitely generated ideal whose closure $\bar{I}$ is open, then $I = \bar{I}$. Since $V(J_n^2) = T$ there exists an m such that $J_n^2 \supset J_m$. Thus, we may assume that $J_n^2 \supset J_{n+1}$ for all n by passing to a subsequence. Set $J_n^\wedge = \lim_{k \geq n} J_n/J_k \subset A^\wedge$. Since the closure $\bar{I} = \bigcap (I + J_n^\wedge)$ (Lemma 87.4.2) is open we see that there exists an m such that $I + J_n^\wedge \supset J_m^\wedge$ for all $n \geq m$. Fix such an m. We have

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_{n-1}^\wedge (I + J_{n+1}^\wedge) \supset J_{n-1}^\wedge J_m^\wedge$$

for all $n \geq m+1$. Namely, the first inclusion is trivial and the second was shown above. Because $J_{n-1} J_m \supset J_{n-1}^2 \supset J_n$ these inclusions show that the image of J_n in $A^\wedge$ is contained in the ideal $J_{n-1}^\wedge I + J_{n+1}^\wedge$. Because this ideal is open we conclude that

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_n^\wedge.$$

Say $I = (g_1, \ldots, g_t)$. Pick $f \in J_{m+1}^\wedge$. Using the last displayed inclusion, valid for all $n \geq m+1$, we can write by induction on $c \geq 0$

$$f = \sum f_{i,c} g_i \mod J_{m+1+c}^\wedge$$

with $f_{i,c} \in J_m^\wedge$ and $f_{i,c} \equiv f_{i,c-1} \bmod J_{m+c}^\wedge$. It follows that $I J_m^\wedge \supset J_{m+1}^\wedge$. Combined with $I + J_{m+1}^\wedge \supset J_m^\wedge$ we conclude that I is open.

Proof of (1). Assume $T = V(f_1, f_2, f_3, \ldots)$. Let $I_m \subset A^\wedge$ be the ideal generated by $f_1, \ldots, f_m$. We distinguish two cases.

Case I: For some m the closure of I_m is open. Then I_m is open by the result of the previous paragraph. For any n we have $(J_n)^2 \supset J_{n+1}$ by design, so the closure of $(J_n^\wedge)^2$ contains $J_{n+1}^\wedge$ and thus is open. Taking n large, it follows that the closure of the product of any two open ideals in $A^\wedge$ is open. Let us prove I_m^k is open for $k \geq 1$ by induction on k. The case $k = 1$ is our hypothesis on m in Case I. For $k > 1$, suppose I_m^{k-1} is open. Then $I_m^k = I_m^{k-1} \cdot I_m$ is the product of two open ideals and hence has open closure. But then since I_m^k is finitely generated it follows that I_m^k is open by the previous paragraph (applied to $I = I_m^k$), so we can continue the induction on k. As each element of I_m is topologically nilpotent, we conclude that I_m is an ideal of definition which proves that $A^\wedge$ is adic with a finitely generated ideal of definition, i.e., $X_{/T}$ is adic*.

Case II. For all m the closure $\bar{I}_m$ of I_m is not open. Then the topology on $A^\wedge/\bar{I}_m$ is not discrete. This means we can pick $\phi(m) \geq m$ such that

$$\mathrm{Im}(J_{\phi(m)} \to A/(f_1, \ldots, f_m)) \neq \mathrm{Im}(J_{\phi(m)+1} \to A/(f_1, \ldots, f_m))$$

To see this we have used that $A^\wedge/(\bar{I}_m + J_n^\wedge) = A/((f_1, \ldots, f_m) + J_n)$. Choose exponents $e_i > 0$ such that $f_i^{e_i} \in J_{\phi(m)+1}$ for $0 < m < i$. Let $J = (f_1^{e_1}, f_2^{e_2}, f_3^{e_3}, \ldots)$. Then $V(J) = T$. We claim that $J \not\supset J_n$ for all n which is a contradiction proving Case II does not occur. Namely, the image of J in $A/(f_1, \ldots, f_m)$ is contained in the image of $J_{\phi(m)+1}$ which is properly contained in the image of J_m. $\square$

87.15. Fibre products

0AJ0 Obligatory section about fibre products of formal algebraic spaces.

0AJ1 **Lemma 87.15.1.** Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of maps of sheaves on $(Sch/S)_{fppf}$. Assume (a) X_i is a formal algebraic space over S, (b) $X_i \to X$ is representable by algebraic spaces and étale, and (c) $\coprod X_i \to X$ is a surjection of sheaves. Then X is a formal algebraic space over S.

Proof. For each i pick $\{X_{ij} \to X_i\}_{j \in J_i}$ as in Definition 87.11.1. Then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a family as in Definition 87.11.1 for X. $\square$

0AJ2 **Lemma 87.15.2.** Let S be a scheme. Let X, Y be formal algebraic spaces over S and let Z be a sheaf whose diagonal is representable by algebraic spaces. Let $X \to Z$ and $Y \to Z$ be maps of sheaves. Then $X \times_Z Y$ is a formal algebraic space.

Proof. Choose $\{X_i \to X\}$ and $\{Y_j \to Y\}$ as in Definition 87.11.1. Then $\{X_i \times_Z Y_j \to X \times_Z Y\}$ is a family of maps which are representable by algebraic spaces and étale. Thus Lemma 87.15.1 tells us it suffices to show that $X \times_Z Y$ is a formal algebraic space when X and Y are affine formal algebraic spaces.

Assume X and Y are affine formal algebraic spaces. Write $X = \mathrm{colim}\, X_\lambda$ and $Y = \mathrm{colim}\, Y_\mu$ as in Definition 87.9.1. Then $X \times_Z Y = \mathrm{colim}\, X_\lambda \times_Z Y_\mu$. Each $X_\lambda \times_Z Y_\mu$ is an algebraic space. For $\lambda \leq \lambda'$ and $\mu \leq \mu'$ the morphism

$$X_\lambda \times_Z Y_\mu \to X_\lambda \times_Z Y_{\mu'} \to X_{\lambda'} \times_Z Y_{\mu'}$$

is a thickening as a composition of base changes of thickenings. Thus we conclude by applying Lemma 87.13.1. $\square$

0AJ3 **Lemma 87.15.3.** Let S be a scheme. The category of formal algebraic spaces over S has fibre products.

Proof. Special case of Lemma 87.15.2 because formal algebraic spaces have representable diagonals, see Lemma 87.11.2. $\square$

0CB9 **Lemma 87.15.4.** Let S be a scheme. Let $X \to Z$ and $Y \to Z$ be morphisms of formal algebraic spaces over S. Then $(X \times_Z Y)_{red} = (X_{red} \times_{Z_{red}} Y_{red})_{red}$.

Proof. This follows from the universal property of the reduction in Lemma 87.12.1. $\square$

We have already proved the following lemma (without knowing that fibre products exist).

0AN2 **Lemma 87.15.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The diagonal morphism $\Delta : X \to X \times_Y X$ is representable (by schemes), a monomorphism, locally quasi-finite, locally of finite type, and separated.

Proof. Let T be a scheme and let $T \to X \times_Y X$ be a morphism. Then

$$T \times_{(X \times_Y X)} X = T \times_{(X \times_S X)} X$$

Hence the result follows immediately from Lemma 87.11.2. $\square$

87.16. Separation axioms for formal algebraic spaces

0AJ4 This section is about "absolute" separation conditions on formal algebraic spaces. We will discuss separation conditions for morphisms of formal algebraic spaces later.

0AJ5 **Lemma 87.16.1.** Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent

(1) the reduction of X (Lemma 87.12.1) is a quasi-separated algebraic space,
(2) for $U \to X$, $V \to X$ with U, V quasi-compact schemes the fibre product $U \times_X V$ is quasi-compact,
(3) for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is quasi-compact.

Proof. Observe that $U \times_X V$ is a scheme by Lemma 87.11.2. Let $U_{red}, V_{red}, X_{red}$ be the reduction of U, V, X. Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \to U \times_X V$$

is a thickening of schemes. From this the equivalence of (1) and (2) is clear, keeping in mind the analogous lemma for algebraic spaces, see Properties of Spaces, Lemma 66.3.3. We omit the proof of the equivalence of (2) and (3). $\square$

0AJ6 **Lemma 87.16.2.** Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent

(1) the reduction of X (Lemma 87.12.1) is a separated algebraic space,
(2) for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is affine and
$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$
is surjective.

Proof. If (2) holds, then X_{red} is a separated algebraic space by applying Properties of Spaces, Lemma 66.3.3 to morphisms $U \to X_{red}$ and $V \to X_{red}$ with U, V affine and using that $U \times_{X_{red}} V = U \times_X V$.

Assume (1). Let $U \to X$ and $V \to X$ be as in (2). Observe that $U \times_X V$ is a scheme by Lemma 87.11.2. Let $U_{red}, V_{red}, X_{red}$ be the reduction of U, V, X. Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \to U \times_X V$$

is a thickening of schemes. It follows that $(U \times_X V)_{red} = (U_{red} \times_{X_{red}} V_{red})_{red}$. In particular, we see that $(U \times_X V)_{red}$ is an affine scheme and that

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}((U \times_X V)_{red})$$

is surjective, see Properties of Spaces, Lemma 66.3.3. Then $U \times_X V$ is affine by Limits of Spaces, Proposition 70.15.2. On the other hand, the morphism $U \times_X V \to U \times V$ of affine schemes is the composition

$$U \times_X V = X \times_{(X \times_S X)} (U \times_S V) \to U \times_S V \to U \times V$$

The first morphism is a monomorphism and locally of finite type (Lemma 87.11.2). The second morphism is an immersion (Schemes, Lemma 26.21.9). Hence the composition is a monomorphism which is locally of finite type. On the other hand, the composition is integral as the map on underlying reduced affine schemes is a closed immersion by the above and hence universally closed (use Morphisms, Lemma 29.44.7). Thus the ring map

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$

is an epimorphism which is integral of finite type hence finite hence surjective (use Morphisms, Lemma 29.44.4 and Algebra, Lemma 10.107.6). □

0AJ7 Definition 87.16.3. Let S be a scheme. Let X be a formal algebraic space over S. We say

(1) X is quasi-separated if the equivalent conditions of Lemma 87.16.1 are satisfied.
(2) X is separated if the equivalent conditions of Lemma 87.16.2 are satisfied.

The following lemma implies in particular that the completed tensor product of weakly admissible topological rings is a weakly admissible topological ring.

0AN3 Lemma 87.16.4. Let S be a scheme. Let $X \to Z$ and $Y \to Z$ be morphisms of formal algebraic spaces over S. Assume Z separated.

(1) If X and Y are affine formal algebraic spaces, then so is $X \times_Z Y$.
(2) If X and Y are McQuillan affine formal algebraic spaces, then so is $X \times_Z Y$.
(3) If X, Y, and Z are McQuillan affine formal algebraic spaces corresponding to the weakly admissible topological S-algebras A, B, and C, then $X \times_Z Y$ corresponds to $A \widehat{\otimes}_C B$.

Proof. Write $X = \text{colim}\, X_\lambda$ and $Y = \text{colim}\, Y_\mu$ as in Definition 87.9.1. Then $X \times_Z Y = \text{colim}\, X_\lambda \times_Z Y_\mu$. Since Z is separated the fibre products are affine, hence we see that (1) holds. Assume X and Y corresponds to the weakly admissible topological S-algebras A and B and $X_\lambda = \text{Spec}(A/I_\lambda)$ and $Y_\mu = \text{Spec}(B/J_\mu)$. Then

$$X_\lambda \times_Z Y_\mu \to X_\lambda \times Y_\mu \to \text{Spec}(A \otimes B)$$

is a closed immersion. Thus one of the conditions of Lemma 87.9.6 holds and we conclude that $X \times_Z Y$ is McQuillan. If also Z is McQuillan corresponding to C, then

$$X_\lambda \times_Z Y_\mu = \text{Spec}(A/I_\lambda \otimes_C B/J_\mu)$$

hence we see that the weakly admissible topological ring corresponding to $X \times_Z Y$ is the completed tensor product (see Definition 87.4.7). □

0APX Lemma 87.16.5. Let S be a scheme. Let X be a formal algebraic space over S. Let $U \to X$ be a morphism where U is a separated algebraic space over S. Then $U \to X$ is separated.

Proof. The statement makes sense because $U \to X$ is representable by algebraic spaces (Lemma 87.11.3). Let T be a scheme and $T \to X$ a morphism. We have to show that $U \times_X T \to T$ is separated. Since $U \times_X T \to U \times_S T$ is a monomorphism, it suffices to show that $U \times_S T \to T$ is separated. As this is the base change of $U \to S$ this follows. We used in the argument above: Morphisms of Spaces, Lemmas 67.4.4, 67.4.8, 67.10.3, and 67.4.11. □

87.17. Quasi-compact formal algebraic spaces

0AJ8 Here is the characterization of quasi-compact formal algebraic spaces.

0AJ9 Lemma 87.17.1. Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent

(1) the reduction of X (Lemma 87.12.1) is a quasi-compact algebraic space,
(2) we can find $\{X_i \to X\}_{i \in I}$ as in Definition 87.11.1 with I finite,
(3) there exists a morphism $Y \to X$ representable by algebraic spaces which is étale and surjective and where Y is an affine formal algebraic space.

Proof. Omitted. □

0AJA Definition 87.17.2. Let S be a scheme. Let X be a formal algebraic space over S. We say X is quasi-compact if the equivalent conditions of Lemma 87.17.1 are satisfied.

0AJB Lemma 87.17.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent

(1) the induced map $f_{red} : X_{red} \to Y_{red}$ between reductions (Lemma 87.12.1) is a quasi-compact morphism of algebraic spaces,
(2) for every quasi-compact scheme T and morphism $T \to Y$ the fibre product $X \times_Y T$ is a quasi-compact formal algebraic space,
(3) for every affine scheme T and morphism $T \to Y$ the fibre product $X \times_Y T$ is a quasi-compact formal algebraic space, and
(4) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j$ is a quasi-compact formal algebraic space.

Proof. Omitted. □

0AJC Definition 87.17.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. We say f is quasi-compact if the equivalent conditions of Lemma 87.17.3 are satisfied.

This agrees with the already existing notion when the morphism is representable by algebraic spaces (and in particular when it is representable).

0AM2 Lemma 87.17.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces. Then f is quasi-compact in the sense of Definition 87.17.4 if and only if f is quasi-compact in the sense of Bootstrap, Definition 80.4.1.

Proof. This is immediate from the definitions and Lemma 87.17.3. □

87.18. Quasi-compact and quasi-separated formal algebraic spaces

0AJD The following result is due to Yasuda, see [Yas09, Proposition 3.32].

0AJE **Lemma 87.18.1.** Let S be a scheme. Let X be a quasi-compact and quasi-separated formal algebraic space over S. Then $X = \mathrm{colim}\, X_\lambda$ for a system of algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over a directed set Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening. [Yas09, Proposition 3.32]

Proof. By Lemma 87.17.1 we may choose an affine formal algebraic space Y and a representable surjective étale morphism $Y \to X$. Write $Y = \mathrm{colim}\, Y_\lambda$ as in Definition 87.9.1.

Pick $\lambda \in \Lambda$. Then $Y_\lambda \times_X Y$ is a scheme by Lemma 87.9.11. The reduction (Lemma 87.12.1) of $Y_\lambda \times_X Y$ is equal to the reduction of $Y_{red} \times_{X_{red}} Y_{red}$ which is quasi-compact as X is quasi-separated and Y_{red} is affine. Therefore $Y_\lambda \times_X Y$ is a quasi-compact scheme. Hence there exists a $\mu \geq \lambda$ such that $\mathrm{pr}_2 : Y_\lambda \times_X Y \to Y$ factors through Y_μ, see Lemma 87.9.4. Let Z_λ be the scheme theoretic image of the morphism $\mathrm{pr}_2 : Y_\lambda \times_X Y \to Y_\mu$. This is independent of the choice of μ and we can and will think of $Z_\lambda \subset Y$ as the scheme theoretic image of the morphism $\mathrm{pr}_2 : Y_\lambda \times_X Y \to Y$. Observe that Z_λ is also equal to the scheme theoretic image of the morphism $\mathrm{pr}_1 : Y \times_X Y_\lambda \to Y$ since this is isomorphic to the morphism used to define Z_λ. We claim that $Z_\lambda \times_X Y = Y \times_X Z_\lambda$ as subfunctors of $Y \times_X Y$. Namely, since $Y \to X$ is étale we see that $Z_\lambda \times_X Y$ is the scheme theoretic image of the morphism

$$\mathrm{pr}_{13} = \mathrm{pr}_1 \times \mathrm{id}_Y : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

by Morphisms of Spaces, Lemma 67.16.3. By the same token, $Y \times_X Z_\lambda$ is the scheme theoretic image of the morphism

$$\mathrm{pr}_{13} = \mathrm{id}_Y \times \mathrm{pr}_2 : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

The claim follows. Then $R_\lambda = Z_\lambda \times_X Y = Y \times_X Z_\lambda$ together with the morphism $R_\lambda \to Z_\lambda \times_S Z_\lambda$ defines an étale equivalence relation. In this way we obtain an algebraic space $X_\lambda = Z_\lambda / R_\lambda$. By construction the diagram

$$\begin{tikzcd} Z_\lambda \arrow[r] \arrow[d] & Y \arrow[d] \\ X_\lambda \arrow[r] & X \end{tikzcd}$$

is cartesian (because X is the coequalizer of the two projections $R = Y \times_X Y \to Y$, because $Z_\lambda \subset Y$ is R-invariant, and because R_λ is the restriction of R to Z_λ). Hence $X_\lambda \to X$ is representable and a closed immersion, see Spaces, Lemma 65.11.5. On the other hand, since $Y_\lambda \subset Z_\lambda$ we see that $(X_\lambda)_{red} = X_{red}$, in other words, $X_\lambda \to X$ is a thickening. Finally, we claim that

$$X = \mathrm{colim}\, X_\lambda$$

We have $Y \times_X X_\lambda = Z_\lambda \supset Y_\lambda$. Every morphism $T \to X$ where T is a scheme over S lifts étale locally to a morphism into Y which lifts étale locally into a morphism into some Y_λ. Hence $T \to X$ lifts étale locally on T to a morphism into X_λ. This finishes the proof. □

0AJF **Remark 87.18.2.** In this remark we translate the statement and proof of Lemma 87.18.1 into the language of formal schemes à la EGA. Looking at Remark 87.13.4 we see that the lemma can be translated as follows

(*) Every quasi-compact and quasi-separated formal scheme has a fundamental system of ideals of definition.

To prove this we first use the induction principle (reformulated for quasi-compact and quasi-separated formal schemes) of Cohomology of Schemes, Lemma 30.4.1 to reduce to the following situation: $\mathfrak{X} = \mathfrak{U} \cup \mathfrak{V}$ with $\mathfrak{U}$, $\mathfrak{V}$ open formal subschemes, with $\mathfrak{V}$ affine, and the result is true for $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{U} \cap \mathfrak{V}$. Pick any ideals of definition $\mathcal{I} \subset \mathcal{O}_\mathfrak{U}$ and $\mathcal{J} \subset \mathcal{O}_\mathfrak{V}$. By our assumption that we have a fundamental system of ideals of definition on $\mathfrak{U}$ and $\mathfrak{V}$ and because $\mathfrak{U} \cap \mathfrak{V}$ is quasi-compact, we can find ideals of definition $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$ such that

$$\mathcal{I}'|_{\mathfrak{U} \cap \mathfrak{V}} \subset \mathcal{J}|_{\mathfrak{U} \cap \mathfrak{V}} \quad \text{and} \quad \mathcal{J}'|_{\mathfrak{U} \cap \mathfrak{V}} \subset \mathcal{I}|_{\mathfrak{U} \cap \mathfrak{V}}$$

Let $U \to U' \to \mathfrak{U}$ and $V \to V' \to \mathfrak{V}$ be the closed immersions determined by the ideals of definition $\mathcal{I}' \subset \mathcal{I} \subset \mathcal{O}_\mathfrak{U}$ and $\mathcal{J}' \subset \mathcal{J} \subset \mathcal{O}_\mathfrak{V}$. Let $\mathfrak{U} \cap V$ denote the open subscheme of V whose underlying topological space is that of $\mathfrak{U} \cap \mathfrak{V}$. By our choice of $\mathcal{I}'$ there is a factorization $\mathfrak{U} \cap V \to U'$. We define similarly $U \cap \mathfrak{V}$ which factors through V'. Then we consider

$$Z_U = \text{scheme theoretic image of } U \amalg (\mathfrak{U} \cap V) \longrightarrow U'$$

and

$$Z_V = \text{scheme theoretic image of } (U \cap \mathfrak{V}) \amalg V \longrightarrow V'$$

Since taking scheme theoretic images of quasi-compact morphisms commutes with restriction to opens (Morphisms, Lemma 29.6.3) we see that $Z_U \cap \mathfrak{V} = \mathfrak{U} \cap Z_V$. Thus Z_U and Z_V glue to a scheme Z which comes equipped with a morphism $Z \to \mathfrak{X}$. Analogous to the discussion in Remark 87.13.3 we see that Z corresponds to a weak ideal of definition $\mathcal{I}_Z \subset \mathcal{O}_\mathfrak{X}$. Note that $Z_U \subset U'$ and that $Z_V \subset V'$. Thus the collection of all $\mathcal{I}_Z$ constructed in this manner forms a fundamental system of weak ideals of definition. Hence a subfamily gives a fundamental system of ideals of definition, see Remark 87.13.4.

0DE8 Lemma 87.18.3. Let S be a scheme. Let X be a formal algebraic space over S. Then X is an affine formal algebraic space if and only if its reduction X_{red} (Lemma 87.12.1) is affine.

Proof. By Lemmas 87.16.1 and 87.17.1 and Definitions 87.16.3 and 87.17.2 we see that X is quasi-compact and quasi-separated. By Yasuda's lemma (Lemma 87.18.1) we can write $X = \mathrm{colim}\, X_\lambda$ as a filtered colimit of thickenings of algebraic spaces. However, each X_λ is affine by Limits of Spaces, Lemma 70.15.3 because $(X_\lambda)_{red} = X_{red}$. Hence X is an affine formal algebraic space by definition. $\square$

87.19. Morphisms representable by algebraic spaces

0AJG Let $f : X \to Y$ be a morphism of formal algebraic spaces which is representable by algebraic spaces. For these types of morphisms we have a lot of theory at our disposal, thanks to the work done in the chapters on algebraic spaces.

0APY Lemma 87.19.1. The composition of morphisms representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).

Proof. See Bootstrap, Lemma 80.3.8. $\square$

0APZ Lemma 87.19.2. A base change of a morphism representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).

Proof. See Bootstrap, Lemma 80.3.3. □

0AQ0 **Lemma 87.19.3.** Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. If $g \circ f : X \to Z$ is representable by algebraic spaces, then $f : X \to Y$ is representable by algebraic spaces.

Proof. Note that the diagonal of $Y \to Z$ is representable by Lemma 87.15.5. Thus $X \to Y$ is representable by algebraic spaces by Bootstrap, Lemma 80.3.10. □

The property of being representable by algebraic spaces is local on the source and the target.

0AN4 **Lemma 87.19.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent:

(1) the morphism f is representable by algebraic spaces,
(2) there exists a commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are formal algebraic spaces, the vertical arrows are representable by algebraic spaces, $U \to X$ is surjective étale, and $U \to V$ is representable by algebraic spaces,
(3) for any commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are formal algebraic spaces and the vertical arrows are representable by algebraic spaces, the morphism $U \to V$ is representable by algebraic spaces,
(4) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that $X_{ji} \to Y_j$ is representable by algebraic spaces for each j and i,
(5) there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces, such that $X_i \to Y_i$ is representable by algebraic spaces, and
(6) add more here.

Proof. It is clear that (1) implies (2) because we can take $U = X$ and $V = Y$. Conversely, (2) implies (1) by Bootstrap, Lemma 80.11.4 applied to $U \to X \to Y$.

Assume (1) is true and consider a diagram as in (3). Then $U \to Y$ is representable by algebraic spaces (as the composition $U \to X \to Y$, see Bootstrap, Lemma 80.3.8) and factors through V. Thus $U \to V$ is representable by algebraic spaces by Lemma 87.19.3.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product $Y_j \times_Y X$ is a formal algebraic space by Lemma 87.15.3. It is clear that (4) implies (5).

Assume $X_i \to Y_i \to Y$ as in (5). Then we set $V = \coprod Y_i$ and $U = \coprod X_i$ to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j any covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. Then $X_{ij} \to Y_j$ is representable by algebraic spaces by (3) and we see that (4) is true. This concludes the proof. □

0AJH Lemma 87.19.5. Let S be a scheme. Let Y be an affine formal algebraic space over S. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Then X is a formal algebraic space.

Proof. Write $Y = \operatorname{colim} Y_\lambda$ as in Definition 87.9.1. For each λ the fibre product $X \times_Y Y_\lambda$ is an algebraic space. Hence $X = \operatorname{colim} X \times_Y Y_\lambda$ is a formal algebraic space by Lemma 87.13.1. □

0AJI Lemma 87.19.6. Let S be a scheme. Let Y be a formal algebraic space over S. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Then X is a formal algebraic space.

Proof. Let $\{Y_i \to Y\}$ be as in Definition 87.11.1. Then $X \times_Y Y_i \to X$ is a family of morphisms representable by algebraic spaces, étale, and jointly surjective. Thus it suffices to show that $X \times_Y Y_i$ is a formal algebraic space, see Lemma 87.15.1. This follows from Lemma 87.19.5. □

0AKN Lemma 87.19.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then f is representable (by schemes) and affine.

Proof. We will show that f is affine; it will then follow that f is representable and affine by Morphisms of Spaces, Lemma 67.20.3. Write $Y = \operatorname{colim} Y_\mu$ and $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Let $T \to Y$ be a morphism where T is a scheme over S. We have to show that $X \times_Y T \to T$ is affine, see Bootstrap, Definition 80.4.1. To do this we may assume that T is affine and we have to prove that $X \times_Y T$ is affine. In this case $T \to Y$ factors through $Y_\mu \to Y$ for some μ, see Lemma 87.9.4. Since f is quasi-compact we see that $X \times_Y T$ is quasi-compact (Lemma 87.17.3). Hence $X \times_Y T \to X$ factors through X_λ for some λ. Similarly $X_\lambda \to Y$ factors through Y_μ after increasing μ. Then $X \times_Y T = X_\lambda \times_{Y_\mu} T$. We conclude as fibre products of affine schemes are affine. □

0AN5 Lemma 87.19.8. Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent

(1) $\operatorname{Spf}(\varphi) : \operatorname{Spf}(B) \to \operatorname{Spf}(A)$ is representable by algebraic spaces,
(2) $\operatorname{Spf}(\varphi) : \operatorname{Spf}(B) \to \operatorname{Spf}(A)$ is representable (by schemes),
(3) φ is taut, see Definition 87.5.1.

Proof. Parts (1) and (2) are equivalent by Lemma 87.19.7.

Assume the equivalent conditions (1) and (2) hold. If $I \subset A$ is a weak ideal of definition, then $\operatorname{Spec}(A/I) \to \operatorname{Spf}(A)$ is representable and a thickening (this is clear from the construction of the formal spectrum but it also follows from Lemma

87.9.6). Then $\mathrm{Spec}(A/I) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) \to \mathrm{Spf}(B)$ is representable and a thickening as a base change. Hence by Lemma 87.9.6 there is a weak ideal of definition $J(I) \subset B$ such that $\mathrm{Spec}(A/I) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) = \mathrm{Spec}(B/J(I))$ as subfunctors of $\mathrm{Spf}(B)$. We obtain a cartesian diagram

$$\begin{tikzcd} \mathrm{Spec}(B/J(I)) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[d] \\ \mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

By Lemma 87.16.4 we see that $B/J(I) = B\widehat{\otimes}_A A/I$. It follows that $J(I)$ is the closure of the ideal $\varphi(I)B$, see Lemma 87.4.11. Since $\mathrm{Spf}(A) = \mathrm{colim}\,\mathrm{Spec}(A/I)$ with I as above, we find that $\mathrm{Spf}(B) = \mathrm{colim}\,\mathrm{Spec}(B/J(I))$. Thus the ideals $J(I)$ form a fundamental system of weak ideals of definition (see Lemma 87.9.6). Hence (3) holds.

Assume (3) holds. We are essentially just going to reverse the arguments given in the previous paragraph. Let $I \subset A$ be a weak ideal of definition. By Lemma 87.16.4 we get a cartesian diagram

$$\begin{tikzcd} \mathrm{Spf}(B\widehat{\otimes}_A A/I) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[d] \\ \mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

If $J(I)$ is the closure of IB, then $J(I)$ is open in B by tautness of φ. Hence if J is open in B and $J \subset J(B)$, then $B/J \otimes_A A/I = B/(IB + J) = B/J(I)$ because $J(I) = \bigcap_{J \subset B \text{ open}} (IB + J)$ by Lemma 87.4.2. Hence the limit defining the completed tensor product collapses to give $B\widehat{\otimes}_A A/I = B/J(I)$. Thus $\mathrm{Spf}(B\widehat{\otimes}_A A/I) = \mathrm{Spec}(B/J(I))$. This proves that $\mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spec}(A/I)$ is representable for every weak ideal of definition $I \subset A$. Since every morphism $T \to \mathrm{Spf}(A)$ with T quasi-compact factors through $\mathrm{Spec}(A/I)$ for some weak ideal of definition I (Lemma 87.9.4) we conclude that $\mathrm{Spf}(\varphi)$ is representable, i.e., (2) holds. This finishes the proof. $\square$

0AKP Lemma 87.19.9. Let S be a scheme. Let Y be an affine formal algebraic space. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable and affine. Then

(1) X is an affine formal algebraic space,
(2) if Y is countably indexed, then X is countably indexed,
(3) if Y is countably indexed and classical, then X is countably indexed and classical,
(4) if Y is weakly adic, then X is weakly adic,
(5) if Y is adic*, then X is adic*, and
(6) if Y is Noetherian and f is (locally) of finite type, then X is Noetherian.

Proof. Proof of (1). Write $Y = \mathrm{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 87.9.1. Since f is representable and affine, the fibre products $X_\lambda = Y_\lambda \times_Y X$ are affine. And $X = \mathrm{colim}\, Y_\lambda \times_Y X$. Thus X is an affine formal algebraic space.

Proof of (2). If Y is countably indexed, then in the argument above we may assume Λ is countable. Then we immediately see that X is countably indexed too.

Proof of (3), (4), and (5). In each of these cases the assumptions imply that Y is a countably indexed affine formal algebraic space (Lemma 87.10.3) and hence X is too by (2). Thus we may write $X = \text{Spf}(A)$ and $Y = \text{Spf}(B)$ for some weakly admissible topological S-algebras A and B, see Lemma 87.10.4. By Lemma 87.9.10 the morphism f corresponds to a continuous S-algebra homomorphism $\varphi : B \to A$. We see from Lemma 87.19.8 that φ is taut. We conclude that (3) follows from Lemma 87.5.9, (4) follows from Lemma 87.7.5, and (5) follows from Lemma 87.6.5.

Proof of (6). Combining (3) with Lemma 87.10.3 we see that X is adic*. Thus we can use the criterion of Lemma 87.10.5. First, it tells us the affine schemes Y_λ are Noetherian. Then $X_\lambda \to Y_\lambda$ is of finite type, hence X_λ is Noetherian too (Morphisms, Lemma 29.15.6). Then the criterion tells us X is Noetherian and the proof is complete. □

0AKQ Lemma 87.19.10. Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then

(1) if Y is countably indexed, then X is countably indexed,
(2) if Y is countably indexed and classical, then X is countably indexed and classical,
(3) if Y is weakly adic, then X is weakly adic,
(4) if Y is adic*, then X is adic*, and
(5) if Y is Noetherian and f is (locally) of finite type, then X is Noetherian.

Proof. Combine Lemmas 87.19.7 and 87.19.9. □

0AN6 Example 87.19.11. Let B be a weakly admissible topological ring. Let $B \to A$ be a ring map (no topology). Then we can consider

$$A^\wedge = \lim A/JA$$

where the limit is over all weak ideals of definition J of B. Then $A^\wedge$ (endowed with the limit topology) is a complete linearly topologized ring. The (open) kernel I of the surjection $A^\wedge \to A/JA$ is the closure of $JA^\wedge$, see Lemma 87.4.2. By Lemma 87.4.10 we see that I consists of topologically nilpotent elements. Thus I is a weak ideal of definition of $A^\wedge$ and we conclude $A^\wedge$ is a weakly admissible topological ring. Thus $\varphi : B \to A^\wedge$ is taut map of weakly admissible topological rings and

$$\text{Spf}(A^\wedge) \longrightarrow \text{Spf}(B)$$

is a special case of the phenomenon studied in Lemma 87.19.8.

0AN7 Remark 87.19.12 (Warning). The discussion in Lemmas 87.19.8, 87.19.9, and 87.19.10 is sharp in the following two senses:

(1) If A and B are weakly admissible rings and $\varphi : A \to B$ is a continuous map, then $\text{Spf}(\varphi) : \text{Spf}(B) \to \text{Spf}(A)$ is in general not representable.
(2) If $f : Y \to X$ is a representable morphism of affine formal algebraic spaces and $X = \text{Spf}(A)$ is McQuillan, then it does not follow that Y is McQuillan.

An example for (1) is to take $A = k$ a field (with discrete topology) and $B = k[[t]]$ with the t-adic topology. An example for (2) is given in Examples, Section 110.74.

The warning above notwithstanding, we do have the following result.

0AN8 **Lemma 87.19.13.** Let S be a scheme. Let Y be a McQuillan affine formal algebraic space over S, i.e., $Y = \text{Spf}(B)$ for some weakly admissible topological S-algebra B. Then there is an equivalence of categories between

(1) the category of morphisms $f : X \to Y$ of affine formal algebraic spaces which are representable by algebraic spaces and étale, and
(2) the category of topological B-algebras of the form $A^\wedge$ where A is an étale B-algebra and $A^\wedge = \lim A/JA$ with $J \subset B$ running over the weak ideals of definition of B.

The equivalence is given by sending $A^\wedge$ to $X = \text{Spf}(A^\wedge)$. In particular, any X as in (1) is McQuillan.

Proof. Let A be an étale B-algebra. Then $B/J \to A/JA$ is étale for every open ideal $J \subset B$. Hence the morphism $\text{Spf}(A^\wedge) \to Y$ is representable and étale. The functor Spf is fully faithful by Lemma 87.9.10. To finish the proof we will show in the next paragraph that any $X \to Y$ as in (1) is in the essential image.

Choose a weak ideal of definition $J_0 \subset B$. Set $Y_0 = \text{Spec}(B/J_0)$ and $X_0 = Y_0 \times_Y X$. Then $X_0 \to Y_0$ is an étale morphism of affine schemes (see Lemma 87.19.7). Say $X_0 = \text{Spec}(A_0)$. By Algebra, Lemma 10.143.10 we can find an étale algebra map $B \to A$ such that $A_0 \cong A/J_0A$. Consider an ideal of definition $J \subset J_0$. As above we may write $\text{Spec}(B/J) \times_Y X = \text{Spec}(\bar{A})$ for some étale ring map $B/J \to \bar{A}$. Then both $B/J \to \bar{A}$ and $B/J \to A/JA$ are étale ring maps lifting the étale ring map $B/J_0 \to A_0$. By More on Algebra, Lemma 15.11.2 there is a unique B/J-algebra isomorphism $\varphi_J : A/JA \to \bar{A}$ lifting the identification modulo J_0. Since the maps φ_J are unique they are compatible for varying J. Thus

$$X = \text{colim}\, \text{Spec}(B/J) \times_Y X = \text{colim}\, \text{Spec}(A/JA) = \text{Spf}(A)$$

and we see that the lemma holds. $\square$

0AN9 **Lemma 87.19.14.** With notation and assumptions as in Lemma 87.19.13 let $f : X \to Y$ correspond to $B \to A^\wedge$. The following are equivalent

(1) $f : X \to Y$ is surjective,
(2) $B \to A$ is faithfully flat,
(3) for every weak ideal of definition $J \subset B$ the ring map $B/J \to A/JA$ is faithfully flat, and
(4) for some weak ideal of definition $J \subset B$ the ring map $B/J \to A/JA$ is faithfully flat.

Proof. Let $J \subset B$ be a weak ideal of definition. As every element of J is topologically nilpotent, we see that every element of $1 + J$ is a unit. It follows that J is contained in the Jacobson radical of B (Algebra, Lemma 10.19.1). Hence a flat ring map $B \to A$ is faithfully flat if and only if $B/J \to A/JA$ is faithfully flat (Algebra, Lemma 10.39.16). In this way we see that (2) – (4) are equivalent. If (1) holds, then for every weak ideal of definition $J \subset B$ the morphism $\text{Spec}(A/JA) = \text{Spec}(B/J) \times_Y X \to \text{Spec}(B/J)$ is surjective which implies (3). Conversely, assume (3). A morphism $T \to Y$ with T quasi-compact factors through $\text{Spec}(B/J)$ for some ideal of definition J of B (Lemma 87.9.4). Hence $X \times_Y T = \text{Spec}(A/JA) \times_{\text{Spec}(B/J)} T \to T$ is surjective as a base change of the surjective morphism $\text{Spec}(A/JA) \to \text{Spec}(B/J)$. Thus (1) holds. $\square$

87.20. Types of formal algebraic spaces

0AKR In this section we define "locally Noetherian", "locally adic*", "locally weakly adic", "locally countably indexed and classical", and "locally countably indexed" formal algebraic spaces. The types "locally adic", "locally classical", and "locally McQuillan" are missing as we do not know how to prove the analogue of the following lemmas for those cases (it would suffice to prove the analogue of these lemmas for étale coverings between affine formal algebraic spaces).

0AKS **Lemma 87.20.1.** Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is countably indexed if and only if Y is countably indexed.

Proof. Assume X is countably indexed. We write $X = \operatorname{colim} X_n$ as in Lemma 87.10.1. Write $Y = \operatorname{colim} Y_\lambda$ as in Definition 87.9.1. For every n we can pick a λ_n such that $X_n \to Y$ factors through Y_{λ_n}, see Lemma 87.9.4. On the other hand, for every λ the scheme $Y_\lambda \times_Y X$ is affine (Lemma 87.19.7) and hence $Y_\lambda \times_Y X \to X$ factors through X_n for some n (Lemma 87.9.4). Picture

$$\begin{tikzcd} Y_\lambda \times_Y X \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X \arrow[d] \\ Y_\lambda \arrow[r, dotted] \arrow[rr, bend right] & Y_{\lambda_n} \arrow[r] & Y \end{tikzcd}$$

If we can show the dotted arrow exists, then we conclude that $Y = \operatorname{colim} Y_{\lambda_n}$ and Y is countably indexed. To do this we pick a μ with $\mu \geq \lambda$ and $\mu \geq \lambda_n$. Thus both $Y_\lambda \to Y$ and $Y_{\lambda_n} \to Y$ factor through $Y_\mu \to Y$. Say $Y_\mu = \operatorname{Spec}(B_\mu)$, the closed subscheme Y_λ corresponds to $J \subset B_\mu$, and the closed subscheme Y_{λ_n} corresponds to $J' \subset B_\mu$. We are trying to show that $J' \subset J$. By the diagram above we know $J'A_\mu \subset JA_\mu$ where $Y_\mu \times_Y X = \operatorname{Spec}(A_\mu)$. Since $X \to Y$ is surjective and flat the morphism $Y_\lambda \times_Y X \to Y_\lambda$ is a faithfully flat morphism of affine schemes, hence $B_\mu \to A_\mu$ is faithfully flat. Thus $J' \subset J$ as desired.

Assume Y is countably indexed. Then X is countably indexed by Lemma 87.19.10. $\square$

0GXR **Lemma 87.20.2.** Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is countably indexed and classical if and only if Y is countably indexed and classical.

Proof. We have already seen the implication in one direction in Lemma 87.19.10. For the other direction, note that by Lemma 87.20.1 we may assume both X and Y are countably indexed. Thus $X = \operatorname{Spf}(A)$ and $Y = \operatorname{Spf}(B)$ for some weakly admissible topological S-algebras A and B, see Lemma 87.10.4. By Lemma 87.9.10 the morphism $X \to Y$ corresponds to a continuous S-algebra homomorphism $\varphi : B \to A$. We see from Lemma 87.19.8 that φ is taut. Let $J \subset B$ be an open ideal and let $I \subset A$ be the closure of JA. By Lemmas 87.16.4 and 87.4.11 we see that $\operatorname{Spec}(B/J) \times_Y X = \operatorname{Spec}(A/I)$. Hence $B/J \to A/I$ is faithfully flat (since $X \to Y$ is surjective and flat). This means that $\varphi : B \to A$ is as in Section 87.8 (with the roles of A and B swapped). We conclude that the lemma holds by Lemma 87.8.2. $\square$

0GXS Lemma 87.20.3. Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is weakly adic if and only if Y is weakly adic.

Proof. The proof is exactly the same as the proof of Lemma 87.20.2 except that at the end we use Lemma 87.8.4. □

0AKT Lemma 87.20.4. Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is adic* if and only if Y is adic*.

Proof. The proof is exactly the same as the proof of Lemma 87.20.2 except that at the end we use Lemma 87.8.5. □

0AKW Lemma 87.20.5. Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and (locally) of finite type. Then X is Noetherian if and only if Y is Noetherian.

Proof. Observe that a Noetherian affine formal algebraic space is adic*, see Lemma 87.10.3. Thus by Lemma 87.20.4 we may assume that both X and Y are adic*. We will use the criterion of Lemma 87.10.5 to see that the lemma holds. Namely, write $Y = \mathrm{colim}\, Y_n$ as in Lemma 87.10.1. For each n set $X_n = Y_n \times_Y X$. Then X_n is an affine scheme (Lemma 87.19.7) and $X = \mathrm{colim}\, X_n$. Each of the morphisms $X_n \to Y_n$ is faithfully flat and of finite type. Thus the lemma follows from the fact that in this situation X_n is Noetherian if and only if Y_n is Noetherian, see Algebra, Lemma 10.164.1 (to go down) and Algebra, Lemma 10.31.1 (to go up). □

0AKX Lemma 87.20.6. Let S be a scheme. Let

$$P \in \left\{\begin{matrix} \textit{countably indexed}, \\ \textit{countably indexed and classical}, \\ \textit{weakly adic},\ \textit{adic}*,\ \textit{Noetherian} \end{matrix}\right\}$$

Let X be a formal algebraic space over S. The following are equivalent

(1) if Y is an affine formal algebraic space and $f : Y \to X$ is representable by algebraic spaces and étale, then Y has property P,
(2) for some $\{X_i \to X\}_{i \in I}$ as in Definition 87.11.1 each X_i has property P.

Proof. It is clear that (1) implies (2). Assume (2) and let $Y \to X$ be as in (1). Since the fibre products $X_i \times_X Y$ are formal algebraic spaces (Lemma 87.15.2) we can pick coverings $\{X_{ij} \to X_i \times_X Y\}$ as in Definition 87.11.1. Since Y is quasi-compact, there exist $(i_1, j_1), \ldots, (i_n, j_n)$ such that

$$X_{i_1 j_1} \amalg \ldots \amalg X_{i_n j_n} \longrightarrow Y$$

is surjective and étale. Then $X_{i_k j_k} \to X_{i_k}$ is representable by algebraic spaces and étale hence $X_{i_k j_k}$ has property P by Lemma 87.19.10. Then $X_{i_1 j_1} \amalg \ldots \amalg X_{i_n j_n}$ is an affine formal algebraic space with property P (small detail omitted on finite disjoint unions of affine formal algebraic spaces). Hence we conclude by applying one of Lemmas 87.20.1, 87.20.2, 87.20.3, 87.20.4, and 87.20.5. □

The previous lemma clears the way for the following definition.

0AKY Definition 87.20.7. Let S be a scheme. Let X be a formal algebraic space over S. We say X is locally countably indexed, locally countably indexed and classical, locally weakly adic, locally adic*, or locally Noetherian if the equivalent conditions of Lemma 87.20.6 hold for the corresponding property.

The formal completion of a locally Noetherian algebraic space along a closed subset is a locally Noetherian formal algebraic space.

0AQ1 Lemma 87.20.8. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $X_{/T}$ be the formal completion of X along T.

(1) If $X \setminus T \to X$ is quasi-compact, then $X_{/T}$ is locally adic*.
(2) If X is locally Noetherian, then $X_{/T}$ is locally Noetherian.

Proof. Choose a surjective étale morphism $U \to X$ with $U = \coprod U_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. Let $T_i \subset U_i$ be the inverse image of T. We have $X_{/T} \times_X U_i = (U_i)_{/T_i}$ (Lemma 87.14.4). Hence $\{(U_i)_{/T_i} \to X_{/T}\}$ is a covering as in Definition 87.11.1. Moreover, if $X \setminus T \to X$ is quasi-compact, so is $U_i \setminus T_i \to U_i$ and if X is locally Noetherian, so is U_i. Thus the lemma follows from the affine case which is Lemma 87.14.6. $\square$

0CKX Remark 87.20.9 (Warning). Suppose $X = \text{Spec}(A)$ and $T \subset X$ is the zero locus of a finitely generated ideal $I \subset A$. Let $J = \sqrt{I}$ be the radical of I. Then from the definitions we see that $X_{/T} = \text{Spf}(A^\wedge)$ where $A^\wedge = \lim A/I^n$ is the I-adic completion of A. On the other hand, the map $A^\wedge \to \lim A/J^n$ from the I-adic completion to the J-adic completion can fail to be a ring isomorphisms. As an example let

$$A = \bigcup\nolimits_{n \geq 1} \mathbf{C}[t^{1/n}]$$

and $I = (t)$. Then $J = \mathfrak m$ is the maximal ideal of the valuation ring A and $J^2 = J$. Thus the J-adic completion of A is $\mathbf{C}$ whereas the I-adic completion is the valuation ring described in Example 87.13.2 (but in particular it is easy to see that $A \subset A^\wedge$).

0GBB Lemma 87.20.10. Let S be a scheme. Let $X \to Y$ and $Z \to Y$ be morphisms of formal algebraic space over S. Then

(1) If X and Z are locally countably indexed, then $X \times_Y Z$ is locally countably indexed.
(2) If X and Z are locally countably indexed and classical, then $X \times_Y Z$ is locally countably indexed and classical.
(3) If X and Z are weakly adic, then $X \times_Y Z$ is weakly adic.
(4) If X and Z are locally adic*, then $X \times_Y Z$ is locally adic*.
(5) If X and Z are locally Noetherian and $X_{red} \to Y_{red}$ is locally of finite type, then $X \times_Y Z$ is locally Noetherian.

Proof. Choose a covering $\{Y_j \to Y\}$ as in Definition 87.11.1. For each j choose a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. For each j choose a covering $\{Z_{jk} \to Y_j \times_Y Z\}$ as in Definition 87.11.1. Observe that $X_{ji} \times_{Y_j} Z_{jk}$ is an affine formal algebraic space by Lemma 87.16.4. Hence

$$\{X_{ji} \times_{Y_j} Z_{jk} \to X \times_Y Z\}$$

is a covering as in Definition 87.11.1. Thus it suffices to prove (1), (2), (3), and (4) in case X, Y, and Z are affine formal algebraic spaces.

Assume X and Z are countably indexed. Say $X = \operatorname{colim} X_n$ and $Z = \operatorname{colim} Z_m$ as in Lemma 87.10.1. Write $Y = \operatorname{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 87.9.1. For each n and m we can find $\lambda_{n,m} \in \Lambda$ such that $X_n \to Y$ and $Z_m \to Y$ factor through $Y_{\lambda_{n,m}}$ (for example see Lemma 87.9.4). Pick $\lambda_0 \in \Lambda$. By induction for $t \geq 1$ pick an element $\lambda_t \in \Lambda$ such that $\lambda_t \geq \lambda_{n,m}$ for all $1 \leq n, m \leq t$ and $\lambda_t \geq \lambda_{t-1}$. Set $Y' = \operatorname{colim} Y_{\lambda_t}$. Then $Y' \to Y$ is a monomorphism such that $X \to Y$ and $Z \to Y$ factor through Y'. Hence we may replace Y by Y', i.e., we may assume that Y is countably indexed.

Assume X, Y, and Z are countably indexed. By Lemma 87.10.4 we can write $X = \text{Spf}(A)$, $Y = \text{Spf}(B)$, $Z = \text{Spf}(C)$ for some weakly admissible topological rings A, B, and C. The morphsms $X \to Y$ and $Z \to Y$ are given by continuous ring maps $B \to A$ and $B \to C$, see Lemma 87.9.10. By Lemma 87.16.4 we see that $X \times_Y Z = \text{Spf}(A \widehat{\otimes}_B C)$ and that $A \widehat{\otimes}_B C$ is a weakly admissible topological ring. In particular, we see that $X \times_Y Z$ is countably indexed by Lemma 87.4.12 part (3). This proves (1).

Proof of (2). In this case X and Z are countably indexed and hence the arguments above show that $X \times_Y Z$ is the formal spectrum of $A \widehat{\otimes}_B C$ where A and C are admissible. Then $A \widehat{\otimes}_B C$ is admissible by Lemma 87.4.12 part (2).

Proof of (3). As before we conclude that $X \times_Y Z$ is the formal spectrum of $A \widehat{\otimes}_B C$ where A and C are weakly adic. Then $A \widehat{\otimes}_B C$ is weakly adic by Lemma 87.7.6.

Proof of (4). Arguing as above, this follows from Lemma 87.4.12 part (4).

Proof of (5). To deduce case (5) from Lemma 87.4.12 part (5) we need to show the hypotheses match. Namely, with notation as in the first parapgrah of the proof, if $X_{red} \to Y_{red}$ is locally of finite type, then $(X_{ji})_{red} \to (Y_j)_{red}$ is locally of finite type. This follows from Morphisms of Spaces, Lemma 67.23.4 and the fact that in the commutative diagram

$$\begin{tikzcd} (X_{ji})_{red} \arrow[r] \arrow[d] & (Y_j)_{red} \arrow[d] \\ X_{red} \arrow[r] & Y_{red} \end{tikzcd}$$

the vertical morphisms are étale. Namely, we have $(X_{ji})_{red} = X_{ij} \times_X X_{red}$ and $(Y_j)_{red} = Y_j \times_Y Y_{red}$ by Lemma 87.12.3. Thus as above we reduce to the case where X, Y, Z are affine formal algebraic spaces, X, Z are Noetherian, and $X_{red} \to Y_{red}$ is of finite type. Next, in the second paragraph of the proof we replaced Y by Y' but by construction $Y_{red} = Y'_{red}$, hence the finite type assumption is preserved by this replacement. Then we see that X, Y, Z correspond to A, B, C and $X \times_Y Z$ to $A \widehat{\otimes}_B C$ with A, C Noetherian adic. Finally, taking the reduction corresponds to dividing by the ideal of topologically nilpotent elements (Example 87.12.2) hence the fact that $X_{red} \to Y_{red}$ is of finite type does indeed mean that $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type and the proof is complete. □

0GHM Lemma 87.20.11. Let S be a scheme. Let X be a locally Noetherian formal algebraic space over S. Then $X = \operatorname{colim} X_n$ for a system $X_1 \to X_2 \to X_3 \to \ldots$ of finite order thickenings of locally Noetherian algebraic spaces over S where $X_1 = X_{red}$ and X_n is the nth infinitesimal neighbourhood of X_1 in X_m for all $m \geq n$.

Proof. We only sketch the proof and omit some of the details. Set $X_1 = X_{red}$. Define $X_n \subset X$ as the subfunctor defined by the rule: a morphism $f : T \to X$ where T is a scheme factors through X_n if and only if the nth power of the ideal sheaf of the closed immersion $X_1 \times_X T \to T$ is zero. Then $X_n \subset X$ is a subsheaf as vanishing of quasi-coherent modules can be checked fppf locally. We claim that $X_n \to X$ is representable by schemes, a closed immersion, and that $X = \text{colim}\, X_n$ (as fppf sheaves). To check this we may work étale locally on X. Hence we may assume $X = \text{Spf}(A)$ is a Noetherian affine formal algebraic space. Then $X_1 = \text{Spec}(A/\mathfrak{a})$ where $\mathfrak{a} \subset A$ is the ideal of topologically nilpotent elements of the Noetherian adic topological ring A. Then $X_n = \text{Spec}(A/\mathfrak{a}^n)$ and we obtain what we want. $\square$

87.21. Morphisms and continuous ring maps

0ANA In this section we denote WAdm the category of weakly admissible topological rings and continuous ring homomorphisms. We define full subcategories

$$\text{WAdm} \supset \text{WAdm}^{count} \supset \text{WAdm}^{cic} \supset \text{WAdm}^{weakly\ adic} \supset \text{WAdm}^{adic*} \supset \text{WAdm}^{Noeth}$$

whose objects are

(1) WAdm^{count}: those weakly admissible topological rings A which have a countable fundamental system of open ideals,
(2) WAdm^{cic}: the admissible topological rings A which have a countable fundamental system of open ideals,
(3) $\text{WAdm}^{weakly\ adic}$: the weakly adic topological rings (Section 87.7),
(4) WAdm^{adic*}: the adic topological rings which have a finitely generated ideal of definition, and
(5) WAdm^{Noeth}: the adic topological rings which are Noetherian.

Clearly, the formal spectra of these types of rings are the basic building blocks of locally countably indexed, locally countably indexed and classical, locally weakly adic, locally adic*, and locally Noetherian formal algebraic spaces.

We briefly review the relationship between morphisms of countably indexed, affine formal algebraic spaces and morphisms of WAdm^{count}. Let S be a scheme. Let X and Y be countably indexed, affine formal algebraic spaces. Write $X = \text{Spf}(A)$ and $Y = \text{Spf}(B)$ topological S-algebras A and B in WAdm^{count}, see Lemma 87.10.4. By Lemma 87.9.10 there is a 1-to-1 correspondence between morphisms $f : X \to Y$ and continuous maps

$$\varphi : B \longrightarrow A$$

of topological S-algebras. The relationship is given by $f \mapsto f^\sharp$ and $\varphi \mapsto \text{Spf}(\varphi)$.

Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces and $U \to X$ and $V \to Y$ representable by algebraic spaces and étale. By Definition 87.20.7 (and hence via Lemma 87.20.6) we see that U and V are countably indexed affine formal algebraic spaces. By the discussion in the previous paragraph we see that $U \to V$ is isomorphic to $\text{Spf}(\varphi)$ for some continuous map

$$\varphi : B \longrightarrow A$$

of topological S-algebras in WAdm^{count}.

0ANB Lemma 87.21.1. Let $A \in \text{Ob}(\text{WAdm})$. Let $A \to A'$ be a ring map (no topology). Let $(A')^\wedge = \lim_{I \subset A \text{ w.i.d}} A'/IA'$ be the object of WAdm constructed in Example 87.19.11.

(1) If A is in WAdm^{count}, so is $(A')^\wedge$.
(2) If A is in WAdm^{cic}, so is $(A')^\wedge$.
(3) If A is in $\text{WAdm}^{weakly\ adic}$, so is $(A')^\wedge$.
(4) If A is in WAdm^{adic*}, so is $(A')^\wedge$.
(5) If A is in WAdm^{Noeth} and A' is Noetherian, then $(A')^\wedge$ is in WAdm^{Noeth}.

Proof. Recall that $A \to (A')^\wedge$ is taut, see discussion in Example 87.19.11. Hence statements (1), (2), (3), and (4) follow from Lemmas 87.5.7, 87.5.9, 87.7.5, and 87.6.5. Finally, assume that A is Noetherian and adic. By (4) we know that $(A')^\wedge$ is adic. By Algebra, Lemma 10.97.6 we see that $(A')^\wedge$ is Noetherian. Hence (5) holds. $\square$

0CBA Situation 87.21.2. Let P be a property of morphisms of WAdm^{count}. Consider commutative diagrams

0ANC (87.21.2.1)
$$\begin{tikzcd} A \arrow[r] & (A')^\wedge \\ B \arrow[u, "\varphi"] \arrow[r] & (B')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions

(1) A and B are objects of WAdm^{count},
(2) $A \to A'$ and $B \to B'$ are étale ring maps,
(3) $(A')^\wedge = \lim A'/IA'$, resp. $(B')^\wedge = \lim B'/JB'$ where $I \subset A$, resp. $J \subset B$ runs through the weakly admissible ideals of definition of A, resp. B,
(4) $\varphi : B \to A$ and $\varphi' : (B')^\wedge \to (A')^\wedge$ are continuous.

By Lemma 87.21.1 the topological rings $(A')^\wedge$ and $(B')^\wedge$ are objects of WAdm^{count}. We say P is a local property if the following axioms hold:

0AND (1) for any diagram (87.21.2.1) we have $P(\varphi) \Rightarrow P(\varphi')$,
0ANE (2) for any diagram (87.21.2.1) with $A \to A'$ faithfully flat we have $P(\varphi') \Rightarrow P(\varphi)$,
0ANF (3) if $P(B \to A_i)$ for $i = 1, \ldots, n$, then $P(B \to \prod_{i=1,\ldots,n} A_i)$.

Axiom (3) makes sense as WAdm^{count} has finite products.

0ANG Lemma 87.21.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. Let P be a local property of morphisms of WAdm^{count}. The following are equivalent

(1) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a morphism of WAdm^{count} with property P,

(2) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a morphism of WAdm^{count} with property P, and

(3) there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a morphism of WAdm^{count} with property P.

Proof. It is clear that (1) implies (2) and that (2) implies (3). Assume $\{X_i \to X\}$ and $X_i \to Y_i \to Y$ as in (3) and let a diagram as in (1) be given. Since $Y_i \times_Y V$ is a formal algebraic space (Lemma 87.15.2) we may pick coverings $\{Y_{ij} \to Y_i \times_Y V\}$ as in Definition 87.11.1. For each (i,j) we may similarly choose coverings $\{X_{ijk} \to Y_{ij} \times_{Y_i} X_i \times_X U\}$ as in Definition 87.11.1. Since U is quasi-compact we can choose $(i_1, j_1, k_1), \ldots, (i_n, j_n, k_n)$ such that

$$X_{i_1j_1k_1} \amalg \ldots \amalg X_{i_nj_nk_n} \longrightarrow U$$

is surjective. For $s = 1, \ldots, n$ consider the commutative diagram

$$\begin{tikzcd} & & & X_{i_sj_sk_s} \arrow[dl] \arrow[d] \arrow[dr] & & & \\ X \arrow[d] & X_{i_s} \arrow[l] \arrow[d] & X_{i_s} \times_X U \arrow[l] \arrow[d] & Y_{i_sj_s} \arrow[dl] \arrow[dr] & X_{i_s} \times_X U \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\ Y & Y_{i_s} \arrow[l] & Y_{i_s} \times_Y V \arrow[l] & & Y_{i_s} \times_Y V \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

Let us say that P holds for a morphism of countably indexed affine formal algebraic spaces if it holds for the corresponding morphism of WAdm^{count}. Observe that the maps $X_{i_sj_sk_s} \to X_{i_s}$, $Y_{i_sj_s} \to Y_{i_s}$ are given by completions of étale ring maps, see Lemma 87.19.13. Hence we see that $P(X_{i_s} \to Y_{i_s})$ implies $P(X_{i_sj_sk_s} \to Y_{i_sj_s})$ by axiom (1). Observe that the maps $Y_{i_sj_s} \to V$ are given by completions of étale rings maps (same lemma as before). By axiom (2) applied to the diagram

$$\begin{tikzcd} X_{i_sj_sk_s} \arrow[r, equal] \arrow[d] & X_{i_sj_sk_s} \arrow[d] \\ Y_{i_sj_s} \arrow[r] & V \end{tikzcd}$$

(this is permissible as identities are faithfully flat ring maps) we conclude that $P(X_{i_sj_sk_s} \to V)$ holds. By axiom (3) we find that $P(\coprod_{s=1,\ldots,n} X_{i_sj_sk_s} \to V)$ holds.

Since the morphism $\coprod X_{i_s j_s k_s} \to U$ is surjective by construction, the corresponding morphism of WAdm^{count} is the completion of a faithfully flat étale ring map, see Lemma 87.19.14. One more application of axiom (2) (with $B' = B$) implies that $P(U \to V)$ is true as desired. $\square$

0ANH Remark 87.21.4 (Variant for adic-star). Let P be a property of morphisms of WAdm^{adic*}. We say P is a local property if axioms (1), (2), (3) of Situation 87.21.2 hold for morphisms of WAdm^{adic*}. In exactly the same way we obtain a variant of Lemma 87.21.3 for morphisms between locally adic* formal algebraic spaces over S.

0ANI Remark 87.21.5 (Variant for Noetherian). Let P be a property of morphisms of WAdm^{Noeth}. We say P is a local property if axioms (1), (2), (3), of Situation 87.21.2 hold for morphisms of WAdm^{Noeth}. In exactly the same way we obtain a variant of Lemma 87.21.3 for morphisms between locally Noetherian formal algebraic spaces over S.

0GBC Situation 87.21.6. Let P be a local property of morphisms of WAdm^{count}, see Situation 87.21.2. We say P is stable under base change if given $B \to A$ and $B \to C$ in WAdm^{count} we have $P(B \to A) \Rightarrow P(C \to A\widehat{\otimes}_B C)$. This makes sense as $A\widehat{\otimes}_B C$ is an object of WAdm^{count} by Lemma 87.4.12.

0GBD Lemma 87.21.7. Let S be a scheme. Let P be a local property of morphisms of WAdm^{count} which is stable under base change. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally countably indexed formal algebraic spaces over S. If f satisfies the equivalent conditions of Lemma 87.21.3 then so does $\text{pr}_2 : X \times_Y Z \to Z$.

Proof. Choose a covering $\{Y_j \to Y\}$ as in Definition 87.11.1. For each j choose a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. For each j choose a covering $\{Z_{jk} \to Y_j \times_Y Z\}$ as in Definition 87.11.1. Observe that $X_{ji} \times_{Y_j} Z_{jk}$ is an affine formal algebraic space which is countably indexed, see Lemma 87.20.10. Then we see that

$$\{X_{ji} \times_{Y_j} Z_{jk} \to X \times_Y Z\}$$

is a covering as in Definition 87.11.1. Moreover, the morphisms $X_{ji} \times_{Y_j} Z_{jk} \to Z$ factor through Z_{jk}. By assumption we know that $X_{ji} \to Y_j$ corresponds to a morphism $B_j \to A_{ji}$ of WAdm^{count} having property P. The morphisms $Z_{jk} \to Y_j$ correspond to morphisms $B_j \to C_{jk}$ in WAdm^{count}. Since $X_{ji} \times_{Y_j} Z_{jk} = \text{Spf}(A_{ji}\widehat{\otimes}_{B_j} C_{jk})$ by Lemma 87.16.4 we see that it suffices to show that $C_{jk} \to A_{ji}\widehat{\otimes}_{B_j} C_{jk}$ has property P which is exactly what the condition that P is stable under base change guarantees. $\square$

0GBE Remark 87.21.8 (Variant for adic-star). Let P be a local property of morphisms of WAdm^{adic*}, see Remark 87.21.4. We say P is stable under base change if given $B \to A$ and $B \to C$ in WAdm^{adic*} we have $P(B \to A) \Rightarrow P(C \to A\widehat{\otimes}_B C)$. This makes sense as $A\widehat{\otimes}_B C$ is an object of WAdm^{adic*} by Lemma 87.4.12. In exactly the same way we obtain a variant of Lemma 87.21.7 for morphisms between locally adic* formal algebraic spaces over S.

0GBF Remark 87.21.9 (Variant for Noetherian). Let P be a local property of morphisms of WAdm^{Noeth}, see Remark 87.21.5. We say P is stable under base change if given $B \to A$ and $B \to C$ in WAdm^{Noeth} the property $P(B \to A)$ implies both that

$A\widehat{\otimes}_B C$ is adic Noetherian[7] and that $P(C \to A\widehat{\otimes}_B C)$. In exactly the same way we obtain a variant of Lemma 87.21.7 for morphisms between locally Noetherian formal algebraic spaces over S.

0GBG Remark 87.21.10 (Another variant for Noetherian). Let P and Q be local properties of morphisms of WAdm^{Noeth}, see Remark 87.21.5. We say P is stable under base change by Q if given $B \to A$ and $B \to C$ in WAdm^{Noeth} satisfying $P(B \to A)$ and $Q(B \to C)$, then $A\widehat{\otimes}_B C$ is adic Noetherian and $P(C \to A\widehat{\otimes}_B C)$ holds. Arguing exactly as in the proof of Lemma 87.21.7 we obtain the following statement: given morphisms $f : X \to Y$ and $g : Y \to Z$ of locally Noetherian formal algebraic spaces over S such that

(1) the equivalent conditions of Lemma 87.21.3 hold for f and P,
(2) the equivalent conditions of Lemma 87.21.3 hold for g and Q,

then the equivalent conditions of Lemma 87.21.3 hold for $\text{pr}_2 : X \times_Y Z \to Z$ and P.

0GBH Situation 87.21.11. Let P be a local property of morphisms of WAdm^{count}, see Situation 87.21.2. We say P is stable under composition if given $B \to A$ and $C \to B$ in WAdm^{count} we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$.

0GBI Lemma 87.21.12. Let S be a scheme. Let P be a local property of morphisms of WAdm^{count} which is stable under composition. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally countably indexed formal algebraic spaces over S. If f and g satisfies the equivalent conditions of Lemma 87.21.3 then so does $g \circ f : X \to Z$.

Proof. Choose a covering $\{Z_k \to Z\}$ as in Definition 87.11.1. For each k choose a covering $\{Y_{kj} \to Z_k \times_Z Y\}$ as in Definition 87.11.1. For each k and j choose a covering $\{X_{kji} \to Y_{kj} \times_Y X\}$ as in Definition 87.11.1. If f and g satisfies the equivalent conditions of Lemma 87.21.3 then $X_{kji} \to Y_{jk}$ and $Y_{jk} \to Z_k$ correspond to arrows $B_{kj} \to A_{kji}$ and $C_k \to B_{kj}$ of WAdm^{count} having property P. Hence the compositions do too and we conclude. $\square$

0GBJ Remark 87.21.13 (Variant for adic-star). Let P be a local property of morphisms of WAdm^{adic*}, see Remark 87.21.4. We say P is stable under composition if given $B \to A$ and $C \to B$ in WAdm^{adic*} we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally adic* formal algebraic spaces over S.

0GBK Remark 87.21.14 (Variant for Noetherian). Let P be a local property of morphisms of WAdm^{Noeth}, see Remark 87.21.5. We say P is stable under composition if given $B \to A$ and $C \to B$ in WAdm^{Noeth} we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally Noetherian formal algebraic spaces over S.

0GBL Situation 87.21.15. Let P be a local property of morphisms of WAdm^{count}, see Situation 87.21.2. We say P has the cancellation property if given $B \to A$ and $C \to B$ in WAdm^{count} we have $P(C \to B) \wedge P(C \to A) \Rightarrow P(B \to A)$.

0GBM Lemma 87.21.16. Let S be a scheme. Let P be a local property of morphisms of WAdm^{count} which has the cancellation property. Let $f : X \to Y$ and $g : Y \to Z$ be

[7]See Lemma 87.4.12 for a criterion.

morphisms of locally countably indexed formal algebraic spaces over S. If $g \circ f$ and g satisfies the equivalent conditions of Lemma 87.21.3 then so does $f : X \to Y$.

Proof. Choose a covering $\{Z_k \to Z\}$ as in Definition 87.11.1. For each k choose a covering $\{Y_{kj} \to Z_k \times_Z Y\}$ as in Definition 87.11.1. For each k and j choose a covering $\{X_{kji} \to Y_{kj} \times_Y X\}$ as in Definition 87.11.1. Let $X_{kji} \to Y_{jk}$ and $Y_{jk} \to Z_k$ correspond to arrows $B_{kj} \to A_{kji}$ and $C_k \to B_{kj}$ of WAdm^{count}. If $g \circ f$ and g satisfies the equivalent conditions of Lemma 87.21.3 then $C_k \to B_{kj}$ and $C_k \to A_{kji}$ satisfy P. Hence $B_{kj} \to A_{kji}$ does too and we conclude. □

0GBN Remark 87.21.17 (Variant for adic-star). Let P be a local property of morphisms of WAdm^{adic*}, see Remark 87.21.4. We say P has the cancellation property if given $B \to A$ and $C \to B$ in WAdm^{adic*} we have $P(C \to A) \wedge P(C \to B) \Rightarrow P(B \to A)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally adic* formal algebraic spaces over S.

0GBP Remark 87.21.18 (Variant for Noetherian). Let P be a local property of morphisms of WAdm^{Noeth}, see Remark 87.21.5. We say P has the cancellation property if given $B \to A$ and $C \to B$ in WAdm^{Noeth} we have $P(C \to B) \wedge P(C \to A) \Rightarrow P(C \to B)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally Noetherian formal algebraic spaces over S.

87.22. Taut ring maps and representability by algebraic spaces

0GBQ In this section we briefly show that morphisms between locally countably index formal algebraic spaces correspond étale locally to taut continuous ring homomorphisms between weakly admissible topological rings having countable fundamental systems of open ideals. In fact, this is rather clear from Lemma 87.19.8 and we encourage the reader to skip this section.

0ANJ Lemma 87.22.1. Let $B \to A$ be an arrow of WAdm^{count}. The following are equivalent

(a) $B \to A$ is taut (Definition 87.5.1),
(b) for $B \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ a fundamental system of weak ideals of definitions there exist a commutative diagram

$$\begin{tikzcd} A \arrow[r] & \ldots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \\ B \arrow[r] \arrow[u] & \ldots \arrow[r] & B/J_3 \arrow[r] \arrow[u] & B/J_2 \arrow[r] \arrow[u] & B/J_1 \arrow[u] \end{tikzcd}$$

such that $A_{n+1}/J_n A_{n+1} = A_n$ and $A = \lim A_n$ as topological ring.

Moreover, these equivalent conditions define a local property, i.e., they satisfy axioms (1), (2), (3).

Proof. The equivalence of (a) and (b) is immediate. Below we will give an algebraic proof of the axioms, but it turns out we've already proven them. Namely, using Lemma 87.19.8 the equivalent conditions (a) and (b) translate to saying the corresponding morphism of affine formal algebraic spaces is representable by algebraic spaces. Since this condition is "étale local on the source and target" by Lemma 87.19.4 we immediately get axioms (1), (2), and (3).

Direct algebraic proof of (1), (2), (3). Let a diagram (87.21.2.1) as in Situation 87.21.2 be given. By Example 87.19.11 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a) and (b).

Assume (a) and (b) hold for φ. Let $J \subset B$ be a weak ideal of definition. Then the closure of JA, resp. $J(B')^\wedge$ is a weak ideal of definition $I \subset A$, resp. $J' \subset (B')^\wedge$. Then the closure of $I(A')^\wedge$ is a weak ideal of definition $I' \subset (A')^\wedge$. A topological argument shows that I' is also the closure of $J(A')^\wedge$ and of $J'(A')^\wedge$. Finally, as J runs over a fundamental system of weak ideals of definition of B so do the ideals I and I' in A and $(A')^\wedge$. It follows that (a) holds for φ'. This proves (1).

Assume $A \to A'$ is faithfully flat and that (a) and (b) hold for φ'. Let $J \subset B$ be a weak ideal of definition. Using (a) and (b) for the maps $B \to (B')^\wedge \to (A')^\wedge$ we find that the closure I' of $J(A')^\wedge$ is a weak ideal of definition. In particular, I' is open and hence the inverse image of I' in A is open. Now we have (explanation below)

$$\begin{aligned} A \cap I' &= A \cap \bigcap (J(A')^\wedge + \mathrm{Ker}((A')^\wedge \to A'/I_0A')) \\ &= A \cap \bigcap \mathrm{Ker}((A')^\wedge \to A'/JA' + I_0A') \\ &= \bigcap (JA + I_0) \end{aligned}$$

which is the closure of JA by Lemma 87.4.2. The intersections are over weak ideals of definition $I_0 \subset A$. The first equality because a fundamental system of neighbourhoods of 0 in $(A')^\wedge$ are the kernels of the maps $(A')^\wedge \to A'/I_0A'$. The second equality is trivial. The third equality because $A \to A'$ is faithfully flat, see Algebra, Lemma 10.82.11. Thus the closure of JA is open. By Lemma 87.4.10 the closure of JA is a weak ideal of definition of A. Finally, given a weak ideal of definition $I \subset A$ we can find J such that $J(A')^\wedge$ is contained in the closure of $I(A')^\wedge$ by property (a) for $B \to (B')^\wedge$ and φ'. Thus we see that (a) holds for φ. This proves (2).

We omit the proof of (3). $\square$

0ANK Lemma 87.22.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent

(1) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a taut map $B \to A$ of WAdm^{count},

(2) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a taut ring map in WAdm^{count},

(3) there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space,

$Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a taut ring map in WAdm^{count}, and
(4) f is representable by algebraic spaces.

Proof. The property of a map in WAdm^{count} being "taut" is a local property by Lemma 87.22.1. Thus Lemma 87.21.3 exactly tells us that (1), (2), and (3) are equivalent. On the other hand, by Lemma 87.19.8 being "taut" on maps in WAdm^{count} corresponds exactly to being "representable by algebraic spaces" for the corresponding morphisms of countably indexed affine formal algebraic spaces. Thus the implication (1) $\Rightarrow$ (2) of Lemma 87.19.4 shows that (4) implies (1) of the current lemma. Similarly, the implication (4) $\Rightarrow$ (1) of Lemma 87.19.4 shows that (2) implies (4) of the current lemma. $\square$

87.23. Adic morphisms

0AQ2 This section matches the occasionally used notion of an "adic morphism" $f : X \to Y$ of locally adic* formal algebraic spaces X and Y on the one hand with representability of f by algebraic spaces and on the other hand with our notion of taut continuous ring homomorphisms. First we recall that tautness is equivalent to adicness for adic rings with finitely generated ideal of definition.

0GBS Lemma 87.23.1. Let A and B be pre-adic topological rings. Let $\varphi : A \to B$ be a continuous ring homomorphism.

(1) If φ is adic, then φ is taut.
(2) If B is complete, A has a finitely generated ideal of definition, and φ is taut, then φ is adic.

In particular the conditions "φ is adic" and "φ is taut" are equivalent on the category WAdm^{adic*}.

Proof. Part (1) is Lemma 87.6.4. Part (2) is Lemma 87.6.5. The final statement is a consequence of (1) and (2). $\square$

Let S be a scheme. Let $f : X \to Y$ be a morphism of locally adic* formal algebraic spaces over S. By Lemma 87.22.2 the following are equivalent

(1) f is representable by algebraic spaces (in other words, the equivalent conditions of Lemma 87.19.4 hold),
(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an adic[8] map in WAdm^{adic*}.

In this situation we will say that f is an adic morphism (the formal definition is below). This notion/terminology will only be defined/used for morphisms between formal algebraic spaces which are locally adic* since otherwise we don't have the equivalence between (1) and (2) above.

[8]Equivalently taut by Lemma 87.23.1.

0AQ3 Definition 87.23.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume X and Y are locally adic*. We say f is an adic morphism if f is representable by algebraic spaces. See discussion above.

87.24. Morphisms of finite type

0AM3 Due to how things are setup in the Stacks project, the following is really the correct thing to do and stronger notions should have a different name.

0AM4 Definition 87.24.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S.

(1) We say f is locally of finite type if f is representable by algebraic spaces and is locally of finite type in the sense of Bootstrap, Definition 80.4.1.
(2) We say f is of finite type if f is locally of finite type and quasi-compact (Definition 87.17.4).

We will discuss the relationship between finite type morphisms of certain formal algebraic spaces and continuous ring maps $A \to B$ which are topologically of finite type in Section 87.29.

0AJJ Lemma 87.24.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent

(1) f is of finite type,
(2) f is representable by algebraic spaces and is of finite type in the sense of Bootstrap, Definition 80.4.1.

Proof. This follows from Bootstrap, Lemma 80.4.5, the implication "quasi-compact + locally of finite type ⇒ finite type" for morphisms of algebraic spaces, and Lemma 87.17.5. □

0AQ4 Lemma 87.24.3. The composition of finite type morphisms is of finite type. The same holds for locally of finite type.

Proof. See Bootstrap, Lemma 80.4.3 and use Morphisms of Spaces, Lemma 67.23.2. □

0AQ5 Lemma 87.24.4. A base change of a finite type morphism is finite type. The same holds for locally of finite type.

Proof. See Bootstrap, Lemma 80.4.2 and use Morphisms of Spaces, Lemma 67.23.3. □

0AQ6 Lemma 87.24.5. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.

Proof. By Lemma 87.19.3 we see that f is representable by algebraic spaces. Let T be a scheme and let $T \to Z$ be a morphism. Then we can apply Morphisms of Spaces, Lemma 67.23.6 to the morphisms $T \times_Z X \to T \times_Z Y \to T$ of algebraic spaces to conclude. □

Being locally of finite type is local on the source and the target.

0ANL Lemma 87.24.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent:

(1) the morphism f is locally of finite type,
(2) there exists a commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are formal algebraic spaces, the vertical arrows are representable by algebraic spaces and étale, $U \to X$ is surjective, and $U \to V$ is locally of finite type,
(3) for any commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where U, V are formal algebraic spaces and vertical arrows representable by algebraic spaces and étale, the morphism $U \to V$ is locally of finite type,
(4) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that $X_{ji} \to Y_j$ is locally of finite type for each j and i,
(5) there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, such that $X_i \to Y_i$ is locally of finite type, and
(6) add more here.

Proof. In each of the 5 cases the morphism $f : X \to Y$ is representable by algebraic spaces, see Lemma 87.19.4. We will use this below without further mention.

It is clear that (1) implies (2) because we can take $U = X$ and $V = Y$. Conversely, assume given a diagram as in (2). Let T be a scheme and let $T \to Y$ be a morphism. Then we can consider
$$\begin{tikzcd} U \times_Y T \arrow[r] \arrow[d] & V \times_Y T \arrow[d] \\ X \times_Y T \arrow[r] & T \end{tikzcd}$$
The vertical arrows are étale and the top horizontal arrow is locally of finite type as base changes of such morphisms. Hence by Morphisms of Spaces, Lemma 67.23.4 we conclude that $X \times_Y T \to T$ is locally of finite type. In other words (1) holds.

Assume (1) is true and consider a diagram as in (3). Then $U \to Y$ is locally of finite type (as the composition $U \to X \to Y$, see Bootstrap, Lemma 80.4.3). Let T be a scheme and let $T \to V$ be a morphism. Then the projection $T \times_V U \to T$ factors as
$$T \times_V U = (T \times_Y U) \times_{(V \times_Y V)} V \to T \times_Y U \to T$$
The second arrow is locally of finite type (as a base change of the composition $U \to X \to Y$) and the first is the base change of the diagonal $V \to V \times_Y V$ which is locally of finite type by Lemma 87.15.5.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product $Y_j \times_Y X$ is a formal algebraic space by Lemma 87.15.3. It is clear that (4) implies (5).

Assume $X_i \to Y_i \to Y$ as in (5). Then we set $V = \coprod Y_i$ and $U = \coprod X_i$ to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j any covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. Then $X_{ij} \to Y_j$ is locally of finite type by (3) and we see that (4) is true. This concludes the proof. □

0ANM Example 87.24.7. Let S be a scheme. Let A be a weakly admissible topological ring over S. Let $A \to A'$ be a finite type ring map. Then

$$(A')^\wedge = \lim_{I \subset A \text{ } w.i.d.} A'/IA'$$

is a weakly admissible ring and the corresponding morphism $\text{Spf}((A')^\wedge) \to \text{Spf}(A)$ is representable, see Example 87.19.11. If $T \to \text{Spf}(A)$ is a morphism where T is a quasi-compact scheme, then this factors through $\text{Spec}(A/I)$ for some weak ideal of definition $I \subset A$ (Lemma 87.9.4). Then $T \times_{\text{Spf}(A)} \text{Spf}((A')^\wedge)$ is equal to $T \times_{\text{Spec}(A/I)} \text{Spec}(A'/IA')$ and we see that $\text{Spf}((A')^\wedge) \to \text{Spf}(A)$ is of finite type.

0AQ7 Lemma 87.24.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. If Y is locally Noetherian and f locally of finite type, then X is locally Noetherian.

Proof. Pick $\{Y_j \to Y\}$ and $\{X_{ij} \to Y_j \times_Y X\}$ as in Lemma 87.24.6. Then it follows from Lemma 87.19.9 that each X_{ij} is Noetherian. This proves the lemma. □

0AQ8 Lemma 87.24.9. Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. If Z is locally Noetherian and f locally of finite type, then $Z \times_Y X$ is locally Noetherian.

Proof. The morphism $Z \times_Y X \to Z$ is locally of finite type by Lemma 87.24.4. Hence this follows from Lemma 87.24.8. □

87.25. Surjective morphisms

0GHN By Lemma 87.12.4 the following definition does not clash with the already existing definitions for morphisms of algebraic spaces or morphisms of formal algebraic spaces which are representable by algebraic spaces.

0GHP Definition 87.25.1. Let S be a scheme. A morphism $f : X \to Y$ of formal algebraic spaces over S is said to be surjective if it induces a surjective morphism $X_{red} \to Y_{red}$ on underlying reduced algebraic spaces.

0GHQ Lemma 87.25.2. The composition of two surjective morphisms is a surjective morphism.

Proof. Omitted. □

0GHR Lemma 87.25.3. A base change of a surjective morphism is a surjective morphism.

Proof. Omitted. □

0GHS **Lemma 87.25.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent

(1) f is surjective,
(2) for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a surjective morphism of formal algebraic spaces,
(3) for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a surjective morphism of formal algebraic spaces,
(4) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j \to Y_j$ is a surjective morphism of formal algebraic spaces,
(5) there exists a surjective morphism $Z \to Y$ of formal algebraic spaces such that $X \times_Y Z \to Z$ is surjective, and
(6) add more here.

Proof. Omitted. $\square$

87.26. Monomorphisms

0AQA Here is the definition.

0AQB **Definition 87.26.1.** Let S be a scheme. A morphism of formal algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves.

An example is the following. Let X be an algebraic space and let $T \subset |X|$ be a closed subset. Then the morphism $X_{/T} \to X$ from the formal completion of X along T to X is a monomorphism. In particular, monomorphisms of formal algebraic spaces are in general not representable.

0GHT **Lemma 87.26.2.** The composition of two monomorphisms is a monomorphism.

Proof. Omitted. $\square$

0GHU **Lemma 87.26.3.** A base change of a monomorphism is a monomorphism.

Proof. Omitted. $\square$

0GHV **Lemma 87.26.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent

(1) f is a monomorphism,
(2) for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a monomorphism of formal algebraic spaces,
(3) for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a monomorphism of formal algebraic spaces,
(4) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j \to Y_j$ is a monomorphism of formal algebraic spaces, and
(5) there exists a family of morphisms $\{Y_j \to Y\}$ such that $\coprod Y_j \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ such that each $X \times_Y Y_j \to Y_j$ is a monomorphism for all j,
(6) there exists a morphism $Z \to Y$ of formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and locally of finite presentation such that $X \times_Y Z \to X$ is a monomorphism, and
(7) add more here.

Proof. Omitted. $\square$

87.27. Closed immersions

0ANN Here is the definition.

0ANP Definition 87.27.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S. We say f is a *closed immersion* if f is representable by algebraic spaces and a closed immersion in the sense of Bootstrap, Definition 80.4.1.

Please skip the initial the obligatory lemmas when reading this section.

0GHW Lemma 87.27.2. The composition of two closed immersions is a closed immersion.

Proof. Omitted. □

0GHX Lemma 87.27.3. A base change of a closed immersion is a closed immersion.

Proof. Omitted. □

0GHY Lemma 87.27.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent

(1) f is a closed immersion,
(2) for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a closed immersion,
(3) for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a closed immersion,
(4) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j \to Y_j$ is a closed immersion, and
(5) there exists a morphism $Z \to Y$ of formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and locally of finite presentation such that $X \times_Y Z \to X$ is a closed immersion, and
(6) add more here.

Proof. Omitted. □

0ANQ Lemma 87.27.5. Let S be a scheme. Let X be a McQuillan affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a McQuillan affine formal algebraic space and f corresponds to a continuous homomorphism $A \to B$ of weakly admissible topological S-algebras which is taut, has closed kernel, and has dense image.

Proof. Write $X = \text{Spf}(A)$ where A is a weakly admissible topological ring. Let I_λ be a fundamental system of weakly admissible ideals of definition in A. Then $Y \times_X \text{Spec}(A/I_\lambda)$ is a closed subscheme of $\text{Spec}(A/I_\lambda)$ and hence affine (Definition 87.27.1). Say $Y \times_X \text{Spec}(A/I_\lambda) = \text{Spec}(B_\lambda)$. The ring map $A/I_\lambda \to B_\lambda$ is surjective. Hence the projections

$$B = \lim B_\lambda \longrightarrow B_\lambda$$

are surjective as the compositions $A \to B \to B_\lambda$ are surjective. It follows that Y is McQuillan by Lemma 87.9.6. The ring map $A \to B$ is taut by Lemma 87.19.8. The kernel is closed because B is complete and $A \to B$ is continuous. Finally, as $A \to B_\lambda$ is surjective for all λ we see that the image of A in B is dense. □

Even though we have the result above, in general we do not know how closed immersions behave when the target is a McQuillan affine formal algebraic space, see Remark 87.29.4.

0ANR Example 87.27.6. Let S be a scheme. Let A be a weakly admissible topological ring over S. Let $K \subset A$ be a closed ideal. Setting

$$B = (A/K)^\wedge = \lim_{I \subset A \ w.i.d.} A/(I + K)$$

the morphism $\text{Spf}(B) \to \text{Spf}(A)$ is representable, see Example 87.19.11. If $T \to \text{Spf}(A)$ is a morphism where T is a quasi-compact scheme, then this factors through $\text{Spec}(A/I)$ for some weak ideal of definition $I \subset A$ (Lemma 87.9.4). Then $T \times_{\text{Spf}(A)} \text{Spf}(B)$ is equal to $T \times_{\text{Spec}(A/I)} \text{Spec}(A/(K + I))$ and we see that $\text{Spf}(B) \to \text{Spf}(A)$ is a closed immersion. The kernel of $A \to B$ is K as K is closed, but beware that in general the ring map $A \to B = (A/K)^\wedge$ need not be surjective.

0GHZ Lemma 87.27.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume

(1) f is representable by algebraic spaces,
(2) f is a monomorphism,
(3) the inclusion $Y_{red} \to Y$ factors through f, and
(4) f is locally of finite type or Y is locally Noetherian.

Then f is a closed immersion.

Proof. Assumptions (2) and (3) imply that $X_{red} = X \times_Y Y_{red} = Y_{red}$. We will use this without further mention.

If $Y' \to Y$ is an étale morphism of formal algebraic spaces over S, then the base change $f' : X \times_Y Y' \to Y'$ satisfies conditions (1) – (4). Hence by Lemma 87.27.4 we may assume Y is an affine formal algebraic space.

Say $Y = \text{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 87.9.1. Then $X_\lambda = X \times_Y Y_\lambda$ is an algebraic space endowed with a monomorphism $f_\lambda : X_\lambda \to Y_\lambda$ which induces an isomorphism $X_{\lambda,red} \to Y_{\lambda,red}$. Thus X_λ is an affine scheme by Limits of Spaces, Proposition 70.15.2 (as $X_{\lambda,red} \to X_\lambda$ is surjective and integral). To finish the proof it suffices to show that $X_\lambda \to Y_\lambda$ is a closed immersion which we will do in the next paragraph.

Let $X \to Y$ be a monomorphism of affine schemes such that $X_{red} = X \times_Y Y_{red} = Y_{red}$. In general, this does not imply that $X \to Y$ is a closed immersion, see Examples, Section 110.35. However, under our assumption (4) we know that in the previous parapgrah either $X_\lambda \to Y_\lambda$ is of finite type or Y_λ is Noetherian. This means that $X \to Y$ corresponds to a ring map $R \to A$ such that $R/I \to A/IA$ is an isomorphism where $I \subset R$ is the nil radical (ie., the maximal locally nilpotent ideal of R) and either $R \to A$ is of finite type or R is Noetherian. In the first case $R \to A$ is surjective by Algebra, Lemma 10.126.9 and in the second case I is finitely generated, hence nilpotent, hence $R \to A$ is surjective by Nakayama's lemma, see Algebra, Lemma 10.20.1 part (11). □

87.28. Restricted power series

0AKZ Let A be a topological ring complete with respect to a linear topology (More on Algebra, Definition 15.36.1). Let I_λ be a fundamental system of open ideals. Let $r \geq 0$ be an integer. In this setting one often denotes

$$A\{x_1, \ldots, x_r\} = \lim_\lambda A/I_\lambda[x_1, \ldots, x_r] = \lim_\lambda (A[x_1, \ldots, x_r]/I_\lambda A[x_1, \ldots, x_r])$$

endowed with the limit topology. In other words, this is the completion of the polynomial ring with respect to the ideals I_λ. We can think of elements of $A\{x_1, \ldots, x_r\}$

as power series

$$f = \sum\nolimits_{E=(e_1,\ldots,e_r)} a_E x_1^{e_1} \ldots x_r^{e_r}$$

in $x_1, \ldots, x_r$ with coefficients $a_E \in A$ which tend to zero in the topology of A. In other words, for any λ all but a finite number of a_E are in I_λ. For this reason elements of $A\{x_1, \ldots, x_r\}$ are sometimes called restricted power series. Sometimes this ring is denoted $A\langle x_1, \ldots, x_r\rangle$; we will refrain from using this notation.

0AJM Remark 87.28.1 (Universal property restricted power series). Let $A \to C$ be a continuous map of complete linearly topologized rings. Then any A-algebra map $A[x_1, \ldots x_r] \to C$ extends uniquely to a continuous map $A\{x_1, \ldots, x_r\} \to C$ on restricted power series. [DG67, Chapter 0, 7.5.3]

0AL0 Remark 87.28.2. Let A be a ring and let $I \subset A$ be an ideal. If A is I-adically complete, then the I-adic completion $A[x_1, \ldots, x_r]^\wedge$ of $A[x_1, \ldots, x_r]$ is the restricted power series ring over A as a ring. However, it is not clear that $A[x_1, \ldots, x_r]^\wedge$ is I-adically complete. We think of the topology on $A\{x_1, \ldots, x_r\}$ as the limit topology (which is always complete) whereas we often think of the topology on $A[x_1, \ldots, x_r]^\wedge$ as the I-adic topology (not always complete). If I is finitely generated, then $A\{x_1, \ldots, x_r\} = A[x_1, \ldots, x_r]^\wedge$ as topological rings, see Algebra, Lemma 10.96.3.

87.29. Algebras topologically of finite type

0ALL Here is our definition. This definition is not generally agreed upon. Many authors impose further conditions, often because they are only interested in specific types of rings and not the most general case.

0ANS Definition 87.29.1. Let $A \to B$ be a continuous map of topological rings (More on Algebra, Definition 15.36.1). We say B is topologically of finite type over A if there exists an A-algebra map $A[x_1, \ldots, x_n] \to B$ whose image is dense in B.

If A is a complete, linearly topologized ring, then the restricted power series ring $A\{x_1, \ldots, x_r\}$ is topologically of finite type over A. If k is a field, then the power series ring $k[[x_1, \ldots, x_r]]$ is topologically of finite type over k.

For continuous taut maps of weakly admissible topological rings, being topologically of finite type corresponds exactly to morphisms of finite type between the associated affine formal algebraic spaces.

0ANT Lemma 87.29.2. Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent

(1) $\text{Spf}(\varphi) : Y = \text{Spf}(B) \to \text{Spf}(A) = X$ is of finite type,
(2) φ is taut and B is topologically of finite type over A.

Proof. We can use Lemma 87.19.8 to relate tautness of φ to representability of $\text{Spf}(\varphi)$. We will use this without further mention below. It follows that $X = \text{colim}\,\text{Spec}(A/I)$ and $Y = \text{colim}\,\text{Spec}(B/J(I))$ where $I \subset A$ runs over the weak ideals of definition of A and $J(I)$ is the closure of IB in B.

Assume (2). Choose a ring map $A[x_1, \ldots, x_r] \to B$ whose image is dense. Then $A[x_1, \ldots, x_r] \to B \to B/J(I)$ has dense image too which means that it is surjective. Therefore $B/J(I)$ is of finite type over A/I. Let $T \to X$ be a morphism with T a quasi-compact scheme. Then $T \to X$ factors through $\text{Spec}(A/I)$ for some I

(Lemma 87.9.4). Then $T \times_X Y = T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(B/J(I))$, see proof of Lemma 87.19.8. Hence $T \times_Y X \to T$ is of finite type as the base change of the morphism $\mathrm{Spec}(B/J(I)) \to \mathrm{Spec}(A/I)$ which is of finite type. Thus (1) is true.

Assume (1). Pick any $I \subset A$ as above. Since $\mathrm{Spec}(A/I) \times_X Y = \mathrm{Spec}(B/J(I))$ we see that $A/I \to B/J(I)$ is of finite type. Choose $b_1, \ldots, b_r \in B$ mapping to generators of $B/J(I)$ over A/I. We claim that the image of the ring map $A[x_1, \ldots, x_r] \to B$ sending x_i to b_i is dense. To prove this, let $I' \subset I$ be a second weak ideal of definition. Then we have

$$B/(J(I') + IB) = B/J(I)$$

because $J(I)$ is the closure of IB and because $J(I')$ is open. Hence we may apply Algebra, Lemma 10.126.9 to see that $(A/I')[x_1, \ldots, x_r] \to B/J(I')$ is surjective. Thus (2) is true, concluding the proof. □

Let A be a topological ring complete with respect to a linear topology. Let (I_λ) be a fundamental system of open ideals. Let $\mathcal{C}$ be the category of inverse systems (B_λ) where

(1) B_λ is a finite type A/I_λ-algebra, and
(2) $B_\mu \to B_\lambda$ is an A/I_μ-algebra homomorphism which induces an isomorphism $B_\mu/I_\lambda B_\mu \to B_\lambda$.

Morphisms in $\mathcal{C}$ are given by compatible systems of homomorphisms.

0AL1 Lemma 87.29.3. Let S be a scheme. Let X be an affine formal algebraic space over S. Assume X is McQuillan and let A be the weakly admissible topological ring associated to X. Then there is an anti-equivalence of categories between

(1) the category $\mathcal{C}$ introduced above, and
(2) the category of maps $Y \to X$ of finite type of affine formal algebraic spaces.

Proof. Let (I_λ) be a fundamental system of weakly admissible ideals of definition in A. Consider Y as in (2). Then $Y \times_X \mathrm{Spec}(A/I_\lambda)$ is affine (Definition 87.24.1 and Lemma 87.19.7). Say $Y \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$. The ring map $A/I_\lambda \to B_\lambda$ is of finite type because $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A/I_\lambda)$ is of finite type (by Definition 87.24.1). Then (B_λ) is an object of $\mathcal{C}$.

Conversely, given an object (B_λ) of $\mathcal{C}$ we can set $Y = \mathrm{colim}\,\mathrm{Spec}(B_\lambda)$. This is an affine formal algebraic space. We claim that

$$Y \times_X \mathrm{Spec}(A/I_\lambda) = (\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$$

To show this it suffices we get the same values if we evaluate on a quasi-compact scheme U. A morphism $U \to (\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda)$ comes from a morphism $U \to \mathrm{Spec}(B_\mu) \times_{\mathrm{Spec}(A/I_\mu)} \mathrm{Spec}(A/I_\lambda)$ for some $\mu \geq \lambda$ (use Lemma 87.9.4 two times). Since $\mathrm{Spec}(B_\mu) \times_{\mathrm{Spec}(A/I_\mu)} \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$ by our second assumption on objects of $\mathcal{C}$ this proves what we want. Using this we can show the morphism $Y \to X$ is of finite type. Namely, we note that for any morphism $U \to X$ with U a quasi-compact scheme, we get a factorization $U \to \mathrm{Spec}(A/I_\lambda) \to X$ for some λ (see lemma cited above). Hence

$$Y \times_X U = Y \times_X \mathrm{Spec}(A/I_\lambda)) \times_{\mathrm{Spec}(A/I_\lambda)} U = \mathrm{Spec}(B_\lambda) \times_{\mathrm{Spec}(A/I_\lambda)} U$$

is a scheme of finite type over U as desired. Thus the construction $(B_\lambda) \mapsto \mathrm{colim}\,\mathrm{Spec}(B_\lambda)$ does give a functor from category (1) to category (2).

To finish the proof we show that the above constructions define quasi-inverse functors between the categories (1) and (2). In one direction you have to show that

$$(\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$$

for any object (B_λ) in the category $\mathcal{C}$. This we proved above. For the other direction you have to show that

$$Y = \mathrm{colim}(Y \times_X \mathrm{Spec}(A/I_\lambda))$$

given Y in the category (2). Again this is true by evaluating on quasi-compact test objects and because $X = \mathrm{colim}\,\mathrm{Spec}(A/I_\lambda)$. □

0AJK Remark 87.29.4. Let A be a weakly admissible topological ring and let (I_λ) be a fundamental system of weak ideals of definition. Let $X = \mathrm{Spf}(A)$, in other words, X is a McQuillan affine formal algebraic space. Let $f : Y \to X$ be a morphism of affine formal algebraic spaces. In general it will not be true that Y is McQuillan. More specifically, we can ask the following questions:

(1) Assume that $f : Y \to X$ is a closed immersion. Then Y is McQuillan and f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings which is taut, whose kernel $K \subset A$ is a closed ideal, and whose image $\varphi(A)$ is dense in B, see Lemma 87.27.5. What conditions on A guarantee that $B = (A/K)^\wedge$ as in Example 87.27.6?
(2) What conditions on A guarantee that closed immersions $f : Y \to X$ correspond to quotients A/K of A by closed ideals, in other words, the corresponding continuous map φ is surjective and open?
(3) Suppose that $f : Y \to X$ is of finite type. Then we get $Y = \mathrm{colim}\,\mathrm{Spec}(B_\lambda)$ where (B_λ) is an object of $\mathcal{C}$ by Lemma 87.29.3. In this case it is true that there exists a fixed integer r such that B_λ is generated by r elements over A/I_λ for all λ (the argument is essentially already given in the proof of (1) ⇒ (2) in Lemma 87.29.2). However, it is not clear that the projections $\lim B_\lambda \to B_\lambda$ are surjective, i.e., it is not clear that Y is McQuillan. Is there an example where Y is not McQuillan?
(4) Suppose that $f : Y \to X$ is of finite type and Y is McQuillan. Then f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings. In fact φ is taut and B is topologically of finite type over A, see Lemma 87.29.2. In other words, f factors as
$$Y \longrightarrow \mathbf{A}^r_X \longrightarrow X$$
where the first arrow is a closed immersion of McQuillan affine formal algebraic spaces. However, then questions (1) and (2) are in force for $Y \to \mathbf{A}^r_X$.

Below we will answer these questions when X is countably indexed, i.e., when A has a countable fundamental system of open ideals. If you have answers to these questions in greater generality, or if you have counter examples, please email stacks.project@gmail.com.

0AQI Lemma 87.29.5. Let S be a scheme. Let X be a countably indexed affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a countably indexed affine formal algebraic space and f corresponds to $A \to A/K$ where A is an object of WAdm^{count} (Section 87.21) and $K \subset A$ is a closed ideal.

Proof. By Lemma 87.10.4 we see that $X = \mathrm{Spf}(A)$ where A is an object of WAdm^{count}. Since a closed immersion is representable and affine, we conclude by Lemma 87.19.9 that Y is an affine formal algebraic space and countably index. Thus applying Lemma 87.10.4 again we see that $Y = \mathrm{Spf}(B)$ with B an object of WAdm^{count}. By Lemma 87.27.5 we conclude that f is given by a morphism $A \to B$ of WAdm^{count} which is taut and has dense image. To finish the proof we apply Lemma 87.5.10. $\square$

0ANU Lemma 87.29.6. Let $B \to A$ be an arrow of WAdm^{count}, see Section 87.21. The following are equivalent

(a) $B \to A$ is taut and $B/J \to A/I$ is of finite type for every weak ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA,
(b) $B \to A$ is taut and $B/J_\lambda \to A/I_\lambda$ is of finite type for a cofinal system (J_λ) of weak ideals of definition of B where $I_\lambda \subset A$ is the closure of $J_\lambda A$,
(c) $B \to A$ is taut and A is topologically of finite type over B,
(d) A is isomorphic as a topological B-algebra to a quotient of $B\{x_1, \ldots, x_n\}$ by a closed ideal.

Moreover, these equivalent conditions define a local property, i.e., they satisfy Axioms (1), (2), (3).

Proof. The implications (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (c) are straightforward from the definitions. Assume (b) holds and let $J \subset B$ and $I \subset A$ be as in (a). Choose a commutative diagram

$$\begin{tikzcd} A \arrow[r] & \ldots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \\ B \arrow[u] \arrow[r] & \ldots \arrow[r] & B/J_3 \arrow[u] \arrow[r] & B/J_2 \arrow[u] \arrow[r] & B/J_1 \arrow[u] \end{tikzcd}$$

such that $A_{n+1}/J_nA_{n+1} = A_n$ and such that $A = \lim A_n$ as in Lemma 87.22.1. For every m there exists a λ such that $J_\lambda \subset J_m$. Since $B/J_\lambda \to A/I_\lambda$ is of finite type, this implies that $B/J_m \to A/I_m$ is of finite type. Let $\alpha_1, \ldots, \alpha_n \in A_1$ be generators of A_1 over B/J_1. Since A is a countable limit of a system with surjective transition maps, we can find $a_1, \ldots, a_n \in A$ mapping to $\alpha_1, \ldots, \alpha_n$ in A_1. By Remark 87.28.1 we find a continuous map $B\{x_1, \ldots, x_n\} \to A$ mapping x_i to a_i. This map induces surjections $(B/J_m)[x_1, \ldots, x_n] \to A_m$ by Algebra, Lemma 10.126.9. For $m \geq 1$ we obtain a short exact sequence

$$0 \to K_m \to (B/J_m)[x_1, \ldots, x_n] \to A_m \to 0$$

The induced transition maps $K_{m+1} \to K_m$ are surjective because $A_{m+1}/J_mA_{m+1} = A_m$. Hence the inverse limit of these short exact sequences is exact, see Algebra, Lemma 10.86.4. Since $B\{x_1, \ldots, x_n\} = \lim (B/J_m)[x_1, \ldots, x_n]$ and $A = \lim A_m$ we conclude that $B\{x_1, \ldots, x_n\} \to A$ is surjective and open. As A is complete the kernel is a closed ideal. In this way we see that (a), (b), (c), and (d) are equivalent.

Let a diagram (87.21.2.1) as in Situation 87.21.2 be given. By Example 87.24.7 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a), (b), (c), and (d). Moreover, by Lemma 87.22.1 in order to prove Axioms (1) and (2) we may assume both $B \to A$ and $(B')^\wedge \to (A')^\wedge$ are taut. Now pick a weak ideal of definition $J \subset B$. Let $J' \subset (B')^\wedge$, $I \subset A$, $I' \subset (A')^\wedge$ be the closure of $J(B')^\wedge$, JA, $J(A')^\wedge$. By what was said above, it suffices to consider the commutative diagram

$$\begin{tikzcd} A/I \arrow[r] & (A')^\wedge/I' \\ B/J \arrow[r] \arrow[u, "\overline{\varphi}"] & (B')^\wedge/J' \arrow[u, "\overline{\varphi}'"'] \end{tikzcd}$$

and to show (1) $\overline{\varphi}$ finite type $\Rightarrow \overline{\varphi}'$ finite type, and (2) if $A \to A'$ is faithfully flat, then $\overline{\varphi}'$ finite type $\Rightarrow \overline{\varphi}$ finite type. Note that $(B')^\wedge/J' = B'/JB'$ and $(A')^\wedge/I' = A'/IA'$ by the construction of the topologies on $(B')^\wedge$ and $(A')^\wedge$. In particular the horizontal maps in the diagram are étale. Part (1) now follows from Algebra, Lemma 10.6.2 and part (2) from Descent, Lemma 35.14.2 as the ring map $A/I \to (A')^\wedge/I' = A'/IA'$ is faithfully flat and étale.

We omit the proof of Axiom (3). $\square$

0CB6 Lemma 87.29.7. In Lemma 87.29.6 if B is admissible (for example adic), then the equivalent conditions (a) – (d) are also equivalent to

(e) $B \to A$ is taut and $B/J \to A/I$ is of finite type for some ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA.

Proof. It is enough to show that (e) implies (a). Let $J' \subset B$ be a weak ideal of definition and let $I' \subset A$ be the closure of $J'A$. We have to show that $B/J' \to A/I'$ is of finite type. If the corresponding statement holds for the smaller weak ideal of definition $J'' = J' \cap J$, then it holds for J'. Thus we may assume $J' \subset J$. As J is an ideal of definition (and not just a weak ideal of definition), we get $J^n \subset J'$ for some $n \geq 1$. Thus we can consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & I/I' \arrow[r] & A/I' \arrow[r] & A/I \arrow[r] & 0 \\ 0 \arrow[r] & J/J' \arrow[r] \arrow[u] & B/J' \arrow[r] \arrow[u] & B/J \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Since $I' \subset A$ is open and since I is the closure of JA we see that $I/I' = (J/J') \cdot A/I'$. Because J/J' is a nilpotent ideal and as $B/J \to A/I$ is of finite type, we conclude from Algebra, Lemma 10.126.8 that A/I' is of finite type over B/J' as desired. $\square$

0ANV Lemma 87.29.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces. Assume Y countably indexed. The following are equivalent

(1) f is locally of finite type,
(2) f is of finite type,
(3) f corresponds to a morphism $B \to A$ of WAdm^{count} (Section 87.21) satisfying the equivalent conditions of Lemma 87.29.6.

Proof. Since X and Y are affine it is clear that conditions (1) and (2) are equivalent. In cases (1) and (2) the morphism f is representable by algebraic spaces by definition, hence affine by Lemma 87.19.7. Thus if (1) or (2) holds we see that X is countably indexed by Lemma 87.19.9. Write $X = \text{Spf}(A)$ and $Y = \text{Spf}(B)$ for topological S-algebras A and B in WAdm^{count}, see Lemma 87.10.4. By Lemma 87.9.10 we see that f corresponds to a continuous map $B \to A$. Hence now the result follows from Lemma 87.29.2. $\square$

0ANW Lemma 87.29.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent

(1) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a morphism of WAdm^{count} which is taut and topologically of finite type,

(2) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a morphism of WAdm^{count} which is taut and topologically of finite type,

(3) there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a morphism of WAdm^{count} which is, taut and topologically of finite type, and

(4) f is locally of finite type.

Proof. By Lemma 87.29.6 the property $P(\varphi)=$"φ is taut and topologically of finite type" is local on WAdm^{count}. Hence by Lemma 87.21.3 we see that conditions (1), (2), and (3) are equivalent. On the other hand, by Lemma 87.29.8 the condition P on morphisms of WAdm^{count} corresponds exactly to morphisms of countably indexed, affine formal algebraic spaces being locally of finite type. Thus the implication (1) $\Rightarrow$ (3) of Lemma 87.24.6 shows that (4) implies (1) of the current lemma. Similarly, the implication (4) $\Rightarrow$ (1) of Lemma 87.24.6 shows that (2) implies (4) of the current lemma. $\square$

87.30. Separation axioms for morphisms

0ARM This section is the analogue of Morphisms of Spaces, Section 67.4 for morphisms of formal algebraic spaces.

0ARN Definition 87.30.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is separated if $\Delta_{X/Y}$ is a closed immersion.
(2) We say f is quasi-separated if $\Delta_{X/Y}$ is quasi-compact.

Since $\Delta_{X/Y}$ is representable (by schemes) by Lemma 87.15.5 we can test this by considering morphisms $T \to X \times_Y X$ from affine schemes T and checking whether

$$E = T \times_{X \times_Y X} X \longrightarrow T$$

is quasi-compact or a closed immersion, see Lemma 87.17.5 or Definition 87.27.1. Note that the scheme E is the equalizer of two morphisms $a, b : T \to X$ which agree as morphisms into Y and that $E \to T$ is a monomorphism and locally of finite type.

0ARP Lemma 87.30.2. All of the separation axioms listed in Definition 87.30.1 are stable under base change.

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of formal algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. Each of the properties of the diagonal used in Definition 87.30.1 is stable under base change. Hence the lemma is true. $\square$

0ARQ Lemma 87.30.3. Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of formal algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then

(1) i is representable (by schemes), locally of finite type, locally quasi-finite, separated, and a monomorphism,
(2) if $Z \to T$ is separated, then i is a closed immersion, and
(3) if $Z \to T$ is quasi-separated, then i is quasi-compact.

Proof. By general category theory the following diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "i"'] \arrow[d] & X \times_T Y \arrow[d] \\ Z \arrow[r, "\Delta_{Z/T}"] & Z \times_T Z \end{tikzcd}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 87.15.5. $\square$

0ARR Lemma 87.30.4. All of the separation axioms listed in Definition 87.30.1 are stable under composition of morphisms.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 87.30.3 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$. $\square$

0ARS Lemma 87.30.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 87.30.1. The following are equivalent

(1) f is $\mathcal{P}$,
(2) for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,
(3) for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,
(4) for every affine scheme Z and every morphism $Z \to Y$ the formal algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Definition 87.16.3),
(5) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that the base change $Y_j \times_Y X \to Y_j$ has $\mathcal{P}$ for all j.

Proof. We will repeatedly use Lemma 87.30.2 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Assume (3) and let $Z \to Y$ be a morphism where Z is an affine scheme. Let U, V be affine schemes and let $a : U \to Z \times_Y X$ and $b : V \to Z \times_Y X$ be morphisms. Then

$$U \times_{Z \times_Y X} V = (Z \times_Y X) \times_{\Delta, (Z \times_Y X) \times_Z (Z \times_Y X)} (U \times_Z V)$$

and we see that this is quasi-compact if $\mathcal{P} =$"quasi-separated" or an affine scheme equipped with a closed immersion into $U \times_Z V$ if $\mathcal{P} =$"separated". Thus (4) holds.

Assume (4) and let $Z \to Y$ be a morphism where Z is an affine scheme. Let U, V be affine schemes and let $a : U \to Z \times_Y X$ and $b : V \to Z \times_Y X$ be morphisms. Reading the argument above backwards, we see that $U \times_{Z \times_Y X} V \to U \times_Z V$ is quasi-compact if $\mathcal{P} =$"quasi-separated" or a closed immersion if $\mathcal{P} =$"separated". Since we can choose U and V as above such that U varies through an étale covering of $Z \times_Y X$, we find that the corresponding morphisms

$$U \times_Z V \to (Z \times_Y X) \times_Z (Z \times_Y X)$$

form an étale covering by affines. Hence we conclude that $\Delta : (Z \times_Y X) \to (Z \times_Y X) \times_Z (Z \times_Y X)$ is quasi-compact, resp. a closed immersion. Thus (3) holds.

Let us prove that (3) implies (5). Assume (3) and let $\{Y_j \to Y\}$ be as in Definition 87.11.1. We have to show that the morphisms

$$\Delta_j : Y_j \times_Y X \longrightarrow (Y_j \times_Y X) \times_{Y_j} (Y_j \times_Y X) = Y_j \times_Y X \times_Y X$$

has the corresponding property (i.e., is quasi-compact or a closed immersion). Write $Y_j = \operatorname{colim} Y_{j,\lambda}$ as in Definition 87.9.1. Replacing Y_j by $Y_{j,\lambda}$ in the formula above, we have the property by our assumption that (3) holds. Since the displayed arrow is the colimit of the arrows $\Delta_{j,\lambda}$ and since we can test whether Δ_j has the corresponding property by testing after base change by affine schemes mapping into $Y_j \times_Y X \times_Y X$, we conclude by Lemma 87.9.4.

Let us prove that (5) implies (1). Let $\{Y_j \to Y\}$ be as in (5). Then we have the fibre product diagram

$$\begin{tikzcd} \coprod Y_j \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod Y_j \times_Y X \times_Y X \arrow[r] & X \times_Y X \end{tikzcd}$$

By assumption the left vertical arrow is quasi-compact or a closed immersion. It follows from Spaces, Lemma 65.5.6 that also the right vertical arrow is quasi-compact or a closed immersion. $\square$

87.31. Proper morphisms

0AM5 Here is the definition we will use.

0AM6 Definition 87.31.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S. We say f is *proper* if f is representable by algebraic spaces and is proper in the sense of Bootstrap, Definition 80.4.1.

It follows from the definitions that a proper morphism is of finite type.

0ART Lemma 87.31.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent

(1) f is proper,

(2) for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is proper,
(3) for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is proper,
(4) for every affine scheme Z and every morphism $Z \to Y$ the formal algebraic space $Z \times_Y X$ is an algebraic space proper over Z,
(5) there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that the base change $Y_j \times_Y X \to Y_j$ is proper for all j.

Proof. Omitted. □

0GBT Lemma 87.31.3. Proper morphisms of formal algebraic spaces are preserved by base change.

Proof. This is an immediate consequence of Lemma 87.31.2 and transitivity of base change. □

87.32. Formal algebraic spaces and fpqc coverings

0AQC This section is the analogue of Properties of Spaces, Section 66.17. Please read that section first.

0AQD Lemma 87.32.1. Let S be a scheme. Let X be a formal algebraic space over S. Then X satisfies the sheaf property for the fpqc topology.

Proof. The proof is identical to the proof of Properties of Spaces, Proposition 66.17.1. Since X is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines $f : T' \to T$ we have: $X(T)$ is the equalizer of the two maps $X(T') \to X(T' \times_T T')$. See Topologies, Lemma 34.9.13.

Let $a, b : T \to X$ be two morphisms such that $a \circ f = b \circ f$. We have to show $a = b$. Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Lemma 87.11.2 the morphism $\Delta_{X/S}$ is a representable monomorphism. Hence $E \to T$ is a monomorphism of schemes. Our assumption that $a \circ f = b \circ f$ implies that $T' \to T$ factors (uniquely) through E. Consider the commutative diagram

$$\begin{tikzcd} T' \times_T E \arrow[r] \arrow[d] & E \arrow[d] \\ T' \arrow[r] \arrow[ur] \arrow[u, bend left] & T \end{tikzcd}$$

Since the projection $T' \times_T E \to T'$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $E \to T$ is an isomorphism by Descent, Lemma 35.23.17. This means $a = b$ as desired.

Next, let $c : T' \to X$ be a morphism such that the two compositions $T' \times_T T' \to T' \to X$ are the same. We have to find a morphism $a : T \to X$ whose composition with $T' \to T$ is c. Choose a formal affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |X_{red}|$ contains the image of $|c| : |T'| \to |X_{red}|$. This is possible by Definition 87.11.1, Properties of Spaces, Lemma 66.4.6, the fact that a finite union of formal affine algebraic spaces is a formal affine algebraic space, and the fact that $|T'|$ is quasi-compact (small argument omitted). The morphism $U \to X$ is representable by schemes (Lemma 87.9.11) and separated (Lemma 87.16.5). Thus

$$V = U \times_{X,c} T' \longrightarrow T'$$

is an étale and separated morphism of schemes. It is also surjective by our choice of $U \to X$ (if you do not want to argue this you can replace U by a disjoint union of formal affine algebraic spaces so that $U \to X$ is surjective everything else still works as well). The fact that $c \circ \text{pr}_0 = c \circ \text{pr}_1$ means that we obtain a descent datum on $V/T'/T$ (Descent, Definition 35.34.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, c\text{pr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{c\text{pr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism $V \to T'$ is ind-quasi-affine by More on Morphisms, Lemma 37.66.8 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 29.36.6). By More on Groupoids, Lemma 40.15.3 the descent datum is effective. Say $W \to T$ is a morphism such that there is an isomorphism $\alpha : T' \times_T W \to V$ compatible with the given descent datum on V and the canonical descent datum on $T' \times_T W$. Then $W \to T$ is surjective and étale (Descent, Lemmas 35.23.7 and 35.23.29). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions $b' \circ (\text{pr}_0, 1), b' \circ (\text{pr}_1, 1) : (T' \times_T T') \times_T W \to T' \times_T W \to U$ agree by our choice of α and the corresponding property of c (computation omitted). Hence b' descends to a morphism $b : W \to U$ by Descent, Lemma 35.13.7. The diagram

$$\begin{tikzcd} T' \times_T W \arrow[r] \arrow[d] & W \arrow[r, "b"'] & U \arrow[d] \\ T' \arrow[rr, "c"] & & X \end{tikzcd}$$

is commutative. What this means is that we have proved the existence of a étale locally on T, i.e., we have an $a' : W \to X$. However, since we have proved uniqueness in the first paragraph, we find that this étale local solution satisfies the glueing condition, i.e., we have $\text{pr}_0^*a' = \text{pr}_1^*a'$ as elements of $X(W \times_T W)$. Since X is an étale sheaf we find an unique $a \in X(T)$ restricting to a' on W. $\square$

87.33. Maps out of affine formal schemes

0AQE We prove a few results that will be useful later. In the paper [Bha16] the reader can find very general results of a similar nature.

0AQF **Lemma 87.33.1.** Let S be a scheme. Let A be a weakly admissible topological S-algebra. Let X be an affine scheme over S. Then the natural map

$$\text{Mor}_S(\text{Spec}(A), X) \longrightarrow \text{Mor}_S(\text{Spf}(A), X)$$

is bijective.

Proof. If X is affine, say $X = \text{Spec}(B)$, then we see from Lemma 87.9.10 that morphisms $\text{Spf}(A) \to \text{Spec}(B)$ correspond to continuous S-algebra maps $B \to A$ where B has the discrete topology. These are just S-algebra maps, which correspond to morphisms $\text{Spec}(A) \to \text{Spec}(B)$. $\square$

0AQG Lemma 87.33.2. Let S be a scheme. Let A be a weakly admissible topological S-algebra such that A/I is a local ring for some weak ideal of definition $I \subset A$. Let X be a scheme over S. Then the natural map

$$\mathrm{Mor}_S(\mathrm{Spec}(A), X) \longrightarrow \mathrm{Mor}_S(\mathrm{Spf}(A), X)$$

is bijective.

Proof. Let $\varphi : \mathrm{Spf}(A) \to X$ be a morphism. Since $\mathrm{Spec}(A/I)$ is local we see that φ maps $\mathrm{Spec}(A/I)$ into an affine open $U \subset X$. However, this then implies that $\mathrm{Spec}(A/J)$ maps into U for every ideal of definition J. Hence we may apply Lemma 87.33.1 to see that φ comes from a morphism $\mathrm{Spec}(A) \to X$. This proves surjectivity of the map. We omit the proof of injectivity. $\square$

0AQH Lemma 87.33.3. Let S be a scheme. Let R be a complete local Noetherian S-algebra. Let X be an algebraic space over S. Then the natural map

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \mathrm{Mor}_S(\mathrm{Spf}(R), X)$$

is bijective.

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. We have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for R as above.

Injectivity: Let $x, x' : \mathrm{Spec}(R) \to X$ be two morphisms mapping to the same element in the right hand side. Consider the fibre product

$$T = \mathrm{Spec}(R) \times_{(x,x'), X \times_S X, \Delta} X$$

Then T is a scheme and $T \to \mathrm{Spec}(R)$ is locally of finite type, monomorphism, separated, and locally quasi-finite, see Morphisms of Spaces, Lemma 67.4.1. In particular T is locally Noetherian, see Morphisms, Lemma 29.15.6. Let $t \in T$ be the unique point mapping to the closed point of $\mathrm{Spec}(R)$ which exists as x and x' agree over $R/\mathfrak{m}$. Then $R \to \mathcal{O}_{T,t}$ is a local ring map of Noetherian rings such that $R/\mathfrak{m}^n \to \mathcal{O}_{T,t}/\mathfrak{m}^n\mathcal{O}_{T,t}$ is an isomorphism for all n (because x and x' agree over $\mathrm{Spec}(R/\mathfrak{m}^n)$ for all n). Since $\mathcal{O}_{T,t}$ maps injectively into its completion (see Algebra, Lemma 10.51.4) we conclude that $R = \mathcal{O}_{T,t}$. Hence x and x' agree over R.

Surjectivity: Let (x_n) be an element of the right hand side. Choose a scheme U and a surjective étale morphism $U \to X$. Denote $x_0 : \mathrm{Spec}(k) \to X$ the morphism induced on the residue field $k = R/\mathfrak{m}$. The morphism of schemes $U \times_{X,x_0} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale. Thus $U \times_{X,x_0} \mathrm{Spec}(k)$ is a nonempty disjoint union of spectra of finite separable field extensions of k, see Morphisms, Lemma 29.36.7. Hence we can find a finite separable field extension k'/k and a k'-point $u_0 : \mathrm{Spec}(k') \to U$ such that

$$\begin{tikzcd}
\mathrm{Spec}(k') \arrow[r, "u_0"'] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, "x_0"] & X
\end{tikzcd}$$

commutes. Let $R \subset R'$ be the finite étale extension of Noetherian complete local rings which induces k'/k on residue fields (see Algebra, Lemmas 10.153.7 and

10.153.9). Denote x'_n the restriction of x_n to $\mathrm{Spec}(R'/\mathfrak{m}^nR')$. By More on Morphisms of Spaces, Lemma 76.16.8 we can find an element $(u'_n) \in \lim \mathrm{Mor}_S(\mathrm{Spec}(R'/\mathfrak{m}^nR'), U)$ mapping to (x'_n). By Lemma 87.33.2 the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote $x' : \mathrm{Spec}(R') \to X$ the composition. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Hence the diagram

$$\begin{tikzcd} \mathrm{Spec}(R' \otimes_R R') \arrow[r] \arrow[d] & \mathrm{Spec}(R') \arrow[d, "x'"] \\ \mathrm{Spec}(R') \arrow[r, "x'"] & X \end{tikzcd}$$

is commutative by the injectivity shown above and the fact that x'_n is the restriction of x_n which is defined over $R/\mathfrak{m}^n$. Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to a morphism $x : \mathrm{Spec}(R) \to X$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. □

0GBU Lemma 87.33.4. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is quasi-compact. Let R be a complete local Noetherian S-algebra. Then an adic morphism $p : \mathrm{Spf}(R) \to X_{/T}$ corresponds to a unique morphism $g : \mathrm{Spec}(R) \to X$ such that $g^{-1}(T) = \{\mathfrak{m}_R\}$.

Proof. The statement makes sense because $X_{/T}$ is adic* by Lemma 87.20.8 (and hence we're allowed to use the terminology adic for morphisms, see Definition 87.23.2). Let p be given. By Lemma 87.33.3 we get a unique morphism $g : \mathrm{Spec}(R) \to X$ corresponding to the composition $\mathrm{Spf}(R) \to X_{/T} \to X$. Let $Z \subset X$ be the reduced induced closed subspace structure on T. The incusion morphism $Z \to X$ corresponds to a morphism $Z \to X_{/T}$. Since p is adic it is representable by algebraic spaces and we find

$$\mathrm{Spf}(R) \times_{X_{/T}} Z = \mathrm{Spf}(R) \times_X Z$$

is an algebraic space endowed with a closed immersion to $\mathrm{Spf}(R)$. (Equality holds because $X_{/T} \to X$ is a monomorphism.) Thus this fibre product is equal to $\mathrm{Spec}(R/J)$ for some ideal $J \subset R$ wich contains $\mathfrak{m}_R^{n_0}$ for some $n_0 \geq 1$. This implies that $\mathrm{Spec}(R) \times_X Z$ is a closed subscheme of $\mathrm{Spec}(R)$, say $\mathrm{Spec}(R) \times_X Z = \mathrm{Spec}(R/I)$, whose intersection with $\mathrm{Spec}(R/\mathfrak{m}_R^n)$ for $n \geq n_0$ is equal to $\mathrm{Spec}(R/J)$. In algebraic terms this says $I + \mathfrak{m}_R^n = J + \mathfrak{m}_R^n = J$ for all $n \geq n_0$. By Krull's intersection theorem this implies $I = J$ and we conclude. □

87.34. The small étale site of a formal algebraic space

0DE9 The motivation for the following definition comes from classical formal schemes: the underlying topological space of a formal scheme $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is the underlying topological space of the reduction $\mathfrak{X}_{red}$.

An important remark is the following. Suppose that X is an algebraic space with reduction X_{red} (Properties of Spaces, Definition 66.12.5). Then we have

$$X_{spaces,étale} = X_{red,spaces,étale}, \quad X_{étale} = X_{red,étale}, \quad X_{affine,étale} = X_{red,affine,étale}$$

by More on Morphisms of Spaces, Theorem 76.8.1 and Lemma 76.8.2. Therefore the following definition does not conflict with the already existing notion in case our formal algebraic space happens to be an algebraic space.

0DEA Definition 87.34.1. Let S be a scheme. Let X be a formal algebraic space with reduction X_{red} (Lemma 87.12.1).

(1) The small étale site $X_{étale}$ of X is the site $X_{red,étale}$ of Properties of Spaces, Definition 66.18.1.
(2) The site $X_{spaces,étale}$ is the site $X_{red,spaces,étale}$ of Properties of Spaces, Definition 66.18.2.
(3) The site $X_{affine,étale}$ is the site $X_{red,affine,étale}$ of Properties of Spaces, Lemma 66.18.6.

In Lemma 87.34.6 we will see that $X_{spaces,étale}$ can be described by in terms of morphisms of formal algebraic spaces which are representable by algebraic spaces and étale. By Properties of Spaces, Lemmas 66.18.3 and 66.18.6 we have identifications

0DEB (87.34.1.1) $$Sh(X_{étale}) = Sh(X_{spaces,étale}) = Sh(X_{affine,étale})$$

We will call this the (small) étale topos of X.

0DEC Lemma 87.34.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S.

(1) There is a continuous functor $Y_{spaces,étale} \to X_{spaces,étale}$ which induces a morphism of sites
$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$
(2) The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 7.14.5).
(3) The morphism of topoi associated to $f_{spaces,étale}$ induces, via (87.34.1.1), a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.

Proof. The only point here is that f induces a morphism of reductions $X_{red} \to Y_{red}$ by Lemma 87.12.1. Hence this lemma is immediate from the corresponding lemma for morphisms of algebraic spaces (Properties of Spaces, Lemma 66.18.8). □

If the morphism of formal algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

0DED Lemma 87.34.3. Let S be a scheme, and let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume f is representable by algebraic spaces and étale. In this case there is a cocontinuous functor $j : X_{étale} \to Y_{étale}$. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 7.21.1. Moreover, j is continuous as well, hence Sites, Lemma 7.21.5 applies.

Proof. This will follow immediately from the case of algebraic spaces (Properties of Spaces, Lemma 66.18.11) if we can show that the induced morphism $X_{red} \to Y_{red}$ is étale. Observe that $X \times_Y Y_{red}$ is an algebraic space, étale over the reduced algebraic space Y_{red}, and hence reduced itself (by our definition of reduced algebraic spaces in Properties of Spaces, Section 66.7. Hence $X_{red} = X \times_Y Y_{red}$ as desired. □

0DEE Lemma 87.34.4. Let S be a scheme. Let X be an affine formal algebraic space over S. Then $X_{affine,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that

(1) U is an affine formal algebraic space,
(2) φ is representable by algebraic spaces and étale.

Proof. Denote $\mathcal{C}$ the category introduced in the lemma. Observe that for $\varphi : U \to X$ in $\mathcal{C}$ the morphism φ is representable (by schemes) and affine, see Lemma 87.19.7. Recall that $X_{affine,étale} = X_{red,affine,étale}$. Hence we can define a functor

$$\mathcal{C} \longrightarrow X_{affine,étale}, \quad (U \to X) \longmapsto U \times_X X_{red}$$

because $U \times_X X_{red}$ is an affine scheme.

To finish the proof we will construct a quasi-inverse. Namely, write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. For each λ we have $X_{red} \subset X_\lambda$ is a thickening. Thus for every λ we have an equivalence

$$X_{red,affine,étale} = X_{\lambda,affine,étale}$$

for example by More on Algebra, Lemma 15.11.2. Hence if $U_{red} \to X_{red}$ is an étale morphism with U_{red} affine, then we obtain a system of étale morphisms $U_\lambda \to X_\lambda$ of affine schemes compatible with the transition morphisms in the system defining X. Hence we can take

$$U = \operatorname{colim} U_\lambda$$

as our affine formal algebraic space over X. The construction gives that $U \times_X X_\lambda = U_\lambda$. This shows that $U \to X$ is representable and étale. We omit the verification that the constructions are mutually inverse to each other. □

0DEF Lemma 87.34.5. Let S be a scheme. Let X be an affine formal algebraic space over S. Assume X is McQuillan, i.e., equal to $\text{Spf}(A)$ for some weakly admissible topological S-algebra A. Then $(X_{affine,étale})^{opp}$ is equivalent to the category whose

(1) objects are A-algebras of the form $B^\wedge = \lim B/JB$ where $A \to B$ is an étale ring map and J runs over the weak ideals of definition of A, and
(2) morphisms are continuous A-algebra homomorphisms.

Proof. Combine Lemmas 87.34.4 and 87.19.13. □

0DEG Lemma 87.34.6. Let S be a scheme. Let X be a formal algebraic space over S. Then $X_{spaces,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that φ is representable by algebraic spaces and étale.

Proof. Denote $\mathcal{C}$ the category introduced in the lemma. Recall that $X_{spaces,étale} = X_{red,spaces,étale}$. Hence we can define a functor

$$\mathcal{C} \longrightarrow X_{spaces,étale}, \quad (U \to X) \longmapsto U \times_X X_{red}$$

because $U \times_X X_{red}$ is an algebraic space étale over X_{red}.

To finish the proof we will construct a quasi-inverse. Choose an object $\psi : V \to X_{red}$ of $X_{red,spaces,étale}$. Consider the functor $U_{V,\psi} : (Sch/S)_{fppf} \to Sets$ given by

$$U_{V,\psi}(T) = \{(a, b) \mid a : T \to X,\ b : T \times_{a,X} X_{red} \to V,\ \psi \circ b = a|_{T \times_{a,X} X_{red}}\}$$

We claim that the transformation $U_{V,\psi} \to X$, $(a, b) \mapsto a$ defines an object of the category $\mathcal{C}$. First, let's prove that $U_{V,\psi}$ is a formal algebraic space. Observe that $U_{V,\psi}$ is a sheaf for the fppf topology (some details omitted). Next, suppose that $X_i \to X$ is an étale covering by affine formal algebraic spaces as in Definition 87.11.1. Set $V_i = V \times_{X_{red}} X_{i,red}$ and denote $\psi_i : V_i \to X_{i,red}$ the projection. Then we have

$$U_{V,\psi} \times_X X_i = U_{V_i,\psi_i}$$

by a formal argument because $X_{i,red} = X_i \times_X X_{red}$ (as $X_i \to X$ is representable by algebraic spaces and étale). Hence it suffices to show that U_{V_i,ψ_i} is an affine formal algebraic space, because then we will have a covering $U_{V_i,\psi_i} \to U_{V,\psi}$ as in Definition 87.11.1. On the other hand, we have seen in the proof of Lemma 87.34.3 that $\psi_i : V_i \to X_i$ is the base change of a representable and étale morphism $U_i \to X_i$ of affine formal algebraic spaces. Then it is not hard to see that $U_i = U_{V_i,\psi_i}$ as desired.

We omit the verification that $U_{V,\psi} \to X$ is representable by algebraic spaces and étale. Thus we obtain our functor $(V, \psi) \mapsto (U_{V,\psi} \to X)$ in the other direction. We omit the verification that the constructions are mutually inverse to each other. □

0DEH Lemma 87.34.7. Let S be a scheme. Let X be a formal algebraic space over S. Then $X_{affine,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that

(1) U is an affine formal algebraic space,
(2) φ is representable by algebraic spaces and étale.

Proof. This follows by combining Lemmas 87.34.6 and 87.18.3. □

87.35. The structure sheaf

0DEI Let X be a formal algebraic space. A structure sheaf for X is a sheaf of topological rings $\mathcal{O}_X$ on the étale site $X_{étale}$ (which we defined in Section 87.34) such that

$$\mathcal{O}_X(U_{red}) = \lim \Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$$

as topological rings whenever

(1) $\varphi : U \to X$ is a morphism of formal algebraic spaces,
(2) U is an affine formal algebraic space,
(3) φ is representable by algebraic spaces and étale,
(4) $U_{red} \to X_{red}$ is the corresponding affine object of $X_{étale}$, see Lemma 87.34.7,
(5) $U = \text{colim}\, U_\lambda$ is a colimit representation for U as in Definition 87.9.1.

Structure sheaves exist but may behave in unexpected manner.

0DEJ Lemma 87.35.1. Every formal algebraic space has a structure sheaf.

Proof. Let S be a scheme. Let X be a formal algebraic space over S. By (87.34.1.1) it suffices to construct $\mathcal{O}_X$ as a sheaf of topological rings on $X_{affine,étale}$. Denote $\mathcal{C}$ the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that U is an affine formal algebraic space and φ is representable by algebraic spaces and étale. By Lemma 87.34.7 the functor $U \mapsto U_{red}$ is an equivalence of categories $\mathcal{C} \to X_{affine,étale}$. Hence by the rule given above the lemma, we already have $\mathcal{O}_X$ as a presheaf of topological rings on $X_{affine,étale}$. Thus it suffices to check the sheaf condition.

By definition of $X_{affine,étale}$ a covering corresponds to a finite family $\{g_i : U_i \to U\}_{i=1,\ldots,n}$ of morphisms of $\mathcal{C}$ such that $\{U_{i,red} \to U_{red}\}$ is an étale covering. The morphisms g_i are representably by algebraic spaces (Lemma 87.19.3) hence affine (Lemma 87.19.7). Then g_i is étale (follows formally from Properties of Spaces, Lemma 66.16.6 as U_i and U are étale over X in the sense of Bootstrap, Section 80.4). Finally, write $U = \text{colim}\, U_\lambda$ as in Definition 87.9.1.

With these preparations out of the way, we can prove the sheaf property as follows. For each λ we set $U_{i,\lambda} = U_i \times_U U_\lambda$ and $U_{ij,\lambda} = (U_i \times_U U_j) \times_U U_\lambda$. By the above, these are affine schemes, $\{U_{i,\lambda} \to U_\lambda\}$ is an étale covering, and $U_{ij,\lambda} = U_{i,\lambda} \times_{U_\lambda} U_{j,\lambda}$. Also we have $U_i = \operatorname{colim} U_{i,\lambda}$ and $U_i \times_U U_j = \operatorname{colim} U_{ij,\lambda}$. For each λ we have an exact sequence

$$0 \to \Gamma(U_\lambda, \mathcal{O}_{U_\lambda}) \to \prod\nolimits_i \Gamma(U_{i,\lambda}, \mathcal{O}_{U_{i,\lambda}}) \to \prod\nolimits_{i,j} \Gamma(U_{ij,\lambda}, \mathcal{O}_{U_{ij,\lambda}})$$

as we have the sheaf condition for the structure sheaf on U_λ and the étale topology (see Étale Cohomology, Proposition 59.17.1). Since limits commute with limits, the inverse limit of these exact sequences is an exact sequence

$$0 \to \lim \Gamma(U_\lambda, \mathcal{O}_{U_\lambda}) \to \prod\nolimits_i \lim \Gamma(U_{i,\lambda}, \mathcal{O}_{U_{i,\lambda}}) \to \prod\nolimits_{i,j} \lim \Gamma(U_{ij,\lambda}, \mathcal{O}_{U_{ij,\lambda}})$$

which exactly means that

$$0 \to \mathcal{O}_X(U_{red}) \to \prod\nolimits_i \mathcal{O}_X(U_{i,red}) \to \prod\nolimits_{i,j} \mathcal{O}_X((U_i \times_U U_j)_{red})$$

is exact and hence the sheaf propery holds as desired. □

0DEK Remark 87.35.2. The structure sheaf does not always have "enough sections". In Examples, Section 110.74 we have seen that there exist affine formal algebraic spaces which aren't McQuillan and there are even examples whose points are not separated by regular functions.

In the next lemma we prove that the structure sheaf on a countably indexed affine formal scheme has vanishing higher cohomology. For non-countably indexed ones, presumably this generally doesn't hold.

0DEL Lemma 87.35.3. If X is a countably indexed affine formal algebraic space, then we have $H^n(X_{étale}, \mathcal{O}_X) = 0$ for $n > 0$.

Proof. We may work with $X_{affine,étale}$ as this gives the same topos. We will apply Cohomology on Sites, Lemma 21.10.9 to show we have vanishing. Since $X_{affine,étale}$ has finite disjoint unions, this reduces us to the Čech complex of a covering given by a single arrow $\{U_{red} \to V_{red}\}$ in $X_{affine,étale} = X_{red,affine,étale}$ (see Étale Cohomology, Lemma 59.22.1). Thus we have to show that

$$0 \to \mathcal{O}_X(V_{red}) \to \mathcal{O}_X(U_{red}) \to \mathcal{O}_X(U_{red} \times_{V_{red}} U_{red}) \to \ldots$$

is exact. We will do this below in the case $V_{red} = X_{red}$. The general case is proven in exactly the same way.

Recall that $X = \mathrm{Spf}(A)$ where A is a weakly admissible topological ring having a countable fundamental system of weak ideals of definition. We have seen in Lemmas 87.34.4 and 87.34.5 that the object U_{red} in $X_{affine,étale}$ corresponds to a morphism $U \to X$ of affine formal algebraic spaces which is representable by algebraic space and étale and $U = \mathrm{Spf}(B^\wedge)$ where B is an étale A-algebra. By our rule for the structure sheaf we see

$$\mathcal{O}_X(U_{red}) = B^\wedge$$

We recall that $B^\wedge = \lim B/JB$ where the limit is over weak ideals of definition $J \subset A$. Working through the definitions we obtain

$$\mathcal{O}_X(U_{red} \times_{X_{red}} U_{red}) = (B \otimes_A B)^\wedge$$

and so on. Since $U \to X$ is a covering the map $A \to B$ is faithfully flat, see Lemma 87.19.14. Hence the complex

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

is universally exact, see Descent, Lemma 35.3.6. Our goal is to show that

$$H^n(0 \to A^\wedge \to B^\wedge \to (B \otimes_A B)^\wedge \to (B \otimes_A B \otimes_A B)^\wedge \to \ldots)$$

is zero for $n > 0$. To see what is going on, let's split our exact complex (before completion) into short exact sequences

$$0 \to A \to B \to M_1 \to 0, \quad 0 \to M_i \to B^{\otimes_A i+1} \to M_{i+1} \to 0$$

By what we said above, these are universally exact short exact sequences. Hence $JM_i = M_i \cap J(B^{\otimes_A i+1})$ for every ideal J of A. In particular, the topology on M_i as a submodule of $B^{\otimes_A i+1}$ is the same as the topology on M_i as a quotient module of $B^{\otimes_A i}$. Therefore, since there exists a countable fundamental system of weak ideals of definition in A, the sequences

$$0 \to A^\wedge \to B^\wedge \to M_1^\wedge \to 0, \quad 0 \to M_i^\wedge \to (B^{\otimes_A i+1})^\wedge \to M_{i+1}^\wedge \to 0$$

remain exact by Lemma 87.4.5. This proves the lemma. □

0DEM Remark 87.35.4. Even if the structure sheaf has good properties, this does not mean there is a good theory of quasi-coherent modules. For example, in Examples, Section 110.13 we have seen that for almost any Noetherian affine formal algebraic spaces the most natural notion of a quasi-coherent module leads to a category of modules which is not abelian.

87.36. Colimits of formal algebraic spaces

0GVL In this section we generalize the result of Section 87.13 to the case of systems of morphisms of formal algebraic spaces. We remark that in the lemmas below the condition "$f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$" can be reformulated as "$f_{\lambda\mu}$ is representable and a thickening".

0GVM Lemma 87.36.1. Let S be a scheme. Suppose given a directed set Λ and a system of affine formal algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$. Then $X = \text{colim}_{\lambda\in\Lambda} X_\lambda$ is an affine formal algebraic space over S.

Proof. We may write $X_\lambda = \text{colim}_{\omega\in\Omega_\lambda} X_{\lambda,\omega}$ as the colimit of affine schemes over a directed set Ω_λ such that the transition morphisms $X_{\lambda,\omega} \to X_{\lambda,\omega'}$ are thickenings. For each $\lambda, \mu \in \Lambda$ and $\omega \in \Omega_\lambda$, with $\mu \geq \lambda$ there exists an $\omega' \in \Omega_\mu$ such that the morphism $X_{\lambda,\omega} \to X_\mu$ factors through $X_{\mu,\omega'}$, see Lemma 87.9.4. Then the morphism $X_{\lambda,\omega} \to X_{\mu,\omega'}$ is a closed immersion inducing an isomorphism on reductions and hence a thickening. Set $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ and say $(\lambda, \omega) \leq (\mu, \omega')$ if and only if $\lambda \leq \mu$ and $X_{\lambda,\omega} \to X_\mu$ factors through $X_{\mu,\omega'}$. It follows from the above that Ω is a directed set and that $X = \text{colim}_{\lambda\in\Lambda} X_\lambda = \text{colim}_{(\lambda,\omega)\in\Omega} X_{\lambda,\omega}$. This finishes the proof. □

0GVN Lemma 87.36.2. Let S be a scheme. Suppose given a directed set Λ and a system of formal algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$. Then $X = \text{colim}_{\lambda\in\Lambda} X_\lambda$ is a formal algebraic space over S.

Proof. Since we take the colimit in the category of fppf sheaves, we see that X is a sheaf. Choose and fix $\lambda \in \Lambda$. Choose a covering $\{X_{i,\lambda} \to X_\lambda\}$ as in Definition 87.11.1. In particular, we see that $\{X_{i,\lambda,red} \to X_{\lambda,red}\}$ is an étale covering by affine schemes. For each $\mu \geq \lambda$ there exists a cartesian diagram

$$\begin{tikzcd} X_{i,\lambda} \arrow[r] \arrow[d] & X_{i,\mu} \arrow[d] \\ X_\lambda \arrow[r] & X_\mu \end{tikzcd}$$

with étale vertical arrows. Namely, the étale morphism $X_{i,\lambda,red} \to X_{\lambda,red} = X_{\mu,red}$ corresponds to an étale morphism $X_{i,\mu} \to X_\mu$ of formal algebraic spaces with $X_{i,\mu}$ an affine formal algebraic space, see Lemma 87.34.4. The same lemma implies the base change of $X_{i,\mu}$ to X_λ agrees with $X_{i,\lambda}$. It also follows that $X_{i,\mu} = X_\mu \times_{X_{\mu'}} X_{i,\mu'}$ for $\mu' \geq \mu \geq \lambda$. Set $X_i = \operatorname{colim} X_{i,\mu}$. Then $X_{i,\mu} = X_i \times_X X_\mu$ (as functors). Since any morphism $T \to X = \operatorname{colim} X_\mu$ from an affine (or quasi-compact) scheme T maps into X_μ for some μ, we see conclude that $\operatorname{colim} X_{i,\mu} \to \operatorname{colim} X_\mu$ is étale. Thus, if we can show that $\operatorname{colim} X_{i,\mu}$ is an affine formal algebraic space, then the lemma holds. Note that the morphisms $X_{i,\mu} \to X_{i,\mu'}$ are closed immersions as a base change of the closed immersion $X_\mu \to X_{\mu'}$. Finally, the morphism $X_{i,\mu,red} \to X_{i,\mu',red}$ is an isomorphism as $X_{\mu,red} \to X_{\mu',red}$ is an isomorphism. Hence this reduces us to the case discussed in Lemma 87.36.1. $\square$

87.37. Recompletion

0GVP In this section we define the completion of a formal algebraic space along a closed subset of its reduction. It is the natural generalization of Section 87.14.

0GVQ **Lemma 87.37.1.** Let S be a scheme. Let X be an affine formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. Then the functor

$$X_{/T} : (Sch/S)_{fppf} \longrightarrow \textit{Sets}, \quad U \longmapsto \{f : U \to X : f(|U|) \subset T\}$$

is an affine formal algebraic space.

Proof. Write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Then $X_{\lambda,red} = X_{red}$ and we may and do view T as a closed subset of $|X_\lambda| = |X_{\lambda,red}|$. By Lemma 87.14.1 for each λ the completion $(X_\lambda)_{/T}$ is an affine formal algebraic space. The transition morphisms $(X_\lambda)_{/T} \to (X_\mu)_{/T}$ are closed immersions as base changes of the transition morphisms $X_\lambda \to X_\mu$, see Lemma 87.14.4. Also the morphisms $((X_\lambda)_{/T})_{red} \to ((X_\mu)_{/}T)_{red}$ are isomorphisms by Lemma 87.14.5. Since $X_{/T} = \operatorname{colim}(X_\lambda)_{/T}$ we conclude by Lemma 87.36.1. $\square$

0GVR **Lemma 87.37.2.** Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. Then the functor

$$X_{/T} : (Sch/S)_{fppf} \longrightarrow \textit{Sets}, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a formal algebraic space.

Proof. The functor $X_{/T}$ is an fppf sheaf since if $\{U_i \to U\}$ is an fppf covering, then $\coprod |U_i| \to |U|$ is surjective.

Choose a covering $\{g_i : X_i \to X\}_{i \in I}$ as in Definition 87.11.1. The morphisms $X_i \times_X X_{/T} \to X_{/T}$ are étale (see Spaces, Lemma 65.5.5) and the map $\coprod X_i \times_X$

$X_{/T} \to X_{/T}$ is a surjection of sheaves. Thus it suffices to prove that $X_{/T} \times_X X_i$ is an affine formal algebraic space. A U-valued point of $X_i \times_X X_{/T}$ is a morphism $U \to X_i$ whose image is contained in the closed subset $|g_{i,red}|^{-1}(T) \subset |X_{i,red}|$. Thus this follows from Lemma 87.37.1. □

0GVS Definition 87.37.3. Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. The formal algebraic space $X_{/T}$ of Lemma 87.14.2 is called the completion of X along T.

Let $f : X \to X'$ be a morphism of formal algebraic spaces over a scheme S. Suppose that $T \subset |X_{red}|$ and $T' \subset |X'_{red}|$ are closed subsets such that $|f_{red}|(T) \subset T'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

0GVT Lemma 87.37.4. Let S be a scheme. Let $f : X' \to X$ be a morphism of formal algebraic spaces over S. Let $T \subset |X_{red}|$ be a closed subset and let $T' = |f_{red}|^{-1}(T) \subset |X'_{red}|$. Then

$$\begin{tikzcd} X'_{/T'} \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ X_{/T} \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of formal algebraic spaces over S.

Proof. Namely, observe that the horizontal arrows are monomorphisms by construction. Thus it suffices to show that a morphism $g : U \to X'$ from a scheme U defines a point of $X'_{/T}$ if and only if $f \circ g$ defines a point of $X_{/T}$. In other words, we have to show that $g(U)$ is contained in $T' \subset |X'_{red}|$ if and only if $(f \circ g)(U)$ is contained in $T \subset |X_{red}|$. This follows immediately from our choice of T' as the inverse image of T. □

0GVU Lemma 87.37.5. Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. The reduction $(X_{/T})_{red}$ of the completion $X_{/T}$ of X along T is the reduced induced closed subspace Z of X_{red} corresponding to T.

Proof. It follows from Lemma 87.12.1, Properties of Spaces, Definition 66.12.5 (which uses Properties of Spaces, Lemma 66.12.3 to construct Z), and the definition of $X_{/T}$ that Z and $(X_{/T})_{red}$ are reduced algebraic spaces characterized the same mapping property: a morphism $g : Y \to X$ whose source is a reduced algebraic space factors through them if and only if $|Y|$ maps into $T \subset |X|$. □

0GVV Lemma 87.37.6. Let S be a scheme. Let X be an affine formal algebraic space over S. Let $T \subset X_{red}$ be a closed subset and let $X_{/T}$ be the formal completion of X along T. Then

(1) $X_{/T}$ is an affine formal algebraic space,
(2) if X is McQuillan, then $X_{/T}$ is McQuillan,
(3) if $|X_{red}| \setminus T$ is quasi-compact and X is countably indexed, then $X_{/T}$ is countably indexed,
(4) if $|X_{red}| \setminus T$ is quasi-compact and X is adic*, then $X_{/T}$ is adic*,

(5) if X is Noetherian, then $X_{/T}$ is Noetherian.

Proof. Part (1) is Lemma 87.37.1. If X is McQuillan, then $X = \text{Spf}(A)$ for some weakly admissible topological ring A. Then $X_{/T} \to X \to \text{Spec}(A)$ satisfies property (2) of Lemma 87.9.6 and hence $X_{/T}$ is McQuillan, see Definition 87.9.7.

Assume X and T are as in (3). Then $X = \text{Spf}(A)$ where A has a fundamental system $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of weak ideals of definition, see Lemma 87.10.4. By Algebra, Lemma 10.29.1 we can find a finitely generated ideal $\overline{J} = (\overline{f}_1, \ldots, \overline{f}_r) \subset A/I_1$ such that T is cut out by $\overline{J}$ inside $\text{Spec}(A/I_1) = |X_{red}|$. Choose $f_i \in A$ lifting $\overline{f}_i$. If $Z = \text{Spec}(B)$ is an affine scheme and $g : Z \to X$ is a morphism with $g(Z) \subset T$ (set theoretically), then $g^\sharp : A \to B$ factors through A/I_n for some n and $g^\sharp(f_i)$ is nilpotent in B for each i. Thus $J_{m,n} = (f_1, \ldots, f_r)^m + I_n$ maps to zero in B for some $n, m \geq 1$. It follows that $X_{/T}$ is the formal spectrum of $\lim_{n,m} A/J_{m,n}$ and hence countably indexed. This proves (3).

Proof of (4). Here the argument is the same as in (3). However, here we may choose $I_n = I^n$ for some finitely generated ideal $I \subset A$. Then it is clear that $X_{/T}$ is the formal spectrum of $\lim A/J^n$ where $J = (f_1, \ldots, f_r) + I$. Some details omitted.

Proof of (5). In this case X_{red} is the spectrum of a Noetherian ring and hence the assumption that $|X_{red}| \setminus T$ is quasi-compact is satisfied. Thus as in the proof of (4) we see that $X_{/T}$ is the spectrum of $\lim A/J^n$ which is a Noetherian adic topological ring, see Algebra, Lemma 10.97.6. □

0GVW Lemma 87.37.7. Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset X_{red}$ be a closed subset and let $X_{/T}$ be the formal completion of X along T. Then

(1) if $X_{red} \setminus T \to X_{red}$ is quasi-compact and X is locally countably indexed, then $X_{/T}$ is locally countably indexed,
(2) if $X_{red} \setminus T \to X_{red}$ is quasi-compact and X is locally adic*, then $X_{/T}$ is locally adic*, and
(3) if X is locally Noetherian, then $X_{/T}$ is locally Noetherian.

Proof. Choose a covering $\{X_i \to X\}$ as in Definition 87.11.1. Let $T_i \subset X_{i,red}$ be the inverse image of T. We have $X_i \times_X X_{/T} = (X_i)_{/T_i}$ (Lemma 87.37.4). Hence $\{(X_i)_{/T_i} \to X_{/T}\}$ is a covering as in Definition 87.11.1. Moreover, if $X_{red} \setminus T \to X_{red}$ is quasi-compact, so is $X_{i,red} \setminus T_i \to X_{i,red}$ and if X is locally countably indexed, or locally adic*, pr locally Noetherian, the is X_i is countably index, or adic*, or Noetherian. Thus the lemma follows from the affine case which is Lemma 87.37.6. □

87.38. Completion along a closed subspace

0GXT This section is the analgue of Section 87.14 for completions with respect to a closed subspace.

0GXU Definition 87.38.1. Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace and denote $Z_n \subset X$ the nth order infinitesimal neighbourhood. The formal algebraic space

$$X^\wedge_Z = \text{colim}\, Z_n$$

(see Lemma 87.36.2) is called the completion of X along Z.

Observe that if $T = |Z|$ then there is a canonical morphism $X_Z^\wedge \to X_{/T}$ comparing the completions along Z and T (Section 87.14) which need not be an isomorphism.

Let $f : X \to X'$ be a morphism of algebraic spaces over a scheme S. Suppose that $Z \subset X$ and $Z' \subset X'$ are closed subspaces such that $f|_Z$ maps Z into Z' inducing a morphism $Z \to Z'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_Z^\wedge \longrightarrow (X')_{Z'}^\wedge$$

between the completions.

0GXV **Lemma 87.38.2.** Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace and let $Z' = f^{-1}(Z) = X' \times_X Z$. Then

$$\begin{tikzcd} (X')_{Z'}^\wedge \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ X_Z^\wedge \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of sheaves. In particular, the morphism $(X')_{Z'}^\wedge \to X_Z^\wedge$ is representable by algebraic spaces.

Proof. Namely, suppose that $Y \to X$ is a morphism from a scheme into X such that $Y \to X$ factors through Z. Then $Y \times_X X' \to X$ is a morphism of algebraic spaces such that $Y \times_X X' \to X'$ factors through Z'. Since $Z'_n = X' \times_X Z_n$ for all $n \geq 1$ the same is true for the infinitesimal neighbourhoods. Hence the cartesian square of functors follows from the formulas $X_Z^\wedge = \text{colim}\, Z_n$ and $(X')_{Z'}^\wedge = \text{colim}\, Z'_n$. $\square$

0GXW **Lemma 87.38.3.** Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The reduction $(X_Z^\wedge)_{red}$ of the completion $X_Z^\wedge$ of X along Z is Z_{red}.

Proof. Omitted. $\square$

0GXX **Lemma 87.38.4.** Let S be a scheme. Let $X = \text{Spec}(A)$ be an affine scheme over S. Let $Z \subset X$ be a closed subscheme. Let $X_Z^\wedge$ be the formal completion of X along Z.

(1) The affine formal algebraic space $X_Z^\wedge$ is weakly adic.
(2) If $Z \to X$ is of finite presentation, then $X_Z^\wedge$ is adic*.
(3) If $Z = V(I)$ for some finitely generated ideal $I \subset A$, then $X_Z^\wedge = \text{Spf}(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A.
(4) If X is Noetherian, then $X_Z^\wedge$ is Noetherian.

Proof. Omitted. $\square$

0GXY **Lemma 87.38.5.** Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $X_Z^\wedge$ be the formal completion of X along Z.

(1) The formal algebraic space $X_Z^\wedge$ is locally weakly adic.
(2) If $Z \to X$ is of finite presentation, then $X_Z^\wedge$ is locally adic*.
(3) If X is locally Noetherian, then X_Z is locally Noetherian.

Proof. Omitted. $\square$

87.39. Other chapters

CHAPTER 88

Algebraization of Formal Spaces

0AM7

88.1. Introduction

0AM8 The main goal of this chapter is to prove Artin's theorem on dilatations, see Theorem 88.29.1; the result on contractions will be discussed in Artin's Axioms, Section 98.27. Both results use some material on formal algebraic spaces, hence in the middle part of this chapter, we continue the discussion of formal algebraic spaces from the previous chapter, see Formal Spaces, Section 87.1. The first part of this chapter is dedicated to algebraic preliminaries, mostly dealing with algebraization of rig-étale algebras.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. In the first part of this chapter (Sections 88.2 – 88.10) we discuss the category of I-adically complete algebras B topologically of finite type over a Noetherian ring A. It is shown that $B = A\{x_1, \ldots, x_n\}/J$ for some (closed) ideal J in the restricted power series ring (where A is endowed with the I-adic topology). We show there is a good notion of a naive cotangent complex $NL^\wedge_{B/A}$. If some power of I annihilates $NL^\wedge_{B/A}$, then we say B is a rig-étale algebra over (A, I); there is a similar notion of rig-smooth algebras. If A is a G-ring, then we can show, using Popescu's theorem, that any rig-smooth algebra B over (A, I) is the completion of a finite type A-algebra; informally we say that we can "algebraize" B. However, the main result of the first part is that any rig-étale algebra B over (A, I) can be algebraized, see Lemma 88.10.2. One thing to note here is that we prove this without assuming the ring A is a G-ring.

Many of the results discussed in the first part can be found in the paper [Elk73]. Other general references for this part are [DG67], [Abb10], and [FK].

In the second part of this chapter (Sections 88.12 – 88.24) we talk about types of morphisms of formal algebraic spaces in a reasonable level of generality (mostly for locally Noetherian formal algebraic spaces). The most interesting of these is the notion of a "formal modification" in the last section. We carefully check that our definition agrees with Artin's definition in [Art70].

Finally, in the third and last part of this chapter (Sections 88.25 – 88.30) we prove the main theorem and we give a few applications. In fact, we deduce Artin's theorem from a stronger result, namely, Theorem 88.27.4. This theorem says very roughly: if $f : \mathfrak{X} \to \mathfrak{X}'$ is a rig-étale morphism and $\mathfrak{X}'$ is the formal completion of a locally Noetherian algebraic space, then so is $\mathfrak{X}$. In Artin's work the morphism f is assumed proper and rig-surjective.

88.2. Two categories

0AL2

Let A be a ring and let $I \subset A$ be an ideal. In this section ${}^\wedge$ will mean I-adic completion. Set $A_n = A/I^n$ so that the I-adic completion of A is $A^\wedge = \lim A_n$. Let $\mathcal{C}$ be the category

0AL3 (88.2.0.1)
$$\mathcal{C} = \left\{ \begin{matrix} \text{inverse systems } \ldots \to B_3 \to B_2 \to B_1 \\ \text{where } B_n \text{ is a finite type } A_n\text{-algebra,} \\ B_{n+1} \to B_n \text{ is an } A_{n+1}\text{-algebra map} \\ \text{which induces } B_{n+1}/I^nB_{n+1} \cong B_n \end{matrix} \right\}$$

Morphisms in $\mathcal{C}$ are given by systems of homomorphisms. Let $\mathcal{C}'$ be the category

0AL4 (88.2.0.2)
$$\mathcal{C}' = \left\{ \begin{matrix} A\text{-algebras } B \text{ which are } I\text{-adically complete} \\ \text{such that } B/IB \text{ is of finite type over } A/I \end{matrix} \right\}$$

Morphisms in $\mathcal{C}'$ are A-algebra maps. There is a functor

0AJN (88.2.0.3)
$$\mathcal{C}' \longrightarrow \mathcal{C}, \quad B \longmapsto (B/I^nB)$$

Indeed, since B/IB is of finite type over A/I the ring maps $A_n = A/I^n \to B/I^nB$ are of finite type by Algebra, Lemma 10.126.8.

0AJP **Lemma 88.2.1.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor
$$\mathcal{C} \longrightarrow \mathcal{C}', \quad (B_n) \longmapsto B = \lim B_n$$
is a quasi-inverse to (88.2.0.3). The completions $A[x_1, \ldots, x_r]^\wedge$ are in $\mathcal{C}'$ and any object of $\mathcal{C}'$ is of the form
$$B = A[x_1, \ldots, x_r]^\wedge/J$$
for some ideal $J \subset A[x_1, \ldots, x_r]^\wedge$.

Proof. Let (B_n) be an object of $\mathcal{C}$. By Algebra, Lemma 10.98.2 we see that $B = \lim B_n$ is I-adically complete and $B/I^nB = B_n$. Hence we see that B is an object of $\mathcal{C}'$ and that we can recover the object (B_n) by taking the quotients. Conversely, if B is an object of $\mathcal{C}'$, then $B = \lim B/I^nB$ by assumption. Thus $B \mapsto (B/I^nB)$ is a quasi-inverse to the functor of the lemma.

Since $A[x_1, \ldots, x_r]^\wedge = \lim A_n[x_1, \ldots, x_r]$ it is an object of $\mathcal{C}'$ by the first statement of the lemma. Finally, let B be an object of $\mathcal{C}'$. Choose $b_1, \ldots, b_r \in B$ whose images in B/IB generate B/IB as an algebra over A/I. Since B is I-adically complete, the A-algebra map $A[x_1, \ldots, x_r] \to B$, $x_i \mapsto b_i$ extends to an A-algebra map $A[x_1, \ldots, x_r]^\wedge \to B$. To finish the proof we have to show this map is surjective which follows from Algebra, Lemma 10.96.1 as our map $A[x_1, \ldots, x_r] \to B$ is surjective modulo I and as $B = B^\wedge$. $\square$

We warn the reader that, in case A is not Noetherian, the quotient of an object of $\mathcal{C}'$ may not be an object of $\mathcal{C}'$. See Examples, Lemma 110.8.1. Next we show this does not happen when A is Noetherian.

0AJQ **Lemma 88.2.2.** Let A be a Noetherian ring and let $I \subset A$ be an ideal. Then [GD60, Proposition 7.5.5]

(1) every object of the category $\mathcal{C}'$ (88.2.0.2) is Noetherian,
(2) if $B \in \mathrm{Ob}(\mathcal{C}')$ and $J \subset B$ is an ideal, then B/J is an object of $\mathcal{C}'$,
(3) for a finite type A-algebra C the I-adic completion $C^\wedge$ is in $\mathcal{C}'$,
(4) in particular the completion $A[x_1, \ldots, x_r]^\wedge$ is in $\mathcal{C}'$.

Proof. Part (4) follows from Algebra, Lemma 10.97.6 as $A[x_1, \ldots, x_r]$ is Noetherian (Algebra, Lemma 10.31.1). To see (1) by Lemma 88.2.1 we reduce to the case of the completion of the polynomial ring which we just proved. Part (2) follows from Algebra, Lemma 10.97.1 which tells us that ever finite B-module is IB-adically complete. Part (3) follows in the same manner as part (4). □

0AL5 Remark 88.2.3 (Base change). Let $\varphi : A_1 \to A_2$ be a ring map and let $I_i \subset A_i$ be ideals such that $\varphi(I_1^c) \subset I_2$ for some $c \geq 1$. This induces ring maps $A_{1,cn} = A_1/I_1^{cn} \to A_2/I_2^n = A_{2,n}$ for all $n \geq 1$. Let $\mathcal{C}_i$ be the category (88.2.0.1) for (A_i, I_i). There is a base change functor

0AJZ (88.2.3.1)
$$\mathcal{C}_1 \longrightarrow \mathcal{C}_2, \quad (B_n) \longmapsto (B_{cn} \otimes_{A_{1,cn}} A_{2,n})$$

Let $\mathcal{C}_i'$ be the category (88.2.0.2) for (A_i, I_i). If I_2 is finitely generated, then there is a base change functor

0AK0 (88.2.3.2)
$$\mathcal{C}_1' \longrightarrow \mathcal{C}_2', \quad B \longmapsto (B \otimes_{A_1} A_2)^\wedge$$

because in this case the completion is complete (Algebra, Lemma 10.96.3). If both I_1 and I_2 are finitely generated, then the two base change functors agree via the functors (88.2.0.3) which are equivalences by Lemma 88.2.1.

0AL6 Remark 88.2.4 (Base change by closed immersion). Let A be a Noetherian ring and $I \subset A$ an ideal. Let $\mathfrak{a} \subset A$ be an ideal. Denote $\bar{A} = A/\mathfrak{a}$. Let $\bar{I} \subset \bar{A}$ be an ideal such that $I^c\bar{A} \subset \bar{I}$ and $\bar{I}^d \subset I\bar{A}$ for some $c, d \geq 1$. In this case the base change functor (88.2.3.2) for (A, I) to $(\bar{A}, \bar{I})$ is given by $B \mapsto \bar{B} = B/\mathfrak{a}B$. Namely, we have

0AK1 (88.2.4.1)
$$\bar{B} = (B \otimes_A \bar{A})^\wedge = (B/\mathfrak{a}B)^\wedge = B/\mathfrak{a}B$$

the last equality because any finite B-module is I-adically complete by Algebra, Lemma 10.97.1 and if annihilated by $\mathfrak{a}$ also $\bar{I}$-adically complete by Algebra, Lemma 10.96.9.

88.3. A naive cotangent complex

0AJL Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an A-algebra which is I-adically complete such that $A/I \to B/IB$ is of finite type, i.e., an object of (88.2.0.2). By Lemma 88.2.2 we can write

$$B = A[x_1, \ldots, x_r]^\wedge/J$$

for some finitely generated ideal J. For a choice of presentation as above we define the naive cotangent complex in this setting by the formula

0AJR (88.3.0.1)
$$NL_{B/A}^\wedge = (J/J^2 \longrightarrow \bigoplus B\mathrm{d}x_i)$$

with terms sitting in degrees -1 and 0 where the map sends the residue class of $g \in J$ to the differential $\mathrm{d}g = \sum(\partial g/\partial x_i)\mathrm{d}x_i$. Here the partial derivative is taken by thinking of g as a power series. The following lemma shows that $NL_{B/A}^\wedge$ is well defined up to homotopy.

0GAE Lemma 88.3.1. Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an object of (88.2.0.2). The naive cotangent complex $NL_{B/A}^\wedge$ is well defined in $K(B)$.

Proof. The lemma signifies that given a second presentation $B = A[y_1, \ldots, y_s]^\wedge/K$ the complexes of B-modules

$$(J/J^2 \to B\mathrm{d}x_i) \quad \text{and} \quad (K/K^2 \to \bigoplus B\mathrm{d}y_j)$$

are homotopy equivalent. To see this, we can argue exactly as in the proof of Algebra, Lemma 10.134.2.

Step 1. If we choose $g_i(y_1, \ldots, y_s) \in A[y_1, \ldots, y_s]^\wedge$ mapping to the image of x_i in B, then we obtain a (unique) continuous A-algebra homomorphism

$$A[x_1, \ldots, x_r]^\wedge \to A[y_1, \ldots, y_s]^\wedge, \quad x_i \mapsto g_i(y_1, \ldots, y_s)$$

compatible with the given surjections to B. Such a map is called a morphism of presentations. It induces a map from J into K and hence induces a B-module map $J/J^2 \to K/K^2$. Sending $\mathrm{d}x_i$ to $\sum(\partial g_i/\partial y_j)\mathrm{d}y_j$ we obtain a map of complexes

$$(J/J^2 \to \bigoplus B\mathrm{d}x_i) \longrightarrow (K/K^2 \to \bigoplus B\mathrm{d}y_j)$$

Of course we can do the same thing with the roles of the two presentations exchanged to get a map of complexes in the other direction.

Step 2. The construction above is compatible with compositions of morphsms of presentations. Hence to finish the proof it suffices to show: given $g_i(x_1, \ldots, x_r) \in A[x_1, \ldots, x_n]^\wedge$ mapping to the image of x_i in B, the induced map of complexes

$$(J/J^2 \to \bigoplus B\mathrm{d}x_i) \longrightarrow (J/J^2 \to \bigoplus B\mathrm{d}x_i)$$

is homotopic to the identity map. To see this consider the map $h : \bigoplus B\mathrm{d}x_i \to J/J^2$ given by the rule $\mathrm{d}x_i \mapsto g_i(x_1, \ldots, x_n) - x_i$ and compute. $\square$

0GAF **Lemma 88.3.2.** Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let $A \to B$ be a finite type ring map. Choose a presentation $\alpha : A[x_1, \ldots, x_n] \to B$. Then $NL^\wedge_{B^\wedge/A} = \lim NL(\alpha) \otimes_B B^\wedge$ as complexes and $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes^{\mathbf{L}}_B B^\wedge$ in $D(B^\wedge)$.

Proof. The statement makes sense as $B^\wedge$ is an object of (88.2.0.2) by Lemma 88.2.2. Let $J = \mathrm{Ker}(\alpha)$. The functor of taking I-adic completion is exact on finite modules over $A[x_1, \ldots, x_n]$ and agrees with the functor $M \mapsto M \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]^\wedge$, see Algebra, Lemmas 10.97.1 and 10.97.2. Moreover, the ring maps $A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]^\wedge$ and $B \to B^\wedge$ are flat. Hence $B^\wedge = A[x_1, \ldots, x_n]^\wedge/J^\wedge$ and

$$(J/J^2) \otimes_B B^\wedge = (J/J^2)^\wedge = J^\wedge/(J^\wedge)^2$$

Since $NL(\alpha) = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$, see Algebra, Section 10.134, we conclude the complex $NL^\wedge_{B^\wedge/A}$ is equal to $NL(\alpha) \otimes_B B^\wedge$. The final statement follows as $NL_{B/A}$ is homotopy equivalent to $NL(\alpha)$ and because the ring map $B \to B^\wedge$ is flat (so derived base change along $B \to B^\wedge$ is just base change). $\square$

0AJS **Lemma 88.3.3.** Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an object of (88.2.0.2). Then

1. the pro-objects $\{NL^\wedge_{B/A} \otimes_B B/I^nB\}$ and $\{NL_{B_n/A_n}\}$ of $D(B)$ are strictly isomorphic (see proof for elucidation),
2. $NL^\wedge_{B/A} = R\lim NL_{B_n/A_n}$ in $D(B)$.

Here B_n and A_n are as in Section 88.2.

Proof. The statement means the following: for every n we have a well defined complex NL_{B_n/A_n} of B_n-modules and we have transition maps $NL_{B_{n+1}/A_{n+1}} \to NL_{B_n/A_n}$. See Algebra, Section 10.134. Thus we can consider

$$\ldots \to NL_{B_3/A_3} \to NL_{B_2/A_2} \to NL_{B_1/A_1}$$

as an inverse system of complexes of B-modules and a fortiori as an inverse system in $D(B)$. Furthermore $R\lim NL_{B_n/A_n}$ is a homotopy limit of this inverse system, see Derived Categories, Section 13.34.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. This defines presentations

$$B_n = B/I^nB = A_n[x_1, \ldots, x_r]/J_n$$

where

$$J_n = JA_n[x_1, \ldots, x_r] = J/(J \cap I^nA[x_1, \ldots, x_r]^\wedge)$$

The two term complex $J_n/J_n^2 \longrightarrow \bigoplus B_n\mathrm{d}x_i$ represents NL_{B_n/A_n}, see Algebra, Section 10.134. By Artin-Rees (Algebra, Lemma 10.51.2) in the Noetherian ring $A[x_1, \ldots, x_r]^\wedge$ (Lemma 88.2.2) we find a $c \geq 0$ such that we have canonical surjections

$$J/I^nJ \to J_n \to J/I^{n-c}J \to J_{n-c}, \quad n \geq c$$

for all $n \geq c$. A moment's thought shows that these maps are compatible with differentials and we obtain maps of complexes

$$NL^\wedge_{B/A} \otimes_B B/I^nB \to NL_{B_n/A_n} \to NL^\wedge_{B/A} \otimes_B B/I^{n-c}B \to NL_{B_{n-c}/A_{n-c}}$$

compatible with the transition maps of the inverse systems $\{NL^\wedge_{B/A} \otimes_B B/I^nB\}$ and $\{NL_{B_n/A_n}\}$. This proves part (1) of the lemma.

By part (1) and since pro-isomorphic systems have the same $R\lim$ in order to prove (2) it suffices to show that $NL^\wedge_{B/A}$ is equal to $R\lim NL^\wedge_{B/A} \otimes_B B/I^nB$. However, $NL^\wedge_{B/A}$ is a two term complex $M^\bullet$ of finite B-modules which are I-adically complete for example by Algebra, Lemma 10.97.1. Hence $M^\bullet = \lim M^\bullet/I^nM^\bullet = R\lim M^\bullet/I^nM^\bullet$, see More on Algebra, Lemma 15.87.1 and Remark 15.87.6. $\square$

0GAG Lemma 88.3.4. Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 88.2.3 with A_1 and A_2 Noetherian. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. Then there is a canonical map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}.

Proof. Choose a presentation $B_1 = A_1[x_1, \ldots, x_r]^\wedge/J_1$. Since $A_2/I_2^n[x_1, \ldots, x_r] = A_1/I_1^{cn}[x_1, \ldots, x_r] \otimes_{A_1/I_1^{cn}} A_2/I_2^n$ we have

$$A_2[x_1, \ldots, x_r]^\wedge = (A_1[x_1, \ldots, x_r]^\wedge \otimes_{A_1} A_2)^\wedge$$

where we use I_2-adic completion on both sides (but of course I_1-adic completion for $A_1[x_1, \ldots, x_r]^\wedge$). Set $J_2 = J_1A_2[x_1, \ldots, x_r]^\wedge$. Arguing similarly we get the presentation

$$\begin{aligned} B_2 &= (B_1 \otimes_{A_1} A_2)^\wedge \\ &= \lim \frac{A_1/I_1^{cn}[x_1, \ldots, x_r]}{J_1(A_1/I_1^{cn}[x_1, \ldots, x_r])} \otimes_{A_1/I_1^{cn}} A_2/I_2^n \\ &= \lim \frac{A_2/I_2^n[x_1, \ldots, x_r]}{J_2(A_2/I_2^n[x_1, \ldots, x_r])} \\ &= A_2[x_1, \ldots, x_r]^\wedge/J_2 \end{aligned}$$

for B_2 over A_2. As a consequence obtain a commutative diagram

$$\begin{tikzcd} NL^\wedge_{B_1/A_1}: \arrow[d] & J_1/J_1^2 \arrow[r, "\mathrm{d}"'] \arrow[d] & \bigoplus B_1 \mathrm{d}x_i \arrow[d] \\ NL^\wedge_{B_2/A_2}: & J_2/J_2^2 \arrow[r, "\mathrm{d}"] & \bigoplus B_2 \mathrm{d}x_i \end{tikzcd}$$

The induced arrow $J_1/J_1^2 \otimes_{B_1} B_2 \to J_2/J_2^2$ is surjective because J_2 is generated by the image of J_1. This determines the arrow displayed in the lemma. We omit the proof that this arrow is well defined up to homotopy (i.e., indepedent of the choice of the presentations up to homotopy). The statement about the induced map on cohomology modules follows easily from the discussion (details omitted). $\square$

0ALM Lemma 88.3.5. Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let $B \to C$ be morphism of (88.2.0.2). Then there is an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] & \end{tikzcd}$$

See proof for elucidation.

Proof. Observe that taking the tensor product $NL^\wedge_{B/A} \otimes_B C$ makes sense as $NL^\wedge_{B/A}$ is well defined up to homotopy by Lemma 88.3.1. Also, (B, IB) is pair where B is a Noetherian ring (Lemma 88.2.2) and C is in the corresponding category (88.2.0.2). Thus all the terms in the 6-term sequence are (well) defined.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a presentation $C = B[y_1, \ldots, y_s]^\wedge/J'$. Combinging these presentations gives a presentation

$$C = A[x_1, \ldots, x_r, y_1, \ldots, y_s]^\wedge/K$$

Then the reader verifies that we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus C\mathrm{d}x_i \arrow[r] & \bigoplus C\mathrm{d}x_i \oplus \bigoplus C\mathrm{d}y_j \arrow[r] & \bigoplus C\mathrm{d}y_j \arrow[r] & 0 \\ & J/J^2 \otimes_B C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J'/(J')^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Note that the vertical arrow on the left hand side is the tensor product of the arrow defining $NL^\wedge_{B/A}$ with id_C. The lemma follows by applying the snake lemma (Algebra, Lemma 10.4.1). $\square$

0AQJ Lemma 88.3.6. With assumptions as in Lemma 88.3.5 assume that $B/I^nB \to C/I^nC$ is a local complete intersection homomorphism for all n. Then $H^{-1}(NL^\wedge_{B/A} \otimes_B C) \to H^{-1}(NL^\wedge_{C/A})$ is injective.

Proof. For each $n \geq 1$ we set $A_n = A/I^n$, $B_n = B/I^nB$, and $C_n = C/I^nC$. We have

$$\begin{aligned} H^{-1}(NL^\wedge_{B/A} \otimes_B C) &= \lim H^{-1}(NL^\wedge_{B/A} \otimes_B C_n) \\ &= \lim H^{-1}(NL^\wedge_{B/A} \otimes_B B_n \otimes_{B_n} C_n) \\ &= \lim H^{-1}(NL_{B_n/A_n} \otimes_{B_n} C_n) \end{aligned}$$

The first equality follows from More on Algebra, Lemma 15.100.1 and the fact that $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$ is a finite C-module and hence I-adically complete for example by Algebra, Lemma 10.97.1. The second equality is trivial. The third holds by Lemma 88.3.3. The maps $H^{-1}(NL_{B_n/A_n} \otimes_{B_n} C_n) \to H^{-1}(NL_{C_n/A_n})$ are injective by More on Algebra, Lemma 15.33.6. The proof is finished because we also have $H^{-1}(NL^\wedge_{C/A}) = \lim H^{-1}(NL_{C_n/A_n})$ similarly to the above. □

88.4. Rig-smooth algebras

0GAH As motivation for the following definition, please take a look at More on Algebra, Remark 15.84.2.

0GAI Definition 88.4.1. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). We say B is rig-smooth over (A, I) if there exists an integer $c \geq 0$ such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for every B-module N.

Let us work out what this means.

0GAJ Lemma 88.4.2. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Write $B = A[x_1, \ldots, x_r]^\wedge/J$ (Lemma 88.2.2) and let $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$ be its naive cotangent complex (88.3.0.1). The following are equivalent

(1) B is rig-smooth over (A, I),
(2) the object $NL^\wedge_{B/A}$ of $D(B)$ satisfies the equivalent conditions (1) – (4) of More on Algebra, Lemma 15.84.10 with respect to the ideal IB,
(3) there exists a $c \geq 0$ such that for all $a \in I^c$ there is a map $h : \bigoplus B\mathrm{d}x_i \to J/J^2$ such that $a : J/J^2 \to J/J^2$ is equal to $h \circ \mathrm{d}$,
(4) there exist $b_1, \ldots, b_s \in B$ such that $V(b_1, \ldots, b_s) \subset V(IB)$ and such that for every $l = 1, \ldots, s$ there exist $m \geq 0$, $f_1, \ldots, f_m \in J$, and subset $T \subset \{1, \ldots, n\}$ with $|T| = m$ such that
  (a) $\det_{i \in T, j \leq m}(\partial f_j/\partial x_i)$ divides b_l in B, and
  (b) $b_l J \subset (f_1, \ldots, f_m) + J^2$.

Proof. The equivalence of (1), (2), and (3) is immediate from More on Algebra, Lemma 15.84.10.

Assume $b_1, \ldots, b_s$ are as in (4). Since B is Noetherian the inclusion $V(b_1, \ldots, b_s) \subset V(IB)$ implies $I^cB \subset (b_1, \ldots, b_s)$ for some $c \geq 0$ (for example by Algebra, Lemma 10.62.4). Pick $1 \leq l \leq s$ and $m \geq 0$ and $f_1, \ldots, f_m \in J$ and $T \subset \{1, \ldots, n\}$ with $|T| = m$ satisfying (4)(a) and (b). Then if we invert b_l we see that

$$NL^\wedge_{B/A} \otimes_B B_{b_l} = \left(\bigoplus\nolimits_{j \leq m} B_{b_l} f_j \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B_{b_l}\mathrm{d}x_i\right)$$

and moreover the arrow is isomorphic to the inclusion of the direct summand $\bigoplus_{i\in T} B_{b_l}\mathrm{d}x_i$. We conclude that $H^{-1}(NL^\wedge_{B/A})$ is b_l-power torsion and that $H^0(NL^\wedge_{B/A})$ becomes finite free after inverting b_l. Combined with the inclusion $I^cB \subset (b_1, \ldots, b_s)$ we see that $H^{-1}(NL^\wedge_{B/A})$ is IB-power torsion. Hence we see that condition (4) of More on Algebra, Lemma 15.84.10 holds. In this way we see that (4) implies (2).

Assume the equivalent conditions (1), (2), and (3) hold. We will prove that (4) holds, but we strongly urge the reader to convince themselves of this. The complex $NL^\wedge_{B/A}$ determines an object of $D^b_{\mathrm{Coh}}(\mathrm{Spec}(B))$ whose restriction to the Zariski open $U = \mathrm{Spec}(B) \setminus V(IB)$ is a finite locally free module $\mathcal{E}$ placed in degree 0 (this follows for example from the the fourth equivalent condition in More on Algebra, Lemma 15.84.10). Choose generators $f_1, \ldots, f_M$ for J. This determines an exact sequence

$$\bigoplus\nolimits_{j=1,\ldots,M} \mathcal{O}_U \cdot f_j \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \cdot \mathrm{d}x_i \to \mathcal{E} \to 0$$

Let $U = \bigcup_{l=1,\ldots,s} U_l$ be a finite affine open covering such that $\mathcal{E}|_{U_l}$ is free of rank $r_l = n - m_l$ for some integer $n \geq m_l \geq 0$. After replacing each U_l by an affine open covering we may assume there exists a subset $T_l \subset \{1, \ldots, n\}$ such that the elements $\mathrm{d}x_i$, $i \in \{1, \ldots, n\} \setminus T_l$ map to a basis for $\mathcal{E}|_{U_l}$. Repeating the argument, we may assume there exists a subset $T'_l \subset \{1, \ldots, M\}$ of cardinality m_l such that f_j, $j \in T'_l$ map to a basis of the kernel of $\mathcal{O}_{U_l} \cdot \mathrm{d}x_i \to \mathcal{E}|_{U_l}$. Finally, since the open covering $U = \bigcup U_l$ may be refined by a open covering by standard opens (Algebra, Lemma 10.17.2) we may assume $U_l = D(g_l)$ for some $g_l \in B$. In particular we have $V(g_1, \ldots, g_s) = V(IB)$. A linear algebra argument using our choices above shows that $\det_{i\in T_l, j\in T'_l}(\partial f_j/\partial x_i)$ maps to an invertible element of B_{b_l}. Similarly, the vanishing of cohomology of $NL^\wedge_{B/A}$ in degree -1 over U_l shows that $J/J^2 + (f_j; j \in T')$ is annihilated by a power of b_l. After replacing each g_l by a suitable power we obtain conditions (4)(a) and (4)(b) of the lemma. Some details omitted. □

0GAK **Lemma 88.4.3.** Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.

1. If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus V(I)$, then $B^\wedge$ is rig-smooth over (A, I).
2. If $B^\wedge$ is rig-smooth over (A, I), then there exists $g \in 1 + IB$ such that $\mathrm{Spec}(B_g)$ is smooth over $\mathrm{Spec}(A) \setminus V(I)$.

Proof. We will use Lemma 88.4.2 without further mention.

Assume (1). Recall that formation of $NL_{B/A}$ commutes with localization, see Algebra, Lemma 10.134.13. Hence by the very definition of smooth ring maps (in terms of the naive cotangent complex being quasi-isomorphic to a finite projective module placed in degree 0), we see that $NL_{B/A}$ satisfies the fourth equivalent condition of More on Algebra, Lemma 15.84.10 with respect to the ideal IB (small detail omitted). Since $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B B^\wedge$ by Lemma 88.3.2 we conclude (2) holds by More on Algebra, Lemma 15.84.7.

Assume (2). Choose a presentation $B = A[x_1, \ldots, x_n]/J$, set $N = J/J^2$, and consider the element $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, J/J^2)$ determined by the identity map on J/J^2. Using again that $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B B^\wedge$ we find that our assumption implies the image

$$\xi \otimes 1 \in \mathrm{Ext}^1_{B^\wedge}(NL_{B/A} \otimes_B B^\wedge, N \otimes_B B^\wedge) = \mathrm{Ext}^1_{B^\wedge}(NL_{B/A}, N) \otimes_B B^\wedge$$

is annihilated by I^c for some integer $c \geq 0$. The equality holds for example by More on Algebra, Lemma 15.99.2 (but can also easily be deduced from the much simpler More on Algebra, Lemma 15.65.4). Thus $M = I^c B\xi \subset \text{Ext}^1_B(NL_{B/A}, N)$ is a finite submodule which maps to zero in $\text{Ext}^1_B(NL_{B/A}, N) \otimes_B B^\wedge$. Since $B \to B^\wedge$ is flat this means that $M \otimes_B B^\wedge$ is zero. By Nakayama's lemma (Algebra, Lemma 10.20.1) this means that $M = I^c B\xi$ is annihilated by an element of the form $g = 1 + x$ with $x \in IB$. This implies that for every $b \in I^c B$ there is a B-linear dotted arrow making the diagram commute

$$\begin{tikzcd} J/J^2 \arrow[r] \arrow[d, "b"] & \bigoplus B\mathrm{d}x_i \arrow[d, dotted, "h"] \\ J/J^2 \arrow[r] & (J/J^2)_g \end{tikzcd}$$

Thus $(NL_{B/A})_{gb}$ is quasi-isomorphic to a finite projective module; small detail omitted. Since $(NL_{B/A})_{gb} = NL_{B_{gb}/A}$ in $D(B_{gb})$ this shows that B_{gb} is smooth over $\text{Spec}(A)$. As this holds for all $b \in I^c B$ we conclude that $\text{Spec}(B_g) \to \text{Spec}(A)$ is smooth over $\text{Spec}(A) \setminus V(I)$ as desired. □

0GAL Lemma 88.4.4. Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 88.2.3 with A_1 and A_2 Noetherian. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. Let $f_1 \in B_1$ with image $f_2 \in B_2$. If $\text{Ext}^1_{B_1}(NL^\wedge_{B_1/A_1}, N_1)$ is annihilated by f_1 for every B_1-module N_1, then $\text{Ext}^1_{B_2}(NL^\wedge_{B_2/A_2}, N_2)$ is annihilated by f_2 for every B_2-module N_2.

Proof. By Lemma 88.3.4 there is a map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}. Thus the result by More on Algebra, Lemmas 15.84.6, 15.84.7, and 15.84.9 the last two applied with the principal ideals $(f_1) \subset B_1$ and $(f_2) \subset B_2$. □

0GAM Lemma 88.4.5. Let $A_1 \to A_2$ be a map of Noetherian rings. Let $I_i \subset A_i$ be an ideal such that $V(I_1 A_2) = V(I_2)$. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1 as in Remark 88.2.3. If B_1 is rig-smooth over (A_1, I_1), then B_2 is rig-smooth over (A_2, I_2).

Proof. Follows from Lemma 88.4.4 and Definition 88.4.1 and the fact that I_2^c is contained in $I_1 A_2$ for some $c \geq 0$ as A_2 is Noetherian. □

88.5. Deformations of ring homomorphisms

0GAN Some work on lifting ring homomorphisms from rig-smooth algebras.

0AK3 Remark 88.5.1 (Linear approximation). Let A be a ring and $I \subset A$ be a finitely generated ideal. Let C be an I-adically complete A-algebra. Let $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ be a continuous A-algebra map. Suppose given $\delta_i \in C$, $i = 1, \ldots, r$. Then we can consider

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) + \delta_i$$

see Formal Spaces, Remark 87.28.1. Then we have

$$\psi'(g) = \psi(g) + \sum \psi(\partial g/\partial x_i)\delta_i + \xi$$

with error term $\xi \in (\delta_i\delta_j)$. This follows by writing g as a power series and working term by term. Convergence is automatic as the coefficients of g tend to zero. Details omitted.

0GAP Remark 88.5.2 (Lifting maps). Let A be a Noetherian ring and $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Let C be an I-adically complete A-algebra. Let $\psi_n : B \to C/I^nC$ be an A-algebra homomorphism. The obstruction to lifting ψ_n to an A-algebra homomorphism into $C/I^{2n}C$ is an element

$$o(\psi_n) \in \mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)$$

as we will explain. Namely, choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a lift $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ of ψ_n. Since $\psi(J) \subset I^nC$ we get $\psi(J^2) \subset I^{2n}C$ and hence we get a B-linear homomorphism

$$o(\psi) : J/J^2 \longrightarrow I^nC/I^{2n}C, \quad g \longmapsto \psi(g)$$

which of course extends to a C-linear map $J/J^2 \otimes_B C \to I^nC/I^{2n}C$. Since $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$ we get $o(\psi_n)$ as the image of $o(\psi)$ by the identification

$$\begin{aligned} &\mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C) \\ &= \mathrm{Coker}\left(\mathrm{Hom}_B(\bigoplus B\mathrm{d}x_i, I^nC/I^{2n}C) \to \mathrm{Hom}_B(J/J^2, I^nC/I^{2n}C)\right) \end{aligned}$$

See More on Algebra, Lemma 15.84.4 part (1) for the equality.

Suppose that $o(\psi_n)$ maps to zero in $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^{n'}C/I^{2n'}C)$ for some integer n' with $n > n' > n/2$. We claim that this means we can find an A-algebra homomorphism $\psi'_{2n'} : B \to C/I^{2n'}C$ which agrees with ψ_n as maps into $C/I^{n'}C$. The extreme case $n' = n$ explains why we previously said $o(\psi_n)$ is the obstruction to lifting ψ_n to $C/I^{2n}C$. Proof of the claim: the hypothesis that $o(\psi_n)$ maps to zero tells us we can find a B-module map

$$h : \bigoplus B\mathrm{d}x_i \longrightarrow I^{n'}C/I^{2n'}C$$

such that $o(\psi)$ and $h \circ \mathrm{d}$ agree as maps into $I^{n'}C/I^{2n'}C$. Say $h(\mathrm{d}x_i) = \delta_i \bmod I^{2n'}C$ for some $\delta_i \in I^{n'}C$. Then we look at the map

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) - \delta_i$$

A computation with power series shows that $\psi'(J) \subset I^{2n'}C$. Namely, for $g \in J$ we get

$$\psi'(g) \equiv \psi(g) - \sum \psi(\partial g/\partial x_i)\delta_i \equiv o(\psi)(g) - (h \circ \mathrm{d})(g) \equiv 0 \bmod I^{2n'}C$$

See Remark 88.5.1 for the first equality. Hence ψ' induces an A-algebra homomorphism $\psi'_{2n'} : B \to C/I^{2n'}C$ as desired.

0GAQ Lemma 88.5.3. Assume given the following data

(1) an integer $c \geq 0$,
(2) an ideal I of a Noetherian ring A,
(3) B in (88.2.0.2) for (A, I) such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for any B-module N,
(4) a Noetherian I-adically complete A-algebra C; denote $d = d(\mathrm{Gr}_I(C))$ and $q_0 = q(\mathrm{Gr}_I(C))$ the integers found in Local Cohomology, Section 51.22,
(5) an integer $n \geq \max(q_0 + (d+1)c, 2(d+1)c + 1)$, and

(6) an A-algebra homomorphism $\psi_n : B \to C/I^nC$.

Then there exists a map $\varphi : B \to C$ of A-algebras such that $\psi_n \bmod I^{n-(d+1)c} = \varphi \bmod I^{n-(d+1)c}$.

Proof. Consider the obstruction class

$$o(\psi_n) \in \text{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)$$

of Remark 88.5.2. For any C/I^nC-module N we have

$$\begin{aligned}\text{Ext}^1_B(NL^\wedge_{B/A}, N) &= \text{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes^{\mathbf{L}}_B C/I^nC, N) \\ &= \text{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, N)\end{aligned}$$

The first equality by More on Algebra, Lemma 15.99.1 and the second one by More on Algebra, Lemma 15.84.6. In particular, we see that $\text{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, N)$ is annihilated by I^cC for all C/I^nC-modules N. It follows that we may apply Local Cohomology, Lemma 51.22.7 to see that $o(\psi_n)$ maps to zero in

$$\text{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, I^{n'}C/I^{2n'}C) = \text{Ext}^1_B(NL^\wedge_{B/A}, I^{n'}C/I^{2n'}C) =$$

where $n' = n - (d+1)c$. By the discussion in Remark 88.5.2 we obtain a map

$$\psi'_{2n'} : B \to C/I^{2n'}C$$

which agrees with ψ_n modulo $I^{n'}$. Observe that $2n' > n$ because $n \geq 2(d+1)c+1$.

We may repeat this procedure. Starting with $n_0 = n$ and $\psi^0 = \psi_n$ we end up getting a strictly increasing sequence of integers

$$n_0 < n_1 < n_2 < \ldots$$

and A-algebra homorphisms $\psi^i : B \to C/I^{n_i}C$ such that ψ^{i+1} and ψ^i agree modulo $I^{n_i - tc}$. Since C is I-adically complete we can take φ to be the limit of the maps $\psi^i \bmod I^{n_i-(d+1)c} : B \to C/I^{n_i-(d+1)c}C$ and the lemma follows. $\square$

We suggest the reader skip ahead to the next section. Namely, the following two lemmas are consequences of the result above if the algebra C in them is assumed Noetherian.

0AK6 Lemma 88.5.4. Let $I = (a)$ be a principal ideal of a Noetherian ring A. Let B be an object of (88.2.0.2). Assume given an integer $c \geq 0$ such that $\text{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by a^c for all B-modules N. Let C be an I-adically complete A-algebra such that a is a nonzerodivisor on C. Let $n > 2c$. For any A-algebra map $\psi_n : B \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c}C = \varphi \bmod a^{n-c}C$.

Proof. Consider the obstruction class

$$o(\psi_n) \in \text{Ext}^1_B(NL^\wedge_{B/A}, a^nC/a^{2n}C)$$

of Remark 88.5.2. Since a is a nonzerodivisor on C the map $a^c : a^nC/a^{2n}C \to a^nC/a^{2n}C$ is isomorphic to the map $a^nC/a^{2n}C \to a^{n-c}C/a^{2n-c}C$ in the category of C-modules. Hence by our assumption on $NL^\wedge_{B/A}$ we conclude that the class $o(\psi_n)$ maps to zero in

$$\text{Ext}^1_B(NL^\wedge_{B/A}, a^{n-c}C/a^{2n-c}C)$$

and a fortiori in

$$\text{Ext}^1_B(NL^\wedge_{B/A}, a^{n-c}C/a^{2n-2c}C)$$

By the discussion in Remark 88.5.2 we obtain a map

$$\psi_{2n-2c} : B \to C/a^{2n-2c}C$$

which agrees with ψ_n modulo $a^{n-c}C$. Observe that $2n - 2c > n$ because $n > 2c$.

We may repeat this procedure. Starting with $n_0 = n$ and $\psi^0 = \psi_n$ we end up getting a strictly increasing sequence of integers

$$n_0 < n_1 < n_2 < \ldots$$

and A-algebra homorphisms $\psi^i : B \to C/a^{n_i}C$ such that ψ^{i+1} and ψ^i agree modulo $a^{n_i-c}C$. Since C is I-adically complete we can take φ to be the limit of the maps $\psi^i \bmod a^{n_i-c}C : B \to C/a^{n_i-c}C$ and the lemma follows. □

0AK7 Lemma 88.5.5. Let $I = (a)$ be a principal ideal of a Noetherian ring A. Let B be an object of (88.2.0.2). Assume given an integer $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by a^c for all B-modules N. Let C be an I-adically complete A-algebra. Assume given an integer $d \geq 0$ such that $C[a^\infty] \cap a^dC = 0$. Let $n > \max(2c, c+d)$. For any A-algebra map $\psi_n : B \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.

If C is Noetherian we have $C[a^\infty] = C[a^e]$ for some $e \geq 0$. By Artin-Rees (Algebra, Lemma 10.51.2) there exists an integer f such that $a^nC \cap C[a^\infty] \subset a^{n-f}C[a^\infty]$ for all $n \geq f$. Then $d = e + f$ is an integer as in the lemma. This argument works in particular if C is an object of (88.2.0.2) by Lemma 88.2.2.

Proof. Let $C \to C'$ be the quotient of C by $C[a^\infty]$. The A-algebra C' is I-adically complete by Algebra, Lemma 10.96.10 and the fact that $\bigcap(C[a^\infty] + a^nC) = C[a^\infty]$ because for $n \geq d$ the sum $C[a^\infty] + a^nC$ is direct. For $m \geq d$ the diagram

$$\begin{tikzcd} 0 \arrow[r] & C[a^\infty] \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C[a^\infty] \arrow[r] & C/a^mC \arrow[r] & C'/a^mC' \arrow[r] & 0 \end{tikzcd}$$

has exact rows. Thus C is the fibre product of C' and C/a^mC over C'/a^mC' for all $m \geq d$. By Lemma 88.5.4 we can choose a homomorphism $\varphi' : B \to C'$ such that φ' and ψ_n agree as maps into $C'/a^{n-c}C'$. We obtain a homomorphism $(\varphi', \psi_n \bmod a^{n-c}C) : B \to C' \times_{C'/a^{n-c}C'} C/a^{n-c}C$. Since $n - c \geq d$ this is the same thing as a homomorphism $\varphi : B \to C$. This finishes the proof. □

88.6. Algebraization of rig-smooth algebras over G-rings

0ALU If the base ring A is a Noetherian G-ring, then we can prove [Elk73, III Theorem 7] for arbitrary rig-smooth algebras with respect to any ideal $I \subset A$ (not necessarily principal).

0GAR Lemma 88.6.1. Let I be an ideal of a Noetherian ring A. Let $r \geq 0$ and write $P = A[x_1, \ldots, x_r]$ the I-adic completion. Consider a resolution

$$P^{\oplus t} \xrightarrow{K} P^{\oplus m} \xrightarrow{g_1, \ldots, g_m} P \to B \to 0$$

of a quotient of P. Assume B is rig-smooth over (A, I). Then there exists an integer n such that for any complex

$$P^{\oplus t} \xrightarrow{K'} P^{\oplus m} \xrightarrow{g'_1, \ldots, g'_m} P$$

with $g_i - g'_i \in I^n P$ and $K - K' \in I^n \text{Mat}(m \times t, P)$ there exists an isomorphism $B \to B'$ of A-algebras where $B' = P/(g'_1, \ldots, g'_m)$.

Proof. (A) By Definition 88.4.1 we can choose a $c \geq 0$ such that I^c annihilates $\text{Ext}^1_B(NL^\wedge_{B/A}, N)$ for all B-modules N.

(B) By More on Algebra, Lemmas 15.4.1 and 15.4.2 there exists a constant $c_1 = c(g_1, \ldots, g_m, K)$ such that for $n \geq c_1 + 1$ the complex

$$P^{\oplus t} \xrightarrow{K'} P^{\oplus m} \xrightarrow{g'_1, \ldots, g'_m} P \to B' \to 0$$

is exact and $\text{Gr}_I(B) \cong \text{Gr}_I(B')$.

(C) Let $d_0 = d(\text{Gr}_I(B))$ and $q_0 = q(\text{Gr}_I(B))$ be the integers found in Local Cohomology, Section 51.22.

We claim that $n = \max(c_1 + 1, q_0 + (d_0 + 1)c, 2(d_0 + 1)c + 1)$ works where c is as in (A), c_1 is as in (B), and q_0, d_0 are as in (C).

Let $g'_1, \ldots, g'_m$ and K' be as in the lemma. Since $g_i = g'_i \in I^n P$ we obtain a canonical A-algebra homomorphism

$$\psi_n : B \longrightarrow B'/I^n B'$$

which induces an isomorphism $B/I^n B \to B'/I^n B'$. Since $\text{Gr}_I(B) \cong \text{Gr}_I(B')$ we have $d_0 = d(\text{Gr}_I(B'))$ and $q_0 = q(\text{Gr}_I(B'))$ and since $n \geq \max(q_0 + (1 + d_0)c, 2(d_0 + 1)c + 1)$ we may apply Lemma 88.5.3 to find an A-algebra homomorphism

$$\varphi : B \longrightarrow B'$$

such that $\varphi \bmod I^{n-(d_0+1)c} B' = \psi_n \bmod I^{n-(d_0+1)c} B'$. Since $n - (d_0 + 1)c > 0$ we see that φ is an A-algebra homomorphism which modulo I induces the isomorphism $B/IB \to B'/IB'$ we found above. The rest of the proof shows that these facts force φ to be an isomorphism; we suggest the reader find their own proof of this.

Namely, it follows that φ is surjective for example by applying Algebra, Lemma 10.96.1 part (1) using the fact that B and B' are complete. Thus φ induces a surjection $\text{Gr}_I(B) \to \text{Gr}_I(B')$ which has to be an isomorphism because the source and target are isomorphic Noetherian rings, see Algebra, Lemma 10.31.10 (of course you can show φ induces the isomorphism we found above but that would need a tiny argument). Thus φ induces injective maps $I^e B/I^{e+1} B \to I^e B'/I^{e+1} B'$ for all $e \geq 0$. This implies φ is injective since for any $b \in B$ there exists an $e \geq 0$ such that $b \in I^e B$, $b \notin I^{e+1} B$ by Krull's intersection theorem (Algebra, Lemma 10.51.4). This finishes the proof. □

0GAS Lemma 88.6.2. Let I be an ideal of a Noetherian ring A. Let C^h be the henselization of a finite type A-algebra C with respect to the ideal IC. Let $J \subset C^h$ be an ideal. Then there exists a finite type A-algebra B such that $B^\wedge \cong (C^h/J)^\wedge$.

Proof. By More on Algebra, Lemma 15.12.4 the ring C^h is Noetherian. Say $J = (g_1, \ldots, g_m)$. The ring C^h is a filtered colimit of étale C algebras C' such that $C/IC \to C'/IC'$ is an isomorphism (see proof of More on Algebra, Lemma 15.12.1).

Pick an C' such that $g_1, \ldots, g_m$ are the images of $g'_1, \ldots, g'_m \in C'$. Setting $B = C'/(g'_1, \ldots, g'_m)$ we get a finite type A-algebra. Of course (C, IC) and $C', IC')$ have the same henselizations and the same completions. It follows easily from this that $B^\wedge = (C^h/J)^\wedge$. □

0GAT Proposition 88.6.3. Let I be an ideal of a Noetherian G-ring A. Let B be an object of (88.2.0.2). If B is rig-smooth over (A, I), then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$ of A-algebras.

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Write $P = A[x_1, \ldots, x_r]^\wedge$. Choose generators $g_1, \ldots, g_m \in J$. Choose generators $k_1, \ldots, k_t$ of the module of relations between $g_1, \ldots, g_m$, i.e., such that

$$P^{\oplus t} \xrightarrow{k_1, \ldots, k_t} P^{\oplus m} \xrightarrow{g_1, \ldots, g_m} P \to B \to 0$$

is a resolution. Write $k_i = (k_{i1}, \ldots, k_{im})$ so that we have

0AKB (88.6.3.1) $$\sum\nolimits_j k_{ij} g_j = 0$$

for $i = 1, \ldots, t$. Denote $K = (k_{ij})$ the $m \times t$-matrix with entries k_{ij}.

Let $A[x_1, \ldots, x_r]^h$ be the henselization of the pair $(A[x_1, \ldots, x_r], IA[x_1, \ldots, x_r])$, see More on Algebra, Lemma 15.12.1. We may and do think of $A[x_1, \ldots, x_r]^h$ as a subring of $P = A[x_1, \ldots, x_r]^\wedge$, see More on Algebra, Lemma 15.12.4. Since A is a Noetherian G-ring, so is $A[x_1, \ldots, x_r]$, see More on Algebra, Proposition 15.50.10. Hence we have approximation for the map $A[x_1, \ldots, x_r]^h \to A[x_1, \ldots, x_r]^\wedge = P$ with respect to the ideal generated by I, see Smoothing Ring Maps, Lemma 16.14.1. Choose a large enough integer n as in Lemma 88.6.1. By the approximation property we may choose $g'_1, \ldots, g'_m \in A[x_1, \ldots, x_r]^h$ and a matrix $K' = (k'_{ij}) \in \text{Mat}(m \times t, A[x_1, \ldots, x_r]^h)$ such that $\sum_j k'_{ij} g'_j = 0$ in $A[x_1, \ldots, x_r]^h$ and such that $g_i - g'_i \in I^n P$ and $K - K' \in I^n \text{Mat}(m \times t, P)$. By our choice of n we conclude that there is an isomorphism

$$B \to P/(g'_1, \ldots, g'_m) = \left(A[x_1, \ldots, x_r]^h/(g'_1, \ldots, g'_m)\right)^\wedge$$

This finishes the proof by Lemma 88.6.2. □

The following lemma isn't true in general if A is not a G-ring but just Noetherian. Namely, if $(A, \mathfrak m)$ is local and $I = \mathfrak m$, then the lemma is equivalent to Artin approximation for A^h (as in Smoothing Ring Maps, Theorem 16.13.1) which does not hold for every Noetherian local ring.

0AK4 Lemma 88.6.4. Let A be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let B, C be finite type A-algebras. For any A-algebra map $\varphi : B^\wedge \to C^\wedge$ of I-adic completions and any $N \geq 1$ there exist

(1) an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$,
(2) an A-algebra map $\varphi : B \to C'$

such that φ and ψ agree modulo I^N into $C^\wedge = (C')^\wedge$.

Proof. The statement of the lemma makes sense as $C \to C'$ is flat (Algebra, Lemma 10.143.3) hence induces an isomorphism $C/I^nC \to C'/I^nC'$ for all n (More on Algebra, Lemma 15.89.2) and hence an isomorphism on completions. Let C^h be the henselization of the pair (C, IC), see More on Algebra, Lemma 15.12.1. Then

C^h is the filtered colimit of the algebras C' and the maps $C \to C' \to C^h$ induce isomorphism on completions (More on Algebra, Lemma 15.12.4). Thus it suffices to prove there exists an A-algebra map $B \to C^h$ which is congruent to ψ modulo I^N. Write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The ring map ψ corresponds to elements $\hat{c}_1, \ldots, \hat{c}_n \in C^\wedge$ with $f_j(\hat{c}_1, \ldots, \hat{c}_n) = 0$ for $j = 1, \ldots, m$. Namely, as A is a Noetherian G-ring, so is C, see More on Algebra, Proposition 15.50.10. Thus Smoothing Ring Maps, Lemma 16.14.1 applies to give elements $c_1, \ldots, c_n \in C^h$ such that $f_j(c_1, \ldots, c_n) = 0$ for $j = 1, \ldots, m$ and such that $\hat{c}_i - c_i \in I^N C^h$. This determines the map $B \to C^h$ as desired. □

88.7. Algebraization of rig-smooth algebras

0GAU It turns out that if the rig-smooth algebra has a specific presentation, then it is straightforward to algebraize it. Please also see Remark 88.7.3 for a discussion.

0GAV Lemma 88.7.1. Let A be a ring. Let $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ and set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Assume $m \leq n$ and set $g = \det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$. Then

(1) g annihilates $\text{Ext}^1_B(NL_{B/A}, N)$ for every B-module N,
(2) if $n = m$, then multiplication by g on $NL_{B/A}$ is 0 in $D(B)$.

Proof. Let T be the $m \times m$ matrix with entries $\partial f_j/\partial x_i$ for $1 \leq i, j \leq n$. Let $K \in D(B)$ be represented by the complex $T : B^{\oplus m} \to B^{\oplus m}$ with terms sitting in degrees -1 and 0. By More on Algebra, Lemmas 15.84.12 we have $g : K \to K$ is zero in $D(B)$. Set $J = (f_1, \ldots, f_m)$. Recall that $NL_{B/A}$ is homotopy equivalent to $J/J^2 \to \bigoplus_{i=1,\ldots,n} B\text{d}x_i$, see Algebra, Section 10.134. Denote L the complex $J/J^2 \to \bigoplus_{i=1,\ldots,m} B\text{d}x_i$ to that we have the quotient map $NL_{B/A} \to L$. We also have a surjective map of complexes $K \to L$ by sending the jth basis element in the term $B^{\oplus m}$ in degree -1 to the class of f_j in J/J^2. Picture

$$NL_{B/A} \to L \leftarrow K$$

From More on Algebra, Lemma 15.84.8 we conclude that multiplication by g on L is 0 in $D(B)$. On the other hand, the distinguished triangle $B^{\oplus n-m}[0] \to NL_{B/A} \to L$ shows that $\text{Ext}^1_B(L, N) \to \text{Ext}^1_B(NL_{B/A}, N)$ is surjective for every B-module N and hence annihilated by g. This proves part (1). If $n = m$ then $NL_{B/A} = L$ and we see that (2) holds. □

0GAW Lemma 88.7.2. Let I be an ideal of a Noetherian ring A. Let B be an object of (88.2.0.2). Let $B = A[x_1, \ldots, x_r]^\wedge/J$ be a presentation. Assume there exists an element $b \in B$, $0 \leq m \leq r$, and $f_1, \ldots, f_m \in J$ such that

(1) $V(b) \subset V(IB)$ in $\text{Spec}(B)$,
(2) the image of $\Delta = \det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$ in B divides b, and
(3) $bJ \subset (f_1, \ldots, f_m) + J^2$.

Then there exists a finite type A-algebra C and an A-algebra isomorphism $B \cong C^\wedge$.

Proof. The conditions imply that B is rig-smooth over (A, I), see Lemma 88.4.2. Write $b'\Delta = b$ in B for some $b' \in B$. Say $I = (a_1, \ldots, a_t)$. Since $V(b) \subset V(IB)$ there exists an integer $c \geq 0$ such that $I^c B \subset bB$. Write $bb_i = a_i^c$ in B for some $b_i \in B$.

Choose an integer $n \gg 0$ (we will see later how large). Choose polynomials $f'_1, \ldots, f'_m \in A[x_1, \ldots, x_r]$ such that $f_i - f'_i \in I^n A[x_1, \ldots, x_r]^\wedge$. We set $\Delta' =$

$\det_{1 \leq i,j \leq m}(\partial f'_j/\partial x_i)$ and we consider the finite type A-algebra

$$C = A[x_1, \ldots, x_r, z_1, \ldots, z_t]/(f'_1, \ldots, f'_m, z_1\Delta' - a_1^c, \ldots, z_t\Delta' - a_t^c)$$

We will apply Lemma 88.7.1 to C. We compute

$$\det \left(\begin{matrix} \text{matrix of partials of} \\ f'_1, \ldots, f'_m, z_1\Delta' - a_1^c, \ldots, z_t\Delta' - a_t^c \\ \text{with respect to the variables} \\ x_1, \ldots, x_m, z_1, \ldots, z_t \end{matrix} \right) = (\Delta')^{t+1}$$

Hence we see that $\mathrm{Ext}^1_C(NL_{C/A}, N)$ is annihilated by $(\Delta')^{t+1}$ for all C-modules N. Since a_i^c is divisible by Δ' in C we see that $a_i^{(t+1)c}$ annihilates these Ext^1's also. Thus I^{c_1} annihilates $\mathrm{Ext}^1_C(NL_{C/A}, N)$ for all C-modules N where $c_1 = 1 + t((t+1)c - 1)$. The exact value of c_1 doesn't matter for the rest of the argument; what matters is that it is independent of n.

Since $NL^\wedge_{C^\wedge/A} = NL_{C/A} \otimes_C C^\wedge$ by Lemma 88.3.2 we conclude that multiplication by I^{c_1} is zero on $\mathrm{Ext}^1_{C^\wedge}(NL^\wedge_{C^\wedge/A}, N)$ for any $C^\wedge$-module N as well, see More on Algebra, Lemmas 15.84.7 and 15.84.6. In particular $C^\wedge$ is rig-smooth over (A, I).

Observe that we have a surjective A-algebra homomorphism

$$\psi_n : C \longrightarrow B/I^nB$$

sending the class of x_i to the class of x_i and sending the class of z_i to the class of b_ib'. This works because of our choices of b' and b_i in the first paragraph of the proof.

Let $d = d(\mathrm{Gr}_I(B))$ and $q_0 = q(\mathrm{Gr}_I(B))$ be the integers found in Local Cohomology, Section 51.22. By Lemma 88.5.3 if we take $n \geq \max(q_0 + (d+1)c_1, 2(d+1)c_1 + 1)$ we can find a homomorphism $\varphi : C^\wedge \to B$ of A-algebras which is congruent to ψ_n modulo $I^{n-(d+1)c_1}B$.

Since $\varphi : C^\wedge \to B$ is surjective modulo I we see that it is surjective (for example use Algebra, Lemma 10.96.1). To finish the proof it suffices to show that $\mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$ is annihilated by a power of I, see More on Algebra, Lemma 15.108.4.

Since φ is surjective we see that $NL^\wedge_{B/C^\wedge}$ has cohomology modules $H^0(NL^\wedge_{B/C^\wedge}) = 0$ and $H^{-1}(NL^\wedge_{B/C^\wedge}) = \mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$. We have an exact sequence

$$H^{-1}(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^{-1}(NL^\wedge_{B/A}) \to H^{-1}(NL^\wedge_{B/C^\wedge}) \to H^0(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^0(NL^\wedge_{B/A}) \to 0$$

by Lemma 88.3.5. The first two modules are annihilated by a power of I as B and $C^\wedge$ are rig-smooth over (A, I). Hence it suffices to show that the kernel of the surjective map $H^0(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^0(NL^\wedge_{B/A})$ is annihilated by a power of I. For this it suffices to show that it is annihilated by a power of b. In other words, it suffices to show that

$$H^0(NL^\wedge_{C^\wedge/A}) \otimes_{C^\wedge} B[1/b] \longrightarrow H^0(NL^\wedge_{B/A}) \otimes_B B[1/b]$$

is an isomorphism. However, both are free $B[1/b]$ modules of rank $r - m$ with basis $\mathrm{d}x_{m+1}, \ldots, \mathrm{d}x_r$ and we conclude the proof. $\square$

0GAX Remark 88.7.3. Let I be an ideal of a Noetherian ring A. Let B be an object of (88.2.0.2) which is rig-smooth over (A, I). As far as we know, it is an open question as to whether B is isomorphic to the I-adic completion of a finite type A-algebra. Here are some things we do know:

(1) If A is a G-ring, then the answer is yes by Proposition 88.6.3.
(2) If B is rig-étale over (A, I), then the answer is yes by Lemma 88.10.2.
(3) If I is principal, then the answer is yes by [Elk73, III Theorem 7].
(4) In general there exists an ideal $J = (b_1, \ldots, b_s) \subset B$ such that $V(J) \subset V(IB)$ and such that the I-adic completion of each of the affine blowup algebras $B[\frac{J}{b_i}]$ are isomorphic to the I-adic completion of a finite type A-algebra.

To see the last statement, choose $b_1, \ldots, b_s$ as in Lemma 88.4.2 part (4) and use the properties mentioned there to see that Lemma 88.7.2 applies to each completion $(B[\frac{J}{b_i}])^\wedge$. Part (4) tells us that "rig-locally a rig-smooth formal algebraic space is the completion of a finite type scheme over A" and it tells us that "there is an admissible formal blowing up of $\text{Spf}(B)$ which is affine locally algebraizable".

88.8. Rig-étale algebras

0ALP In view of our definition of rig-smooth algebras (Definition 88.4.1), the following definition should not come as a surprise.

0GAY Definition 88.8.1. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). We say B is rig-étale over (A, I) if there exists an integer $c \geq 0$ such that for all $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$.

Condition (7) in the next lemma is one of the conditions used in [Art70] to define formal modifications. We have added it to the list of conditions to facilitate comparison with our conditions later on.

0AJU Lemma 88.8.2. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Write $B = A[x_1, \ldots, x_r]^\wedge/J$ (Lemma 88.2.2) and let $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\text{d}x_i)$ be its naive cotangent complex (88.3.0.1). The following are equivalent

(1) B is rig-étale over (A, I),
0AJV (2) there exists a $c \geq 0$ such that for all $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$,
0AJW (3) there exits a $c \geq 0$ such that $H^i(NL^\wedge_{B/A})$, $i = -1, 0$ is annihilated by I^c,
0AJX (4) there exists a $c \geq 0$ such that $H^i(NL_{B_n/A_n})$, $i = -1, 0$ is annihilated by I^c for all $n \geq 1$ where $A_n = A/I^n$ and $B_n = B/I^nB$,
0GAZ (5) for every $a \in I$ there exists a $c \geq 0$ such that
   (a) a^c annihilates $H^0(NL^\wedge_{B/A})$, and
   (b) there exist $f_1, \ldots, f_r \in J$ such that $a^cJ \subset (f_1, \ldots, f_r) + J^2$.
0GB0 (6) for every $a \in I$ there exist $f_1, \ldots, f_r \in J$ and $c \geq 0$ such that
   (a) $\det_{1 \leq i,j \leq r}(\partial f_j/\partial x_i)$ divides a^c in B, and
   (b) $a^cJ \subset (f_1, \ldots, f_r) + J^2$.
0AJY (7) choosing generaters $f_1, \ldots, f_t$ for J we have
   (a) the Jacobian ideal of B over A, namely the ideal in B generated by the $r \times r$ minors of the matrx $(\partial f_j/\partial x_i)_{1 \leq i \leq r, 1 \leq j \leq t}$, contains the ideal I^cB for some c, and

(b) the Cramer ideal of B over A, namely the ideal in B generated by the image in B of the rth Fitting ideal of J as an $A[x_1, \ldots, x_r]^\wedge$-module, contains I^cB for some c.

Proof. The equivalence of (1) and (2) is a restatement of Definition 88.8.1.

The equivalence of (2) and (3) follows from More on Algebra, Lemma 15.84.11.

The equivalence of (3) and (4) follows from the fact that the systems $\{NL_{B_n/A_n}\}$ and $NL^\wedge_{B/A} \otimes_B B_n$ are strictly isomorphic, see Lemma 88.3.3. Some details omitted.

Assume (2). Let $a \in I$. Let c be such that multiplication by a^c is zero on $NL^\wedge_{B/A}$. By More on Algebra, Lemma 15.84.4 part (1) there exists a map $\alpha : \bigoplus B\text{d}x_i \to J/J^2$ such that $\text{d} \circ \alpha$ and $\alpha \circ \text{d}$ are both multiplication by a^c. Let $f_i \in J$ be an element whose class modulo J^2 is equal to $\alpha(\text{d}x_i)$. A simple calculation gives that (6)(a), (b) hold.

We omit the verification that (6) implies (5); it is just a statement on two term complexes over B of the form $M \to B^{\oplus r}$.

Assume (5) holds. Say $I = (a_1, \ldots, a_t)$. Let $c_i \geq 0$ be the integer such that (5)(a), (b) hold for $a_i^{c_i}$. Then we see that $I^{\sum c_i}$ annihilates $H^0(NL^\wedge_{B/A})$. Let $f_{i,1}, \ldots, f_{i,r} \in J$ be as in (5)(b) for a_i. Consider the composition

$$B^{\oplus r} \to J/J^2 \to \bigoplus B\text{d}x_i$$

where the jth basis vector is mapped to the class of $f_{i,j}$ in J/J^2. By (5)(a) and (b) the cokernel of the composition is annihilated by $a_i^{2c_i}$. Thus this map is surjective after inverting $a_i^{c_i}$, and hence an isomorphism (Algebra, Lemma 10.16.4). Thus the kernel of $B^{\oplus r} \to \bigoplus B\text{d}x_i$ is a_i-power torsion, and hence $H^{-1}(NL^\wedge_{B/A}) = \text{Ker}(J/J^2 \to \bigoplus B\text{d}x_i)$ is a_i-power torsion. Since B is Noetherian (Lemma 88.2.2), all modules including $H^{-1}(NL^\wedge_{B/A})$ are finite. Thus $a_i^{d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ for some $d_i \geq 0$. It follows that $I^{\sum d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ and we see that (3) holds.

Thus conditions (2), (3), (4), (5), and (6) are equivalent. Thus it remains to show that these conditions are equivalent with (7). Observe that the Cramer ideal $\text{Fit}_r(J)B$ is equal to $\text{Fit}_r(J/J^2)$ as $J/J^2 = J \otimes_{A[x_1,\ldots,x_r]^\wedge} B$, see More on Algebra, Lemma 15.8.4 part (3). Also, observe that the Jacobian ideal is just $\text{Fit}_0(H^0(NL^\wedge_{B/A}))$. Thus we see that the equivalence of (3) and (7) is a purely algebraic question which we discuss in the next paragraph.

Let R be a Noetherian ring and let $I \subset R$ be an ideal. Let $M \xrightarrow{d} R^{\oplus r}$ be a two term complex. We have to show that the following are equivalent

(A) the cohomology of $M \to R^{\oplus r}$ is annihilated by a power of I, and
(B) the ideals $\text{Fit}_r(M)$ and $\text{Fit}_0(\text{Coker}(d))$ contain a power of I.

Since R is Noetherian, we can reformulate part (2) as an inclusion of the corresponding closed subschemes, see Algebra, Lemmas 10.17.2 and 10.32.5. On the other hand, over the complement of $V(\text{Fit}_0(\text{Coker}(d)))$ the cokernel of d vanishes and over the complement of $V(\text{Fit}_r(M))$ the module M is locally generated by r elements, see More on Algebra, Lemma 15.8.6. Thus (B) is equivalent to

(C) away from $V(I)$ the cokernel of d vanishes and the module M is locally generated by $\leq r$ elements.

Of course this is equivalent to the condition that $M \to R^{\oplus r}$ has vanishing cohomology over $\text{Spec}(R) \setminus V(I)$ which in turn is equivalent to (A). This finishes the proof. $\square$

0GB1 Lemma 88.8.3. Let A be a Noetherian ring and let I be an ideal. Let B be an object of (88.2.0.2). If B is rig-étale over (A, I), then B is rig-smooth over (A, I).

Proof. Immediate from Definitions 88.4.1 and 88.8.1. $\square$

0ALQ Lemma 88.8.4. Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.

(1) If $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$, then $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2.
(2) If $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2, then there exists $g \in 1 + IB$ such that $\text{Spec}(B_g)$ is étale over $\text{Spec}(A) \setminus V(I)$.

Proof. Assume $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2. The naive cotangent complex $NL_{B/A}$ is a complex of finite type B-modules and hence H^{-1} and H^0 are finite B-modules. Completion is an exact functor on finite B-modules (Algebra, Lemma 10.97.2) and $NL^\wedge_{B^\wedge/A}$ is the completion of the complex $NL_{B/A}$ (this is easy to see by choosing presentations). Hence the assumption implies there exists a $c \geq 0$ such that H^{-1}/I^nH^{-1} and H^0/I^nH^0 are annihilated by I^c for all n. By Nakayama's lemma (Algebra, Lemma 10.20.1) this means that I^cH^{-1} and I^cH^0 are annihilated by an element of the form $g = 1 + x$ with $x \in IB$. After inverting g (which does not change the quotients B/I^nB) we see that $NL_{B/A}$ has cohomology annihilated by I^c. Thus $A \to B$ is étale at any prime of B not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 10.143.1.

Conversely, assume that $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists a $c \geq 0$ such that multiplication by a^c is zero $NL_{B/A}$. Since $NL^\wedge_{B^\wedge/A}$ is the derived completion of $NL_{B/A}$ (see Lemma 88.3.3) it follows that $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2. $\square$

0AK2 Lemma 88.8.5. Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 88.2.3 with A_1 and A_2 Noetherian. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. If multiplication by $f_1 \in B_1$ on $NL^\wedge_{B_1/A_1}$ is zero in $D(B_1)$, then multiplication by the image $f_2 \in B_2$ on $NL^\wedge_{B_2/A_2}$ is zero in $D(B_2)$.

Proof. By Lemma 88.3.4 there is a map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}. Thus the result by More on Algebra, Lemma 15.84.8. $\square$

0GB2 Lemma 88.8.6. Let $A_1 \to A_2$ be a map of Noetherian rings. Let $I_i \subset A_i$ be an ideal such that $V(I_1A_2) = V(I_2)$. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1 as in Remark 88.2.3. If B_1 is rig-étale over (A_1, I_1), then B_2 is rig-étale over (A_2, I_2).

Proof. Follows from Lemma 88.8.5 and Definition 88.8.1 and the fact that $I_2^c \subset I_1A_2$ for some $c \geq 0$ as A_2 is Noetherian. $\square$

0AKJ **Lemma 88.8.7.** Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be a finite type A-algebra such that $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Let C be a Noetherian A-algebra. Then any A-algebra map $B^\wedge \to C^\wedge$ of I-adic completions comes from a unique A-algebra map

$$B \longrightarrow C^h$$

where C^h is the henselization of the pair (C, IC) as in More on Algebra, Lemma 15.12.1. Moreover, any A-algebra homomorphism $B \to C^h$ factors through some étale C-algebra C' such that $C/IC \to C'/IC'$ is an isomorphism.

Proof. Uniqueness follows from the fact that C^h is a subring of $C^\wedge$, see for example More on Algebra, Lemma 15.12.4. The final assertion follows from the fact that C^h is the filtered colimit of these C-algebras C', see proof of More on Algebra, Lemma 15.12.1. Having said this we now turn to the proof of existence.

Let $\varphi : B^\wedge \to C^\wedge$ be the given map. This defines a section

$$\sigma : (B \otimes_A C)^\wedge \longrightarrow C^\wedge$$

of the completion of the map $C \to B \otimes_A C$. We may replace (A, I, B, C, φ) by $(C, IC, B \otimes_A C, C, \sigma)$. In this way we see that we may assume that $A = C$.

Proof of existence in the case $A = C$. In this case the map $\varphi : B^\wedge \to A^\wedge$ is necessarily surjective. By Lemmas 88.8.4 and 88.3.5 we see that the cohomology groups of $NL^\wedge_{A^\wedge/_\varphi B^\wedge}$ are annihilated by a power of I. Since φ is surjective, this implies that $\text{Ker}(\varphi)/\text{Ker}(\varphi)^2$ is annihilated by a power of I. Hence $\varphi : B^\wedge \to A^\wedge$ is the completion of a finite type B-algebra $B \to D$, see More on Algebra, Lemma 15.108.4. Hence $A \to D$ is a finite type algebra map which induces an isomorphism $A^\wedge \to D^\wedge$. By Lemma 88.8.4 we may replace D by a localization and assume that $A \to D$ is étale away from $V(I)$. Since $A^\wedge \to D^\wedge$ is an isomorphism, we see that $\text{Spec}(D) \to \text{Spec}(A)$ is also étale in a neighbourhood of $V(ID)$ (for example by More on Morphisms, Lemma 37.12.3). Thus $\text{Spec}(D) \to \text{Spec}(A)$ is étale. Therefore D maps to A^h and the lemma is proved. $\square$

88.9. A pushout argument

0AK8 The only goal in this section is to prove the following lemma which will play a key role in algebraization of rig-étale algebras. We will use a bit of the theory of algebraic spaces to prove this lemma; an earlier version of this chapter gave a (much longer) proof using algebra and a bit of deformation theory that the interested reader can find in the history of the Stacks project.

0ALT **Lemma 88.9.1.** Let A be a Noetherian ring and $I \subset A$ an ideal. Let $J \subset A$ be a nilpotent ideal. Consider a commutative diagram

$$\begin{tikzcd} C \arrow[r] & C_0 \arrow[r, equal] & C/JC \\ & B_0 \arrow[u] & \\ A \arrow[uu] \arrow[r] & A_0 \arrow[u] \arrow[r, equal] & A/J \end{tikzcd}$$

whose vertical arrows are of finite type such that

(1) $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$,
(2) $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ is étale over $\mathrm{Spec}(A_0) \setminus V(IA_0)$, and
(3) $B_0 \to C_0$ is étale and induces an isomorphism $B_0/IB_0 = C_0/IC_0$.

Then we can fill in the diagram above to a commutative diagram

$$\begin{tikzcd} C \arrow[r] & C/JC \\ B \arrow[u] \arrow[r] & B_0 \arrow[u] \\ A \arrow[u] \arrow[r] & A/J \arrow[u] \end{tikzcd}$$

with $A \to B$ of finite type, $B/JB = B_0$, $B \to C$ étale, and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ étale over $\mathrm{Spec}(A) \setminus V(I)$.

Proof. Set $X = \mathrm{Spec}(A)$, $X_0 = \mathrm{Spec}(A_0)$, $Y_0 = \mathrm{Spec}(B_0)$, $Z = \mathrm{Spec}(C)$, $Z_0 = \mathrm{Spec}(C_0)$. Furthermore, denote $U \subset X$, $U_0 \subset X_0$, $V_0 \subset Y_0$, $W \subset Z$, $W_0 \subset Z_0$ the complement of the vanishing set of I. Here is a picture to help visualize the situation:

$$\begin{tikzcd} Z \arrow[dd] & Z_0 \arrow[l] \arrow[d] & & W \arrow[dd] & W_0 \arrow[l] \arrow[d] \\ & Y_0 \arrow[d] & & & V_0 \arrow[d] \\ X & X_0 \arrow[l] & & U & U_0 \arrow[l] \end{tikzcd}$$

The conditions in the lemma guarantee that

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ V_0 \arrow[r] & Y_0 \end{tikzcd}$$

is an elementary distinguished square, see Derived Categories of Spaces, Definition 75.9.1. In addition we know that $W_0 \to U_0$ and $V_0 \to U_0$ are étale. The morphism $X_0 \subset X$ is a finite order thickening as J is assumed nilpotent. By the topological invariance of the étale site we can find a unique étale morphism $V \to X$ of schemes with $V_0 = V \times_X X_0$ and we can lift the given morphism $W_0 \to V_0$ to a unique morphism $W \to V$ over X. See Étale Morphisms, Theorem 41.15.2. Since $W_0 \to V_0$ is separated, the morphism $W \to V$ is separated too, see for example More on Morphisms, Lemma 37.10.3. By Pushouts of Spaces, Lemma 81.9.2 we can construct an elementary distinguished square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

in the category of algebraic spaces over X. Since the base change of an elementary distinguished square is an elementary distinguished square (Derived Categories of

Spaces, Lemma 75.9.2) we see that

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ V_0 \arrow[r] & Y \times_X X_0 \end{tikzcd}$$

is an elementary distinguished square. It follows that there is a unique isomorphism $Y \times_X X_0 = Y_0$ compatible with the two squares involving these spaces because elementary distinguished squares are pushouts (Pushouts of Spaces, Lemma 81.9.1). It follows that Y is affine by Limits of Spaces, Proposition 70.15.2. Write $Y = \mathrm{Spec}(B)$. It is clear that B fits into the desired diagram and satisfies all the properties required of it. □

88.10. Algebraization of rig-étale algebras

0AK5 The main goal is to prove algebraization for rig-étale algebras when the underlying Noetherian ring A is not assumed to be a G-ring and when the ideal $I \subset A$ is arbitrary – not necessarily principal. We first prove the principal ideal case and then use the result of Section 88.9 to finish the proof.

0ALS **Lemma 88.10.1.** Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (88.2.0.2) which is rig-étale over (A, I). Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$. The rig-étale case of [Elk73, III Theorem 7]

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. By Lemma 88.8.2 part (6) we can find $c \geq 0$ and $f_1, \ldots, f_r \in J$ such that $\det_{1 \leq i,j \leq r}(\partial f_j/\partial x_i)$ divides a^c in B and $a^c J \subset (f_1, \ldots, f_r) + J^2$. Hence Lemma 88.7.2 applies. This finishes the proof, but we'd like to point out that in this case the use of Lemma 88.5.3 can be replaced by the much easier Lemma 88.5.5. □

0AKA **Lemma 88.10.2.** Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2) which is rig-étale over (A, I). Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.

Proof. We prove this lemma by induction on the number of generators of I. Say $I = (a_1, \ldots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 88.10.1. Assume $t > 1$.

For any $m \geq 1$ set $\bar{A}_m = A/(a_t^m)$. Consider the ideal $\bar{I}_m = (\bar{a}_1, \ldots, \bar{a}_{t-1})$ in $\bar{A}_m$. Observe that $V(I\bar{A}_m) = V(\bar{I}_m)$. Let $B_m = B/(a_t^m)$ be the base change of B for the map $(A, I) \to (\bar{A}_m, \bar{I}_m)$, see Remark 88.2.4. By Lemma 88.8.6 we find that B_m is rig-étale over $(\bar{A}_m, \bar{I}_m)$.

By induction hypothesis (on t) we can find a finite type $\bar{A}_m$-algebra C_m and a map $C_m \to B_m$ which induces an isomorphism $C_m^\wedge \cong B_m$ where the completion is with respect to $\bar{I}_m$. By Lemma 88.8.4 we may assume that $\mathrm{Spec}(C_m) \to \mathrm{Spec}(\bar{A}_m)$ is étale over $\mathrm{Spec}(\bar{A}_m) \setminus V(\bar{I}_m)$.

We claim that we may choose $A_m \to C_m \to B_m$ as in the previous paragraph such that moreover there are isomorphisms $C_m/(a_t^{m-1}) \to C_{m-1}$ compatible with the given A-algebra structure and the maps to $B_{m-1} = B_m/(a_t^{m-1})$. Namely, first fix a choice of $A_1 \to C_1 \to B_1$. Suppose we have found $C_{m-1} \to C_{m-2} \to \ldots \to C_1$ with the desired properties. Note that $C_m/(a_t^{m-1})$ is étale over $\mathrm{Spec}(\bar{A}_{m-1}) \setminus V(\bar{I}_{m-1})$.

Hence by Lemma 88.8.7 there exists an étale extension $C_{m-1} \to C'_{m-1}$ which induces an isomorphism modulo $\bar{I}_{m-1}$ and an $\bar{A}_{m-1}$-algebra map $C_m/(a_t^{m-1}) \to C'_{m-1}$ inducing the isomorphism $B_m/(a_t^{m-1}) \to B_{m-1}$ on completions. Note that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale over the complement of $V(\bar{I}_{m-1})$ by Morphisms, Lemma 29.36.18 and over $V(\bar{I}_{m-1})$ induces an isomorphism on completions hence is étale there too (for example by More on Morphisms, Lemma 37.12.3). Thus $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale. By the topological invariance of étale morphisms (Étale Morphisms, Theorem 41.15.2) there exists an étale ring map $C_m \to C'_m$ such that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is isomorphic to $C_m/(a_t^{m-1}) \to C'_m/(a_t^{m-1})$. Observe that the $\bar{I}_m$-adic completion of C'_m is equal to the $\bar{I}_m$-adic completion of C_m, i.e., to B_m (details omitted). We apply Lemma 88.9.1 to the diagram

$$
\begin{tikzcd}
 & C'_m \arrow[r] & C'_m/(a_t^{m-1}) \\
C''_m \arrow[ur, dotted] \arrow[rr, dotted] & & C_{m-1} \arrow[u] \\
 & \bar{A}_m \arrow[r] \arrow[uu] \arrow[ul, dotted] & \bar{A}_{m-1} \arrow[u]
\end{tikzcd}
$$

to see that there exists a "lift" of C''_m of C_{m-1} to an algebra over $\bar{A}_m$ with all the desired properties.

By construction (C_m) is an object of the category (88.2.0.1) for the principal ideal (a_t). Thus the inverse limit $B' = \lim C_m$ is an (a_t)-adically complete A-algebra such that B'/a_tB' is of finite type over $A/(a_t)$, see Lemma 88.2.1. By construction the I-adic completion of B' is isomorphic to B (details omitted). Consider the complex $NL^\wedge_{B'/A}$ constructed using the (a_t)-adic topology. Choosing a presentation for B' (which induces a similar presentation for B) the reader immediately sees that $NL^\wedge_{B'/A} \otimes_{B'} B = NL^\wedge_{B/A}$. Since $a_t \in I$ and since the cohomology modules of $NL^\wedge_{B'/A}$ are finite B'-modules (hence complete for the a_t-adic topology), we conclude that a_t^c acts as zero on these cohomologies as the same thing is true by assumption for $NL^\wedge_{B/A}$. Thus B' is rig-étale over $(A, (a_t))$ by Lemma 88.8.2. Hence finally, we may apply Lemma 88.10.1 to B' over $(A, (a_t))$ to finish the proof. $\square$

0AKG Lemma 88.10.3. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra with $A/I \to B/IB$ of finite type. The equivalent conditions of Lemma 88.8.2 are also equivalent to

0AKH (8) there exists a finite type A-algebra C such that $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$ and such that $B \cong C^\wedge$.

Proof. Combine Lemmas 88.8.2, 88.10.2, and 88.8.4. Small detail omitted. $\square$

88.11. Finite type morphisms

0GBV In Formal Spaces, Section 87.24 we have defined finite type morphisms of formal algebraic spaces. In this section we study the corresponding types of continuous ring maps of adic topological rings which have a finitely generated ideal of definition. We strongly suggest the reader skip this section.

0GBW Lemma 88.11.1. Let A and B be adic topological rings which have a finitely generated ideal of definition. Let $\varphi : A \to B$ be a continuous ring homomorphism. The following are equivalent:

(1) φ is adic and B is topologically of finite type over A,
(2) φ is taut and B is topologically of finite type over A,
(3) there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology and there exist an ideal of definition $I' \subset A$ such that $A/I' \to B/I'B$ is of finite type,
(4) for all ideals of definition $I \subset A$ the topology on B is the I-adic topology and $A/I \to B/IB$ is of finite type,
(5) there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology and B is in the category (88.2.0.2),
(6) for all ideals of definition $I \subset A$ the topology on B is the I-adic topology and B is in the category (88.2.0.2),
(7) B as a topological A-algebra is the quotient of $A\{x_1, \ldots, x_r\}$ by a closed ideal,
(8) B as a topological A-algebra is the quotient of $A[x_1, \ldots, x_r]^\wedge$ by a closed ideal where $A[x_1, \ldots, x_r]^\wedge$ is the completion of $A[x_1, \ldots, x_r]$ with respect to some ideal of definition of A, and
(9) add more here.

Moreover, these equivalent conditions define a local property of morphisms of WAdm^{adic*} as defined in Formal Spaces, Remark 87.21.4.

Proof. Taut ring homomorphisms are defined in Formal Spaces, Definition 87.5.1. Adic ring homomorphisms are defined in Formal Spaces, Definition 87.6.1. The lemma follows from a combination of Formal Spaces, Lemmas 87.29.6, 87.29.7, and 87.23.1. We omit the details. To be sure, there is no difference between the topological rings $A[x_1, \ldots, x_n]^\wedge$ and $A\{x_1, \ldots, x_r\}$, see Formal Spaces, Remark 87.28.2. $\square$

0GBX Remark 88.11.2. Let $A \to B$ be an arrow of WAdm^{adic*} which is adic and topologically of finite type (see Lemma 88.11.1). Write $B = A\{x_1, \ldots, x_r\}/J$. Then we can set[1]

$$NL^\wedge_{B/A} = \left(J/J^2 \longrightarrow \bigoplus B\text{d}x_i\right)$$

Exactly as in the proof of Lemma 88.3.1 the reader can show that this complex of B-modules is well defined up to (unique isomorphism) in the homotopy category $K(B)$. Now, if A is Noetherian and $I \subset A$ is an ideal of definition, then this construction reproduces the naive cotangent complex of B over (A, I) defined by Equation (88.3.0.1) in Section 88.3 simply because $A[x_1, \ldots, x_n]^\wedge$ agrees with $A\{x_1, \ldots, x_r\}$ by Formal Spaces, Remark 87.28.2. In particular, we find that, still when A is an adic Noetherian topological ring, the object $NL^\wedge_{B/A}$ is independent of the choice of the ideal of definition $I \subset A$.

0GBY Lemma 88.11.3. Consider the property P on arrows of WAdm^{adic*} defined in Lemma 88.11.1. Then P is stable under base change as defined in Formal Spaces, Remark 87.21.8.

[1] In fact, this construction works for arrows of WAdm^{count} satisfying the equivalent conditions of Formal Spaces, Lemma 87.29.6.

Proof. The statement makes sense by Lemma 88.11.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in WAdm^{adic*} and that as a topological B-algebra we have $A = B\{x_1, \ldots, x_r\}/J$ for some closed ideal J. Then $A\widehat{\otimes}_B C$ is isomorphic to the quotient of $C\{x_1, \ldots, x_r\}/J'$ where J' is the closure of $JC\{x_1, \ldots, x_r\}$. Some details omitted. □

0GBZ Lemma 88.11.4. Consider the property P on arrows of WAdm^{adic*} defined in Lemma 88.11.1. Then P is stable under composition as defined in Formal Spaces, Remark 87.21.13.

Proof. The statement makes sense by Lemma 88.11.1. The easiest way to prove it is true is to show that (a) compositions of adic ring maps between adic topological rings are adic and (b) that compositions of continuous ring maps preserves the property of being topologically of finite type. We omit the details. □

The following lemma says that morphisms of adic* formal algebraic spaces are locally of finite type if and only if they are étale locally given by the types of maps of topological rings described in Lemma 88.11.1.

0GC0 Lemma 88.11.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally adic* formal algebraic spaces over S. The following are equivalent

(1) for every commutative diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an arrow of WAdm^{adic*} which is adic and topologically of finite type,
(2) there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to an arrow of WAdm^{adic*} which is adic and topologically of finite type,
(3) there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to an arrow of WAdm^{adic*} which is adic and topologically of finite type, and
(4) f is locally of finite type.

Proof. Immediate consequence of the equivalence of (1) and (2) in Lemma 88.11.1 and Formal Spaces, Lemma 87.29.9. □

88.12. Finite type on reductions

0GC1 In this section we talk a little bit about morphisms $X \to Y$ of locally countably indexed formal algebraic spaces such that $X_{red} \to Y_{red}$ is locally of finite type. We will translate this into an algebraic condition. To understand this algebraic condition it pays to keep in mind the following:

- If A is a weakly admissible topological ring, then the set $\mathfrak{a} \subset A$ of topological nilpotent elements is an open, radical ideal and $\mathrm{Spf}(A)_{red} = \mathrm{Spec}(A/\mathfrak{a})$.

See Formal Spaces, Definition 87.4.8, Lemma 87.4.10, and Example 87.12.2.

0GC2 Lemma 88.12.1. For an arrow $\varphi : A \to B$ in WAdm^{count} consider the property $P(\varphi)$ ="the induced ring homomorphism $A/\mathfrak{a} \to B/\mathfrak{b}$ is of finite type" where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the ideals of topologically nilpotent elements. Then P is a local property as defined in Formal Spaces, Situation 87.21.2.

Proof. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

as in Formal Spaces, Situation 87.21.2. Taking Spf of this diagram we obtain

$$\begin{tikzcd} \mathrm{Spf}(B) \arrow[d] & \mathrm{Spf}((B')^\wedge) \arrow[l] \arrow[d] \\ \mathrm{Spf}(A) & \mathrm{Spf}((A')^\wedge) \arrow[l] \end{tikzcd}$$

of affine formal algebraic spaces whose horizontal arrows are representable by algebraic spaces and étale by Formal Spaces, Lemma 87.19.13. Hence we obtain a commutative diagram of affine schemes

$$\begin{tikzcd} \mathrm{Spf}(B)_{red} \arrow[d, "f"] & \mathrm{Spf}((B')^\wedge)_{red} \arrow[l, "g"'] \arrow[d, "f'"] \\ \mathrm{Spf}(A)_{red} & \mathrm{Spf}((A')^\wedge)_{red} \arrow[l] \end{tikzcd}$$

whose horizontal arrows are étale by Formal Spaces, Lemma 87.12.3. By Formal Spaces, Example 87.12.2 and Lemma 87.19.14 conditions (1), (2), and (3) of Formal Spaces, Situation 87.21.2 translate into the following statements

1. if f is locally of finite type, then f' is locally of finite type,
2. if f' is locally of finite type and g is surjective, then f is locally of finite type, and
3. if $T_i \to S$, $i = 1, \ldots, n$ are locally of finite type, then $\coprod_{i=1,\ldots,n} T_i \to S$ is locally of finite type.

Properties (1) and (2) follow from the fact that being locally of finite type is local on the source and target in the étale topology, see discussion in Morphisms of Spaces, Section 67.23. Property (3) is a straightforward consequence of the definition. □

0GC3 Lemma 88.12.2. Consider the property P on arrows of WAdm^{count} defined in Lemma 88.12.1. Then P is stable under base change (Formal Spaces, Situation 87.21.6).

Proof. The statement makes sense by Lemma 88.12.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in WAdm^{count} such that $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type where $\mathfrak{b} \subset B$ and $\mathfrak{a} \subset A$ are the ideals of topologically nilpotent elements. Since A and B are weakly admissible, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are open. Let $\mathfrak{c} \subset C$ be the (open) ideal of topologically nilpotent elements. Then we find a surjection $A\widehat{\otimes}_B C \to A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$ whose kernel is a weak ideal of definition and hence consists of topologically nilpotent elements (please compare with the proof of Formal Spaces, Lemma 87.4.12). Since already $C/\mathfrak{c} \to A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$ is of finite type as a base change of $B/\mathfrak{b} \to A/\mathfrak{a}$ we conclude. □

0GC4 Lemma 88.12.3. Consider the property P on arrows of WAdm^{count} defined in Lemma 88.12.1. Then P is stable under composition (Formal Spaces, Situation 87.21.11).

Proof. Omitted. Hint: compositions of finite type ring maps are of finite type. □

0GC5 Lemma 88.12.4. Let $\varphi : A \to B$ be an arrow of WAdm^{count}. If φ is taut and topologically of finite type, then φ satisfies the condition defined in Lemma 88.12.1.

Proof. This is an easy consequence of the definitions. □

0GC6 Lemma 88.12.5. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} satisfying the condition defined in Lemma 88.12.1. Then $A \to B$ is topologically of finite type.

Proof. Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements. Choose $b_1, \ldots, b_r \in B$ which map to generators of $B/\mathfrak{b}$ over A. Choose generators $b_{r+1}, \ldots, b_s$ of the ideal $\mathfrak{b}$. We claim that the image of

$$\varphi : A[x_1, \ldots, x_s] \longrightarrow B, \quad x_i \longmapsto b_i$$

has dense image. Namely, if $b \in \mathfrak{b}^n$ for some $n \geq 0$, then we can write $b = \sum b_E b_{r+1}^{e_{r+1}} \ldots b_s^{e_s}$ for multiindices $E = (e_{r+1}, \ldots, e_s)$ with $|E| = \sum e_i = n$ and $b_E \in B$. Next, we can write $b_E = f_E(b_1, \ldots, b_r) + b'_E$ with $b'_E \in \mathfrak{b}$ and $f_E \in A[x_1, \ldots, x_r]$. Combined we obtain $b \in \text{Im}(\varphi) + \mathfrak{b}^{n+1}$. By induction we see that $B = \text{Im}(\varphi) + \mathfrak{b}^n$ for all $n \geq 0$ which mplies what we want as $\mathfrak{b}$ is an ideal of definition of B. □

0GG6 Lemma 88.12.6. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth}. If φ is adic the following are equivalent

(1) φ satisfies the condition defined in Lemma 88.12.1 and
(2) φ satisfies the condition defined in Lemma 88.11.1.

Proof. Omitted. Hint: For the proof of (1) ⇒ (2) use Lemma 88.12.5. □

0GC7 Lemma 88.12.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent

(1) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an arrow of WAdm^{count} satisfying the property defined in Lemma 88.12.1,

(2) there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to an arrow of WAdm^{count} satisfying the property defined in Lemma 88.12.1,
(3) there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to an arrow of WAdm^{count} satisfying the property defined in Lemma 88.12.1, and
(4) the morphism $f_{red} : X_{red} \to Y_{red}$ is locally of finite type.

Proof. The equivalence of (1), (2), and (3) follows from Lemma 88.12.1 and an application of Formal Spaces, Lemma 87.21.3. Let Y_j and X_{ji} be as in (2). Then

- The families $\{Y_{j,red} \to Y_{red}\}$ and $\{X_{ji,red} \to X_{red}\}$ are étale coverings by affine schemes. This follows from the discussion in the proof of Formal Spaces, Lemma 87.12.1 or directly from Formal Spaces, Lemma 87.12.3.
- If $X_{ji} \to Y_j$ corresponds to the morphism $B_j \to A_{ji}$ of WAdm^{count}, then $X_{ji,red} \to Y_{j,red}$ corresponds to the ring map $B_j/\mathfrak{b}_j \to A_{ji}/\mathfrak{a}_{ji}$ where $\mathfrak{b}_j$ and $\mathfrak{a}_{ji}$ are the ideals of topologically nilpotent elements. This follows from Formal Spaces, Example 87.12.2. Hence $X_{ji,red} \to Y_{j,red}$ is locally of finite type if and only if $B_j \to A_{ji}$ satisfies the property defined in Lemma 88.12.1.

The equivalence of (2) and (4) follows from these remarks because being locally of finite type is a property of morphisms of algebraic spaces which is étale local on source and target, see discussion in Morphisms of Spaces, Section 67.23. ☐

88.13. Flat morphisms

0GC8 In this section we define flat morphisms of locally Noetherian formal algebraic spaces.

0GC9 Lemma 88.13.1. The property $P(\varphi)$ ="φ is flat" on arrows of WAdm^{Noeth} is a local property as defined in Formal Spaces, Remark 87.21.5.

Proof. Let us recall what the statement signifies. First, WAdm^{Noeth} is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is flat $\Rightarrow \varphi'$ is flat,
(2) if $B \to B'$ faithfully flat, then φ' is flat $\Rightarrow \varphi$ is flat, and

(3) if $A \to B_i$ is flat for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is flat.

We will use without further mention that completions of Noetherian rings are flat (Algebra, Lemma 10.97.2). Since of course $A \to A'$ and $B \to B'$ are flat, we see in particular that the horizontal arrows in the diagram are flat.

Proof of (1). If φ is flat, then the composition $A \to (A')^\wedge \to (B')^\wedge$ is flat. Hence $A' \to (B')^\wedge$ is flat by More on Flatness, Lemma 38.2.3. Hence we see that $(A')^\wedge \to (B')^\wedge$ is flat by applying More on Algebra, Lemma 15.27.5 with $R = A'$, with ideal $I(A')$, and with $M = (B')^\wedge = M^\wedge$.

Proof of (2). Assume φ' is flat and $B \to B'$ is faithfully flat. Then the composition $A \to (A')^\wedge \to (B')^\wedge$ is flat. Also we see that $B \to (B')^\wedge$ is faithfully flat by Formal Spaces, Lemma 87.19.14. Hence by Algebra, Lemma 10.39.9 we find that $\varphi : A \to B$ is flat.

Proof of (3). Omitted. □

0GCA Lemma 88.13.2. Denote P the property of arrows of WAdm^{Noeth} defined in Lemma 88.13.1. Denote Q the property defined in Lemma 88.12.1 viewed as a property of arrows of WAdm^{Noeth}. Denote R the property defined in Lemma 88.11.1 viewed as a property of arrows of WAdm^{Noeth}. Then

(1) P is stable under base change by Q (Formal Spaces, Remark 87.21.10), and
(2) $P + R$ is stable under base change (Formal Spaces, Remark 87.21.9).

Proof. The statement makes sense as each of the properties P, Q, and R is a local property of morphisms of WAdm^{Noeth}. Let $\varphi : B \to A$ and $\psi : B \to C$ be morphisms of WAdm^{Noeth}. If either $Q(\varphi)$ or $Q(\psi)$ then we see that $A\widehat{\otimes}_B C$ is Noetherian by Formal Spaces, Lemma 87.4.12. Since R implies Q (Lemma 88.12.4), we find that this holds in both cases (1) and (2). This is the first thing we have to check. It remains to show that $C \to A\widehat{\otimes}_B C$ is flat.

Proof of (1). Fix ideals of definition $I \subset A$ and $J \subset B$. By Lemma 88.12.5 the ring map $B \to C$ is topologically of finite type. Hence $B \to C/J^n$ is of finite type for all $n \geq 1$. Hence $A \otimes_B C/J^n$ is Noetherian as a ring (because it is of finite type over A and A is Noetherian). Thus the I-adic completion $A\widehat{\otimes}_B C/J^n$ of $A \otimes_B C/J^n$ is flat over C/J^n because $C/J^n \to A \otimes_B C/J^n$ is flat as a base change of $B \to A$ and because $A \otimes_B C/J^n \to A\widehat{\otimes}_B C/J^n$ is flat by Algebra, Lemma 10.97.2 Observe that $A\widehat{\otimes}_B C/J^n = (A\widehat{\otimes}_B C)/J^n(A\widehat{\otimes}_B C)$; details omitted. We conclude that $M = A\widehat{\otimes}_B C$ is a C-module which is complete with respect to the J-adic topology such that M/J^nM is flat over C/J^n for all $n \geq 1$. This implies that M is flat over C by More on Algebra, Lemma 15.27.4.

Proof of (2). In this case $B \to A$ is adic and hence we have just $A\widehat{\otimes}_B C = \lim A \otimes_B C/J^n$. The rings $A \otimes_B C/J^n$ are Noetherian by an application of Formal Spaces, Lemma 87.4.12 with C replaced by C/J^n. We conclude in the same manner as before. □

0GCB Lemma 88.13.3. Denote P the property of arrows of WAdm^{Noeth} defined in Lemma 88.13.1. Then P is stable under composition (Formal Spaces, Remark 87.21.14).

Proof. This is true because compositions of flat ring maps are flat. □

0GCC Definition 88.13.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is flat if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a flat map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on the source and target.

0GCD Lemma 88.13.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent

(1) f is flat,
(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a flat map in WAdm^{Noeth},
(3) there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a flat map in WAdm^{Noeth}, and
(4) there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a flat map in WAdm^{Noeth}.

Proof. The equivalence of (1) and (2) is Definition 88.13.4. The equivalence of (2), (3), and (4) follows from the fact that being flat is a local property of arrows of WAdm^{Noeth} by Lemma 88.13.1 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

0GCE Lemma 88.13.6. Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S.

(1) If f is flat and $g_{red} : Z_{red} \to Y_{red}$ is locally of finite type, then the base change $X \times_Y Z \to Z$ is flat.
(2) If f is flat and locally of finite type, then the base change $X \times_Y Z \to Z$ is flat and locally of finite type.

Proof. Part (1) follows from a combination of Formal Spaces, Remark 87.21.10, Lemma 88.13.2 part (1), Lemma 88.13.5, and Lemma 88.12.7.

Part (2) follows from a combination of Formal Spaces, Remark 87.21.9, Lemma 88.13.2 part (2), Lemma 88.13.5, and Lemma 88.11.5. □

0GCF Lemma 88.13.7. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are flat, then so is $g \circ f$.

Proof. Combine Formal Spaces, Remark 87.21.14 and Lemma 88.13.3. □

0GCG Lemma 88.13.8. Let S be a scheme. Let $f : X \to Y$ be a morphisms of locally Noetherian formal algebraic spaces over S. If f is representable by algebraic spaces and flat in the sense of Bootstrap, Definition 80.4.1, then f is flat in the sense of Definition 88.13.4.

Proof. This is a sanity check whose proof should be trivial but isn't quite. We urge the reader to skip the proof. Assume f is representable by algebraic spaces and flat in the sense of Bootstrap, Definition 80.4.1. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale. Then the morphism $U \to V$ corresponds to a taut map $B \to A$ of WAdm^{Noeth} by Formal Spaces, Lemma 87.22.2. Observe that this means $B \to A$ is adic (Formal Spaces, Lemma 87.23.1) and in particular for any ideal of definition $J \subset B$ the topology on A is the J-adic topology and the diagrams

$$\begin{tikzcd} \mathrm{Spec}(A/J^nA) \arrow[r] \arrow[d] & \mathrm{Spec}(B/J^n) \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

are cartesian.

Let $T \to V$ is a morphism where T is a scheme. Then

$$\begin{aligned} X \times_Y T \to T \text{ is flat} &\Rightarrow U \times_Y T \to T \text{ is flat} \\ &\Rightarrow U \times_V V \times_Y T \to T \text{ is flat} \\ &\Rightarrow U \times_V V \times_Y T \to V \times_Y T \text{ is flat} \\ &\Rightarrow U \times_V T \to T \text{ is flat} \end{aligned}$$

The first statement is the assumption on f. The first implication because $U \to X$ is étale and hence flat and compositions of flat morphisms of algebraic spaces are flat. The second impliciation because $U \times_Y T = U \times_V V \times_Y T$. The third implication by More on Flatness, Lemma 38.2.3. The fourth implication because we can pullback by the morphism $T \to V \times_Y T$. We conclude that $U \to V$ is flat in the sense of Bootstrap, Definition 80.4.1. In terms of the continuous ring map $B \to A$ this means the ring maps $B/J^n \to A/J^nA$ are flat (see diagram above).

Finally, we can conclude that $B \to A$ is flat for example by More on Algebra, Lemma 15.27.4. □

88.14. Rig-closed points

0GG7 We develop just enough theory to be able to use this for testing rig-flatness in a later section. The reader can find more theory in [BL93] who discuss (among other things) the case of locally Noetherian formal schemes.

0GG8 **Lemma 88.14.1.** Let A be a Noetherian adic topological ring. Let $\mathfrak{q} \subset A$ be a prime ideal. The following are equivalent

(1) for some ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$ and $\mathfrak{q}$ is maximal with respect to this property,
(2) for some ideal of definition $I \subset A$ the prime $\mathfrak{q}$ defines a closed point of $\mathrm{Spec}(A) \setminus V(I)$,
(3) for any ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$ and $\mathfrak{q}$ is maximal with respect to this property,
(4) for any ideal of definition $I \subset A$ the prime $\mathfrak{q}$ defines a closed point of $\mathrm{Spec}(A) \setminus V(I)$,
(5) $\dim(A/\mathfrak{q}) = 1$ and for some ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$,
(6) $\dim(A/\mathfrak{q}) = 1$ and for any ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$,
(7) $\dim(A/\mathfrak{q}) = 1$ and the induced topology on $A/\mathfrak{q}$ is nontrivial,
(8) $A/\mathfrak{q}$ is a 1-dimensional Noetherian complete local domain whose maximal ideal is the radical of the image of any ideal of definition of A, and
(9) add more here.

Proof. It is clear that (1) and (2) are equivalent and for the same reason that (3) and (4) are equivalent. Since $V(I)$ is independent of the choice of the ideal of definition I of A, we see that (2) and (4) are equivalent.

Assume the equivalent conditions (1) – (4) hold. If $\dim(A/\mathfrak{q}) > 1$ we can choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m} \subset A$ such that $\dim((A/\mathfrak{q})_\mathfrak{m}) > 1$. Then $\mathrm{Spec}((A/\mathfrak{q})_\mathfrak{m}) - V(I(A/\mathfrak{q})_\mathfrak{m})$ would be infinite by Algebra, Lemma 10.61.1. This contradicts the fact that $\mathfrak{q}$ is closed in $\mathrm{Spec}(A) \setminus V(I)$. Hence we see that (6) holds. Trivially (6) implies (5).

Conversely, assume (5) holds. Let $I \subset A$ be an ideal of definition. Since $A/\mathfrak{q}$ is complete with respect to $I(A/\mathfrak{q})$ (for example by Algebra, Lemma 10.97.1) we see that all closed points of $\mathrm{Spec}(A/\mathfrak{q})$ are contained in $V(IA/\mathfrak{q})$ by Algebra, Lemma 10.96.6. Since $\dim(A/\mathfrak{q}) = 1$ and since $I \not\subset \mathfrak{q}$ we conclude two things: (a) $V(IA/\mathfrak{q})$ must contain all points distinct from the generic point of $\mathrm{Spec}(A/\mathfrak{q})$, and (b) $V(IA/\mathfrak{q})$ must be a (finite) discrete set. From (a) we see that $\mathfrak{q}$ is a closed point of $\mathrm{Spec}(A) \setminus V(I)$ and we conclude that (2) holds.

Continuing to assume (5) we see that the finite discrete space $V(IA/\mathfrak{q})$ must be a singleton by More on Algebra, Lemma 15.11.16 for example (and the fact that complete pairs are henselian pairs, see More on Algebra, Lemma 15.11.4). Hence we see that (8) is true. Conversely, it is clear that (8) implies (5).

At this point we know that (1) – (6) and (8) are equivalent. We omit the verification that these are also equivalent to (7). $\square$

In order to comfortably talk about such primes we introduce the following nonstandard notation.

0GG9 Definition 88.14.2. Let A be a Noetherian adic topological ring. Let $\mathfrak{q} \subset A$ be a prime ideal. We say $\mathfrak{q}$ is rig-closed if the equivalent conditions of Lemma 88.14.1 are satisfied.

We will need a few lemmas which essentially tell us there are plenty of rig-closed primes even in a relative settting.

0GGA Lemma 88.14.3. Let $\varphi : A \to B$ in WAdm^{Noeth}. Denote $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ the ideals of topologically nilpotent elements. Assume $A/\mathfrak{a} \to B/\mathfrak{b}$ is of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. The residue field κ of the local ring $B/\mathfrak{q}$ is a finite type $A/\mathfrak{a}$-algebra.

Proof. Let $\mathfrak{q} \subset \mathfrak{m} \subset B$ be the unique maximal ideal containing $\mathfrak{q}$. Then $\mathfrak{b} \subset \mathfrak{m}$. Hence $A/\mathfrak{a} \to B/\mathfrak{b} \to B/\mathfrak{m} = \kappa$ is of finite type. $\square$

0GGB Lemma 88.14.4. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. Let $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subset A$. Let $\mathfrak{a} \subset A$ be the ideal of topologically nilpotent elements. The following are equivalent

(1) the residue field κ of $B/\mathfrak{q}$ is finite over $A/\mathfrak{a}$,
(2) $\mathfrak{p} \subset A$ is rig-closed,
(3) $A/\mathfrak{p} \subset B/\mathfrak{q}$ is a finite extension of rings.

Proof. Assume (1). Recall that $B/\mathfrak{q}$ is a Noetherian local ring of dimension 1 whose topology is the adic topology coming from the maximal ideal. Since φ is adic, we see that $A \to B/\mathfrak{q}$ is adic. Hence $\varphi(\mathfrak{a})$ is a nonzero ideal in $B/\mathfrak{q}$. Hence $B/\mathfrak{q} + \varphi(\mathfrak{a})$ has finite length. Hence $B/\mathfrak{q} + \varphi(\mathfrak{a})$ is finite as an $A/\mathfrak{a}$-module by our assumption. Thus $B/\mathfrak{q}$ is finite over A by Algebra, Lemma 10.96.12. Thus (3) holds.

Assume (3). Then $\text{Spec}(B/\mathfrak{q}) \to \text{Spec}(A/\mathfrak{p})$ is surjective by Algebra, Lemma 10.36.17. This implies (2).

Assume (2). Denote κ' the residue field of $A/\mathfrak{p}$. By Lemma 88.14.3 (and Lemma 88.12.4) the extension κ/κ' is finitely generated as an algebra. By the Hilbert Nullstellensatz (Algebra, Lemma 10.34.2) we see that κ/κ' is a finite extension. Hence we see that (1) holds. $\square$

0GGC Lemma 88.14.5. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. If A/I is Jacobson for some ideal of definition $I \subset A$, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subset A$ is rig-closed.

Proof. By Lemma 88.14.3 (combined with Lemma 88.12.4) the residue field κ of $B/\mathfrak{q}$ is of finite type over $A/\mathfrak{a}$. Since $A/\mathfrak{a}$ is Jacobson, we see that κ is finite over $A/\mathfrak{a}$ by Algebra, Lemma 10.35.18. We conclude by Lemma 88.14.4. $\square$

0GGD Lemma 88.14.6. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} which is adic and topologically of finite type. Let $\mathfrak{p} \subset A$ be rig-closed. Let $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ be the ideals of topologically nilpotent elements. If φ is flat, then the following are equivalent

(1) the maximal ideal of $A/\mathfrak{p}$ is in the image of $\text{Spec}(B/\mathfrak{b}) \to \text{Spec}(A/\mathfrak{a})$,
(2) there exists a rig-closed prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.

and if so then φ, $\mathfrak{p}$, and $\mathfrak{q}$ satisfy the conclusions of Lemma 88.14.4.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). To prove the existence of $\mathfrak q$ we may replace A by $A/\mathfrak p$ and B by $B/\mathfrak pB$ (some details omitted). Thus we may assume $(A, \mathfrak m, \kappa)$ is a local complete 1-dimensional Noetherian ring, $\mathfrak m = \mathfrak a$, and $\mathfrak p = (0)$. Condition (1) just says that $B_0 = B \otimes_A \kappa = B/\mathfrak mB = B/\mathfrak aB$ is nonzero. Note that B_0 is of finite type over κ. Hence we can use induction on $\dim(B_0)$. If $\dim(B_0) = 0$, then any minimal prime $\mathfrak q \subset B$ will do (flatness of $A \to B$ insures that $\mathfrak q$ will lie over $\mathfrak p = (0)$). If $\dim(B_0) > 0$ then we can find an element $b \in B$ which maps to an element $b_0 \in B_0$ which is a nonzerodivisor and a nonunit, see Algebra, Lemma 10.63.20. By Algebra, Lemma 10.99.2 the ring $B' = B/bB$ is flat over A. Since $B'_0 = B' \otimes_A \kappa = B_0/(b_0)$ is not zero, we may apply the induction hypothesis to B' and conclude. The final statement of the lemma is clear from Lemma 88.14.4. □

We introduce some notation.

0GGE Definition 88.14.7. Let A be an adic topological ring which has a finitely generated ideal of definition. Let $f \in A$. The completed principal localization $A_{\{f\}}$ of A is the completion of $A_f = A[1/f]$ of the principal localization of A at f with respect to any ideal of definition of A.

To be sure, if f is topologically nilpotent, then $A_{\{f\}}$ is the zero ring.

0GGF Lemma 88.14.8. Let A be an adic Noetherian topological ring. Let $\mathfrak p \subset A$ be a prime ideal. Let $f \in A$ be an element mapping to a unit in $A/\mathfrak p$. Then

$$\mathfrak pA_{\{f\}} = \mathfrak p(A_f)^\wedge = \mathfrak p \otimes_A (A_f)^\wedge = (\mathfrak p_f)^\wedge$$

is a prime ideal with quotient

$$A/\mathfrak p = (A/\mathfrak p) \otimes_A (A_f)^\wedge = (A_f)^\wedge/\mathfrak p(A_f)^\wedge = A_{\{f\}}/\mathfrak pA_{\{f\}}$$

Proof. Since A_f is Noetherian the ring map $A \to A_f \to (A_f)^\wedge$ is flat. For any finite A-module M we see that $M \otimes_A (A_f)^\wedge$ is the completion of M_f. If f is a unit on M, then $M_f = M$ is already complete. See discussion in Algebra, Section 10.97. From these observations the results follow easily. □

0GGG Lemma 88.14.9. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} which is adic and topologically of finite type. Let $\mathfrak q \subset B$ be rig-closed. There exists an $f \in A$ which maps to a unit in $B/\mathfrak q$ such that we obtain a diagram

$$\begin{tikzcd} B \arrow[r] & B_{\{f\}} \\ A \arrow[u, "\varphi"] \arrow[r] & A_{\{f\}} \arrow[u, "\varphi_{\{f\}}"'] \end{tikzcd} \quad \text{with primes} \quad \begin{tikzcd} \mathfrak q \arrow[r, dash] \arrow[d, dash] & \mathfrak q' \arrow[r, equal] \arrow[d, dash] & \mathfrak qB_{\{f\}} \\ \mathfrak p \arrow[r, dash] & \mathfrak p' \end{tikzcd}$$

such that $\mathfrak p'$ is rig-closed, i.e., the map $A_{\{f\}} \to B_{\{f\}}$ and the prime ideals $\mathfrak q'$ and $\mathfrak p'$ satisfy the equivalent conditions of Lemma 88.14.4.

Proof. Please see Lemma 88.14.8 for the description of $\mathfrak q'$. The only assertion the lemma makes is that for a suitable choice of f the prime ideal $\mathfrak p'$ has the property $\dim((A_f)^\wedge/\mathfrak p') = 1$. By Lemma 88.14.4 this in turn just means that the residue field κ of $B/\mathfrak q = (B_f)^\wedge/\mathfrak q'$ is finite over $(A_f)^\wedge/\mathfrak a' = (A/\mathfrak a)_f$. By Lemma 88.14.3 we know that $A/\mathfrak a \to \kappa$ is a finite type algebra homomorphism. By the Hilbert Nullstellensatz in the form of Algebra, Lemma 10.34.2 we can find an $f \in A$ which maps to a unit in κ such that κ is finite over A_f. This finishes the proof. □

0GGH Lemma 88.14.10. Let A be a Noetherian adic topological ring. Denote $A\{x_1, \ldots, x_n\}$ the restricted power series over A. Let $\mathfrak q \subset A\{x_1, \ldots, x_n\}$ be a prime ideal. Set $\mathfrak q' = A[x_1, \ldots, x_n] \cap \mathfrak q$ and $\mathfrak p = A \cap \mathfrak q$. If $\mathfrak q$ and $\mathfrak p$ are rig-closed, then the map

$$A[x_1, \ldots, x_n]_{\mathfrak q'} \to A\{x_1, \ldots, x_n\}_{\mathfrak q}$$

defines an isomorphism on completions with respect to their maximal ideals.

Proof. By Lemma 88.14.4 the ring map $A/\mathfrak p \to A\{x_1, \ldots, x_n\}/\mathfrak q$ is finite. For every $m \geq 1$ the module $\mathfrak q^m/\mathfrak q^{m+1}$ is finite over A as it is a finite $A\{x_1, \ldots, x_n\}/\mathfrak q$-module. Hence $A\{x_1, \ldots x_n\}/\mathfrak q^m$ is a finite A-module. Hence $A[x_1, \ldots, x_n] \to A\{x_1, \ldots, x_n\}/\mathfrak q^m$ is surjective (as the image is dense and an A-submodule). It follows in a straightforward manner that $A[x_1, \ldots, x_n]/(\mathfrak q')^m \to A\{x_1, \ldots, x_n\}/\mathfrak q^m$ is an isomorphism for all m. From this the lemma easily follows. Hint: Pick a topologically nilpotent $g \in A$ which is not contained in $\mathfrak p$. Then the map of completions is the map

$$\lim_m \left(A[x_1, \ldots, x_n]/(\mathfrak q')^m\right)_g \longrightarrow \left(A\{x_1, \ldots, x_n\}/\mathfrak q^m\right)_g$$

Some details omitted. □

0GGI Lemma 88.14.11. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth}. Assume φ is adic, topologically of finite type, flat, and $A/I \to B/IB$ is étale for some (resp. any) ideal of definition $I \subset A$. Let $\mathfrak q \subset B$ be rig-closed such that $\mathfrak p = A \cap \mathfrak q$ is rig-closed as well. Then $\mathfrak p B_{\mathfrak q} = \mathfrak q B_{\mathfrak q}$.

Proof. Let κ be the residue field of the 1-dimensional complete local ring $A/\mathfrak p$. Since $A/I \to B/IB$ is étale, we see that $B \otimes_A \kappa$ is a finite product of finite separable extensions of κ, see Algebra, Lemma 10.143.4. One of these is the residue field of $B/\mathfrak q$. By Algebra, Lemma 10.96.12 we see that $B/\mathfrak p B$ is a finite $A/\mathfrak p$-algebra. It is also flat. Combining the above we see that $A/\mathfrak p \to B/\mathfrak p B$ is finite étale, see Algebra, Lemma 10.143.7. Hence $B/\mathfrak p B$ is reduced, which implies the statement of the lemma (details omitted). □

0GGJ Lemma 88.14.12. Let A be an adic Noetherian topological ring. Let $\mathfrak p \subset A$ be a rig-closed prime. For any $n \geq 1$ the ring map

$$A/\mathfrak p \longrightarrow A\{x_1, \ldots, x_n\} \otimes_A A/\mathfrak p = A/\mathfrak p\{x_1, \ldots, x_n\}$$

is regular. In particular, the algebra $A\{x_1, \ldots, x_n\} \otimes_A \kappa(\mathfrak p)$ is geometrically regular over $\kappa(\mathfrak p)$.

Proof. We will use some fact on regular ring maps the reader can find in More on Algebra, Section 15.41. Since $A/\mathfrak p$ is a complete local Noetherian ring it is excellent (More on Algebra, Proposition 15.52.3). Hence $A/\mathfrak p[x_1, \ldots, x_n]$ is excellent (by the same reference). Hence $A/\mathfrak p[x_1, \ldots, x_n] \to A/\mathfrak p\{x_1, \ldots, x_n\}$ is a regular ring homomorphism by More on Algebra, Lemma 15.50.14. Of course $A/\mathfrak p \to A/\mathfrak p[x_1, \ldots, x_n]$ is smooth and hence regular. Since the composition of regular ring maps is regular the proof is complete. □

88.15. Rig-flat homomorphisms

0GGK In this section we define rig-flat homomorphisms of adic Noetherian topological rings.

0GGL Lemma 88.15.1. Let $\varphi : A \to B$ be a morphism in WAdm^{adic*} (Formal Spaces, Section 87.21). Assume φ is adic. The following are equivalent:

(1) B_f is flat over A for all topologically nilpotent $f \in A$,
(2) B_g is flat over A for all topologically nilpotent $g \in B$,
(3) $B_{\mathfrak q}$ is flat over A for all primes $\mathfrak q \subset B$ which do not contain an ideal of definition,
(4) $B_{\mathfrak q}$ is flat over A for every rig-closed prime $\mathfrak q \subset B$, and
(5) add more here.

Proof. Follows from the definitions and Algebra, Lemma 10.39.18. $\square$

0GGM Definition 88.15.2. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of WAdm^{Noeth}. We say φ is naively rig-flat if φ is adic, topologically of finite type, and satisfies the equivalent conditions of Lemma 88.15.1.

The example below shows that this notion does not "localize".

0GGN Example 88.15.3. By Examples, Lemma 110.17.1 there exists a local Noetherian 2-dimensional domain $(A, \mathfrak m)$ complete with respect to a principal ideal $I = (a)$ and an element $f \in \mathfrak m$, $f \not\in I$ with the following property: the ring $A_{\{f\}}[1/a]$ is nonreduced. Here $A_{\{f\}}$ is the I-adic completion $(A_f)^\wedge$ of the principal localization A_f. To be sure the ring $A_{\{f\}}[1/a]$ is nonzero. Let $B = A_{\{f\}}/\text{nil}(A_{\{f\}})$ be the quotient by its nilradical. Observe that $A \to B$ is adic and topologically of finite type. In fact, B is a quotient of $A\{x\} = A[x]^\wedge$ by the map sending x to the image of $1/f$ in B. Every prime $\mathfrak q$ of B not containing a must lie over $(0) \subset A$[2]. Hence $B_{\mathfrak q}$ is flat over A as it is a module over the fraction field of A. Thus $A \to B$ is naively rig-flat. On the other hand, the map

$$A_{\{f\}} \longrightarrow B_{\{f\}} = (B_f)^\wedge = B = A_{\{f\}}/\text{nil}(A_{\{f\}})$$

is not flat after inverting a because we get the nontrivial surjection $A_{\{f\}}[1/a] \to A_{\{f\}}[1/a]/\text{nil}(A_{\{f\}}[1/a])$. Hence $A_{\{f\}} \to B^\wedge_{\{f\}}$ is not naively rig-flat!

It turns out that it is easy to work around this problem by using the following definition.

0GGP Definition 88.15.4. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of WAdm^{Noeth}. We say φ is rig-flat if φ is adic, topologically of finite type, and for all $f \in A$ the induced map

$$A_{\{f\}} \longrightarrow B_{\{f\}}$$

is naively rig-flat (Definition 88.15.2).

Setting $f = 1$ in the definition above we see that rig-flatness implies naive rig-flatness. The example shows the converse is false. However, in many situations we don't need to worry about the difference between rig-flatness and its naive version as the next lemma shows.

[2]Namely, we can find $\mathfrak q \subset \mathfrak q' \subset B$ with $a \in \mathfrak q'$ because B is a-adically complete. Then $\mathfrak p' = A \cap \mathfrak q'$ contains a but not f hence is a height 1 prime. Then $\mathfrak p = A \cap \mathfrak q$ must be strictly contained in $\mathfrak p'$ as $a \not\in \mathfrak p$. Since $\dim(A) = 2$ we see that $\mathfrak p = (0)$.

0GGQ Lemma 88.15.5. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth}. If A/I is Jacobson for some (equivalently any) ideal of definition $I \subset A$ and φ is naively rig-flat, then φ is rig-flat.

Proof. Assume φ is naively rig-flat. We first state some obvious consequences of the assumptions. Namely, let $f \in A$. Then $A, B, A_{\{f\}}, B_{\{f\}}$ are Noetherian adic topological rings. The maps $A \to A_{\{f\}} \to B_{\{f\}}$ and $A \to B \to B_{\{f\}}$ are adic and topologically of finite type. The ring maps $A \to A_{\{f\}}$ and $B \to B_{\{f\}}$ are flat as compositions of $A \to A_f$ and $B \to B_f$ and the completion maps which are flat by Algebra, Lemma 10.97.2. The quotients of each of the rings $A, B, A_{\{f\}}, B_{\{f\}}$ by I is of finite type over A/I and hence Jacobson too (Algebra, Proposition 10.35.19).

Let $\mathfrak q' \subset B_{\{f\}}$ be rig-closed. It suffices to prove that $(B_{\{f\}})_{\mathfrak q'}$ is flat over $A_{\{f\}}$, see Lemma 88.15.1. By Lemma 88.14.5 the primes $\mathfrak q \subset B$ and $\mathfrak p' \subset A_{\{f\}}$ and $\mathfrak p \subset A$ lying under $\mathfrak q'$ are rig-closed. We are going to apply Algebra, Lemma 10.100.2 to the diagram

$$\begin{tikzcd} B_{\mathfrak q} \arrow[r] & (B_{\{f\}})_{\mathfrak q'} \\ A_{\mathfrak p} \arrow[r] \arrow[u] & (A_{\{f\}})_{\mathfrak p'} \arrow[u] \end{tikzcd}$$

with $M = B_{\mathfrak q}$. The only assumption that hasn't been checked yet is the fact that $\mathfrak p$ generates the maximal ideal of $(A_{\{f\}})_{\mathfrak p'}$. This follows from Lemma 88.14.8; here we use that $\mathfrak p$ and $\mathfrak p'$ are rig-closed to see that f maps to a unit of $A/\mathfrak p$ (this is the only step in the proof that fails without the Jacobson assumption). Namely, this tells us that $A/\mathfrak p \to A_{\{f\}}/\mathfrak p'$ is a finite inclusion of local rings (Lemma 88.14.4) and f maps to a unit in the second one. ☐

0GGR Lemma 88.15.6. Let $\varphi : A \to B$ and $A \to C$ be arrows of WAdm^{Noeth}. Assume φ is rig-flat and $A \to C$ adic and topologically of finite type. Then $C \to B\widehat{\otimes}_A C$ is rig-flat.

Proof. Assume φ is rig-flat. Let $f \in C$ be an element. We have to show that $C_{\{f\}} \to B\widehat{\otimes}_A C_{\{f\}}$ is naively rig-flat. Since we can replace C by $C_{\{f\}}$ we it suffices to show that $C \to B\widehat{\otimes}_A C$ is naively rig-flat.

If $A \to C$ is surjective or more generally if C is finite as an A-module, then $B \otimes_A C = B\widehat{\otimes}_A C$ as a finite module over a complete Noetherian ring is complete, see Algebra, Lemma 10.97.1. By the usual base change for flatness (Algebra, Lemma 10.39.7) we see that naive rig-flatness of φ implies naive rig-flatness for $C \to B \times_A C$ in this case.

In the general case, we can factor $A \to C$ as $A \to A\{x_1, \ldots, x_n\} \to C$ where $A\{x_1, \ldots, x_n\}$ is the restricted power series ring and $A\{x_1, \ldots, x_n\} \to C$ is surjective. Thus it suffices to show $C \to B\widehat{\otimes}_A B$ is naively rig-flat in case $C = A\{x_1, \ldots, x_n\}$. Since $A\{x_1, \ldots, x_n\} = A\{x_1, \ldots, x_{n-1}\}\{x_n\}$ by induction on n we reduce to the case discussed in the next paragraph.

Here $C = A\{x\}$. Note that $B\widehat{\otimes}_A C = B\{x\}$. We have to show that $A\{x\} \to B\{x\}$ is naively rig-flat. Let $\mathfrak q \subset B\{x\}$ be a rig-closed prime ideal. We have to show that $B\{x\}_{\mathfrak q}$ is flat over $A\{x\}$. Set $\mathfrak p = A \cap \mathfrak q$. By Lemma 88.14.9 we can find an $f \in A$ such that f maps to a unit in $B\{x\}/\mathfrak q$ and such that the prime ideal $\mathfrak p'$ in $A_{\{f\}}$

induced is rig-closed. Below we will use that $A_{\{f\}}\{x\} = A\{x\}_{\{f\}}$ and similarly for B; details omitted. Consider the diagram

$$\begin{tikzcd} (B\{x\})_{\mathfrak q} \arrow[r] & (B_{\{f\}}\{x\})_{\mathfrak q'} \\ A\{x\} \arrow[r] \arrow[u] & A_{\{f\}}\{x\} \arrow[u] \end{tikzcd}$$

We want to show that the left vertical arrow is flat. The top horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $B_{\{f\}}\{x\}$ is the completion of a localization of the Noetherian ring $B\{x\}$. Similarly the bottom horizontal arrow is flat. Hence it suffices to prove that the right vertical arrow is flat. This reduces us to the case discussed in the next paragraph.

Here $C = A\{x\}$, we have a rig-closed prime ideal $\mathfrak q \subset B\{x\}$ such that $\mathfrak p = A \cap \mathfrak q$ is rig-closed as well. This implies, via Lemma 88.14.4, that the intermediate primes $B \cap \mathfrak q$ and $A\{x\} \cap \mathfrak q$ are rig-closed as well. Consider the diagram

$$\begin{tikzcd} (B[x])_{B[x]\cap \mathfrak q} \arrow[r] & (B\{x\})_{\mathfrak q} \\ (A[x])_{A[x]\cap \mathfrak q} \arrow[r] \arrow[u] & (A\{x\})_{A\{x\}\cap \mathfrak q} \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian local rings. By Lemma 88.14.10 the horizontal arrows define isomorphisms on completions. We already know that the left vertical arrow is flat (as $A \to B$ is naively rig-flat and hence $A[x] \to B[x]$ is flat away from the closed locus defined by an ideal of definition). Hence we finally conclude by More on Algebra, Lemma 15.43.8. □

0GGS Lemma 88.15.7. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[r] \arrow[u, "\varphi"] & A' \arrow[u, "\varphi'"'] \end{tikzcd}$$

in WAdm^{Noeth} with all arrows adic and topologically of finite type. Assume $A \to A'$ and $B \to B'$ are flat. Let $I \subset A$ be an ideal of definition. If φ is rig-flat and $A/I \to A'/IA'$ is étale, then φ' is rig-flat.

Proof. Given $f \in A'$ the assumptions of the lemma remain true for the digram

$$\begin{tikzcd} B \arrow[r] & (B')_{\{f\}} \\ A \arrow[r] \arrow[u, "\varphi"] & (A')_{\{f\}} \arrow[u] \end{tikzcd}$$

Hence it suffices to prove that φ' is naively rig-flat.

Take a rig-closed prime ideal $\mathfrak q' \subset B'$. We have to show that $(B')_{\mathfrak q'}$ is flat over A'. We can choose an $f \in A$ which maps to a unit of $B'/\mathfrak q'$ such that the induced prime ideal $\mathfrak p''$ of $A_{\{f\}}$ is rig-closed, see Lemma 88.14.9. To be precise, here $\mathfrak q'' = \mathfrak q' B'_{\{f\}}$ and $\mathfrak p'' = A_{\{f\}} \cap \mathfrak q''$. Consider the diagram

$$\begin{tikzcd} B'_{\mathfrak q'} \arrow[r] & (B'_{\{f\}})_{\mathfrak q''} \\ A \arrow[u] \arrow[r] & A_{\{f\}} \arrow[u] \end{tikzcd}$$

We want to show that the left vertical arrow is flat. The top horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $B'_{\{f\}}$ is the completion of a localization of the Noetherian ring B'_f. Similarly the bottom horizontal arrow is flat. Hence it suffices to prove that the right vertical arrow is flat. Finally, all the assumptions of the lemma remain true for the diagram

$$\begin{tikzcd} B_{\{f\}} \arrow[r] & B'_{\{f\}} \\ A_{\{f\}} \arrow[u] \arrow[r] & A'_{\{f\}} \arrow[u] \end{tikzcd}$$

This reduces us to the case discussed in the next paragraph.

Take a rig-closed prime ideal $\mathfrak q' \subset B'$ and assume $\mathfrak p = A \cap \mathfrak q'$ is rig-closed as well. This implies also the primes $\mathfrak q = B \cap \mathfrak q'$ and $\mathfrak p' = A' \cap \mathfrak q'$ are rig-closed, see Lemma 88.14.4. We are going to apply Algebra, Lemma 10.100.2 to the diagram

$$\begin{tikzcd} B_{\mathfrak q} \arrow[r] & B'_{\mathfrak q'} \\ A_{\mathfrak p} \arrow[u] \arrow[r] & A'_{\mathfrak p'} \arrow[u] \end{tikzcd}$$

with $M = B_{\mathfrak q}$. The only assumption that hasn't been checked yet is the fact that $\mathfrak p$ generates the maximal ideal of $A'_{\mathfrak p'}$. This follows from Lemma 88.14.11. $\square$

0GGT Lemma 88.15.8. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u, "\varphi"] \arrow[r] & A' \arrow[u, "\varphi'"'] \end{tikzcd}$$

in WAdm^{Noeth} with all arrows adic and topologically of finite type. Assume $A \to A'$ flat and $B \to B'$ faithfully flat. If φ' is rig-flat, then φ is rig-flat.

Proof. Given $f \in A$ the assumptions of the lemma remain true for the digram

$$\begin{tikzcd} B_{\{f\}} \arrow[r] & (B')_{\{f\}} \\ A_{\{f\}} \arrow[u, "\varphi"] \arrow[r] & (A')_{\{f\}} \arrow[u] \end{tikzcd}$$

(To check the condition on faithful flatness: faithful flatness of $B \to B'$ is equivalent to $B \to B'$ being flat and $\mathrm{Spec}(B'/IB') \to \mathrm{Spec}(B/IB)$ being surjective for some ideal of definition $I \subset A$.) Hence it suffices to prove that φ is naively rig-flat. However, we know that φ' is naively rig-flat and that $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is surjective. From this the result follows immediately. $\square$

Finally, we can show that rig-flatness is a local property.

0GGU Lemma 88.15.9. The property $P(\varphi)$ ="φ is rig-flat" on arrows of WAdm^{Noeth} is a local property as defined in Formal Spaces, Remark 87.21.4.

Proof. Let us recall what the statement signifies. First, WAdm^{Noeth} is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is rig-flat $\Rightarrow \varphi'$ is rig-flat,
(2) if $B \to B'$ faithfully flat, then φ' is rig-flat $\Rightarrow \varphi$ is rig-flat, and
(3) if $A \to B_i$ is rig-flat for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-flat.

Being adic and topologically of finite type satisfies conditions (1), (2), and (3), see Lemma 88.11.1. Thus in verifying (1), (2), and (3) for the property "rig-flat" we may already assume our ring maps are all adic and topologically of finite type. Then (1) and (2) follow from Lemmas 88.15.7 and 88.15.8. We omit the trivial proof of (3). $\square$

0GGV Lemma 88.15.10. The property $P(\varphi)$ ="φ is rig-flat" on arrows of WAdm^{Noeth} is stable under composition as defined in Formal Spaces, Remark 87.21.14.

Proof. The statement makes sense by Lemma 88.15.9. To see that it is true assume we have rig-flat morphisms $A \to B$ and $B \to C$ in WAdm^{Noeth}. Then $A \to C$ is adic and topologically of finite type by Lemma 88.11.4. To finish the proof we have to show that for all $f \in A$ the map $A_{\{f\}} \to C_{\{f\}}$ is naively rig-flat. Since $A_{\{f\}} \to B_{\{f\}}$ and $B_{\{f\}} \to C_{\{f\}}$ are naively rig-flat, it suffices to show that compositions of naively rig-flat maps are naively rig-flat. This is a consequence of Algebra, Lemma 10.39.4. $\square$

88.16. Rig-flat morphisms

0GGW In this section we use the work done in Section 88.15 to define rig-flat morphisms of locally Noetherian algebraic spaces.

0GGX Definition 88.16.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is rig-flat if for every

commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-flat map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

0GGY Lemma 88.16.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent

(1) f is rig-flat,
(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-flat map in WAdm^{Noeth},
(3) there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-flat map in WAdm^{Noeth}, and
(4) there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-flat map in WAdm^{Noeth}.

Proof. The equivalence of (1) and (2) is Definition 88.16.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-flat is a local property of arrows of WAdm^{Noeth} by Lemma 88.15.9 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

0GGZ Lemma 88.16.3. Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-flat and g is locally of finite type, then the base change $X \times_Y Z \to Z$ is rig-flat.

Proof. By Formal Spaces, Remark 87.21.10 and the discussion in Formal Spaces, Section 87.23, this follows from Lemma 88.15.6. □

0GH0 Lemma 88.16.4. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-flat, then so is $g \circ f$.

Proof. By Formal Spaces, Remark 87.21.14 this follows from Lemma 88.15.10. □

88.17. Rig-smooth homomorphisms

0GCH In this section we prove some properties of rig-smooth homomorphisms of adic Noetherian topological rings which are needed to introduce rig-smooth morpisms of locally Noetherian formal algebraic spaces.

0GCI **Lemma 88.17.1.** Let $A \to B$ be a morphism in WAdm^{Noeth} (Formal Spaces, Section 87.21). The following are equivalent:

(a) $A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and there exists an ideal of definition $I \subset B$ such that B is rig-smooth over (A, I), and
(b) $A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and for all ideals of definition $I \subset A$ the algebra B is rig-smooth over (A, I).

Proof. Let I and I' be ideals of definitions of A. Then there exists an integer $c \geq 0$ such that $I^c \subset I'$ and $(I')^c \subset I$. Hence B is rig-smooth over (A, I) if and only if B is rig-smooth over (A, I'). This follows from Definition 88.4.1, the inclusions $I^c \subset I'$ and $(I')^c \subset I$, and the fact that the naive cotangent complex $NL^\wedge_{B/A}$ is independent of the choice of ideal of definition of A by Remark 88.11.2. □

0GCJ **Definition 88.17.2.** Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of WAdm^{Noeth}. We say φ is *rig-smooth* if the equivalent conditions of Lemma 88.17.1 hold.

This defines a local property.

0GCK **Lemma 88.17.3.** The property $P(\varphi)$ ="φ is rig-smooth" on arrows of WAdm^{Noeth} is a local property as defined in Formal Spaces, Remark 87.21.5.

Proof. Let us recall what the statement signifies. First, WAdm^{Noeth} is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is rig-smooth $\Rightarrow \varphi'$ is rig-smooth,
(2) if $B \to B'$ faithfully flat, then φ' is rig-smooth $\Rightarrow \varphi$ is rig-smooth, and
(3) if $A \to B_i$ is rig-smooth for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-smooth.

The equivalent conditions of Lemma 88.11.1 satisfy conditions (1), (2), and (3). Thus in verifying (1), (2), and (3) for the property "rig-smooth" we may already assume our ring maps satisfy the equivalent conditions of Lemma 88.11.1 in each case.

Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology and B, $(A')^\wedge$, and $(B')^\wedge$ are in the category (88.2.0.2) for (A, I). Since $A \to A'$ and $B \to B'$ are étale the complexes $NL_{A'/A}$ and $NL_{B'/B}$ are zero and hence $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero by Lemma 88.3.2. Applying Lemma 88.3.5 to $A \to (A')^\wedge \to (B')^\wedge$ we get isomorphisms

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

Thus $NL^\wedge_{(B')^\wedge/A} \to NL_{(B')^\wedge/(A')^\wedge}$ is a quasi-isomorphism. The ring maps $B/I^nB \to B'/I^nB'$ are étale and hence are local complete intersections (Algebra, Lemma 10.143.2). Hence we may apply Lemmas 88.3.5 and 88.3.6 to $A \to B \to (B')^\wedge$ and we get isomorphisms

$$H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

We conclude that $NL^\wedge_{B/A} \otimes_B (B')^\wedge \to NL^\wedge_{(B')^\wedge/A}$ is a quasi-isomorphism. Combining these two observations we obtain that

$$NL^\wedge_{(B')^\wedge/(A')^\wedge} \cong NL^\wedge_{B/A} \otimes_B (B')^\wedge$$

in $D((B')^\wedge)$. With these preparations out of the way we can start the actual proof.

Proof of (1). Assume φ is rig-smooth. Then there exists a $c \geq 0$ such that $\text{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by I^c for every B-module N. By More on Algebra, Lemmas 15.84.6 and 15.84.7 this property is preserved under base change by $B \to (B')^\wedge$. Hence $\text{Ext}^1_{(B')^\wedge}(NL^\wedge_{(B')^\wedge/(A')^\wedge}, N)$ is annihilated by $I^c(A')^\wedge$ for all $(B')^\wedge$-modules N which tells us that φ' is rig-smooth. This proves (1).

To prove (2) assume $B \to B'$ is faithfully flat and that φ' is rig-smooth. Then there exists a $c \geq 0$ such that $\text{Ext}^1_{(B')^\wedge}(NL^\wedge_{(B')^\wedge/(A')^\wedge}, N')$ is annihilated by $I^c(B')^\wedge$ for every $(B')^\wedge$-module N'. The composition $B \to B' \to (B')^\wedge$ is flat (Algebra, Lemma 10.97.2) hence for any B-module N we have

$$\text{Ext}^1_B(NL^\wedge_{B/A}, N) \otimes_B (B')^\wedge = \text{Ext}^1_{(B')^\wedge}(NL^\wedge_{B/A} \otimes_B (B')^\wedge, N \otimes_B (B')^\wedge)$$

by More on Algebra, Lemma 15.99.2 part (3) (minor details omitted). Thus we see that this module is annihilated by I^c. However, $B \to (B')^\wedge$ is actually faithfully flat by our assumption that $B \to B'$ is faithfully flat (Formal Spaces, Lemma 87.19.14). Thus we conclude that $\text{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by I^c. Hence φ is rig-smooth. This proves (2).

To prove (3), setting $B = \prod_{i=1,\ldots,n} B_i$ we just observe that $NL^\wedge_{B/A}$ is the direct sum of the complexes $NL^\wedge_{B_i/A}$ viewed as complexes of B-modules. $\square$

0GCL Lemma 88.17.4. Consider the properties $P(\varphi)$ =“φ is rig-smooth” and $Q(\varphi)$=“φ is adic” on arrows of WAdm^{Noeth}. Then P is stable under base change by Q as defined in Formal Spaces, Remark 87.21.10.

Proof. The statement makes sense by Lemma 88.17.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in WAdm^{Noeth} and that $B \to A$ is rig-smooth and $B \to C$ is adic (Formal Spaces, Definition 87.6.1). Then we can choose an ideal of definition $I \subset B$ such that the topology on A and C is the I-adic topology. In this situation it follows immediately that $A \widehat{\otimes}_B C$ is rig-smooth over (C, IC) by Lemma 88.4.5. $\square$

0GCM Lemma 88.17.5. The property $P(\varphi) =$"φ is rig-smooth" on arrows of WAdm^{Noeth} is stable under composition as defined in Formal Spaces, Remark 87.21.14.

Proof. We strongly urge the reader to find their own proof and not read the proof that follows. The statement makes sense by Lemma 88.17.1. To see that it is true assume we have rig-smooth morphisms $A \to B$ and $B \to C$ in WAdm^{Noeth}. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 88.3.5 we obtain an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] \end{tikzcd}$$

Observe that $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$ and $H^{-1}(NL^\wedge_{C/B})$ are annihilated by a power of I; this follows from Lemma 88.4.2 part (2) combined with More on Algebra, Lemmas 15.84.6 and 15.84.7 (to deal with the base change by $B \to C$). Hence $H^{-1}(NL^\wedge_{C/A})$ is annihilated by a power of I. Next, by the characterization of rig-smooth algebras in Lemma 88.4.2 part (2) which in turn refers to More on Algebra, Lemma 15.84.10 part (5) we can choose $f_1, \ldots, f_s \in IB$ and $g_1, \ldots, g_t \in IC$ such that $V(f_1, \ldots, f_s) = V(IB)$ and $V(g_1, \ldots, g_t) = V(IC)$ and such that $H^0(NL^\wedge_{B/A})_{f_i}$ is a finite projective B_{f_i}-module and $H^0(NL^\wedge_{C/B})_{g_j}$ is a finite projective C_{g_j}-module. Since the cohomologies in degree -1 vanish upon localization at $f_i g_j$ we get a short exact sequence

$$0 \to (C \otimes_B H^0(NL^\wedge_{B/A}))_{f_i g_j} \to H^0(NL^\wedge_{C/A})_{f_i g_j} \to H^0(NL^\wedge_{C/B})_{f_i g_j} \to 0$$

and we conclude that $H^0(NL^\wedge_{C/A})_{f_i g_j}$ is a finite projective $C_{f_i g_j}$-module as an extension of same. Thus by the criterion in Lemma 88.4.2 part (2) and via that the criterion in More on Algebra, Lemma 15.84.10 part (4) we conclude that C is rig-smooth over (A, I). □

The following lemma can be interpreted as saying that a rig-smooth homomorphism is "rig-syntomic" or "rig-flat+rig-lci".

0GH1 Lemma 88.17.6. Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth}. If φ is rig-smooth, then φ is rig-flat, and for any presentation $B = A\{x_1, \ldots, x_n\}/J$ and prime $J \subset \mathfrak q \subset A\{x_1, \ldots, x_n\}$ not containing an ideal of definition the ideal $J_\mathfrak q \subset A\{x_1, \ldots, x_n\}_\mathfrak q$ is generated by a regular sequence.

Proof. Let $f \in A$. To prove that φ is rig-flat we have to show that $\varphi_{\{f\}} : A_{\{f\}} \to B_{\{f\}}$ is naively rig-flat. Now either by viewing $\varphi_{\{f\}}$ as a base change of φ and using Lemma 88.17.4 or by using the fact that being rig-smooth is a local property (Lemma 88.17.3) we see that $\varphi_{\{f\}}$ is rig-smooth. Hence it suffices to show that φ is naively rig-flat.

Choose a presentation $B = A\{x_1, \ldots, x_n\}/J$. In order to check the second part of the lemma it suffices to check $J_\mathfrak q \subset A\{x_1, \ldots, x_n\}_\mathfrak q$ is generated by a regular sequence for $J \subset \mathfrak q$ for $\mathfrak q$ maximal with respect to not containing an ideal of definition, see Algebra, Lemma 10.68.6 (which shows that the set of primes in $V(J)$ where there is a regular sequence generating J is open). In other words, we may assume $\mathfrak{q}$ is rig-closed in $A\{x_1, \ldots, x_n\}$. And to check that B is naively rig-flat, it also suffices to check that the corresponding localizations $B_\mathfrak{q}$ are flat over A.

Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be rig-closed with $J \subset \mathfrak{q}$. By Lemma 88.14.9 we may choose an $f \in A$ mapping to a unit in $A\{x_1, \ldots, x_n\}/\mathfrak{q}$ and such that the prime ideal $\mathfrak{p}'$ in $A_{\{f\}}$ induced is rig-closed. Below we will use that $A_{\{f\}}\{x_1, \ldots, x_n\} = A\{x_1, \ldots, x_n\}_{\{f\}}$; details omitted. Consider the diagram

$$\begin{tikzcd} A\{x_1, \ldots, x_n\}_\mathfrak{q}/J_\mathfrak{q} \arrow[r] & A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}/JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'} \\ A\{x_1, \ldots, x_n\}_\mathfrak{q} \arrow[r] \arrow[u] & A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'} \arrow[u] \\ A \arrow[r] \arrow[u] & A_{\{f\}} \arrow[u] \end{tikzcd}$$

The middle horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $A_{\{f\}}\{x_1, \ldots, x_n\}$ is the completion of a localization of the Noetherian ring $A\{x_1, \ldots, x_n\}$. Similarly the bottom horizontal arrow is flat. Hence to show that $J_\mathfrak{q}$ is generated by a regular sequence and that $A \to A\{x_1, \ldots, x_n\}_\mathfrak{q}/J_\mathfrak{q}$ is flat, it suffices to prove the same things for $JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}$ and $A_{\{f\}} \to A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}/JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}$. See Algebra, Lemma 10.68.5 or More on Algebra, Lemma 15.32.4 for the statement on regular sequences. Finally, we have already seen that $A_{\{f\}} \to B_{\{f\}}$ is rig-smooth. This reduces us to the case discussed in the next paragraph.

Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be rig-closed with $J \subset \mathfrak{q}$ such that moreover $\mathfrak{p} = A \cap \mathfrak{q}$ is rig-closed as well. By the characterization of rig-smooth algebras given in Lemma 88.4.2 after reordering the variables $x_1, \ldots, x_n$ we can find $m \geq 0$ and $f_1, \ldots, f_m \in J$ such that

(1) $J_\mathfrak{q}$ is generated by $f_1, \ldots, f_m$, and
(2) $\det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$ maps to a unit in $A\{x_1, \ldots, x_n\}_\mathfrak{q}$.

By Lemma 88.14.12 the fibre ring

$$F = A\{x_1, \ldots, x_n\} \otimes_A \kappa(\mathfrak{p})$$

is regular. Observe that the A-derivations $\partial/\partial x_i$ extend (uniquely) to derivations $D_i : F \to F$. By More on Algebra, Lemma 15.48.3 we see that $f_1, \ldots, f_m$ map to a regular sequence in $F_\mathfrak{q}$. By flatness of $A \to A\{x_1, \ldots, x_n\}$ and Algebra, Lemma 10.99.3 this shows that $f_1, \ldots, f_m$ map to a regular sequence in $A\{x_1, \ldots, x_m\}_\mathfrak{q}$ and the quotient by these elements is flat over A. This finishes the proof. □

0GH2 Lemma 88.17.7. Let $A \to B \to C$ be arrows in WAdm^{Noeth} which are adic and topologically of finite type. If $B \to C$ is rig-smooth, then the kernel of the map

$$H^{-1}(NL^\wedge_{B/A} \otimes_B C) \to H^{-1}(NL^\wedge_{C/A})$$

(see Lemma 88.3.5) is annihilated by an ideal of definition.

Proof. Let $\overline{\mathfrak{q}} \subset C$ be a prime ideal which does not contain an ideal of definition. Since the modules in question are finite it suffices to show that

$$H^{-1}(NL^\wedge_{B/A} \otimes_B C)_{\overline{\mathfrak{q}}} \to H^{-1}(NL^\wedge_{C/A})_{\overline{\mathfrak{q}}}$$

is injective. As in the proof of Lemma 88.3.5 choose presentations $B = A\{x_1, \ldots, x_r\}/J$, $C = B\{y_1, \ldots, y_s\}/J'$, and $C = A\{x_1, \ldots, x_r, y_1, \ldots, y_s\}/K$. Looking at the diagram in the proof of Lemma 88.3.5 we see that it suffices to show that $J/J^2 \otimes_B C \to K/K^2$ is injective after localization at the prime ideal $\mathfrak q \subset A\{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ corresponding to $\overline{\mathfrak q}$. Please compare with More on Algebra, Lemma 15.33.6 and its proof. This is the same as asking $J/KJ \to K/K^2$ to be injective after localization at $\mathfrak q$. Equivalently, we have to show that $J_{\mathfrak q} \cap K_{\mathfrak q}^2 = (KJ)_{\mathfrak q}$. By Lemma 88.17.6 we know that $(K/J)_{\mathfrak q} = J'_{\mathfrak q}$ is generated by a regular sequence. Hence the desired intersection property follows from More on Algebra, Lemma 15.32.5 (and the fact that an ideal generated by a regular sequence is H_1-regular, see More on Algebra, Section 15.32). ☐

88.18. Rig-smooth morphisms

0GCN In this section we use the work done in Section 88.17 to define rig-smooth morphisms of locally Noetherian algebraic spaces.

0GCP Definition 88.18.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is rig-smooth if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-smooth map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

0GCQ Lemma 88.18.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent

(1) f is rig-smooth,
(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-smooth map in WAdm^{Noeth},
(3) there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-smooth map in WAdm^{Noeth}, and
(4) there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-smooth map in WAdm^{Noeth}.

Proof. The equivalence of (1) and (2) is Definition 88.18.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-smooth is a local property of arrows of WAdm^{Noeth} by Lemma 88.17.3 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

0GCR Lemma 88.18.3. Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-smooth and g is adic, then the base change $X \times_Y Z \to Z$ is rig-smooth.

Proof. By Formal Spaces, Remark 87.21.10 and the discussion in Formal Spaces, Section 87.23, this follows from Lemma 88.17.4. □

0GCS Lemma 88.18.4. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-smooth, then so is $g \circ f$.

Proof. By Formal Spaces, Remark 87.21.14 this follows from Lemma 88.17.5. □

0GH3 Lemma 88.18.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. If f is rig-smooth, then f is rig-flat.

Proof. Follows immediately from Lemma 88.17.6 and the definitions. □

88.19. Rig-étale homomorphisms

0GCT In this section we prove some properties of rig-étale homomorphisms of adic Noetherian topological rings which are needed to introduce rig-étale morphisms of locally Noetherian algebraic spaces.

0GCU Lemma 88.19.1. Let $A \to B$ be a morphism in WAdm^{Noeth} (Formal Spaces, Section 87.21). The following are equivalent:

(a) $A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and there exists an ideal of definition $I \subset B$ such that B is rig-étale over (A, I), and
(b) $A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and for all ideals of definition $I \subset A$ the algebra B is rig-étale over (A, I).

Proof. Let I and I' be ideals of definitions of A. Then there exists an integer $c \geq 0$ such that $I^c \subset I'$ and $(I')^c \subset I$. Hence B is rig-étale over (A, I) if and only if B is rig-étale over (A, I'). This follows from Definition 88.8.1, the inclusions $I^c \subset I'$ and $(I')^c \subset I$, and the fact that the naive cotangent complex $NL^\wedge_{B/A}$ is independent of the choice of ideal of definition of A by Remark 88.11.2. □

0GCV Definition 88.19.2. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of WAdm^{Noeth}. We say φ is rig-etale if the equivalent conditions of Lemma 88.19.1 hold.

This defines a local property.

0AQL Lemma 88.19.3. The property $P(\varphi) =$"φ is rig-étale" on arrows of WAdm^{Noeth} is a local property as defined in Formal Spaces, Remark 87.21.5.

Proof. This proof is exactly the same as the proof of Lemma 88.17.3. Let us recall what the statement signifies. First, WAdm^{Noeth} is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is rig-étale $\Rightarrow \varphi'$ is rig-étale,
(2) if $B \to B'$ faithfully flat, then φ' is rig-étale $\Rightarrow \varphi$ is rig-étale, and
(3) if $A \to B_i$ is rig-étale for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-étale.

The equivalent conditions of Lemma 88.11.1 satisfy conditions (1), (2), and (3). Thus in verifying (1), (2), and (3) for the property "rig-étale" we may already assume our ring maps satisfy the equivalent conditions of Lemma 88.11.1 in each case.

Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology and B, $(A')^\wedge$, and $(B')^\wedge$ are in the category (88.2.0.2) for (A, I). Since $A \to A'$ and $B \to B'$ are étale the complexes $NL_{A'/A}$ and $NL_{B'/B}$ are zero and hence $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero by Lemma 88.3.2. Applying Lemma 88.3.5 to $A \to (A')^\wedge \to (B')^\wedge$ we get isomorphisms

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

Thus $NL^\wedge_{(B')^\wedge/A} \to NL_{(B')^\wedge/(A')^\wedge}$ is a quasi-isomorphism. The ring maps $B/I^nB \to B'/I^nB'$ are étale and hence are local complete intersections (Algebra, Lemma 10.143.2). Hence we may apply Lemmas 88.3.5 and 88.3.6 to $A \to B \to (B')^\wedge$ and we get isomorphisms

$$H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

We conclude that $NL^\wedge_{B/A} \otimes_B (B')^\wedge \to NL^\wedge_{(B')^\wedge/A}$ is a quasi-isomorphism. Combining these two observations we obtain that

$$NL^\wedge_{(B')^\wedge/(A')^\wedge} \cong NL^\wedge_{B/A} \otimes_B (B')^\wedge$$

in $D((B')^\wedge)$. With these preparations out of the way we can start the actual proof.

Proof of (1). Assume φ is rig-étale. Then there exists a $c \geq 0$ such that multiplication by $a \in I^c$ is zero on $NL^\wedge_{B/A}$ in $D(B)$. This property is preserved under base change by $B \to (B')^\wedge$, see More on Algebra, Lemmas 15.84.6. By the isomorphism above we find that φ' is rig-étale. This proves (1).

To prove (2) assume $B \to B'$ is faithfully flat and that φ' is rig-étale. Then there exists a $c \geq 0$ such that multiplication by $a \in I^c$ is zero on $NL^\wedge_{(B')^\wedge/(A')^\wedge}$ in $D((B')^\wedge)$. By the isomorphism above we see that a^c annihilates the cohomology modules of $NL^\wedge_{B/A} \otimes_B (B')^\wedge$. The composition $B \to (B')^\wedge$ is faithfully flat by our assumption that $B \to B'$ is faithfully flat, see Formal Spaces, Lemma 87.19.14. Hence the cohomology modules of $NL^\wedge_{B/A}$ are annihilated by I^c. It follows from Lemma 88.8.2 that φ is rig-étale. This proves (2).

To prove (3), setting $B = \prod_{i=1,\ldots,n} B_i$ we just observe that $NL^\wedge_{B/A}$ is the direct sum of the complexes $NL^\wedge_{B_i/A}$ viewed as complexes of B-modules. $\square$

0GCW Lemma 88.19.4. Consider the properties $P(\varphi)$ =“φ is rig-étale” and $Q(\varphi)$=“φ is adic” on arrows of WAdm^{Noeth}. Then P is stable under base change by Q as defined in Formal Spaces, Remark 87.21.10.

Proof. The statement makes sense by Lemma 88.19.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in WAdm^{Noeth} and that $B \to A$ is rig-étale and $B \to C$ is adic (Formal Spaces, Definition 87.6.1). Then we can choose an ideal of definition $I \subset B$ such that the topology on A and C is the I-adic topology. In this situation it follows immediately that $A \widehat{\otimes}_B C$ is rig-étale over (C, IC) by Lemma 88.8.6. $\square$

0GCX Lemma 88.19.5. The property $P(\varphi)$ =“φ is rig-étale” on arrows of WAdm^{Noeth} is stable under composition as defined in Formal Spaces, Remark 87.21.14.

Proof. The statement makes sense by Lemma 88.19.1. To see that it is true assume we have rig-étale morphisms $A \to B$ and $B \to C$ in WAdm^{Noeth}. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 88.3.5 we obtain an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] \end{tikzcd}$$

There exists a $c \geq 0$ such that for all $a \in I$ multiplication by a^c is zero on $NL^\wedge_{B/A}$ in $D(B)$ and $NL^\wedge_{C/B}$ in $D(C)$. Then of course multiplication by a^c is zero on $NL^\wedge_{B/A} \otimes_B C$ in $D(C)$ too. Hence $H^0(NL^\wedge_{B/A}) \otimes_A C$, $H^0(NL^\wedge_{C/B})$, $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$, and $H^{-1}(NL^\wedge_{C/B})$ are annihilated by a^c. From the exact sequence we obtain that multiplication by a^{2c} is zero on $H^0(NL^\wedge_{C/A})$ and $H^{-1}(NL^\wedge_{C/A})$. It follows from Lemma 88.8.2 that C is rig-étale over (A, I) as desired. $\square$

0GCY Lemma 88.19.6. The property $P(\varphi)$ =“φ is rig-étale” on arrows of WAdm^{Noeth} has the cancellation property as defined in Formal Spaces, Remark 87.21.18.

Proof. The statement makes sense by Lemma 88.19.1. To see that it is true assume we have maps $A \to B$ and $B \to C$ in WAdm^{Noeth} with $A \to C$ and $A \to B$ rig-étale. We have to show that $B \to C$ is rig-étale. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By

Lemma 88.3.5 we obtain an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] \end{tikzcd}$$

There exists a $c \geq 0$ such that for all $a \in I$ multiplication by a^c is zero on $NL^\wedge_{B/A}$ in $D(B)$ and $NL^\wedge_{C/A}$ in $D(C)$. Hence $H^0(NL^\wedge_{B/A}) \otimes_A C$, $H^0(NL^\wedge_{C/A})$, and $H^{-1}(NL^\wedge_{C/A})$ are annihilated by a^c. From the exact sequence we obtain that multiplication by a^{2c} is zero on $H^0(NL^\wedge_{C/B})$ and $H^{-1}(NL^\wedge_{C/B})$. It follows from Lemma 88.8.2 that C is rig-étale over (B, IB) as desired. $\square$

88.20. Rig-étale morphisms

0AQK In this section we use the work done in Section 88.19 to define rig-étale morphisms of locally Noetherian algebraic spaces.

0AQM Definition 88.20.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is rig-étale if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-étale map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

0GCZ Lemma 88.20.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent

(1) f is rig-étale,
(2) for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-étale map in WAdm^{Noeth},
(3) there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-étale map in WAdm^{Noeth}, and
(4) there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-étale map in WAdm^{Noeth}.

Proof. The equivalence of (1) and (2) is Definition 88.20.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-étale is a local property of arrows of WAdm^{Noeth} by Lemma 88.19.3 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

To be sure, a rig-étale morphism is locally of finite type.

0AQN Lemma 88.20.3. A rig-étale morphism of locally Noetherian formal algebraic spaces is locally of finite type.

Proof. The property P in Lemma 88.19.3 implies the equivalent conditions (a), (b), (c), and (d) in Formal Spaces, Lemma 87.29.6. Hence this follows from Formal Spaces, Lemma 87.29.9. □

0GD0 Lemma 88.20.4. A rig-étale morphism of locally Noetherian formal algebraic spaces is rig-smooth.

Proof. Follows from the definitions and Lemma 88.8.3. □

0GD1 Lemma 88.20.5. Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-étale and g is adic, then the base change $X \times_Y Z \to Z$ is rig-étale.

Proof. By Formal Spaces, Remark 87.21.10 and the discussion in Formal Spaces, Section 87.23, this follows from Lemma 88.19.4. □

0GD2 Lemma 88.20.6. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-étale, then so is $g \circ f$.

Proof. By Formal Spaces, Remark 87.21.14 this follows from Lemma 88.19.5. □

0GD3 Lemma 88.20.7. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be a morphism of locally Noetherian formal algebraic spaces over S. If $g \circ f$ and g are rig-étale, then so is f.

Proof. By Formal Spaces, Remark 87.21.18 this follows from Lemma 88.19.6. □

0GH4 Lemma 88.20.8. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If $g \circ f$ is rig-étale and g is an adic monomorphism, then f is rig-étale.

Proof. Use Lemma 88.20.5 and that f is the base change of $g \circ f$ by g. □

0GD4 Lemma 88.20.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent

(1) f is rig-smooth,
(2) f is rig-étale,
(3) for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} whose kernel J has the following property: $I(J/J^2) = 0$ for some ideal of definition I of B.

Proof. Let us observe that given V and $V \to Y$ as in (2) without any further assumption on f we see that the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} by Formal Spaces, Lemma 87.29.5.

We have (2) $\Rightarrow$ (1) by Lemma 88.20.4.

Proof of (3) $\Rightarrow$ (2). Assume (3). By Lemma 88.20.2 it suffices to show that the ring maps $B \to A$ occuring in (3) are rig-étale in the sense of Definition 88.19.2. Let I be as in (3). The naive cotangent complex $NL^\wedge_{A/B}$ of A over (B, I) is the complex of A-modules given by putting J/J^2 in degree -1. Hence A is rig-étale over (B, I) by Definition 88.8.1.

Assume (1) and let V and $B \to A$ be as in (3). By Definition 88.18.1 we see that $B \to A$ is rig-smooth. Choose any ideal of definition $I \subset B$. Then A is rig-smooth over (B, I). As above the complex $NL^\wedge_{A/B}$ is given by putting J/J^2 in degree -1. Hence by Lemma 88.4.2 we see that J/J^2 is annihilated by a power I^n for some $n \geq 1$. Since B is adic, we see that I^n is an ideal of definition of B and the proof is complete. $\square$

88.21. Rig-surjective morphisms

0AQP For morphisms locally of finite type between locally Noetherian formal algebraic spaces a definition borrowed from [Art70] can be used. See Remark 88.21.2 for a discussion of what to do in more general cases.

0AQQ Definition 88.21.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume that X and Y are locally Noetherian and that f is locally of finite type. We say f is rig-surjective if for every solid diagram

$$\begin{tikzcd} \text{Spf}(R') \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d, "f"] \\ \text{Spf}(R) \arrow[r, "p"] & Y \end{tikzcd}$$

where R is a complete discrete valuation ring and where p is an adic morphism there exists an extension of complete discrete valuation rings $R \subset R'$ and a morphism $\text{Spf}(R') \to X$ making the displayed diagram commute.

We will see in the lemmas below that this notion behaves reasonably well in the context of locally Noetherian formal algebraic spaces and morphisms which are locally of finite type. In the next remark we discuss options for modifying this definition to a wider class of morphisms of formal algebraic spaces.

0AQZ Remark 88.21.2. The condition as formulated in Definition 88.21.1 is not right even for morphisms of finite type of locally adic* formal algebraic spaces. For example, if $A = (\bigcup_{n \geq 1} k[t^{1/n}])^\wedge$ where the completion is the t-adic completion, then there are no adic morphisms $\text{Spf}(R) \to \text{Spf}(A)$ where R is a complete discrete valuation ring. Thus any morphism $X \to \text{Spf}(A)$ would be rig-surjective, but since A is a domain and $t \in A$ is not zero, we want to think of A as having at least one "rig-point", and we do not want to allow $X = \emptyset$. To cover this particular case, one can consider adic morphisms

$$\text{Spf}(R) \longrightarrow Y$$

where R is a valuation ring complete with respect to a principal ideal J whose radical is $\mathfrak{m}_R = \sqrt{J}$. In this case the value group of R can be embedded into $(\mathbf{R}, +)$ and one obtains the point of view used by Berkovich in defining an analytic space associated to Y, see [Ber90]. Another approach is championed by Huber. In his theory, one drops the hypothesis that $\text{Spec}(R/J)$ is a singleton, see [Hub93a].

0AQR Lemma 88.21.3. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. Then if f and g are rig-surjective, so is $g \circ f$.

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemma 87.24.3). □

0AQS Lemma 88.21.4. Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. If f is rig-surjective, then the base change $Z \times_Y X \to Z$ is too.

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemmas 87.24.9 and 87.24.4). □

0GH5 Lemma 88.21.5. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of finite type of locally Noetherian formal algebraic spaces over S. If $g \circ f$ is rig-surjective and g is a monomorphism, then f is rig-surjective.

Proof. Use Lemma 88.21.4 and that f is the base change of $g \circ f$ by g. □

0AQT Lemma 88.21.6. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z locally Noetherian and f and g locally of finite type. If $g \circ f : X \to Z$ is rig-surjective, so is $g : Y \to Z$.

Proof. Immediate from the definition. □

0AQU Lemma 88.21.7. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is representable by algebraic spaces, étale, and surjective. Then f is rig-surjective.

Proof. Let $p : \text{Spf}(R) \to Y$ be an adic morphism where R is a complete discrete valuation ring. Let $Z = \text{Spf}(R) \times_Y X$. Then $Z \to \text{Spf}(R)$ is representable by algebraic spaces, étale, and surjective. Hence Z is nonempty. Pick a nonempty affine formal algebraic space V and an étale morphism $V \to Z$ (possible by our definitions). Then $V \to \text{Spf}(R)$ corresponds to $R \to A^\wedge$ where $R \to A$ is an étale ring map, see Formal Spaces, Lemma 87.19.13. Since $A^\wedge \neq 0$ (as $V \neq \emptyset$) we can find a maximal ideal $\mathfrak{m}$ of A lying over $\mathfrak{m}_R$. Then $A_\mathfrak{m}$ is a discrete valuation ring (More on Algebra, Lemma 15.44.4). Then $R' = A_\mathfrak{m}^\wedge$ is a complete discrete valuation ring (More on Algebra, Lemma 15.43.5). Applying Formal Spaces, Lemma 87.9.10. we find the desired morphism $\text{Spf}(R') \to V \to Z \to X$. □

The upshot of the lemmas above is that we may check whether $f : X \to Y$ is rig-surjective, étale locally on Y.

0AQV Lemma 88.21.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is locally of finite type. Let $\{g_i : Y_i \to Y\}$ be a family of morphisms of formal algebraic spaces which are representable by algebraic spaces and étale such that $\coprod g_i$ is surjective. Then f is rig-surjective if and only if each $f_i : X \times_Y Y_i \to Y_i$ is rig-surjective.

Proof. Namely, if f is rig-surjective, so is any base change (Lemma 88.21.4). Conversely, if all f_i are rig-surjective, so is $\coprod f_i : \coprod X \times_Y Y_i \to \coprod Y_i$. By Lemma 88.21.7 the morphism $\coprod g_i : \coprod Y_i \to Y$ is rig-surjective. Hence $\coprod X \times_Y Y_i \to Y$ is rig-surjective (Lemma 88.21.3). Since this morphism factors through $X \to Y$ we see that $X \to Y$ is rig-surjective by Lemma 88.21.6. $\square$

0AQX Lemma 88.21.9. Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. If $A/I^n \to B/I^nB$ is of finite type and flat for all n and faithfully flat for $n = 1$, then $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective.

Proof. We will use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 87.9.10. Let $\varphi : A \to R$ be a continuous map into a complete discrete valuation ring A. This implies that $\varphi(I) \subset \mathfrak{m}_R$. On the other hand, since we only need to produce the lift $\varphi' : B' \to R'$ in the case that φ corresponds to an adic morphism, we may assume that $\varphi(I) \neq 0$. Thus we may consider the base change $C = B\widehat{\otimes}_A R$, see Remark 88.2.3 for example. Then C is an $\mathfrak{m}_R$-adically complete R-algebra such that $C/\mathfrak{m}_R^nC$ is of finite type and flat over $R/\mathfrak{m}_R^n$ and such that $C/\mathfrak{m}_RC$ is nonzero. Pick any maximal ideal $\mathfrak{m} \subset C$ lying over $\mathfrak{m}_R$. By flatness (which implies going down) we see that $\text{Spec}(C_\mathfrak{m}) \setminus V(\mathfrak{m}_RC_\mathfrak{m})$ is a nonempty open. Hence We can pick a prime $\mathfrak{q} \subset \mathfrak{m}$ such that $\mathfrak{q}$ defines a closed point of $\text{Spec}(C_\mathfrak{m}) \setminus \{\mathfrak{m}\}$ and such that $\mathfrak{q} \notin V(IC_\mathfrak{m})$, see Properties, Lemma 28.6.4. Then $C/\mathfrak{q}$ is a dimension 1-local domain and we can find $C/\mathfrak{q} \subset R'$ with R' a discrete valuation ring (Algebra, Lemma 10.119.13). By construction $\mathfrak{m}_RR' \subset \mathfrak{m}_{R'}$ and we see that $C \to R'$ extends to a continuous map $C \to (R')^\wedge$ (in fact we can pick R' such that $R' = (R')^\wedge$ in our current situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{tikzcd} (R')^\wedge & C \arrow[l] & B \arrow[l] \\ R \arrow[u] & R \arrow[l] \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

which gives the desired lift. $\square$

0AQY Lemma 88.21.10. Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. Assume that

(1) the I-torsion in A is 0,
(2) $A/I^n \to B/I^nB$ is flat and of finite type for all n.

Then $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective if and only if $A/I \to B/IB$ is faithfully flat.

Proof. Faithful flatness implies rig-surjectivity by Lemma 88.21.9. To prove the converse we will use without further mention that the vanishing of I-torsion is equivalent to the vanishing of I-power torsion (More on Algebra, Lemma 15.88.3). We will also use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 87.9.10.

Assume $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective. Choose a maximal ideal $I \subset \mathfrak{m} \subset A$. The open $U = \text{Spec}(A_\mathfrak{m}) \setminus V(I_\mathfrak{m})$ of $\text{Spec}(A_\mathfrak{m})$ is nonempty as the $I_\mathfrak{m}$-torsion of $A_\mathfrak{m}$ is zero (use Algebra, Lemma 10.62.4). Thus we can find a prime $\mathfrak{q} \subset A_\mathfrak{m}$ which defines a point of U (i.e., $IA_\mathfrak{m} \not\subset \mathfrak{q}$) and which corresponds to a closed point of $\text{Spec}(A_\mathfrak{m}) \setminus \{\mathfrak{m}\}$, see Properties, Lemma 28.6.4. Then $A_\mathfrak{m}/\mathfrak{q}$ is a dimension 1 local domain. Thus we can find an injective local homomorphism of local rings $A_\mathfrak{m}/\mathfrak{q} \subset R$ where R is a discrete valuation ring (Algebra, Lemma 10.119.13). By construction $IR \subset \mathfrak{m}_R$ and we see that $A \to R$ extends to a continuous map $A \to R^\wedge$. Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{tikzcd} R' & B \arrow[l] \\ R^\wedge \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Thus we find a prime ideal of B lying over $\mathfrak{m}$. It follows that $\text{Spec}(B/IB) \to \text{Spec}(A/I)$ is surjective, whence $A/I \to B/IB$ is faithfully flat (Algebra, Lemma 10.39.16). $\square$

0AR0 Lemma 88.21.11. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume X and Y are locally Noetherian, f locally of finite type, and f a monomorphism. Then f is rig surjective if and only if every adic morphism $\text{Spf}(R) \to Y$ where R is a complete discrete valuation ring factors through X.

Proof. One direction is trivial. For the other, suppose that $\text{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\text{Spf}(R') \to \text{Spf}(R) \to X$ factoring through Y. Then $\text{Spec}(R'/\mathfrak{m}_R^n R') \to \text{Spec}(R/\mathfrak{m}_R^n)$ is surjective and flat, hence the morphisms $\text{Spec}(R/\mathfrak{m}_R^n) \to X$ factor through X as X satisfies the sheaf condition for fpqc coverings, see Formal Spaces, Lemma 87.32.1. In other words, $\text{Spf}(R) \to Y$ factors through X. $\square$

0GD5 Lemma 88.21.12. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent

(1) f is rig-surjective, and
(2) for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} whose kernel J has the following property: $IJ^n = 0$ for some ideal of definition I of B and some $n \geq 1$.

Proof. Let us observe that given V and $V \to Y$ as in (2) without any further assumption on f we see that the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} by Formal Spaces, Lemma 87.29.5.

Assume (1). By Lemma 88.21.4 we see that $\text{Spf}(A) \to \text{Spf}(B)$ is rig-surjective. Let $I \subset B$ be an ideal of definition. Since B is adic, $I^m \subset B$ is an ideal of definition for all $m \geq 1$. If $I^m J^n \neq 0$ for all $n, m \geq 1$, then IJ is not nilpotent, hence $V(IJ) \neq \text{Spec}(B)$. Thus we can find a prime ideal $\mathfrak{p} \subset B$ with $\mathfrak{p} \notin V(I) \cup V(J)$. Observe that $I(B/\mathfrak{p}) \neq B/\mathfrak{p}$ hence we can find a maximal ideal $\mathfrak{p} + I \subset \mathfrak{m} \subset B$. By Algebra, Lemma 10.119.13 we can find a discrete valuation ring R and an injective local ring homomorphism $(B/\mathfrak{p})_\mathfrak{m} \to R$. Clearly, the ring map $B \to R$ cannot factor through $A = B/J$. According to Lemma 88.21.11 this contradicts the fact that $\text{Spf}(A) \to \text{Spf}(B)$ is rig-surjective. Hence for some n, m we do have $I^n J^m = 0$ which shows that (2) holds.

Assume (2). By Lemma 88.21.8 it suffices to show that $\text{Spf}(A) \to \text{Spf}(B)$ is rig-surjective. Pick an ideal of definition $I \subset B$ and an integer n such that $IJ^n = 0$. Consider a ring map $B \to R$ where R is a discrete valuation ring and the image of I is nonzero. Since R is a domain, we conclude the image of J in R is zero. Hence $B \to R$ factors through the surjection $B \to A$ and we are done by definition of rig-surjective morphisms. □

0GD6 Lemma 88.21.13. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent

(1) f is rig-smooth and rig-surjective,
(2) f is rig-étale and rig-surjective, and
(3) for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} whose kernel J has the following property: $IJ = 0$ for some ideal of definition I of B.

Proof. Let I and J be ideals of a ring B such that $IJ^n = 0$ and $I(J/J^2) = 0$. Then $I^n J = 0$ (proof omitted). Hence this lemma follows from a trivial combination of Lemmas 88.20.9 and 88.21.12. □

0GH6 Lemma 88.21.14. Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. Assume

(1) g is locally of finite type,
(2) f is rig-smooth (resp. rig-étale) and rig-surjective,
(3) $g \circ f$ is rig-smooth (resp. rig-étale)

then g is rig-smooth (resp. rig-étale).

Proof. We will prove this in the rig-smooth case and indicate the necessary changes to prove the rig-étale case at the end of the proof. Consider a commutative diagram

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with V and W affine formal algebraic spaces, $V \to Y$ and $W \to Z$ representable by algebraic spaces and étale. We have to show that $V \to W$ corresponds to a rig-smooth map of adic Noetherian topological rings, see Definition 88.18.1. We may write $V = \text{Spf}(B)$ and $W = \text{Spf}(C)$ and that $V \to W$ corresponds to an adic ring map $C \to B$ which is topologically of finite type, see Lemma 88.11.5.

We will use below without further mention that $X \times_Y V \to V$ is rig-smooth and rig-surjective, see Lemmas 88.18.3 and 88.21.4. Also, the composition $X \times_Y V \to V \to W$ is rig-smooth since $g \circ f$ is rig-smooth.

Let $I \subset C$ be an ideal of definition. The module Assume $C \to B$ is not rig-smooth to get a contradiction. This means that there exists a prime ideal $\mathfrak{q} \subset B$ not containing IB such that either $H^{-1}(NL^\wedge_{B/C})_\mathfrak{p}$ is nonzero or $H^0(NL^\wedge_{B/C})_\mathfrak{p}$ is not a finite free $B_\mathfrak{q}$-module. See Lemma 88.4.2; some details omitted. We may choose a maximal ideal $IB + \mathfrak{q} \subset \mathfrak{m}$. By Algebra, Lemma 10.119.13 we can find a complete discrete valuation ring R and an injective local ring homomorphism $(B/\mathfrak{q})_\mathfrak{m} \to R$.

After replacing R by an extension, we may assume given a lift $\text{Spf}(R) \to X \times_Y V$ of the adic morphism $\text{Spf}(R) \to V = \text{Spf}(B)$. Choose an étale covering $\{\text{Spf}(A_i) \to X \times_Y V\}$ as in Formal Spaces, Definition 87.11.1. By Lemma 88.21.7 we may assume $\text{Spf}(R) \to X \times_Y V$ lifts to a morphism $\text{Spf}(R) \to \text{Spf}(A_i)$ for some i (this might require replacing R by another extension). Set $A = A_i$. Consider the ring maps

$$C \to B \to A \to R$$

Let $\mathfrak{p} \subset A$ be the kernel of the map $A \to R$ and note that $\mathfrak{p}$ lies over $\mathfrak{q}$. We know that $C \to A$ and $B \to A$ are rig-smooth. In particular the ring map $B_\mathfrak{q} \to A_\mathfrak{p}$ is flat by Lemma 88.17.6. Consider the associated exact sequence

$$\begin{tikzcd}
 & H^0(NL^\wedge_{B/C}) \otimes_B A_\mathfrak{p} \arrow[r] & H^0(NL^\wedge_{A/C})_\mathfrak{p} \arrow[r] & H^0(NL^\wedge_{A/B})_\mathfrak{p} \arrow[r] & 0 \\
0 \arrow[r] & H^{-1}(NL^\wedge_{B/C} \otimes_B A)_\mathfrak{p} \arrow[r] & H^{-1}(NL^\wedge_{A/C})_\mathfrak{p} \arrow[r] & H^{-1}(NL^\wedge_{A/B})_\mathfrak{p} \arrow[ull] &
\end{tikzcd}$$

of Lemmas 88.3.5 and 88.17.7. Given the rig-smoothness of $C \to A$ and $B \to A$ we conclude that $H^{-1}(NL^\wedge_{B/C} \otimes_B A)_\mathfrak{p} = 0$ and that $H^0(NL^\wedge_{B/C}) \otimes_B A_\mathfrak{p}$ is finite free as a kernel of a surjection of finite free $A_\mathfrak{p}$-modules. Since $B_\mathfrak{q} \to A_\mathfrak{p}$ is flat and hence faithfully flat, this implies that $H^{-1}(NL^\wedge_{B/C})_\mathfrak{q} = 0$ and that $H^0(NL^\wedge_{B/C})_\mathfrak{q}$ is finite free which is the contradiction we were looking for.

In the rig-étale case one argues in exactly the same manner but the conclusion obtained is that both $H^{-1}(NL^\wedge_{B/C})_\mathfrak{q}$ and $H^0(NL^\wedge_{B/C})_\mathfrak{q}$ are zero. □

88.22. Formal algebraic spaces over cdvrs

0GD7 In this section we will use the following terminology: if A is a weakly admissible topological ring, then we say "X is a formal algebraic space over A" to mean that X is a formal algebraic space which comes equipped with a morphism $p : X \to \text{Spf}(A)$ of formal algebraic spaces. In this situation we will call p the structure morphism.

0GD8 **Lemma 88.22.1.** Let X be a locally Noetherian formal algebraic space over a complete discrete valuation ring A. Then there exists a closed immersion $X' \to X$ of formal algebraic spaces such that X' is flat over A and such that any morphism $Y \to X$ of locally Noetherian formal algebraic spaces with Y flat over A factors through X'.

Proof. Let $\pi \in A$ be the uniformizer. Recall that an A-module is flat if and only if the π-power torsion is 0.

First assume that X is an affine formal algebraic space. Then $X = \text{Spf}(B)$ with B an adic Noetherian A-algebra. In this case we set $X' = \text{Spf}(B')$ where $B' = B/\pi$-power torsion. It is clear that X' is flat over A and that $X' \to X$ is a closed immersion. Let $g : Y \to X$ be a morphism of locally Noetherian formal algebraic spaces with Y flat over A. Choose a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1. Then $Y_j = \mathrm{Spf}(C_j)$ with C_j flat over A. Hence the morphism $Y_j \to X$, which correspond to a continuous R-algebra map $B \to C_j$, factors through X' as clearly $B \to C_j$ kills the π-power torsion. Since $\{Y_j \to Y\}$ is a covering and since $X' \to X$ is a monomorphism, we conclude that g factors through X'.

Let X and $\{X_i \to X\}_{i \in I}$ be as in Formal Spaces, Definition 87.11.1. For each i let $X'_i \to X_i$ be the flat part as constructed above. For $i, j \in I$ the projection $X'_i \times_X X_j \to X'_i$ is an étale (by assumption) morphism of schemes (by Formal Spaces, Lemma 87.9.11). Hence $X'_i \times_X X_j$ is flat over A as morphisms representable by algebraic spaces and étale are flat (Lemma 88.13.8). Thus the projection $X'_i \times_X X_j \to X_j$ factors through X'_j by the universal property. We conclude that

$$R_{ij} = X'_i \times_X X_j = X'_i \times_X X'_j = X_i \times_X X'_j$$

because the morphisms $X'_i \to X_i$ are injections of sheaves. Set $U = \coprod X'_i$, set $R = \coprod R_{ij}$, and denote $s, t : R \to U$ the two projections. As a sheaf $R = U \times_X U$ and s and t are étale. Then $(t, s) : R \to U$ defines an étale equivalence relation by our observations above. Thus $X' = U/R$ is an algebraic space by Spaces, Theorem 65.10.5. By construction the diagram

$$\begin{tikzcd} \coprod X'_i \arrow[r] \arrow[d] & \coprod X_i \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a closed immersion we conclude that $X' \to X$ is representable by Bootstrap, Lemma 80.5.2. Then we can use Spaces, Lemma 65.5.6 to conclude that $X' \to X$ is a closed immersion.

Finally, suppose that $Y \to X$ is a morphism with Y a locally Noetherian formal algebraic space flat over A. Then each $X_i \times_X Y$ is étale over Y and therefore flat over A (see above). Then $X_i \times_X Y \to X_i$ factors through X'_i. Hence $Y \to X$ factors through X' because $\{X_i \times_X Y \to Y\}$ is an étale covering. □

0GD9 Lemma 88.22.2. Let X be a locally Noetherian formal algebraic space which is locally of finite type over a complete discrete valuation ring A. Let $X' \subset X$ be as in Lemma 88.22.1. If $X \to X \times_{\mathrm{Spf}(A)} X$ is rig-étale and rig-surjective, then $X' = \mathrm{Spf}(A)$ or $X' = \emptyset$.

Proof. (Aside: the diagonal is always locally of finite type by Formal Spaces, Lemma 87.15.5 and $X \times_{\mathrm{Spf}(A)} X$ is locally Noetherian by Formal Spaces, Lemmas 87.24.4 and 87.24.8. Thus imposing the conditions on the diagonal morphism makes sense.) The diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X' \times_{\mathrm{Spf}(A)} X' \arrow[d] \\ X \arrow[r] & X \times_{\mathrm{Spf}(A)} X \end{tikzcd}$$

is cartesian. Hence $X' \to X' \times_{\mathrm{Spf}(A)} X'$ is rig-étale and rig-surjective by Lemma 88.21.4. Choose an affine formal algebraic space U and a morphism $U \to X'$ which is representable by algebraic spaces and étale. Then $U = \text{Spf}(B)$ where B is an adic Noetherian topological ring which is a flat A-algebra, whose topology is the π-adic topology where $\pi \in A$ is a uniformizer, and such that $A/\pi^n A \to B/\pi^n B$ is of finite type for each n. For later use, we remark that this in particular implies: if $B \neq 0$, then the map $\text{Spf}(B) \to \text{Spf}(A)$ is a surjection of sheaves (please recall that we are using the fppf topology as always). Repeating the argument above, we see that

$$W = U \times_{X'} U = X' \times_{X' \times_{\text{Spf}(A)} X'} (U \times_{\text{Spf}(A)} U) \longrightarrow U \times_{\text{Spf}(A)} U$$

is a closed immersion and rig-étale and rig-surjective. We have $U \times_{\text{Spf}(A)} U = \text{Spf}(B \widehat{\otimes}_A B)$ by Formal Spaces, Lemma 87.16.4. Then $B \widehat{\otimes}_A B$ is a flat A-algebra as the π-adic completion of the flat A-algebra $B \otimes_A B$. Hence $W = U \times_{\text{Spf}(A)} U$ by Lemma 88.21.13. In other words, we have $U \times_{X'} U = U \times_{\text{Spf}(A)} U$ which in turn means that the image of $U \to X'$ (as a map of sheaves) maps injectively to $\text{Spf}(A)$. Choose a covering $\{U_i \to X'\}$ as in Formal Spaces, Definition 87.11.1. In particular $\coprod U_i \to X'$ is a surjection of sheaves. By applying the above to $U_i \coprod U_j \to X'$ (using the fact that $U_i \amalg U_j$ is an affine formal algebraic space as well) we see that $X' \to \text{Spf}(A)$ is an injective map of fppf sheaves. Since X' is flat over A, either X' is empty (if U_i is empty for all i) or the map is an isomorphism (if U_i is nonempty for some i when we have seen that $U_i \to \text{Spf}(A)$ is a surjective map of sheaves) and the proof is complete. $\square$

0GDA Lemma 88.22.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume

(1) X and Y are locally Noetherian,
(2) f locally of finite type,
(3) $\Delta_f : X \to X \times_Y X$ is rig-étale and rig-surjective.

Then f is rig surjective if and only if every adic morphism $\text{Spf}(R) \to Y$ where R is a complete discrete valuation ring lifts to a morphism $\text{Spf}(R) \to X$.

Proof. One direction is trivial. For the other, suppose that $\text{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\text{Spf}(R') \to \text{Spf}(R) \to X$ factoring through Y. Consider the fibre product diagram

$$\begin{tikzcd} \text{Spf}(R') \arrow[r] \arrow[dr] & \text{Spf}(R) \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\ & \text{Spf}(R) \arrow[r] & Y \end{tikzcd}$$

The morphism p is locally of finite type as a base change of f, see Formal Spaces, Lemma 87.24.4. The diagonal morphism Δ_p is the base change of Δ_f and hence is rig-étale and rig-surjective. By Lemma 88.22.2 the flat locus of $\text{Spf}(R) \times_Y X$ over R is either $\emptyset$ or equal to $\text{Spf}(R)$. However, since $\text{Spf}(R')$ factors through it we conclude it is not empty and hence we get a morphism $\text{Spf}(R) \to \text{Spf}(R) \times_Y X \to X$ as desired. $\square$

88.23. The completion functor

0GDB In this section we consider the following situation. First we fix a base scheme S. All rings, topological rings, schemes, algebraic spaces, and formal algebraic spaces and morphisms between these will be over S. Next, we fix an algebraic space X and a closed subset $T \subset |X|$. We denote $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. Picture

$$U \to X \quad |X| = |U| \amalg T$$

In this situation, given an algebraic space X' over X, i.e., an algebraic space X' endowed with a morphism $f : X' \to X$, then we denote $T' \subset |X'|$ the inverse image of T and we let $U' \subset X'$ be the open subspace with $|U'| = |X'| \setminus T'$. Picture

$$U' = f^{-1}U \qquad \begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d, "f"'] \\ U \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} {|U'|} \arrow[r] \arrow[d] & {|X'|} \arrow[d, "|f|"] & T' \arrow[l] \arrow[d] \\ {|U|} \arrow[r] & {|X|} & T \arrow[l] \end{tikzcd} \qquad T' = |f|^{-1}T$$

We will relate properties of f to properties of the induced morphism

$$f_{/T} : X'_{/T'} \longrightarrow X_{/T}$$

of formal completions. As indicated in the displayed formula, we will denote this morphism $f_{/T}$. We have already seen that $f_{/T}$ is representable by algebraic spaces in Formal Spaces, Lemma 87.14.4. In fact, as the proof of that lemma shows, the diagram

$$\begin{tikzcd} X'_{/T'} \arrow[r] \arrow[d, "f_{/T}"'] & X' \arrow[d, "f"] \\ X_{/T} \arrow[r] & X \end{tikzcd}$$

is cartesian. Please keep this fact in mind whilst reading the lemmas stated and proved below.

0AQ9 Lemma 88.23.1. In the situation above. If f is locally of finite type, then $f_{/T}$ is locally of finite type.

Proof. (Finite type morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.24.) Namely, suppose that $Z \to X$ is a morphism from a scheme into X such that $|Z|$ maps into T. From the cartesian square above we see that $Z \times_X X'$ is an algebraic space representing $Z \times_{X_{/T}} X'_{/T'}$. Since $Z \times_X X' \to Z$ is locally of finite type by Morphisms of Spaces, Lemma 67.23.3 we conclude. $\square$

0GI0 Lemma 88.23.2. In the situation above. If f is étale, then $f_{/T}$ is étale.

Proof. By the same argument as in the proof of Lemma 88.23.1 this follows from Morphisms of Spaces, Lemma 67.39.4. $\square$

0GDC Lemma 88.23.3. In the situation above. If f is a closed immersion, then $f_{/T}$ is a closed immersion.

Proof. (Closed immersions of formal algebraic spaces are discussed in Formal Spaces, Section 87.27.) By the same argument as in the proof of Lemma 88.23.1 this follows from Spaces, Lemma 65.12.3. $\square$

0GDD Lemma 88.23.4. In the situation above. If f is proper, then $f_{/T}$ is proper.

Proof. (Proper morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.31.) By the same argument as in the proof of Lemma 88.23.1 this follows from Morphisms of Spaces, Lemma 67.40.3. $\square$

0GDE Lemma 88.23.5. In the situation above. If f is quasi-compact, then $f_{/T}$ is quasi-compact.

Proof. (Quasi-compact morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.17.) We have to show that $(X'_{/T'})_{red} \to (X_{/T})_{red}$ is a quasi-compact morphism of algebraic spaces. By Formal Spaces, Lemma 87.14.5 this is the morphism $Z' \to Z$ where $Z' \subset X'$, resp. $Z \subset X$ is the reduced induced algebraic space structure on T', resp. T. It follows that $Z' \to f^{-1}Z = Z \times_X X'$ is a thickening (a closed immersion defining an isomorphism on underlying topological spaces). Since $Z \times_X X' \to Z$ is quasi-compact as a base change of f (Morphisms of Spaces, Lemma 67.8.4) we conclude that $Z' \to Z$ is too by More on Morphisms of Spaces, Lemma 76.10.1. $\square$

0GDF Remark 88.23.6. In the situation above consider the diagonal morphisms $\Delta_f : X' \to X' \times_X X'$ and $\Delta_{f_{/T}} : X'_{/T'} \to X'_{/T'} \times_{X_{/T}} X'_{/T'}$. It is easy to see that

$$X'_{/T'} \times_{X_{/T}} X'_{/T'} = (X' \times_X X')_{/T''}$$

as subfunctors of $X' \times_X X'$ where $T'' \subset |X' \times_X X'|$ is the inverse image of T. Hence we see that $\Delta_{f_{/T}} = (\Delta_f)_{/T''}$. We will use this below to show that properties of Δ_f are inherited by $\Delta_{f_{/T}}$.

0GDG Lemma 88.23.7. In the situation above. If f is (quasi-)separated, then $f_{/T}$ is too.

Proof. (Separation conditions on morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.30.) We have to show that if Δ_f is quasi-compact, resp. a closed immersion, then the same is true for $\Delta_{f_{/T}}$. This follows from the discussion in Remark 88.23.6 and Lemmas 88.23.5 and 88.23.3. $\square$

0GDH Lemma 88.23.8. In the situation above. If X is locally Noetherian, f is locally of finite type, and $U' \to U$ is smooth, then $f_{/T}$ is rig-smooth.

Proof. The strategy of the proof is this: reduce to the case where X and X' are affine, translate the affine case into algebra, and finally apply Lemma 88.4.3. We urge the reader to skip the details.

Choose a surjective étale morphism $W \to X$ with $W = \coprod W_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. For each i choose a surjective étale morphism $W'_i \to W_i \times_X X'$ where $W'_i = \coprod W'_{ij}$ is a disjoint union of affines. In particular $\coprod W'_{ij} \to X'$ is surjective and étale. Denote $f_{ij} : W_{ij} \to W_i$ the given morphism. Denote $T_i \subset W_i$ and $T'_{ij} \subset W_{ij}$ the inverse images of T. Since taking the completion along the inverse image of T produces cartesian diagrams (see above) we have $(W_i)_{/T_i} = W_i \times_X X_{/T}$ and similarly $(W'_{ij})_{/T'_{ij}} = W'_{ij} \times_{X'} X'_{/T'}$. Moreover, recall that $(W_i)_{/T_i}$ and $(W'_{ij})_{/T'_{ij}}$ are affine formal algebraic spaces. Hence $\{(W'_{ij})_{/T'_{ij}} \to X'_{/T'}\}$ is a covering as in Formal Spaces, Definition 87.11.1. By Lemma 88.18.2 we see that it suffices to prove that

$$(W'_{ij})_{/T'_{ij}} \longrightarrow (W_i)_{/T_i}$$

is rig-smooth. Observe that $W'_{ij} \to W_i$ is locally of finite type and induces a smooth morphism $W'_{ij} \setminus T'_{ij} \to W_i \setminus T_i$ (as this is true for f and these properties of morphisms are étale local on the source and target). Observe that W_i is locally Noetherian (as X is locally Noetherian and this property is étale local on the algebraic space). Hence it suffices to prove the lemma when X and X' are affine schemes.

Assume $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ are affine schemes. Since X is Noetherian, we see that A is Noetherian. The morphism f is given by a ring map $A \to A'$ of finite type. Let $I \subset A$ be an ideal cutting out T. Then IA' cuts out T'. Also $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus T$. Let $A^\wedge$ and $(A')^\wedge$ be the I-adic completions. We have $X_{/T} = \mathrm{Spf}(A^\wedge)$ and $X'_{/T'} = \mathrm{Spf}((A')^\wedge)$, see proof of Formal Spaces, Lemma 87.20.8. By Lemma 88.4.3 we see that $(A')^\wedge$ is rig-smooth over $(A.I)$ which in turn means that $A^\wedge \to (A')^\wedge$ is rig-smooth which finally implies that $X'_{/T'} \to X_{/T}$ is rig smooth by Lemma 88.18.2. ☐

0AR2 Lemma 88.23.9. In the situation above. If X is locally Noetherian, f is locally of finite type, and $U' \to U$ is étale, then $f_{/T}$ is rig-étale.

Proof. The proof is exactly the same as the proof of Lemma 88.23.8 except with Lemmas 88.4.3 and 88.18.2 replaced by Lemmas 88.8.4 and 88.20.2 ☐

0AQW Lemma 88.23.10. In the situation above. If X is locally Noetherian, f is proper, and $U' \to U$ is surjective, then $f_{/T}$ is rig-surjective.

Proof. (The statement makes sense by Lemma 88.23.1 and Formal Spaces, Lemma 87.20.8.) Let R be a complete discrete valuation ring with fraction field K. Let $p : \mathrm{Spf}(R) \to X_{/T}$ be an adic morphism of formal algebraic spaces. By Formal Spaces, Lemma 87.33.4 the composition $\mathrm{Spf}(R) \to X_{/T} \to X$ corresponds to a morphism $q : \mathrm{Spec}(R) \to X$ which maps $\mathrm{Spec}(K)$ into U. Since $U' \to U$ is proper and surjective we see that $\mathrm{Spec}(K) \times_U U'$ is nonempty and proper over K. Hence we can choose a field extension K'/K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & U \arrow[r] & X \end{tikzcd}$$

Let $R' \subset K'$ be a discrete valuation ring dominating R with fraction field K', see Algebra, Lemma 10.119.13. Since $\mathrm{Spec}(K) \to X$ extends to $\mathrm{Spec}(R) \to X$ we see by the valuative criterion of properness (Morphisms of Spaces, Lemma 67.44.1) that we can extend our K'-point of U' to a morphism $\mathrm{Spec}(R') \to X'$ over $\mathrm{Spec}(R) \to X$. It follows that the inverse image of T' in $\mathrm{Spec}(R')$ is the closed point and we find an adic morphism $\mathrm{Spf}((R')^\wedge) \to X'_{/T'}$ lifting p as desired (note that $(R')^\wedge$ is a complete discrete valuation ring by More on Algebra, Lemma 15.43.5). ☐

0GDI Lemma 88.23.11. In the situation above. If X is locally Noetherian, f is separated and locally of finite type, and $U' \to U$ is a monomorphism, then $\Delta_{f_{/T}}$ is rig-surjective.

Proof. The diagonal $\Delta_f : X' \to X' \times_X X'$ is a closed immersion and the restriction $U' \to U' \times_U U'$ of Δ_f is surjective. Hence the lemma follows from the discussion in Remark 88.23.6 and Lemma 88.23.10. ☐

88.24. Formal modifications

0GDJ In this section we define and study Artin's notion of a formal modification of locally Noetherian formal algebraic spaces. First, here is the definition.

0GDK Definition 88.24.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is a formal modification if

(1) f is a proper morphism (Formal Spaces, Definition 87.31.1),
(2) f is rig-étale,
(3) f is rig-surjective,
(4) $\Delta_f : X \to X \times_Y X$ is rig-surjective.

A typical example is given in Lemma 88.24.3 and indeed we will later show that every formal modification is "formal locally" of this type, see Lemma 88.29.2. Let us compare these conditions with those in Artin's paper.

0GDL Remark 88.24.2. In [Art70, Definition 1.7] a formal modification is defined as a proper morphism $f : X \to Y$ of locally Noetherian formal algebraic spaces satisfying the following three conditions[3]

(i) the Cramer and Jacobian ideal of f each contain an ideal of definition of X,
(ii) the ideal defining the diagonal map $\Delta : X \to X \times_Y X$ is annihilated by an ideal of definition of $X \times_Y X$, and
(iii) any adic morphism $\text{Spf}(R) \to Y$ lifts to $\text{Spf}(R) \to X$ whenever R is a complete discrete valuation ring.

Let us compare these to our list of conditions above.

Ad (i). Property (i) agrees with our condition that f be a rig-étale morphism: this follows from Lemma 88.8.2 part (7).

Ad (ii). Assume f is rig-étale. Then $\Delta_f : X \to X \times_Y X$ is rig-étale as a morphism of locally Noetherian formal algebraic spaces which are rig-étale over X (via id_X for the first one and via pr_1 for the second one). See Lemmas 88.20.5 and 88.20.7. Hence property (ii) agrees with our condition that Δ_f be rig-surjective by Lemma 88.21.13.

Ad (iii). Property (iii) does not quite agree with our notion of a rig-surjective morphism, as Artin requires all adic morphisms $\text{Spf}(R) \to Y$ to lift to morphisms into X whereas our notion of rig-surjective only asserts the existence of a lift after replacing R by an extension. However, since we already have that Δ_f is rig-étale and rig-surjective by (i) and (ii), these conditions are equivalent by Lemma 88.22.3.

0GDM Lemma 88.24.3. Let S, $f : X' \to X$, $T \subset |X|$, $U \subset X$, $T' \subset |X'|$, and $U' \subset X'$ be as in Section 88.23. If X is locally Noetherian, f is proper, and $U' \to U$ is an isomorphism, then $f_{/T} : X'_{/T'} \to X_{/T}$ is a formal modification.

Proof. By Formal Spaces, Lemmas 87.20.8 the source and target of the arrow are locally Noetherian formal algebraic spaces. The other conditions follow from Lemmas 88.23.4, 88.23.9, 88.23.10, and 88.23.11. $\square$

0GDN Lemma 88.24.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S which is a formal modification. Then for any adic morphism $Y' \to Y$ of locally Noetherian formal algebraic spaces, the base change $f' : X \times_Y Y' \to Y'$ is a formal modification.

[3]We will not completely translate these conditions into the language developed in the Stacks project. We hope nonetheless the discussion here will be useful to the reader.

Proof. The morphism f' is proper by Formal Spaces, Lemma 87.31.3. The morphism f' is rig-etale by Lemma 88.20.5. Then morphism f' is rig-surjective by Lemma 88.21.4. Set $X' = X \times_Y'$. The morphism $\Delta_{f'}$ is the base change of Δ_f by the adic morphism $X' \times_{Y'} X' \to X \times_Y X$. Hence $\Delta_{f'}$ is rig-surjective by Lemma 88.21.4. $\square$

88.25. Completions and morphisms, I

0GDP In this section we put some preliminary results on completions which we will use in the proof of Theorem 88.27.4. Although the lemmas stated and proved here are not trivial (some are based on our work on algebraization of rig-étale algebras), we still suggest the reader skip this section on a first reading.

0AR4 Lemma 88.25.1. Let $T \subset X$ be a closed subset of a Noetherian affine scheme X. Let W be a Noetherian affine formal algebraic space. Let $g : W \to X_{/T}$ be a rig-étale morphism. Then there exists an affine scheme X' and a finite type morphism $f : X' \to X$ étale over $X \setminus T$ such that there is an isomorphism $X'_{/f^{-1}T} \cong W$ compatible with $f_{/T}$ and g. Moreover, if $W \to X_{/T}$ is étale, then $X' \to X$ is étale.

Proof. The existence of X' is a restatement of Lemma 88.10.3. The final statement follows from More on Morphisms, Lemma 37.12.3. $\square$

0AR3 Lemma 88.25.2. Assume we have

(1) Noetherian affine schemes X, X', and Y,
(2) a closed subset $T \subset |X|$,
(3) a morphism $f : X' \to X$ locally of finite type and étale over $X \setminus T$,
(4) a morphism $h : Y \to X$,
(5) a morphism $\alpha : Y_{/T} \to X'_{/T}$ over $X_{/T}$ (see proof for notation).

Then there exists an étale morphism $b : Y' \to Y$ of affine schemes which induces an isomorphism $b_{/T} : Y'_{/T} \to Y_{/T}$ and a morphism $a : Y' \to X'$ over X such that $\alpha = a_{/T} \circ b_{/T}^{-1}$.

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism of schemes $g : V \to X$ associates the morphism $g_{/T} : V_{/g^{-1}T} \to X_{/T}$ of formal algebraic spaces; it is a functor from the category of schemes over X to the category of formal algebraic spaces over $X_{/T}$, see Section 88.23. Having said this, the lemma is just a reformulation of Lemma 88.8.7. $\square$

0AR6 Lemma 88.25.3. Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces. Let $T \subset |X|$ be closed. Assume that

(1) X is locally Noetherian,
(2) g is a monomorphism and locally of finite type,
(3) $f|_{X \setminus T} : X \setminus T \to Y$ factors through g, and
(4) $f_{/T} : X_{/T} \to Y$ factors through g,

then f factors through g.

Proof. Consider the fibre product $E = X \times_Y Z \to X$. By assumption the open immersion $X \setminus T \to X$ factors through E and any morphism $\varphi : X' \to X$ with $|\varphi|(|X'|) \subset T$ factors through E as well, see Formal Spaces, Section 87.14. By More on Morphisms of Spaces, Lemma 76.20.3 this implies that $E \to X$ is étale at every point of E mapping to a point of T. Hence $E \to X$ is an étale monomorphism, hence an open immersion (Morphisms of Spaces, Lemma 67.51.2). Then it follows that $E = X$ since our assumptions imply that $|X| = |E|$. $\square$

0GI1 Lemma 88.25.4. Let S be a scheme. Let X, W be algebraic spaces over S with X locally Noetherian. Let $T \subset |X|$ be a closed subset. Let $a, b : X \to W$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$ and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to W$. Then $a = b$.

Proof. Let E be the equalizer of a and b. Then E is an algebraic space and $E \to X$ is locally of finite type and a monomorphism, see Morphisms of Spaces, Lemma 67.4.1. Our assumptions imply we can apply Lemma 88.25.3 to the two morphisms $f = \text{id} : X \to X$ and $g : E \to X$ and the closed subset T of $|X|$. $\square$

0AR7 Lemma 88.25.5. Let S be a scheme. Let X, Y be locally Noetherian algebraic spaces over S. Let $T \subset |X|$ and $T' \subset |Y|$ be closed subsets. Let $a, b : X \to Y$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$, such that $|a|(T) \subset T'$ and $|b|(T) \subset T'$, and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to Y_{/T'}$. Then $a = b$.

Proof. Consequence of the more general Lemma 88.25.4. $\square$

0AR8 Lemma 88.25.6. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $s, t : R \to U$ be two morphisms of algebraic spaces over X. Assume

(1) R, U are locally of finite type over X,
(2) the base change of s and t to $X \setminus T$ is an étale equivalence relation, and
(3) the formal completion $(t_{/T}, s_{/T}) : R_{/T} \to U_{/T} \times_{X_{/T}} U_{/T}$ is an equivalence relation too (see proof for notation).

Then $(t, s) : R \to U \times_X U$ is an étale equivalence relation.

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism $f : X' \to X$ of algebraic spaces associates the morphism $f_{/T} : X'_{/f^{-1}T} \to X_{/T}$ of formal algebraic spaces, see Section 88.23. The morphisms $s, t : R \to U$ are étale over $X \setminus T$ by assumption. Since the formal completions of the maps $s, t : R \to U$ are étale, we see that s and t are étale for example by More on Morphisms, Lemma 37.12.3. Applying Lemma 88.25.3 to the morphisms $\text{id} : R \times_{U \times_X U} R \to R \times_{U \times_X U} R$ and $\Delta : R \to R \times_{U \times_X U} R$ we conclude that (t, s) is a monomorphism. Applying it again to $(t \circ \text{pr}_0, s \circ \text{pr}_1) : R \times_{s,U,t} R \to U \times_X U$ and $(t, s) : R \to U \times_X U$ we find that "transitivity" holds. We omit the proof of the other two axioms of an equivalence relation. $\square$

0AR9 Lemma 88.25.7. Let S be a scheme. Let X be a locally Noetherian algebraic space over S and let $T \subset |X|$ be a closed subset. Let $f : X' \to X$ be a morphism of algebraic spaces which is locally of finite type and étale outside of T. There exists a factorization

$$X' \longrightarrow X'' \longrightarrow X$$

of f with the following properties: $X'' \to X$ is locally of finite type, $X'' \to X$ is an isomorphism over $X \setminus T$, and $X'_{/T} \to X''_{/T}$ is an isomorphism (see proof for notation).

Proof. The notation using the subscript ${}_{/T}$ in the statement refers to the construction which to a morphism $f : X' \to X$ of algebraic spaces associates the morphism $f_{/T} : X'_{/f^{-1}T} \to X_{/T}$ of formal algebraic spaces, see Section 88.23. We will also use the notion $U \subset X$ and $U' \subset X'$ to denote the open subspaces with $|U| = |X| \setminus T$ and $U' = |X'| \setminus f^{-1}T$ introduced in Section 88.23.

After replacing X' by $X' \amalg U$ we may and do assume the image of $X' \to X$ contains U. Let

$$R = X' \amalg_{U'} (U' \times_U U')$$

be the pushout of $U' \to X'$ and the diagonal morphism $U' \to U' \times_U U' = U' \times_X U'$. Since $U' \to X$ is étale, this diagonal is an open immersion and we see that R is an algebraic space (this follows for example from Spaces, Lemma 65.8.5). The two projections $U' \times_U U' \to U'$ extend to R and we obtain two étale morphisms $s, t : R \to X'$. Checking on each piece separatedly we find that R is an étale equivalence relation on X'. Set $X'' = X'/R$ which is an algebraic space by Bootstrap, Theorem 80.10.1. By construction have the factorization as in the lemma and the morphism $X'' \to X$ is locally of finite type (as this can be checked étale locally, i.e., on X'). Since $U' \to U$ is a surjective étale morphism and since $s^{-1}(U') = t^{-1}(U') = U' \times_U U'$ we see that $U'' = U \times_X X'' \to U$ is an isomorphism. Finally, we have to show the morphism $X' \to X''$ induces an isomorphism $X'_{/T} \to X''_{/T}$. To see this, note that the formal completion of R along the inverse image of T is equal to the formal completion of X' along the inverse image of T by our choice of R! By our construction of the formal completion in Formal Spaces, Section 87.14 we have $X''_{/T} = (X'_{/T})/(R_{/T})$ as sheaves. Since $X'_{/T} = R_{/T}$ we conclude that $X'_{/T} = X''_{/T}$ and this finishes the proof. □

88.26. Rig glueing of morphisms

0GI2 Let X, W be algebraic spaces with X Noetherian. Let $Z \subset X$ be a closed subspace with open complement U. The proposition below says roughly speaking that

$$\{\text{morphisms } X \to W\} = \{\text{compatible morphisms } U \to W \text{ and } X_{/Z} \to W\}$$

where compatibility of $a : U \to W$ and $b : X_{/Z} \to W$ means that a and b define the same "morphism of rig-spaces". To introduce the category of "rig-spaces" requires a lot of work, but we don't need to do so in order to state precisely what the condition means in this case.

0GI3 Proposition 88.26.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset with complementary open subspace $U \subset X$. Let $f : X' \to X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \to U$ is an isomorphism. For any algebraic space W over S the map

$$\mathrm{Mor}_S(X, W) \longrightarrow \mathrm{Mor}_S(X', W) \times_{\mathrm{Mor}_S(X'_{/T}, W)} \mathrm{Mor}_S(X_{/T}, W)$$

is bijective.

Proof. Let $w' : X' \to W$ and $\hat{w} : X_{/T} \to W$ be morphisms which determine the same morphism $X'_{/T} \to W$ by composition with $X'_{/T} \to X$ and $X'_{/T} \to X_{/T}$. We have to prove there exists a unique morphism $w : X \to W$ whose composition with $X' \to X$ and $X_{/T} \to X$ recovers w' and $\hat{w}$. The uniqueness is immediate from Lemma 88.25.4.

The assumptions on T and f are preserved by base change by any étale morphism $X_1 \to X$ of algebraic spaces. Since formal algebraic spaces are sheaves for the étale topology and since we aready have the uniqueness, it suffices to prove existence after replacing X by the members of an étale covering. Thus we may assume X is an affine Noetherian scheme.

Assume X is an affine Noetherian scheme. We will construct the morphism $w : X \to W$ using the material in Pushouts of Spaces, Section 81.13. It makes sense to read a little bit of the material in that section before continuing the read the proof.

Set $X'' = X' \times_X X'$ and consider the two morphisms $a = w' \circ \text{pr}_1 : X'' \to W$ and $b = w' \circ \text{pr}_2 : X'' \to W$. Then we see that a and b agree over the open U and that $a_{/T}$ and $b_{a/T}$ agree (as these are both equal to the composition $X''_{/T} \to X_{/T} \to W$ where the second arrow is $\hat{w}$). Thus by Lemma 88.25.4 we see $a = b$.

Denote $Z \subset X$ the reduced induced closed subscheme structure on T. For $n \geq 1$ denote $Z_n \subset X$ the nth infinitesimal neighbourhood of Z. Denote $w_n = \hat{w}|_{Z_n} : Z_n \to W$ so that we have $\hat{w} = \text{colim}\, w_n$ on $X_{/T} = \text{colim}\, Z_n$. Set $Y_n = X' \amalg Z_n$. Consider the two projections

$$s_n, t_n : R_n = Y_n \times_X Y_n \longrightarrow Y_n$$

Let $Y_n \to X_n \to X$ be the coequalizer of s_n and t_n as in Pushouts of Spaces, Section 81.13 (in particular this coequalizer exists, has good properties, etc, see Pushouts of Spaces, Lemma 81.13.1). By the result $a = b$ of the previous parapgraph and the agreement of w' and $\hat{w}$ over $X'_{/T}$ we see that the morphism

$$w' \amalg w_n : Y_n \longrightarrow W$$

equalizes the morphisms s_n and t_n. Hence we see that for all $n \geq 1$ there is a morphism $w^n : X_n \to W$ compatible with w' and w_n. Moreover, for $m \geq 1$ the composition

$$X_n \to X_{n+m} \xrightarrow{w^{n+m}} W$$

is equal to w^n by construction (as the corresponding statement holds for $w' \amalg w_{n+m}$ and $w' \amalg w_n$). By Pushouts of Spaces, Lemma 81.13.4 and Remark 81.13.5 the system of algebraic spaces X_n is essentially constant with value X and we conclude. $\square$

88.27. Algebraization of rig-étale morphisms

0AR1 In this section we prove a generalization of the result on dilatations from the paper of Artin [Art70].

The notation in this section will agree with the notation in Section 88.23 except our algebraic spaces and formal algebraic spaces will be locally Noetherian.

Thus, we first fix a base scheme S. All rings, topological rings, schemes, algebraic spaces, and formal algebraic spaces and morphisms between these will be over S. Next, we fix a locally Noetherian algebraic space X and a closed subset $T \subset |X|$. We denote $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. Picture

$$U \to X \quad |X| = |U| \amalg T$$

Given a morphism of algebraic spaces $f : X' \to X$, we will use the notation $U' = f^{-1}U$, $T' = |f|^{-1}(T)$, and $f_{/T} : X'_{/T'} \to X_{/T}$ as in Section 88.23. We will sometimes write $X'_{/T}$ in stead of $X'_{/T'}$ and more generally for a morphism $a : X' \to X''$ of algebraic spaces over X we will denote $a_{/T} : X'_{/T} \to X''_{/T}$ the induced morphism of formal algebraic spaces obtained by completing the morphism a along the inverse images of T in X' and X''.

Given this setup we will consider the functor
(88.27.0.1)

0AR5
$$\left\{\begin{matrix} \text{morphisms of algebraic spaces} \\ f : X' \to X \text{ which are locally} \\ \text{of finite type and such that} \\ U' \to U \text{ is an isomorphism} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{morphisms } g : W \to X_{/T} \\ \text{of formal algebraic spaces} \\ \text{with } W \text{ locally Noetherian} \\ \text{and } g \text{ rig-étale} \end{matrix}\right\}$$

sending $f : X' \to X$ to $f_{/T} : X'_{/T'} \to X_{/T}$. This makes sense because $f_{/T}$ is rig-étale by Lemma 88.23.9.

0GDQ Lemma 88.27.1. In the situation above, let $X_1 \to X$ be a morphism of algebraic spaces with X_1 locally Noetherian. Denote $T_1 \subset |X_1|$ the inverse image of T and $U_1 \subset X_1$ the inverse image of U. We denote

(1) $\mathcal{C}_{X,T}$ the category whose objects are morphisms of algebraic spaces $f : X' \to X$ which are locally of finite type and such that $U' = f^{-1}U \to U$ is an isomorphism,
(2) $\mathcal{C}_{X_1,T_1}$ the category whose objects are morphisms of algebraic spaces $f_1 : X'_1 \to X_1$ which are locally of finite type and such that $f_1^{-1}U_1 \to U_1$ is an isomorphism,
(3) $\mathcal{C}_{X_{/T}}$ the category whose objects are morphisms $g : W \to X_{/T}$ of formal algebraic spaces with W locally Noetherian and g rig-étale,
(4) $\mathcal{C}_{X_{1,/T_1}}$ the category whose objects are morphisms $g_1 : W_1 \to X_{1,/T_1}$ of formal algebraic spaces with W_1 locally Noetherian and g_1 rig-étale.

Then the diagram

$$\begin{tikzcd} \mathcal{C}_{X,T} \arrow[r] \arrow[d] & \mathcal{C}_{X_{/T}} \arrow[d] \\ \mathcal{C}_{X_1,T_1} \arrow[r] & \mathcal{C}_{X_{1,/T_1}} \end{tikzcd}$$

is commutative where the horizonal arrows are given by (88.27.0.1) and the vertical arrows by base change along $X_1 \to X$ and along $X_{1,/T_1} \to X_{/T}$.

Proof. This follows immediately from the fact that the completion functor $(h : Y \to X) \mapsto Y_{/T} = Y_{/|h|^{-1}T}$ on the category of algebraic spaces over X commutes with fibre products. $\square$

0GDR Lemma 88.27.2. In the situation above. Let $f : X' \to X$ be a morphism of algebraic spaces which is locally of finite type and an isomorphism over U. Let $g : Y \to X$ be a morphism with Y locally Noetherian. Then completion defines a bijection

$$\mathrm{Mor}_X(Y, X') \longrightarrow \mathrm{Mor}_{X_{/T}}(Y_{/T}, X'_{/T})$$

In particular, the functor (88.27.0.1) is fully faithful.

Proof. Let $a, b : Y \to X'$ be morphisms over X such that $a_{/T} = b_{/T}$. Then we see that a and b agree over the open subspace $g^{-1}U$ and after completion along $g^{-1}T$.

Hence $a = b$ by Lemma 88.25.5. In other words, the completion map is always injective.

Let $\alpha : Y_{/T} \to X'_{/T}$ be a morphism of formal algebraic spaces over $X_{/T}$. We have to prove there exists a morphism $a : Y \to X'$ over X such that $\alpha = a_{/T}$. The proof proceeds by a standard but cumbersome reduction to the affine case and then applying Lemma 88.25.2.

Let $\{h_i : Y_i \to Y\}$ be an étale covering of algebraic spaces. If we can find for each i a morphism $a_i : Y_i \to X'$ over X whose completion $(a_i)_{/T} : (Y_i)_{/T} \to X'_{/T}$ is equal to $\alpha \circ (h_i)_{/T}$, then we get a morphism $a : Y \to X'$ with $\alpha = a_{/T}$. Namely, we first observe that $(a_i)_{/T} \circ \text{pr}_1 = (a_j)_{/T} \circ \text{pr}_2$ as morphisms $(Y_i \times_Y Y_j)_{/T} \to X'_{/T}$ by the agreement with α (this uses that completion ${}_{/T}$ commutes with fibre products). By the injectivity already proven this shows that $a_i \circ \text{pr}_1 = a_j \circ \text{pr}_2$ as morphisms $Y_i \times_Y Y_j \to X'$. Since X' is an fppf sheaf this means that the collection of morphisms a_i descends to a morphism $a : Y \to X'$. We have $\alpha = a_{/T}$ because $\{(a_i)_{/T} : (Y_i)_{/T} \to X'_{/T}\}$ is an étale covering.

By the result of the previous paragraph, to prove existence, we may assume that Y is affine and that $g : Y \to X$ factors as $g_1 : Y \to X_1$ and an étale morphism $X_1 \to X$ with X_1 affine. Then we can consider $T_1 \subset |X_1|$ the inverse image of T and we can set $X'_1 = X' \times_X X_1$ with projection $f_1 : X'_1 \to X_1$ and

$$\alpha_1 = (\alpha, (g_1)_{/T_1}) : Y_{/T_1} = Y_{/T} \longrightarrow X'_{/T} \times_{X_{/T}} (X_1)_{/T_1} = (X'_1)_{/T_1}$$

We conclude that it suffices to prove the existence for α_1 over X_1, in other words, we may replace $X, T, X', Y, f, g, \alpha$ by $X_1, T_1, X'_1, Y, g_1, \alpha_1$. This reduces us to the case described in the next paragraph.

Assume Y and X are affine. Recall that $(Y_{/T})_{red}$ is an affine scheme (isomorphic to the reduced induced scheme structure on $g^{-1}T \subset Y$, see Formal Spaces, Lemma 87.14.5). Hence $\alpha_{red} : (Y_{/T})_{red} \to (X'_{/T})_{red}$ has quasi-compact image E in $f^{-1}T$ (this is the underlying topological space of $(X'_{/T})_{red}$ by the same lemma as above). Thus we can find an affine scheme V and an étale morpism $h : V \to X'$ such that the image of h contains E. Choose a solid cartesian diagram

$$\begin{tikzcd}
Y'_{/T} \arrow[r, dotted] \arrow[rd, dotted] & W \arrow[r] \arrow[d] & V_{/T} \arrow[d, "h_{/T}"] \\
 & Y_{/T} \arrow[r, "\alpha"] & X'_{/T}
\end{tikzcd}$$

By construction, the morphism $W \to Y_{/T}$ is representable by algebraic spaces, étale, and surjective (surjectivity can be seen by looking at the reductions, see Formal Spaces, Lemma 87.12.4). By Lemma 88.25.1 we can write $W = Y'_{/T}$ for $Y' \to Y$ étale and Y' affine. This gives the dotted arrows in the diagram. Since $W \to Y_{/T}$ is surjective, we see that the image of $Y' \to Y$ contains $g^{-1}T$. Hence $\{Y' \to Y, Y \setminus g^{-1}T \to Y\}$ is an étale covering. As f is an isomorphism over U we have a (unique) morphism $Y \setminus g^{-1}T \to X'$ over X agreeing with α on completions (as the completion of $Y \setminus g^{-1}T$ is empty). Thus it suffices to prove the existence for Y' which reduces us to the case studied in the next paragraph.

By the result of the previous paragraph, we may assume that Y is affine and that α factors as $Y_{/T} \to V_{/T} \to X'_{/T}$ where V is an affine scheme étale over X'. We may still replace Y by the members of an affine étale covering. By Lemma 88.25.2 we may find an étale morphism $b : Y' \to Y$ of affine schemes which induces an isomorphism $b_{/T} : Y'_{/T} \to Y_{/T}$ and a morphism $c : Y' \to V$ such that $c_{/T} \circ b_{/T}^{-1}$ is the given morphism $Y_{/T} \to V_{/T}$. Setting $a' : Y' \to X'$ equal to the composition of c and $V \to X'$ we find that $a'_{/T} = \alpha \circ b_{/T}$, in other words, we have existence for Y' and $\alpha \circ b_{/T}$. Then we are done by replacing considering once more the étale covering $\{Y' \to Y, Y \setminus g^{-1}T \to Y\}$. □

0ARA Lemma 88.27.3. In the situation above. Assume X is affine. Then the functor (88.27.0.1) is an equivalence.

Before we prove this lemma let us discuss an example. Suppose that $S = \text{Spec}(k)$, $X = \mathbf{A}^1_k$, and $T = \{0\}$. Then $X_{/T} = \text{Spf}(k[[x]])$. Let $W = \text{Spf}(k[[x]] \times k[[x]])$. Then the corresponding $f : X' \to X$ is the affine line with zero doubled mapping to the affine line (Schemes, Example 26.14.3). Moreover, this is the output of the construction in Lemma 88.25.7 starting with $X \amalg X$ over X.

Proof. We already know the functor is fully faithful, see Lemma 88.27.2. Essential surjectivity. Let $g : W \to X_{/T}$ be a morphism of formal algebraic spaces with W locally Noetherian and g rig-étale. We will prove W is in the essential image in a number of steps.

Step 1: W is an affine formal algebraic space. Then we can find $U \to X$ of finite type and étale over $X \setminus T$ such that $U_{/T}$ is isomorphic to W, see Lemma 88.25.1. Thus we see that W is in the essential image by Lemma 88.25.7.

Step 2: W is separated. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 87.11.1. By Step 1 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Say $W_i = (X'_i)_{/T}$ and $W_i \times_W W_j = (X'_{ij})_{/T}$. By fully faithfulness we obtain morphisms $t_{ij} : X'_{ij} \to X'_i$ and $s_{ij} : X'_{ij} \to X'_j$ matching the projections $W_i \times_W W_j \to W_i$ and $W_i \times_W W_j \to W_j$. Consider the structure

$$R = \coprod X'_{ij}, \quad V = \coprod X'_i, \quad s = \coprod s_{ij}, \quad t = \coprod t_{ij}$$

(We can't use the letter U as it has already been used.) Applying Lemma 88.25.6 we find that $(t, s) : R \to V \times_X V$ defines an étale equivalence relation on V over X. Thus we can take the quotient $X' = V/R$ and it is an algebraic space, see Bootstrap, Theorem 80.10.1. Since completion commutes with fibre products and taking quotient sheaves, we find that $X'_{/T} \cong W$ as formal algebraic spaces over $X_{/T}$.

Step 3: W is general. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 87.11.1. The formal algebraic spaces W_i and $W_i \times_W W_j$ are separated. Hence by Step 2 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Then we argue exactly as in the previous paragraph to see that W is in the essential image as well. This concludes the proof. □

0ARB Theorem 88.27.4. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. The completion functor (88.27.0.1)

$$\left\{\begin{matrix} \text{morphisms of algebraic spaces} \\ f : X' \to X \text{ which are locally} \\ \text{of finite type and such that} \\ f^{-1}U \to U \text{ is an isomorphism} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{morphisms } g : W \to X_{/T} \\ \text{of formal algebraic spaces} \\ \text{with } W \text{ locally Noetherian} \\ \text{and } g \text{ rig-étale} \end{matrix}\right\}$$

sending $f : X' \to X$ to $f_{/T} : X'_{/T'} \to X_{/T}$ is an equivalence.

Proof. The functor is fully faithful by Lemma 88.27.2. Let $g : W \to X_{/T}$ be a morphism of formal algebraic spaces with W locally Noetherian and g rig-étale. We will prove W is in the essential image to finish the proof.

Choose an étale covering $\{X_i \to X\}$ with X_i affine for all i. Denote $U_i \subset X_i$ the inverse image of U and denote $T_i \subset X_i$ the inverse image of T. Recall that $(X_i)_{/T_i} = (X_i)_{/T} = (X_i \times_X X)_{/T}$ and $W_i = X_i \times_X W = (X_i)_{/T} \times_{X_{/T}} W$, see Lemma 88.27.1. Observe that we obtain isomorphisms

$$\alpha_{ij} : W_i \times_{X_{/T}} (X_j)_{/T} \longrightarrow (X_i)_{/T} \times_{X_{/T}} W_j$$

satisfying a suitable cocycle condition. By Lemma 88.27.3 applied to $X_i, T_i, U_i, W_i \to (X_i)_{/T}$ there exists a morphism $X'_i \to X_i$ of algebraic spaces which is locally of finite type and an isomorphism over U_i and an isomorphism $\beta_i : (X'_i)_{/T} \cong W_i$ over $(X_i)_{/T}$. By fully faithfullness we find an isomorphism

$$a_{ij} : X'_i \times_X X_j \longrightarrow X_i \times_X X'_j$$

over $X_i \times_X X_j$ such that $\alpha_{ij} = \beta_j|_{X_i \times_X X_j} \circ (a_{ij})_{/T} \circ \beta_i^{-1}|_{X_i \times_X X_j}$. By fully faithfulness again (this time over $X_i \times_X X_j \times_X X_k$) we see that these morphisms a_{ij} satisfy the same cocycle condition as satisfied by the α_{ij}. In other words, we obtain a descent datum (as in Descent on Spaces, Definition 74.22.3) (X'_i, a_{ij}) relative to the family $\{X_i \to X\}$. By Bootstrap, Lemma 80.11.3, this descent datum is effective. Thus we find a morphism $f : X' \to X$ of algebraic spaces and isomorphisms $h_i : X' \times_X X_i \to X'_i$ over X_i such that $a_{ij} = h_j|_{X_i \times_X X_j} \circ h_i^{-1}|_{X_i \times_X X_j}$. The reader can check that the ensuing isomorphisms

$$(X' \times_X X_i)_{/T} \xrightarrow{\beta_i \circ (h_i)_{/T}} W_i$$

over X_i glue to an isomorphism $X'_{/T} \to W$ over $X_{/T}$; some details omitted. $\square$

88.28. Completions and morphisms, II

0GDS To obtain Artin's theorem on dilatations, we need to match formal modifications with actual modifications in the correspondence given by Theorem 88.27.4. We urge the reader to skip this section.

0ARU Lemma 88.28.1. With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is quasi-compact if and only if g is quasi-compact.

Proof. If f is quasi-compact, then g is quasi-compact by Lemma 88.23.5. Conversely, assume g is quasi-compact. Choose an étale covering $\{X_i \to X\}$ with X_i affine. It suffices to prove that the base change $X' \times_X X_i \to X_i$ is quasi-compact, see Morphisms of Spaces, Lemma 67.8.8. By Formal Spaces, Lemma 87.17.3 the base changes $W_i \times_{X_{/T}} (X_i)_{/T} \to (X_i)_{/T}$ are quasi-compact. By Lemma 88.27.1 we reduce to the case described in the next paragraph.

Assume X is affine and $g : W \to X_{/T}$ quasi-compact. We have to show that X' is quasi-compact. Let $V \to X'$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then $V_{/T} \to X'_{/T} = W$ is a surjective étale morphism. Since W is quasi-compact, then we can find a finite subset $J' \subset J$ such that $\coprod_{j \in J'} (V_j)_{/T} \to W$ is surjective. Then it follows that

$$U \amalg \coprod\nolimits_{j \in J'} V_j \longrightarrow X'$$

is surjective (and hence X' is quasi-compact). Namely, we have $|X'| = |U| \amalg |W_{red}|$ as $X'_{/T} = W$. □

0ARV Lemma 88.28.2. With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is quasi-separated if and only if g is so.

Proof. If f is quasi-separated, then g is quasi-separated by Lemma 88.23.7. Conversely, assume g is quasi-separated. We have to show that f is quasi-separated. Exactly as in the proof of Lemma 88.28.1 we may check this over the members of a étale covering of X by affine schemes using Morphisms of Spaces, Lemma 67.4.12 and Formal Spaces, Lemma 87.30.5. Thus we may and do assume X is affine.

Let $V \to X'$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. To show that X' is quasi-separated, it suffices to show that $V_j \times_{X'} V_{j'}$ is quasi-compact for all $j, j' \in J$. Since W is quasi-separated the fibre products $(V_j \times_Y V_{j'})_{/T} = (V_j)_{/T} \times_{X'_{/T}} (V_{j'})_{/T}$ are quasi-compact for all $j, j' \in J$. Since X is Noetherian affine and $U' \to U$ is an isomorphism, we see that

$$(V_j \times_{X'} V_{j'}) \times_X U = (V_j \times_X V_{j'}) \times_X U$$

is quasi-compact. Hence we conclude by the equality

$$|V_j \times_{X'} V_{j'}| = |(V_j \times_{X'} V_{j'}) \times_X U| \amalg |(V_j \times_{X'} V_{j'})_{/T,red}|$$

and the fact that a formal algebraic space is quasi-compact if and only if its associated reduced algebraic space is so. □

0ARW Lemma 88.28.3. With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is separated $\Leftrightarrow$ g is separated and $\Delta_g : W \to W \times_{X_{/T}} W$ is rig-surjective.

Proof. If f is separated, then g is separated and Δ_g is rig-surjective by Lemmas 88.23.7 and 88.23.11. Assume g is separated and Δ_g is rig-surjective. Exactly as in the proof of Lemma 88.28.1 we may check this over the members of a étale covering of X by affine schemes using Morphisms of Spaces, Lemma 67.4.4 (locality on the base of being separated for morphisms of algebraic spaces), Formal Spaces, Lemma 87.30.2 (being separated for morphisms of formal algebraic spaces is preserved by base change), and Lemma 88.21.4 (being rig-surjective is preserved by base change). Thus we may and do assume X is affine. Furthermore, we already know that $f : X' \to X$ is quasi-separated by Lemma 88.28.2.

By Cohomology of Spaces, Lemma 69.19.1 and Remark 69.19.3 it suffices to show that given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "p"] \arrow[ur, dashed] & X' \times_X X' \end{tikzcd}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute (as this will give the uniqueness part of the valuative criterion). Let $h : \mathrm{Spec}(R) \to X$ be the composition of p with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset U$. This case is trivial because $U' = X' \times_X U \to U$ is an isomorphism. Case II: h maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $g : W \to X_{/T}$ is separated. Namely, if Z denotes the reduced induced closed subspace structure on T, then h factors through Z and

$$W \times_{X_{/T}} Z = X' \times_X Z \longrightarrow Z$$

is separated by assumption (and for example Formal Spaces, Lemma 87.30.5) which implies we get the lifting property by Cohomology of Spaces, Lemma 69.19.1 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$p_{/T} : \mathrm{Spf}(R) \longrightarrow (X' \times_X X')_{/T} = W \times_{X_{/T}} W$$

is an adic morphism (by Formal Spaces, Lemma 87.14.4 and Definition 87.23.2). Hence our assumption that $\Delta_g : W \to W \times_{X_{/T}} W$ is rig-surjective implies we can lift $p_{/T}$ to a morphism $\mathrm{Spf}(R) \to W = X'_{/T}$, see Lemma 88.21.11. Algebraizing the composition $\mathrm{Spf}(R) \to X'$ using Formal Spaces, Lemma 87.33.3 we find a morphism $\mathrm{Spec}(R) \to X'$ lifting p as desired. □

0ARX Lemma 88.28.4. With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is proper if and only if g is a formal modification (Definition 88.24.1).

Proof. If f is proper, then g is a formal modification by Lemma 88.24.3. Assume g is a formal modification. By Lemmas 88.28.1 and 88.28.3 we see that f is quasi-compact and separated.

By Cohomology of Spaces, Lemma 69.19.2 and Remark 69.19.3 it suffices to show that given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r, "p"] \arrow[ur, dashed] & X \end{tikzcd}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute. There are three cases: Case I: $p(\mathrm{Spec}(R)) \subset U$. This case is trivial because $U' \to U$ is an isomorphism. Case II: p maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $g : W \to X_{/T}$ is proper. Namely, if Z denotes the reduced induced closed subspace structure on T, then p factors through Z and

$$W \times_{X_{/T}} Z = X' \times_X Z \longrightarrow Z$$

is proper by assumption which implies we get the lifting property by Cohomology of Spaces, Lemma 69.19.2 applied to the displayed arrow. Case III: $p(\mathrm{Spec}(K))$ is not in T but p maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$p_{/T} : \mathrm{Spf}(R) \longrightarrow X'_{/T} = W$$

is an adic morphism (by Formal Spaces, Lemma 87.14.4 and Definition 87.23.2). Hence our assumption that $g : W \to X_{/T}$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $\mathrm{Spf}(R') \to W = X'_{/T}$ for some extension of complete discrete valuation rings $R \subset R'$. Algebraizing the composition $\mathrm{Spf}(R') \to X'$ using Formal Spaces, Lemma 87.33.3 we find a morphism $\mathrm{Spec}(R') \to X'$ lifting p as desired. □

0GI4 Lemma 88.28.5. With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is étale if and only if g is étale.

Proof. If f is étale, then g is étale by Lemma 88.23.2. Conversely, assume g is étale. Since f is an isomorphism over U we see that f is étale over U. Thus it suffices to prove that f is étale at any point of X' lying over T. Denote $Z \subset X$ the reduced closed subspace whose underlying topological space is $|Z| = T \subset |X|$, see Properties of Spaces, Definition 66.12.5. Letting $Z_n \subset X$ be the nth infinitesimal neighbourhood we have $X_{/T} = \mathrm{colim}\, Z_n$. Since $X'_{/T} = W \to X_{/T}$ we conclude that $f^{-1}(Z_n) = X' \times_X Z_n \to Z_n$ is étale by the assumed étaleness of g. By More on Morphisms of Spaces, Lemma 76.20.3 we conclude that f is étale at points lying over T. □

88.29. Artin's theorem on dilatations

0GDT In this section we use a different font for formal algebraic spaces to stress the similarity of the statements with the corresponding statements in [Art70]. Here is the first main theorem of this chapter.

0GDU Theorem 88.29.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $\mathfrak{X} = X_{/T}$ be the formal completion of X along T. Let [Art70, Theorem 3.2]

$$\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$$

be a formal modification (Definition 88.24.1). Then there exists a unique proper morphism $f : X' \to X$ which is an isomorphism over the complement of T in X whose completion $f_{/T}$ recovers $\mathfrak{f}$.

Proof. This follows from Theorem 88.27.4 and Lemma 88.28.4. □

Here is the characterization of formal modifcations as promised in Section 88.24.

0GDV Lemma 88.29.2. Let S be a scheme. Let $\mathfrak{X}' \to \mathfrak{X}$ be a formal modification (Definition 88.24.1) of locally Noetherian formal algebraic spaces over S. Given

(1) any adic Noetherian topological ring A,
(2) any adic morphism $\mathrm{Spf}(A) \longrightarrow \mathfrak{X}$

there exists a proper morphism $X \to \mathrm{Spec}(A)$ of algebraic spaces and an isomorphism

$$\mathrm{Spf}(A) \times_{\mathfrak{X}} \mathfrak{X}' \longrightarrow X_{/Z}$$

over $\mathrm{Spf}(A)$ of the base change of $\mathfrak{X}$ with the formal completion of X along the "closed fibre" $Z = X \times_{\mathrm{Spec}(A)} \mathrm{Spf}(A)_{red}$ of X over A.

Proof. The morphism $\mathrm{Spf}(A) \times_{\mathfrak{X}} \mathfrak{X}' \to \mathrm{Spf}(A)$ is a formal modification by Lemma 88.24.4. Hence this follows from Theorem 88.29.1. $\square$

88.30. Application to modifications

0AS1 Let A be a Noetherian ring and let $I \subset A$ be an ideal. We set $X = \mathrm{Spec}(A)$ and $U = X \setminus V(I)$. In this section we will consider the category

0AS2 (88.30.0.1)
$$\left\{ f : X' \longrightarrow X \;\middle|\; \begin{array}{c} X' \text{ is an algebraic space} \\ f \text{ is locally of finite type} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from X'/X to X''/X will be a morphism of algebraic spaces $X' \to X''$ over X.

Let $A \to B$ be a homomorphism of Noetherian rings and let $J \subset B$ be an ideal such that $J = \sqrt{IB}$. Then base change along the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives a functor from the category (88.30.0.1) for A to the category (88.30.0.1) for B.

0AE5 Lemma 88.30.1. Let $A \to B$ be a ring homomorphism of Noetherian rings inducing an isomorphism on I-adic completions for some ideal $I \subset A$ (for example if B is the I-adic completion of A). Then base change defines an equivalence of categories between the category (88.30.0.1) for (A, I) with the category (88.30.0.1) for (B, IB).

Proof. Set $X = \mathrm{Spec}(A)$ and $T = V(I)$. Set $X_1 = \mathrm{Spec}(B)$ and $T_1 = V(IB)$. By Theorem 88.27.4 (in fact we only need the affine case treated in Lemma 88.27.3) the category (88.30.0.1) for X and T is equivalent to the the category of rig-étale morphisms $W \to X_{/T}$ of locally Noetherian formal algebraic spaces. Similarly, the the category (88.30.0.1) for X_1 and T_1 is equivalent to the category of rig-étale morphisms $W_1 \to X_{1,/T_1}$ of locally Noetherian formal algebraic spaces. Since $X_{/T} = \mathrm{Spf}(A^\wedge)$ and $X_{1,/T_1} = \mathrm{Spf}(B^\wedge)$ (Formal Spaces, Lemma 87.14.6) we see that these categories are equivalent by our assumption that $A^\wedge \to B^\wedge$ is an isomorphism. We omit the verification that this equivalence is given by base change. $\square$

0BH5 Lemma 88.30.2. Notation and assumptions as in Lemma 88.30.1. Let $f : X' \to \mathrm{Spec}(A)$ correspond to $g : Y' \to \mathrm{Spec}(B)$ via the equivalence. Then f is quasi-compact, quasi-separated, separated, proper, finite, and add more here if and only if g is so.

Proof. You can deduce this for the statements quasi-compact, quasi-separated, separated, and proper by using Lemmas 88.28.1 88.28.2, 88.28.3, 88.28.2, and 88.28.4 to translate the corresponding property into a property of the formal completion and using the argument of the proof of Lemma 88.30.1. However, there is a direct argument using fpqc descent as follows. First, you can reduce to proving the lemma for $A \to A^\wedge$ and $B \to B^\wedge$ since $A^\wedge \to B^\wedge$ is an isomorphism. Then note that $\{U \to \mathrm{Spec}(A), \mathrm{Spec}(A^\wedge) \to \mathrm{Spec}(A)\}$ is an fpqc covering with $U = \mathrm{Spec}(A) \setminus V(I)$

as before. The base change of f by $U \to \mathrm{Spec}(A)$ is id_U by definition of our category (88.30.0.1). Let P be a property of morphisms of algebraic spaces which is fpqc local on the base (Descent on Spaces, Definition 74.10.1) such that P holds for identity morphisms. Then we see that P holds for f if and only if P holds for g. This applies to P equal to quasi-compact, quasi-separated, separated, proper, and finite by Descent on Spaces, Lemmas 74.11.1, 74.11.2, 74.11.18, 74.11.19, and 74.11.23. □

0AF7 **Lemma 88.30.3.** Let $A \to B$ be a local map of local Noetherian rings such that

(1) $A \to B$ is flat,
(2) $\mathfrak{m}_B = \mathfrak{m}_A B$, and
(3) $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$

Then the base change functor from the category (88.30.0.1) for $(A, \mathfrak{m}_A)$ to the category (88.30.0.1) for $(B, \mathfrak{m}_B)$ is an equivalence.

Proof. The conditions signify that $A \to B$ induces an isomorphism on completions, see More on Algebra, Lemma 15.43.9. Hence this lemma is a special case of Lemma 88.30.1. □

0AE6 **Lemma 88.30.4.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f : X \to S$ be an object of (88.30.0.1). Then there exists a U-admissible blowup $S' \to S$ which dominates X.

Proof. Special case of More on Morphisms of Spaces, Lemma 76.39.5. □

88.31. Other chapters

CHAPTER 89

Resolution of Surfaces Revisited

0BH6 89.1. Introduction

0BH7 This chapter discusses resolution of singularities of Noetherian algebraic spaces of dimension 2. We have already discussed resolution of surfaces for schemes following Lipman [Lip78] in an earlier chapter. See Resolution of Surfaces, Section 54.1. Most of the results in this chapter are straightforward consequences of the results on schemes.

Unless specifically mentioned otherwise all geometric objects in this chapter will be algebraic spaces. Thus if we say "let $f : X \to Y$ be a modification" then this means that f is a morphism as in Spaces over Fields, Definition 72.8.1. Similarly for proper morphism, etc, etc.

89.2. Modifications

0BH8 Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. We set $S = \mathrm{Spec}(A)$ and $U = S \setminus \{\mathfrak{m}\}$. In this section we will consider the category

0AE2 (89.2.0.1)
$$\left\{ f : X \longrightarrow S \;\middle|\; \begin{array}{c} X \text{ is an algebraic space} \\ f \text{ is a proper morphism} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from X/S to X'/S will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over S. In Algebraization of Formal Spaces, Section 88.30 we have seen that this category only depends on the completion of A and we have proven some elementary properties of objects in this category. In this section we specifically study cases where $\dim(A) \leq 2$ or where the dimension of the closed fibre is at most 1.

0AE3 **Lemma 89.2.1.** Let $(A, \mathfrak{m}, \kappa)$ be a 2-dimensional Noetherian local domain such that $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is a normal scheme. Then any modification $f : X \to \mathrm{Spec}(A)$ is a morphism as in (89.2.0.1).

Proof. Let $f : X \to S$ be a modification. We have to show that $f^{-1}(U) \to U$ is an isomorphism. Since every closed point u of U has codimension 1, this follows from Spaces over Fields, Lemma 72.3.3. $\square$

0AGM **Lemma 89.2.2.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $g : X \to Y$ be a morphism in the category (89.2.0.1). If the induced morphism $X_\kappa \to Y_\kappa$ of special fibres is a closed immersion, then g is a closed immersion.

Proof. This is a special case of More on Morphisms of Spaces, Lemma 76.49.3. $\square$

0AYJ Lemma 89.2.3. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain of dimension ≥ 1. Let $f : X \to \mathrm{Spec}(A)$ be a morphism of algebraic spaces. Assume at least one of the following conditions is satisfied

(1) f is a modification (Spaces over Fields, Definition 72.8.1),
(2) f is an alteration (Spaces over Fields, Definition 72.8.3),
(3) f is locally of finite type, quasi-separated, X is integral, and there is exactly one point of $|X|$ mapping to the generic point of $\mathrm{Spec}(A)$,
(4) f is locally of finite type, X is decent, and the points of $|X|$ mapping to the generic point of $\mathrm{Spec}(A)$ are the generic points of irreducible components of $|X|$,
(5) add more here.

Then $\dim(X_\kappa) \leq \dim(A) - 1$.

Proof. Cases (1), (2), and (3) are special cases of (4). Choose an affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. The ring map $A \to B$ is of finite type. We have to show that $\dim(U_\kappa) \leq \dim(A) - 1$. Since X is decent, the generic points of irreducible components of U are the points lying over generic points of irreducible components of $|X|$, see Decent Spaces, Lemma 68.20.1. Hence the fibre of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over (0) is the (finite) set of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of B. Thus $A_f \to B_f$ is finite for some nonzero $f \in A$ (Algebra, Lemma 10.122.10). We conclude $\kappa(\mathfrak{q}_i)$ is a finite extension of the fraction field of A. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. Then

$$\dim(B_\mathfrak{q}) = \max \dim((B/\mathfrak{q}_i)_\mathfrak{q}) \leq \dim(A)$$

the inequality by the dimension formula for $A \subset B/\mathfrak{q}_i$, see Algebra, Lemma 10.113.1. However, the dimension of $B_\mathfrak{q}/\mathfrak{m}B_\mathfrak{q}$ (which is the local ring of U_κ at the corresponding point) is at least one less because the minimal primes $\mathfrak{q}_i$ are not in $V(\mathfrak{m})$. We conclude by Properties, Lemma 28.10.2. $\square$

0AGN Lemma 89.2.4. If $(A, \mathfrak{m}, \kappa)$ is a complete Noetherian local domain of dimension 2, then every modification of $\mathrm{Spec}(A)$ is projective over A.

Proof. By More on Morphisms of Spaces, Lemma 76.43.6 it suffices to show that the special fibre of any modification X of $\mathrm{Spec}(A)$ has dimension ≤ 1. This follows from Lemma 89.2.3. $\square$

89.3. Strategy

0BH9 Let S be a scheme. Let X be a decent algebraic space over S. Let $x_1, \ldots, x_n \in |X|$ be pairwise distinct closed points. For each i we pick an elementary étale neighbourhood $(U_i, u_i) \to (X, x_i)$ as in Decent Spaces, Lemma 68.11.4. This means that U_i is an affine scheme, $U_i \to X$ is étale, u_i is the unique point of U_i lying over x_i, and $\mathrm{Spec}(\kappa(u_i)) \to X$ is a monomorphism representing x_i. After shrinking U_i we may and do assume that for $j \neq i$ there does not exist a point of U_i mapping to x_j. Observe that $u_i \in U_i$ is a closed point.

Denote $\mathcal{C}_{X, \{x_1, \ldots, x_n\}}$ the category of morphisms of algebraic spaces $f : Y \to X$ which induce an isomorphism $f^{-1}(X \setminus \{x_1, \ldots, x_n\}) \to X \setminus \{x_1, \ldots, x_n\}$. For each i denote $\mathcal{C}_{U_i, u_i}$ the category of morphisms of algebraic spaces $g_i : Y_i \to U_i$ which induce an isomorphism $g_i^{-1}(U_i \setminus \{u_i\}) \to U_i \setminus \{u_i\}$. Base change defines an functor

0BHA (89.3.0.1) $$F : \mathcal{C}_{X, \{x_1, \ldots, x_n\}} \longrightarrow \mathcal{C}_{U_1, u_1} \times \ldots \times \mathcal{C}_{U_n, u_n}$$

To reduce at least some of the problems in this chapter to the case of schemes we have the following lemma.

0BHB Lemma 89.3.1. The functor F (89.3.0.1) is an equivalence.

Proof. For $n = 1$ this is Limits of Spaces, Lemma 70.19.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $g_i : Y_i \to U_i$ are objects of $\mathcal{C}_{U_i, u_i}$. Then by the case $n = 1$ we can find $f'_i : Y'_i \to X$ which are isomorphisms over $X \setminus \{x_i\}$ and whose base change to U_i is f_i. Then we can set

$$f : Y = Y'_1 \times_X \ldots \times_X Y'_n \to X$$

This is an object of $\mathcal{C}_{X, \{x_1, \ldots, x_n\}}$ whose base change by $U_i \to X$ recovers g_i. Thus the functor is essentially surjective. We omit the proof of fully faithfulness. $\square$

0BHC Lemma 89.3.2. Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1). If $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F, then f is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, if and only if g_i is so for $i = 1, \ldots, n$.

Proof. Follows from Limits of Spaces, Lemma 70.19.2. $\square$

0BHD Lemma 89.3.3. Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1). If $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F, then $Y_{x_i} \cong (Y_i)_{u_i}$ as algebraic spaces.

Proof. This is clear because $u_i \to x_i$ is an isomorphism. $\square$

89.4. Dominating by quadratic transformations

0AHG We define the blowup of a space at a point only if X is decent.

0BHE Definition 89.4.1. Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 68.14.6 we can represent x by a closed immersion $i : \text{Spec}(k) \to X$. The blowing up $X' \to X$ of X at x means the blowing up of X in the closed subspace $Z = i(\text{Spec}(k)) \subset X$.

In this generality the blowing up of X at x is not necessarily proper. However, if X is locally Noetherian, then it follows from Divisors on Spaces, Lemma 71.17.11 that the blowing up is proper. Recall that a locally Noetherian algebraic space is Noetherian if and only if it is quasi-compact and quasi-separated. Moreover, for a locally Noetherian algebraic space, being quasi-separated is equivalent to being decent (Decent Spaces, Lemma 68.14.1).

0BHF Lemma 89.4.2. Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1) and assume $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F. Then there exists a factorization

$$Y = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$$

of f where $Z_{j+1} \to Z_j$ is the blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over u_i.

Proof. A blowing up is a representable morphism. Hence in either case we inductively see that $Z_j \to X$ or $Z_{i,j} \to U_i$ is representable. Whence each Z_j or $Z_{i,j}$ is a decent algebraic space by Decent Spaces, Lemma 68.6.5. This shows that the assertions make sense (since blowing up is only defined for decent spaces). To prove the equivalence, let's start with a sequence of blowups $Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$. The first morphism $Z_1 \to X$ is given by blowing up one of the x_i, say x_1. Applying F to $Z_1 \to X$ we find a blowup $Z_{1,1} \to U_1$ at u_1 is the blowing up at u_1 and otherwise $Z_{i,0} = U_i$ for $i > 1$. In the next step, we either blow up one of the x_i, $i \geq 2$ on Z_1 or we pick a closed point z_1 of the fibre of $Z_1 \to X$ over x_1. In the first case it is clear what to do and in the second case we use that $(Z_1)_{x_1} \cong (Z_{1,1})_{u_1}$ (Lemma 89.3.3) to get a closed point $z_{1,1} \in Z_{1,1}$ corresponding to z_1. Then we set $Z_{1,2} \to Z_{1,1}$ equal to the blowing up in $z_{1,1}$. Continuing in this manner we construct the factorizations of each g_i.

Conversely, given sequences of blowups $Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$ we construct the sequence of blowing ups of X in exactly the same manner. $\square$

0BHG Lemma 89.4.3. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{j+1} \to X_j$ is the blowing up of X_j at a closed point x_j lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Pick elementary étale neighbourhoods $(U_i, u_i) \to (X, x_i)$ as in Section 89.3. For each i the restriction $\mathcal{I}_i = \mathcal{I}|_{U_i} \subset \mathcal{O}_{U_i}$ is a quasi-coherent sheaf of ideals supported at u_i. The local ring of U_i at u_i is regular and has dimension 2. Thus we may apply Resolution of Surfaces, Lemma 54.4.1 to find a sequence

$$X_{i,m_i} \to X_{i,m_i-1} \to \ldots \to X_1 \to X_{i,0} = U_i$$

of blowing ups in closed points lying over u_i such that $\mathcal{I}_i\mathcal{O}_{X_{i,m_i}}$ is invertible. By Lemma 89.4.2 we find a sequence of blowing ups

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

as in the statement of the lemma whose base change to our U_i produces the given sequences. It follows that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. Namely, we know this is true over $X \setminus \{x_1, \ldots, x_n\}$ and in an étale neighbourhood of the fibre of each x_i it is true by construction. $\square$

0BHH Lemma 89.4.4. Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $f : Y \to X$ be a proper morphism of algebraic spaces which is an isomorphism over $U = X \setminus T$. Then there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Morphisms of Spaces, Lemma 76.39.5 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 89.4.3 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors on Spaces, Lemma 71.17.5) we find the desired factorization. □

89.5. Dominating by normalized blowups

0BHI In this section we prove that a modification of a surface can be dominated by a sequence of normalized blowups in points.

0BHJ Definition 89.5.1. Let S be a scheme. Let X be a decent algebraic space over S satisfying the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1. Let $x \in |X|$ be a closed point. The normalized blowup of X at x is the composition $X'' \to X' \to X$ where $X' \to X$ is the blowup of X at x (Definition 89.4.1) and $X'' \to X'$ is the normalization of X'.

Here the normalization $X'' \to X'$ is defined as the algebraic space X' satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1 by Divisors on Spaces, Lemma 71.17.8. See Morphisms of Spaces, Definition 67.49.6 for the definition of the normalization.

In general the normalized blowing up need not be proper even when X is Noetherian. Recall that an algebraic space is Nagata if it has an étale covering by affines which are spectra of Nagata rings (Properties of Spaces, Definition 66.7.2 and Remark 66.7.3 and Properties, Definition 28.13.1).

0BHK Lemma 89.5.2. In Definition 89.5.1 if X is Nagata, then the normalized blowing up of X at x is a normal Nagata algebraic space proper over X.

Proof. The blowup morphism $X' \to X$ is proper (as X is locally Noetherian we may apply Divisors on Spaces, Lemma 71.17.11). Thus X' is Nagata (Morphisms of Spaces, Lemma 67.26.1). Therefore the normalization $X'' \to X'$ is finite (Morphisms of Spaces, Lemma 67.49.9) and we conclude that $X'' \to X$ is proper as well (Morphisms of Spaces, Lemmas 67.45.9 and 67.40.4). It follows that the normalized blowing up is a normal (Morphisms of Spaces, Lemma 67.49.8) Nagata algebraic space. □

Here is the analogue of Lemma 89.4.2 for normalized blowups.

0BHL Lemma 89.5.3. Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1) and assume $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F. Assume X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1. Then there exists a factorization

$$Y = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$$

of f where $Z_{j+1} \to Z_j$ is the normalized blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the normalized blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over u_i.

Proof. This follows by the exact same argument as used to prove Lemma 89.4.2. □

A Nagata algebraic space is locally Noetherian.

0BHM Lemma 89.5.4. Let S be a scheme. Let X be a Noetherian Nagata algebraic space over S with $\dim(X) = 2$. Let $f : Y \to X$ be a proper birational morphism. Then there exists a commutative diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] & \ldots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[d] \\ Y \arrow[rrrr] & & & & X \end{tikzcd}$$

where $X_0 \to X$ is the normalization and where $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point.

Proof. Although one can prove this lemma directly for algebraic spaces, we will continue the approach used above to reduce it to the case of schemes.

We will use that Noetherian algebraic spaces are quasi-separated and hence points have well defined residue fields (for example by Decent Spaces, Lemma 68.11.4). We will use the results of Morphisms of Spaces, Sections 67.26, 67.35, and 67.49 without further mention. We may replace Y by its normalization. Let $X_0 \to X$ be the normalization. The morphism $Y \to X$ factors through X_0. Thus we may assume that both X and Y are normal.

Assume X and Y are normal. The morphism $f : Y \to X$ is an isomorphism over an open which contains every point of codimension 0 and 1 in Y and every point of Y over which the fibre is finite, see Spaces over Fields, Lemma 72.3.3. Hence we see that there is a finite set of closed points $T \subset |X|$ such that f is an isomorphism over $X \setminus T$. By More on Morphisms of Spaces, Lemma 76.39.5 there exists an $X \setminus T$-admissible blowup $Y' \to X$ which dominates Y. After replacing Y by the normalization of Y' we see that we may assume that $Y \to X$ is representable.

Say $T = \{x_1, \ldots, x_r\}$. Pick elementary étale neighbourhoods $(U_i, u_i) \to (X, x_i)$ as in Section 89.3. For each i the morphism $Y_i = Y \times_X U_i \to U_i$ is a proper birational morphism which is an isomorphism over $U_i \setminus \{u_i\}$. Thus we may apply Resolution of Surfaces, Lemma 54.5.3 to find a sequence

$$X_{i,m_i} \to X_{i,m_i-1} \to \ldots \to X_1 \to X_{i,0} = U_i$$

of normalized blowing ups in closed points lying over u_i such that X_{i,m_i} dominates Y_i. By Lemma 89.5.3 we find a sequence of normalized blowing ups

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

as in the statement of the lemma whose base change to our U_i produces the given sequences. It follows that X_m dominates Y by the equivalence of categories of Lemma 89.3.1. □

89.6. Base change to the completion

0BHN The following simple lemma will turn out to be a useful tool in what follows.

0BHP Lemma 89.6.1. Let $(A, \mathfrak{m}, \kappa)$ be a local ring with finitely generated maximal ideal $\mathfrak{m}$. Let X be a decent algebraic space over A. Let $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ where $A^\wedge$ is the $\mathfrak{m}$-adic completion of A. For a point $q \in |Y|$ with image $p \in |X|$ lying over the closed point of $\mathrm{Spec}(A)$ the map $\mathcal{O}^h_{X,p} \to \mathcal{O}^h_{Y,q}$ of henselian local rings induces an isomorphism on completions.

Proof. This follows immediately from the case of schemes by choosing an elementary étale neighbourhood $(U, u) \to (X, p)$ as in Decent Spaces, Lemma 68.11.4, setting $V = U \times_X Y = U \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ and $v = (u, q)$. The case of schemes is Resolution of Surfaces, Lemma 54.11.1. $\square$

0BHQ Lemma 89.6.2. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type with X a decent algebraic space. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $y \in |Y|$ with image $x \in |X|$. Then

(1) if $\mathcal{O}^h_{Y,y}$ is regular, then $\mathcal{O}^h_{X,x}$ is regular,
(2) if y is in the closed fibre, then $\mathcal{O}^h_{Y,y}$ is regular $\Leftrightarrow \mathcal{O}^h_{X,x}$ is regular, and
(3) If X is proper over A, then X is regular if and only if Y is regular.

Proof. By étale localization the first two statements follow immediately from the counter part to this lemma for schemes, see Resolution of Surfaces, Lemma 54.11.2. For part (3), since $Y \to X$ is surjective (as $A \to A^\wedge$ is faithfully flat) we see that Y regular implies X regular by part (1). Conversely, if X is regular, then the henselian local rings of Y are regular for all points of the special fibre. Let $y \in |Y|$ be a general point. Since $|Y| \to |\mathrm{Spec}(A^\wedge)|$ is closed in the proper case, we can find a specialization $y \rightsquigarrow y_0$ with y_0 in the closed fibre. Choose an elementary étale neighbourhood $(V, v_0) \to (Y, y_0)$ as in Decent Spaces, Lemma 68.11.4. Since Y is decent we can lift $y \rightsquigarrow y_0$ to a specialization $v \rightsquigarrow v_0$ in V (Decent Spaces, Lemma 68.12.2). Then we conclude that $\mathcal{O}_{V,v}$ is a localization of $\mathcal{O}_{V,v_0}$ hence regular and the proof is complete. $\square$

0BHR Lemma 89.6.3. Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let X be an algebraic space over A. Assume

(1) A is analytically unramified (Algebra, Definition 10.162.9),
(2) X is locally of finite type over A,
(3) $X \to \mathrm{Spec}(A)$ is étale at every point of codimension 0 in X.

Then the normalization of X is finite over X.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Then $U \to \mathrm{Spec}(A)$ satisfies the assumptions and hence the conclusions of Resolution of Surfaces, Lemma 54.11.5. $\square$

89.7. Implied properties

0BHS In this section we prove that for a Noetherian integral algebraic space the existence of a regular alteration has quite a few consequences. This section should be skipped by those not interested in "bad" Noetherian algebraic spaces.

0BHT Lemma 89.7.1. Let S be a scheme. Let Y be a Noetherian integral algebraic space over S. Assume there exists an alteration $f : X \to Y$ with X regular. Then the normalization $Y^\nu \to Y$ is finite and Y has a dense open which is regular.

Proof. By étale localization, it suffices to prove this when $Y = \mathrm{Spec}(A)$ where A is a Noetherian domain. Let B be the integral closure of A in its fraction field. Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Spaces, Lemma 69.20.2 we see that C is a finite A-module. As X is normal (Properties of Spaces, Lemma 66.25.4) we see that C is normal domain (Spaces over Fields, Lemma 72.4.6). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian.

There exists a nonempty open $V \subset Y$ such that $f^{-1}V \to V$ is finite, see Spaces over Fields, Definition 72.8.3. After shrinking V we may assume that $f^{-1}V \to V$ is flat (Morphisms of Spaces, Proposition 67.32.1). Thus $f^{-1}V \to V$ is faithfully flat. Then V is regular by Algebra, Lemma 10.164.4. □

0BHU **Lemma 89.7.2.** Let $(A, \mathfrak{m}, \kappa)$ be a local Noetherian domain. Assume there exists an alteration $f : X \to \mathrm{Spec}(A)$ with X regular. Then

(1) there exists a nonzero $f \in A$ such that A_f is regular,
(2) the integral closure B of A in its fraction field is finite over A,
(3) the $\mathfrak{m}$-adic completion of B is a normal ring, i.e., the completions of B at its maximal ideals are normal domains, and
(4) the generic formal fibre of A is regular.

Proof. Parts (1) and (2) follow from Lemma 89.7.1. We have to redo part of the proof of that lemma in order to set up notation for the proof of (3). Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Spaces, Lemma 69.20.2 we see that C is a finite A-module. As X is normal (Properties of Spaces, Lemma 66.25.4) we see that C is normal domain (Spaces over Fields, Lemma 72.4.6). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian. By Resolution of Surfaces, Lemma 54.13.2 in order to prove (3) it suffices to show that the $\mathfrak{m}$-adic completion $C^\wedge$ is normal.

By Algebra, Lemma 10.97.8 the completion $C^\wedge$ is the product of the completions of C at the prime ideals of C lying over $\mathfrak{m}$. There are finitely many of these and these are the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of C. (The corresponding result for B explains the final statement of the lemma.) Thus replacing A by $C_{\mathfrak{m}_i}$ and X by $X_i = X \times_{\mathrm{Spec}(C)} \mathrm{Spec}(C_{\mathfrak{m}_i})$ we reduce to the case discussed in the next paragraph. (Note that $\Gamma(X_i, \mathcal{O}) = C_{\mathfrak{m}_i}$ by Cohomology of Spaces, Lemma 69.11.2.)

Here A is a Noetherian local normal domain and $f : X \to \mathrm{Spec}(A)$ is a regular alteration with $\Gamma(X, \mathcal{O}_X) = A$. We have to show that the completion $A^\wedge$ of A is a normal domain. By Lemma 89.6.2 $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular. Since $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ by Cohomology of Spaces, Lemma 69.11.2. We conclude that $A^\wedge$ is normal as before. Namely, Y is normal by Properties of Spaces, Lemma 66.25.4. It is connected because $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is local. Hence Y is normal and integral (as connected and normal implies integral for separated algebraic spaces). Thus $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is a normal domain by Spaces over Fields, Lemma 72.4.6. This proves (3).

Proof of (4). Let $\eta \in \mathrm{Spec}(A)$ denote the generic point and denote by a subscript η the base change to η. Since f is an alteration, the scheme X_η is finite and faithfully flat over η. Since $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular by Lemma 89.6.2 we see that Y_η is regular (as a limit of opens in Y). Then $Y_\eta \to \mathrm{Spec}(A^\wedge \otimes_A \kappa(\eta))$ is finite faithfully flat onto the generic formal fibre. We conclude by Algebra, Lemma 10.164.4. □

89.8. Resolution

0BHV Here is a definition.

0BHW Definition 89.8.1. Let S be a scheme. Let Y be a Noetherian integral algebraic space over S. A resolution of singularities of X is a modification $f : X \to Y$ such that X is regular.

In the case of surfaces we sometimes want a bit more information.

0BHX Definition 89.8.2. Let S be a scheme. Let Y be a 2-dimensional Noetherian integral algebraic space over S. We say Y has a resolution of singularities by normalized blowups if there exists a sequence

$$Y_n \to X_{n-1} \to \ldots \to Y_1 \to Y_0 \to Y$$

where

(1) Y_i is proper over Y for $i = 0, \ldots, n$,
(2) $Y_0 \to Y$ is the normalization,
(3) $Y_i \to Y_{i-1}$ is a normalized blowup for $i = 1, \ldots, n$, and
(4) Y_n is regular.

Observe that condition (1) implies that the normalization Y_0 of Y is finite over Y and that the normalizations used in the normalized blowing ups are finite as well. We finally come to the main theorem of this chapter.

0BHY Theorem 89.8.3. Let S be a scheme. Let Y be a two dimensional integral Noetherian algebraic space over S. The following are equivalent

(1) there exists an alteration $X \to Y$ with X regular,
(2) there exists a resolution of singularities of Y,
(3) Y has a resolution of singularities by normalized blowups,
(4) the normalization $Y^\nu \to Y$ is finite, Y^ν has finitely many singular points $y_1, \ldots, y_m \in |Y|$, and for each i the completion of the henselian local ring $\mathcal{O}^h_{Y^\nu, y_i}$ is normal.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate.

Let $X \to Y$ be an alteration with X regular. Then $Y^\nu \to Y$ is finite by Lemma 89.7.1. Consider the factorization $f : X \to Y^\nu$ from Morphisms of Spaces, Lemma 67.49.8. The morphism f is finite over an open $V \subset Y^\nu$ containing every point of codimension ≤ 1 in Y^ν by Spaces over Fields, Lemma 72.3.2. Then f is flat over V by Algebra, Lemma 10.128.1 and the fact that a normal local ring of dimension ≤ 2 is Cohen-Macaulay by Serre's criterion (Algebra, Lemma 10.157.4). Then V is regular by Algebra, Lemma 10.164.4. As Y^ν is Noetherian we conclude that $Y^\nu \setminus V = \{y_1, \ldots, y_m\}$ is finite. For each i let $\mathcal{O}^h_{Y^\nu, y_i}$ be the henselian local ring. Then $X \times_Y \mathrm{Spec}(\mathcal{O}^h_{Y^\nu, y_i})$ is a regular alteration of $\mathrm{Spec}(\mathcal{O}^h_{Y^\nu, y_i})$ (some details omitted). By Lemma 89.7.2 the completion of $\mathcal{O}^h_{Y^\nu, y_i}$ is normal. In this way we see that (1) $\Rightarrow$ (4).

Assume (4). We have to prove (3). We may immediately replace Y by its normalization. Let $y_1, \ldots, y_m \in |Y|$ be the singular points. Choose a collection of elementary étale neighbourhoods $(V_i, v_i) \to (Y, y_i)$ as in Section 89.3. For each i the henselian local ring $\mathcal{O}^h_{Y^\nu, y_i}$ is the henselization of $\mathcal{O}_{V_i, v_i}$. Hence these rings have isomorphic completions. Thus by the result for schemes (Resolution of Surfaces, Theorem 54.14.5) we see that there exist finite sequences of normalized blowups

$$X_{i,n_i} \to X_{i,n_i-1} \to \ldots \to V_i$$

blowing up only in points lying over v_i such that X_{i,n_i} is regular. By Lemma 89.5.3 there is a sequence of normalized blowing ups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to Y$$

and of course X_n is regular too (look at the local rings). This completes the proof. □

89.9. Examples

0AE8 Some examples related to the results earlier in this chapter.

0AE9 Example 89.9.1. Let k be a field. The ring $A = k[x, y, z]/(x^r + y^s + z^t)$ is a UFD for r, s, t pairwise coprime integers. Namely, since $x^r + y^s + z^t$ is irreducible A is a domain. The element z is a prime element, i.e., generates a prime ideal in A. On the other hand, if $t = 1 + ers$ for some e, then [Sam68, 4(c)]

$$A[1/z] \cong k[x', y', 1/z]$$

where $x' = x/z^{es}$, $y' = y/z^{er}$ and $z = (x')^r + (y')^s$. Thus $A[1/z]$ is a localization of a polynomial ring and hence a UFD. It follows from an argument of Nagata that A is a UFD. See Algebra, Lemma 10.120.7. A similar argument can be given if t is not congruent to 1 modulo rs.

0AEA Example 89.9.2. The ring $A = \mathbf{C}[[x, y, z]]/(x^r + y^s + z^t)$ is not a UFD when $1 < r < s < t$ are pairwise coprime integers and not equal to $2, 3, 5$. For example consider the special case $A = \mathbf{C}[[x, y, z]]/(x^2 + y^5 + z^7)$. Consider the maps (See [Bri68] and [Lip69] for nonvanishing of local Picard groups in general.)

$$\psi_\zeta : \mathbf{C}[[x, y, z]]/(x^2 + y^5 + z^7) \to \mathbf{C}[[t]]$$

given by

$$x \mapsto t^7, \quad y \mapsto t^3, \quad z \mapsto -\zeta t^2(1+t)^{1/7}$$

where ζ is a 7th root of unity. The kernel $\mathfrak{p}_\zeta$ of ψ_ζ is a height one prime, hence if A is a UFD, then it is principal, say given by $f_\zeta \in \mathbf{C}[[x, y, z]]$. Note that $V(x^3 - y^7) = \bigcup V(\mathfrak{p}_\zeta)$ and $A/(x^3 - y^7)$ is reduced away from the closed point. Hence, still assuming A is a UFD, we would obtain

$$\prod\nolimits_\zeta f_\zeta = u(x^3 - y^7) + a(x^2 + y^5 + z^7) \quad \text{in} \quad \mathbf{C}[[x, y, z]]$$

for some unit $u \in \mathbf{C}[[x, y, z]]$ and some element $a \in \mathbf{C}[[x, y, z]]$. After scaling by a constant we may assume $u(0, 0, 0) = 1$. Note that the left hand side vanishes to order 7. Hence $a = -x \bmod \mathfrak{m}^2$. But then we get a term xy^5 on the right hand side which does not occur on the left hand side. A contradiction.

0AEB Example 89.9.3. There exists an excellent 2-dimensional Noetherian local ring and a modification $X \to S = \mathrm{Spec}(A)$ which is not a scheme. We sketch a construction. Let X be a normal surface over $\mathbf{C}$ with a unique singular point $x \in X$. Assume that there exists a resolution $\pi : X' \to X$ such that the exceptional fibre $C = \pi^{-1}(x)_{red}$ is a smooth projective curve. Furthermore, assume there exists a point $c \in C$ such that if $\mathcal{O}_C(nc)$ is in the image of $\mathrm{Pic}(X') \to \mathrm{Pic}(C)$, then $n = 0$. Then we let $X'' \to X'$ be the blowing up in the nonsingular point c. Let $C' \subset X''$ be the strict transform of C and let $E \subset X''$ be the exceptional fibre. By Artin's results

([Art70]; use for example [Mum61] to see that the normal bundle of C' is negative) we can blow down the curve C' in X'' to obtain an algebraic space X'''. Picture

$$\begin{tikzcd} & X'' \arrow[dl] \arrow[dr] & \\ X' \arrow[dr] & & X''' \arrow[dl] \\ & X & \end{tikzcd}$$

We claim that X''' is not a scheme. This provides us with our example because X''' is a scheme if and only if the base change of X''' to $A = \mathcal{O}_{X,x}$ is a scheme (details omitted). If X''' where a scheme, then the image of C' in X''' would have an affine neighbourhood. The complement of this neighbourhood would be an effective Cartier divisor on X''' (because X''' is nonsingular apart from 1 point). This effective Cartier divisor would correspond to an effective Cartier divisor on X'' meeting E and avoiding C'. Taking the image in X' we obtain an effective Cartier divisor meeting C (set theoretically) in c. This is impossible as no multiple of c is the restriction of a Cartier divisor by assumption.

To finish we have to find such a singular surface X. We can just take X to be the affine surface given by

$$x^3 + y^3 + z^3 + x^4 + y^4 + z^4 = 0$$

in $\mathbf{A}^3_{\mathbf{C}} = \text{Spec}(\mathbf{C}[x, y, z])$ and singular point $(0, 0, 0)$. Then $(0, 0, 0)$ is the only singular point. Blowing up X in the maximal ideal corresponding to $(0, 0, 0)$ we find three charts each isomorphic to the smooth affine surface

$$1 + s^3 + t^3 + x(1 + s^4 + t^4) = 0$$

which is nonsingular with exceptional divisor C given by $x = 0$. The reader will recognize C as an elliptic curve. Finally, the surface X is rational as projection from $(0, 0, 0)$ shows, or because in the equation for the blowup we can solve for x. Finally, the Picard group of a nonsingular rational surface is countable, whereas the Picard group of an elliptic curve over the complex numbers is uncountable. Hence we can find a closed point c as indicated.

89.10. Other chapters

Part 6

Deformation Theory

CHAPTER 90

Formal Deformation Theory

06G7 90.1. Introduction

06G8 This chapter develops formal deformation theory in a form applicable later in the Stacks project, closely following Rim [GRR72, Exposee VI] and Schlessinger [Sch68]. We strongly encourage the reader new to this topic to read the paper by Schlessinger first, as it is sufficiently general for most applications, and Schlessinger's results are indeed used in most papers that use this kind of formal deformation theory.

Let Λ be a complete Noetherian local ring with residue field k, and let $\mathcal{C}_\Lambda$ denote the category of Artinian local Λ-algebras with residue field k. Given a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ such that $F(k)$ is a one element set, Schlessinger's paper introduced conditions (H1)-(H4) such that:

1. F has a "hull" if and only if (H1)-(H3) hold.
2. F is prorepresentable if and only if (H1)-(H4) hold.

The purpose of this chapter is to generalize these results in two ways exactly as is done in Rim's paper:

- (A) The functor F is replaced by a category $\mathcal{F}$ cofibered in groupoids over $\mathcal{C}_\Lambda$, see Section 90.3.
- (B) We let Λ be a Noetherian ring and $\Lambda \to k$ a finite ring map to a field. The category $\mathcal{C}_\Lambda$ is the category of Artinian local Λ-algebras A endowed with a given identification $A/\mathfrak{m}_A = k$.

The analogue of the condition that $F(k)$ is a one element set is that $\mathcal{F}(k)$ is the trivial groupoid. If $\mathcal{F}$ satisfies this condition then we say it is a predeformation category, but in general we do not make this assumption. Rim's paper [GRR72, Exposee VI] is the original source for the results in this document. We also mention the useful paper [TV13], which discusses deformation theory with groupoids but in less generality than we do here.

An important role is played by the "completion" $\widehat{\mathcal{C}}_\Lambda$ of the category $\mathcal{C}_\Lambda$. An object of $\widehat{\mathcal{C}}_\Lambda$ is a Noetherian complete local Λ-algebra R whose residue field is identified with k, see Section 90.4. On the one hand $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is a strictly full subcategory and on the other hand $\widehat{\mathcal{C}}_\Lambda$ is a full subcategory of the category of pro-objects of $\mathcal{C}_\Lambda$. A functor $\mathcal{C}_\Lambda \to \textit{Sets}$ is prorepresentable if it is isomorphic to the restriction of a representable functor $\underline{R} = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, -)$ to $\mathcal{C}_\Lambda$ where $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$.

Categories cofibred in groupoids are dual to categories fibred in groupoids; we introduce them in Section 90.5. A smooth morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ is one that satisfies the infinitesimal lifting criterion for objects, see Section 90.8. This is analogous to the definition of a formally smooth ring map, see Algebra, Definition 10.138.1 and is exactly dual to the notion in Criteria for Representability, Section 97.6. This is an important notion as we eventually want to prove that certain kinds of categories cofibred in groupoids have a smooth prorepresentable presentation, much like the characterization of algebraic stacks in Algebraic Stacks, Sections 94.16 and 94.17. A versal formal object of a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$ is an object $\xi \in \widehat{\mathcal{F}}(R)$ of the completion such that the associated morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth.

In Section 90.10, we define conditions (S1) and (S2) on $\mathcal{F}$ generalizing Schlessinger's (H1) and (H2). The analogue of Schlessinger's (H3)—the condition that $\mathcal{F}$ has finite dimensional tangent space—is not given a name. A key step in the development of the theory is the existence of versal formal objects for predeformation categories satisfying (S1), (S2) and (H3), see Lemma 90.13.4. Schlessinger's notion of a hull for a functor $F : \mathcal{C}_\Lambda \to \text{Sets}$ is, in our terminology, a versal formal object $\xi \in \widehat{F}(R)$ such that the induced map of tangent spaces $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ is an isomorphism. In the literature a hull is often called a "miniversal" object. We do not do so, and here is why. It can happen that a functor has a versal formal object without having a hull. Moreover, we show in Section 90.14 that if a predeformation category has a versal formal object, then it always has a minimal one (as defined in Definition 90.14.4) which is unique up to isomorphism, see Lemma 90.14.5. But it can happen that the minimal versal formal object does not induce an isomorphism on tangent spaces! (See Examples 90.15.3 and 90.15.8.)

Keeping in mind the differences pointed out above, Theorem 90.15.5 is the direct generalization of (1) above: it recovers Schlessinger's result in the case that $\mathcal{F}$ is a functor and it characterizes minimal versal formal objects, in the presence of conditions (S1) and (S2), in terms of the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ on tangent spaces.

In Section 90.16, we define Rim's condition (RS) on $\mathcal{F}$ generalizing Schlessinger's (H4). A deformation category is defined as a predeformation category satisfying (RS). The analogue to prorepresentable functors are the categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, see Definitions 90.21.1, 90.22.1, and 90.23.1. This notion of a presentation takes into account the groupoid structure of the fibers of $\mathcal{F}$. In Theorem 90.26.4 we prove that $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid in functors if and only if $\mathcal{F}$ has a finite dimensional tangent space and finite dimensional infinitesimal automorphism space. This is the generalization of (2) above: it reduces to Schlessinger's result in the case that $\mathcal{F}$ is a functor. There is a final Section 90.27 where we discuss how to use minimal versal formal objects to produce a (unique up to isomorphism) minimal presentation by a smooth prorepresentable groupoid in functors.

We also find the following conceptual explanation for Schlessinger's conditions. If a predeformation category $\mathcal{F}$ satisfies (RS), then the associated functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \text{Sets}$ satisfies (H1) and (H2) (Lemmas 90.16.6 and 90.10.5). Conversely, if a functor $F : \mathcal{C}_\Lambda \to \text{Sets}$ arises naturally as the functor of isomorphism classes of a category $\mathcal{F}$ cofibered in groupoids, then it seems to happen in practice that an argument showing F satisfies (H1) and (H2) will also show $\mathcal{F}$ satisfies (RS). Examples are discussed in Deformation Problems, Section 93.1. Moreover, if $\mathcal{F}$ satisfies (RS), then condition (H4) for $\overline{\mathcal{F}}$ has a simple interpretation in terms of extending automorphisms of objects of $\mathcal{F}$ (Lemma 90.16.7). These observations suggest that (RS) should be regarded as the fundamental deformation theoretic glueing condition.

90.2. Notation and Conventions

06G9 A ring is commutative with 1. The maximal ideal of a local ring A is denoted by $\mathfrak{m}_A$. The set of positive integers is denoted by $\mathbf{N} = \{1, 2, 3, \ldots\}$. If U is an object of a category $\mathcal{C}$, we denote by $\underline{U}$ the functor $\mathrm{Mor}_\mathcal{C}(U, -) : \mathcal{C} \to \mathrm{Sets}$, see Remarks 90.5.2 (12). Warning: this may conflict with the notation in other chapters where we sometimes use $\underline{U}$ to denote $h_U(-) = \mathrm{Mor}_\mathcal{C}(-, U)$.

Throughout this chapter Λ is a Noetherian ring and $\Lambda \to k$ is a finite ring map from Λ to a field. The kernel of this map is denoted $\mathfrak{m}_\Lambda$ and the image $k' \subset k$. It turns out that $\mathfrak{m}_\Lambda$ is a maximal ideal, $k' = \Lambda/\mathfrak{m}_\Lambda$ is a field, and the extension k/k' is finite. See discussion surrounding (90.3.3.1).

90.3. The base category

06GB Motivation. An important application of formal deformation theory is to criteria for representability by algebraic spaces. Suppose given a locally Noetherian base S and a functor $F : (Sch/S)_{fppf}^{opp} \to \mathrm{Sets}$. Let k be a finite type field over S, i.e., we are given a finite type morphism $\mathrm{Spec}(k) \to S$. One of Artin's criteria is that for any element $x \in F(\mathrm{Spec}(k))$ the predeformation functor associated to the triple (S, k, x) should be prorepresentable. By Morphisms, Lemma 29.16.1 the condition that k is of finite type over S means that there exists an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra. This motivates why we work throughout this chapter with a base category as follows.

06GC Definition 90.3.1. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\mathcal{C}_\Lambda$ to be the category with

(1) objects are pairs (A, φ) where A is an Artinian local Λ-algebra and where $\varphi : A/\mathfrak{m}_A \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

We say we are in the classical case if Λ is a Noetherian complete local ring and k is its residue field.

Note that if $\Lambda \to k$ is surjective and if A is an Artinian local Λ-algebra, then the identification φ, if it exists, is unique. Moreover, in this case any Λ-algebra map $A \to B$ is going to be compatible with the identifications. Hence in this case $\mathcal{C}_\Lambda$ is just the category of local Artinian Λ-algebras whose residue field "is" k. By abuse of notation we also denote objects of $\mathcal{C}_\Lambda$ simply A in the general case. Moreover, we will often write $A/\mathfrak{m} = k$, i.e., we will pretend all rings in $\mathcal{C}_\Lambda$ have residue field k (since all ring maps in $\mathcal{C}_\Lambda$ are compatible with the given identifications this should never cause any problems). Throughout the rest of this chapter the base ring Λ and the field k are fixed. The category $\mathcal{C}_\Lambda$ will be the base category for the cofibered categories considered below.

06GD Definition 90.3.2. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is a small extension if it is surjective and $\mathrm{Ker}(f)$ is a nonzero principal ideal which is annihilated by $\mathfrak{m}_B$.

By the following lemma we can often reduce arguments involving surjective ring maps in $\mathcal{C}_\Lambda$ to the case of small extensions.

06GE **Lemma 90.3.3.** Let $f : B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Then f can be factored as a composition of small extensions.

Proof. Let I be the kernel of f. The maximal ideal $\mathfrak{m}_B$ is nilpotent since B is Artinian, say $\mathfrak{m}_B^n = 0$. Hence we get a factorization

$$B = B/I\mathfrak{m}_B^{n-1} \to B/I\mathfrak{m}_B^{n-2} \to \ldots \to B/I \cong A$$

of f into a composition of surjective maps whose kernels are annihilated by the maximal ideal. Thus it suffices to prove the lemma when f itself is such a map, i.e. when I is annihilated by $\mathfrak{m}_B$. In this case I is a k-vector space, which has finite dimension, see Algebra, Lemma 10.53.6. Take a basis $x_1, \ldots, x_n$ of I as a k-vector space to get a factorization

$$B \to B/(x_1) \to \ldots \to B/(x_1, \ldots, x_n) \cong A$$

of f into a composition of small extensions. $\square$

The next lemma says that we can compute the length of a module over a local Λ-algebra with residue field k in terms of the length over Λ. To explain the notation in the statement, let $k' \subset k$ be the image of our fixed finite ring map $\Lambda \to k$. Note that $k' \subset k$ is a finite extension of rings. Hence k' is a field and k/k' is a finite extension of fields, see Algebra, Lemma 10.36.18. Moreover, as $\Lambda \to k'$ is surjective we see that its kernel is a maximal ideal $\mathfrak{m}_\Lambda$. Thus

06S2 (90.3.3.1) $$[k : k'] = [k : \Lambda/\mathfrak{m}_\Lambda] < \infty$$

and in the classical case we have $k = k'$. The notation $k' = \Lambda/\mathfrak{m}_\Lambda$ will be fixed throughout this chapter.

06GG **Lemma 90.3.4.** Let A be a local Λ-algebra with residue field k. Let M be an A-module. Then $[k : k']\text{length}_A(M) = \text{length}_\Lambda(M)$. In the classical case we have $\text{length}_A(M) = \text{length}_\Lambda(M)$.

Proof. If M is a simple A-module then $M \cong k$ as an A-module, see Algebra, Lemma 10.52.10. In this case $\text{length}_A(M) = 1$ and $\text{length}_\Lambda(M) = [k' : k]$, see Algebra, Lemma 10.52.6. If $\text{length}_A(M)$ is finite, then the result follows on choosing a filtration of M by A-submodules with simple quotients using additivity, see Algebra, Lemma 10.52.3. If $\text{length}_A(M)$ is infinite, the result follows from the obvious inequality $\text{length}_A(M) \leq \text{length}_\Lambda(M)$. $\square$

06S3 **Lemma 90.3.5.** Let $A \to B$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent

1. f is surjective,
2. $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and
3. $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.

Proof. For any ring map $f : A \to B$ in $\mathcal{C}_\Lambda$ we have $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ for example because $\mathfrak{m}_A$, $\mathfrak{m}_B$ is the set of nilpotent elements of A, B. Suppose f is surjective. Let $y \in \mathfrak{m}_B$. Choose $x \in A$ with $f(x) = y$. Since f induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ we see that $x \in \mathfrak{m}_A$. Hence the induced map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. In this way we see that (1) implies (2).

It is clear that (2) implies (3). The map $A \to B$ gives rise to a canonical commutative diagram

$$\begin{tikzcd} \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k \arrow[r] \arrow[d] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r] \arrow[d] & \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \arrow[r] \arrow[d] & 0 \\ \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k \arrow[r] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r] & \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Hence if (3) holds, then so does (2).

Assume (2). To show that $A \to B$ is surjective it suffices by Nakayama's lemma (Algebra, Lemma 10.20.1) to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. (Note that $\mathfrak{m}_A$ is a nilpotent ideal.) As $k = A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. Applying Nakayama's lemma once more we see that it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective which is what we assumed. $\square$

If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then the map $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is the map on relative cotangent spaces. Here is a formal definition.

06GY Definition 90.3.6. Let $R \to S$ be a local homomorphism of local rings. The relative cotangent space[1] of R over S is the $S/\mathfrak{m}_S$-vector space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$.

If $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are two ring maps, then the fiber product $A_1 \times_A A_2$ is the subring of $A_1 \times A_2$ consisting of elements whose two projections to A are equal. Throughout this chapter we will be considering conditions involving such a fiber product when f_1 and f_2 are in $\mathcal{C}_\Lambda$. It isn't always the case that the fibre product is an object of $\mathcal{C}_\Lambda$.

06S4 Example 90.3.7. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Let $A = k[\epsilon] = k[x]/(x^2)$. Then A is an object of $\mathcal{C}_\Lambda$. Suppose that $D : k \to k$ is a derivation of k over Λ, for example $D = \partial/\partial x_i$. Then the map

$$f_D : k \longrightarrow k[\epsilon], \quad a \mapsto a + D(a)\epsilon$$

is a morphism of $\mathcal{C}_\Lambda$. Set $A_1 = A_2 = k$ and set $f_1 = f_{\partial/\partial x_1}$ and $f_2(a) = a$. Then $A_1 \times_A A_2 = \{a \in k \mid \partial/\partial x_1(a) = 0\}$ which does not surject onto k. Hence the fibre product isn't an object of $\mathcal{C}_\Lambda$.

It turns out that this problem can only occur if the residue field extension k/k' (90.3.3.1) is inseparable and neither f_1 nor f_2 is surjective.

06GH Lemma 90.3.8. Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$. Then:

(1) If f_1 or f_2 is surjective, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.
(2) If f_2 is a small extension, then so is $A_1 \times_A A_2 \to A_1$.
(3) If the field extension k/k' is separable, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.

Proof. The ring $A_1 \times_A A_2$ is a Λ-algebra via the map $\Lambda \to A_1 \times_A A_2$ induced by the maps $\Lambda \to A_1$ and $\Lambda \to A_2$. It is a local ring with unique maximal ideal

$$\mathfrak{m}_{A_1} \times_{\mathfrak{m}_A} \mathfrak{m}_{A_2} = \mathrm{Ker}(A_1 \times_A A_2 \longrightarrow k)$$

[1] Caution: We will see later that in our general setting the tangent space of an object $A \in \mathcal{C}_\Lambda$ over Λ should not be defined simply as the k-linear dual of the relative cotangent space. In fact, the correct definition of the relative cotangent space is $\Omega_{S/R} \otimes_S S/\mathfrak{m}_S$.

A ring is Artinian if and only if it has finite length as a module over itself, see Algebra, Lemma 10.53.6. Since A_1 and A_2 are Artinian, Lemma 90.3.4 implies $\text{length}_\Lambda(A_1)$ and $\text{length}_\Lambda(A_2)$, and hence $\text{length}_\Lambda(A_1 \times A_2)$, are all finite. As $A_1 \times_A A_2 \subset A_1 \times A_2$ is a Λ-submodule, this implies $\text{length}_{A_1 \times_A A_2}(A_1 \times_A A_2) \leq \text{length}_\Lambda(A_1 \times_A A_2)$ is finite. So $A_1 \times_A A_2$ is Artinian. Thus the only thing that is keeping $A_1 \times_A A_2$ from being an object of $\mathcal{C}_\Lambda$ is the possibility that its residue field maps to a proper subfield of k via the map $A_1 \times_A A_2 \to A \to A/\mathfrak{m}_A = k$ above.

Proof of (1). If f_2 is surjective, then the projection $A_1 \times_A A_2 \to A_1$ is surjective. Hence the composition $A_1 \times_A A_2 \to A_1 \to A_1/\mathfrak{m}_{A_1} = k$ is surjective and we conclude that $A_1 \times_A A_2$ is an object of $\mathcal{C}_\Lambda$.

Proof of (2). If f_2 is a small extension then $A_2 \to A$ and $A_1 \times_A A_2 \to A_1$ are both surjective with the same kernel. Hence the kernel of $A_1 \times_A A_2 \to A_1$ is a 1-dimensional k-vector space and we see that $A_1 \times_A A_2 \to A_1$ is a small extension.

Proof of (3). Choose $\overline{x} \in k$ such that $k = k'(\overline{x})$ (see Fields, Lemma 9.19.1). Let $P'(T) \in k'[T]$ be the minimal polynomial of $\overline{x}$ over k'. Since k/k' is separable we see that $\mathrm{d}P/\mathrm{d}T(\overline{x}) \neq 0$. Choose a monic $P \in \Lambda[T]$ which maps to P' under the surjective map $\Lambda[T] \to k'[T]$. Because A, A_1, A_2 are henselian, see Algebra, Lemma 10.153.10, we can find $x, x_1, x_2 \in A, A_1, A_2$ with $P(x) = 0, P(x_1) = 0, P(x_2) = 0$ and such that the image of x, x_1, x_2 in k is $\overline{x}$. Then $(x_1, x_2) \in A_1 \times_A A_2$ because x_1, x_2 map to $x \in A$ by uniqueness, see Algebra, Lemma 10.153.2. Hence the residue field of $A_1 \times_A A_2$ contains a generator of k over k' and we win. □

Next we define essential surjections in $\mathcal{C}_\Lambda$. A necessary and sufficient condition for a surjection in $\mathcal{C}_\Lambda$ to be essential is given in Lemma 90.3.12.

06GF Definition 90.3.9. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is an essential surjection if it has the following properties:

(1) f is surjective.
(2) If $g : C \to B$ is a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective, then g is surjective.

Using Lemma 90.3.5, we can characterize essential surjections in $\mathcal{C}_\Lambda$ as follows.

06S5 Lemma 90.3.10. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent

(1) f is an essential surjection,
(2) the map $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is an essential surjection, and
(3) the map $B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an essential surjection.

Proof. Assume (3). Let $C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $C \to A$ is surjective. Then $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective too. We conclude that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective by our assumption. Hence $C \to B$ is surjective by applying Lemma 90.3.5 (2 times).

Assume (1). Let $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ be a morphism of $\mathcal{C}_\Lambda$ such that $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective. Set $C' = C \times_{B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)} B$ which is an object of $\mathcal{C}_\Lambda$ by Lemma 90.3.8. Note that $C' \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is still surjective, hence $C' \to A$ is surjective by Lemma 90.3.5. Thus $C' \to B$ is surjective by our assumption. This implies that $C' \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective, which implies by the construction of C' that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.

In the first paragraph we proved (3) $\Rightarrow$ (1) and in the second paragraph we proved (1) $\Rightarrow$ (3). The equivalence of (2) and (3) is a special case of the equivalence of (1) and (3), hence we are done. $\square$

To analyze essential surjections in $\mathcal{C}_\Lambda$ a bit more we introduce some notation. Suppose that A is an object of $\mathcal{C}_\Lambda$ or more generally any Λ-algebra equipped with a Λ-algebra surjection $A \to k$. There is a canonical exact sequence

06S6 (90.3.10.1)
$$\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\text{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0$$

see Algebra, Lemma 10.131.9. Note that $\Omega_{k/\Lambda} = \Omega_{k/k'}$ with k' as in (90.3.3.1). Let $H_1(L_{k/\Lambda})$ be the first homology module of the naive cotangent complex of k over Λ, see Algebra, Definition 10.134.1. Then we can extend (90.3.10.1) to the exact sequence

06S7 (90.3.10.2)
$$H_1(L_{k/\Lambda}) \to \mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\text{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0,$$

see Algebra, Lemma 10.134.4. If $B \to A$ is a ring map in $\mathcal{C}_\Lambda$ or more generally a map of Λ-algebras equipped with Λ-algebra surjections onto k, then we obtain a commutative diagram

06S8 (90.3.10.3)
$$\begin{tikzcd}
H_1(L_{k/\Lambda}) \arrow[r] \arrow[d, equal] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r, "\text{d}_B"'] \arrow[d] & \Omega_{B/\Lambda} \otimes_B k \arrow[r] \arrow[d] & \Omega_{k/\Lambda} \arrow[r] \arrow[d, equal] & 0 \\
H_1(L_{k/\Lambda}) \arrow[r] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r, "\text{d}_A"] & \Omega_{A/\Lambda} \otimes_A k \arrow[r] & \Omega_{k/\Lambda} \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

06S9 **Lemma 90.3.11.** There is a canonical map

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}).$$

If $k' \subset k$ is separable (for example if the characteristic of k is zero), then this map induces an isomorphism $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. If $k = k'$ (for example in the classical case), then $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 = H_1(L_{k/\Lambda})$. The composition

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}) \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$$

comes from the canonical map $\mathfrak{m}_\Lambda \to \mathfrak{m}_A$.

Proof. Note that $H_1(L_{k'/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$ as $\Lambda \to k'$ is surjective with kernel $\mathfrak{m}_\Lambda$. The map arises from functoriality of the naive cotangent complex. If $k' \subset k$ is separable, then $k' \to k$ is an étale ring map, see Algebra, Lemma 10.143.4. Thus its naive cotangent complex has trivial homology groups, see Algebra, Definition 10.143.1. Then Algebra, Lemma 10.134.4 applied to the ring maps $\Lambda \to k' \to k$ implies that $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. We omit the proof of the final statement. $\square$

06H0 **Lemma 90.3.12.** Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Notation as in (90.3.10.3).

(1) The equivalent conditions of Lemma 90.3.5 characterizing when f is surjective are also equivalent to
  (a) $\text{Im}(\text{d}_B) \to \text{Im}(\text{d}_A)$ is surjective, and
  (b) the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective.
(2) The following are equivalent
  (a) f is an essential surjection (see Lemma 90.3.10),
  (b) the map $\text{Im}(\text{d}_B) \to \text{Im}(\text{d}_A)$ is an isomorphism, and

(c) the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is an isomorphism.

(3) If k/k' is separable, then f is an essential surjection if and only if the map $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an isomorphism.

(4) If f is a small extension, then f is not essential if and only if f has a section $s : A \to B$ in $\mathcal{C}_\Lambda$ with $f \circ s = \mathrm{id}_A$.

Proof. Proof of (1). It follows from (90.3.10.3) that (1)(a) and (1)(b) are equivalent. Also, if $A \to B$ is surjective, then (1)(a) and (1)(b) hold. Assume (1)(a). Since the kernel of d_A is the image of $H_1(L_{k/\Lambda})$ which also maps to $\mathfrak{m}_B/\mathfrak{m}_B^2$ we conclude that $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Hence $B \to A$ is surjective by Lemma 90.3.5. This finishes the proof of (1).

Proof of (2). The equivalence of (2)(b) and (2)(c) is immediate from (90.3.10.3).

Assume (2)(b). Let $g : C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective. We conclude that $\mathfrak{m}_C/\mathfrak{m}_C^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective by Lemma 90.3.5. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective and by the assumption we see that $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_B)$ is surjective. It follows that $C \to B$ is surjective by (1).

Assume (2)(a). Then f is surjective and we see that $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective. Let K be the kernel. Note that $K = \mathrm{d}_B(\mathrm{Ker}(\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2))$ by (90.3.10.3). Choose a splitting

$$\Omega_{B/\Lambda} \otimes_B k = \Omega_{A/\Lambda} \otimes_A k \oplus K$$

of k-vector space. The map $\mathrm{d} : B \to \Omega_{B/\Lambda}$ induces via the projection onto K a map $D : B \to K$. Set $C = \{b \in B \mid D(b) = 0\}$. The Leibniz rule shows that this is a Λ-subalgebra of B. Let $\overline{x} \in k$. Choose $x \in B$ mapping to $\overline{x}$. If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ such that $D(y) = D(x)$. Hence $x - y \in C$ is an element which maps to $\overline{x}$. Thus $C \to k$ is surjective and C is an object of $\mathcal{C}_\Lambda$. Similarly, pick $\omega \in \mathrm{Im}(\mathrm{d}_A)$. We can find $x \in \mathfrak{m}_B$ such that $\mathrm{d}_B(x)$ maps to ω by (1). If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ which maps to zero in $\mathfrak{m}_A/\mathfrak{m}_A^2$ such that $D(y) = D(x)$. Hence $z = x - y$ is an element of $\mathfrak{m}_C$ whose image $\mathrm{d}_C(z) \in \Omega_{C/k} \otimes_C k$ maps to ω. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective. We conclude that $C \to A$ is surjective by (1). Hence $C \to B$ is surjective by assumption. Hence $D = 0$, i.e., $K = 0$, i.e., (2)(c) holds. This finishes the proof of (2).

Proof of (3). If k'/k is separable, then $H_1(L_{k/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k$, see Lemma 90.3.11. Hence $\mathrm{Im}(\mathrm{d}_A) = \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ and similarly for B. Thus (3) follows from (2).

Proof of (4). A section s of f is not surjective (by definition a small extension has nontrivial kernel), hence f is not essentially surjective. Conversely, assume f is a small extension but not an essential surjection. Choose a ring map $C \to B$ in $\mathcal{C}_\Lambda$ which is not surjective, such that $C \to A$ is surjective. Let $C' \subset B$ be the image of $C \to B$. Then $C' \neq B$ but C' surjects onto A. Since $f : B \to A$ is a small extension, $\mathrm{length}_C(B) = \mathrm{length}_C(A) + 1$. Thus $\mathrm{length}_C(C') \leq \mathrm{length}_C(A)$ since C' is a proper subring of B. But $C' \to A$ is surjective, so in fact we must have $\mathrm{length}_C(C') = \mathrm{length}_C(A)$ and $C' \to A$ is an isomorphism which gives us our section. □

06SA Example 90.3.13. Let $\Lambda = k[[x]]$ be the power series ring in 1 variable over k. Set $A = k$ and $B = \Lambda/(x^2)$. Then $B \to A$ is an essential surjection by Lemma 90.3.12

because it is a small extension and the map $B \to A$ does not have a right inverse (in the category $\mathcal{C}_\Lambda$). But the map

$$k \cong \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 = 0$$

is not an isomorphism. Thus in Lemma 90.3.12 (3) it is necessary to consider the map of relative cotangent spaces $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$.

90.4. The completed base category

06GV The following "completion" of the category $\mathcal{C}_\Lambda$ will serve as the base category of the completion of a category cofibered in groupoids over $\mathcal{C}_\Lambda$ (Section 90.7).

06GW Definition 90.4.1. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\widehat{\mathcal{C}}_\Lambda$ to be the category with

(1) objects are pairs (R, φ) where R is a Noetherian complete local Λ-algebra and where $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

As in the discussion following Definition 90.3.1 we will usually denote an object of $\widehat{\mathcal{C}}_\Lambda$ simply R, with the identification $R/\mathfrak{m}_R = k$ understood. In this section we discuss some basic properties of objects and morphisms of the category $\widehat{\mathcal{C}}_\Lambda$ paralleling our discussion of the category $\mathcal{C}_\Lambda$ in the previous section.

Our first observation is that any object $A \in \mathcal{C}_\Lambda$ is an object of $\widehat{\mathcal{C}}_\Lambda$ as an Artinian local ring is always Noetherian and complete with respect to its maximal ideal (which is after all a nilpotent ideal). Moreover, it is clear from the definitions that $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is the strictly full subcategory consisting of all Artinian rings. As it turns out, conversely every object of $\widehat{\mathcal{C}}_\Lambda$ is a limit of objects of $\mathcal{C}_\Lambda$.

Suppose that R is an object of $\widehat{\mathcal{C}}_\Lambda$. Consider the rings $R_n = R/\mathfrak{m}_R^n$ for $n \in \mathbf{N}$. These are Noetherian local rings with a unique nilpotent prime ideal, hence Artinian, see Algebra, Proposition 10.60.7. The ring maps

$$\ldots \to R_{n+1} \to R_n \to \ldots \to R_2 \to R_1 = k$$

are all surjective. Completeness of R by definition means that $R = \lim R_n$. If $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$ then we obtain a system of ring maps $f_n : R_n \to S_n$ whose limit is the given map.

06GZ Lemma 90.4.2. Let $f : R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. The following are equivalent

(1) f is surjective,
(2) the map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective, and
(3) the map $\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to \mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$ is surjective.

Proof. Note that for $n \geq 2$ we have the equality of relative cotangent spaces

$$\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) = \mathfrak{m}_{R_n}/(\mathfrak{m}_\Lambda R_n + \mathfrak{m}_{R_n}^2)$$

and similarly for S. Hence by Lemma 90.3.5 we see that $R_n \to S_n$ is surjective for all n. Now let K_n be the kernel of $R_n \to S_n$. Then the sequences

$$0 \to K_n \to R_n \to S_n \to 0$$

form an exact sequence of directed inverse systems. The system (K_n) is Mittag-Leffler since each K_n is Artinian. Hence by Algebra, Lemma 10.86.4 taking limits preserves exactness. So $\lim R_n \to \lim S_n$ is surjective, i.e., f is surjective. □

06SB Lemma 90.4.3. The category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.

Proof. Let $R \to S_1$ and $R \to S_2$ be morphisms of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = S_1 \otimes_R S_2$. This ring has a finitely generated maximal ideal $\mathfrak{m} = \mathfrak{m}_{S_1} \otimes S_2 + S_1 \otimes \mathfrak{m}_{S_2}$ with residue field k. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ whose residue field is identified with k, see Algebra, Lemma 10.97.5. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $S_1 \to C^\wedge$ and $S_2 \to C^\wedge$ turn $C^\wedge$ into a pushout over R in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

We will not need the following lemma.

06H1 Lemma 90.4.4. The category $\widehat{\mathcal{C}}_\Lambda$ admits coproducts of pairs of objects.

Proof. Let R and S be objects of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = R \otimes_\Lambda S$. There is a canonical surjective map $C \to R \otimes_\Lambda S \to k \otimes_\Lambda k \to k$ where the last map is the multiplication map. The kernel of $C \to k$ is a maximal ideal $\mathfrak{m}$. Note that $\mathfrak{m}$ is generated by $\mathfrak{m}_R C$, $\mathfrak{m}_S C$ and finitely many elements of C which map to generators of the kernel of $k \otimes_\Lambda k \to k$. Hence $\mathfrak{m}$ is a finitely generated ideal. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ with residue field k, see Algebra, Lemma 10.97.5. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $R \to C^\wedge$ and $S \to C^\wedge$ turn $C^\wedge$ into a coproduct in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

An empty coproduct in a category is an initial object of the category. In the classical case $\widehat{\mathcal{C}}_\Lambda$ has an initial object, namely Λ itself. More generally, if $k' = k$, then the completion $\Lambda^\wedge$ of Λ with respect to $\mathfrak{m}_\Lambda$ is an initial object. More generally still, if $k' \subset k$ is separable, then $\widehat{\mathcal{C}}_\Lambda$ has an initial object too. Namely, choose a monic polynomial $P \in \Lambda[T]$ such that $k \cong k'[T]/(P')$ where $p' \in k'[T]$ is the image of P. Then $R = \Lambda^\wedge[T]/(P)$ is an initial object, see proof of Lemma 90.3.8.

If R is an initial object as above, then we have $\mathcal{C}_\Lambda = \mathcal{C}_R$ and $\widehat{\mathcal{C}}_\Lambda = \widehat{\mathcal{C}}_R$ which effectively brings the whole discussion in this chapter back to the classical case. But, if $k' \subset k$ is inseparable, then an initial object does not exist.

06SC Lemma 90.4.5. Let S be an object of $\widehat{\mathcal{C}}_\Lambda$. Then $\dim_k \mathrm{Der}_\Lambda(S, k) < \infty$.

Proof. Let $x_1, \ldots, x_n \in \mathfrak{m}_S$ map to a k-basis for the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$. Choose $y_1, \ldots, y_m \in S$ whose images in k generate k over k'. We claim that $\dim_k \mathrm{Der}_\Lambda(S, k) \leq n + m$. To see this it suffices to prove that if $D(x_i) = 0$ and $D(y_j) = 0$, then $D = 0$. Let $a \in S$. We can find a polynomial $P = \sum \lambda_J y^J$ with $\lambda_J \in \Lambda$ whose image in k is the same as the image of a in k. Then we see that $D(a - P) = D(a) - D(P) = D(a)$ by our assumption that $D(y_j) = 0$ for all j. Thus we may assume $a \in \mathfrak{m}_S$. Write $a = \sum a_i x_i$ with $a_i \in S$. By the Leibniz rule

$$D(a) = \sum x_i D(a_i) + \sum a_i D(x_i) = \sum x_i D(a_i)$$

as we assumed $D(x_i) = 0$. We have $\sum x_i D(a_i) = 0$ as multiplication by x_i is zero on k. □

06SD Lemma 90.4.6. Let $f : R \to S$ be a morphism of $\widehat{\mathcal{C}}_\Lambda$. If $\text{Der}_\Lambda(S, k) \to \text{Der}_\Lambda(R, k)$ is injective, then f is surjective.

Proof. If f is not surjective, then $\mathfrak m_S/(\mathfrak m_R S + \mathfrak m_S^2)$ is nonzero by Lemma 90.4.2. Then also $Q = S/(f(R) + \mathfrak m_R S + \mathfrak m_S^2)$ is nonzero. Note that Q is a $k = R/\mathfrak m_R$-vector space via f. We turn Q into an S-module via $S \to k$. The quotient map $D : S \to Q$ is an R-derivation: if $a_1, a_2 \in S$, we can write $a_1 = f(b_1) + a_1'$ and $a_2 = f(b_2) + a_2'$ for some $b_1, b_2 \in R$ and $a_1', a_2' \in \mathfrak m_S$. Then b_i and a_i have the same image in k for $i = 1, 2$ and

$$
\begin{aligned}
a_1a_2 &= (f(b_1) + a_1')(f(b_2) + a_2') \\
&= f(b_1)a_2' + f(b_2)a_1' \\
&= f(b_1)(f(b_2) + a_2') + f(b_2)(f(b_1) + a_1') \\
&= f(b_1)a_2 + f(b_2)a_1
\end{aligned}
$$

in Q which proves the Leibniz rule. Hence $D : S \to Q$ is a Λ-derivation which is zero on composing with $R \to S$. Since $Q \neq 0$ there also exist derivations $D : S \to k$ which are zero on composing with $R \to S$, i.e., $\text{Der}_\Lambda(S, k) \to \text{Der}_\Lambda(R, k)$ is not injective. □

06SE Lemma 90.4.7. Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. Let (J_n) be a decreasing sequence of ideals such that $\mathfrak m_R^n \subset J_n$. Set $J = \bigcap J_n$. Then the sequence (J_n/J) defines the $\mathfrak m_{R/J}$-adic topology on R/J.

Proof. It is clear that $\mathfrak m_{R/J}^n \subset J_n/J$. Thus it suffices to show that for every n there exists an N such that $J_N/J \subset \mathfrak m_{R/J}^n$. This is equivalent to $J_N \subset \mathfrak m_R^n + J$. For each n the ring $R/\mathfrak m_R^n$ is Artinian, hence there exists a N_n such that

$$J_{N_n} + \mathfrak m_R^n = J_{N_n+1} + \mathfrak m_R^n = \ldots$$

Set $E_n = (J_{N_n} + \mathfrak m_R^n)/\mathfrak m_R^n$. Set $E = \lim E_n \subset \lim R/\mathfrak m_R^n = R$. Note that $E \subset J$ as for any $f \in E$ and any m we have $f \in J_m + \mathfrak m_R^n$ for all $n \gg 0$, so $f \in J_m$ by Krull's intersection theorem, see Algebra, Lemma 10.51.4. Since the transition maps $E_n \to E_{n-1}$ are all surjective, we see that J surjects onto E_n. Hence for $N = N_n$ works. □

06SF Lemma 90.4.8. Let $\ldots \to A_3 \to A_2 \to A_1$ be a sequence of surjective ring maps in $\mathcal{C}_\Lambda$. If $\dim_k(\mathfrak m_{A_n}/\mathfrak m_{A_n}^2)$ is bounded, then $S = \lim A_n$ is an object in $\widehat{\mathcal{C}}_\Lambda$ and the ideals $I_n = \text{Ker}(S \to A_n)$ define the $\mathfrak m_S$-adic topology on S.

Proof. We will use freely that the maps $S \to A_n$ are surjective for all n. Note that the maps $\mathfrak m_{A_{n+1}}/\mathfrak m_{A_{n+1}}^2 \to \mathfrak m_{A_n}/\mathfrak m_{A_n}^2$ are surjective, see Lemma 90.4.2. Hence for n sufficiently large the dimension $\dim_k(\mathfrak m_{A_n}/\mathfrak m_{A_n}^2)$ stabilizes to an integer, say r. Thus we can find $x_1, \ldots, x_r \in \mathfrak m_S$ whose images in A_n generate $\mathfrak m_{A_n}$. Moreover, pick $y_1, \ldots, y_t \in S$ whose images in k generate k over Λ. Then we get a ring map $P = \Lambda[z_1, \ldots, z_{r+t}] \to S$, $z_i \mapsto x_i$ and $z_{r+j} \mapsto y_j$ such that the composition $P \to S \to A_n$ is surjective for all n. Let $\mathfrak m \subset P$ be the kernel of $P \to k$. Let $R = P^\wedge$ be the $\mathfrak m$-adic completion of P; this is an object of $\widehat{\mathcal{C}}_\Lambda$. Since we still have the compatible system of (surjective) maps $R \to A_n$ we get a map $R \to S$. Set $J_n = \text{Ker}(R \to A_n)$. Set $J = \bigcap J_n$. By Lemma 90.4.7 we see that $R/J = \lim R/J_n = \lim A_n = S$ and that the ideals $J_n/J = I_n$ define the $\mathfrak m$-adic topology.

(Note that for each n we have $\mathfrak{m}_R^{N_n} \subset J_n$ for some N_n and not necessarily $N_n = n$, so a renumbering of the ideals J_n may be necessary before applying the lemma.) $\square$

06SG Lemma 90.4.9. Let $R', R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Suppose that $R = R' \oplus I$ for some ideal I of R. Let $x_1, \ldots, x_r \in I$ map to a basis of $I/\mathfrak{m}_R I$. Set $S = R'[[X_1, \ldots, X_r]]$ and consider the R'-algebra map $S \to R$ mapping X_i to x_i. Assume that for every $n \gg 0$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ has a left inverse in $\mathcal{C}_\Lambda$. Then $S \to R$ is an isomorphism.

Proof. As $R = R' \oplus I$ we have

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus I/\mathfrak{m}_R I$$

and similarly

$$\mathfrak{m}_S/\mathfrak{m}_S^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus \bigoplus kX_i$$

Hence for $n > 1$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ induces an isomorphism on cotangent spaces. Thus a left inverse $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ is surjective by Lemma 90.4.2. Since h_n is injective as a left inverse it is an isomorphism. Thus the canonical surjections $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ are all isomorphisms and we win. $\square$

90.5. Categories cofibered in groupoids

06GA In developing the theory we work with categories cofibered in groupoids. We assume as known the definition and basic properties of categories fibered in groupoids, see Categories, Section 4.35.

06GJ Definition 90.5.1. Let $\mathcal{C}$ be a category. A category cofibered in groupoids over $\mathcal{C}$ is a category $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}^{opp}$ is a category fibered in groupoids over $\mathcal{C}^{opp}$ via $p^{opp} : \mathcal{F}^{opp} \to \mathcal{C}^{opp}$.

Explicitly, $p : \mathcal{F} \to \mathcal{C}$ is cofibered in groupoids if the following two conditions hold:

(1) For every morphism $f : U \to V$ in $\mathcal{C}$ and every object x lying over U, there is a morphism $x \to y$ of $\mathcal{F}$ lying over f.
(2) For every pair of morphisms $a : x \to y$ and $b : x \to z$ of $\mathcal{F}$ and any morphism $f : p(y) \to p(z)$ such that $p(b) = f \circ p(a)$, there exists a unique morphism $c : y \to z$ of $\mathcal{F}$ lying over f such that $b = c \circ a$.

06GK Remarks 90.5.2. Everything about categories fibered in groupoids translates directly to the cofibered setting. The following remarks are meant to fix notation. Let $\mathcal{C}$ be a category.

(1) We often omit the functor $p : \mathcal{F} \to \mathcal{C}$ from the notation.
(2) The fiber category over an object U in $\mathcal{C}$ is denoted by $\mathcal{F}(U)$. Its objects are those of $\mathcal{F}$ lying over U and its morphisms are those of $\mathcal{F}$ lying over id_U. If x, y are objects of $\mathcal{F}(U)$, we sometimes write $\mathrm{Mor}_U(x, y)$ for $\mathrm{Mor}_{\mathcal{F}(U)}(x, y)$.
(3) The fibre categories $\mathcal{F}(U)$ are groupoids, see Categories, Lemma 4.35.2. Hence the morphisms in $\mathcal{F}(U)$ are all isomorphisms. We sometimes write $\mathrm{Aut}_U(x)$ for $\mathrm{Mor}_{\mathcal{F}(U)}(x, x)$.

06SH (4) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$, let $f : U \to V$ be a morphism in $\mathcal{C}$, and let $x \in \mathrm{Ob}(\mathcal{F}(U))$. A pushforward of x along f is a morphism $x \to y$ of $\mathcal{F}$ lying over f. A pushforward is unique up to unique isomorphism (see the discussion following Categories, Definition 4.33.1). We sometimes write $x \to f_*x$ for "the" pushforward of x along f.

(5) A choice of pushforwards for $\mathcal{F}$ is the choice of a pushforward of x along f for every pair (x, f) as above. We can make such a choice of pushforwards for $\mathcal{F}$ by the axiom of choice.

(6) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$. Given a choice of pushforwards for $\mathcal{F}$, there is an associated pseudo-functor $\mathcal{C} \to \text{Groupoids}$. We will never use this construction so we give no details.

06GL (7) A morphism of categories cofibered in groupoids over $\mathcal{C}$ is a functor commuting with the projections to $\mathcal{C}$. If $\mathcal{F}$ and $\mathcal{F}'$ are categories cofibered in groupoids over $\mathcal{C}$, we denote the morphisms from $\mathcal{F}$ to $\mathcal{F}'$ by $\text{Mor}_\mathcal{C}(\mathcal{F}, \mathcal{F}')$.

06GM (8) Categories cofibered in groupoids form a $(2,1)$-category $\text{Cof}(\mathcal{C})$. Its 1-morphisms are the morphisms described in (7). If $p : \mathcal{F} \to C$ and $p' : \mathcal{F}' \to \mathcal{C}$ are categories cofibered in groupoids and $\varphi, \psi : \mathcal{F} \to \mathcal{F}'$ are 1-morphisms, then a 2-morphism $t : \varphi \to \psi$ is a morphism of functors such that $p'(t_x) = \text{id}_{p(x)}$ for all $x \in \text{Ob}(\mathcal{F})$.

06GN (9) Let $F : \mathcal{C} \to \text{Groupoids}$ be a functor. There is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}$ associated to F as follows. An object of $\mathcal{F}$ is a pair (U, x) where $U \in \text{Ob}(\mathcal{C})$ and $x \in \text{Ob}(F(U))$. A morphism $(U, x) \to (V, y)$ is a pair (f, a) where $f \in \text{Mor}_\mathcal{C}(U, V)$ and $a \in \text{Mor}_{F(V)}(F(f)(x), y)$. The functor $\mathcal{F} \to \mathcal{C}$ sends (U, x) to U. See Categories, Section 4.37.

07W5 (10) Let $\mathcal{F}$ be cofibered in groupoids over $\mathcal{C}$. For $U \in \text{Ob}(\mathcal{C})$ set $\overline{\mathcal{F}}(U)$ equal to the set of isomorphisms classes of the category $\mathcal{F}(U)$. If $f : U \to V$ is a morphism of $\mathcal{C}$, then we obtain a map of sets $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$ by mapping the isomorphism class of x to the isomorphism class of a pushforward f_*x of x see (4). Then $\overline{\mathcal{F}} : \mathcal{C} \to \text{Sets}$ is a functor. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of cofibered categories, we denote by $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ the associated morphism of functors.

06GP (11) Let $F : \mathcal{C} \to \text{Sets}$ be a functor. We can think of a set as a discrete category, i.e., as a groupoid with only identity morphisms. Then the construction (9) associates to F a category cofibered in sets. This defines a fully faithful embedding of the category of functors $\mathcal{C} \to \text{Sets}$ to the category of categories cofibered in groupoids over $\mathcal{C}$. We identify the category of functors with its image under this embedding. Hence if $F : \mathcal{C} \to \text{Sets}$ is a functor, we denote the associated category cofibered in sets also by F; and if $\varphi : F \to G$ is a morphism of functors, we denote still by φ the corresponding morphism of categories cofibered in sets, and vice-versa. See Categories, Section 4.38.

06GQ (12) Let U be an object of $\mathcal{C}$. We write $\underline{U}$ for the functor $\text{Mor}_\mathcal{C}(U, -) : \mathcal{C} \to \text{Sets}$. This defines a fully faithful embedding of $\mathcal{C}^{opp}$ into the category of functors $\mathcal{C} \to \text{Sets}$. Hence, if $f : U \to V$ is a morphism, we are justified in denoting still by f the induced morphism $\underline{V} \to \underline{U}$, and vice-versa.

06SI (13) Fiber products of categories cofibered in groupoids: If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$, then a construction of their 2-fiber product is given by the construction for their 2-fiber product as categories over $\mathcal{C}_\Lambda$, as described in Categories, Lemma 4.32.3.

0DZJ (14) Products of categories cofibered in groupoids: If $\mathcal{F}$ and $\mathcal{G}$ are categories cofibered in groupoids over $\mathcal{C}_\Lambda$ then their product is defined to be the 2-fiber product $\mathcal{F} \times_{\mathcal{C}_\Lambda} \mathcal{G}$ as described in Categories, Lemma 4.32.3.

06GR (15) Restricting the base category: Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The restriction $\mathcal{F}|_{\mathcal{C}'}$ is the full subcategory of $\mathcal{F}$ whose objects lie over objects of $\mathcal{C}'$. It is a category cofibered in groupoids via the functor $p|_{\mathcal{C}'} : \mathcal{F}|_{\mathcal{C}'} \to \mathcal{C}'$.

90.6. Prorepresentable functors and predeformation categories

06GI Our basic goal is to understand categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}}_\Lambda$. Since $\mathcal{C}_\Lambda$ is a full subcategory of $\widehat{\mathcal{C}}_\Lambda$ we can restrict categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$, see Remarks 90.5.2 (15). In particular we can do this with functors, in particular with representable functors. The functors on $\mathcal{C}_\Lambda$ one obtains in this way are called prorepresentable functors.

06GX Definition 90.6.1. Let $F : \mathcal{C}_\Lambda \to \textit{Sets}$ be a functor. We say F is prorepresentable if there exists an isomorphism $F \cong \underline{R}|_{\mathcal{C}_\Lambda}$ of functors for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$.

Note that if $F : \mathcal{C}_\Lambda \to \textit{Sets}$ is prorepresentable by $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$, then

$$F(k) = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, k) = \{*\}$$

is a singleton. The categories cofibered in groupoids over $\mathcal{C}_\Lambda$ that are arise in deformation theory will often satisfy an analogous condition.

06GS Definition 90.6.2. A predeformation category $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to a category with a single object and a single morphism, i.e., $\mathcal{F}(k)$ contains at least one object and there is a unique morphism between any two objects. A morphism of predeformation categories is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

A feature of a predeformation category is the following. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then every object of $\mathcal{F}$ comes equipped with a unique morphism to x_0. Namely, if x is an object of $\mathcal{F}$ over A, then we can choose a pushforward $x \to q_*x$ where $q : A \to k$ is the quotient map. There is a unique isomorphism $q_*x \to x_0$ and the composition $x \to q_*x \to x_0$ is the desired morphism.

06GT Remark 90.6.3. We say that a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ is a predeformation functor if the associated cofibered set is a predeformation category, i.e. if $F(k)$ is a one element set. Thus if $\mathcal{F}$ is a predeformation category, then $\overline{\mathcal{F}}$ is a predeformation functor.

06GU Remark 90.6.4. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. We denote by $\mathcal{F}_x$ the category of objects over x. An object of $\mathcal{F}_x$ is an arrow $y \to x$. A morphism $(y \to x) \to (z \to x)$ in $\mathcal{F}_x$ is a commutative diagram

$$\begin{tikzcd} y \arrow[rr] \arrow[dr] & & z \arrow[dl] \\ & x & \end{tikzcd}$$

There is a forgetful functor $\mathcal{F}_x \to \mathcal{F}$. We define the functor $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ as the composition $\mathcal{F}_x \to \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$. Then $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ is a predeformation category (proof omitted). In this way we can pass from an arbitrary category cofibered in groupoids over $\mathcal{C}_\Lambda$ to a predeformation category at any $x \in \mathrm{Ob}(\mathcal{F}(k))$.

90.7. Formal objects and completion categories

06H2 In this section we discuss how to go between categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ and vice versa.

06H3 **Definition 90.7.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$ is the category with the following objects and morphisms.

1. A formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$ consists of an object R of $\widehat{\mathcal{C}}_\Lambda$, and a collection indexed by $n \in \mathbf{N}$ of objects ξ_n of $\mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$.
2. Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. A morphism $a : \xi \to \eta$ of formal objects consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, such that for every n the diagram
$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"'] \arrow[d, "a_{n+1}"'] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$
commutes.

The category of formal objects comes with a functor $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ which sends an object (R, ξ_n, f_n) to R and a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ to the map $R \to S$.

06H4 **Lemma 90.7.2.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Then $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is a category cofibered in groupoids.

Proof. Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Let (R, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}$. For each n choose a pushforward $\xi_n \to \eta_n$ of ξ_n along $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. For each n there exists a unique morphism $g_n : \eta_{n+1} \to \eta_n$ in $\mathcal{F}$ lying over $S/\mathfrak{m}_S^{n+1} \to S/\mathfrak{m}_S^n$ such that
$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"'] \arrow[d] & \xi_n \arrow[d] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$
commutes (by the first axiom of a category cofibred in groupoids). Hence we obtain a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ lying over $R \to S$, i.e., the first axiom of a category cofibred in groupoids holds for $\widehat{\mathcal{F}}$. To see the second axiom suppose that we have morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ and $b : (R, \xi_n, f_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ and a morphism $c_0 : S \to T$ in $\widehat{\mathcal{C}}_\Lambda$ such that $c_0 \circ a_0 = b_0$. By the second axiom of a category cofibred in groupoids for $\mathcal{F}$ we obtain unique maps $c_n : \eta_n \to \theta_n$ lying over $S/\mathfrak{m}_S^n \to T/\mathfrak{m}_T^n$ such that $c_n \circ a_n = b_n$. Setting $c = (c_n)_{n \geq 0}$ gives the desired morphism $c : (S, \eta_n, g_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ (we omit the verification that $h_n \circ c_{n+1} = c_n \circ g_n$). □

06H5 **Definition 90.7.3.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The category cofibered in groupoids $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is called the *completion* of $\mathcal{F}$.

If $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$, we have defined $\widehat{\mathcal{F}}(R)$ for $R \in \text{Ob}(\widehat{\mathcal{C}}_\Lambda)$ in terms of the filtration of R by powers of its maximal ideal. But suppose

$\mathcal{I} = (I_n)$ is a filtration of R by ideals inducing the $\mathfrak{m}_R$-adic topology. We define $\widehat{\mathcal{F}}_{\mathcal{I}}(R)$ to be the category with the following objects and morphisms:

(1) An object is a collection $(\xi_n, f_n)_{n \in \mathbf{N}}$ of objects ξ_n of $\mathcal{F}(R/I_n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projections $R/I_{n+1} \to R/I_n$.
(2) A morphism $a : (\xi_n, f_n) \to (\eta_n, g_n)$ consists of a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}(R/I_n)$, such that for every n the diagram

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"] \arrow[d, "a_{n+1}"'] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes.

06H6 Lemma 90.7.4. In the situation above, $\widehat{\mathcal{F}}_{\mathcal{I}}(R)$ is equivalent to the category $\widehat{\mathcal{F}}(R)$.

Proof. An equivalence $\widehat{\mathcal{F}}_{\mathcal{I}}(R) \to \widehat{\mathcal{F}}(R)$ can be defined as follows. For each n, let $m(n)$ be the least m that $I_m \subset \mathfrak{m}_R^n$. Given an object (ξ_n, f_n) of $\widehat{\mathcal{F}}_{\mathcal{I}}(R)$, let η_n be the pushforward of $\xi_{m(n)}$ along $R/I_{m(n)} \to R/\mathfrak{m}_R^n$. Let $g_n : \eta_{n+1} \to \eta_n$ be the unique morphism of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ such that

$$\begin{tikzcd} \xi_{m(n+1)} \arrow[r, "f_{m(n)} \circ \ldots \circ f_{m(n+1)-1}"'] \arrow[d] & \xi_{m(n)} \arrow[d] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes (existence and uniqueness is guaranteed by the axioms of a cofibred category). The functor $\widehat{\mathcal{F}}_{\mathcal{I}}(R) \to \widehat{\mathcal{F}}(R)$ sends (ξ_n, f_n) to (R, η_n, g_n). We omit the verification that this is indeed an equivalence of categories. $\square$

06H7 Remark 90.7.5. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Suppose that for each $R \in \text{Ob}(\widehat{\mathcal{C}}_\Lambda)$ we are given a filtration $\mathcal{I}_R$ of R by ideals. If $\mathcal{I}_R$ induces the $\mathfrak{m}_R$-adic topology on R for all R, then one can define a category $\widehat{\mathcal{F}}_{\mathcal{I}}$ by mimicking the definition of $\widehat{\mathcal{F}}$. This category comes equipped with a morphism $\widehat{p}_{\mathcal{I}} : \widehat{\mathcal{F}}_{\mathcal{I}} \to \widehat{\mathcal{C}}_\Lambda$ making it into a category cofibered in groupoids such that $\widehat{\mathcal{F}}_{\mathcal{I}}(R)$ is isomorphic to $\widehat{\mathcal{F}}_{\mathcal{I}_R}(R)$ as defined above. The categories cofibered in groupoids $\widehat{\mathcal{F}}_{\mathcal{I}}$ and $\widehat{\mathcal{F}}$ are equivalent, by using over an object $R \in \text{Ob}(\widehat{\mathcal{C}}_\Lambda)$ the equivalence of Lemma 90.7.4.

06H8 Remark 90.7.6. Let $F : \mathcal{C}_\Lambda \to \text{Sets}$ be a functor. Identifying functors with cofibered sets, the completion of F is the functor $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \text{Sets}$ given by $\widehat{F}(S) = \lim F(S/\mathfrak{m}_S^n)$. This agrees with the definition in Schlessinger's paper [Sch68].

06SJ Remark 90.7.7. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. We claim that there is a canonical equivalence

$$can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F}.$$

Namely, let $A \in \text{Ob}(\mathcal{C}_\Lambda)$ and let (A, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda}(A)$. Since A is Artinian there is a minimal $m \in \mathbf{N}$ such that $\mathfrak{m}_A^m = 0$. Then can sends (A, ξ_n, f_n) to ξ_m. This functor is an equivalence of categories cofibered in groupoids by Categories, Lemma 4.35.9 because it is an equivalence on all fibre categories by Lemma 90.7.4 and the fact that the $\mathfrak{m}_A$-adic topology on a local Artinian ring A comes from the zero ideal. We will frequently identify $\mathcal{F}$ with a full subcategory of $\widehat{\mathcal{F}}$ via a quasi-inverse to the functor *can*.

06H9 Remark 90.7.8. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then there is an induced morphism $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of categories cofibered in groupoids over $\widehat{\mathcal{C}}_\Lambda$. It sends an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ to $(R, \varphi(\xi_n), \varphi(f_n))$, and it sends a morphism $(a_0 : R \to S, a_n : \xi_n \to \eta_n)$ between objects ξ and η of $\widehat{\mathcal{F}}$ to $(a_0 : R \to S, \varphi(a_n) : \varphi(\xi_n) \to \varphi(\eta_n))$. Finally, if $t : \varphi \to \varphi'$ is a 2-morphism between 1-morphisms $\varphi, \varphi' : \mathcal{F} \to \mathcal{G}$ of categories cofibred in groupoids, then we obtain a 2-morphism $\widehat{t} : \widehat{\varphi} \to \widehat{\varphi}'$. Namely, for $\xi = (R, \xi_n, f_n)$ as above we set $\widehat{t}_\xi = (t_{\varphi(\xi_n)})$. Hence completion defines a functor between 2-categories

$$\widehat{\ } : \mathrm{Cof}(\mathcal{C}_\Lambda) \longrightarrow \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda)$$

from the 2-category of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to the 2-category of categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$.

06HA Remark 90.7.9. We claim the completion functor of Remark 90.7.8 and the restriction functor $|_{\mathcal{C}_\Lambda} : \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{Cof}(\mathcal{C}_\Lambda)$ of Remarks 90.5.2 (15) are "2-adjoint" in the following precise sense. Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Cof}(\mathcal{C}_\Lambda))$ and let $\mathcal{G} \in \mathrm{Ob}(\mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda))$. Then there is an equivalence of categories

$$\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$$

To describe this equivalence, we define canonical morphisms $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ and $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ as follows

(1) Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda))$ and let ξ be an object of the fiber category $\mathcal{G}(R)$. Choose a pushforward $\xi \to \xi_n$ of ξ to $R/\mathfrak{m}_R^n$ for each $n \in \mathbf{N}$, and let $f_n : \xi_{n+1} \to \xi_n$ be the induced morphism. Then $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ sends ξ to (R, ξ_n, f_n).
(2) This is the equivalence $can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 90.7.7.

Having said this, the equivalence $\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \to \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$ sends a morphism $\varphi : \mathcal{G}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to

$$\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}} \xrightarrow{\widehat{\varphi}} \widehat{\mathcal{F}}$$

There is a quasi-inverse $\Psi : \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}}) \to \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F})$ to Φ which sends $\psi : \mathcal{G} \to \widehat{\mathcal{F}}$ to

$$\mathcal{G}|_{\mathcal{C}_\Lambda} \xrightarrow{\psi|_{\mathcal{C}_\Lambda}} \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}.$$

We omit the verification that Φ and Ψ are quasi-inverse. We also do not address functoriality of Φ (because it would lead into 3-category territory which we want to avoid at all cost).

06HB Remark 90.7.10. For a category $\mathcal{C}$ we denote by $\mathrm{CofSet}(\mathcal{C})$ the category of cofibered sets over $\mathcal{C}$. It is a 1-category isomorphic the category of functors $\mathcal{C} \to \mathit{Sets}$. See Remarks 90.5.2 (11). The completion and restriction functors restrict to functors $\widehat{\ } : \mathrm{CofSet}(\mathcal{C}_\Lambda) \to \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda)$ and $|_{\mathcal{C}_\Lambda} : \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{CofSet}(\mathcal{C}_\Lambda)$ which we denote by the same symbols. As functors on the categories of cofibered sets, completion and restriction are adjoints in the usual 1-categorical sense: the same construction as in Remark 90.7.9 defines a functorial bijection

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(G|_{\mathcal{C}_\Lambda}, F) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(G, \widehat{F})$$

for $F \in \mathrm{Ob}(\mathrm{CofSet}(\mathcal{C}_\Lambda))$ and $G \in \mathrm{Ob}(\mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda))$. Again the map $\widehat{F}|_{\mathcal{C}_\Lambda} \to F$ is an isomorphism.

06HE Remark 90.7.11. Let $G : \widehat{\mathcal{C}}_\Lambda \to \mathrm{Sets}$ be a functor that commutes with limits. Then the map $G \to \widehat{G|_{\mathcal{C}_\Lambda}}$ described in Remark 90.7.9 is an isomorphism. Indeed, if S is an object of $\widehat{\mathcal{C}}_\Lambda$, then we have canonical bijections

$$\widehat{G|_{\mathcal{C}_\Lambda}}(S) = \lim_n G(S/\mathfrak{m}_S^n) = G(\lim_n S/\mathfrak{m}_S^n) = G(S).$$

In particular, if R is an object of $\widehat{\mathcal{C}}_\Lambda$ then $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ because the representable functor $\underline{R}$ commutes with limits by definition of limits.

06HC Remark 90.7.12. Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. It defines a functor $\underline{R} : \widehat{\mathcal{C}}_\Lambda \to \mathrm{Sets}$ as described in Remarks 90.5.2 (12). As usual we identify this functor with the associated cofibered set. If $\mathcal{F}$ is a cofibered category over $\mathcal{C}_\Lambda$, then there is an equivalence of categories

06SK (90.7.12.1) $$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \widehat{\mathcal{F}}(R).$$

It is given by the composition

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \xrightarrow{\Phi} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\sim} \widehat{\mathcal{F}}(R)$$

where Φ is as in Remark 90.7.9 and the second equivalence comes from the 2-Yoneda lemma (the cofibered analogue of Categories, Lemma 4.41.2). Explicitly, the equivalence sends a morphism $\varphi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to the formal object $(R, \varphi(R \to R/\mathfrak{m}_R^n), \varphi(f_n))$ in $\widehat{\mathcal{F}}(R)$, where $f_n : R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ is the projection.

Assume a choice of pushforwards for $\mathcal{F}$ has been made. Given any $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we construct an explicit $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ which maps to ξ under (90.7.12.1). Namely, say $\xi = (R, \xi_n, f_n)$. An object α in $\underline{R}|_{\mathcal{C}_\Lambda}$ is the same thing as a morphism $\alpha : R \to A$ of $\widehat{\mathcal{C}}_\Lambda$ with A Artinian. Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. Then α factors through a unique $\alpha_m : R/\mathfrak{m}_R^m \to A$ and we can set $\underline{\xi}(\alpha) = \alpha_{m,*}\xi_m$. We omit the description of $\underline{\xi}$ on morphisms and we omit the proof that $\underline{\xi}$ maps to ξ via (90.7.12.1).

Assume a choice of pushforwards for $\widehat{\mathcal{F}}$ has been made. In this case the proof of Categories, Lemma 4.41.2 gives an explicit quasi-inverse

$$\iota : \widehat{\mathcal{F}}(R) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}})$$

to the 2-Yoneda equivalence which takes ξ to the morphism $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ sending $f \in \underline{R}(S) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(R, S)$ to $f_*\xi$. A quasi-inverse to (90.7.12.1) is then

$$\widehat{\mathcal{F}}(R) \xrightarrow{\iota} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\Psi} \mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F})$$

where Ψ is as in Remark 90.7.9. Given $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we have $\Psi(\iota(\xi)) \cong \underline{\xi}$ where $\underline{\xi}$ is as in the previous paragraph, because both are mapped to ξ under the equivalence of categories (90.7.12.1). Using $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ (see Remark 90.7.11) and unwinding the definitions of Φ and Ψ we conclude that $\iota(\xi)$ is isomorphic to the completion of $\underline{\xi}$.

06SL Remark 90.7.13. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. Let $a = (a_n) : \xi \to \eta$ be a morphism of formal objects, i.e., a morphism of $\widehat{\mathcal{F}}$. Let $f = \widehat{p}(a) = a_0 : R \to S$ be the projection of a in $\widehat{\mathcal{C}}_\Lambda$. Then we obtain a 2-commutative diagram

$$\begin{tikzcd} \underline{R}|_{\mathcal{C}_\Lambda} \arrow[dr, "\underline{\xi}"'] & & \underline{S}|_{\mathcal{C}_\Lambda} \arrow[ll, "f"'] \arrow[dl, "\underline{\eta}"] \\ & \mathcal{F} & \end{tikzcd}$$

where $\underline{\xi}$ and $\underline{\eta}$ are the morphisms constructed in Remark 90.7.12. To see this let $\alpha : S \to A$ be an object of $\underline{S}|_{\mathcal{C}_\Lambda}$ (see loc. cit.). Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. We get a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"] & R/\mathfrak{m}_R^m \arrow[d, "f_m"'] \arrow[dr, "\beta_m"] & \\ S \arrow[r] & S/\mathfrak{m}_S^m \arrow[r, "\alpha_m"] & A \end{tikzcd}$$

such that the bottom arrows compose to give α. Then $\underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$ and $\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m$. The morphism $a_m : \xi_m \to \eta_m$ lies over f_m hence we obtain a canonical morphism

$$\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m \longrightarrow \underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$$

lying over id_A such that

$$\begin{tikzcd} \xi_m \arrow[r] \arrow[d, "a_m"] & \beta_{m,*}\xi_m \arrow[d] \\ \eta_m \arrow[r] & \alpha_{m,*}\eta_m \end{tikzcd}$$

commutes by the axioms of a category cofibred in groupoids. This defines a transformation of functors $\underline{\xi} \circ f \to \underline{\eta}$ which witnesses the 2-commutativity of the first diagram of this remark.

06HD Remark 90.7.14. According to Remark 90.7.12, giving a formal object ξ of $\mathcal{F}$ is equivalent to giving a prorepresentable functor $U : \mathcal{C}_\Lambda \to \textit{Sets}$ and a morphism $U \to \mathcal{F}$.

90.8. Smooth morphisms

06HF In this section we discuss smooth morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

06HG Definition 90.8.1. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say φ is *smooth* if it satisfies the following condition: Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B)), x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. Then there exists $x' \in \mathrm{Ob}(\mathcal{F}(B))$, a morphism $x' \to x$ lying over $B \to A$, and a morphism $\varphi(x') \to y$ lying over $\mathrm{id} : B \to B$, such that the diagram

$$\begin{tikzcd} \varphi(x') \arrow[r] \arrow[dr] & y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes.

06HH Lemma 90.8.2. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then φ is smooth if the condition in Definition 90.8.1 is assumed to hold only for small extensions $B \to A$.

Proof. Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B))$, $x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. By Lemma 90.3.3 we can factor $B \to A$ into small extensions $B = B_n \to B_{n-1} \to \ldots \to B_0 = A$. We argue by induction on n. If $n = 1$ the result is true by assumption. If $n > 1$, then denote $f : B = B_n \to B_{n-1}$ and denote $g : B_{n-1} \to B_0 = A$. Choose a pushforward $y \to f_*y$ of y along f, so that the morphism $y \to \varphi(x)$ factors as $y \to f_*y \to \varphi(x)$. By the induction hypothesis we can find $x_{n-1} \to x$ lying over $g : B_{n-1} \to A$ and $a : \varphi(x_{n-1}) \to f_*y$ lying over $\mathrm{id} : B_{n-1} \to B_{n-1}$ such that

$$\begin{tikzcd}
\varphi(x_{n-1}) \arrow[r, "a"'] \arrow[dr] & f_*y \arrow[d] \\
& \varphi(x)
\end{tikzcd}$$

commutes. We can apply the assumption to the composition $y \to \varphi(x_{n-1})$ of $y \to f_*y$ with $a^{-1} : f_*y \to \varphi(x_{n-1})$. We obtain $x_n \to x_{n-1}$ lying over $B_n \to B_{n-1}$ and $\varphi(x_n) \to y$ lying over $\mathrm{id} : B_n \to B_n$ so that the diagram

$$\begin{tikzcd}
\varphi(x_n) \arrow[r] \arrow[d] & y \arrow[d] \\
\varphi(x_{n-1}) \arrow[r, "a"] \arrow[dr] & f_*y \arrow[d] \\
& \varphi(x)
\end{tikzcd}$$

commutes. Then the composition $x_n \to x_{n-1} \to x$ and $\varphi(x_n) \to y$ are the morphisms required by the definition of smoothness. $\square$

06HI Remark 90.8.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Choices of pushforwards along $B \to A$ for objects in the fiber categories $\mathcal{F}(B)$ and $\mathcal{G}(B)$ determine functors $\mathcal{F}(B) \to \mathcal{F}(A)$ and $\mathcal{G}(B) \to \mathcal{G}(A)$ fitting into a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{F}(B) \arrow[r, "\varphi"] \arrow[d] & \mathcal{G}(B) \arrow[d] \\
\mathcal{F}(A) \arrow[r, "\varphi"] & \mathcal{G}(A).
\end{tikzcd}$$

Hence there is an induced functor $\mathcal{F}(B) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(B)$. Unwinding the definitions shows that $\varphi : \mathcal{F} \to \mathcal{G}$ is smooth if and only if this induced functor is essentially surjective whenever $B \to A$ is surjective (or equivalently, by Lemma 90.8.2, whenever $B \to A$ is a small extension).

06HJ Remark 90.8.4. The characterization of smooth morphisms in Remark 90.8.3 is analogous to Schlessinger's notion of a smooth morphism of functors, cf. [Sch68, Definition 2.2.]. In fact, when $\mathcal{F}$ and $\mathcal{G}$ are cofibered in sets then our notion is equivalent to Schlessinger's. Namely, in this case let $F, G : \mathcal{C}_\Lambda \to \textit{Sets}$ be the corresponding functors, see Remarks 90.5.2 (11). Then $F \to G$ is smooth if and only if for every surjection of rings $B \to A$ in $\mathcal{C}_\Lambda$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

06HK Remark 90.8.5. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then the morphism $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth. Namely, suppose that $f : B \to A$ is a ring map in $\mathcal{C}_\Lambda$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$ and let $\overline{y} \in \overline{\mathcal{F}}(B)$ be the isomorphism class of $y \in \mathrm{Ob}(\mathcal{F}(B))$ such that $\overline{f_* y} = \overline{x}$. Then we simply take $x' = y$, the implied morphism $x' = y \to x$ over $B \to A$, and the equality $\overline{x'} = \overline{y}$ as the solution to the problem posed in Definition 90.8.1.

If $R \to S$ is a ring map $\widehat{\mathcal{C}}_\Lambda$, then there is an induced morphism $\underline{S} \to \underline{R}$ between the functors $\underline{S}, \underline{R} : \widehat{\mathcal{C}}_\Lambda \to \textit{Sets}$. In this situation, smoothness of the restriction $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is a familiar notion:

06HL Lemma 90.8.6. Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Then the induced morphism $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth if and only if S is a power series ring over R.

Proof. Assume S is a power series ring over R. Say $S = R[[x_1, \ldots, x_n]]$. Smoothness of $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ means the following (see Remark 90.8.4): Given a surjective ring map $B \to A$ in $\mathcal{C}_\Lambda$, a ring map $R \to B$, a ring map $S \to A$ such that the solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dotted] & A \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

is commutative then a dotted arrow exists making the diagram commute. (Note the similarity with Algebra, Definition 10.138.1.) To construct the dotted arrow choose elements $b_i \in B$ whose images in A are equal to the images of x_i in A. Note that $b_i \in \mathfrak{m}_B$ as x_i maps to an element of $\mathfrak{m}_A$. Hence there is a unique R-algebra map $R[[x_1, \ldots, x_n]] \to B$ which maps x_i to b_i and which can serve as our dotted arrow.

Conversely, assume $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Let $x_1, \ldots, x_n \in S$ be elements whose images form a basis in the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ of S over R. Set $T = R[[X_1, \ldots, X_n]]$. Note that both

$$S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \cong R/\mathfrak{m}_R[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/(\mathfrak{m}_R T + \mathfrak{m}_T^2) \cong R/\mathfrak{m}_R[X_1, \ldots, X_n]/(X_i X_j).$$

Let $S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the local R-algebra isomorphism given by mapping the class of x_i to the class of X_i. Let $f_1 : S \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the composition $S \to S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$. The assumption that $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means we can lift f_1 to a map $f_2 : S \to T/\mathfrak{m}_T^2$, then to a map $f_3 : S \to T/\mathfrak{m}_T^3$, and so on, for all $n \geq 1$. Thus we get an induced map $f : S \to T = \lim T/\mathfrak{m}_T^n$ of local R-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to \mathfrak{m}_T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ of relative cotangent spaces. Hence f is surjective by Lemma 90.4.2 (where we think of f as a map in $\widehat{\mathcal{C}}_R$). Choose preimages $y_i \in S$ of $X_i \in T$ under f. As T is a power series ring over R there exists a local R-algebra homomorphism $s : T \to S$ mapping X_i to

y_i. By construction $f \circ s = \mathrm{id}$. Then s is injective. But s induces an isomorphism on relative cotangent spaces since f does, so it is also surjective by Lemma 90.4.2 again. Hence s and f are isomorphisms. □

Smooth morphisms satisfy the following functorial properties.

06HM Lemma 90.8.7. Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

(1) If φ and ψ are smooth, then $\psi \circ \varphi$ is smooth.
(2) If φ is essentially surjective and $\psi \circ \varphi$ is smooth, then ψ is smooth.
(3) If $\mathcal{G}' \to \mathcal{G}$ is a morphism of categories cofibered in groupoids and φ is smooth, then $\mathcal{F} \times_\mathcal{G} \mathcal{G}' \to \mathcal{G}'$ is smooth.

Proof. Statements (1) and (2) follow immediately from the definitions. Proof of (3) omitted. Hints: use the formulation of smoothness given in Remark 90.8.3 and use that $\mathcal{F} \times_\mathcal{G} \mathcal{G}'$ is the 2-fibre product, see Remarks 90.5.2 (13). □

06HN Lemma 90.8.8. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a smooth morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ is essentially surjective. Then $\varphi : \mathcal{F} \to \mathcal{G}$ and $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ are essentially surjective.

Proof. Let y be an object of $\mathcal{G}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. Let $y \to y_0$ be a pushforward of y along $A \to k$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ there exist an object x_0 of $\mathcal{F}$ lying over k and an isomorphism $y_0 \to \varphi(x_0)$. Smoothness of φ implies there exists an object x of $\mathcal{F}$ over A whose image $\varphi(x)$ is isomorphic to y. Thus $\varphi : \mathcal{F} \to \mathcal{G}$ is essentially surjective.

Let $\eta = (R, \eta_n, g_n)$ be an object of $\widehat{\mathcal{G}}$. We construct an object ξ of $\widehat{\mathcal{F}}$ with an isomorphism $\eta \to \varphi(\xi)$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$, there exists a morphism $\eta_1 \to \varphi(\xi_1)$ in $\mathcal{G}(k)$ for some $\xi_1 \in \mathrm{Ob}(\mathcal{F}(k))$. The morphism $\eta_2 \xrightarrow{g_1} \eta_1 \to \varphi(\xi_1)$ lies over the surjective ring map $R/\mathfrak{m}_R^2 \to k$, hence by smoothness of φ there exists $\xi_2 \in \mathrm{Ob}(\mathcal{F}(R/\mathfrak{m}_R^2))$, a morphism $f_1 : \xi_2 \to \xi_1$ lying over $R/\mathfrak{m}_R^2 \to k$, and a morphism $\eta_2 \to \varphi(\xi_2)$ such that

$$\begin{tikzcd}
\varphi(\xi_2) \arrow[r, "\varphi(f_1)"] & \varphi(\xi_1) \\
\eta_2 \arrow[u] \arrow[r, "g_1"] & \eta_1 \arrow[u]
\end{tikzcd}$$

commutes. Continuing in this way we construct an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ and a morphism $\eta \to \varphi(\xi) = (R, \varphi(\xi_n), \varphi(f_n))$ in $\widehat{\mathcal{G}}(R)$. □

Later we are interested in producing smooth morphisms from prorepresentable functors to predeformation categories $\mathcal{F}$. By the discussion in Remark 90.7.12 these morphisms correspond to certain formal objects of $\mathcal{F}$. More precisely, these are the so-called versal formal objects of $\mathcal{F}$.

06HR Definition 90.8.9. Let $\mathcal{F}$ be a category cofibered in groupoids. Let ξ be a formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We say ξ is versal if the corresponding morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 90.7.12 is smooth.

06HS Remark 90.8.10. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let ξ be a formal object of $\mathcal{F}$. It follows from the definition of smoothness that versality of ξ is equivalent to the following condition: If

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd}$$

is a diagram in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \to A$ of Artinian rings (we may assume it is a small extension), then there exists a morphism $\xi \to y$ such that

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur] & x \end{tikzcd}$$

commutes. In particular, the condition that ξ be versal does not depend on the choices of pushforwards made in the construction of $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ in Remark 90.7.12.

06HT Lemma 90.8.11. Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$. For any formal object η of $\widehat{\mathcal{F}}$, there exists a morphism $\xi \to \eta$.

Proof. By assumption the morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth. Then $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ is the completion of $\underline{\xi}$, see Remark 90.7.12. By Lemma 90.8.8 there exists an object f of $\underline{R}$ such that $\iota(\xi)(f) = \eta$. Then f is a ring map $f : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$. And $\iota(\xi)(f) = \eta$ means that $f_*\xi \cong \eta$ which means exactly that there is a morphism $\xi \to \eta$ lying over f. □

90.9. Smooth or unobstructed categories

0DYK Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We can consider $\mathcal{C}_\Lambda$ as a category cofibered in groupoids over $\mathcal{C}_\Lambda$ using the identity functor. In this way $p : \mathcal{F} \longrightarrow \mathcal{C}_\Lambda$ becomes a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

06HP Definition 90.9.1. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We say $\mathcal{F}$ is smooth or unobstructed if its structure morphism p is smooth in the sense of Definition 90.8.1.

This is the "absolute" notion of smoothness for a category cofibered in groupoids over $\mathcal{C}_\Lambda$, although it would be more correct to say that $\mathcal{F}$ is smooth over Λ. One has to be careful with the phrase "$\mathcal{F}$ is unobstructed": it may happen that $\mathcal{F}$ has an obstruction theory with nonvanishing obstruction spaces even though $\mathcal{F}$ is smooth.

06HQ Remark 90.9.2. Suppose $\mathcal{F}$ is a predeformation category admitting a smooth morphism $\varphi : \mathcal{U} \to \mathcal{F}$ from a predeformation category $\mathcal{U}$. Then by Lemma 90.8.8 φ is essentially surjective, so by Lemma 90.8.7 $p : \mathcal{F} \to \mathcal{C}_\Lambda$ is smooth if and only if the composition $\mathcal{U} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$ is smooth, i.e. $\mathcal{F}$ is smooth if and only if $\mathcal{U}$ is smooth.

0DYL Lemma 90.9.3. Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. The following are equivalent

(1) $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth,
(2) $\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology,

(3) $\Lambda \to R$ is flat and $R \otimes_\Lambda k'$ is geometrically regular over k', and
(4) $\Lambda \to R$ is flat and $k' \to R \otimes_\Lambda k'$ is formally smooth in the $\mathfrak m_R$-adic topology.

In the classical case, these are also equivalent to

(5) R is isomorphic to $\Lambda[[x_1, \ldots, x_n]]$ for some n.

Proof. Smoothness of $p : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{C}_\Lambda$ means that given $B \to A$ surjective in $\mathcal{C}_\Lambda$ and given $R \to A$ we can find the dotted arrow in the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[rd, dashed] & A \\ \Lambda \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

This is certainly true if $\Lambda \to R$ is formally smooth in the $\mathfrak m_R$-adic topology, see More on Algebra, Definitions 15.37.3 and 15.37.1. Conversely, if this holds, then we see that $\Lambda \to R$ is formally smooth in the $\mathfrak m_R$-adic topology by More on Algebra, Lemma 15.38.1. Thus (1) and (2) are equivalent.

The equivalence of (2), (3), and (4) is More on Algebra, Proposition 15.40.5. The equivalence with (5) follows for example from Lemma 90.8.6 and the fact that $\mathcal{C}_\Lambda$ is the same as $\underline{\Lambda}|_{\mathcal{C}_\Lambda}$ in the classical case. □

0DZK Lemma 90.9.4. Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. The following are equivalent

(1) $\mathcal{F}$ is unobstructed, and
(2) $\Lambda \to R$ is formally smooth in the $\mathfrak m_R$-adic topology.

In the classical case these are also equivalent to

(3) $R \cong \Lambda[[x_1, \ldots, x_n]]$ for some n.

Proof. If (1) holds, i.e., if $\mathcal{F}$ is unobstructed, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \xrightarrow{\xi} \mathcal{F} \to \mathcal{C}_\Lambda$$

is smooth, see Lemma 90.8.7. Hence we see that (2) holds by Lemma 90.9.3. Conversely, if (2) holds, then the composition is smooth and moreover the first arrow is essentially surjective by Lemma 90.8.11. Hence we find that the second arrow is smooth by Lemma 90.8.7 which means that $\mathcal{F}$ is unobstructed by definition. The equivalence with (3) in the classical case follows from Lemma 90.9.3. □

06SM Lemma 90.9.5. There exists an $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that the equivalent conditions of Lemma 90.9.3 hold and moreover $H_1(L_{k/\Lambda}) = \mathfrak m_R/\mathfrak m_R^2$ and $\Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$.

Proof. In the classical case we choose $R = \Lambda$. More generally, if the residue field extension k/k' is separable, then there exists a unique finite étale extension $\Lambda^\wedge \to R$ (Algebra, Lemmas 10.153.9 and 10.153.7) of the completion $\Lambda^\wedge$ of Λ inducing the extension k/k' on residue fields.

In the general case we proceed as follows. Choose a smooth Λ-algebra P and a Λ-algebra surjection $P \to k$. (For example, let P be a polynomial algebra.) Denote $\mathfrak m_P$ the kernel of $P \to k$. The Jacobi-Zariski sequence, see (90.3.10.2) and Algebra, Lemma 10.134.4, is an exact sequence

$$0 \to H_1(NL_{k/\Lambda}) \to \mathfrak m_P/\mathfrak m_P^2 \to \Omega_{P/\Lambda} \otimes_P k \to \Omega_{k/\Lambda} \to 0$$

We have the 0 on the left because P/k is smooth, hence $NL_{P/\Lambda}$ is quasi-isomorphic to a finite projective module placed in degree 0, hence $H_1(NL_{P/\Lambda} \otimes_P k) = 0$. Suppose $f \in \mathfrak{m}_P$ maps to a nonzero element of $\Omega_{P/\Lambda} \otimes_P k$. Setting $P' = P/(f)$ we have a Λ-algebra surjection $P' \to k$. Observe that P' is smooth at $\mathfrak{m}_{P'}$: this follows from More on Morphisms, Lemma 37.38.1. Thus after replacing P by a principal localization of P', we see that $\dim(\mathfrak{m}_P/\mathfrak{m}_P^2)$ decreases. Repeating finitely many times, we may assume the map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{P/\Lambda} \otimes_P k$ is zero so that the exact sequence breaks into isomorphisms $H_1(L_{k/\Lambda}) = \mathfrak{m}_P/\mathfrak{m}_P^2$ and $\Omega_{P/\Lambda} \otimes_P k = \Omega_{k/\Lambda}$.

Let R be the $\mathfrak{m}_P$-adic completion of P. Then R is an object of $\widehat{\mathcal{C}}_\Lambda$. Namely, it is a complete local Noetherian ring (see Algebra, Lemma 10.97.6) and its residue field is identified with k. We claim that R works.

First observe that the map $P \to R$ induces isomorphisms $\mathfrak{m}_P/\mathfrak{m}_P^2 = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{P/\Lambda} \otimes_P k = \Omega_{R/\Lambda} \otimes_R k$. This is true because both $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $\Omega_{P/\Lambda} \otimes_P k$ only depend on the Λ-algebra $P/\mathfrak{m}_P^2$, see Algebra, Lemma 10.131.11, the same holds for R and we have $P/\mathfrak{m}_P^2 = R/\mathfrak{m}_R^2$. Using the functoriality of the Jacobi-Zariski sequence (90.3.10.3) we deduce that $H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$ as the same is true for P.

Finally, since $\Lambda \to P$ is smooth we see that $\Lambda \to P$ is formally smooth by Algebra, Proposition 10.138.13. Then $\Lambda \to P$ is formally smooth for the $\mathfrak{m}_P$-adic topology by More on Algebra, Lemma 15.37.2. This property is inherited by the completion R by More on Algebra, Lemma 15.37.4 and the proof is complete. In fact, it turns out that whenever $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, then R is isomorphic to a completion of a smooth algebra over Λ, but we won't use this. ☐

06SN Example 90.9.6. Here is a more explicit example of an R as in Lemma 90.9.5. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Then we can take

$$R = \Lambda[x_1, \ldots, x_n]^\wedge_{(x_1^p - t_1, \ldots, x_n^p - t_n)}$$

We cannot do "better" in this example, i.e., we cannot approximate $\mathcal{C}_\Lambda$ by a smaller smooth object of $\widehat{\mathcal{C}}_\Lambda$ (one can argue that the dimension of R has to be at least n since the map $\Omega_{R/\Lambda} \otimes_R k \to \Omega_{k/\Lambda}$ is surjective). We will discuss this phenomenon later in more detail.

90.10. Schlessinger's conditions

06HV In the following we often consider fibre products $A_1 \times_A A_2$ of rings in the category $\mathcal{C}_\Lambda$. We have seen in Example 90.3.7 that such a fibre product may not always be an object of $\mathcal{C}_\Lambda$. However, in virtually all cases below one of the two maps $A_i \to A$ is surjective and $A_1 \times_A A_2$ will be an object of $\mathcal{C}_\Lambda$ by Lemma 90.3.8. We will use this result without further mention.

We denote by $k[\epsilon]$ the ring of dual numbers over k. More generally, for a k-vector space V, we denote by $k[V]$ the k-algebra whose underlying vector space is $k \oplus V$ and whose multiplication is given by $(a, v) \cdot (a', v') = (aa', av' + a'v)$. When $V = k$, $k[V]$ is the ring of dual numbers over k. For any finite dimensional k-vector space V the ring $k[V]$ is in $\mathcal{C}_\Lambda$.

06HW Definition 90.10.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We define conditions (S1) and (S2) on $\mathcal{F}$ as follows:

(S1) Every diagram in $\mathcal{F}$

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective can be completed to a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

(S2) The condition of (S1) holds for diagrams in $\mathcal{F}$ lying over a diagram in $\mathcal{C}_\Lambda$ of the form

$$\begin{tikzcd} & k[\epsilon] \arrow[d] \\ A \arrow[r] & k. \end{tikzcd}$$

Moreover, if we have two commutative diagrams in $\mathcal{F}$

$$\begin{tikzcd} y \arrow[r, "c"'] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"'] \arrow[d, "a'"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

then there exists a morphism $b : y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$ such that $a = a' \circ b$.

We can partly explain the meaning of conditions (S1) and (S2) in terms of fibre categories. Suppose that $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. Denote $p_i : A_1 \times_A A_2 \to A_i$ the projection maps. Assume a choice of pushforwards for $\mathcal{F}$ has been made. Then the commutative diagram of rings translates into a 2-commutative diagram

$$\begin{tikzcd} \mathcal{F}(A_1 \times_A A_2) \arrow[r, "p_{2,*}"'] \arrow[d, "p_{1,*}"'] & \mathcal{F}(A_2) \arrow[d, "f_{2,*}"] \\ \mathcal{F}(A_1) \arrow[r, "f_{1,*}"] & \mathcal{F}(A) \end{tikzcd}$$

of fibre categories whence a functor

06SP (90.10.1.1) $$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

into the 2-fibre product of categories. Condition (S1) requires that this functor be essentially surjective. The first part of condition (S2) requires that this functor be a essentially surjective if f_2 equals the map $k[\epsilon] \to k$. Moreover in this case, the second part of (S2) implies that two objects which become isomorphic in the target are isomorphic in the source (but it is not equivalent to this statement). The advantage of stating the conditions as in the definition is that no choices have to be made.

06HX **Lemma 90.10.2.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (S1) if the condition of (S1) is assumed to hold only when $A_2 \to A$ is a small extension.

Proof. Proof omitted. Hints: apply Lemma 90.3.3 and use induction similar to the proof of Lemma 90.8.2. □

06HY **Remark 90.10.3.** When $\mathcal{F}$ is cofibered in sets, conditions (S1) and (S2) are exactly conditions (H1) and (H2) from Schlessinger's paper [Sch68]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$, conditions (S1) and (S2) state:

(S1) If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is surjective.

(S2) If $A \to k$ is a map in $\mathcal{C}_\Lambda$, then the induced map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ is bijective.

The injectivity of the map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ comes from the second part of condition (S2) and the fact that morphisms are identities.

06HZ **Lemma 90.10.4.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}$ satisfies (S2), then the condition of (S2) also holds when $k[\epsilon]$ is replaced by $k[V]$ for any finite dimensional k-vector space V.

Proof. In the case that $\mathcal{F}$ is cofibred in sets, i.e., corresponds to a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$ this follows from the description of (S2) for F in Remark 90.10.3 and the fact that $k[V] \cong k[\epsilon] \times_k \ldots \times_k k[\epsilon]$ with $\dim_k V$ factors. The case of functors is what we will use in the rest of this chapter.

We prove the general case by induction on $\dim(V)$. If $\dim(V) = 1$, then $k[V] \cong k[\epsilon]$ and the result holds by assumption. If $\dim(V) > 1$ we write $V = V' \oplus k\epsilon$. Pick a diagram

$$\begin{tikzcd} & x_V \arrow[d] & & & k[V] \arrow[d] \\ x \arrow[r] & x_0 & \text{lying over} & A \arrow[r] & k \end{tikzcd}$$

Choose a morphism $x_V \to x_{V'}$ lying over $k[V] \to k[V']$ and a morphism $x_V \to x_\epsilon$ lying over $k[V] \to k[\epsilon]$. Note that the morphism $x_V \to x_0$ factors as $x_V \to x_{V'} \to x_0$ and as $x_V \to x_\epsilon \to x_0$. By induction hypothesis we can find a diagram

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x_{V'} \arrow[d] & & A \times_k k[V'] \arrow[r] \arrow[d] & k[V'] \arrow[d] \\ x \arrow[r] & x_0 & \text{lying over} & A \arrow[r] & k \end{tikzcd}$$

This gives us a commutative diagram

$$\begin{tikzcd} & x_\epsilon \arrow[d] & & & k[\epsilon] \arrow[d] \\ y' \arrow[r] & x_0 & \text{lying over} & A \times_k k[V'] \arrow[r] & k \end{tikzcd}$$

Hence by (S2) we get a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y' \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} (A \times_k k[V']) \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \times_k k[V'] \arrow[r] & k \end{tikzcd}$$

Note that $(A \times_k k[V']) \times_k k[\epsilon] = A \times_k k[V' \oplus k\epsilon] = A \times_k k[V]$. We claim that y fits into the correct commutative diagram. To see this we let $y \to y_V$ be a morphism lying over $A \times_k k[V] \to k[V]$. We can factor the morphisms $y \to y' \to x_{V'}$ and $y \to x_\epsilon$ through the morphism $y \to y_V$ (by the axioms of categories cofibred in groupoids). Hence we see that both y_V and x_V fit into commutative diagrams

$$\begin{tikzcd} y_V \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x_V \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x_{V'} \arrow[r] & x_0 \end{tikzcd}$$

and hence by the second part of (S2) there exists an isomorphism $y_V \to x_V$ compatible with $y_V \to x_{V'}$ and $x_V \to x_{V'}$ and in particular compatible with the maps to x_0. The composition $y \to y_V \to x_V$ then fits into the required commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V] \arrow[r] \arrow[d] & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

In this way we see that the first part of $(S2)$ holds with $k[\epsilon]$ replaced by $k[V]$.

To prove the second part suppose given two commutative diagrams

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V] \arrow[r] \arrow[d] & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

We will use the morphisms $x_V \to x_{V'} \to x_0$ and $x_V \to x_\epsilon \to x_0$ introduced in the first paragraph of the proof. Choose morphisms $y \to y_{V'}$ and $y' \to y'_{V'}$ lying over $A \times_k k[V] \to A \times_k k[V']$. The axioms of a cofibred category imply we can find commutative diagrams

$$\begin{tikzcd} y_{V'} \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y'_{V'} \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V'] \arrow[r] \arrow[d] & k[V'] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

By induction hypothesis we obtain an isomorphism $b : y_{V'} \to y'_{V'}$ compatible with the morphisms $y_{V'} \to x$ and $y'_{V'} \to x$, in particular compatible with the morphisms to x_0. Then we have commutative diagrams

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y'_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y'_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

where the morphism $y \to y'_{V'}$ is the composition $y \to y_{V'} \xrightarrow{b} y'_{V'}$ and where the morphisms $y \to x_\epsilon$ and $y' \to x_\epsilon$ are the compositions of the maps $y \to x_V$ and $y' \to x_V$ with the morphism $x_V \to x_\epsilon$. Then the second part of (S2) guarantees the existence of an isomorphism $y \to y'$ compatible with the maps to $y'_{V'}$, in particular compatible with the maps to x (because b was compatible with the maps to x). $\square$

06I0 **Lemma 90.10.5.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$.

(1) If $\mathcal{F}$ satisfies (S1), then so does $\overline{\mathcal{F}}$.
(2) If $\mathcal{F}$ satisfies (S2), then so does $\overline{\mathcal{F}}$ provided at least one of the following conditions is satisfied
   (a) $\mathcal{F}$ is a predeformation category,
   (b) the category $\mathcal{F}(k)$ is a set or a setoid, or
   (c) for any morphism $x_\epsilon \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$ the pushforward map $\mathrm{Aut}_{k[\epsilon]}(x_\epsilon) \to \mathrm{Aut}_k(x_0)$ is surjective.

Proof. Assume $\mathcal{F}$ has (S1). Suppose we have ring maps $f_i : A_i \to A$ in $\mathcal{C}_\Lambda$ with f_2 surjective. Let $x_i \in \mathcal{F}(A_i)$ such that the pushforwards $f_{1,*}(x_1)$ and $f_{2,*}(x_2)$ are isomorphic. Then we can denote x an object of $\mathcal{F}$ over A isomorphic to both of these and we obtain a diagram as in (S1). Hence we find an object y of $\mathcal{F}$ over $A_1 \times_A A_2$ whose pushforward to A_1, resp. A_2 is isomorphic to x_1, resp. x_2. In this way we see that (S1) holds for $\overline{\mathcal{F}}$.

Assume $\mathcal{F}$ has (S2). The first part of (S2) for $\overline{\mathcal{F}}$ follows as in the argument above. The second part of (S2) for $\overline{\mathcal{F}}$ signifies that the map

$$\overline{\mathcal{F}}(A \times_k k[\epsilon]) \to \overline{\mathcal{F}}(A) \times_{\overline{\mathcal{F}}(k)} \overline{\mathcal{F}}(k[\epsilon])$$

is injective for any ring A in $\mathcal{C}_\Lambda$. Suppose that $y, y' \in \mathcal{F}(A \times_k k[\epsilon])$. Using the axioms of cofibred categories we can choose commutative diagrams

$$\begin{tikzcd} y \arrow[r, "c"'] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"'] \arrow[d, "a'"'] & x'_\epsilon \arrow[d, "e'"] \\ x' \arrow[r, "d'"] & x'_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

Assume that there exist isomorphisms $\alpha : x \to x'$ in $\mathcal{F}(A)$ and $\beta : x_\epsilon \to x'_\epsilon$ in $\mathcal{F}(k[\epsilon])$. This also means there exists an isomorphism $\gamma : x_0 \to x'_0$ compatible with α. To prove (S2) for $\overline{\mathcal{F}}$ we have to show that there exists an isomorphism $y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$. By (S2) for $\mathcal{F}$ such a morphism will exist if we can choose the isomorphisms α and β and γ such that

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\alpha"] & x_0 \arrow[d, "\gamma"] & x_\epsilon \arrow[l, "e"'] \arrow[d, "\beta"] \\ x' \arrow[r] & x'_0 & x'_\epsilon \arrow[l, "e'"] \end{tikzcd}$$

is commutative (because then we can replace x by x' and x_ϵ by x'_ϵ in the previous displayed diagram). The left hand square commutes by our choice of γ. We can factor $e' \circ \beta$ as $\gamma' \circ e$ for some second map $\gamma' : x_0 \to x'_0$. Now the question is whether we can arrange it so that $\gamma = \gamma'$? This is clear if $\mathcal{F}(k)$ is a set, or a setoid. Moreover, if $\text{Aut}_{k[\epsilon]}(x_\epsilon) \to \text{Aut}_k(x_0)$ is surjective, then we can adjust the choice of β by precomposing with an automorphism of x_ϵ whose image is $\gamma^{-1} \circ \gamma'$ to make things work. □

06SQ Lemma 90.10.6. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 90.6.4.

(1) If $\mathcal{F}$ satisfies (S1), then so does $\mathcal{F}_{x_0}$.
(2) If $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$.

Proof. Any diagram as in Definition 90.10.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of condition (S1) or (S2) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

06IS Lemma 90.10.7. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Consider a diagram of $\mathcal{F}$

$$\begin{tikzcd} y \arrow[r] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k. \end{tikzcd}$$

in $\mathcal{C}_\Lambda$. Assume $\mathcal{F}$ satisfies (S2). Then there exists a morphism $s : x \to y$ with $a \circ s = \text{id}_x$ if and only if there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$.

Proof. The "only if" direction is clear. Conversely, assume there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$. Note that $p(s_\epsilon) : A \to k[\epsilon]$ is a ring map compatible with the map $A \to k$. Hence we obtain

$$\sigma = (\text{id}_A, p(s_\epsilon)) : A \to A \times_k k[\epsilon].$$

Choose a pushforward $x \to \sigma_* x$. By construction we can factor s_ϵ as $x \to \sigma_* x \to x_\epsilon$. Moreover, as σ is a section of $A \times_k k[\epsilon] \to A$, we get a morphism $\sigma_* x \to x$ such that $x \to \sigma_* x \to x$ is id_x. Because $e \circ s_\epsilon = d$ we find that the diagram

$$\begin{tikzcd} \sigma_* x \arrow[r] \arrow[d] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd}$$

is commutative. Hence by (S2) we obtain a morphism $\sigma_* x \to y$ such that $\sigma_* x \to y \to x$ is the given map $\sigma_* x \to x$. The solution to the problem is now to take $a : x \to y$ equal to the composition $x \to \sigma_* x \to y$. □

06IT Lemma 90.10.8. Consider a commutative diagram in a predeformation category $\mathcal{F}$

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d, "a_2"] \\ x_1 \arrow[r, "a_1"] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d, "f_2"] \\ A_1 \arrow[r, "f_1"] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ where $f_2 : A_2 \to A$ is a small extension. Assume there is a map $h : A_1 \to A_2$ such that $f_2 = f_1 \circ h$. Let $I = \mathrm{Ker}(f_2)$. Consider the ring map

$$g : A_1 \times_A A_2 \longrightarrow k[I] = k \oplus I, \quad (u, v) \longmapsto \overline{u} \oplus (v - h(u))$$

Choose a pushforward $y \to g_*y$. Assume $\mathcal{F}$ satisfies (S2). If there exists a morphism $x_1 \to g_*y$, then there exists a morphism $b : x_1 \to x_2$ such that $a_1 = a_2 \circ b$.

Proof. Note that $\mathrm{id}_{A_1} \times g : A_1 \times_A A_2 \to A_1 \times_k k[I]$ is an isomorphism and that $k[I] \cong k[\epsilon]$. Hence we have a diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & g_*y \arrow[d] \\ x_1 \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A_1 \arrow[r] & k. \end{tikzcd}$$

where x_0 is an object of $\mathcal{F}$ lying over k (every object of $\mathcal{F}$ has a unique morphism to x_0, see discussion following Definition 90.6.2). If we have a morphism $x_1 \to g_*y$ then Lemma 90.10.7 provides us with a section $s : x_1 \to y$ of the map $y \to x_1$. Composing this with the map $y \to x_2$ we obtain $b : x_1 \to x_2$ which has the property that $a_1 = a_2 \circ b$ because the diagram of the lemma commutes and because s is a section. □

90.11. Tangent spaces of functors

06I2 Let R be a ring. We write Mod_R for the category of R-modules and Mod_R^{fg} for the category of finitely generated R-modules.

06I3 Definition 90.11.1. Let $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$, resp. $L : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be a functor. We say that L is R-linear if for every pair of objects M, N of Mod_R^{fg}, resp. Mod_R the map

$$L : \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(L(M), L(N))$$

is a map of R-modules.

06I4 Remark 90.11.2. One can define the notion of an R-linearity for any functor between categories enriched over Mod_R. We made the definition specifically for functors $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$ and $L : \mathrm{Mod}_R \to \mathrm{Mod}_R$ because these are the cases that we have needed so far.

06I5 Remark 90.11.3. If $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$ is an R-linear functor, then L preserves finite products and sends the zero module to the zero module, see Homology, Lemma 12.3.7. On the other hand, if a functor $\mathrm{Mod}_R^{fg} \to \mathrm{Sets}$ preserves finite products and sends the zero module to a one element set, then it has a unique lift to a R-linear functor, see Lemma 90.11.4.

06I6 Lemma 90.11.4. Let $L : \mathrm{Mod}_R^{fg} \to \mathrm{Sets}$, resp. $L : \mathrm{Mod}_R \to \mathrm{Sets}$ be a functor. Suppose $L(0)$ is a one element set and L preserves finite products. Then there exists a unique R-linear functor $\widetilde{L} : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$, resp. $\widetilde{L} : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$, such that

$$\begin{tikzcd} & \mathrm{Mod}_R \arrow[dr, "\text{forget}"] & \\ \mathrm{Mod}_R^{fg} \arrow[ur, "\widetilde{L}"] \arrow[rr, "L"] & & \mathrm{Sets} \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} & \mathrm{Mod}_R \arrow[dr, "\text{forget}"] & \\ \mathrm{Mod}_R \arrow[ur, "\widetilde{L}"] \arrow[rr, "L"] & & \mathrm{Sets} \end{tikzcd}$$

commutes.

Proof. We only prove this in case $L : \text{Mod}_R^{fg} \to \text{Sets}$. Let M be a finitely generated R-module. We define $\widetilde{L}(M)$ to be the set $L(M)$ with the following R-module structure.

Multiplication: If $r \in R$, multiplication by r on $L(M)$ is defined to be the map $L(M) \to L(M)$ induced by the multiplication map $r\cdot : M \to M$.

Addition: The sum map $M \times M \to M : (m_1, m_2) \mapsto m_1 + m_2$ induces a map $L(M \times M) \to L(M)$. By assumption $L(M \times M)$ is canonically isomorphic to $L(M) \times L(M)$. Addition on $L(M)$ is defined by the map $L(M) \times L(M) \cong L(M \times M) \to L(M)$.

Zero: There is a unique map $0 \to M$. The zero element of $L(M)$ is the image of $L(0) \to L(M)$.

We omit the verification that this defines an R-module $\widetilde{L}(M)$, the unique such that is R-linearly functorial in M. $\square$

06I7 Lemma 90.11.5. Let $L_1, L_2 : \text{Mod}_R^{fg} \to \text{Sets}$ be functors that take 0 to a one element set and preserve finite products. Let $t : L_1 \to L_2$ be a morphism of functors. Then t induces a morphism $\widetilde{t} : \widetilde{L}_1 \to \widetilde{L}_2$ between the functors guaranteed by Lemma 90.11.4, which is given simply by $\widetilde{t}_M = t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ for each $M \in \text{Ob}(\text{Mod}_R^{fg})$. In other words, $t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ is a map of R-modules.

Proof. Omitted. $\square$

In the case $R = K$ is a field, a K-linear functor $L : \text{Mod}_K^{fg} \to \text{Mod}_K$ is determined by its value $L(K)$.

06I8 Lemma 90.11.6. Let K be a field. Let $L : \text{Mod}_K^{fg} \to \text{Mod}_K$ be a K-linear functor. Then L is isomorphic to the functor $L(K) \otimes_K - : \text{Mod}_K^{fg} \to \text{Mod}_K$.

Proof. For $V \in \text{Ob}(\text{Mod}_K^{fg})$, the isomorphism $L(K) \otimes_K V \to L(V)$ is given on pure tensors by $x \otimes v \mapsto L(f_v)(x)$, where $f_v : K \to V$ is the K-linear map sending $1 \mapsto v$. When $V = K$, this is the isomorphism $L(K) \otimes_K K \to L(K)$ given by multiplication by K. For general V, it is an isomorphism by the case $V = K$ and the fact that L commutes with finite products (Remark 90.11.3). $\square$

For a ring R and an R-module M, let $R[M]$ be the R-algebra whose underlying R-module is $R \oplus M$ and whose multiplication is given by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. When $M = R$ this is the ring of dual numbers over R, which we denote by $R[\epsilon]$.

Now let S be a ring and assume R is an S-algebra. Then the assignment $M \mapsto R[M]$ determines a functor $\text{Mod}_R \to S\text{-Alg}/R$, where $S\text{-Alg}/R$ denotes the category of S-algebras over R. Note that $S\text{-Alg}/R$ admits finite products: if $A_1 \to R$ and $A_2 \to R$ are two objects, then $A_1 \times_R A_2$ is a product.

06I9 Lemma 90.11.7. Let R be an S-algebra. Then the functor $\text{Mod}_R \to S\text{-Alg}/R$ described above preserves finite products.

Proof. This is merely the statement that if M and N are R-modules, then the map $R[M \times N] \to R[M] \times_R R[N]$ is an isomorphism in $S\text{-Alg}/R$. $\square$

06IA Lemma 90.11.8. Let R be an S-algebra, and let $\mathcal{C}$ be a strictly full subcategory of $S\text{-Alg}/R$ containing $R[M]$ for all $M \in \text{Ob}(\text{Mod}_R^{fg})$. Let $F : \mathcal{C} \to \text{Sets}$ be a functor. Suppose that $F(R)$ is a one element set and that for any $M, N \in \text{Ob}(\text{Mod}_R^{fg})$, the induced map

$$F(R[M] \times_R R[N]) \to F(R[M]) \times F(R[N])$$

is a bijection. Then $F(R[M])$ has a natural R-module structure for any $M \in \text{Ob}(\text{Mod}_R^{fg})$.

Proof. Note that $R \cong R[0]$ and $R[M] \times_R R[N] \cong R[M \times N]$ hence R and $R[M] \times_R R[N]$ are objects of $\mathcal{C}$ by our assumptions on $\mathcal{C}$. Thus the conditions on F make sense. The functor $\text{Mod}_R \to S\text{-Alg}/R$ of Lemma 90.11.7 restricts to a functor $\text{Mod}_R^{fg} \to \mathcal{C}$ by the assumption on $\mathcal{C}$. Let L be the composition $\text{Mod}_R^{fg} \to \mathcal{C} \to \text{Sets}$, i.e., $L(M) = F(R[M])$. Then L preserves finite products by Lemma 90.11.7 and the assumption on F. Hence Lemma 90.11.4 shows that $L(M) = F(R[M])$ has a natural R-module structure for any $M \in \text{Ob}(\text{Mod}_R^{fg})$. $\square$

06IB Definition 90.11.9. Let $\mathcal{C}$ be a category as in Lemma 90.11.8. Let $F : \mathcal{C} \to \text{Sets}$ be a functor such that $F(R)$ is a one element set. The tangent space TF of F is $F(R[\epsilon])$.

When $F : \mathcal{C} \to \text{Sets}$ satisfies the hypotheses of Lemma 90.11.8, the tangent space TF has a natural R-module structure.

06SR Example 90.11.10. Since $\mathcal{C}_\Lambda$ contains all $k[V]$ for finite dimensional vector spaces V we see that Definition 90.11.9 applies with $S = \Lambda$, $R = k$, $\mathcal{C} = \mathcal{C}_\Lambda$, and $F : \mathcal{C}_\Lambda \to \text{Sets}$ a predeformation functor. The tangent space is $TF = F(k[\epsilon])$.

06IC Example 90.11.11. Let us work out the tangent space of Example 90.11.10 when $F : \mathcal{C}_\Lambda \to \text{Sets}$ is a prorepresentable functor, say $F = \underline{S}|_{\mathcal{C}_\Lambda}$ for $S \in \text{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Then F commutes with arbitrary limits and thus satisfies the hypotheses of Lemma 90.11.8. We compute

$$TF = F(k[\epsilon]) = \text{Mor}_{\mathcal{C}_\Lambda}(S, k[\epsilon]) = \text{Der}_\Lambda(S, k)$$

and more generally for a finite dimensional k-vector space V we have

$$F(k[V]) = \text{Mor}_{\mathcal{C}_\Lambda}(S, k[V]) = \text{Der}_\Lambda(S, V).$$

Explicitly, a Λ-algebra map $f : S \to k[V]$ compatible with the augmentations $q : S \to k$ and $k[V] \to k$ corresponds to the derivation D defined by $s \mapsto f(s) - q(s)$. Conversely, a Λ-derivation $D : S \to V$ corresponds to $f : S \to k[V]$ in $\mathcal{C}_\Lambda$ defined by the rule $f(s) = q(s) + D(s)$. Since these identifications are functorial we see that the k-vector spaces structures on TF and $\text{Der}_\Lambda(S, k)$ correspond (see Lemma 90.11.5). It follows that $\dim_k TF$ is finite by Lemma 90.4.5.

06SS Example 90.11.12. The computation of Example 90.11.11 simplifies in the classical case. Namely, in this case the tangent space of the functor $F = \underline{S}|_{\mathcal{C}_\Lambda}$ is simply the relative cotangent space of S over Λ, in a formula $TF = T_{S/\Lambda}$. In fact, this works more generally when the field extension k/k' is separable. See Exercises, Exercise 111.35.2.

06ID Lemma 90.11.13. Let $F, G : \mathcal{C} \to \text{Sets}$ be functors satisfying the hypotheses of Lemma 90.11.8. Let $t : F \to G$ be a morphism of functors. For any $M \in \text{Ob}(\text{Mod}_R^{fg})$, the map $t_{R[M]} : F(R[M]) \to G(R[M])$ is a map of R-modules, where

$F(R[M])$ and $G(R[M])$ are given the R-module structure from Lemma 90.11.8. In particular, $t_{R[\epsilon]} : TF \to TG$ is a map of R-modules.

Proof. Follows from Lemma 90.11.5. $\square$

06ST Example 90.11.14. Suppose that $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$. Set $F = \underline{R}|_{\mathcal{C}_\Lambda}$ and $G = \underline{S}|_{\mathcal{C}_\Lambda}$. The ring map f induces a transformation of functors $G \to F$. By Lemma 90.11.13 we get a k-linear map $TG \to TF$. This is the map

$$TG = \text{Der}_\Lambda(S, k) \longrightarrow \text{Der}_\Lambda(R, k) = TF$$

as follows from the canonical identifications $F(k[V]) = \text{Der}_\Lambda(R, V)$ and $G(k[V]) = \text{Der}_\Lambda(S, V)$ of Example 90.11.11 and the rule for computing the map on tangent spaces.

06IE Lemma 90.11.15. Let $F : \mathcal{C} \to \text{Sets}$ be a functor satisfying the hypotheses of Lemma 90.11.8. Assume $R = K$ is a field. Then $F(K[V]) \cong TF \otimes_K V$ for any finite dimensional K-vector space V.

Proof. Follows from Lemma 90.11.6. $\square$

90.12. Tangent spaces of predeformation categories

06I1 We will define tangent spaces of predeformation functors using the general Definition 90.11.9. We have spelled this out in Example 90.11.10. It applies to predeformation categories by looking at the associated functor of isomorphism classes.

06IG Definition 90.12.1. Let $\mathcal{F}$ be a predeformation category. The tangent space $T\mathcal{F}$ of $\mathcal{F}$ is the set $\overline{\mathcal{F}}(k[\epsilon])$ of isomorphism classes of objects in the fiber category $\mathcal{F}(k[\epsilon])$.

Thus $T\mathcal{F}$ is nothing but the tangent space of the associated functor $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \text{Sets}$. It has a natural vector space structure when $\mathcal{F}$ satisfies (S2), or, in fact, as long as $\overline{\mathcal{F}}$ does.

06IH Lemma 90.12.2. Let $\mathcal{F}$ be a predeformation category such that $\overline{\mathcal{F}}$ satisfies (S2)[2]. Then $T\mathcal{F}$ has a natural k-vector space structure. For any finite dimensional vector space V we have $\overline{\mathcal{F}}(k[V]) = T\mathcal{F} \otimes_k V$ functorially in V.

Proof. Let us write $F = \overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \text{Sets}$. This is a predeformation functor and F satisfies (S2). By Lemma 90.10.4 (and the translation of Remark 90.10.3) we see that

$$F(A \times_k k[V]) \longrightarrow F(A) \times F(k[V])$$

is a bijection for every finite dimensional vector space V and every $A \in \text{Ob}(\mathcal{C}_\Lambda)$. In particular, if $A = k[W]$ then we see that $F(k[W] \times_k k[V]) = F(k[W]) \times F(k[V])$. In other words, the hypotheses of Lemma 90.11.8 hold and we see that $TF = T\mathcal{F}$ has a natural k-vector space structure. The final assertion follows from Lemma 90.11.15. $\square$

A morphism of predeformation categories induces a map on tangent spaces.

06II Definition 90.12.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. The differential $d\varphi : T\mathcal{F} \to T\mathcal{G}$ of φ is the map obtained by evaluating the morphism of functors $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ at $A = k[\epsilon]$.

[2]For example if $\mathcal{F}$ satisfies (S2), see Lemma 90.10.5.

06IJ Lemma 90.12.4. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of predeformation categories. Assume $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ both satisfy (S2). Then $d\varphi : T\mathcal{F} \to T\mathcal{G}$ is k-linear.

Proof. In the proof of Lemma 90.12.2 we have seen that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ satisfy the hypotheses of Lemma 90.11.8. Hence the lemma follows from Lemma 90.11.13. □

06IK Remark 90.12.5. We can globalize the notions of tangent space and differential to arbitrary categories cofibered in groupoids as follows. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. As in Remark 90.6.4, we get a predeformation category $\mathcal{F}_x$. We define

$$T_x\mathcal{F} = T\mathcal{F}_x$$

to be the tangent space of $\mathcal{F}$ at x. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(k))$, then there is an induced morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_{\varphi(x)}$. We define the differential $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ of φ at x to be the map $d\varphi_x : T\mathcal{F}_x \to T\mathcal{G}_{\varphi(x)}$. If both $\mathcal{F}$ and $\mathcal{G}$ satisfy (S2) then all of these tangent spaces have a natural k-vector space structure and all the differentials $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ are k-linear (use Lemmas 90.10.6 and 90.12.4).

The following observations are uninteresting in the classical case or when k/k' is a separable field extension, because then $\mathrm{Der}_\Lambda(k, k)$ and $\mathrm{Der}_\Lambda(k, V)$ are zero. There is a canonical identification

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[\epsilon]) = \mathrm{Der}_\Lambda(k, k).$$

Namely, for $D \in \mathrm{Der}_\Lambda(k, k)$ let $f_D : k \to k[\epsilon]$ be the map $a \mapsto a + D(a)\epsilon$. More generally, given a finite dimensional vector space V over k we have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[V]) = \mathrm{Der}_\Lambda(k, V)$$

and we will use the same notation f_D for the map associated to the derivation D. We also have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k[W], k[V]) = \mathrm{Hom}_k(V, W) \oplus \mathrm{Der}_\Lambda(k, V)$$

where (φ, D) corresponds to the map $f_{\varphi,D} : a + w \mapsto a + \varphi(w) + D(a)$. We will sometimes write $f_{1,D} : a+v \to a+v+D(a)$ for the automorphism of $k[V]$ determined by the derivation $D : k \to V$. Note that $f_{1,D} \circ f_{1,D'} = f_{1,D+D'}$.

Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. By the above there is a canonical map

$$\gamma_V : \mathrm{Der}_\Lambda(k, V) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $D \mapsto f_{D,*}(x_0)$. Moreover, there is an action

$$a_V : \mathrm{Der}_\Lambda(k, V) \times \overline{\mathcal{F}}(k[V]) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $(D, x) \mapsto f_{1,D,*}(x)$. These two maps are compatible, i.e., $f_{1,D,*}f_{D',*}x_0 = f_{D+D',*}x_0$ as follows from a computation of the compositions of these maps. Note that the maps γ_V and a_V are independent of the choice of x_0 as there is a unique x_0 up to isomorphism.

06SU Lemma 90.12.6. Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. If $\overline{\mathcal{F}}$ has (S2) then the maps γ_V are k-linear and we have $a_V(D, x) = x + \gamma_V(D)$.

Proof. In the proof of Lemma 90.12.2 we have seen that the functor $V \mapsto \overline{\mathcal{F}}(k[V])$ transforms 0 to a singleton and products to products. The same is true of the functor $V \mapsto \text{Der}_\Lambda(k, V)$. Hence γ_V is linear by Lemma 90.11.5. Let $D : k \to V$ be a Λ-derivation. Set $D_1 : k \to V^{\oplus 2}$ equal to $a \mapsto (D(a), 0)$. Then

$$\begin{tikzcd} k[V \times V] \arrow[r, "+"'] \arrow[d, "f_{1,D_1}"] & k[V] \arrow[d, "f_{1,D}"] \\ k[V \times V] \arrow[r, "+"] & k[V] \end{tikzcd}$$

commutes. Unwinding the definitions and using that $\overline{F}(V \times V) = \overline{F}(V) \times \overline{F}(V)$ this means that $a_D(x_1) + x_2 = a_D(x_1 + x_2)$ for all $x_1, x_2 \in \overline{F}(V)$. Thus it suffices to show that $a_V(D, 0) = 0 + \gamma_V(D)$ where $0 \in \overline{F}(V)$ is the zero vector. By definition this is the element $f_{0,*}(x_0)$. Since $f_D = f_{1,D} \circ f_0$ the desired result follows. $\square$

A special case of the constructions above are the map

06SV (90.12.6.1) $$\gamma : \text{Der}_\Lambda(k, k) \longrightarrow T\mathcal{F}$$

and the action

06SW (90.12.6.2) $$a : \text{Der}_\Lambda(k, k) \times T\mathcal{F} \longrightarrow T\mathcal{F}$$

defined for any predeformation category $\mathcal{F}$. Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of predeformation categories, then we get commutative diagrams

$$\begin{tikzcd} \text{Der}_\Lambda(k, k) \arrow[r, "\gamma"'] \arrow[rd, "\gamma"'] & T\mathcal{F} \arrow[d, "d\varphi"'] \\ & T\mathcal{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \text{Der}_\Lambda(k, k) \times T\mathcal{F} \arrow[r, "a"] \arrow[d, "1 \times d\varphi"'] & T\mathcal{F} \arrow[d, "d\varphi"] \\ \text{Der}_\Lambda(k, k) \times T\mathcal{G} \arrow[r, "a"] & T\mathcal{G} \end{tikzcd}$$

90.13. Versal formal objects

06SX The existence of a versal formal object forces $\mathcal{F}$ to have property (S1).

06SY **Lemma 90.13.1.** *Let $\mathcal{F}$ be a predeformation category. Assume $\mathcal{F}$ has a versal formal object. Then $\mathcal{F}$ satisfies (S1).*

Proof. Let ξ be a versal formal object of $\mathcal{F}$. Let

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd}$$

be a diagram in $\mathcal{F}$ such that $x_2 \to x$ lies over a surjective ring map. Since the natural morphism $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \xrightarrow{\sim} \mathcal{F}$ is an equivalence (see Remark 90.7.7), we can consider this diagram also as a diagram in $\widehat{\mathcal{F}}$. By Lemma 90.8.11 there exists a morphism $\xi \to x_1$, so by Remark 90.8.10 we also get a morphism $\xi \to x_2$ making the diagram

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd}$$

commute. If $x_1 \to x$ and $x_2 \to x$ lie above ring maps $A_1 \to A$ and $A_2 \to A$ then taking the pushforward of ξ to $A_1 \times_A A_2$ gives an object y as required by (S1). $\square$

In the case that our cofibred category satisfies (S1) and (S2) we can characterize the versal formal objects as follows.

06IU Lemma 90.13.2. Let $\mathcal{F}$ be a predeformation category satisfying (S1) and (S2). Let ξ be a formal object of $\mathcal{F}$ corresponding to $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$, see Remark 90.7.12. Then ξ is versal if and only if the following two conditions hold:

1. the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to T\mathcal{F}$ on tangent spaces is surjective, and
2. given a diagram in $\widehat{\mathcal{F}}$

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ R \arrow[r] & A \end{tikzcd}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $B \to A$ a small extension of Artinian rings, then there exists a ring map $R \to B$ such that

$$\begin{tikzcd} & B \arrow[d, "f"] \\ R \arrow[r] \arrow[ur] & A \end{tikzcd}$$

commutes.

Proof. If ξ is versal then (1) holds by Lemma 90.8.8 and (2) holds by Remark 90.8.10. Assume (1) and (2) hold. By Remark 90.8.10 we must show that given a diagram in $\widehat{\mathcal{F}}$ as in (2), there exists $\xi \to y$ such that

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur] & x \end{tikzcd}$$

commutes. Let $b : R \to B$ be the map guaranteed by (2). Denote $y' = b_*\xi$ and choose a factorization $\xi \to y' \to x$ lying over $R \to B \to A$ of the given morphism $\xi \to x$. By (S1) we obtain a commutative diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & y \arrow[d] \\ y' \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B \times_A B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B \arrow[r, "f"] & A. \end{tikzcd}$$

Set $I = \text{Ker}(f)$. Let $\overline{g} : B \times_A B \to k[I]$ be the ring map $(u, v) \mapsto \overline{u} \oplus (v - u)$, cf. Lemma 90.10.8. By (1) there exists a morphism $\xi \to \overline{g}_*z$ which lies over a ring map $i : R \to k[\epsilon]$. Choose an Artinian quotient $b_1 : R \to B_1$ such that both $b : R \to B$ and $i : R \to k[\epsilon]$ factor through $R \to B_1$, i.e., giving $h : B_1 \to B$ and $i' : B_1 \to k[\epsilon]$. Choose a pushforward $y_1 = b_{1,*}\xi$, a factorization $\xi \to y_1 \to y'$ lying over $R \to B_1 \to B$ of $\xi \to y'$, and a factorization $\xi \to y_1 \to \overline{g}_*z$ lying over

$R \to B_1 \to k[\epsilon]$ of $\xi \to \overline{g}_* z$. Applying (S1) once more we obtain

$$\begin{tikzcd} z_1 \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & y \arrow[d] \\ y_1 \arrow[r] & y' \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B_1 \times_A B \arrow[r] \arrow[d] & B \times_A B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B_1 \arrow[r] & B \arrow[r] & A. \end{tikzcd}$$

Note that the map $g : B_1 \times_A B \to k[I]$ of Lemma 90.10.8 (defined using h) is the composition of $B_1 \times_A B \to B \times_A B$ and the map $\overline{g}$ above. By construction there exists a morphism $y_1 \to g_* z_1 \cong \overline{g}_* z$! Hence Lemma 90.10.8 applies (to the outer rectangles in the diagrams above) to give a morphism $y_1 \to y$ and precomposing with $\xi \to y_1$ gives the desired morphism $\xi \to y$. $\square$

If $\mathcal{F}$ has property (S1) then the "largest quotient where a lift exists" exists. Here is a precise statement.

06SZ **Lemma 90.13.3.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let $B \to A$ be a surjection in $\mathcal{C}_\Lambda$ with kernel I annihilated by $\mathfrak{m}_B$. Let $x \in \mathcal{F}(A)$. The set of ideals

$$\mathcal{J} = \{J \subset I \mid \text{there exists an } y \to x \text{ lying over } B/J \to A\}$$

has a smallest element.

Proof. Note that $\mathcal{J}$ is nonempty as $I \in \mathcal{J}$. Also, if $J \in \mathcal{J}$ and $J \subset J' \subset I$ then $J' \in \mathcal{J}$ because we can pushforward the object y to an object y' over B/J'. Let J and K be elements of the displayed set. We claim that $J \cap K \in \mathcal{J}$ which will prove the lemma. Since I is a k-vector space we can find an ideal $J \subset J' \subset I$ such that $J \cap K = J' \cap K$ and such that $J' + K = I$. By the above we may replace J by J' and assume that $J + K = I$. In this case

$$A/(J \cap K) = A/J \times_{A/I} A/K.$$

Hence the existence of an element $z \in \mathcal{F}(A/(J \cap K))$ mapping to x follows, via (S1), from the existence of the elements we have assumed exist over A/J and A/K. $\square$

We will improve on the following result later.

06IW **Lemma 90.13.4.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume the following conditions hold:

(1) $\mathcal{F}$ is a predeformation category.
(2) $\mathcal{F}$ satisfies (S1).
(3) $\mathcal{F}$ satisfies (S2).
(4) $\dim_k T\mathcal{F}$ is finite.

Then $\mathcal{F}$ has a versal formal object.

Proof. Assume (1), (2), (3), and (4) hold. Choose an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. See Lemma 90.9.5. Let $r = \dim_k T\mathcal{F}$ and put $S = R[[X_1, \ldots, X_r]]$.

We are going to inductively construct for $n \geq 2$ pairs $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ where $J_n \subset S$ is an decreasing sequence of ideals and $f_{n-1} : \xi_n \to \xi_{n-1}$ is a morphism of $\mathcal{F}$ lying over the projection $S/J_n \to S/J_{n-1}$.

Step 1. Let $J_1 = \mathfrak{m}_S$. Let ξ_1 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over $k = S/J_1 = S/\mathfrak{m}_S$

Step 2. Let $J_2 = \mathfrak{m}_S^2 + \mathfrak{m}_R S$. Then $S/J_2 = k[V]$ with $V = kX_1 \oplus \ldots \oplus kX_r$ By (S2) for $\overline{\mathcal{F}}$ we get a bijection

$$\overline{\mathcal{F}}(S/J_2) \longrightarrow T\mathcal{F} \otimes_k V,$$

see Lemmas 90.10.5 and 90.12.2. Choose a basis $\theta_1, \ldots, \theta_r$ for $T\mathcal{F}$ and set $\xi_2 = \sum \theta_i \otimes X_i \in \mathrm{Ob}(\mathcal{F}(S/J_2))$. The point of this choice is that

$$d\xi_2 : \mathrm{Mor}_{\mathcal{C}_\Lambda}(S/J_2, k[\epsilon]) \longrightarrow T\mathcal{F}$$

is surjective. Let $f_1 : \xi_2 \to \xi_1$ be the unique morphism.

Induction step. Assume $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ has been constructed for some $n \geq 2$. There is a minimal element J_{n+1} of the set of ideals $J \subset S$ satisfying: (a) $\mathfrak{m}_S J_n \subset J \subset J_n$ and (b) there exists a morphism $\xi_{n+1} \to \xi_n$ lying over $S/J \to S/J_n$, see Lemma 90.13.3. Let $f_n : \xi_{n+1} \to \xi_n$ be any morphism of $\mathcal{F}$ lying over $S/J_{n+1} \to S/J_n$.

Set $J = \bigcap J_n$. Set $\overline{S} = S/J$. Set $\overline{J}_n = J_n/J$. By Lemma 90.4.7 the sequence of ideals $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology on $\overline{S}$. Since (ξ_n, f_n) is an object of $\widehat{\mathcal{F}}_{\mathcal{I}}(\overline{S})$, where $\mathcal{I}$ is the filtration $(\overline{J}_n)$ of $\overline{S}$, we see that (ξ_n, f_n) induces an object ξ of $\widehat{\mathcal{F}}(\overline{S})$. see Lemma 90.7.4.

We prove ξ is versal. For versality it suffices to check conditions (1) and (2) of Lemma 90.13.2. Condition (1) follows from our choice of ξ_2 in Step 2 above. Suppose given a diagram in $\widehat{\mathcal{F}}$

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ \overline{S} \arrow[r] & A \end{tikzcd}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $f : B \to A$ a small extension of Artinian rings. We have to show there is a map $\overline{S} \to B$ fitting into the diagram on the right. Choose n such that $\overline{S} \to A$ factors through $\overline{S} \to S/J_n$. This is possible as the sequence $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology as we saw above. The pushforward of ξ along $\overline{S} \to S/J_n$ is ξ_n. We may factor $\xi \to x$ as $\xi \to \xi_n \to x$ hence we get a diagram in $\mathcal{F}$

$$\begin{tikzcd} & y \arrow[d] \\ \xi_n \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ S/J_n \arrow[r] & A. \end{tikzcd}$$

To check condition (2) of Lemma 90.13.2 it suffices to complete the diagram

$$\begin{tikzcd} S/J_{n+1} \arrow[r, dashed] \arrow[d] & B \arrow[d, "f"] \\ S/J_n \arrow[r] & A \end{tikzcd}$$

or equivalently, to complete the diagram

$$\begin{tikzcd} & S/J_n \times_A B \arrow[d, "p_1"] \\ S/J_{n+1} \arrow[r] \arrow[ru, dashed] & S/J_n. \end{tikzcd}$$

If p_1 has a section we are done. If not, by Lemma 90.3.8 (2) p_1 is a small extension, so by Lemma 90.3.12 (4) p_1 is an essential surjection. Recall that $S = R[[X_1, \ldots, X_r]]$ and that we chose R such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Hence there exists a map $h : R \to B$ lifting the map $R \to S \to S/J_n \to A$. By the universal property of a power series ring there is an R-algebra map $h : S = R[[X_1, \ldots, X_r]] \to B$ lifting the given map $S \to S/J_n \to A$. This induces a map $g : S \to S/J_n \times_A B$ making the solid square in the diagram

$$\begin{tikzcd} S \arrow[r, "g"'] \arrow[d] & S/J_n \times_A B \arrow[d, "p_1"] \\ S/J_{n+1} \arrow[r] \arrow[ru, dashed] & S/J_n \end{tikzcd}$$

commute. Then g is a surjection since p_1 is an essential surjection. We claim the ideal $K = \text{Ker}(g)$ of S satisfies conditions (a) and (b) of the construction of J_{n+1} in the induction step above. Namely, $K \subset J_n$ is clear and $\mathfrak{m}_S J_n \subset K$ as p_1 is a small extension; this proves (a). By (S1) applied to

$$\begin{tikzcd} & y \arrow[d] \\ \xi_n \arrow[r] & x, \end{tikzcd}$$

there exists a lifting of ξ_n to $S/K \cong S/J_n \times_A B$, so (b) holds. Since J_{n+1} was the minimal ideal with properties (a) and (b) this implies $J_{n+1} \subset K$. Thus the desired map $S/J_{n+1} \to S/K \cong S/J_n \times_A B$ exists. $\square$

0D3G Remark 90.13.5. Let $F : \mathcal{C}_\Lambda \to \textit{Sets}$ be a predeformation functor satisfying (S1) and (S2). The condition $\dim_k TF < \infty$ is precisely condition (H3) from Schlessinger's paper. Recall that (S1) and (S2) correspond to conditions (H1) and (H2), see Remark 90.10.3. Thus Lemma 90.13.4 tells us

$$(H1) + (H2) + (H3) \Rightarrow \text{ there exists a versal formal object}$$

for predeformation functors. We will make the link with hulls in Remark 90.15.6.

90.14. Minimal versal formal objects

06T0 We do a little bit of work to try and understand (non)uniqueness of versal formal objects. It turns out that if a predeformation category has a versal formal object, then it has a minimal versal formal object and any two such are isomorphic. Moreover, all versal formal objects are "more or less" the same up to replacing the base ring by a power series extension.

Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. For every object x of $\mathcal{F}$ lying over $A \in \text{Ob}(\mathcal{C}_\Lambda)$ consider the category $\mathcal{S}_x$ with objects

$$\text{Ob}(\mathcal{S}_x) = \{x' \to x \mid x' \to x \text{ lies over } A' \subset A\}$$

and morphisms are morphisms over x. For every $y \to x$ in $\mathcal{F}$ lying over $f : B \to A$ in $\mathcal{C}_\Lambda$ there is a functor $f_* : \mathcal{S}_y \to \mathcal{S}_x$ defined as follows: Given $y' \to y$ lying over $B' \subset B$ set $A' = f(B')$ and let $y' \to x'$ be over $B' \to f(B')$ be the pushforward of y'. By the axioms of a category cofibred in groupoids we obtain a unique morphism $x' \to x$ lying over $f(B') \to A$ such that

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

commutes. Then $x' \to x$ is an object of $\mathcal{S}_x$. We say an object $x' \to x$ of $\mathcal{S}_x$ is minimal if any morphism $(x'_1 \to x) \to (x' \to x)$ in $\mathcal{S}_x$ is an isomorphism, i.e., x' and x'_1 are defined over the same subring of A. Since A has finite length as a Λ-module we see that minimal objects always exist.

06T1 Lemma 90.14.1. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1).

(1) For $y \to x$ in $\mathcal{F}$ a minimal object in $\mathcal{S}_y$ maps to a minimal object of $\mathcal{S}_x$.
(2) For $y \to x$ in $\mathcal{F}$ lying over a surjection $f : B \to A$ in $\mathcal{C}_\Lambda$ every minimal object of $\mathcal{S}_x$ is the image of a minimal object of $\mathcal{S}_y$.

Proof. Proof of (1). Say $y \to x$ lies over $f : B \to A$. Let $y' \to y$ lying over $B' \subset B$ be a minimal object of $\mathcal{S}_y$. Let

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B' \arrow[r] \arrow[d] & f(B') \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

be as in the construction of f_* above. Suppose that $(x'' \to x) \to (x' \to x)$ is a morphism of $\mathcal{S}_x$ with $x'' \to x'$ lying over $A'' \subset f(B')$. By (S1) there exists $y'' \to y'$ lying over $B' \times_{f(B')} A'' \to B'$. Since $y' \to y$ is minimal we conclude that $B' \times_{f(B')} A'' \to B'$ is an isomorphism, which implies that $A'' = f(B')$, i.e., $x' \to x$ is minimal.

Proof of (2). Suppose $f : B \to A$ is surjective and $y \to x$ lies over f. Let $x' \to x$ be a minimal object of $\mathcal{S}_x$ lying over $A' \subset A$. By (S1) there exists $y' \to y$ lying over $B' = f^{-1}(A') = B \times_A A' \to B$ whose image in $\mathcal{S}_x$ is $x' \to x$. So $f_*(y' \to y) = x' \to x$. Choose a morphism $(y'' \to y) \to (y' \to y)$ in $\mathcal{S}_y$ with $y'' \to y$ a minimal object (this is possible by the remark on lengths above the lemma). Then $f_*(y'' \to y)$ is an object of $\mathcal{S}_x$ which maps to $x' \to x$ (by functoriality of f_*) hence is isomorphic to $x' \to x$ by minimality of $x' \to x$. $\square$

06T2 Lemma 90.14.2. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. There exists a morphism $\xi' \to \xi$ lying over $R' \subset R$ with the following minimality properties

(1) for every $f : R \to A$ with $A \in \text{Ob}(\mathcal{C}_\Lambda)$ the pushforwards

$$\begin{tikzcd} \xi' \arrow[r] \arrow[d] & x' \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} R' \arrow[r] \arrow[d] & f(R') \arrow[d] \\ R \arrow[r] & A \end{tikzcd}$$

produce a minimal object $x' \to x$ of $\mathcal{S}_x$, and
(2) for any morphism of formal objects $\xi'' \to \xi'$ the corresponding morphism $R'' \to R'$ is surjective.

Proof. Write $\xi = (R, \xi_n, f_n)$. Set $R'_1 = k$ and $\xi'_1 = \xi_1$. Suppose that we have constructed minimal objects $\xi'_m \to \xi_m$ of $\mathcal{S}_{\xi_m}$ lying over $R'_m \subset R/\mathfrak{m}_R^m$ for $m \leq n$ and morphisms $f'_m : \xi'_{m+1} \to \xi'_m$ compatible with f_m for $m \leq n-1$. By Lemma 90.14.1 (2) there exists a minimal object $\xi'_{n+1} \to \xi_{n+1}$ lying over $R'_{n+1} \subset R/\mathfrak{m}_R^{n+1}$ whose image is $\xi'_n \to \xi_n$ over $R'_n \subset R/\mathfrak{m}_R^n$. This produces the commutative diagram

$$\begin{tikzcd} \xi'_{n+1} \arrow[r, "f'_n"'] \arrow[d] & \xi'_n \arrow[d] \\ \xi_{n+1} \arrow[r, "f_n"] & \xi_n \end{tikzcd}$$

by construction. Moreover the ring map $R'_{n+1} \to R'_n$ is surjective. Set $R' = \lim_n R'_n$. Then $R' \to R$ is injective.

However, it isn't a priori clear that R' is Noetherian. To prove this we use that ξ is versal. Namely, versality implies that there exists a morphism $\xi \to \xi'_n$ in $\widehat{\mathcal{F}}$, see Lemma 90.8.11. The corresponding map $R \to R'_n$ has to be surjective (as $\xi'_n \to \xi_n$ is minimal in $\mathcal{S}_{\xi_n}$). Thus the dimensions of the cotangent spaces are bounded and Lemma 90.4.8 implies R' is Noetherian, i.e., an object of $\widehat{\mathcal{C}}_\Lambda$. By Lemma 90.7.4 (plus the result on filtrations of Lemma 90.4.8) the sequence of elements ξ'_n defines a formal object ξ' over R' and we have a map $\xi' \to \xi$.

By construction (1) holds for $R \to R/\mathfrak{m}_R^n$ for each n. Since each $R \to A$ as in (1) factors through $R \to R/\mathfrak{m}_R^n \to A$ we see that (1) for $x' \to x$ over $f(R) \subset A$ follows from the minimality of $\xi'_n \to \xi_n$ over $R'_n \to R/\mathfrak{m}_R^n$ by Lemma 90.14.1 (1).

If $R'' \to R'$ as in (2) is not surjective, then $R'' \to R' \to R'_n$ would not be surjective for some n and $\xi'_n \to \xi_n$ wouldn't be minimal, a contradiction. This contradiction proves (2). ☐

06T3 Lemma 90.14.3. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. Let $\xi' \to \xi$ be a morphism of formal objects lying over $R' \subset R$ as constructed in Lemma 90.14.2. Then

$$R \cong R'[[x_1, \ldots, x_r]]$$

is a power series ring over R'. Moreover, ξ' is a versal formal object too.

Proof. By Lemma 90.8.11 there exists a morphism $\xi \to \xi'$. By Lemma 90.14.2 the corresponding map $f : R \to R'$ induces a surjection $f|_{R'} : R' \to R'$. This is an isomorphism by Algebra, Lemma 10.31.10. Hence $I = \text{Ker}(f)$ is an ideal of R such that $R = R' \oplus I$. Let $x_1, \ldots, x_n \in I$ be elements which form a basis for $I/\mathfrak{m}_R I$. Consider the map $S = R'[[X_1, \ldots, X_r]] \to R$ mapping X_i to x_i. For every $n \geq 1$ we get a surjection of Artinian R'-algebras $B = S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n = A$. Denote $y \in \text{Ob}(\mathcal{F}(B)$, resp. $x \in \text{Ob}(\mathcal{F}(A))$ the pushforward of ξ' along $R' \to S \to B$, resp. $R' \to S \to A$. Note that x is also the pushforward of ξ along $R \to A$ as ξ is the pushforward of ξ' along $R' \to R$. Thus we have a solid diagram

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur, dotted] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & S/\mathfrak{m}_S^n \arrow[d] \\ R \arrow[r] \arrow[ur, dotted] & R/\mathfrak{m}_R^n \end{tikzcd}$$

Because ξ is versal, using Remark 90.8.10 we obtain the dotted arrows fitting into these diagrams. In particular, the maps $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ have sections $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. It follows from Lemma 90.4.9 that $S \to R$ is an isomorphism.

As ξ is a pushforward of ξ' along $R' \to R$ we obtain from Remark 90.7.13 a commutative diagram

$$\begin{tikzcd} \underline{R}|_{\mathcal{C}_\Lambda} \arrow[rr] \arrow[dr, "\underline{\xi}"'] & & \underline{R'}|_{\mathcal{C}_\Lambda} \arrow[dl, "\underline{\xi'}"] \\ & \mathcal{F} & \end{tikzcd}$$

Since $R' \to R$ has a left inverse (namely $R \to R/I = R'$) we see that $\underline{R}|_{\mathcal{C}_\Lambda} \to \underline{R'}|_{\mathcal{C}_\Lambda}$ is essentially surjective. Hence by Lemma 90.8.7 we see that $\underline{\xi'}$ is smooth, i.e., ξ' is a versal formal object. $\square$

Motivated by the preceding lemmas we make the following definition.

06T4 Definition 90.14.4. Let $\mathcal{F}$ be a predeformation category. We say a versal formal object ξ of $\mathcal{F}$ is minimal[3] if for any morphism of formal objects $\xi' \to \xi$ the underlying map on rings is surjective. Sometimes a minimal versal formal object is called miniversal.

The work in this section shows this definition is reasonable. First of all, the existence of a versal formal object implies that $\mathcal{F}$ has (S1). Then the preceding lemmas show there exists a minimal versal formal object. Finally, any two minimal versal formal objects are isomorphic. Here is a summary of our results (with detailed proofs).

06T5 Lemma 90.14.5. Let $\mathcal{F}$ be a predeformation category which has a versal formal object. Then

(1) $\mathcal{F}$ has a minimal versal formal object,
(2) minimal versal objects are unique up to isomorphism, and
(3) any versal object is the pushforward of a minimal versal object along a power series ring extension.

Proof. Suppose $\mathcal{F}$ has a versal formal object ξ over R. Then it satisfies (S1), see Lemma 90.13.1. Let $\xi' \to \xi$ over $R' \subset R$ be any of the morphisms constructed in Lemma 90.14.2. By Lemma 90.14.3 we see that ξ' is versal, hence it is a minimal versal formal object (by construction). This proves (1). Also, $R \cong R'[[x_1, \ldots, x_n]]$ which proves (3).

Suppose that ξ_i/R_i are two minimal versal formal objects. By Lemma 90.8.11 there exist morphisms $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$. The corresponding ring maps $f : R_1 \to R_2$ and $g : R_2 \to R_1$ are surjective by minimality. Hence the compositions $g \circ f : R_1 \to R_1$ and $f \circ g : R_2 \to R_2$ are isomorphisms by Algebra, Lemma 10.31.10. Thus f

[3]This may be nonstandard terminology. Many authors tie this notion in with properties of tangent spaces. We will make the link in Section 90.15.

and g are isomorphisms whence the maps $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$ are isomorphisms (because $\widehat{\mathcal{F}}$ is cofibred in groupoids by Lemma 90.7.2). This proves (2) and finishes the proof of the lemma. $\square$

90.15. Miniversal formal objects and tangent spaces

06IL The general notion of minimality introduced in Definition 90.14.4 can sometimes be deduced from the behaviour on tangent spaces. Let ξ be a formal object of the predeformation category $\mathcal{F}$ and let $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ be the corresponding morphism. Then we can consider the following the condition

06IM (90.15.0.1)
$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective}$$

and the condition

06T6 (90.15.0.2)
$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective on } \mathrm{Der}_\Lambda(k, k)\text{-orbits.}$$

Here we are using the identification $T\underline{R}|_{\mathcal{C}_\Lambda} = \mathrm{Der}_\Lambda(R, k)$ of Example 90.11.11 and the action (90.12.6.2) of derivations on the tangent spaces. If $k' \subset k$ is separable, then $\mathrm{Der}_\Lambda(k, k) = 0$ and the two conditions are equivalent. It turns out that, in the presence of condition (S2) a versal formal object is minimal if and only if $\underline{\xi}$ satisfies (90.15.0.2). Moreover, if $\underline{\xi}$ satisfies (90.15.0.1), then $\mathcal{F}$ satisfies (S2).

06IR **Lemma 90.15.1.** Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (90.15.0.2) holds. Then ξ is a minimal versal formal object. In particular, such ξ are unique up to isomorphism.

Proof. If ξ is not minimal, then there exists a morphism $\xi' \to \xi$ lying over $R' \to R$ such that $R = R'[[x_1, \ldots, x_n]]$ with $n > 0$, see Lemma 90.14.5. Thus $d\underline{\xi}$ factors as

$$\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R', k) \to T\mathcal{F}$$

and we see that (90.15.0.2) cannot hold because $D : f \mapsto \partial/\partial x_1(f) \bmod \mathfrak{m}_R$ is an element of the kernel of the first arrow which is not in the image of $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$. $\square$

06IV **Lemma 90.15.2.** Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (90.15.0.1) holds. Then

(1) $\mathcal{F}$ satisfies (S1).
(2) $\mathcal{F}$ satisfies (S2).
(3) $\dim_k T\mathcal{F}$ is finite.

Proof. Condition (S1) holds by Lemma 90.13.1. The first part of (S2) holds since (S1) holds. Let

$$\begin{tikzcd} y \arrow[r, "c"] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"] \arrow[d, "a'"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

be diagrams as in the second part of (S2). As above we can find morphisms $b : \xi \to y$ and $b' : \xi \to y'$ such that

$$\begin{tikzcd} \xi \arrow[r, "b'"] \arrow[d, "b"'] & y' \arrow[d, "a'"] \\ y \arrow[r, "a"] & x \end{tikzcd}$$

commutes. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ denote the structure morphism. Say $\widehat{p}(\xi) = R$, i.e., ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We see that the pushforward of ξ via $p(c) \circ p(b)$ is x_ϵ and that the pushforward of ξ via $p(c') \circ p(b')$ is x_ϵ. Since ξ satisfies (90.15.0.1), we see that $p(c) \circ p(b) = p(c') \circ p(b')$ as maps $R \to k[\epsilon]$. Hence $p(b) = p(b')$ as maps from $R \to A \times_k k[\epsilon]$. Thus we see that y and y' are isomorphic to the pushforward of ξ along this map and we get a unique morphism $y \to y'$ over $A \times_k k[\epsilon]$ compatible with b and b' as desired.

Finally, by Example 90.11.11 we see $\dim_k T\mathcal{F} = \dim_k T\underline{R}|_{\mathcal{C}_\Lambda}$ is finite. $\square$

06T7 Example 90.15.3. There exist predeformation categories which have a versal formal object satisfying (90.15.0.2) but which do not satisfy (S2). A quick example is to take $F = \underline{k[\epsilon]}/G$ where $G \subset \mathrm{Aut}_{\mathcal{C}_\Lambda}(k[\epsilon])$ is a finite nontrivial subgroup. Namely, the map $\underline{k[\epsilon]} \to F$ is smooth, but the tangent space of F does not have a natural k-vector space structure (as it is a quotient of a k-vector space by a finite group).

06T8 Lemma 90.15.4. Let $\mathcal{F}$ be a predeformation category satisfying (S2) which has a versal formal object. Then its minimal versal formal object satisfies (90.15.0.2).

Proof. Let ξ be a minimal versal formal object for $\mathcal{F}$, see Lemma 90.14.5. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. In order to parse (90.15.0.2) we point out that $T\mathcal{F}$ has a natural k-vector space structure (see Lemma 90.12.2), that $d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$ is linear (see Lemma 90.12.4), and that the action of $\mathrm{Der}_\Lambda(k, k)$ is given by addition (see Lemma 90.12.6). Consider the diagram

$$\begin{tikzcd} & \mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k) & \\ K \arrow[r] & \mathrm{Der}_\Lambda(R, k) \arrow[u] \arrow[r, "d\underline{\xi}"] & T\mathcal{F} \\ & \mathrm{Der}_\Lambda(k, k) \arrow[u] \arrow[ur] & \end{tikzcd}$$

The vector space K is the kernel of $d\underline{\xi}$. Note that the middle column is exact in the middle as it is dual to the sequence (90.3.10.1). If (90.15.0.2) fails, then we can find a nonzero element $D \in K$ which does not map to zero in $\mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k)$. This means there exists an $t \in \mathfrak{m}_R$ such that $D(t) = 1$. Set $R' = \{a \in R \mid D(a) = 0\}$. As D is a derivation this is a subring of R. Since $D(t) = 1$ we see that $R' \to k$ is surjective (compare with the proof of Lemma 90.3.12). Note that $\mathfrak{m}_{R'} = \mathrm{Ker}(D : \mathfrak{m}_R \to k)$ is an ideal of R and $\mathfrak{m}_R^2 \subset \mathfrak{m}_{R'}$. Hence

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_R^2 + k\bar{t}$$

which implies that the map

$$R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \to R/\mathfrak{m}_R^2$$

sending ϵ to $\bar{t}$ is an isomorphism. In particular there is a map $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$.

Let $\xi \to y$ be a morphism lying over $R \to R/\mathfrak{m}_R^2$. Let $y \to x$ be a morphism lying over $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$. Let $y \to x_\epsilon$ be a morphism lying over $R/\mathfrak{m}_R^2 \to k[\epsilon]$. Let x_0 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over k. By the axioms of a category cofibred in groupoids we obtain a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ R'/\mathfrak{m}_R^2 \arrow[r] & k. \end{tikzcd}$$

Because $D \in K$ we see that x_ϵ is isomorphic to $0 \in \mathcal{F}(k[\epsilon])$, i.e., x_ϵ is the pushforward of x_0 via $k \to k[\epsilon], a \mapsto a$. Hence by Lemma 90.10.7 we see that there exists a morphism $x \to y$. Since $\text{length}_\Lambda(R'/\mathfrak{m}_R^2) < \text{length}_\Lambda(R/\mathfrak{m}_R^2)$ the corresponding ring map $R'/\mathfrak{m}_R^2 \to R/\mathfrak{m}_R^2$ is not surjective. This contradicts the minimality of ξ/R, see part (1) of Lemma 90.14.2. This contradiction shows that such a D cannot exist, hence we win. □

06IX Theorem 90.15.5. Let $\mathcal{F}$ be a predeformation category. Consider the following conditions

(1) $\mathcal{F}$ has a minimal versal formal object satisfying (90.15.0.1),
(2) $\mathcal{F}$ has a minimal versal formal object satisfying (90.15.0.2),
(3) the following conditions hold:
   (a) $\mathcal{F}$ satisfies (S1).
   (b) $\mathcal{F}$ satisfies (S2).
   (c) $\dim_k T\mathcal{F}$ is finite.

We always have

$$(1) \Rightarrow (3) \Rightarrow (2).$$

If $k' \subset k$ is separable, then all three are equivalent.

Proof. Lemma 90.15.2 shows that (1) ⇒ (3). Lemmas 90.13.4 and 90.15.4 show that (3) ⇒ (2). If $k' \subset k$ is separable then $\text{Der}_\Lambda(k, k) = 0$ and we see that (90.15.0.1) = (90.15.0.2), i.e., (1) is the same as (2).

An alternative proof of (3) ⇒ (1) in the classical case is to add a few words to the proof of Lemma 90.13.4 to see that one can right away construct a versal object which satisfies (90.15.0.1) in this case. This avoids the use of Lemma 90.13.4 in the classical case. Details omitted. □

06IY Remark 90.15.6. Let $F : \mathcal{C}_\Lambda \to \textit{Sets}$ be a predeformation functor satisfying (S1) and (S2) and $\dim_k TF < \infty$. Recall that these conditions correspond to the conditions (H1), (H2), and (H3) from Schlessinger's paper, see Remark 90.13.5. Now, in the classical case (or if $k' \subset k$ is separable) following Schlessinger we introduce the notion of a hull: a hull is a versal formal object $\xi \in \widehat{F}(R)$ such that $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ is an isomorphism, i.e., (90.15.0.1) holds. Thus Theorem 90.15.5 tells us

$$(H1) + (H2) + (H3) \Rightarrow \text{ there exists a hull}$$

in the classical case. In other words, our theorem recovers Schlessinger's theorem on the existence of hulls.

06IZ Remark 90.15.7. Let $\mathcal{F}$ be a predeformation category. Recall that $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth, see Remark 90.8.5. Hence if $\xi \in \widehat{\mathcal{F}}(R)$ is a versal formal object, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F} \longrightarrow \overline{\mathcal{F}}$$

is smooth (Lemma 90.8.7) and we conclude that the image $\overline{\xi}$ of ξ in $\overline{\mathcal{F}}$ is a versal formal object. If (90.15.0.1) holds, then $\overline{\xi}$ induces an isomorphism $T\underline{R}|_{\mathcal{C}_\Lambda} \to T\overline{\mathcal{F}}$ because $\mathcal{F} \to \overline{\mathcal{F}}$ identifies tangent spaces. Hence in this case $\overline{\xi}$ is a hull for $\overline{\mathcal{F}}$, see Remark 90.15.6. By Theorem 90.15.5 we can always find such a ξ if $k' \subset k$ is separable and $\mathcal{F}$ is a predeformation category satisfying (S1), (S2), and $\dim_k T\mathcal{F} < \infty$.

06T9 Example 90.15.8. In Lemma 90.9.5 we constructed objects $R \in \widehat{\mathcal{C}}_\Lambda$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth and such that

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2 \quad \text{and} \quad \Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$$

Let us reinterpret this using the theorem above. Namely, consider $\mathcal{F} = \mathcal{C}_\Lambda$ as a category cofibred in groupoids over itself (using the identity functor). Then $\mathcal{F}$ is a predeformation category, satisfies (S1) and (S2), and we have $T\mathcal{F} = 0$. Thus $\mathcal{F}$ satisfies condition (3) of Theorem 90.15.5. The theorem implies that (2) holds, i.e., we can find a minimal versal formal object $\xi \in \widehat{\mathcal{F}}(S)$ over some $S \in \widehat{\mathcal{C}}_\Lambda$ satisfying (90.15.0.2). Lemma 90.9.3 shows that $\Lambda \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology (because $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F} = \mathcal{C}_\Lambda$ is smooth). Now condition (90.15.0.2) tells us that $\mathrm{Der}_\Lambda(S, k) \to 0$ is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. This means the injection $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(S, k)$ is also surjective. In other words, we have $\Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}$. Since $\Lambda \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology, we can apply More on Algebra, Lemma 15.40.4 to conclude the exact sequence (90.3.10.2) turns into a pair of identifications

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_S/\mathfrak{m}_S^2 \quad \text{and} \quad \Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}$$

Reading the argument backwards, we find that the R constructed in Lemma 90.9.5 carries a minimal versal object. By the uniqueness of minimal versal objects (Lemma 90.14.5) we also conclude $R \cong S$, i.e., the two constructions give the same answer.

90.16. Rim-Schlessinger conditions and deformation categories

06J1 There is a very natural property of categories fibred in groupoids over $\mathcal{C}_\Lambda$ which is easy to check in practice and which implies Schlessinger's properties (S1) and (S2) we have introduced earlier.

06J2 Definition 90.16.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We say that $\mathcal{F}$ satisfies condition (RS) if for every diagram in $\mathcal{F}$

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in $\mathcal{F}$ such that the diagram

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lies over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

06J3 **Lemma 90.16.2.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Given a commutative diagram in $\mathcal{F}$

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

with $A_2 \to A$ surjective, then it is a fiber square.

Proof. Since $\mathcal{F}$ satisfies (RS), there exists a fiber product diagram

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

The induced map $y \to x_1 \times_x x_2$ lies over $\mathrm{id} : A_1 \times_A A_1 \to A_1 \times_A A_1$, hence it is an isomorphism. $\square$

06J4 **Lemma 90.16.3.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (RS) if the condition in Definition 90.16.1 is assumed to hold only when $A_2 \to A$ is a small extension.

Proof. Apply Lemma 90.3.3. The proof is similar to that of Lemma 90.8.2. $\square$

06J5 **Lemma 90.16.4.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The following are equivalent

(1) $\mathcal{F}$ satisfies (RS),
(2) the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ see (90.10.1.1) is an equivalence of categories whenever $A_2 \to A$ is surjective, and
(3) same as in (2) whenever $A_2 \to A$ is a small extension.

Proof. Assume (1). By Lemma 90.16.2 we see that every object of $\mathcal{F}(A_1 \times_A A_2)$ is of the form $x_1 \times_x x_2$. Moreover

$$\mathrm{Mor}_{A_1 \times_A A_2}(x_1 \times_x x_2, y_1 \times_y y_2) = \mathrm{Mor}_{A_1}(x_1, y_1) \times_{\mathrm{Mor}_A(x,y)} \mathrm{Mor}_{A_2}(x_2, y_2).$$

Hence we see that $\mathcal{F}(A_1 \times_A A_2)$ is a 2-fibre product of $\mathcal{F}(A_1)$ with $\mathcal{F}(A_2)$ over $\mathcal{F}(A)$ by Categories, Remark 4.31.5. In other words, we see that (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate.

Assume (3). Let $q_1 : A_1 \to A$ and $q_2 : A_2 \to A$ be given with q_2 a small extension. We will use the description of the 2-fibre product $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ from Categories, Remark 4.31.5. Hence let $y \in \mathcal{F}(A_1 \times_A A_2)$ correspond to

$(x_1, x_2, x, a_1 : x_1 \to x, a_2 : x_2 \to x)$. Let z be an object of $\mathcal{F}$ lying over C. Then

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{F}}(z, y) &= \{(f, \alpha) \mid f : C \to A_1 \times_A A_2, \alpha : f_* z \to y\} \\
&= \{(f_1, f_2, \alpha_1, \alpha_2) \mid f_i : C \to A_i,\ \alpha_i : f_{i,*} z \to x_i, \\
&\qquad q_1 \circ f_1 = q_2 \circ f_2,\ q_{1,*}\alpha_1 = q_{2,*}\alpha_2\} \\
&= \mathrm{Mor}_{\mathcal{F}}(z, x_1) \times_{\mathrm{Mor}_{\mathcal{F}}(z,x)} \mathrm{Mor}_{\mathcal{F}}(z, x_2)
\end{aligned}$$

whence y is a fibre product of x_1 and x_2 over x. Thus we see that $\mathcal{F}$ satisfies (RS) in case $A_2 \to A$ is a small extension. Hence (RS) holds by Lemma 90.16.3. □

06J6 Remark 90.16.5. When $\mathcal{F}$ is cofibered in sets, condition (RS) is exactly condition (H4) from Schlessinger's paper [Sch68, Theorem 2.11]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \mathrm{Sets}$, condition (RS) states: If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is bijective.

06J7 Lemma 90.16.6. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The condition (RS) for $\mathcal{F}$ implies both (S1) and (S2) for $\mathcal{F}$.

Proof. Using the reformulation of Lemma 90.16.4 and the explanation of (S1) following Definition 90.10.1 it is immediate that (RS) implies (S1). This proves the first part of (S2). The second part of (S2) follows because Lemma 90.16.2 tells us that $y = x_1 \times_{d, x_0, e} x_2 = y'$ if y, y' are as in the second part of the definition of (S2) in Definition 90.10.1. (In fact the morphism $y \to y'$ is compatible with both a, a' and c, c'!) □

The following lemma is the analogue of Lemma 90.10.5. Recall that if $\mathcal{F}$ is a category cofibred in groupoids over $\mathcal{C}_\Lambda$ and x is an object of $\mathcal{F}$ lying over A, then we denote $\mathrm{Aut}_A(x) = \mathrm{Mor}_A(x, x) = \mathrm{Mor}_{\mathcal{F}(A)}(x, x)$. If $x' \to x$ is a morphism of $\mathcal{F}$ lying over $A' \to A$ then there is a well defined map of groups $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$.

06J8 Lemma 90.16.7. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). The following conditions are equivalent:

(1) $\overline{\mathcal{F}}$ satisfies (RS).
(2) Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. The induced map of sets of isomorphism classes
$$\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$
is injective.
(3) For every morphism $x' \to x$ in $\mathcal{F}$ lying over a surjective ring map $A' \to A$, the map $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$ is surjective.
(4) For every morphism $x' \to x$ in $\mathcal{F}$ lying over a small extension $A' \to A$, the map $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$ is surjective.

Proof. We prove that (1) is equivalent to (2) and (2) is equivalent to (3). The equivalence of (3) and (4) follows from Lemma 90.3.3.

Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. By Remark 90.16.5 we see $\overline{\mathcal{F}}$ satisfies (RS) if and only if the map

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$

is bijective for any such f_1, f_2. This map is at least surjective since that is the condition of (S1) and $\overline{\mathcal{F}}$ satisfies (S1) by Lemmas 90.16.6 and 90.10.5. Moreover, this map factors as

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \longrightarrow \overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \longrightarrow \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2),$$

where the first map is a bijection since

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence by (RS) for $\mathcal{F}$. Hence (1) is equivalent to (2).

Assume (2) holds. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map $f : A' \to A$. Let $a \in \mathrm{Aut}_A(x)$. The objects

$$(x', x', a : x \to x),\ (x', x', \mathrm{id} : x \to x)$$

of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$ have the same image in $\overline{\mathcal{F}}(A') \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A')$. By (2) there exists maps $b_1, b_2 : x' \to x'$ such that

$$\begin{tikzcd} x \arrow[r, "a"'] \arrow[d, "f_*b_1"'] & x \arrow[d, "f_*b_2"] \\ x \arrow[r, "\mathrm{id}"] & x \end{tikzcd}$$

commutes. Hence $b_2^{-1} \circ b_1 \in \mathrm{Aut}_{A'}(x')$ has image $a \in \mathrm{Aut}_A(x)$. Hence (3) holds.

Assume (3) holds. Suppose

$$(x_1, x_2, a : (f_1)_*x_1 \to (f_2)_*x_2),\ (x_1', x_2', a' : (f_1)_*x_1' \to (f_2)_*x_2')$$

are objects of $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ with the same image in $\overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$. Then there are morphisms $b_1 : x_1 \to x_1'$ in $\mathcal{F}(A_1)$ and $b_2 : x_2 \to x_2'$ in $\mathcal{F}(A_2)$. By (3) we can modify b_2 by an automorphism of x_2 over A_2 so that the diagram

$$\begin{tikzcd} (f_1)_*x_1 \arrow[r, "a"'] \arrow[d, "(f_1)_*b_1"'] & (f_2)_*x_2 \arrow[d, "(f_2)_*b_2"] \\ (f_1)_*x_1' \arrow[r, "a'"] & (f_2)_*x_2'. \end{tikzcd}$$

commutes. This proves $(x_1, x_2, a) \cong (x_1', x_2', a')$ in $\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)}$. Hence (2) holds. □

Finally we define the notion of a deformation category.

06J9 Definition 90.16.8. A deformation category is a predeformation category $\mathcal{F}$ satisfying (RS). A morphism of deformation categories is a morphism of categories over $\mathcal{C}_\Lambda$.

06JA Remark 90.16.9. We say that a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is a deformation functor if the associated cofibered set is a deformation category, i.e. if $F(k)$ is a one element set and F satisfies (RS). If $\mathcal{F}$ is a deformation category, then $\overline{\mathcal{F}}$ is a predeformation functor but not necessarily a deformation functor, as Lemma 90.16.7 shows.

06JB Example 90.16.10. A prorepresentable functor F is a deformation functor. Namely, suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ and $F(A) = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, A)$. There is a unique morphism $R \to k$, so $F(k)$ is a one element set. Since

$$\mathrm{Hom}_\Lambda(R, A_1 \times_A A_2) = \mathrm{Hom}_\Lambda(R, A_1) \times_{\mathrm{Hom}_\Lambda(R,A)} \mathrm{Hom}_\Lambda(R, A_2)$$

the same is true for maps in $\widehat{\mathcal{C}}_\Lambda$ and we see that F has (RS).

The following is one of our typical remarks on passing from a category cofibered in groupoids to the predeformation category at a point over k: it says that this process preserves (RS).

06JC Lemma 90.16.11. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 90.6.4. If $\mathcal{F}$ satisfies (RS), then so does $\mathcal{F}_{x_0}$. In particular, $\mathcal{F}_{x_0}$ is a deformation category.

Proof. Any diagram as in Definition 90.16.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of (RS) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. $\square$

The following lemma is the analogue of the fact that 2-fibre products of algebraic stacks are algebraic stacks.

06L4 Lemma 90.16.12. Let

$$\begin{tikzcd} \mathcal{H} \times_\mathcal{F} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{H} \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

be 2-fibre product of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ all satisfy (RS), then $\mathcal{H} \times_\mathcal{F} \mathcal{G}$ satisfies (RS).

Proof. If A is an object of $\mathcal{C}_\Lambda$, then an object of the fiber category of $\mathcal{H} \times_\mathcal{F} \mathcal{G}$ over A is a triple (u, v, a) where $u \in \mathcal{H}(A)$, $v \in \mathcal{G}(A)$, and $a : f(u) \to g(v)$ is a morphism in $\mathcal{F}(A)$. Consider a diagram in $\mathcal{H} \times_\mathcal{F} \mathcal{G}$

$$\begin{tikzcd} & (u_2, v_2, a_2) \arrow[d] & & & A_2 \arrow[d] \\ (u_1, v_1, a_1) \arrow[r] & (u, v, a) & \text{lying over} & A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective. Since $\mathcal{H}$ and $\mathcal{G}$ satisfy (RS), there are fiber products $u_1 \times_u u_2$ and $v_1 \times_v v_2$ lying over $A_1 \times_A A_2$. Since $\mathcal{F}$ satisfies (RS), Lemma 90.16.2 shows

$$\begin{tikzcd} f(u_1 \times_u u_2) \arrow[r] \arrow[d] & f(u_2) \arrow[d] & & g(v_1 \times_v v_2) \arrow[r] \arrow[d] & g(v_2) \arrow[d] \\ f(u_1) \arrow[r] & f(u) & \text{and} & g(v_1) \arrow[r] & g(v) \end{tikzcd}$$

are both fiber squares in $\mathcal{F}$. Thus we can view $a_1 \times_a a_2$ as a morphism from $f(u_1 \times_u u_2)$ to $g(v_1 \times_v v_2)$ over $A_1 \times_A A_2$. It follows that

$$\begin{tikzcd} (u_1 \times_u u_2, v_1 \times_v v_2, a_1 \times_a a_2) \arrow[r] \arrow[d] & (u_2, v_2, a_2) \arrow[d] \\ (u_1, v_1, a_1) \arrow[r] & (u, v, a) \end{tikzcd}$$

is a fiber square in $\mathcal{H} \times_\mathcal{F} \mathcal{G}$ as desired. $\square$

90.17. Lifts of objects

06JD The content of this section is that the tangent space has a principal homogeneous action on the set of lifts along a small extension in the case of a deformation category.

06JE **Definition 90.17.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $f : A' \to A$ be a map in $\mathcal{C}_\Lambda$. Let $x \in \mathcal{F}(A)$. The category $\mathrm{Lift}(x, f)$ of lifts of x along f is the category with the following objects and morphisms.

(1) Objects: A lift of x along f is a morphism $x' \to x$ lying over f.
(2) Morphisms: A morphism of lifts from $a_1 : x'_1 \to x$ to $a_2 : x'_2 \to x$ is a morphism $b : x'_1 \to x'_2$ in $\mathcal{F}(A')$ such that $a_2 = a_1 \circ b$.

The set $\mathrm{Lift}(x, f)$ of lifts of x along f is the set of isomorphism classes of $\mathrm{Lift}(x, f)$.

06JF **Remark 90.17.2.** When the map $f : A' \to A$ is clear from the context, we may write $\mathrm{Lift}(x, A')$ and $\mathrm{Lift}(x, A')$ in place of $\mathrm{Lift}(x, f)$ and $\mathrm{Lift}(x, f)$.

06JG **Remark 90.17.3.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let V be a finite dimensional vector space. Then $\mathrm{Lift}(x_0, k[V])$ is the set of isomorphism classes of $\mathcal{F}_{x_0}(k[V])$ where $\mathcal{F}_{x_0}$ is the predeformation category of objects in $\mathcal{F}$ lying over x_0, see Remark 90.6.4. Hence if $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$ (see Lemma 90.10.6) and by Lemma 90.12.2 we see that

$$\mathrm{Lift}(x_0, k[V]) = T\mathcal{F}_{x_0} \otimes_k V$$

as k-vector spaces.

06JH **Remark 90.17.4.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let

$$\begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

be a fibre square in $\mathcal{C}_\Lambda$ such that either $A_1 \to A$ or $A_2 \to A$ is surjective. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Given lifts $x_1 \to x$ and $x_2 \to x$ of x to A_1 and A_2, we get by (RS) a lift $x_1 \times_x x_2 \to x$ of x to $A_1 \times_A A_2$. Conversely, by Lemma 90.16.2 any lift of x to $A_1 \times_A A_2$ is of this form. Hence a bijection

$$\mathrm{Lift}(x, A_1) \times \mathrm{Lift}(x, A_2) \longrightarrow \mathrm{Lift}(x, A_1 \times_A A_2).$$

Similarly, if $x_1 \to x$ is a fixed lifting of x to A_1, then there is a bijection

$$\mathrm{Lift}(x_1, A_1 \times_A A_2) \longrightarrow \mathrm{Lift}(x, A_2).$$

Now let

$$\begin{tikzcd} A'_1 \times_A A_2 \arrow[r] \arrow[d] & A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A'_1 \arrow[r] & A_1 \arrow[r] & A \end{tikzcd}$$

be a composition of fibre squares in $\mathcal{C}_\Lambda$ with both $A'_1 \to A_1$ and $A_1 \to A$ surjective. Let $x_1 \to x$ be a morphism lying over $A_1 \to A$. Then by the above we have bijections

$$\begin{aligned}\mathrm{Lift}(x_1, A'_1 \times_A A_2) &= \mathrm{Lift}(x_1, A'_1) \times \mathrm{Lift}(x_1, A_1 \times_A A_2)\\ &= \mathrm{Lift}(x_1, A'_1) \times \mathrm{Lift}(x, A_2).\end{aligned}$$

06JI Lemma 90.17.5. Let $\mathcal{F}$ be a deformation category. Let $A' \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$ whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. If $\mathrm{Lift}(x, A')$ is nonempty, then there is a free and transitive action of $T\mathcal{F} \otimes_k I$ on $\mathrm{Lift}(x, A')$.

Proof. Consider the ring map $g : A' \times_A A' \to k[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$ (compare with Lemma 90.10.8). There is an isomorphism

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections of $A' \times_A A'$ and $A' \times_k k[I]$ to A. Thus there is a bijection

06TA (90.17.5.1) $$\mathrm{Lift}(x, A' \times_A A') \longrightarrow \mathrm{Lift}(x, A' \times_k k[I])$$

By Remark 90.17.4 there is a bijection

06TB (90.17.5.2) $$\mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(x, A' \times_A A')$$

There is a commutative diagram

$$\begin{tikzcd} A' \times_k k[I] \arrow[r] \arrow[d] & A \times_k k[I] \arrow[r] \arrow[d] & k[I] \arrow[d] \\ A' \arrow[r] & A \arrow[r] & k. \end{tikzcd}$$

Thus if we choose a pushforward $x \to x_0$ of x along $A \to k$, we obtain by the end of Remark 90.17.4 a bijection

06TC (90.17.5.3) $$\mathrm{Lift}(x, A' \times_k k[I]) \longrightarrow \mathrm{Lift}(x, A') \times \mathrm{Lift}(x_0, k[I])$$

Composing (90.17.5.2), (90.17.5.1), and (90.17.5.3) we get a bijection

$$\Phi : \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(x, A') \times \mathrm{Lift}(x_0, k[I]).$$

This bijection commutes with the projections on the first factors. By Remark 90.17.3 we see that $\mathrm{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. If pr_2 is the second projection of $\mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A')$, then we get a map

$$a = \mathrm{pr}_2 \circ \Phi^{-1} : \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \longrightarrow \mathrm{Lift}(x, A').$$

Unwinding all the above we see that $a(x' \to x, \theta)$ is the unique lift $x'' \to x$ such that $g_*(x', x'') = \theta$ in $\mathrm{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. To see this is an action of $T\mathcal{F} \otimes_k I$ on $\mathrm{Lift}(x, A')$ we have to show the following: if x', x'', x''' are lifts of x and $g_*(x', x'') = \theta$, $g_*(x'', x''') = \theta'$, then $g_*(x', x''') = \theta + \theta'$. This follows from the commutative diagram

$$\begin{tikzcd} A' \times_A A' \times_A A' \arrow[r, "(a_1,a_2,a_3) \mapsto (g(a_1,a_2), g(a_2,a_3))"'] \arrow[dr, "(a_1,a_2,a_3) \mapsto g(a_1,a_3)"'] & k[I] \times_k k[I] = k[I \times I] \arrow[d, "+"] \\ & k[I] \end{tikzcd}$$

The action is free and transitive because Φ is bijective. $\square$

06JJ Remark 90.17.6. The action of Lemma 90.17.5 is functorial. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of deformation categories. Let $A' \to A$ be a surjective ring map whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. In this situation φ induces the vertical arrows in the following commutative diagram

$$\begin{tikzcd} \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \arrow[r] \arrow[d, "(\varphi, d\varphi \otimes \mathrm{id}_I)"'] & \mathrm{Lift}(x, A') \arrow[d, "\varphi"] \\ \mathrm{Lift}(\varphi(x), A') \times (T\mathcal{G} \otimes_k I) \arrow[r] & \mathrm{Lift}(\varphi(x), A') \end{tikzcd}$$

The commutativity follows as each of the maps (90.17.5.2), (90.17.5.1), and (90.17.5.3) of the proof of Lemma 90.17.5 gives rise to a similar commutative diagram.

90.18. Schlessinger's theorem on prorepresentable functors

06JK We deduce Schlessinger's theorem characterizing prorepresentable functors on $\mathcal{C}_\Lambda$.

06JL Lemma 90.18.1. Let $F, G : \mathcal{C}_\Lambda \to \textit{Sets}$ be deformation functors. Let $\varphi : F \to G$ be a smooth morphism which induces an isomorphism $d\varphi : TF \to TG$ of tangent spaces. Then φ is an isomorphism.

Proof. We prove $F(A) \to G(A)$ is a bijection for all $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ by induction on $\mathrm{length}_A(A)$. For $A = k$ the statement follows from the assumption that F and G are deformation functors. Suppose that the statement holds for rings of length less than n and let A' be a ring of length n. Choose a small extension $f : A' \to A$. We have a commutative diagram

$$\begin{tikzcd} F(A') \arrow[r] \arrow[d, "F(f)"'] & G(A') \arrow[d, "G(f)"] \\ F(A) \arrow[r, "\sim"] & G(A) \end{tikzcd}$$

where the map $F(A) \to G(A)$ is a bijection. By smoothness of $F \to G$, $F(A') \to G(A')$ is surjective (Lemma 90.8.8). Thus we can check bijectivity by checking it on fibers $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty. These fibers are precisely $\mathrm{Lift}(x, A')$ and $\mathrm{Lift}(\varphi(x), A')$ and by assumption we have an isomorphism $d\varphi \otimes \mathrm{id} : TF \otimes_k \mathrm{Ker}(f) \to TG \otimes_k \mathrm{Ker}(f)$. Thus, by Lemma 90.17.5 and Remark 90.17.6, for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty the map $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ is a map of sets commuting with free transitive actions by $TF \otimes_k \mathrm{Ker}(f)$. Hence it is bijective. $\square$

Note that in case $k' \subset k$ is separable condition (c) in the theorem below is empty.

06JM Theorem 90.18.2. Let $F : \mathcal{C}_\Lambda \to \textit{Sets}$ be a functor. Then F is prorepresentable if and only if (a) F is a deformation functor, (b) $\dim_k TF$ is finite, and (c) $\gamma : \mathrm{Der}_\Lambda(k, k) \to TF$ is injective.

Proof. Assume F is prorepresentable by $R \in \widehat{\mathcal{C}}_\Lambda$. We see F is a deformation functor by Example 90.16.10. We see $\dim_k TF$ is finite by Example 90.11.11. Finally, $\mathrm{Der}_\Lambda(k, k) \to TF$ is identified with $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$ by Example 90.11.14 which is injective because $R \to k$ is surjective.

Conversely, assume (a), (b), and (c) hold. By Lemma 90.16.6 we see that (S1) and (S2) hold. Hence by Theorem 90.15.5 there exists a minimal versal formal object ξ of F such that (90.15.0.2) holds. Say ξ lies over R. The map

$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$$

is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. Since the action of $\mathrm{Der}_\Lambda(k, k)$ on the left hand side is free by (c) and Lemma 90.12.6 we see that the map is bijective. Thus we see that $\underline{\xi}$ is an isomorphism by Lemma 90.18.1. □

90.19. Infinitesimal automorphisms

06JN Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Given a morphism $x' \to x$ in $\mathcal{F}$ lying over $A' \to A$, there is an induced homomorphism

$$\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x).$$

Lemma 90.16.7 says that the cokernel of this homomorphism determines whether condition (RS) on $\mathcal{F}$ passes to $\overline{\mathcal{F}}$. In this section we study the kernel of this homomorphism. We will see that it also gives a measure of how far $\mathcal{F}$ is from $\overline{\mathcal{F}}$.

06JP Definition 90.19.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over $A' \to A$. The kernel

$$\mathrm{Inf}(x'/x) = \mathrm{Ker}(\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x))$$

is the group of infinitesimal automorphisms of x' over x.

06JQ Definition 90.19.2. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Assume a choice of pushforward $x_0 \to x'_0$ of x_0 along the map $k \to k[\epsilon], a \mapsto a$ has been made. Then there is a unique map $x'_0 \to x_0$ such that $x_0 \to x'_0 \to x_0$ is the identity on x_0. Then

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Inf}(x'_0/x_0)$$

is the group of infinitesimal automorphisms of x_0

06JR Remark 90.19.3. Up to canonical isomorphism $\mathrm{Inf}_{x_0}(\mathcal{F})$ does not depend on the choice of pushforward $x_0 \to x'_0$ because any two pushforwards are canonically isomorphic. Moreover, if $y_0 \in \mathcal{F}(k)$ and $x_0 \cong y_0$ in $\mathcal{F}(k)$, then $\mathrm{Inf}_{x_0}(\mathcal{F}) \cong \mathrm{Inf}_{y_0}(\mathcal{F})$ where the isomorphism depends (only) on the choice of an isomorphism $x_0 \to y_0$. In particular, $\mathrm{Aut}_k(x_0)$ acts on $\mathrm{Inf}_{x_0}(\mathcal{F})$.

06JS Remark 90.19.4. Assume $\mathcal{F}$ is a predeformation category. Then

(1) for $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ the automorphism group $\mathrm{Aut}_k(x_0)$ is trivial and hence $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Aut}_{k[\epsilon]}(x'_0)$, and
(2) for $x_0, y_0 \in \mathrm{Ob}(\mathcal{F}(k))$ there is a unique isomorphism $x_0 \to y_0$ and hence a canonical identification $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Inf}_{y_0}(\mathcal{F})$.

Since $\mathcal{F}(k)$ is nonempty, choosing $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and setting

$$\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_{x_0}(\mathcal{F})$$

we get a well defined group of infinitesimal automorphisms of $\mathcal{F}$. With this notation we have $\mathrm{Inf}(\mathcal{F}_{x_0}) = \mathrm{Inf}_{x_0}(\mathcal{F})$. Please compare with the equality $T\mathcal{F}_{x_0} = T_{x_0}\mathcal{F}$ in Remark 90.12.5.

We will see that $\mathrm{Inf}_{x_0}(\mathcal{F})$ has a natural k-vector space structure when $\mathcal{F}$ satisfies (RS). At the same time, we will see that if $\mathcal{F}$ satisfies (RS), then the infinitesimal automorphisms $\mathrm{Inf}(x'/x)$ of a morphism $x' \to x$ lying over a small extension are governed by $\mathrm{Inf}_{x_0}(\mathcal{F})$, where x_0 is a pushforward of x to $\mathcal{F}(k)$. In order to do this, we introduce the automorphism functor for any object $x \in \mathrm{Ob}(\mathcal{F})$ as follows.

06JT Definition 90.19.5. Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids over an arbitrary base category $\mathcal{C}$. Assume a choice of pushforwards has been made. Let $x \in \mathrm{Ob}(\mathcal{F})$ and let $U = p(x)$. Let $U/\mathcal{C}$ denote the category of objects under U. The automorphism functor of x is the functor $Aut(x) : U/\mathcal{C} \to \mathrm{Sets}$ sending an object $f : U \to V$ to $\mathrm{Aut}_V(f_*x)$ and sending a morphism

$$\begin{tikzcd} V' \arrow[rr] & & V \\ & U \arrow[ul, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$

to the homomorphism $\mathrm{Aut}_{V'}(f'_*x) \to \mathrm{Aut}_V(f_*x)$ coming from the unique morphism $f'_*x \to f_*x$ lying over $V' \to V$ and compatible with $x \to f'_*x$ and $x \to f_*x$.

We will be concerned with the automorphism functors of objects in a category cofibered in groupoids $\mathcal{F}$ over $\mathcal{C}_\Lambda$. If $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$, then the category $A/\mathcal{C}_\Lambda$ is nothing but the category $\mathcal{C}_A$, i.e. the category defined in Section 90.3 where we take $\Lambda = A$ and $k = A/\mathfrak{m}_A$. Hence the automorphism functor of an object $x \in \mathrm{Ob}(\mathcal{F}(A))$ is a functor $Aut(x) : \mathcal{C}_A \to \mathrm{Sets}$.

The following lemma could be deduced from Lemma 90.16.12 by thinking about the "inertia" of a category cofibred in groupoids, see for example Stacks, Section 8.7 and Categories, Section 4.34. However, it is easier to see it directly.

06JU Lemma 90.19.6. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Then $Aut(x) : \mathcal{C}_A \to \mathrm{Sets}$ satisfies (RS).

Proof. It follows that $Aut(x)$ satisfies (RS) from the fully faithfulness of the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ in Lemma 90.16.4. □

06JV Lemma 90.19.7. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$.

(1) $T_{\mathrm{id}_{x_0}} Aut(x)$ has a natural k-vector space structure such that addition agrees with composition in $T_{\mathrm{id}_{x_0}} Aut(x)$. In particular, composition in $T_{\mathrm{id}_{x_0}} Aut(x)$ is commutative.
(2) There is a canonical isomorphism $T_{\mathrm{id}_{x_0}} Aut(x) \to T_{\mathrm{id}_{x_0}} Aut(x_0)$ of k-vector spaces.

Proof. We apply Remark 90.6.4 to the functor $Aut(x) : \mathcal{C}_A \to \mathrm{Sets}$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x)_{\mathrm{id}_{x_0}}$. By Lemmas 90.19.6 and 90.16.11 F is a deformation functor. By definition $T_{\mathrm{id}_{x_0}} Aut(x) = TF = F(k[\epsilon])$ which has a natural k-vector space structure specified by Lemma 90.11.8.

Addition is defined as the composition

$$F(k[\epsilon]) \times F(k[\epsilon]) \longrightarrow F(k[\epsilon] \times_k k[\epsilon]) \longrightarrow F(k[\epsilon])$$

where the first map is the inverse of the bijection guaranteed by (RS) and the second is induced by the k-algebra map $k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$ which maps $(\epsilon, 0)$ and $(0, \epsilon)$ to ϵ. If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then $F(A) \to F(B)$ is a homomorphism where $F(A) = Aut(x)_{\mathrm{id}_{x_0}}(A)$ and $F(B) = Aut(x)_{\mathrm{id}_{x_0}}(B)$ are groups under composition. We conclude that $+ : F(k[\epsilon]) \times F(k[\epsilon]) \to F(k[\epsilon])$ is a homomorphism where $F(k[\epsilon])$ is regarded as a group under composition. With $\mathrm{id} \in F(k[\epsilon])$ the unit element we see that $+(v, \mathrm{id}) = +(\mathrm{id}, v) = v$ for any $v \in F(k[\epsilon])$ because (id, v) is the pushforward of v along the ring map $k[\epsilon] \to k[\epsilon] \times_k k[\epsilon]$ with $\epsilon \mapsto (\epsilon, 0)$. In general, given a group G with multiplication $\circ$ and $+ : G \times G \to G$ is a homomorphism such that $+(g, 1) = +(1, g) = g$, where 1 is the identity of G, then $+ = \circ$. This shows addition in the k-vector space structure on $F(k[\epsilon])$ agrees with composition.

Finally, (2) is a matter of unwinding the definitions. Namely $T_{\mathrm{id}_{x_0}} Aut(x)$ is the set of automorphisms α of the pushforward of x along $A \to k \to k[\epsilon]$ which are trivial modulo ϵ. On the other hand $T_{\mathrm{id}_{x_0}} Aut(x_0)$ is the set of automorphisms of the pushforward of x_0 along $k \to k[\epsilon]$ which are trivial modulo ϵ. Since x_0 is the pushforward of x along $A \to k$ the result is clear. □

06JW Remark 90.19.8. We point out some basic relationships between infinitesimal automorphism groups, liftings, and tangent spaces to automorphism functors. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism lying over a ring map $A' \to A$. Then from the definitions we have an equality

$$\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$$

where the liftings are of id_x as an object of $Aut(x')$. If $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and x_0' is the pushforward to $\mathcal{F}(k[\epsilon])$, then applying this to $x_0' \to x_0$ we get

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Lift}(\mathrm{id}_{x_0}, k[\epsilon]) = T_{\mathrm{id}_{x_0}} Aut(x_0),$$

the last equality following directly from the definitions.

06JX Lemma 90.19.9. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then $\mathrm{Inf}_{x_0}(\mathcal{F})$ is equal as a set to $T_{\mathrm{id}_{x_0}} Aut(x_0)$, and so has a natural k-vector space structure such that addition agrees with composition of automorphisms.

Proof. The equality of sets is as in the end of Remark 90.19.8 and the statement about the vector space structure follows from Lemma 90.19.7. □

07W6 Lemma 90.19.10. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then φ induces a k-linear map $\mathrm{Inf}_{x_0}(\mathcal{F}) \to \mathrm{Inf}_{\varphi(x_0)}(\mathcal{G})$.

Proof. It is clear that φ induces a morphism from $Aut(x_0) \to Aut(\varphi(x_0))$ which maps the identity to the identity. Hence this follows from the result for tangent spaces, see Lemma 90.12.4. □

06JY Lemma 90.19.11. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism lying over a surjective ring map $A' \to A$ with kernel I annihilated by $\mathfrak{m}_{A'}$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then $\mathrm{Inf}(x'/x)$ has a free and transitive action by $T_{\mathrm{id}_{x_0}} Aut(x') \otimes_k I = \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k I$.

Proof. This is just the analogue of Lemma 90.17.5 in the setting of automorphism sheaves. To be precise, we apply Remark 90.6.4 to the functor $Aut(x') : \mathcal{C}_{A'} \to \text{Sets}$ and the element $\text{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x')_{\text{id}_{x_0}}$. By Lemmas 90.19.6 and 90.16.11 F is a deformation functor. Hence Lemma 90.17.5 gives a free and transitive action of $TF \otimes_k I$ on $\text{Lift}(\text{id}_x, A')$, because as $\text{Lift}(\text{id}_x, A')$ is a group it is always nonempty. Note that we have equalities of vector spaces

$$TF = T_{\text{id}_{x_0}} Aut(x') \otimes_k I = \text{Inf}_{x_0}(\mathcal{F}) \otimes_k I$$

by Lemma 90.19.7. The equality $\text{Inf}(x'/x) = \text{Lift}(\text{id}_x, A')$ of Remark 90.19.8 finishes the proof. $\square$

06JZ **Lemma 90.19.12.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. If $\text{Inf}_{x_0}(\mathcal{F}) = 0$ then $\text{Inf}(x'/x) = 0$.

Proof. Follows from Lemmas 90.3.3 and 90.19.11. $\square$

06K0 **Lemma 90.19.13.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \text{Ob}(\mathcal{F}(k))$. Then $\text{Inf}_{x_0}(\mathcal{F}) = 0$ if and only if the natural morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ of categories cofibered in groupoids is an equivalence.

Proof. The morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ is an equivalence if and only if $\mathcal{F}_{x_0}$ is fibered in setoids, cf. Categories, Section 4.39 (a setoid is by definition a groupoid in which the only automorphism of any object is the identity). We prove that $\text{Inf}_{x_0}(\mathcal{F}) = 0$ if and only if this condition holds for $\mathcal{F}_{x_0}$. Obviously if $\mathcal{F}_{x_0}$ is fibered in setoids then $\text{Inf}_{x_0}(\mathcal{F}) = 0$. Conversely assume $\text{Inf}_{x_0}(\mathcal{F}) = 0$. Let A be an object of $\mathcal{C}_\Lambda$. Then by Lemma 90.19.12, $\text{Inf}(x/x_0) = 0$ for any object $x \to x_0$ of $\mathcal{F}_{x_0}(A)$. Since by definition $\text{Inf}(x/x_0)$ equals the group of automorphisms of $x \to x_0$ in $\mathcal{F}_{x_0}(A)$, this proves $\mathcal{F}_{x_0}(A)$ is a setoid. $\square$

90.20. Applications

0DYM We collect some results on deformation categories we will use later.

06L5 **Lemma 90.20.1.** Let $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{F}$ be 1-morphisms of deformation categories. Then

(1) $\mathcal{W} = \mathcal{H} \times_\mathcal{F} \mathcal{G}$ is a deformation category, and
(2) we have a 6-term exact sequence of vector spaces

$$0 \to \text{Inf}(\mathcal{W}) \to \text{Inf}(\mathcal{H}) \oplus \text{Inf}(\mathcal{G}) \to \text{Inf}(\mathcal{F}) \to T\mathcal{W} \to T\mathcal{H} \oplus T\mathcal{G} \to T\mathcal{F}$$

Proof. Part (1) follows from Lemma 90.16.12 and the fact that $\mathcal{W}(k)$ is the fibre product of two setoids with a unique isomorphism class over a setoid with a unique isomorphism class.

Part (2). Let $w_0 \in \text{Ob}(\mathcal{W}(k))$ and let x_0, y_0, z_0 be the image of w_0 in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. Then $\text{Inf}(\mathcal{W}) = \text{Inf}_{w_0}(\mathcal{W})$ and simlarly for $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{F}$, see Remark 90.19.4. We apply Lemmas 90.12.4 and 90.19.10 to get all the linear maps except for the "boundary map" $\delta : \text{Inf}_{x_0}(\mathcal{F}) \to T\mathcal{W}$. We will insert suitable signs later.

Construction of δ. Choose a pushforward $w_0 \to w_0'$ along $k \to k[\epsilon]$. Denote x_0', y_0', z_0' the images of w_0' in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. In particular we obtain isomorphisms $b' : f(y_0') \to x_0'$ and $c' : x_0' \to g(z_0')$. Denote $b : f(y_0) \to x_0$ and $c : x_0 \to g(z_0)$ the pushforwards along $k[\epsilon] \to k$. Observe that this means $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ and $w_0 =$

$(k, y_0, z_0, c \circ b)$ in terms of the explicit form of the fibre product of categories, see Remarks 90.5.2 (13). Given $\alpha : x'_0 \to x'_0$ we set $\delta(\alpha) = (k[\epsilon], y'_0, z'_0, c' \circ \alpha \circ b')$ which is indeed an object of $\mathcal{W}$ over $k[\epsilon]$ and comes with a morphism $(k[\epsilon], y'_0, z'_0, c' \circ \alpha \circ b') \to w_0$ over $k[\epsilon] \to k$ as α pushes forward to the identity over k. More generally, for any k-vector space V we can define a map

$$\mathrm{Lift}(\mathrm{id}_{x_0}, k[V]) \longrightarrow \mathrm{Lift}(w_0, k[V])$$

using exactly the same formulae. This construction is functorial in the vector space V (details omitted). Hence δ is k-linear by an application of Lemma 90.11.5.

Having constructed these maps it is straightforward to show the sequence is exact. Injectivity of the first map comes from the fact that $f \times g : \mathcal{W} \to \mathcal{H} \times \mathcal{G}$ is faithful. If $(\beta, \gamma) \in \mathrm{Inf}_{y_0}(\mathcal{H}) \oplus \mathrm{Inf}_{z_0}(\mathcal{G})$ map to the same element of $\mathrm{Inf}_{x_0}(\mathcal{F})$ then (β, γ) defines an automorphism of $w'_0 = (k[\epsilon], y'_0, z'_0, c' \circ b')$ whence exactness at the second spot. If α as above gives the trivial deformation $(k[\epsilon], y'_0, z'_0, c' \circ \alpha \circ b')$ of w_0, then the isomorphism $w'_0 = (k[\epsilon], y'_0, z'_0, c' \circ b') \to (k[\epsilon], y'_0, z'_0, c' \circ \alpha \circ b')$ produces a pair (β, γ) which is a preimage of α. If $w = (k[\epsilon], y, z, \phi)$ is a deformation of w_0 such that $y'_0 \cong y$ and $z \cong z'_0$ then the map

$$f(y'_0) \to f(y) \xrightarrow{\phi} g(z) \to g(z'_0)$$

is an α which maps to w under δ. Finally, if y and z are deformations of y_0 and z_0 and there exists an isomorphism $\phi : f(y) \to g(z)$ of deformations of $f(y_0) = x_0 = g(z_0)$ then we get a preimage $w = (k[\epsilon], y, z, \phi)$ of (x, y) in $T\mathcal{W}$. This finishes the proof. $\square$

0DYN Lemma 90.20.2. Let $\mathcal{H}_1 \to \mathcal{G}$, $\mathcal{H}_2 \to \mathcal{G}$, and $\mathcal{G} \to \mathcal{F}$ be maps of categories cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume

(1) $\mathcal{F}$ and $\mathcal{G}$ are deformation categories,
(2) $T\mathcal{G} \to T\mathcal{F}$ is injective, and
(3) $\mathrm{Inf}(\mathcal{G}) \to \mathrm{Inf}(\mathcal{F})$ is surjective.

Then $\mathcal{H}_1 \times_\mathcal{G} \mathcal{H}_2 \to \mathcal{H}_1 \times_\mathcal{F} \mathcal{H}_2$ is smooth.

Proof. Denote $p_i : \mathcal{H}_i \to \mathcal{G}$ and $q : \mathcal{G} \to \mathcal{F}$ be the given maps. Let $A' \to A$ be a small extension in $\mathcal{C}_\Lambda$. An object of $\mathcal{H}_1 \times_\mathcal{F} \mathcal{H}_2$ over A' is a triple (x'_1, x'_2, a') where x'_i is an object of $\mathcal{H}_i$ over A' and $a' : q(p_1(x'_1)) \to q(p_2(x'_2))$ is a morphism of the fibre category of $\mathcal{F}$ over A'. By pushforward along $A' \to A$ we get (x_1, x_2, a). Lifting this to an object of $\mathcal{H}_1 \times_\mathcal{G} \mathcal{H}_2$ over A means finding a morphism $b : p_1(x_1) \to p_2(x_2)$ over A with $q(b) = a$. Thus we have to show that we can lift b to a morphism $b' : p_1(x'_1) \to p_2(x'_2)$ whose image under q is a'.

Observe that we can think of

$$p_1(x'_1) \to p_1(x_1) \xrightarrow{b} p_2(x_2) \quad \text{and} \quad p_2(x'_2) \to p_2(x_2)$$

as two objects of $\mathrm{Lift}(p_2(x_2), A' \to A)$. The functor q sends these objects to the two objects

$$q(p_1(x'_1)) \to q(p_1(x_1)) \xrightarrow{b} q(p_2(x_2)) \quad \text{and} \quad q(p_2(x'_2)) \to q(p_2(x_2))$$

of $\mathrm{Lift}(q(p_2(x_2)), A' \to A)$ which are isomorphic using the map $a' : q(p_1(x'_1)) \to q(p_2(x'_2))$. On the other hand, the functor

$$q : \mathrm{Lift}(p_2(x_2), A' \to A) \to \mathrm{Lift}(q(p_2(x_2)), A' \to A)$$

defines a injection on isomorphism classes by Lemma 90.17.5 and our assumption on tangent spaces. Thus we see that there is a morphism $b' : p_1(x'_1) \to p_2(x'_2)$ whose pushforward to A is b. However, we may need to adjust our choice of b' to achieve $q(b') = a'$. For this it suffices to see that $q : \text{Inf}(p_2(x'_2)/p_2(x_2)) \to \text{Inf}(q(p_2(x'_2))/q(p_2(x_2)))$ is surjective. This follows from our assumption on infinitesimal automorphisms and Lemma 90.19.11. $\square$

0DYP Lemma 90.20.3. Let $f : \mathcal{F} \to \mathcal{G}$ be a map of deformation categories. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$ with image $y_0 \in \text{Ob}(\mathcal{G}(k))$. If

(1) the map $T\mathcal{F} \to T\mathcal{G}$ is surjective, and
(2) for every small extension $A' \to A$ in $\mathcal{C}_\Lambda$ and $x \in \mathcal{F}(A)$ with image $y \in \mathcal{G}(A)$ if there is a lift of y to A', then there is a lift of x to A',

then $\mathcal{F} \to \mathcal{G}$ is smooth (and vice versa).

Proof. Let $A' \to A$ be a small extension. Let $x \in \mathcal{F}(A)$. Let $y' \to f(x)$ be a morphism in $\mathcal{G}$ over $A' \to A$. Consider the functor $\text{Lift}(A', x) \to \text{Lift}(A', f(x))$ induced by f. We have to show that there exists an object $x' \to x$ of $\text{Lift}(A', x)$ mapping to $y' \to f(x)$, see Lemma 90.8.2. By condition (2) we know that $\text{Lift}(A', x)$ is not the empty category. By condition (2) and Lemma 90.17.5 we conlude that the map on isomorphism classes is surjective as desired. $\square$

0E3R Lemma 90.20.4. Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be maps of categories cofibred in groupoids over $\mathcal{C}_\Lambda$. If

(1) $\mathcal{F}$, $\mathcal{G}$ are deformation categories
(2) the map $T\mathcal{F} \to T\mathcal{G}$ is surjective, and
(3) $\mathcal{F} \to \mathcal{H}$ is smooth.

Then $\mathcal{F} \to \mathcal{G}$ is smooth.

Proof. Let $A' \to A$ be a small extension in $\mathcal{C}_\Lambda$ and let $x \in \mathcal{F}(A)$ with image $y \in \mathcal{G}(A)$. Assume there is a lift $y' \in \mathcal{G}(A')$. According to Lemma 90.20.3 all we have to do is check that x has a lift too. Take the image $z' \in \mathcal{H}(A')$ of y'. Since $\mathcal{F} \to \mathcal{H}$ is smooth, there is an $x' \in \mathcal{F}(A')$ mapping to both $x \in \mathcal{F}(A)$ and $z' \in \mathcal{H}(A')$, see Definition 90.8.1. This finishes the proof. $\square$

90.21. Groupoids in functors on an arbitrary category

06K2 We begin with generalities on groupoids in functors on an arbitrary category. In the next section we will pass to the category $\mathcal{C}_\Lambda$. For clarity we shall sometimes refer to an ordinary groupoid, i.e., a category whose morphisms are all isomorphisms, as a groupoid category.

06K3 Definition 90.21.1. Let $\mathcal{C}$ be a category. The category of groupoids in functors on $\mathcal{C}$ is the category with the following objects and morphisms.

(1) Objects: A groupoid in functors on $\mathcal{C}$ is a quintuple (U, R, s, t, c) where $U, R : \mathcal{C} \to \textit{Sets}$ are functors and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms with the following property: For any object T of $\mathcal{C}$, the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

(2) Morphisms: A morphism $(U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in functors on $\mathcal{C}$ consists of morphisms $U \to U'$ and $R \to R'$ with the following property: For any object T of $\mathcal{C}$, the induced maps $U(T) \to U'(T)$ and $R(T) \to R'(T)$ define a functor between groupoid categories
$$(U(T), R(T), s, t, c) \to (U'(T), R'(T), s', t', c').$$

06K4 Remark 90.21.2. A groupoid in functors on $\mathcal{C}$ amounts to the data of a functor $\mathcal{C} \to \text{Groupoids}$, and a morphism of groupoids in functors on $\mathcal{C}$ amounts to a morphism of the corresponding functors $\mathcal{C} \to \text{Groupoids}$ (where Groupoids is regarded as a 1-category). However, for our purposes it is more convenient to use the terminology of groupoids in functors. In fact, thinking of a groupoid in functors as the corresponding functor $\mathcal{C} \to \text{Groupoids}$, or equivalently as the category cofibered in groupoids associated to that functor, can lead to confusion (Remark 90.23.2).

06K5 Remark 90.21.3. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. There are unique morphisms $e : U \to R$ and $i : R \to R$ such that for every object T of $\mathcal{C}$, $e : U(T) \to R(T)$ sends $x \in U(T)$ to the identity morphism on x and $i : R(T) \to R(T)$ sends $a \in U(T)$ to the inverse of a in the groupoid category $(U(T), R(T), s, t, c)$. We will sometimes refer to s, t, c, e, and i as "source", "target", "composition", "identity", and "inverse".

06K6 Definition 90.21.4. Let $\mathcal{C}$ be a category. A groupoid in functors on $\mathcal{C}$ is representable if it is isomorphic to one of the form $(\underline{U}, \underline{R}, s, t, c)$ where U and R are objects of $\mathcal{C}$ and the pushout $R \amalg_{s,U,t} R$ exists.

06K7 Remark 90.21.5. Hence a representable groupoid in functors on $\mathcal{C}$ is given by objects U and R of $\mathcal{C}$ and morphisms $s, t : U \to R$ and $c : R \to R \amalg_{s,U,t} R$ such that $(\underline{U}, \underline{R}, s, t, c)$ satisfies the condition of Definition 90.21.1. The reason for requiring the existence of the pushout $R \amalg_{s,U,t} R$ is so that the composition morphism c is defined at the level of morphisms in $\mathcal{C}$. This requirement will always be satisfied below when we consider representable groupoids in functors on $\widehat{\mathcal{C}}_\Lambda$, since by Lemma 90.4.3 the category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.

06K8 Remark 90.21.6. We will say "let $(\underline{U}, \underline{R}, s, t, c)$ be a groupoid in functors on $\mathcal{C}$" to mean that we have a representable groupoid in functors. Thus this means that U and R are objects of $\mathcal{C}$, there are morphisms $s, t : U \to R$, the pushout $R \amalg_{s,U,t} R$ exists, there is a morphism $c : R \to R \amalg_{s,U,t} R$, and $(\underline{U}, \underline{R}, s, t, c)$ is a groupoid in functors on $\mathcal{C}$.

We introduce notation for restriction of groupoids in functors. This will be relevant below in situations where we restrict from $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$.

06K9 Definition 90.21.7. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. The restriction $(U, R, s, t, c)|_{\mathcal{C}'}$ of (U, R, s, t, c) to $\mathcal{C}'$ is the groupoid in functors on $\mathcal{C}'$ given by $(U|_{\mathcal{C}'}, R|_{\mathcal{C}'}, s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'})$.

06KA Remark 90.21.8. In the situation of Definition 90.21.7, we often denote $s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'}$ simply by s, t, c.

06KB Definition 90.21.9. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$.

(1) The assignment $T \mapsto (U(T), R(T), s, t, c)$ determines a functor $\mathcal{C} \to \text{Groupoids}$. The quotient category cofibered in groupoids $[U/R] \to \mathcal{C}$ is the category cofibered in groupoids over $\mathcal{C}$ associated to this functor (as in Remarks 90.5.2 (9)).

(2) The quotient morphism $U \to [U/R]$ is the morphism of categories cofibered in groupoids over $\mathcal{C}$ defined by the rules
   (a) $x \in U(T)$ maps to the object $(T, x) \in \mathrm{Ob}([U/R](T))$, and
   (b) $x \in U(T)$ and $f : T \to T'$ give rise to the morphism $(f, \mathrm{id}_{U(f)(x)}) : (T, x) \to (T, U(f)(x))$ lying over $f : T \to T'$.

90.22. Groupoids in functors on the base category

06KC In this section we discuss groupoids in functors on $\mathcal{C}_\Lambda$. Our eventual goal is to show that prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$ serve as "presentations" for well-behaved deformation categories in the same way that smooth groupoids in algebraic spaces serve as presentations for algebraic stacks, cf. Algebraic Stacks, Section 94.16.

06KD Definition 90.22.1. A groupoid in functors on $\mathcal{C}_\Lambda$ is prorepresentable if it is isomorphic to $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ for some representable groupoid in functors $(\underline{R_0}, \underline{R_1}, s, t, c)$ on the category $\widehat{\mathcal{C}}_\Lambda$.

Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Taking completions, we get a quintuple $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$. By Remark 90.7.10 completion as a functor on $\mathrm{CofSet}(\mathcal{C}_\Lambda)$ is a right adjoint, so it commutes with limits. In particular, there is a canonical isomorphism

$$R \widehat{\times_{s,U,t}} R \longrightarrow \widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R},$$

so $\widehat{c}$ can be regarded as a functor $\widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R} \to \widehat{R}$. Then $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ is a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$, with identity and inverse morphisms being the completions of those of (U, R, s, t, c).

06KE Definition 90.22.2. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The completion $(U, R, s, t, c)^\wedge$ of (U, R, s, t, c) is the groupoid in functors $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ on $\widehat{\mathcal{C}}_\Lambda$ described above.

06KF Remark 90.22.3. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Then there is a canonical isomorphism $(U, R, s, t, c)^\wedge|_{\mathcal{C}_\Lambda} \cong (U, R, s, t, c)$, see Remark 90.7.7. On the other hand, let (U, R, s, t, c) be a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$ such that $U, R : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ both commute with limits, e.g. if U, R are representable. Then there is a canonical isomorphism $((U, R, s, t, c)|_{\mathcal{C}_\Lambda})^\wedge \cong (U, R, s, t, c)$. This follows from Remark 90.7.11.

06KG Lemma 90.22.4. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$.
(1) (U, R, s, t, c) is prorepresentable if and only if its completion is representable as a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$.
(2) (U, R, s, t, c) is prorepresentable if and only if U and R are prorepresentable.

Proof. Part (1) follows from Remark 90.22.3. For (2), the "only if" direction is clear from the definition of a prorepresentable groupoid in functors. Conversely, assume U and R are prorepresentable, say $U \cong \underline{R_0}|_{\mathcal{C}_\Lambda}$ and $R \cong \underline{R_1}|_{\mathcal{C}_\Lambda}$ for objects R_0 and R_1 of $\widehat{\mathcal{C}}_\Lambda$. Since $\underline{R_0} \cong \widehat{\underline{R_0}|_{\mathcal{C}_\Lambda}}$ and $\underline{R_1} \cong \widehat{\underline{R_1}|_{\mathcal{C}_\Lambda}}$ by Remark 90.7.11 we see that the completion $(U, R, s, t, c)^\wedge$ is a groupoid in functors of the form $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$. By Lemma 90.4.3 the pushout $\underline{R_1} \times_{\widehat{s},\underline{R_1},\widehat{t}} \underline{R_1}$ exists. Hence $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$ is a representable groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$. Finally, the restriction $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ gives back (U, R, s, t, c) by Remark 90.22.3 hence (U, R, s, t, c) is prorepresentable by definition. □

90.23. Smooth groupoids in functors on the base category

06KH The notion of smoothness for groupoids in functors on $\mathcal{C}_\Lambda$ is defined as follows.

06KI Definition 90.23.1. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. We say (U, R, s, t, c) is smooth if $s, t : R \to U$ are smooth.

06KJ Remark 90.23.2. We note that this terminology is potentially confusing: if (U, R, s, t, c) is a smooth groupoid in functors, then the quotient $[U/R]$ need not be a smooth category cofibred in groupoids as defined in Definition 90.9.1. However smoothness of (U, R, s, t, c) does imply (in fact is equivalent to) smoothness of the quotient morphism $U \to [U/R]$ as we shall see in Lemma 90.23.4.

06KK Remark 90.23.3. Let $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ be a prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Then $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ is smooth if and only if R_1 is a power series over R_0 via both s and t. This follows from Lemma 90.8.6.

06KL Lemma 90.23.4. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The following are equivalent:

(1) The groupoid in functors (U, R, s, t, c) is smooth.
(2) The morphism $s : R \to U$ is smooth.
(3) The morphism $t : R \to U$ is smooth.
(4) The quotient morphism $U \to [U/R]$ is smooth.

Proof. Statement (2) is equivalent to (3) since the inverse $i : R \to R$ of (U, R, s, t, c) is an isomorphism and $t = s \circ i$. By definition (1) is equivalent to (2) and (3) together, hence it is equivalent to either of them individually.

Finally we prove (2) is equivalent to (4). Unwinding the definitions:

(2) Smoothness of $s : R \to U$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$, $a \in R(A)$, and $y \in U(B)$ such that $s(a) = U(f)(y)$, then there exists $a' \in R(B)$ such that $R(f)(a') = a$ and $s(a') = y$.
(4) Smoothness of $U \to [U/R]$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$ and $(f, a) : (B, y) \to (A, x)$ is a morphism of $[U/R]$, then there exists $x' \in U(B)$ and $b \in R(B)$ with $s(b) = x'$, $t(b) = y$ such that $c(a, R(f)(b)) = e(x)$. Here $e : U \to R$ denotes the identity and the notation (f, a) is as in Remarks 90.5.2 (9); in particular $a \in R(A)$ with $s(a) = U(f)(y)$ and $t(a) = x$.

If (4) holds and f, a, y as in (2) are given, let $x = t(a)$ so that we have a morphism $(f, a) : (B, y) \to (A, x)$. Then (4) produces x' and b, and $a' = i(b)$ satisfies the requirements of (2). Conversely, assume (2) holds and let $(f, a) : (B, y) \to (A, x)$ as in (4) be given. Then (2) produces $a' \in R(B)$, and $x' = t(a')$ and $b = i(a')$ satisfy the requirements of (4). □

90.24. Deformation categories as quotients of groupoids in functors

06KS

We discuss conditions on a groupoid in functors on $\mathcal{C}_\Lambda$ which guarantee that the quotient is a deformation category, and we calculate the tangent and infinitesimal automorphism spaces of such a quotient.

06KT Lemma 90.24.1. Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R satisfy (RS). Then $[U/R]$ satisfies (RS).

Proof. Let

$$\begin{tikzcd} & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

be a diagram in $[U/R]$ such that $f_2 : A_2 \to A$ is surjective. The notation is as in Remarks 90.5.2 (9). Hence $f_1 : A_1 \to A$, $f_2 : A_2 \to A$ are maps in $\mathcal{C}_\Lambda$, $x \in U(A)$, $x_1 \in U(A_1)$, $x_2 \in U(A_2)$, and $a_1, a_2 \in R(A)$ with $s(a_1) = U(f_1)(x_1)$, $t(a_1) = x$ and $s(a_2) = U(f_2)(x_2)$, $t(a_2) = x$. We construct a fiber product lying over $A_1 \times_A A_2$ for this diagram in $[U/R]$ as follows.

Let $a = c(i(a_1), a_2)$, where $i : R \to R$ is the inverse morphism. Then $a \in R(A)$, $x_2 \in U(A_2)$ and $s(a) = U(f_2)(x_2)$. Hence an element $(a, x_2) \in R(A) \times_{s, U(A), U(f_2)} U(A_2)$. By smoothness of $s : R \to U$ there is an element $\widetilde{a} \in R(A_2)$ with $R(f_2)(\widetilde{a}) = a$ and $s(\widetilde{a}) = x_2$. In particular $U(f_2)(t(\widetilde{a})) = t(a) = U(f_1)(x_1)$. Thus x_1 and $t(\widetilde{a})$ define an element

$$(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2).$$

By the assumption that U satisfies (RS), we have an identification $U(A_1) \times_{U(A)} U(A_2) = U(A_1 \times_A A_2)$. Let us denote $x_1 \times t(\widetilde{a}) \in U(A_1 \times_A A_2)$ the element corresponding to $(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2)$. Let p_1, p_2 be the projections of $A_1 \times_A A_2$. We claim

$$\begin{tikzcd} (A_1 \times_A A_2, x_1 \times t(\widetilde{a})) \arrow[r, "(p_2, i(\widetilde{a}))"'] \arrow[d, "(p_1, e(x_1))"'] & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

is a fiber square in $[U/R]$. (Note $e : U \to R$ denotes the identity.)

The diagram is commutative because $c(a_2, R(f_2)(i(\widetilde{a}))) = c(a_2, i(a)) = a_1$. To check it is a fiber square, let

$$\begin{tikzcd} (B, z) \arrow[r, "(g_2, b_2)"'] \arrow[d, "(g_1, b_1)"'] & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

be a commutative diagram in $[U/R]$. We will show there is a unique morphism $(g, b) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ compatible with the morphisms to (A_1, x_1) and (A_2, x_2). We must take $g = (g_1, g_2) : B \to A_1 \times_A A_2$. Since by assumption R satisfies (RS), we have an identification $R(A_1 \times_A A_2) = R(A_1) \times_{R(A)} R(A_2)$. Hence we can write $b = (b'_1, b'_2)$ for some $b'_1 \in R(A_1)$, $b'_2 \in R(A_2)$ which agree in $R(A)$. Then $((g_1, g_2), (b'_1, b'_2)) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ will commute with the projections if and only if $b'_1 = b_1$ and $b'_2 = c(\widetilde{a}, b_2)$ proving unicity and existence. □

06KU Lemma 90.24.2. Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R are deformation functors. Then:

(1) The quotient $[U/R]$ is a deformation category.
(2) The tangent space of $[U/R]$ is
$$T[U/R] = \text{Coker}(ds - dt : TR \to TU).$$
(3) The space of infinitesimal automorphisms of $[U/R]$ is
$$\text{Inf}([U/R]) = \text{Ker}(ds \oplus dt : TR \to TU \oplus TU).$$

Proof. Since U and R are deformation functors $[U/R]$ is a predeformation category. Since (RS) holds for deformation functors by definition we see that (RS) holds for [U/R] by Lemma 90.24.1. Hence $[U/R]$ is a deformation category. Statements (2) and (3) follow directly from the definitions. □

90.25. Presentations of categories cofibered in groupoids

06KW A presentation is defined as follows.

06KX Definition 90.25.1. Let $\mathcal{F}$ be a category cofibered in groupoids over a category $\mathcal{C}$. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}$. A presentation of $\mathcal{F}$ by (U, R, s, t, c) is an equivalence $\varphi : [U/R] \to \mathcal{F}$ of categories cofibered in groupoids over $\mathcal{C}$.

The following two general lemmas will be used to get presentations.

06KY Lemma 90.25.2. Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to \textit{Sets}$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Define R, s, t, c as follows:

(1) $R : \mathcal{C} \to \textit{Sets}$ is the functor $U \times_{f, \mathcal{F}, f} U$.
(2) $t, s : R \to U$ are the first and second projections, respectively.
(3) $c : R \times_{s, U, t} R \to R$ is the morphism given by projection onto the first and last factors of $U \times_{f, \mathcal{F}, f} U \times_{f, \mathcal{F}, f} U$ under the canonical isomorphism $R \times_{s, U, t} R \to U \times_{f, \mathcal{F}, f} U \times_{f, \mathcal{F}, f} U$.

Then (U, R, s, t, c) is a groupoid in functors on $\mathcal{C}$.

Proof. Omitted. □

06KZ Lemma 90.25.3. Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to \textit{Sets}$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2. Then there is a natural morphism $[f] : [U/R] \to \mathcal{F}$ such that:

(1) $[f] : [U/R] \to \mathcal{F}$ is fully faithful.
(2) $[f] : [U/R] \to \mathcal{F}$ is an equivalence if and only if $f : U \to \mathcal{F}$ is essentially surjective.

Proof. Omitted. □

90.26. Presentations of deformation categories

06L0 According to the next lemma, a smooth morphism from a predeformation functor to a predeformation category $\mathcal{F}$ gives rise to a presentation of $\mathcal{F}$ by a smooth groupoid in functors.

06L1 **Lemma 90.26.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $U : \mathcal{C}_\Lambda \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a smooth morphism of categories cofibered in groupoids. Then:

(1) If (U, R, s, t, c) is the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2, then (U, R, s, t, c) is smooth.
(2) If $f : U(k) \to \mathcal{F}(k)$ is essentially surjective, then the morphism $[f] : [U/R] \to \mathcal{F}$ of Lemma 90.25.3 is an equivalence.

Proof. From the construction of Lemma 90.25.2 we have a commutative diagram

$$\begin{tikzcd} R = U \times_{f,\mathcal{F},f} U \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d, "f"] \\ U \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

where t, s are the first and second projections. So t, s are smooth by Lemma 90.8.7. Hence (1) holds.

If the assumption of (2) holds, then by Lemma 90.8.8 the morphism $f : U \to \mathcal{F}$ is essentially surjective. Hence by Lemma 90.25.3 the morphism $[f] : [U/R] \to \mathcal{F}$ is an equivalence. □

06L6 **Lemma 90.26.2.** Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a deformation functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Then $U \times_{f,\mathcal{F},f} U$ is a deformation functor with tangent space fitting into an exact sequence of k-vector spaces

$$0 \to \text{Inf}(\mathcal{F}) \to T(U \times_{f,\mathcal{F},f} U) \to TU \oplus TU$$

Proof. Follows from Lemma 90.20.1 and the fact that $\text{Inf}(U) = (0)$. □

06L7 **Lemma 90.26.3.** Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2. If $\dim_k \text{Inf}(\mathcal{F}) < \infty$, then (U, R, s, t, c) is prorepresentable.

Proof. Note that U is a deformation functor by Example 90.16.10. By Lemma 90.26.2 we see that $R = U \times_{f,\mathcal{F},f} U$ is a deformation functor whose tangent space $TR = T(U \times_{f,\mathcal{F},f} U)$ sits in an exact sequence $0 \to \text{Inf}(\mathcal{F}) \to TR \to TU \oplus TU$. Since we have assumed the first space has finite dimension and since TU has finite dimension by Example 90.11.11 we see that $\dim TR < \infty$. The map $\gamma : \text{Der}_\Lambda(k, k) \to TR$ see (90.12.6.1) is injective because its composition with $TR \to TU$ is injective by Theorem 90.18.2 for the prorepresentable functor U. Thus R is prorepresentable by Theorem 90.18.2. It follows from Lemma 90.22.4 that (U, R, s, t, c) is prorepresentable. □

06L8 **Theorem 90.26.4.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ if and only if the following conditions hold:

(1) $\mathcal{F}$ is a deformation category.
(2) $\dim_k T\mathcal{F}$ is finite.
(3) $\dim_k \text{Inf}(\mathcal{F})$ is finite.

Proof. Recall that a prorepresentable functor is a deformation functor, see Example 90.16.10. Thus if $\mathcal{F}$ is equivalent to a smooth prorepresentable groupoid in functors, then conditions (1), (2), and (3) follow from Lemma 90.24.2 (1), (2), and (3).

Conversely, assume conditions (1), (2), and (3) hold. Condition (1) implies that (S1) and (S2) are satisfied, see Lemma 90.16.6. By Lemma 90.13.4 there exists a versal formal object ξ. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 90.25.2 from the map $\underline{\xi}$. By Lemma 90.26.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 90.26.3 we see that (U, R, s, t, c) is prorepresentable. Hence $[U/R] \to \mathcal{F}$ is the desired presentation of $\mathcal{F}$. □

90.27. Remarks regarding minimality

06TD The main theorem of this chapter is Theorem 90.26.4 above. It describes completely those categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a presentation by a smooth prorepresentable groupoid in functors. In this section we briefly discuss how the minimality discussed in Sections 90.14 and 90.15 can be used to obtain a "minimal" smooth prorepresentable presentation.

06KM Definition 90.27.1. Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.

(1) We say (U, R, s, t, c) is normalized if the groupoid $(U(k[\epsilon]), R(k[\epsilon]), s, t, c)$ is totally disconnected, i.e., there are no morphisms between distinct objects.
(2) We say (U, R, s, t, c) is minimal if the $U \to [U/R]$ is given by a minimal versal formal object of $[U/R]$.

The difference between the two notions is related to the difference between conditions (90.15.0.1) and (90.15.0.2) and disappears when $k' \subset k$ is separable. Also a normalized smooth prorepresentable groupoid in functors is minimal as the following lemma shows. Here is a precise statement.

06KN Lemma 90.27.2. Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.

(1) (U, R, s, t, c) is normalized if and only if the morphism $U \to [U/R]$ induces an isomorphism on tangent spaces, and
(2) (U, R, s, t, c) is minimal if and only if the kernel of $TU \to T[U/R]$ is contained in the image of $\text{Der}_\Lambda(k, k) \to TU$.

Proof. Part (1) follows immediately from the definitions. To see part (2) set $\mathcal{F} = [U/R]$. Since $\mathcal{F}$ has a presentation it is a deformation category, see Theorem 90.26.4. In particular it satisfies (RS), (S1), and (S2), see Lemma 90.16.6. Recall that minimal versal formal objects are unique up to isomorphism, see Lemma 90.14.5. By Theorem 90.15.5 a minimal versal object induces a map $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ satisfying (90.15.0.2). Since $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ over $\mathcal{F}$ we see that $TU \to T\mathcal{F} = T[U/R]$ satisfies the property as stated in the lemma. □

The quotient of a minimal prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ does not admit autoequivalences which are not automorphisms. To prove this, we first note the following lemma.

06KP Lemma 90.27.3. Let $U : \mathcal{C}_\Lambda \to \textit{Sets}$ be a prorepresentable functor. Let $\varphi : U \to U$ be a morphism such that $d\varphi : TU \to TU$ is an isomorphism. Then φ is an isomorphism.

Proof. If $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ for some $R \in \text{Ob}(\widehat{\mathcal{C}}_\Lambda)$, then completing φ gives a morphism $\underline{R} \to \underline{R}$. If $f : R \to R$ is the corresponding morphism in $\widehat{\mathcal{C}}_\Lambda$, then f induces an isomorphism $\text{Der}_\Lambda(R, k) \to \text{Der}_\Lambda(R, k)$, see Example 90.11.14. In particular f is a surjection by Lemma 90.4.6. As a surjective endomorphism of a Noetherian ring is an isomorphism (see Algebra, Lemma 10.31.10) we conclude f, hence $\underline{R} \to \underline{R}$, hence $\varphi : U \to U$ is an isomorphism. $\square$

06KQ Lemma 90.27.4. Let (U, R, s, t, c) be a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U/R]$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.

Proof. A morphism $\varphi : [U/R] \to [U/R]$ is the same thing as a morphism $\varphi : (U, R, s, t, c) \to (U, R, s, t, c)$ of groupoids in functors over $\mathcal{C}_\Lambda$ as defined in Definition 90.21.1. Denote $\phi : U \to U$ and $\psi : R \to R$ the corresponding morphisms. Because the diagram

$$\begin{tikzcd} & \text{Der}_\Lambda(k,k) \arrow[dl, "\gamma"'] \arrow[dr, "\gamma"] & \\ TU \arrow[rr, "d\phi"'] \arrow[d] & & TU \arrow[d] \\ T[U/R] \arrow[rr, "d\varphi"] & & T[U/R] \end{tikzcd}$$

is commutative, since $d\varphi$ is bijective, and since we have the characterization of minimality in Lemma 90.27.2 we conclude that $d\phi$ is injective (hence bijective by dimension reasons). Thus $\phi : U \to U$ is an isomorphism by Lemma 90.27.3. We can use a similar argument, using the exact sequence

$$0 \to \text{Inf}([U/R]) \to TR \to TU \oplus TU$$

of Lemma 90.26.2 to prove that $\psi : R \to R$ is an isomorphism. But is also a consequence of the fact that $R = U \times_{[U/R]} U$ and that φ and ϕ are isomorphisms. $\square$

06KR Lemma 90.27.5. Let (U, R, s, t, c) and (U', R', s', t', c') be minimal smooth prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U'/R']$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.

Proof. Let $\psi : [U'/R'] \to [U/R]$ be a quasi-inverse to φ. Then $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms by Lemma 90.27.4, hence φ and ψ are isomorphisms. $\square$

The following lemma summarizes some of the things we have seen earlier in this chapter.

06L2 Lemma 90.27.6. Let $\mathcal{F}$ be a deformation category such that $\dim_k T\mathcal{F} < \infty$ and $\dim_k \text{Inf}(\mathcal{F}) < \infty$. Then there exists a minimal versal formal object ξ of $\mathcal{F}$. Say ξ lies over $R \in \text{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Let $U = \underline{R}|_{\mathcal{C}_\Lambda}$. Let $f = \underline{\xi} : U \to \mathcal{F}$ be the associated morphism. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2. Then (U, R, s, t, c) is a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ and there is an equivalence $[U/R] \to \mathcal{F}$.

Proof. As $\mathcal{F}$ is a deformation category it satisfies (S1) and (S2), see Lemma 90.16.6. By Lemma 90.13.4 there exists a versal formal object. By Lemma 90.14.5 there exists a minimal versal formal object ξ/R as in the statement of the lemma. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 90.25.2 from the map $\underline{\xi}$. By Lemma 90.26.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 90.26.3 we see that (U, R, s, t, c) is prorepresentable. Finally, (U, R, s, t, c) is minimal because $U \to [U/R] = \mathcal{F}$ corresponds to the minimal versal formal object ξ. □

Presentations by minimal prorepresentable groupoids in functors satisfy the following uniqueness property.

06L3 Lemma 90.27.7. Let $\mathcal{F}$ be category cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume there exist presentations of $\mathcal{F}$ by minimal smooth prorepresentable groupoids in functors (U, R, s, t, c) and (U', R', s', t', c'). Then (U, R, s, t, c) and (U', R', s', t', c') are isomorphic.

Proof. Follows from Lemma 90.27.5 and the observation that a morphism $[U/R] \to [U'/R']$ is the same thing as a morphism of groupoids in functors (by our explicit construction of $[U/R]$ in Definition 90.21.9). □

In summary we have proved the following theorem.

06TE Theorem 90.27.8. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Consider the following conditions

1. (1) $\mathcal{F}$ admits a presentation by a normalized smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,
2. (2) $\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,
3. (3) $\mathcal{F}$ admits a presentation by a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, and
4. (4) $\mathcal{F}$ satisfies the following conditions
   - (a) $\mathcal{F}$ is a deformation category.
   - (b) $\dim_k T\mathcal{F}$ is finite.
   - (c) $\dim_k \text{Inf}(\mathcal{F})$ is finite.

Then (2), (3), (4) are equivalent and are implied by (1). If $k' \subset k$ is separable, then (1), (2), (3), (4) are all equivalent. Furthermore, the minimal smooth prorepresentable groupoids in functors which provide a presentation of $\mathcal{F}$ are unique up to isomorphism.

Proof. We see that (1) implies (3) and is equivalent to (3) if $k' \subset k$ is separable from Lemma 90.27.2. It is clear that (3) implies (2). We see that (2) implies (4) by Theorem 90.26.4. We see that (4) implies (3) by Lemma 90.27.6. This proves all the implications. The final uniqueness statement follows from Lemma 90.27.7. □

90.28. Uniqueness of versal rings

0DQA Given R, S in $\widehat{\mathcal{C}}_\Lambda$ we say maps $f, g : R \to S$ are formally homotopic if there exists an $r \geq 0$ and maps $h : R \to R[[t_1, \ldots, t_r]]$ and $k : R[[t_1, \ldots, t_r]] \to S$ in $\widehat{\mathcal{C}}_\Lambda$ such that for all $a \in R$ we have

1. $h(a) \bmod (t_1, \ldots, t_r) = a$,
2. $f(a) = k(a)$,
3. $g(a) = k(h(a))$.

We will say (r, h, k) is a formal homotopy between f and g.

0DQB **Lemma 90.28.1.** Being formally homotopic is an equivalence relation on sets of morphisms in $\widehat{\mathcal{C}}_\Lambda$.

Proof. Suppose we have any $r \geq 1$ and two maps $h_1, h_2 : R \to R[[t_1, \ldots, t_r]]$ such that $h_1(a) \bmod (t_1, \ldots, t_r) = h_2(a) \bmod (t_1, \ldots, t_r) = a$ for all $a \in R$ and a map $k : R[[t_1, \ldots, t_r]] \to S$. Then we claim $k \circ h_1$ is formally homotopic to $k \circ h_2$. The symmetric inherent in this claim will show that our notion of formally homotopic is symmetric. Namely, the map

$$\Psi : R[[t_1, \ldots, t_r]] \longrightarrow R[[t_1, \ldots, t_r]], \quad \sum a_I t^I \longmapsto \sum h_1(a_I) t^I$$

is an isomorphism. Set $h(a) = \Psi^{-1}(h_2(a))$ for $a \in R$ and $k' = k \circ \Psi$, then we see that (r, h, k') is a formal homotopy between $k \circ h_1$ and $k \circ h_2$, proving the claim

Say we have three maps $f_1, f_2, f_3 : R \to S$ as above and a formal homotopy (r_1, h_1, k_1) between f_1 and f_2 and a formal homotopy (r_2, h_2, k_2) between f_3 and f_2 (!). After relabeling the coordinates we may assume $h_2 : R \to R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]]$ and $k_2 : R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \to S$. By choosing a suitable isomorphism

$$R[[t_1, \ldots, t_{r_1+r_2}]] \longrightarrow R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \widehat{\otimes}_{h_2, R, h_1} R[[t_1, \ldots, t_{r_1}]]$$

we may fit these maps into a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "h_1"'] \arrow[d, "h_2"] & R[[t_1, \ldots, t_{r_1}]] \arrow[d, "h_2'"] \\
R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \arrow[r, "h_1'"] & R[[t_1, \ldots, t_{r_1+r_2}]]
\end{tikzcd}$$

with $h_2'(t_i) = t_i$ for $1 \leq i \leq r_1$ and $h_1'(t_i) = t_i$ for $r_1 + 1 \leq i \leq r_2$. Some details omitted. Since this diagram is a pushout in the category $\widehat{\mathcal{C}}_\Lambda$ (see proof of Lemma 90.4.3) and since $k_1 \circ h_1 = f_2 = k_2 \circ h_2$ we conclude there exists a map

$$k : R[[t_1, \ldots, t_{r_1+r_2}]] \to S$$

with $k_1 = k \circ h_2'$ and $k_2 = k \circ h_1'$. Denote $h = h_1' \circ h_2 = h_2' \circ h_1$. Then we have

1. $k(h_1'(a)) = k_2(a) = f_3(a)$, and
2. $k(h_2'(a)) = k_1(a) = f_1(a)$.

By the claim in the first paragraph of the proof this shows that f_1 and f_3 are formally homotopic. $\square$

0DQC **Lemma 90.28.2.** In the category $\widehat{\mathcal{C}}_\Lambda$, if $f_1, f_2 : R \to S$ are formally homotopic and $g : S \to S'$ is a morphism, then $g \circ f_1$ and $g \circ f_2$ are formally homotopic.

Proof. Namely, if (r, h, k) is a formal homotopy between f_1 and f_2, then $(r, h, g \circ k)$ is a formal homotopy between $g \circ f_1$ and $g \circ f_2$. $\square$

0DQD **Lemma 90.28.3.** Let $\mathcal{F}$ be a deformation category over $\mathcal{C}_\Lambda$ with $\dim_k T\mathcal{F} < \infty$ and $\dim_k \text{Inf}(\mathcal{F}) < \infty$. Let ξ be a versal formal object lying over R. Let η be a formal object lying over S. Then any two maps

$$f, g : R \to S$$

such that $f_*\xi \cong \eta \cong g_*\xi$ are formally homotopic.

Proof. By Theorem 90.26.4 and its proof, $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid

$$(\underline{R}, \underline{R_1}, s, t, c, e, i)|_{\mathcal{C}_\Lambda}$$

in functors on $\mathcal{C}_\lambda$ such that $\mathcal{F}$. Then the maps $s : R \to R_1$ and $t : R \to R_1$ are formally smooth ring maps and $e : R_1 \to R$ is a section. In particular, we can choose an isomorphism $R_1 = R[[t_1, \ldots, t_r]]$ for some $r \geq 0$ such that s is the embedding $R \subset R[[t_1, \ldots, t_r]]$ and t corresponds to a map $h : R \to R[[t_1, \ldots, t_r]]$ with $h(a) \bmod (t_1, \ldots, t_r) = a$ for all $a \in R$. The existence of the isomorphism $\alpha : f_*\xi \to g_*\xi$ means exactly that there is a map $k : R_1 \to S$ such that $f = k \circ s$ and $g = k \circ t$. This exactly means that (r, h, k) is a formal homotopy between f and g. $\square$

0DQE **Lemma 90.28.4.** In the category $\widehat{\mathcal{C}}_\Lambda$, if $f_1, f_2 : R \to S$ are formally homotopic and $\mathfrak{p} \subset R$ is a minimal prime ideal, then $f_1(\mathfrak{p})S = f_2(\mathfrak{p})S$ as ideals.

Proof. Suppose (r, h, k) is a formal homotopy between f_1 and f_2. We claim that $\mathfrak{p}R[[t_1, \ldots, t_r]] = h(\mathfrak{p})R[[t_1, \ldots, t_r]]$. The claim implies the lemma by further composing with k. To prove the claim, observe that the map $\mathfrak{p} \mapsto \mathfrak{p}R[[t_1, \ldots, t_r]]$ is a bijection between the minimal prime ideals of R and the minimal prime ideals of $R[[t_1, \ldots, t_r]]$. Finally, $h(\mathfrak{p})R[[t_1, \ldots, t_r]]$ is a minimal prime as h is flat, and hence of the form $\mathfrak{q}R[[t_1, \ldots, t_r]]$ for some minimal prime $\mathfrak{q} \subset R$ by what we just said. But since $h \bmod (t_1, \ldots, t_r) = \text{id}_R$ by definition of a formal homotopy, we conclude that $\mathfrak{q} = \mathfrak{p}$ as desired. $\square$

90.29. Change of residue field

07W7 In this section we quickly discuss what happens if we replace the residue field k by a finite extension. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. Throughout this whole chapter we have used $\mathcal{C}_\Lambda$ to denote the category of Artinian local Λ-algebras whose residue field is identified with k, see Definition 90.3.1. However, since in this section we will discuss what happen when we change k we will instead use the notation $\mathcal{C}_{\Lambda,k}$ to indicate the dependence on k.

07W8 **Situation 90.29.1.** Let Λ be a Noetherian ring and let $\Lambda \to k \to l$ be a finite ring maps where k and l are fields. Thus l/k is a finite extensions of fields. A typical object of $\mathcal{C}_{\Lambda,l}$ will be denoted B and a typical object of $\mathcal{C}_{\Lambda,k}$ will be denoted A. We define

07W9 (90.29.1.1) $$\mathcal{C}_{\Lambda,l} \longrightarrow \mathcal{C}_{\Lambda,k}, \quad B \longmapsto B \times_l k$$

Given a category cofibred in groupoids $p : \mathcal{F} \to \mathcal{C}_{\Lambda,k}$ we obtain an associated category cofibred in groupoids

$$p_{l/k} : \mathcal{F}_{l/k} \longrightarrow \mathcal{C}_{\Lambda,l}$$

by setting $\mathcal{F}_{l/k}(B) = \mathcal{F}(B \times_l k)$.

The functor (90.29.1.1) makes sense: because $B \times_l k \subset B$ we have

$$\begin{aligned}[k : k'] \text{length}_{B \times_l k}(B \times_l k) &= \text{length}_\Lambda(B \times_l k) \\ &\leq \text{length}_\Lambda(B) \\ &= [l : k'] \text{length}_B(B) < \infty\end{aligned}$$

(see Lemma 90.3.4) hence $B \times_l k$ is Artinian (see Algebra, Lemma 10.53.6). Thus $B \times_l k$ is an Artinian local ring with residue field k. Note that (90.29.1.1) commutes with fibre products

$$(B_1 \times_B B_2) \times_l k = (B_1 \times_l k) \times_{(B \times_l k)} (B_2 \times_l k)$$

and transforms surjective ring maps into surjective ring maps. We use the "expensive" notation $\mathcal{F}_{l/k}$ to prevent confusion with the construction of Remark 90.6.4. Here are some elementary observations.

07WA Lemma 90.29.2. With notation and assumptions as in Situation 90.29.1.

(1) We have $\overline{\mathcal{F}_{l/k}} = (\overline{\mathcal{F}})_{l/k}$.
(2) If $\mathcal{F}$ is a predeformation category, then $\mathcal{F}_{l/k}$ is a predeformation category.
(3) If $\mathcal{F}$ satisfies (S1), then $\mathcal{F}_{l/k}$ satisfies (S1).
(4) If $\mathcal{F}$ satisfies (S2), then $\mathcal{F}_{l/k}$ satisfies (S2).
(5) If $\mathcal{F}$ satisfies (RS), then $\mathcal{F}_{l/k}$ satisfies (RS).

Proof. Part (1) is immediate from the definitions.

Since $\mathcal{F}_{l/k}(l) = \mathcal{F}(k)$ part (2) follows from the definition, see Definition 90.6.2.

Part (3) follows as the functor (90.29.1.1) commutes with fibre products and transforms surjective maps into surjective maps, see Definition 90.10.1.

Part (4). To see this consider a diagram

$$\begin{tikzcd} & l[\epsilon] \arrow[d] \\ B \arrow[r] & l \end{tikzcd}$$

in $\mathcal{C}_{\Lambda,l}$ as in Definition 90.10.1. Applying the functor (90.29.1.1) we obtain

$$\begin{tikzcd} & k[l\epsilon] \arrow[d] \\ B \times_l k \arrow[r] & k \end{tikzcd}$$

where $l\epsilon$ denotes the finite dimensional k-vector space $l\epsilon \subset l[\epsilon]$. According to Lemma 90.10.4 the condition of (S2) for $\mathcal{F}$ also holds for this diagram. Hence (S2) holds for $\mathcal{F}_{l/k}$.

Part (5) follows from the characterization of (RS) in Lemma 90.16.4 part (2) and the fact that (90.29.1.1) commutes with fibre products. □

The following lemma applies in particular when $\mathcal{F}$ satisfies (S2) and is a predeformation category, see Lemma 90.10.5.

07WB Lemma 90.29.3. With notation and assumptions as in Situation 90.29.1. Assume $\mathcal{F}$ is a predeformation category and $\overline{\mathcal{F}}$ satisfies (S2). Then there is a canonical l-vector space isomorphism

$$T\mathcal{F} \otimes_k l \longrightarrow T\mathcal{F}_{l/k}$$

of tangent spaces.

Proof. By Lemma 90.29.2 we may replace $\mathcal{F}$ by $\overline{\mathcal{F}}$. Moreover we see that $T\mathcal{F}$, resp. $T\mathcal{F}_{l/k}$ has a canonical k-vector space structure, resp. l-vector space structure, see Lemma 90.12.2. Then

$$T\mathcal{F}_{l/k} = \mathcal{F}_{l/k}(l[\epsilon]) = \mathcal{F}(k[l\epsilon]) = T\mathcal{F} \otimes_k l$$

the last equality by Lemma 90.12.2. More generally, given a finite dimensional l-vector space V we have

$$\mathcal{F}_{l/k}(l[V]) = \mathcal{F}(k[V_k]) = T\mathcal{F} \otimes_k V_k$$

where V_k denotes V seen as a k-vector space. We conclude that the functors $V \mapsto \mathcal{F}_{l/k}(l[V])$ and $V \mapsto T\mathcal{F} \otimes_k V_k$ are canonically identified as functors to the category of sets. By Lemma 90.11.4 we see there is at most one way to turn either functor into an l-linear functor. Hence the isomorphisms are compatible with the l-vector space structures and we win. $\square$

07WC Lemma 90.29.4. With notation and assumptions as in Situation 90.29.1. Assume $\mathcal{F}$ is a deformation category. Then there is a canonical l-vector space isomorphism

$$\text{Inf}(\mathcal{F}) \otimes_k l \longrightarrow \text{Inf}(\mathcal{F}_{l/k})$$

of infinitesimal automorphism spaces.

Proof. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$ and denote $x_{l,0}$ the corresponding object of $\mathcal{F}_{l/k}$ over l. Recall that $\text{Inf}(\mathcal{F}) = \text{Inf}_{x_0}(\mathcal{F})$ and $\text{Inf}(\mathcal{F}_{l/k}) = \text{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$, see Remark 90.19.4. Recall that the vector space structure on $\text{Inf}_{x_0}(\mathcal{F})$ comes from identifying it with the tangent space of the functor $Aut(x_0)$ which is defined on the category $\mathcal{C}_{k,k}$ of Artinian local k-algebras with residue field k. Similarly, $\text{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$ is the tangent space of $Aut(x_{l,0})$ which is defined on the category $\mathcal{C}_{l,l}$ of Artinian local l-algebras with residue field l. Unwinding the definitions we see that $Aut(x_{l,0})$ is the restriction of $Aut(x_0)_{l/k}$ (which lives on $\mathcal{C}_{k,l}$) to $\mathcal{C}_{l,l}$. Since there is no difference between the tangent space of $Aut(x_0)_{l/k}$ seen as a functor on $\mathcal{C}_{k,l}$ or $\mathcal{C}_{l,l}$, the lemma follows from Lemma 90.29.3 and the fact that $Aut(x_0)$ satisfies (RS) by Lemma 90.19.6 (whence we have (S2) by Lemma 90.16.6). $\square$

07WD Lemma 90.29.5. With notation and assumptions as in Situation 90.29.1. If $\mathcal{F} \to \mathcal{G}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,k}$, then $\mathcal{F}_{l/k} \to \mathcal{G}_{l/k}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$.

Proof. This follows immediately from the definitions and the fact that (90.29.1.1) preserves surjections. $\square$

There are many more things you can say about the relationship between $\mathcal{F}$ and $\mathcal{F}_{l/k}$ (in particular about the relationship between versal deformations) and we will add these here as needed.

0DQF Lemma 90.29.6. With notation and assumptions as in Situation 90.29.1. Let ξ be a versal formal object for $\mathcal{F}$ lying over $R \in \text{Ob}(\widehat{\mathcal{C}}_{\Lambda,k})$. Then there exist

(1) an $S \in \text{Ob}(\widehat{\mathcal{C}}_{\Lambda,l})$ and a local Λ-algebra homomorphism $R \to S$ which is formally smooth in the $\mathfrak{m}_S$-adic topology and induces the given field extension l/k on residue fieds, and
(2) a versal formal object of $\mathcal{F}_{l/k}$ lying over S.

Proof. Construction of S. Choose a surjection $R[x_1, \ldots, x_n] \to l$ of R-algebras. The kernel is a maximal ideal $\mathfrak{m}$. Set S equal to the $\mathfrak{m}$-adic completion of the Noetherian ring $R[x_1, \ldots, x_n]$. Then S is in $\widehat{\mathcal{C}}_{\Lambda,l}$ by Algebra, Lemma 10.97.6. The map $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by More on Algebra, Lemmas 15.37.2 and 15.37.4 and the fact that $R \to R[x_1, \ldots, x_n]$ is formally smooth. (Compare with the proof Lemma 90.9.5.)

Since ξ is versal, the transformation $\underline{\xi} : \underline{R}|_{\mathcal{C}_{\Lambda,k}} \to \mathcal{F}$ is smooth. By Lemma 90.29.5 the induced map

$$(\underline{R}|_{\mathcal{C}_{\Lambda,k}})_{l/k} \longrightarrow \mathcal{F}_{l/k}$$

is smooth. Thus it suffices to construct a smooth morphism $\underline{S}|_{\mathcal{C}_{\Lambda,l}} \to (\underline{R}|_{\mathcal{C}_{\Lambda,k}})_{l/k}$. To give such a map means for every object B of $\mathcal{C}_{\Lambda,l}$ a map of sets

$$\text{Mor}_{\widehat{\mathcal{C}}_{\Lambda,l}}(S, B) \longrightarrow \text{Mor}_{\widehat{\mathcal{C}}_{\Lambda,k}}(R, B \times_l k)$$

functorial in B. Given an element $\varphi : S \to B$ on the left hand side we send it to the composition $R \to S \to B$ whose image is contained in the sub Λ-algebra $B \times_l k$. Smoothness of the map means that given a surjection $B' \to B$ and a commutative diagram

$$\begin{tikzcd}
S \arrow[r] & B \arrow[r, equal] & B \\
R \arrow[u] \arrow[r] & B' \times_l k \arrow[u] \arrow[r] & B' \arrow[u]
\end{tikzcd}$$

we have to find a ring map $S \to B'$ fitting into the outer rectangle. The existence of this map is guaranteed as we chose $R \to S$ to be formally smooth in the $\mathfrak{m}_S$-adic topology, see More on Algebra, Lemma 15.37.5. $\square$

90.30. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
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Schemes

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CHAPTER 91

Deformation Theory

08KW 91.1. Introduction

08KX The goal of this chapter is to give a (relatively) gentle introduction to deformation theory of modules, morphisms, etc. In this chapter we deal with those results that can be proven using the naive cotangent complex. In the chapter on the cotangent complex we will extend these results a little bit. The advanced reader may wish to consult the treatise by Illusie on this subject, see [Ill72].

91.2. Deformations of rings and the naive cotangent complex

08S3 In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08S4 (91.2.0.1)
$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & ? \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u, "c"] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is an ideal of square zero and is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a solution to (91.2.0.1).

08S5 **Lemma 91.2.1.** Given a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & N_2 \arrow[r] & B'_2 \arrow[r] & B_2 \arrow[r] & 0 \\
 & 0 \arrow[r] & I_2 \arrow[r] \arrow[u, "c_2"'] & A'_2 \arrow[r] \arrow[u] & A_2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[uur] & B'_1 \arrow[r] & B_1 \arrow[r] \arrow[uur] & 0 & \\
0 \arrow[r] & I_1 \arrow[r] \arrow[u, "c_1"] \arrow[uur] & A'_1 \arrow[r] \arrow[u] \arrow[uur] & A_1 \arrow[r] \arrow[u] \arrow[uur] & 0 &
\end{tikzcd}$$

with front and back solutions to (91.2.0.1) we have

(1) There exist a canonical element in $\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2)$ whose vanishing is a necessary and sufficient condition for the existence of a ring map $B_1' \to B_2'$ fitting into the diagram.
(2) If there exists a map $B_1' \to B_2'$ fitting into the diagram the set of all such maps is a principal homogeneous space under $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$.

Proof. Let $E = B_1$ viewed as a set. Consider the surjection $A_1[E] \to B_1$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B_1/A_1} = (J/J^2 \to \Omega_{A_1[E]/A_1} \otimes_{A_1[E]} B_1)$$

in Algebra, Section 10.134. Since $\Omega_{A_1[E]/A_1} \otimes B_1$ is a free B_1-module we have

$$\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2) = \frac{\mathrm{Hom}_{B_1}(J/J^2, N_2)}{\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1} \otimes B_1, N_2)}$$

We will construct an obstruction in the module on the right. Let $J' = \mathrm{Ker}(A_1'[E] \to B_1)$. Note that there is a surjection $J' \to J$ whose kernel is $I_1 A_1[E]$. For every $e \in E$ denote $x_e \in A_1[E]$ the corresponding variable. Choose a lift $y_e \in B_1'$ of the image of x_e in B_1 and a lift $z_e \in B_2'$ of the image of x_e in B_2. These choices determine A_1'-algebra maps

$$A_1'[E] \to B_1' \quad \text{and} \quad A_1'[E] \to B_2'$$

The first of these gives a map $J' \to N_1$, $f' \mapsto f'(y_e)$ and the second gives a map $J' \to N_2$, $f' \mapsto f'(z_e)$. A calculation shows that these maps annihilate $(J')^2$. Because the left square of the diagram (involving c_1 and c_2) commutes we see that these maps agree on $I_1 A_1[E]$ as maps into N_2. Observe that B_1' is the pushout of $J' \to A_1'[B_1]$ and $J' \to N_1$. Thus, if the maps $J' \to N_1 \to N_2$ and $J' \to N_2$ agree, then we obtain a map $B_1' \to B_2'$ fitting into the diagram. Thus we let the obstruction be the class of the map

$$J/J^2 \to N_2, \quad f \mapsto f'(z_e) - \nu(f'(y_e))$$

where $\nu : N_1 \to N_2$ is the given map and where $f' \in J'$ is a lift of f. This is well defined by our remarks above. Note that we have the freedom to modify our choices of z_e into $z_e + \delta_{2,e}$ and y_e into $y_e + \delta_{1,e}$ for some $\delta_{i,e} \in N_i$. This will modify the map above into

$$f \mapsto f'(z_e + \delta_{2,e}) - \nu(f'(y_e + \delta_{1,e})) = f'(z_e) - \nu(f'(z_e)) + \sum (\delta_{2,e} - \nu(\delta_{1,e}))\frac{\partial f}{\partial x_e}$$

This means exactly that we are modifying the map $J/J^2 \to N_2$ by the composition $J/J^2 \to \Omega_{A_1[E]/A_1} \otimes B_1 \to N_2$ where the second map sends $\mathrm{d}x_e$ to $\delta_{2,e} - \nu(\delta_{1,e})$. Thus our obstruction is well defined and is zero if and only if a lift exists.

Part (2) comes from the observation that given two maps $\varphi, \psi : B_1' \to B_2'$ fitting into the diagram, then $\varphi - \psi$ factors through a map $D : B_1 \to N_2$ which is an A_1-derivation:

$$\begin{aligned}
D(fg) &= \varphi(f'g') - \psi(f'g') \\
&= \varphi(f')\varphi(g') - \psi(f')\psi(g') \\
&= (\varphi(f') - \psi(f'))\varphi(g') + \psi(f')(\varphi(g') - \psi(g')) \\
&= gD(f) + fD(g)
\end{aligned}$$

Thus D corresponds to a unique B_1-linear map $\Omega_{B_1/A_1} \to N_2$. Conversely, given such a linear map we get a derivation D and given a ring map $\psi : B_1' \to B_2'$ fitting into the diagram the map $\psi + D$ is another ring map fitting into the diagram. □

08S7 Lemma 91.2.2. If there exists a solution to (91.2.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. We observe right away that given two solutions B_1' and B_2' to (91.2.0.1) we obtain by Lemma 91.2.1 an obstruction element $o(B_1', B_2') \in \mathrm{Ext}^1_B(NL_{B/A}, N)$ to the existence of a map $B_1' \to B_2'$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution B' and an element $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, N)$ we can find a second solution B'_ξ such that $o(B', B'_\xi) = \xi$.

Let $E = B$ viewed as a set. Consider the surjection $A[E] \to B$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B/A} = (J/J^2 \to \Omega_{A[E]/A} \otimes_{A[E]} B)$$

in Algebra, Section 10.134. Since $\Omega_{A[E]/A} \otimes B$ is a free B-module we have

$$\mathrm{Ext}^1_B(NL_{B/A}, N) = \frac{\mathrm{Hom}_B(J/J^2, N)}{\mathrm{Hom}_B(\Omega_{A[E]/A} \otimes B, N)}$$

Thus we may represent ξ as the class of a morphism $\delta : J/J^2 \to N$.

For every $e \in E$ denote $x_e \in A[E]$ the corresponding variable. Choose a lift $y_e \in B'$ of the image of x_e in B. These choices determine an A'-algebra map $\varphi : A'[E] \to B'$. Let $J' = \mathrm{Ker}(A'[E] \to B)$. Observe that φ induces a map $\varphi|_{J'} : J' \to N$ and that B' is the pushout, as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & J' \arrow[r] \arrow[u, "\varphi|_{J'}"] & A'[E] \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "="'] & 0 \end{tikzcd}$$

Let $\psi : J' \to N$ be the sum of the map $\varphi|_{J'}$ and the composition

$$J' \to J'/(J')^2 \to J/J^2 \xrightarrow{\delta} N.$$

Then the pushout along ψ is an other ring extension B'_ξ fitting into a diagram as above. A calculation shows that $o(B', B'_\xi) = \xi$ as desired. □

0GPT Lemma 91.2.3. Let A be a ring. Let B be an A-algebra. Let N be a B-module. The set of isomorphism classes of extensions of A-algebras

$$0 \to N \to B' \to B \to 0$$

where N is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. To prove this we apply the previous results to the case where (91.2.0.1) is given by the diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & ? \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & A \arrow[r, "\mathrm{id}"] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Thus our lemma follows from Lemma 91.2.2 and the fact that there exists a solution, namely $N \oplus B$. (See remark below for a direct construction of the bijection.) $\square$

0GPU Remark 91.2.4. Let $A \to B$ and N be as in Lemma 91.2.3. Let $\alpha : P \to B$ be a presentation of B over A, see Algebra, Section 10.134. With $J = \mathrm{Ker}(\alpha)$ the naive cotangent complex $NL(\alpha)$ associated to α is the complex $J/J^2 \to \Omega_{P/A} \otimes_P B$. We have

$$\mathrm{Ext}^1_B(NL(\alpha), N) = \mathrm{Coker}\left(\mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, N) \to \mathrm{Hom}_B(J/J^2, N)\right)$$

because $\Omega_{P/A}$ is a free module. Consider a extension $0 \to N \to B' \to B \to 0$ as in the lemma. Since P is a polynomial algebra over A we can lift α to an A-algebra map $\alpha' : P' \to B'$. Then $\alpha'|_J : J \to N$ factors as $J \to J/J^2 \to N$ as N has square zero in B'. The lemma sends our extension to the class of this map $J/J^2 \to N$ in the displayed cokernel.

0GPV Lemma 91.2.5. Given ring maps $A \to B \to C$, a B-module M, a C-module N, a B-linear map $c : M \to N$, and extensions of A-algebras with square zero kernels

(a) $0 \to M \to B' \to B \to 0$ corresponding to $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, M)$, and
(b) $0 \to N \to C' \to C \to 0$ corresponding to $\zeta \in \mathrm{Ext}^1_C(NL_{C/A}, N)$.

See Lemma 91.2.3. Then there is an A-algebra map $B' \to C'$ compatible with $B \to C$ and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_B(NL_{B/A}, M) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$$

using the map $M \to N$ and

$$\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$$

where the first arrows uses the restriction map $D(C) \to D(B)$ and the second arrow uses the canonical map of complexes $NL_{B/A} \to NL_{C/A}$. The statement of the lemma can be deduced from Lemma 91.2.1 applied to the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & N \arrow[r] & C' \arrow[r] & C \arrow[r] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[uur] & B' \arrow[r] & B \arrow[r] \arrow[uur] & 0 & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[uur] & A \arrow[r] \arrow[u] \arrow[uur] & A \arrow[r] \arrow[u] \arrow[uur] & 0 &
\end{tikzcd}$$

and a compatibility between the constructions in the proofs of Lemmas 91.2.3 and 91.2.1 whose statement and proof we omit. (See remark below for a direct argument.) $\square$

0GPW Remark 91.2.6. Let $A \to B \to C$, M, N, $c : M \to N$, $0 \to M \to B' \to B \to 0$, $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, M)$, $0 \to N \to C' \to C \to 0$, and $\zeta \in \mathrm{Ext}^1_C(NL_{C/A}, N)$ be as in

Lemma 91.2.5. Using pushout along $c : M \to N$ we can construct an extension

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & B'_1 \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[u, "c"] & B' \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

by setting $B'_1 = (N \times B')/M$ where M is antidiagonally embedded. Using pullback along $B \to C$ we can construct an extension

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & C' \arrow[r] & C \arrow[r] & 0 \\ 0 \arrow[r] & N \arrow[r] \arrow[u] & B'_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

by setting $B'_2 = C' \times_C B$ (fibre product of rings). A simple diagram chase tells us that there exists an A-algebra map $B' \to C'$ compatible with $B \to C$ and c if and only if B'_1 is isomorphic to B'_2 as A-algebra extensions of B by N. Thus to see Lemma 91.2.5 is true, it suffices to show that B'_1 corresponds via the bijection of Lemma 91.2.3 to the image of ξ by the map $\mathrm{Ext}^1_B(NL_{B/A}, M) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$ and that B'_2 correspond to the image of ζ by the map $\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$. The first of these two statements is immediate from the construction of the class in Remark 91.2.4. For the second, choose a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\beta"'] & C \\ P \arrow[r, "\alpha"] \arrow[u, "\varphi"] & B \arrow[u] \end{tikzcd}$$

of A-algebras, such that α is a presentation of B over A and β is a presentation of C over A. See Remark 91.2.4 and references therein. Set $J = \mathrm{Ker}(\alpha)$ and $K = \mathrm{Ker}(\beta)$. The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ and in particular $\bar{\varphi} : J/J^2 \to K/K^2$. Choose A-algebra homomorphism $\beta' : Q \to C'$ which is a lift of β. Then $\alpha' = (\beta' \circ \varphi, \alpha) : P \to B'_2 = C' \times_C B$ is a lift of α. With these choices the composition of the map $K/K^2 \to N$ induced by β' and the map $\bar{\varphi} : J/J^2 \to K/K^2$ is the restriction of α' to J/J^2. Unwinding the constructions of our classes in Remark 91.2.4 this indeed shows that B'_2 correspond to the image of ζ by the map $\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$.

0GPX Lemma 91.2.7. Let $0 \to I \to A' \to A \to 0$, $A \to B$, and $c : I \to N$ be as in (91.2.0.1). Denote $\xi \in \mathrm{Ext}^1_A(NL_{A/A'}, I)$ the element corresponding to the extension A' of A by I via Lemma 91.2.3. The set of isomorphism classes of solutions is canonically bijective to the fibre of

$$\mathrm{Ext}^1_B(NL_{B/A'}, N) \to \mathrm{Ext}^1_A(NL_{A'/A}, N)$$

over the image of ξ.

Proof. By Lemma 91.2.3 applied to $A' \to B$ and the B-module N we see that elements ζ of $\mathrm{Ext}^1_B(NL_{B/A'}, N)$ parametrize extensions $0 \to N \to B' \to B \to 0$ of A'-algebras. By Lemma 91.2.5 applied to $A' \to A \to B$ and $c : I \to N$ we see that there is an A'-algebra map $A' \to B'$ compatible with c and $A \to B$ if and only if ζ maps to ξ. Of course this is the same thing as saying B' is a solution of (91.2.0.1). $\square$

0GPY Remark 91.2.8. Observe that in the situation of Lemma 91.2.7 we have

$$\mathrm{Ext}^1_A(NL_{A'/A}, N) = \mathrm{Ext}^1_B(NL_{A'/A} \otimes_A^{\mathbf{L}} B, N) = \mathrm{Ext}^1_B(NL_{A'/A} \otimes_A B, N)$$

The first equality by More on Algebra, Lemma 15.60.3 and the second by More on Algebra, Lemma 15.85.1. We have maps of complexes

$$NL_{A'/A} \otimes_A B \to NL_{B/A'} \to NL_{B/A}$$

which is close to being a distinguished triangle, see Algebra, Lemma 10.134.4. If it were a distinguished triangle we would conclude that the image of ξ in $\mathrm{Ext}^2_B(NL_{B/A}, N)$ would be the obstruction to the existence of a solution to (91.2.0.1).

If our ring map $A \to B$ is a local complete intersection, then there is a solutuion. This is a kind of lifting result; observe that for syntomic ring maps we have proved a rather strong lifting result in Smoothing Ring Maps, Proposition 16.3.2.

08S6 Lemma 91.2.9. If $A \to B$ is a local complete intersection ring map, then there exists a solution to (91.2.0.1).

First proof. Write $B = A[x_1, \ldots, x_n]/J$. By More on Algebra, Definition 15.33.2 the ideal J is Koszul-regular. This implies J is H_1-regular and quasi-regular, see More on Algebra, Section 15.32. Let $J' \subset A'[x_1, \ldots, x_n]$ be the inverse image of J. Denote $I[x_1, \ldots, x_n]$ the kernel of $A'[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$. By More on Algebra, Lemma 15.32.5 we have $I[x_1, \ldots, x_n] \cap (J')^2 = J'I[x_1, \ldots, x_n] = JI[x_1, \ldots, x_n]$. Hence we obtain a short exact sequence

$$0 \to I \otimes_A B \to J'/(J')^2 \to J/J^2 \to 0$$

Since J/J^2 is projective (More on Algebra, Lemma 15.32.3) we can choose a splitting of this sequence

$$J'/(J')^2 = I \otimes_A B \oplus J/J^2$$

Let $(J')^2 \subset J'' \subset J'$ be the elements which map to the second summand in the decomposition above. Then

$$0 \to I \otimes_A B \to A'[x_1, \ldots, x_n]/J'' \to B \to 0$$

is a solution to (91.2.0.1) with $N = I \otimes_A B$. The general case is obtained by doing a pushout along the given map $I \otimes_A B \to N$. $\square$

Second proof. Please read Remark 91.2.8 before reading this proof. By More on Algebra, Lemma 15.33.6 the maps $NL_{A'/A} \otimes_A B \to NL_{B/A'} \to NL_{B/A}$ do form a distinguished triangle in $D(B)$. Hence it suffices to show that $\mathrm{Ext}^2_{B/A}(NL_{B/A}, N)$ vanishes. By More on Algebra, Lemma 15.85.4 the complex $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$. This implies our Ext^2 vanishes for example by More on Algebra, Lemma 15.76.1 part (1). $\square$

91.3. Thickenings of ringed spaces

08KY In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ on a ringed space $(X', \mathcal{O}_{X'})$ is locally nilpotent if any local section of $\mathcal{I}$ is locally nilpotent. Compare with Algebra, Item 29.
(2) A thickening of ringed spaces is a morphism $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that
    (a) i induces a homeomorphism $X \to X'$,

(b) the map $i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X$ is surjective, and
(c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.

(3) A first order thickening of ringed spaces is a thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define morphisms of thickenings, morphisms of thickenings over a base ringed space, etc.

If $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is a thickening of ringed spaces then we identify the underlying topological spaces and think of $\mathcal{O}_X$, $\mathcal{O}_{X'}$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $X = X'$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. By Modules, Lemma 17.13.4 the category of $\mathcal{O}_X$-modules is equivalent to the category of $\mathcal{O}_{X'}$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}_X$-module.

08KZ Situation 91.3.1. A morphism of thickenings (f, f') is given by a commutative diagram

08L0 (91.3.1.1)
$$\begin{tikzcd} (X, \mathcal{O}_X) \arrow[r, "i"'] \arrow[d, "f"'] & (X', \mathcal{O}_{X'}) \arrow[d, "f'"] \\ (S, \mathcal{O}_S) \arrow[r, "t"] & (S', \mathcal{O}_{S'}) \end{tikzcd}$$

of ringed spaces whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. As $f = f'$ on underlying topological spaces we will identify the (topological) pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{S'} \to \mathcal{O}_{X'}$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}_{X'}$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

08L1 Definition 91.3.2. In Situation 91.3.1 we say that (f, f') is a strict morphism of thickenings if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

The following lemma in particular shows that a morphism $(f, f') : (X \subset X') \to (S \subset S')$ of thickenings of schemes is strict if and only if $X = S \times_{S'} X'$.

08L2 Lemma 91.3.3. In Situation 91.3.1 the morphism (f, f') is a strict morphism of thickenings if and only if (91.3.1.1) is cartesian in the category of ringed spaces.

Proof. Omitted. $\square$

91.4. Modules on first order thickenings of ringed spaces

08L3 In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We will freely use the notation introduced in Section 91.3, in particular we will identify the underlying topological spaces. In this section we consider short exact sequences

08L4 (91.4.0.1)
$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}_X$-modules and $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module. In this situation we have a canonical $\mathcal{O}_X$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

08L5 Lemma 91.4.1. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) If there exists an $\mathcal{O}_{X'}$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram

$$\begin{tikzcd} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_{\mathcal{F}'}"'] \arrow[d, "1\otimes\varphi"'] & \mathcal{K} \arrow[d, "\psi"] \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r, "c_{\mathcal{G}'}"] & \mathcal{L} \end{tikzcd}$$

is commutative.

(2) The set of $\mathcal{O}_{X'}$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$.

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ by Modules, Lemma 17.13.4. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. $\square$

08L6 Lemma 91.4.2. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{tikzcd} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_{\mathcal{F}'}"'] \arrow[d, "1\otimes\varphi"'] & \mathcal{K} \arrow[d, "\psi"] \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r, "c_{\mathcal{G}'}"] & \mathcal{L} \end{tikzcd}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1,-\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi,1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, "\psi"] & \mathcal{F}' \arrow[r] \arrow[d, "\varphi'"] & \mathcal{F} \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

as desired. □

08L7 Lemma 91.4.3. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (91.4.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$.

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 12.6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}_X$-modules, see Modules, Lemma 17.13.4. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}_{X'}$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. □

08L8 Lemma 91.4.4. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. Then there exists an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (91.4.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}_X$-module, then there does exist a sequence (91.4.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}_{X'}$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules, Lemma 17.17.6). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{tikzcd}
\mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} \arrow[r, hook] \arrow[d] & \mathcal{J}/\mathcal{I}\mathcal{J} \arrow[d, dotted, "\gamma"'] \\
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c"] & \mathcal{K}
\end{tikzcd}$$

a diagram of $\mathcal{O}_X$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}_X$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r] \arrow[d, "\gamma'"'] & \mathcal{H}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}_X$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (91.4.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (91.4.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}_X$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 91.4.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\square$

08L9 Remark 91.4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. A first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is said to be trivial if there exists a morphism of ringed spaces $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ which is a left inverse to i. The choice of such a morphism π is called a trivialization of the first order thickening. Given π we obtain a splitting

08LA (91.4.5.1)
$$\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{I}$$

as sheaves of algebras on X by using $\pi^\sharp$ to split the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(X, \mathcal{O}_X)$ is equivalent to the category of $\mathcal{O}_X$-modules.

08LB Remark 91.4.6. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a trivial first order thickening of ringed spaces and let $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}_X$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$ we may set
$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$
and use the splitting (91.4.5.1) associated to π and the map c to define the $\mathcal{O}_{X'}$-module structure and obtain an extension (91.4.0.1). We will call $\mathcal{F}'_{c,triv}$ the trivial extension of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (91.4.0.1) we can use $\pi^\sharp : \mathcal{O}_X \to \mathcal{O}_{X'}$ to think of $\mathcal{F}'$ as an $\mathcal{O}_X$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$. Lemma 91.4.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection
$$\left\{\begin{matrix} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (91.4.0.1) with } c = c_{\mathcal{F}'} \end{matrix}\right\} \longrightarrow \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$
Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

08LC Remark 91.4.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be a morphism of first order thickenings of $(X, \mathcal{O}_X)$. Picture
$$\begin{tikzcd} & (X, \mathcal{O}_X) \arrow[ld] \arrow[rd] & \\ (X'_1, \mathcal{O}_{X'_1}) \arrow[rr, "h"] & & (X'_2, \mathcal{O}_{X'_2}) \end{tikzcd}$$
Observe that $h^\sharp : \mathcal{O}_{X'_2} \to \mathcal{O}_{X'_1}$ in particular induces an $\mathcal{O}_X$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that
$$\begin{tikzcd} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd}$$
is commutative. Then there is a canonical functoriality
$$\left\{\begin{matrix} \mathcal{F}'_2 \text{ as in (91.4.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \mathcal{F}'_1 \text{ as in (91.4.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{matrix}\right\}$$

Namely, thinking of all sheaves $\mathcal{O}_X$, $\mathcal{O}_{X'_i}$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on X, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[d] & \mathcal{F}'_2 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \mathcal{F}'_1 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

We omit the construction of the $\mathcal{O}_{X'_1}$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

08LD Remark 91.4.8. Let $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $\mathcal{I}_i$, and $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be as in Remark 91.4.7. Assume that we are given trivializations $\pi_i : X'_i \to X$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickening of $(X, \mathcal{O}_X)$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{tikzcd} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd}$$

is commutative. In this situation the construction of Remark 91.4.6 induces a commutative diagram

$$\begin{tikzcd} \{\mathcal{F}'_2 \text{ as in (91.4.0.1) with } c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\} \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_2) \arrow[d] \\ \{\mathcal{F}'_1 \text{ as in (91.4.0.1) with } c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\} \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \end{tikzcd}$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 91.4.7.

08LE Remark 91.4.9. Let $(X, \mathcal{O}_X)$ be a ringed space. We define a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ to be a complex if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}_X$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(X'_1, \mathcal{O}_{X'_1}) \to (X'_3, \mathcal{O}_{X'_3})$ factors through $(X, \mathcal{O}_X) \to (X'_3, \mathcal{O}_{X'_3})$, i.e., the first order thickening $(X'_1, \mathcal{O}_{X'_1})$ of $(X, \mathcal{O}_X)$ is trivial and comes with a canonical trivialization $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$.

We say a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ is a short exact sequence if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}_X$-modules.

08LF Remark 91.4.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

be a complex first order thickenings of $(X, \mathcal{O}_X)$, see Remark 91.4.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{tikzcd} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{I}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_3"'] \arrow[d] & \mathcal{K}_3 \arrow[d] \\ \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "c_2"] & \mathcal{K}_2 \end{tikzcd}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

as in (91.4.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_{X'_2}$-modules with $c_{\mathcal{F}'_2} = c_2$. In this situation we can apply the functoriality of Remark 91.4.7 to obtain an extension $\mathcal{F}'_1$ on X'_1 (we'll describe $\mathcal{F}'_1$ in this special case below). By Remark 91.4.6 using the canonical splitting $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$ of Remark 91.4.9 we obtain $\xi_{\mathcal{F}'_1} \in \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 91.4.4. In this situation we claim that the canonical map

$$\partial : \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}'_1}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}_X$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}'_2 = j'_*\mathcal{F}'_2$ equal to the pushout of $\mathcal{F}'_2$ by j' so that $c_{\mathcal{E}'_2} = j' \circ c_2$. Picture:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[d, "j'"'] & \mathcal{F}'_2 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{E}'_2 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

Set $\mathcal{E}'_3 = \mathcal{E}'_2$ but viewed as an $\mathcal{O}_{X'_3}$-module via $\mathcal{O}_{X'_3} \to \mathcal{O}_{X'_2}$. Then $c_{\mathcal{E}'_3} = j \circ c_3$. The proof of Lemma 91.4.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}'_3/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}'_1 = \mathcal{F}'_2/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}'_1}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}'_2/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}_X$-modules using $\pi^\sharp$ where $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2/\mathcal{K}_3 \arrow[r] \arrow[d] & \mathcal{F}'_2/\mathcal{K}_3 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}/\mathcal{K}_3 \arrow[r] & \mathcal{E}'_3/\mathcal{K}_3 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

which is $\mathcal{O}_X$-linear (with the $\mathcal{O}_X$-module structures given above).

91.5. Infinitesimal deformations of modules on ringed spaces

08LG Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We freely use the notation introduced in Section 91.3. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}_X$-module structure. Thus the sequence above is an extension as in (91.4.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}_{X'}$ we have $i^*\mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathcal{I}\mathcal{O}_{X'} = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

08LH **Lemma 91.5.1.** Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.

Proof. This is a special case of Lemma 91.4.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}_X$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 12.5.8. □

08LI **Lemma 91.5.2.** Let (f, f') be a morphism of first order thickenings of ringed spaces as in Situation 91.3.1. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over S and that (f, f') is a strict morphism of thickenings (Definition 91.3.2). Then the following are equivalent

1. $\mathcal{F}'$ is flat over S', and
2. the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). We may check these conditions at stalks. Let $x \in X \subset X'$ be a point with image $s = f(x) \in S \subset S'$. Set $A' = \mathcal{O}_{S',s}$, $B' = \mathcal{O}_{X',x}$, $A = \mathcal{O}_{S,s}$, and $B = \mathcal{O}_{X,x}$. Then $A = A'/J$ and $B = B'/I$ for some square zero ideals. Since (f, f') is a strict morphism of thickenings we have $I = JB'$. Let $M' = \mathcal{F}'_x$ and $M = \mathcal{F}_x$. Then M' is a B'-module and M is a B-module. Since $\mathcal{F} = i^*\mathcal{F}'$ we see that the kernel of the surjection $M' \to M$ is $IM' = JM'$. Thus we have a short exact sequence

$$0 \to JM' \to M' \to M \to 0$$

Using Sheaves, Lemma 6.26.4 and Modules, Lemma 17.16.1 to identify stalks of pullbacks and tensor products we see that the stalk at x of the canonical map of the lemma is the map

$$(J \otimes_A B) \otimes_B M = J \otimes_A M = J \otimes_{A'} M' \longrightarrow JM'$$

The assumption that $\mathcal{F}$ is flat over S signifies that M is a flat A-module.

Assume (1). Flatness implies $\text{Tor}_1^{A'}(M', A) = 0$ by Algebra, Lemma 10.75.8. This means $J \otimes_{A'} M' \to M'$ is injective by Algebra, Remark 10.75.9. Hence $J \otimes_A M \to JM'$ is an isomorphism.

Assume (2). Then $J \otimes_{A'} M' \to M'$ is injective. Hence $\text{Tor}_1^{A'}(M', A) = 0$ by Algebra, Remark 10.75.9. Hence M' is flat over A' by Algebra, Lemma 10.99.8. □

08LJ Lemma 91.5.3. Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{G}'$ is flat over S' and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Proof. Combine Lemmas 91.5.1 and 91.5.2. □

08LK Lemma 91.5.4. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. There exists an element

$$o(\varphi) \in \text{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 91.5.1 that the vanishing of the boundary of φ via the map

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) \longrightarrow \text{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\text{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \text{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology, Lemma 20.28.1). □

08LL Lemma 91.5.5. Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over S' and that (f, f') is a strict morphism of thickenings. There exists an element

$$o(\varphi) \in \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 91.5.4 as we claim that under the assumptions of the lemma we have

$$\operatorname{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J}$ by Lemma 91.5.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \operatorname{Tor}_1^{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{O}_X)$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 91.5.2. Thus $\tau_{\geq -1}Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\operatorname{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \operatorname{Ext}^1_{\mathcal{O}_X}(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 13.16.1. $\square$

Second proof. We can apply Lemma 91.4.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G}$ by Lemma 91.5.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. $\square$

08LM Lemma 91.5.6. Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$.

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism by Lemma 91.5.2. Apply Lemma 91.4.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $c = 1$. By Lemma 91.5.2 the corresponding extensions $\mathcal{F}'$ are all flat over S'. $\square$

08LN Lemma 91.5.7. Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if

(1) the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an isomorphism, and
(2) the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 91.4.4 is zero.

Proof. This follows immediately from the characterization of $\mathcal{O}_{X'}$-modules flat over S' of Lemma 91.5.2 and Lemma 91.4.4. $\square$

91.6. Application to flat modules on flat thickenings of ringed spaces

08VQ Consider a commutative diagram

$$\begin{tikzcd} (X, \mathcal{O}_X) \arrow[r, "i"'] \arrow[d, "f"'] & (X', \mathcal{O}_{X'}) \arrow[d, "f'"] \\ (S, \mathcal{O}_S) \arrow[r, "t"] & (S', \mathcal{O}_{S'}) \end{tikzcd}$$

of ringed spaces whose horizontal arrows are first order thickenings as in Situation 91.3.1. Set $\mathcal{I} = \text{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \text{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over S.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 91.5.2 to $\mathcal{O}_{X'}$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

08VR **Lemma 91.6.1.** In the situation above.

(1) There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 91.4.4 is zero.
(2) If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.
(3) Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\text{id}_\mathcal{F}$ is canonically isomorphic to $\text{Ext}^0_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.

Proof. Part (1) follows from Lemma 91.5.7 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 91.5.6. Part (3) follows from Lemma 91.5.3. □

08VS **Situation 91.6.2.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a commutative diagram

$$\begin{tikzcd} (X'_1, \mathcal{O}'_1) \arrow[r, "h"'] \arrow[d, "f'_1"'] & (X'_2, \mathcal{O}'_2) \arrow[r] \arrow[d, "f'_2"'] & (X'_3, \mathcal{O}'_3) \arrow[d, "f'_3"'] \\ (S'_1, \mathcal{O}_{S'_1}) \arrow[r] & (S'_2, \mathcal{O}_{S'_2}) \arrow[r] & (S'_3, \mathcal{O}_{S'_3}) \end{tikzcd}$$

where (a) the top row is a short exact sequence of first order thickenings of X, (b) the lower row is a short exact sequence of first order thickenings of S, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over S'_2 and set $\mathcal{F} = \mathcal{F}'_2|_X$. Let $\pi : X'_1 \to X$ be the canonical splitting (Remark 91.4.9).

08VT **Lemma 91.6.3.** In Situation 91.6.2 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over S'_1 restricting to $\mathcal{F}$ on X. Their difference (Lemma 91.6.1) is an element θ of $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F})$ whose boundary in $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F})$ equals the obstruction (Lemma 91.6.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over S'_3.

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on X and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence flatness by Lemma 91.5.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 91.6.2 and the fact that $\mathcal{F}$ is flat over S. The statement on the obstruction class is a direct translation of the result of Remark 91.4.10 to this particular situation. □

91.7. Deformations of ringed spaces and the naive cotangent complex

08U6 In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces. We denote $\mathcal{J} = \text{Ker}(t^\sharp)$ and we identify the underlying topological spaces of S and S'. Moreover we assume given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules (Sheaves, Definition 6.21.7 and Section 6.26). In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08U7 (91.7.0.1)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & ? \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r] \arrow[u, "c"] & \mathcal{O}_{S'} \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

(where the vertical arrows are f-maps) and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in (91.3.1.1) where $\text{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a solution to (91.7.0.1).

08U8 **Lemma 91.7.1.** Assume given a commutative diagram of morphisms of ringed spaces

08U9 (91.7.1.1)
$$\begin{tikzcd}
 & & (X_2, \mathcal{O}_{X_2}) \arrow[r, "i_2"'] \arrow[d, "f_2"] \arrow[lldd, "g"'] & (X'_2, \mathcal{O}_{X'_2}) \arrow[d, "f'_2"] \\
 & & (S_2, \mathcal{O}_{S_2}) \arrow[r, "t_2"] \arrow[lldd] & (S'_2, \mathcal{O}_{S'_2}) \arrow[lldd] \\
(X_1, \mathcal{O}_{X_1}) \arrow[r, "i_1"'] \arrow[d, "f_1"'] & (X'_1, \mathcal{O}_{X'_1}) \arrow[d, "f'_1"] & & \\
(S_1, \mathcal{O}_{S_1}) \arrow[r, "t_1"] & (S'_1, \mathcal{O}_{S'_1}) & &
\end{tikzcd}$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = \mathrm{Ker}(i_j^\sharp)$ and assume given a g-map $\nu : \mathcal{G}_1 \to \mathcal{G}_2$ of modules giving rise to the commutative diagram

08UA (91.7.1.2)
$$\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{G}_2 \arrow[r] & \mathcal{O}_{X'_2} \arrow[r] & \mathcal{O}_{X_2} \arrow[r] & 0 \\
 & 0 \arrow[r] & \mathcal{J}_2 \arrow[r] \arrow[u, "c_2"] & \mathcal{O}_{S'_2} \arrow[r] \arrow[u] & \mathcal{O}_{S_2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{G}_1 \arrow[r] \arrow[uur] & \mathcal{O}_{X'_1} \arrow[r] & \mathcal{O}_{X_1} \arrow[r] \arrow[uur] & 0 & \\
0 \arrow[r] & \mathcal{J}_1 \arrow[r] \arrow[u, "c_1"] \arrow[uur] & \mathcal{O}_{S'_1} \arrow[r] \arrow[u] \arrow[uur] & \mathcal{O}_{S_1} \arrow[r] \arrow[u] \arrow[uur] & 0 &
\end{tikzcd}$$

with front and back solutions to (91.7.0.1).

(1) There exist a canonical element in $\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$ whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed spaces $X'_2 \to X'_1$ fitting into (91.7.1.1) compatibly with ν.
(2) If there exists a morphism $X'_2 \to X'_1$ fitting into (91.7.1.1) compatibly with ν the set of all such morphisms is a principal homogeneous space under

$$\mathrm{Hom}_{\mathcal{O}_{X_1}}(\Omega_{X_1/S_1}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_{X_2}}(g^*\Omega_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^0_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2).$$

Proof. The naive cotangent complex NL_{X_1/S_1} is defined in Modules, Definition 17.31.6. The equalities in the last statement of the lemma follow from the fact that g^* is adjoint to g_*, the fact that $H^0(NL_{X_1/S_1}) = \Omega_{X_1/S_1}$ (by construction of the naive cotangent complex) and the fact that Lg^* is the left derived functor of g^*. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topological spaces of all ringed spaces in the lemma is the same.

To do this, observe that $g^{-1} NL_{X_1/S_1}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{S_1} \to g^{-1}\mathcal{O}_{X_1}$, see Modules, Lemma 17.31.3. Moreover, the degree 0 term of NL_{X_1/S_1} is a flat $\mathcal{O}_{X_1}$-module, hence the canonical map

$$Lg^* NL_{X_1/S_1} \longrightarrow g^{-1} NL_{X_1/S_1} \otimes_{g^{-1}\mathcal{O}_{X_1}} \mathcal{O}_{X_2}$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(g^{-1} NL_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(NL_{g^{-1}\mathcal{O}_{X_1}/g^{-1}f_1^{-1}\mathcal{O}_{S_1}}, \mathcal{G}_2)$$

The set of morphism of ringed spaces $X'_2 \to X'_1$ fitting into (91.7.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{S'_1}$-algebras $g^{-1}\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (91.7.1.2) of sheaves on X and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (91.7.1.2) of sheaves on a space X and we are looking for homomorphisms of sheaves of rings $\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into it. As ext groups we will use $\text{Ext}^k_{\mathcal{O}_{X_1}}(NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}_{X'_1} \times_{\mathcal{O}_{X_2}} \mathcal{O}_{X'_2}$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_{X_1}$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}$, see Modules, Lemma 17.31.2. Set

$$\mathcal{I}' = \text{Ker}(\mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X_1}) \quad \text{and} \quad \mathcal{I} = \text{Ker}(\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{O}_{X_1})$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{S'_2}$-algebras

$$a : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_1} \quad \text{and} \quad b : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_2}$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (91.7.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_{X_1}$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \text{Ext}^1_{\mathcal{O}_{X_1}}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{S_1}[\mathcal{E}]/\mathcal{O}_{S_1}} \otimes_{\mathcal{O}_{S_1}[\mathcal{E}]} \mathcal{O}_{X_1}$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into (91.7.1.2). In this case consider the map $\mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_{X_1}$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_2}$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into (91.7.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 91.2.1. □

08UB Lemma 91.7.2. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \text{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \text{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules, Lemma 17.31.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_\mathcal{B}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_\mathcal{B}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{H} \arrow[r] & \Omega \end{tikzcd}$$

This finishes the proof. □

08UC **Lemma 91.7.3.** *If there exists a solution to (91.7.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.*

Proof. We observe right away that given two solutions X'_1 and X'_2 to (91.7.0.1) we obtain by Lemma 91.7.1 an obstruction element $o(X'_1, X'_2) \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ to the existence of a map $X'_1 \to X'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution X' and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ we can find a second solution X'_ξ such that $o(X', X'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}_X$ as in Lemma 91.7.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X)$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}_X} \mathcal{O}_{X'}$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}_{X'}$.

These choices determine an $f^{-1}\mathcal{O}_{S'}$-algebra map $\varphi : \mathcal{O}_{S'}[\mathcal{E}] \to \mathcal{O}_{X'}$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}_{X'}$ is the pushout, as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{I}' \arrow[r] \arrow[u, "\varphi|_{\mathcal{I}'}"] & f^{-1}\mathcal{O}_{S'}[\mathcal{E}] \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u, "="'] & 0 \end{tikzcd}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}_{X'_\xi}$ fitting into a diagram as above. A calculation (omitted) shows that $o(X', X'_\xi) = \xi$ as desired. □

0GPZ **Lemma 91.7.4.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a $\mathcal{O}_X$-module. The set of isomorphism classes of extensions of $f^{-1}\mathcal{O}_S$-algebras

$$0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $\mathcal{G}$ is an ideal of square zero[1] is canonically bijective to $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.

Proof. To prove this we apply the previous results to the case where (91.7.0.1) is given by the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] & ? \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r, "\mathrm{id}"] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Thus our lemma follows from Lemma 91.7.3 and the fact that there exists a solution, namely $\mathcal{G} \oplus \mathcal{O}_X$. (See remark below for a direct construction of the bijection.) $\square$

0GQ0 **Remark 91.7.5.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $\mathcal{G}$ be as in Lemma 91.7.4. Consider an extension $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ as in the lemma. We can choose a sheaf of sets $\mathcal{E}$ and a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\alpha'"'] \arrow[rd, "\alpha"] & \\ \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

such that $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ is surjective with kernel $\mathcal{J}$. (For example you can take any sheaf of sets surjecting onto $\mathcal{O}_{X'}$.) Then

$$NL_{X/S} \cong NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X\right)$$

See Modules, Section 17.31 and in particular Lemma 17.31.2. Of course α' determines a map $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_{X'}$ which in turn determines a map

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{G}$$

which in turn determines the element of $\mathrm{Ext}^1_{\mathcal{O}_X}(NL(\alpha), \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ corresponding to $\mathcal{O}_{X'}$ by the bijection of the lemma.

0GQ1 **Lemma 91.7.6.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be morphisms of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. Let $c : \mathcal{F} \to \mathcal{G}$ be a g-map. Finally, consider

(a) $0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ an extension of $f^{-1}\mathcal{O}_S$-algebras corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F})$, and
(b) $0 \to \mathcal{G} \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$ an extension of $g^{-1}f^{-1}\mathcal{O}_S$-algebras corresponding to $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G})$.

See Lemma 91.7.4. Then there is an S-morphism $g' : Y' \to X'$ compatible with g and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$.

[1] In other words, the set of isomorphism classes of first order thickenings $i : X \to X'$ over S endowed with an isomorphism $\mathcal{G} \to \mathrm{Ker}(i^\sharp)$ of $\mathcal{O}_X$-modules.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, Lg^*\mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

using the map $Lg^*\mathcal{F} \to g^*\mathcal{F} \xrightarrow{c} \mathcal{G}$ and

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

using the map $Lg^* NL_{X/S} \to NL_{Y/S}$. The statement of the lemma can be deduced from Lemma 91.7.1 applied to the diagram

$$\begin{tikzcd}
& 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}_{Y'} \arrow[r] & \mathcal{O}_Y \arrow[r] & 0 \\
& 0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[uur] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[uur] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[uur] & \mathcal{O}_S \arrow[r] \arrow[u] \arrow[uur] & \mathcal{O}_S \arrow[r] \arrow[u] \arrow[uur] & 0
\end{tikzcd}$$

and a compatibility between the constructions in the proofs of Lemmas 91.7.4 and 91.7.1 whose statement and proof we omit. (See remark below for a direct argument.) $\square$

0GQ2 **Remark 91.7.7.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, $\mathcal{F}$, $\mathcal{G}$, $c : \mathcal{F} \to \mathcal{G}$, $0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$, $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F})$, $0 \to \mathcal{G} \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$, and $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G})$ be as in Lemma 91.7.6. Using pushout along $c : g^{-1}\mathcal{F} \to \mathcal{G}$ we can construct an extension

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}'_1 \arrow[r] & g^{-1}\mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & g^{-1}\mathcal{F} \arrow[r] \arrow[u, "c"] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_X \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Using pullback along $g^\sharp : g^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ we can construct an extension

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}_{Y'} \arrow[r] & \mathcal{O}_Y \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[u, equal] & \mathcal{O}'_2 \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

A diagram chase tells us that there exists an S-morphism $Y' \to X'$ compatible with g and c if and only if $\mathcal{O}'_1$ is isomorphic to $\mathcal{O}'_2$ as $g^{-1}f^{-1}\mathcal{O}_S$-algebra extensions of $g^{-1}\mathcal{O}_X$ by $\mathcal{G}$. By Lemma 91.7.4 these extensions are classified by the LHS of

$$\mathrm{Ext}^1_{g^{-1}\mathcal{O}_X}(NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

Here the equality comes from tensor-hom adjunction and the equalities

$$NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S} = g^{-1} NL_{X/S} \quad \text{and} \quad Lg^* NL_{X/S} = g^{-1} NL_{X/S} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_X} \mathcal{O}_Y$$

For the first of these see Modules, Lemma 17.31.3; the second follows from the definition of derived pullback. Thus, in order to see that Lemma 91.7.6 is true, it suffices to show that $\mathcal{O}'_1$ corresponds to the image of ξ and that $\mathcal{O}'_2$ correspond to the image of ζ. The correspondence between ξ and $\mathcal{O}'_1$ is immediate from the construction of the class ξ in Remark 91.7.5. For the correspondence between ζ and $\mathcal{O}'_2$, we first choose a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\beta'"'] \arrow[rd, "\beta"] & \\ \mathcal{O}_{Y'} \arrow[r] & \mathcal{O}_Y \end{tikzcd}$$

such that $g^{-1}f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_Y$ is surjective with kernel $\mathcal{K}$. Next choose a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\beta'"'] & \mathcal{E}' \arrow[l, "\varphi"'] \arrow[d, "\alpha'"'] \arrow[rd, "\alpha"] & \\ \mathcal{O}_{Y'} & \mathcal{O}'_2 \arrow[l] \arrow[r] & g^{-1}\mathcal{O}_X \end{tikzcd}$$

such that $g^{-1}f^{-1}\mathcal{O}_S[\mathcal{E}'] \to g^{-1}\mathcal{O}_X$ is surjective with kernel $\mathcal{J}$. (For example just take $\mathcal{E}' = \mathcal{E} \amalg \mathcal{O}'_2$ as a sheaf of sets.) The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ (notation as in Modules, Section 17.31) and in particular $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. Then $NL(\alpha) \cong NL_{Y/S}$ and $NL(\beta) \cong NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S}$ and the map of complexes $NL(\alpha) \to NL(\beta)$ represents the map $Lg^* NL_{X/S} \to NL_{Y/S}$ used in the statement of Lemma 91.7.6 (see first part of its proof). Now ζ corresponds to the class of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β', see Remark 91.7.5. Similarly, the extension $\mathcal{O}'_2$ corresponds to the map $\mathcal{J}/\mathcal{J}^2 \to \mathcal{G}$ induced by α'. The commutative diagram above shows that this map is the composition of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β' with the map $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. This proves the compatibility we were looking for.

0GQ3 Lemma 91.7.8. Let $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$, $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $\mathcal{G}$, and $c : \mathcal{J} \to \mathcal{G}$ be as in (91.7.0.1). Denote $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/S'}, \mathcal{J})$ the element corresponding to the extension $\mathcal{O}_{S'}$ of $\mathcal{O}_S$ by $\mathcal{J}$ via Lemma 91.7.4. The set of isomorphism classes of solutions is canonically bijective to the fibre of

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/S'}, \mathcal{G})$$

over the image of ξ.

Proof. By Lemma 91.7.4 applied to $X \to S'$ and the $\mathcal{O}_X$-module $\mathcal{G}$ we see that elements ζ of $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G})$ parametrize extensions $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ of $f^{-1}\mathcal{O}_{S'}$-algebras. By Lemma 91.7.6 applied to $X \to S \to S'$ and $c : \mathcal{J} \to \mathcal{G}$ we see that there is an S'-morphism $X' \to S'$ compatible with c and $f : X \to S$ if and only if ζ maps to ξ. Of course this is the same thing as saying $\mathcal{O}_{X'}$ is a solution of (91.7.0.1). □

0GQ4 Remark 91.7.9. In the situation of Lemma 91.7.8 we have maps of complexes

$$Lf^* NL_{S'/S} \to NL_{X/S'} \to NL_{X/S}$$

These maps are closed to forming a distinguished triangle, see Modules, Lemma 17.31.7. If it were a distinguished triangle we would conclude that the image of ξ in $\mathrm{Ext}^2_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ would be the obstruction to the existence of a solution to (91.7.0.1).

91.8. Deformations of schemes

0D13 In this section we spell out what the results in Section 91.7 mean for deformations of schemes.

0D14 Lemma 91.8.1. Let $S \subset S'$ be a first order thickening of schemes. Let $f : X \to S$ be a flat morphism of schemes. If there exists a flat morphism $f' : X' \to S'$ of schemes and an isomorphism $a : X \to X' \times_{S'} S$ over S, then

(1) the set of isomorphism classes of pairs $(f' : X' \to S', a)$ is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, f^*\mathcal{C}_{S/S'})$, and
(2) the set of automorphisms of $\varphi : X' \to X'$ over S' which reduce to the identity on $X' \times_{S'} S$ is $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/S}, f^*\mathcal{C}_{S/S'})$.

Proof. First we observe that thickenings of schemes as defined in More on Morphisms, Section 37.2 are the same things as morphisms of schemes which are thickenings in the sense of Section 91.3. We may think of X as a closed subscheme of X' so that $(f, f') : (X \subset X') \to (S \subset S')$ is a morphism of first order thickenings. Then we see from More on Morphisms, Lemma 37.10.1 (or from the more general Lemma 91.5.2) that the ideal sheaf of X in X' is equal to $f^*\mathcal{C}_{S/S'}$. Hence we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & f^*\mathcal{C}_{S/S'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{C}_{S/S'} \arrow[r] \arrow[u] & \mathcal{O}_{S'} \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where the vertical arrows are f-maps; please compare with (91.7.0.1). Thus part (1) follows from Lemma 91.7.3 and part (2) from part (2) of Lemma 91.7.1. (Note that $NL_{X/S}$ as defined for a morphism of schemes in More on Morphisms, Section 37.13 agrees with $NL_{X/S}$ as used in Section 91.7.) $\square$

91.9. Thickenings of ringed topoi

08M6 This section is the analogue of Section 91.3 for ringed topoi. In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}'$ on a ringed topos $(Sh(\mathcal{D}), \mathcal{O}')$ is locally nilpotent if any local section of $\mathcal{I}$ is locally nilpotent.
(2) A thickening of ringed topoi is a morphism $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that
    (a) i_* is an equivalence $Sh(\mathcal{C}) \to Sh(\mathcal{D})$,
    (b) the map $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective, and
    (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
(3) A first order thickening of ringed topoi is a thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define morphisms of thickenings of ringed topoi, morphisms of thickenings of ringed topoi over a base ringed topos, etc.

If $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a thickening of ringed topoi then we identify the underlying topoi and think of $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $\mathcal{C}$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

of $\mathcal{O}'$-modules. By Modules on Sites, Lemma 18.25.1 the category of $\mathcal{O}$-modules is equivalent to the category of $\mathcal{O}'$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}$-module.

08M7 Situation 91.9.1. A morphism of thickenings of ringed topoi (f, f') is given by a commutative diagram

08M8 (91.9.1.1)
$$\begin{tikzcd} (Sh(\mathcal{C}), \mathcal{O}) \arrow[r, "i"'] \arrow[d, "f"'] & (Sh(\mathcal{D}), \mathcal{O}') \arrow[d, "f'"] \\ (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \arrow[r, "t"] & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \end{tikzcd}$$

of ringed topoi whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \text{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \text{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. As $f = f'$ on underlying topoi we will identify the pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{\mathcal{B}'} \to \mathcal{O}'$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}'$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

08M9 Definition 91.9.2. In Situation 91.9.1 we say that (f, f') is a strict morphism of thickenings if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

91.10. Modules on first order thickenings of ringed topoi

08MA In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (Sh(\mathcal{C}, \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We will freely use the notation introduced in Section 91.9, in particular we will identify the underlying topological topoi. In this section we consider short exact sequences

08MB (91.10.0.1)
$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}$-modules and $\mathcal{F}'$ is an $\mathcal{O}'$-module. In this situation we have a canonical $\mathcal{O}$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \text{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

08MC Lemma 91.10.1. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.10.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) If there exists an $\mathcal{O}'$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram

$$\begin{tikzcd} \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \arrow[r, "c_{\mathcal{F}'}"'] \arrow[d, "1\otimes\varphi"'] & \mathcal{K} \arrow[d, "\psi"] \\ \mathcal{I} \otimes_\mathcal{O} \mathcal{G} \arrow[r, "c_{\mathcal{G}'}"] & \mathcal{L} \end{tikzcd}$$

is commutative.

(2) The set of $\mathcal{O}'$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$.

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$ by Modules on Sites, Lemma 18.25.1. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. $\square$

08MD Lemma 91.10.2. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.10.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{tikzcd} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_{\mathcal{F}'}"'] \arrow[d, "1\otimes\varphi"'] & \mathcal{K} \arrow[d, "\psi"] \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} \arrow[r, "c_{\mathcal{G}'}"] & \mathcal{L} \end{tikzcd}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1,-\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi,1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, "\psi"'] & \mathcal{F}' \arrow[r] \arrow[d, "\varphi'"'] & \mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & 0 \\ 0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & 0 \end{tikzcd}$$

as desired. $\square$

08ME **Lemma 91.10.3.** Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (91.10.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$.

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 12.6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}$-modules, see Modules on Sites, Lemma 18.25.1. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}'$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. $\square$

08MF **Lemma 91.10.4.** Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. Then there exists an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (91.10.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}$-module, then there does exist a sequence (91.10.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}'$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules on Sites, Lemma 18.28.8). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{tikzcd} \mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} \arrow[r, hook] \arrow[d] & \mathcal{J}/\mathcal{I}\mathcal{J} \arrow[d, dotted, "\gamma"] \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c"] & \mathcal{K} \end{tikzcd}$$

a diagram of $\mathcal{O}$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r] \arrow[d, "\gamma'"] & \mathcal{H}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (91.10.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (91.10.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 91.10.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\square$

08MG Remark 91.10.5. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. A first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is said to be trivial if there exists a morphism of ringed topoi $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ which is a left inverse to i. The choice of such a morphism π is called a trivialization of the first order thickening. Given π we obtain a splitting

08MH (91.10.5.1) $$\mathcal{O}' = \mathcal{O} \oplus \mathcal{I}$$

as sheaves of algebras on $\mathcal{C}$ by using $\pi^\sharp$ to split the surjection $\mathcal{O}' \to \mathcal{O}$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to the category of $\mathcal{O}$-modules.

08MI Remark 91.10.6. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a trivial first order thickening of ringed topoi and let $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (91.10.5.1) associated to π and the map c to define the $\mathcal{O}'$-module structure and obtain an extension (91.10.0.1). We will call $\mathcal{F}'_{c,triv}$ the trivial extension of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (91.10.0.1) we can use $\pi^\sharp : \mathcal{O} \to \mathcal{O}'$ to think of $\mathcal{F}'$ as an $\mathcal{O}$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\text{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{K})$. Lemma 91.10.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{\begin{matrix} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (91.10.0.1) with } c = c_{\mathcal{F}'} \end{matrix}\right\} \longrightarrow \text{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

08MJ Remark 91.10.7. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Picture

$$\begin{tikzcd} & (Sh(\mathcal{C}), \mathcal{O}) \arrow[dl] \arrow[dr] & \\ (Sh(\mathcal{D}_1), \mathcal{O}'_1) \arrow[rr, "h"] & & (Sh(\mathcal{D}_2), \mathcal{O}'_2) \end{tikzcd}$$

Observe that $h^\sharp : \mathcal{O}'_2 \to \mathcal{O}'_1$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_\mathcal{O} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{tikzcd} \mathcal{I}_2 \otimes_\mathcal{O} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_\mathcal{O} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd}$$

is commutative. Then there is a canonical functoriality

$$\left\{\begin{matrix} \mathcal{F}'_2 \text{ as in (91.10.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \mathcal{F}'_1 \text{ as in (91.10.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{matrix}\right\}$$

Namely, thinking of all sheaves $\mathcal{O}$, $\mathcal{O}'_i$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on $\mathcal{C}$, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[d] & \mathcal{F}'_2 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \mathcal{F}'_1 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

We omit the construction of the $\mathcal{O}'_1$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

08MK Remark 91.10.8. Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be as in Remark 91.10.7. Assume that we are given trivializations $\pi_i : (Sh(\mathcal{D}_i), \mathcal{O}'_i) \to (Sh(\mathcal{C}), \mathcal{O})$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_\mathcal{O} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{tikzcd} \mathcal{I}_2 \otimes_\mathcal{O} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_\mathcal{O} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd}$$

is commutative. In this situation the construction of Remark 91.10.6 induces a commutative diagram

$$\begin{tikzcd} \{\mathcal{F}'_2 \text{ as in (91.10.0.1) with } c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\} \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \arrow[d] \\ \{\mathcal{F}'_1 \text{ as in (91.10.0.1) with } c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\} \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \end{tikzcd}$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 91.10.7.

0CYC Remark 91.10.9. Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be as in Remark 91.10.7. Observe that $h^\sharp : \mathcal{O}'_2 \to \mathcal{O}'_1$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{tikzcd} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd}$$

is commutative. Then we claim the map

$$\mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$$

sends $o(\mathcal{F}, \mathcal{K}_2, c_2)$ to $o(\mathcal{F}, \mathcal{K}_1, c_1)$.

To prove this claim choose an embedding $j_2 : \mathcal{K}_2 \to \mathcal{K}'_2$ where $\mathcal{K}'_2$ is an injective $\mathcal{O}$-module. As in the proof of Lemma 91.10.4 we can choose an extension of $\mathcal{O}_2$-modules

$$0 \to \mathcal{K}'_2 \to \mathcal{E}_2 \to \mathcal{F} \to 0$$

such that $c_{\mathcal{E}_2} = j_2 \circ c_2$. The proof of Lemma 91.10.4 constructs $o(\mathcal{F}, \mathcal{K}_2, c_2)$ as the Yoneda extension class (in the sense of Derived Categories, Section 13.27) of the exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_2 \to \mathcal{K}'_2 \to \mathcal{E}_2/\mathcal{K}_2 \to \mathcal{F} \to 0$$

Let $\mathcal{K}'_1$ be the cokernel of $\mathcal{K}_2 \to \mathcal{K}_1 \oplus \mathcal{K}'_2$. There is an injection $j_1 : \mathcal{K}_1 \to \mathcal{K}'_1$ and a map $\mathcal{K}'_2 \to \mathcal{K}'_1$ forming a commutative square. We form the pushout:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}'_2 \arrow[r] \arrow[d] & \mathcal{E}_2 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}'_1 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

There is a canonical $\mathcal{O}_1$-module structure on $\mathcal{E}_1$ and for this structure we have $c_{\mathcal{E}_1} = j_1 \circ c_1$ (this uses the commutativity of the diagram involving c_1 and c_2 above). The procedure of Lemma 91.10.4 tells us that $o(\mathcal{F}, \mathcal{K}_1, c_1)$ is the Yoneda extension class of the exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_1 \to \mathcal{K}'_1 \to \mathcal{E}_1/\mathcal{K}_1 \to \mathcal{F} \to 0$$

Since we have maps of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[d] & \mathcal{K}'_2 \arrow[r] \arrow[d] & \mathcal{E}_2/\mathcal{K}_2 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K}_2 \arrow[r] & \mathcal{K}'_2 \arrow[r] & \mathcal{E}_2/\mathcal{K}_2 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

we conclude that the claim is true.

08ML Remark 91.10.10. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. We define a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)$$

of $(Sh(\mathcal{C}), \mathcal{O})$ to be a complex if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)$ factors through $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)$, i.e., the first order thickening $(Sh(\mathcal{D}_1), \mathcal{O}'_1)$ of $(Sh(\mathcal{C}), \mathcal{O})$ is trivial and comes with a canonical trivialization $\pi : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$.

We say a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)$$

of $(Sh(\mathcal{C}), \mathcal{O})$ is a short exact sequence if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}$-modules.

08MM Remark 91.10.11. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let

$$(Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)$$

be a complex first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, see Remark 91.10.10. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{tikzcd} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_2"'] \arrow[d] & \mathcal{K}_2 \arrow[d] \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_1"] & \mathcal{K}_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{I}_3 \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_3"'] \arrow[d] & \mathcal{K}_3 \arrow[d] \\ \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \arrow[r, "c_2"] & \mathcal{K}_2 \end{tikzcd}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

as in (91.10.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}'_2$-modules with $c_{\mathcal{F}'_2} = c_2$. In this situation we can apply the functoriality of Remark 91.10.7 to obtain an extension $\mathcal{F}'_1$ of $\mathcal{O}'_1$-modules (we'll describe $\mathcal{F}'_1$ in this special case below). By Remark 91.10.6 using the canonical splitting $\pi : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ of Remark 91.10.10 we obtain $\xi_{\mathcal{F}'_1} \in \text{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \text{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 91.10.4. In this situation we claim that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}'_1}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}'_2 = j'_*\mathcal{F}'_2$ equal to the pushout of $\mathcal{F}'_2$ by j' so that $c_{\mathcal{E}'_2} = j' \circ c_2$. Picture:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[d, "j'"'] & \mathcal{F}'_2 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{E}'_2 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

Set $\mathcal{E}'_3 = \mathcal{E}'_2$ but viewed as an $\mathcal{O}'_3$-module via $\mathcal{O}'_3 \to \mathcal{O}'_2$. Then $c_{\mathcal{E}'_3} = j \circ c_3$. The proof of Lemma 91.10.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}'_3/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}'_1 = \mathcal{F}'_2/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}'_1}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}'_2/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}$-modules using $\pi^\sharp$ where $\pi : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2/\mathcal{K}_3 \arrow[r] \arrow[d] & \mathcal{F}'_2/\mathcal{K}_3 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}/\mathcal{K}_3 \arrow[r] & \mathcal{E}'_3/\mathcal{K}_3 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

which is $\mathcal{O}$-linear (with the $\mathcal{O}$-module structures given above).

91.11. Infinitesimal deformations of modules on ringed topoi

08MN Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We freely use the notation introduced in Section 91.9. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}'$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}$-module structure. Thus the sequence above is an extension as in (91.10.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}'$ we have $i^*\mathcal{O}' = \mathcal{O}$ and $\mathcal{I}\mathcal{O}' = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

08MP **Lemma 91.11.1.** Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.

Proof. This is a special case of Lemma 91.10.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}'$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 12.5.8. □

08MQ Lemma 91.11.2. Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 91.9.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$ and that (f, f') is a strict morphism of thickenings (Definition 91.9.2). Then the following are equivalent

(1) $\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$, and
(2) the canonical map $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). By definition flatness over $\mathcal{O}_\mathcal{B}$ means flatness over $f^{-1}\mathcal{O}_\mathcal{B}$. Similarly for flatness over $f^{-1}\mathcal{O}_{\mathcal{B}'}$. Note that the strictness of (f, f') and the assumption that $\mathcal{F} = i^*\mathcal{F}'$ imply that

$$\mathcal{F} = \mathcal{F}'/(f^{-1}\mathcal{J})\mathcal{F}'$$

as sheaves on $\mathcal{C}$. Moreover, observe that $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}} \mathcal{F}$. Hence the equivalence of (1) and (2) follows from Modules on Sites, Lemma 18.28.15. □

08VU Lemma 91.11.3. Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 91.9.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. Then the following are equivalent

(1) $\mathcal{F}'$ is an $\mathcal{O}'$-module of finite presentation, and
(2) $\mathcal{F}$ is an $\mathcal{O}$-module of finite presentation.

Proof. The implication (1) $\Rightarrow$ (2) follows from Modules on Sites, Lemma 18.23.4. For the converse, assume $\mathcal{F}$ of finite presentation. We may and do assume that $\mathcal{C} = \mathcal{C}'$. By Lemma 91.11.2 we have a short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0$$

Let U be an object of $\mathcal{C}$ such that $\mathcal{F}|_U$ has a presentation

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

After replacing U by the members of a covering we may assume the map $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ lifts to a map $(\mathcal{O}'_U)^{\oplus n} \to \mathcal{F}'|_U$. The induced map $\mathcal{I}^{\oplus n} \to \mathcal{I} \otimes \mathcal{F}$ is surjective by right exactness of $\otimes$. Thus after replacing U by the members of a covering we can find a lift $(\mathcal{O}'|_U)^{\oplus m} \to (\mathcal{O}'|_U)^{\oplus n}$ of the given map $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n}$ such that

$$(\mathcal{O}'_U)^{\oplus m} \to (\mathcal{O}'_U)^{\oplus n} \to \mathcal{F}'|_U \to 0$$

is a complex. Using right exactness of $\otimes$ once more it is seen that this complex is exact. $\square$

08MR Lemma 91.11.4. Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{G}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Proof. Combine Lemmas 91.11.1 and 91.11.2. $\square$

08MS Lemma 91.11.5. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. There exists an element

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 91.11.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology on Sites, Lemma 21.19.1). $\square$

08MT Lemma 91.11.6. Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. There exists an element

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 91.11.5 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J}$ by Lemma 91.11.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}'}(\mathcal{F}', \mathcal{O})$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 91.11.2. Thus $\tau_{\geq -1} Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\text{Ext}^1_\mathcal{O}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \text{Ext}^1_\mathcal{O}(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 13.16.1. □

Second proof. We can apply Lemma 91.10.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_\mathcal{O} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_\mathcal{O} \mathcal{G}$ by Lemma 91.11.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. □

08MU Lemma 91.11.7. Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_\mathcal{B}$. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\text{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{I} \otimes_\mathcal{O} \mathcal{F})$.

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F}$$

is an isomorphism by Lemma 91.11.2. Apply Lemma 91.10.3 with $\mathcal{K} = \mathcal{I} \otimes_\mathcal{O} \mathcal{F}$ and $c = 1$. By Lemma 91.11.2 the corresponding extensions $\mathcal{F}'$ are all flat over $\mathcal{O}_{\mathcal{B}'}$. □

08MV Lemma 91.11.8. Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_\mathcal{B}$. There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if

(1) the canonical map $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F}$ is an isomorphism, and
(2) the class $o(\mathcal{F}, \mathcal{I} \otimes_\mathcal{O} \mathcal{F}, 1) \in \text{Ext}^2_\mathcal{O}(\mathcal{F}, \mathcal{I} \otimes_\mathcal{O} \mathcal{F})$ of Lemma 91.10.4 is zero.

Proof. This follows immediately from the characterization of $\mathcal{O}'$-modules flat over $\mathcal{O}_{\mathcal{B}'}$ of Lemma 91.11.2 and Lemma 91.10.4. □

91.12. Application to flat modules on flat thickenings of ringed topoi

08VV Consider a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{C}), \mathcal{O}) \arrow[r, "i"'] \arrow[d, "f"'] & (Sh(\mathcal{D}), \mathcal{O}') \arrow[d, "f'"] \\ (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \arrow[r, "t"] & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \end{tikzcd}$$

of ringed topoi whose horizontal arrows are first order thickenings as in Situation 91.9.1. Set $\mathcal{I} = \text{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \text{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 91.11.2 to $\mathcal{O}'$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

08VW Lemma 91.12.1. In the situation above.

(1) There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F}, 1) \in \mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$ of Lemma 91.10.4 is zero.
(2) If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\mathrm{Ext}^1_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$.
(3) Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\mathrm{id}_\mathcal{F}$ is canonically isomorphic to $\mathrm{Ext}^0_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$.

Proof. Part (1) follows from Lemma 91.11.8 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 91.11.7. Part (3) follows from Lemma 91.11.4. $\square$

0CYD Situation 91.12.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Consider a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \arrow[r, "h"'] \arrow[d, "f'_1"'] & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) \arrow[d, "f'_2"'] \\ (Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) \arrow[r] & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) \end{tikzcd}$$

where h is a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, the lower horizontal arrow is a morphism of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, each f'_i restricts to f, both pairs (f, f'_i) are strict morphisms of thickenings, and both f'_i are flat. Finally, let $\mathcal{F}$ be an $\mathcal{O}$-module flat over $\mathcal{O}_\mathcal{B}$.

0CYE Lemma 91.12.3. In Situation 91.12.2 the obstruction class $o(\mathcal{F}, f^*\mathcal{J}_2 \otimes_\mathcal{O} \mathcal{F}, 1)$ maps to the obstruction class $o(\mathcal{F}, f^*\mathcal{J}_1 \otimes_\mathcal{O} \mathcal{F}, 1)$ under the canonical map

$$\mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J}_2 \otimes_\mathcal{O} \mathcal{F}) \to \mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_\mathcal{O} \mathcal{F})$$

Proof. Follows from Remark 91.10.9. $\square$

08VX Situation 91.12.4. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Consider a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \arrow[r, "h"'] \arrow[d, "f'_1"'] & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) \arrow[r] \arrow[d, "f'_2"'] & (Sh(\mathcal{C}'_3), \mathcal{O}'_3) \arrow[d, "f'_3"'] \\ (Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) \arrow[r] & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) \arrow[r] & (Sh(\mathcal{B}'_3), \mathcal{O}_{\mathcal{B}'_3}) \end{tikzcd}$$

where (a) the top row is a short exact sequence of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, (b) the lower row is a short exact sequence of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over $\mathcal{O}_{\mathcal{B}'_2}$ and set $\mathcal{F} = \mathcal{F}'_2 \otimes \mathcal{O}$. Let $\pi : (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ be the canonical splitting (Remark 91.10.10).

08VY Lemma 91.12.5. In Situation 91.12.4 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over $\mathcal{O}_{\mathcal{B}'_1}$ restricting to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$. Their difference (Lemma 91.12.1) is an element θ of $\mathrm{Ext}^1_\mathcal{O}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_\mathcal{O} \mathcal{F})$ whose boundary in $\mathrm{Ext}^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_\mathcal{O} \mathcal{F})$ equals the obstruction (Lemma 91.12.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over $\mathcal{O}_{\mathcal{B}'_3}$.

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$ and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F}$. Hence flatness by Lemma 91.11.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 91.12.4 and the fact that $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$. The statement on the obstruction class is a direct translation of the result of Remark 91.10.11 to this particular situation. $\square$

91.13. Deformations of ringed topoi and the naive cotangent complex

08UE In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi. We denote $\mathcal{J} = \text{Ker}(t^\sharp)$ and we identify the underlying topoi of $\mathcal{B}$ and $\mathcal{B}'$. Moreover we assume given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_{\mathcal{B}}$-modules. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08UF (91.13.0.1)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & ? \arrow[r] & \mathcal{O} \arrow[r] & 0 \\
0 \arrow[r] & f^{-1}\mathcal{J} \arrow[r] \arrow[u, "c"] & f^{-1}\mathcal{O}_{\mathcal{B}'} \arrow[r] \arrow[u] & f^{-1}\mathcal{O}_{\mathcal{B}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in (91.9.1.1) where $\text{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a solution to (91.13.0.1).

08UG **Lemma 91.13.1.** *Assume given a commutative diagram of morphisms ringed topoi*

08UH (91.13.1.1)
$$\begin{tikzcd}
 & & (Sh(\mathcal{C}_2), \mathcal{O}_2) \arrow[r, "i_2"] \arrow[d, "f_2"] \arrow[ddll, "g"'] & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) \arrow[d, "f'_2"] \\
 & & (Sh(\mathcal{B}_2), \mathcal{O}_{\mathcal{B}_2}) \arrow[r, "t_2"] \arrow[ddll] & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) \arrow[ddll] \\
(Sh(\mathcal{C}_1), \mathcal{O}_1) \arrow[r, "i_1"] \arrow[d, "f_1"] & (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \arrow[d, "f'_1"] & & \\
(Sh(\mathcal{B}_1), \mathcal{O}_{\mathcal{B}_1}) \arrow[r, "t_1"] & (Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & &
\end{tikzcd}$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = \text{Ker}(i_j^\sharp)$ and assume given a map of $g^{-1}\mathcal{O}_1$-modules $\nu : g^{-1}\mathcal{G}_1 \to \mathcal{G}_2$ giving rise to the commutative

diagram

08UI (91.13.1.2)
$$
\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{G}_2 \arrow[r] & \mathcal{O}'_2 \arrow[r] & \mathcal{O}_2 \arrow[r] & 0 \\
 & 0 \arrow[r] & f_2^{-1}\mathcal{J}_2 \arrow[r] \arrow[u, "c_2"'] & f_2^{-1}\mathcal{O}_{\mathcal{B}'_2} \arrow[r] \arrow[u] & f_2^{-1}\mathcal{O}_{\mathcal{B}_2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{G}_1 \arrow[r] \arrow[uur] & \mathcal{O}'_1 \arrow[r] & \mathcal{O}_1 \arrow[r] \arrow[uur] & 0 & \\
0 \arrow[r] & f_1^{-1}\mathcal{J}_1 \arrow[r] \arrow[u, "c_1"] \arrow[uur] & f_1^{-1}\mathcal{O}_{\mathcal{B}'_1} \arrow[r] \arrow[u] \arrow[uur] & f_1^{-1}\mathcal{O}_{\mathcal{B}_1} \arrow[r] \arrow[u] \arrow[uur] & 0 &
\end{tikzcd}
$$

with front and back solutions to (91.13.0.1). (The north-north-west arrows are maps on $\mathcal{C}_2$ after applying g^{-1} to the source.)

(1) There exist a canonical element in $\mathrm{Ext}^1_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$ whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed topoi $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ fitting into (91.13.1.1) compatibly with ν.

(2) If there exists a morphism $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ fitting into (91.13.1.1) compatibly with ν the set of all such morphisms is a principal homogeneous space under

$$\mathrm{Hom}_{\mathcal{O}_1}(\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_2}(g^*\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^0_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2).$$

Proof. The proof of this lemma is identical to the proof of Lemma 91.7.1. We urge the reader to read that proof instead of this one. We will identify the underlying topoi for every thickening in sight (we have already used this convention in the statement). The equalities in the last statement of the lemma are immediate from the definitions. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topos of all ringed topoi in the lemma is the same.

To do this, observe that $g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1} \to g^{-1}\mathcal{O}_1$, see Modules on Sites, Lemma 18.33.5. Moreover, the degree 0 term of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is a flat $\mathcal{O}_1$-module, hence the canonical map

$$Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \longrightarrow g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \otimes_{g^{-1}\mathcal{O}_1} \mathcal{O}_2$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(NL_{g^{-1}\mathcal{O}_1/g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$$

The set of morphism of ringed topoi $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ fitting into (91.13.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}'_1}$-algebras $g^{-1}\mathcal{O}'_1 \to \mathcal{O}'_2$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (91.13.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \mathrm{id}$ on underlying topoi) and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (91.13.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \mathrm{id}$ on underlying topoi) and we are looking for homomorphisms of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_1}(NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}'_1 \times_{\mathcal{O}_2} \mathcal{O}'_2$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_1$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$, see Modules on Sites, Lemma 18.35.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}_1) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{O}_1)$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{\mathcal{B}'_2}$-algebras

$$a : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_1 \quad \text{and} \quad b : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_2$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (91.13.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_1$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_1}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]/\mathcal{O}_{\mathcal{B}_1}} \otimes_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]} \mathcal{O}_1$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (91.13.1.2). In this case consider the map $\mathcal{O}'_1[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_1$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \longrightarrow \mathcal{O}'_2$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (91.13.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 91.2.1. □

08UJ Lemma 91.13.2. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules on Sites, Lemma 18.35.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{H} \arrow[r] & \Omega \end{tikzcd}$$

This finishes the proof. □

08UK **Lemma 91.13.3.** If there exists a solution to (91.13.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$.

Proof. We observe right away that given two solutions $\mathcal{O}'_1$ and $\mathcal{O}'_2$ to (91.13.0.1) we obtain by Lemma 91.13.1 an obstruction element $o(\mathcal{O}'_1, \mathcal{O}'_2) \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ to the existence of a map $\mathcal{O}'_1 \to \mathcal{O}'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution $\mathcal{O}'$ and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ we can find a second solution $\mathcal{O}'_\xi$ such that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}$ as in Lemma 91.13.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}] \to \mathcal{O}$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]/f^{-1}\mathcal{O}_{\mathcal{B}}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]} \mathcal{O})$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}} \mathcal{O}'$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}'$.

These choices determine an $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebra map $\varphi : \mathcal{O}_{\mathcal{B}'}[\mathcal{E}] \to \mathcal{O}'$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}'$ is the pushout, as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}' \arrow[r] & \mathcal{O} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{I}' \arrow[r] \arrow[u, "\varphi|_{\mathcal{I}'}"] & f^{-1}\mathcal{O}_{\mathcal{B}'}[\mathcal{E}] \arrow[r] \arrow[u] & \mathcal{O} \arrow[r] \arrow[u, "="'] & 0 \end{tikzcd}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}'_\xi$ fitting into a diagram as above. A calculation (omitted) shows that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$ as desired. □

0GQ5 Lemma 91.13.4. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Let $\mathcal{G}$ be an $\mathcal{O}$-module. The set of isomorphism classes of extensions of $f^{-1}\mathcal{O}_\mathcal{B}$-algebras

$$0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$$

where $\mathcal{G}$ is an ideal of square zero[2] is canonically bijective to $\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.

Proof. To prove this we apply the previous results to the case where (91.13.0.1) is given by the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & ? \arrow[r] & \mathcal{O} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & f^{-1}\mathcal{O}_\mathcal{B} \arrow[r, "\mathrm{id}"] \arrow[u] & f^{-1}\mathcal{O}_\mathcal{B} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Thus our lemma follows from Lemma 91.13.3 and the fact that there exists a solution, namely $\mathcal{G} \oplus \mathcal{O}$. (See remark below for a direct construction of the bijection.) □

0GQ6 Remark 91.13.5. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (\mathcal{B}, \mathcal{O}_\mathcal{B})$ and $\mathcal{G}$ be as in Lemma 91.13.4. Consider an extension $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ as in the lemma. We can choose a sheaf of sets $\mathcal{E}$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "\alpha'"'] \arrow[dr, "\alpha"] & \\
\mathcal{O}' \arrow[r] & \mathcal{O}
\end{tikzcd}$$

such that $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}$ is surjective with kernel $\mathcal{J}$. (For example you can take any sheaf of sets surjecting onto $\mathcal{O}'$.) Then

$$NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}} \cong NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]/f^{-1}\mathcal{O}_\mathcal{B}} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]} \mathcal{O}\right)$$

See Modules on Sites, Section 18.35 and in particular Lemma 18.35.2. Of course α' determines a map $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}'$ which in turn determines a map

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{G}$$

which in turn determines the element of $\mathrm{Ext}^1_\mathcal{O}(NL(\alpha), \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$ corresponding to $\mathcal{O}'$ by the bijection of the lemma.

0GQ7 Lemma 91.13.6. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be a $\mathcal{O}_\mathcal{C}$-module. Let $\mathcal{G}$ be a $\mathcal{O}_\mathcal{D}$-module. Let $c : g^*\mathcal{F} \to \mathcal{G}$ be a $\mathcal{O}_\mathcal{D}$-linear map. Finally, consider

(a) $0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_\mathcal{C} \to 0$ an extension of $f^{-1}\mathcal{O}_\mathcal{B}$-algebras corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{F})$, and
(b) $0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_\mathcal{D} \to 0$ an extension of $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}$-algebras corresponding to $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{D}}(NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.

[2]In other words, the set of isomorphism classes of first order thickenings $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}), \mathcal{O}')$ over $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ endowed with an isomorphism $\mathcal{G} \to \mathrm{Ker}(i^\sharp)$ of $\mathcal{O}$-modules.

See Lemma 91.13.4. Then there is a morphism

$$g' : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}'}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}'})$$

of ringed topoi over $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ compatible with g and c if and only if ξ and ζ map to the same element of $\text{Ext}^1_{\mathcal{O}_\mathcal{D}}(Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.

Proof. The stament makes sense as we have the maps

$$\text{Ext}^1_{\mathcal{O}_\mathcal{C}}(NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{F}) \to \text{Ext}^1_{\mathcal{O}_\mathcal{D}}(Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, Lg^*\mathcal{F}) \to \text{Ext}^1_{\mathcal{O}_\mathcal{D}}(Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$$

using the map $Lg^*\mathcal{F} \to g^*\mathcal{F} \xrightarrow{c} \mathcal{G}$ and

$$\text{Ext}^1_{\mathcal{O}_Y}(NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}, \mathcal{G}) \to \text{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$$

using the map $Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \to NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$. The statement of the lemma can be deduced from Lemma 91.13.1 applied to the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}_{\mathcal{D}'} \arrow[r] & \mathcal{O}_\mathcal{D} \arrow[r] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] \arrow[u] & g^{-1}f^{-1}\mathcal{O}_\mathcal{B} \arrow[r] \arrow[u] & g^{-1}f^{-1}\mathcal{O}_\mathcal{B} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[uur] & \mathcal{O}_{\mathcal{C}'} \arrow[r] & \mathcal{O}_\mathcal{C} \arrow[r] \arrow[uur] & 0 & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[uur] & f^{-1}\mathcal{O}_\mathcal{B} \arrow[r] \arrow[u] \arrow[uur] & f^{-1}\mathcal{O}_\mathcal{B} \arrow[r] \arrow[u] \arrow[uur] & 0 &
\end{tikzcd}$$

and a compatibility between the constructions in the proofs of Lemmas 91.13.4 and 91.13.1 whose statement and proof we omit. (See remark below for a direct argument.) □

0GQ8 Remark 91.13.7. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$, $\mathcal{F}$, $\mathcal{G}$, $c : g^*\mathcal{F} \to \mathcal{G}$, $0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_\mathcal{C} \to 0$, $\xi \in \text{Ext}^1_{\mathcal{O}_\mathcal{C}}(NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{F})$, $0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_\mathcal{D} \to 0$, and $\zeta \in \text{Ext}^1_{\mathcal{O}_\mathcal{D}}(NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$ be as in Lemma 91.13.6. Using pushout along $c : g^{-1}\mathcal{F} \to \mathcal{G}$ we can construct an extension

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}'_1 \arrow[r] & g^{-1}\mathcal{O}_\mathcal{C} \arrow[r] & 0 \\
0 \arrow[r] & g^{-1}\mathcal{F} \arrow[r] \arrow[u, "c"] & g^{-1}\mathcal{O}_{\mathcal{C}'} \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_\mathcal{C} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Using pullback along $g^\sharp : g^{-1}\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ we can construct an extension

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}_{\mathcal{D}'} \arrow[r] & \mathcal{O}_\mathcal{D} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[u, equal] & \mathcal{O}'_2 \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_\mathcal{C} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

A diagram chase tells us that there exists a morphism $g' : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}'}) \to (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}'})$ over $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ compatible with g and c if and only if $\mathcal{O}'_1$ is isomorphic to $\mathcal{O}'_2$ as

$g^{-1}f^{-1}\mathcal{O}_\mathcal{B}$-algebra extensions of $g^{-1}\mathcal{O}_\mathcal{C}$ by $\mathcal{G}$. By Lemma 91.13.4 these extensions are classified by the LHS of

$$\mathrm{Ext}^1_{g^{-1}\mathcal{O}_\mathcal{C}}(NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_\mathcal{D}}(Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$$

Here the equality comes from tensor-hom adjunction and the equalities

$$NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}} = g^{-1} NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \quad \text{and} \quad Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} = g^{-1} NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_X} \mathcal{O}_Y$$

For the first of these see Modules on Sites, Lemma 18.35.3; the second follows from the definition of derived pullback. Thus, in order to see that Lemma 91.13.6 is true, it suffices to show that $\mathcal{O}'_1$ corresponds to the image of ξ and that $\mathcal{O}'_2$ correspond to the image of ζ. The correspondence between ξ and $\mathcal{O}'_1$ is immediate from the construction of the class ξ in Remark 91.13.5. For the correspondence between ζ and $\mathcal{O}'_2$, we first choose a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\beta'"'] \arrow[dr, "\beta"] & \\ \mathcal{O}_{\mathcal{D}'} \arrow[r] & \mathcal{O}_\mathcal{D} \end{tikzcd}$$

such that $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}_\mathcal{D}$ is surjective with kernel $\mathcal{K}$. Next choose a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\beta'"'] & \mathcal{E}' \arrow[l, "\varphi"'] \arrow[d, "\alpha'"'] \arrow[dr, "\alpha"] & \\ \mathcal{O}_{\mathcal{D}'} & \mathcal{O}'_2 \arrow[l] \arrow[r] & g^{-1}\mathcal{O}_\mathcal{C} \end{tikzcd}$$

such that $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}'] \to g^{-1}\mathcal{O}_\mathcal{C}$ is surjective with kernel $\mathcal{J}$. (For example just take $\mathcal{E}' = \mathcal{E} \amalg \mathcal{O}'_2$ as a sheaf of sets.) The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ (notation as in Modules, Section 17.31) and in particular $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. Then $NL(\alpha) \cong NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$ and $NL(\beta) \cong NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}}$ and the map of complexes $NL(\alpha) \to NL(\beta)$ represents the map $Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \to NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$ used in the statement of Lemma 91.13.6 (see first part of its proof). Now ζ corresponds to the class of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β', see Remark 91.13.5. Similarly, the extension $\mathcal{O}'_2$ corresponds to the map $\mathcal{J}/\mathcal{J}^2 \to \mathcal{G}$ induced by α'. The commutative diagram above shows that this map is the composition of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β' with the map $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. This proves the compatibility we were looking for.

0GQ9 Lemma 91.13.8. Let $t : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$, $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, $\mathcal{G}$, and $c : \mathcal{J} \to \mathcal{G}$ be as in (91.13.0.1). Denote $\xi \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{J})$ the element corresponding to the extension $\mathcal{O}_{\mathcal{B}'}$ of $\mathcal{O}_\mathcal{B}$ by $\mathcal{J}$ via Lemma 91.13.4. The set of isomorphism classes of solutions is canonically bijective to the fibre of

$$\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_\mathcal{O}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

over the image of ξ.

Proof. By Lemma 91.13.4 applied to $t \circ f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ and the $\mathcal{O}$-module $\mathcal{G}$ we see that elements ζ of $\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$ parametrize extensions $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ of $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebras. By Lemma 91.13.6 applied to

$$(Sh(\mathcal{C}), \mathcal{O}) \xrightarrow{f} (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \xrightarrow{t} (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$$

and $c : \mathcal{J} \to \mathcal{G}$ we see that there is an morphism

$$f' : (Sh(\mathcal{C}), \mathcal{O}') \longrightarrow (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$$

over $(Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ compatible with c and f if and only if ζ maps to ξ. Of course this is the same thing as saying $\mathcal{O}'$ is a solution of (91.13.0.1). □

91.14. Deformations of algebraic spaces

0D15 In this section we spell out what the results in Section 91.13 mean for deformations of algebraic spaces.

0D16 Lemma 91.14.1. Let S be a scheme. Let $i : Z \to Z'$ be a morphism of algebraic spaces over S. The following are equivalent

(1) i is a thickening of algebraic spaces as defined in More on Morphisms of Spaces, Section 76.9, and
(2) the associated morphism $i_{small} : (Sh(Z_{étale}), \mathcal{O}_Z) \to (Sh(Z'_{étale}), \mathcal{O}_{Z'})$ of ringed topoi (Properties of Spaces, Lemma 66.21.3) is a thickening in the sense of Section 91.9.

Proof. We stress that this is not a triviality.

Assume (1). By More on Morphisms of Spaces, Lemma 76.9.6 the morphism i induces an equivalence of small étale sites and in particular of topoi. Of course $i^\sharp$ is surjective with locally nilpotent kernel by definition of thickenings.

Assume (2). (This direction is less important and more of a curiosity.) For any étale morphism $Y' \to Z'$ we see that $Y = Z \times_{Z'} Y'$ has the same étale topos as Y'. In particular, Y' is quasi-compact if and only if Y is quasi-compact because being quasi-compact is a topos theoretic notion (Sites, Lemma 7.17.3). Having said this we see that Y' is quasi-compact and quasi-separated if and only if Y is quasi-compact and quasi-separated (because you can characterize Y' being quasi-separated by saying that for all Y'_1, Y'_2 quasi-compact algebraic spaces étale over Y' we have that $Y'_1 \times_{Y'} Y'_2$ is quasi-compact). Take Y' affine. Then the algebraic space Y is quasi-compact and quasi-separated. For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ we have $H^q(Y, \mathcal{F}) = H^q(Y', (Y \to Y')_*\mathcal{F})$ because the étale topoi are the same. Then $H^q(Y', (Y \to Y')_*\mathcal{F}) = 0$ because the pushforward is quasi-coherent (Morphisms of Spaces, Lemma 67.11.2) and Y is affine. It follows that Y' is affine by Cohomology of Spaces, Proposition 69.16.7 (there surely is a proof of this direction of the lemma avoiding this reference). Hence i is an affine morphism. In the affine case it follows easily from the conditions in Section 91.9 that i is a thickening of algebraic spaces. □

0D17 Lemma 91.14.2. Let S be a scheme. Let $Y \subset Y'$ be a first order thickening of algebraic spaces over S. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. If there exists a flat morphism $f' : X' \to Y'$ of algebraic spaces over S and an isomorphsm $a : X \to X' \times_{Y'} Y$ over Y, then

(1) the set of isomorphism classes of pairs $(f' : X' \to Y', a)$ is principal homogeneous under $\text{Ext}^1_{\mathcal{O}_X}(NL_{X/Y}, f^*\mathcal{C}_{Y/Y'})$, and
(2) the set of automorphisms of $\varphi : X' \to X'$ over Y' which reduce to the identity on $X' \times_{Y'} Y$ is $\text{Ext}^0_{\mathcal{O}_X}(NL_{X/Y}, f^*\mathcal{C}_{Y/Y'})$.

Proof. We will apply the material on deformations of ringed topoi to the small étale topoi of the algebraic spaces in the lemma. We may think of X as a closed subspace of X' so that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of first order thickenings. By Lemma 91.14.1 this translates into a morphism of thickenings of ringed topoi. Then we see from More on Morphisms of Spaces, Lemma 76.18.1 (or from the more general Lemma 91.11.2) that the ideal sheaf of X in X' is equal to $f^*\mathcal{C}_{Y'/Y}$ and this is in fact equivalent to flatness of X' over Y'. Hence we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & f^*\mathcal{C}_{Y/Y'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & f_{small}^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & f_{small}^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u] & f_{small}^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Please compare with (91.13.0.1). Observe that automorphisms φ as in (2) give automorphisms $\varphi^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ fitting in the diagram above. Conversely, an automorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ fitting into the diagram of sheaves above is equal to $\varphi^\sharp$ for some automorphism φ as in (2) by More on Morphisms of Spaces, Lemma 76.9.2. Finally, by More on Morphisms of Spaces, Lemma 76.9.7 if we find another sheaf of rings $\mathcal{A}$ on $X_{\acute{e}tale}$ fitting into the diagram

$$\begin{tikzcd} 0 \arrow[r] & f^*\mathcal{C}_{Y/Y'} \arrow[r] & \mathcal{A} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & f_{small}^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & f_{small}^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u] & f_{small}^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

then there exists a first order thickening $X \subset X''$ with $\mathcal{O}_{X''} = \mathcal{A}$ and applying More on Morphisms of Spaces, Lemma 76.9.2 once more, we obtain a morphism $(f, f'') : (X \subset X'') \to (Y \subset Y')$ with all the desired properties. Thus part (1) follows from Lemma 91.13.3 and part (2) from part (2) of Lemma 91.13.1. (Note that $NL_{X/Y}$ as defined for a morphism of algebraic spaces in More on Morphisms of Spaces, Section 76.21 agrees with $NL_{X/Y}$ as used in Section 91.13.) $\square$

Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules (not necessarily quasi-coherent). Consider the functor

$$F : \left\{ \begin{matrix} \text{extensions of } f^{-1}\mathcal{O}_B \text{ algebras} \\ 0 \to \mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X \to 0 \\ \text{where } \mathcal{F} \text{ is an ideal of square zero} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{extensions of } f^{-1}\mathcal{O}_B \text{ algebras} \\ 0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O}_X \to 0 \\ \text{where } \mathcal{G} \text{ is an ideal of square zero} \end{matrix} \right\}$$

given by pushout.

0D3P Lemma 91.14.3. In the situation above assume that X is quasi-compact and quasi-separated and that $DQ_X(\mathcal{F}) \to DQ_X(\mathcal{G})$ (Derived Categories of Spaces, Section 75.19) is an isomorphism. Then the functor F is an equivalence of categories.

Proof. Recall that $NL_{X/B}$ is an object of $D_{QCoh}(\mathcal{O}_X)$, see More on Morphisms of Spaces, Lemma 76.21.4. Hence our assumption implies the maps

$$\mathrm{Ext}^i_X(NL_{X/B}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_X(NL_{X/B}, \mathcal{G})$$

are isomorphisms for all i. This implies our functor is fully faithful by Lemma 91.13.1. On the other hand, the functor is essentially surjective by Lemma 91.13.3 because we have the solutions $\mathcal{O}_X \oplus \mathcal{F}$ and $\mathcal{O}_X \oplus \mathcal{G}$ in both categories. $\square$

Let S be a scheme. Let $B \subset B'$ be a first order thickening of algebraic spaces over S with ideal sheaf $\mathcal{J}$ which we view either as a quasi-coherent $\mathcal{O}_B$-module or as a quasi-coherent sheaf of ideals on B', see More on Morphisms of Spaces, Section 76.9. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules (not necessarily quasi-coherent). Let $c : f^{-1}\mathcal{J} \to \mathcal{F}$ be a map of $f^{-1}\mathcal{O}_B$-modules and denote $c' : f^{-1}\mathcal{J} \to \mathcal{G}$ the composition. Consider the functor

$$FT : \{\text{solutions to (91.13.0.1) for } \mathcal{F} \text{ and } c\} \longrightarrow \{\text{solutions to (91.13.0.1) for } \mathcal{G} \text{ and } c'\}$$

given by pushout.

0D3Q Lemma 91.14.4. In the situation above assume that X is quasi-compact and quasi-separated and that $DQ_X(\mathcal{F}) \to DQ_X(\mathcal{G})$ (Derived Categories of Spaces, Section 75.19) is an isomorphism. Then the functor FT is an equivalence of categories.

Proof. A solution of (91.13.0.1) for $\mathcal{F}$ in particular gives an extension of $f^{-1}\mathcal{O}_{B'}$-algebras

$$0 \to \mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X \to 0$$

where $\mathcal{F}$ is an ideal of square zero. Similarly for $\mathcal{G}$. Moreover, given such an extension, we obtain a map $c_{\mathcal{O}'} : f^{-1}\mathcal{J} \to \mathcal{F}$. Thus we are looking at the full subcategory of such extensions of $f^{-1}\mathcal{O}_{B'}$-algebras with $c = c_{\mathcal{O}'}$. Clearly, if $\mathcal{O}'' = F(\mathcal{O}')$ where F is the equivalence of Lemma 91.14.3 (applied to $X \to B'$ this time), then $c_{\mathcal{O}''}$ is the composition of $c_{\mathcal{O}'}$ and the map $\mathcal{F} \to \mathcal{G}$. This proves the lemma. $\square$

91.15. Deformations of complexes

0DYQ This section is a warmup for the next one. We will use as much as possible the material in the chapters on commutative algebra.

0DYR Lemma 91.15.1. Let $R' \to R$ be a surjection of rings whose kernel is an ideal I of square zero. For every $K \in D^-(R)$ there is a canonical map

$$\omega(K) : K \longrightarrow K \otimes_R^{\mathbf{L}} I[2]$$

in $D(R)$ with the following properties

(1) $\omega(K) = 0$ if and only if there exists $K' \in D(R')$ with $K' \otimes_{R'}^{\mathbf{L}} R = K$,
(2) given $K \to L$ in $D^-(R)$ the diagram

$$\begin{tikzcd} K \arrow[r, "\omega(K)"'] \arrow[d] & K \otimes_R^{\mathbf{L}} I[2] \arrow[d] \\ L \arrow[r, "\omega(L)"] & L \otimes_R^{\mathbf{L}} I[2] \end{tikzcd}$$

commutes, and
(3) formation of $\omega(K)$ is compatible with ring maps $R' \to S'$ (see proof for a precise statement).

Proof. Choose a bounded above complex $K^\bullet$ of free R-modules representing K. Then we can choose free R'-modules $(K')^n$ lifting K^n. We can choose R'-module maps $(d')^n_K : (K')^n \to (K')^{n+1}$ lifting the differentials $d^n_K : K^n \to K^{n+1}$ of $K^\bullet$. Although the compositions

$$(d')^{n+1}_K \circ (d')^n_K : (K')^n \to (K')^{n+2}$$

may not be zero, they do factor as

$$(K')^n \to K^n \xrightarrow{\omega^n_K} K^{n+2} \otimes_R I = I(K')^{n+2} \to (K')^{n+2}$$

because $d^{n+1} \circ d^n = 0$. A calculation shows that ω^n_K defines a map of complexes. This map of complexes defines $\omega(K)$.

Let us prove this construction is compatible with a map of complexes $\alpha^\bullet : K^\bullet \to L^\bullet$ of bounded above free R-modules and given choices of lifts $(K')^n, (L')^n, (d')^n_K, (d')^n_L$. Namely, choose $(\alpha')^n : (K')^n \to (L')^n$ lifting the components $\alpha^n : K^n \to L^n$. As before we get a factorization

$$(K')^n \to K^n \xrightarrow{h^n} L^{n+1} \otimes_R I = I(L')^{n+1} \to (L')^{n+2}$$

of $(d')^n_L \circ (\alpha')^n - (\alpha')^{n+1} \circ (d')^n_K$. Then it is an pleasant calculation to show that

$$\omega^n_L \circ \alpha^n = (d^{n+1}_L \otimes \mathrm{id}_I) \circ h^n + h^{n+1} \circ d^n_K + (\alpha^{n+2} \otimes \mathrm{id}_I) \circ \omega^n_K$$

This proves the commutativity of the diagram in (2) of the lemma in this particular case. Using this for two different choices of bounded above free complexes representing K, we find that $\omega(K)$ is well defined! And of course (2) holds in general as well.

If K lifts to K' in $D^-(R')$, then we can represent K' by a bounded above complex of free R'-modules and we see immediately that $\omega(K) = 0$. Conversely, going back to our choices $K^\bullet$, $(K')^n$, $(d')^n_K$, if $\omega(K) = 0$, then we can find $g^n : K^n \to K^{n+1} \otimes_R I$ with

$$\omega^n = (d^{n+1}_K \otimes \mathrm{id}_I) \circ g^n + g^{n+1} \circ d^n_K$$

This means that with differentials $(d')^n_K + g^n : (K')^n \to (K')^{n+1}$ we obtain a complex of free R'-modules lifting $K^\bullet$. This proves (1).

Finally, part (3) means the following: Let $R' \to S'$ be a map of rings. Set $S = S' \otimes_{R'} R$ and denote $J = IS' \subset S'$ the square zero kernel of $S' \to S$. Then given $K \in D^-(R)$ the statement is that we get a commutative diagram

$$\begin{tikzcd}
K \otimes^{\mathbf{L}}_R S \arrow[r, "\omega(K)\otimes\mathrm{id}"'] \arrow[d] & (K \otimes^{\mathbf{L}}_R I[2]) \otimes^{\mathbf{L}}_R S \arrow[d] \\
K \otimes^{\mathbf{L}}_R S \arrow[r, "\omega(K\otimes^{\mathbf{L}}_R S)"] & (K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S J[2]
\end{tikzcd}$$

Here the right vertical arrow comes from

$$(K \otimes^{\mathbf{L}}_R I[2]) \otimes^{\mathbf{L}}_R S = (K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S (I \otimes^{\mathbf{L}}_R S)[2] \longrightarrow (K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S J[2]$$

Choose $K^\bullet$, $(K')^n$, and $(d')^n_K$ as above. Then we can use $K^\bullet \otimes_R S$, $(K')^n \otimes_{R'} S'$, and $(d')^n_K \otimes \mathrm{id}_{S'}$ for the construction of $\omega(K \otimes^{\mathbf{L}}_R S)$. With these choices commutativity is immediately verified on the level of maps of complexes. $\square$

91.16. Deformations of complexes on ringed topoi

0DIS This material is taken from [Lie06a].

The material in this section works in the setting of a first order thickening of ringed topoi as defined in Section 91.9. However, in order to simplify the notation we will assume the underlying sites $\mathcal{C}$ and $\mathcal{D}$ are the same. Moreover, the surjective homomorphism $\mathcal{O}' \to \mathcal{O}$ of sheaves of rings will be denoted $\mathcal{O} \to \mathcal{O}_0$ as is perhaps more customary in the literature.

0DIT **Lemma 91.16.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings. Assume given the following data

(1) flat $\mathcal{O}$-modules $\mathcal{G}^n$,
(2) maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{G}^{n+1}$,
(3) a complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules,
(4) maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{K}_0^n$

such that

(a) $H^n(\mathcal{K}_0^\bullet) = 0$ for $n \gg 0$,
(b) $\mathcal{G}^n = 0$ for $n \gg 0$,
(c) with $\mathcal{G}_0^n = \mathcal{G}^n \otimes_\mathcal{O} \mathcal{O}_0$ the induced maps determine a complex $\mathcal{G}_0^\bullet$ and a map of complexes $\mathcal{G}_0^\bullet \to \mathcal{K}_0^\bullet$.

Then there exist

(i) flat $\mathcal{O}$-modules $\mathcal{F}^n$,
(ii) maps of $\mathcal{O}$-modules $\mathcal{F}^n \to \mathcal{F}^{n+1}$,
(iii) maps of $\mathcal{O}$-modules $\mathcal{F}^n \to \mathcal{K}_0^n$,
(iv) maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{F}^n$,

such that $\mathcal{F}^n = 0$ for $n \gg 0$, such that the diagrams

$$\begin{tikzcd}
\mathcal{G}^n \arrow[r] \arrow[d] & \mathcal{G}^{n+1} \arrow[d] \\
\mathcal{F}^n \arrow[r] & \mathcal{F}^{n+1}
\end{tikzcd}$$

commute for all n, such that the composition $\mathcal{G}^n \to \mathcal{F}^n \to \mathcal{K}_0^n$ is the given map $\mathcal{G}^n \to \mathcal{K}_0^n$, and such that with $\mathcal{F}_0^n = \mathcal{F}^n \otimes_\mathcal{O} \mathcal{O}_0$ we obtain a complex $\mathcal{F}_0^\bullet$ and map of complexes $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ which is a quasi-isomorphism.

Proof. We will prove by descending induction on e that we can find $\mathcal{F}^n$, $\mathcal{G}^n \to \mathcal{F}^n$, and $\mathcal{F}^n \to \mathcal{F}^{n+1}$ for $n \geq e$ fitting into a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathcal{G}^{e-1} \arrow[r] \arrow[dd, bend right] & \mathcal{G}^e \arrow[r] \arrow[d] \arrow[dd, bend right] & \mathcal{G}^{e+1} \arrow[r] \arrow[d] \arrow[dd, bend right] & \ldots \\
 & & \mathcal{F}^e \arrow[r] \arrow[d] & \mathcal{F}^{e+1} \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \mathcal{K}_0^{e-1} \arrow[r] & \mathcal{K}_0^e \arrow[r] & \mathcal{K}_0^{e+1} \arrow[r] & \ldots
\end{tikzcd}$$

such that $\mathcal{F}_0^\bullet$ is a complex, the induced map $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ induces an isomorphism on H^n for $n > e$ and a surjection for $n = e$. For $e \gg 0$ this is true because we can take $\mathcal{F}^n = 0$ for $n \geq e$ in that case by assumptions (a) and (b).

Induction step. We have to construct $\mathcal{F}^{e-1}$ and the maps $\mathcal{G}^{e-1} \to \mathcal{F}^{e-1}$, $\mathcal{F}^{e-1} \to \mathcal{F}^e$, and $\mathcal{F}^{e-1} \to \mathcal{K}_0^{e-1}$. We will choose $\mathcal{F}^{e-1} = A \oplus B \oplus C$ as a direct sum of three pieces.

For the first we take $A = \mathcal{G}^{e-1}$ and we choose our map $\mathcal{G}^{e-1} \to \mathcal{F}^{e-1}$ to be the inclusion of the first summand. The maps $A \to \mathcal{K}_0^{e-1}$ and $A \to \mathcal{F}^e$ will be the obvious ones.

To choose B we consider the surjection (by induction hypothesis)

$$\gamma : \mathrm{Ker}(\mathcal{F}_0^e \to \mathcal{F}_0^{e+1}) \longrightarrow \mathrm{Ker}(\mathcal{K}_0^e \to \mathcal{K}_0^{e+1})/\mathrm{Im}(\mathcal{K}_0^{e-1} \to \mathcal{K}_0^e)$$

We can choose a set I, for each $i \in I$ an object U_i of $\mathcal{C}$, and sections $s_i \in \mathcal{F}^e(U_i)$, $t_i \in \mathcal{K}_0^{e-1}(U_i)$ such that

(1) s_i maps to a section of $\mathrm{Ker}(\gamma) \subset \mathrm{Ker}(\mathcal{F}_0^e \to \mathcal{F}_0^{e+1})$,
(2) s_i and t_i map to the same section of $\mathcal{K}_0^e$,
(3) the sections s_i generate $\mathrm{Ker}(\gamma)$ as an $\mathcal{O}_0$-module.

We omit giving the full justification for this; one uses that $\mathcal{F}^e \to \mathcal{F}_0^e$ is a surjective maps of sheaves of sets. Then we set to put

$$B = \bigoplus\nolimits_{i \in I} j_{U_i!}\mathcal{O}_{U_i}$$

and define the maps $B \to \mathcal{F}^e$ and $B \to \mathcal{K}_0^{e-1}$ by using s_i and t_i to determine where to send the summand $j_{U_i!}\mathcal{O}_{U_i}$.

With $\mathcal{F}^{e-1} = A \oplus B$ and maps as above, this produces a diagram as above for $e-1$ such that $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ induces an isomorphism on H^n for $n \geq e$. To get the map to be surjective on H^{e-1} we choose the summand C as follows. Choose a set J, for each $j \in J$ an object U_j of $\mathcal{C}$ and a section t_j of $\mathrm{Ker}(\mathcal{K}_0^{e-1} \to \mathcal{K}_0^e)$ over U_j such that these sections generate this kernel over $\mathcal{O}_0$. Then we put

$$C = \bigoplus\nolimits_{j \in J} j_{U_j!}\mathcal{O}_{U_j}$$

and the zero map $C \to \mathcal{F}^e$ and the map $C \to \mathcal{K}_0^{e-1}$ by using s_j to determine where to the summand $j_{U_j!}\mathcal{O}_{U_j}$. This finishes the induction step by taking $\mathcal{F}^{e-1} = A \oplus B \oplus C$ and maps as indicated. $\square$

0DIU Lemma 91.16.2. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. For every object K_0 in $D^-(\mathcal{O}_0)$ there is a canonical map

$$\omega(K_0) : K_0 \longrightarrow K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I}[2]$$

in $D(\mathcal{O}_0)$ such that for any map $K_0 \to L_0$ in $D^-(\mathcal{O}_0)$ the diagram

$$\begin{tikzcd} K_0 \arrow[r, "\omega(K_0)"'] \arrow[d] & (K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})[2] \arrow[d] \\ L_0 \arrow[r, "\omega(L_0)"] & (L_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})[2] \end{tikzcd}$$

commutes.

Proof. Represent K_0 by any complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules. Apply Lemma 91.16.1 with $\mathcal{G}^n = 0$ for all n. Denote $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ the maps produced by the lemma. Then we see that $d \circ d : \mathcal{F}^n \to \mathcal{F}^{n+2}$ is zero modulo $\mathcal{I}$. Since $\mathcal{F}^n$ is flat, we see that $\mathcal{I}\mathcal{F}^n = \mathcal{F}^n \otimes_\mathcal{O} \mathcal{I} = \mathcal{F}_0^n \otimes_{\mathcal{O}_0} \mathcal{I}$. Hence we obtain a canonical map of complexes

$$d \circ d : \mathcal{F}_0^\bullet \longrightarrow (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]$$

Since $\mathcal{F}_0^\bullet$ is a bounded above complex of flat $\mathcal{O}_0$-modules, it is K-flat and may be used to compute derived tensor product. Moreover, the map of complexes $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ is a quasi-isomorphism by construction. Therefore the source and target of the map just constructed represent K_0 and $K_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I}[2]$ and we obtain our map $\omega(K_0)$.

Let us show that this procedure is compatible with maps of complexes. Namely, let $\mathcal{L}_0^\bullet$ represent another object of $D^-(\mathcal{O}_0)$ and suppose that

$$\mathcal{K}_0^\bullet \longrightarrow \mathcal{L}_0^\bullet$$

is a map of complexes. Apply Lemma 91.16.1 for the complex $\mathcal{L}_0^\bullet$, the flat modules $\mathcal{F}^n$, the maps $\mathcal{F}^n \to \mathcal{F}^{n+1}$, and the compositions $\mathcal{F}^n \to \mathcal{K}_0^n \to \mathcal{L}_0^n$ (we apologize for the reversal of letters used). We obtain flat modules $\mathcal{G}^n$, maps $\mathcal{F}^n \to \mathcal{G}^n$, maps $\mathcal{G}^n \to \mathcal{G}^{n+1}$, and maps $\mathcal{G}^n \to \mathcal{L}_0^n$ with all properties as in the lemma. Then it is clear that

$$\begin{tikzcd} \mathcal{F}_0^\bullet \arrow[r] \arrow[d] & (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \arrow[d] \\ \mathcal{G}_0^\bullet \arrow[r] & (\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \end{tikzcd}$$

is a commutative diagram of complexes.

To see that $\omega(K_0)$ is well defined, suppose that we have two complexes $\mathcal{K}_0^\bullet$ and $(\mathcal{K}_0')^\bullet$ of $\mathcal{O}_0$-modules representing K_0 and two systems $(\mathcal{F}^n, d : \mathcal{F}^n \to \mathcal{F}^{n+1}, \mathcal{F}^n \to \mathcal{K}_0^n)$ and $((\mathcal{F}')^n, d : (\mathcal{F}')^n \to (\mathcal{F}')^{n+1}, (\mathcal{F}')^n \to \mathcal{K}_0^n)$ as above. Then we can choose a complex $(\mathcal{K}_0'')^\bullet$ and quasi-isomorphisms $\mathcal{K}_0^\bullet \to (\mathcal{K}_0'')^\bullet$ and $(\mathcal{K}_0')^\bullet \to (\mathcal{K}_0'')^\bullet$ realizing the fact that both complexes represent K_0 in the derived category. Next, we apply the result of the previous paragraph to

$$(\mathcal{K}_0)^\bullet \oplus (\mathcal{K}_0')^\bullet \longrightarrow (\mathcal{K}_0'')^\bullet$$

This produces a commutative diagram

$$\begin{tikzcd} \mathcal{F}_0^\bullet \oplus (\mathcal{F}_0')^\bullet \arrow[r] \arrow[d] & (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \oplus ((\mathcal{F}_0')^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \arrow[d] \\ \mathcal{G}_0^\bullet \arrow[r] & (\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \end{tikzcd}$$

Since the vertical arrows give quasi-isomorphisms on the summands we conclude the desired commutativity in $D(\mathcal{O}_0)$.

Having established well-definedness, the statement on compatibility with maps is a consequence of the result in the second paragraph. $\square$

0DIV Lemma 91.16.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : K \to L$ be a map of $D^-(\mathcal{O})$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $n \in \mathbf{Z}$.

(1) If $H^i(\alpha)$ is an isomorphism for $i \geq n$, then $H^i(\alpha \otimes_\mathcal{O}^{\mathbf{L}} \mathrm{id}_\mathcal{F})$ is an isomorphism for $i \geq n$.

(2) If $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, then $H^i(\alpha \otimes_{\mathcal{O}}^{\mathbf{L}} \text{id}_{\mathcal{F}})$ is an isomorphism for $i > n$ and surjective for $i = n$.

Proof. Choose a distinguished triangle

$$K \to L \to C \to K[1]$$

In case (2) we see that $H^i(C) = 0$ for $i \geq n$. Hence $H^i(C \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) = 0$ for $i \geq n$ by (the dual of) Derived Categories, Lemma 13.16.1. This in turn shows that $H^i(\alpha \otimes_{\mathcal{O}}^{\mathbf{L}} \text{id}_{\mathcal{F}})$ is an isomorphism for $i > n$ and surjective for $i = n$. In case (1) we moreover see that $H^{n-1}(L) \to H^{n-1}(C)$ is surjective. Considering the diagram

$$\begin{tikzcd} H^{n-1}(L) \otimes_{\mathcal{O}} \mathcal{F} \arrow[r] \arrow[d] & H^{n-1}(C) \otimes_{\mathcal{O}} \mathcal{F} \arrow[d, equal] \\ H^{n-1}(L \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) \arrow[r] & H^{n-1}(C \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) \end{tikzcd}$$

we conclude the lower horizontal arrow is surjective. Combined with what was said before this implies that $H^n(\alpha \otimes_{\mathcal{O}}^{\mathbf{L}} \text{id}_{\mathcal{F}})$ is an isomorphism. □

0DIW Lemma 91.16.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. For every object K_0 in $D^-(\mathcal{O}_0)$ the following are equivalent

(1) the class $\omega(K_0) \in \text{Ext}^2_{\mathcal{O}_0}(K_0, K_0 \otimes_{\mathcal{O}_0} \mathcal{I})$ constructed in Lemma 91.16.2 is zero,
(2) there exists $K \in D^-(\mathcal{O})$ with $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0 = K_0$ in $D(\mathcal{O}_0)$.

Proof. Let K be as in (2). Then we can represent K by a bounded above complex $\mathcal{F}^\bullet$ of flat $\mathcal{O}$-modules. Then $\mathcal{F}_0^\bullet = \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$ represents K_0 in $D(\mathcal{O}_0)$. Since $d_{\mathcal{F}^\bullet} \circ d_{\mathcal{F}^\bullet} = 0$ as $\mathcal{F}^\bullet$ is a complex, we see from the very construction of $\omega(K_0)$ that it is zero.

Assume (1). Let $\mathcal{F}^n$, $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ be as in the construction of $\omega(K_0)$. The nullity of $\omega(K_0)$ implies that the map

$$\omega = d \circ d : \mathcal{F}_0^\bullet \longrightarrow (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]$$

is zero in $D(\mathcal{O}_0)$. By definition of the derived category as the localization of the homotopy category of complexes of $\mathcal{O}_0$-modules, there exists a quasi-isomorphism $\alpha : \mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$ such that there exist $\mathcal{O}_0$-modules maps $h^n : \mathcal{G}_0^n \to \mathcal{F}_0^{n+1} \otimes_{\mathcal{O}} \mathcal{I}$ with

$$\omega \circ \alpha = d_{\mathcal{F}_0^\bullet \otimes \mathcal{I}} \circ h + h \circ d_{\mathcal{G}_0^\bullet}$$

We set

$$\mathcal{H}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{G}_0^n$$

and we define

$$d' : \mathcal{H}^n \longrightarrow \mathcal{H}^{n+1}, \quad (f^n, g_0^n) \longmapsto (d(f^n) - h^n(g_0^n), d(g_0^n))$$

with obvious notation using that $\mathcal{F}_0^{n+1} \otimes_{\mathcal{O}_0} \mathcal{I} = \mathcal{F}^{n+1} \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{I}\mathcal{F}^{n+1} \subset \mathcal{F}^{n+1}$. Then one checks $d' \circ d' = 0$ by our choice of h^n and definition of ω. Hence $\mathcal{H}^\bullet$ defines an object in $D(\mathcal{O})$. On the other hand, there is a short exact sequence of complexes of $\mathcal{O}$-modules

$$0 \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} \to \mathcal{H}^\bullet \to \mathcal{G}_0^\bullet \to 0$$

We still have to show that $\mathcal{H}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0$ is isomorphic to K_0. Choose a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{H}^\bullet$ where $\mathcal{E}^\bullet$ is a bounded above complex of flat $\mathcal{O}$-modules. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I} \arrow[r] \arrow[d, "\beta"] & \mathcal{E}^\bullet \arrow[r] \arrow[d, "\gamma"] & \mathcal{E}_0^\bullet \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} \arrow[r] & \mathcal{H}^\bullet \arrow[r] & \mathcal{G}_0^\bullet \arrow[r] & 0 \end{tikzcd}$$

We claim that δ is a quasi-isomorphism. Since $H^i(\delta)$ is an isomorphism for $i \gg 0$, we can use descending induction on n such that $H^i(\delta)$ is an isomorphism for $i \geq n$. Observe that $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}$ represents $\mathcal{E}_0^\bullet \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I}$, that $\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}$ represents $\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I}$, and that $\beta = \delta \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathrm{id}_{\mathcal{I}}$ as maps in $D(\mathcal{O}_0)$. This is true because $\beta = (\alpha \otimes \mathrm{id}_{\mathcal{I}}) \circ (\delta \otimes \mathrm{id}_{\mathcal{I}})$. Suppose that $H^i(\delta)$ is an isomorphism in degrees $\geq n$. Then the same is true for β by what we just said and Lemma 91.16.3. Then we can look at the diagram

$$\begin{tikzcd} H^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}) \arrow[r] \arrow[d, "H^{n-1}(\beta)"] & H^{n-1}(\mathcal{E}^\bullet) \arrow[r] \arrow[d] & H^{n-1}(\mathcal{E}_0^\bullet) \arrow[r] \arrow[d, "H^{n-1}(\delta)"] & H^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}) \arrow[r] \arrow[d, "H^n(\beta)"] & H^n(\mathcal{E}^\bullet) \arrow[d] \\ H^{n-1}(\mathcal{F}_0^\bullet \otimes_{\mathcal{O}} \mathcal{I}) \arrow[r] & H^{n-1}(\mathcal{H}^\bullet) \arrow[r] & H^{n-1}(\mathcal{G}_0^\bullet) \arrow[r] & H^n(\mathcal{F}_0^\bullet \otimes_{\mathcal{O}} \mathcal{I}) \arrow[r] & H^n(\mathcal{H}^\bullet) \end{tikzcd}$$

Using Homology, Lemma 12.5.19 we see that $H^{n-1}(\delta)$ is surjective. This in turn implies that $H^{n-1}(\beta)$ is surjective by Lemma 91.16.3. Using Homology, Lemma 12.5.19 again we see that $H^{n-1}(\delta)$ is an isomorphism. The claim holds by induction, so δ is a quasi-isomorphism which is what we wanted to show. $\square$

0DIX Lemma 91.16.5. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings. Assume given the following data

(1) a complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$,
(2) a complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules,
(3) a quasi-isomorphism $\mathcal{K}_0^\bullet \to \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$,

Then there exist a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ such that the map of complexes $\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0 \to \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$ factors through $\mathcal{K}_0^\bullet$ in the homotopy category of complexes of $\mathcal{O}_0$-modules.

Proof. Set $\mathcal{F}_0^\bullet = \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$. By Derived Categories, Lemma 13.9.8 there exists a factorization

$$\mathcal{K}_0^\bullet \to \mathcal{L}_0^\bullet \to \mathcal{F}_0^\bullet$$

of the given map such that the first arrow has an inverse up to homotopy and the second arrow is termwise split surjective. Hence we may assume that $\mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet$ is termwise surjective. In that case we take

$$\mathcal{G}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{K}_0^n$$

and everything is clear. $\square$

0DIY Lemma 91.16.6. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. Let $K, L \in D^-(\mathcal{O})$. Set $K_0 = K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0$ and $L_0 = L \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0$ in $D^-(\mathcal{O}_0)$. Given $\alpha_0 : K_0 \to L_0$ in $D(\mathcal{O}_0)$ there is a canonical element

$$o(\alpha_0) \in \mathrm{Ext}^1_{\mathcal{O}_0}(K_0, L_0 \otimes_{\mathcal{O}_0}^{\mathbf{L}} \mathcal{I})$$

whose vanishing is necessary and sufficient for the existence of a map $\alpha : K \to L$ in $D(\mathcal{O})$ with $\alpha_0 = \alpha \otimes^{\mathbf{L}}_{\mathcal{O}} \text{id}$.

Proof. Finding $\alpha : K \to L$ lifing α_0 is the same as finding $\alpha : K \to L$ such that the composition $K \xrightarrow{\alpha} L \to L_0$ is equal to the composition $K \to K_0 \xrightarrow{\alpha_0} L_0$. The short exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}_0 \to 0$ gives rise to a canonical distinguished triangle

$$L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I} \to L \to L_0 \to (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]$$

in $D(\mathcal{O})$. By Derived Categories, Lemma 13.4.2 the composition

$$K \to K_0 \xrightarrow{\alpha_0} L_0 \to (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]$$

is zero if and only if we can find $\alpha : K \to L$ lifting α_0. The composition is an element in

$$\text{Hom}_{D(\mathcal{O})}(K, (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]) = \text{Hom}_{D(\mathcal{O}_0)}(K_0, (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]) = \text{Ext}^1_{\mathcal{O}_0}(K_0, L_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$

by adjunction. □

0DIZ Lemma 91.16.7. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. Let $K_0 \in D^-(\mathcal{O})$. A lift of K_0 is a pair (K, α_0) consisting of an object K in $D^-(\mathcal{O})$ and an isomorphism $\alpha_0 : K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 \to K_0$ in $D(\mathcal{O}_0)$.

(1) Given a lift (K, α) the group of automorphism of the pair is canonically the cokernel of a map
$$\text{Ext}^{-1}_{\mathcal{O}_0}(K_0, K_0) \longrightarrow \text{Hom}_{\mathcal{O}_0}(K_0, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$
(2) If there is a lift, then the set of isomorphism classes of lifts is principal homogenenous under $\text{Ext}^1_{\mathcal{O}_0}(K_0, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$.

Proof. An automorphism of (K, α) is a map $\varphi : K \to K$ in $D(\mathcal{O})$ with $\varphi \otimes_{\mathcal{O}} \text{id}_{\mathcal{O}_0} = \text{id}$. This is the same thing as saying that

$$K \xrightarrow{\varphi - \text{id}} K \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$$

is zero. We conclude the group of automorphisms is the cokernel of a map

$$\text{Hom}_{\mathcal{O}}(K, K_0[-1]) \longrightarrow \text{Hom}_{\mathcal{O}}(K, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$

by the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I} \to K \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 \to (K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]$$

in $D(\mathcal{O})$ and Derived Categories, Lemma 13.4.2. To translate into the groups in the lemma use adjunction of the restriction functor $D(\mathcal{O}_0) \to D(\mathcal{O})$ and $- \otimes_{\mathcal{O}} \mathcal{O}_0 : D(\mathcal{O}) \to D(\mathcal{O}_0)$. This proves (1).

Proof of (2). Assume that $K_0 = K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$ in $D(\mathcal{O})$. By Lemma 91.16.6 the map sending a lift (K', α_0) to the obstruction $o(\alpha_0)$ to lifting α_0 defines a canonical injective map from the set of isomomorphism classes of pairs to $\text{Ext}^1_{\mathcal{O}_0}(K_0, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$. To finish the proof we show that it is surjective. Pick $\xi : K_0 \to (K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})[1]$ in the Ext^1 of the lemma. Choose a bounded above complex $\mathcal{F}^\bullet$ of flat $\mathcal{O}$-modules representing K. The map ξ can be represented as $t \circ s^{-1}$ where $s : \mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet$ is a quasi-isomorphism and $t : \mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}[1]$ is a map of complexes. By Lemma 91.16.5 we can assume there exists a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ of complexes of $\mathcal{O}$-modules such that $\mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$ factors through s up to homotopy. We may and do replace $\mathcal{G}^\bullet$ by a bounded above complex of flat $\mathcal{O}$-modules (by picking a qis from such to $\mathcal{G}^\bullet$ and replacing). Then we see that ξ is represented by a map of complexes $t : \mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}[1]$ and the quasi-isomorphism $\mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$. Set

$$\mathcal{H}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{G}_0^n$$

with differentials

$$\mathcal{H}^n \to \mathcal{H}^{n+1}, \quad (f^n, g_0^n) \mapsto (d(f^n) + t(g_0^n), d(g_0^n))$$

This makes sense as $\mathcal{F}_0^{n+1} \otimes_{\mathcal{O}_0} \mathcal{I} = \mathcal{F}^{n+1} \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{I}\mathcal{F}^{n+1} \subset \mathcal{F}^{n+1}$. We omit the computation that shows that $\mathcal{H}^\bullet$ is a complex of $\mathcal{O}$-modules. By construction there is a short exact sequence

$$0 \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} \to \mathcal{H}^\bullet \to \mathcal{G}_0^\bullet \to 0$$

of complexes of $\mathcal{O}$-modules. Exactly as in the proof of Lemma 91.16.4 one shows that this sequence induces an isomorphism $\alpha_0 : \mathcal{H}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}_0 \to \mathcal{G}_0^\bullet$ in $D(\mathcal{O}_0)$. In other words, we have produced a pair $(\mathcal{H}^\bullet, \alpha_0)$. We omit the verification that $o(\alpha_0) = \xi$; hint: $o(\alpha_0)$ can be computed explitly in this case as we have maps $\mathcal{H}^n \to \mathcal{F}^n$ (not compatible with differentials) lifting the components of α_0. This finishes the proof. $\square$

91.17. Other chapters

Preliminaries

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(4) Categories
(5) Topology
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(7) Sites and Sheaves
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(9) Fields
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Schemes

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Topics in Scheme Theory

(42) Chow Homology
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CHAPTER 92

The Cotangent Complex

08P5 92.1. Introduction

08P6 The goal of this chapter is to construct the cotangent complex of a ring map, of a morphism of schemes, and of a morphism of algebraic spaces. Some references are the notes [Qui], the paper [Qui70], and the books [And67] and [Ill72].

92.2. Advice for the reader

08UM In writing this chapter we have tried to minimize the use of simplicial techniques. We view the choice of a resolution $P_\bullet$ of a ring B over a ring A as a tool to calculating the homology of abelian sheaves on the category $\mathcal{C}_{B/A}$, see Remark 92.5.5. This is similar to the role played by a "good cover" to compute cohomology using the Čech complex. To read a bit on homology on categories, please visit Cohomology on Sites, Section 21.39. The derived lower shriek functor $L\pi_!$ is to homology what $R\Gamma(\mathcal{C}_{B/A}, -)$ is to cohomology. The category $\mathcal{C}_{B/A}$, studied in Section 92.4, is the opposite of the category of factorizations $A \to P \to B$ where P is a polynomial algebra over A. This category comes with maps of sheaves of rings

$$\underline{A} \longrightarrow \mathcal{O} \longrightarrow \underline{B}$$

where over the object $U = (P \to B)$ we have $\mathcal{O}(U) = P$. It turns out that we obtain the cotangent complex of B over A as

$$L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{B})$$

see Lemma 92.4.3. We have consistently tried to use this point of view to prove the basic properties of cotangent complexes of ring maps. In particular, all of the results can be proven without relying on the existence of standard resolutions, although we have not done so. The theory is quite satisfactory, except that perhaps the proof of the fundamental triangle (Proposition 92.7.4) uses just a little bit more theory on derived lower shriek functors. To provide the reader with an alternative, we give a rather complete sketch of an approach to this result based on simple properties of standard resolutions in Remarks 92.7.5 and 92.7.6.

Our approach to the cotangent complex for morphisms of ringed topoi, morphisms of schemes, morphisms of algebraic spaces, etc is to deduce as much as possible from the case of "plain ring maps" discussed above.

92.3. The cotangent complex of a ring map

08PL Let A be a ring. Let Alg_A be the category of A-algebras. Consider the pair of adjoint functors (U, V) where $V : \text{Alg}_A \to \text{Sets}$ is the forgetful functor and $U : \text{Sets} \to \text{Alg}_A$ assigns to a set E the polynomial algebra $A[E]$ on E over A. Let

$X_\bullet$ be the simplicial object of $\text{Fun}(\text{Alg}_A, \text{Alg}_A)$ constructed in Simplicial, Section 14.34.

Consider an A-algebra B. Denote $P_\bullet = X_\bullet(B)$ the resulting simplicial A-algebra. Recall that $P_0 = A[B]$, $P_1 = A[A[B]]$, and so on. In particular each term P_n is a polynomial A-algebra. Recall also that there is an augmentation

$$\epsilon : P_\bullet \longrightarrow B$$

where we view B as a constant simplicial A-algebra.

08PM Definition 92.3.1. Let $A \to B$ be a ring map. The standard resolution of B over A is the augmentation $\epsilon : P_\bullet \to B$ with terms

$$P_0 = A[B], \quad P_1 = A[A[B]], \quad \ldots$$

and maps as constructed above.

It will turn out that we can use the standard resolution to compute left derived functors in certain settings.

08PN Definition 92.3.2. The cotangent complex $L_{B/A}$ of a ring map $A \to B$ is the complex of B-modules associated to the simplicial B-module

$$\Omega_{P_\bullet/A} \otimes_{P_\bullet, \epsilon} B$$

where $\epsilon : P_\bullet \to B$ is the standard resolution of B over A.

In Simplicial, Section 14.23 we associate a chain complex to a simplicial module, but here we work with cochain complexes. Thus the term $L_{B/A}^{-n}$ in degree $-n$ is the B-module $\Omega_{P_n/A} \otimes_{P_n, \epsilon_n} B$ and $L_{B/A}^m = 0$ for $m > 0$.

08PP Remark 92.3.3. Let $A \to B$ be a ring map. Let $\mathcal{A}$ be the category of arrows $\psi : C \to B$ of A-algebras and let $\mathcal{S}$ be the category of maps $E \to B$ where E is a set. There are adjoint functors $V : \mathcal{A} \to \mathcal{S}$ (the forgetful functor) and $U : \mathcal{S} \to \mathcal{A}$ which sends $E \to B$ to $A[E] \to B$. Let $X_\bullet$ be the simplicial object of $\text{Fun}(\mathcal{A}, \mathcal{A})$ constructed in Simplicial, Section 14.34. The diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left] \arrow[d] & \mathcal{S} \arrow[l, shift left] \arrow[d] \\ \text{Alg}_A \arrow[r, shift left] & \text{Sets} \arrow[l, shift left] \end{tikzcd}$$

commutes. It follows that $X_\bullet(\text{id}_B : B \to B)$ is equal to the standard resolution of B over A.

08S9 Lemma 92.3.4. Let $A_i \to B_i$ be a system of ring maps over a directed index set I. Then $\text{colim}\, L_{B_i/A_i} = L_{\text{colim}\, B_i/\,\text{colim}\, A_i}$.

Proof. This is true because the forgetful functor $V : A\text{-Alg} \to \text{Sets}$ and its adjoint $U : \text{Sets} \to A\text{-Alg}$ commute with filtered colimits. Moreover, the functor $B/A \mapsto \Omega_{B/A}$ does as well (Algebra, Lemma 10.131.5). $\square$

92.4. Simplicial resolutions and derived lower shriek

08PQ Let $A \to B$ be a ring map. Consider the category whose objects are A-algebra maps $\alpha : P \to B$ where P is a polynomial algebra over A (in some set[1] of variables) and whose morphisms $s : (\alpha : P \to B) \to (\alpha' : P' \to B)$ are A-algebra homomorphisms $s : P \to P'$ with $\alpha' \circ s = \alpha$. Let $\mathcal{C} = \mathcal{C}_{B/A}$ denote the opposite of this category. The reason for taking the opposite is that we want to think of objects (P, α) as corresponding to the diagram of affine schemes

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(P) \arrow[ld] \\ \mathrm{Spec}(A) & \end{tikzcd}$$

We endow $\mathcal{C}$ with the chaotic topology (Sites, Example 7.6.6), i.e., we endow $\mathcal{C}$ with the structure of a site where coverings are given by identities so that all presheaves are sheaves. Moreover, we endow $\mathcal{C}$ with two sheaves of rings. The first is the sheaf $\mathcal{O}$ which sends to object (P, α) to P. Then second is the constant sheaf B, which we will denote $\underline{B}$. We obtain the following diagram of morphisms of ringed topoi

08PR (92.4.0.1)
$$\begin{tikzcd} (Sh(\mathcal{C}), \underline{B}) \arrow[r, "i"'] \arrow[d, "\pi"'] & (Sh(\mathcal{C}), \mathcal{O}) \\ (Sh(*), B) & \end{tikzcd}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{B}$ is the obvious map. The map π is as in Cohomology on Sites, Example 21.39.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{B})$ left adjoint to $Ri_* = i_* : D(\underline{B}) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{B}) \longrightarrow D(B)$ left adjoint to $\pi^* = \pi^{-1} : D(B) \to D(\underline{B})$.

08PS **Lemma 92.4.1.** With notation as above let $P_\bullet$ be a simplicial A-algebra endowed with an augmentation $\epsilon : P_\bullet \to B$. Assume each P_n is a polynomial algebra over A and ϵ is a trivial Kan fibration on underlying simplicial sets. Then

$$L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)$$

in $D(\mathrm{Ab})$, resp. $D(B)$ functorially in $\mathcal{F}$ in $\mathrm{Ab}(\mathcal{C})$, resp. $\mathrm{Mod}(\underline{B})$.

Proof. We will use the criterion of Cohomology on Sites, Lemma 21.39.7 to prove this. Given an object $U = (Q, \beta)$ of $\mathcal{C}$ we have to show that

$$S_\bullet = \mathrm{Mor}_\mathcal{C}((Q, \beta), (P_\bullet, \epsilon))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set E (this is possible by our choice of the category $\mathcal{C}$). We see that

$$S_\bullet = \mathrm{Mor}_{\mathrm{Sets}}((E, \beta|_E), (P_\bullet, \epsilon))$$

[1]It suffices to consider sets of cardinality at most the cardinality of B.

Let $*$ be the constant simplicial set on a singleton. For $b \in B$ let $F_{b,\bullet}$ be the simplicial set defined by the cartesian diagram

$$\begin{tikzcd} F_{b,\bullet} \arrow[r] \arrow[d] & P_\bullet \arrow[d, "\epsilon"] \\ * \arrow[r, "b"] & B \end{tikzcd}$$

With this notation $S_\bullet = \prod_{e \in E} F_{\beta(e),\bullet}$. Since we assumed ϵ is a trivial Kan fibration we see that $F_{b,\bullet} \to *$ is a trivial Kan fibration (Simplicial, Lemma 14.30.3). Thus $S_\bullet \to *$ is a trivial Kan fibration (Simplicial, Lemma 14.30.6). Therefore $S_\bullet$ is homotopy equivalent to $*$ (Simplicial, Lemma 14.30.8). □

In particular, we can use the standard resolution of B over A to compute derived lower shriek.

08PT Lemma 92.4.2. Let $A \to B$ be a ring map. Let $\epsilon : P_\bullet \to B$ be the standard resolution of B over A. Let π be as in (92.4.0.1). Then

$$L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)$$

in $D(\textit{Ab})$, resp. $D(B)$ functorially in $\mathcal{F}$ in $\textit{Ab}(\mathcal{C})$, resp. $\text{Mod}(\underline{B})$.

First proof. We will apply Lemma 92.4.1. Since the terms P_n are polynomial algebras we see the first assumption of that lemma is satisfied. The second assumption is proved as follows. By Simplicial, Lemma 14.34.3 the map ϵ is a homotopy equivalence of underlying simplicial sets. By Simplicial, Lemma 14.31.9 this implies ϵ induces a quasi-isomorphism of associated complexes of abelian groups. By Simplicial, Lemma 14.31.8 this implies that ϵ is a trivial Kan fibration of underlying simplicial sets. □

Second proof. We will use the criterion of Cohomology on Sites, Lemma 21.39.7. Let $U = (Q, \beta)$ be an object of $\mathcal{C}$. We have to show that

$$S_\bullet = \text{Mor}_\mathcal{C}((Q, \beta), (P_\bullet, \epsilon))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set E (this is possible by our choice of the category $\mathcal{C}$). Using the notation of Remark 92.3.3 we see that

$$S_\bullet = \text{Mor}_\mathcal{S}((E \to B), i(P_\bullet \to B))$$

By Simplicial, Lemma 14.34.3 the map $i(P_\bullet \to B) \to i(B \to B)$ is a homotopy equivalence in $\mathcal{S}$. Hence $S_\bullet$ is homotopy equivalent to

$$\text{Mor}_\mathcal{S}((E \to B), (B \to B)) = \{*\}$$

as desired. □

08PU Lemma 92.4.3. Let $A \to B$ be a ring map. Let π and i be as in (92.4.0.1). There is a canonical isomorphism

$$L_{B/A} = L\pi_!(Li^*\Omega_{\mathcal{O}/A}) = L\pi_!(i^*\Omega_{\mathcal{O}/A}) = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_\mathcal{O} \underline{B})$$

in $D(B)$.

Proof. For an object $\alpha : P \to B$ of the category $\mathcal{C}$ the module $\Omega_{P/A}$ is a free P-module. Thus $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module. Hence $Li^*\Omega_{\mathcal{O}/A} = i^*\Omega_{\mathcal{O}/A}$ is the sheaf of $\underline{B}$-modules which associates to $\alpha : P \to A$ the B-module $\Omega_{P/A} \otimes_{P,\alpha} B$. By Lemma 92.4.2 we see that the right hand side is computed by the value of this sheaf on the standard resolution which is our definition of the left hand side (Definition 92.3.2). □

08QE **Lemma 92.4.4.** If $A \to B$ is a ring map, then $L\pi_!(\pi^{-1}M) = M$ with π as in (92.4.0.1).

Proof. This follows from Lemma 92.4.1 which tells us $L\pi_!(\pi^{-1}M)$ is computed by $(\pi^{-1}M)(P_\bullet, \epsilon)$ which is the constant simplicial object on M. □

08QF **Lemma 92.4.5.** If $A \to B$ is a ring map, then $H^0(L_{B/A}) = \Omega_{B/A}$.

Proof. We will prove this by a direct calculation. We will use the identification of Lemma 92.4.3. There is clearly a map from $\Omega_{\mathcal{O}/A} \otimes \underline{B}$ to the constant sheaf with value $\Omega_{B/A}$. Thus this map induces a map

$$H^0(L_{B/A}) = H^0(L\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})) = \pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B}) \to \Omega_{B/A}$$

By choosing an object $P \to B$ of $\mathcal{C}_{B/A}$ with $P \to B$ surjective we see that this map is surjective (by Algebra, Lemma 10.131.6). To show that it is injective, suppose that $P \to B$ is an object of $\mathcal{C}_{B/A}$ and that $\xi \in \Omega_{P/A} \otimes_P B$ is an element which maps to zero in $\Omega_{B/A}$. We first choose factorization $P \to P' \to B$ such that $P' \to B$ is surjective and P' is a polynomial algebra over A. We may replace P by P'. If $B = P/I$, then the kernel $\Omega_{P/A} \otimes_P B \to \Omega_{B/A}$ is the image of I/I^2 (Algebra, Lemma 10.131.9). Say ξ is the image of $f \in I$. Then we consider the two maps $a, b : P' = P[x] \to P$, the first of which maps x to 0 and the second of which maps x to f (in both cases $P[x] \to B$ maps x to zero). We see that ξ and 0 are the image of $\mathrm{d}x \otimes 1$ in $\Omega_{P'/A} \otimes_{P'} B$. Thus ξ and 0 have the same image in the colimit (see Cohomology on Sites, Example 21.39.1) $\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})$ as desired. □

08QG **Lemma 92.4.6.** If B is a polynomial algebra over the ring A, then with π as in (92.4.0.1) we have that $\pi_!$ is exact and $\pi_!\mathcal{F} = \mathcal{F}(B \to B)$.

Proof. This follows from Lemma 92.4.1 which tells us the constant simplicial algebra on B can be used to compute $L\pi_!$. □

08QH **Lemma 92.4.7.** If B is a polynomial algebra over the ring A, then $L_{B/A}$ is quasi-isomorphic to $\Omega_{B/A}[0]$.

Proof. Immediate from Lemmas 92.4.3 and 92.4.6. □

92.5. Constructing a resolution

08PV In the Noetherian finite type case we can construct a "small" simplicial resolution for finite type ring maps.

08PW **Lemma 92.5.1.** Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let $\mathcal{A}$ be the category of A-algebra maps $C \to B$. Let $n \geq 0$ and let $P_\bullet$ be a simplicial object of $\mathcal{A}$ such that

(1) $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,
(2) P_k is finite type over A for $k \leq n$,
(3) $P_\bullet = \mathrm{cosk}_n \mathrm{sk}_n P_\bullet$ as simplicial objects of $\mathcal{A}$.

Then P_{n+1} is a finite type A-algebra.

Proof. Although the proof we give of this lemma is straightforward, it is a bit messy. To clarify the idea we explain what happens for low n before giving the proof in general. For example, if $n = 0$, then (3) means that $P_1 = P_0 \times_B P_0$. Since the ring map $P_0 \to B$ is surjective, this is of finite type over A by More on Algebra, Lemma 15.5.1.

If $n = 1$, then (3) means that

$$P_2 = \{(f_0, f_1, f_2) \in P_1^3 \mid d_0 f_0 = d_0 f_1,\ d_1 f_0 = d_0 f_2,\ d_1 f_1 = d_1 f_2\}$$

where the equalities take place in P_0. Observe that the triple

$$(d_0 f_0, d_1 f_0, d_1 f_1) = (d_0 f_1, d_0 f_2, d_1 f_2)$$

is an element of the fibre product $P_0 \times_B P_0 \times_B P_0$ over B because the maps $d_i : P_1 \to P_0$ are morphisms over B. Thus we get a map

$$\psi : P_2 \longrightarrow P_0 \times_B P_0 \times_B P_0$$

The fibre of ψ over an element $(g_0, g_1, g_2) \in P_0 \times_B P_0 \times_B P_0$ is the set of triples (f_0, f_1, f_2) of 1-simplices with $(d_0, d_1)(f_0) = (g_0, g_1)$, $(d_0, d_1)(f_1) = (g_0, g_2)$, and $(d_0, d_1)(f_2) = (g_1, g_2)$. As $P_\bullet \to B$ is a trivial Kan fibration the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus we see that P_2 fits into the cartesian diagram

$$\begin{tikzcd} P_2 \arrow[r] \arrow[d] & P_1^3 \arrow[d] \\ P_0 \times_B P_0 \times_B P_0 \arrow[r] & (P_0 \times_B P_0)^3 \end{tikzcd}$$

By More on Algebra, Lemma 15.5.2 we conclude. The general case is similar, but requires a bit more notation.

The case $n > 1$. By Simplicial, Lemma 14.19.14 the condition $P_\bullet = \text{cosk}_n \text{sk}_n P_\bullet$ implies the same thing is true in the category of simplicial A-algebras and hence in the category of sets (as the forgetful functor from A-algebras to sets commutes with limits). Thus

$$P_{n+1} = \text{Mor}(\Delta[n+1], P_\bullet) = \text{Mor}(\text{sk}_n \Delta[n+1], \text{sk}_n P_\bullet)$$

by Simplicial, Lemma 14.11.3 and Equation (14.19.0.1). We will prove by induction on $1 \leq k < m \leq n+1$ that the ring

$$Q_{k,m} = \text{Mor}(\text{sk}_k \Delta[m], \text{sk}_k P_\bullet)$$

is of finite type over A. The case $k = 1$, $1 < m \leq n+1$ is entirely similar to the discussion above in the case $n = 1$. Namely, there is a cartesian diagram

$$\begin{tikzcd} Q_{1,m} \arrow[r] \arrow[d] & P_1^N \arrow[d] \\ P_0 \times_B \ldots \times_B P_0 \arrow[r] & (P_0 \times_B P_0)^N \end{tikzcd}$$

where $N = \binom{m+1}{2}$. We conclude as before.

Let $1 \leq k_0 \leq n$ and assume $Q_{k,m}$ is of finite type over A for all $1 \leq k \leq k_0$ and $k < m \leq n+1$. For $k_0 + 1 < m \leq n+1$ we claim there is a cartesian square

$$\begin{tikzcd} Q_{k_0+1,m} \arrow[r] \arrow[d] & P_{k_0+1}^N \arrow[d] \\ Q_{k_0,m} \arrow[r] & Q_{k_0,k_0+1}^N \end{tikzcd}$$

where N is the number of nondegenerate (k_0+1)-simplices of $\Delta[m]$. Namely, to see this is true, think of an element of $Q_{k_0+1,m}$ as a function f from the (k_0+1)-skeleton of $\Delta[m]$ to $P_\bullet$. We can restrict f to the k_0-skeleton which gives the left vertical map of the diagram. We can also restrict to each nondegenerate (k_0+1)-simplex which gives the top horizontal arrow. Moreover, to give such an f is the same thing as giving its restriction to k_0-skeleton and to each nondegenerate (k_0+1)-face, provided these agree on the overlap, and this is exactly the content of the diagram. Moreover, the fact that $P_\bullet \to B$ is a trivial Kan fibration implies that the map

$$P_{k_0} \to Q_{k_0,k_0+1} = \mathrm{Mor}(\partial\Delta[k_0+1], P_\bullet)$$

is surjective as every map $\partial\Delta[k_0+1] \to B$ can be extended to $\Delta[k_0+1] \to B$ for $k_0 \geq 1$ (small argument about constant simplicial sets omitted). Since by induction hypothesis the rings $Q_{k_0,m}$, Q_{k_0,k_0+1} are finite type A-algebras, so is $Q_{k_0+1,m}$ by More on Algebra, Lemma 15.5.2 once more. □

08PX Proposition 92.5.2. Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. There exists a simplicial A-algebra $P_\bullet$ with an augmentation $\epsilon : P_\bullet \to B$ such that each P_n is a polynomial algebra of finite type over A and such that ϵ is a trivial Kan fibration of simplicial sets.

Proof. Let $\mathcal{A}$ be the category of A-algebra maps $C \to B$. In this proof our simplicial objects and skeleton and coskeleton functors will be taken in this category.

Choose a polynomial algebra P_0 of finite type over A and a surjection $P_0 \to B$. As a first approximation we take $P_\bullet = \mathrm{cosk}_0(P_0)$. In other words, $P_\bullet$ is the simplicial A-algebra with terms $P_n = P_0 \times_A \ldots \times_A P_0$. (In the final paragraph of the proof this simplicial object will be denoted $P_\bullet^0$.) By Simplicial, Lemma 14.32.3 the map $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets. Also, observe that $P_\bullet = \mathrm{cosk}_0\mathrm{sk}_0 P_\bullet$.

Suppose for some $n \geq 0$ we have constructed $P_\bullet$ (in the final paragraph of the proof this will be $P_\bullet^n$) such that

(a) $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,
(b) P_k is a finitely generated polynomial algebra for $0 \leq k \leq n$, and
(c) $P_\bullet = \mathrm{cosk}_n\mathrm{sk}_n P_\bullet$

By Lemma 92.5.1 we can find a finitely generated polynomial algebra Q over A and a surjection $Q \to P_{n+1}$. Since P_n is a polynomial algebra the A-algebra maps $s_i : P_n \to P_{n+1}$ lift to maps $s'_i : P_n \to Q$. Set $d'_j : Q \to P_n$ equal to the composition of $Q \to P_{n+1}$ and $d_j : P_{n+1} \to P_n$. We obtain a truncated simplicial object $P'_\bullet$ of $\mathcal{A}$ by setting $P'_k = P_k$ for $k \leq n$ and $P'_{n+1} = Q$ and morphisms $d'_i = d_i$ and $s'_i = s_i$ in degrees $k \leq n-1$ and using the morphisms d'_j and s'_i in degree n. Extend this to a full simplicial object $P'_\bullet$ of $\mathcal{A}$ using cosk_{n+1}. By functoriality of the coskeleton functors there is a morphism $P'_\bullet \to P_\bullet$ of simplicial objects extending the given morphism of $(n+1)$-truncated simplicial objects. (This morphism will be denoted $P_\bullet^{n+1} \to P_\bullet^n$ in the final paragraph of the proof.)

Note that conditions (b) and (c) are satisfied for $P'_\bullet$ with n replaced by $n+1$. We claim the map $P'_\bullet \to P_\bullet$ satisfies assumptions (1), (2), (3), and (4) of Simplicial, Lemmas 14.32.1 with $n+1$ instead of n. Conditions (1) and (2) hold by construction. By Simplicial, Lemma 14.19.14 we see that we have $P_\bullet = \text{cosk}_{n+1}\text{sk}_{n+1}P_\bullet$ and $P'_\bullet = \text{cosk}_{n+1}\text{sk}_{n+1}P'_\bullet$ not only in $\mathcal{A}$ but also in the category of A-algebras, whence in the category of sets (as the forgetful functor from A-algebras to sets commutes with all limits). This proves (3) and (4). Thus the lemma applies and $P'_\bullet \to P_\bullet$ is a trivial Kan fibration. By Simplicial, Lemma 14.30.4 we conclude that $P'_\bullet \to B$ is a trivial Kan fibration and (a) holds as well.

To finish the proof we take the inverse limit $P_\bullet = \lim P_\bullet^n$ of the sequence of simplicial algebras

$$\ldots \to P_\bullet^2 \to P_\bullet^1 \to P_\bullet^0$$

constructed above. The map $P_\bullet \to B$ is a trivial Kan fibration by Simplicial, Lemma 14.30.5. However, the construction above stabilizes in each degree to a fixed finitely generated polynomial algebra as desired. □

08PY Lemma 92.5.3. Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let π, $\underline{B}$ be as in (92.4.0.1). If $\mathcal{F}$ is an $\underline{B}$-module such that $\mathcal{F}(P, \alpha)$ is a finite B-module for all $\alpha : P = A[x_1, \ldots, x_n] \to B$, then the cohomology modules of $L\pi_!(\mathcal{F})$ are finite B-modules.

Proof. By Lemma 92.4.1 and Proposition 92.5.2 we can compute $L\pi_!(\mathcal{F})$ by a complex constructed out of the values of $\mathcal{F}$ on finite type polynomial algebras. □

08PZ Lemma 92.5.4. Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Then $H^n(L_{B/A})$ is a finite B-module for all $n \in \mathbf{Z}$.

Proof. Apply Lemmas 92.4.3 and 92.5.3. □

08QI Remark 92.5.5 (Resolutions). Let $A \to B$ be any ring map. Let us call an augmented simplicial A-algebra $\epsilon : P_\bullet \to B$ a resolution of B over A if each P_n is a polynomial algebra and ϵ is a trivial Kan fibration of simplicial sets. If $P_\bullet \to B$ is an augmentation of a simplicial A-algebra with each P_n a polynomial algebra surjecting onto B, then the following are equivalent

(1) $\epsilon : P_\bullet \to B$ is a resolution of B over A,
(2) $\epsilon : P_\bullet \to B$ is a quasi-isomorphism on associated complexes,
(3) $\epsilon : P_\bullet \to B$ induces a homotopy equivalence of simplicial sets.

To see this use Simplicial, Lemmas 14.30.8, 14.31.9, and 14.31.8. A resolution $P_\bullet$ of B over A gives a cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$ as in Cohomology on Sites, Lemma 21.39.7 and it follows that

$$L\pi_!\mathcal{F} = \mathcal{F}(P_\bullet)$$

functorially in $\mathcal{F}$, see Lemma 92.4.1. The (formal part of the) proof of Proposition 92.5.2 shows that resolutions exist. We also have seen in the first proof of Lemma 92.4.2 that the standard resolution of B over A is a resolution (so that this terminology doesn't lead to a conflict). However, the argument in the proof of Proposition 92.5.2 shows the existence of resolutions without appealing to the simplicial computations in Simplicial, Section 14.34. Moreover, for any choice of resolution we have a canonical isomorphism

$$L_{B/A} = \Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} B$$

in $D(B)$ by Lemma 92.4.3. The freedom to choose an arbitrary resolution can be quite useful.

08QJ Lemma 92.5.6. Let $A \to B$ be a ring map. Let π, $\mathcal{O}$, $\underline{B}$ be as in (92.4.0.1). For any $\mathcal{O}$-module $\mathcal{F}$ we have

$$L\pi_!(\mathcal{F}) = L\pi_!(Li^*\mathcal{F}) = L\pi_!(\mathcal{F} \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$$

in $D(\textit{Ab})$.

Proof. It suffices to verify the assumptions of Cohomology on Sites, Lemma 21.39.12 hold for $\mathcal{O} \to \underline{B}$ on $\mathcal{C}_{B/A}$. We will use the results of Remark 92.5.5 without further mention. Choose a resolution $P_\bullet$ of B over A to get a suitable cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$. Since $P_\bullet \to B$ induces a quasi-isomorphism on associated complexes of abelian groups we see that $L\pi_!\mathcal{O} = B$. On the other hand $L\pi_!\underline{B}$ is computed by $\underline{B}(U_\bullet) = B$. This verifies the second assumption of Cohomology on Sites, Lemma 21.39.12 and we are done with the proof. ☐

08QK Lemma 92.5.7. Let $A \to B$ be a ring map. Let π, $\mathcal{O}$, $\underline{B}$ be as in (92.4.0.1). We have

$$L\pi_!(\mathcal{O}) = L\pi_!(\underline{B}) = B \quad \text{and} \quad L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = L\pi_!(\Omega_{\mathcal{O}/A})$$

in $D(\textit{Ab})$.

Proof. This is just an application of Lemma 92.5.6 (and the first equality on the right is Lemma 92.4.3). ☐

Here is a special case of the fundamental triangle that is easy to prove.

08SA Lemma 92.5.8. Let $A \to B \to C$ be ring maps. If B is a polynomial algebra over A, then there is a distinguished triangle $L_{B/A} \otimes_B^{\mathbf{L}} C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes_B^{\mathbf{L}} C[1]$ in $D(C)$.

Proof. We will use the observations of Remark 92.5.5 without further mention. Choose a resolution $\epsilon : P_\bullet \to C$ of C over B (for example the standard resolution). Since B is a polynomial algebra over A we see that $P_\bullet$ is also a resolution of C over A. Hence $L_{C/A}$ is computed by $\Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} C$ and $L_{C/B}$ is computed by $\Omega_{P_\bullet/B} \otimes_{P_\bullet,\epsilon} C$. Since for each n we have the short exact sequence $0 \to \Omega_{B/A} \otimes_B P_n \to \Omega_{P_n/A} \to \Omega_{P_n/B}$ (Algebra, Lemma 10.138.9) and since $L_{B/A} = \Omega_{B/A}[0]$ (Lemma 92.4.7) we obtain the result. ☐

09D4 Example 92.5.9. Let $A \to B$ be a ring map. In this example we will construct an "explicit" resolution $P_\bullet$ of B over A of length 2. To do this we follow the procedure of the proof of Proposition 92.5.2, see also the discussion in Remark 92.5.5.

We choose a surjection $P_0 = A[u_i] \to B$ where u_i is a set of variables. Choose generators $f_t \in P_0$, $t \in T$ of the ideal $\text{Ker}(P_0 \to B)$. We choose $P_1 = A[u_i, x_t]$ with face maps d_0 and d_1 the unique A-algebra maps with $d_j(u_i) = u_i$ and $d_0(x_t) = 0$ and $d_1(x_t) = f_t$. The map $s_0 : P_0 \to P_1$ is the unique A-algebra map with $s_0(u_i) = u_i$. It is clear that

$$P_1 \xrightarrow{d_0 - d_1} P_0 \to B \to 0$$

is exact, in particular the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus, if $P_\bullet$ denotes the 1-truncated simplicial A-algebra given by P_0, P_1, d_0, d_1, and s_0, then the augmentation $\text{cosk}_1(P_\bullet) \to B$ is a trivial Kan fibration. The next step of the procedure in the proof of Proposition 92.5.2 is to choose a polynomial algebra P_2 and a surjection

$$P_2 \longrightarrow \text{cosk}_1(P_\bullet)_2$$

Recall that

$$\text{cosk}_1(P_\bullet)_2 = \{(g_0, g_1, g_2) \in P_1^3 \mid d_0(g_0) = d_0(g_1), d_1(g_0) = d_0(g_2), d_1(g_1) = d_1(g_2)\}$$

Thinking of $g_i \in P_1$ as a polynomial in x_t the conditions are

$$g_0(0) = g_1(0), \quad g_0(f_t) = g_2(0), \quad g_1(f_t) = g_2(f_t)$$

Thus $\text{cosk}_1(P_\bullet)_2$ contains the elements $y_t = (x_t, x_t, f_t)$ and $z_t = (0, x_t, x_t)$. Every element G in $\text{cosk}_1(P_\bullet)_2$ is of the form $G = H + (0, 0, g)$ where H is in the image of $A[u_i, y_t, z_t] \to \text{cosk}_1(P_\bullet)_2$. Here $g \in P_1$ is a polynomial with vanishing constant term such that $g(f_t) = 0$ in P_0. Observe that

(1) $g = x_t x_{t'} - f_t x_{t'}$ and
(2) $g = \sum r_t x_t$ with $r_t \in P_0$ if $\sum r_t f_t = 0$ in P_0

are elements of P_1 of the desired form. Let

$$Rel = \text{Ker}(\bigoplus\nolimits_{t \in T} P_0 \longrightarrow P_0), \quad (r_t) \longmapsto \sum r_t f_t$$

We set $P_2 = A[u_i, y_t, z_t, v_r, w_{t,t'}]$ where $r = (r_t) \in Rel$, with map

$$P_2 \longrightarrow \text{cosk}_1(P_\bullet)_2$$

given by $y_t \mapsto (x_t, x_t, f_t)$, $z_t \mapsto (0, x_t, x_t)$, $v_r \mapsto (0, 0, \sum r_t x_t)$, and $w_{t,t'} \mapsto (0, 0, x_t x_{t'} - f_t x_{t'})$. A calculation (omitted) shows that this map is surjective. Our choice of the map displayed above determines the maps $d_0, d_1, d_2 : P_2 \to P_1$. Finally, the procedure in the proof of Proposition 92.5.2 tells us to choose the maps $s_0, s_1 : P_1 \to P_2$ lifting the two maps $P_1 \to \text{cosk}_1(P_\bullet)_2$. It is clear that we can take s_i to be the unique A-algebra maps determined by $s_0(x_t) = y_t$ and $s_1(x_t) = z_t$.

92.6. Functoriality

08QL In this section we consider a commutative square

08QM (92.6.0.1)
$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of ring maps. We claim there is a canonical B-linear map of complexes

$$L_{B/A} \longrightarrow L_{B'/A'}$$

associated to this diagram. Namely, if $P_\bullet \to B$ is the standard resolution of B over A and $P'_\bullet \to B'$ is the standard resolution of B' over A', then there is a canonical map $P_\bullet \to P'_\bullet$ of simplicial A-algebras compatible with the augmentations $P_\bullet \to B$ and $P'_\bullet \to B'$. This can be seen in terms of the construction of standard resolutions in Simplicial, Section 14.34 but in the special case at hand it probably suffices to say simply that the maps

$$P_0 = A[B] \longrightarrow A'[B'] = P'_0, \quad P_1 = A[A[B]] \longrightarrow A'[A'[B']] = P'_1,$$

and so on are given by the given maps $A \to A'$ and $B \to B'$. The desired map $L_{B/A} \to L_{B'/A'}$ then comes from the associated maps $\Omega_{P_n/A} \to \Omega_{P'_n/A'}$.

Another description of the functoriality map can be given as follows. Let $\mathcal{C} = \mathcal{C}_{B/A}$ and $\mathcal{C}' = \mathcal{C}'_{B'/A}$ be the categories considered in Section 92.4. There is a functor

$$u : \mathcal{C} \longrightarrow \mathcal{C}', \quad (P, \alpha) \longmapsto (P \otimes_A A', c \circ (\alpha \otimes 1))$$

where $c : B \otimes_A A' \to B'$ is the obvious map. As discussed in Cohomology on Sites, Example 21.39.3 we obtain a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ and a commutative diagram of maps of ringed topoi

08QN (92.6.0.2)
$$\begin{tikzcd}
(Sh(\mathcal{C}'), \underline{B}) \arrow[d, "\pi"] & (Sh(\mathcal{C}'), \underline{B'}) \arrow[l, "h"'] \arrow[d, "\pi"] & (Sh(\mathcal{C}), \underline{B'}) \arrow[l, "g"'] \arrow[d, "\pi'"] \\
(Sh(*), B) & (Sh(*), B') \arrow[l, "f"] & (Sh(*), B') \arrow[l]
\end{tikzcd}$$

Here h is the identity on underlying topoi and given by the ring map $B \to B'$ on sheaves of rings. By Cohomology on Sites, Remark 21.38.7 given $\mathcal{F}$ on $\mathcal{C}$ and $\mathcal{F}'$ on $\mathcal{C}'$ and a transformation $t : \mathcal{F} \to g^{-1}\mathcal{F}'$ we obtain a canonical map $L\pi_!(\mathcal{F}) \to L\pi'_!(\mathcal{F}')$. If we apply this to the sheaves

$$\mathcal{F} : (P, \alpha) \mapsto \Omega_{P/A} \otimes_P B, \quad \mathcal{F}' : (P', \alpha') \mapsto \Omega_{P'/A'} \otimes_{P'} B',$$

and the transformation t given by the canonical maps

$$\Omega_{P/A} \otimes_P B \longrightarrow \Omega_{P \otimes_A A'/A'} \otimes_{P \otimes_A A'} B'$$

to get a canonical map

$$L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \longrightarrow L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$$

By Lemma 92.4.3 this gives $L_{B/A} \to L_{B'/A'}$. We omit the verification that this map agrees with the map defined above in terms of simplicial resolutions.

08QP Lemma 92.6.1. Assume (92.6.0.1) induces a quasi-isomorphism $B \otimes_A^{\mathbf{L}} A' = B'$. Then, with notation as in (92.6.0.2) and $\mathcal{F}' \in \text{Ab}(\mathcal{C}')$, we have $L\pi_!(g^{-1}\mathcal{F}') = L\pi'_!(\mathcal{F}')$.

Proof. We use the results of Remark 92.5.5 without further mention. We will apply Cohomology on Sites, Lemma 21.39.8. Let $P_\bullet \to B$ be a resolution. If we can show that $u(P_\bullet) = P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism, then we are done. The complex of A-modules $s(P_\bullet)$ associated to $P_\bullet$ (viewed as a simplicial A-module) is a free A-module resolution of B. Namely, P_n is a free A-module and $s(P_\bullet) \to B$ is a quasi-isomorphism. Thus $B \otimes_A^{\mathbf{L}} A'$ is computed by $s(P_\bullet) \otimes_A A' = s(P_\bullet \otimes_A A')$. Therefore the assumption of the lemma signifies that $\epsilon' : P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism. $\square$

The following lemma in particular applies when $A \to A'$ is flat and $B' = B \otimes_A A'$ (flat base change).

08QQ Lemma 92.6.2. If (92.6.0.1) induces a quasi-isomorphism $B \otimes_A^{\mathbf{L}} A' = B'$, then the functoriality map induces an isomorphism

$$L_{B/A} \otimes_B^{\mathbf{L}} B' \longrightarrow L_{B'/A'}$$

Proof. We will use the notation introduced in Equation (92.6.0.2). We have

$$L_{B/A} \otimes_B^{\mathbf{L}} B' = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \otimes_B^{\mathbf{L}} B' = L\pi_!(Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}))$$

the first equality by Lemma 92.4.3 and the second by Cohomology on Sites, Lemma 21.39.6. Since $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module, we see that $\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ is a flat $\underline{B}$-module. Thus $Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B'}$ which is equal to $g^{-1}(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$ by inspection. we conclude by Lemma 92.6.1 and the fact that $L_{B'/A'}$ is computed by $L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$. $\square$

08SB **Remark 92.6.3.** Suppose that we are given a square (92.6.0.1) such that there exists an arrow $\kappa : B \to A'$ making the diagram commute:

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[rd, "\kappa"] & B' \\ A \arrow[u] \arrow[r, "\alpha"] & A' \arrow[u] \end{tikzcd}$$

In this case we claim the functoriality map $P_\bullet \to P'_\bullet$ is homotopic to the composition $P_\bullet \to B \to A' \to P'_\bullet$. Namely, using κ the functoriality map factors as

$$P_\bullet \to P_{A'/A',\bullet} \to P'_\bullet$$

where $P_{A'/A',\bullet}$ is the standard resolution of A' over A'. Since A' is the polynomial algebra on the empty set over A' we see from Simplicial, Lemma 14.34.3 that the augmentation $\epsilon_{A'/A'} : P_{A'/A',\bullet} \to A'$ is a homotopy equivalence of simplicial rings. Observe that the homotopy inverse map $c : A' \to P_{A'/A',\bullet}$ constructed in the proof of that lemma is just the structure morphism, hence we conclude what we want because the two compositions

$$\begin{tikzcd} P_\bullet \arrow[r] & P_{A'/A',\bullet} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "c\circ\epsilon_{A'/A'}"'] & P_{A'/A',\bullet} \arrow[r] & P'_\bullet \end{tikzcd}$$

are the two maps discussed above and these are homotopic (Simplicial, Remark 14.26.5). Since the second map $P_\bullet \to P'_\bullet$ induces the zero map $\Omega_{P_\bullet/A} \to \Omega_{P'_\bullet/A'}$ we conclude that the functoriality map $L_{B/A} \to L_{B'/A'}$ is homotopic to zero in this case.

08SC **Lemma 92.6.4.** Let $A \to B$ and $A \to C$ be ring maps. Then the map $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ is an isomorphism in $D(B \times C)$.

Proof. Although this lemma can be deduced from the fundamental triangle we will give a direct and elementary proof of this now. Factor the ring map $A \to B \times C$ as $A \to A[x] \to B \times C$ where $x \mapsto (1, 0)$. By Lemma 92.5.8 we have a distinguished triangle

$$L_{A[x]/A} \otimes_{A[x]}^{\mathbf{L}} (B \times C) \to L_{B\times C/A} \to L_{B\times C/A[x]} \to L_{A[x]/A} \otimes_{A[x]}^{\mathbf{L}} (B \times C)[1]$$

in $D(B \times C)$. Similarly we have the distinguished triangles

$$\begin{matrix} L_{A[x]/A} \otimes_{A[x]}^{\mathbf{L}} B \to L_{B/A} \to L_{B/A[x]} \to L_{A[x]/A} \otimes_{A[x]}^{\mathbf{L}} B[1] \\ L_{A[x]/A} \otimes_{A[x]}^{\mathbf{L}} C \to L_{C/A} \to L_{C/A[x]} \to L_{A[x]/A} \otimes_{A[x]}^{\mathbf{L}} C[1] \end{matrix}$$

Thus it suffices to prove the result for $B\times C$ over $A[x]$. Note that $A[x] \to A[x, x^{-1}]$ is flat, that $(B\times C)\otimes_{A[x]} A[x, x^{-1}] = B\otimes_{A[x]} A[x, x^{-1}]$, and that $C\otimes_{A[x]} A[x, x^{-1}] = 0$. Thus by base change (Lemma 92.6.2) the map $L_{B\times C/A[x]} \to L_{B/A[x]} \oplus L_{C/A[x]}$

becomes an isomorphism after inverting x. In the same way one shows that the map becomes an isomorphism after inverting $x - 1$. This proves the lemma. $\square$

92.7. The fundamental triangle

08QR In this section we consider a sequence of ring maps $A \to B \to C$. It is our goal to show that this triangle gives rise to a distinguished triangle

08QS (92.7.0.1)
$$L_{B/A} \otimes_B^{\mathbf{L}} C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes_B^{\mathbf{L}} C[1]$$

in $D(C)$. This will be proved in Proposition 92.7.4. For an alternative approach see Remark 92.7.5.

Consider the category $\mathcal{C}_{C/B/A}$ wich is the opposite of the category whose objects are $(P \to B, Q \to C)$ where

(1) P is a polynomial algebra over A,
(2) $P \to B$ is an A-algebra homomorphism,
(3) Q is a polynomial algebra over P, and
(4) $Q \to C$ is a P-algebra-homomorphism.

We take the opposite as we want to think of $(P \to B, Q \to C)$ as corresponding to the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(C) \arrow[r] \arrow[d] & \mathrm{Spec}(Q) \arrow[d] \\
\mathrm{Spec}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(P) \arrow[ld] \\
\mathrm{Spec}(A) &
\end{tikzcd}$$

Let $\mathcal{C}_{B/A}$, $\mathcal{C}_{C/A}$, $\mathcal{C}_{C/B}$ be the categories considered in Section 92.4. There are functors

$$\begin{matrix}
u_1 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{B/A}, & (P \to B, Q \to C) \mapsto (P \to B) \\
u_2 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/A}, & (P \to B, Q \to C) \mapsto (Q \to C) \\
u_3 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}, & (P \to B, Q \to C) \mapsto (Q \otimes_P B \to C)
\end{matrix}$$

These functors induce corresponding morphisms of topoi g_i. Let us denote $\mathcal{O}_i = g_i^{-1}\mathcal{O}$ so that we get morphisms of ringed topoi

08QT (92.7.0.2)
$$\begin{aligned}
g_1 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_1) \longrightarrow (Sh(\mathcal{C}_{B/A}), \mathcal{O}) \\
g_2 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_2) \longrightarrow (Sh(\mathcal{C}_{C/A}), \mathcal{O}) \\
g_3 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_3) \longrightarrow (Sh(\mathcal{C}_{C/B}), \mathcal{O})
\end{aligned}$$

Let us denote $\pi : Sh(\mathcal{C}_{C/B/A}) \to Sh(*)$, $\pi_1 : Sh(\mathcal{C}_{B/A}) \to Sh(*)$, $\pi_2 : Sh(\mathcal{C}_{C/A}) \to Sh(*)$, and $\pi_3 : Sh(\mathcal{C}_{C/B}) \to Sh(*)$, so that $\pi = \pi_i \circ g_i$. We will obtain our distinguished triangle from the identification of the cotangent complex in Lemma 92.4.3 and the following lemmas.

08QU **Lemma 92.7.1.** *With notation as in (92.7.0.2) set*

$$\begin{aligned}
\Omega_1 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B} \text{ on } \mathcal{C}_{B/A} \\
\Omega_2 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{C} \text{ on } \mathcal{C}_{C/A} \\
\Omega_3 &= \Omega_{\mathcal{O}/B} \otimes_{\mathcal{O}} \underline{C} \text{ on } \mathcal{C}_{C/B}
\end{aligned}$$

Then we have a canonical short exact sequence of sheaves of $\underline{C}$-modules

$$0 \to g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C} \to g_2^{-1}\Omega_2 \to g_3^{-1}\Omega_3 \to 0$$

on $\mathcal{C}_{C/B/A}$.

Proof. Recall that g_i^{-1} is gotten by simply precomposing with u_i. Given an object $U = (P \to B, Q \to C)$ we have a split short exact sequence

$$0 \to \Omega_{P/A} \otimes Q \to \Omega_{Q/A} \to \Omega_{Q/P} \to 0$$

for example by Algebra, Lemma 10.138.9. Tensoring with C over Q we obtain a short exact sequence

$$0 \to \Omega_{P/A} \otimes C \to \Omega_{Q/A} \otimes C \to \Omega_{Q/P} \otimes C \to 0$$

We have $\Omega_{P/A} \otimes C = \Omega_{P/A} \otimes B \otimes C$ whence this is the value of $g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}$ on U. The module $\Omega_{Q/A} \otimes C$ is the value of $g_2^{-1}\Omega_2$ on U. We have $\Omega_{Q/P} \otimes C = \Omega_{Q \otimes_P B/B} \otimes C$ by Algebra, Lemma 10.131.12 hence this is the value of $g_3^{-1}\Omega_3$ on U. Thus the short exact sequence of the lemma comes from assigning to U the last displayed short exact sequence. $\square$

08QV **Lemma 92.7.2.** With notation as in (92.7.0.2) suppose that C is a polynomial algebra over B. Then $L\pi_!(g_3^{-1}\mathcal{F}) = L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$

Proof. Write $C = B[E]$ for some set E. Choose a resolution $P_\bullet \to B$ of B over A. For every n consider the object $U_n = (P_n \to B, P_n[E] \to C)$ of $\mathcal{C}_{C/B/A}$. Then $U_\bullet$ is a cosimplicial object of $\mathcal{C}_{C/B/A}$. Note that $u_3(U_\bullet)$ is the constant cosimplicial object of $\mathcal{C}_{C/B}$ with value $(C \to C)$. We will prove that the object $U_\bullet$ of $\mathcal{C}_{C/B/A}$ satisfies the hypotheses of Cohomology on Sites, Lemma 21.39.7. This implies the lemma as it shows that $L\pi_!(g_3^{-1}\mathcal{F})$ is computed by the constant simplicial abelian group $\mathcal{F}(C \to C)$ which is the value of $L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ by Lemma 92.4.6.

Let $U = (\beta : P \to B, \gamma : Q \to C)$ be an object of $\mathcal{C}_{C/B/A}$. We may write $P = A[S]$ and $Q = A[S \amalg T]$ by the definition of our category $\mathcal{C}_{C/B/A}$. We have to show that

$$\mathrm{Mor}_{\mathcal{C}_{C/B/A}}(U_\bullet, U)$$

is homotopy equivalent to a singleton simplicial set $*$. Observe that this simplicial set is the product

$$\prod\nolimits_{s \in S} F_s \times \prod\nolimits_{t \in T} F'_t$$

where F_s is the corresponding simplicial set for $U_s = (A[\{s\}] \to B, A[\{s\}] \to C)$ and F'_t is the corresponding simplicial set for $U_t = (A \to B, A[\{t\}] \to C)$. Namely, the object U is the product $\prod U_s \times \prod U_t$ in $\mathcal{C}_{C/B/A}$. It suffices each F_s and F'_t is homotopy equivalent to $*$, see Simplicial, Lemma 14.26.10. The case of F_s follows as $P_\bullet \to B$ is a trivial Kan fibration (as a resolution) and F_s is the fibre of this map over $\beta(s)$. (Use Simplicial, Lemmas 14.30.3 and 14.30.8). The case of F'_t is more interesting. Here we are saying that the fibre of

$$P_\bullet[E] \longrightarrow C = B[E]$$

over $\gamma(t) \in C$ is homotopy equivalent to a point. In fact we will show this map is a trivial Kan fibration. Namely, $P_\bullet \to B$ is a trivial can fibration. For any ring R we have

$$R[E] = \mathrm{colim}_{\Sigma \subset \mathrm{Map}(E, \mathbf{Z}_{\geq 0}) \text{ finite}} \prod\nolimits_{I \in \Sigma} R$$

(filtered colimit). Thus the displayed map of simplicial sets is a filtered colimit of trivial Kan fibrations, whence a trivial Kan fibration by Simplicial, Lemma 14.30.7. □

08QW Lemma 92.7.3. With notation as in (92.7.0.2) we have $Lg_{i,!} \circ g_i^{-1} = \text{id}$ for $i = 1, 2, 3$ and hence also $L\pi_! \circ g_i^{-1} = L\pi_{i,!}$ for $i = 1, 2, 3$.

Proof. Proof for $i = 1$. We claim the functor $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{B/A}$ Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(P' \to B) \to (P \to B)$ of $\mathcal{C}_{B/A}$. Recall that this means we have an A-algebra homomorphism $P \to P'$ compatible with maps to B. Then we set $Q' = Q \otimes_P P'$ with induced map to C and the morphism

$$(P' \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

in $\mathcal{C}_{C/B/A}$ (note reversal arrows again) is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{B/A}$. Moreover, observe that the fibre category of u_1 over $P \to B$ is the category $\mathcal{C}_{C/P}$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{B/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 21.40.2. Thus $L_n g_{1,!} g_1^{-1}\mathcal{F}$ is the (pre)sheaf which assigns to $U \in \text{Ob}(\mathcal{C}_{B/A})$ the nth homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over U. Since these restrictions are constant the desired result follows from Lemma 92.4.4 via our identifications of fibre categories above.

The case $i = 2$. We claim $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{C/A}$ is a fibred category. Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(Q' \to C) \to (Q \to C)$ of $\mathcal{C}_{C/A}$. Recall that this means we have a B-algebra homomorphism $Q \to Q'$ compatible with maps to C. Then

$$(P \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{C/A}$. Note that the fibre category of u_2 over $Q \to C$ has an final (beware reversal arrows) object, namely, $(A \to B, Q \to C)$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{C/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 21.40.2. Thus $L_n g_{2,!} g_2^{-1}\mathcal{F}$ is the (pre)sheaf which assigns to $U \in \text{Ob}(\mathcal{C}_{C/A})$ the nth homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over U. Since these restrictions are constant the desired result follows from Cohomology on Sites, Lemma 21.39.5 because the fibre categories all have final objects.

The case $i = 3$. In this case we will apply Cohomology on Sites, Lemma 21.40.3 to $u = u_3 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}$ and $\mathcal{F}' = g_3^{-1}\mathcal{F}$ for some abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$. Suppose $U = (\overline{Q} \to C)$ is an object of $\mathcal{C}_{C/B}$. Then $\mathcal{I}_U = \mathcal{C}_{\overline{Q}/B/A}$ (again beware of reversal of arrows). The sheaf $\mathcal{F}'_U$ is given by the rule $(P \to B, Q \to \overline{Q}) \mapsto \mathcal{F}(Q \otimes_P B \to C)$. In other words, this sheaf is the pullback of a sheaf on $\mathcal{C}_{\overline{Q}/C}$ via the morphism $Sh(\mathcal{C}_{\overline{Q}/B/A}) \to Sh(\mathcal{C}_{\overline{Q}/B})$. Thus Lemma 92.7.2 shows that $H_n(\mathcal{I}_U, \mathcal{F}'_U) = 0$ for $n > 0$ and equal to $\mathcal{F}(\overline{Q} \to C)$ for $n = 0$. The aforementioned Cohomology on Sites, Lemma 21.40.3 implies that $Lg_{3,!}(g_3^{-1}\mathcal{F}) = \mathcal{F}$ and the proof is done. □

08QX Proposition 92.7.4. Let $A \to B \to C$ be ring maps. There is a canonical distinguished triangle

$$L_{B/A} \otimes_B^{\mathbf{L}} C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes_B^{\mathbf{L}} C[1]$$

in $D(C)$.

Proof. Consider the short exact sequence of sheaves of Lemma 92.7.1 and apply the derived functor $L\pi_!$ to obtain a distinguished triangle

$$L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}) \to L\pi_!(g_2^{-1}\Omega_2) \to L\pi_!(g_3^{-1}\Omega_3) \to L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C})[1]$$

in $D(C)$. Using Lemmas 92.7.3 and 92.4.3 we see that the second and third terms agree with $L_{C/A}$ and $L_{C/B}$ and the first one equals

$$L\pi_{1,!}(\Omega_1 \otimes_{\underline{B}} \underline{C}) = L\pi_{1,!}(\Omega_1) \otimes_B^{\mathbf{L}} C = L_{B/A} \otimes_B^{\mathbf{L}} C$$

The first equality by Cohomology on Sites, Lemma 21.39.6 (and flatness of Ω_1 as a sheaf of modules over $\underline{B}$) and the second by Lemma 92.4.3. $\square$

08SD **Remark 92.7.5.** We sketch an alternative, perhaps simpler, proof of the existence of the fundamental triangle. Let $A \to B \to C$ be ring maps and assume that $B \to C$ is injective. Let $P_\bullet \to B$ be the standard resolution of B over A and let $Q_\bullet \to C$ be the standard resolution of C over B. Picture

$$\begin{tikzcd}
P_\bullet : & A[A[A[B]]] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d] & A[A[B]] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & A[B] \arrow[r] \arrow[d] & B \\
Q_\bullet : & A[A[A[C]]] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A[A[C]] \arrow[r, shift left=1] \arrow[r, shift right=1] & A[C] \arrow[r] & C
\end{tikzcd}$$

Observe that since $B \to C$ is injective, the ring Q_n is a polynomial algebra over P_n for all n. Hence we obtain a cosimplicial object in $\mathcal{C}_{C/B/A}$ (beware reversal arrows). Now set $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B$. The key to the proof of Proposition 92.7.4 is to show that $\overline{Q}_\bullet$ is a resolution of C over B. This follows from Cohomology on Sites, Lemma 21.39.12 applied to $\mathcal{C} = \Delta$, $\mathcal{O} = P_\bullet$, $\mathcal{O}' = B$, and $\mathcal{F} = Q_\bullet$ (this uses that Q_n is flat over P_n; see Cohomology on Sites, Remark 21.39.11 to relate simplicial modules to sheaves). The key fact implies that the distinguished triangle of Proposition 92.7.4 is the distinguished triangle associated to the short exact sequence of simplicial C-modules

$$0 \to \Omega_{P_\bullet/A} \otimes_{P_\bullet} C \to \Omega_{Q_\bullet/A} \otimes_{Q_\bullet} C \to \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C \to 0$$

which is deduced from the short exact sequences $0 \to \Omega_{P_n/A} \otimes_{P_n} Q_n \to \Omega_{Q_n/A} \to \Omega_{Q_n/P_n} \to 0$ of Algebra, Lemma 10.138.9. Namely, by Remark 92.5.5 and the key fact the complex on the right hand side represents $L_{C/B}$ in $D(C)$.

If $B \to C$ is not injective, then we can use the above to get a fundamental triangle for $A \to B \to B \times C$. Since $L_{B \times C/B} \to L_{B/B} \oplus L_{C/B}$ and $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ are quasi-isomorphism in $D(B \times C)$ (Lemma 92.6.4) this induces the desired distinguished triangle in $D(C)$ by tensoring with the flat ring map $B \times C \to C$.

08SE **Remark 92.7.6.** Let $A \to B \to C$ be ring maps with $B \to C$ injective. Recall the notation $P_\bullet$, $Q_\bullet$, $\overline{Q}_\bullet$ of Remark 92.7.5. Let $R_\bullet$ be the standard resolution of C over B. In this remark we explain how to get the canonical identification of $\Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$ with $L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C$. Let $S_\bullet \to B$ be the standard resolution of B over B. Note that the functoriality map $S_\bullet \to R_\bullet$ identifies R_n as a polynomial algebra over S_n because $B \to C$ is injective. For example in degree 0 we have the map $B[B] \to B[C]$, in degree 1 the map $B[B[B]] \to B[B[C]]$, and so on. Thus $\overline{R}_\bullet = R_\bullet \otimes_{S_\bullet} B$ is a simplicial polynomial algebra over B as well and it follows (as in Remark 92.7.5) from Cohomology on Sites, Lemma 21.39.12 that $\overline{R}_\bullet \to C$ is a resolution. Since we have a commutative diagram

$$\begin{tikzcd} Q_\bullet \arrow[r] & R_\bullet & \\ P_\bullet \arrow[u] \arrow[r] & S_\bullet \arrow[u] \arrow[r] & B \end{tikzcd}$$

we obtain a canonical map $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B \to \overline{R}_\bullet$. Thus the maps

$$L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C \longrightarrow \Omega_{\overline{R}_\bullet/B} \otimes_{\overline{R}_\bullet} C \longleftarrow \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$$

are quasi-isomorphisms (Remark 92.5.5) and composing one with the inverse of the other gives the desired identification.

92.8. Localization and étale ring maps

08QY In this section we study what happens if we localize our rings. Let $A \to A' \to B$ be ring maps such that $B = B \otimes_A^{\mathbf{L}} A'$. This happens for example if $A' = S^{-1}A$ is the localization of A at a multiplicative subset $S \subset A$. In this case for an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}_{B/A'}$ the homology of $g^{-1}\mathcal{F}'$ over $\mathcal{C}_{B/A}$ agrees with the homology of $\mathcal{F}'$ over $\mathcal{C}_{B/A'}$, see Lemma 92.6.1 for a precise statement.

08QZ **Lemma 92.8.1.** Let $A \to A' \to B$ be ring maps such that $B = B \otimes_A^{\mathbf{L}} A'$. Then $L_{B/A} = L_{B/A'}$ in $D(B)$.

Proof. According to the discussion above (i.e., using Lemma 92.6.1) and Lemma 92.4.3 we have to show that the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A} \otimes_P B$ on $\mathcal{C}_{B/A}$ is the pullback of the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A'} \otimes_P B$. The pullback functor g^{-1} is given by precomposing with the functor $u : \mathcal{C}_{B/A} \to \mathcal{C}_{B/A'}$, $(P \to B) \mapsto (P \otimes_A A' \to B)$. Thus we have to show that

$$\Omega_{P/A} \otimes_P B = \Omega_{P \otimes_A A'/A'} \otimes_{(P \otimes_A A')} B$$

By Algebra, Lemma 10.131.12 the right hand side is equal to

$$(\Omega_{P/A} \otimes_A A') \otimes_{(P \otimes_A A')} B$$

Since P is a polynomial algebra over A the module $\Omega_{P/A}$ is free and the equality is obvious. $\square$

08R0 **Lemma 92.8.2.** Let $A \to B$ be a ring map such that $B = B \otimes_A^{\mathbf{L}} B$. Then $L_{B/A} = 0$ in $D(B)$.

Proof. This is true because $L_{B/A} = L_{B/B} = 0$ by Lemmas 92.8.1 and 92.4.7. $\square$

08R1 **Lemma 92.8.3.** Let $A \to B$ be a ring map such that $\text{Tor}_i^A(B, B) = 0$ for $i > 0$ and such that $L_{B/B \otimes_A B} = 0$. Then $L_{B/A} = 0$ in $D(B)$.

Proof. By Lemma 92.6.2 we see that $L_{B/A} \otimes_B^{\mathbf{L}} (B \otimes_A B) = L_{B \otimes_A B/B}$. Now we use the distinguished triangle (92.7.0.1)

$$L_{B \otimes_A B/B} \otimes_{(B \otimes_A B)}^{\mathbf{L}} B \to L_{B/B} \to L_{B/B \otimes_A B} \to L_{B \otimes_A B/B} \otimes_{(B \otimes_A B)}^{\mathbf{L}} B[1]$$

associated to the ring maps $B \to B \otimes_A B \to B$ and the vanishing of $L_{B/B}$ (Lemma 92.4.7) and $L_{B/B \otimes_A B}$ (assumed) to see that

$$0 = L_{B \otimes_A B/B} \otimes_{(B \otimes_A B)}^{\mathbf{L}} B = L_{B/A} \otimes_B^{\mathbf{L}} (B \otimes_A B) \otimes_{(B \otimes_A B)}^{\mathbf{L}} B = L_{B/A}$$

as desired. □

08R2 **Lemma 92.8.4.** The cotangent complex $L_{B/A}$ is zero in each of the following cases:

(1) $A \to B$ and $B \otimes_A B \to B$ are flat, i.e., $A \to B$ is weakly étale (More on Algebra, Definition 15.104.1),
(2) $A \to B$ is a flat epimorphism of rings,
(3) $B = S^{-1}A$ for some multiplicative subset $S \subset A$,
(4) $A \to B$ is unramified and flat,
(5) $A \to B$ is étale,
(6) $A \to B$ is a filtered colimit of ring maps for which the cotangent complex vanishes,
(7) B is a henselization of a local ring of A,
(8) B is a strict henselization of a local ring of A, and
(9) add more here.

Proof. In case (1) we may apply Lemma 92.8.2 to the surjective flat ring map $B \otimes_A B \to B$ to conclude that $L_{B/B\otimes_A B} = 0$ and then we use Lemma 92.8.3 to conclude. The cases (2) – (5) are each special cases of (1). Part (6) follows from Lemma 92.3.4. Parts (7) and (8) follows from the fact that (strict) henselizations are filtered colimits of étale ring extensions of A, see Algebra, Lemmas 10.155.7 and 10.155.11. □

08R3 **Lemma 92.8.5.** Let $A \to B \to C$ be ring maps such that $L_{C/B} = 0$. Then $L_{C/A} = L_{B/A} \otimes_B^{\mathbf{L}} C$.

Proof. This is a trivial consequence of the distinguished triangle (92.7.0.1). □

08SF **Lemma 92.8.6.** Let $A \to B$ be ring maps and $S \subset A$, $T \subset B$ multiplicative subsets such that S maps into T. Then $L_{T^{-1}B/S^{-1}A} = L_{B/A} \otimes_B T^{-1}B$ in $D(T^{-1}B)$.

Proof. Lemma 92.8.5 shows that $L_{T^{-1}B/A} = L_{B/A} \otimes_B T^{-1}B$ and Lemma 92.8.1 shows that $L_{T^{-1}B/A} = L_{T^{-1}B/S^{-1}A}$. □

08UN **Lemma 92.8.7.** Let $A \to B$ be a local ring homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced maps of henselizations, resp. strict henselizations. Then

$$L_{B^h/A^h} = L_{B^h/A} = L_{B/A} \otimes_B^{\mathbf{L}} B^h \quad \text{resp.} \quad L_{B^{sh}/A^{sh}} = L_{B^{sh}/A} = L_{B/A} \otimes_B^{\mathbf{L}} B^{sh}$$

in $D(B^h)$, resp. $D(B^{sh})$.

Proof. The complexes $L_{A^h/A}$, $L_{A^{sh}/A}$, $L_{B^h/B}$, and $L_{B^{sh}/B}$ are all zero by Lemma 92.8.4. Using the fundamental distinguished triangle (92.7.0.1) for $A \to B \to B^h$ we obtain $L_{B^h/A} = L_{B/A} \otimes_B^{\mathbf{L}} B^h$. Using the fundamental triangle for $A \to A^h \to B^h$ we obtain $L_{B^h/A^h} = L_{B^h/A}$. Similarly for strict henselizations. □

92.9. Smooth ring maps

08R4 Let $C \to B$ be a surjection of rings with kernel I. Let us call such a ring map "weakly quasi-regular" if I/I^2 is a flat B-module and $\text{Tor}_*^C(B, B)$ is the exterior algebra on I/I^2. The generalization to "smooth ring maps" of what is done in Lemma 92.8.4 for "étale ring maps" is to look at flat ring maps $A \to B$ such that the multiplication map $B \otimes_A B \to B$ is weakly quasi-regular. For the moment we just stick to smooth ring maps.

08R5 **Lemma 92.9.1.** If $A \to B$ is a smooth ring map, then $L_{B/A} = \Omega_{B/A}[0]$.

Proof. We have the agreement in cohomological degree 0 by Lemma 92.4.5. Thus it suffices to prove the other cohomology groups are zero. It suffices to prove this locally on $\mathrm{Spec}(B)$ as $L_{B_g/A} = (L_{B/A})_g$ for $g \in B$ by Lemma 92.8.5. Thus we may assume that $A \to B$ is standard smooth (Algebra, Lemma 10.137.10), i.e., that we can factor $A \to B$ as $A \to A[x_1, \ldots, x_n] \to B$ with $A[x_1, \ldots, x_n] \to B$ étale. In this case Lemmas 92.8.4 and Lemma 92.8.5 show that $L_{B/A} = L_{A[x_1,\ldots,x_n]/A} \otimes B$ whence the conclusion by Lemma 92.4.7. $\square$

92.10. Positive characteristic

0G5X In this section we fix a prime number p. If A is a ring with $p = 0$ in A, then $F_A : A \to A$ denotes the Frobenius endomorphism $a \mapsto a^p$.

0G5Y **Lemma 92.10.1.** Let $A \to B$ be a ring map with $p = 0$ in A. Let $P_\bullet$ be the standard resolution of B over A. The map $P_\bullet \to P_\bullet$ induced by the diagram

$$\begin{tikzcd} B \arrow[r, "F_B"'] & B \\ A \arrow[u] \arrow[r, "F_A"] & A \arrow[u] \end{tikzcd}$$

discussed in Section 92.6 is homotopic to the Frobenius endomorphism $P_\bullet \to P_\bullet$ given by Frobenius on each P_n.

Proof. Let $\mathcal{A}$ be the category of $\mathbf{F}_p$-algebra maps $A \to B$. Let $\mathcal{S}$ be the category of pairs (A, E) where A is an $\mathbf{F}_p$-algebra and E is a set. Consider the adjoint functors

$$V : \mathcal{A} \to \mathcal{S}, \quad (A \to B) \mapsto (A, B)$$

and

$$U : \mathcal{S} \to \mathcal{A}, \quad (A, E) \mapsto (A \to A[E])$$

Let X be the simplicial object in in the category of functors from $\mathcal{A}$ to $\mathcal{A}$ constructed in Simplicial, Section 14.34. It is clear that $P_\bullet = X(A \to B)$ because if we fix A then.

Set $Y = U \circ V$. Recall that X is constructed from Y and certain maps and has terms $X_n = Y \circ \ldots \circ Y$ with $n + 1$ terms; the construction is given in Simplicial, Example 14.33.1 and please see proof of Simplicial, Lemma 14.34.2 for details.

Let $f : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}}$ be the Frobenius endomorphism of the identity functor. In other words, we set $f_{A \to B} = (F_A, F_B) : (A \to B) \to (A \to B)$. Then our two maps on $X(A \to B)$ are given by the natural transformations $f \star 1_X$ and $1_X \star f$. Details omitted. Thus we conclude by Simplicial, Lemma 14.33.6. $\square$

0G5Z **Lemma 92.10.2.** Let p be a prime number. Let $A \to B$ be a ring homomorphism and assume that $p = 0$ in A. The map $L_{B/A} \to L_{B/A}$ of Section 92.6 induced by the Frobenius maps F_A and F_B is homotopic to zero.

Proof. Let $P_\bullet$ be the standard resolution of B over A. By Lemma 92.10.1 the map $P_\bullet \to P_\bullet$ induced by F_A and F_B is homotopic to the map $F_{P_\bullet} : P_\bullet \to P_\bullet$ given by Frobenius on each term. Hence we obtain what we want as clearly $F_{P_\bullet}$ induces the zero zero map $\Omega_{P_n/A} \to \Omega_{P_n/A}$ (since the derivative of a pth power is zero). $\square$

0G60 Lemma 92.10.3. Let p be a prime number. Let $A \to B$ be a ring homomorphism and assume that $p = 0$ in A. If A and B are perfect, then $L_{B/A}$ is zero in $D(B)$.

Proof. The map $(F_A, F_B) : (A \to B) \to (A \to B)$ is an isomorphism hence induces an isomorphism on $L_{B/A}$ and on the other hand induces zero on $L_{B/A}$ by Lemma 92.10.2. $\square$

92.11. Comparison with the naive cotangent complex

08R6 The naive cotangent complex was introduced in Algebra, Section 10.134.

08R7 Remark 92.11.1. Let $A \to B$ be a ring map. Working on $\mathcal{C}_{B/A}$ as in Section 92.4 let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Note that $L\pi_!(\mathcal{J}) = 0$ by Lemma 92.5.7. Set $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ so that $L_{B/A} = L\pi_!(\Omega)$ by Lemma 92.4.3. It follows that $L\pi_!(\mathcal{J} \to \Omega) = L\pi_!(\Omega) = L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

08R8 (92.11.1.1) $$(J \to \Omega_{P/A} \otimes_P B) \longrightarrow L_{B/A}$$

where $J = \mathrm{Ker}(P \to B)$ in $D(A)$, see Cohomology on Sites, Remark 21.39.4. Continuing in this manner, note that $L\pi_!(\mathcal{J} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}) = L\pi_!(\mathcal{J}) = 0$ by Lemma 92.5.6. Since $\mathrm{Tor}_0^{\mathcal{O}}(\mathcal{J}, \underline{B}) = \mathcal{J}/\mathcal{J}^2$ the spectral sequence

$$H_p(\mathcal{C}_{B/A}, \mathrm{Tor}_q^{\mathcal{O}}(\mathcal{J}, \underline{B})) \Rightarrow H_{p+q}(\mathcal{C}_{B/A}, \mathcal{J} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}) = 0$$

(dual of Derived Categories, Lemma 13.21.3) implies that $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$ and $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$. It follows that the complex of $\underline{B}$-modules $\mathcal{J}/\mathcal{J}^2 \to \Omega$ satisfies $\tau_{\geq -1} L\pi_!(\mathcal{J}/\mathcal{J}^2 \to \Omega) = \tau_{\geq -1} L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

08R9 (92.11.1.2) $$(J/J^2 \to \Omega_{P/A} \otimes_P B) \longrightarrow \tau_{\geq -1} L_{B/A}$$

in $D(B)$, see Cohomology on Sites, Remark 21.39.4.

The first case is where we have a surjection of rings.

08RA Lemma 92.11.2. Let $A \to B$ be a surjective ring map with kernel I. Then $H^0(L_{B/A}) = 0$ and $H^{-1}(L_{B/A}) = I/I^2$. This isomorphism comes from the map (92.11.1.2) for the object $(A \to B)$ of $\mathcal{C}_{B/A}$.

Proof. We will show below (using the surjectivity of $A \to B$) that there exists a short exact sequence

$$0 \to \pi^{-1}(I/I^2) \to \mathcal{J}/\mathcal{J}^2 \to \Omega \to 0$$

of sheaves on $\mathcal{C}_{B/A}$. Taking $L\pi_!$ and the associated long exact sequence of homology, and using the vanishing of $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ and $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ shown in Remark 92.11.1 we obtain what we want using Lemma 92.4.4.

What is left is to verify the local statement mentioned above. For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. The value on U of the short sequence of sheaves above is the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{P/A} \otimes_P B \to 0$$

Verification omitted (hint: the only tricky point is that $IP \cap J^2 = IJ$; which follows for example from More on Algebra, Lemma 15.30.9). $\square$

08RB **Lemma 92.11.3.** Let $A \to B$ be a ring map. Then $\tau_{\geq -1}L_{B/A}$ is canonically quasi-isomorphic to the naive cotangent complex.

Proof. Consider $P = A[B] \to B$ with kernel I. The naive cotangent complex $NL_{B/A}$ of B over A is the complex $I/I^2 \to \Omega_{P/A} \otimes_P B$, see Algebra, Definition 10.134.1. Observe that in (92.11.1.2) we have already constructed a canonical map

$$c : NL_{B/A} \longrightarrow \tau_{\geq -1}L_{B/A}$$

Consider the distinguished triangle (92.7.0.1)

$$L_{P/A} \otimes_P^{\mathbf{L}} B \to L_{B/A} \to L_{B/P} \to (L_{P/A} \otimes_P^{\mathbf{L}} B)[1]$$

associated to the ring maps $A \to A[B] \to B$. We know that $L_{P/A} = \Omega_{P/A}[0] = NL_{P/A}$ in $D(P)$ (Lemma 92.4.7 and Algebra, Lemma 10.134.3) and that $\tau_{\geq -1}L_{B/P} = I/I^2[1] = NL_{B/P}$ in $D(B)$ (Lemma 92.11.2 and Algebra, Lemma 10.134.6). To show c is a quasi-isomorphism it suffices by Algebra, Lemma 10.134.4 and the long exact cohomology sequence associated to the distinguished triangle to show that the maps $L_{P/A} \to L_{B/A} \to L_{B/P}$ are compatible on cohomology groups with the corresponding maps $NL_{P/A} \to NL_{B/A} \to NL_{B/P}$ of the naive cotangent complex. We omit the verification. $\square$

08UP **Remark 92.11.4.** We can make the comparison map of Lemma 92.11.3 explicit in the following way. Let $P_\bullet$ be the standard resolution of B over A. Let $I = \mathrm{Ker}(A[B] \to B)$. Recall that $P_0 = A[B]$. The map of the lemma is given by the commutative diagram

$$\begin{tikzcd}
L_{B/A} \arrow[d] & \ldots \arrow[r] & \Omega_{P_2/A} \otimes_{P_2} B \arrow[r] \arrow[d] & \Omega_{P_1/A} \otimes_{P_1} B \arrow[r] \arrow[d] & \Omega_{P_0/A} \otimes_{P_0} B \arrow[d] \\
NL_{B/A} & \ldots \arrow[r] & 0 \arrow[r] & I/I^2 \arrow[r] & \Omega_{P_0/A} \otimes_{P_0} B
\end{tikzcd}$$

We construct the downward arrow with target I/I^2 by sending $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in I/I^2. Here $d_i : P_1 \to P_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps P_1 into $I = \mathrm{Ker}(P_0 \to B)$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{P_1/A} \otimes_{P_1} B \to \Omega_{P_0/A} \otimes_{P_0} B$ as this differential maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{P_2/A} \otimes_{P_2} B \to \Omega_{P_1/A} \otimes_{P_1} B$ maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes. We omit the verification that this is the same as the map of Lemma 92.11.3.

09D5 **Remark 92.11.5.** Adopt notation as in Remark 92.11.1. The arguments given there show that the differential

$$H_2(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \longrightarrow H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))$$

of the spectral sequence is an isomorphism. Let $\mathcal{C}'_{B/A}$ denote the full subcategory of $\mathcal{C}_{B/A}$ consisting of surjective maps $P \to B$. The agreement of the cotangent complex with the naive cotangent complex (Lemma 92.11.3) shows that we have an exact sequence of sheaves

$$0 \to \underline{H_1(L_{B/A})} \to \mathcal{J}/\mathcal{J}^2 \xrightarrow{\mathrm{d}} \Omega \to \underline{H_2(L_{B/A})} \to 0$$

on $\mathcal{C}'_{B/A}$. It follows that $\mathrm{Ker}(d)$ and $\mathrm{Coker}(d)$ on the whole category $\mathcal{C}_{B/A}$ have vanishing higher homology groups, since these are computed by the homology groups of constant simplicial abelian groups by Lemma 92.4.1. Hence we conclude that

$$H_n(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \to H_n(L_{B/A})$$

is an isomorphism for all $n \geq 2$. Combined with the remark above we obtain the formula $H_2(L_{B/A}) = H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))$.

92.12. A spectral sequence of Quillen

08RC In this section we discuss a spectral sequence relating derived tensor product to the cotangent complex.

08RD **Lemma 92.12.1.** Notation and assumptions as in Cohomology on Sites, Example 21.39.1. Assume $\mathcal{C}$ has a cosimplicial object as in Cohomology on Sites, Lemma 21.39.7. Let $\mathcal{F}$ be a flat $\underline{B}$-module such that $H_0(\mathcal{C}, \mathcal{F}) = 0$. Then $H_l(\mathcal{C}, \mathrm{Sym}^k_{\underline{B}}(\mathcal{F})) = 0$ for $l < k$.

Proof. We drop the subscript $\underline{B}$ from tensor products, wedge powers, and symmetric powers. We will prove the lemma by induction on k. The cases $k = 0, 1$ follow from the assumptions. If $k > 1$ consider the exact complex

$$\ldots \to \wedge^2\mathcal{F} \otimes \mathrm{Sym}^{k-2}\mathcal{F} \to \mathcal{F} \otimes \mathrm{Sym}^{k-1}\mathcal{F} \to \mathrm{Sym}^k\mathcal{F} \to 0$$

with differentials as in the Koszul complex. If we think of this as a resolution of $\mathrm{Sym}^k\mathcal{F}$, then this gives a first quadrant spectral sequence

$$E_1^{p,q} = H_p(\mathcal{C}, \wedge^{q+1}\mathcal{F} \otimes \mathrm{Sym}^{k-q-1}\mathcal{F}) \Rightarrow H_{p+q}(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$$

By Cohomology on Sites, Lemma 21.39.10 we have

$$L\pi_!(\wedge^{q+1}\mathcal{F} \otimes \mathrm{Sym}^{k-q-1}\mathcal{F}) = L\pi_!(\wedge^{q+1}\mathcal{F}) \otimes^{\mathbf{L}}_B L\pi_!(\mathrm{Sym}^{k-q-1}\mathcal{F}))$$

It follows (from the construction of derived tensor products) that the induction hypothesis combined with the vanishing of $H_0(\mathcal{C}, \wedge^{q+1}(\mathcal{F})) = 0$ will prove what we want. This is true because $\wedge^{q+1}(\mathcal{F})$ is a quotient of $\mathcal{F}^{\otimes q+1}$ and $H_0(\mathcal{C}, \mathcal{F}^{\otimes q+1})$ is a quotient of $H_0(\mathcal{C}, \mathcal{F})^{\otimes q+1}$ which is zero. $\square$

08SG **Remark 92.12.2.** In the situation of Lemma 92.12.1 one can show that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F})) = \wedge^k_B(H_1(\mathcal{C}, \mathcal{F}))$. Namely, it can be deduced from the proof that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$ is the S_k-coinvariants of

$$H^{-k}(L\pi_!(\mathcal{F}) \otimes^{\mathbf{L}}_B L\pi_!(\mathcal{F}) \otimes^{\mathbf{L}}_B \ldots \otimes^{\mathbf{L}}_B L\pi_!(\mathcal{F})) = H_1(\mathcal{C}, \mathcal{F})^{\otimes k}$$

Thus our claim is that this action is given by the usual action of S_k on the tensor product multiplied by the sign character. To prove this one has to work through the sign conventions in the definition of the total complex associated to a multi-complex. We omit the verification.

08RE **Lemma 92.12.3.** Let A be a ring. Let $P = A[E]$ be a polynomial ring. Set $I = (e; e \in E) \subset P$. The maps $\mathrm{Tor}_i^P(A, I^{n+1}) \to \mathrm{Tor}_i^P(A, I^n)$ are zero for all i and n.

Proof. Denote $x_e \in P$ the variable corresponding to $e \in E$. A free resolution of A over P is given by the Koszul complex $K_\bullet$ on the x_e. Here K_i has basis given by wedges $e_1 \wedge \ldots \wedge e_i$, $e_1, \ldots, e_i \in E$ and $d(e) = x_e$. Thus $K_\bullet \otimes_P I^n = I^n K_\bullet$ computes $\mathrm{Tor}_i^P(A, I^n)$. Observe that everything is graded with $\deg(x_e) = 1$, $\deg(e) = 1$, and

$\deg(a) = 0$ for $a \in A$. Suppose $\xi \in I^{n+1}K_i$ is a cocycle homogeneous of degree m. Note that $m \geq i + 1 + n$. Then $\xi = d\eta$ for some $\eta \in K_{i+1}$ as $K_\bullet$ is exact in degrees > 0. (The case $i = 0$ is left to the reader.) Now $\deg(\eta) = m \geq i + 1 + n$. Hence writing η in terms of the basis we see the coordinates are in I^n. Thus ξ maps to zero in the homology of $I^n K_\bullet$ as desired. $\square$

08RF Theorem 92.12.4 (Quillen spectral sequence). Let $A \to B$ be a surjective ring map. Consider the sheaf $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ of $\underline{B}$-modules on $\mathcal{C}_{B/A}$, see Section 92.4. Then there is a spectral sequence with E_1-page

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, \mathrm{Sym}^p_{\underline{B}}(\Omega)) \Rightarrow \mathrm{Tor}^A_{-p-q}(B, B)$$

with d_r of bidegree $(r, -r+1)$. Moreover, $H_i(\mathcal{C}_{B/A}, \mathrm{Sym}^k_{\underline{B}}(\Omega)) = 0$ for $i < k$.

Proof. Let $I \subset A$ be the kernel of $A \to B$. Let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Then $I\mathcal{O} \subset \mathcal{J}$. Set $\mathcal{K} = \mathcal{J}/I\mathcal{O}$ and $\overline{\mathcal{O}} = \mathcal{O}/I\mathcal{O}$.

For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. Moreover $\overline{\mathcal{O}}(U) = B[E]$ and $K = \mathcal{K}(U) = (e; e \in E)$ is the ideal generated by the variables in the polynomial ring $B[E]$. In particular it is clear that

$$K/K^2 \xrightarrow{d} \Omega_{P/A} \otimes_P B$$

is a bijection. In other words, $\Omega = \mathcal{K}/\mathcal{K}^2$ and $\mathrm{Sym}^k_B(\Omega) = \mathcal{K}^k/\mathcal{K}^{k+1}$. Note that $\pi_!(\Omega) = \Omega_{B/A} = 0$ (Lemma 92.4.5) as $A \to B$ is surjective (Algebra, Lemma 10.131.4). By Lemma 92.12.1 we conclude that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^k/\mathcal{K}^{k+1}) = H_i(\mathcal{C}_{B/A}, \mathrm{Sym}^k_{\underline{B}}(\Omega)) = 0$$

for $i < k$. This proves the final statement of the theorem.

The approach to the theorem is to note that

$$B \otimes^{\mathbf{L}}_A B = L\pi_!(\mathcal{O}) \otimes^{\mathbf{L}}_A B = L\pi_!(\mathcal{O} \otimes^{\mathbf{L}}_{\underline{A}} \underline{B}) = L\pi_!(\overline{\mathcal{O}})$$

The first equality by Lemma 92.5.7, the second equality by Cohomology on Sites, Lemma 21.39.6, and the third equality as $\mathcal{O}$ is flat over $\underline{A}$. The sheaf $\overline{\mathcal{O}}$ has a filtration

$$\ldots \subset \mathcal{K}^3 \subset \mathcal{K}^2 \subset \mathcal{K} \subset \overline{\mathcal{O}}$$

This induces a filtration F on a complex C representing $L\pi_!(\overline{\mathcal{O}})$ with F^pC representing $L\pi_!(\mathcal{K}^p)$ (construction of C and F omitted). Consider the spectral sequence of Homology, Section 12.24 associated to (C, F). It has E_1-page

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, \mathcal{K}^p/\mathcal{K}^{p+1}) \quad \Rightarrow \quad H_{-p-q}(\mathcal{C}_{B/A}, \overline{\mathcal{O}}) = \mathrm{Tor}^A_{-p-q}(B, B)$$

and differentials $E_r^{p,q} \to E_r^{p+r, q-r+1}$. To show convergence we will show that for every k there exists a c such that $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < k$ and $n > c$[2].

Given $k \geq 0$ set $c = k^2$. We claim that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$$

is zero for $i < k$ and all $n \geq 0$. Note that $\mathcal{K}^n/\mathcal{K}^{n+c}$ has a finite filtration whose successive quotients $\mathcal{K}^m/\mathcal{K}^{m+1}$, $n \leq m < n + c$ have $H_i(\mathcal{C}_{B/A}, \mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for

[2]A posteriori the "correct" vanishing $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < n$ can be concluded.

$i < n$ (see above). Hence the claim implies $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) = 0$ for $i < k$ and all $n \geq k$ which is what we need to show.

Proof of the claim. Recall that for any $\mathcal{O}$-module $\mathcal{F}$ the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}}^{\mathbf{L}} B$ induces an isomorphism on applying $L\pi_!$, see Lemma 92.5.6. Consider the map

$$\mathcal{K}^{n+k} \otimes_{\mathcal{O}}^{\mathbf{L}} B \longrightarrow \mathcal{K}^n \otimes_{\mathcal{O}}^{\mathbf{L}} B$$

We claim that this map induces the zero map on cohomology sheaves in degrees $0, -1, \ldots, -k+1$. If this second claim holds, then the k-fold composition

$$\mathcal{K}^{n+c} \otimes_{\mathcal{O}}^{\mathbf{L}} B \longrightarrow \mathcal{K}^n \otimes_{\mathcal{O}}^{\mathbf{L}} B$$

factors through $\tau_{\leq -k}\mathcal{K}^n \otimes_{\mathcal{O}}^{\mathbf{L}} B$ hence induces zero on $H_i(\mathcal{C}_{B/A}, -) = L_i\pi_!(-)$ for $i < k$, see Derived Categories, Lemma 13.12.5. By the remark above this means the same thing is true for $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$ which proves the (first) claim.

Proof of the second claim. The statement is local, hence we may work over an object $U = (P \to B)$ as above. We have to show the maps

$$\text{Tor}_i^P(B, K^{n+k}) \to \text{Tor}_i^P(B, K^n)$$

are zero for $i < k$. There is a spectral sequence

$$\text{Tor}_a^P(P/IP, \text{Tor}_b^{P/IP}(B, K^n)) \Rightarrow \text{Tor}_{a+b}^P(B, K^n),$$

see More on Algebra, Example 15.62.2. Thus it suffices to prove the maps

$$\text{Tor}_i^{P/IP}(B, K^{n+1}) \to \text{Tor}_i^{P/IP}(B, K^n)$$

are zero for all i. This is Lemma 92.12.3. □

08RG Remark 92.12.5. In the situation of Theorem 92.12.4 let $I = \text{Ker}(A \to B)$. Then $H^{-1}(L_{B/A}) = H_1(\mathcal{C}_{B/A}, \Omega) = I/I^2$, see Lemma 92.11.2. Hence $H_k(\mathcal{C}_{B/A}, \text{Sym}^k(\Omega)) = \wedge_B^k(I/I^2)$ by Remark 92.12.2. Thus the E_1-page looks like

$$\begin{array}{llll}
B & & & \\
0 & & & \\
0 & I/I^2 & & \\
0 & H^{-2}(L_{B/A}) & & \\
0 & H^{-3}(L_{B/A}) & \wedge^2(I/I^2) & \\
0 & H^{-4}(L_{B/A}) & H_3(\mathcal{C}_{B/A}, \text{Sym}^2(\Omega)) & \\
0 & H^{-5}(L_{B/A}) & H_4(\mathcal{C}_{B/A}, \text{Sym}^2(\Omega)) & \wedge^3(I/I^2)
\end{array}$$

with horizontal differential. Thus we obtain edge maps $\text{Tor}_i^A(B, B) \to H^{-i}(L_{B/A})$, $i > 0$ and $\wedge_B^i(I/I^2) \to \text{Tor}_i^A(B, B)$. Finally, we have $\text{Tor}_1^A(B, B) = I/I^2$ and there is a five term exact sequence

$$\text{Tor}_3^A(B, B) \to H^{-3}(L_{B/A}) \to \wedge_B^2(I/I^2) \to \text{Tor}_2^A(B, B) \to H^{-2}(L_{B/A}) \to 0$$

of low degree terms.

09D6 Remark 92.12.6. Let $A \to B$ be a ring map. Let $P_\bullet$ be a resolution of B over A (Remark 92.5.5). Set $J_n = \text{Ker}(P_n \to B)$. Note that

$$\text{Tor}_2^{P_n}(B, B) = \text{Tor}_1^{P_n}(J_n, B) = \text{Ker}(J_n \otimes_{P_n} J_n \to J_n^2).$$

Hence $H_2(L_{B/A})$ is canonically equal to

$$\text{Coker}(\text{Tor}_2^{P_1}(B, B) \to \text{Tor}_2^{P_0}(B, B))$$

by Remark 92.11.5. To make this more explicit we choose P_2, P_1, P_0 as in Example 92.5.9. We claim that

$$\mathrm{Tor}_2^{P_1}(B, B) = \wedge^2(\bigoplus\nolimits_{t \in T} B) \ \oplus \ \bigoplus\nolimits_{t \in T} J_0 \ \oplus \ \mathrm{Tor}_2^{P_0}(B, B)$$

Namely, the basis elements $x_t \wedge x_{t'}$ of the first summand corresponds to the element $x_t \otimes x_{t'} - x_{t'} \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. For $f \in J_0$ the element $x_t \otimes f$ of the second summand corresponds to the element $x_t \otimes s_0(f) - s_0(f) \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. Finally, the map $\mathrm{Tor}_2^{P_0}(B, B) \to \mathrm{Tor}_2^{P_1}(B, B)$ is given by s_0. The map $d_0 - d_1 : \mathrm{Tor}_2^{P_1}(B, B) \to \mathrm{Tor}_2^{P_0}(B, B)$ is zero on the last summand, maps $x_t \otimes f$ to $f \otimes f_t - f_t \otimes f$, and maps $x_t \wedge x_{t'}$ to $f_t \otimes f_{t'} - f_{t'} \otimes f_t$. All in all we conclude that there is an exact sequence

$$\wedge^2_B(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B, B) \to H^{-2}(L_{B/A}) \to 0$$

In this way we obtain a direct proof of a consequence of Quillen's spectral sequence discussed in Remark 92.12.5.

92.13. Comparison with Lichtenbaum-Schlessinger

09AM Let $A \to B$ be a ring map. In [LS67] there is a fairly explicit determination of $\tau_{\geq -2} L_{B/A}$ which is often used in calculations of versal deformation spaces of singularities. The construction follows. Choose a polynomial algebra P over A and a surjection $P \to B$ with kernel I. Choose generators f_t, $t \in T$ for I which induces a surjection $F = \bigoplus_{t \in T} P \to I$ with F a free P-module. Let $Rel \subset F$ be the kernel of $F \to I$, in other words Rel is the set of relations among the f_t. Let $TrivRel \subset Rel$ be the submodule of trivial relations, i.e., the submodule of Rel generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$. Consider the complex of B-modules

09CD (92.13.0.1)
$$Rel/TrivRel \longrightarrow F \otimes_P B \longrightarrow \Omega_{P/A} \otimes_P B$$

where the last term is placed in degree 0. The first map is the obvious one and the second map sends the basis element corresponding to $t \in T$ to $\mathrm{d}f_t \otimes 1$.

09CE Definition 92.13.1. Let $A \to B$ be a ring map. Let M be a (B, B)-bimodule over A. An A-biderivation is an A-linear map $\lambda : B \to M$ such that $\lambda(xy) = x\lambda(y) + \lambda(x)y$.

For a polynomial algebra the biderivations are easy to describe.

09CF Lemma 92.13.2. Let $P = A[S]$ be a polynomial ring over A. Let M be a (P, P)-bimodule over A. Given $m_s \in M$ for $s \in S$, there exists a unique A-biderivation $\lambda : P \to M$ mapping s to m_s for $s \in S$.

Proof. We set

$$\lambda(s_1 \ldots s_t) = \sum s_1 \ldots s_{i-1} m_{s_i} s_{i+1} \ldots s_t$$

in M. Extending by A-linearity we obtain a biderivation. ☐

Here is the comparison statement. The reader may also read about this in [And74, page 206, Proposition 12] or in the paper [DRGV92] which extends the complex (92.13.0.1) by one term and the comparison to $\tau_{\geq -3}$.

09CG Lemma 92.13.3. In the situation above denote L the complex (92.13.0.1). There is a canonical map $L_{B/A} \to L$ in $D(B)$ which induces an isomorphism $\tau_{\geq -2} L_{B/A} \to L$ in $D(B)$.

Proof. Let $P_\bullet \to B$ be a resolution of B over A (Remark 92.5.5). We will identify $L_{B/A}$ with $\Omega_{P_\bullet/A} \otimes B$. To construct the map we make some choices.

Choose an A-algebra map $\psi : P_0 \to P$ compatible with the given maps $P_0 \to B$ and $P \to B$.

Write $P_1 = A[S]$ for some set S. For $s \in S$ we may write

$$\psi(d_0(s) - d_1(s)) = \sum p_{s,t} f_t$$

for some $p_{s,t} \in P$. Think of $F = \bigoplus_{t \in T} P$ as a (P_1, P_1)-bimodule via the maps $(\psi \circ d_0, \psi \circ d_1)$. By Lemma 92.13.2 we obtain a unique A-biderivation $\lambda : P_1 \to F$ mapping s to the vector with coordinates $p_{s,t}$. By construction the composition

$$P_1 \longrightarrow F \longrightarrow P$$

sends $f \in P_1$ to $\psi(d_0(f) - d_1(f))$ because the map $f \mapsto \psi(d_0(f) - d_1(f))$ is an A-biderivation agreeing with the composition on generators.

For $g \in P_2$ we claim that $\lambda(d_0(g) - d_1(g) + d_2(g))$ is an element of Rel. Namely, by the last remark of the previous paragraph the image of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in P is

$$\psi((d_0 - d_1)(d_0(g) - d_1(g) + d_2(g)))$$

which is zero by Simplicial, Section 14.23).

The choice of ψ determines a map

$$\mathrm{d}\psi \otimes 1 : \Omega_{P_0/A} \otimes B \longrightarrow \Omega_{P/A} \otimes B$$

Composing λ with the map $F \to F \otimes B$ gives a usual A-derivation as the two P_1-module structures on $F \otimes B$ agree. Thus λ determines a map

$$\overline{\lambda} : \Omega_{P_1/A} \otimes B \longrightarrow F \otimes B$$

Finally, We obtain a B-linear map

$$q : \Omega_{P_2/A} \otimes B \longrightarrow Rel/TrivRel$$

by mapping $\mathrm{d}g$ to the class of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in the quotient.

The diagram

$$\begin{tikzcd}
\Omega_{P_3/A} \otimes B \arrow[r] \arrow[d] & \Omega_{P_2/A} \otimes B \arrow[r] \arrow[d, "q"'] & \Omega_{P_1/A} \otimes B \arrow[r] \arrow[d, "\overline{\lambda}"'] & \Omega_{P_0/A} \otimes B \arrow[d, "\mathrm{d}\psi\otimes 1"'] \\
0 \arrow[r] & Rel/TrivRel \arrow[r] & F \otimes B \arrow[r] & \Omega_{P/A} \otimes B
\end{tikzcd}$$

commutes (calculation omitted) and we obtain the map of the lemma. By Remark 92.11.4 and Lemma 92.11.3 we see that this map induces isomorphisms $H_1(L_{B/A}) \to H_1(L)$ and $H_0(L_{B/A}) \to H_0(L)$.

It remains to see that our map $L_{B/A} \to L$ induces an isomorphism $H_2(L_{B/A}) \to H_2(L)$. Choose a resolution of B over A with $P_0 = P = A[u_i]$ and then P_1 and P_2 as in Example 92.5.9. In Remark 92.12.6 we have constructed an exact sequence

$$\wedge^2_B(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B, B) \to H^{-2}(L_{B/A}) \to 0$$

where $P_0 = P$ and $J_0 = \mathrm{Ker}(P \to B) = I$. Calculating the Tor group using the short exact sequences $0 \to I \to P \to B \to 0$ and $0 \to Rel \to F \to I \to 0$ we find that $\mathrm{Tor}_2^P(B, B) = \mathrm{Ker}(Rel \otimes B \to F \otimes B)$. The image of the map $\wedge^2_B(I/I^2) \to$

$\mathrm{Tor}_2^P(B,B)$ under this identification is exactly the image of $TrivRel \otimes B$. Thus we see that $H_2(L_{B/A}) \cong H_2(L)$.

Finally, we have to check that our map $L_{B/A} \to L$ actually induces this isomorphism. We will use the notation and results discussed in Example 92.5.9 and Remarks 92.12.6 and 92.11.5 without further mention. Pick an element ξ of $\mathrm{Tor}_2^{P_0}(B,B) = \mathrm{Ker}(I \otimes_P I \to I^2)$. Write $\xi = \sum h_{t',t} f_{t'} \otimes f_t$ for some $h_{t',t} \in P$. Tracing through the exact sequences above we find that ξ corresponds to the image in $Rel \otimes B$ of the element $r \in Rel \subset F = \bigoplus_{t \in T} P$ with tth coordinate $r_t = \sum_{t' \in T} h_{t',t} f_{t'}$. On the other hand, ξ corresponds to the element of $H_2(L_{B/A}) = H_2(\Omega)$ which is the image via $\mathrm{d} : H_2(\mathcal{J}/\mathcal{J}^2) \to H_2(\Omega)$ of the boundary of ξ under the 2-extension

$$0 \to \mathrm{Tor}_2^{\mathcal{O}}(\underline{B}, \underline{B}) \to \mathcal{J} \otimes_{\mathcal{O}} \mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$$

We compute the successive transgressions of our element. First we have

$$\xi = (d_0 - d_1)(-\sum s_0(h_{t',t} f_{t'}) \otimes x_t)$$

and next we have

$$\sum s_0(h_{t',t} f_{t'}) x_t = d_0(v_r) - d_1(v_r) + d_2(v_r)$$

by our choice of the variables v in Example 92.5.9. We may choose our map λ above such that $\lambda(u_i) = 0$ and $\lambda(x_t) = -e_t$ where $e_t \in F$ denotes the basis vector corresponding to $t \in T$. Hence the construction of our map q above sends $\mathrm{d}v_r$ to

$$\lambda(\sum s_0(h_{t',t} f_{t'}) x_t) = \sum\nolimits_t \left(\sum\nolimits_{t'} h_{t',t} f_{t'}\right) e_t$$

matching the image of ξ in $Rel \otimes B$ (the two minus signs we found above cancel out). This agreement finishes the proof. $\square$

09D7 Remark 92.13.4 (Functoriality of the Lichtenbaum-Schlessinger complex). Consider a commutative square

$$\begin{tikzcd} A' \arrow[r] & B' \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

of ring maps. Choose a factorization

$$\begin{tikzcd} A' \arrow[r] & P' \arrow[r] & B' \\ A \arrow[u] \arrow[r] & P \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

with P a polynomial algebra over A and P' a polynomial algebra over A'. Choose generators f_t, $t \in T$ for $\mathrm{Ker}(P \to B)$. For $t \in T$ denote f'_t the image of f_t in P'. Choose $f'_s \in P'$ such that the elements f'_t for $t \in T' = T \amalg S$ generate the kernel of $P' \to B'$. Set $F = \bigoplus_{t \in T} P$ and $F' = \bigoplus_{t' \in T'} P'$. Let $Rel = \mathrm{Ker}(F \to P)$ and $Rel' = \mathrm{Ker}(F' \to P')$ where the maps are given by multiplication by f_t, resp. f'_t on the coordinates. Finally, set $TrivRel$, resp. $TrivRel'$ equal to the submodule of Rel, resp. $TrivRel$ generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$ for

$t, t' \in T$, resp. T'. Having made these choices we obtain a canonical commutative diagram

$$\begin{tikzcd} L' : & Rel'/TrivRel' \arrow[r] & F' \otimes_{P'} B' \arrow[r] & \Omega_{P'/A'} \otimes_{P'} B' \\ L : \arrow[u] & Rel/TrivRel \arrow[r] \arrow[u] & F \otimes_P B \arrow[r] \arrow[u] & \Omega_{P/A} \otimes_P B \arrow[u] \end{tikzcd}$$

Moreover, tracing through the choices made in the proof of Lemma 92.13.3 the reader sees that one obtains a commutative diagram

$$\begin{tikzcd} L_{B'/A'} \arrow[r] & L' \\ L_{B/A} \arrow[r] \arrow[u] & L \arrow[u] \end{tikzcd}$$

92.14. The cotangent complex of a local complete intersection

08SH If $A \to B$ is a local complete intersection map, then $L_{B/A}$ is a perfect complex. The key to proving this is the following lemma.

08SI **Lemma 92.14.1.** Let $A = \mathbf{Z}[x_1, \ldots, x_n] \to B = \mathbf{Z}$ be the ring map which sends x_i to 0 for $i = 1, \ldots, n$. Let $I = (x_1, \ldots, x_n) \subset A$. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.

Proof. There are several ways to prove this. For example one can explicitly construct a resolution of B over A and compute. We will use (92.7.0.1). Namely, consider the distinguished triangle

$$L_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}} \otimes_{\mathbf{Z}[x_1,\ldots,x_n]} \mathbf{Z} \to L_{\mathbf{Z}/\mathbf{Z}} \to L_{\mathbf{Z}/\mathbf{Z}[x_1,\ldots,x_n]} \to L_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}} \otimes_{\mathbf{Z}[x_1,\ldots,x_n]} \mathbf{Z}[1]$$

The complex $L_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}}$ is quasi-isomorphic to $\Omega_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}}$ by Lemma 92.4.7. The complex $L_{\mathbf{Z}/\mathbf{Z}}$ is zero in $D(\mathbf{Z})$ by Lemma 92.8.4. Thus we see that $L_{B/A}$ has only one nonzero cohomology group which is as described in the lemma by Lemma 92.11.2. □

08SJ **Lemma 92.14.2.** Let $A \to B$ be a surjective ring map whose kernel I is generated by a Koszul-regular sequence (for example a regular sequence). Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.

Proof. Let $f_1, \ldots, f_r \in I$ be a Koszul regular sequence generating I. Consider the ring map $\mathbf{Z}[x_1, \ldots, x_r] \to A$ sending x_i to f_i. Since $x_1, \ldots, x_r$ is a regular sequence in $\mathbf{Z}[x_1, \ldots, x_r]$ we see that the Koszul complex on $x_1, \ldots, x_r$ is a free resolution of $\mathbf{Z} = \mathbf{Z}[x_1, \ldots, x_r]/(x_1, \ldots, x_r)$ over $\mathbf{Z}[x_1, \ldots, x_r]$ (see More on Algebra, Lemma 15.30.2). Thus the assumption that $f_1, \ldots, f_r$ is Koszul regular exactly means that $B = A \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]} \mathbf{Z}$. Hence $L_{B/A} = L_{\mathbf{Z}/\mathbf{Z}[x_1,\ldots,x_r]} \otimes^{\mathbf{L}}_{\mathbf{Z}} B$ by Lemmas 92.6.2 and 92.14.1. □

08SK **Lemma 92.14.3.** Let $A \to B$ be a surjective ring map whose kernel I is Koszul. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.

Proof. Locally on $\mathrm{Spec}(A)$ the ideal I is generated by a Koszul regular sequence, see More on Algebra, Definition 15.32.1. Hence this follows from Lemma 92.6.2. □

08SL Proposition 92.14.4. Let $A \to B$ be a local complete intersection map. Then $L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$.

Proof. Choose a surjection $P = A[x_1, \ldots, x_n] \to B$ with kernel J. By Lemma 92.11.3 we see that $J/J^2 \to \bigoplus B\text{d}x_i$ is quasi-isomorphic to $\tau_{\geq -1}L_{B/A}$. Note that J/J^2 is finite projective (More on Algebra, Lemma 15.32.3), hence $\tau_{\geq -1}L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$. Thus it suffices to show that $H^i(L_{B/A}) = 0$ for $i \not\in [-1, 0]$. This follows from (92.7.0.1)

$$L_{P/A} \otimes_P^{\mathbf{L}} B \to L_{B/A} \to L_{B/P} \to L_{P/A} \otimes_P^{\mathbf{L}} B[1]$$

and Lemma 92.14.3 to see that $H^i(L_{B/P})$ is zero unless $i \in \{-1, 0\}$. (We also use Lemma 92.4.7 for the term on the left.) $\square$

92.15. Tensor products and the cotangent complex

09D8 Let R be a ring and let A, B be R-algebras. In this section we discuss $L_{A \otimes_R B/R}$. Most of the information we want is contained in the following diagram

09D9 (92.15.0.1)

$$\begin{tikzcd}
L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) \arrow[r] & L_{A \otimes_R B/B} \arrow[r] & E \\
L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) \arrow[r] \arrow[u, equal] & L_{A \otimes_R B/R} \arrow[r] \arrow[u] & L_{A \otimes_R B/A} \arrow[u] \\
 & L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) \arrow[r, equal] \arrow[u] & L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) \arrow[u]
\end{tikzcd}$$

Explanation: The middle row is the fundamental triangle (92.7.0.1) for the ring maps $R \to A \to A \otimes_R B$. The middle column is the fundamental triangle (92.7.0.1) for the ring maps $R \to B \to A \otimes_R B$. Next, E is an object of $D(A \otimes_R B)$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an E exists by Derived Categories, Proposition 13.4.23 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define E as the cone (Derived Categories, Definition 13.9.1) of the map of complexes

$$L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) \oplus L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) \longrightarrow L_{A \otimes_R B/R}$$

and get the two maps with target E by an application of TR3. In the Tor independent case the object E is zero.

09DA Lemma 92.15.1. If A and B are Tor independent R-algebras, then the object E in (92.15.0.1) is zero. In this case we have

$$L_{A \otimes_R B/R} = L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) \oplus L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B)$$

which is represented by the complex $L_{A/R} \otimes_R B \oplus L_{B/R} \otimes_R A$ of $A \otimes_R B$-modules.

Proof. The first two statements are immediate from Lemma 92.6.2. The last statement follows as $L_{A/R}$ is a complex of free A-modules, hence $L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B)$ is represented by $L_{A/R} \otimes_A (A \otimes_R B) = L_{A/R} \otimes_R B$ $\square$

In general we can say this about the object E.

09DB Lemma 92.15.2. Let R be a ring and let A, B be R-algebras. The object E in (92.15.0.1) satisfies

$$H^i(E) = \begin{cases} 0 & \text{if} \quad i \geq -1 \\ \mathrm{Tor}_1^R(A, B) & \text{if} \quad i = -2 \end{cases}$$

Proof. We use the description of E as the cone on $L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) \to L_{A \otimes_R B/A}$. By Lemma 92.13.3 the canonical truncations $\tau_{\geq -2} L_{B/R}$ and $\tau_{\geq -2} L_{A \otimes_R B/A}$ are computed by the Lichtenbaum-Schlessinger complex (92.13.0.1). These isomorphisms are compatible with functoriality (Remark 92.13.4). Thus in this proof we work with the Lichtenbaum-Schlessinger complexes.

Choose a polynomial algebra P over R and a surjection $P \to B$. Choose generators $f_t \in P$, $t \in T$ of the kernel of this surjection. Let $Rel \subset F = \bigoplus_{t \in T} P$ be the kernel of the map $F \to P$ which maps the basis vector corresponding to t to f_t. Set $P_A = A \otimes_R P$ and $F_A = A \otimes_R F = P_A \otimes_P F$. Let Rel_A be the kernel of the map $F_A \to P_A$. Using the exact sequence

$$0 \to Rel \to F \to P \to B \to 0$$

and standard short exact sequences for Tor we obtain an exact sequence

$$A \otimes_R Rel \to Rel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

Note that $P_A \to A \otimes_R B$ is a surjection whose kernel is generated by the elements $1 \otimes f_t$ in P_A. Denote $TrivRel_A \subset Rel_A$ the P_A-submodule generated by the elements $(\ldots, 1 \otimes f_{t'}, 0, \ldots, 0, -1 \otimes f_t \otimes 1, 0, \ldots)$. Since $TrivRel \otimes_R A \to TrivRel_A$ is surjective, we find a canonical exact sequence

$$A \otimes_R (Rel/TrivRel) \to Rel_A/TrivRel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

The map of Lichtenbaum-Schlessinger complexes is given by the diagram

$$\begin{tikzcd} Rel_A/TrivRel_A \arrow[r] & F_A \otimes_{P_A} (A \otimes_R B) \arrow[r] & \Omega_{P_A/A \otimes_R B} \otimes_{P_A} (A \otimes_R B) \\ Rel/TrivRel \arrow[r] \arrow[u, "-2"] & F \otimes_P B \arrow[r] \arrow[u, "-1"] & \Omega_{P/A} \otimes_P B \arrow[u, "0"] \end{tikzcd}$$

Note that vertical maps -1 and -0 induce an isomorphism after applying the functor $A \otimes_R - = P_A \otimes_P -$ to the source and the vertical map -2 gives exactly the map whose cokernel is the desired Tor module as we saw above. $\square$

92.16. Deformations of ring maps and the cotangent complex

08SM This section is the continuation of Deformation Theory, Section 91.2 which we urge the reader to read first. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08SN (92.16.0.1)
$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & ? \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u, "c"] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is an ideal of square zero and is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a solution to (92.16.0.1).

08SP Lemma 92.16.1. In the situation above we have

(1) There is a canonical element $\xi \in \text{Ext}^2_B(L_{B/A}, N)$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (92.16.0.1).
(2) If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\text{Ext}^1_B(L_{B/A}, N)$.
(3) Given a solution B', the set of automorphisms of B' fitting into (92.16.0.1) is canonically isomorphic to $\text{Ext}^0_B(L_{B/A}, N)$.

Proof. Via the identifications $NL_{B/A} = \tau_{\geq -1}L_{B/A}$ (Lemma 92.11.3) and $H^0(L_{B/A}) = \Omega_{B/A}$ (Lemma 92.4.5) we have seen parts (2) and (3) in Deformation Theory, Lemmas 91.2.1 and 91.2.2.

Proof of (1). Roughly speaking, this follows from the discussion in Deformation Theory, Remark 91.2.8 by replacing the naive cotangent complex by the full cotangent complex. Here is a more detailed explanation. By Deformation Theory, Lemma 91.2.7 and Remark 91.2.8 there exists an element

$$\xi' \in \text{Ext}^1_A(NL_{A/A'}, N) = \text{Ext}^1_B(NL_{A/A'} \otimes^{\mathbf{L}}_A B, N) = \text{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N)$$

(for the equalities see Deformation Theory, Remark 91.2.8 and use that $NL_{A'/A} = \tau_{\geq -1}L_{A'/A}$) such that a solution exists if and only if this element is in the image of the map

$$\text{Ext}^1_B(NL_{B/A'}, N) = \text{Ext}^1_B(L_{B/A'}, N) \longrightarrow \text{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N)$$

The distinguished triangle (92.7.0.1) for $A' \to A \to B$ gives rise to a long exact sequence

$$\ldots \to \text{Ext}^1_B(L_{B/A'}, N) \to \text{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N) \to \text{Ext}^2_B(L_{B/A}, N) \to \ldots$$

Hence taking ξ the image of ξ' works. $\square$

92.17. The Atiyah class of a module

09DC Let $A \to B$ be a ring map. Let M be a B-module. Let $P \to B$ be an object of $\mathcal{C}_{B/A}$ (Section 92.4). Consider the extension of principal parts

$$0 \to \Omega_{P/A} \otimes_P M \to P^1_{P/A}(M) \to M \to 0$$

see Algebra, Lemma 10.133.6. This sequence is functorial in P by Algebra, Remark 10.133.7. Thus we obtain a short exact sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M} \to P^1_{\mathcal{O}/\underline{A}}(M) \to \underline{M} \to 0$$

on $\mathcal{C}_{B/A}$. We have $L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M}) = L_{B/A} \otimes_B M = L_{B/A} \otimes^{\mathbf{L}}_B M$ by Lemma 92.4.2 and the flatness of the terms of $L_{B/A}$. We have $L\pi_!(\underline{M}) = M$ by Lemma 92.4.4. Thus a distinguished triangle

09DD (92.17.0.1) $$L_{B/A} \otimes^{\mathbf{L}}_B M \to L\pi_!\left(P^1_{\mathcal{O}/\underline{A}}(M)\right) \to M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1]$$

in $D(B)$. Here we use Cohomology on Sites, Remark 21.39.13 to get a distinguished triangle in $D(B)$ and not just in $D(A)$.

09DE Definition 92.17.1. Let $A \to B$ be a ring map. Let M be a B-module. The map $M \to L_{B/A} \otimes_B^{\mathbf{L}} M[1]$ in (92.17.0.1) is called the Atiyah class of M.

92.18. The cotangent complex

08UQ In this section we discuss the cotangent complex of a map of sheaves of rings on a site. In later sections we specialize this to obtain the cotangent complex of a morphism of ringed topoi, a morphism of ringed spaces, a morphism of schemes, a morphism of algebraic space, etc.

Let $\mathcal{C}$ be a site and let $Sh(\mathcal{C})$ denote the associated topos. Let $\mathcal{A}$ denote a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A}$-Alg be the category of $\mathcal{A}$-algebras. Consider the pair of adjoint functors (U, V) where $V : \mathcal{A}\text{-Alg} \to Sh(\mathcal{C})$ is the forgetful functor and $U : Sh(\mathcal{C}) \to \mathcal{A}\text{-Alg}$ assigns to a sheaf of sets $\mathcal{E}$ the polynomial algebra $\mathcal{A}[\mathcal{E}]$ on $\mathcal{E}$ over $\mathcal{A}$. Let $X_\bullet$ be the simplicial object of $\text{Fun}(\mathcal{A}\text{-Alg}, \mathcal{A}\text{-Alg})$ constructed in Simplicial, Section 14.34.

Now assume that $\mathcal{A} \to \mathcal{B}$ is a homomorphism of sheaves of rings. Then $\mathcal{B}$ is an object of the category $\mathcal{A}$-Alg. Denote $\mathcal{P}_\bullet = X_\bullet(\mathcal{B})$ the resulting simplicial $\mathcal{A}$-algebra. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$, $\mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]]$, and so on. Recall also that there is an augmentation

$$\epsilon : \mathcal{P}_\bullet \longrightarrow \mathcal{B}$$

where we view $\mathcal{B}$ as a constant simplicial $\mathcal{A}$-algebra.

08SR Definition 92.18.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The standard resolution of $\mathcal{B}$ over $\mathcal{A}$ is the augmentation $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ with terms

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}], \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]], \quad \ldots$$

and maps as constructed above.

With this definition in hand the cotangent complex of a map of sheaves of rings is defined as follows. We will use the module of differentials as defined in Modules on Sites, Section 18.33.

08SS Definition 92.18.2. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The cotangent complex $L_{\mathcal{B}/\mathcal{A}}$ is the complex of $\mathcal{B}$-modules associated to the simplicial module

$$\Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet, \epsilon} \mathcal{B}$$

where $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. We usually think of $L_{\mathcal{B}/\mathcal{A}}$ as an object of $D(\mathcal{B})$.

These constructions satisfy a functoriality similar to that discussed in Section 92.6. Namely, given a commutative diagram

08ST (92.18.2.1)
$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$L_{\mathcal{B}/\mathcal{A}} \longrightarrow L_{\mathcal{B}'/\mathcal{A}'}$$

constructed as follows. If $\mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$ is the standard resolution of $\mathcal{B}'$ over $\mathcal{A}'$, then there is a canonical map

$\mathcal{P}_\bullet \to \mathcal{P}'_\bullet$ of simplicial $\mathcal{A}$-algebras compatible with the augmentations $\mathcal{P}_\bullet \to \mathcal{B}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$. The maps

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{A}'[\mathcal{B}'] = \mathcal{P}'_0, \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]] \longrightarrow \mathcal{A}'[\mathcal{A}'[\mathcal{B}']] = \mathcal{P}'_1$$

and so on are given by the given maps $\mathcal{A} \to \mathcal{A}'$ and $\mathcal{B} \to \mathcal{B}'$. The desired map $L_{\mathcal{B}/\mathcal{A}} \to L_{\mathcal{B}'/\mathcal{A}'}$ then comes from the associated maps on sheaves of differentials.

08SV Lemma 92.18.3. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $f^{-1}L_{\mathcal{B}/\mathcal{A}} = L_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.

Proof. The diagram

$$\begin{tikzcd}
\mathcal{A}\text{-Alg} \arrow[r, shift left] \arrow[d, "f^{-1}"'] & Sh(\mathcal{C}) \arrow[l, shift left] \arrow[d, "f^{-1}"] \\
f^{-1}\mathcal{A}\text{-Alg} \arrow[r, shift left] & Sh(\mathcal{D}) \arrow[l, shift left]
\end{tikzcd}$$

commutes. □

08SW Lemma 92.18.4. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^i(L_{\mathcal{B}/\mathcal{A}})$ is the sheaf associated to the presheaf $U \mapsto H^i(L_{\mathcal{B}(U)/\mathcal{A}(U)})$.

Proof. Let $\mathcal{C}'$ be the site we get by endowing $\mathcal{C}$ with the chaotic topology (presheaves are sheaves). There is a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ where f_* is the inclusion of sheaves into presheaves and f^{-1} is sheafification. By Lemma 92.18.3 it suffices to prove the result for $\mathcal{C}'$, i.e., in case $\mathcal{C}$ has the chaotic topology.

If $\mathcal{C}$ carries the chaotic topology, then $L_{\mathcal{B}/\mathcal{A}}(U)$ is equal to $L_{\mathcal{B}(U)/\mathcal{A}(U)}$ because

$$\begin{tikzcd}
\mathcal{A}\text{-Alg} \arrow[r, shift left] \arrow[d, "\text{sections over } U"'] & Sh(\mathcal{C}) \arrow[l, shift left] \arrow[d, "\text{sections over } U"] \\
\mathcal{A}(U)\text{-Alg} \arrow[r, shift left] & \text{Sets} \arrow[l, shift left]
\end{tikzcd}$$

commutes. □

08SX Remark 92.18.5. It is clear from the proof of Lemma 92.18.4 that for any $U \in \mathrm{Ob}(\mathcal{C})$ there is a canonical map $L_{\mathcal{B}(U)/\mathcal{A}(U)} \to L_{\mathcal{B}/\mathcal{A}}(U)$ of complexes of $\mathcal{B}(U)$-modules. Moreover, these maps are compatible with restriction maps and the complex $L_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the rule $U \mapsto L_{\mathcal{B}(U)/\mathcal{A}(U)}$.

08UR Lemma 92.18.6. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^0(L_{\mathcal{B}/\mathcal{A}}) = \Omega_{\mathcal{B}/\mathcal{A}}$.

Proof. Follows from Lemmas 92.18.4 and 92.4.5 and Modules on Sites, Lemma 18.33.4. □

08SY Lemma 92.18.7. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}'$ be homomorphisms of sheaves of rings on $\mathcal{C}$. Then

$$L_{\mathcal{B}\times\mathcal{B}'/\mathcal{A}} \longrightarrow L_{\mathcal{B}/\mathcal{A}} \oplus L_{\mathcal{B}'/\mathcal{A}}$$

is an isomorphism in $D(\mathcal{B} \times \mathcal{B}')$.

Proof. By Lemma 92.18.4 it suffices to prove this for ring maps. In the case of rings this is Lemma 92.6.4. □

The fundamental triangle for the cotangent complex of sheaves of rings is an easy consequence of the result for homomorphisms of rings.

08SZ Lemma 92.18.8. Let $\mathcal{D}$ be a site. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be homomorphisms of sheaves of rings on $\mathcal{D}$. There is a canonical distinguished triangle

$$L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{C} \to L_{\mathcal{C}/\mathcal{A}} \to L_{\mathcal{C}/\mathcal{B}} \to L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{C}[1]$$

in $D(\mathcal{C})$.

Proof. We will use the method described in Remarks 92.7.5 and 92.7.6 to construct the triangle; we will freely use the results mentioned there. As in those remarks we first construct the triangle in case $\mathcal{B} \to \mathcal{C}$ is an injective map of sheaves of rings. In this case we set

(1) $\mathcal{P}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$,
(2) $\mathcal{Q}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{A}$,
(3) $\mathcal{R}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{B}$,
(4) $\mathcal{S}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{B}$,
(5) $\overline{\mathcal{Q}}_\bullet = \mathcal{Q}_\bullet \otimes_{\mathcal{P}_\bullet} \mathcal{B}$, and
(6) $\overline{\mathcal{R}}_\bullet = \mathcal{R}_\bullet \otimes_{\mathcal{S}_\bullet} \mathcal{B}$.

The distinguished triangle is the distinguished triangle associated to the short exact sequence of simplicial $\mathcal{C}$-modules

$$0 \to \Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet} \mathcal{C} \to \Omega_{\mathcal{Q}_\bullet/\mathcal{A}} \otimes_{\mathcal{Q}_\bullet} \mathcal{C} \to \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C} \to 0$$

The first two terms are equal to the first two terms of the triangle of the statement of the lemma. The identification of the last term with $L_{\mathcal{C}/\mathcal{B}}$ uses the quasi-isomorphisms of complexes

$$L_{\mathcal{C}/\mathcal{B}} = \Omega_{\mathcal{R}_\bullet/\mathcal{B}} \otimes_{\mathcal{R}_\bullet} \mathcal{C} \longrightarrow \Omega_{\overline{\mathcal{R}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{R}}_\bullet} \mathcal{C} \longleftarrow \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C}$$

All the constructions used above can first be done on the level of presheaves and then sheafified. Hence to prove sequences are exact, or that map are quasi-isomorphisms it suffices to prove the corresponding statement for the ring maps $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ which are known. This finishes the proof in the case that $\mathcal{B} \to \mathcal{C}$ is injective.

In general, we reduce to the case where $\mathcal{B} \to \mathcal{C}$ is injective by replacing $\mathcal{C}$ by $\mathcal{B} \times \mathcal{C}$ if necessary. This is possible by the argument given in Remark 92.7.5 by Lemma 92.18.7. □

08T0 Lemma 92.18.9. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. If p is a point of $\mathcal{C}$, then $(L_{\mathcal{B}/\mathcal{A}})_p = L_{\mathcal{B}_p/\mathcal{A}_p}$.

Proof. This is a special case of Lemma 92.18.3. □

For the construction of the naive cotangent complex and its properties we refer to Modules on Sites, Section 18.35.

08US Lemma 92.18.10. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. There is a canonical map $L_{\mathcal{B}/\mathcal{A}} \to NL_{\mathcal{B}/\mathcal{A}}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1} L_{\mathcal{B}/\mathcal{A}}$.

Proof. Let $\mathcal{P}_\bullet$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{A}[\mathcal{B}] \to \mathcal{B})$. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$. The map of the lemma is given by the commutative diagram

$$\begin{tikzcd} L_{\mathcal{B}/\mathcal{A}} \arrow[d] & \ldots \arrow[r] & \Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} \arrow[r] \arrow[d] & \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} \arrow[r] \arrow[d] & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \arrow[d] \\ NL_{\mathcal{B}/\mathcal{A}} & \ldots \arrow[r] & 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \end{tikzcd}$$

We construct the downward arrow with target $\mathcal{I}/\mathcal{I}^2$ by sending a local section $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in $\mathcal{I}/\mathcal{I}^2$. Here $d_i : \mathcal{P}_1 \to \mathcal{P}_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps $\mathcal{P}_1$ into $\mathcal{I} = \mathrm{Ker}(\mathcal{P}_0 \to \mathcal{B})$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} \to \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B}$ as this differential maps a local section $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} \to \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B}$ maps a local section $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes.

To see that our map induces an isomorphism on the cohomology sheaves H^0 and H^{-1} we argue as follows. Let $\mathcal{C}'$ be the site with the same underlying category as $\mathcal{C}$ but endowed with the chaotic topology. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be the morphism of topoi whose pullback functor is sheafification. Let $\mathcal{A}' \to \mathcal{B}'$ be the given map, but thought of as a map of sheaves of rings on $\mathcal{C}'$. The construction above gives a map $L_{\mathcal{B}'/\mathcal{A}'} \to NL_{\mathcal{B}'/\mathcal{A}'}$ on $\mathcal{C}'$ whose value over any object U of $\mathcal{C}'$ is just the map

$$L_{\mathcal{B}(U)/\mathcal{A}(U)} \to NL_{\mathcal{B}(U)/\mathcal{A}(U)}$$

of Remark 92.11.4 which induces an isomorphism on H^0 and H^{-1}. Since $f^{-1}L_{\mathcal{B}'/\mathcal{A}'} = L_{\mathcal{B}/\mathcal{A}}$ (Lemma 92.18.3) and $f^{-1} NL_{\mathcal{B}'/\mathcal{A}'} = NL_{\mathcal{B}/\mathcal{A}}$ (Modules on Sites, Lemma 18.35.3) the lemma is proved. □

92.19. The Atiyah class of a sheaf of modules

09DF Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Let $\mathcal{P}_\bullet \to \mathcal{B}$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ (Section 92.18). For every $n \geq 0$ consider the extension of principal parts

09DG (92.19.0.1) $$0 \to \Omega_{\mathcal{P}_n/\mathcal{A}} \otimes_{\mathcal{P}_n} \mathcal{F} \to \mathcal{P}^1_{\mathcal{P}_n/\mathcal{A}}(\mathcal{F}) \to \mathcal{F} \to 0$$

see Modules on Sites, Lemma 18.34.6. The functoriality of this construction (Modules on Sites, Remark 18.34.7) tells us (92.19.0.1) is the degree n part of a short exact sequence of simplicial $\mathcal{P}_\bullet$-modules (Cohomology on Sites, Section 21.41). Using the functor $L\pi_! : D(\mathcal{P}_\bullet) \to D(\mathcal{B})$ of Cohomology on Sites, Remark 21.41.3 (here we use that $\mathcal{P}_\bullet \to \mathcal{A}$ is a resolution) we obtain a distinguished triangle

09DH (92.19.0.2) $$L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F} \to L\pi_!\left(\mathcal{P}^1_{\mathcal{P}_\bullet/\mathcal{A}}(\mathcal{F})\right) \to \mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F}[1]$$

in $D(\mathcal{B})$.

09DI Definition 92.19.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. The map $\mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F}[1]$ in (92.19.0.2) is called the Atiyah class of $\mathcal{F}$.

92.20. The cotangent complex of a morphism of ringed spaces

08UT The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

08UU Definition 92.20.1. Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. The cotangent complex L_f of f is $L_f = L_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$. We will also use the notation $L_f = L_{X/S} = L_{\mathcal{O}_X/\mathcal{O}_S}$.

More precisely, this means that we consider the cotangent complex (Definition 92.18.2) of the homomorphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ of sheaves of rings on the site associated to the topological space X (Sites, Example 7.6.4).

08UV Lemma 92.20.2. Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Then $H^0(L_{X/S}) = \Omega_{X/S}$.

Proof. Special case of Lemma 92.18.6. $\square$

08T4 Lemma 92.20.3. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then there is a canonical distinguished triangle

$$Lf^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to Lf^*L_{Y/Z}[1]$$

in $D(\mathcal{O}_X)$.

Proof. Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_Z = f^{-1}g^{-1}\mathcal{O}_Z$. By Lemma 92.18.3 we have $f^{-1}L_{Y/Z} = L_{f^{-1}\mathcal{O}_Y/h^{-1}\mathcal{O}_Z}$ and this is a complex of flat $f^{-1}\mathcal{O}_Y$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 92.18.8 with $\mathcal{A} = h^{-1}\mathcal{O}_Z$, $\mathcal{B} = f^{-1}\mathcal{O}_Y$, and $\mathcal{C} = \mathcal{O}_X$. $\square$

08UW Lemma 92.20.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical map $L_{X/Y} \to NL_{X/Y}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_{X/Y}$.

Proof. Special case of Lemma 92.18.10. $\square$

92.21. Deformations of ringed spaces and the cotangent complex

08UX This section is the continuation of Deformation Theory, Section 91.7 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules. We ask whether we can find the question mark fitting into the following diagram

08UY (92.21.0.1)
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] & ? \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{J} \arrow[r] \arrow[u, "c"] & \mathcal{O}_{S'} \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in Deformation Theory, Equation (91.3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a solution to (92.21.0.1).

08UZ Lemma 92.21.1. In the situation above we have

(1) There is a canonical element $\xi \in \text{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (92.21.0.1).
(2) If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\text{Ext}^1_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.
(3) Given a solution X', the set of automorphisms of X' fitting into (92.21.0.1) is canonically isomorphic to $\text{Ext}^0_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.

Proof. Via the identifications $NL_{X/S} = \tau_{\geq -1}L_{X/S}$ (Lemma 92.20.4) and $H^0(L_{X/S}) = \Omega_{X/S}$ (Lemma 92.20.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 91.7.1 and 91.7.3.

Proof of (1). Roughly speaking, this follows from the discussion in Deformation Theory, Remark 91.7.9 by replacing the naive cotangent complex by the full cotangent complex. Here is a more detailed explanation. By Deformation Theory, Lemma 91.7.8 there exists an element

$$\xi' \in \text{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/S'}, \mathcal{G}) = \text{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G})$$

such that a solution exists if and only if this element is in the image of the map

$$\text{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G}) = \text{Ext}^1_{\mathcal{O}_X}(L_{X/S'}, \mathcal{G}) \longrightarrow \text{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G})$$

The distinguished triangle of Lemma 92.20.3 for $X \to S \to S'$ gives rise to a long exact sequence

$$\ldots \to \text{Ext}^1_{\mathcal{O}_X}(L_{X/S'}, \mathcal{G}) \to \text{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G}) \to \text{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G}) \to \ldots$$

Hence taking ξ the image of ξ' works. $\square$

92.22. The cotangent complex of a morphism of ringed topoi

08SQ The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

08SU Definition 92.22.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The cotangent complex L_f of f is $L_f = L_{\mathcal{O}_\mathcal{C}/f^{-1}\mathcal{O}_\mathcal{D}}$. We sometimes write $L_f = L_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{D}}$.

This definition applies to many situations, but it doesn't always produce the thing one expects. For example, if $f : X \to Y$ is a morphism of schemes, then f induces a morphism of big étale sites $f_{big} : (Sch/X)_{étale} \to (Sch/Y)_{étale}$ which is a morphism of ringed topoi (Descent, Remark 35.8.4). However, $L_{f_{big}} = 0$ since $(f_{big})^\sharp$ is an isomorphism. On the other hand, if we take L_f where we think of f as a morphism between the underlying Zariski ringed topoi, then L_f does agree with the cotangent complex $L_{X/Y}$ (as defined below) whose zeroth cohomology sheaf is $\Omega_{X/Y}$.

08V0 Lemma 92.22.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Then $H^0(L_f) = \Omega_f$.

Proof. Special case of Lemma 92.18.6. $\square$

08V1 Lemma 92.22.3. Let $f : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $g : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then there is a canonical distinguished triangle

$$Lf^* L_g \to L_{g \circ f} \to L_f \to Lf^* L_g[1]$$

in $D(\mathcal{O}_1)$.

Proof. Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_3 = f^{-1}g^{-1}\mathcal{O}_3$. By Lemma 92.18.3 we have $f^{-1}L_g = L_{f^{-1}\mathcal{O}_2/h^{-1}\mathcal{O}_3}$ and this is a complex of flat $f^{-1}\mathcal{O}_2$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 92.18.8 with $\mathcal{A} = h^{-1}\mathcal{O}_3$, $\mathcal{B} = f^{-1}\mathcal{O}_2$, and $\mathcal{C} = \mathcal{O}_1$. $\square$

08V2 **Lemma 92.22.4.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. There is a canonical map $L_f \to NL_f$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_f$.

Proof. Special case of Lemma 92.18.10. $\square$

92.23. Deformations of ringed topoi and the cotangent complex

08V3 This section is the continuation of Deformation Theory, Section 91.13 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_\mathcal{B}$-modules. We ask whether we can find the question mark fitting into the following diagram

08V4 (92.23.0.1)
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] & ? \arrow[r] & \mathcal{O} \arrow[r] & 0 \\ 0 \arrow[r] & f^{-1}\mathcal{J} \arrow[r] \arrow[u, "c"] & f^{-1}\mathcal{O}_{\mathcal{B}'} \arrow[r] \arrow[u] & f^{-1}\mathcal{O}_\mathcal{B} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in Deformation Theory, Equation (91.9.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a solution to (92.23.0.1).

08V5 **Lemma 92.23.1.** In the situation above we have

(1) There is a canonical element $\xi \in \mathrm{Ext}^2_\mathcal{O}(L_f, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (92.23.0.1).
(2) If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_\mathcal{O}(L_f, \mathcal{G})$.
(3) Given a solution X', the set of automorphisms of X' fitting into (92.23.0.1) is canonically isomorphic to $\mathrm{Ext}^0_\mathcal{O}(L_f, \mathcal{G})$.

Proof. Via the identifications $NL_f = \tau_{\geq -1}L_f$ (Lemma 92.22.4) and $H^0(L_f) = \Omega_f$ (Lemma 92.22.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 91.13.1 and 91.13.3.

Proof of (1). To match notation with Deformation Theory, Section 91.13 we will write $NL_f = NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}$ and $L_f = L_{\mathcal{O}/\mathcal{O}_\mathcal{B}}$ and similarly for the morphisms t and $t \circ f$. By Deformation Theory, Lemma 91.13.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_\mathcal{O}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(Lf^* L_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

such that a solution exists if and only if this element is in the image of the map

$$\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1_\mathcal{O}(Lf^* L_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

The distinguished triangle of Lemma 92.22.3 for f and t gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}}(Lf^*L_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$$

Hence taking ξ the image of ξ' works. □

92.24. The cotangent complex of a morphism of schemes

08T1 As promised above we define the cotangent complex of a morphism of schemes as follows.

08T2 **Definition 92.24.1.** Let $f : X \to Y$ be a morphism of schemes. The cotangent complex $L_{X/Y}$ of X over Y is the cotangent complex of f as a morphism of ringed spaces (Definition 92.20.1).

In particular, the results of Section 92.20 apply to cotangent complexes of morphisms of schemes. The next lemma shows this definition is compatible with the definition for ring maps and it also implies that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

08T3 **Lemma 92.24.2.** Let $f : X \to Y$ be a morphism of schemes. Let $U = \mathrm{Spec}(A) \subset X$ and $V = \mathrm{Spec}(B) \subset Y$ be affine opens such that $f(U) \subset V$. There is a canonical map

$$\widetilde{L_{B/A}} \longrightarrow L_{X/Y}|_U$$

of complexes which is an isomorphism in $D(\mathcal{O}_U)$. This map is compatible with restricting to smaller affine opens of X and Y.

Proof. By Remark 92.18.5 there is a canonical map of complexes $L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to L_{X/Y}(U)$ of $B = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $A \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $L_{B/A} \to L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$ of complexes of B-modules. Using the universal property of the $\widetilde{\ }$ functor (see Schemes, Lemma 26.7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows immediately from Lemmas 92.18.9 and 92.8.6 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \mathrm{Spec}(B)$ as $B_\mathfrak{p}$ and $A_\mathfrak{q}$ where $\mathfrak{q} = A \cap \mathfrak{p}$; references used are Schemes, Lemma 26.5.4 and Sheaves, Lemma 6.21.5). □

08V6 **Lemma 92.24.3.** Let Λ be a ring. Let X be a scheme over Λ. Then

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

where $\underline{\Lambda}$ is the constant sheaf with value Λ on X.

Proof. Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda) \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 92.20.3 it suffices to show that $L_q = 0$. To see this it suffices to show for $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ that

$$(L_q)_\mathfrak{p} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda),\mathfrak{p}}/\Lambda} = L_{\Lambda_\mathfrak{p}/\Lambda}$$

(Lemma 92.18.9) is zero which follows from Lemma 92.8.4. □

92.25. The cotangent complex of a scheme over a ring

08V7 Let Λ be a ring and let X be a scheme over Λ. Write $L_{X/\operatorname{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 92.24.3. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 92.4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over X_{Zar} and endow it with a sheaf of (polynomial) Λ-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let Λ be a ring. Let X be a scheme over Λ. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

08V8 (92.25.0.1)

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \\ \operatorname{Spec}(\Lambda) & \mathbf{A} \arrow[l] \end{tikzcd}$$

of schemes where

(1) U is an open subscheme of X,
(2) there exists an isomorphism $\mathbf{A} = \operatorname{Spec}(P)$ where P is a polynomial algebra over Λ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\operatorname{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that X_{Zar} denotes the small Zariski site X. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{Zar},\ (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over U is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 92.4.

08V9 **Lemma 92.25.1.** *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over X_{Zar}.*

Proof. Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of X_{Zar} consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over X_{Zar}. $\square$

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from X_{Zar} (see Stacks, Section 8.10). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{Zar})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

08VA (92.25.1.1)

$$\begin{tikzcd} (Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) \arrow[r, "i"'] \arrow[d, "\pi"'] & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\ (Sh(X_{Zar}), \mathcal{O}_X) & \end{tikzcd}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

08VB Remark 92.25.2. In the situation above, for every $U \subset X$ open let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over Λ. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over U. Moreover, as discussed in Remark 92.5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 21.39.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in U, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 21.40.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

08T9 Lemma 92.25.3. In the situation above there is a canonical isomorphism

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

in $D(\mathcal{O}_X)$.

Proof. We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over Λ. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 92.25.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 92.25.2. Now Remark 92.18.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on X. This is what we want by Lemma 92.24.3. □

92.26. The cotangent complex of a morphism of algebraic spaces

08VC We define the cotangent complex of a morphism of algebraic spaces using the associated morphism between the small étale sites.

08VD Definition 92.26.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The cotangent complex $L_{X/Y}$ of X over Y is the cotangent complex of the morphism of ringed topoi f_{small} between the small étale sites of X and Y (see Properties of Spaces, Lemma 66.21.3 and Definition 92.22.1).

In particular, the results of Section 92.22 apply to cotangent complexes of morphisms of algebraic spaces. The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

08VE Lemma 92.26.2. Let S be a scheme. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of algebraic spaces over S with p and q étale. Then there is a canonical identification $L_{X/Y}|_{U_{\acute{e}tale}} = L_{U/V}$ in $D(\mathcal{O}_U)$.

Proof. Formation of the cotangent complex commutes with pullback (Lemma 92.18.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{\acute{e}tale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}} = \mathcal{O}_{V_{\acute{e}tale}}$ (Properties of Spaces, Lemma 66.26.1). Tracing through the definitions we conclude that $L_{X/Y}|_{U_{\acute{e}tale}} = L_{U/V}$. $\square$

08VF Lemma 92.26.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y representable by schemes X_0 and Y_0. Then there is a canonical identification $L_{X/Y} = \epsilon^* L_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where ϵ is as in Derived Categories of Spaces, Section 75.4 and L_{X_0/Y_0} is as in Definition 92.24.1.

Proof. Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to f. There is a canonical map $\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0} \to f_{small}^{-1}\mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1}\mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$\begin{tikzcd} X_{0,Zar} \arrow[d, "f_0"] & X_{\acute{e}tale} \arrow[l, "\epsilon"] \arrow[d, "f"] \\ Y_{0,Zar} & Y_{\acute{e}tale} \arrow[l, "\epsilon"] \end{tikzcd}$$

see Derived Categories of Spaces, Remark 75.6.3. Thus we obtain a canonical map

$$\epsilon^{-1}L_{X_0/Y_0} = \epsilon^{-1}L_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = L_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \longrightarrow L_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = L_{X/Y}$$

by the functoriality discussed in Section 92.18 and Lemma 92.18.3. To see that the induced map $\epsilon^*L_{X_0/Y_0} \to L_{X/Y}$ is an isomorphism we may check on stalks at geometric points (Properties of Spaces, Theorem 66.19.12). We will use Lemma 92.18.9 to compute the stalks. Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$L_{X/Y,\overline{x}} = L_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

and

$$(\epsilon^*L_{X_0/Y_0})_{\overline{x}} = L_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = L_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 7.34.2, as well as the corresponding result for modules, see Modules on Sites, Lemma 18.36.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 66.22.1). Thus the result follows from Lemma 92.8.7. $\square$

08VG Lemma 92.26.4. Let Λ be a ring. Let X be an algebraic space over Λ. Then

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

where $\underline{\Lambda}$ is the constant sheaf with value Λ on $X_{\acute{e}tale}$.

Proof. Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda)_{étale} \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 92.22.3 it suffices to show that $L_q = 0$. To see this it suffices to show (Properties of Spaces, Theorem 66.19.12) for a geometric point $\overline{t} : \mathrm{Spec}(k) \to \mathrm{Spec}(\Lambda)$ that

$$(L_q)_{\overline{t}} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda)_{étale}, \overline{t}}/\Lambda}$$

(Lemma 92.18.9) is zero. Since $\mathcal{O}_{\mathrm{Spec}(\Lambda)_{étale}, \overline{t}}$ is a strict henselization of a local ring of Λ (Properties of Spaces, Lemma 66.22.1) this follows from Lemma 92.8.4. □

92.27. The cotangent complex of an algebraic space over a ring

08VH Let Λ be a ring and let X be an algebraic space over Λ. Write $L_{X/\mathrm{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 92.26.4. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 92.4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over $X_{étale}$ and endow it with a sheaf of (polynomial) Λ-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let Λ be a ring. Let X be an algebraic space over Λ. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

08VI (92.27.0.1)
$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \\ \mathrm{Spec}(\Lambda) & \mathbf{A} \arrow[l] \end{tikzcd}$$

of schemes where

(1) U is a scheme,
(2) $U \to X$ is étale,
(3) there exists an isomorphism $\mathbf{A} = \mathrm{Spec}(P)$ where P is a polynomial algebra over Λ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\mathrm{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that $X_{étale}$ denotes the small étale site of X whose objects are schemes étale over X. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{étale}, \quad (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over U is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 92.4.

08VJ **Lemma 92.27.1.** *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over $X_{étale}$.*

Proof. Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of $X_{étale}$ consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over $X_{étale}$. □

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from $X_{étale}$ (see Stacks, Section 8.10). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{étale})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.

(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

08VK (92.27.1.1)
$$\begin{tikzcd} (Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) \arrow[r, "i"'] \arrow[d, "\pi"'] & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\ (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \end{tikzcd}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

08VL Remark 92.27.2. In the situation above, for every object $U \to X$ of $X_{\acute{e}tale}$ let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over Λ. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over U. Moreover, as discussed in Remark 92.5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 21.39.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in U, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 21.40.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

08VM Lemma 92.27.3. In the situation above there is a canonical isomorphism

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

in $D(\mathcal{O}_X)$.

Proof. We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over Λ. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 92.27.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 92.27.2. Now Remark 92.18.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on $X_{\acute{e}tale}$. This is what we want by Lemma 92.26.4. $\square$

92.28. Fibre products of algebraic spaces and the cotangent complex

09DJ Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. Consider the fibre product $X \times_B Y$ with projection morphisms $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$. In this section we discuss $L_{X \times_B Y/B}$. Most of the information we want is contained in the following diagram

09DK (92.28.0.1)
$$\begin{tikzcd}
Lp^*L_{X/B} \arrow[r] & L_{X \times_B Y/Y} \arrow[r] & E \\
Lp^*L_{X/B} \arrow[r] \arrow[u, equal] & L_{X \times_B Y/B} \arrow[r] \arrow[u] & L_{X \times_B Y/X} \arrow[u] \\
 & Lq^*L_{Y/B} \arrow[r, equal] \arrow[u] & Lq^*L_{Y/B} \arrow[u]
\end{tikzcd}$$

Explanation: The middle row is the fundamental triangle of Lemma 92.22.3 for the morphisms $X \times_B Y \to X \to B$. The middle column is the fundamental triangle for the morphisms $X \times_B Y \to Y \to B$. Next, E is an object of $D(\mathcal{O}_{X \times_B Y})$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an E exists by Derived Categories, Proposition 13.4.23 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define E as the cone (Derived Categories, Definition 13.9.1) of the map of complexes

$$Lp^*L_{X/B} \oplus Lq^*L_{Y/B} \longrightarrow L_{X \times_B Y/B}$$

and get the two maps with target E by an application of TR3. In the Tor independent case the object E is zero.

09DL **Lemma 92.28.1.** *In the situation above, if X and Y are Tor independent over B, then the object E in (92.28.0.1) is zero. In this case we have*

$$L_{X \times_B Y/B} = Lp^*L_{X/B} \oplus Lq^*L_{Y/B}$$

Proof. Choose a scheme W and a surjective étale morphism $W \to B$. Choose a scheme U and a surjective étale morphism $U \to X \times_B W$. Choose a scheme V and a surjective étale morphism $V \to Y \times_B W$. Then $U \times_W V \to X \times_B Y$ is surjective étale too. Hence it suffices to prove that the restriction of E to $U \times_W V$ is zero. By Lemma 92.26.3 and Derived Categories of Spaces, Lemma 75.20.3 this reduces us to the case of schemes. Taking suitable affine opens we reduce to the case of affine schemes. Using Lemma 92.24.2 we reduce to the case of a tensor product of rings, i.e., to Lemma 92.15.1. □

In general we can say the following about the object E.

09DM **Lemma 92.28.2.** *Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. The object E in (92.28.0.1) satisfies $H^i(E) = 0$ for $i = 0, -1$ and for a geometric point $(\overline{x}, \overline{y}) : \mathrm{Spec}(k) \to X \times_B Y$ we have*

$$H^{-2}(E)_{(\overline{x},\overline{y})} = \mathrm{Tor}_1^R(A, B) \otimes_{A \otimes_R B} C$$

where $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{Y,\overline{y}}$, and $C = \mathcal{O}_{X \times_B Y,(\overline{x},\overline{y})}$.

Proof. The formation of the cotangent complex commutes with taking stalks and pullbacks, see Lemmas 92.18.9 and 92.18.3. Note that C is a henselization of $A \otimes_R B$. $L_{C/R} = L_{A\otimes_R B/R} \otimes_{A\otimes_R B} C$ by the results of Section 92.8. Thus the stalk of E at our geometric point is the cone of the map $L_{A/R} \otimes C \to L_{A\otimes_R B/R} \otimes C$. Therefore the results of the lemma follow from the case of rings, i.e., Lemma 92.15.2. □

92.29. Other chapters

CHAPTER 93

Deformation Problems

0DVK 93.1. Introduction

0DVL The goal of this chapter is to work out examples of the general theory developed in the chapters Formal Deformation Theory, Deformation Theory, The Cotangent Complex.

Section 3 of the paper [Sch68] by Schlessinger discusses some examples as well.

93.2. Examples of deformation problems

06LA List of things that should go here:

(1) Deformations of schemes:
   (a) The Rim-Schlessinger condition.
   (b) Computing the tangent space.
   (c) Computing the infinitesimal deformations.
   (d) The deformation category of an affine hypersurface.
(2) Deformations of sheaves (for example fix X/S, a finite type point s of S, and a quasi-coherent sheaf $\mathcal{F}_s$ over X_s).
(3) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
(4) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
(5) Add more here.

93.3. General outline

0DVM This section lays out the procedure for discussing the next few examples.

Step I. For each section we fix a Noetherian ring Λ and we fix a finite ring map $\Lambda \to k$ where k is a field. As usual we let $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda,k}$ be our base category, see Formal Deformation Theory, Definition 90.3.1.

Step II. In each section we define a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$. Occassionally we will consider instead a functor $F : \mathcal{C}_\Lambda \to \text{Sets}$.

Step III. We explain to what extent $\mathcal{F}$ satisfies the Rim-Schlesssinger condition (RS) discussed in Formal Deformation Theory, Section 90.16. Similarly, we may discuss to what extent our $\mathcal{F}$ satisfies (S1) and (S2) or to what extent F satisfies the corresponding Schlessinger's conditions (H1) and (H2). See Formal Deformation Theory, Section 90.10.

Step IV. Let x_0 be an object of $\mathcal{F}(k)$, in other words an object of $\mathcal{F}$ over k. In this chapter we will use the notation

$$\mathcal{D}ef_{x_0} = \mathcal{F}_{x_0}$$

to denote the predeformation category constructed in Formal Deformation Theory, Remark 90.6.4. If $\mathcal{F}$ satisfies (RS), then $\mathcal{D}ef_{x_0}$ is a deformation category (Formal Deformation Theory, Lemma 90.16.11) and satisfies (S1) and (S2) (Formal Deformation Theory, Lemma 90.16.6). If (S1) and (S2) are satisfied, then an important question is whether the tangent space

$$T\mathcal{D}ef_{x_0} = T_{x_0}\mathcal{F} = T\mathcal{F}_{x_0}$$

(see Formal Deformation Theory, Remark 90.12.5 and Definition 90.12.1) is finite dimensional. Namely, this insures that $\mathcal{D}ef_{x_0}$ has a versal formal object (Formal Deformation Theory, Lemma 90.13.4).

Step V. If $\mathcal{F}$ passes Step IV, then the next question is whether the k-vector space

$$\mathrm{Inf}(\mathcal{D}ef_{x_0}) = \mathrm{Inf}_{x_0}(\mathcal{F})$$

of infinitesimal automorphisms of x_0 is finite dimensional. Namely, if true, this implies that $\mathcal{D}ef_{x_0}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Theorem 90.26.4.

93.4. Finite projective modules

0DVN This section is just a warmup. Of course finite projective modules should not have any “moduli”.

0D3I Example 93.4.1 (Finite projective modules). Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, M) consisting of an object A of $\mathcal{C}_\Lambda$ and a finite projective A-module M, and
(2) a morphism $(f, g) : (B, N) \to (A, M)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V, let $x_0 = (k, V)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_V = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 10.78.2) we see that

$$\begin{matrix}\text{isomorphism classes}\\ \text{of objects of } \mathcal{F}(A)\end{matrix} = \coprod\nolimits_{n \geq 0} \{*\}$$

Although this means that the deformation theory of $\mathcal{F}$ is essentially trivial, we still work through the steps outlined in Section 93.3 to provide an easy example.

0DVP Lemma 93.4.2. Example 93.4.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_V$ is a deformation category for any finite dimensional vector space V over k.

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Thus we have to show that the category of finite projective modules over $A_1 \times_A A_2$ is equivalent to the fibre product of the categories of finite projective modules over

A_1 and A_2 over the category of finite projective modules over A. This is a special case of More on Algebra, Lemma 15.6.9. We recall that the inverse functor sends the triple (M_1, M_2, φ) where M_1 is a finite projective A_1-module, M_2 is a finite projective A_2-module, and $\varphi : M_1 \otimes_{A_1} A \to M_2 \otimes_{A_2} A$ is an isomorphism of A-module, to the finite projective $A_1 \times_A A_2$-module $M_1 \times_\varphi M_2$. $\square$

0DVQ Lemma 93.4.3. In Example 93.4.1 let V be a finite dimensional k-vector space. Then

$$T\mathcal{D}ef_V = (0) \quad \text{and} \quad \mathrm{Inf}(\mathcal{D}ef_V) = \mathrm{End}_k(V)$$

are finite dimensional.

Proof. With $\mathcal{F}$ as in Example 93.4.1 set $x_0 = (k, V) \in \mathrm{Ob}(\mathcal{F}(k))$. Recall that $T\mathcal{D}ef_V = T_{x_0}\mathcal{F}$ is the set of isomorphism classes of pairs (x, α) consisting of an object x of $\mathcal{F}$ over the dual numbers $k[\epsilon]$ and a morphism $\alpha : x \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$.

Up to isomorphism, there is a unique pair (M, α) consisting of a finite projective module M over $k[\epsilon]$ and $k[\epsilon]$-linear map $\alpha : M \to V$ which induces an isomorphism $M \otimes_{k[\epsilon]} k \to V$. For example, if $V = k^{\oplus n}$, then we take $M = k[\epsilon]^{\oplus n}$ with the obvious map α.

Similarly, $\mathrm{Inf}(\mathcal{D}ef_V) = \mathrm{Inf}_{x_0}(\mathcal{F})$ is the set of automorphisms of the trivial deformation x'_0 of x_0 over $k[\epsilon]$. See Formal Deformation Theory, Definition 90.19.2 for details.

Given (M, α) as in the second paragraph, we see that an element of $\mathrm{Inf}_{x_0}(\mathcal{F})$ is an automorphism $\gamma : M \to M$ with $\gamma \bmod \epsilon = \mathrm{id}$. Then we can write $\gamma = \mathrm{id}_M + \epsilon\psi$ where $\psi : M/\epsilon M \to M/\epsilon M$ is k-linear. Using α we can think of ψ as an element of $\mathrm{End}_k(V)$ and this finishes the proof. $\square$

93.5. Representations of a group

0DVR The deformation theory of representations can be very interesting.

0D3J Example 93.5.1 (Representations of a group). Let Γ be a group. Let $\mathcal{F}$ be the category defined as follows

(1) an object is a triple (A, M, ρ) consisting of an object A of $\mathcal{C}_\Lambda$, a finite projective A-module M, and a homomorphism $\rho : \Gamma \to \mathrm{GL}_A(M)$, and
(2) a morphism $(f, g) : (B, N, \tau) \to (A, M, \rho)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and Γ-equivariant and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M, ρ) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V and a representation $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$, let $x_0 = (k, V, \rho_0)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{V,\rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 10.78.2) we see that

$$\begin{matrix}\text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A)\end{matrix} = \coprod\nolimits_{n \geq 0} \begin{matrix}\mathrm{GL}_n(A)\text{-conjugacy classes of} \\ \text{homomorphisms } \rho : \Gamma \to \mathrm{GL}_n(A)\end{matrix}$$

This is already more interesting than the discussion in Section 93.4.

0DVS **Lemma 93.5.2.** Example 93.5.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{V,\rho_0}$ is a deformation category for any finite dimensional representation $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$.

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Consider an object

$$((A_1, M_1, \rho_1), (A_2, M_2, \rho_2), (\mathrm{id}_A, \varphi))$$

of the category $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$. Then, as seen in the proof of Lemma 93.4.2, we can consider the finite projective $A_1 \times_A A_2$-module $M_1 \times_\varphi M_2$. Since φ is compatible with the given actions we obtain

$$\rho_1 \times \rho_2 : \Gamma \longrightarrow \mathrm{GL}_{A_1 \times_A A_2}(M_1 \times_\varphi M_2)$$

Then $(M_1 \times_\varphi M_2, \rho_1 \times \rho_2)$ is an object of $\mathcal{F}(A_1 \times_A A_2)$. This construction determines a quasi-inverse to our functor. $\square$

0DVT **Lemma 93.5.3.** In Example 93.5.1 let $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$ be a finite dimensional representation. Then

$$T\mathcal{D}ef_{V,\rho_0} = \mathrm{Ext}^1_{k[\Gamma]}(V, V) = H^1(\Gamma, \mathrm{End}_k(V)) \quad \text{and} \quad \mathrm{Inf}(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, \mathrm{End}_k(V))$$

Thus $\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0})$ is always finite dimensional and $T\mathcal{D}ef_{V,\rho_0}$ is finite dimensional if Γ is finitely generated.

Proof. We first deal with the infinitesimal automorphisms. Let $M = V \otimes_k k[\epsilon]$ with induced action $\rho_0' : \Gamma \to \mathrm{GL}_n(M)$. Then an infinitesimal automorphism, i.e., an element of $\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0})$, is given by an automorphism $\gamma = \mathrm{id} + \epsilon\psi : M \to M$ as in the proof of Lemma 93.4.3, where moreover ψ has to commute with the action of Γ (given by ρ_0). Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, \mathrm{End}_k(V))$$

as predicted in the lemma.

Next, let $(k[\epsilon], M, \rho)$ be an object of $\mathcal{F}$ over $k[\epsilon]$ and let $\alpha : M \to V$ be a Γ-equivariant map inducing an isomorphism $M/\epsilon M \to V$. Since M is free as a $k[\epsilon]$-module we obtain an extension of Γ-modules

$$0 \to V \to M \xrightarrow{\alpha} V \to 0$$

We omit the detailed construction of the map on the left. Conversely, if we have an extension of Γ-modules as above, then we can use this to make a $k[\epsilon]$-module structure on M and get an object of $\mathcal{F}(k[\epsilon])$ together with a map α as above. It follows that

$$T\mathcal{D}ef_{V,\rho_0} = \mathrm{Ext}^1_{k[\Gamma]}(V, V)$$

as predicted in the lemma. This is equal to $H^1(\Gamma, \mathrm{End}_k(V))$ by Étale Cohomology, Lemma 59.57.4.

The statement on dimensions follows from Étale Cohomology, Lemma 59.57.5. $\square$

In Example 93.5.1 if Γ is finitely generated and (V, ρ_0) is a finite dimensional representation of Γ over k, then $\mathcal{D}ef_{V,\rho_0}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.5.2 and 93.5.3 and the general discussion in Section 93.3.

0ET1 Lemma 93.5.4. In Example 93.5.1 assume Γ finitely generated. Let $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$ be a finite dimensional representation. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor

$$F : \mathcal{C}_\Lambda \longrightarrow \text{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{V,\rho_0}(A))/\cong$$

of isomorphism classes of objects has a hull. If $H^0(\Gamma, \mathrm{End}_k(V)) = k$, then F is prorepresentable.

Proof. The existence of a hull follows from Lemmas 93.5.2 and 93.5.3 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $H^0(\Gamma, \mathrm{End}_k(V)) = k$. To see that F is prorepresentable it suffices to show that F is a deformation functor, see Formal Deformation Theory, Theorem 90.18.2. In other words, we have to show F satisfies (RS). For this we can use the criterion of Formal Deformation Theory, Lemma 90.16.7. The required surjectivity of automorphism groups will follow if we show that

$$A \cdot \mathrm{id}_M = \mathrm{End}_{A[\Gamma]}(M)$$

for any object (A, M, ρ) of $\mathcal{F}$ such that $M \otimes_A k$ is isomorphic to V as a representation of Γ. Since the left hand side is contained in the right hand side, it suffices to show $\mathrm{length}_A \mathrm{End}_{A[\Gamma]}(M) \leq \mathrm{length}_A A$. Choose pairwise distinct ideals $(0) = I_n \subset \ldots \subset I_1 \subset A$ with $n = \mathrm{length}(A)$. By correspondingly filtering M, we see that it suffices to prove $\mathrm{Hom}_{A[\Gamma]}(M, I_t M/I_{t+1}M)$ has length 1. Since $I_t M/I_{t+1}M \cong M \otimes_A k$ and since any $A[\Gamma]$-module map $M \to M \otimes_A k$ factors uniquely through the quotient map $M \to M \otimes_A k$ to give an element of

$$\mathrm{End}_{A[\Gamma]}(M \otimes_A k) = \mathrm{End}_{k[\Gamma]}(V) = k$$

we conclude. $\square$

93.6. Continuous representations

0DVU A very interesting thing one can do is to take an infinite Galois group and study the deformation theory of its representations, see [Maz89].

0D3K Example 93.6.1 (Representations of a topological group). Let Γ be a topological group. Let $\mathcal{F}$ be the category defined as follows

(1) an object is a triple (A, M, ρ) consisting of an object A of $\mathcal{C}_\Lambda$, a finite projective A-module M, and a continuous homomorphism $\rho : \Gamma \to \mathrm{GL}_A(M)$ where $\mathrm{GL}_A(M)$ is given the discrete topology[1], and
(2) a morphism $(f, g) : (B, N, \tau) \to (A, M, \rho)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and Γ-equivariant and induces an isomorpism $N \otimes_{B,f} A \cong M$.

[1]An alternative would be to require the A-module M with G-action given by ρ is an A-G-module as defined in Étale Cohomology, Definition 59.57.1. However, since M is a finite A-module, this is equivalent.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M, ρ) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V and a continuous representation $\rho_0 : \Gamma \to \text{GL}_k(V)$, let $x_0 = (k, V, \rho_0)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}\!\mathit{ef}_{V,\rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 10.78.2) we see that

$$\begin{matrix}\text{isomorphism classes}\\ \text{of objects of } \mathcal{F}(A)\end{matrix} = \coprod\nolimits_{n \geq 0} \begin{matrix}\text{GL}_n(A)\text{-conjugacy classes of}\\ \text{continuous homomorphisms } \rho : \Gamma \to \text{GL}_n(A)\end{matrix}$$

0DVV Lemma 93.6.2. Example 93.6.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}\!\mathit{ef}_{V,\rho_0}$ is a deformation category for any finite dimensional continuous representation $\rho_0 : \Gamma \to \text{GL}_k(V)$.

Proof. The proof is exactly the same as the proof of Lemma 93.5.2. $\square$

0DVW Lemma 93.6.3. In Example 93.6.1 let $\rho_0 : \Gamma \to \text{GL}_k(V)$ be a finite dimensional continuous representation. Then

$$T\mathcal{D}\!\mathit{ef}_{V,\rho_0} = H^1(\Gamma, \text{End}_k(V)) \quad \text{and} \quad \text{Inf}(\mathcal{D}\!\mathit{ef}_{V,\rho_0}) = H^0(\Gamma, \text{End}_k(V))$$

Thus $\text{Inf}(\mathcal{D}\!\mathit{ef}_{V,\rho_0})$ is always finite dimensional and $T\mathcal{D}\!\mathit{ef}_{V,\rho_0}$ is finite dimensional if Γ is topologically finitely generated.

Proof. The proof is exactly the same as the proof of Lemma 93.5.3. $\square$

In Example 93.6.1 if Γ is topologically finitely generated and (V, ρ_0) is a finite dimensional continuous representation of Γ over k, then $\mathcal{D}\!\mathit{ef}_{V,\rho_0}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.6.2 and 93.6.3 and the general discussion in Section 93.3.

0ET2 Lemma 93.6.4. In Example 93.6.1 assume Γ is topologically finitely generated. Let $\rho_0 : \Gamma \to \text{GL}_k(V)$ be a finite dimensional representation. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor

$$F : \mathcal{C}_\Lambda \longrightarrow \text{Sets}, \quad A \longmapsto \text{Ob}(\mathcal{D}\!\mathit{ef}_{V,\rho_0}(A))/\cong$$

of isomorphism classes of objects has a hull. If $H^0(\Gamma, \text{End}_k(V)) = k$, then F is prorepresentable.

Proof. The proof is exactly the same as the proof of Lemma 93.5.4. $\square$

93.7. Graded algebras

0DVX We will use the example in this section in the proof that the stack of polarized proper schemes is an algebraic stack. For this reason we will consider commutative graded algebras whose homogeneous parts are finite projective modules (sometimes called “locally finite”).

0D3L Example 93.7.1 (Graded algebras). Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, P) consisting of an object A of $\mathcal{C}_\Lambda$ and a graded A-algebra P such that P_d is a finite projective A-module for all $d \geq 0$, and

(2) a morphism $(f, g) : (B, Q) \to (A, P)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : Q \to P$ which is f-linear and induces an isomorpism $Q \otimes_{B,f} A \cong P$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, P) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a graded k-algebra P with $\dim_k(P_d) < \infty$ for all $d \geq 0$, let $x_0 = (k, P)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DVY Lemma 93.7.2. Example 93.7.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_P$ is a deformation category for any graded k-algebra P.

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Consider an object

$$((A_1, P_1), (A_2, P_2), (\mathrm{id}_A, \varphi))$$

of the category $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$. Then we consider $P_1 \times_\varphi P_2$. Since $\varphi : P_1 \otimes_{A_1} A \to P_2 \otimes_{A_2} A$ is an isomorphism of graded algebras, we see that the graded pieces of $P_1 \times_\varphi P_2$ are finite projective $A_1 \times_A A_2$-modules, see proof of Lemma 93.4.2. Thus $P_1 \times_\varphi P_2$ is an object of $\mathcal{F}(A_1 \times_A A_2)$. This construction determines a quasi-inverse to our functor and the proof is complete. $\square$

0DVZ Lemma 93.7.3. In Example 93.7.1 let P be a graded k-algebra. Then

$$T\mathcal{D}ef_P \quad \text{and} \quad \mathrm{Inf}(\mathcal{D}ef_P) = \mathrm{Der}_k(P, P)$$

are finite dimensional if P is finitely generated over k.

Proof. We first deal with the infinitesimal automorphisms. Let $Q = P \otimes_k k[\epsilon]$. Then an element of $\mathrm{Inf}(\mathcal{D}ef_P)$ is given by an automorphism $\gamma = \mathrm{id} + \epsilon\delta : Q \to Q$ as above where now $\delta : P \to P$. The fact that γ is graded implies that δ is homogeneous of degree 0. The fact that γ is k-linear implies that δ is k-linear. The fact that γ is multiplicative implies that δ is a k-derivation. Conversely, given a k-derivation $\delta : P \to P$ homogeneous of degree 0, we obtain an automorphism $\gamma = \mathrm{id} + \epsilon\delta$ as above. Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_P) = \mathrm{Der}_k(P, P)$$

as predicted in the lemma. Clearly, if P is generated in degrees P_i, $0 \leq i \leq N$, then δ is determined by the linear maps $\delta_i : P_i \to P_i$ for $0 \leq i \leq N$ and we see that

$$\dim_k \mathrm{Der}_k(P, P) < \infty$$

as desired.

To finish the proof of the lemma we show that there is a finite dimensional deformation space. To do this we choose a presentation

$$k[X_1, \ldots, X_n]/(F_1, \ldots, F_m) \longrightarrow P$$

of graded k-algebras where $\deg(X_i) = d_i$ and F_j is homogeneous of degree e_j. Let Q be any graded $k[\epsilon]$-algebra finite free in each degree which comes with an isomorphsm $\alpha : Q/\epsilon Q \to P$ so that (Q, α) defines an element of $T\mathcal{D}ef_P$. Choose a homogeneous element $q_i \in Q$ of degree d_i mapping to the image of X_i in P. Then we obtain

$$k[\epsilon][X_1, \ldots, X_n] \longrightarrow Q, \quad X_i \longmapsto q_i$$

and since $P = Q/\epsilon Q$ this map is surjective by Nakayama's lemma. A small diagram chase shows we can choose homogeneous elements $F_{\epsilon,j} \in k[\epsilon][X_1, \ldots, X_n]$ of degree e_j mapping to zero in Q and mapping to F_j in $k[X_1, \ldots, X_n]$. Then

$$k[\epsilon][X_1, \ldots, X_n]/(F_{\epsilon,1}, \ldots, F_{\epsilon,m}) \longrightarrow Q$$

is a presentation of Q by flatness of Q over $k[\epsilon]$. Write

$$F_{\epsilon,j} = F_j + \epsilon G_j$$

There is some ambiguity in the vector $(G_1, \ldots, G_m)$. First, using different choices of $F_{\epsilon,j}$ we can modify G_j by an arbitrary element of degree e_j in the kernel of $k[X_1, \ldots, X_n] \to P$. Hence, instead of $(G_1, \ldots, G_m)$, we remember the element

$$(g_1, \ldots, g_m) \in P_{e_1} \oplus \ldots \oplus P_{e_m}$$

where g_j is the image of G_j in P_{e_j}. Moreover, if we change our choice of q_i into $q_i + \epsilon p_i$ with p_i of degree d_i then a computation (omitted) shows that g_j changes into

$$g_j^{new} = g_j - \sum\nolimits_{i=1}^{n} p_i \partial F_j / \partial X_i$$

We conclude that the isomorphism class of Q is determined by the image of the vector $(G_1, \ldots, G_m)$ in the k-vector space

$$W = \text{Coker}(P_{d_1} \oplus \ldots \oplus P_{d_n} \xrightarrow{(\frac{\partial F_j}{\partial X_i})} P_{e_1} \oplus \ldots \oplus P_{e_m})$$

In this way we see that we obtain an injection

$$T\mathcal{D}ef_P \longrightarrow W$$

Since W visibly has finite dimension, we conclude that the lemma is true. □

In Example 93.7.1 if P is a finitely generated graded k-algebra, then $\mathcal{D}ef_P$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.7.2 and 93.7.3 and the general discussion in Section 93.3.

0ET3 **Lemma 93.7.4.** In Example 93.7.1 assume P is a finitely generated graded k-algebra. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor

$$F : \mathcal{C}_\Lambda \longrightarrow \text{Sets}, \quad A \longmapsto \text{Ob}(\mathcal{D}ef_P(A))/\cong$$

of isomorphism classes of objects has a hull.

Proof. This follows immediately from Lemmas 93.7.2 and 93.7.3 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7. □

93.8. Rings

0DY0 The deformation theory of rings is the same as the deformation theory of affine schemes. For rings and schemes when we talk about deformations it means we are thinking about flat deformations.

0DY1 **Example 93.8.1** (Rings). Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, P) consisting of an object A of $\mathcal{C}_\Lambda$ and a flat A-algebra P, and

(2) a morphism $(f, g) : (B, Q) \to (A, P)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : Q \to P$ which is f-linear and induces an isomorpism $Q \otimes_{B,f} A \cong P$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, P) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a k-algebra P, let $x_0 = (k, P)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DY2 Lemma 93.8.2. Example 93.8.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_P$ is a deformation category for any k-algebra P.

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. This is a special case of More on Algebra, Lemma 15.7.7. □

0DY3 Lemma 93.8.3. In Example 93.8.1 let P be a k-algebra. Then

$$T\mathcal{D}ef_P = \text{Ext}^1_P(NL_{P/k}, P) \quad \text{and} \quad \text{Inf}(\mathcal{D}ef_P) = \text{Der}_k(P, P)$$

Proof. Recall that $\text{Inf}(\mathcal{D}ef_P)$ is the set of automorphisms of the trivial deformation $P[\epsilon] = P \otimes_k k[\epsilon]$ of P to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 91.2.1 this is equal to $\text{Hom}_P(\Omega_{P/k}, P)$ which in turn is equal to $\text{Der}_k(P, P)$ by Algebra, Lemma 10.131.3.

Recall that $T\mathcal{D}ef_P$ is the set of isomorphism classes of flat deformations Q of P to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_P(k[\epsilon])$. Recall that a $k[\epsilon]$-algebra Q with $Q/\epsilon Q = P$ is flat over $k[\epsilon]$ if and only if

$$0 \to P \xrightarrow{\epsilon} Q \to P \to 0$$

is exact. This is proven in More on Morphisms, Lemma 37.10.1 and more generally in Deformation Theory, Lemma 91.5.2. Thus we may apply Deformation Theory, Lemma 91.2.2 to see that the set of isomorphism classes of such deformations is equal to $\text{Ext}^1_P(NL_{P/k}, P)$. □

0DZL Lemma 93.8.4. In Example 93.8.1 let P be a smooth k-algebra. Then $T\mathcal{D}ef_P = (0)$.

Proof. By Lemma 93.8.3 we have to show $\text{Ext}^1_P(NL_{P/k}, P) = (0)$. Since $k \to P$ is smooth $NL_{P/k}$ is quasi-isomorphic to the complex consisting of a finite projective P-module placed in degree 0. □

0DY4 Lemma 93.8.5. In Lemma 93.8.3 if P is a finite type k-algebra, then

(1) $\text{Inf}(\mathcal{D}ef_P)$ is finite dimensional if and only if $\dim(P) = 0$, and
(2) $T\mathcal{D}ef_P$ is finite dimensional if $\text{Spec}(P) \to \text{Spec}(k)$ is smooth except at a finite number of points.

Proof. Proof of (1). We view $\text{Der}_k(P, P)$ as a P-module. If it has finite dimension over k, then it has finite length as a P-module, hence it is supported in finitely many closed points of $\text{Spec}(P)$ (Algebra, Lemma 10.52.11). Since $\text{Der}_k(P, P) = \text{Hom}_P(\Omega_{P/k}, P)$ we see that $\text{Der}_k(P, P)_\mathfrak{p} = \text{Der}_k(P_\mathfrak{p}, P_\mathfrak{p})$ for any prime $\mathfrak{p} \subset P$ (this uses Algebra, Lemmas 10.131.8, 10.131.15, and 10.10.2). Let $\mathfrak{p}$ be a minimal prime ideal of P corresponding to an irreducible component of dimension $d > 0$. Then $P_\mathfrak{p}$ is an Artinian local ring essentially of finite type over k with residue field and $\Omega_{P_\mathfrak{p}/k}$

is nonzero for example by Algebra, Lemma 10.140.3. Any nonzero finite module over an Artinian local ring has both a sub and a quotient module isomorphic to the residue field. Thus we find that $\mathrm{Der}_k(P_\mathfrak{p}, P_\mathfrak{p}) = \mathrm{Hom}_{P_\mathfrak{p}}(\Omega_{P_\mathfrak{p}/k}, P_\mathfrak{p})$ is nonzero too. Combining all of the above we find that (1) is true.

Proof of (2). For a prime $\mathfrak{p}$ of P we will use that $NL_{P_\mathfrak{p}/k} = (NL_{P/k})_\mathfrak{p}$ (Algebra, Lemma 10.134.13) and we will use that $\mathrm{Ext}^1_P(NL_{P/k}, P)_\mathfrak{p} = \mathrm{Ext}^1_{P_\mathfrak{p}}(NL_{P_\mathfrak{p}/k}, P_\mathfrak{p})$ (More on Algebra, Lemma 15.65.4). Given a prime $\mathfrak{p} \subset P$ then $k \to P$ is smooth at $\mathfrak{p}$ if and only if $(NL_{P/k})_\mathfrak{p}$ is quasi-isomorphic to a finite projective module placed in degree 0 (this follows immediately from the definition of a smooth ring map but it also follows from the stronger Algebra, Lemma 10.137.12).

Assume that P is smooth over k at all but finitely many primes. Then these "bad" primes are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset P$ by Algebra, Lemma 10.61.3 and the fact that the "bad" primes form a closed subset of $\mathrm{Spec}(P)$. For $\mathfrak{p} \not\in \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ we have $\mathrm{Ext}^1_P(NL_{P/k}, P)_\mathfrak{p} = 0$ by the results above. Thus $\mathrm{Ext}^1_P(NL_{P/k}, P)$ is a finite P-module whose support is contained in $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. By Algebra, Proposition 10.63.6 for example, we find that the dimension over k of $\mathrm{Ext}^1_P(NL_{P/k}, P)$ is a finite integer combination of $\dim_k \kappa(\mathfrak{m}_i)$ and hence finite by the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1). $\square$

In Example 93.8.1, let P be a finite type k-algebra. Then $\mathcal{D}ef_P$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ if and only if $\dim(P) = 0$. Furthermore, $\mathcal{D}ef_P$ has a versal formal object if $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ has finitely many singular points. This follows from Lemmas 93.8.2 and 93.8.5 and the general discussion in Section 93.3.

0ET4 Lemma 93.8.6. In Example 93.8.1 assume P is a finite type k-algebra such that $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth except at a finite number of points. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor

$$F : \mathcal{C}_\Lambda \longrightarrow \mathit{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_P(A))/\cong$$

of isomorphism classes of objects has a hull.

Proof. This follows immediately from Lemmas 93.8.2 and 93.8.5 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7. $\square$

0DYS Lemma 93.8.7. In Example 93.8.1 let P be a k-algebra. Let $S \subset P$ be a multiplicative subset. There is a natural functor

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

of deformation categories.

Proof. Given a deformation of P we can take the localization of it to get a deformation of the localization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Let $S_Q \subset Q$ be the inverse image of S. Then Hence $(A, S_Q^{-1}Q) \to (k, S^{-1}P)$ is the desired object of $\mathcal{D}ef_{S^{-1}P}$. $\square$

0DYT Lemma 93.8.8. In Example 93.8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote (P^h, J^h) the henselization of the pair (P, J). There is a natural functor

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

of deformation categories.

Proof. Given a deformation of P we can take the henselization of it to get a deformation of the henselization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Denote $J_Q \subset Q$ the inverse image of J in Q. Let (Q^h, J_Q^h) be the henselization of the pair (Q, J_Q). Recall that $Q \to Q^h$ is flat (More on Algebra, Lemma 15.12.2) and hence Q^h is flat over A. By More on Algebra, Lemma 15.12.7 we see that the map $Q^h \to P^h$ induces an isomorphism $Q^h \otimes_A k = Q^h \otimes_Q P = P^h$. Hence $(A, Q^h) \to (k, P^h)$ is the desired object of $\mathcal{D}ef_{P^h}$. $\square$

0DYU Lemma 93.8.9. In Example 93.8.1 let P be a k-algebra. Assume P is a local ring and let P^{sh} be a strict henselization of P. There is a natural functor

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^{sh}}$$

of deformation categories.

Proof. Given a deformation of P we can take the strict henselization of it to get a deformation of the strict henselization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Since the kernel of the surjection $Q \to P$ is nilpotent, we find that Q is a local ring with the same residue field as P. Let Q^{sh} be the strict henselization of Q. Recall that $Q \to Q^{sh}$ is flat (More on Algebra, Lemma 15.45.1) and hence Q^{sh} is flat over A. By Algebra, Lemma 10.156.4 we see that the map $Q^{sh} \to P^{sh}$ induces an isomorphism $Q^{sh} \otimes_A k = Q^{sh} \otimes_Q P = P^{sh}$. Hence $(A, Q^{sh}) \to (k, P^{sh})$ is the desired object of $\mathcal{D}ef_{P^{sh}}$. $\square$

0DYV Lemma 93.8.10. In Example 93.8.1 let P be a k-algebra. Assume P Noetherian and let $J \subset P$ be an ideal. Denote $P^\wedge$ the J-adic completion. There is a natural functor

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

of deformation categories.

Proof. Given a deformation of P we can take the completion of it to get a deformation of the completion; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Observe that Q is a Noetherian ring: the kernel of the surjective ring map $Q \to P$ is nilpotent and finitely generated and P is Noetherian; apply Algebra, Lemma 10.97.5. Denote $J_Q \subset Q$ the inverse image of J in Q. Let $Q^\wedge$ be the J_Q-adic completion of Q. Recall that $Q \to Q^\wedge$ is flat (Algebra, Lemma 10.97.2) and hence $Q^\wedge$ is flat over A. The induced map $Q^\wedge \to P^\wedge$ induces an isomorphism $Q^\wedge \otimes_A k = Q^\wedge \otimes_Q P = P^\wedge$ by Algebra, Lemma 10.97.1 for example. Hence $(A, Q^\wedge) \to (k, P^\wedge)$ is the desired object of $\mathcal{D}ef_{P^\wedge}$. $\square$

0DY5 Lemma 93.8.11. In Lemma 93.8.3 if $P = k[[x_1, \ldots, x_n]]/(f)$ for some nonzero $f \in (x_1, \ldots, x_n)^2$, then

(1) $\text{Inf}(\mathcal{D}ef_P)$ is finite dimensional if and only if $n = 1$, and
(2) $T\mathcal{D}ef_P$ is finite dimensional if

$$\sqrt{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)} = (x_1, \ldots, x_n)$$

Proof. Proof of (1). Consider the derivations $\partial/\partial x_i$ of $k[[x_1, \ldots, x_n]]$ over k. Write $f_i = \partial f/\partial x_i$. The derivation

$$\theta = \sum h_i \partial/\partial x_i$$

of $k[[x_1, \ldots, x_n]]$ induces a derivation of $P = k[[x_1, \ldots, x_n]]/(f)$ if and only if $\sum h_i f_i \in (f)$. Moreover, the induced derivation of P is zero if and only if $h_i \in (f)$ for $i = 1, \ldots, n$. Thus we find

$$\mathrm{Ker}((f_1, \ldots, f_n) : P^{\oplus n} \longrightarrow P) \subset \mathrm{Der}_k(P, P)$$

The left hand side is a finite dimensional k-vector space only if $n = 1$; we omit the proof. We also leave it to the reader to see that the right hand side has finite dimension if $n = 1$. This proves (1).

Proof of (2). Let Q be a flat deformation of P over $k[\epsilon]$ as in the proof of Lemma 93.8.3. Choose lifts $q_i \in Q$ of the image of x_i in P. Then Q is a complete local ring with maximal ideal generated by $q_1, \ldots, q_n$ and ϵ (small argument omitted). Thus we get a surjection

$$k[\epsilon][[x_1, \ldots, x_n]] \longrightarrow Q, \quad x_i \longmapsto q_i$$

Choose an element of the form $f + \epsilon g \in k[\epsilon][[x_1, \ldots, x_n]]$ mapping to zero in Q. Observe that g is well defined modulo (f). Since Q is flat over $k[\epsilon]$ we get

$$Q = k[\epsilon][[x_1, \ldots, x_n]]/(f + \epsilon g)$$

Finally, if we changing the choice of q_i amounts to changing the coordinates x_i into $x_i + \epsilon h_i$ for some $h_i \in k[[x_1, \ldots, x_n]]$. Then $f + \epsilon g$ changes into $f + \epsilon(g + \sum h_i f_i)$ where $f_i = \partial f / \partial x_i$. Thus we see that the isomorphism class of the deformation Q is determined by an element of

$$k[[x_1, \ldots, x_n]]/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$$

This has finite dimension over k if and only if its support is the closed point of $k[[x_1, \ldots, x_n]]$ if and only if $\sqrt{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)} = (x_1, \ldots, x_n)$. $\square$

93.9. Schemes

0DY6 The deformation theory of schemes.

0DY7 Example 93.9.1 (Schemes). Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, X) consisting of an object A of $\mathcal{C}_\Lambda$ and a scheme X flat over A, and
(2) a morphism $(f, g) : (B, Y) \to (A, X)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a morphism $g : X \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "f"] & \mathrm{Spec}(B) \end{tikzcd}$$

is a cartesian commutative diagram of schemes.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, X) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a scheme X over k, let $x_0 = (k, X)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0DY8 Lemma 93.9.2. Example 93.9.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_X$ is a deformation category for any scheme X over k.

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(A_2) \arrow[d] \\ \mathrm{Spec}(A_1) \arrow[r] & \mathrm{Spec}(A_1 \times_A A_2) \end{tikzcd}$$

is a pushout diagram as in More on Morphisms, Lemma 37.14.3. Thus the lemma is a special case of More on Morphisms, Lemma 37.14.6. ☐

0DY9 Lemma 93.9.3. In Example 93.9.1 let X be a scheme over k. Then

$$\mathrm{Inf}(\mathcal{D}ef_X) = \mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_X)$ is the set of automorphisms of the trivial deformation $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$ of X to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 91.8.1 this is equal to $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$. The equality $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ follows from More on Morphisms, Lemma 37.13.3. The equality $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ follows from Morphisms, Lemma 29.32.2.

Recall that $T_{x_0}\mathcal{D}ef_X$ is the set of isomorphism classes of flat deformations X' of X to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_X(k[\epsilon])$. Thus the second statement of the lemma follows from Deformation Theory, Lemma 91.8.1. ☐

0DYA Lemma 93.9.4. In Lemma 93.9.3 if X is proper over k, then $\mathrm{Inf}(\mathcal{D}ef_X)$ and $T\mathcal{D}ef_X$ are finite dimensional.

Proof. By the lemma we have to show $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ and $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ are finite dimensional. By More on Morphisms, Lemma 37.13.4 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1. By Derived Categories of Schemes, Lemma 36.11.7 the displayed Ext-groups are finite k-vector spaces and the proof is complete. ☐

In Example 93.9.1 if X is a proper scheme over k, then $\mathcal{D}ef_X$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.9.2 and 93.9.4 and the general discussion in Section 93.3.

0ET5 Lemma 93.9.5. In Example 93.9.1 assume X is a proper k-scheme. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor

$$F : \mathcal{C}_\Lambda \longrightarrow \mathrm{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_X(A))/\cong$$

of isomorphism classes of objects has a hull. If $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 93.9.2 and 93.9.4 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. Then $\mathcal{D}ef_X$ and F are equivalent by Formal Deformation Theory, Lemma 90.19.13. Hence F is a deformation functor (because $\mathcal{D}ef_X$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 90.18.2. □

0DYW Lemma 93.9.6. In Example 93.9.1 let X be a scheme over k. Let $U \subset X$ be an open subscheme. There is a natural functor

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_U$$

of deformation categories.

Proof. Given a deformation of X we can take the corresponding open of it to get a deformation of U. We omit the details. □

0DYX Lemma 93.9.7. In Example 93.9.1 let $X = \mathrm{Spec}(P)$ be an affine scheme over k. With $\mathcal{D}ef_P$ as in Example 93.8.1 there is a natural equivalence

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_P$$

of deformation categories.

Proof. The functor sends (A, Y) to $\Gamma(Y, \mathcal{O}_Y)$. This works because any deformation of X is affine by More on Morphisms, Lemma 37.2.3. □

0DZM Lemma 93.9.8. In Example 93.9.1 let X be a scheme over k Let $p \in X$ be a point. With $\mathcal{D}ef_{\mathcal{O}_{X,p}}$ as in Example 93.8.1 there is a natural functor

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{\mathcal{O}_{X,p}}$$

of deformation categories.

Proof. Choose an affine open $U = \mathrm{Spec}(P) \subset X$ containing p. Then $\mathcal{O}_{X,p}$ is a localization of P. We combine the functors from Lemmas 93.9.6, 93.9.7, and 93.8.7. □

0DYY Situation 93.9.9. Let $\Lambda \to k$ be as in Section 93.3. Let X be a scheme over k which has an affine open covering $X = U_1 \cup U_2$ with $U_{12} = U_1 \cap U_2$ affine too. Write $U_1 = \mathrm{Spec}(P_1)$, $U_2 = \mathrm{Spec}(P_2)$ and $U_{12} = \mathrm{Spec}(P_{12})$. Let $\mathcal{D}ef_X$, $\mathcal{D}ef_{U_1}$, $\mathcal{D}ef_{U_2}$, and $\mathcal{D}ef_{U_{12}}$ be as in Example 93.9.1 and let $\mathcal{D}ef_{P_1}$, $\mathcal{D}ef_{P_2}$, and $\mathcal{D}ef_{P_{12}}$ be as in Example 93.8.1.

0DYZ Lemma 93.9.10. In Situation 93.9.9 there is an equivalence

$$\mathcal{D}ef_X = \mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2}$$

of deformation categories, see Examples 93.9.1 and 93.8.1.

Proof. It suffices to show that the functors of Lemma 93.9.6 define an equivalence

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times_{\mathcal{D}ef_{U_{12}}} \mathcal{D}ef_{U_2}$$

because then we can apply Lemma 93.9.7 to translate into rings. To do this we construct a quasi-inverse. Denote $F_i : \mathcal{D}ef_{U_i} \to \mathcal{D}ef_{U_{12}}$ the functor of Lemma 93.9.6. An object of the RHS is given by an A in $\mathcal{C}_\Lambda$, objects $(A, V_1) \to (k, U_1)$ and $(A, V_2) \to (k, U_2)$, and a morphism

$$g : F_1(A, V_1) \to F_2(A, V_2)$$

Now $F_i(A, V_i) = (A, V_{i,3-i})$ where $V_{i,3-i} \subset V_i$ is the open subscheme whose base change to k is $U_{12} \subset U_i$. The morphism g defines an isomorphism $V_{1,2} \to V_{2,1}$ of schemes over A compatible with $\text{id} : U_{12} \to U_{12}$ over k. Thus $(\{1,2\}, V_i, V_{i,3-i}, g, g^{-1})$ is a glueing data as in Schemes, Section 26.14. Let Y be the glueing, see Schemes, Lemma 26.14.1. Then Y is a scheme over A and the compatibilities mentioned above show that there is a canonical isomorphism $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = X$. Thus $(A, Y) \to (k, X)$ is an object of $\mathcal{D}ef_X$. We omit the verification that this construction is a functor and is quasi-inverse to the given one. □

93.10. Morphisms of Schemes

0E3S The deformation theory of morphisms of schemes. Of course this is just an example of deformations of diagrams of schemes.

0E3T **Example 93.10.1** (Morphisms of schemes). Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair $(A, X \to Y)$ consisting of an object A of $\mathcal{C}_\Lambda$ and a morphism $X \to Y$ of schemes over A with both X and Y flat over A, and
(2) a morphism $(f, g, h) : (A', X' \to Y') \to (A, X \to Y)$ consists of a morphism $f : A' \to A$ in $\mathcal{C}_\Lambda$ together with morphisms of schemes $g : X \to X'$ and $h : Y \to Y'$ such that

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "h"'] \arrow[d] & Y' \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "f"] & \mathrm{Spec}(A') \end{tikzcd}$$

is a commutative diagram of schemes where both squares are cartesian.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends $(A, X \to Y)$ to A and (f, g, h) to f. It is clear that p is cofibred in groupoids. Given a morphism of schemes $X \to Y$ over k, let $x_0 = (k, X \to Y)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{X \to Y} = \mathcal{F}_{x_0}$$

0E3U **Lemma 93.10.2.** *Example 93.10.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{X\to Y}$ is a deformation category for any morphism of schemes $X \to Y$ over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(A_2) \arrow[d] \\ \mathrm{Spec}(A_1) \arrow[r] & \mathrm{Spec}(A_1 \times_A A_2) \end{tikzcd}$$

is a pushout diagram as in More on Morphisms, Lemma 37.14.3. Thus the lemma follows immediately from More on Morphisms, Lemma 37.14.6 as this describes the category of schemes flat over $A_1 \times_A A_2$ as the fibre product of the category of schemes flat over A_1 with the category of schemes flat over A_2 over the category of schemes flat over A. □

0E3V Lemma 93.10.3. In Example 93.9.1 let $f : X \to Y$ be a morphism of schemes over k. There is a canonical exact sequence of k-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Inf}(\mathcal{D}\!\mathit{ef}_{X \to Y}) \arrow[r] & \mathrm{Inf}(\mathcal{D}\!\mathit{ef}_X \times \mathcal{D}\!\mathit{ef}_Y) \arrow[r] & \mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) \arrow[dll] \\ & T\mathcal{D}\!\mathit{ef}_{X \to Y} \arrow[r] & T(\mathcal{D}\!\mathit{ef}_X \times \mathcal{D}\!\mathit{ef}_Y) \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{Y/k}, \mathcal{O}_X) \end{tikzcd}$$

Proof. The obvious map of deformation categories $\mathcal{D}\!\mathit{ef}_{X \to Y} \to \mathcal{D}\!\mathit{ef}_X \times \mathcal{D}\!\mathit{ef}_Y$ gives two of the arrows in the exact sequence of the lemma. Recall that $\mathrm{Inf}(\mathcal{D}\!\mathit{ef}_{X \to Y})$ is the set of automorphisms of the trivial deformation

$$f' : X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \xrightarrow{f \times \mathrm{id}} Y' = Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$$

of $X \to Y$ to $k[\epsilon]$ equal to the identity modulo ϵ. This is clearly the same thing as pairs $(\alpha, \beta) \in \mathrm{Inf}(\mathcal{D}\!\mathit{ef}_X \times \mathcal{D}\!\mathit{ef}_Y)$ of infinitesimal automorphisms of X and Y compatible with f', i.e., such that $f' \circ \alpha = \beta \circ f'$. By Deformation Theory, Lemma 91.7.1 for an arbitrary pair (α, β) the difference between the morphism $f' : X' \to Y'$ and the morphism $\beta^{-1} \circ f' \circ \alpha : X' \to Y'$ defines an elment in

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/k}, f_*\mathcal{O}_X)$$

Equality by More on Morphisms, Lemma 37.13.3. This defines the last top horizontal arrow and shows exactness in the first two places. For the map

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) \to T\mathcal{D}\!\mathit{ef}_{X \to Y}$$

we interpret elements of the source as morphisms $f_\epsilon : X' \to Y'$ over $\mathrm{Spec}(k[\epsilon])$ equal to f modulo ϵ using Deformation Theory, Lemma 91.7.1. We send f_ϵ to the isomorphism class of $(f_\epsilon : X' \to Y')$ in $T\mathcal{D}\!\mathit{ef}_{X \to Y}$. Note that $(f_\epsilon : X' \to Y')$ is isomorphic to the trivial deformation $(f' : X' \to Y')$ exactly when $f_\epsilon = \beta^{-1} \circ f \circ \alpha$ for some pair (α, β) which implies exactness in the third spot. Clearly, if some first order deformation $(f_\epsilon : X_\epsilon \to Y_\epsilon)$ maps to zero in $T(\mathcal{D}\!\mathit{ef}_X \times \mathcal{D}\!\mathit{ef}_Y)$, then we can choose isomorphisms $X' \to X_\epsilon$ and $Y' \to Y_\epsilon$ and we conclude we are in the image of the south-west arrow. Therefore we have exactness at the fourth spot. Finally, given two first order deformations X_ϵ, Y_ϵ of X, Y there is an obstruction in

$$ob(X_\epsilon, Y_\epsilon) \in \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{Y/k}, \mathcal{O}_X)$$

which vanishes if and only if $f : X \to Y$ lifts to $X_\epsilon \to Y_\epsilon$, see Deformation Theory, Lemma 91.7.1. This finishes the proof. □

0E3W Lemma 93.10.4. In Lemma 93.10.3 if X and Y are both proper over k, then $\mathrm{Inf}(\mathcal{D}\!\mathit{ef}_{X \to Y})$ and $T\mathcal{D}\!\mathit{ef}_{X \to Y}$ are finite dimensional.

Proof. Omitted. Hint: argue as in Lemma 93.9.4 and use the exact sequence of the lemma. □

In Example 93.10.1 if $X \to Y$ is a morphism of proper schemes over k, then $\mathcal{D}\!\mathit{ef}_{X \to Y}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.10.2 and 93.10.4 and the general discussion in Section 93.3.

0ET6 Lemma 93.10.5. In Example 93.10.1 assume $X \to Y$ is a morphism of proper k-schemes. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor

$$F : \mathcal{C}_\Lambda \longrightarrow \text{Sets}, \quad A \longmapsto \text{Ob}(\mathcal{D}\!\mathit{ef}_{X\to Y}(A))/\cong$$

of isomorphism classes of objects has a hull. If $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \text{Der}_k(\mathcal{O}_Y, \mathcal{O}_Y) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 93.10.2 and 93.10.4 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \text{Der}_k(\mathcal{O}_Y, \mathcal{O}_Y) = 0$. Then the exact sequence of Lemma 93.10.3 combined with Lemma 93.9.3 shows that $\text{Inf}(\mathcal{D}\!\mathit{ef}_{X\to Y}) = 0$. Then $\mathcal{D}\!\mathit{ef}_{X\to Y}$ and F are equivalent by Formal Deformation Theory, Lemma 90.19.13. Hence F is a deformation functor (because $\mathcal{D}\!\mathit{ef}_{X\to Y}$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 90.18.2. ☐

0E3X Lemma 93.10.6. In Example 93.9.1 let $f : X \to Y$ be a morphism of schemes over k. If $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^1f_*\mathcal{O}_X = 0$, then the morphism of deformation categories

This is discussed in [Vak06, Section 5.3] and [Ran89, Theorem 3.3].

$$\mathcal{D}\!\mathit{ef}_{X\to Y} \to \mathcal{D}\!\mathit{ef}_X$$

is an equivalence.

Proof. We construct a quasi-inverse to the forgetful functor of the lemma. Namely, suppose that (A, U) is an object of $\mathcal{D}\!\mathit{ef}_X$. The given map $X \to U$ is a finite order thickening and we can use it to identify the underlying topological spaces of U and X, see More on Morphisms, Section 37.2. Thus we may and do think of $\mathcal{O}_U$ as a sheaf of A-algebras on X; moreover the fact that $U \to \text{Spec}(A)$ is flat, means that $\mathcal{O}_U$ is flat as a sheaf of A-modules. In particular, we have a filtration

$$0 = \mathfrak{m}_A^n\mathcal{O}_U \subset \mathfrak{m}_A^{n-1}\mathcal{O}_U \subset \ldots \subset \mathfrak{m}_A^2\mathcal{O}_U \subset \mathfrak{m}_A\mathcal{O}_U \subset \mathcal{O}_U$$

with subquotients equal to $\mathcal{O}_X \otimes_k \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$ by flatness, see More on Morphisms, Lemma 37.10.1 or the more general Deformation Theory, Lemma 91.5.2. Set

$$\mathcal{O}_V = f_*\mathcal{O}_U$$

viewed as sheaf of A-algebras on Y. Since $R^1f_*\mathcal{O}_X = 0$ we find by the description above that $R^1f_*(\mathfrak{m}_A^i\mathcal{O}_U/\mathfrak{m}_A^{i+1}\mathcal{O}_U) = 0$ for all i. This implies that the sequences

$$0 \to (f_*\mathcal{O}_X) \otimes_k \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1} \to f_*(\mathcal{O}_U/\mathfrak{m}_A^{i+1}\mathcal{O}_U) \to f_*(\mathcal{O}_U/\mathfrak{m}_A^i\mathcal{O}_U) \to 0$$

are exact for all i. Reading the references given above backwards (and using induction) we find that $\mathcal{O}_V$ is a flat sheaf of A-algebras with $\mathcal{O}_V/\mathfrak{m}_A\mathcal{O}_V = \mathcal{O}_Y$. Using More on Morphisms, Lemma 37.2.2 we find that $(Y, \mathcal{O}_V)$ is a scheme, call it V. The equality $\mathcal{O}_V = f_*\mathcal{O}_U$ defines a morphism of ringed spaces $U \to V$ which is easily seen to be a morphism of schemes. This finishes the proof by the flatness already established. ☐

93.11. Algebraic spaces

0E3Y The deformation theory of algebraic spaces.

0E3Z Example 93.11.1 (Algebraic spaces). Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, X) consisting of an object A of $\mathcal{C}_\Lambda$ and an algebraic space X flat over A, and

(2) a morphism $(f, g) : (B, Y) \to (A, X)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a morphism $g : X \to Y$ of algebraic spaces over Λ such that

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "f"] & \mathrm{Spec}(B) \end{tikzcd}$$

is a cartesian commutative diagram of algebraic spaces.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, X) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given an algebraic space X over k, let $x_0 = (k, X)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0E40 Lemma 93.11.2. Example 93.11.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_X$ is a deformation category for any algebraic space X over k.

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(A_2) \arrow[d] \\ \mathrm{Spec}(A_1) \arrow[r] & \mathrm{Spec}(A_1 \times_A A_2) \end{tikzcd}$$

is a pushout diagram as in Pushouts of Spaces, Lemma 81.6.2. Thus the lemma is a special case of Pushouts of Spaces, Lemma 81.6.7. $\square$

0E41 Lemma 93.11.3. In Example 93.11.1 let X be an algebraic space over k. Then

$$\mathrm{Inf}(\mathcal{D}ef_X) = \mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_X)$ is the set of automorphisms of the trivial deformation $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$ of X to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 91.14.2 this is equal to $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$. The equality $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ follows from More on Morphisms of Spaces, Lemma 76.21.4. The equality $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ follows from More on Morphisms of Spaces, Definition 76.7.2 and Modules on Sites, Definition 18.33.3.

Recall that $T_{x_0}\mathcal{D}ef_X$ is the set of isomorphism classes of flat deformations X' of X to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_X(k[\epsilon])$. Thus the second statement of the lemma follows from Deformation Theory, Lemma 91.14.2. $\square$

0E42 Lemma 93.11.4. In Lemma 93.11.3 if X is proper over k, then $\mathrm{Inf}(\mathcal{D}ef_X)$ and $T\mathcal{D}ef_X$ are finite dimensional.

Proof. By the lemma we have to show $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ and $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ are finite dimensional. By More on Morphisms of Spaces, Lemma 76.21.5 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1. By Derived Categories of Spaces, Lemma 75.8.4 the displayed Ext-groups are finite k-vector spaces and the proof is complete. $\square$

In Example 93.11.1 if X is a proper algebraic space over k, then $\mathcal{D}ef_X$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.11.2 and 93.11.4 and the general discussion in Section 93.3.

0ET7 **Lemma 93.11.5.** *In Example 93.11.1 assume X is a proper algebraic space over k. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow \textit{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_X(A))/\cong$$

of isomorphism classes of objects has a hull. If $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 93.11.2 and 93.11.4 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. Then $\mathcal{D}ef_X$ and F are equivalent by Formal Deformation Theory, Lemma 90.19.13. Hence F is a deformation functor (because $\mathcal{D}ef_X$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 90.18.2. $\square$

93.12. Deformations of completions

0DZ0 In this section we compare the deformation problem posed by an algebra and its completion. We first discuss "liftability".

0DZ1 **Lemma 93.12.1.** *Let $A' \to A$ be a surjection of rings with nilpotent kernel. Let $A' \to P'$ be a flat ring map. Set $P = P' \otimes_{A'} A$. Let M be an A-flat P-module. Then the following are equivalent*

(1) *there is an A'-flat P'-module M' with $M' \otimes_{P'} P = M$, and*
(2) *there is an object $K' \in D^-(P')$ with $K' \otimes^{\mathbf{L}}_{P'} P = M$.*

Proof. Suppose that M' is as in (1). Then

$$M = M' \otimes_P P' = M' \otimes_{A'} A = M' \otimes_A^{\mathbf{L}} A' = M' \otimes_{P'}^{\mathbf{L}} P$$

The first two equalities are clear, the third holds because M' is flat over A', and the fourth holds by More on Algebra, Lemma 15.61.2. Thus (2) holds. Conversely, suppose K' is as in (2). We may and do assume M is nonzero. Let t be the largest integer such that $H^t(K')$ is nonzero (exists because M is nonzero). Then $H^t(K') \otimes_{P'} P = H^t(K' \otimes^{\mathbf{L}}_{P'} P)$ is zero if $t > 0$. Since the kernel of $P' \to P$ is nilpotent this implies $H^t(K') = 0$ by Nakayama's lemma a contradiction. Hence $t = 0$ (the case $t < 0$ is absurd as well). Then $M' = H^0(K')$ is a P'-module such that $M = M' \otimes_{P'} P$ and the spectral sequence for Tor gives an injective map

$$\mathrm{Tor}_1^{P'}(M', P) \to H^{-1}(M' \otimes^{\mathbf{L}}_{P'} P) = 0$$

By the reference on derived base change above $0 = \mathrm{Tor}_1^{P'}(M', P) = \mathrm{Tor}_1^{A'}(M', A)$. We conclude that M' is A'-flat by Algebra, Lemma 10.99.8. $\square$

0DZ2 **Lemma 93.12.2.** Consider a commutative diagram of Noetherian rings

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & Q' \arrow[d] \\ A \arrow[r] & P \arrow[r] & Q \end{tikzcd}$$

with cartesian squares, with flat horizontal arrows, and with surjective vertial arrows whose kernels are nilpotent. Let $J' \subset P'$ be an ideal such that $P'/J' = Q'/J'Q'$. Let M be an A-flat P-module. Assume for all $g \in J'$ there exists an A'-flat $(P')_g$-module lifting M_g. Then the following are equivalent

1. M has an A'-flat lift to a P'-module, and
2. $M \otimes_P Q$ has an A'-flat lift to a Q'-module.

Proof. Let $I = \text{Ker}(A' \to A)$. By induction on the integer $n > 1$ such that $I^n = 0$ we reduce to the case where I is an ideal of square zero; details omitted. We translate the condition of liftability of M into the problem of finding an object of $D^-(P')$ as in Lemma 93.12.1. The obstruction to doing this is the element

$$\omega(M) \in \text{Ext}^2_P(M, M \otimes^{\mathbf{L}}_P IP) = \text{Ext}^2_P(M, M \otimes_P IP)$$

constructed in Deformation Theory, Lemma 91.15.1. The equality in the displayed formula holds as $M \otimes^{\mathbf{L}}_P IP = M \otimes_P IP$ since M and P are A-flat[2]. The obstruction for lifting $M \otimes_P Q$ is similarly the element

$$\omega(M \otimes_P Q) \in \text{Ext}^2_Q(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ)$$

which is the image of $\omega(M)$ by the functoriality of the construction $\omega(-)$ of Deformation Theory, Lemma 91.15.1. By More on Algebra, Lemma 15.99.2 we have

$$\text{Ext}^2_Q(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ) = \text{Ext}^2_P(M, M \otimes_P IP) \otimes_P Q$$

here we use that P is Noetherian and M finite. Our assumption on $P' \to Q'$ guarantees that for an P-module E the map $E \to E \otimes_P Q$ is bijective on J'-power torsion, see More on Algebra, Lemma 15.89.3. Thus we conclude that it suffices to show $\omega(M)$ is J'-power torsion. In other words, it suffices to show that $\omega(M)$ dies in

$$\text{Ext}^2_P(M, M \otimes_P IP)_g = \text{Ext}^2_{P_g}(M_g, M_g \otimes_{P_g} IP_g)$$

for all $g \in J'$. However, by the compatibility of formation of $\omega(M)$ with base change again, we conclude that this is true as M_g is assumed to have a lift (of course you have to use the whole string of equivalences again). $\square$

0DZ3 **Lemma 93.12.3.** Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $\mathfrak b \subset B$ be an ideal such that $\text{Spec}(B) \to \text{Spec}(A)$ is syntomic on the complement of $V(\mathfrak b)$. Then B has a flat lift to A' if and only if the $\mathfrak b$-adic completion $B^\wedge$ has a flat lift to A'.

Proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $\mathfrak p \subset P$ be the inverse image of $\mathfrak b$. Set $P' = A'[x_1, \ldots, x_n]$ and denote $\mathfrak p' \subset P'$ the inverse image of $\mathfrak p$. (Of course $\mathfrak p$ and $\mathfrak p'$ do not designate prime ideals here.) We will denote $P^\wedge$ and $(P')^\wedge$ the respective completions.

[2] Choose a resolution $F_\bullet \to I$ by free A-modules. Since $A \to P$ is flat, $P \otimes_A F_\bullet$ is a free resolution of IP. Hence $M \otimes^{\mathbf{L}}_P IP$ is represented by $M \otimes_P P \otimes_A F_\bullet = M \otimes_A F_\bullet$. This only has cohomology in degree 0 as M is A-flat.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = B^\wedge$ and $\mathfrak c = \mathfrak b C$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Then C' is complete with respect to the inverse image $\mathfrak c'$ of $\mathfrak c$ (Algebra, Lemma 10.97.10). We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. These maps pass through completions to give surjections $P^\wedge \to C$ and $(P')^\wedge \to C'$ (for the second again using Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $(P')^\wedge$-module structure lifting C as an A-flat $P^\wedge$-module. Conversely, if we can lift C to a $(P')^\wedge$-module N' flat over A', then N' is a cyclic module $N' \cong (P')^\wedge/\tilde J$ (using Nakayama again) and setting $C' = (P')^\wedge/\tilde J$ we find a flat lift of C as an algebra.

Observe that $P' \to (P')^\wedge$ is a flat ring map which induces an isomorphism $P'/\mathfrak p' = (P')^\wedge/\mathfrak p'(P')^\wedge$. We conclude that our lemma is a consequence of Lemma 93.12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in \mathfrak p'$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 16.3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. $\square$

Notation. Let $A \to B$ be a ring map. Let N be a B-module. We denote $\mathrm{Exal}_A(B, N)$ the set of isomorphism classes of extensions

$$0 \to N \to C \to B \to 0$$

of A-algebras such that N is an ideal of square zero in C. Given a second such $0 \to N \to C' \to B \to 0$ an isomorphism is a A-algebra isomorpism $C \to C'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}"'] & C \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\ 0 \arrow[r] & N \arrow[r] & C' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

commutes. The assignment $N \mapsto \mathrm{Exal}_A(B, N)$ is a functor which transforms products into products. Hence this is an additive functor and $\mathrm{Exal}_A(B, N)$ has a natural B-module structure. In fact, by Deformation Theory, Lemma 91.2.2 we have $\mathrm{Exal}_A(B, N) = \mathrm{Ext}^1_B(NL_{B/A}, N)$.

0DZ4 Lemma 93.12.4. Let k be a field. Let B be a finite type k-algebra. Let $J \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$. Let N be a finite B-module. Then there is a canonical bijection

$$\mathrm{Exal}_k(B, N) \to \mathrm{Exal}_k(B^\wedge, N^\wedge)$$

Here $B^\wedge$ and $N^\wedge$ are the J-adic completions.

Proof. The map is given by completion: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the completion $C^\wedge$ of C with respect to the inverse image of J. Compare with the proof of Lemma 93.8.10.

Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. By More on Algebra, Lemma 15.102.1 we conclude that

$$\mathrm{Ext}^1_B(NL_{B/k}, N) \longrightarrow \lim_n \mathrm{Ext}^1_B(NL_{B/k}, N/J^nN)$$

is an isomorphism for any finite B-module N.

Injectivity of the map. Suppose that $0 \to N \to C \to B \to 0$ is in $\mathrm{Exal}_k(B, N)$ and maps to zero in $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. Choose a splitting $C^\wedge = B^\wedge \oplus N^\wedge$. Then the induced map $C \to C^\wedge \to N^\wedge$ gives maps $C \to N/J^nN$ for all n. Hence we see that our element is in the kernel of the maps

$$\mathrm{Ext}^1_B(NL_{B/k}, N) \to \mathrm{Ext}^1_B(NL_{B/k}, N/J^nN)$$

for all n. By the previous paragraph we conclude that our element is zero.

Surjectivity of the map. Let $0 \to N^\wedge \to C' \to B^\wedge \to 0$ be an element of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. Pulling back by $B \to B^\wedge$ we get an element $0 \to N^\wedge \to C'' \to B \to 0$ in $\mathrm{Exal}_k(B, N^\wedge)$. we have

$$\mathrm{Ext}^1_B(NL_{B/k}, N^\wedge) = \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B B^\wedge = \mathrm{Ext}^1_B(NL_{B/k}, N)$$

The first equality as $N^\wedge = N \otimes_B B^\wedge$ (Algebra, Lemma 10.97.1) and More on Algebra, Lemma 15.65.3. The second equality because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion (see above), $B \to B^\wedge$ is flat and induces an isomorphism $B/J \to B^\wedge/JB^\wedge$, and More on Algebra, Lemma 15.89.3. Thus we can find a $C \in \mathrm{Exal}_k(B, N)$ mapping to C'' in $\mathrm{Exal}_k(B, N^\wedge)$. Thus

$$0 \to N^\wedge \to C' \to B^\wedge \to 0 \quad \text{and} \quad 0 \to N^\wedge \to C^\wedge \to B^\wedge \to 0$$

are two elements of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$ mapping to the same element of $\mathrm{Exal}_k(B, N^\wedge)$. Taking the difference we get an element $0 \to N^\wedge \to C' \to B^\wedge \to 0$ of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$ whose image in $\mathrm{Exal}_k(B, N^\wedge)$ is zero. This means there exists

$$\begin{tikzcd} 0 \arrow[r] & N^\wedge \arrow[r] & C' \arrow[r] & B^\wedge \arrow[r] & 0 \\ & & B \arrow[u, "\sigma"] \arrow[ur] & & \end{tikzcd}$$

Let $J' \subset C'$ be the inverse image of $JB^\wedge \subset B^\wedge$. To finish the proof it suffices to note that σ is continuous for the J-adic topology on B and the J'-adic topology on C' and that C' is J'-adically complete by Algebra, Lemma 10.97.10 (here we also use that C' is Noetherian; small detail omitted). Namely, this means that σ factors through the completion $B^\wedge$ and $C' = 0$ in $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. ☐

0DZ5 Lemma 93.12.5. In Example 93.8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote $P^\wedge$ the J-adic completion. If

(1) $k \to P$ is of finite type, and
(2) $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$.

then the functor between deformation categories of Lemma 93.8.10

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{P^\wedge}$ are deformation categories by Lemma 93.8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 90.20.3. The property on liftability is proven in Lemma 93.12.3 and the isomorphism on tangent spaces is the special case of Lemma 93.12.4 where $N = B$. □

93.13. Deformations of localizations

0DZ6 In this section we compare the deformation problem posed by an algebra and its localization at a multiplicative subset. We first discuss "liftability".

0DZ7 Lemma 93.13.1. Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $S \subset B$ be a multiplicative subset such that if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not syntomic at $\mathfrak{q}$, then $S \cap \mathfrak{q} = \emptyset$. Then B has a flat lift to A' if and only if $S^{-1}B$ has a flat lift to A'.

Proof. This proof is the same as the proof of Lemma 93.12.3 but easier. We suggest the reader to skip the proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $S_P \subset P$ be the inverse image of S. Set $P' = A'[x_1, \ldots, x_n]$ and denote $S_{P'} \subset P'$ the inverse image of S_P.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = S^{-1}B$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Elements of C' which map to invertible elements of C are invertible. We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. By the remark above these maps pass through localizations to give surjections $S_P^{-1}P \to C$ and $S_{P'}^{-1}P' \to C'$ (for the second use Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $S_{P'}^{-1}P'$-module structure lifting C as an A-flat $S_P^{-1}P$-module. Conversely, if we can lift C to a $S_{P'}^{-1}P'$-module N' flat over A', then N' is a cyclic module $N' \cong S_{P'}^{-1}P'/\tilde{J}$ (using Nakayama again) and setting $C' = S_{P'}^{-1}P'/\tilde{J}$ we find a flat lift of C as an algebra.

The syntomic locus of a morphism of schemes is open by definition. Let $J_B \subset B$ be an ideal cutting out the set of points in $\mathrm{Spec}(B)$ where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not syntomic. Denote $J_P \subset P$ and $J_{P'} \subset P'$ the corresponding ideals. Observe that $P' \to S_{P'}^{-1}P'$ is a flat ring map which induces an isomorphism $P'/J_{P'} = S_{P'}^{-1}P'/J_{P'}S_{P'}^{-1}P'$ by our assumption on S in the lemma, namely, the assumption in the lemma is exactly that $B/J_B = S^{-1}(B/J_B)$. We conclude that our lemma is a consequence of Lemma 93.12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in J_B$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 16.3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. $\square$

0DZ8 Lemma 93.13.2. Let k be a field. Let B be a finite type k-algebra. Let $S \subset B$ be a multiplicative subset ideal such that if $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is not smooth at $\mathfrak q$ then $S \cap \mathfrak q = \emptyset$. Let N be a finite B-module. Then there is a canonical bijection

$$\mathrm{Exal}_k(B, N) \to \mathrm{Exal}_k(S^{-1}B, S^{-1}N)$$

Proof. This proof is the same as the proof of Lemma 93.12.4 but easier. We suggest the reader to skip the proof. The map is given by localization: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the localization $S_C^{-1}C$ of C with respect to the inverse image $S_C \subset C$ of S. Compare with the proof of Lemma 93.8.7.

The smooth locus of a morphism of schemes is open by definition. Let $J \subset B$ be an ideal cutting out the set of points in $\mathrm{Spec}(B)$ where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not smooth. Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. Finally, we have

$$\mathrm{Ext}^1_{S^{-1}B}(NL_{S^{-1}B/k}, S^{-1}N) = \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B S^{-1}B = \mathrm{Ext}^1_B(NL_{B/k}, N)$$

The first equality by More on Algebra, Lemma 15.99.2 and Algebra, Lemma 10.134.13. The second because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion and elements of S act invertibly on J-power torsion modules. This concludes the proof by the description of $\mathrm{Exal}_A(B, N)$ as $\mathrm{Ext}^1_B(NL_{B/A}, N)$ given just above Lemma 93.12.4. $\square$

0DZ9 Lemma 93.13.3. In Example 93.8.1 let P be a k-algebra. Let $S \subset P$ be a multiplicative subset. If

(1) $k \to P$ is of finite type, and
(2) $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth at all points of $V(g)$ for all $g \in S$.

then the functor between deformation categories of Lemma 93.8.7

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{S^{-1}P}$ are deformation categories by Lemma 93.8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 90.20.3. The property on liftability is proven in Lemma 93.13.1 and the isomorphism on tangent spaces is the special case of Lemma 93.13.2 where $N = B$. $\square$

93.14. Deformations of henselizations

0DZA In this section we compare the deformation problem posed by an algebra and its completion. We first discuss "liftability".

0DZB **Lemma 93.14.1.** Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $\mathfrak{b} \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is syntomic on the complement of $V(\mathfrak{b})$. Let $(B^h, \mathfrak{b}^h)$ be the henselization of the pair $(B, \mathfrak{b})$. Then B has a flat lift to A' if and only if B^h has a flat lift to A'.

First proof. This proof is a cheat. Namely, if B has a flat lift B', then taking the henselization $(B')^h$ we obtain a flat lift of B^h (compare with the proof of Lemma 93.8.8). Conversely, suppose that C' is an A'-flat lift of $(B')^h$. Then let $\mathfrak{c}' \subset C'$ be the inverse image of the ideal $\mathfrak{b}^h$. Then the completion $(C')^\wedge$ of C' with respect to $\mathfrak{c}'$ is a lift of $B^\wedge$ (details omitted). Hence we see that B has a flat lift by Lemma 93.12.3. □

Second proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $\mathfrak{p} \subset P$ be the inverse image of $\mathfrak{b}$. Set $P' = A'[x_1, \ldots, x_n]$ and denote $\mathfrak{p}' \subset P'$ the inverse image of $\mathfrak{p}$. (Of course $\mathfrak{p}$ and $\mathfrak{p}'$ do not designate prime ideals here.) We will denote P^h and $(P')^h$ the respective henselizations. We will use that taking henselizations is functorial and that the henselization of a quotient is the corresponding quotient of the henselization, see More on Algebra, Lemmas 15.11.16 and 15.12.7.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = B^h$ and $\mathfrak{c} = \mathfrak{b}C$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Then C' is henselian with respect to the inverse image $\mathfrak{c}'$ of $\mathfrak{c}$ (by More on Algebra, Lemma 15.11.9 and the fact that the kernel of $C' \to C$ is nilpotent). We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. These maps pass through henselizations to give surjections $P^h \to C$ and $(P')^h \to C'$ (for the second again using Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $(P')^h$-module structure lifting C as an A-flat P^h-module. Conversely, if we can lift C to a $(P')^h$-module N' flat over A', then N' is a cyclic module $N' \cong (P')^h/\tilde{J}$ (using Nakayama again) and setting $C' = (P')^h/\tilde{J}$ we find a flat lift of C as an algebra.

Observe that $P' \to (P')^h$ is a flat ring map which induces an isomorphism $P'/\mathfrak{p}' = (P')^h/\mathfrak{p}'(P')^h$ (More on Algebra, Lemma 15.12.2). We conclude that our lemma is a consequence of Lemma 93.12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in \mathfrak{p}'$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 16.3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

0DZC **Lemma 93.14.2.** Let k be a field. Let B be a finite type k-algebra. Let $J \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$. Let N be a finite B-module. Then there is a canonical bijection

$$\mathrm{Exal}_k(B, N) \to \mathrm{Exal}_k(B^h, N^h)$$

Here (B^h, J^h) is the henselization of (B, J) and $N^h = N \otimes_B B^h$.

Proof. This proof is the same as the proof of Lemma 93.12.4 but easier. We suggest the reader to skip the proof. The map is given by henselization: given $0 \to N \to C \to B \to 0$ in $\text{Exal}_k(B, N)$ we send it to the henselization C^h of C with respect to the inverse image $J_C \subset C$ of J. Compare with the proof of Lemma 93.8.8.

Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\text{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. Finally, we have

$$\begin{aligned} \text{Ext}^1_{B^h}(NL_{B^h/k}, N^h) &= \text{Ext}^1_{B^h}(NL_{B/k} \otimes_B B^h, N \otimes_B B^h) \\ &= \text{Ext}^1_B(NL_{B/k}, N) \otimes_B B^h \\ &= \text{Ext}^1_B(NL_{B/k}, N) \end{aligned}$$

The first equality by More on Algebra, Lemma 15.33.8 (or rather its analogue for henselizations of pairs). The second by More on Algebra, Lemma 15.99.2. The third because $\text{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion, the map $B \to B^h$ is flat and induces an isomorphism $B/J \to B^h/JB^h$ (More on Algebra, Lemma 15.12.2), and More on Algebra, Lemma 15.89.3. This concludes the proof by the description of $\text{Exal}_A(B, N)$ as $\text{Ext}^1_B(NL_{B/A}, N)$ given just above Lemma 93.12.4. □

0DZD Lemma 93.14.3. In Example 93.8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote (P^h, J^h) the henselization of the pair (P, J). If

(1) $k \to P$ is of finite type, and
(2) $\text{Spec}(P) \to \text{Spec}(k)$ is smooth on the complement of $V(J)$,

then the functor between deformation categories of Lemma 93.8.8

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{P^h}$ are deformation categories by Lemma 93.8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 90.20.3. The property on liftability is proven in Lemma 93.14.1 and the isomorphism on tangent spaces is the special case of Lemma 93.14.2 where $N = B$. □

93.15. Application to isolated singularities

0DZE We apply the discussion above to study the deformation theory of a finite type algebra with finitely many singular points.

0DZF Lemma 93.15.1. In Example 93.8.1 let P be a k-algebra. Assume that $k \to P$ is of finite type and that $\text{Spec}(P) \to \text{Spec}(k)$ is smooth except at the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of P. Let $P_{\mathfrak{m}_i}$, $P^h_{\mathfrak{m}_i}$, $P^\wedge_{\mathfrak{m}_i}$ be the local ring, henselization, completion. Then the maps of deformation categories

$$\mathcal{D}ef_P \to \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}} \to \prod \mathcal{D}ef_{P^h_{\mathfrak{m}_i}} \to \prod \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$$

are smooth and induce isomorphisms on their finite dimensional tangent spaces.

Proof. The tangent space is finite dimensional by Lemma 93.8.5. The functors between the categories are constructed in Lemmas 93.8.7, 93.8.8, and 93.8.10 (we omit some verifications of the form: the completion of the henselization is the completion).

Set $J = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ and apply Lemma 93.12.5 to get that $\mathcal{D}ef_P \to \mathcal{D}ef_{P^\wedge}$ is smooth and induces an isomorphism on tangent spaces where $P^\wedge$ is the J-adic completion of P. However, since $P^\wedge = \prod P^\wedge_{\mathfrak{m}_i}$ we see that the map $\mathcal{D}ef_P \to \prod \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$ is smooth and induces an isomorphism on tangent spaces.

Let (P^h, J^h) be the henselization of the pair (P, J). Then $P^h = \prod P^h_{\mathfrak{m}_i}$ (look at idempotents and use More on Algebra, Lemma 15.11.6). Hence we can apply Lemma 93.14.3 to conclude as in the case of completion.

To get the final case it suffices to show that $\mathcal{D}ef_{P_{\mathfrak{m}_i}} \to \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$ is smooth and induce isomorphisms on tangent spaces for each i separately. To do this, we may replace P by a principal localization whose only singular point is a maximal ideal $\mathfrak{m}$ (corresponding to $\mathfrak{m}_i$ in the original P). Then we can apply Lemma 93.13.3 with multiplicative subset $S = P \setminus \mathfrak{m}$ to conclude. Minor details omitted. □

93.16. Unobstructed deformation problems

0DZG Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibred in groupoids. Recall that we say $\mathcal{F}$ is smooth or unobstructed if p is smooth. This means that given a surjection $\varphi : A' \to A$ in $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(A))$ there exists a morphism $f : x' \to x$ in $\mathcal{F}$ with $p(f) = \varphi$. See Formal Deformation Theory, Section 90.9. In this section we give some geometrically meaningful examples.

0DZH Lemma 93.16.1. In Example 93.8.1 let P be a local complete intersection over k (Algebra, Definition 10.135.1). Then $\mathcal{D}ef_P$ is unobstructed.

Proof. Let $(A, Q) \to (k, P)$ be an object of $\mathcal{D}ef_P$. Then we see that $A \to Q$ is a syntomic ring map by Algebra, Definition 10.136.1. Hence for any surjection $A' \to A$ in $\mathcal{C}_\Lambda$ we see that there is a morphism $(A', Q') \to (A, Q)$ lifting $A' \to A$ by Smoothing Ring Maps, Proposition 16.3.2. This proves the lemma. □

0DZN Lemma 93.16.2. In Situation 93.9.9 if $U_{12} \to \mathrm{Spec}(k)$ is smooth, then the morphism

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2}$$

is smooth. If in addition U_1 is a local complete intersection over k, then

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_2}$$

is smooth.

Proof. The equality signs hold by Lemma 93.9.7. Let us think of $\mathcal{C}_\Lambda$ as a deformation category over $\mathcal{C}_\Lambda$ as in Formal Deformation Theory, Section 90.9. Then

$$\mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2} = \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2},$$

see Formal Deformation Theory, Remarks 90.5.2 (14). Using Lemma 93.9.10 the first statement is that the functor

$$\mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2} \longrightarrow \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2}$$

is smooth. This follows from Formal Deformation Theory, Lemma 90.20.2 as long as we can show that $T\mathcal{D}ef_{P_{12}} = (0)$. This vanishing follows from Lemma 93.8.4 as

P_{12} is smooth over k. For the second statement it suffices to show that $\mathcal{D}ef_{P_1} \to \mathcal{C}_\Lambda$ is smooth, see Formal Deformation Theory, Lemma 90.8.7. In other words, we have to show $\mathcal{D}ef_{P_1}$ is unobstructed, which is Lemma 93.16.1. □

0DZP Lemma 93.16.3. In Example 93.9.1 let X be a scheme over k. Assume

(1) X is separated, finite type over k and $\dim(X) \leq 1$,
(2) $X \to \text{Spec}(k)$ is smooth except at the closed points $p_1, \ldots, p_n \in X$.

Let $\mathcal{O}_{X,p_1}$, $\mathcal{O}^h_{X,p_1}$, $\mathcal{O}^\wedge_{X,p_1}$ be the local ring, henselization, completion. Consider the maps of deformation categories

$$\mathcal{D}ef_X \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}^h_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}^\wedge_{X,p_i}}$$

The first arrow is smooth and the second and third arrows are smooth and induce isomorphisms on tangent spaces.

Proof. Choose an affine open $U_2 \subset X$ containing $p_1, \ldots, p_n$ and the generic point of every irreducible component of X. This is possible by Varieties, Lemma 33.43.3 and Properties, Lemma 28.29.5. Then $X \setminus U_2$ is finite and we can choose an affine open $U_1 \subset X \setminus \{p_1, \ldots, p_n\}$ such that $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$. Then U_1 and U_{12} are smooth affine schemes over k. We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 93.16.2. Applying Lemmas 93.9.7 and 93.15.1 we win. □

0DZQ Lemma 93.16.4. In Example 93.9.1 let X be a scheme over k. Assume

(1) X is separated, finite type over k and $\dim(X) \leq 1$,
(2) X is a local complete intersection over k, and
(3) $X \to \text{Spec}(k)$ is smooth except at finitely many points.

Then $\mathcal{D}ef_X$ is unobstructed.

Proof. Let $p_1, \ldots, p_n \in X$ be the points where $X \to \text{Spec}(k)$ isn't smooth. Choose an affine open $U_2 \subset X$ containing $p_1, \ldots, p_n$ and the generic point of every irreducible component of X. This is possible by Varieties, Lemma 33.43.3 and Properties, Lemma 28.29.5. Then $X \setminus U_2$ is finite and we can choose an affine open $U_1 \subset X \setminus \{p_1, \ldots, p_n\}$ such that $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$. Then U_1 and U_{12} are smooth affine schemes over k. We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 93.16.2. Applying Lemmas 93.9.7 and 93.16.1 we win. □

93.17. Smoothings

0E7S Suppose given a finite type scheme or algebraic space X over a field k. It is often useful to find a flat morphism of finite type $Y \to \text{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X. Such a thing is called a smoothing of X. In this section we will find a smoothing for 1-dimensional separated X which have isolated local complete intersection singularities.

0E7T Lemma 93.17.1. Let k be a field. Set $S = \text{Spec}(k[[t]])$ and $S_n = \text{Spec}(k[t]/(t^n))$. Let $Y \to S$ be a proper, flat morphism of schemes whose special fibre X is Cohen-Macaulay and equidimensional of dimension d. Denote $X_n = Y \times_S S_n$. If for some $n \geq 1$ the dth Fitting ideal of Ω_{X_n/S_n} contains t^{n-1}, then the generic fibre of $Y \to S$ is smooth.

Proof. By More on Morphisms, Lemma 37.22.7 we see that $Y \to S$ is a Cohen-Macaulay morphism. By Morphisms, Lemma 29.29.4 we see that $Y \to S$ has relative dimension d. By Divisors, Lemma 31.10.3 the dth Fitting ideal $\mathcal{I} \subset \mathcal{O}_Y$ of $\Omega_{Y/S}$ cuts out the singular locus of the morphism $Y \to S$. In other words, $V(\mathcal{I}) \subset Y$ is the closed subset of points where $Y \to S$ is not smooth. By Divisors, Lemma 31.10.1 formation of this Fitting ideal commutes with base change. By assumption we see that t^{n-1} is a section of $\mathcal{I} + t^n\mathcal{O}_Y$. Thus for every $x \in X = V(t) \subset Y$ we conclude that $t^{n-1} \in \mathcal{I}_x$ where $\mathcal{I}_x$ is the stalk at x. This implies that $V(\mathcal{I}) \subset V(t)$ in an open neighbourhood of X in Y. Since $Y \to S$ is proper, this implies $V(\mathcal{I}) \subset V(t)$ as desired. □

0E7U **Lemma 93.17.2.** *Let k be a field. Let $1 \leq c \leq n$ be integers. Let $f_1, \ldots, f_c \in k[x_1, \ldots x_n]$ be elements. Let a_{ij}, $0 \leq i \leq n$, $1 \leq j \leq c$ be variables. Consider*

$$g_j = f_j + a_{0j} + a_{1j}x_1 + \ldots + a_{nj}x_n \in k[a_{ij}][x_1, \ldots, x_n]$$

Denote $Y \subset \mathbf{A}_k^{n+c(n+1)}$ the closed subscheme cut out by $g_1, \ldots, g_c$. Denote $\pi : Y \to \mathbf{A}_k^{c(n+1)}$ the projection onto the affine space with variables a_{ij}. Then there is a nonempty Zariski open of $\mathbf{A}_k^{c(n+1)}$ over which π is smooth.

Proof. Recall that the set of points where π is smooth is open. Thus the complement, i.e., the singular locus, is closed. By Chevalley's theorem (in the form of Morphisms, Lemma 29.22.2) the image of the singular locus is constructible. Hence if the generic point of $\mathbf{A}_k^{c(n+1)}$ is not in the image of the singular locus, then the lemma follows (by Topology, Lemma 5.15.15 for example). Thus we have to show there is no point $y \in Y$ where π is not smooth mapping to the generic point of $\mathbf{A}_k^{c(n+1)}$. Consider the matrix of partial derivatives

$$(\frac{\partial g_j}{\partial x_i}) = (\frac{\partial f_j}{\partial x_i} + a_{ij})$$

The image of this matrix in $\kappa(y)$ must have rank $< c$ since otherwise π would be smooth at y, see discussion in Smoothing Ring Maps, Section 16.2. Thus we can find $\lambda_1, \ldots, \lambda_c \in \kappa(y)$ not all zero such that the vector $(\lambda_1, \ldots, \lambda_c)$ is in the kernel of this matrix. After renumbering we may assume $\lambda_1 \neq 0$. Dividing by λ_1 we may assume our vector has the form $(1, \lambda_2, \ldots, \lambda_c)$. Then we obtain

$$a_{i1} = -\frac{\partial f_j}{\partial x_1} - \sum\nolimits_{j=2,\ldots,c} \lambda_j(\frac{\partial f_j}{\partial x_i} + a_{ij})$$

in $\kappa(y)$ for $i = 1, \ldots, n$. Moreover, since $y \in Y$ we also have

$$a_{0j} = -f_j - a_{1j}x_1 - \ldots - a_{nj}x_n$$

in $\kappa(y)$. This means that the subfield of $\kappa(y)$ generated by a_{ij} is contained in the subfield of $\kappa(y)$ generated by the images of $x_1, \ldots, x_n, \lambda_2, \ldots, \lambda_c$, and a_{ij} except for a_{i1} and a_{0j}. We count and we see that the transcendence degree of this is at most $c(n + 1) - 1$. Hence y cannot map to the generic point as desired. □

0E7V **Lemma 93.17.3.** *Let k be a field. Let A be a global complete interesection over k. There exists a flat finite type ring map $k[[t]] \to B$ with $B/tB \cong A$ such that $B[1/t]$ is smooth over $k((t))$.*

Proof. Write $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in Algebra, Definition 10.135.1. We are going to choose $a_{ij} \in (t) \subset k[[t]]$ and set

$$g_j = f_j + a_{0j} + a_{1j}x_1 + \ldots + a_{nj}x_n \in k[[t]][x_1, \ldots, x_n]$$

After doing this we take $B = k[[t]][x_1, \ldots, x_n]/(g_1, \ldots, g_c)$. We claim that $k[[t]] \to B$ is flat at every prime ideal lying over (t). Namely, the elements $f_1, \ldots, f_c$ form a regular sequence in the local ring at any prime ideal $\mathfrak{p}$ of $k[x_1, \ldots, x_n]$ containing $f_1, \ldots, f_c$ (Algebra, Lemma 10.135.4). Thus $g_1, \ldots, g_c$ is locally a lift of a regular sequence and we can apply Algebra, Lemma 10.99.3. Flatness at primes lying over $(0) \subset k[[t]]$ is automatic because $k((t)) = k[[t]]_{(0)}$ is a field. Thus B is flat over $k[[t]]$.

All that remains is to show that for suitable choices of a_{ij} the generic fibre $B_{(0)}$ is smooth over $k((t))$. For this we have to show that we can choose our a_{ij} so that the induced morphism

$$(a_{ij}) : \text{Spec}(k[[t]]) \longrightarrow \mathbf{A}_k^{c(n+1)}$$

maps into the nonempty Zariski open of Lemma 93.17.2. This is clear because there is no nonzero polynomial in the a_{ij} which vanishes on $(t)^{\oplus c(n+1)}$. (We leave this as an exercise to the reader.) $\square$

0E7W Lemma 93.17.4. Let k be a field. Let A be a finite dimensional k-algebra which is a local complete intersection over k. Then there is a finite flat $k[[t]]$-algebra B with $B/tB \cong A$ and $B[1/t]$ étale over $k((t))$.

Proof. Since A is Artinian (Algebra, Lemma 10.53.2), we can write A as a product of local Artinian rings (Algebra, Lemma 10.53.6). Thus it suffices to prove the lemma if A is local (this uses that being a local complete intersection is preserved under taking principal localizations, see Algebra, Lemma 10.135.2). In this case A is a global complete intersection. Consider the algebra B constructed in Lemma 93.17.3. Then $k[[t]] \to B$ is quasi-finite at the unique prime of B lying over (t) (Algebra, Definition 10.122.3). Observe that $k[[t]]$ is a henselian local ring (Algebra, Lemma 10.153.9). Thus $B = B' \times C$ where B' is finite over $k[[t]]$ and C has no prime lying over (t), see Algebra, Lemma 10.153.3. Then B' is the ring we are looking for (recall that étale is the same thing as smooth of relative dimension 0). $\square$

0E7X Lemma 93.17.5. Let k be a field. Let A be a k-algebra. Assume

(1) A is a local ring essentially of finite type over k,
(2) A is a complete intersection over k (Algebra, Definition 10.135.5).

Set $d = \dim(A) + \text{trdeg}_k(\kappa)$ where κ is the residue field of A. Then there exists an integer n and a flat, essentially of finite type ring map $k[[t]] \to B$ with $B/tB \cong A$ such that t^n is in the dth Fitting ideal of $\Omega_{B/k[[t]]}$.

Proof. By Algebra, Lemma 10.135.7 we can write A as the localization at a prime $\mathfrak{p}$ of a global complete intersection P over k. Observe that $\dim(P) = d$ by Algebra, Lemma 10.116.3. By Lemma 93.17.3 we can find a flat, finite type ring map $k[[t]] \to Q$ such that $P \cong Q/tQ$ and such that $k((t)) \to Q[1/t]$ is smooth. It follows from the construction of Q in the lemma that $k[[t]] \to Q$ is a relative global complete intersection of relative dimension d; alternatively, Algebra, Lemma 10.136.15 tells us that Q or a suitable principal localization of Q is such a global complete intersection. Hence by Divisors, Lemma 31.10.3 the dth Fitting ideal $I \subset Q$ of $\Omega_{Q/k[[t]]}$ cuts out the singular locus of $\mathrm{Spec}(Q) \to \mathrm{Spec}(k[[t]])$. Thus $t^n \in I$ for some n. Let $\mathfrak q \subset Q$ be the inverse image of $\mathfrak p$. Set $B = Q_{\mathfrak q}$. The lemma is proved. □

0E7Y Lemma 93.17.6. Let X be a scheme over a field k. Assume

(1) X is proper over k,
(2) X is a local complete intersection over k,
(3) X has dimension ≤ 1, and
(4) $X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.

Then there exists a flat projective morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X.

Proof. Observe that X is Cohen-Macaulay, see Algebra, Lemma 10.135.3. Thus $X = X' \amalg X''$ with $\dim(X') = 0$ and X'' equidimensional of dimension 1, see Morphisms, Lemma 29.29.4. Since X' is finite over k (Varieties, Lemma 33.20.2) we can find $Y' \to \mathrm{Spec}(k[[t]])$ with special fibre X' and generic fibre smooth by Lemma 93.17.4. Thus it suffices to prove the lemma for X''. After replacing X by X'' we have X is Cohen-Macaulay and equidimensional of dimension 1.

We are going to use deformation theory for the situation $\Lambda = k \to k$. Let $p_1, \ldots, p_r \in X$ be the closed singular points of X, i.e., the points where $X \to \mathrm{Spec}(k)$ isn't smooth. For each i we pick an integer n_i and a flat, essentially of finite type ring map

$$k[[t]] \longrightarrow B_i$$

with $B_i/tB_i \cong \mathcal{O}_{X,p_i}$ such that t^{n_i} is in the 1st Fitting ideal of $\Omega_{B_i/k[[t]]}$. This is possible by Lemma 93.17.5. Observe that the system $(B_i/t^n B_i)$ defines a formal object of $\mathcal{D}ef_{\mathcal{O}_{X,p_i}}$ over $k[[t]]$. By Lemma 93.16.3 the map

$$\mathcal{D}ef_X \longrightarrow \prod\nolimits_{i=1,\ldots,r} \mathcal{D}ef_{\mathcal{O}_{X,p_i}}$$

is a smooth map between deformation categories. Hence by Formal Deformation Theory, Lemma 90.8.8 there exists a formal object (X_n) in $\mathcal{D}ef_X$ mapping to the formal object $\prod_i (B_i/t^n)$ by the arrow above. By More on Morphisms of Spaces, Lemma 76.43.5 there exists a projective scheme Y over $k[[t]]$ and compatible isomorphisms $Y \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k[t]/(t^n)) \cong X_n$. By More on Morphisms, Lemma 37.12.4 we see that $Y \to \mathrm{Spec}(k[[t]])$ is flat. Since X is Cohen-Macaulay and equidimensional of dimension 1 we may apply Lemma 93.17.1 to check Y has smooth generic fibre[3]. Choose n strictly larger than the maximum of the integers n_i found above. It we can show t^{n-1} is in the first Fitting ideal of Ω_{X_n/S_n} with $S_n = \mathrm{Spec}(k[t]/(t^n))$, then the proof is done. To do this it suffices to prove this is true in each of the local rings of X_n at closed points p. However, if p corresponds to a smooth point for $X \to \mathrm{Spec}(k)$, then $\Omega_{X_n/S_n,p}$ is free of rank 1 and the first Fitting ideal is equal to the local ring. If $p = p_i$ for some i, then

$$\Omega_{X_n/S_n,p_i} = \Omega_{(B_i/t^nB_i)/(k[t]/(t^n))} = \Omega_{B_i/k[[t]]}/t^n\Omega_{B_i/k[[t]]}$$

Since taking Fitting ideals commutes with base change (with already used this but in this algebraic setting it follows from More on Algebra, Lemma 15.8.4), and since $n - 1 \geq n_i$ we see that t^{n-1} is in the Fitting ideal of this module over B_i/t^nB_i as desired. □

[3]Warning: in general it is not true that the local ring of Y at the point p_i is isomorphic to B_i. We only know that this is true after dividing by t^n on both sides!

0E7Z Lemma 93.17.7. Let k be a field and let X be a scheme over k. Assume

(1) X is separated, finite type over k and $\dim(X) \leq 1$,
(2) X is a local complete intersection over k, and
(3) $X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.

Then there exists a flat, separated, finite type morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X.

Proof. If X is reduced, then we can choose an embedding $X \subset \overline{X}$ as in Varieties, Lemma 33.43.6. Writing $X = \overline{X} \setminus \{x_1, \ldots, x_n\}$ we see that $\mathcal{O}_{\overline{X},x_i}$ is a discrete valuation ring and hence in particular a local complete intersection (Algebra, Definition 10.135.5). Thus $\overline{X}$ is a local complete intersection over k because this holds over the open X and at the points x_i by Algebra, Lemma 10.135.7. Thus we may apply Lemma 93.17.6 to find a projective flat morphism $\overline{Y} \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is $\overline{X}$. Then we remove $x_1, \ldots, x_n$ from $\overline{Y}$ to obtain Y.

In the general case, write $X = X' \amalg X''$ where with $\dim(X') = 0$ and X'' equidimensional of dimension 1. Then X'' is reduced and the first paragraph applies to it. On the other hand, X' can be dealt with as in the proof of Lemma 93.17.6. Some details omitted. $\square$

93.18. Other chapters

Part 7

Algebraic Stacks

CHAPTER 94

Algebraic Stacks

026K 94.1. Introduction

026L This is where we define algebraic stacks and make some very elementary observations. The general philosophy will be to have no separation conditions whatsoever and add those conditions necessary to make lemmas, propositions, theorems true/provable. Thus the notions discussed here differ slightly from those in other places in the literature, e.g., [LMB00].

This chapter is not an introduction to algebraic stacks. For an informal discussion of algebraic stacks, please take a look at Introducing Algebraic Stacks, Section 105.1.

94.2. Conventions

026M The conventions we use in this chapter are the same as those in the chapter on algebraic spaces. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We discuss what changes if you change the big fppf site in Section 94.18.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points over S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 35.13.7. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

94.3. Notation

0400 We use the letters S, T, U, V, X, Y to indicate schemes. We use the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to indicate categories (fibred, fibred in groupoids, stacks, ...) over $(Sch/S)_{fppf}$. We use small case letters f, g for functors such as $f : \mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We use capital F, G, H for algebraic spaces over S, and more generally for presheaves of sets on $(Sch/S)_{fppf}$. (In future chapters we will revert to using also X, Y, etc for algebraic spaces.)

The reason for these choices is that we want to clearly distinguish between the different types of objects in this chapter, to build the foundations.

94.4. Representable categories fibred in groupoids

02ZQ Let S be a scheme contained in Sch_{fppf}. The basic object of study in this chapter will be a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$, see Categories, Definition 4.35.1. We will often simply say "let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$" to indicate this situation. A 1-morphism $\mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ will be a 1-morphism in the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Categories, Definition 4.35.6. It is simply a functor $\mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We recall this is really a $(2,1)$-category and that all 2-fibre products exist.

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Recall that $\mathcal{X}$ is said to be representable if there exists a scheme $U \in \text{Ob}((Sch/S)_{fppf})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (Sch/U)_{fppf}$$

of categories over $(Sch/S)_{fppf}$, see Categories, Definition 4.40.1. We will sometimes say that $\mathcal{X}$ is representable by a scheme to distinguish from the case where $\mathcal{X}$ is representable by an algebraic space (see below).

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by U, V, then we have

04SR (94.4.0.1) $$\text{Mor}_{\text{Cat}/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y})\Big/\text{2-isomorphism} = \text{Mor}_{Sch/S}(U, V)$$

see Categories, Lemma 4.40.3. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $U \to V$. Conversely, given a morphism of schemes $U \to V$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $U \to V$ and which is unique up to unique 2-isomorphism.

94.5. The 2-Yoneda lemma

04SS Let $U \in \text{Ob}((Sch/S)_{fppf})$, and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We will frequently use the 2-Yoneda lemma, see Categories, Lemma 4.41.2. Technically it says that there is an equivalence of categories

$$\text{Mor}_{\text{Cat}/(Sch/S)_{fppf}}((Sch/U)_{fppf}, \mathcal{X}) \longrightarrow \mathcal{X}_U, \quad f \longmapsto f(U/U).$$

It says that 1-morphisms $(Sch/U)_{fppf} \to \mathcal{X}$ correspond to objects x of the fibre category $\mathcal{X}_U$. Namely, given a 1-morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$ we obtain the object $x = f(U/U) \in \text{Ob}(\mathcal{X}_U)$. Conversely, given a choice of pullbacks for $\mathcal{X}$ as in Categories, Definition 4.33.6, and an object x of $\mathcal{X}_U$, we obtain a functor $(Sch/U)_{fppf} \to \mathcal{X}$ defined by the rule

$$(\varphi : V \to U) \longmapsto \varphi^* x$$

on objects. By abuse of notation we use $x : (Sch/U)_{fppf} \to \mathcal{X}$ to indicate this functor. It indeed has the property that $x(U/U) = x$ and moreover, given any other functor f with $f(U/U) = x$ there exists a unique 2-isomorphism $x \to f$. In other words the functor x is well determined by the object x up to unique 2-isomorphism.

We will use this without further mention in the following.

94.6. Representable morphisms of categories fibred in groupoids

04ST Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism, see Categories, Definition 4.42.3. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable. Choose a representing object V_y and an equivalence

$$(Sch/V_y)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}.$$

The projection $(Sch/V_y)_{fppf} \to (Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{Y} \to (Sch/U)_{fppf}$ comes from a morphism of schemes $f_y : V_y \to U$, see Section 94.4. We represent this by the diagram

0401 (94.6.0.1)
$$\begin{tikzcd} V_y \arrow[r, squiggly] \arrow[d, "f_y"] & (Sch/V_y)_{fppf} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ U \arrow[r, squiggly] & (Sch/U)_{fppf} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

where the squiggly arrows represent the 2-Yoneda embedding. Here are some lemmas about this notion that work in great generality (namely, they work for categories fibred in groupoids over any base category which has fibre products).

02ZR **Lemma 94.6.1.** Let $f : X \to Y$ be a morphism of $(Sch/S)_{fppf}$. Then the 1-morphism induced by f

$$(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$$

is a representable 1-morphism.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. $\square$

0456 **Lemma 94.6.2.** Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable if and only if f' is representable.

Proof. Omitted. $\square$

02ZS **Lemma 94.6.3.** Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be representable 1-morphisms. Then

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a representable 1-morphism.

Proof. This is entirely formal and works in any category. $\square$

02ZT Lemma 94.6.4. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

Then the base change f' is a representable 1-morphism.

Proof. This is entirely formal and works in any category. $\square$

02ZU Lemma 94.6.5. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be representable 1-morphisms. Then

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a representable 1-morphism.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 94.6.3 and 94.6.4. $\square$

94.7. Split categories fibred in groupoids

04SU Let S be a scheme contained in Sch_{fppf}. Recall that given a "presheaf of groupoids"

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow \text{Groupoids}$$

we get a category fibred in groupoids $\mathcal{S}_F$ over $(Sch/S)_{fppf}$, see Categories, Example 4.37.1. Any category fibred in groupoids isomorphic (!) to one of these is called a split category fibred in groupoids. Any category fibred in groupoids is equivalent to a split one.

If F is a presheaf of sets then $\mathcal{S}_F$ is fibred in sets, see Categories, Definition 4.38.2, and Categories, Example 4.38.5. The rule $F \mapsto \mathcal{S}_F$ is in some sense fully faithful on presheaves, see Categories, Lemma 4.38.6. If F, G are presheaves, then

$$\mathcal{S}_{F \times G} = \mathcal{S}_F \times_{(Sch/S)_{fppf}} \mathcal{S}_G$$

and if $F \to H$ and $G \to H$ are maps of presheaves of sets, then

$$\mathcal{S}_{F \times_H G} = \mathcal{S}_F \times_{\mathcal{S}_H} \mathcal{S}_G$$

where the right hand sides are 2-fibre products. This is immediate from the definitions as the fibre categories of $\mathcal{S}_F, \mathcal{S}_G, \mathcal{S}_H$ have only identity morphisms.

An even more special case is where $F = h_X$ is a representable presheaf. In this case we have $\mathcal{S}_{h_X} = (Sch/X)_{fppf}$, see Categories, Example 4.38.7.

We will use the notation $\mathcal{S}_F$ without further mention in the following.

94.8. Categories fibred in groupoids representable by algebraic spaces

02ZV A slightly weaker notion than being representable is the notion of being representable by algebraic spaces which we discuss in this section. This discussion might have been avoided had we worked with some category Spaces_{fppf} of algebraic spaces instead of the category Sch_{fppf}. However, it seems to us natural to consider the category of schemes as the natural collection of "test objects" over which the fibre categories of an algebraic stack are defined.

In analogy with Categories, Definitions 4.40.1 we make the following definition.

04SV **Definition 94.8.1.** Let S be a scheme contained in Sch_{fppf}. A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ is called *representable by an algebraic space over* S if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(Sch/S)_{fppf}$.

We continue our abuse of notation in suppressing the equivalence j whenever we encounter such a situation. It follows formally from the above that if $\mathcal{X}$ is representable (by a scheme), then it is representable by an algebraic space. Here is the analogue of Categories, Lemma 4.40.2.

02ZX **Lemma 94.8.2.** Let S be a scheme contained in Sch_{fppf}. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Then $\mathcal{X}$ is representable by an algebraic space over S if and only if the following conditions are satisfied:

(1) $\mathcal{X}$ is fibred in setoids[1], and
(2) the presheaf $U \mapsto \text{Ob}(\mathcal{X}_U)/\cong$ is an algebraic space.

Proof. Omitted, but see Categories, Lemma 4.40.2. □

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by algebraic spaces F, G over S, then we have

04SW (94.8.2.1) $$\text{Mor}_{\text{Cat}/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y})\Big/2\text{-isomorphism} = \text{Mor}_{Sch/S}(F, G)$$

see Categories, Lemma 4.39.6. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $F \to G$. Conversely, given a morphism of sheaves $F \to G$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $F \to G$ and which is unique up to unique 2-isomorphism.

94.9. Morphisms representable by algebraic spaces

04SX In analogy with Categories, Definition 4.42.3 we make the following definition.

02ZW **Definition 94.9.1.** Let S be a scheme contained in Sch_{fppf}. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ is called *representable by algebraic spaces* if for any $U \in \text{Ob}((Sch/S)_{fppf})$ and any $y : (Sch/U)_{fppf} \to \mathcal{Y}$ the category fibred in groupoids

$$(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

over $(Sch/U)_{fppf}$ is representable by an algebraic space over U.

[1]This means that it is fibred in groupoids and objects in the fibre categories have no nontrivial automorphisms, see Categories, Definition 4.38.2.

Choose an algebraic space F_y over U which represents $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$. We may think of F_y as an algebraic space over S which comes equipped with a canonical morphism $f_y : F_y \to U$ over S, see Spaces, Section 65.16. Here is the diagram

0402 (94.9.1.1)
$$\begin{tikzcd} F_y \arrow[d, "f_y"'] & (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \arrow[l, squiggly] \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_0"'] & \mathcal{X} \arrow[d, "f"] \\ U & (Sch/U)_{fppf} \arrow[l, squiggly] \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The right square is 2-commutative, and is a 2-fibre product square.

Here is the analogue of Categories, Lemma 4.42.5.

02ZY Lemma 94.9.2. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are necessary and sufficient conditions for f to be representable by algebraic spaces:

(1) for each scheme U/S the functor $f_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and
(2) for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf
$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$
is an algebraic space over U.

Here we have made a choice of pullbacks for $\mathcal{Y}$.

Proof. This follows from the description of fibre categories of the 2-fibre products $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ in Categories, Lemma 4.42.1 combined with Lemma 94.8.2. $\square$

Here are some lemmas about this notion that work in great generality.

0457 Lemma 94.9.3. Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram
$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$
of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable by algebraic spaces if and only if f' is representable by algebraic spaces.

Proof. Omitted. $\square$

02ZZ Lemma 94.9.4. Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over S. If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces over S, then the 1-morphism f is representable by algebraic spaces.

Proof. Omitted. This relies only on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 65.7.3. $\square$

0458 Lemma 94.9.5. Let S be an object of Sch_{fppf}. Let $a : F \to G$ be a map of presheaves of sets on $(Sch/S)_{fppf}$. Denote $a' : \mathcal{S}_F \to \mathcal{S}_G$ the associated map of categories fibred in sets. Then a is representable by algebraic spaces (see Bootstrap, Definition 80.3.1) if and only if a' is representable by algebraic spaces.

Proof. Omitted. □

04SY Lemma 94.9.6. Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a is representable by algebraic spaces (see Bootstrap, Definition 80.3.1) if and only if f is representable by algebraic spaces.

Proof. Omitted. Hint: Combine Lemmas 94.9.3 and 94.9.5. □

0302 Lemma 94.9.7. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

Then the base change f' is a 1-morphism representable by algebraic spaces.

Proof. This is formal. □

0300 Lemma 94.9.8. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms. Assume

(1) f is representable by algebraic spaces, and
(2) $\mathcal{Z}$ is representable by an algebraic space over S.

Then the 2-fibre product $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space.

Proof. This is a reformulation of Bootstrap, Lemma 80.3.6. First note that $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is fibred in setoids over $(Sch/S)_{fppf}$. Hence it is equivalent to $\mathcal{S}_F$ for some presheaf F on $(Sch/S)_{fppf}$, see Categories, Lemma 4.39.5. Moreover, let G be an algebraic space which represents $\mathcal{Z}$. The 1-morphism $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces by Lemma 94.9.7. And $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $F \to G$ by Categories, Lemma 4.39.6. Then $F \to G$ is representable by algebraic spaces by Lemma 94.9.6. Hence Bootstrap, Lemma 80.3.6 implies that F is an algebraic space as desired. □

Let S, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, f, g be as in Lemma 94.9.8. Let F and G be algebraic spaces over S such that F represents $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ and G represents $\mathcal{Z}$. The 1-morphism $f' : \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $f' : F \to G$ of algebraic spaces by (94.8.2.1). Thus we have the following diagram

0403 (94.9.8.1)
$$\begin{tikzcd} F \arrow[d, "f'"'] & \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[l, squiggly] \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ G & \mathcal{Z} \arrow[l, squiggly] \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects.

0301 Lemma 94.9.9. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. If $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms representable by algebraic spaces, then

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a 1-morphism representable by algebraic spaces.

Proof. This follows from Lemma 94.9.8. Details omitted. □

0303 Lemma 94.9.10. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. Then

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a 1-morphism representable by algebraic spaces.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 94.9.9 and 94.9.7. □

0CKY Lemma 94.9.11. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces and $\mathcal{Y}$ is a stack in groupoids, then $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack in groupoids.

Lemma in an email of Matthew Emerton dated June 15, 2016

Proof. The property of a morphism being representable by algebraic spaces is preserved under base-change (Lemma 94.9.8), and so, passing to the base-change $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over $\mathcal{Y}$, we may reduce to the case of a morphism of categories fibred in groupoids $\mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces, and whose target is a stack in groupoids; our goal is then to prove that $\mathcal{X}$ is also a stack in groupoids. This follows from Stacks, Lemma 8.6.11 whose assumptions are satisfied as a result of Lemma 94.9.2. □

94.10. Properties of morphisms representable by algebraic spaces

03YJ Here is the definition that makes this work.

03YK Definition 94.10.1. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume f is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 74.10.1.

In this case we say that f has property $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathcal{Y}_U$ the resulting morphism of algebraic spaces $f_y : F_y \to U$, see diagram (94.9.1.1), has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the target. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

0459 Lemma 94.10.2. Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 94.10.1. Consider a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences and f (or equivalently f') is representable by algebraic spaces. Then f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$.

Proof. Note that this makes sense by Lemma 94.9.3. Proof omitted. □

Here is a sanity check.

045A Lemma 94.10.3. Let S be a scheme contained in Sch_{fppf}. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be as in Definition 94.10.1. Assume a is representable by algebraic spaces. Then $a : F \to G$ has property $\mathcal{P}$ (see Bootstrap, Definition 80.4.1) if and only if the corresponding morphism $\mathcal{S}_F \to \mathcal{S}_G$ of categories fibred in groupoids has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 94.9.5. Proof omitted. □

04TC Lemma 94.10.4. Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 94.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a has $\mathcal{P}$ if and only if f has $\mathcal{P}$.

Proof. The lemma makes sense by Lemma 94.9.6. The lemma follows on combining Lemmas 94.10.2 and 94.10.3. □

045B Lemma 94.10.5. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 94.10.1 which is stable under composition. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms which are representable by algebraic spaces. If f and g have property $\mathcal{P}$ so does $g \circ f : \mathcal{X} \to \mathcal{Z}$.

Proof. Note that the lemma makes sense by Lemma 94.9.9. Proof omitted. □

045C Lemma 94.10.6. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 94.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the 2-fibre product diagram

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

If f has $\mathcal{P}$, then the base change f' has $\mathcal{P}$.

Proof. The lemma makes sense by Lemma 94.9.7. Proof omitted. □

045D Lemma 94.10.7. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 94.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

Assume that for every scheme U and object x of $\mathcal{Y}_U$, there exists an fppf covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of the functor $g : \mathcal{Z}_{U_i} \to \mathcal{Y}_{U_i}$. In this case, if f' has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. Proof omitted. Hint: Compare with the proof of Spaces, Lemma 65.5.6. □

045E Lemma 94.10.8. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{P}$ be a property as in Definition 94.10.1 which is stable under composition. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. If f_1 and f_2 have property $\mathcal{P}$ so does $f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$.

Proof. The lemma makes sense by Lemma 94.9.10. Proof omitted. □

045F Lemma 94.10.9. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 94.10.1. Suppose that for any morphism of algebraic spaces $a : F \to G$ we have $\mathcal{P}(a) \Rightarrow \mathcal{P}'(a)$. If f has property $\mathcal{P}$ then f has property $\mathcal{P}'$.

Proof. Formal. □

05UK Lemma 94.10.10. Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism (see Definition 94.10.1 and Descent on Spaces, Lemma 74.11.30). Then j is fully faithful on fibre categories.

Proof. We have seen in Lemma 94.9.2 that j is faithful on fibre categories. Consider a scheme U, two objects u, v of $\mathcal{X}_U$, and an isomorphism $t : j(u) \to j(v)$ in $\mathcal{Y}_U$. We have to construct an isomorphism in $\mathcal{X}_U$ between u and v. By the 2-Yoneda lemma (see Section 94.5) we think of u, v as 1-morphisms $u, v : (Sch/U)_{fppf} \to \mathcal{X}$ and we consider the 2-fibre product

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X}.$$

By assumption this is representable by an algebraic space $F_{j \circ v}$, over U and the morphism $F_{j \circ v} \to U$ is a monomorphism. But since $(1_U, v, 1_{j(v)})$ gives a 1-morphism of $(Sch/U)_{fppf}$ into the displayed 2-fibre product, we see that $F_{j \circ v} = U$ (here we use that if $V \to U$ is a monomorphism of algebraic spaces which has a section, then $V = U$). Therefore the 1-morphism projecting to the first coordinate

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X} \to (Sch/U)_{fppf}$$

is an equivalence of fibre categories. Since $(1_U, u, t)$ and $(1_U, v, 1_{j(v)})$ give two objects in $((Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X})_U$ which have the same first coordinate, there must be a 2-morphism between them in the 2-fibre product. This is by definition a morphism $\tilde{t} : u \to v$ such that $j(\tilde{t}) = t$. □

Here is a characterization of those categories fibred in groupoids for which the diagonal is representable by algebraic spaces.

045G Lemma 94.10.11. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent:

(1) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(2) for every scheme U over S, and any $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the sheaf $Isom(x, y)$ is an algebraic space over U,
(3) for every scheme U over S, and any $x \in \mathrm{Ob}(\mathcal{X}_U)$ the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is representable by algebraic spaces,
(4) for every pair of schemes T_1, T_2 over S, and any $x_i \in \mathrm{Ob}(\mathcal{X}_{T_i})$, $i = 1, 2$ the 2-fibre product $(Sch/T_1)_{fppf} \times_{x_1,\mathcal{X},x_2} (Sch/T_2)_{fppf}$ is representable by an algebraic space,
(5) for every representable category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,
(6) for every pair $\mathcal{T}_1, \mathcal{T}_2$ of representable categories fibred in groupoids over $(Sch/S)_{fppf}$ and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$, $i = 1, 2$ the 2-fibre product $\mathcal{T}_1 \times_{x_1,\mathcal{X},x_2} \mathcal{T}_2$ is representable by an algebraic space,
(7) for every category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,
(8) for every pair $\mathcal{T}_1, \mathcal{T}_2$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ which are representable by algebraic spaces, and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$ the 2-fibre product $\mathcal{T}_1 \times_{x_1,\mathcal{X},x_2} \mathcal{T}_2$ is representable by an algebraic space.

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.2.5 and the definitions. Let us prove the equivalence of (1) and (3). Write $\mathcal{C} = (Sch/S)_{fppf}$ for the base category. We will use some of the observations of the proof of the similar Categories, Lemma 4.42.6. We will use the symbol $\cong$ to mean "equivalence of categories fibred in groupoids over $\mathcal{C} = (Sch/S)_{fppf}$". Assume (1). Suppose given U and x as in (3). For any scheme V and $y \in \mathrm{Ob}(\mathcal{X}_V)$ we see (compare reference above) that

$$\mathcal{C}/U \times_{x,\mathcal{X},y} \mathcal{C}/V \cong (\mathcal{C}/U \times_S V) \times_{(x,y),\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$$

which is representable by an algebraic space by assumption. Conversely, assume (3). Consider any scheme U over S and a pair (x, x') of objects of $\mathcal{X}$ over U. We have to show that $\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},(x,x')} U$ is representable by an algebraic space. This is clear because (compare reference above)

$$\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},(x,x')} \mathcal{C}/U \cong (\mathcal{C}/U \times_{x,\mathcal{X},x'} \mathcal{C}/U) \times_{\mathcal{C}/U\times_S U,\Delta} \mathcal{C}/U$$

and the right hand side is representable by an algebraic space by assumption and the fact that the category of algebraic spaces over S has fibre products and contains U and S.

The equivalences (3) ⇔ (4), (5) ⇔ (6), and (7) ⇔ (8) are formal. The equivalences (3) ⇔ (5) and (4) ⇔ (6) follow from Lemma 94.9.3. Assume (3), and let $\mathcal{U} \to \mathcal{X}$ be as in (7). To prove (7) we have to show that for every scheme V and 1-morphism $y : (Sch/V)_{fppf} \to \mathcal{X}$ the 2-fibre product $(Sch/V)_{fppf} \times_{y,\mathcal{X}} \mathcal{U}$ is representable by an algebraic space. Property (3) tells us that y is representable by algebraic spaces hence Lemma 94.9.8 implies what we want. Finally, (7) directly implies (3). □

In the situation of the lemma, for any 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ as in the lemma, it makes sense to say that x has property $\mathcal{P}$, for any property as in Definition 94.10.1. In particular this holds for $\mathcal{P}$ = "surjective", $\mathcal{P}$ = "smooth", and $\mathcal{P}$ = "étale", see Descent on Spaces, Lemmas 74.11.6, 74.11.26, and 74.11.28. We will use these three cases in the definitions of algebraic stacks below.

94.11. Stacks in groupoids

0304 Let S be a scheme contained in Sch_{fppf}. Recall that a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ over $(Sch/S)_{fppf}$ is said to be a stack in groupoids (see Stacks, Definition 8.5.1) if and only if

1. $p : \mathcal{X} \to (Sch/S)_{fppf}$ is fibred in groupoids over $(Sch/S)_{fppf}$,
2. for all $U \in \mathrm{Ob}((Sch/S)_{fppf})$, for all $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $(Sch/U)_{fppf}$, and
3. for all coverings $\mathcal{U} = \{U_i \to U\}$ in $(Sch/S)_{fppf}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

For examples see Examples of Stacks, Section 95.9 ff.

94.12. Algebraic stacks

026N Here is the definition of an algebraic stack. We remark that condition (2) implies we can make sense out of the condition in part (3) that $(Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective, see discussion following Lemma 94.10.11.

026O Definition 94.12.1. Let S be a base scheme contained in Sch_{fppf}. An algebraic stack over S is a category

$$p : \mathcal{X} \to (Sch/S)_{fppf}$$

over $(Sch/S)_{fppf}$ with the following properties:

1. The category $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$.
2. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.
3. There exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$ which is surjective and smooth[2].

There are some differences with other definitions found in the literature.

The first is that we require $\mathcal{X}$ to be a stack in groupoids in the fppf topology, whereas in many references the étale topology is used. It somehow seems to us that the fppf topology is the natural topology to work with. In the end the resulting 2-category of algebraic stacks ends up being the same. This is explained in Criteria for Representability, Section 97.19.

The second is that we only require the diagonal map of $\mathcal{X}$ to be representable by algebraic spaces, whereas in most references some other conditions are imposed. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of $\mathcal{X}$ be representable by algebraic spaces, and simply add an additional hypothesis wherever this is necessary. It has the added benefit

[2] In future chapters we will denote this simply $U \to \mathcal{X}$ as is customary in the literature. Another good alternative would be to formulate this condition as the existence of a representable category fibred in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$.

that any algebraic space (as defined in Spaces, Definition 65.6.1) gives rise to an algebraic stack.

The third is that in some papers it is required that there exists a scheme U and a surjective and étale morphism $U \to \mathcal{X}$. In the groundbreaking paper [DM69] where algebraic stacks were first introduced Deligne and Mumford used this definition and showed that the moduli stack of stable genus $g > 1$ curves is an algebraic stack which has an étale covering by a scheme. Michael Artin, see [Art74], realized that many natural results on algebraic stacks generalize to the case where one only assume a smooth covering by a scheme. Hence our choice above. To distinguish the two cases one sees the terms "Deligne-Mumford stack" and "Artin stack" used in the literature. We will reserve the term "Artin stack" for later use (insert future reference here), and continue to use "algebraic stack", but we will use "Deligne-Mumford stack" to indicate those algebraic stacks which have an étale covering by a scheme.

03YO Definition 94.12.2. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. We say $\mathcal{X}$ is a Deligne-Mumford stack if there exists a scheme U and a surjective étale morphism $(Sch/U)_{fppf} \to \mathcal{X}$.

We will compare our notion of a Deligne-Mumford stack with the notion as defined in the paper by Deligne and Mumford later (see insert future reference here).

The category of algebraic stacks over S forms a 2-category. Here is the precise definition.

03YP Definition 94.12.3. Let S be a scheme contained in Sch_{fppf}. The 2-category of algebraic stacks over S is the sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ (see Categories, Definition 4.35.6) defined as follows:

(1) Its objects are those categories fibred in groupoids over $(Sch/S)_{fppf}$ which are algebraic stacks over S.
(2) Its 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ are any functors of categories over $(Sch/S)_{fppf}$, as in Categories, Definition 4.32.1.
(3) Its 2-morphisms are transformations between functors over $(Sch/S)_{fppf}$, as in Categories, Definition 4.32.1.

In other words this 2-category is the full sub 2-category of $\text{Cat}/(Sch/S)_{fppf}$ whose objects are algebraic stacks. Note that every 2-morphism is automatically an isomorphism. Hence this is actually a $(2, 1)$-category and not just a 2-category.

We will see later (insert future reference here) that this 2-category has 2-fibre products.

Similar to the remark above the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$. It turns out that it is closed under equivalences. Here is the precise statement.

03YQ Lemma 94.12.4. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are equivalent as categories over $(Sch/S)_{fppf}$. Then $\mathcal{X}$ is an algebraic stack if and only if $\mathcal{Y}$ is an algebraic stack. Similarly, $\mathcal{X}$ is a Deligne-Mumford stack if and only if $\mathcal{Y}$ is a Deligne-Mumford stack.

Proof. Assume $\mathcal{X}$ is an algebraic stack (resp. a Deligne-Mumford stack). By Stacks, Lemma 8.5.4 this implies that $\mathcal{Y}$ is a stack in groupoids over Sch_{fppf}. Choose an equivalence $f : \mathcal{X} \to \mathcal{Y}$ over Sch_{fppf}. This gives a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\Delta_\mathcal{X}"] & \mathcal{Y} \arrow[d, "\Delta_\mathcal{Y}"] \\ \mathcal{X} \times \mathcal{X} \arrow[r, "f \times f"] & \mathcal{Y} \times \mathcal{Y} \end{tikzcd}$$

whose horizontal arrows are equivalences. This implies that $\Delta_\mathcal{Y}$ is representable by algebraic spaces according to Lemma 94.9.3. Finally, let U be a scheme over S, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a 1-morphism which is surjective and smooth (resp. étale). Considering the diagram

$$\begin{tikzcd} (Sch/U)_{fppf} \arrow[r, "\mathrm{id}"] \arrow[d, "x"] & (Sch/U)_{fppf} \arrow[d, "f \circ x"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

and applying Lemma 94.10.2 we conclude that $f \circ x$ is surjective and smooth (resp. étale) as desired. □

94.13. Algebraic stacks and algebraic spaces

03YR In this section we discuss some simple criteria which imply that an algebraic stack is an algebraic space. The main result is that this happens exactly when objects of fibre categories have no nontrivial automorphisms. This is not a triviality! Before we come to this we first do a sanity check.

03YS **Lemma 94.13.1.** Let S be a scheme contained in Sch_{fppf}.

(1) A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which is representable by an algebraic space is a Deligne-Mumford stack.
(2) If F is an algebraic space over S, then the associated category fibred in groupoids $p : \mathcal{S}_F \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.
(3) If $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $(Sch/X)_{fppf} \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.

Proof. It is clear that (2) implies (3). Parts (1) and (2) are equivalent by Lemma 94.12.4. Hence it suffices to prove (2). First, we note that $\mathcal{S}_F$ is stack in sets since F is a sheaf (Stacks, Lemma 8.6.3). A fortiori it is a stack in groupoids. Second the diagonal morphism $\mathcal{S}_F \to \mathcal{S}_F \times \mathcal{S}_F$ is the same as the morphism $\mathcal{S}_F \to \mathcal{S}_{F \times F}$ which comes from the diagonal of F. Hence this is representable by algebraic spaces according to Lemma 94.9.4. Actually it is even representable (by schemes), as the diagonal of an algebraic space is representable, but we do not need this. Let U be a scheme and let $h_U \to F$ be a surjective étale morphism. We may think of this as a surjective étale morphism of algebraic spaces. Hence by Lemma 94.10.3 the corresponding 1-morphism $(Sch/U)_{fppf} \to \mathcal{S}_F$ is surjective and étale. □

The following result says that a Deligne-Mumford stack whose inertia is trivial "is" an algebraic space. This lemma will be obsoleted by the stronger Proposition 94.13.3 below which says that this holds more generally for algebraic stacks...

045H **Lemma 94.13.2.** Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent

(1) $\mathcal{X}$ is a Deligne-Mumford stack and is a stack in setoids,

(2) $\mathcal{X}$ is a Deligne-Mumford stack such that the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and
(3) $\mathcal{X}$ is representable by an algebraic space.

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 94.13.1. Finally, assume (1). By Stacks, Lemma 8.6.3 there exists a sheaf F on $(Sch/S)_{fppf}$ and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$. By Lemma 94.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective étale morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 80.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 94.10.4 we conclude that $h_U \to F$ is surjective and étale. Finally, we apply Bootstrap, Theorem 80.6.1 to see that F is an algebraic space. □

04SZ Proposition 94.13.3. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent

(1) $\mathcal{X}$ is a stack in setoids,
(2) the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and
(3) $\mathcal{X}$ is representable by an algebraic space.

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 94.13.2. Finally, assume (1). By Stacks, Lemma 8.6.3 there exists an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ where F is a sheaf on $(Sch/S)_{fppf}$. By Lemma 94.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective smooth morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 80.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 94.10.4 we conclude that $h_U \to F$ is surjective and smooth. In particular it is surjective, flat and locally of finite presentation (by Lemma 94.10.9 and the fact that a smooth morphism of algebraic spaces is flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 67.37.5 and 67.37.7). Finally, we apply Bootstrap, Theorem 80.10.1 to see that F is an algebraic space. □

94.14. 2-Fibre products of algebraic stacks

04TD The 2-category of algebraic stacks has products and 2-fibre products. The first lemma is really a special case of Lemma 94.14.3 but its proof is slightly easier.

04TE Lemma 94.14.1. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Then $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is an algebraic stack, and is a product in the 2-category of algebraic stacks over S.

Proof. An object of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T is just a pair (x, y) where x is an object of $\mathcal{X}_T$ and y is an object of $\mathcal{Y}_T$. Hence it is immediate from the definitions that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is a stack in groupoids. If (x, y) and (x', y') are two objects of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T, then

$$Isom((x, y), (x', y')) = Isom(x, x') \times Isom(y, y').$$

Hence it follows from the equivalences in Lemma 94.10.11 and the fact that the category of algebraic spaces has products that the diagonal of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is representable by algebraic spaces. Finally, suppose that $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/V)_{fppf} \to \mathcal{Y}$. Note that

$$(Sch/U \times_S V)_{fppf} = (Sch/U)_{fppf} \times_{(Sch/S)_{fppf}} (Sch/V)_{fppf}.$$

The object $(\mathrm{pr}_U^*x, \mathrm{pr}_V^*y)$ of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over $(Sch/U \times_S V)_{fppf}$ thus defines a 1-morphism

$$(Sch/U \times_S V)_{fppf} \longrightarrow \mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$$

which is the composition of base changes of x and y, hence is surjective and smooth, see Lemmas 94.10.6 and 94.10.5. We conclude that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is indeed an algebraic stack. We omit the verification that it really is a product. $\square$

04TF Lemma 94.14.2. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{Z}$ be a stack in groupoids over $(Sch/S)_{fppf}$ whose diagonal is representable by algebraic spaces. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack.

Proof. We have to check conditions (1), (2), and (3) of Definition 94.12.1. The first condition follows from Stacks, Lemma 8.5.6.

The second condition we have to check is that the *Isom*-sheaves are representable by algebraic spaces. To do this, suppose that T is a scheme over S, and u, v are objects of $(\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y})_T$. By our construction of 2-fibre products (which goes all the way back to Categories, Lemma 4.32.3) we may write $u = (x, y, \alpha)$ and $v = (x', y', \alpha')$. Here $\alpha : f(x) \to g(y)$ and similarly for α'. Then it is clear that

$$\begin{tikzcd} Isom(u,v) \arrow[r] \arrow[d] & Isom(y,y') \arrow[d, "\phi \mapsto g(\phi)\circ\alpha"] \\ Isom(x,x') \arrow[r, "\psi \mapsto \alpha' \circ f(\psi)"] & Isom(f(x), g(y')) \end{tikzcd}$$

is a cartesian diagram of sheaves on $(Sch/T)_{fppf}$. Since by assumption the sheaves $Isom(y, y')$, $Isom(x, x')$, $Isom(f(x), g(y'))$ are algebraic spaces (see Lemma 94.10.11) we see that $Isom(u, v)$ is an algebraic space.

Let $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/V)_{fppf} \to \mathcal{Y}$. Consider the morphism

$$(Sch/U)_{fppf} \times_{f\circ x,\mathcal{Z},g\circ y} (Sch/V)_{fppf} \longrightarrow \mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}.$$

As the diagonal of $\mathcal{Z}$ is representable by algebraic spaces the source of this arrow is representable by an algebraic space F, see Lemma 94.10.11. Moreover, the morphism is the composition of base changes of x and y, hence surjective and smooth, see Lemmas 94.10.6 and 94.10.5. Choosing a scheme W and a surjective étale morphism $W \to F$ we see that the composition of the displayed 1-morphism with the corresponding 1-morphism

$$(Sch/W)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{f\circ x,\mathcal{Z},g\circ y} (Sch/V)_{fppf}$$

is surjective and smooth which proves the last condition. $\square$

04T2 Lemma 94.14.3. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of algebraic stacks. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack. It is also the 2-fibre product in the 2-category of algebraic stacks over $(Sch/S)_{fppf}$.

Proof. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack follows from the stronger Lemma 94.14.2. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is a 2-fibre product in the 2-category of algebraic stacks over S follows formally from the fact that the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of stacks in groupoids over $(Sch/S)_{fppf}$. □

94.15. Algebraic stacks, overhauled

04T0 Some basic results on algebraic stacks.

04T1 Lemma 94.15.1. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. Let $V \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $y : (Sch/V)_{fppf} \to \mathcal{Y}$ be surjective and smooth. Then there exists an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 2-commutative diagram

$$\begin{tikzcd} (Sch/U)_{fppf} \arrow[r, "a"] \arrow[d, "x"] & (Sch/V)_{fppf} \arrow[d, "y"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with x surjective and smooth.

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective smooth 1-morphism $z : (Sch/W)_{fppf} \to \mathcal{X}$. As $\mathcal{Y}$ is an algebraic stack we may choose an equivalence

$$j : \mathcal{S}_F \longrightarrow (Sch/W)_{fppf} \times_{f\circ z,\mathcal{Y},y} (Sch/V)_{fppf}$$

where F is an algebraic space. By Lemma 94.10.6 the morphism $\mathcal{S}_F \to (Sch/W)_{fppf}$ is surjective and smooth as a base change of y. Hence by Lemma 94.10.5 we see that $\mathcal{S}_F \to \mathcal{X}$ is surjective and smooth. Choose an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$. Then applying Lemma 94.10.5 once more we obtain the desired properties. □

This lemma is a generalization of Proposition 94.13.3.

04Y5 Lemma 94.15.2. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. The following are equivalent:

(1) for $U \in \mathrm{Ob}((Sch/S)_{fppf})$ the functor $f : \mathcal{X}_U \to \mathcal{Y}_U$ is faithful,
(2) the functor f is faithful, and
(3) f is representable by algebraic spaces.

Proof. Parts (1) and (2) are equivalent by general properties of 1-morphisms of categories fibred in groupoids, see Categories, Lemma 4.35.9. We see that (3) implies (2) by Lemma 94.9.2. Finally, assume (2). Let U be a scheme. Let $y \in \mathrm{Ob}(\mathcal{Y}_U)$. We have to prove that

$$\mathcal{W} = (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

is representable by an algebraic space over U. Since $(Sch/U)_{fppf}$ is an algebraic stack we see from Lemma 94.14.3 that $\mathcal{W}$ is an algebraic stack. On the other hand the explicit description of objects of $\mathcal{W}$ as triples $(V, x, \alpha : y(V) \to f(x))$ and the fact that f is faithful, shows that the fibre categories of $\mathcal{W}$ are setoids. Hence Proposition 94.13.3 guarantees that $\mathcal{W}$ is representable by an algebraic space. $\square$

05UL Lemma 94.15.3. Let S be a scheme contained in Sch_{fppf}. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If

(1) $\mathcal{U}$ is representable by an algebraic space, and
(2) u is representable by algebraic spaces, surjective and smooth,

then $\mathcal{X}$ is an algebraic stack over S.

Proof. We have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, see Definition 94.12.1. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Lemma 94.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 8.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 8.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

$$\begin{tikzcd}
& \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 \arrow[rr] \arrow[dd] & & \mathcal{T}_1 \arrow[dd] \\
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{T}_1' \arrow[ur] \arrow[dd] & \\
& \mathcal{T}_2 \arrow[rr] & & \mathcal{X} \\
\mathcal{T}_2' \arrow[ur] \arrow[rr] & & \mathcal{U} \arrow[ur] &
\end{tikzcd}$$

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{tikzcd}
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\
\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 94.9.6, 94.9.7, 94.10.4, and 94.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 80.10.1 and we win. $\square$

An application of Lemma 94.15.3 is that something which is an algebraic space over an algebraic stack is an algebraic stack. This is the analogue of Bootstrap, Lemma 80.3.6. Actually, it suffices to assume the morphism $\mathcal{X} \to \mathcal{Y}$ is "algebraic", as we will see in Criteria for Representability, Lemma 97.8.2.

05UM **Lemma 94.15.4.** Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that

(1) $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, and
(2) $\mathcal{Y}$ is an algebraic stack over S.

Then $\mathcal{X}$ is an algebraic stack over S.

Proof. Let $\mathcal{V} \to \mathcal{Y}$ be a surjective smooth 1-morphism from a representable stack in groupoids to $\mathcal{Y}$. This exists by Definition 94.12.1. Then the 2-fibre product $\mathcal{U} = \mathcal{V} \times_{\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space by Lemma 94.9.8. The 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 94.9.7 and 94.10.6. By Lemma 94.15.3 we conclude that $\mathcal{X}$ is an algebraic stack. □

05UN **Lemma 94.15.5.** Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces. Then, if $\mathcal{Y}$ is a stack in groupoids (resp. an algebraic stack), so is $\mathcal{X}$.

Removing the hypothesis that j is a monomorphism was observed in an email from Matthew Emerton dates June 15, 2016

Proof. The statement on algebraic stacks will follow from the statement on stacks in groupoids by Lemma 94.15.4. If j is representable by algebraic spaces, then j is faithful on fibre categories and for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf

$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$

is an algebraic space over U. See Lemma 94.9.2. In particular this presheaf is a sheaf and the conclusion follows from Stacks, Lemma 8.6.11. □

94.16. From an algebraic stack to a presentation

04T3 Given an algebraic stack over S we obtain a groupoid in algebraic spaces over S whose associated quotient stack is the algebraic stack.

Recall that if (U, R, s, t, c) is a groupoid in algebraic spaces over S then $[U/R]$ denotes the quotient stack associated to this datum, see Groupoids in Spaces, Definition 78.20.1. In general $[U/R]$ is not an algebraic stack. In particular the stack $[U/R]$ occurring in the following lemma is in general not algebraic.

04T4 **Lemma 94.16.1.** Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{U}$ be an algebraic stack over S which is representable by an algebraic space. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism. Then

(1) the 2-fibre product $\mathcal{R} = \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U}$ is representable by an algebraic space,
(2) there is a canonical equivalence
$$\mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} = \mathcal{R} \times_{\mathrm{pr}_1,\mathcal{U},\mathrm{pr}_0} \mathcal{R},$$
(3) the projection pr_{02} induces via (2) a 1-morphism
$$\mathrm{pr}_{02} : \mathcal{R} \times_{\mathrm{pr}_1,\mathcal{U},\mathrm{pr}_0} \mathcal{R} \longrightarrow \mathcal{R}$$
(4) let U, R be the algebraic spaces representing $\mathcal{U}, \mathcal{R}$ and $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ are the morphisms corresponding to the 1-morphisms $\mathrm{pr}_0, \mathrm{pr}_1 : \mathcal{R} \to \mathcal{U}$ and $\mathrm{pr}_{02} : \mathcal{R} \times_{\mathrm{pr}_1,\mathcal{U},\mathrm{pr}_0} \mathcal{R} \to \mathcal{R}$ above, then the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over S,
(5) the morphism f induces a canonical 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ of stacks in groupoids over $(Sch/S)_{fppf}$, and

(6) the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful.

Proof. Proof of (1). By definition $\Delta_\mathcal{X}$ is representable by algebraic spaces so Lemma 94.10.11 applies to show that $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence the result follows from Lemma 94.9.8.

Let T be a scheme over S. By construction of the 2-fibre product (see Categories, Lemma 4.32.3) we see that the objects of the fibre category $\mathcal{R}_T$ are triples (a, b, α) where $a, b \in \mathrm{Ob}(\mathcal{U}_T)$ and $\alpha : f(a) \to f(b)$ is a morphism in the fibre category $\mathcal{X}_T$.

Proof of (2). The equivalence comes from repeatedly applying Categories, Lemmas 4.31.8 and 4.31.10. Let us identify $\mathcal{U} \times_\mathcal{X} \mathcal{U} \times_\mathcal{X} \mathcal{U}$ with $(\mathcal{U} \times_\mathcal{X} \mathcal{U}) \times_\mathcal{X} \mathcal{U}$. If T is a scheme over S, then on fibre categories over T this equivalence maps the object $((a, b, \alpha), c, \beta)$ on the left hand side to the object $((a, b, \alpha), (b, c, \beta))$ of the right hand side.

Proof of (3). The 1-morphism pr_{02} is constructed in the proof of Categories, Lemma 4.31.9. In terms of the description of objects of the fibre category above we see that $((a, b, \alpha), (b, c, \beta))$ maps to $(a, c, \beta \circ \alpha)$.

Unfortunately, this is not compatible with our conventions on groupoids where we always have $j = (t, s) : R \to U$, and we "think" of a T-valued point r of R as a morphism $r : s(r) \to t(r)$. However, this does not affect the proof of (4), since the opposite of a groupoid is a groupoid. But in the proof of (5) it is responsible for the inverses in the displayed formula below.

Proof of (4). Recall that the sheaf U is isomorphic to the sheaf $T \mapsto \mathrm{Ob}(\mathcal{U}_T)/\cong$, and similarly for R, see Lemma 94.8.2. It follows from Categories, Lemma 4.39.8 that this description is compatible with 2-fibre products so we get a similar matching of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ and $R \times_{s, U, t} R$. The morphisms $t, s : R \to U$ and $c : R \times_{s, U, t} R \to R$ we get from the general equality (94.8.2.1). Explicitly these maps are the transformations of functors that come from letting $\mathrm{pr}_0, \mathrm{pr}_0, \mathrm{pr}_{02}$ act on isomorphism classes of objects of fibre categories. Hence to show that we obtain a groupoid in algebraic spaces it suffices to show that for every scheme T over S the structure

$$(\mathrm{Ob}(\mathcal{U}_T)/\cong, \mathrm{Ob}(\mathcal{R}_T)/\cong, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

is a groupoid which is clear from our description of objects of $\mathcal{R}_T$ above.

Proof of (5). We will eventually apply Groupoids in Spaces, Lemma 78.23.2 to obtain the functor $[U/R] \to \mathcal{X}$. Consider the 1-morphism $f : \mathcal{U} \to \mathcal{X}$. We have a 2-arrow $\tau : f \circ \mathrm{pr}_1 \to f \circ \mathrm{pr}_0$ by definition of $\mathcal{R}$ as the 2-fibre product. Namely, on an object (a, b, α) of $\mathcal{R}$ over T it is the map $\alpha^{-1} : b \to a$. We claim that

$$\tau \circ \mathrm{id}_{\mathrm{pr}_{02}} = (\tau \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\tau \star \mathrm{id}_{\mathrm{pr}_1}).$$

This identity says that given an object $((a, b, \alpha), (b, c, \beta))$ of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ over T, then the composition of

$$c \xrightarrow{\beta^{-1}} b \xrightarrow{\alpha^{-1}} a$$

is the same as the arrow $(\beta \circ \alpha)^{-1} : a \to c$. This is clearly true, hence the claim holds. In this way we see that all the assumption of Groupoids in Spaces, Lemma 78.23.2 are satisfied for the structure $(\mathcal{U}, \mathcal{R}, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ and the 1-morphism f and the 2-morphism τ. Except, to apply the lemma we need to prove this holds for the structure $(\mathcal{S}_U, \mathcal{S}_R, s, t, c)$ with suitable morphisms.

Now there should be some general abstract nonsense argument which transfer these data between the two, but it seems to be quite long. Instead, we use the following trick. Pick a quasi-inverse $j^{-1} : \mathcal{S}_U \to \mathcal{U}$ of the canonical equivalence $j : \mathcal{U} \to \mathcal{S}_U$ which comes from $U(T) = \mathrm{Ob}(\mathcal{U}_T)/\cong$. This just means that for every scheme T/S and every object $a \in \mathcal{U}_T$ we have picked out a particular element of its isomorphism class, namely $j^{-1}(j(a))$. Using j^{-1} we may therefore see $\mathcal{S}_U$ as a subcategory of $\mathcal{U}$. Having chosen this subcategory we can consider those objects (a, b, α) of $\mathcal{R}_T$ such that a, b are objects of $(\mathcal{S}_U)_T$, i.e., such that $j^{-1}(j(a)) = a$ and $j^{-1}(j(b)) = b$. Then it is clear that this forms a subcategory of $\mathcal{R}$ which maps isomorphically to $\mathcal{S}_R$ via the canonical equivalence $\mathcal{R} \to \mathcal{S}_R$. Moreover, this is clearly compatible with forming the 2-fibre product $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$. Hence we see that we may simply restrict f to $\mathcal{S}_U$ and restrict τ to a transformation between functors $\mathcal{S}_R \to \mathcal{X}$. Hence it is clear that the displayed equality of Groupoids in Spaces, Lemma 78.23.2 holds since it holds even as an equality of transformations of functors $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R} \to \mathcal{X}$ before restricting to the subcategory $\mathcal{S}_{R \times_{s, U, t} R}$.

This proves that Groupoids in Spaces, Lemma 78.23.2 applies and we get our desired morphism of stacks $f_{can} : [U/R] \to \mathcal{X}$. We briefly spell out how f_{can} is defined in this special case. On an object a of $\mathcal{S}_U$ over T we have $f_{can}(a) = f(a)$, where we think of $\mathcal{S}_U \subset \mathcal{U}$ by the chosen embedding above. If a, b are objects of $\mathcal{S}_U$ over T, then a morphism $\varphi : a \to b$ in $[U/R]$ is by definition an object of the form $\varphi = (b, a, \alpha)$ of $\mathcal{R}$ over T. (Note switch.) And the rule in the proof of Groupoids in Spaces, Lemma 78.23.2 is that

04TG (94.16.1.1)
$$f_{can}(\varphi) = \left(f(a) \xrightarrow{\alpha^{-1}} f(b) \right).$$

Proof of (6). Both $[U/R]$ and $\mathcal{X}$ are stacks. Hence given a scheme T/S and objects a, b of $[U/R]$ over T we obtain a transformation of fppf sheaves

$$Isom(a, b) \longrightarrow Isom(f_{can}(a), f_{can}(b))$$

on $(Sch/T)_{fppf}$. We have to show that this is an isomorphism. We may work fppf locally on T, hence we may assume that a, b come from morphisms $a, b : T \to U$. By the embedding $\mathcal{S}_U \subset \mathcal{U}$ above we may also think of a, b as objects of $\mathcal{U}$ over T. In Groupoids in Spaces, Lemma 78.22.1 we have seen that the left hand sheaf is represented by the algebraic space

$$R \times_{(t,s), U \times_S U, (b,a)} T$$

over T. On the other hand, the right hand side is by Stacks, Lemma 8.2.5 equal to the sheaf associated to the following stack in setoids:

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f \circ b, f \circ a)} T = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f,f)} (\mathcal{U} \times \mathcal{U}) \times_{\mathcal{U} \times \mathcal{U}, (b,a)} T = \mathcal{R} \times_{(\mathrm{pr}_0, \mathrm{pr}_1), \mathcal{U} \times \mathcal{U}, (b,a)} T$$

which is representable by the fibre product displayed above. At this point we have shown that the two *Isom*-sheaves are isomorphic. Our 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ induces this isomorphism on *Isom*-sheaves by Equation (94.16.1.1). $\square$

We can use the previous very abstract lemma to produce presentations.

04T5 **Lemma 94.16.2.** Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let U be an algebraic space over S. Let $f : \mathcal{S}_U \to \mathcal{X}$ be a surjective smooth morphism. Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ be the result of applying Lemma 94.16.1 to U and f. Then

(1) the morphisms s, t are smooth, and
(2) the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence.

Proof. The morphisms s, t are smooth by Lemmas 94.10.2 and 94.10.3. As the 1-morphism f is smooth and surjective it is clear that given any scheme T and any object $a \in \mathrm{Ob}(\mathcal{X}_T)$ there exists a smooth and surjective morphism $T' \to T$ such that $a|'_T$ comes from an object of $[U/R]_{T'}$. Since $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful, we deduce that $[U/R] \to \mathcal{X}$ is essentially surjective as descent data on objects are effective on both sides, see Stacks, Lemma 8.4.8. □

04WY Remark 94.16.3. If the morphism $f : \mathcal{S}_U \to \mathcal{X}$ of Lemma 94.16.2 is only assumed surjective, flat and locally of finite presentation, then it will still be the case that $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence. In this case the morphisms s, t will be flat and locally of finite presentation, but of course not smooth in general.

Lemma 94.16.2 suggests the following definitions.

04TH Definition 94.16.4. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. We say (U, R, s, t, c) is a smooth groupoid[3] if $s, t : R \to U$ are smooth morphisms of algebraic spaces.

04TI Definition 94.16.5. Let $\mathcal{X}$ be an algebraic stack over S. A presentation of $\mathcal{X}$ is given by a smooth groupoid (U, R, s, t, c) in algebraic spaces over S, and an equivalence $f : [U/R] \to \mathcal{X}$.

We have seen above that every algebraic stack has a presentation. Our next task is to show that every smooth groupoid in algebraic spaces over S gives rise to an algebraic stack.

94.17. The algebraic stack associated to a smooth groupoid

04TJ In this section we start with a smooth groupoid in algebraic spaces and we show that the associated quotient stack is an algebraic stack.

04WZ Lemma 94.17.1. Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Then the diagonal of $[U/R]$ is representable by algebraic spaces.

Proof. It suffices to show that the *Isom*-sheaves are algebraic spaces, see Lemma 94.10.11. This follows from Bootstrap, Lemma 80.11.5. □

04X0 Lemma 94.17.2. Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the morphism $\mathcal{S}_U \to [U/R]$ is smooth and surjective.

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Lemmas 94.17.1 and 94.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective and smooth.

[3] This terminology might be a bit confusing: it does not imply that $[U/R]$ is smooth over anything.

Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.31.10 and the second equality by Groupoids in Spaces, Lemma 78.22.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is smooth as the base change of the smooth morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the main result of this section.

04TK Theorem 94.17.3. Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the quotient stack $[U/R]$ is an algebraic stack over S.

Proof. We check the three conditions of Definition 94.12.1. By construction we have that $[U/R]$ is a stack in groupoids which is the first condition.

The second condition follows from the stronger Lemma 94.17.1.

Finally, we have to show there exists a scheme W over S and a surjective smooth 1-morphism $(Sch/W)_{fppf} \longrightarrow \mathcal{X}$. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to U$. Note that this gives a surjective étale morphism $\mathcal{S}_W \to \mathcal{S}_U$ of categories fibred in sets, see Lemma 94.10.3. Of course then $\mathcal{S}_W \to \mathcal{S}_U$ is also surjective and smooth, see Lemma 94.10.9. Hence $\mathcal{S}_W \to \mathcal{S}_U \to [U/R]$ is surjective and smooth by a combination of Lemmas 94.17.2 and 94.10.5. □

94.18. Change of big site

04X1 In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. We encourage the reader to skip this section.

Pullbacks of stacks is defined in Stacks, Section 8.12.

04X2 Lemma 94.18.1. Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 34.12. Let S be an object of Sch_{fppf}. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : (Sch/S)_{fppf} \to (Sch'/S)_{fppf}$. Let $\mathcal{X}$ be a stack in groupoids over $(Sch/S)_{fppf}$.

(1) if $\mathcal{X}$ is representable by some $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $f^{-1}\mathcal{X}$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$,
(2) if $\mathcal{X}$ is representable by $F \in Sh((Sch/S)_{fppf})$ which is an algebraic space, then $f^{-1}\mathcal{X}$ is representable by the algebraic space $f^{-1}F$,

(3) if $\mathcal{X}$ is an algebraic stack, then $f^{-1}\mathcal{X}$ is an algebraic stack, and
(4) if $\mathcal{X}$ is a Deligne-Mumford stack, then $f^{-1}\mathcal{X}$ is a Deligne-Mumford stack too.

Proof. Let us prove (3). By Lemma 94.16.2 we may write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c). By Groupoids in Spaces, Lemma 78.28.1 we see that $f^{-1}[U/R] = [f^{-1}U/f^{-1}R]$. Of course $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a smooth groupoid in algebraic spaces too. Hence (3) is proved.

Now the other cases (1), (2), (4) each mean that $\mathcal{X}$ has a presentation $[U/R]$ of a particular kind, and hence translate into the same kind of presentation for $f^{-1}\mathcal{X} = [f^{-1}U/f^{-1}R]$. Whence the lemma is proved. □

It is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

04X3 Lemma 94.18.2. Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote Algebraic-Stacks$/S$ the 2-category of algebraic stacks over S defined using Sch_{fppf}. Similarly, denote Algebraic-Stacks$'/S$ the 2-category of algebraic stacks over S defined using Sch'_{fppf}. The rule $\mathcal{X} \mapsto f^{-1}\mathcal{X}$ of Lemma 94.18.1 defines a functor of 2-categories

$$\text{Algebraic-Stacks}/S \longrightarrow \text{Algebraic-Stacks}'/S$$

which defines equivalences of morphism categories

$$\mathrm{Mor}_{\text{Algebraic-Stacks}/S}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Mor}_{\text{Algebraic-Stacks}'/S}(f^{-1}\mathcal{X}, f^{-1}\mathcal{Y})$$

for every objects $\mathcal{X}, \mathcal{Y}$ of Algebraic-Stacks$/S$. An object $\mathcal{X}'$ of Algebraic-Stacks$'/S$ is equivalence to $f^{-1}\mathcal{X}$ for some $\mathcal{X}$ in Algebraic-Stacks$/S$ if and only if it has a presentation $\mathcal{X} = [U'/R']$ with U', R' isomorphic to $f^{-1}U$, $f^{-1}R$ for some $U, R \in$ Spaces$/S$.

Proof. The statement on morphism categories is a consequence of the more general Stacks, Lemma 8.12.12. The characterization of the "essential image" follows from the description of f^{-1} in the proof of Lemma 94.18.1. □

94.19. Change of base scheme

04X4 In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic stack over S can be viewed as an algebraic stack over S'.

04X5 Lemma 94.19.1. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The constructions A and B of Stacks, Section 8.13 above give isomorphisms of 2-categories

$$\left\{\begin{matrix}\text{2-category of algebraic} \\ \text{stacks } \mathcal{X} \text{ over } S\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{2-category of pairs } (\mathcal{X}', f) \text{ consisting of an} \\ \text{algebraic stack } \mathcal{X}' \text{ over } S' \text{ and a morphism} \\ f : \mathcal{X}' \to (Sch/S)_{fppf} \text{ of algebraic stacks over } S'\end{matrix}\right\}$$

Proof. The statement makes sense as the functor $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ is the localization functor associated to the object S/S' of $(Sch/S')_{fppf}$. By Stacks, Lemma 8.13.2 the only thing to show is that the constructions A and B preserve the subcategories of algebraic stacks. For example, if $\mathcal{X} = [U/R]$ then construction A applied to $\mathcal{X}$ just produces $\mathcal{X}' = \mathcal{X}$. Conversely, if $\mathcal{X}' = [U'/R']$ the morphism p induces morphisms of algebraic spaces $U' \to S$ and $R' \to S$, and then $\mathcal{X} = [U'/R']$ but now viewed as a stack over S. Hence the lemma is clear. □

04X6 Definition 94.19.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. If $p : \mathcal{X} \to (Sch/S)_{fppf}$ is an algebraic stack over S, then $\mathcal{X}$ viewed as an algebraic stack over S' is the algebraic stack

$$\mathcal{X} \longrightarrow (Sch/S')_{fppf}$$

gotten by applying construction A of Lemma 94.19.1 to $\mathcal{X}$.

Conversely, what if we start with an algebraic stack $\mathcal{X}'$ over S' and we want to get an algebraic stack over S? Well, then we consider the 2-fibre product

$$\mathcal{X}'_S = (Sch/S)_{fppf} \times_{(Sch/S')_{fppf}} \mathcal{X}'$$

which is an algebraic stack over S' according to Lemma 94.14.3. Moreover, it comes equipped with a natural 1-morphism $p : \mathcal{X}'_S \to (Sch/S)_{fppf}$ and hence by Lemma 94.19.1 it corresponds in a canonical way to an algebraic stack over S.

04X7 Definition 94.19.3. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let $\mathcal{X}'$ be an algebraic stack over S'. The change of base of $\mathcal{X}'$ is the algebraic stack $\mathcal{X}'_S$ over S described above.

94.20. Other chapters

CHAPTER 95

Examples of Stacks

04SL 95.1. Introduction

04SM This is a discussion of examples of stacks in algebraic geometry. Some of them are algebraic stacks, some are not. We will discuss which are algebraic stacks in a later chapter. This means that in this chapter we mainly worry about the descent conditions. See [Vis04] for example.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

95.2. Notation

04SN In this chapter we fix a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

95.3. Examples of stacks

04SQ We first give some important examples of stacks over $(Sch/S)_{fppf}$.

95.4. Quasi-coherent sheaves

03YL We define a category $\mathcal{QC}oh$ as follows:

(1) An object of $\mathcal{QC}oh$ is a pair $(X, \mathcal{F})$, where X/S is an object of $(Sch/S)_{fppf}$, and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and
(2) a morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ is a pair consisting of a morphism $f : Y \to X$ of schemes over S and an f-map (see Sheaves, Section 6.26) $\varphi : \mathcal{F} \to \mathcal{G}$.
(3) The composition of morphisms

$$(Z, \mathcal{H}) \xrightarrow{(g,\psi)} (Y, \mathcal{G}) \xrightarrow{(f,\phi)} (X, \mathcal{F})$$

is $(f \circ g, \psi \circ \phi)$ where $\psi \circ \phi$ is the composition of f-maps.

Thus $\mathcal{QC}oh$ is a category and

$$p : \mathcal{QC}oh \to (Sch/S)_{fppf}, \quad (X, \mathcal{F}) \mapsto X$$

is a functor. Note that the fibre category of $\mathcal{QC}oh$ over a scheme X is the opposite of the category $QCoh(\mathcal{O}_X)$ of quasi-coherent $\mathcal{O}_X$-modules. We remark for later use that given $(X, \mathcal{F}), (Y, \mathcal{G}) \in \mathrm{Ob}(\mathcal{QC}oh)$ we have

04U2 (95.4.0.1)
$$\mathrm{Mor}_{\mathcal{QC}oh}((Y, \mathcal{G}), (X, \mathcal{F})) = \coprod\nolimits_{f \in \mathrm{Mor}_S(Y, X)} \mathrm{Mor}_{QCoh(\mathcal{O}_Y)}(f^*\mathcal{F}, \mathcal{G})$$

See the discussion on f-maps of modules in Sheaves, Section 6.26.

The category $\mathcal{QC}oh$ is not a stack over $(Sch/S)_{fppf}$ because its collection of objects is a proper class. On the other hand we will see that it does satisfy all the axioms of a stack. We will get around the set theoretical issue in Section 95.5.

04U3 Lemma 95.4.1. A morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of $\mathcal{QC}oh$ is strongly cartesian if and only if the map φ induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$.

Proof. Let $(X, \mathcal{F}) \in \mathrm{Ob}(\mathcal{QC}oh)$. Let $f : Y \to X$ be a morphism of $(Sch/S)_{fppf}$. Note that there is a canonical f-map $c : \mathcal{F} \to f^*\mathcal{F}$ and hence we get a morphism $(f, c) : (Y, f^*\mathcal{F}) \to (X, \mathcal{F})$. We claim that (f, c) is strongly cartesian. Namely, for any object $(Z, \mathcal{H})$ of $\mathcal{QC}oh$ we have

$$\begin{aligned} \mathrm{Mor}_{\mathcal{QC}oh}((Z, \mathcal{H}), (Y, f^*\mathcal{F})) &= \coprod\nolimits_{g \in \mathrm{Mor}_S(Z, Y)} \mathrm{Mor}_{QCoh(\mathcal{O}_Z)}(g^*f^*\mathcal{F}, \mathcal{H}) \\ &= \coprod\nolimits_{g \in \mathrm{Mor}_S(Z, Y)} \mathrm{Mor}_{QCoh(\mathcal{O}_Z)}((f \circ g)^*\mathcal{F}, \mathcal{H}) \\ &= \mathrm{Mor}_{\mathcal{QC}oh}((Z, \mathcal{H}), (X, \mathcal{F})) \times_{\mathrm{Mor}_S(Z, X)} \mathrm{Mor}_S(Z, Y) \end{aligned}$$

where we have used Equation (95.4.0.1) twice. This proves that the condition of Categories, Definition 4.33.1 holds for (f, c), and hence our claim is true. Now by Categories, Lemma 4.33.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the “if” part of the lemma. For the converse, note that given $(X, \mathcal{F})$ and $f : Y \to X$, if there exists a strongly cartesian morphism lifting f with target $(X, \mathcal{F})$ then it has to be isomorphic to (f, c) (see discussion following Categories, Definition 4.33.1). Hence the "only if" part of the lemma holds. □

03YM Lemma 95.4.2. The functor $p : \mathcal{QC}oh \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 8.4.1.

Proof. It is clear from Lemma 95.4.1 that $\mathcal{QC}oh$ is a fibred category over $(Sch/S)_{fppf}$. Given covering $\mathcal{U} = \{X_i \to X\}_{i \in I}$ of $(Sch/S)_{fppf}$ the functor

$$QCoh(\mathcal{O}_X) \longrightarrow DD(\mathcal{U})$$

is fully faithful and essentially surjective, see Descent, Proposition 35.5.2. Hence Stacks, Lemma 8.4.2 applies to show that $\mathcal{QC}oh$ satisfies all the axioms of a stack. □

95.5. The stack of finitely generated quasi-coherent sheaves

0404 It turns out that we can get a stack of quasi-coherent sheaves if we only consider finite type quasi-coherent modules. Let us denote

$$p_{fg} : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$$

the full subcategory of $\mathcal{QC}oh$ over $(Sch/S)_{fppf}$ consisting of pairs $(T, \mathcal{F})$ such that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_T$-module of finite type.

04U4 Lemma 95.5.1. The functor $p_{fg} : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 8.4.1.

Proof. We will verify assumptions (1), (2), (3) of Stacks, Lemma 8.4.3 to prove this. By Lemma 95.4.1 a morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ is strongly cartesian if and only if it induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$. By Modules, Lemma 17.9.2 the pullback of a finite type $\mathcal{O}_X$-module is of finite type. Hence assumption (1) of Stacks, Lemma 8.4.3 holds. Assumption (2) holds trivially. Finally, to prove assumption (3) we have to show: If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\{f_i : X_i \to X\}$ is an fppf covering such that each $f_i^*\mathcal{F}$ is of finite type, then $\mathcal{F}$ is of finite type. Considering the restriction of $\mathcal{F}$ to an affine open of X this reduces to the following algebra statement: Suppose that $R \to S$ is a finitely presented, faithfully flat ring map and M an R-module. If $M \otimes_R S$ is a finitely generated S-module, then M is a finitely generated R-module. A stronger form of the algebra fact can be found in Algebra, Lemma 10.83.2. □

04U5 Lemma 95.5.2. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) The category of finite type $\mathcal{O}_X$-modules has a set of isomorphism classes.
(2) The category of finite type quasi-coherent $\mathcal{O}_X$-modules has a set of isomorphism classes.

Proof. Part (2) follows from part (1) as the category in (2) is a full subcategory of the category in (1). Consider any open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Denote $j_i : U_i \to X$ the inclusion maps. Consider any map $r : I \to \mathbf{N}$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module whose restriction to U_i is generated by at most $r(i)$ sections from $\mathcal{F}(U_i)$, then $\mathcal{F}$ is a quotient of the sheaf

$$\mathcal{H}_{\mathcal{U},r} = \bigoplus\nolimits_{i \in I} j_{i,!}\mathcal{O}_{U_i}^{\oplus r(i)}$$

By definition, if $\mathcal{F}$ is of finite type, then there exists some open covering with $\mathcal{U}$ whose index set is $I = X$ such that this condition is true. Hence it suffices to show that there is a set of possible choices for $\mathcal{U}$ (obvious), a set of possible choices for $r : I \to \mathbf{N}$ (obvious), and a set of possible quotient modules of $\mathcal{H}_{\mathcal{U},r}$ for each $\mathcal{U}$ and r. In other words, it suffices to show that given an $\mathcal{O}_X$-module $\mathcal{H}$ there is at most a set of isomorphism classes of quotients. This last assertion becomes obvious by thinking of the kernels of a quotient map $\mathcal{H} \to \mathcal{F}$ as being parametrized by a subset of the power set of $\prod_{U \subset X \text{ open}} \mathcal{H}(U)$. □

04U6 Lemma 95.5.3. There exists a subcategory $\mathcal{QC}oh_{fg,small} \subset \mathcal{QC}oh_{fg}$ with the following properties:

(1) the inclusion functor $\mathcal{QC}oh_{fg,small} \to \mathcal{QC}oh_{fg}$ is fully faithful and essentially surjective, and
(2) the functor $p_{fg,small} : \mathcal{QC}oh_{fg,small} \to (Sch/S)_{fppf}$ turns $\mathcal{QC}oh_{fg,small}$ into a stack over $(Sch/S)_{fppf}$.

Proof. We have seen in Lemmas 95.5.1 and 95.5.2 that $p_{fg} : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 8.4.1 as well as the additional condition (4) of Stacks, Remark 8.4.9. Hence we obtain $\mathcal{QC}oh_{fg,small}$ from the discussion in that remark. □

We will often perform the replacement

$$\mathcal{QC}oh_{fg} \leadsto \mathcal{QC}oh_{fg,small}$$

without further remarking on it, and by abuse of notation we will simply denote $\mathcal{QC}oh_{fg}$ this replacement.

04U7 Remark 95.5.4. Note that the whole discussion in this section works if we want to consider those quasi-coherent sheaves which are locally generated by at most κ sections, for some infinite cardinal κ, e.g., $\kappa = \aleph_0$.

95.6. Finite étale covers

0BLY We define a category FÉt as follows:

(1) An object of FÉt is a finite étale morphism $Y \to X$ of schemes (by our conventions this means a finite étale morphism in $(Sch/S)_{fppf}$),
(2) A morphism $(b, a) : (Y \to X) \to (Y' \to X')$ of FÉt is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "b"'] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "a"'] & X' \end{tikzcd}$$

in the category of schemes.

Thus FÉt is a category and

$$p : \text{FÉt} \to (Sch/S)_{fppf}, \quad (Y \to X) \mapsto X$$

is a functor. Note that the fibre category of FÉt over a scheme X is just the category FÉt_X studied in Fundamental Groups, Section 58.5.

0BLZ Lemma 95.6.1. The functor

$$p : \text{FÉt} \longrightarrow (Sch/S)_{fppf}$$

defines a stack over $(Sch/S)_{fppf}$.

Proof. Fppf descent for finite étale morphisms follows from Descent, Lemmas 35.37.1, 35.23.23, and 35.23.29. Details omitted. $\square$

95.7. Algebraic spaces

04SP We define a category $\mathcal{S}paces$ as follows:

(1) An object of $\mathcal{S}paces$ is a morphism $X \to U$ of algebraic spaces over S, where U is representable by an object of $(Sch/S)_{fppf}$, and
(2) a morphism $(f, g) : (X \to U) \to (Y \to V)$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ U \arrow[r, "g"] & V \end{tikzcd}$$

of morphisms of algebraic spaces over S.

Thus $\mathcal{S}paces$ is a category and

$$p : \mathcal{S}paces \to (Sch/S)_{fppf}, \quad (X \to U) \mapsto U$$

is a functor. Note that the fibre category of $\mathcal{S}paces$ over a scheme U is just the category Spaces/U of algebraic spaces over U (see Topologies on Spaces, Section 73.2). Hence we sometimes think of an object of $\mathcal{S}paces$ as a pair X/U consisting of a scheme U and an algebraic space X over U. We remark for later use that given $(X/U), (Y/V) \in \mathrm{Ob}(\mathcal{S}paces)$ we have

04U8 (95.7.0.1) $$\mathrm{Mor}_{\mathcal{S}paces}(X/U, Y/V) = \coprod\nolimits_{g \in \mathrm{Mor}_S(U,V)} \mathrm{Mor}_{\mathrm{Spaces}/U}(X, U \times_{g,V} Y)$$

The category $\mathcal{S}paces$ is almost, but not quite a stack over $(Sch/S)_{fppf}$. The problem is a set theoretical issue as we will explain below.

04U9 **Lemma 95.7.1.** A morphism $(f, g) : X/U \to Y/V$ of $\mathcal{S}paces$ is strongly cartesian if and only if the map f induces an isomorphism $X \to U \times_{g,V} Y$.

Proof. Let $Y/V \in \mathrm{Ob}(\mathcal{S}paces)$. Let $g : U \to V$ be a morphism of $(Sch/S)_{fppf}$. Note that the projection $p : U \times_{g,V} Y \to Y$ gives rise a morphism $(p, g) : U \times_{g,V} Y/U \to Y/V$ of $\mathcal{S}paces$. We claim that (p, g) is strongly cartesian. Namely, for any object Z/W of $\mathcal{S}paces$ we have

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{S}paces}(Z/W, U \times_{g,V} Y/U) &= \coprod\nolimits_{h \in \mathrm{Mor}_S(W,U)} \mathrm{Mor}_{\mathrm{Spaces}/W}(Z, W \times_{h,U} U \times_{g,V} Y) \\
&= \coprod\nolimits_{h \in \mathrm{Mor}_S(W,U)} \mathrm{Mor}_{\mathrm{Spaces}/W}(Z, W \times_{g\circ h,V} Y) \\
&= \mathrm{Mor}_{\mathcal{S}paces}(Z/W, Y/V) \times_{\mathrm{Mor}_S(W,V)} \mathrm{Mor}_S(W, U)
\end{aligned}$$

where we have used Equation (95.7.0.1) twice. This proves that the condition of Categories, Definition 4.33.1 holds for (p, g), and hence our claim is true. Now by Categories, Lemma 4.33.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the "if" part of the lemma. For the converse, note that given Y/V and $g : U \to V$, if there exists a strongly cartesian morphism lifting g with target Y/V then it has to be isomorphic to (p, g) (see discussion following Categories, Definition 4.33.1). Hence the "only if" part of the lemma holds. $\square$

04UA **Lemma 95.7.2.** The functor $p : \mathcal{S}paces \to (Sch/S)_{fppf}$ satisfies conditions (1) and (2) of Stacks, Definition 8.4.1.

Proof. It is follows from Lemma 95.7.1 that $\mathcal{S}paces$ is a fibred category over $(Sch/S)_{fppf}$ which proves (1). Suppose that $\{U_i \to U\}_{i \in I}$ is a covering of $(Sch/S)_{fppf}$. Suppose that X, Y are algebraic spaces over U. Finally, suppose that $\varphi_i : X_{U_i} \to Y_{U_i}$ are morphisms of Spaces/U_i such that φ_i and φ_j restrict to the same morphisms $X_{U_i \times_U U_j} \to Y_{U_i \times_U U_j}$ of algebraic spaces over $U_i \times_U U_j$. To prove (2) we have to show that there exists a unique morphism $\varphi : X \to Y$ over U whose base change to U_i is equal to φ_i. As a morphism from X to Y is the same thing as a map of sheaves this follows directly from Sites, Lemma 7.26.1. $\square$

04UB **Remark 95.7.3.** Ignoring set theoretical difficulties[1] $\mathcal{S}paces$ also satisfies descent for objects and hence is a stack. Namely, we have to show that given

(1) an fppf covering $\{U_i \to U\}_{i \in I}$,
(2) for each $i \in I$ an algebraic space X_i/U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

[1]The difficulty is not that $\mathcal{S}paces$ is a proper class, since by our definition of an algebraic space over S there is only a set worth of isomorphism classes of algebraic spaces over S. It is rather that arbitrary disjoint unions of algebraic spaces may end up being too large, hence lie outside of our chosen "partial universe" of sets.

there exists an algebraic space X/U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. First, note that by Sites, Lemma 7.26.4 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij}. Then by Bootstrap, Lemma 80.11.1 we see that X is an algebraic space (if we ignore the set theoretic condition of that lemma). We will use this argument in the next section to show that if we consider only algebraic spaces of finite type, then we obtain a stack.

95.8. The stack of finite type algebraic spaces

04UC It turns out that we can get a stack of spaces if we only consider spaces of finite type. Let us denote

$$p_{ft} : \mathit{Spaces}_{ft} \to (Sch/S)_{fppf}$$

the full subcategory of Spaces over $(Sch/S)_{fppf}$ consisting of pairs X/U such that $X \to U$ is a morphism of finite type.

04UD Lemma 95.8.1. The functor $p_{ft} : \mathit{Spaces}_{ft} \to (Sch/S)_{fppf}$ satisfies the conditions (1), (2) and (3) of Stacks, Definition 8.4.1.

Proof. We are going to write this out in ridiculous detail (which may make it hard to see what is going on).

We have seen in Lemma 95.7.1 that a morphism $(f, g) : X/U \to Y/V$ of Spaces is strongly cartesian if the induced morphism $f : X \to U \times_V Y$ is an isomorphism. Note that if $Y \to V$ is of finite type then also $U \times_V Y \to U$ is of finite type, see Morphisms of Spaces, Lemma 67.23.3. So if $(f, g) : X/U \to Y/V$ of Spaces is strongly cartesian in Spaces and Y/V is an object of Spaces_{ft} then automatically also X/U is an object of Spaces_{ft}, and of course (f, g) is also strongly cartesian in Spaces_{ft}. In this way we conclude that Spaces_{ft} is a fibred category over $(Sch/S)_{fppf}$. This proves (1).

The argument above also shows that the inclusion functor $\mathit{Spaces}_{ft} \to \mathit{Spaces}$ transforms strongly cartesian morphisms into strongly cartesian morphisms. In other words $\mathit{Spaces}_{ft} \to \mathit{Spaces}$ is a 1-morphism of fibred categories over $(Sch/S)_{fppf}$.

Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Let X, Y be algebraic spaces of finite type over U. By Stacks, Lemma 8.2.3 we obtain a map of presheaves

$$\mathit{Mor}_{\mathit{Spaces}_{ft}}(X, Y) \longrightarrow \mathit{Mor}_{\mathit{Spaces}}(X, Y)$$

which is an isomorphism as Spaces_{ft} is a full subcategory of Spaces. Hence the left hand side is a sheaf, because in Lemma 95.7.2 we showed the right hand side is a sheaf. This proves (2).

To prove condition (3) of Stacks, Definition 8.4.1 we have to show the following: Given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(Sch/S)_{fppf}$,
(2) for each $i \in I$ an algebraic space X_i of finite type over U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. This follows from Bootstrap, Lemma 80.11.3 part (2). By Descent on Spaces, Lemma 74.11.10 we see that $X \to U$ is of finite type which concludes the proof. $\square$

04UE **Lemma 95.8.2.** There exists a subcategory $\mathit{Spaces}_{ft,small} \subset \mathit{Spaces}_{ft}$ with the following properties:

(1) the inclusion functor $\mathit{Spaces}_{ft,small} \to \mathit{Spaces}_{ft}$ is fully faithful and essentially surjective, and
(2) the functor $p_{ft,small} : \mathit{Spaces}_{ft,small} \to (Sch/S)_{fppf}$ turns $\mathit{Spaces}_{ft,small}$ into a stack over $(Sch/S)_{fppf}$.

Proof. We have seen in Lemmas 95.8.1 that $p_{ft} : \mathit{Spaces}_{ft} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 8.4.1. The additional condition (4) of Stacks, Remark 8.4.9 holds because every algebraic space X over S is of the form U/R for $U, R \in \mathrm{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 65.9.1. Thus there is only a set worth of isomorphism classes of objects. Hence we obtain $\mathit{Spaces}_{ft,small}$ from the discussion in that remark. $\square$

We will often perform the replacement

$$\mathit{Spaces}_{ft} \rightsquigarrow \mathit{Spaces}_{ft,small}$$

without further remarking on it, and by abuse of notation we will simply denote Spaces_{ft} this replacement.

04UF **Remark 95.8.3.** Note that the whole discussion in this section works if we want to consider those algebraic spaces X/U which are locally of finite type such that the inverse image in X of an affine open of U can be covered by countably many affines. If needed we can also introduce the notion of a morphism of κ-type (meaning some bound on the number of generators of ring extensions and some bound on the cardinality of the affines over a given affine in the base) where κ is a cardinal, and then we can produce a stack

$$\mathit{Spaces}_{\kappa} \longrightarrow (Sch/S)_{fppf}$$

in exactly the same manner as above (provided we make sure that Sch is large enough depending on κ).

95.9. Examples of stacks in groupoids

04UG The examples above are examples of stacks which are not stacks in groupoids. In the rest of this chapter we give algebraic geometric examples of stacks in groupoids.

95.10. The stack associated to a sheaf

0305 Let $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ be a presheaf. We obtain a category fibred in sets

$$p_F : \mathcal{S}_F \to (Sch/S)_{fppf},$$

see Categories, Example 4.38.5. This is a stack in sets if and only if F is a sheaf, see Stacks, Lemma 8.6.3.

95.11. The stack in groupoids of finitely generated quasi-coherent sheaves

03YN Let $p : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$ be the stack introduced in Section 95.5 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : \mathcal{QC}oh'_{fg} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.35.3, see Stacks, Lemma 8.5.3. In this particular case this simply means $\mathcal{QC}oh'_{fg}$ has the same objects as $\mathcal{QC}oh_{fg}$ but the morphisms are pairs $(f, g) : (U, \mathcal{F}) \to (U', \mathcal{F}')$ where g is an isomorphism $g : f^*\mathcal{F}' \to \mathcal{F}$.

95.12. The stack in groupoids of finite type algebraic spaces

04UH Let $p : \mathcal{S}paces_{ft} \to (Sch/S)_{fppf}$ be the stack introduced in Section 95.8 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : \mathcal{S}paces'_{ft} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.35.3, see Stacks, Lemma 8.5.3. In this particular case this simply means $\mathcal{S}paces'_{ft}$ has the same objects as $\mathcal{S}paces_{ft}$, i.e., finite type morphisms $X \to U$ where X is an algebraic space over S and U is a scheme over S. But the morphisms $(f, g) : X/U \to Y/V$ are now commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ U \arrow[r, "g"] & V \end{tikzcd}$$

which are cartesian.

95.13. Quotient stacks

04UI Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. In this case the quotient stack

$$[U/R] \longrightarrow (Sch/S)_{fppf}$$

is a stack in groupoids by construction, see Groupoids in Spaces, Definition 78.20.1. It is even the case that the *Isom*-sheaves are representable by algebraic spaces, see Bootstrap, Lemma 80.11.5. These quotient stacks are of fundamental importance to the theory of algebraic stacks.

A special case of the construction above is the quotient stack

$$[X/G] \longrightarrow (Sch/S)_{fppf}$$

associated to a datum $(B, G/B, m, X/B, a)$. Here

(1) B is an algebraic space over S,
(2) (G, m) is a group algebraic space over B,
(3) X is an algebraic space over B, and
(4) $a : G \times_B X \to X$ is an action of G on X over B.

Namely, by Groupoids in Spaces, Definition 78.20.1 the stack in groupoids $[X/G]$ is the quotient stack $[X/G \times_B X]$ given above. It behooves us to spell out what the category $[X/G]$ really looks like. We will do this in Section 95.15.

95.14. Classifying torsors

036Z We want to carefuly explain a number of variants of what it could mean to study the stack of torsors for a group algebraic space G or a sheaf of groups $\mathcal{G}$.

04UJ 95.14.1. Torsors for a sheaf of groups. Let $\mathcal{G}$ be a sheaf of groups on $(Sch/S)_{fppf}$. For $U \in \mathrm{Ob}((Sch/S)_{fppf})$ we denote $\mathcal{G}|_U$ the restriction of $\mathcal{G}$ to $(Sch/U)_{fppf}$. We define a category $\mathcal{G}$-Torsors as follows:

(1) An object of $\mathcal{G}$-Torsors is a pair $(U, \mathcal{F})$ where U is an object of $(Sch/S)_{fppf}$ and $\mathcal{F}$ is a $\mathcal{G}|_U$-torsor, see Cohomology on Sites, Definition 21.4.1.
(2) A morphism $(U, \mathcal{F}) \to (V, \mathcal{H})$ is given by a pair (f, α), where $f : U \to V$ is a morphism of schemes over S, and $\alpha : f^{-1}\mathcal{H} \to \mathcal{F}$ is an isomorphism of $\mathcal{G}|_U$-torsors.

Thus $\mathcal{G}$-Torsors is a category and

$$p : \mathcal{G}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}, \quad (U, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}$-Torsors over U is the category of $\mathcal{G}|_U$-torsors which is a groupoid.

04UK Lemma 95.14.2. Up to a replacement as in Stacks, Remark 8.4.9 the functor

$$p : \mathcal{G}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a $\mathcal{G}|_{U_i}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $\mathcal{G}|_{U_i \times_U U_j}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 7.26 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the equivalence of categories in Sites, Lemma 7.26.5 the action maps $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $a : \mathcal{G}|_U \times \mathcal{F} \to \mathcal{F}$. Now we have to show that a is an action and that $\mathcal{F}$ becomes a $\mathcal{G}|_U$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}$-Torsors. Some details omitted. □

04UL 95.14.3. Variant on torsors for a sheaf. The construction of Subsection 95.14.1 can be generalized slightly. Namely, let $\mathcal{G} \to \mathcal{B}$ be a map of sheaves on $(Sch/S)_{fppf}$ and let

$$m : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \longrightarrow \mathcal{G}$$

be a group law on $\mathcal{G}/\mathcal{B}$. In other words, the pair $(\mathcal{G}, m)$ is a group object of the topos $Sh((Sch/S)_{fppf})/\mathcal{B}$. See Sites, Section 7.30 for information regarding localizations of topoi. In this setting we can define a category $\mathcal{G}/\mathcal{B}$-Torsors as follows (where we use the Yoneda embedding to think of schemes as sheaves):

(1) An object of $\mathcal{G}/\mathcal{B}$-Torsors is a triple $(U, b, \mathcal{F})$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to \mathcal{B}$ is a section of $\mathcal{B}$ over U, and
   (c) $\mathcal{F}$ is a $U \times_{b,\mathcal{B}} \mathcal{G}$-torsor over U.
(2) A morphism $(U, b, \mathcal{F}) \to (U', b', \mathcal{F}')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over S such that $b = b' \circ f$, and $g : f^{-1}\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors.

Thus $\mathcal{G}/\mathcal{B}$-Torsors is a category and

$$p : \mathcal{G}/\mathcal{B}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}/\mathcal{B}$-Torsors over U is the disjoint union over $b : U \to \mathcal{B}$ of the categories of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors, hence is a groupoid.

In the special case $\mathcal{B} = S$ we recover the category $\mathcal{G}$-Torsors introduced in Subsection 95.14.1.

04UM Lemma 95.14.4. Up to a replacement as in Stacks, Remark 8.4.9 the functor

$$p : \mathcal{G}/\mathcal{B}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. This proof is a repeat of the proof of Lemma 95.14.2. The reader is encouraged to read that proof first since the notation is less cumbersome. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a pair $(b_i, \mathcal{F}_i)$ consisting of a morphism $b_i : U_i \to \mathcal{B}$ and a $U_i \times_{b_i, \mathcal{B}} \mathcal{G}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ we have $b_i|_{U_i \times_U U_j} = b_j|_{U_i \times_U U_j}$ and we are given an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $(U_i \times_U U_j) \times_{\mathcal{B}} \mathcal{G}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 7.26 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the sheaf axiom for $\mathcal{B}$ the morphisms b_i come from a unique morphism $b : U \to \mathcal{B}$. By the equivalence of categories in Sites, Lemma 7.26.5 the action maps $(U_i \times_{b_i, \mathcal{B}} \mathcal{G}) \times_{U_i} \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $(U \times_{b, \mathcal{B}} \mathcal{G}) \times \mathcal{F} \to \mathcal{F}$. Now we have to show that this is an action and that $\mathcal{F}$ becomes a $U \times_{b, \mathcal{B}} \mathcal{G}$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions on the $\mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}/\mathcal{B}$-Torsors. Some details omitted. □

04UN 95.14.5. Principal homogeneous spaces. Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category G-Principal as follows:

(1) An object of G-Principal is a triple (U, b, X) where
  (a) U is an object of $(Sch/S)_{fppf}$,
  (b) $b : U \to B$ is a morphism over S, and
  (c) X is a principal homogeneous G_U-space over U where $G_U = U \times_{b,B} G$.
  See Groupoids in Spaces, Definition 78.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of principal homogeneous G_U-spaces.

Thus G-Principal is a category and

$$p : G\text{-Principal} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, X) \longmapsto U$$

is a functor. Note that the fibre category of G-Principal over U is the disjoint union over $b : U \to B$ of the categories of principal homogeneous $U \times_{b,B} G$-spaces, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is a principal homogeneous G_U-space over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & X' \arrow[d] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

where g is G-equivariant.

04UP Remark 95.14.6. We conjecture that up to a replacement as in Stacks, Remark 8.4.9 the functor

$$p : G\text{-Principal} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$. This would follow if one could show that given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(Sch/S)_{fppf}$,

(2) an group algebraic space H over U,
(3) for every i a principal homogeneous H_{U_i}-space X_i over U_i, and
(4) H-equivariant isomorphisms $\varphi_{ij} : X_{i,U_i \times_U U_j} \to X_{j,U_i \times_U U_j}$ satisfying the cocycle condition,

there exists a principal homogeneous H-space X over U which recovers (X_i, φ_{ij}). The technique of the proof of Bootstrap, Lemma 80.11.8 reduces this to a set theoretical question, so the reader who ignores set theoretical questions will "know" that the result is true. In `https://math.columbia.edu/~dejong/wordpress/?p=591` there is a suggestion as to how to approach this problem.

04UQ 95.14.7. Variant on principal homogeneous spaces. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory G-Principal-Schemes $\subset$ G-Principal whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is a principal homogeneous G-space over U which is representable, i.e., a scheme.

It is in general not the case that G-Principal-Schemes is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist principal homogeneous spaces which are not schemes, hence descent for objects will not be satisfied in general.

04UR 95.14.8. Torsors in fppf topology. Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category G-Torsors as follows:

(1) An object of G-Torsors is a triple (U, b, X) where
  (a) U is an object of $(Sch/S)_{fppf}$,
  (b) $b : U \to B$ is a morphism, and
  (c) X is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$.
  See Groupoids in Spaces, Definition 78.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of G_U-torsors.

Thus G-Torsors is a category and

$$p : G\text{-Torsors} \longrightarrow (Sch/S)_{fppf}, \quad (U, a, X) \longmapsto U$$

is a functor. Note that the fibre category of G-Torsors over U is the disjoint union over $b : U \to B$ of the categories of fppf $U \times_{b,B} G$-torsors, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is an fppf G_U-torsor over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d] & X' \arrow[d] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

where g is G-equivariant.

04US **Lemma 95.14.9.** *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : G\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects, which is Bootstrap, Lemma 80.11.8. We omit the proof of axioms (1) and (2) of Stacks, Definition 8.5.1. □

04UT Lemma 95.14.10. Let B be an algebraic space over S. Let G be a group algebraic space over B. Denote $\mathcal{G}$, resp. $\mathcal{B}$ the algebraic space G, resp. B seen as a sheaf on $(Sch/S)_{fppf}$. The functor

$$G\text{-Torsors} \longrightarrow \mathcal{G}/\mathcal{B}\text{-Torsors}$$

which associates to a triple (U, b, X) the triple $(U, b, \mathcal{X})$ where $\mathcal{X}$ is X viewed as a sheaf is an equivalence of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We will use the result of Stacks, Lemma 8.4.8 to prove this. The functor is fully faithful since the category of algebraic spaces over S is a full subcategory of the category of sheaves on $(Sch/S)_{fppf}$. Moreover, all objects (on both sides) are locally trivial torsors so condition (2) of the lemma referenced above holds. Hence the functor is an equivalence. □

04UU 95.14.11. Variant on torsors in fppf topology. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory G-Torsors-Schemes $\subset$ G-Torsors whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is an fppf G-torsor over U which is representable, i.e., a scheme.

It is in general not the case that G-Torsors-Schemes is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist fppf G-torsors which are not schemes, hence descent for objects will not be satisfied in general.

95.15. Quotients by group actions

04UV At this point we have introduced enough notation that we can work out in more detail what the stacks $[X/G]$ of Section 95.13 look like.

04WL Situation 95.15.1. Here

1. S is a scheme contained in Sch_{fppf},
2. B is an algebraic space over S,
3. (G, m) is a group algebraic space over B,
4. $\pi : X \to B$ is an algebraic space over B, and
5. $a : G \times_B X \to X$ is an action of G on X over B.

In this situation we construct a category $[[X/G]]$[2] as follows:

1. An object of $[[X/G]]$ consists of a quadruple $(U, b, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S,
   (c) P is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$, and

[2]The notation $[[X/G]]$ with double brackets serves to distinguish this category from the stack $[X/G]$ introduced earlier. In Proposition 95.15.3 we show that the two are canonically equivalent. Afterwards we will use the notation $[X/G]$ to indicate either.

(d) $\varphi : P \to X$ is a G-equivariant morphism fitting into the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\ U \arrow[r, "b"] & B \end{tikzcd}$$

(2) A morphism of $[[X/G]]$ is a pair $(f, g) : (U, b, P, \varphi) \to (U', b', P', \varphi')$ where $f : U \to U'$ is a morphism of schemes over B and $g : P \to P'$ is a G-equivariant morphism over f which induces an isomorphism $P \cong U \times_{f,U'} P'$, and has the property that $\varphi = \varphi' \circ g$. In other words (f, g) fits into the following commutative diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d] \arrow[rrd, "\varphi"] & P' \arrow[d] \arrow[rd, "\varphi'"] & \\ U \arrow[r, "f"] \arrow[rrd, "b"] & U' \arrow[rd, "b'"] & X \arrow[d] \\ & & B \end{tikzcd}$$

Thus $[[X/G]]$ is a category and

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}, \quad (U, b, P, \varphi) \longmapsto U$$

is a functor. Note that the fibre category of $[[X/G]]$ over U is the disjoint union over $b \in \mathrm{Mor}_S(U, B)$ of fppf $U \times_{b,B} G$-torsors P endowed with a G-equivariant morphism to X. Hence the fibre categories of $[[X/G]]$ are groupoids.

Note that the functor

$$[[X/G]] \longrightarrow G\text{-Torsors}, \quad (U, b, P, \varphi) \longmapsto (U, b, P)$$

is a 1-morphism of categories over $(Sch/S)_{fppf}$.

0370 Lemma 95.15.2. Up to a replacement as in Stacks, Remark 8.4.9 the functor

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Suppose that $\{U_i \to U\}_{i \in I}$ is a covering in $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, P_i, \varphi_i)$ be objects of $[[X/G]]$ over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This in particular implies that we get a descent datum on the triples (U_i, b_i, P_i) for the stack in groupoids G-Torsors by applying the functor $[[X/G]] \to G$-Torsors above. We have seen that G-Torsors is a stack in groupoids (Lemma 95.14.9). Hence we may assume that $b_i = b|_{U_i}$ for some morphism $b : U \to B$, and that $P_i = U_i \times_U P$ for some fppf $G_U = U \times_{b,B} G$-torsor P over U. The morphisms φ_i are compatible with the canonical descent datum on the restrictions $U_i \times_U P$ and hence define a morphism $\varphi : P \to X$. (For example you can use Sites, Lemma 7.26.5 or you can use Descent on Spaces, Lemma 74.7.2 to get φ.) This proves descent for objects. We omit the proof of axioms (1) and (2) of Stacks, Definition 8.5.1. $\square$

04WM **Proposition 95.15.3.** In Situation 95.15.1 there exists a canonical equivalence

$$[X/G] \longrightarrow [[X/G]]$$

of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We write this out in detail, to make sure that all the definitions work out in exactly the correct manner. Recall that $[X/G]$ is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$, see Groupoids in Spaces, Definition 78.20.1. This means that $[X/G]$ is the stackification of the category fibred in groupoids $[X/_pG]$ associated to the functor

$$(Sch/S)_{fppf} \longrightarrow \text{Groupoids}, \quad U \longmapsto (X(U), G(U) \times_{B(U)} X(U), s, t, c)$$

where $s(g, x) = x$, $t(g, x) = a(g, x)$, and $c((g, x), (g', x')) = (m(g, g'), x')$. By the construction of Categories, Example 4.37.1 an object of $[X/_pG]$ is a pair (U, x) with $x \in X(U)$ and a morphism $(f, g) : (U, x) \to (U', x')$ of $[X/_pG]$ is given by a morphism of schemes $f : U \to U'$ and an element $g \in G(U)$ such that $a(g, x) = x' \circ f$. Hence we can define a 1-morphism of stacks in groupoids

$$F_p : [X/_pG] \longrightarrow [[X/G]]$$

by the following rules: On objects we set

$$F_p(U, x) = (U, \pi \circ x, G \times_{B, \pi \circ x} U, a \circ (\mathrm{id}_G \times x))$$

This makes sense because the diagram

$$\begin{tikzcd}
G \times_{B, \pi \circ x} U \arrow[r, "\mathrm{id}_G \times x"'] \arrow[d] & G \times_{B, \pi} X \arrow[r, "a"'] & X \arrow[d, "\pi"] \\
U \arrow[rr, "\pi \circ x"] & & B
\end{tikzcd}$$

commutes, and the two horizontal arrows are G-equivariant if we think of the fibre products as trivial G-torsors over U, resp. X. On morphisms $(f, g) : (U, x) \to (U', x')$ we set $F_p(f, g) = (f, R_{g^{-1}})$ where $R_{g^{-1}}$ denotes right translation by the inverse of g. More precisely, the morphism $F_p(f, g) : F_p(U, x) \to F_p(U', x')$ is given by the cartesian diagram

$$\begin{tikzcd}
G \times_{B, \pi \circ x} U \arrow[r, "R_{g^{-1}}"'] \arrow[d] & G \times_{B, \pi \circ x'} U' \arrow[d] \\
U \arrow[r, "f"] & U'
\end{tikzcd}$$

where $R_{g^{-1}}$ on T-valued points is given by

$$R_{g^{-1}}(g', u) = (m(g', i(g(u))), f(u))$$

To see that this works we have to verify that

$$a \circ (\mathrm{id}_G \times x) = a \circ (\mathrm{id}_G \times x') \circ R_{g^{-1}}$$

which is true because the right hand side applied to the T-valued point (g', u) gives the desired equality

$$\begin{aligned}
a((\mathrm{id}_G \times x')(m(g', i(g(u))), f(u))) &= a(m(g', i(g(u))), x'(f(u))) \\
&= a(g', a(i(g(u)), x'(f(u)))) \\
&= a(g', x(u))
\end{aligned}$$

because $a(g, x) = x' \circ f$ and hence $a(i(g), x' \circ f) = x$.

By the universal property of stackification from Stacks, Lemma 8.9.2 we obtain a canonical extension $F : [X/G] \to [[X/G]]$ of the 1-morphism F_p above. We first prove that F is fully faithful. To do this, since both source and target are stacks in groupoids, it suffices to prove that the *Isom*-sheaves are identified under F. Pick a scheme U and objects ξ, ξ' of $[X/G]$ over U. We want to show that

$$F : \mathit{Isom}_{[X/G]}(\xi, \xi') \longrightarrow \mathit{Isom}_{[[X/G]]}(F(\xi), F(\xi'))$$

is an isomorphism of sheaves. To do this it suffices to work locally on U, and hence we may assume that ξ, ξ' come from objects (U, x), (U, x') of $[X/_pG]$ over U; this follows directly from the construction of the stackification, and it is also worked out in detail in Groupoids in Spaces, Section 78.24. Either by directly using the description of morphisms in $[X/_pG]$ above, or using Groupoids in Spaces, Lemma 78.22.1 we see that in this case

$$\mathit{Isom}_{[X/G]}(\xi, \xi') = U \times_{(x,x'), X \times_S X, (s,t)} (G \times_B X)$$

A T-valued point of this fibre product corresponds to a pair (u, g) with $u \in U(T)$, and $g \in G(T)$ such that $a(g, x \circ u) = x' \circ u$. (Note that this implies $\pi \circ x \circ u = \pi \circ x' \circ u$.) On the other hand, a T-valued point of $\mathit{Isom}_{[[X/G]]}(F(\xi), F(\xi'))$ by definition corresponds to a morphism $u : T \to U$ such that $\pi \circ x \circ u = \pi \circ x' \circ u : T \to B$ and an isomorphism

$$R : G \times_{B, \pi \circ x \circ u} T \longrightarrow G \times_{B, \pi \circ x' \circ u} T$$

of trivial G_T-torsors compatible with the given maps to X. Since the torsors are trivial we see that $R = R_{g^{-1}}$ (right multiplication) by some $g \in G(T)$. Compatibility with the maps $a \circ (1_G, x \circ u), a \circ (1_G, x' \circ u) : G \times_B T \to X$ is equivalent to the condition that $a(g, x \circ u) = x' \circ u$. Hence we obtain the desired equality of *Isom*-sheaves.

Now that we know that F is fully faithful we see that Stacks, Lemma 8.4.8 applies. Thus to show that F is an equivalence it suffices to show that objects of $[[X/G]]$ are fppf locally in the essential image of F. This is clear as fppf torsors are fppf locally trivial, and hence we win. ☐

0CQJ Lemma 95.15.4. Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Then the stacks in groupoids

$$[B/G], \quad [[B/G]], \quad G\text{-Torsors}, \quad \mathcal{G}/\mathcal{B}\text{-Torsors}$$

are all canonically equivalent. If $G \to B$ is flat and locally of finite presentation, then these are also equivalent to G-Principal.

Proof. The equivalence G-Torsors $\to$ $\mathcal{G}/\mathcal{B}$-Torsors is given in Lemma 95.14.10. The equivalence $[B/G] \to [[B/G]]$ is given in Proposition 95.15.3. Unwinding the definition of $[[B/G]]$ given in Section 95.15 we see that $[[B//G]] = G$-Torsors.

Finally, assume $G \to B$ is flat and locally of finite presentation. To show that the natural functor G-Torsors $\to$ G-Principal is an equivalence it suffices to show that for a scheme U over B a principal homogeneous G_U-space $X \to U$ is fppf locally trivial. By our definition of principal homogeneous spaces (Groupoids in Spaces, Definition 78.9.3) there exists an fpqc covering $\{U_i \to U\}$ such that $U_i \times_U X \cong G \times_B U_i$ as algebraic spaces over U_i. This implies that $X \to U$ is surjective, flat, and locally of finite presentation, see Descent on Spaces, Lemmas 74.11.6, 74.11.13, and 74.11.10. Choose a scheme W and a surjective étale morphism $W \to X$. Then it follows from what we just said that $\{W \to U\}$ is an fppf covering such that $X_W \to W$ has a section. Hence X is an fppf G_U-torsor. $\square$

0371 Remark 95.15.5. Let S be a scheme. Let G be an abstract group. Let X be an algebraic space over S. Let $G \to \mathrm{Aut}_S(X)$ be a group homomorphism. In this setting we can define $[[X/G]]$ similarly to the above as follows:

1. An object of $[[X/G]]$ consists of a triple $(U, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) P is a sheaf on $(Sch/U)_{fppf}$ which comes with an action of G that turns it into a torsor under the constant sheaf with value G, and
   (c) $\varphi : P \to X$ is a G-equivariant map of sheaves.
2. A morphism $(f, g) : (U, P, \varphi) \to (U', P', \varphi')$ is given by a morphism of schemes $f : T \to T'$ and a G-equivariant isomorphism $g : P \to f^{-1}P'$ such that $\varphi = \varphi' \circ g$.

In exactly the same manner as above we obtain a functor

$$[[X/G]] \longrightarrow (Sch/S)_{fppf}$$

which turns $[[X/G]]$ into a stack in groupoids over $(Sch/S)_{fppf}$. The constant sheaf $\underline{G}$ is (provided the cardinality of G is not too large) representable by G_S on $(Sch/S)_{fppf}$ and this version of $[[X/G]]$ is equivalent to the stack $[[X/G_S]]$ introduced above.

95.16. The Picard stack

0372 In this section we introduce the Picard stack in complete generality. In the chapter on Quot and Hilb we will show that it is an algebraic stack under suitable hypotheses, see Quot, Section 99.10.

Let S be a scheme. Let $\pi : X \to B$ be a morphism of algebraic spaces over S. We define a category $\mathcal{P}ic_{X/B}$ as follows:

1. An object is a triple $(U, b, \mathcal{L})$, where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S, and
   (c) $\mathcal{L}$ is in invertible sheaf on the base change $X_U = U \times_{b,B} X$.
2. A morphism $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ is given by a morphism of schemes $f : U \to U'$ over B and an isomorphism $g : f^*\mathcal{L}' \to \mathcal{L}$.

The composition of $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ with $(f', g') : (U', b', \mathcal{L}') \to (U'', b'', \mathcal{L}'')$ is given by $(f \circ f', g \circ f^*(g'))$. Thus we get a category $\mathcal{P}ic_{X/B}$ and

$$p : \mathcal{P}ic_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{L}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{P}ic_{X/B}$ over U is the disjoint union over $b \in \mathrm{Mor}_S(U, B)$ of the categories of invertible sheaves on $X_U = U \times_{b,B} X$. Hence the fibre categories are groupoids.

04WN Lemma 95.16.1. Up to a replacement as in Stacks, Remark 8.4.9 the functor

$$\mathcal{P}ic_{X/B} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, \mathcal{L}_i)$ be an object of $\mathcal{P}ic_{X/B}$ lying over U_i, and let $\varphi_{ij} : \text{pr}_0^*\xi_i \to \text{pr}_1^*\xi_j$ be a descent datum. This implies in particular that the morphisms b_i are the restrictions of a morphism $b : U \to B$. Write $X_U = U \times_{b,B} X$ and $X_i = U_i \times_{b_i,B} X = U_i \times_U U \times_{b,B} X = U_i \times_U X_U$. Observe that $\mathcal{L}_i$ is an invertible $\mathcal{O}_{X_i}$-module. Note that $\{X_i \to X_U\}$ forms an fppf covering as well. Moreover, the descent datum φ_{ij} translates into a descent datum on the invertible sheaves $\mathcal{L}_i$ relative to the fppf covering $\{X_i \to X_U\}$. Hence by Descent on Spaces, Proposition 74.4.1 we obtain a unique invertible sheaf $\mathcal{L}$ on X_U which recovers $\mathcal{L}_i$ and the descent data over X_i. The triple $(U, b, \mathcal{L})$ is therefore the object of $\mathcal{P}ic_{X/B}$ over U we were looking for. Details omitted. □

95.17. Examples of inertia stacks

0373 Here are some examples of inertia stacks.

0374 **Example 95.17.1.** Let S be a scheme. Let G be a commutative group. Let $X \to S$ be a scheme over S. Let $a : G \times X \to X$ be an action of G on X. For $g \in G$ we denote $g : X \to X$ the corresponding automorphism. In this case the inertia stack of $[X/G]$ (see Remark 95.15.5) is given by

$$I_{[X/G]} = \coprod\nolimits_{g \in G} [X^g/G],$$

where, given an element g of G, the symbol X^g denotes the scheme $X^g = \{x \in X \mid g(x) = x\}$. In a formula X^g is really the fibre product

$$X^g = X \times_{(1,1), X \times_S X, (g,1)} X.$$

Indeed, for any S-scheme T, a T-point on the inertia stack of $[X/G]$ consists of a triple $(P/T, \phi, \alpha)$ consisting of an fppf G-torsor $P \to T$ together with a G-equivariant morphism $\phi : P \to X$, together with an automorphism α of $P \to T$ over T such that $\phi \circ \alpha = \phi$. Since G is a sheaf of commutative groups, α is, locally in the fppf topology over T, given by multiplication by some element g of G. The condition that $\phi \circ \alpha = \phi$ means that ϕ factors through the inclusion of X^g in X, i.e., ϕ is obtained by composing that inclusion with a morphism $P \to X^\gamma$. The above discussion allows us to define a morphism of fibred categories $I_{[X/G]} \to \coprod_{g \in G}[X^g/G]$ given on T-points by the discussion above. We omit showing that this is an equivalence.

0375 **Example 95.17.2.** Let $f : X \to S$ be a morphism of schemes. Assume that for any $T \to S$ the base change $f_T : X_T \to T$ has the property that the map $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. (This implies that f is cohomologically flat in dimension 0 (insert future reference here) but is stronger.) Consider the Picard stack $\mathcal{P}ic_{X/S}$, see Section 95.16. The points of its inertia stack over an S-scheme T consist of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on X_T and α is an automorphism of that line bundle. I.e., we can think of α as an element of $H^0(X_T, \mathcal{O}_{X_T})^\times = H^0(T, \mathcal{O}_T^*)$ by our condition. Note that $H^0(T, \mathcal{O}_T^*) = \mathbf{G}_{m,S}(T)$, see Groupoids, Example 39.5.1. Hence the inertia stack of $\mathcal{P}ic_{X/S}$ is

$$I_{\mathcal{P}ic_{X/S}} = \mathbf{G}_{m,S} \times_S \mathcal{P}ic_{X/S}.$$

as a stack over $(Sch/S)_{fppf}$.

95.18. Finite Hilbert stacks

05WA We formulate this in somewhat greater generality than is perhaps strictly needed. Fix a 1-morphism

$$F : \mathcal{X} \longrightarrow \mathcal{Y}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. For each integer $d \geq 1$ consider a category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ defined as follows:

(1) An object (U, Z, y, x, α) where U, Z are objects of in $(Sch/S)_{fppf}$ and Z is a finite locally free of degree d over U, where $y \in \mathrm{Ob}(\mathcal{Y}_U)$, $x \in \mathrm{Ob}(\mathcal{X}_Z)$ and $\alpha : y|_Z \to F(x)$ is an isomorphism[3].
(2) A morphism $(U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is given by a morphism of schemes $f : U \to U'$, a morphism of schemes $g : Z \to Z'$ which induces an isomorphism $Z \to Z' \times_U U'$, and isomorphisms $b : y \to f^*y'$, $a : x \to g^*x'$ inducing a commutative diagram

$$\begin{tikzcd} y|_Z \arrow[r, "\alpha"'] \arrow[d, "b|_Z"'] & F(x) \arrow[d, "F(a)"] \\ f^*y'|_Z \arrow[r, "\alpha'"] & F(g^*x') \end{tikzcd}$$

It is clear from the definitions that there is a canonical forgetful functor

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow (Sch/S)_{fppf}$$

which assigns to the quintuple (U, Z, y, x, α) the scheme U and to the morphism $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ the morphism $f : U \to U'$.

05WB **Lemma 95.18.1.** *The category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ endowed with the functor p above defines a stack in groupoids over $(Sch/S)_{fppf}$.*

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, Z_i, y_i, x_i, \alpha_i)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ lying over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. First, observe that φ_{ij} induces a descent datum $(Z_i/U_i, \varphi_{ij})$ which is effective by Descent, Lemma 35.37.1 This produces a scheme Z/U which is finite locally free of degree d by Descent, Lemma 35.23.30. From now on we identify Z_i with $Z \times_U U_i$. Next, the objects y_i in the fibre categories $\mathcal{Y}_{U_i}$ descend to an object y in $\mathcal{Y}_U$ because $\mathcal{Y}$ is a

[3]This means the data gives rise, via the 2-Yoneda lemma (Categories, Lemma 4.41.2), to a 2-commutative diagram

$$\begin{tikzcd} (Sch/Z)_{fppf} \arrow[r, "x"'] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ (Sch/U)_{fppf} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. Alternatively, we may picture α as a 2-morphism

$$\begin{tikzcd} (Sch/Z)_{fppf} \arrow[r, bend left, "y\circ(Z\to U)"{name=A}] \arrow[r, bend right, "F\circ x"'{name=B}] & \mathcal{Y}. \arrow[from=A, to=B, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

stack in groupoids. Similarly the objects x_i in the fibre categories $\mathcal{X}_{Z_i}$ descend to an object x in $\mathcal{X}_Z$ because $\mathcal{X}$ is a stack in groupoids. Finally, the given isomorphisms

$$\alpha_i : (y|_Z)_{Z_i} = y_i|_{Z_i} \longrightarrow F(x_i) = F(x|_{Z_i})$$

glue to a morphism $\alpha : y|_Z \to F(x)$ as the $\mathcal{Y}$ is a stack and hence $\mathit{Isom}_{\mathcal{Y}}(y|_Z, F(x))$ is a sheaf. Details omitted. $\square$

05WC Definition 95.18.2. We will denote $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ the degree d finite Hilbert stack of $\mathcal{X}$ over $\mathcal{Y}$ constructed above. If $\mathcal{Y} = S$ we write $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y} = S$ we denote it $\mathcal{H}_d$.

Note that given $F : \mathcal{X} \to \mathcal{Y}$ as above we have the following natural 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$:

05WD (95.18.2.1)
$$\begin{tikzcd} \mathcal{H}_d(\mathcal{X}) \arrow[dr] & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \arrow[l] \arrow[r] \arrow[d] & \mathcal{Y} \\ & \mathcal{H}_d & \end{tikzcd}$$

Each of the arrows is given by a "forgetful functor".

05XV Lemma 95.18.3. The 1-morphism $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful.

Proof. To check that $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful it suffices to prove that it is faithful on fibre categories. Suppose that $\xi = (U, Z, y, x, \alpha)$ and $\xi' = (U, Z', y', x', \alpha')$ are two objects of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over the scheme U. Let $(g, b, a), (g', b', a') : \xi \to \xi'$ be two morphisms in the fibre category of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. The image of these morphisms in $\mathcal{H}_d(\mathcal{X})$ agree if and only if $g = g'$ and $a = a'$. Then the commutative diagram

$$\begin{tikzcd} y|_Z \arrow[r, "\alpha"'] \arrow[d, "b|_Z,\ b'|_Z"'] & F(x) \arrow[d, "F(a)=F(a')"] \\ y'|_Z \arrow[r, "\alpha'"] & F(g^*x') = F((g')^*x') \end{tikzcd}$$

implies that $b|_Z = b'|_Z$. Since $Z \to U$ is finite locally free of degree d we see $\{Z \to U\}$ is an fppf covering, hence $b = b'$. $\square$

95.19. Other chapters

Schemes

Topics in Scheme Theory

Algebraic Spaces

Topics in Geometry

Deformation Theory

Algebraic Stacks

CHAPTER 96

Sheaves on Algebraic Stacks

06TF

96.1. Introduction

06TG There is a myriad of ways to think about sheaves on algebraic stacks. In this chapter we discuss one approach, which is particularly well adapted to our foundations for algebraic stacks. Whenever we introduce a type of sheaves we will indicate the precise relationship with similar notions in the literature. The goal of this chapter is to state those results that are either obviously true or straightforward to prove and leave more intricate constructions till later.

In fact, it turns out that to develop a fully fledged theory of constructible étale sheaves and/or an adequate discussion of derived categories of complexes $\mathcal{O}$-modules whose cohomology sheaves are quasi-coherent takes a significant amount of work, see [Ols07b]. We will return to this in Cohomology of Stacks, Section 103.1.

In the literature and in research papers on sheaves on algebraic stacks the lisse-étale site of an algebraic stack often plays a prominent role. However, it is a problematic beast, because it turns out that a morphism of algebraic stacks does not induce a morphism of lisse-étale topoi. We have therefore made the design decision to avoid any mention of the lisse-étale site as long as possible. Arguments that traditionally use the lisse-étale site will be replaced by an argument using a Čech covering in the site $\mathcal{X}_{smooth}$ defined below.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

96.2. Conventions

06TH The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 94.2. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We record what changes if you change the big fppf site elsewhere (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

96.3. Presheaves

06TI

In this section we define presheaves on categories fibred in groupoids over $(Sch/S)_{fppf}$, but most of the discussion works for categories over any base category. This section also serves to introduce the notation we will use later on.

06TJ Definition 96.3.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A presheaf on $\mathcal{X}$ is a presheaf on the underlying category of $\mathcal{X}$.
(2) A morphism of presheaves on $\mathcal{X}$ is a morphism of presheaves on the underlying category of $\mathcal{X}$.

We denote $\mathrm{PSh}(\mathcal{X})$ the category of presheaves on $\mathcal{X}$.

This defines presheaves of sets. Of course we can also talk about presheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of abelian presheaves, i.e., presheaves of abelian groups, is denoted $\mathrm{PAb}(\mathcal{X})$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Recall that this means just that f is a functor over $(Sch/S)_{fppf}$. The material in Sites, Section 7.19 provides us with a pair of adjoint functors[1]

06TK (96.3.1.1) $$f^p : \mathrm{PSh}(\mathcal{Y}) \longrightarrow \mathrm{PSh}(\mathcal{X}) \quad \text{and} \quad {}_pf : \mathrm{PSh}(\mathcal{X}) \longrightarrow \mathrm{PSh}(\mathcal{Y}).$$

The adjointness is

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F})$$

where $\mathcal{F} \in \mathrm{Ob}(\mathrm{PSh}(\mathcal{X}))$ and $\mathcal{G} \in \mathrm{Ob}(\mathrm{PSh}(\mathcal{Y}))$. We call $f^p\mathcal{G}$ the pullback of $\mathcal{G}$. It follows from the definitions that

$$f^p\mathcal{G}(x) = \mathcal{G}(f(x))$$

for any $x \in \mathrm{Ob}(\mathcal{X})$. The presheaf ${}_pf\mathcal{F}$ is called the pushforward of $\mathcal{F}$. It is described by the formula

$$({}_pf\mathcal{F})(y) = \lim_{f(x)\to y} \mathcal{F}(x).$$

The rest of this section should probably be moved to the chapter on sites and in any case should be skipped on a first reading.

06TL Lemma 96.3.2. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Then $(g \circ f)^p = f^p \circ g^p$ and there is a canonical isomorphism ${}_p(g \circ f) \to {}_pg \circ {}_pf$ compatible with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$.

Proof. Let $\mathcal{H}$ be a presheaf on $\mathcal{Z}$. Then $(g \circ f)^p\mathcal{H} = f^p(g^p\mathcal{H})$ is given by the equalities

$$(g \circ f)^p\mathcal{H}(x) = \mathcal{H}((g \circ f)(x)) = \mathcal{H}(g(f(x))) = f^p(g^p\mathcal{H})(x).$$

We omit the verification that this is compatible with restriction maps.

Next, we define the transformation ${}_p(g\circ f) \to {}_pg\circ{}_pf$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. If z is an object of $\mathcal{Z}$ then we get a category $\mathcal{J}$ of quadruples $(x, f(x) \to y, y, g(y) \to z)$ and a category $\mathcal{I}$ of pairs $(x, g(f(x)) \to z)$. There is a canonical functor $\mathcal{J} \to \mathcal{I}$

[1] These functors will be denoted f^{-1} and f_* after Lemma 96.4.4 has been proved.

sending the object $(x, \alpha : f(x) \to y, y, \beta : g(y) \to z)$ to $(x, \beta \circ f(\alpha) : g(f(x)) \to z)$. This gives the arrow in

$$\begin{aligned}({}_p(g \circ f)\mathcal{F})(z) &= \lim\nolimits_{g(f(x)) \to z} \mathcal{F}(x) \\ &= \lim\nolimits_{\mathcal{I}} \mathcal{F} \\ &\to \lim\nolimits_{\mathcal{J}} \mathcal{F} \\ &= \lim\nolimits_{g(y) \to z} \left(\lim\nolimits_{f(x) \to y} \mathcal{F}(x) \right) \\ &= ({}_pg \circ {}_pf\mathcal{F})(x)\end{aligned}$$

by Categories, Lemma 4.14.9. We omit the verification that this is compatible with restriction maps. An alternative to this direct construction is to define ${}_p(g \circ f) \cong {}_pg \circ {}_pf$ as the unique map compatible with the adjointness properties. This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$ means that given presheaves $\mathcal{H}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathrm{PSh}(\mathcal{X})}(f^pg^p\mathcal{H}, \mathcal{F}) \arrow[r, equal] \arrow[d, equal] & \mathrm{Mor}_{\mathrm{PSh}(\mathcal{Y})}(g^p\mathcal{H}, {}_pf\mathcal{F}) \arrow[r, equal] & \mathrm{Mor}_{\mathrm{PSh}(\mathcal{Y})}(\mathcal{H}, {}_pg_pf\mathcal{F}) \\
\mathrm{Mor}_{\mathrm{PSh}(\mathcal{X})}((g \circ f)^p\mathcal{G}, \mathcal{F}) \arrow[rr, equal] & & \mathrm{Mor}_{\mathrm{PSh}(\mathcal{Y})}(\mathcal{G}, {}_p(g \circ f)\mathcal{F}) \arrow[u]
\end{tikzcd}$$

Proof omitted. □

06TM Lemma 96.3.3. Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $t : f \to g$ be a 2-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assigned to t there are canonical isomorphisms of functors

$$t^p : g^p \longrightarrow f^p \quad \text{and} \quad {}_pt : {}_pf \longrightarrow {}_pg$$

which compatible with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ and with vertical and horizontal composition of 2-morphisms.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then $t^p : g^p\mathcal{G} \to f^p\mathcal{G}$ is given by the family of maps

$$g^p\mathcal{G}(x) = \mathcal{G}(g(x)) \xrightarrow{\mathcal{G}(t_x)} \mathcal{G}(f(x)) = f^p\mathcal{G}(x)$$

parametrized by $x \in \mathrm{Ob}(\mathcal{X})$. This makes sense as $t_x : f(x) \to g(x)$ and $\mathcal{G}$ is a contravariant functor. We omit the verification that this is compatible with restriction mappings.

To define the transformation ${}_pt$ for $y \in \mathrm{Ob}(\mathcal{Y})$ define ${}^f_y\mathcal{I}$, resp. ${}^g_y\mathcal{I}$ to be the category of pairs $(x, \psi : f(x) \to y)$, resp. $(x, \psi : g(x) \to y)$, see Sites, Section 7.19. Note that t defines a functor ${}_yt : {}^g_y\mathcal{I} \to {}^f_y\mathcal{I}$ given by the rule

$$(x, g(x) \to y) \longmapsto (x, f(x) \xrightarrow{t_x} g(x) \to y).$$

Note that for $\mathcal{F}$ a presheaf on $\mathcal{X}$ the composition of ${}_yt$ with $\mathcal{F} : {}^f_y\mathcal{I}^{opp} \to \mathrm{Sets}$, $(x, f(x) \to y) \mapsto \mathcal{F}(x)$ is equal to $\mathcal{F} : {}^g_y\mathcal{I}^{opp} \to \mathrm{Sets}$. Hence by Categories, Lemma 4.14.9 we get for every $y \in \mathrm{Ob}(\mathcal{Y})$ a canonical map

$$({}_pf\mathcal{F})(y) = \lim\nolimits_{{}^f_y\mathcal{I}} \mathcal{F} \longrightarrow \lim\nolimits_{{}^g_y\mathcal{I}} \mathcal{F} = ({}_pg\mathcal{F})(y)$$

We omit the verification that this is compatible with restriction mappings. An alternative to this direct construction is to define ${}_pt$ as the unique map compatible with the adjointness properties of the pairs $(f^p, {}_pf)$ and $(g^p, {}_pg)$ (see below). This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ means that given presheaves $\mathcal{G}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathrm{PSh}(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) \arrow[r, equal] \arrow[d, "- \circ t^p"'] & \mathrm{Mor}_{\mathrm{PSh}(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F}) \arrow[d, "{}_pt \circ -"] \\ \mathrm{Mor}_{\mathrm{PSh}(\mathcal{X})}(g^p\mathcal{G}, \mathcal{F}) \arrow[r, equal] & \mathrm{Mor}_{\mathrm{PSh}(\mathcal{Y})}(\mathcal{G}, {}_pg\mathcal{F}) \end{tikzcd}$$

Proof omitted. Hint: Work through the proof of Sites, Lemma 7.19.2 and observe the compatibility from the explicit description of the horizontal and vertical maps in the diagram.

We omit the verification that this is compatible with vertical and horizontal compositions. Hint: The proof of this for t^p is straightforward and one can conclude that this holds for the ${}_pt$ maps using compatibility with adjointness. $\square$

96.4. Sheaves

06TN We first make an observation that is important and trivial (especially for those readers who do not worry about set theoretical issues).

Consider a big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6 and denote its underlying category Sch_α. Besides being the underlying category of a fppf site, the category Sch_α can also can serve as the underlying category for a big Zariski site, a big étale site, a big smooth site, and a big syntomic site, see Topologies, Remark 34.11.1. We denote these sites Sch_{Zar}, $Sch_{étale}$, Sch_{smooth}, and $Sch_{syntomic}$. In this situation, since we have defined the big Zariski site $(Sch/S)_{Zar}$ of S, the big étale site $(Sch/S)_{étale}$ of S, the big smooth site $(Sch/S)_{smooth}$ of S, the big syntomic site $(Sch/S)_{syntomic}$ of S, and the big fppf site $(Sch/S)_{fppf}$ of S as the localizations (see Sites, Section 7.25) Sch_{Zar}/S, $Sch_{étale}/S$, Sch_{smooth}/S, $Sch_{syntomic}/S$, and Sch_{fppf}/S of these (absolute) big sites we see that all of these have the same underlying category, namely Sch_α/S.

It follows that if we have a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids, then $\mathcal{X}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology, see Stacks, Definition 8.10.2.

06TP Definition 96.4.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) The associated Zariski site, denoted $\mathcal{X}_{Zar}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{Zar}$.
(2) The associated étale site, denoted $\mathcal{X}_{étale}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{étale}$.
(3) The associated smooth site, denoted $\mathcal{X}_{smooth}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{smooth}$.
(4) The associated syntomic site, denoted $\mathcal{X}_{syntomic}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{syntomic}$.
(5) The associated fppf site, denoted $\mathcal{X}_{fppf}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{fppf}$.

This definition makes sense by the discussion above. If $\mathcal{X}$ is an algebraic stack, the literature calls $\mathcal{X}_{fppf}$ (or a site equivalent to it) the big fppf site of $\mathcal{X}$ and similarly for the other ones. We may occasionally use this terminology to distinguish this construction from others.

06TQ Remark 96.4.2. We only use this notation when the symbol $\mathcal{X}$ refers to a category fibred in groupoids, and not a scheme, an algebraic space, etc. In this way we will avoid confusion with the small étale site of a scheme, or algebraic space which is denoted $X_{étale}$ (in which case we use a roman capital instead of a calligraphic one).

Now that we have these topologies defined we can say what it means to have a sheaf on $\mathcal{X}$, i.e., define the corresponding topoi.

06TR Definition 96.4.3. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) We say $\mathcal{F}$ is a Zariski sheaf, or a sheaf for the Zariski topology if $\mathcal{F}$ is a sheaf on the associated Zariski site $\mathcal{X}_{Zar}$.
(2) We say $\mathcal{F}$ is an étale sheaf, or a sheaf for the étale topology if $\mathcal{F}$ is a sheaf on the associated étale site $\mathcal{X}_{étale}$.
(3) We say $\mathcal{F}$ is a smooth sheaf, or a sheaf for the smooth topology if $\mathcal{F}$ is a sheaf on the associated smooth site $\mathcal{X}_{smooth}$.
(4) We say $\mathcal{F}$ is a syntomic sheaf, or a sheaf for the syntomic topology if $\mathcal{F}$ is a sheaf on the associated syntomic site $\mathcal{X}_{syntomic}$.
(5) We say $\mathcal{F}$ is an fppf sheaf, or a sheaf, or a sheaf for the fppf topology if $\mathcal{F}$ is a sheaf on the associated fppf site $\mathcal{X}_{fppf}$.

A morphism of sheaves is just a morphism of presheaves. We denote these categories of sheaves $Sh(\mathcal{X}_{Zar})$, $Sh(\mathcal{X}_{étale})$, $Sh(\mathcal{X}_{smooth})$, $Sh(\mathcal{X}_{syntomic})$, and $Sh(\mathcal{X}_{fppf})$.

Of course we can also talk about sheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of abelian sheaves, i.e., sheaves of abelian groups, is denoted $\mathrm{Ab}(\mathcal{X}_{fppf})$ and similarly for the other topologies. If $\mathcal{X}$ is an algebraic stack, then $Sh(\mathcal{X}_{fppf})$ is equivalent (modulo set theoretical problems) to what in the literature would be termed the category of sheaves on the big fppf site of $\mathcal{X}$. Similar for other topologies. We may occasionally use this terminology to distinguish this construction from others.

Since the topologies are listed in increasing order of strength we have the following strictly full inclusions

$$Sh(\mathcal{X}_{fppf}) \subset Sh(\mathcal{X}_{syntomic}) \subset Sh(\mathcal{X}_{smooth}) \subset Sh(\mathcal{X}_{étale}) \subset Sh(\mathcal{X}_{Zar}) \subset \mathrm{PSh}(\mathcal{X})$$

We sometimes write $Sh(\mathcal{X}_{fppf}) = Sh(\mathcal{X})$ and $\mathrm{Ab}(\mathcal{X}_{fppf}) = \mathrm{Ab}(\mathcal{X})$ in accordance with our terminology that a sheaf on $\mathcal{X}$ is an fppf sheaf on $\mathcal{X}$.

With this setup functoriality of these topoi is straightforward, and moreover, is compatible with the inclusion functors above.

06TS Lemma 96.4.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functors ${}_pf$ and f^p of (96.3.1.1) transform τ sheaves into τ sheaves and define a morphism of topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$.

Proof. This follows immediately from Stacks, Lemma 8.10.3. $\square$

In other words, pushforward and pullback of presheaves as defined in Section 96.3 also produces pushforward and pullback of τ-sheaves. Having said all of the above we see that we can write $f^p = f^{-1}$ and ${}_pf = f_*$ without any possibility of confusion.

06TT Definition 96.4.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We denote

$$f = (f^{-1}, f_*) : Sh(\mathcal{X}_{fppf}) \longrightarrow Sh(\mathcal{Y}_{fppf})$$

the associated morphism of fppf topoi constructed above. Similarly for the associated Zariski, étale, smooth, and syntomic topoi.

As discussed in Sites, Section 7.44 the same formula (on the underlying sheaf of sets) defines pushforward and pullback for sheaves (for one of our topologies) of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc.

96.5. Computing pushforward

06W5 Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $y \in \mathrm{Ob}(\mathcal{Y})$. We can compute $f_*\mathcal{F}(y)$ in the following way. Suppose that y lies over the scheme V and using the 2-Yoneda lemma think of y as a 1-morphism. Consider the projection

$$\mathrm{pr} : (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X}$$

Then we have a canonical identification

06W6 (96.5.0.1)
$$f_*\mathcal{F}(y) = \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

Namely, objects of the 2-fibre product are triples $(h : U \to V, x, f(x) \to h^*y)$. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $\alpha : f(x) \to y$ of $\mathcal{Y}$. Since $f_*\mathcal{F}(y) = \lim_{f(x)\to y} \mathcal{F}(x)$ by definition the equality follows.

As a consequence we have the following "base change" result for pushforwards. This result is trivial and hinges on the fact that we are using "big" sites.

075B Lemma 96.5.1. Let S be a scheme. Let

$$\begin{tikzcd} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of categories fibred in groupoids over S. Then we have a canonical isomorphism

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

functorial in the presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Given an object y' of $\mathcal{Y}'$ over V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

Hence by (96.5.0.1) a bijection $g^{-1}f_*\mathcal{F}(y') \to f'_*(g')^{-1}\mathcal{F}(y')$. We omit the verification that this is compatible with restriction mappings. $\square$

In the case of a representable morphism of categories fibred in groupoids this formula (96.5.0.1) simplifies. We suggest the reader skip the rest of this section.

06W7 Lemma 96.5.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent

(1) f is representable, and
(2) for every $y \in \mathrm{Ob}(\mathcal{Y})$ the functor $\mathcal{X}^{opp} \to \textit{Sets}$, $x \mapsto \mathrm{Mor}_\mathcal{Y}(f(x), y)$ is representable.

Proof. According to the discussion in Algebraic Stacks, Section 94.6 we see that f is representable if and only if for every $y \in \mathrm{Ob}(\mathcal{Y})$ lying over U the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable, i.e., of the form $(Sch/V_y)_{fppf}$ for some scheme V_y over U. Objects in this 2-fibre products are triples $(h : V \to U, x, \alpha : f(x) \to h^*y)$ where α lies over id_V. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $f(x) \to y$. Hence the 2-fibre product is representable by V_y and $f(x_y) \to y$ where x_y is an object of $\mathcal{X}$ over V_y if and only if the functor in (2) is representable by x_y with universal object a map $f(x_y) \to y$. $\square$

Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"'] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

be a 1-morphism of categories fibred in groupoids. Assume f is representable. For every $y \in \mathrm{Ob}(\mathcal{Y})$ we choose an object $u(y) \in \mathrm{Ob}(\mathcal{X})$ representing the functor $x \mapsto \mathrm{Mor}_\mathcal{Y}(f(x), y)$ of Lemma 96.5.2 (this is possible by the axiom of choice). The objects come with canonical morphisms $f(u(y)) \to y$ by construction. For every morphism $\beta : y' \to y$ in $\mathcal{Y}$ we obtain a unique morphism $u(\beta) : u(y') \to u(y)$ in $\mathcal{X}$ such that the diagram

$$\begin{tikzcd} f(u(y')) \arrow[r, "f(u(\beta))"'] \arrow[d] & f(u(y)) \arrow[d] \\ y' \arrow[r] & y \end{tikzcd}$$

commutes. In other words, $u : \mathcal{Y} \to \mathcal{X}$ is a functor. In fact, we can say a little bit more. Namely, suppose that $V' = q(y')$, $V = q(y)$, $U' = p(u(y'))$ and $U = p(u(y))$. Then

$$\begin{tikzcd} U' \arrow[r, "p(u(\beta))"'] \arrow[d] & U \arrow[d] \\ V' \arrow[r, "q(\beta)"] & V \end{tikzcd}$$

is a fibre product square. This is true because $U' \to U$ represents the base change $(Sch/V')_{fppf} \times_{y',\mathcal{Y}} \mathcal{X} \to (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ of $V' \to V$.

06W8 Lemma 96.5.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Then the functor $u : \mathcal{Y}_\tau \to \mathcal{X}_\tau$ is continuous and defines a morphism of sites $\mathcal{X}_\tau \to \mathcal{Y}_\tau$ which induces the same morphism of topoi $Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ as the morphism f

constructed in Lemma 96.4.4. Moreover, $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ for any presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Let $\{y_i \to y\}$ be a τ-covering in $\mathcal{Y}$. By definition this simply means that $\{q(y_i) \to q(y)\}$ is a τ-covering of schemes. By the final remark above the lemma we see that $\{p(u(y_i)) \to p(u(y))\}$ is the base change of the τ-covering $\{q(y_i) \to q(y)\}$ by $p(u(y)) \to q(y)$, hence is itself a τ-covering by the axioms of a site. Hence $\{u(y_i) \to u(y)\}$ is a τ-covering of $\mathcal{X}$. This proves that u is continuous.

Let's use the notation u_p, u_s, u^p, u^s of Sites, Sections 7.5 and 7.13. If we can show the final assertion of the lemma, then we see that $f_* = u^p = u^s$ (by continuity of u seen above) and hence by adjointness $f^{-1} = u_s$ which will prove u_s is exact, hence that u determines a morphism of sites, and the equality will be clear as well. To see that $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ note that by definition

$$f_*\mathcal{F}(y) = ({}_pf\mathcal{F})(y) = \lim\nolimits_{f(x)\to y} \mathcal{F}(x).$$

Since $u(y)$ is a final object in the category the limit is taken over we conclude. $\square$

96.6. The structure sheaf

06TU Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ has a final object, namely, $\text{id} : (Sch/S)_{fppf} \to (Sch/S)_{fppf}$ and p is a 1-morphism from $\mathcal{X}$ to this final object. Hence any presheaf $\mathcal{G}$ on $(Sch/S)_{fppf}$ gives a presheaf $p^{-1}\mathcal{G}$ on $\mathcal{X}$ defined by the rule $p^{-1}\mathcal{G}(x) = \mathcal{G}(p(x))$. Moreover, the discussion in Section 96.4 shows that $p^{-1}\mathcal{G}$ is a τ sheaf whenever $\mathcal{G}$ is a τ-sheaf.

Recall that the site $(Sch/S)_{fppf}$ is a ringed site with structure sheaf $\mathcal{O}$ defined by the rule

$$(Sch/S)^{opp} \longrightarrow \text{Rings}, \quad U/S \longmapsto \Gamma(U, \mathcal{O}_U)$$

see Descent, Definition 35.8.2.

06TV Definition 96.6.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The structure sheaf of $\mathcal{X}$ is the sheaf of rings $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$.

For an object x of $\mathcal{X}$ lying over U we have $\mathcal{O}_\mathcal{X}(x) = \mathcal{O}(U) = \Gamma(U, \mathcal{O}_U)$. Needless to say $\mathcal{O}_\mathcal{X}$ is also a Zariski, étale, smooth, and syntomic sheaf, and hence each of the sites $\mathcal{X}_{Zar}$, $\mathcal{X}_{étale}$, $\mathcal{X}_{smooth}$, $\mathcal{X}_{syntomic}$, and $\mathcal{X}_{fppf}$ is a ringed site. This construction is functorial as well.

06TW Lemma 96.6.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a canonical identification $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ which turns $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ into a morphism of ringed topoi.

Proof. Denote $p : \mathcal{X} \to (Sch/S)_{fppf}$ and $q : \mathcal{Y} \to (Sch/S)_{fppf}$ the structural functors. Then $p = q \circ f$, hence $p^{-1} = f^{-1} \circ q^{-1}$ by Lemma 96.3.2. Since $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$ and $\mathcal{O}_\mathcal{Y} = q^{-1}\mathcal{O}$ the result follows. $\square$

06TX Remark 96.6.3. In the situation of Lemma 96.6.2 the morphism of ringed topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ is flat as is clear from the equality $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. This is a bit counter intuitive, for example because a closed immersion of algebraic stacks is typically not flat (as a morphism of algebraic stacks). However, exactly the same thing happens when taking a closed immersion $i : X \to Y$ of schemes: in this case the associated morphism of big τ-sites $i : (Sch/X)_\tau \to (Sch/Y)_\tau$ also is flat.

96.7. Sheaves of modules

06WA Since we have a structure sheaf we have modules.

06WB **Definition 96.7.1.** Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) A *presheaf of modules* on $\mathcal{X}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules. The category of presheaves of modules is denoted $\text{PMod}(\mathcal{O}_\mathcal{X})$.
(2) We say a presheaf of modules $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-*module*, or more precisely a *sheaf of $\mathcal{O}_\mathcal{X}$-modules* if $\mathcal{F}$ is an fppf sheaf. The category of $\mathcal{O}_\mathcal{X}$-modules is denoted $\text{Mod}(\mathcal{O}_\mathcal{X})$.

These (pre)sheaves of modules occur in the literature as (pre)sheaves of $\mathcal{O}_\mathcal{X}$-modules on the big fppf site of $\mathcal{X}$. We will occasionally use this terminology if we want to distinguish these categories from others. We will also encounter presheaves of modules which are sheaves in the Zariski, étale, smooth, or syntomic topologies (without necessarily being sheaves). If need be these will be denoted $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and similarly for the other topologies.

Next, we address functoriality – first for presheaves of modules. Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

be a 1-morphism of categories fibred in groupoids. The functors f^{-1}, f_* on abelian presheaves extend to functors

(96.7.1.1)
06WD
$$f^{-1} : \text{PMod}(\mathcal{O}_\mathcal{Y}) \longrightarrow \text{PMod}(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : \text{PMod}(\mathcal{O}_\mathcal{X}) \longrightarrow \text{PMod}(\mathcal{O}_\mathcal{Y})$$

This is immediate for f^{-1} because $f^{-1}\mathcal{G}(x) = \mathcal{G}(f(x))$ which is a module over $\mathcal{O}_\mathcal{Y}(f(x)) = \mathcal{O}(q(f(x))) = \mathcal{O}(p(x)) = \mathcal{O}_\mathcal{X}(x)$. Alternatively it follows because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ and because f^{-1} commutes with limits (on presheaves). Since f_* is a right adjoint it commutes with all limits (on presheaves) in particular products. Hence we can extend f_* to a functor on presheaves of modules as in the proof of Modules on Sites, Lemma 18.12.1. We claim that the functors (96.7.1.1) form an adjoint pair of functors:

$$\text{Mor}_{\text{PMod}(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{PMod}(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

As $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ this follows from Modules on Sites, Lemma 18.12.3 by endowing $\mathcal{X}$ and $\mathcal{Y}$ with the chaotic topology.

Next, we discuss functoriality for modules, i.e., for sheaves of modules in the fppf topology. Denote by f also the induced morphism of ringed topoi, see Lemma 96.6.2 (for the fppf topologies right now). Note that the functors f^{-1} and f_* of (96.7.1.1) preserve the subcategories of sheaves of modules, see Lemma 96.4.4. Hence it follows immediately that

06WE (96.7.1.2) $$f^{-1} : \text{Mod}(\mathcal{O}_\mathcal{Y}) \longrightarrow \text{Mod}(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : \text{Mod}(\mathcal{O}_\mathcal{X}) \longrightarrow \text{Mod}(\mathcal{O}_\mathcal{Y})$$

form an adjoint pair of functors:

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

By uniqueness of adjoints we conclude that $f^* = f^{-1}$ where f^* is as defined in Modules on Sites, Section 18.13 for the morphism of ringed topoi f above. Of course we could have seen this directly because $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$ and because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$.

Similarly for sheaves of modules in the Zariski, étale, smooth, syntomic topology.

96.8. Representable categories

076N In this short section we compare our definitions with what happens in case the algebraic stacks in question are representable.

075I Lemma 96.8.1. Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over (Sch/S). Assume $\mathcal{X}$ is representable by a scheme X. For $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ there is a canonical equivalence

$$(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) = ((Sch/X)_\tau, \mathcal{O}_X)$$

of ringed sites.

Proof. This follows by choosing an equivalence $(Sch/X)_\tau \to \mathcal{X}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ and using the functoriality of the construction $\mathcal{X} \leadsto \mathcal{X}_\tau$. $\square$

075J Lemma 96.8.2. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by schemes X, Y. Let $f : X \to Y$ be the morphism of schemes corresponding to f. For $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ the morphism of ringed topoi $f : (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \to (Sh(\mathcal{Y}_\tau), \mathcal{O}_\mathcal{Y})$ agrees with the morphism of ringed topoi $f : (Sh((Sch/X)_\tau), \mathcal{O}_X) \to (Sh((Sch/Y)_\tau), \mathcal{O}_Y)$ via the identifications of Lemma 96.8.1.

Proof. Follows by unwinding the definitions. $\square$

96.9. Restriction

075C A trivial but useful observation is that the localization of a category fibred in groupoids at an object is equivalent to the big site of the scheme it lies over.

06W0 Lemma 96.9.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}_\tau/x \to (Sch/U)_\tau$.

Proof. Special case of Stacks, Lemma 8.10.4. $\square$

We use the lemma above to talk about the pullback and the restriction of a (pre)sheaf to a scheme.

06W1 Definition 96.9.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) The pullback $x^{-1}\mathcal{F}$ of $\mathcal{F}$ is the restriction $\mathcal{F}|_{(\mathcal{X}/x)}$ viewed as a presheaf on $(Sch/U)_{fppf}$ via the equivalence $\mathcal{X}/x \to (Sch/U)_{fppf}$ of Lemma 96.9.1.
(2) The restriction of $\mathcal{F}$ to $U_{\acute{e}tale}$ is $x^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$, abusively written $\mathcal{F}|_{U_{\acute{e}tale}}$.

This notation makes sense because to the object x the 2-Yoneda lemma, see Algebraic Stacks, Section 94.5 associates a 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}/x$ which is quasi-inverse to $p : \mathcal{X}/x \to (Sch/U)_{fppf}$. Hence $x^{-1}\mathcal{F}$ truly is the pullback of $\mathcal{F}$ via this 1-morphism. In particular, by the material above, if $\mathcal{F}$ is a sheaf (or a Zariski, étale, smooth, syntomic sheaf), then $x^{-1}\mathcal{F}$ is a sheaf on $(Sch/U)_{fppf}$ (or on $(Sch/U)_{Zar}$, $(Sch/U)_{\acute{e}tale}$, $(Sch/U)_{smooth}$, $(Sch/U)_{syntomic}$).

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\varphi : x \to y$ be a morphism of $\mathcal{X}$ lying over the morphism of schemes $a : U \to V$. Recall that a induces a morphism of small étale sites $a_{small} : U_{\acute{e}tale} \to V_{\acute{e}tale}$, see Étale Cohomology, Section 59.34. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $\mathcal{F}|_{U_{\acute{e}tale}}$ and $\mathcal{F}|_{V_{\acute{e}tale}}$ be the restrictions of $\mathcal{F}$ via x and y. There is a natural comparison map

06W2 (96.9.2.1)
$$c_\varphi : \mathcal{F}|_{V_{\acute{e}tale}} \longrightarrow a_{small,*}(\mathcal{F}|_{U_{\acute{e}tale}})$$

of presheaves on $U_{\acute{e}tale}$. Namely, if $V' \to V$ is étale, set $U' = V' \times_V U$ and define c_φ on sections over V' via

$$\begin{tikzcd} a_{small,*}(\mathcal{F}|_{U_{\acute{e}tale}})(V') \arrow[r, equal] & \mathcal{F}|_{U_{\acute{e}tale}}(U') \arrow[r, equal] & \mathcal{F}(x') \\ \mathcal{F}|_{V_{\acute{e}tale}}(V') \arrow[u, "c_\varphi"] \arrow[rr, equal] & & \mathcal{F}(y') \arrow[u, "\mathcal{F}(\varphi')"'] \end{tikzcd}$$

Here $\varphi' : x' \to y'$ is a morphism of $\mathcal{X}$ fitting into a commutative diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "\varphi'"'] & x \arrow[d, "\varphi"] \\ y' \arrow[r] & y \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d, "a"] \\ V' \arrow[r] & V \end{tikzcd}$$

The existence and uniqueness of φ' follow from the axioms of a category fibred in groupoids. We omit the verification that c_φ so defined is indeed a map of presheaves (i.e., compatible with restriction mappings) and that it is functorial in $\mathcal{F}$. In case $\mathcal{F}$ is a sheaf for the étale topology we obtain a comparison map

06W3 (96.9.2.2)
$$c_\varphi : a_{small}^{-1}(\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}}$$

which is also denoted c_φ as indicated (this is the customary abuse of notation in not distinguishing between adjoint maps).

075D **Lemma 96.9.3.** Let $\mathcal{F}$ be an étale sheaf on $\mathcal{X} \to (Sch/S)_{fppf}$.

(1) If $\varphi : x \to y$ and $\psi : y \to z$ are morphisms of $\mathcal{X}$ lying over $a : U \to V$ and $b : V \to W$, then the composition
$$a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}})) \xrightarrow{a_{small}^{-1}c_\psi} a_{small}^{-1}(\mathcal{F}|_{V_{\acute{e}tale}}) \xrightarrow{c_\varphi} \mathcal{F}|_{U_{\acute{e}tale}}$$
is equal to $c_{\psi\circ\varphi}$ via the identification
$$(b \circ a)_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}}) = a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}})).$$
(2) If $\varphi : x \to y$ lies over an étale morphism of schemes $a : U \to V$, then (96.9.2.2) is an isomorphism.
(3) Suppose $f : \mathcal{Y} \to \mathcal{X}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ and y is an object of $\mathcal{Y}$ lying over the scheme U with image $x = f(y)$. Then there is a canonical identification $f^{-1}\mathcal{F}|_{U_{\acute{e}tale}} = \mathcal{F}|_{U_{\acute{e}tale}}$.

(4) Moreover, given $\psi : y' \to y$ in $\mathcal{Y}$ lying over $a : U' \to U$ the comparison map $c_\psi : a_{small}^{-1}(f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}) \to f^{-1}\mathcal{F}|_{U'_{\acute{e}tale}}$ is equal to the comparison map $c_{f(\psi)} : a_{small}^{-1}\mathcal{F}|_{U_{\acute{e}tale}} \to \mathcal{F}|_{U'_{\acute{e}tale}}$ via the identifications in (3).

Proof. The verification of these properties is omitted. □

Next, we turn to the restriction of (pre)sheaves of modules.

06W9 Lemma 96.9.4. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The equivalence of Lemma 96.9.1 extends to an equivalence of ringed sites $(\mathcal{X}_\tau/x, \mathcal{O}_\mathcal{X}|_x) \to ((Sch/U)_\tau, \mathcal{O})$.

Proof. This is immediate from the construction of the structure sheaves. □

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a (pre)sheaf of modules on $\mathcal{X}$ as in Definition 96.7.1. Let x be an object of $\mathcal{X}$ lying over U. Then Lemma 96.9.4 guarantees that the restriction $x^{-1}\mathcal{F}$ is a (pre)sheaf of modules on $(Sch/U)_{fppf}$. We will sometimes write $x^*\mathcal{F} = x^{-1}\mathcal{F}$ in this case. Similarly, if $\mathcal{F}$ is a sheaf for the Zariski, étale, smooth, or syntomic topology, then $x^{-1}\mathcal{F}$ is as well. Moreover, the restriction $\mathcal{F}|_{U_{\acute{e}tale}} = x^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$ to U is a presheaf of $\mathcal{O}_{U_{\acute{e}tale}}$-modules. If $\mathcal{F}$ is a sheaf for the étale topology, then $\mathcal{F}|_{U_{\acute{e}tale}}$ is a sheaf of modules. Moreover, if $\varphi : x \to y$ is a morphism of $\mathcal{X}$ lying over $a : U \to V$ then the comparison map (96.9.2.2) is compatible with $a_{small}^\sharp$ (see Descent, Remark 35.8.4) and induces a comparison map

06WC (96.9.4.1)
$$c_\varphi : a_{small}^*(\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}}$$

of $\mathcal{O}_{U_{\acute{e}tale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 96.9.3 hold in the setting of étale sheaves of modules as well. We will use this in the following without further mention.

06W4 Lemma 96.9.5. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. The site $\mathcal{X}_\tau$ has enough points.

Proof. By Sites, Lemma 7.38.5 we have to show that there exists a family of objects x of $\mathcal{X}$ such that $\mathcal{X}_\tau/x$ has enough points and such that the sheaves $h_x^\#$ cover the final object of the category of sheaves. By Lemma 96.9.1 and Étale Cohomology, Lemma 59.30.1 we see that $\mathcal{X}_\tau/x$ has enough points for every object x and we win. □

96.10. Restriction to algebraic spaces

076P In this section we consider sheaves on categories representable by algebraic spaces. The following lemma is the analogue of Topologies, Lemma 34.4.14 for algebraic spaces.

073M Lemma 96.10.1. Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representable by an algebraic space F. Then there exists a continuous and cocontinuous functor $F_{\acute{e}tale} \to \mathcal{X}_{\acute{e}tale}$ which induces a morphism of ringed sites

$$\pi_F : (\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \longrightarrow (F_{\acute{e}tale}, \mathcal{O}_F)$$

and a morphism of ringed topoi

$$i_F : (Sh(F_{\acute{e}tale}), \mathcal{O}_F) \longrightarrow (Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_\mathcal{X})$$

such that $\pi_F \circ i_F = \mathrm{id}$. Moreover $\pi_{F,*} = i_F^{-1}$.

Proof. Choose an equivalence $j : \mathcal{S}_F \to \mathcal{X}$, see Algebraic Stacks, Sections 94.7 and 94.8. An object of $F_{\acute{e}tale}$ is a scheme U together with an étale morphism $\varphi : U \to F$. Then φ is an object of $\mathcal{S}_F$ over U. Hence $j(\varphi)$ is an object of $\mathcal{X}$ over U. In this way j induces a functor $u : F_{\acute{e}tale} \to \mathcal{X}$. It is clear that u is continuous and cocontinuous for the étale topology on $\mathcal{X}$. Since j is an equivalence, the functor u is fully faithful. Also, fibre products and equalizers exist in $F_{\acute{e}tale}$ and u commutes with them because these are computed on the level of underlying schemes in $F_{\acute{e}tale}$. Thus Sites, Lemmas 7.21.5, 7.21.6, and 7.21.7 apply. In particular u defines a morphism of topoi $i_F : Sh(F_{\acute{e}tale}) \to Sh(\mathcal{X}_{\acute{e}tale})$ and there exists a left adjoint $i_{F,!}$ of i_F^{-1} which commutes with fibre products and equalizers.

We claim that $i_{F,!}$ is exact. If this is true, then we can define π_F by the rules $\pi_F^{-1} = i_{F,!}$ and $\pi_{F,*} = i_F^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_{F,!}$ is right exact and preserves fibre products. Hence it suffices to show that $i_{F,!}* = *$ where $*$ indicates the final object in the category of sheaves of sets. Let U be a scheme and let $\varphi : U \to F$ be surjective and étale. Set $R = U \times_F U$. Then

$$h_R \rightrightarrows h_U \longrightarrow *$$

is a coequalizer diagram in $Sh(F_{\acute{e}tale})$. Using the right exactness of $i_{F,!}$, using $i_{F,!} = (u_p\ )^\#$, and using Sites, Lemma 7.5.6 we see that

$$h_{u(R)} \rightrightarrows h_{u(U)} \longrightarrow i_{F,!}*$$

is a coequalizer diagram in $Sh(F_{\acute{e}tale})$. Using that j is an equivalence and that $F = U/R$ it follows that the coequalizer in $Sh(\mathcal{X}_{\acute{e}tale})$ of the two maps $h_{u(R)} \to h_{u(U)}$ is $*$. We omit the proof that these morphisms are compatible with structure sheaves. □

0GQA Remark 96.10.2. The constructions in Lemma 96.10.1 are compatible with étale localization. Here is a precise formulation. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G, and that the induced morphism $f : F \to G$ of algebraic spaces is étale. Denote $f_{small} : F_{\acute{e}tale} \to G_{\acute{e}tale}$ the corresponding morphism of ringed topoi. Then

$$\begin{tikzcd}
(Sh(F_{\acute{e}tale}), \mathcal{O}_F) \arrow[r, "f_{small}"'] \arrow[d, "i_F"'] & (Sh(G_{\acute{e}tale}), \mathcal{O}_G) \arrow[d, "i_G"] \\
(Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_\mathcal{X}) \arrow[r, "f"'] \arrow[d, "\pi_F"'] & (Sh(\mathcal{Y}_{\acute{e}tale}), \mathcal{O}_\mathcal{Y}) \arrow[d, "\pi_G"] \\
(Sh(F_{\acute{e}tale}), \mathcal{O}_F) \arrow[r, "f_{small}"] & (Sh(G_{\acute{e}tale}), \mathcal{O}_G)
\end{tikzcd}$$

is a commutative diagram of ringed topoi. We omit the details.

Assume $\mathcal{X}$ is an algebraic stack represented by the algebraic space F. Let $j : \mathcal{S}_F \to \mathcal{X}$ be an equivalence and denote $u : F_{\acute{e}tale} \to \mathcal{X}_{\acute{e}tale}$ the functor of the proof of Lemma 96.10.1 above. Given a sheaf $\mathcal{F}$ on $\mathcal{X}_{\acute{e}tale}$ we have

$$\pi_{F,*}\mathcal{F}(U) = i_F^{-1}\mathcal{F}(U) = \mathcal{F}(u(U)).$$

This is why we often think of i_F^{-1} as a restriction functor similarly to Definition 96.9.2 and to the restriction of a sheaf on the big étale site of a scheme to the small étale site of a scheme. We often use the notation

075K (96.10.2.1)
$$\mathcal{F}|_{F_{\acute{e}tale}} = i_F^{-1}\mathcal{F} = \pi_{F,*}\mathcal{F}$$

in this situation.

073N **Lemma 96.10.3.** Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces, and $f_{small} : F_{\acute{e}tale} \to G_{\acute{e}tale}$ the corresponding morphism of ringed topoi. Then

$$\begin{tikzcd}
(Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_\mathcal{X}) \arrow[r, "f"'] \arrow[d, "\pi_F"'] & (Sh(\mathcal{Y}_{\acute{e}tale}), \mathcal{O}_\mathcal{Y}) \arrow[d, "\pi_G"] \\
(Sh(F_{\acute{e}tale}), \mathcal{O}_F) \arrow[r, "f_{small}"] & (Sh(G_{\acute{e}tale}), \mathcal{O}_G)
\end{tikzcd}$$

is a commutative diagram of ringed topoi.

Proof. This is similar to Topologies, Lemma 34.4.17 (3) but there is a small snag due to the fact that $F \to G$ may not be representable by schemes. In particular we don't get a commutative diagram of ringed sites, but only a commutative diagram of ringed topoi.

Before we start the proof proper, we choose equivalences $j : \mathcal{S}_F \to \mathcal{X}$ and $j' : \mathcal{S}_G \to \mathcal{Y}$ which induce functors $u : F_{\acute{e}tale} \to \mathcal{X}$ and $u' : G_{\acute{e}tale} \to \mathcal{Y}$ as in the proof of Lemma 96.10.1. Because of the 2-functoriality of sheaves on categories fibred in groupoids over Sch_{fppf} (see discussion in Section 96.3) we may assume that $\mathcal{X} = \mathcal{S}_F$ and $\mathcal{Y} = \mathcal{S}_G$ and that $f : \mathcal{S}_F \to \mathcal{S}_G$ is the functor associated to the morphism $f : F \to G$. Correspondingly we will omit u and u' from the notation, i.e., given an object $U \to F$ of $F_{\acute{e}tale}$ we denote U/F the corresponding object of $\mathcal{X}$. Similarly for G.

Let $\mathcal{G}$ be a sheaf on $\mathcal{X}_{\acute{e}tale}$. To prove (2) we compute $\pi_{G,*}f_*\mathcal{G}$ and $f_{small,*}\pi_{F,*}\mathcal{G}$. To do this let $V \to G$ be an object of $G_{\acute{e}tale}$. Then

$$\pi_{G,*}f_*\mathcal{G}(V) = f_*\mathcal{G}(V/G) = \Gamma\Big((Sch/V)_{fppf} \times_\mathcal{Y} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big)$$

see (96.5.0.1). The fibre product in the formula is

$$(Sch/V)_{fppf} \times_\mathcal{Y} \mathcal{X} = (Sch/V)_{fppf} \times_{\mathcal{S}_G} \mathcal{S}_F = \mathcal{S}_{V \times_G F}$$

i.e., it is the split category fibred in groupoids associated to the algebraic space $V \times_G F$. And $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf on $\mathcal{S}_{V \times_G F}$ for the étale topology.

In particular, if $V \times_G F$ is representable, i.e., if it is a scheme, then $\pi_{G,*}f_*\mathcal{G}(V) = \mathcal{G}(V \times_G F/F)$ and also

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{F,*}\mathcal{G}(V \times_G F) = \mathcal{G}(V \times_G F/F)$$

which proves the desired equality in this special case.

In general, choose a scheme U and a surjective étale morphism $U \to V \times_G F$. Set $R = U \times_{V \times_G F} U$. Then $U/V \times_G F$ and $R/V \times_G F$ are objects of the fibre product category above. Since $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf for the étale topology on $\mathcal{S}_{V \times_G F}$ the diagram

$$\Gamma\Big((Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big) \longrightarrow \mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) \rightrightarrows \mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F)$$

is an equalizer diagram. Note that $\mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) = \mathcal{G}(U/F)$ and $\mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F) = \mathcal{G}(R/F)$ by the definition of pullbacks. Moreover, by the material in Properties of Spaces, Section 66.18 (especially, Properties of Spaces, Remark 66.18.4 and Lemma 66.18.8) we see that there is an equalizer diagram

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) \longrightarrow \pi_{F,*}\mathcal{G}(U/F) \rightrightarrows \pi_{F,*}\mathcal{G}(R/F)$$

Since we also have $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ and $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ we obtain a canonical identification $f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{G,*}f_*\mathcal{G}(V)$. We omit the proof that this is compatible with restriction mappings and that it is functorial in $\mathcal{G}$. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ and $f : F \to G$ be as in the second part of the lemma above. A consequence of the lemma, using (96.10.2.1), is that

075M (96.10.3.1)
$$(f_*\mathcal{F})|_{G_{\acute{e}tale}} = f_{small,*}(\mathcal{F}|_{F_{\acute{e}tale}})$$

for any sheaf $\mathcal{F}$ on $\mathcal{X}_{\acute{e}tale}$. Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (96.10.3.1) is an isomorphism of $\mathcal{O}_G$-modules on $G_{\acute{e}tale}$.

Finally, suppose that we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "a"] \arrow[rd, "f"'] & \mathcal{V} \arrow[d, "g"] \arrow[ld, phantom, "\Swarrow\varphi"] \\ & \mathcal{X} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$, that $\mathcal{F}$ is a sheaf on $\mathcal{X}_{\acute{e}tale}$, and that $\mathcal{U}, \mathcal{V}$ are representable by algebraic spaces U, V. Then we obtain a comparison map

076Q (96.10.3.2)
$$c_\varphi : a_{small}^{-1}(g^{-1}\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$$

where $a : U \to V$ denotes the morphism of algebraic spaces corresponding to a. This is the analogue of (96.9.2.2). We define c_φ as the adjoint to the map

$$g^{-1}\mathcal{F}|_{V_{\acute{e}tale}} \longrightarrow a_{small,*}(f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}) = (a_*f^{-1}\mathcal{F})|_{V_{\acute{e}tale}}$$

(equality by (96.10.3.1)) which is the restriction to V (96.10.2.1) of the map

$$g^{-1}\mathcal{F} \to a_*a^{-1}g^{-1}\mathcal{F} = a_*f^{-1}\mathcal{F}$$

where the last equality uses the 2-commutativity of the diagram above. In case $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules c_φ induces a comparison map

076R (96.10.3.3)
$$c_\varphi : a_{small}^*(g^*\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow f^*\mathcal{F}|_{U_{\acute{e}tale}}$$

of $\mathcal{O}_{U_{\acute{e}tale}}$-modules. This is the analogue of (96.9.4.1). Note that the properties (1), (2), (3), and (4) of Lemma 96.9.3 hold in this setting as well.

96.11. Quasi-coherent modules

06WF At this point we can apply the general definition of a quasi-coherent module to the situation discussed in this chapter.

06WG Definition 96.11.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. A quasi-coherent module on $\mathcal{X}$, or a quasi-coherent $\mathcal{O}_\mathcal{X}$-module is a quasi-coherent module on the ringed site $(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ as in Modules on Sites, Definition 18.23.1. The category of quasi-coherent sheaves on $\mathcal{X}$ is denoted $QCoh(\mathcal{O}_\mathcal{X})$.

If $\mathcal{X}$ is an algebraic stack, then this definition agrees with all definitions in the literature in the sense that $QCoh(\mathcal{O}_\mathcal{X})$ is equivalent (modulo set theoretic issues) to any variant of this category defined in the literature. For example, we will match our definition with the definition in [Ols07b, Definition 6.1] in Cohomology on Stacks, Lemma 96.12.2. We will also see alternative constructions of this category later on.

In general (as is the case for morphisms of schemes) the pushforward of quasi-coherent sheaf along a 1-morphism is not quasi-coherent. Pullback does preserve quasi-coherence.

06WH Lemma 96.11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : \text{Mod}(\mathcal{O}_\mathcal{Y}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$ preserves quasi-coherent sheaves.

Proof. This is a general fact, see Modules on Sites, Lemma 18.23.4. ☐

It turns out that quasi-coherent sheaves have a very simple characterization in terms of their pullbacks. See also Lemma 96.12.2 for a characterization in terms of restrictions.

06WI Lemma 96.11.3. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if $x^*\mathcal{F}$ is a quasi-coherent sheaf on $(Sch/U)_{fppf}$ for every object x of $\mathcal{X}$ with $U = p(x)$.

Proof. By Lemma 96.11.2 the condition is necessary. Conversely, since $x^*\mathcal{F}$ is just the restriction to $\mathcal{X}_{fppf}/x$ we see that it is sufficient directly from the definition of a quasi-coherent sheaf (and the fact that the notion of being quasi-coherent is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18.18). ☐

0EM8 Lemma 96.11.4. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of modules on $\mathcal{X}$. The following are equivalent

(1) $\mathcal{F}$ is an object of $\text{Mod}(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$ in the sense of Modules on Sites, Definition 18.23.1,
(2) $\mathcal{F}$ is an object of $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ in the sense of Modules on Sites, Definition 18.23.1, and
(3) $\mathcal{F}$ is a quasi-coherent module on $\mathcal{X}$ in the sense of Definition 96.11.1.

Proof. Assume either (1), (2), or (3) holds. Let x be an object of $\mathcal{X}$ lying over the scheme U. Recall that $x^*\mathcal{F} = x^{-1}\mathcal{F}$ is just the restriction to $\mathcal{X}/x = (Sch/U)_\tau$ where $\tau = fppf$, $\tau = étale$, or $\tau = Zar$, see Section 96.9. By the definition of quasi-coherent modules on a ringed site this restriction is quasi-coherent provided $\mathcal{F}$ is. By Descent, Proposition 35.8.9 we see that $x^*\mathcal{F}$ is the sheaf associated to a quasi-coherent $\mathcal{O}_U$-module and is therefore a quasi-coherent module in the fppf, étale, and Zariski topology; here we also use Descent, Lemma 35.8.1 and Definition

35.8.2. Since this holds for every object x of $\mathcal{X}$, we see that $\mathcal{F}$ is a sheaf in any of the three topologies. Moreover, we find that $\mathcal{F}$ is quasi-coherent in any of the three topologies directly from the definition of being quasi-coherent and the fact that x is an arbitrary object of $\mathcal{X}$. □

96.12. Locally quasi-coherent modules

0GQB Although there is a variant for the Zariski topology, it seems that the étale topology is the natural topology to use in the following definition.

06WJ **Definition 96.12.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We say $\mathcal{F}$ is *locally quasi-coherent*[2] if $\mathcal{F}$ is a sheaf for the étale topology and for every object x of $\mathcal{X}$ the restriction $x^*\mathcal{F}|_{U_{\acute{e}tale}}$ is a quasi-coherent sheaf. Here $U = p(x)$.

We use $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ to indicate the category of locally quasi-coherent modules. We now have the following diagram of categories of modules

$$\begin{tikzcd} QCoh(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d] & \text{Mod}(\mathcal{O}_\mathcal{X}) \arrow[d] \\ \text{LQCoh}(\mathcal{O}_\mathcal{X}) \arrow[r] & \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \end{tikzcd}$$

where the arrows are strictly full embeddings. It turns out that many results for quasi-coherent sheaves have a counter part for locally quasi-coherent modules. Moreover, from many points of view (as we shall see later) this is a natural category to consider. For example the quasi-coherent sheaves are exactly those locally quasi-coherent modules that are "cartesian", i.e., satisfy the second condition of the lemma below.

06WK **Lemma 96.12.2.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if the following two conditions hold

(1) $\mathcal{F}$ is locally quasi-coherent, and
(2) for any morphism $\varphi : x \to y$ of $\mathcal{X}$ lying over $f : U \to V$ the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (96.9.4.1) is an isomorphism.

Proof. Assume $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F}$ is a sheaf for the fppf topology, hence a sheaf for the étale topology. Moreover, any pullback of $\mathcal{F}$ to a ringed topos is quasi-coherent, hence the restrictions $x^*\mathcal{F}|_{U_{\acute{e}tale}}$ are quasi-coherent. This proves $\mathcal{F}$ is locally quasi-coherent. Let y be an object of $\mathcal{X}$ with $V = p(y)$. We have seen that $\mathcal{X}/y = (Sch/V)_{fppf}$. By Descent, Proposition 35.8.9 it follows that $y^*\mathcal{F}$ is the quasi-coherent module associated to a (usual) quasi-coherent module $\mathcal{F}_V$ on the scheme V. Hence certainly the comparison maps (96.9.4.1) are isomorphisms.

Conversely, suppose that $\mathcal{F}$ satisfies (1) and (2). Let y be an object of $\mathcal{X}$ with $V = p(y)$. Denote $\mathcal{F}_V$ the quasi-coherent module on the scheme V corresponding to the restriction $y^*\mathcal{F}|_{V_{\acute{e}tale}}$ which is quasi-coherent by assumption (1), see Descent, Proposition 35.8.9. Condition (2) now signifies that the restrictions $x^*\mathcal{F}|_{U_{\acute{e}tale}}$ for x over y are each isomorphic to the (étale sheaf associated to the) pullback of $\mathcal{F}_V$ via the corresponding morphism of schemes $U \to V$. Hence $y^*\mathcal{F}$ is the sheaf on

[2] This is nonstandard notation.

$(Sch/V)_{fppf}$ associated to $\mathcal{F}_V$. Hence it is quasi-coherent (by Descent, Proposition 35.8.9 again) and we see that $\mathcal{F}$ is quasi-coherent on $\mathcal{X}$ by Lemma 96.11.3. □

06WL Lemma 96.12.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : \text{Mod}(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y}) \to \text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ preserves locally quasi-coherent sheaves.

Proof. Let $\mathcal{G}$ be locally quasi-coherent on $\mathcal{Y}$. Choose an object x of $\mathcal{X}$ lying over the scheme U. The restriction $x^*f^*\mathcal{G}|_{U_{étale}}$ equals $(f \circ x)^*\mathcal{G}|_{U_{étale}}$ hence is a quasi-coherent sheaf by assumption on $\mathcal{G}$. □

06WM Lemma 96.12.4. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) The category $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.
(2) The category $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ is abelian with kernels and cokernels computed in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, in other words the inclusion functor is exact.
(3) Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ if two out of three are locally quasi-coherent so is the third.
(4) Given $\mathcal{F}, \mathcal{G}$ in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is an object of $\text{LQCoh}(\mathcal{O}_\mathcal{X})$.
(5) Given $\mathcal{F}, \mathcal{G}$ in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation on $\mathcal{X}_{étale}$ the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is an object of $\text{LQCoh}(\mathcal{O}_\mathcal{X})$.

Proof. In the arguments below x denotes an arbitrary object of $\mathcal{X}$ lying over the scheme U. To show that an object $\mathcal{H}$ of $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ we will show that the restriction $x^*\mathcal{H}|_{U_{étale}} = \mathcal{H}|_{U_{étale}}$ is a quasi-coherent object of $\text{Mod}(U_{étale}, \mathcal{O}_U)$.

Proof of (1). Let $\mathcal{I} \to \text{LQCoh}(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram. Consider the object $\mathcal{F} = \text{colim}_i \mathcal{F}_i$ of $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. The pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = \text{colim}_i x^*\mathcal{F}_i$. Similarly we have $x^*\mathcal{F}|_{U_{étale}} = \text{colim}_i x^*\mathcal{F}_i|_{U_{étale}}$. Now by assumption each $x^*\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent. Hence $\text{colim}_i x^*\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent by Descent, Lemma 35.10.3. Thus $x^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent as desired.

Proof of (2). It follows from (1) that cokernels exist in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ and agree with the cokernels computed in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ and let $\mathcal{K} = \text{Ker}(\varphi)$ computed in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. If we can show that $\mathcal{K}$ is a locally quasi-coherent module, then the proof of (2) is complete. To see this, note that kernels are computed in the category of presheaves (no sheafification necessary). Hence $\mathcal{K}|_{U_{étale}}$ is the kernel of the map $\mathcal{F}|_{U_{étale}} \to \mathcal{G}|_{U_{étale}}$, i.e., is the kernel of a map of quasi-coherent sheaves on $U_{étale}$ whence quasi-coherent by Descent, Lemma 35.10.3. This proves (2).

Proof of (3). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Since we are using the étale topology, the restriction $0 \to \mathcal{F}_1|_{U_{étale}} \to \mathcal{F}_2|_{U_{étale}} \to \mathcal{F}_3|_{U_{étale}} \to 0$ is a short exact sequence too. Hence (3) follows from the corresponding statement in Descent, Lemma 35.10.3.

Proof of (4). Let $\mathcal{F}$ and $\mathcal{G}$ be in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$. Since restriction to $U_{étale}$ is given by pullback along the morphism of ringed topoi $U_{étale} \to (Sch/U)_{étale} \to \mathcal{X}_{étale}$ we see that the restriction of the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ to $U_{étale}$ is equal to

$\mathcal{F}|_{U_{\acute{e}tale}} \otimes_{\mathcal{O}_U} \mathcal{G}|_{U_{\acute{e}tale}}$, see Modules on Sites, Lemma 18.26.2. Since $\mathcal{F}|_{U_{\acute{e}tale}}$ and $\mathcal{G}|_{U_{\acute{e}tale}}$ are quasi-coherent, so is their tensor product, see Descent, Lemma 35.10.3.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{G}$ be in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation. Since $(Sch/U)_{\acute{e}tale} = \mathcal{X}_{\acute{e}tale}/x$ is a localization of $\mathcal{X}_{\acute{e}tale}$ at an object we see that the restriction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ to $(Sch/U)_{\acute{e}tale}$ is equal to

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}|_{(Sch/U)_{\acute{e}tale}}}(\mathcal{F}|_{(Sch/U)_{\acute{e}tale}}, \mathcal{G}|_{(Sch/U)_{\acute{e}tale}})$$

by Modules on Sites, Lemma 18.27.2. The morphism of ringed topoi $(U_{\acute{e}tale}, \mathcal{O}_U) \to ((Sch/U)_{\acute{e}tale}, \mathcal{O})$ is flat as the pullback of $\mathcal{O}$ is $\mathcal{O}_U$. Hence the pullback of $\mathcal{H}$ by this morphism is equal to $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$ by Modules on Sites, Lemma 18.31.4. In other words, the restriction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ to $U_{\acute{e}tale}$ is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$. Since $\mathcal{F}|_{U_{\acute{e}tale}}$ and $\mathcal{G}|_{U_{\acute{e}tale}}$ are quasi-coherent, so is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$, see Descent, Lemma 35.10.3. We conclude as before. □

In the generality discussed here the category of quasi-coherent sheaves is not abelian. See Examples, Section 110.13. Here is what we can prove without any further work.

06WN Lemma 96.12.5. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the categories $\text{Mod}(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, $\text{Mod}(\mathcal{O}_\mathcal{X})$, and $\text{LQCoh}(\mathcal{O}_\mathcal{X})$.
(2) Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor products $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ computed in $\text{Mod}(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, or $\text{Mod}(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $QCoh(\mathcal{O}_\mathcal{X})$.
(3) Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ finite locally free (in fppf, or equivalently étale, or equivalently Zariski topology) the internal homs $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ computed in $\text{Mod}(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, or $\text{Mod}(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. Let x be an arbitrary object of $\mathcal{X}$ lying over the scheme U. Let $\tau \in \{Zariski, \acute{e}tale, fppf\}$. To show that an object $\mathcal{H}$ of $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ is in $QCoh(\mathcal{O}_\mathcal{X})$ it suffices show that the restriction $x^*\mathcal{H}$ (Section 96.9) is a quasi-coherent object of $\text{Mod}((Sch/U)_\tau, \mathcal{O})$. See Lemmas 96.11.3 and 96.11.4. Similarly for being finite locally free. Recall that $(Sch/U)_\tau = \mathcal{X}_\tau/x$ is a localization of $\mathcal{X}_\tau$ at an object. Hence restriction commutes with colimits, tensor products, and forming internal hom (see Modules on Sites, Lemmas 18.14.3, 18.26.2, and 18.27.2). This reduces the lemma to Descent, Lemma 35.10.6. □

96.13. Stackification and sheaves

06WP It turns out that the category of sheaves on a category fibred in groupoids only "knows about" the stackification.

06WQ Lemma 96.13.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then the morphism of topoi $f : Sh(\mathcal{X}_{fppf}) \to Sh(\mathcal{Y}_{fppf})$ is an equivalence.

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. We claim that $f : \mathcal{X} \to \mathcal{Y}$ is a special cocontinuous functor, see Sites, Definition 7.29.2 which will prove the lemma. By Stacks, Lemma 8.10.3 the functor f is continuous and cocontinuous. By Stacks, Lemma 8.8.1 we see that conditions (3), (4), and (5) of Sites, Lemma 7.29.1 hold. □

06WR Lemma 96.13.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then f^* induces equivalences $\text{Mod}(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{Y})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$.

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. The first assertion is clear from Lemma 96.13.1 and $\mathcal{O}_\mathcal{X} = f^{-1}\mathcal{O}_\mathcal{Y}$. Pullback of quasi-coherent sheaves are quasi-coherent, see Lemma 96.11.2. Hence it suffices to show that if $f^*\mathcal{G}$ is quasi-coherent, then $\mathcal{G}$ is. To see this, let y be an object of $\mathcal{Y}$. Translating the condition that $\mathcal{Y}$ is the stackification of $\mathcal{X}$ we see there exists an fppf covering $\{y_i \to y\}$ in $\mathcal{Y}$ such that $y_i \cong f(x_i)$ for some x_i object of $\mathcal{X}$. Say x_i and y_i lie over the scheme U_i. Then $f^*\mathcal{G}$ being quasi-coherent, means that $x_i^*f^*\mathcal{G}$ is quasi-coherent. Since $x_i^*f^*\mathcal{G}$ is isomorphic to $y_i^*\mathcal{G}$ (as sheaves on $(Sch/U_i)_{fppf}$ we see that $y_i^*\mathcal{G}$ is quasi-coherent. It follows from Modules on Sites, Lemma 18.23.3 that the restriction of $\mathcal{G}$ to $\mathcal{Y}/y$ is quasi-coherent. Hence $\mathcal{G}$ is quasi-coherent by Lemma 96.11.3. □

96.14. Quasi-coherent sheaves and presentations

06WS Let us first match quasi-coherent sheaves with our previously defined notions for schemes and algebraic spaces.

0GQC Lemma 96.14.1. Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids wich is representable by an algebraic space F. If $\mathcal{F}$ is in $\text{LQCoh}(\mathcal{O}_\mathcal{X})$ then the restriction $\mathcal{F}|_{F_{\acute{e}tale}}$ (96.10.2.1) is quasi-coherent.

Proof. Let U be a scheme étale over F. Then $\mathcal{F}|_{U_{\acute{e}tale}} = (\mathcal{F}|_{F_{\acute{e}tale}})|_{U_{\acute{e}tale}}$. This is clear but see also Remark 96.10.2. Thus the assertion follows from the definitions. □

0GQD Lemma 96.14.2. Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids wich is representable by an algebraic space F. The functor (96.10.2.1) defines an equivalence

$$QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_F), \quad \mathcal{F} \longmapsto \mathcal{F}|_{F_{\acute{e}tale}}$$

with quasi-inverse given by $\mathcal{G} \mapsto \pi_F^*\mathcal{G}$. This equivalence is compatible with pullback for morphisms between categories fibred in groupoids representable by algebraic spaces.

Proof. By Lemma 96.11.4 we may work with the étale topology. We will use the notation and results of Lemma 96.10.1 without further mention. Recall that the restriction functor $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \to \text{Mod}(F_{\acute{e}tale}, \mathcal{O}_F)$, $\mathcal{F} \mapsto \mathcal{F}|_{F_{\acute{e}tale}}$ is given by i_F^*. By Lemma 96.14.1 or by Modules on Sites, Lemma 18.23.4 we see that $\mathcal{F}|_{F_{\acute{e}tale}}$ is quasi-coherent if $\mathcal{F}$ is quasi-coherent. Hence we get a functor as indicated in the statement of the lemma and we get a functor π_F^* in the opposite direction. Since $\pi_F \circ i_F = \text{id}$ we see that $i_F^*\pi_F^*\mathcal{G} = \mathcal{G}$.

For $\mathcal{F}$ in $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ there is a canonical map $\pi_F^*(\mathcal{F}|_{F_{\acute{e}tale}}) \to \mathcal{F}$, namely the map adjoint to the identification $\mathcal{F}|_{F_{\acute{e}tale}} = \pi_{F,*}\mathcal{F}$. We will show that this map is an isomorphism if $\mathcal{F}$ is a quasi-coherent module on $\mathcal{X}$. Choose a scheme U and a surjective étale morphism $U \to F$. Denote $x : U \to \mathcal{X}$ the corresponding object of $\mathcal{X}$ over U. It suffices to show that $\pi_F^*(\mathcal{F}|_{F_{\acute{e}tale}}) \to \mathcal{F}$ is an isomorphism after restricting to $\mathcal{X}_{\acute{e}tale}/x = (Sch/U)_{\acute{e}tale}$. Since $U \to F$ is étale, it follows from Remark 96.10.2 that

$$\pi_F^*(\mathcal{F}|_{F_{\acute{e}tale}})|_{\mathcal{X}_{\acute{e}tale}/x} = \pi_U^*(\mathcal{F}|_{U_{\acute{e}tale}})$$

and that the restriction of the map $\pi_F^*(\mathcal{F}|_{F_{\acute{e}tale}}) \to \mathcal{F}$ to $\mathcal{X}_{\acute{e}tale}/x = (Sch/U)_{\acute{e}tale}$ is equal to the corresponding map $\pi_U^*(\mathcal{F}|_{U_{\acute{e}tale}}) \to \mathcal{F}|_{(Sch/U)_{\acute{e}tale}}$. Since we have seen the result is true for schemes in Descent, Section 35.8[3] we conclude.

Compatibility with pullbacks follows from the fact that the quasi-inverse is given by π_F^* and the commutative diagram of ringed topoi in Lemma 96.10.3. □

In Groupoids in Spaces, Definition 78.12.1 we have the defined the notion of a quasi-coherent module on an arbitrary groupoid. The following (formal) proposition tells us that we can study quasi-coherent sheaves on quotient stacks in terms of quasi-coherent modules on presentations.

06WT Proposition 96.14.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Let $\mathcal{X} = [U/R]$ be the quotient stack. The category of quasi-coherent modules on $\mathcal{X}$ is equivalent to the category of quasi-coherent modules on (U, R, s, t, c).

Proof. We will construct quasi-inverse functors

$$QCoh(\mathcal{O}_\mathcal{X}) \longleftrightarrow QCoh(U, R, s, t, c).$$

where $QCoh(U, R, s, t, c)$ denotes the category of quasi-coherent modules on the groupoid (U, R, s, t, c).

Let $\mathcal{F}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. Denote $\mathcal{U}$, $\mathcal{R}$ the categories fibred in groupoids corresponding to U and R. Denote x the (defining) object of $\mathcal{X}$ over U. Recall that we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "s"'] \arrow[d, "t"'] & \mathcal{U} \arrow[d, "x"] \\ \mathcal{U} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

See Groupoids in Spaces, Lemma 78.20.3. By Lemma 96.3.3 the 2-arrow inherent in the diagram induces an isomorphism $\alpha : t^*x^*\mathcal{F} \to s^*x^*\mathcal{F}$ which satisfies the cocycle condition over $\mathcal{R} \times_{s,\mathcal{U},t} \mathcal{R}$; this is a consequence of Groupoids in Spaces, Lemma 78.23.1. Thus if we set $\mathcal{G} = x^*\mathcal{F}|_{U_{\acute{e}tale}}$ then the equivalence of categories in Lemma 96.14.2 (used several times compatibly with pullbacks) gives an isomorphism $\alpha : t_{small}^*\mathcal{G} \to s_{small}^*\mathcal{G}$ satisfying the cocycle condition on $R \times_{s,U,t} R$, i.e., $(\mathcal{G}, \alpha)$ is an object of $QCoh(U, R, s, t, c)$. The rule $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ is our functor from left to right.

Construction of the functor in the other direction. Let $(\mathcal{G}, \alpha)$ be an object of $QCoh(U, R, s, t, c)$. According to Lemma 96.13.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 78.20.1) induces an equivalence of categories of quasi-coherent sheaves. Thus it suffices to construct a quasi-coherent module $\mathcal{F}$ on $[U/_pR]$.

Recall that an object $x = (T, u)$ of $[U/_pR]$ is given by a scheme T and a morphism $u : T \to U$. A morphism $(T, u) \to (T', u')$ is given by a pair (f, r) where $f : T \to T'$ and $r : T \to R$ with $s \circ r = u$ and $t \circ r = u' \circ f$. Let us call a special morphism any

[3] Namely, if U is a scheme and $\mathcal{F}$ is quasi-coherent on $(Sch/U)_{\acute{e}tale}$, then $\mathcal{F} = \mathcal{H}^a$ for some quasi-coherent module $\mathcal{H}$ on the scheme U by Descent, Proposition 35.8.9. In other words, $\mathcal{F} = (\text{id}_{\acute{e}tale,Zar})^*\mathcal{H}$ by Descent, Remark 35.8.6 with notation as in Descent, Lemma 35.8.5. Then we have $\text{id}_{\acute{e}tale,Zar} = \pi_U \circ \text{id}_{small,\acute{e}tale,Zar}$ and hence we see that $\mathcal{F} = \pi_U^*\mathcal{G}$ where $\mathcal{G} = (\text{id}_{small,\acute{e}tale,Zar})^*\mathcal{H}$ is quasi-coherent. Then $\pi_U^* i_U^* \mathcal{F} = \pi_U^* i_U^* \pi_U^* \mathcal{G} = \pi_U^*\mathcal{G} = \mathcal{F}$ as desired.

morphism of the form $(f, e \circ u' \circ f) : (T, u' \circ f) \to (T', u')$. The category of (T, u) with special morphisms is just the category of schemes over U.

With this notation in place, given an object (T, u) of $[U/_pR]$, we set

$$\mathcal{F}(T, u) := \Gamma(T, u_{small}^*\mathcal{G}).$$

Given a morphism $(f, r) : (T, u) \to (T', u')$ we get a map

$$\begin{aligned}\mathcal{F}(T', u') &= \Gamma(T', (u')_{small}^*\mathcal{G}) \\ &\to \Gamma(T, f_{small}^*(u')_{small}^*\mathcal{G}) = \Gamma(T, (u' \circ f)_{small}^*\mathcal{G}) \\ &= \Gamma(T, (t \circ r)_{small}^*\mathcal{G}) = \Gamma(T, r_{small}^* t_{small}^*\mathcal{G}) \\ &\to \Gamma(T, r_{small}^* s_{small}^*\mathcal{G}) = \Gamma(T, (s \circ r)_{small}^*\mathcal{G}) \\ &= \Gamma(T, u_{small}^*\mathcal{G}) \\ &= \mathcal{F}(T, u)\end{aligned}$$

where the first arrow is pullback along f and the second arrow is α. Note that if (T, r) is a special morphism, then this map is just pullback along f as $e_{small}^*\alpha = \text{id}$ by the axioms of a sheaf of quasi-coherent modules on a groupoid. The cocycle condition implies that $\mathcal{F}$ is a presheaf of modules (details omitted). We see that the restriction of $\mathcal{F}$ to $(Sch/T)_{fppf}$ is quasi-coherent by the simple description of the restriction maps of $\mathcal{F}$ in case of a special morphism. Hence $\mathcal{F}$ is a sheaf on $[U/_pR]$ and quasi-coherent (Lemma 96.11.3).

We omit the verification that the functors constructed above are quasi-inverse to each other. $\square$

We finish this section with a technical lemma on maps out of quasi-coherent sheaves. It is an analogue of Schemes, Lemma 26.7.1. We will see later (Criteria for Representability, Theorem 97.17.2) that the assumptions on the groupoid imply that $\mathcal{X}$ is an algebraic stack.

076S Lemma 96.14.4. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $\mathcal{X} = [U/R]$ be the quotient stack. Denote x the object of $\mathcal{X}$ over U. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, and let $\mathcal{H}$ be any object of $\text{Mod}(\mathcal{O}_\mathcal{X})$. The map

$$\text{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{H}) \longrightarrow \text{Hom}_{\mathcal{O}_U}(x^*\mathcal{F}|_{U_{étale}}, x^*\mathcal{H}|_{U_{étale}}), \quad \phi \longmapsto x^*\phi|_{U_{étale}}$$

is injective and its image consists of exactly those $\varphi : x^*\mathcal{F}|_{U_{étale}} \to x^*\mathcal{H}|_{U_{étale}}$ which give rise to a commutative diagram

$$\begin{tikzcd} s_{small}^*(x^*\mathcal{F}|_{U_{étale}}) \arrow[r] \arrow[d, "s_{small}^*\varphi"] & (x \circ s)^*\mathcal{F}|_{R_{étale}} = (x \circ t)^*\mathcal{F}|_{R_{étale}} & t_{small}^*(x^*\mathcal{F}|_{U_{étale}}) \arrow[l] \arrow[d, "t_{small}^*\varphi"'] \\ s_{small}^*(x^*\mathcal{H}|_{U_{étale}}) \arrow[r] & (x \circ s)^*\mathcal{H}|_{R_{étale}} = (x \circ t)^*\mathcal{H}|_{R_{étale}} & t_{small}^*(x^*\mathcal{H}|_{U_{étale}}) \arrow[l] \end{tikzcd}$$

of modules on $R_{étale}$ where the horizontal arrows are the comparison maps (96.10.3.3).

Proof. According to Lemma 96.13.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 78.20.1) induces an equivalence of categories of quasi-coherent sheaves and of fppf $\mathcal{O}$-modules. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$. By Proposition 96.14.3 and its proof there exists a quasi-coherent module $(\mathcal{G}, \alpha)$ on (U, R, s, t, c) such that $\mathcal{F}$ is given by the rule $\mathcal{F}(T, u) = \Gamma(T, u^*\mathcal{G})$.

In particular $x^*\mathcal{F}|_{U_{étale}} = \mathcal{G}$ and it is clear that the map of the statement of the lemma is injective. Moreover, given a map $\varphi : \mathcal{G} \to x^*\mathcal{H}|_{U_{étale}}$ and given any object $y = (T, u)$ of $[U/_pR]$ we can consider the map

$$\mathcal{F}(y) = \Gamma(T, u^*\mathcal{G}) \xrightarrow{u_{small}^*\varphi} \Gamma(T, u_{small}^* x^*\mathcal{H}|_{U_{étale}}) \to \Gamma(T, y^*\mathcal{H}|_{T_{étale}}) = \mathcal{H}(y)$$

where the second arrow is the comparison map (96.9.4.1) for the sheaf $\mathcal{H}$. This assignment is compatible with the restriction mappings of the sheaves $\mathcal{F}$ and $\mathcal{G}$ for morphisms of $[U/_pR]$ if the cocycle condition of the lemma is satisfied. Proof omitted. Hint: the restriction maps of $\mathcal{F}$ are made explicit in terms of $(\mathcal{G}, \alpha)$ in the proof of Proposition 96.14.3. □

96.15. Quasi-coherent sheaves on algebraic stacks

06WU Let $\mathcal{X}$ be an algebraic stack over S. By Algebraic Stacks, Lemma 94.16.2 we can find an equivalence $[U/R] \to \mathcal{X}$ where (U, R, s, t, c) is a smooth groupoid in algebraic spaces. Then

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Proposition 96.14.3. Hence the category of quasi-coherent sheaves on an algebraic stack is equivalent to the category of quasi-coherent modules on a smooth groupoid in algebraic spaces. In particular, by Groupoids in Spaces, Lemma 78.12.6 we see that $QCoh(\mathcal{O}_\mathcal{X})$ is abelian!

There is something slightly disconcerting about our current setup. It is that the fully faithful embedding

$$QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow \text{Mod}(\mathcal{O}_\mathcal{X})$$

is in general not exact. However, exactly the same thing happens for schemes: for most schemes X the embedding

$$QCoh(\mathcal{O}_X) \cong QCoh((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow \text{Mod}((Sch/X)_{fppf}, \mathcal{O}_X)$$

isn't exact, see Descent, Lemma 35.10.2. Parenthetically, the example in the proof of Descent, Lemma 35.10.2 shows that in general the strictly full embedding $QCoh(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}(\mathcal{O}_\mathcal{X})$ isn't exact either.

We collect all the results obtained so far in a single statement.

06WV **Lemma 96.15.1.** Let $\mathcal{X}$ be an algebraic stack over S.

(1) If $[U/R] \to \mathcal{X}$ is a presentation of $\mathcal{X}$ then there is a canonical equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$.
(2) The category $QCoh(\mathcal{O}_\mathcal{X})$ is abelian.
(3) The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$ is right exact but not exact in general.
(4) The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $\text{Mod}(\mathcal{O}_\mathcal{X})$.
(5) Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.
(6) Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ finite locally free the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.
(7) Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $\text{Mod}(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ quasi-coherent, then $\mathcal{F}_2$ is quasi-coherent.

Proof. Properties (4), (5), and (6) were proven in Lemma 96.12.5. Part (1) is Proposition 96.14.3. Part (2) follows from part (1) and Groupoids in Spaces, Lemma 78.12.6 as discussed above. Right exactness of the inclusion functor in (3) follows from (4); please compare with Homology, Lemma 12.7.2. For the nonexactness of the inclusion functor in part (3) see Descent, Lemma 35.10.2. To see (7) observe that it suffices to check the restriction of $\mathcal{F}_2$ to the big site of a scheme is quasi-coherent (Lemma 96.11.3), hence this follows from the corresponding part of Descent, Lemma 35.10.2. $\square$

Next we construct the coherator for modules on an algebraic stack.

0781 Proposition 96.15.2. Let $\mathcal{X}$ be an algebraic stack over S.

(1) The category $QCoh(\mathcal{O}_\mathcal{X})$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_\mathcal{X})$ has enough injectives and all limits.
(2) The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$ has a right adjoint[4]
$$Q : \text{Mod}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$
such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 28.23.4 and the case of algebraic spaces, see Properties of Spaces, Proposition 66.32.2. We advise the reader to read either of those proofs first.

Part (1) means $QCoh(\mathcal{O}_\mathcal{X})$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Lemma 96.15.1 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $\text{Mod}(\mathcal{O}_X)$. By Modules on Sites, Lemma 18.14.2 filtered colimits are exact. Hence (a) and (b) hold.

Choose a presentation $\mathcal{X} = [U/R]$ so that (U, R, s, t, c) is a smooth groupoid in algebraic spaces and in particular s and t are flat morphisms of algebraic spaces. By Lemma 96.15.1 above we have $QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$. By Groupoids in Spaces, Lemma 78.14.2 there exists a set T and a family $(\mathcal{F}_t)_{t \in T}$ of quasi-coherent sheaves on $\mathcal{X}$ such that every quasi-coherent sheaf on $\mathcal{X}$ is the directed colimit of its subsheaves which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.7 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $\text{Mod}(\mathcal{O}_\mathcal{X})$ we consider the functor
$$QCoh(\mathcal{O}_\mathcal{X})^{opp} \longrightarrow \text{Sets}, \quad \mathcal{G} \longmapsto \text{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F})$$
This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\text{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}) = \text{Hom}_\mathcal{X}(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$ is fully faithful. $\square$

[4] This functor is sometimes called the coherator.

96.16. Cohomology

075E Let S be a scheme and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. For any $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the categories $\textit{Ab}(\mathcal{X}_\tau)$ and $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ have enough injectives, see Injectives, Theorems 19.7.4 and 19.8.4. Thus we can use the machinery of Cohomology on Sites, Section 21.2 to define the cohomology groups

$$H^p(\mathcal{X}_\tau, \mathcal{F}) = H^p_\tau(\mathcal{X}, \mathcal{F}) \quad \text{and} \quad H^p(x, \mathcal{F}) = H^p_\tau(x, \mathcal{F})$$

for any $x \in \mathrm{Ob}(\mathcal{X})$ and any object $\mathcal{F}$ of $\textit{Ab}(\mathcal{X}_\tau)$ or $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then we obtain the higher direct images $R^if_*\mathcal{F}$ in $\textit{Ab}(\mathcal{Y}_\tau)$ or $\textit{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$. Of course, as explained in Cohomology on Sites, Section 21.3 there are also derived versions of $H^p(-)$ and R^if_*.

075F **Lemma 96.16.1.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ be an object lying over the scheme U. Let $\mathcal{F}$ be an object of $\textit{Ab}(\mathcal{X}_\tau)$ or $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then

$$H^p_\tau(x, \mathcal{F}) = H^p((Sch/U)_\tau, x^{-1}\mathcal{F})$$

and if $\tau = étale$, then we also have

$$H^p_{étale}(x, \mathcal{F}) = H^p(U_{étale}, \mathcal{F}|_{U_{étale}}).$$

Proof. The first statement follows from Cohomology on Sites, Lemma 21.7.1 and the equivalence of Lemma 96.9.4. The second statement follows from the first combined with Étale Cohomology, Lemma 59.20.3. □

96.17. Injective sheaves

06WW The pushforward of an injective abelian sheaf or module is injective.

06WX **Lemma 96.17.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$.

(1) $f_*\mathcal{I}$ is injective in $\textit{Ab}(\mathcal{Y}_\tau)$ for $\mathcal{I}$ injective in $\textit{Ab}(\mathcal{X}_\tau)$, and
(2) $f_*\mathcal{I}$ is injective in $\textit{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ for $\mathcal{I}$ injective in $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. This follows formally from the fact that f^{-1} is an exact left adjoint of f_*, see Homology, Lemma 12.29.1. □

In the rest of this section we prove that pullback f^{-1} has a left adjoint $f_!$ on abelian sheaves and modules. If f is representable (by schemes or by algebraic spaces), then it will turn out that $f_!$ is exact and f^{-1} will preserve injectives. We first prove a few preliminary lemmas about fibre products and equalizers in categories fibred in groupoids and their behaviour with respect to morphisms.

06WY **Lemma 96.17.2.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) The category $\mathcal{X}$ has fibre products.
(2) If the *Isom*-presheaves of $\mathcal{X}$ are representable by algebraic spaces, then $\mathcal{X}$ has equalizers.
(3) If $\mathcal{X}$ is an algebraic stack (or more generally a quotient stack), then $\mathcal{X}$ has equalizers.

Proof. Part (1) follows Categories, Lemma 4.35.15 as $(Sch/S)_{fppf}$ has fibre products.

Let $a, b : x \to y$ be morphisms of $\mathcal{X}$. Set $U = p(x)$ and $V = p(y)$. The category of schemes has equalizers hence we can let $W \to U$ be the equalizer of $p(a)$ and $p(b)$. Denote $c : z \to x$ a morphism of $\mathcal{X}$ lying over $W \to U$. The equalizer of a and b, if it exists, is the equalizer of $a \circ c$ and $b \circ c$. Thus we may assume that $p(a) = p(b) = f : U \to V$. As $\mathcal{X}$ is fibred in groupoids, there exists a unique automorphism $i : x \to x$ in the fibre category of $\mathcal{X}$ over U such that $a \circ i = b$. Again the equalizer of a and b is the equalizer of id_x and i. Recall that the $Isom_{\mathcal{X}}(x)$ is the presheaf on $(Sch/U)_{fppf}$ which to T/U associates the set of automorphisms of $x|_T$ in the fibre category of $\mathcal{X}$ over T, see Stacks, Definition 8.2.2. If $Isom_{\mathcal{X}}(x)$ is representable by an algebraic space $G \to U$, then we see that id_x and i define morphisms $e, i : U \to G$ over U. Set $M = U \times_{e,G,i} U$, which by Morphisms of Spaces, Lemma 67.4.7 is a scheme. Then it is clear that $x|_M \to x$ is the equalizer of the maps id_x and i in $\mathcal{X}$. This proves (2).

If $\mathcal{X} = [U/R]$ for some groupoid in algebraic spaces (U, R, s, t, c) over S, then the hypothesis of (2) holds by Bootstrap, Lemma 80.11.5. If $\mathcal{X}$ is an algebraic stack, then we can choose a presentation $[U/R] \cong \mathcal{X}$ by Algebraic Stacks, Lemma 94.16.2. □

06WZ **Lemma 96.17.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

(1) The functor f transforms fibre products into fibre products.
(2) If f is faithful, then f transforms equalizers into equalizers.

Proof. By Categories, Lemma 4.35.15 we see that a fibre product in $\mathcal{X}$ is any commutative square lying over a fibre product diagram in $(Sch/S)_{fppf}$. Similarly for $\mathcal{Y}$. Hence (1) is clear.

Let $x \to x'$ be the equalizer of two morphisms $a, b : x' \to x''$ in $\mathcal{X}$. We will show that $f(x) \to f(x')$ is the equalizer of $f(a)$ and $f(b)$. Let $y \to f(x)$ be a morphism of $\mathcal{Y}$ equalizing $f(a)$ and $f(b)$. Say x, x', x'' lie over the schemes U, U', U'' and y lies over V. Denote $h : V \to U'$ the image of $y \to f(x)$ in the category of schemes. The morphism $y \to f(x)$ is isomorphic to $f(h^*x') \to f(x')$ by the axioms of fibred categories. Hence, as f is faithful, we see that $h^*x' \to x'$ equalizes a and b. Thus we obtain a unique morphism $h^*x' \to x$ whose image $y = f(h^*x') \to f(x)$ is the desired morphism in $\mathcal{Y}$. □

06X0 **Lemma 96.17.4.** Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be faithful 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.

(1) the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$ is faithful, and
(2) if $\mathcal{X}, \mathcal{Z}$ have equalizers, so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

Proof. We think of objects in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ as quadruples (U, x, z, α) where $\alpha : f(x) \to g(z)$ is an isomorphism over U, see Categories, Lemma 4.32.3. A morphism $(U, x, z, \alpha) \to (U', x', z', \alpha')$ is a pair of morphisms $a : x \to x'$ and $b : z \to z'$ compatible with α and α'. Thus it is clear that if f and g are faithful, so is the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$. Now, suppose that $(a, b), (a', b') : (U, x, z, \alpha) \to (U', x', z', \alpha')$ are two morphisms of the 2-fibre product. Then consider the equalizer $x'' \to x$ of a and a' and the equalizer $z'' \to z$ of b and b'. Since f commutes with equalizers (by Lemma 96.17.3) we see that $f(x'') \to f(x)$ is the equalizer of $f(a)$ and $f(a')$. Similarly, $g(z'') \to g(z)$ is the equalizer of $g(b)$ and $g(b')$. Picture

$$\begin{tikzcd} f(x'') \arrow[r] \arrow[d, dotted, "\alpha''"'] & f(x) \arrow[r, shift left, "f(a)"] \arrow[r, shift right, "f(a')"'] \arrow[d, "\alpha"'] & f(x') \arrow[d, "\alpha'"] \\ g(z'') \arrow[r] & g(z) \arrow[r, shift left, "g(b)"] \arrow[r, shift right, "g(b')"'] & g(z') \end{tikzcd}$$

It is clear that the dotted arrow exists and is an isomorphism. However, it is not a priori the case that the image of α'' in the category of schemes is the identity of its source. On the other hand, the existence of α'' means that we can assume that x'' and z'' are defined over the same scheme and that the morphisms $x'' \to x$ and $z'' \to z$ have the same image in the category of schemes. Redoing the diagram above we see that the dotted arrow now does project to an identity morphism and we win. Some details omitted. □

As we are working with big sites we have the following somewhat counter intuitive result (which also holds for morphisms of big sites of schemes). Warning: This result isn't true if we drop the hypothesis that f is faithful.

06X1 Lemma 96.17.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^{-1} : \text{Ab}(\mathcal{Y}_\tau) \to \text{Ab}(\mathcal{X}_\tau)$ has a left adjoint $f_! : \text{Ab}(\mathcal{X}_\tau) \to \text{Ab}(\mathcal{Y}_\tau)$. If f is faithful and $\mathcal{X}$ has equalizers, then

(1) $f_!$ is exact, and
(2) $f^{-1}\mathcal{I}$ is injective in $\text{Ab}(\mathcal{X}_\tau)$ for $\mathcal{I}$ injective in $\text{Ab}(\mathcal{Y}_\tau)$.

Proof. By Stacks, Lemma 8.10.3 the functor f is continuous and cocontinuous. Hence by Modules on Sites, Lemma 18.16.2 the functor $f^{-1} : \text{Ab}(\mathcal{Y}_\tau) \to \text{Ab}(\mathcal{X}_\tau)$ has a left adjoint $f_! : \text{Ab}(\mathcal{X}_\tau) \to \text{Ab}(\mathcal{Y}_\tau)$. To see (1) we apply Modules on Sites, Lemma 18.16.3 and to see that the hypotheses of that lemma are satisfied use Lemmas 96.17.2 and 96.17.3 above. Part (2) follows from this formally, see Homology, Lemma 12.29.1. □

06X2 Lemma 96.17.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^* : \text{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y}) \to \text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ has a left adjoint $f_! : \text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ which agrees with the functor $f_!$ of Lemma 96.17.5 on underlying abelian sheaves. If f is faithful and $\mathcal{X}$ has equalizers, then

(1) $f_!$ is exact, and
(2) $f^{-1}\mathcal{I}$ is injective in $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ for $\mathcal{I}$ injective in $\text{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. Recall that f is a continuous and cocontinuous functor of sites and that $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$. Hence Modules on Sites, Lemma 18.41.1 implies f^* has a left adjoint $f_!^{Mod}$. Let x be an object of $\mathcal{X}$ lying over the scheme U. Then f induces an equivalence of ringed sites

$$\mathcal{X}/x \longrightarrow \mathcal{Y}/f(x)$$

as both sides are equivalent to $(Sch/U)_\tau$, see Lemma 96.9.4. Modules on Sites, Remark 18.41.2 shows that $f_!$ agrees with the functor on abelian sheaves.

Assume now that $\mathcal{X}$ has equalizers and that f is faithful. Lemma 96.17.5 tells us that $f_!$ is exact. Finally, Homology, Lemma 12.29.1 implies the statement on pullbacks of injective modules. □

96.18. The Čech complex

06X3 To compute the cohomology of a sheaf on an algebraic stack we compare it to the cohomology of the sheaf restricted to coverings of the given algebraic stack.

Throughout this section the situation will be as follows. We are given a 1-morphism of categories fibred in groupoids

06X4 (96.18.0.1)

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "f"] \arrow[rd, "q"'] & & \mathcal{X} \arrow[ld, "p"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

We are going to think about $\mathcal{U}$ as a "covering" of $\mathcal{X}$. Hence we want to consider the simplicial object

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightrightarrows \mathcal{U}$$

in the category of categories fibred in groupoids over $(Sch/S)_{fppf}$. However, since this is a $(2, 1)$-category and not a category, we should say explicitly what we mean. Namely, we let $\mathcal{U}_n$ be the category with objects $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ where $\alpha_i : f(u_i) \to x$ is an isomorphism in $\mathcal{X}$. We denote $f_n : \mathcal{U}_n \to \mathcal{X}$ the 1-morphism which assigns to $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ the object x. Note that $\mathcal{U}_0 = \mathcal{U}$ and $f_0 = f$. Given a map $\varphi : [m] \to [n]$ we consider the 1-morphism $\mathcal{U}_\varphi : \mathcal{U}_n \longrightarrow \mathcal{U}_n$ given by

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_{\varphi(0)}, \ldots, u_{\varphi(m)}, x, \alpha_{\varphi(0)}, \ldots, \alpha_{\varphi(m)})$$

on objects. All of these 1-morphisms compose correctly on the nose (no 2-morphisms required) and all of these 1-morphisms are 1-morphisms over $\mathcal{X}$. We denote $\mathcal{U}_\bullet$ this simplicial object. If $\mathcal{F}$ is a presheaf of sets on $\mathcal{X}$, then we obtain a cosimplicial set

$$\Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \rightrightarrows \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \Gamma(\mathcal{U}_2, f_2^{-1}\mathcal{F})$$

Here the arrows are the pullback maps along the given morphisms of the simplicial object. If $\mathcal{F}$ is a presheaf of abelian groups, this is a cosimplicial abelian group.

Let $\mathcal{U} \to \mathcal{X}$ be as above and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. The Čech complex associated to the situation is denoted $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$. It is the cochain complex associated to the cosimplicial abelian group above, see Simplicial, Section 14.25. It has terms

$$\check{\mathcal{C}}^n(\mathcal{U} \to \mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}).$$

The boundary maps are the maps

$$d^n = \sum\nolimits_{i=0}^{n+1} (-1)^i \delta_i^{n+1} : \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F})$$

where δ_i^{n+1} corresponds to the map $[n] \to [n + 1]$ omitting the index i. Note that the map $\Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0)$ is in the kernel of the differential d^0. Hence we define the extended Čech complex to be the complex

$$\ldots \to 0 \to \Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0) \to \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}_1) \to \ldots$$

with $\Gamma(\mathcal{X}, \mathcal{F})$ placed in degree -1. The extended Čech complex is acyclic if and only if the canonical map

$$\Gamma(\mathcal{X}, \mathcal{F})[0] \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$$

is a quasi-isomorphism of complexes.

06X5 Lemma 96.18.1. Generalities on Čech complexes.

(1) If

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"'] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

is 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, e^{-1}\mathcal{F})$$

(2) if h and e are equivalences, then the map of (1) is an isomorphism,

(3) if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated Čech complexes are isomorphic.

Proof. In the situation of (1) let $t : f \circ h \to e \circ g$ be a 2-morphism. The map on complexes is given in degree n by pullback along the 1-morphisms $\mathcal{V}_n \to \mathcal{U}_n$ given by the rule

$$(v_0, \ldots, v_n, y, \beta_0, \ldots, \beta_n) \longmapsto (h(v_0), \ldots, h(v_n), e(y), e(\beta_0) \circ t_{v_0}, \ldots, e(\beta_n) \circ t_{v_n}).$$

For (2), note that pullback on global sections is an isomorphism for any presheaf of sets when the pullback is along an equivalence of categories. Part (3) follows on combining (1) and (2). □

06X6 Lemma 96.18.2. If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $\mathrm{id}_\mathcal{X}$ then the extended Čech complex is homotopic to zero.

Proof. Set $\mathcal{U}' = \mathcal{U} \times_\mathcal{X} \mathcal{X}$ equal to the fibre product as described in Categories, Lemma 4.32.3. Set $f' : \mathcal{U}' \to \mathcal{X}$ equal to the second projection. Then $\mathcal{U} \to \mathcal{U}'$, $u \mapsto (u, f(x), 1)$ is an equivalence over $\mathcal{X}$, hence we may replace $(\mathcal{U}, f)$ by $(\mathcal{U}', f')$ by Lemma 96.18.1. The advantage of this is that now f' has a section s' such that $f' \circ s' = \mathrm{id}_\mathcal{X}$ on the nose. Namely, if $t : s \circ f \to \mathrm{id}_\mathcal{X}$ is a 2-isomorphism then we can set $s'(x) = (s(x), x, t_x)$. Thus we may assume that $f \circ s = \mathrm{id}_\mathcal{X}$.

In the case that $f \circ s = \mathrm{id}_\mathcal{X}$ the result follows from general principles. We give the homotopy explicitly. Namely, for $n \geq 0$ define $s_n : \mathcal{U}_n \to \mathcal{U}_{n+1}$ to be the 1-morphism defined by the rule on objects

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_0, \ldots, u_n, s(x), x, \alpha_0, \ldots, \alpha_n, \mathrm{id}_x).$$

Define

$$h^{n+1} : \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F})$$

as pullback along s_n. We also set $s_{-1} = s$ and $h^0 : \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F})$ equal to pullback along s_{-1}. Then the family of maps $\{h^n\}_{n \geq 0}$ is a homotopy between 1 and 0 on the extended Čech complex. □

96.19. The relative Čech complex

06X7 Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ as in (96.18.0.1). Consider the associated simplicial object $\mathcal{U}_\bullet$ and the maps $f_n : \mathcal{U}_n \to \mathcal{X}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Finally, suppose that $\mathcal{F}$ is a sheaf (of sets) on $\mathcal{X}_\tau$. Then

$$
\begin{tikzcd}
f_{0,*}f_0^{-1}\mathcal{F} \arrow[r, shift left] \arrow[r, shift right] & f_{1,*}f_1^{-1}\mathcal{F} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & f_{2,*}f_2^{-1}\mathcal{F}
\end{tikzcd}
$$

is a cosimplicial sheaf on $\mathcal{X}_\tau$ where we use the pullback maps introduced in Sites, Section 7.45. If $\mathcal{F}$ is an abelian sheaf, then $f_{n,*}f_n^{-1}\mathcal{F}$ form a cosimplicial abelian sheaf on $\mathcal{X}_\tau$. The associated complex (see Simplicial, Section 14.25)

$$\ldots \to 0 \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is called the relative Čech complex associated to the situation. We will denote this complex $\mathcal{K}^\bullet(f, \mathcal{F})$. The extended relative Čech complex is the complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

with $\mathcal{F}$ in degree -1. The extended relative Čech complex is acyclic if and only if the map $\mathcal{F}[0] \to \mathcal{K}^\bullet(f, \mathcal{F})$ is a quasi-isomorphism of complexes of sheaves.

06X8 Remark 96.19.1. We can define the complex $\mathcal{K}^\bullet(f, \mathcal{F})$ also if $\mathcal{F}$ is a presheaf, only we cannot use the reference to Sites, Section 7.45 to define the pullback maps. To explain the pullback maps, suppose given a commutative diagram

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr, "h"] \arrow[dr, "g"'] & & \mathcal{U} \arrow[dl, "f"] \\
& \mathcal{X} &
\end{tikzcd}
$$

of categories fibred in groupoids over $(Sch/S)_{fppf}$ and a presheaf $\mathcal{G}$ on $\mathcal{U}$ we can define the pullback map $f_*\mathcal{G} \to g_*h^{-1}\mathcal{G}$ as the composition

$$f_*\mathcal{G} \longrightarrow f_*h_*h^{-1}\mathcal{G} = g_*h^{-1}\mathcal{G}$$

where the map comes from the adjunction map $\mathcal{G} \to h_*h^{-1}\mathcal{G}$. This works because in our situation the functors h_* and h^{-1} are adjoint in presheaves (and agree with their counter parts on sheaves). See Sections 96.3 and 96.4.

06X9 Lemma 96.19.2. Generalities on relative Čech complexes.

(1) If

$$
\begin{tikzcd}
\mathcal{V} \arrow[r, "h"] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\
\mathcal{Y} \arrow[r, "e"] & \mathcal{X}
\end{tikzcd}
$$

is 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$.

(2) if h and e are equivalences, then the map of (1) is an isomorphism,

(3) if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated relative Čech complexes are isomorphic,

Proof. Literally the same as the proof of Lemma 96.18.1 using the pullback maps of Remark 96.19.1. □

06XA Lemma 96.19.3. If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $\text{id}_\mathcal{X}$ then the extended relative Čech complex is homotopic to zero.

Proof. Literally the same as the proof of Lemma 96.18.2. $\square$

06XB Remark 96.19.4. Let us "compute" the value of the relative Čech complex on an object x of $\mathcal{X}$. Say $p(x) = U$. Consider the 2-fibre product diagram (which serves to introduce the notation $g : \mathcal{V} \to \mathcal{Y}$)

$$\begin{tikzcd} \mathcal{V} \arrow[r, equal] \arrow[d, "g"'] & (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, equal] & (Sch/U)_{fppf} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

Note that the morphism $\mathcal{V}_n \to \mathcal{U}_n$ of the proof of Lemma 96.18.1 induces an equivalence $\mathcal{V}_n = (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}_n$. Hence we see from (96.5.0.1) that

$$\Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, x^{-1}\mathcal{F})$$

In words: The value of the relative Čech complex on an object x of $\mathcal{X}$ is the Čech complex of the base change of f to $\mathcal{X}/x \cong (Sch/U)_{fppf}$. This implies for example that Lemma 96.18.2 implies Lemma 96.19.3 and more generally that results on the (usual) Čech complex imply results for the relative Čech complex.

06XC Lemma 96.19.5. Let

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"'] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$ and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. Then the map $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$ of Lemma 96.19.2 is an isomorphism of complexes of abelian presheaves.

Proof. Let y be an object of $\mathcal{Y}$ lying over the scheme T. Set $x = e(y)$. We are going to show that the map induces an isomorphism on sections over y. Note that

$$\Gamma(y, e^{-1}\mathcal{K}^\bullet(f, \mathcal{F})) = \Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \to (Sch/T)_{fppf}, x^{-1}\mathcal{F})$$

by Remark 96.19.4. On the other hand,

$$\Gamma(y, \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf}, y^{-1}e^{-1}\mathcal{F})$$

also by Remark 96.19.4. Note that $y^{-1}e^{-1}\mathcal{F} = x^{-1}\mathcal{F}$ and since the diagram is 2-cartesian the 1-morphism

$$(Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}$$

is an equivalence. Hence the map on sections over y is an isomorphism by Lemma 96.18.1. $\square$

Exactness can be checked on a "covering".

06XD Lemma 96.19.6. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

be a complex in $\text{Ab}(\mathcal{X}_\tau)$. Assume that

(1) for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$, and
(2) $f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ is exact.

Then the sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact.

Proof. Let x be an object of $\mathcal{X}$ lying over the scheme T. Consider the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T)_\tau$. It suffices to show this sequence is exact. By assumption there exists a τ-covering $\{T_i \to T\}$ such that $x|_{T_i}$ is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$ over T_i and moreover the sequence $u_i^{-1}f^{-1}\mathcal{F} \to u_i^{-1}f^{-1}\mathcal{G} \to u_i^{-1}f^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T_i)_\tau$ is exact. Since $u_i^{-1}f^{-1}\mathcal{F} = x^{-1}\mathcal{F}|_{(Sch/T_i)_\tau}$ we conclude that the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ become exact after localizing at each of the members of a covering, hence the sequence is exact. □

06XE Proposition 96.19.7. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. If

(1) $\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$, and
(2) for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,

then the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is exact in $\text{Ab}(\mathcal{X}_\tau)$.

Proof. By Lemma 96.19.6 it suffices to check exactness after pulling back to $\mathcal{U}$. By Lemma 96.19.5 the pullback of the extended relative Čech complex is isomorphic to the extend relative Čech complex for the morphism $\mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{U}$ and an abelian sheaf on $\mathcal{U}_\tau$. Since there is a section $\Delta_{\mathcal{U}/\mathcal{X}} : \mathcal{U} \to \mathcal{U} \times_\mathcal{X} \mathcal{U}$ exactness follows from Lemma 96.19.3. □

Using this we can construct the Čech-to-cohomology spectral sequence as follows. We first give a technical, precise version. In the next section we give a version that applies only to algebraic stacks.

06XF Lemma 96.19.8. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Assume

(1) $\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,
(2) for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,
(3) the category $\mathcal{U}$ has equalizers, and
(4) the functor f is faithful.

Then there is a first quadrant spectral sequence of abelian groups

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. Before we start the proof we make some remarks. By Lemma 96.17.4 (and induction) all of the categories fibred in groupoids $\mathcal{U}_p$ have equalizers and all of the morphisms $f_p : \mathcal{U}_p \to \mathcal{X}$ are faithful. Let $\mathcal{I}$ be an injective object of $\text{Ab}(\mathcal{X}_\tau)$. By Lemma 96.17.5 we see $f_p^{-1}\mathcal{I}$ is an injective object of $\text{Ab}((\mathcal{U}_p)_\tau)$. Hence $f_{p,*}f_p^{-1}\mathcal{I}$ is an injective object of $\textit{Ab}(\mathcal{X}_\tau)$ by Lemma 96.17.1. Hence Proposition 96.19.7 shows that the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{I} \to f_{0,*}f_0^{-1}\mathcal{I} \to f_{1,*}f_1^{-1}\mathcal{I} \to f_{2,*}f_2^{-1}\mathcal{I} \to \ldots$$

is an exact complex in $\textit{Ab}(\mathcal{X}_\tau)$ all of whose terms are injective. Taking global sections of this complex is exact and we see that the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I})$ is quasi-isomorphic to $\Gamma(\mathcal{X}_\tau, \mathcal{I})[0]$.

With these preliminaries out of the way consider the two spectral sequences associated to the double complex (see Homology, Section 12.25)

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I}^\bullet)$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\textit{Ab}(\mathcal{X}_\tau)$. The discussion above shows that Homology, Lemma 12.25.4 applies which shows that $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ is quasi-isomorphic to the total complex associated to the double complex. By our remarks above the complex $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$. Hence the other spectral sequence is as indicated in the lemma. □

To be sure there is a version for modules as well.

06XG **Lemma 96.19.9.** Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume

1. $\mathcal{F}$ is an object of $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,
2. for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,
3. the category $\mathcal{U}$ has equalizers, and
4. the functor f is faithful.

Then there is a first quadrant spectral sequence of $\Gamma(\mathcal{O}_\mathcal{X})$-modules

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. The proof of this lemma is identical to the proof of Lemma 96.19.8 except that it uses an injective resolution in $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 96.17.6 instead of Lemma 96.17.5. □

Here is a lemma that translates a more usual kind of covering in the kinds of coverings we have encountered above.

06XH **Lemma 96.19.10.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

1. Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then for any object y of $\mathcal{Y}$ there exists an fppf covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.
2. Assume that f is representable by algebraic spaces, surjective, and smooth. Then for any object y of $\mathcal{Y}$ there exists an étale covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.

Proof. Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_\mathcal{Y} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective, flat, and locally of finite presentation. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 67.39.7, 67.39.8, 67.5.4, 67.28.2, and 67.30.3). Hence $\{U \to V\}$ is an fppf covering. Denote x the object of $\mathcal{X}$ over U corresponding to the 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$. Then $\{f(x) \to y\}$ is the desired fppf covering of $\mathcal{Y}$.

Proof of (2). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_{\mathcal{Y}} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective and smooth. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective and smooth (see Morphisms of Spaces, Lemmas 67.39.6, 67.5.4, and 67.37.2). Hence $\{U \to V\}$ is a smooth covering. By More on Morphisms, Lemma 37.38.7 there exists an étale covering $\{V_i \to V\}$ such that each $V_i \to V$ factors through U. Denote x_i the object of $\mathcal{X}$ over V_i corresponding to the 1-morphism

$$(Sch/V_i)_{fppf} \to (Sch/U)_{fppf} \to \mathcal{X}.$$

Then $\{f(x_i) \to y\}$ is the desired étale covering of $\mathcal{Y}$. □

072D Lemma 96.19.11. Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Assume

(1) $\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,
(2) for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,
(3) the category $\mathcal{U}$ has equalizers, and
(4) the functor f is faithful.

Then there is a first quadrant spectral sequence of abelian sheaves on $\mathcal{Y}_\tau$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. Note that the assumptions on $f : \mathcal{U} \to \mathcal{X}$ and $\mathcal{F}$ are identical to those in Lemma 96.19.8. Hence the preliminary remarks made in the proof of that lemma hold here also. These remarks imply in particular that

$$0 \to g_*\mathcal{I} \to (g \circ f_0)_* f_0^{-1}\mathcal{I} \to (g \circ f_1)_* f_1^{-1}\mathcal{I} \to \ldots$$

is exact if $\mathcal{I}$ is an injective object of $\textit{Ab}(\mathcal{X}_\tau)$. Having said this, consider the two spectral sequences of Homology, Section 12.25 associated to the double complex $\mathcal{C}^{\bullet,\bullet}$ with terms

$$\mathcal{C}^{p,q} = (g \circ f_p)_*\mathcal{I}^q$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\textit{Ab}(\mathcal{X}_\tau)$. The first spectral sequence implies, via Homology, Lemma 12.25.4, that $g_*\mathcal{I}^\bullet$ is quasi-isomorphic to the total complex associated to $\mathcal{C}^{\bullet,\bullet}$. Since $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$ (see Lemma 96.17.5) the second spectral sequence has terms $E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F}$ as in the statement of the lemma. □

072E Lemma 96.19.12. Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Assume

(1) $\mathcal{F}$ is an object of $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,

(2) for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,
(3) the category $\mathcal{U}$ has equalizers, and
(4) the functor f is faithful.

Then there is a first quadrant spectral sequence in $\textit{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. The proof is identical to the proof of Lemma 96.19.11 except that it uses an injective resolution in $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 96.17.6 instead of Lemma 96.17.5. $\square$

96.20. Cohomology on algebraic stacks

06XI Let $\mathcal{X}$ be an algebraic stack over S. In the sections above we have seen how to define sheaves for the étale, ..., fppf topologies on $\mathcal{X}$. In fact, we have constructed a site $\mathcal{X}_\tau$ for each $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. There is a notion of an abelian sheaf $\mathcal{F}$ on these sites. In the chapter on cohomology of sites we have explained how to define cohomology. Putting all of this together, let's define the derived global sections or total cohomology

$$R\Gamma_{Zar}(\mathcal{X}, \mathcal{F}), R\Gamma_{\acute{e}tale}(\mathcal{X}, \mathcal{F}), \ldots, R\Gamma_{fppf}(\mathcal{X}, \mathcal{F})$$

as $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\textit{Ab}(\mathcal{X}_\tau)$. The ith cohomology group of $\mathcal{F}$ is the ith cohomology of the total cohomology. We will denote this

$$H^i_{Zar}(\mathcal{X}, \mathcal{F}), H^i_{\acute{e}tale}(\mathcal{X}, \mathcal{F}), \ldots, H^i_{fppf}(\mathcal{X}, \mathcal{F}).$$

It will turn out that $H^i_{\acute{e}tale} = H^i_{smooth}$ because of More on Morphisms, Lemma 37.38.7.

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute its total cohomology, resp. cohomology groups; the end result is quasi-isomorphic, resp. isomorphic to the cohomology of $\mathcal{F}$ viewed as a sheaf of abelian groups by the very general Cohomology on Sites, Lemma 21.12.4.

So far our only tool to compute cohomology groups is the result on Čech complexes proved above. We rephrase it here in the language of algebraic stacks for the étale and the fppf topology. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks. Recall that

$$f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \longrightarrow \mathcal{X}$$

is the structure morphism where there are $(p+1)$-factors. Also, recall that a sheaf on $\mathcal{X}$ is a sheaf for the fppf topology. Note that if $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

06XJ Proposition 96.20.1. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks.

(1) Let $\mathcal{F}$ be an abelian étale sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, and smooth. Then there is a spectral sequence

$$E_1^{p,q} = H^q_{\acute{e}tale}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{\acute{e}tale}(\mathcal{X}, \mathcal{F})$$

(2) Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then there is a spectral sequence

$$E_1^{p,q} = H^q_{fppf}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{fppf}(\mathcal{X}, \mathcal{F})$$

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 96.19.8. The 1-morphism f is faithful by Algebraic Stacks, Lemma 94.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 96.17.2. To see (2) apply Lemma 96.19.10. Condition (1) is satisfied by fiat. □

96.21. Higher direct images and algebraic stacks

072F Let $g : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. In the sections above we have constructed a morphism of ringed topoi $g : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. In the chapter on cohomology of sites we have explained how to define higher direct images. Hence the total direct image $Rg_*\mathcal{F}$ is defined as $g_*\mathcal{I}^\bullet$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\text{Ab}(\mathcal{X}_\tau)$. The ith higher direct image $R^ig_*\mathcal{F}$ is the ith cohomology of the total direct image. Important: it matters which topology τ is used here!

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute total direct image and higher direct images.

So far our only tool to compute the higher direct images of g_* is the result on Čech complexes proved above. This requires the choice of a "covering" $f : \mathcal{U} \to \mathcal{X}$. If $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

072G Proposition 96.21.1. Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of algebraic stacks.

(1) Assume that f is representable by algebraic spaces, surjective and smooth.
   (a) If $\mathcal{F}$ is in $\text{Ab}(\mathcal{X}_{étale})$ then there is a spectral sequence
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $\text{Ab}(\mathcal{Y}_{étale})$ with higher direct images computed in the étale topology.
   (b) If $\mathcal{F}$ is in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ then there is a spectral sequence
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $\text{Mod}(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y})$.
(2) Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation.
   (a) If $\mathcal{F}$ is in $\text{Ab}(\mathcal{X})$ then there is a spectral sequence
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $\text{Ab}(\mathcal{Y})$ with higher direct images computed in the fppf topology.
   (b) If $\mathcal{F}$ is in $\text{Mod}(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $\text{Mod}(\mathcal{O}_\mathcal{Y})$.

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 96.19.11 and Lemma 96.19.12. The 1-morphism f is faithful by Algebraic Stacks, Lemma 94.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 96.17.2. To see (2) apply Lemma 96.19.10. Condition (1) is satisfied by fiat in all four cases. $\square$

Here is a description of higher direct images for a morphism of algebraic stacks.

075G **Lemma 96.21.2.** Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks[5] over S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be an object of $\textit{Ab}(\mathcal{X}_\tau)$ or $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \text{pr}^{-1}\mathcal{F}\Big)$$

Here y is an object of $\mathcal{Y}$ lying over the scheme V.

Proof. Choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. By the formula for pushforward (96.5.0.1) we see that $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{matrix} \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \text{pr}^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \text{pr}^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \text{pr}^{-1}\mathcal{I}^{i+1}\Big) \end{matrix}$$

Since pr^{-1} is exact, it suffices to show that pr^{-1} preserves injectives. This follows from Lemmas 96.17.5 and 96.17.6 as well as the fact that pr is a representable morphism of algebraic stacks (so that pr is faithful by Algebraic Stacks, Lemma 94.15.2 and that $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ has equalizers by Lemma 96.17.2). $\square$

Here is a trivial base change result.

075H **Lemma 96.21.3.** Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let

$$\begin{tikzcd} \mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of algebraic stacks over S. Then the base change map is an isomorphism

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

functorial for $\mathcal{F}$ in $\textit{Ab}(\mathcal{X}_\tau)$ or $\mathcal{F}$ in $\textit{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. The isomorphism $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$ is Lemma 96.5.1 (and it holds for arbitrary presheaves). For the total direct images, there is a base change map because the morphisms g and g' are flat, see Cohomology on Sites, Section 21.15.

[5]This result should hold for any 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

To see that this map is a quasi-isomorphism we can use that for an object y' of $\mathcal{Y}'$ over a scheme V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

We conclude that the induced map $g^{-1}R^if_*\mathcal{F} \to R^if'_*(g')^{-1}\mathcal{F}$ is an isomorphism by Lemma 96.21.2. $\square$

96.22. Comparison

073L In this section we collect some results on comparing cohomology defined using stacks and using algebraic spaces.

075L Lemma 96.22.1. Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S representable by the algebraic space F.

(1) If $\mathcal{I}$ injective in $\text{Ab}(\mathcal{X}_{étale})$, then $\mathcal{I}|_{F_{étale}}$ is injective in $\text{Ab}(F_{étale})$,
(2) If $\mathcal{I}^\bullet$ is a K-injective complex in $\text{Ab}(\mathcal{X}_{étale})$, then $\mathcal{I}^\bullet|_{F_{étale}}$ is a K-injective complex in $\text{Ab}(F_{étale})$.

The same does not hold for modules.

Proof. This follows formally from the fact that the restriction functor $\pi_{F,*} = i_F^{-1}$ (see Lemma 96.10.1) is right adjoint to the exact functor π_F^{-1}, see Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9. To see that the lemma does not hold for modules, we refer the reader to Étale Cohomology, Lemma 59.99.1. $\square$

075N Lemma 96.22.2. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces.

(1) For any $\mathcal{F} \in \text{Ab}(\mathcal{X}_{étale})$ we have
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(G_{étale})$.
(2) For any object $\mathcal{F}$ of $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(\mathcal{O}_G)$.

Proof. Part (1) follows immediately from Lemma 96.22.1 and (96.10.3.1) on choosing an injective resolution of $\mathcal{F}$.

Part (2) can be proved as follows. In Lemma 96.10.3 we have seen that $\pi_G \circ f = f_{small} \circ \pi_F$ as morphisms of ringed sites. Hence we obtain $R\pi_{G,*} \circ Rf_* = Rf_{small,*} \circ R\pi_{F,*}$ by Cohomology on Sites, Lemma 21.19.2. Since the restriction functors $\pi_{F,*}$ and $\pi_{G,*}$ are exact, we conclude. $\square$

075P Lemma 96.22.3. Let S be a scheme. Consider a 2-fibre product square

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks over S. Assume that f is representable by algebraic spaces and that $\mathcal{Y}'$ is representable by an algebraic space G'. Then $\mathcal{X}'$ is representable by an algebraic space F' and denoting $f' : F' \to G'$ the induced morphism of algebraic spaces we have

$$g^{-1}(Rf_*\mathcal{F})|_{G'_{\acute{e}tale}} = Rf'_{small,*}((g')^{-1}\mathcal{F}|_{F'_{\acute{e}tale}})$$

for any $\mathcal{F}$ in $\textit{Ab}(\mathcal{X}_{\acute{e}tale})$ or in $\textit{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$

Proof. Follows formally on combining Lemmas 96.21.3 and 96.22.2. $\square$

96.23. Change of topology

075Q Here is a technical lemma which tells us that the fppf cohomology of a locally quasi-coherent sheaf is equal to its étale cohomology provided the comparison maps are isomorphisms for morphisms of $\mathcal{X}$ lying over flat morphisms.

076T **Lemma 96.23.1.** Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Assume

(a) $\mathcal{F}$ is locally quasi-coherent, and
(b) for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (96.9.4.1) is an isomorphism.

Then $\mathcal{F}$ is a sheaf for the fppf topology.

Proof. Let $\{x_i \to x\}$ be an fppf covering of $\mathcal{X}$ lying over the fppf covering $\{f_i : U_i \to U\}$ of schemes over S. By assumption the restriction $\mathcal{G} = \mathcal{F}|_{U_{\acute{e}tale}}$ is quasi-coherent and the comparison maps $f_{i,small}^*\mathcal{G} \to \mathcal{F}|_{U_{i,\acute{e}tale}}$ are isomorphisms. Hence the sheaf condition for $\mathcal{F}$ and the covering $\{x_i \to x\}$ is equivalent to the sheaf condition for $\mathcal{G}^a$ on $(Sch/U)_{fppf}$ and the covering $\{U_i \to U\}$ which holds by Descent, Lemma 35.8.1. $\square$

075R **Lemma 96.23.2.** Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf $\mathcal{O}_\mathcal{X}$-module such that

(a) $\mathcal{F}$ is locally quasi-coherent, and
(b) for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation, the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (96.9.4.1) is an isomorphism.

Then $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module and we have the following

(1) If $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{\acute{e}tale}$ is the comparison morphism, then $R\epsilon_*\mathcal{F} = \epsilon_*\mathcal{F}$.
(2) The cohomology groups $H^p_{fppf}(\mathcal{X}, \mathcal{F})$ are equal to the cohomology groups computed in the étale topology on $\mathcal{X}$. Similarly for the cohomology groups $H^p_{fppf}(x, \mathcal{F})$ and the derived versions $R\Gamma(\mathcal{X}, \mathcal{F})$ and $R\Gamma(x, \mathcal{F})$.
(3) If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ then $R^if_*\mathcal{F}$ is equal to the fppf-sheafification of the higher direct image computed in the étale cohomology. Similarly for derived pullback.

Proof. The assertion that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module follows from Lemma 96.23.1. Note that ϵ is a morphism of sites given by the identity functor on $\mathcal{X}$. The sheaf $R^p\epsilon_*\mathcal{F}$ is therefore the sheaf associated to the presheaf $x \mapsto H^p_{fppf}(x, \mathcal{F})$, see Cohomology on Sites, Lemma 21.7.4. To prove (1) it suffices to show that $H^p_{fppf}(x, \mathcal{F}) = 0$ for $p > 0$ whenever x lies over an affine scheme U. By Lemma 96.16.1 we have $H^p_{fppf}(x, \mathcal{F}) = H^p((Sch/U)_{fppf}, x^{-1}\mathcal{F})$. Combining Descent, Lemma 35.12.4 with

Cohomology of Schemes, Lemma 30.2.2 we see that these cohomology groups are zero.

We have seen above that $\epsilon_*\mathcal{F}$ and $\mathcal{F}$ are the sheaves on $\mathcal{X}_{\acute{e}tale}$ and $\mathcal{X}_{fppf}$ corresponding to the same presheaf on $\mathcal{X}$ (and this is true more generally for any sheaf in the fppf topology on $\mathcal{X}$). We often abusively identify $\mathcal{F}$ and $\epsilon_*\mathcal{F}$ and this is the sense in which parts (2) and (3) of the lemma should be understood. Thus part (2) follows formally from (1) and the Leray spectral sequence, see Cohomology on Sites, Lemma 21.14.6.

Finally we prove (3). The sheaf $R^if_*\mathcal{F}$ (resp. $Rf_{\acute{e}tale,*}\mathcal{F}$) is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

where τ is $fppf$ (resp. *étale*), see Lemma 96.21.2. Note that $\mathrm{pr}^{-1}\mathcal{F}$ satisfies properties (a) and (b) also (by Lemmas 96.12.3 and 96.9.3), hence these two presheaves are equal by (2). This immediately implies (3). □

We will use the following lemma to compare étale cohomology of sheaves on algebraic stacks with cohomology on the lisse-étale topos.

07AK Lemma 96.23.3. Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\tau = \acute{e}tale$ (resp. $\tau = fppf$). Let $\mathcal{X}' \subset \mathcal{X}$ be a full subcategory with the following properties

1. if $x \to x'$ is a morphism of $\mathcal{X}$ which lies over a smooth (resp. flat and locally finitely presented) morphism of schemes and $x' \in \mathrm{Ob}(\mathcal{X}')$, then $x \in \mathrm{Ob}(\mathcal{X}')$, and
2. there exists an object $x \in \mathrm{Ob}(\mathcal{X}')$ lying over a scheme U such that the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective.

We get a site $\mathcal{X}'_\tau$ by declaring a covering of $\mathcal{X}'$ to be any family of morphisms $\{x_i \to x\}$ in $\mathcal{X}'$ which is a covering in $\mathcal{X}_\tau$. Then the inclusion functor $\mathcal{X}' \to \mathcal{X}_\tau$ is fully faithful, cocontinuous, and continuous, whence defines a morphism of topoi

$$g : Sh(\mathcal{X}'_\tau) \longrightarrow Sh(\mathcal{X}_\tau)$$

and $H^p(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) = H^p(\mathcal{X}_\tau, \mathcal{F})$ for all $p \geq 0$ and all $\mathcal{F} \in \mathrm{Ab}(\mathcal{X}_\tau)$.

Proof. Note that assumption (1) implies that if $\{x_i \to x\}$ is a covering of $\mathcal{X}_\tau$ and $x \in \mathrm{Ob}(\mathcal{X}')$, then we have $x_i \in \mathrm{Ob}(\mathcal{X}')$. Hence we see that $\mathcal{X}' \to \mathcal{X}$ is continuous and cocontinuous as the coverings of objects of $\mathcal{X}'_\tau$ agree with their coverings seen as objects of $\mathcal{X}_\tau$. We obtain the morphism g and the functor g^{-1} is identified with the restriction functor, see Sites, Lemma 7.21.5.

In particular, if $\{x_i \to x\}$ is a covering in $\mathcal{X}'_\tau$, then for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}$ then

$$\check{H}^p(\{x_i \to x\}, g^{-1}\mathcal{F}) = \check{H}^p(\{x_i \to x\}, \mathcal{F})$$

Thus if $\mathcal{I}$ is an injective abelian sheaf on $\mathcal{X}_\tau$ then we see that the higher Čech cohomology groups are zero (Cohomology on Sites, Lemma 21.10.2). Hence $H^p(x, g^{-1}\mathcal{I}) = 0$ for all objects x of $\mathcal{X}'$ (Cohomology on Sites, Lemma 21.10.9). In other words injective abelian sheaves on $\mathcal{X}_\tau$ are right acyclic for the functor $H^0(x, g^{-1}-)$. It follows that $H^p(x, g^{-1}\mathcal{F}) = H^p(x, \mathcal{F})$ for all $\mathcal{F} \in \mathrm{Ab}(\mathcal{X})$ and all $x \in \mathrm{Ob}(\mathcal{X}')$.

Choose an object $x \in \mathcal{X}'$ lying over a scheme U as in assumption (2). In particular $\mathcal{X}/x \to \mathcal{X}$ is a morphism of algebraic stacks which representable by algebraic spaces, surjective, and smooth. (Note that $\mathcal{X}/x$ is equivalent to $(Sch/U)_{fppf}$, see Lemma 96.9.1.) The map of sheaves

$$h_x \longrightarrow *$$

in $Sh(\mathcal{X}_\tau)$ is surjective. Namely, for any object x' of $\mathcal{X}$ there exists a τ-covering $\{x'_i \to x'\}$ such that there exist morphisms $x'_i \to x$, see Lemma 96.19.10. Since g is exact, the map of sheaves

$$g^{-1}h_x \longrightarrow * = g^{-1}*$$

in $Sh(\mathcal{X}'_\tau)$ is surjective also. Let $h_{x,n}$ be the $(n+1)$-fold product $h_x \times \ldots \times h_x$. Then we have spectral sequences

07AL (96.23.3.1) $$E_1^{p,q} = H^q(h_{x,p}, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

and

07AM (96.23.3.2) $$E_1^{p,q} = H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}'_\tau, g^{-1}\mathcal{F})$$

see Cohomology on Sites, Lemma 21.13.2.

Case I: $\mathcal{X}$ has a final object x which is also an object of $\mathcal{X}'$. This case follows immediately from the discussion in the second paragraph above.

Case II: $\mathcal{X}$ is representable by an algebraic space F. In this case the sheaves $h_{x,n}$ are representable by an object x_n in $\mathcal{X}$. (Namely, if $\mathcal{S}_F = \mathcal{X}$ and $x : U \to F$ is the given object, then $h_{x,n}$ is representable by the object $U \times_F \ldots \times_F U \to F$ of $\mathcal{S}_F$.) It follows that $H^q(h_{x,p}, \mathcal{F}) = H^q(x_p, \mathcal{F})$. The morphisms $x_n \to x$ lie over smooth morphisms of schemes, hence $x_n \in \mathcal{X}'$ for all n. Hence $H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) = H^q(x_p, g^{-1}\mathcal{F})$. Thus in the two spectral sequences (96.23.3.1) and (96.23.3.2) above the $E_1^{p,q}$ terms agree by the discussion in the second paragraph. The lemma follows in Case II as well.

Case III: $\mathcal{X}$ is an algebraic stack. We claim that in this case the cohomology groups $H^q(h_{x,p}, \mathcal{F})$ and $H^q(g^{-1}h_{x,n}, g^{-1}\mathcal{F})$ agree by Case II above. Once we have proved this the result will follow as before.

Namely, consider the category $\mathcal{X}/h_{x,n}$, see Sites, Lemma 7.30.3. Since $h_{x,n}$ is the $(n+1)$-fold product of h_x an object of this category is an $(n+2)$-tuple $(y, s_0, \ldots, s_n)$ where y is an object of $\mathcal{X}$ and each $s_i : y \to x$ is a morphism of $\mathcal{X}$. This is a category over $(Sch/S)_{fppf}$. There is an equivalence

$$\mathcal{X}/h_{x,n} \longrightarrow (Sch/U)_{fppf} \times_\mathcal{X} \ldots \times_\mathcal{X} (Sch/U)_{fppf} =: \mathcal{U}_n$$

over $(Sch/S)_{fppf}$. Namely, if $x : (Sch/U)_{fppf} \to \mathcal{X}$ also denotes the 1-morphism associated with x and $p : \mathcal{X} \to (Sch/S)_{fppf}$ the structure functor, then we can think of $(y, s_0, \ldots, s_n)$ as $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ where y is an object of $\mathcal{X}$, $f_i : p(y) \to p(x)$ is a morphism of schemes, and $\alpha_i : y \to x(f_i)$ an isomorphism. The category of $2n+3$-tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ is an incarnation of the $(n+1)$-fold fibred product $\mathcal{U}_n$ of algebraic stacks displayed above, as we discussed in Section 96.18. By Cohomology on Sites, Lemma 21.13.3 we have

$$H^p(\mathcal{U}_n, \mathcal{F}|_{\mathcal{U}_n}) = H^p(\mathcal{X}/h_{x,n}, \mathcal{F}|_{\mathcal{X}/h_{x,n}}) = H^p(h_{x,n}, \mathcal{F}).$$

Finally, we discuss the "primed" analogue of this. Namely, $\mathcal{X}'/h_{x,n}$ corresponds, via the equivalence above to the full subcategory $\mathcal{U}'_n \subset \mathcal{U}_n$ consisting of those tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ with $y \in \mathcal{X}'$. Hence certainly property (1) of the statement of the lemma holds for the inclusion $\mathcal{U}'_n \subset \mathcal{U}_n$. To see property (2) choose an object $\xi = (y, s_0, \ldots, s_n)$ which lies over a scheme W such that $(Sch/W)_{fppf} \to$

$\mathcal{U}_n$ is smooth and surjective (this is possible as $\mathcal{U}_n$ is an algebraic stack). Then $(Sch/W)_{fppf} \to \mathcal{U}_n \to (Sch/U)_{fppf}$ is smooth as a composition of base changes of the morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 94.10.6 and 94.10.5. Thus axiom (1) for $\mathcal{X}$ implies that y is an object of $\mathcal{X}'$ whence ξ is an object of $\mathcal{U}'_n$. Using again

$$H^p(\mathcal{U}'_n, \mathcal{F}|_{\mathcal{U}'_n}) = H^p(\mathcal{X}'/h_{x,n}, \mathcal{F}|_{\mathcal{X}'/h_{x,n}}) = H^p(g^{-1}h_{x,n}, g^{-1}\mathcal{F}).$$

we now can use Case II for $\mathcal{U}'_n \subset \mathcal{U}_n$ to conclude. □

96.24. Restricting to affines

0H08 In this section, given a category $\mathcal{X}$ fibred in groupoids over $(Sch/S)_{fppf}$ we will consider the full subcategory $\mathcal{X}_{affine}$ of $\mathcal{X}$ consisting of objects x lying over affine schemes U. We will see how, for any topology τ finer than the Zariski topology, the category of sheaves on $\mathcal{X}$ and $\mathcal{X}_{affine,\tau}$ agree.

0H09 Definition 96.24.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The associated affine site is the full subcategory $\mathcal{X}_{affine}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that U is affine. The topology on $\mathcal{X}_{affine}$ will be the chaotic one, i.e., such that sheaves on $\mathcal{X}_{affine}$ are the same as presheaves.

Thus the functor $p : \mathcal{X} \to (Sch/S)_{fppf}$ restricts to a functor

$$p : \mathcal{X}_{affine} \longrightarrow (\mathrm{Aff}/S)_{fppf}$$

where the notation on the right hand side is the one introduced in Topologies, Definition 34.7.8. It is clear that $\mathcal{X}_{affine}$ is fibred in groupoids over $(\mathrm{Aff}/S)_{fppf}$. It follows that $\mathcal{X}_{affine}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology from $(\mathrm{Aff}/S)_{Zar}$, $(\mathrm{Aff}/S)_{étale}$, $(\mathrm{Aff}/S)_{smooth}$, $(\mathrm{Aff}/S)_{syntomic}$, and $(\mathrm{Aff}/S)_{fppf}$, see Stacks, Definition 8.10.2.

0H0A Definition 96.24.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) The associated affine Zariski site $\mathcal{X}_{affine,Zar}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(\mathrm{Aff}/S)_{Zar}$.
(2) The associated affine étale site $\mathcal{X}_{affine,étale}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(\mathrm{Aff}/S)_{étale}$.
(3) The associated affine smooth site $\mathcal{X}_{affine,smooth}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(\mathrm{Aff}/S)_{smooth}$.
(4) The associated affine syntomic site $\mathcal{X}_{affine,syntomic}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(\mathrm{Aff}/S)_{syntomic}$.
(5) The associated affine fppf site $\mathcal{X}_{affine,fppf}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(\mathrm{Aff}/S)_{fppf}$.

This definition makes sense by the discussion above. For each $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ a family of morphisms $\{x_i \to x\}_{i\in I}$ with fixed target in $\mathcal{X}_{affine}$ is a covering in $\mathcal{X}_{affine,\tau}$ if and only if the family of morphisms $\{p(x_i) \to p(x)\}_{i\in I}$ of affine schemes is a standard τ-covering as defined in Topologies, Definitions 34.3.4, 34.4.5, 34.5.5, 34.6.5, and 34.7.5.

0H0B Lemma 96.24.3. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The functor $\mathcal{X}_{affine,\tau} \to \mathcal{X}_\tau$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh(\mathcal{X}_{affine,\tau})$ to $Sh(\mathcal{X}_\tau)$.

Proof. Omitted. Hint: the proof is exactly the same as the proof of Topologies, Lemmas 34.3.10, 34.4.11, 34.5.9, 34.6.9, and 34.7.11. $\square$

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let us denote $\mathcal{O}$ the restriction of $\mathcal{O}_\mathcal{X}$ to $\mathcal{X}_{affine}$. Then $\mathcal{O}$ is a sheaf in the Zariski, étale, smooth, syntomic, and fppf topologies on $\mathcal{X}_{affine}$. Furthermore, the equivalence of topoi of Lemma 96.24.3 extends to an equivalence

0H0C (96.24.3.1) $$(Sh(\mathcal{X}_{affine,\tau}), \mathcal{O}) \longrightarrow (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X})$$

of ringed topoi for $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$.

96.25. Quasi-coherent modules and affines

0H0D Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. In Section 96.24 we have associated to this a ringed site $(\mathcal{X}_{affine}, \mathcal{O})$.

0H0E Lemma 96.25.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $\mathcal{X}_{affine}$. The following are equivalent

1. for every morphism $x \to x'$ of $\mathcal{X}_{affine}$ the map $\mathcal{F}(x') \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \to \mathcal{F}(x)$ is an isomorphism,
2. $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{affine}, \mathcal{O})$ in the sense of Modules on Sites, Definition 18.23.1,
3. $\mathcal{F}$ is a sheaf for the Zariski topology on $\mathcal{X}_{affine}$ and a quasi-coherent module on $(\mathcal{X}_{affine,Zar}, \mathcal{O})$ in the sense of Modules on Sites, Definition 18.23.1,
4. same as in (3) for the étale topology,
5. same as in (3) for the smooth topology,
6. same as in (3) for the syntomic topology,
7. same as in (3) for the fppf topology, and
8. $\mathcal{F}$ corresponds to a quasi-coherent module on $\mathcal{X}$ via the equivalence (96.24.3.1).

Proof. To make sense out of part (2), recall that $\mathcal{X}_{affine}$ is a site gotten by endowing the category $\mathcal{X}_{affine}$ with the chaotic topology (Definition 96.24.1) and hence a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is the same thing as a presheaf of $\mathcal{O}$-modules. Conditions (1) and (2) are equivalent by Modules on Sites, Lemma 18.24.2. Observe that for $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$ the presheaf $\mathcal{F}$ is a τ-sheaf if and only if for all $x \in \text{Ob}(\mathcal{X}_{affine})$ the restriction to $\mathcal{X}_{affine}/x$ is a τ-sheaf. Set $U = p(x)$. Similarly to the discussion in Section 96.9 the object x of $\mathcal{X}_{affine}$ induces an equivalence $\mathcal{X}_{affine,\acute{e}tale}/x \to (\text{Aff}/U)_{\acute{e}tale}$ of sites. In this way we see that the equivalence of (1) with (3) – (7) follows from Descent, Lemma 35.11.1 applied to each of these sites. The equivalence of (8) and (7) is immediate from the fact that "being quasi-coherent" is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18.18 $\square$

0H0F Lemma 96.25.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $\mathcal{X}_{affine}$. The following are equivalent

1. for every morphism $x \to x'$ of $\mathcal{X}_{affine}$ such that $p(x) \to p(x')$ is an étale morphism (of affine schemes), the map $\mathcal{F}(x') \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \to \mathcal{F}(x)$ is an isomorphism,
2. $\mathcal{F}$ is a sheaf for the étale topology on $\mathcal{X}_{affine}$ and for every object x of $\mathcal{X}_{affine}$ the restriction $x^*\mathcal{F}|_{U_{affine,\acute{e}tale}}$ is quasi-coherent where $U = p(x)$,

(3) $\mathcal{F}$ corresponds to a locally quasi-coherent module on $\mathcal{X}$ via the equivalence (96.24.3.1) for the étale topology.

Proof. To make sense out of condition (2), recall that $U_{affine,étale}$ is the full subcategory of $U_{étale}$ consisting of affine objects, see Topologies, Definition 34.4.8. Similarly to the discussion in Section 96.9 the object x of $\mathcal{X}_{affine}$ induces an equivalence $\mathcal{X}_{affine,étale}/x \to (\text{Aff}/U)_{étale}$ of sites. Then $x^*\mathcal{F}$ is the sheaf of modules on $(\text{Aff}/U)_{étale}$ corresponding to the restriction $\mathcal{F}|_{\mathcal{X}_{affine,étale}/x}$. Finally, using the continuous and cocontinuous inclusion functor $U_{affine,étale} \to (\text{Aff}/U)_{étale}$ we can further restrict and obtain $x^*\mathcal{F}|_{U_{affine,étale}}$.

The equivalence of (1) and (2) follows from the remarks above and Descent, Lemma 35.11.2 applied to the restriction of $\mathcal{F}$ to $U_{affine,étale}$ for every object x of $\mathcal{X}$ lying over an affine scheme U. The equivalence of (2) and (3) is immediate from the definitions and the fact that quasi-coherent modules on $U_{affine,étale}$ and $U_{étale}$ correspond (again by Descent, Lemma 35.11.2 for example). $\square$

96.26. Quasi-coherent objects in the derived category

0H0G Algebraic geometers have contemplated invariants for non-representable functors X (valued in sets or groupoids) on Sch/S for decades. For instance, before the notion of a stack was invented, Mumford defined [Mum65] the Picard groupoid $\text{Pic}(X)$ for the moduli functor X of elliptic curves as the 2-limit $Pic(U)$ over the category of all schemes U equipped with a map to X (i.e., with a family of elliptic curves). Similarly, Beilinson-Drinfeld defined [BD] the category $QCoh(X)$ for an ind-scheme $X = \text{colim}\, X_i$ as the 2-limit $\lim QCoh(X_i)$. This strategy is sufficient for defining 1-categorical invariants like $QCoh(-)$, but inadequate for derived categorical ones (such as the quasi-coherent derived category) as 2-limits of triangulated categories are poorly behaved. With the advent of higher categorical technology and derived algebraic geometry, this problem can be resolved gracefully: one can define the quasi-coherent derived ∞-category $\mathcal{D}_{qc}(X)$ of the functor X as the limit $\lim \mathcal{D}_{qc}(U)$, where U ranges over all derived affines over X (see [Lur04]).

The goal of this section is to attach a triangulated category $QC(X)$ to a functor X (valued in sets or groupoids) as above. In fact, the construction works for any category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids (not just split ones). In good cases, the category $QC(\mathcal{X})$ can be shown to agree with the homotopy category of $\mathcal{D}_{qc}(\mathcal{X})$, though it is outside the scope of this document to explain this comparison. The salient features of the construction are:

(a) $QC(\mathcal{X})$ is a full subcategory of $D(\mathcal{X}_{affine}, \mathcal{O})$ by construction,
(b) $QC(\mathcal{X})$ agrees with $D_{QCoh}(\mathcal{O}_X)$ when $\mathcal{X}$ is representable by the algebraic space X,
(c) $QC(\mathcal{X})$ agrees with $D_{QCoh}(\mathcal{O}_\mathcal{X})$ when $\mathcal{X}$ is an algebraic stack,
(d) when $X = \text{Spf}(A)$ is an affine formal algebraic space attached to a noetherian ring A equipped with the I-adic topology for an ideal I, the triangulated category $QC(X)$ agrees with the full subcategory $D_{comp}(A, I) \subset D(A)$ of derived complete objects.

These results are proven in Proposition 96.26.4, Derived Categories of Stacks, Proposition 104.8.4, and Proposition 96.26.5.

As a motivation for the precise definition of $QC(\mathcal{X})$ we point the reader to the characterization, in Lemma 96.25.1, of quasi-coherent modules on $\mathcal{X}$ as presheaves of $\mathcal{O}$-modules on $\mathcal{X}_{affine}$ which satisfy a kind of base change property.

0H0H Definition 96.26.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{O}$ be the sheaf of rings on $\mathcal{X}_{affine}$ introduced in Section 96.24. We define the triangulated category of quasi-coherent objects in the derived category by the formula

$$QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O})$$

where the right hand side is as defined in Cohomology on Sites, Definition 21.43.1.

Note that this makes sense as $\mathcal{X}_{affine}$ is a category and is viewed as a site by endowing it with the chaotic topology and $\mathcal{O}$ is a sheaf of rings on this category, exactly as required in Cohomology on Sites, Definition 21.43.1.

The relationship of this definition with the category of quasi-coherent modules on $\mathcal{X}$ is not so clear in general! For example, suppose that M is an object of $QC(\mathcal{X})$. Then the cohomology sheaves $H^i(M)$ of M are (pre)sheaves of $\mathcal{O}$-modules on $\mathcal{X}_{affine}$, but in general they are not quasi-coherent. The last nonvanishing cohomology sheaf is quasi-coherent however.

0H0I Lemma 96.26.2. In the situation of Definition 96.26.1 suppose that M is an object of $QC(\mathcal{X})$ and $b \in \mathbf{Z}$ such that $H^i(M) = 0$ for all $i > b$. Then $H^b(M)$ is a quasi-coherent module on $(\mathcal{X}_{affine}, \mathcal{O})$, see Lemma 96.25.1.

Proof. Special case of Cohomology on Sites, Lemma 21.43.3. □

0H0J Lemma 96.26.3. Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The comparision morphism $\epsilon : \mathcal{X}_{affine,\acute{e}tale} \to \mathcal{X}_{affine}$ satisfies the assumptions and conclusions of Cohomology on Sites, Lemma 21.43.12.

Proof. Assumption (1) holds by definition of $\mathcal{X}_{affine}$. For condition (2) we use that for $x \in \text{Ob}(\mathcal{X})$ lying over the affine scheme $U = p(x)$ we have an equivalence $\mathcal{X}_{affine,\acute{e}tale}/x = (\text{Aff}/U)_{\acute{e}tale}$ compatible with structure sheaves; see discussion in Section 96.9. Thus it suffices to show: given an affine scheme $U = \text{Spec}(R)$ and a complex of R-modules $M^\bullet$ the total cohomology of the complex of modules on $(\text{Aff}/U)_{\acute{e}tale}$ associated to $M^\bullet$ is quasi-isomorphic to $M^\bullet$. This follows from a combination of: Derived Categories of Schemes, Lemma 36.3.5 (total cohomology of complexes of modules over affines in the Zariski topology), Derived Categories of Spaces, Remark 75.6.3 (agreement between total cohomology in small Zariski and étale topologies for quasi-coherent complexes of modules), and Étale Cohomology, Lemma 59.99.3 (to see that the étale cohomology of a complex of modules on the big étale site of a scheme may be computed after restricting to the small étale site). □

If we apply the definition in case our category fibred in groupoids $\mathcal{X}$ is representable by an algebraic space X, then we recover $D_{QCoh}(\mathcal{O}_X)$. We will later state and prove the analogous result for algebraic stacks (insert future reference here).

0H0K Proposition 96.26.4. Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representable by an algebraic space X. Then $QC(\mathcal{X})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_X)$.

Proof. Denote X_{affine} the category of affine schemes étale over X endowed with the chaotic topology and its structure sheaf $\mathcal{O}_X$, see Derived Categories of Spaces, Section 75.30. The functor $u : X_{\acute{e}tale} \to \mathcal{X}_{\acute{e}tale}$ of Lemma 96.10.1 gives rise to a functor $X_{affine} \to \mathcal{X}_{affine}$. This is compatible with structure sheaves and produces a functor

$$G : QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O}) \longrightarrow QC(X_{affine}, \mathcal{O}_X)$$

See Cohomology on Sites, Lemma 21.43.10. By Derived Categories of Spaces, Lemma 75.30.1 the triangulated category $QC(X_{affine}, \mathcal{O}_X)$ is equivalent to $D_{QCoh}(\mathcal{O}_X)$. Hence it suffices to prove that G is an equivalence.

Consider the flat comparision morphisms $\epsilon_{\mathcal{X}} : \mathcal{X}_{affine,\acute{e}tale} \to \mathcal{X}_{affine}$ and $\epsilon_X : X_{affine,\acute{e}tale} \to X_{affine}$ of ringed sites. Lemma 96.26.3 and (the proof of) Derived Categories of Spaces, Lemma 75.30.1 show that the functors $\epsilon_{\mathcal{X}}^*$ and ϵ_X^* identify $QC(\mathcal{X}_{affine}, \mathcal{O})$ and $QC(X_{affine}, \mathcal{O}_X)$ with subcategories $Q_{\mathcal{X}} \subset D(\mathcal{X}_{affine,\acute{e}tale}, \mathcal{O})$ and $Q_X \subset D(X_{affine,\acute{e}tale}, \mathcal{O}_X)$. With these identifications the functor G in the first paragraph is induced by the functor

$$Li_X^* = R\pi_{X,*} : D(\mathcal{X}_{affine,\acute{e}tale}, \mathcal{O}) \longrightarrow D(X_{affine,\acute{e}tale}, \mathcal{O}_X)$$

where i_X and π_X are the morphisms from Lemma 96.10.1 but with the étale sites replaced by the corresponding affine ones. The reader can show that this replacement is permissible either by reproving the lemma for the affine sites directly or by using the equivalences of topoi $Sh(\mathcal{X}_{affine,\acute{e}tale}) = Sh(\mathcal{X}_{\acute{e}tale})$ and $Sh(X_{affine,\acute{e}tale}) = Sh(X_{\acute{e}tale})$. The lemma also tells us Li_X^* has a left adjoint

$$L\pi_X^* : D(X_{affine,\acute{e}tale}, \mathcal{O}_X) \longrightarrow D(\mathcal{X}_{affine,\acute{e}tale}, \mathcal{O})$$

and moreover we have $Li_X^* \circ L\pi_X^* = \text{id}$ since $\pi_X \circ i_X$ is the identity. Thus it suffices to show that (a) $L\pi_X^*$ sends Q_X into $Q_{\mathcal{X}}$ and (b) the kernel of Li_X^* is 0. See Derived Categories, Lemma 13.7.2.

Proof of (a). By Derived Categories of Spaces, Lemma 75.30.1 we have $Q_X = D_{QCoh}(X_{affine,\acute{e}tale}, \mathcal{O}_X)$. Let K be an object of Q_X. Let x be an object of $\mathcal{X}_{affine,\acute{e}tale}$ lying over the affine scheme $U = p(x)$. Denote $f : U \to X$ the morphism corresponding to x. Then we see that

$$R\Gamma(x, L\pi_X^*K) = R\Gamma(U, Lf^*K)$$

This follows from transitivity of pullbacks; see discussion in Section 96.10. Next, suppose that $x \to x'$ is a morphism of $\mathcal{X}_{affine,\acute{e}tale}$ lying over the morphism $h : U \to U'$ of affine schemes. As before denote $f : U \to X$ and $f' : U' \to X$ the morphisms corresponding to x and x' so that we have $f = f' \circ h$. Then

$$\begin{aligned} R\Gamma(x, L\pi_X^*K) &= R\Gamma(U, Lf^*K) \\ &= R\Gamma(U, Lh^*L(f')^*K) \\ &= R\Gamma(U', L(f')^*K) \otimes^{\mathbf{L}}_{\mathcal{O}(U')} \mathcal{O}(U) \\ &= R\Gamma(x', L\pi_X^*K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \end{aligned}$$

and hence we have (a) by the footnote in the statement of Cohomology on Sites, Lemma 21.43.12. The third equality is Derived Categories of Schemes, Lemma 36.3.8.

Proof of (b). Let M be an object of $Q_{\mathcal{X}}$ such that $Li_X^*M = 0$. Let x' be an object of $\mathcal{X}_{affine,\acute{e}tale}$ lying over the affine scheme $U' = p(x')$ and assume that the corresponding morphism $f' : U' \to X$ is étale. Then $f' : U' \to X$ is an object of $X_{affine,étale}$ and the condition $Li_X^*M = 0$ implies that $M|_{U'_{étale}} = 0$. In particular, we see that $R\Gamma(x', M) = 0$. However, for an arbitrary object x of the site $\mathcal{X}_{affine,étale}$ there exists a covering $\{x_i \to x\}$ such that for each i there is a morphism $x_i \to x'_i$ with x'_i corresponding to an object of $X_{affine,étale}$. Now since M is in $Q_\mathcal{X}$ we have

$$R\Gamma(x_i, M) = R\Gamma(x'_i, M) \otimes^{\mathbf{L}}_{\mathcal{O}(x'_i)} \mathcal{O}(x_i) = 0$$

and we conclude that M is zero as desired. □

To show that the construction produces an interesting category in another case, let us state and prove a characterization of $QC(\text{Spf}(A))$ for the formal spectrum of a Noetherian adic ring A.

0H0L Proposition 96.26.5. Let S be a scheme. Let $X = \text{Spf}(A)$ where A is an an adic Noetherian topological S-algebra with ideal of definition I, see More on Algebra, Definition 15.36.1 and Formal Spaces, Definition 87.9.9. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ the be category fibred in sets associated to the functor X, see Categories, Example 4.38.5. Then $QC(\mathcal{X})$ is canonically equivalent to the category $D_{comp}(A, I)$ of objects of $D(A)$ which are derived complete with respect to I.

Proof. Recall that $X = \text{colim}\,\text{Spec}(A/I^n)$ as an fppf sheaf. An object of $\mathcal{X}_{affine}$ is the same thing as an affine scheme $U = \text{Spec}(R)$ with a given morphism $f : U \to X$. By Formal Spaces, Lemma 87.9.4 there exists an $n \geq 1$ such that f factors through the monomorphism $\text{Spec}(A/I^n) \to X$. Consider the full subcategory $\mathcal{C} \subset \mathcal{X}_{affine}$ consisting of the objects $\text{Spec}(A/I^n) \to X$. By the remarks just made and Differential Graded Sheaves, Lemma 24.34.1 restriction to $\mathcal{C}$ is an exact equivalence $QC(\mathcal{X}) \to QC(\mathcal{C}, \mathcal{O}|_\mathcal{C})$. For simplicity, let us assume that $I^n \neq I^{n+1}$ for all $n \geq 1$. Then $(\mathcal{C}, \mathcal{O}|_\mathcal{C})$ is isomorphic as a ringed site to the ringed site $(\mathbf{N}, (A/I^n))$, see Differential Graded Sheaves, Section 24.35. Hence we conclude by Differential Graded Sheaves, Proposition 24.35.4. □

The following lemma will be used in comparing $QC(\mathcal{X})$ to $D_{QCoh}(\mathcal{O}_\mathcal{X})$ when $\mathcal{X}$ is an algebraic stack.

0H0X Lemma 96.26.6. Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The comparision morphism $\epsilon : \mathcal{X}_{affine,fppf} \to \mathcal{X}_{affine}$ satisfies the assumptions and conclusions of Cohomology on Sites, Lemma 21.43.12.

Proof. The proof is exactly the same as the proof of Lemma 96.26.3. Assumption (1) holds by definition of $\mathcal{X}_{affine}$. For condition (2) we use that for $x \in \text{Ob}(\mathcal{X})$ lying over the affine scheme $U = p(x)$ we have an equivalence $\mathcal{X}_{affine,étale}/x = (\text{Aff}/U)_{étale}$ compatible with structure sheaves; see discussion in Section 96.9. Thus it suffices to show: given an affine scheme $U = \text{Spec}(R)$ and a complex of R-modules $M^\bullet$ the total cohomology of the complex of modules on $(\text{Aff}/U)_{fppf}$ associated to $M^\bullet$ is quasi-isomorphic to $M^\bullet$. This is Étale Cohomology, Lemma 59.101.3. □

96.27. Other chapters

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(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
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(15) More on Algebra
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(18) Modules on Sites
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(21) Cohomology on Sites
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Schemes

(26) Schemes
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Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
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(70) Limits of Algebraic Spaces
(71) Divisors on Algebraic Spaces
(72) Algebraic Spaces over Fields
(73) Topologies on Algebraic Spaces
(74) Descent and Algebraic Spaces
(75) Derived Categories of Spaces
(76) More on Morphisms of Spaces
(77) Flatness on Algebraic Spaces
(78) Groupoids in Algebraic Spaces
(79) More on Groupoids in Spaces
(80) Bootstrap
(81) Pushouts of Algebraic Spaces

Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
(89) Resolution of Surfaces Revisited

Deformation Theory

CHAPTER 97

Criteria for Representability

05XE 97.1. Introduction

05XF The purpose of this chapter is to find criteria guaranteeing that a stack in groupoids over the category of schemes with the fppf topology is an algebraic stack. Historically, this often involved proving that certain functors were representable, see Grothendieck's lectures [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. This explains the title of this chapter. Another important source of this material comes from the work of Artin, see [Art69b], [Art70], [Art73], [Art71b], [Art71a], [Art69a], [Art69c], and [Art74].

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

97.2. Conventions

05XG The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 94.2.

97.3. What we already know

05XH The analogue of this chapter for algebraic spaces is the chapter entitled "Bootstrap", see Bootstrap, Section 80.1. That chapter already contains some representability results. Moreover, some of the preliminary material treated there we already have worked out in the chapter on algebraic stacks. Here is a list:

(1) We discuss morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 80.3. In Algebraic Stacks, Section 94.9 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
(2) We discuss properties of morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 80.4. In Algebraic Stacks, Section 94.10 we discuss properties of 1-morphisms of categories fibred in groupoids representable by algebraic spaces.
(3) We proved that if F is a sheaf whose diagonal is representable by algebraic spaces and which has an étale covering by an algebraic space, then F is an algebraic space, see Bootstrap, Theorem 80.6.1. (This is a weak version of the result in the next item on the list.)

05XI (4) We proved that if F is a sheaf and if there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat, and locally of finite presentation, then F is an algebraic space, see Bootstrap, Theorem 80.10.1.

(5) We have also proved the "smooth" analogue of (4) for algebraic stacks: If $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ and if there exists a stack in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space and a 1-morphism $u : \mathcal{U} \to \mathcal{X}$ which is representable by algebraic spaces, surjective, and smooth then $\mathcal{X}$ is an algebraic stack, see Algebraic Stacks, Lemma 94.15.3.

Our first task now is to prove the analogue of (4) for algebraic stacks in general; it is Theorem 97.16.1.

97.4. Morphisms of stacks in groupoids

05XJ This section is preliminary and should be skipped on a first reading.

05XK Lemma 97.4.1. Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are representable by algebraic spaces and étale so is $\mathcal{X} \to \mathcal{Y}$.

Proof. Let $\mathcal{U}$ be a representable category fibred in groupoids over S. Let $f : \mathcal{U} \to \mathcal{Y}$ be a 1-morphism. We have to show that $\mathcal{X} \times_\mathcal{Y} \mathcal{U}$ is representable by an algebraic space and étale over $\mathcal{U}$. Consider the composition $h : \mathcal{U} \to \mathcal{Z}$. Then

$$\mathcal{X} \times_\mathcal{Z} \mathcal{U} \longrightarrow \mathcal{Y} \times_\mathcal{Z} \mathcal{U}$$

is a 1-morphism between categories fibres in groupoids which are both representable by algebraic spaces and both étale over $\mathcal{U}$. Hence by Properties of Spaces, Lemma 66.16.6 this is represented by an étale morphism of algebraic spaces. Finally, we obtain the result we want as the morphism f induces a morphism $\mathcal{U} \to \mathcal{Y} \times_\mathcal{Z} \mathcal{U}$ and we have

$$\mathcal{X} \times_\mathcal{Y} \mathcal{U} = (\mathcal{X} \times_\mathcal{Z} \mathcal{U}) \times_{(\mathcal{Y} \times_\mathcal{Z} \mathcal{U})} \mathcal{U}.$$

□

05XL Lemma 97.4.2. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks in groupoids over $(Sch/S)_{fppf}$. Suppose that $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are 1-morphisms. If

(1) $\mathcal{Y}$, $\mathcal{Z}$ are representable by algebraic spaces Y, Z over S,
(2) the associated morphism of algebraic spaces $Y \to Z$ is surjective, flat and locally of finite presentation, and
(3) $\mathcal{Y} \times_\mathcal{Z} \mathcal{X}$ is a stack in setoids,

then $\mathcal{X}$ is a stack in setoids.

Proof. This is a special case of Stacks, Lemma 8.6.10. □

The following lemma is the analogue of Algebraic Stacks, Lemma 94.15.3 and will be superseded by the stronger Theorem 97.16.1.

05XW Lemma 97.4.3. Let S be a scheme. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If

(1) $\mathcal{U}$ is representable by an algebraic space, and
(2) u is representable by algebraic spaces, surjective, flat and locally of finite presentation,

then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable by algebraic spaces.

Proof. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Algebraic Stacks, Lemma 94.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_\mathcal{X} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 8.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_\mathcal{X} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 8.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_\mathcal{U} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

$$\begin{tikzcd}
& \mathcal{T}_1 \times_\mathcal{X} \mathcal{T}_2 \arrow[rr] \arrow[dd] & & \mathcal{T}_1 \arrow[dd] \\
\mathcal{T}_1' \times_\mathcal{U} \mathcal{T}_2' \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{T}_1' \arrow[ur] \arrow[dd] & \\
& \mathcal{T}_2 \arrow[rr] & & \mathcal{X} \\
\mathcal{T}_2' \arrow[ur] \arrow[rr] & & \mathcal{U} \arrow[ur] &
\end{tikzcd}$$

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_\mathcal{U} \mathcal{T}_2' \to \mathcal{T}_1 \times_\mathcal{X} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{tikzcd}
\mathcal{T}_1' \times_\mathcal{U} \mathcal{T}_2' \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\
\mathcal{T}_1 \times_\mathcal{X} \mathcal{T}_2 \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, flat, locally of finite presentation and surjective, see Algebraic Stacks, Lemmas 94.9.6, 94.9.7, 94.10.4, and 94.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 80.10.1 and we win. □

07WG **Lemma 97.4.4.** Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent

(1) $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
(2) for every 1-morphism $\mathcal{V} \to \mathcal{X} \times \mathcal{X}$ with $\mathcal{V}$ representable (by a scheme) the fibre product $\mathcal{Y} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}} \mathcal{V}$ has diagonal representable by algebraic spaces.

Proof. Although this is a bit of a brain twister, it is completely formal. Namely, recall that $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} = \mathcal{I}_\mathcal{X}$ is the inertia of $\mathcal{X}$ and that Δ_Δ is the identity section of $\mathcal{I}_\mathcal{X}$, see Categories, Section 4.34. Thus condition (1) says the following: Given a scheme V, an object x of $\mathcal{X}$ over V, and a morphism $\alpha : x \to x$ of $\mathcal{X}_V$ the condition "$\alpha = \text{id}_x$" defines an algebraic space over V. (In other words, there exists a monomorphism of algebraic spaces $W \to V$ such that a morphism of schemes $f : T \to V$ factors through W if and only if $f^*\alpha = \text{id}_{f^*x}$.)

On the other hand, let V be a scheme and let x, y be objects of $\mathcal{X}$ over V. Then (x, y) define a morphism $\mathcal{V} = (Sch/V)_{fppf} \to \mathcal{X} \times \mathcal{X}$. Next, let $h : V' \to V$ be a morphism of schemes and let $\alpha : h^*x \to h^*y$ and $\beta : h^*x \to h^*y$ be morphisms of $\mathcal{X}_{V'}$. Then (α, β) define a morphism $\mathcal{V}' = (Sch/V)_{fppf} \to \mathcal{Y} \times \mathcal{Y}$. Condition (2) now says that (with any choices as above) the condition "$\alpha = \beta$" defines an algebraic space over V.

To see the equivalence, given (α, β) as in (2) we see that (1) implies that "$\alpha^{-1} \circ \beta = \text{id}_{h^*x}$" defines an algebraic space. The implication (2) $\Rightarrow$ (1) follows by taking $h = \text{id}_V$ and $\beta = \text{id}_x$. $\square$

97.5. Limit preserving on objects

06CT Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is limit preserving on objects if the following condition holds: Given any data consisting of

(1) an affine scheme $U = \lim_{i \in I} U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i,
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y_i|_U$,

then there exists an $i' \geq i$, an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$, an isomorphism $\beta : x_{i'}|_U \to x$, and an isomorphism $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that

06CU (97.5.0.1)
$$\begin{tikzcd} p(x_{i'}|_U) \arrow[r, "\gamma_{i'}|_U"'] \arrow[d, "p(\beta)"'] & (y_i|_{U_{i'}})|_U \arrow[d, equal] \\ p(x) \arrow[r, "\gamma"] & y_i|_U \end{tikzcd}$$

commutes. In this situation we say that "$(i', x_{i'}, \beta, \gamma_{i'})$ is a solution to the problem posed by our data (1), (2), (3), (4)". The motivation for this definition comes from Limits of Spaces, Lemma 70.3.2.

06CV **Lemma 97.5.1.** Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z_i|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z_i|_U$. Set $y_i = q(z_i)$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y_i|_U$. As p is limit preserving on objects there exists an $i' \geq i$ and an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$ as well as isomorphisms $\beta : x_{i'}|_U \to x$ and $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that (97.5.0.1) commutes. Then we consider the object $w_{i'} = (U_{i'}, x_{i'}, z_i|_{U_{i'}}, \gamma_{i'})$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over $U_{i'}$ and define isomorphisms

$$w_{i'}|_U = (U, x_{i'}|_U, z_i|_U, \gamma_{i'}|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w_{i'}) = z_i|_{U_{i'}} \xrightarrow{\text{id}} z_i|_{U_{i'}}.$$

These combine to give a solution to the problem. $\square$

06CW Lemma 97.5.2. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving on objects, then so is the composition $q \circ p$.

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z_i|_U$ be an isomorphism. As q is limit preserving on objects there exist an $i' \geq i$, an object $y_{i'}$ of $\mathcal{Y}$ over $U_{i'}$, an isomorphism $\beta : y_{i'}|_U \to p(x)$, and an isomorphism $\gamma_{i'} : q(y_{i'}) \to z_i|_{U_{i'}}$ such that (97.5.0.1) is commutative. As p is limit preserving on objects there exist an $i'' \geq i'$, an object $x_{i''}$ of $\mathcal{X}$ over $U_{i''}$, an isomorphism $\beta' : x_{i''}|_U \to x$, and an isomorphism $\gamma'_{i''} : p(x_{i''}) \to y_{i'}|_{U_{i''}}$ such that (97.5.0.1) is commutative. The solution is to take $x_{i''}$ over $U_{i''}$ with isomorphism

$$q(p(x_{i''})) \xrightarrow{q(\gamma'_{i''})} q(y_{i'})|_{U_{i''}} \xrightarrow{\gamma_{i'}|_{U_{i''}}} z_i|_{U_{i''}}$$

and isomorphism $\beta' : x_{i''}|_U \to x$. We omit the verification that (97.5.0.1) is commutative. $\square$

06CX Lemma 97.5.3. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:

(1) p is limit preserving on objects, and
(2) p is locally of finite presentation (see Algebraic Stacks, Definition 94.10.1).

Proof. Assume (2). Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let y_i be an object of $\mathcal{Y}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y_i|_U$ be an isomorphism. Let X_{y_i} denote an algebraic space over U_i representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U_i, x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y_i}$ over U_i. By Limits of Spaces, Proposition 70.3.10 there exists an $i' \geq i$ and a morphism $f_{i'} : U_{i'} \to X_{y_i}$ such that f_ξ is the composition of $f_{i'}$ and the projection morphism $U \to U_{i'}$. Also, the 2-Yoneda lemma tells us that $f_{i'}$ corresponds to an object $\xi_{i'} = (U_{i'}, U_{i'} \to U_i, x_{i'}, \alpha)$ of the displayed 2-fibre product over $U_{i'}$ whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x_{i'}|U \to x$. Note that $\alpha : y_i|_{U_{i'}} \to p(x_{i'})$. Hence we see that taking $x_{i'}$, the isomorphism $\gamma : x_{i'}|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x_{i'}) \to y_i|_{U_{i'}}$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}$. We have to show that $X_y \to T$ is locally of finite presentation. To do this we will use the criterion in Limits of Spaces, Remark 70.3.11. Consider an affine scheme $U = \lim_{i \in I} U_i$ written as the directed limit of affine schemes over T. Pick any $i \in I$ and set $y_i = y|_{U_i}$. Also denote i' an element of I which is bigger than or equal to i. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y_i|_U$. Similarly for morphisms

$U_{i'} \to X_y$ over T. Hence (1) guarantees that the canonical map

$$\mathrm{colim}_{i' \geq i} X_y(U_{i'}) \longrightarrow X_y(U)$$

is surjective in this situation. It follows from Limits of Spaces, Lemma 70.3.12 that $X_y \to T$ is locally of finite presentation. □

06CY Lemma 97.5.4. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume p is representable by algebraic spaces and an open immersion. Then p is limit preserving on objects.

Proof. This follows from Lemma 97.5.3 and (via the general principle Algebraic Stacks, Lemma 94.10.9) from the fact that an open immersion of algebraic spaces is locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.11. □

Let S be a scheme. In the following lemma we need the notion of the size of an algebraic space X over S. Namely, given a cardinal κ we will say X has $\text{size}(X) \leq \kappa$ if and only if there exists a scheme U with $\text{size}(U) \leq \kappa$ (see Sets, Section 3.9) and a surjective étale morphism $U \to X$.

07WH Lemma 97.5.5. Let S be a scheme. Let $\kappa = \text{size}(T)$ for some $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ such that

(1) $\mathcal{Y} \to (Sch/S)_{fppf}$ is limit preserving on objects,
(2) for an affine scheme V locally of finite presentation over S and $y \in \mathrm{Ob}(\mathcal{Y}_V)$ the fibre product $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space of size $\leq \kappa$[1],
(3) $\mathcal{X}$ and $\mathcal{Y}$ are stacks for the Zariski topology.

Then f is representable by algebraic spaces.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to prove $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space.

Case I: V is affine and maps into an affine open $\mathrm{Spec}(\Lambda) \subset S$. Then we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\mathrm{Spec}(\Lambda)$, see Algebra, Lemma 10.127.2. Then y comes from an object y_i over V_i for some i by assumption (1). By assumption (3) the fibre product $(Sch/V_i)_{fppf} \times_{y_i,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space Z_i. Then $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by $Z \times_{V_i} V$.

Case II: V is general. Choose an affine open covering $V = \bigcup_{i \in I} V_i$ such that each V_i maps into an affine open of S. We first claim that $\mathcal{Z} = (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is a stack in setoids for the Zariski topology. Namely, it is a stack in groupoids for the Zariski topology by Stacks, Lemma 8.5.6. Then suppose that z is an object of $\mathcal{Z}$ over a scheme T. Denote $g : T \to V$ the morphism corresponding to the projection of z in $(Sch/V)_{fppf}$. Consider the Zariski sheaf $I = Isom_{\mathcal{Z}}(z, z)$. By Case I we see that $I|_{g^{-1}(V_i)} = *$ (the singleton sheaf). Hence $\mathcal{I} = *$. Thus $\mathcal{Z}$ is fibred in setoids. To finish the proof we have to show that the Zariski sheaf $Z : T \mapsto \mathrm{Ob}(\mathcal{Z}_T)/\cong$ is an algebraic space, see Algebraic Stacks, Lemma 94.8.2. There is a map $p : Z \to V$ (transformation of functors) and by Case I we know that $Z_i = p^{-1}(V_i)$ is an algebraic space. The morphisms $Z_i \to Z$ are representable by open immersions and $\coprod Z_i \to Z$ is surjective (in the Zariski topology). Hence Z is

[1]The condition on size can be dropped by those ignoring set theoretic issues.

a sheaf for the fppf topology by Bootstrap, Lemma 80.3.11. Thus Spaces, Lemma 65.8.5 applies and we conclude that Z is an algebraic space[2]. $\square$

07WI Lemma 97.5.6. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces as in Algebraic Stacks, Definition 94.10.1. If

(1) f is representable by algebraic spaces,
(2) $\mathcal{Y} \to (Sch/S)_{fppf}$ is limit preserving on objects,
(3) for an affine scheme V locally of finite presentation over S and $y \in \mathcal{Y}_V$ the resulting morphism of algebraic spaces $f_y : F_y \to V$, see Algebraic Stacks, Equation (94.9.1.1), has property $\mathcal{P}$.

Then f has property $\mathcal{P}$.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to show that $F_y \to V$ has property $\mathcal{P}$. Since $\mathcal{P}$ is fppf local on the base we may assume that V is an affine scheme which maps into an affine open $\text{Spec}(\Lambda) \subset S$. Thus we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\text{Spec}(\Lambda)$, see Algebra, Lemma 10.127.2. Then y comes from an object y_i over V_i for some i by assumption (2). By assumption (3) the morphism $F_{y_i} \to V_i$ has property $\mathcal{P}$. As $\mathcal{P}$ is stable under arbitrary base change and since $F_y = F_{y_i} \times_{V_i} V$ we conclude that $F_y \to V$ has property $\mathcal{P}$ as desired. $\square$

97.6. Formally smooth on objects

06CZ Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is formally smooth on objects if the following condition holds: Given any data consisting of

(1) a first order thickening $U \subset U'$ of affine schemes over S,
(2) an object y' of $\mathcal{Y}$ over U',
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y'|_U$,

then there exists an object x' of $\mathcal{X}$ over U' with an isomorphism $\beta : x'|_U \to x$ and an isomorphism $\gamma' : p(x') \to y'$ such that

06D0 (97.6.0.1)
$$\begin{tikzcd} p(x'|_U) \arrow[r, "\gamma'|_U"] \arrow[d, "p(\beta)"'] & y'|_U \arrow[d, equal] \\ p(x) \arrow[r, "\gamma"] & y'|_U \end{tikzcd}$$

commutes. In this situation we say that "(x', β, γ') is a solution to the problem posed by our data (1), (2), (3), (4)".

06D1 Lemma 97.6.1. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is formally smooth on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.

[2]To see that the set theoretic condition of that lemma is satisfied we argue as follows: First choose the open covering such that $|I| \leq \text{size}(V)$. Next, choose schemes U_i of size $\leq \max(\kappa, \text{size}(V))$ and surjective étale morphisms $U_i \to Z_i$; we can do this by assumption (2) and Sets, Lemma 3.9.6 (details omitted). Then Sets, Lemma 3.9.9 implies that $\coprod U_i$ is an object of $(Sch/S)_{fppf}$. Hence $\coprod Z_i$ is an algebraic space by Spaces, Lemma 65.8.4.

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z'|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z|_U$. Set $y' = q(z')$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y'|_U$. As p is formally smooth on objects there exists an object x' of $\mathcal{X}$ over U' as well as isomorphisms $\beta : x'|_U \to x$ and $\gamma' : p(x') \to y'$ such that (97.6.0.1) commutes. Then we consider the object $w = (U', x', z', \gamma')$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U' and define isomorphisms

$$w'|_U = (U, x'|_U, z'|_U, \gamma'|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w') = z' \xrightarrow{\text{id}} z'.$$

These combine to give a solution to the problem. $\square$

06D2 **Lemma 97.6.2.** Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are formally smooth on objects, then so is the composition $q \circ p$.

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z'|_U$ be an isomorphism. As q is formally smooth on objects there exist an object y' of $\mathcal{Y}$ over U', an isomorphism $\beta : y'|_U \to p(x)$, and an isomorphism $\gamma' : q(y') \to z'$ such that (97.6.0.1) is commutative. As p is formally smooth on objects there exist an object x' of $\mathcal{X}$ over U', an isomorphism $\beta' : x'|_U \to x$, and an isomorphism $\gamma'' : p(x') \to y'$ such that (97.6.0.1) is commutative. The solution is to take x' over U' with isomorphism

$$q(p(x')) \xrightarrow{q(\gamma'')} q(y') \xrightarrow{\gamma'} z'$$

and isomorphism $\beta' : x'|_U \to x$. We omit the verification that (97.6.0.1) is commutative. $\square$

Note that the class of formally smooth morphisms of algebraic spaces is stable under arbitrary base change and local on the target in the fpqc topology, see More on Morphisms of Spaces, Lemma 76.19.3 and 76.19.11. Hence condition (2) in the lemma below makes sense.

06D3 **Lemma 97.6.3.** Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:

(1) p is formally smooth on objects, and
(2) p is formally smooth (see Algebraic Stacks, Definition 94.10.1).

Proof. Assume (2). Let $U \subset U'$ be a first order thickening of affine schemes over S, let y' be an object of $\mathcal{Y}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y'|_U$ be an isomorphism. Let $X_{y'}$ denote an algebraic space over U' representing the 2-fibre product

$$(Sch/U')_{fppf} \times_{y', \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U', x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y'}$ over U'. As

$X_{y'} \to U'$ is formally smooth by assumption there exists a morphism $f' : U' \to X_{y'}$ such that f_ξ is the composition of f' and the morphism $U \to U'$. Also, the 2-Yoneda lemma tells us that f' corresponds to an object $\xi' = (U', U' \to U', x', \alpha)$ of the displayed 2-fibre product over U' whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x'|U \to x$. Note that $\alpha : y' \to p(x')$. Hence we see that taking x', the isomorphism $\gamma : x'|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x') \to y'$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is formally smooth. Hence it suffices to show that given a first order thickening $U \subset U'$ of affine schemes over T, then $X_y(U') \to X_y(U')$ is surjective (morphisms in the category of algebraic spaces over T). Set $y' = y|_{U'}$. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y'|_U$. Similarly for morphisms $U' \to X_y$ over T. Hence (1) guarantees the surjectivity of $X_y(U') \to X_y(U')$ in this situation and we win. □

97.7. Surjective on objects

06D4 Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is surjective on objects if the following condition holds: Given any data consisting of

(1) a field k over S, and
(2) an object y of $\mathcal{Y}$ over $\mathrm{Spec}(k)$,

then there exists an extension K/k of fields over S, an object x of $\mathcal{X}$ over $\mathrm{Spec}(K)$ such that $p(x) \cong y|_{\mathrm{Spec}(K)}$.

06D5 Lemma 97.7.1. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is surjective on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As p is surjective on objects there exists an extension K/k and an object x of $\mathcal{X}$ over K and an isomorphism $\alpha : p(x) \to q(z)|_{\mathrm{Spec}(K)}$. Then $w = (\mathrm{Spec}(K), x, z|_{\mathrm{Spec}(K)}, \alpha)$ is an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over K with $p'(w) = z|_{\mathrm{Spec}(K)}$. □

06D6 Lemma 97.7.2. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are surjective on objects, then so is the composition $q \circ p$.

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As q is surjective on objects there exists a field extension K/k and an object y of $\mathcal{Y}$ over K such that $q(y) \cong x|_{\mathrm{Spec}(K)}$. As p is surjective on objects there exists a field extension L/K and an object x of $\mathcal{X}$ over L such that $p(x) \cong y|_{\mathrm{Spec}(L)}$. Then the field extension L/k and the object x of $\mathcal{X}$ over L satisfy $q(p(x)) \cong z|_{\mathrm{Spec}(L)}$ as desired. □

06D7 Lemma 97.7.3. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:

(1) p is surjective on objects, and
(2) p is surjective (see Algebraic Stacks, Definition 94.10.1).

Proof. Assume (2). Let k be a field and let y be an object of $\mathcal{Y}$ over k. Let X_y denote an algebraic space over k representing the 2-fibre product

$$(Sch/\operatorname{Spec}(k))_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}.$$

As we've assumed that p is surjective we see that X_y is not empty. Hence we can find a field extension K/k and a K-valued point x of X_y. Via the 2-Yoneda lemma this corresponds to an object x of $\mathcal{X}$ over K together with an isomorphism $p(x) \cong y|_{\operatorname{Spec}(K)}$ and we see that (1) holds.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is surjective. By Morphisms of Spaces, Definition 67.5.2 we have to show that $|X_y| \to |T|$ is surjective. This means exactly that given a field k over T and a morphism $t : \operatorname{Spec}(k) \to T$ there exists a field extension K/k and a morphism $x : \operatorname{Spec}(K) \to X_y$ such that

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "x"'] \arrow[d] & X_y \arrow[d] \\ \operatorname{Spec}(k) \arrow[r, "t"] & T \end{tikzcd}$$

commutes. By the 2-Yoneda lemma this means exactly that we have to find $k \subset K$ and an object x of $\mathcal{X}$ over K such that $p(x) \cong t^*y|_{\operatorname{Spec}(K)}$. Hence (1) guarantees that this is the case and we win. □

97.8. Algebraic morphisms

05XX The following notion is occasionally useful.

06CF Definition 97.8.1. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We say that F is *algebraic* if for every scheme T and every object ξ of $\mathcal{Y}$ over T the 2-fibre product

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is an algebraic stack over S.

With this terminology in place we have the following result that generalizes Algebraic Stacks, Lemma 94.15.4.

05XY Lemma 97.8.2. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If

(1) $\mathcal{Y}$ is an algebraic stack, and
(2) F is algebraic (see above),

then $\mathcal{X}$ is an algebraic stack.

Proof. By assumption (1) there exists a scheme T and an object ξ of $\mathcal{Y}$ over T such that the corresponding 1-morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$ is smooth an surjective. Then $\mathcal{U} = (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$ is an algebraic stack by assumption (2). Choose a scheme U and a surjective smooth 1-morphism $(Sch/U)_{fppf} \to \mathcal{U}$. The projection $\mathcal{U} \longrightarrow \mathcal{X}$ is, as the base change of the morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$, surjective and smooth, see Algebraic Stacks, Lemma 94.10.6. Then the composition $(Sch/U)_{fppf} \to \mathcal{U} \to \mathcal{X}$ is surjective and smooth as a composition of surjective and smooth morphisms, see Algebraic Stacks, Lemma 94.10.5. Hence $\mathcal{X}$ is an algebraic stack by Algebraic Stacks, Lemma 94.15.3. $\square$

06CG Lemma 97.8.3. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$ is an algebraic stack and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then F is algebraic.

Proof. Choose a representable stack in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$. Let T be a scheme and let ξ be an object of $\mathcal{Y}$ over T. The morphism of 2-fibre products

$$(Sch/T)_{fppf} \times_{\xi, \mathcal{Y}} \mathcal{U} \longrightarrow (Sch/T)_{fppf} \times_{\xi, \mathcal{Y}} \mathcal{X}$$

is representable by algebraic spaces, surjective, and smooth as a base change of $\mathcal{U} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 94.9.7 and 94.10.6. By our condition on the diagonal of $\mathcal{Y}$ we see that the source of this morphism is representable by an algebraic space, see Algebraic Stacks, Lemma 94.10.11. Hence the target is an algebraic stack by Algebraic Stacks, Lemma 94.15.3. $\square$

0D3R Lemma 97.8.4. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If F is algebraic and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

Proof. Assume F is algebraic and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Take a scheme U over S and two objects x_1, x_2 of $\mathcal{X}$ over U. We have to show that $Isom(x_1, x_2)$ is an algebraic space over U, see Algebraic Stacks, Lemma 94.10.11. Set $y_i = F(x_i)$. We have a morphism of sheaves of sets

$$f : Isom(x_1, x_2) \to Isom(y_1, y_2)$$

and the target is an algebraic space by assumption. Thus it suffices to show that f is representable by algebraic spaces, see Bootstrap, Lemma 80.3.6. Thus we can choose a scheme V over U and an isomorphism $\beta : y_{1,V} \to y_{2,V}$ and we have to show the functor

$$(Sch/V)_{fppf} \to \text{Sets}, \quad T/V \mapsto \{\alpha : x_{1,T} \to x_{2,T} \text{ in } \mathcal{X}_T \mid F(\alpha) = \beta|_T\}$$

is an algebraic space. Consider the objects $z_1 = (V, x_{1,V}, \text{id})$ and $z_2 = (V, x_{2,V}, \beta)$ of

$$\mathcal{Z} = (Sch/V)_{fppf} \times_{y_{1,V}, \mathcal{Y}} \mathcal{X}$$

Then it is straightforward to verify that the functor above is equal to $Isom(z_1, z_2)$ on $(Sch/V)_{fppf}$. Hence this is an algebraic space by our assumption that F is algebraic (and the definition of algebraic stacks). $\square$

97.9. Spaces of sections

05XZ Given morphisms $W \to Z \to U$ we can consider the functor that associates to a scheme U' over U the set of sections $\sigma : Z_{U'} \to W_{U'}$ of the base change $W_{U'} \to Z_{U'}$ of the morphism $W \to Z$. In this section we prove some preliminary lemmas on this functor.

05XQ Lemma 97.9.1. Let $Z \to U$ be a finite morphism of schemes. Let W be an algebraic space and let $W \to Z$ be a surjective étale morphism. Then there exists a surjective étale morphism $U' \to U$ and a section

$$\sigma : Z_{U'} \to W_{U'}$$

of the morphism $W_{U'} \to Z_{U'}$.

Proof. We may choose a separated scheme W' and a surjective étale morphism $W' \to W$. Hence after replacing W by W' we may assume that W is a separated scheme. Write $f : W \to Z$ and $\pi : Z \to U$. Note that $f \circ \pi : W \to U$ is separated as W is separated (see Schemes, Lemma 26.21.13). Let $u \in U$ be a point. Clearly it suffices to find an étale neighbourhood (U', u') of (U, u) such that a section σ exists over U'. Let $z_1, \ldots, z_r$ be the points of Z lying above u. For each i choose a point $w_i \in W$ which maps to z_i. We may pick an étale neighbourhood $(U', u') \to (U, u)$ such that the conclusions of More on Morphisms, Lemma 37.41.5 hold for both $Z \to U$ and the points $z_1, \ldots, z_r$ and $W \to U$ and the points $w_1, \ldots, w_r$. Hence, after replacing (U, u) by (U', u') and relabeling, we may assume that all the field extensions $\kappa(z_i)/\kappa(u)$ and $\kappa(w_i)/\kappa(u)$ are purely inseparable, and moreover that there exist disjoint union decompositions

$$Z = V_1 \amalg \ldots \amalg V_r \amalg A, \quad W = W_1 \amalg \ldots \amalg W_r \amalg B$$

by open and closed subschemes with $z_i \in V_i$, $w_i \in W_i$ and $V_i \to U$, $W_i \to U$ finite. After replacing U by $U \setminus \pi(A)$ we may assume that $A = \emptyset$, i.e., $Z = V_1 \amalg \ldots \amalg V_r$. After replacing W_i by $W_i \cap f^{-1}(V_i)$ and B by $B \cup \bigcup W_i \cap f^{-1}(Z \setminus V_i)$ we may assume that f maps W_i into V_i. Then $f_i = f|_{W_i} : W_i \to V_i$ is a morphism of schemes finite over U, hence finite (see Morphisms, Lemma 29.44.14). It is also étale (by assumption), $f_i^{-1}(\{z_i\}) = w_i$, and induces an isomorphism of residue fields $\kappa(z_i) = \kappa(w_i)$ (because both are purely inseparable extensions of $\kappa(u)$ and $\kappa(w_i)/\kappa(z_i)$ is separable as f is étale). Hence by Étale Morphisms, Lemma 41.14.2 we see that f_i is an isomorphism in a neighbourhood V_i' of z_i. Since $\pi : Z \to U$ is closed, after shrinking U, we may assume that $W_i \to V_i$ is an isomorphism. This proves the lemma. □

05XR Lemma 97.9.2. Let $Z \to U$ be a finite locally free morphism of schemes. Let W be an algebraic space and let $W \to Z$ be an étale morphism. Then the functor

$$F : (Sch/U)_{fppf}^{opp} \longrightarrow \text{Sets},$$

defined by the rule

$$U' \longmapsto F(U') = \{\sigma : Z_{U'} \to W_{U'} \text{ section of } W_{U'} \to Z_{U'}\}$$

is an algebraic space and the morphism $F \to U$ is étale.

Proof. Assume first that $W \to Z$ is also separated. Let U' be a scheme over U and let $\sigma \in F(U')$. By Morphisms of Spaces, Lemma 67.4.7 the morphism σ is a closed immersion. Moreover, σ is étale by Properties of Spaces, Lemma 66.16.6. Hence σ is also an open immersion, see Morphisms of Spaces, Lemma 67.51.2. In other words, $Z_\sigma = \sigma(Z_{U'}) \subset W_{U'}$ is an open subspace such that the morphism $Z_\sigma \to Z_{U'}$ is an isomorphism. In particular, the morphism $Z_\sigma \to U'$ is finite. Hence we obtain a transformation of functors

$$F \longrightarrow (W/U)_{fin}, \quad \sigma \longmapsto (U' \to U, Z_\sigma)$$

where $(W/U)_{fin}$ is the finite part of the morphism $W \to U$ introduced in More on Groupoids in Spaces, Section 79.12. It is clear that this transformation of functors is injective (since we can recover σ from Z_σ as the inverse of the isomorphism $Z_\sigma \to Z_{U'}$). By More on Groupoids in Spaces, Proposition 79.12.11 we know that $(W/U)_{fin}$ is an algebraic space étale over U. Hence to finish the proof in this case it suffices to show that $F \to (W/U)_{fin}$ is representable and an open immersion. To see this suppose that we are given a morphism of schemes $U' \to U$ and an open subspace $Z' \subset W_{U'}$ such that $Z' \to U'$ is finite. Then it suffices to show that there exists an open subscheme $U'' \subset U'$ such that a morphism $T \to U'$ factors through U'' if and only if $Z' \times_{U'} T$ maps isomorphically to $Z \times_{U'} T$. This follows from More on Morphisms of Spaces, Lemma 76.49.6 (here we use that $Z \to B$ is flat and locally of finite presentation as well as finite). Hence we have proved the lemma in case $W \to Z$ is separated as well as étale.

In the general case we choose a separated scheme W' and a surjective étale morphism $W' \to W$. Note that the morphisms $W' \to W$ and $W \to Z$ are separated as their source is separated. Denote F' the functor associated to $W' \to Z \to U$ as in the lemma. In the first paragraph of the proof we showed that F' is representable by an algebraic space étale over U. By Lemma 97.9.1 the map of functors $F' \to F$ is surjective for the étale topology on Sch/U. Moreover, if U' and $\sigma : Z_{U'} \to W_{U'}$ define a point $\xi \in F(U')$, then the fibre product

$$F'' = F' \times_{F,\xi} U'$$

is the functor on Sch/U' associated to the morphisms

$$W'_{U'} \times_{W_{U'},\sigma} Z_{U'} \to Z_{U'} \to U'.$$

Since the first morphism is separated as a base change of a separated morphism, we see that F'' is an algebraic space étale over U' by the result of the first paragraph. It follows that $F' \to F$ is a surjective étale transformation of functors, which is representable by algebraic spaces. Hence F is an algebraic space by Bootstrap, Theorem 80.10.1. Since $F' \to F$ is an étale surjective morphism of algebraic spaces it follows that $F \to U$ is étale because $F' \to U$ is étale. $\square$

97.10. Relative morphisms

05Y0 We continue the discussion started in More on Morphisms, Section 37.68.

Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to T \times_{a,B} X$ is a morphism over T. Picture

05Y1 (97.10.0.1)
$$\begin{tikzcd} T \times_{a,B} Z \arrow[rr, "b"] \arrow[dr] & & T \times_{a,B} X \arrow[dl] & Z \arrow[dr] & & X \arrow[dl] \\ & T \arrow[rrrr, "a"] & & & B & \end{tikzcd}$$

Of course, we can also think of b as a morphism $b : T \times_{a,B} Z \to X$ such that

$$\begin{tikzcd} T \times_{a,B} Z \arrow[r] \arrow[d] \arrow[rr, bend left, "b"] & Z \arrow[dr] & X \arrow[d] \\ T \arrow[rr, "a"] & & B \end{tikzcd}$$

commutes. In this situation we can define a functor

05Y2 (97.10.0.2) $$Mor_B(Z, X) : (Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

05Y3 **Lemma 97.10.1.** Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Then

(1) $Mor_B(Z, X)$ is a sheaf on $(Sch/S)_{fppf}$.
(2) If T is an algebraic space over S, then there is a canonical bijection
$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Mor_B(Z, X)) = \{(a, b) \text{ as in } (97.10.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 73.7). Suppose that $(a_i, b_i) \in Mor_B(Z, X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 74.7.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i, B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to T \times_{a,B} X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in Mor_B(Z, X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair (a, b) fitting into (97.10.0.1).

Let $v : T \to Mor_B(Z, X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in Mor_B(Z, X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \to Mor_B(Z, X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 65.9.1 and since $Mor_B(Z, X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to Mor_B(Z, X)$.

We omit the verification that the two constructions above are mutually inverse. $\square$

05Y4 **Lemma 97.10.2.** Let S be a scheme. Let $Z \to B$, $X \to B$, and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then
$$Mor_{B'}(Z', X') = B' \times_B Mor_B(Z, X)$$
in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 97.10.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). $\square$

05Y5 **Lemma 97.10.3.** Let S be a scheme. Let $Z \to B$ and $X' \to X \to B$ be morphisms of algebraic spaces over S. Assume

(1) $X' \to X$ is étale, and
(2) $Z \to B$ is finite locally free.

Then $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Mor_B(Z, X') \to Mor_B(Z, X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $Mor_B(Z, X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, Mor_B(Z,X)} Mor_B(Z, X')$$

is representable by an algebraic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 97.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 97.9.1 which guarantees that $F \to U$ is surjective in the situation above. $\square$

05Y7 **Proposition 97.10.4.** Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Mor_B(Z, X)$ is an algebraic space.

Proof. Choose a scheme $B' = \coprod B'_i$ which is a disjoint union of affine schemes B'_i and an étale surjective morphism $B' \to B$. We may also assume that $B'_i \times_B Z$ is the spectrum of a ring which is finite free as a $\Gamma(B'_i, \mathcal{O}_{B'_i})$-module. By Lemma 97.10.2 and Spaces, Lemma 65.5.5 the morphism $Mor_{B'}(Z', X') \to Mor_B(Z, X)$ is surjective étale. Hence by Bootstrap, Theorem 80.10.1 it suffices to prove the proposition when $B = B'$ is a disjoint union of affine schemes B'_i so that each $B'_i \times_B Z$ is finite free over B'_i. Then it actually suffices to prove the result for the restriction to each B'_i. Thus we may assume that B is affine and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module.

Choose a scheme X' which is a disjoint union of affine schemes and a surjective étale morphism $X' \to X$. By Lemma 97.10.3 the morphism $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces, étale, and surjective. Hence by Bootstrap, Theorem 80.10.1 it suffices to prove the proposition when X is a disjoint union of affine schemes. This reduces us to the case discussed in the next paragraph.

Assume $X = \coprod_{i \in I} X_i$ is a disjoint union of affine schemes, B is affine, and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module. For any finite subset $E \subset I$ set

$$F_E = Mor_B(Z, \coprod\nolimits_{i \in E} X_i).$$

By More on Morphisms, Lemma 37.68.1 we see that F_E is an algebraic space. Consider the morphism

$$\coprod\nolimits_{E \subset I \text{ finite}} F_E \longrightarrow Mor_B(Z, X)$$

Each of the morphisms $F_E \to Mor_B(Z, X)$ is an open immersion, because it is simply the locus parametrizing pairs (a, b) where b maps into the open subscheme $\coprod_{i \in E} X_i$ of X. Moreover, if T is quasi-compact, then for any pair (a, b) the image of b is contained in $\coprod_{i \in E} X_i$ for some $E \subset I$ finite. Hence the displayed arrow is in fact an open covering and we win[3] by Spaces, Lemma 65.8.5. $\square$

97.11. Restriction of scalars

05Y8 Suppose $X \to Z \to B$ are morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to X$ is a morphism over Z. Picture

05Y9 (97.11.0.1)
$$\begin{tikzcd} & X \arrow[d] \\ T \times_{a,B} Z \arrow[ru, "b"] \arrow[r] \arrow[d] & Z \arrow[d] \\ T \arrow[r, "a"] & B \end{tikzcd}$$

In this situation we can define a functor

05YA (97.11.0.2)
$$\mathrm{Res}_{Z/B}(X) : (Sch/S)^{opp} \longrightarrow \textit{Sets}, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

05YB **Lemma 97.11.1.** Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. Then

(1) $\mathrm{Res}_{Z/B}(X)$ is a sheaf on $(Sch/S)_{fppf}$.
(2) If T is an algebraic space over S, then there is a canonical bijection
$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Res}_{Z/B}(X)) = \{(a, b) \text{ as in (97.11.0.1)}\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 73.7). Suppose that $(a_i, b_i) \in \mathrm{Res}_{Z/B}(X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 74.7.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i,B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in \mathrm{Res}_{Z/B}(X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair (a, b) fitting into (97.11.0.1).

Let $v : T \to \mathrm{Res}_{Z/B}(X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in \mathrm{Res}_{Z/B}(X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

[3]Modulo some set theoretic arguments. Namely, we have to show that $\coprod F_E$ is an algebraic space. This follows because $|I| \leq \text{size}(X)$ and $\text{size}(F_E) \leq \text{size}(X)$ as follows from the explicit description of F_E in the proof of More on Morphisms, Lemma 37.68.1. Some details omitted.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \to \mathrm{Res}_{Z/B}(X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 65.9.1 and since $\mathrm{Res}_{Z/B}(X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to \mathrm{Res}_{Z/B}(X)$.

We omit the verification that the two constructions above are mutually inverse. $\square$

Of course the sheaf $\mathrm{Res}_{Z/B}(X)$ comes with a natural transformation of functors $\mathrm{Res}_{Z/B}(X) \to B$. We will use this without further mention in the following.

05YC Lemma 97.11.2. Let S be a scheme. Let $X \to Z \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then

$$\mathrm{Res}_{Z'/B'}(X') = B' \times_B \mathrm{Res}_{Z/B}(X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 97.11.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). $\square$

05YD Lemma 97.11.3. Let S be a scheme. Let $X' \to X \to Z \to B$ be morphisms of algebraic spaces over S. Assume

(1) $X' \to X$ is étale, and
(2) $Z \to B$ is finite locally free.

Then $\mathrm{Res}_{Z/B}(X') \to \mathrm{Res}_{Z/B}(X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $\mathrm{Res}_{Z/B}(X') \to \mathrm{Res}_{Z/B}(X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $\mathrm{Res}_{Z/B}(X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, \mathrm{Res}_{Z/B}(X)} \mathrm{Res}_{Z/B}(X')$$

is representable by an algebraic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 97.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 97.9.1 which guarantees that $F \to U$ is surjective in the situation above. $\square$

At this point we can use the lemmas above to prove that $\mathrm{Res}_{Z/B}(X)$ is an algebraic space whenever $Z \to B$ is finite locally free in almost exactly the same way as in the proof that $Mor_B(Z, X)$ is an algebraic spaces, see Proposition 97.10.4. Instead we will directly deduce this result from the following lemma and the fact that $Mor_B(Z, X)$ is an algebraic space.

05YE Lemma 97.11.4. Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. The following diagram

$$\begin{tikzcd} \mathit{Mor}_B(Z, X) \arrow[r] & \mathit{Mor}_B(Z, Z) \\ \mathrm{Res}_{Z/B}(X) \arrow[r] \arrow[u] & B \arrow[u, "\mathrm{id}_Z"'] \end{tikzcd}$$

is a cartesian diagram of sheaves on $(Sch/S)_{fppf}$.

Proof. Omitted. Hint: Exercise in the functorial point of view in algebraic geometry. $\square$

05YF Proposition 97.11.5. Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $\mathrm{Res}_{Z/B}(X)$ is an algebraic space.

Proof. By Proposition 97.10.4 the functors $\mathit{Mor}_B(Z, X)$ and $\mathit{Mor}_B(Z, Z)$ are algebraic spaces. Hence this follows from the cartesian diagram of Lemma 97.11.4 and the fact that fibre products of algebraic spaces exist and are given by the fibre product in the underlying category of sheaves of sets (see Spaces, Lemma 65.7.2). $\square$

97.12. Finite Hilbert stacks

05XM In this section we prove some results concerning the finite Hilbert stacks $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ introduced in Examples of Stacks, Section 95.18.

05XN Lemma 97.12.1. Consider a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G"'] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "H"] & \mathcal{Y} \end{tikzcd}$$

of stacks in groupoids over $(Sch/S)_{fppf}$ with a given 2-isomorphism $\gamma : H \circ F' \to F \circ G$. In this situation we obtain a canonical 1-morphism $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. This morphism is compatible with the forgetful 1-morphisms of Examples of Stacks, Equation (95.18.2.1).

Proof. We map the object (U, Z, y', x', α') to the object $(U, Z, H(y'), G(x'), \gamma \star \mathrm{id}_H \star \alpha')$ where $\star$ denotes horizontal composition of 2-morphisms, see Categories, Definition 4.28.1. To a morphism $(f, g, b, a) : (U_1, Z_1, y'_1, x'_1, \alpha'_1) \to (U_2, Z_2, y'_2, x'_2, \alpha'_2)$ we assign $(f, g, H(b), G(a))$. We omit the verification that this defines a functor between categories over $(Sch/S)_{fppf}$. $\square$

05XP Lemma 97.12.2. In the situation of Lemma 97.12.1 assume that the given square is 2-cartesian. Then the diagram

$$\begin{tikzcd} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \arrow[r] \arrow[d] & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is 2-cartesian.

Proof. We get a 2-commutative diagram by Lemma 97.12.1 and hence we get a 1-morphism (i.e., a functor)

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

We indicate why this functor is essentially surjective. Namely, an object of the category on the right hand side is given by a scheme U over S, an object y' of $\mathcal{Y}'_U$, an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U and an isomorphism $H(y') \to y$ in $\mathcal{Y}_U$. The assumption means exactly that there exists an object x' of $\mathcal{X}'_Z$ such that there exist isomorphisms $G(x') \cong x$ and $\alpha' : y'|_Z \to F'(x')$ compatible with α. Then we see that (U, Z, y', x', α') is an object of $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}')$ over U. Details omitted. $\square$

05YG **Lemma 97.12.3.** In the situation of Lemma 97.12.1 assume

(1) $\mathcal{Y}' = \mathcal{Y}$ and $H = \mathrm{id}_{\mathcal{Y}}$,
(2) G is representable by algebraic spaces and étale.

Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If G is also surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

05XT (97.12.3.1)
$$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y})$$

is representable by an algebraic space étale over U. An object of this over U' corresponds to an object x' in the fibre category of $\mathcal{X}'$ over $Z_{U'}$ such that $G(x') \cong x|_{Z_{U'}}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space W such that the projection $W \to Z$ is étale. Then (97.12.3.1) is representable by the algebraic space F parametrizing sections of $W \to Z$ over U introduced in Lemma 97.9.2. Since $F \to U$ is étale we conclude that $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. Finally, if $\mathcal{X}' \to \mathcal{X}$ is surjective also, then $W \to Z$ is surjective, and hence $F \to U$ is surjective by Lemma 97.9.1. Thus in this case $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is also surjective. $\square$

05XS **Lemma 97.12.4.** In the situation of Lemma 97.12.1. Assume that G, H are representable by algebraic spaces and étale. Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If also H is surjective and the induced functor $\mathcal{X}' \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.

Proof. Set $\mathcal{X}'' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 97.4.1 the 1-morphism $\mathcal{X}' \to \mathcal{X}''$ is representable by algebraic spaces and étale (in particular the condition in the second statement of the lemma that $\mathcal{X}' \to \mathcal{X}''$ be surjective makes sense). We obtain a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X}'' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathcal{Y}' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

It follows from Lemma 97.12.2 that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}')$ is the base change of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ by $\mathcal{Y}' \to \mathcal{Y}$. In particular we see that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale, see Algebraic Stacks, Lemma 94.10.6. Moreover, it is also surjective if H is. Hence if we can show that the result holds for the left square in the diagram, then we're done. In this way we reduce to the case where $\mathcal{Y}' = \mathcal{Y}$ which is the content of Lemma 97.12.3. $\square$

05YH Lemma 97.12.5. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Then

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

see Examples of Stacks, Equation (95.18.2.1) is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, p, x, 1)$ be an object of $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/S)$ over U. Here p is just the structure morphism of U. The fifth component 1 exists and is unique since everything is over S. Also, let y be an object of $\mathcal{Y}$ over U. We have to show the 2-fibre product

05YI (97.12.5.1) $$(Sch/U)_{fppf} \times_{\xi \times y, \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by an algebraic space. To explain why this is so we introduce

$$I = Isom_{\mathcal{Y}}(y|_Z, F(x))$$

which is an algebraic space over Z by assumption. Let $a : U' \to U$ be a scheme over U. What does it mean to give an object of the fibre category of (97.12.5.1) over U'? Well, it means that we have an object $\xi' = (U', Z', y', x', \alpha')$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' and isomorphisms $(U', Z', p', x', 1) \cong (U, Z, p, x, 1)|_{U'}$ and $y' \cong y|_{U'}$. Thus ξ' is isomorphic to $(U', U' \times_{a,U} Z, a^*y, x|_{U' \times_{a,U} Z}, \alpha)$ for some morphism

$$\alpha : a^*y|_{U' \times_{a,U} Z} \longrightarrow F(x|_{U' \times_{a,U} Z})$$

in the fibre category of $\mathcal{Y}$ over $U' \times_{a,U} Z$. Hence we can view α as a morphism $b : U' \times_{a,U} Z \to I$. In this way we see that (97.12.5.1) is representable by $\text{Res}_{Z/U}(I)$ which is an algebraic space by Proposition 97.11.5. $\square$

The following lemma is a (partial) generalization of Lemma 97.12.3.

05YJ Lemma 97.12.6. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{X}' \to \mathcal{X}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If G is representable by algebraic spaces, then the 1-morphism

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

05YK (97.12.6.1) $$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y})$$

is representable by an algebraic space étale over U. An object of this over $a : U' \to U$ corresponds to an object x' of $\mathcal{X}'$ over $U' \times_{a,U} Z$ such that $G(x') \cong x|_{U' \times_{a,U} Z}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space X over Z. It follows that (97.12.6.1) is representable by $\text{Res}_{Z/U}(X)$, which is an algebraic space by Proposition 97.11.5. $\square$

06CH **Lemma 97.12.7.** Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces and locally of finite presentation. Then

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is limit preserving on objects.

Proof. This means we have to show the following: Given

(1) an affine scheme $U = \lim_i U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i, and
(3) an object $\Xi = (U, Z, y, x, \alpha)$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U such that $y = y_i|_U$,

then there exists an $i' \geq i$ and an object $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over $U_{i'}$ with $\Xi_{i'}|_U = \Xi$ and $y_{i'} = y_i|_{U_{i'}}$. Namely, the last two equalities will take care of the commutativity of (97.5.0.1).

Let $X_{y_i} \to U_i$ be an algebraic space representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}.$$

Note that $X_{y_i} \to U_i$ is locally of finite presentation by our assumption on F. Write Ξ. It is clear that $\xi = (Z, Z \to U_i, x, \alpha)$ is an object of the 2-fibre product displayed above, hence ξ gives rise to a morphism $f_\xi : Z \to X_{y_i}$ of algebraic spaces over U_i (since X_{y_i} is the functor of isomorphisms classes of objects of $(Sch/U_i)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$, see Algebraic Stacks, Lemma 94.8.2). By Limits, Lemmas 32.10.1 and 32.8.8 there exists an $i' \geq i$ and a finite locally free morphism $Z_{i'} \to U_{i'}$ of degree d whose base change to U is Z. By Limits of Spaces, Proposition 70.3.10 we may, after replacing i' by a bigger index, assume there exists a morphism $f_{i'} : Z_{i'} \to X_{y_i}$ such that

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[rr, bend left, "f_\xi"] & Z_{i'} \arrow[r, "f_{i'}"'] \arrow[d] & X_{y_i} \arrow[d] \\ U \arrow[r] & U_{i'} \arrow[r] & U_i \end{tikzcd}$$

is commutative. We set $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ where

(1) $y_{i'}$ is the object of $\mathcal{Y}$ over $U_{i'}$ which is the pullback of y_i to $U_{i'}$,
(2) $x_{i'}$ is the object of $\mathcal{X}$ over $Z_{i'}$ corresponding via the 2-Yoneda lemma to the 1-morphism

$$(Sch/Z_{i'})_{fppf} \to \mathcal{S}_{X_{y_i}} \to (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} \to \mathcal{X}$$

where the middle arrow is the equivalence which defines X_{y_i} (notation as in Algebraic Stacks, Sections 94.8 and 94.7).
(3) $\alpha_{i'} : y_{i'}|_{Z_{i'}} \to F(x_{i'})$ is the isomorphism coming from the 2-commutativity of the diagram

$$\begin{tikzcd} (Sch/Z_{i'})_{fppf} \arrow[r] \arrow[rd] & (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ & (Sch/U_{i'})_{fppf} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Recall that $f_\xi : Z \to X_{y_i}$ was the morphism corresponding to the object $\xi = (Z, Z \to U_i, x, \alpha)$ of $(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}$ over Z. By construction $f_{i'}$ is the morphism corresponding to the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$. As $f_\xi = f_{i'} \circ (Z \to Z_{i'})$ we see that the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$ pulls back to ξ over Z. Thus $x_{i'}$ pulls back to x and $\alpha_{i'}$ pulls back to α. This means that $\Xi_{i'}$ pulls back to Ξ over U and we win. □

97.13. The finite Hilbert stack of a point

05YL Let $d \geq 1$ be an integer. In Examples of Stacks, Definition 95.18.2 we defined a stack in groupoids $\mathcal{H}_d$. In this section we prove that $\mathcal{H}_d$ is an algebraic stack. We will throughout assume that $S = \mathrm{Spec}(\mathbf{Z})$. The general case will follow from this by base change. Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $\pi : Z \to T$ of degree d. Instead of classifying these directly we first study the quasi-coherent sheaves of algebras $\pi_*\mathcal{O}_Z$.

Let R be a ring. Let us temporarily make the following definition: A free d-dimensional algebra over R is given by a commutative R-algebra structure m on $R^{\oplus d}$ such that $e_1 = (1, 0, \ldots, 0)$ is a unit[4]. We think of m as an R-linear map

$$m : R^{\oplus d} \otimes_R R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $m(e_1, x) = m(x, e_1) = x$ and such that m defines a commutative and associative ring structure. If we write $m(e_i, e_j) = \sum a_{ij}^k e_k$ then we see this boils down to the conditions

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m = \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k = a_{ji}^k & \forall i, j, k \\ a_{i1}^j = \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} is the Kronecker δ-function. OK, so let's define

$$R_{univ} = \mathbf{Z}[a_{ij}^k]/J$$

where the ideal J is the ideal generated by the relations displayed above. Denote

$$m_{univ} : R_{univ}^{\oplus d} \otimes_{R_{univ}} R_{univ}^{\oplus d} \longrightarrow R_{univ}^{\oplus d}$$

the free d-dimensional algebra m over R_{univ} whose structure constants are the classes of a_{ij}^k modulo J. Then it is clear that given any free d-dimensional algebra m over a ring R there exists a unique $\mathbf{Z}$-algebra homomorphism $\psi : R_{univ} \to R$ such that $\psi_* m_{univ} = m$ (this means that m is what you get by applying the base change functor $- \otimes_{R_{univ}} R$ to m_{univ}). In other words, setting $X = \mathrm{Spec}(R_{univ})$ we obtain a canonical identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } R\}$$

for varying $T = \mathrm{Spec}(R)$. By Zariski localization we obtain the following seemingly more general identification

05YM (97.13.0.1) $$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } \Gamma(T, \mathcal{O}_T)\}$$

for any scheme T.

[4]It may be better to think of this as a pair consisting of a multiplication map $m : R^{\oplus d} \otimes_R R^{\oplus d} \to R^{\oplus d}$ and a ring map $\psi : R \to R^{\oplus d}$ satisfying a bunch of axioms.

Next we talk a little bit about isomorphisms of free d-dimensional R-algebras. Namely, suppose that m, m' are two free d-dimensional algebras over a ring R. An isomorphism from m to m' is given by an invertible R-linear map

$$\varphi : R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $\varphi(e_1) = e_1$ and such that

$$m \circ \varphi \otimes \varphi = \varphi \circ m'.$$

Note that we can compose these so that the collection of free d-dimensional algebras over R becomes a category. In this way we obtain a functor

05YN (97.13.0.2)
$$FA_d : Sch_{fppf}^{opp} \longrightarrow \text{Groupoids}$$

from the category of schemes to groupoids: to a scheme T we associate the set of free d-dimensional algebras over $\Gamma(T, \mathcal{O}_T)$ endowed with the structure of a category using the notion of isomorphisms just defined.

The above suggests we consider the functor G in groups which associates to any scheme T the group

$$G(T) = \{g \in \text{GL}_d(\Gamma(T, \mathcal{O}_T)) \mid g(e_1) = e_1\}$$

It is clear that $G \subset \text{GL}_d$ (see Groupoids, Example 39.5.4) is the closed subgroup scheme cut out by the equations $x_{11} = 1$ and $x_{i1} = 0$ for $i > 1$. Hence G is a smooth affine group scheme over $\text{Spec}(\mathbf{Z})$. Consider the action

$$a : G \times_{\text{Spec}(\mathbf{Z})} X \longrightarrow X$$

which associates to a T-valued point (g, m) with $T = \text{Spec}(R)$ on the left hand side the free d-dimensional algebra over R given by

$$a(g, m) = g^{-1} \circ m \circ g \otimes g.$$

Note that this means that g defines an isomorphism $m \to a(g, m)$ of d-dimensional free R-algebras. We omit the verification that a indeed defines an action of the group scheme G on the scheme X.

05YP **Lemma 97.13.1.** The functor in groupoids FA_d defined in (97.13.0.2) is isomorphic (!) to the functor in groupoids which associates to a scheme T the category with

1. set of objects is $X(T)$,
2. set of morphisms is $G(T) \times X(T)$,
3. $s : G(T) \times X(T) \to X(T)$ is the projection map,
4. $t : G(T) \times X(T) \to X(T)$ is $a(T)$, and
5. composition $G(T) \times X(T) \times_{s, X(T), t} G(T) \times X(T) \to G(T) \times X(T)$ is given by $((g, m), (g', m')) \mapsto (gg', m')$.

Proof. We have seen the rule on objects in (97.13.0.1). We have also seen above that $g \in G(T)$ can be viewed as a morphism from m to $a(g, m)$ for any free d-dimensional algebra m. Conversely, any morphism $m \to m'$ is given by an invertible linear map φ which corresponds to an element $g \in G(T)$ such that $m' = a(g, m)$. $\square$

In fact the groupoid $(X, G \times X, s, t, c)$ described in the lemma above is the groupoid associated to the action $a : G \times X \to X$ as defined in Groupoids, Lemma 39.16.1. Since G is smooth over $\text{Spec}(\mathbf{Z})$ we see that the two morphisms $s, t : G \times X \to X$ are smooth: by symmetry it suffices to prove that one of them is, and s is the base change of $G \to \text{Spec}(\mathbf{Z})$. Hence $(G \times X, X, s, t, c)$ is a smooth groupoid scheme, and the quotient stack $[X/G]$ is an algebraic stack by Algebraic Stacks, Theorem 94.17.3.

05YQ Proposition 97.13.2. The stack $\mathcal{H}_d$ is equivalent to the quotient stack $[X/G]$ described above. In particular $\mathcal{H}_d$ is an algebraic stack.

Proof. Note that by Groupoids in Spaces, Definition 78.20.1 the quotient stack $[X/G]$ is the stackification of the category fibred in groupoids associated to the "presheaf in groupoids" which associates to a scheme T the groupoid

$$(X(T), G(T) \times X(T), s, t, c).$$

Since this "presheaf in groupoids" is isomorphic to FA_d by Lemma 97.13.1 it suffices to prove that the $\mathcal{H}_d$ is the stackification of (the category fibred in groupoids associated to the "presheaf in groupoids") FA_d. To do this we first define a functor

$$\text{Spec} : FA_d \longrightarrow \mathcal{H}_d$$

Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $Z \to T$ of degree d. Thus given a scheme T and a free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra m we may assign to this the object

$$Z = \underline{\text{Spec}}_T(\mathcal{A})$$

of $\mathcal{H}_{d,T}$ where $\mathcal{A} = \mathcal{O}_T^{\oplus d}$ endowed with a $\mathcal{O}_T$-algebra structure via m. Moreover, if m' is a second such free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra and if $\varphi : m \to m'$ is an isomorphism of these, then the induced $\mathcal{O}_T$-linear map $\varphi : \mathcal{O}_T^{\oplus d} \to \mathcal{O}_T^{\oplus d}$ induces an isomorphism

$$\varphi : \mathcal{A}' \longrightarrow \mathcal{A}$$

of quasi-coherent $\mathcal{O}_T$-algebras. Hence

$$\underline{\text{Spec}}_T(\varphi) : \underline{\text{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\text{Spec}}_T(\mathcal{A}')$$

is a morphism in the fibre category $\mathcal{H}_{d,T}$. We omit the verification that this construction is compatible with base change so we get indeed a functor $\text{Spec} : FA_d \to \mathcal{H}_d$ as claimed above.

To show that $\text{Spec} : FA_d \to \mathcal{H}_d$ induces an equivalence between the stackification of FA_d and $\mathcal{H}_d$ it suffices to check that

(1) $\mathit{Isom}(m, m') = \mathit{Isom}(\text{Spec}(m), \text{Spec}(m'))$ for any $m, m' \in FA_d(T)$.
(2) for any scheme T and any object $Z \to T$ of $\mathcal{H}_{d,T}$ there exists a covering $\{T_i \to T\}$ such that $Z|_{T_i}$ is isomorphic to $\text{Spec}(m)$ for some $m \in FA_d(T_i)$, and

see Stacks, Lemma 8.9.1. The first statement follows from the observation that any isomorphism

$$\underline{\text{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\text{Spec}}_T(\mathcal{A}')$$

is necessarily given by a global invertible matrix g when $\mathcal{A} = \mathcal{A}' = \mathcal{O}_T^{\oplus d}$ as modules. To prove the second statement let $\pi : Z \to T$ be a finite locally free morphism of degree d. Then $\mathcal{A}$ is a locally free sheaf $\mathcal{O}_T$-modules of rank d. Consider the element $1 \in \Gamma(T, \mathcal{A})$. This element is nonzero in $\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ for every $t \in T$ since the scheme $Z_t = \text{Spec}(\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t))$ is nonempty being of degree $d > 0$ over $\kappa(t)$. Thus $1 : \mathcal{O}_T \to \mathcal{A}$ can locally be used as the first basis element (for example you can use Algebra, Lemma 10.79.4 parts (1) and (2) to see this). Thus, after localizing on T we may assume that there exists an isomorphism $\varphi : \mathcal{A} \to \mathcal{O}_T^{\oplus d}$ such that

$1 \in \Gamma(\mathcal{A})$ corresponds to the first basis element. In this situation the multiplication map $\mathcal{A} \otimes_{\mathcal{O}_T} \mathcal{A} \to \mathcal{A}$ translates via φ into a free d-dimensional algebra m over $\Gamma(T, \mathcal{O}_T)$. This finishes the proof. □

97.14. Finite Hilbert stacks of spaces

05YR The finite Hilbert stack of an algebraic space is an algebraic stack.

05YS Lemma 97.14.1. Let S be a scheme. Let X be an algebraic space over S. Then $\mathcal{H}_d(X)$ is an algebraic stack.

Proof. The 1-morphism

$$\mathcal{H}_d(X) \longrightarrow \mathcal{H}_d$$

is representable by algebraic spaces according to Lemma 97.12.6. The stack $\mathcal{H}_d$ is an algebraic stack according to Proposition 97.13.2. Hence $\mathcal{H}_d(X)$ is an algebraic stack by Algebraic Stacks, Lemma 94.15.4. □

This lemma allows us to bootstrap.

06CI Lemma 97.14.2. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$ such that

(1) $\mathcal{X}$ is representable by an algebraic space, and
(2) F is representable by algebraic spaces, surjective, flat, and locally of finite presentation.

Then $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an algebraic stack.

Proof. Choose a representable stack in groupoids $\mathcal{U}$ over S and a 1-morphism $f : \mathcal{U} \to \mathcal{H}_d(\mathcal{X})$ which is representable by algebraic spaces, smooth, and surjective. This is possible because $\mathcal{H}_d(\mathcal{X})$ is an algebraic stack by Lemma 97.14.1. Consider the 2-fibre product

$$\mathcal{W} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}.$$

Since $\mathcal{U}$ is representable (in particular a stack in setoids) it follows from Examples of Stacks, Lemma 95.18.3 and Stacks, Lemma 8.6.7 that $\mathcal{W}$ is a stack in setoids. The 1-morphism $\mathcal{W} \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, smooth, and surjective as a base change of the morphism f (see Algebraic Stacks, Lemmas 94.9.7 and 94.10.6). Thus, if we can show that $\mathcal{W}$ is representable by an algebraic space, then the lemma follows from Algebraic Stacks, Lemma 94.15.3.

The diagonal of $\mathcal{Y}$ is representable by algebraic spaces according to Lemma 97.4.3. We may apply Lemma 97.12.5 to see that the 1-morphism

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

is representable by algebraic spaces. Consider the 2-fibre product

$$\mathcal{V} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}).$$

The projection morphism $\mathcal{V} \to \mathcal{U} \times \mathcal{X}$ is representable by algebraic spaces as a base change of the last displayed morphism. Hence $\mathcal{V}$ is an algebraic space (see

Bootstrap, Lemma 80.3.6 or Algebraic Stacks, Lemma 94.9.8). The 1-morphism $\mathcal{V} \to \mathcal{U}$ fits into the following 2-cartesian diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{W} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

because

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}) = (\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}) \times_{\mathcal{Y}, F} \mathcal{X}.$$

Hence $\mathcal{V} \to \mathcal{W}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation as a base change of F. It follows that the same thing is true for the corresponding sheaves of sets associated to $\mathcal{V}$ and $\mathcal{W}$, see Algebraic Stacks, Lemma 94.10.4. Thus we conclude that the sheaf associated to $\mathcal{W}$ is an algebraic space by Bootstrap, Theorem 80.10.1. □

97.15. LCI locus in the Hilbert stack

06CJ Please consult Examples of Stacks, Section 95.18 for notation. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that F is representable by algebraic spaces. Fix $d \geq 1$. Consider an object (U, Z, y, x, α) of $\mathcal{H}_d$. There is an induced 1-morphism

$$(Sch/Z)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$$

(by the universal property of 2-fibre products) which is representable by a morphism of algebraic spaces over U. Namely, since F is representable by algebraic spaces, we may choose an algebraic space X_y over U which represents the 2-fibre product $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$. Since $\alpha : y|_Z \to F(x)$ is an isomorphism we see that $\xi = (Z, Z \to U, x, \alpha)$ is an object of the 2-fibre product $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$ over Z. Hence ξ gives rise to a morphism $x_\alpha : Z \to X_y$ of algebraic spaces over U as X_y is the functor of isomorphisms classes of objects of $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$, see Algebraic Stacks, Lemma 94.8.2. Here is a picture

(97.15.0.1)

06CK
$$\begin{tikzcd} Z \arrow[r, "x_\alpha"'] \arrow[dr] & X_y \arrow[d] \\ & U \end{tikzcd} \qquad \begin{tikzcd} (Sch/Z)_{fppf} \arrow[r, "x,\alpha"'] \arrow[dr] & (Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ & (Sch/U)_{fppf} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

We remark that if $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is a morphism between objects of $\mathcal{H}_d$, then the morphism $x'_{\alpha'} : Z' \to X'_{y'}$ is the base change of the morphism x_α by the morphism $g : U' \to U$ (details omitted).

Now assume moreover that F is flat and locally of finite presentation. In this situation we define a full subcategory

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

consisting of those objects (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ such that the corresponding morphism $x_\alpha : Z \to X_y$ is unramified and a local complete intersection morphism (see Morphisms of Spaces, Definition 67.38.1 and More on Morphisms of Spaces, Definition 76.48.1 for definitions).

06CL Lemma 97.15.1. Let S be a scheme. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids and the inclusion functor

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable and an open immersion.

Proof. Let $\Xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d$. It follows from the remark following (97.15.0.1) that the pullback of Ξ by $U' \to U$ belongs to $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ if and only if the base change of x_α is unramified and a local complete intersection morphism. Note that $Z \to U$ is finite locally free (hence flat, locally of finite presentation and universally closed) and that $X_y \to U$ is flat and locally of finite presentation by our assumption on F. Then More on Morphisms of Spaces, Lemmas 76.49.1 and 76.49.7 imply exists an open subscheme $W \subset U$ such that a morphism $U' \to U$ factors through W if and only if the base change of x_α via $U' \to U$ is unramified and a local complete intersection morphism. This implies that

$$(Sch/U)_{fppf} \times_{\Xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is representable by W. Hence the final statement of the lemma holds. The first statement (that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids) follows from this and Algebraic Stacks, Lemma 94.15.5. $\square$

Local complete intersection morphisms are "locally unobstructed". This holds in much greater generality than the special case that we need in this chapter here.

06D8 Lemma 97.15.2. Let $U \subset U'$ be a first order thickening of affine schemes. Let X' be an algebraic space flat over U'. Set $X = U \times_{U'} X'$. Let $Z \to U$ be finite locally free of degree d. Finally, let $f : Z \to X$ be unramified and a local complete intersection morphism. Then there exists a commutative diagram

$$\begin{tikzcd} (Z \subset Z') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (X \subset X') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of algebraic spaces over U' such that $Z' \to U'$ is finite locally free of degree d and $Z = U \times_{U'} Z'$.

Proof. By More on Morphisms of Spaces, Lemma 76.48.12 the conormal sheaf $\mathcal{C}_{Z/X}$ of the unramified morphism $Z \to X$ is a finite locally free $\mathcal{O}_Z$-module and by More on Morphisms of Spaces, Lemma 76.48.13 we have an exact sequence

$$0 \to i^*\mathcal{C}_{X/X'} \to \mathcal{C}_{Z/X'} \to \mathcal{C}_{Z/X} \to 0$$

of conormal sheaves. Since Z is affine this sequence is split. Choose a splitting

$$\mathcal{C}_{Z/X'} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z \subset Z''$ be the universal first order thickening of Z over X' (see More on Morphisms of Spaces, Section 76.15). Denote $\mathcal{I} \subset \mathcal{O}_{Z''}$ the quasi-coherent sheaf of ideals corresponding to $Z \subset Z''$. By definition we have $\mathcal{C}_{Z/X'}$ is $\mathcal{I}$ viewed as a sheaf on Z. Hence the splitting above determines a splitting

$$\mathcal{I} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z' \subset Z''$ be the closed subscheme cut out by $\mathcal{C}_{Z/X} \subset \mathcal{I}$ viewed as a quasi-coherent sheaf of ideals on Z''. It is clear that Z' is a first order thickening of Z and that we obtain a commutative diagram of first order thickenings as in the statement of the lemma.

Since $X' \to U'$ is flat and since $X = U \times_{U'} X'$ we see that $\mathcal{C}_{X/X'}$ is the pullback of $\mathcal{C}_{U/U'}$ to X, see More on Morphisms of Spaces, Lemma 76.18.1. Note that by construction $\mathcal{C}_{Z/Z'} = i^*\mathcal{C}_{X/X'}$ hence we conclude that $\mathcal{C}_{Z/Z'}$ is isomorphic to the pullback of $\mathcal{C}_{U/U'}$ to Z. Applying More on Morphisms of Spaces, Lemma 76.18.1 once again (or its analogue for schemes, see More on Morphisms, Lemma 37.10.1) we conclude that $Z' \to U'$ is flat and that $Z = U \times_{U'} Z'$. Finally, More on Morphisms, Lemma 37.10.3 shows that $Z' \to U'$ is finite locally free of degree d. $\square$

06D9 Lemma 97.15.3. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then

$$p : \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is formally smooth on objects.

Proof. We have to show the following: Given

(1) an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over an affine scheme U,
(2) a first order thickening $U \subset U'$, and
(3) an object y' of $\mathcal{Y}$ over U' such that $y'|_U = y$,

then there exists an object $(U', Z', y', x', \alpha')$ of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. Namely, the last two equalities will take care of the commutativity of (97.6.0.1).

Consider the morphism $x_\alpha : Z \to X_y$ constructed in Equation (97.15.0.1). Denote similarly $X'_{y'}$ the algebraic space over U' representing the 2-fibre product $(Sch/U')_{fppf} \times_{y', \mathcal{Y}, F} \mathcal{X}$. By assumption the morphism $X'_{y'} \to U'$ is flat (and locally of finite presentation). As $y'|_U = y$ we see that $X_y = U \times_{U'} X'_{y'}$. Hence we may apply Lemma 97.15.2 to find $Z' \to U'$ finite locally free of degree d with $Z = U \times_{U'} Z'$ and with $Z' \to X'_{y'}$ extending x_α. By construction the morphism $Z' \to X'_{y'}$ corresponds to a pair (x', α'). It is clear that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. As we've seen in Lemma 97.15.1 that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an "open substack" it follows that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ as desired. $\square$

06DA Lemma 97.15.4. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, surjective, and locally of finite presentation. Then

$$\coprod\nolimits_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is surjective on objects.

Proof. It suffices to prove the following: For any field k and object y of $\mathcal{Y}$ over $\text{Spec}(k)$ there exists an integer $d \geq 1$ and an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ with $U = \text{Spec}(k)$. Namely, in this case we see that p is surjective on objects in the strong sense that an extension of the field is not needed.

Denote X_y the algebraic space over $U = \text{Spec}(k)$ representing the 2-fibre product $(Sch/U')_{fppf} \times_{y', \mathcal{Y}, F} \mathcal{X}$. By assumption the morphism $X_y \to \text{Spec}(k)$ is surjective and locally of finite presentation (and flat). In particular X_y is nonempty. Choose a nonempty affine scheme V and an étale morphism $V \to X_y$. Note that $V \to \mathrm{Spec}(k)$ is (flat), surjective, and locally of finite presentation (by Morphisms of Spaces, Definition 67.28.1). Pick a closed point $v \in V$ where $V \to \mathrm{Spec}(k)$ is Cohen-Macaulay (i.e., V is Cohen-Macaulay at v), see More on Morphisms, Lemma 37.22.7. Applying More on Morphisms, Lemma 37.23.4 we find a regular immersion $Z \to V$ with $Z = \{v\}$. This implies $Z \to V$ is a closed immersion. Moreover, it follows that $Z \to \mathrm{Spec}(k)$ is finite (for example by Algebra, Lemma 10.122.1). Hence $Z \to \mathrm{Spec}(k)$ is finite locally free of some degree d. Now $Z \to X_y$ is unramified as the composition of a closed immersion followed by an étale morphism (see Morphisms of Spaces, Lemmas 67.38.3, 67.39.10, and 67.38.8). Finally, $Z \to X_y$ is a local complete intersection morphism as a composition of a regular immersion of schemes and an étale morphism of algebraic spaces (see More on Morphisms, Lemma 37.62.9 and Morphisms of Spaces, Lemmas 67.39.6 and 67.37.8 and More on Morphisms of Spaces, Lemmas 76.48.6 and 76.48.5). The morphism $Z \to X_y$ corresponds to an object x of $\mathcal{X}$ over Z together with an isomorphism $\alpha : y|_Z \to F(x)$. We obtain an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. By what was said above about the morphism $Z \to X_y$ we see that it actually is an object of the subcategory $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ and we win. $\square$

97.16. Bootstrapping algebraic stacks

06DB The following theorem is one of the main results of this chapter.

06DC Theorem 97.16.1. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If

(1) $\mathcal{X}$ is representable by an algebraic space, and
(2) F is representable by algebraic spaces, surjective, flat and locally of finite presentation,

then $\mathcal{Y}$ is an algebraic stack.

Proof. By Lemma 97.4.3 we see that the diagonal of $\mathcal{Y}$ is representable by algebraic spaces. Hence we only need to verify the existence of a 1-morphism $f : \mathcal{V} \to \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$ with $\mathcal{V}$ representable and f surjective and smooth. By Lemma 97.14.2 we know that

$$\coprod\nolimits_{d \geq 1} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack. It follows from Lemma 97.15.1 and Algebraic Stacks, Lemma 94.15.5 that

$$\coprod\nolimits_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack as well. Choose a representable stack in groupoids $\mathcal{V}$ over $(Sch/S)_{fppf}$ and a surjective and smooth 1-morphism

$$\mathcal{V} \longrightarrow \coprod\nolimits_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}).$$

We claim that the composition

$$\mathcal{V} \longrightarrow \coprod\nolimits_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is smooth and surjective which finishes the proof of the theorem. In fact, the smoothness will be a consequence of Lemmas 97.12.7 and 97.15.3 and the surjectivity a consequence of Lemma 97.15.4. We spell out the details in the following paragraph.

By construction $\mathcal{V} \to \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, surjective, and smooth (and hence also locally of finite presentation and formally smooth by the general principle Algebraic Stacks, Lemma 94.10.9 and More on Morphisms of Spaces, Lemma 76.19.6). Applying Lemmas 97.5.3, 97.6.3, and 97.7.3 we see that $\mathcal{V} \to \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is limit preserving on objects, formally smooth on objects, and surjective on objects. The 1-morphism $\coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ is

(1) limit preserving on objects: this is Lemma 97.12.7 for $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ and we combine it with Lemmas 97.15.1, 97.5.4, and 97.5.2 to get it for $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$,
(2) formally smooth on objects by Lemma 97.15.3, and
(3) surjective on objects by Lemma 97.15.4.

Using Lemmas 97.5.2, 97.6.2, and 97.7.2 we conclude that the composition $\mathcal{V} \to \mathcal{Y}$ is limit preserving on objects, formally smooth on objects, and surjective on objects. Using Lemmas 97.5.3, 97.6.3, and 97.7.3 we see that $\mathcal{V} \to \mathcal{Y}$ is locally of finite presentation, formally smooth, and surjective. Finally, using (via the general principle Algebraic Stacks, Lemma 94.10.9) the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 76.19.6) we see that $\mathcal{V} \to \mathcal{Y}$ is smooth and we win. $\square$

97.17. Applications

06FG Our first task is to show that the quotient stack $[U/R]$ associated to a "flat and locally finitely presented groupoid" is an algebraic stack. See Groupoids in Spaces, Definition 78.20.1 for the definition of the quotient stack. The following lemma is preliminary and is the analogue of Algebraic Stacks, Lemma 94.17.2.

06FH **Lemma 97.17.1.** *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the morphism $\mathcal{S}_U \to [U/R]$ is flat, locally of finite presentation, and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective, flat, and locally of finite presentation. We already know that the left hand side is representable by an algebraic space F, see Algebraic Stacks, Lemmas 94.17.1 and 94.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective, locally of finite presentation, and flat. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.31.10 and the second equality by Groupoids in Spaces, Lemma 78.22.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is flat and locally of finite presentation as the base change of the flat locally finitely presented morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the first main result of this section.

06FI Theorem 97.17.2. Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the quotient stack $[U/R]$ is an algebraic stack over S.

Proof. We check the two conditions of Theorem 97.16.1 for the morphism

$$(Sch/U)_{fppf} \longrightarrow [U/R].$$

The first is trivial (as U is an algebraic space). The second is Lemma 97.17.1. □

97.18. When is a quotient stack algebraic?

06PI In Groupoids in Spaces, Section 78.20 we have defined the quotient stack $[U/R]$ associated to a groupoid (U, R, s, t, c) in algebraic spaces. Note that $[U/R]$ is a stack in groupoids whose diagonal is representable by algebraic spaces (see Bootstrap, Lemma 80.11.5 and Algebraic Stacks, Lemma 94.10.11) and such that there exists an algebraic space U and a 1-morphism $(Sch/U)_{fppf} \to [U/R]$ which is an “fppf surjection” in the sense that it induces a map on presheaves of isomorphism classes of objects which becomes surjective after sheafification. However, it is not the case that $[U/R]$ is an algebraic stack in general. This is not a contradiction with Theorem 97.16.1 as the 1-morphism $(Sch/U)_{fppf} \to [U/R]$ may not be flat and locally of finite presentation.

The easiest way to make examples of non-algebraic quotient stacks is to look at quotients of the form $[S/G]$ where S is a scheme and G is a group scheme over S acting trivially on S. Namely, we will see below (Lemma 97.18.3) that if $[S/G]$ is algebraic, then $G \to S$ has to be flat and locally of finite presentation. An explicit example can be found in Examples, Section 110.52.

06PJ Lemma 97.18.1. Let S be a scheme and let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The quotient stack $[U/R]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $g : U' \to U$ such that

(1) the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves, and
(2) the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via g.

Proof. First, assume that $g : U' \to U$ satisfies (1) and (2). Property (1) implies that $[U'/R'] \to [U/R]$ is an equivalence, see Groupoids in Spaces, Lemma 78.25.2. By Theorem 97.17.2 the quotient stack $[U'/R']$ is an algebraic stack. Hence $[U/R]$ is an algebraic stack too, see Algebraic Stacks, Lemma 94.12.4.

Conversely, assume that $[U/R]$ is an algebraic stack. We may choose a scheme W and a surjective smooth 1-morphism

$$f : (Sch/W)_{fppf} \longrightarrow [U/R].$$

By the 2-Yoneda lemma (Algebraic Stacks, Section 94.5) this corresponds to an object ξ of $[U/R]$ over W. By the description of $[U/R]$ in Groupoids in Spaces, Lemma 78.24.1 we can find a surjective, flat, locally finitely presented morphism $b : U' \to W$ of schemes such that $\xi' = b^*\xi$ corresponds to a morphism $g : U' \to U$. Note that the 1-morphism

$$f' : (Sch/U')_{fppf} \longrightarrow [U/R].$$

corresponding to ξ' is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Lemma 94.10.5. Hence $(Sch/U')_{fppf} \times_{[U/R]} (Sch/U')_{fppf}$ which is represented by the algebraic space

$$\mathit{Isom}_{[U/R]}(\mathrm{pr}_0^*\xi', \mathrm{pr}_1^*\xi') = (U' \times_S U') \times_{(g \circ \mathrm{pr}_0, g \circ \mathrm{pr}_1), U \times_S U} R = R'$$

(see Groupoids in Spaces, Lemma 78.22.1 for the first equality; the second is the definition of restriction) is flat and locally of finite presentation over U' via both s' and t' (by base change, see Algebraic Stacks, Lemma 94.10.6). By this description of R' and by Algebraic Stacks, Lemma 94.16.1 we obtain a canonical fully faithful 1-morphism $[U'/R'] \to [U/R]$. This 1-morphism is essentially surjective because f' is flat, locally of finite presentation, and surjective (see Stacks, Lemma 8.4.8); another way to prove this is to use Algebraic Stacks, Remark 94.16.3. Finally, we can use Groupoids in Spaces, Lemma 78.25.2 to conclude that the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves. $\square$

06PK Lemma 97.18.2. Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. The quotient stack $[X/G]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $\varphi : X' \to X$ such that

(1) $G \times_B X' \to X$, $(g, x') \mapsto a(g, \varphi(x'))$ is a surjection of sheaves, and
(2) the two projections $X'' \to X'$ of the algebraic space X'' given by the rule

$$T \longmapsto \{(x'_1, g, x'_2) \in (X' \times_B G \times_B X')(T) \mid \varphi(x'_1) = a(g, \varphi(x'_2))\}$$

are flat and locally of finite presentation.

Proof. This lemma is a special case of Lemma 97.18.1. Namely, the quotient stack $[X/G]$ is by Groupoids in Spaces, Definition 78.20.1 equal to the quotient stack $[X/G \times_B X]$ of the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ associated to the group action in Groupoids in Spaces, Lemma 78.15.1. There is one small observation that is needed to get condition (1). Namely, the morphism $s : G \times_B X \to X$ is the second projection and the morphism $t : G \times_B X \to X$ is the action morphism a. Hence the morphism $h : U' \times_{g,U,t} R \to R \xrightarrow{s} U$ from Lemma 97.18.1 corresponds to the morphism

$$X' \times_{\varphi, X, a} (G \times_B X) \xrightarrow{\mathrm{pr}_1} X$$

in the current setting. However, because of the symmetry given by the inverse of G this morphism is isomorphic to the morphism

$$(G \times_B X) \times_{\mathrm{pr}_1, X, \varphi} X' \xrightarrow{a} X$$

of the statement of the lemma. Details omitted. $\square$

06PL Lemma 97.18.3. Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. Then the quotient stack $[B/G]$ is an algebraic stack if and only if G is flat and locally of finite presentation over B.

Proof. If G is flat and locally of finite presentation over B, then $[B/G]$ is an algebraic stack by Theorem 97.17.2.

Conversely, assume that $[B/G]$ is an algebraic stack. By Lemma 97.18.2 and because the action is trivial, we see there exists an algebraic space B' and a morphism $B' \to B$ such that (1) $B' \to B$ is a surjection of sheaves and (2) the projections

$$B' \times_B G \times_B B' \to B'$$

are flat and locally of finite presentation. Note that the base change $B' \times_B G \times_B B' \to G \times_B B'$ of $B' \to B$ is a surjection of sheaves also. Thus it follows from Descent on Spaces, Lemma 74.8.1 that the projection $G \times_B B' \to B'$ is flat and locally of finite presentation. By (1) we can find an fppf covering $\{B_i \to B\}$ such that $B_i \to B$ factors through $B' \to B$. Hence $G \times_B B_i \to B_i$ is flat and locally of finite presentation by base change. By Descent on Spaces, Lemmas 74.11.13 and 74.11.10 we conclude that $G \to B$ is flat and locally of finite presentation. □

Later we will see that the quotient stack of a smooth S-space by a group algebraic space G is smooth, even when G is not smooth (Morphisms of Stacks, Lemma 101.33.7).

97.19. Algebraic stacks in the étale topology

076U Let S be a scheme. Instead of working with stacks in groupoids over the big fppf site $(Sch/S)_{fppf}$ we could work with stacks in groupoids over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Stacks, Sections 94.4, 94.5, 94.6, 94.7, 94.8, 94.9, 94.10, and 94.11 makes sense for categories fibred in groupoids over $(Sch/S)_{étale}$. Thus we get a second notion of an algebraic stack by working in the étale topology. This notion is (a priori) weaker than the notion introduced in Algebraic Stacks, Definition 94.12.1 since a stack in the fppf topology is certainly a stack in the étale topology. However, the notions are equivalent as is shown by the following lemma.

076V Lemma 97.19.1. Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Sheaves on Stacks, Section 96.4 and Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let

$$p : \mathcal{X} \to Sch_\alpha/S$$

be a category fibred in groupoids with the following properties:

(1) $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{étale}$,
(2) the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces[5], and
(3) there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and a 1-morphism $(Sch/U)_{étale} \to \mathcal{X}$ which is surjective and smooth.

Then $\mathcal{X}$ is an algebraic stack in the sense of Algebraic Stacks, Definition 94.12.1.

[5] Here we can either mean sheaves in the étale topology whose diagonal is representable and which have an étale surjective covering by a scheme or algebraic spaces as defined in Algebraic Spaces, Definition 65.6.1. Namely, by Bootstrap, Lemma 80.12.1 there is no difference.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Stacks, Definition 94.12.1 are independent of the topology. This is true because these properties involve only the notion of a 2-fibre product of categories fibred in groupoids, 1- and 2-morphisms of categories fibred in groupoids, the notion of a 1-morphism of categories fibred in groupoids representable by algebraic spaces, and what it means for such a 1-morphism to be surjective and smooth. Thus all we have to prove is that an étale stack in groupoids $\mathcal{X}$ with properties (2) and (3) is also an fppf stack in groupoids.

Using (2) let R be an algebraic space representing

$$(Sch_\alpha/U) \times_\mathcal{X} (Sch_\alpha/U)$$

By (3) the projections $s, t : R \to U$ are smooth. Exactly as in the proof of Algebraic Stacks, Lemma 94.16.1 there exists a groupoid in spaces (U, R, s, t, c) and a canonical fully faithful 1-morphism $[U/R]_{étale} \to \mathcal{X}$ where $[U/R]_{étale}$ is the étale stackification of presheaf in groupoids

$$T \longmapsto (U(T), R(T), s(T), t(T), c(T))$$

Claim: If $V \to T$ is a surjective smooth morphism from an algebraic space V to a scheme T, then there exists an étale covering $\{T_i \to T\}$ refining the covering $\{V \to T\}$. This follows from More on Morphisms, Lemma 37.38.7 or the more general Sheaves on Stacks, Lemma 96.19.10. Using the claim and arguing exactly as in Algebraic Stacks, Lemma 94.16.2 it follows that $[U/R]_{étale} \to \mathcal{X}$ is an equivalence.

Next, let $[U/R]$ denote the quotient stack in the fppf topology which is an algebraic stack by Algebraic Stacks, Theorem 94.17.3. Thus we have 1-morphisms

$$U \to [U/R]_{étale} \to [U/R].$$

Both $U \to [U/R]_{étale} \cong \mathcal{X}$ and $U \to [U/R]$ are surjective and smooth (the first by assumption and the second by the theorem) and in both cases the fibre product $U \times_\mathcal{X} U$ and $U \times_{[U/R]} U$ is representable by R. Hence the 1-morphism $[U/R]_{étale} \to [U/R]$ is fully faithful (since morphisms in the quotient stacks are given by morphisms into R, see Groupoids in Spaces, Section 78.24).

Finally, for any scheme T and morphism $t : T \to [U/R]$ the fibre product $V = T \times_{U/R} U$ is an algebraic space surjective and smooth over T. By the claim above there exists an étale covering $\{T_i \to T\}_{i \in I}$ and morphisms $T_i \to V$ over T. This proves that the object t of $[U/R]$ over T comes étale locally from U. We conclude that $[U/R]_{étale} \to [U/R]$ is an equivalence of stacks in groupoids over $(Sch/S)_{étale}$ by Stacks, Lemma 8.4.8. This concludes the proof. □

97.20. Other chapters

Preliminaries

CHAPTER 98

Artin's Axioms

07SZ 98.1. Introduction

07T0 In this chapter we discuss Artin's axioms for the representability of functors by algebraic spaces. As references we suggest the papers [Art69b], [Art70], [Art74].

Some of the notation, conventions, and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let x_0 be an object of $\mathcal{X}$ over a field k of finite type over S. Throughout this chapter an important role is played by the predeformation category (see Formal Deformation Theory, Definition 90.6.2)

$$\mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \{\text{Artinian local } S\text{-algebras with residue field } k\}$$

associated to x_0 over k. We introduce the Rim-Schlessinger condition (RS) for $\mathcal{X}$ and show it guarantees that $\mathcal{F}_{\mathcal{X},k,x_0}$ is a deformation category, i.e., $\mathcal{F}_{\mathcal{X},k,x_0}$ satisies (RS) itself. We discuss how $\mathcal{F}_{\mathcal{X},k,x_0}$ changes if one replaces k by a finite extension and we discuss tangent spaces.

Next, we discuss formal objects $\xi = (\xi_n)$ of $\mathcal{X}$ which are inverse systems of objects lying over the quotients $R/\mathfrak{m}^n$ where R is a Noetherian complete local S-algebra whose residue field is of finite type over S. This is the same thing as having a formal object in $\mathcal{F}_{\mathcal{X},k,x_0}$ for some x_0 and k. A formal object is called effective when there is an object of $\mathcal{X}$ over R which gives rise to the inverse system. A formal object of $\mathcal{X}$ is called versal if it gives rise to a versal formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$. Finally, given a finite type S-scheme U, an object x of $\mathcal{X}$ over U, and a closed point $u_0 \in U$ we say x is versal at u_0 if the induced formal object over the complete local ring $\mathcal{O}_{U,u_0}^\wedge$ is versal.

Having worked through this material we can state Artin's celebrated theorem: our $\mathcal{X}$ is an algebraic stack if the following are true

(1) $\mathcal{O}_{S,s}$ is a G-ring for all $s \in S$,
(2) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(3) $\mathcal{X}$ is a stack for the étale topology,
(4) $\mathcal{X}$ is limit preserving,
(5) $\mathcal{X}$ satisfies (RS),
(6) tangent spaces and spaces of infinitesimal automorphisms of the deformation categories $\mathcal{F}_{\mathcal{X},k,x_0}$ are finite dimensional,
(7) formal objects are effective,

(8) $\mathcal{X}$ satisfies openness of versality.

This is Lemma 98.17.1; see also Proposition 98.17.2 for a slight improvement. There is an analogous proposition characterizing which functors $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ are algebraic spaces, see Section 98.16.

Here is a rough outline of the proof of Artin's theorem. First we show that there are plenty of versal formal objects using (RS) and the finite dimensionality of tangent and aut spaces, see for example Formal Deformation Theory, Lemma 90.27.6. These formal objects are effective by assumption. Effective formal objects can be "approximated" by objects x over finite type S-schemes U, see Lemma 98.10.1. This approximation uses the local rings of S are G-rings and that $\mathcal{X}$ is limit preserving; it is perhaps the most difficult part of the proof relying as it does on general Néron desingularization to approximate formal solutions of algebraic equations over a Noetherian local G-ring by solutions in the henselization. Next openness of versality implies we may (after shrinking U) assume x is versal at every closed point of U. Having done all of this we show that $U \to \mathcal{X}$ is a smooth morphism. Taking sufficiently many $U \to \mathcal{X}$ we show that we obtain a "smooth atlas" for $\mathcal{X}$ which shows that $\mathcal{X}$ is an algebraic stack.

In checking Artin's axioms for a given category $\mathcal{X}$ fibred in groupoids, the most difficult step is often to verify openness of versality. For the discussion that follows, assume that $\mathcal{X}/S$ already satisfies the other conditions listed above. In this chapter we offer two methods that will allow the reader to prove $\mathcal{X}$ satisfies openness of versality:

(1) The first is to assume a stronger Rim-Schlessinger condition, called (RS*) and to assume a stronger version of formal effectiveness, essentially requiring objects over inverse systems of thickenings to be effective. It turns out that under these assumptions, openness of versality comes for free, see Lemma 98.20.3. Please observe that here we are using in an essential manner that $\mathcal{X}$ is defined on that category of all schemes over S, not just the category of Noetherian schemes!
(2) The second, following Artin, is to require $\mathcal{X}$ to come equipped with an obstruction theory. If said obstruction theory "commutes with products" in a suitable sense, then $\mathcal{X}$ satisfies openness of versality, see Lemma 98.22.2.

Obstruction theories can be axiomatized in many different ways and indeed many variants (often adapted to specific moduli stacks) can be found in the literature. We explain a variant using the derived category (which often arises naturally from deformation theory computations done in the literature) in Lemma 98.24.4.

In Section 98.26 we discuss what needs to be modified to make things work for functors defined on the category $(\textit{Noetherian}/S)_{\acute{e}tale}$ of locally Noetherian schemes over S.

In the final section of this chapter as an application of Artin's axioms we prove Artin's theorem on the existence of contractions, see Section 98.27. The theorem says roughly that given an algebraic space X' separated of finite type over S, a closed subset $T' \subset |X'|$, and a formal modification

$$\mathfrak{f} : X'_{/T'} \longrightarrow \mathfrak{X}$$

where $\mathfrak{X}$ is a Noetherian formal algebraic space over S, there exists a proper morphism $f : X' \to X$ which "realizes the contraction". By this we mean that there exists an identification $\mathfrak{X} = X_{/T}$ such that $\mathfrak{f} = f_{/T'} : X'_{/T'} \to X_{/T}$ where $T = f(T')$ and moreover f is an isomorphism over $X \setminus T$. The proof proceeds by defining a functor F on the category of locally Noetherian schemes over S and proving Artin's axioms for F. Amusingly, in this application of Artin's axioms, openness of versality is not the hardest thing to prove, instead the proof that F is limit preserving requires a lot of work and preliminary results.

98.2. Conventions

07T1 The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 94.2. In this chapter the base scheme S will often be locally Noetherian (although we will always reiterate this condition when stating results).

98.3. Predeformation categories

07T2 Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let k be a field and let $\mathrm{Spec}(k) \to S$ be a morphism of finite type (see Morphisms, Lemma 29.16.1). We will sometimes simply say that k is a field of finite type over S. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Given S, $\mathcal{X}$, k, and x_0 we will construct a predeformation category, as defined in Formal Deformation Theory, Definition 90.6.2. The construction will resemble the construction of Formal Deformation Theory, Remark 90.6.4.

First, by Morphisms, Lemma 29.16.1 we may pick an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \to S$ factors through $\mathrm{Spec}(\Lambda)$ and the associated ring map $\Lambda \to k$ is finite. This provides us with the category $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Definition 90.3.1. The category $\mathcal{C}_\Lambda$, up to canonical equivalence, does not depend on the choice of the affine open $\mathrm{Spec}(\Lambda)$ of S. Namely, $\mathcal{C}_\Lambda$ is equivalent to the opposite of the category of factorizations

07T3 (98.3.0.1)
$$\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to S$$

of the structure morphism such that A is an Artinian local ring and such that $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ corresponds to a ring map $A \to k$ which identifies k with the residue field of A.

We let $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ be the category whose

(1) objects are morphisms $x_0 \to x$ of $\mathcal{X}$ where $p(x) = \mathrm{Spec}(A)$ with A an Artinian local ring and $p(x_0) \to p(x) \to S$ a factorization as in (98.3.0.1), and
(2) morphisms $(x_0 \to x) \to (x_0 \to x')$ are commutative diagrams

$$\begin{tikzcd} x & & x' \arrow[ll] \\ & x_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathcal{X}$. (Note the reversal of arrows.)

If $x_0 \to x$ is an object of $\mathcal{F}$ then writing $p(x) = \text{Spec}(A)$ we obtain an object A of $\mathcal{C}_\Lambda$. We often say that $x_0 \to x$ or x lies over A. A morphism of $\mathcal{F}$ between objects $x_0 \to x$ lying over A and $x_0 \to x'$ lying over A' corresponds to a morphism $x' \to x$ of $\mathcal{X}$, hence a morphism $p(x' \to x) : \text{Spec}(A') \to \text{Spec}(A)$ which in turn corresponds to a ring map $A \to A'$. As $\mathcal{X}$ is a category over the category of schemes over S we see that $A \to A'$ is Λ-algebra homomorphism. Thus we obtain a functor

07T4 (98.3.0.2)
$$p : \mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{C}_\Lambda.$$

We will use the notation $\mathcal{F}(A)$ to denote the fibre category over an object A of $\mathcal{C}_\Lambda$. An object of $\mathcal{F}(A)$ is simply a morphism $x_0 \to x$ of $\mathcal{X}$ such that x lies over $\text{Spec}(A)$ and $x_0 \to x$ lies over $\text{Spec}(k) \to \text{Spec}(A)$.

07T5 Lemma 98.3.1. The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ defined above is a predeformation category.

Proof. We have to show that $\mathcal{F}$ is (a) cofibred in groupoids over $\mathcal{C}_\Lambda$ and (b) that $\mathcal{F}(k)$ is a category equivalent to a category with a single object and a single morphism.

Proof of (a). The fibre categories of $\mathcal{F}$ over $\mathcal{C}_\Lambda$ are groupoids as the fibre categories of $\mathcal{X}$ are groupoids. Let $A \to A'$ be a morphism of $\mathcal{C}_\Lambda$ and let $x_0 \to x$ be an object of $\mathcal{F}(A)$. Because $\mathcal{X}$ is fibred in groupoids, we can find a morphism $x' \to x$ lying over $\text{Spec}(A') \to \text{Spec}(A)$. Since the composition $A \to A' \to k$ is equal the given map $A \to k$ we see (by uniqueness of pullbacks up to isomorphism) that the pullback via $\text{Spec}(k) \to \text{Spec}(A')$ of x' is x_0, i.e., that there exists a morphism $x_0 \to x'$ lying over $\text{Spec}(k) \to \text{Spec}(A')$ compatible with $x_0 \to x$ and $x' \to x$. This proves that $\mathcal{F}$ has pushforwards. We conclude by (the dual of) Categories, Lemma 4.35.2.

Proof of (b). If $A = k$, then $\text{Spec}(k) = \text{Spec}(A)$ and since $\mathcal{X}$ is fibred in groupoids over $(Sch/S)_{fppf}$ we see that given any object $x_0 \to x$ in $\mathcal{F}(k)$ the morphism $x_0 \to x$ is an isomorphism. Hence every object of $\mathcal{F}(k)$ is isomorphic to $x_0 \to x_0$. Clearly the only self morphism of $x_0 \to x_0$ in $\mathcal{F}$ is the identity. □

Let S be a locally Noetherian base scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism between categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k is a field of finite type over S. Let x_0 be an object of $\mathcal{X}$ lying over $\text{Spec}(k)$. Set $y_0 = F(x_0)$ which is an object of $\mathcal{Y}$ lying over $\text{Spec}(k)$. Then F induces a functor

07WJ (98.3.1.1)
$$F : \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{F}_{\mathcal{Y},k,y_0}$$

of categories cofibred over $\mathcal{C}_\Lambda$. Namely, to the object $x_0 \to x$ of $\mathcal{F}_{\mathcal{X},k,x_0}(A)$ we associate the object $F(x_0) \to F(x)$ of $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$.

07WK Lemma 98.3.2. Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume either

(1) F is formally smooth on objects (Criteria for Representability, Section 97.6),
(2) F is representable by algebraic spaces and formally smooth, or
(3) F is representable by algebraic spaces and smooth.

Then for every finite type field k over S and object x_0 of $\mathcal{X}$ over k the functor (98.3.1.1) is smooth in the sense of Formal Deformation Theory, Definition 90.8.1.

Proof. Case (1) is a matter of unwinding the definitions. Assumption (2) implies (1) by Criteria for Representability, Lemma 97.6.3. Assumption (3) implies (2) by More on Morphisms of Spaces, Lemma 76.19.6 and the principle of Algebraic Stacks, Lemma 94.10.9. □

07WL Lemma 98.3.3. Let S be a locally Noetherian scheme. Let

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a finite type field over S and w_0 an object of $\mathcal{W}$ over k. Let x_0, z_0, y_0 be the images of w_0 under the morphisms in the diagram. Then

$$\begin{tikzcd} \mathcal{F}_{\mathcal{W},k,w_0} \arrow[r] \arrow[d] & \mathcal{F}_{\mathcal{Z},k,z_0} \arrow[d] \\ \mathcal{F}_{\mathcal{X},k,x_0} \arrow[r] & \mathcal{F}_{\mathcal{Y},k,y_0} \end{tikzcd}$$

is a fibre product of predeformation categories.

Proof. This is a matter of unwinding the definitions. Details omitted. □

98.4. Pushouts and stacks

07WM In this section we show that algebraic stacks behave well with respect to certain pushouts. The results in this section hold over any base scheme.

The following lemma is also correct when Y, X', X, Y' are algebraic spaces, see (insert future reference here).

07WN Lemma 98.4.1. Let S be a scheme. Let

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

be a pushout in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Let $\mathcal{Z}$ be an algebraic stack over S. Then the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

Proof. Let y' be an object of left hand side. The sheaf $Isom(y', y')$ on the category of schemes over Y' is representable by an algebraic space I over Y', see Algebraic Stacks, Lemma 94.10.11. We conclude that the functor of the lemma is fully faithful as Y' is the pushout in the category of algebraic spaces as well as the category of schemes, see Pushouts of Spaces, Lemma 81.6.1.

Let (y, x', f) be an object of the right hand side. Here $f : y|_X \to x'|_X$ is an isomorphism. To finish the proof we have to construct an object y' of $\mathcal{Z}_{Y'}$ whose restrictions to Y and X' agree with y and x' in a manner compatible with f. In fact, it suffices to construct y' fppf locally on Y', see Stacks, Lemma 8.4.8. Choose a representable algebraic stack $\mathcal{W}$ and a surjective smooth morphism $\mathcal{W} \to \mathcal{Z}$. Then

$$(Sch/Y)_{fppf} \times_{y,\mathcal{Z}} \mathcal{W} \quad \text{and} \quad (Sch/X')_{fppf} \times_{x',\mathcal{Z}} \mathcal{W}$$

are algebraic stacks representable by algebraic spaces V and U' smooth over Y and X'. The isomorphism f induces an isomorphism $\varphi : V \times_Y X \to U' \times_{X'} X$ over X. By Pushouts of Spaces, Lemmas 81.6.2 and 81.6.7 we see that the pushout $V' = V \amalg_{V \times_Y X} U'$ is an algebraic space smooth over Y' whose base change to Y and X' recovers V and U' in a manner compatible with φ.

Let W be the algebraic space representing $\mathcal{W}$. The projections $V \to W$ and $U' \to W$ agree as morphisms over $V \times_Y X \cong U' \times_{X'} X$ hence the universal property of the pushout determines a morphism of algebraic spaces $V' \to W$. Choose a scheme Y'_1 and a surjective étale morphism $Y'_1 \to V'$. Set $Y_1 = Y \times_{Y'} Y'_1$, $X'_1 = X' \times_{Y'} Y'_1$, $X_1 = X \times_{Y'} Y'_1$. The composition

$$(Sch/Y'_1) \to (Sch/V') \to (Sch/W) = \mathcal{W} \to \mathcal{Z}$$

corresponds by the 2-Yoneda lemma to an object y'_1 of $\mathcal{Z}$ over Y'_1 whose restriction to Y_1 and X'_1 agrees with $y|_{Y_1}$ and $x'|_{X'_1}$ in a manner compatible with $f|_{X_1}$. Thus we have constructed our desired object smooth locally over Y' and we win. □

98.5. The Rim-Schlessinger condition

06L9 The motivation for the following definition comes from Lemma 98.4.1 and Formal Deformation Theory, Definition 90.16.1 and Lemma 90.16.4.

07WP **Definition 98.5.1.** Let S be a locally Noetherian scheme. Let $\mathcal{Z}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{Z}$ satisfies condition (RS) if for every pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' = Y \amalg_X X' \end{tikzcd}$$

in the category of schemes over S where

(1) X, X', Y, Y' are spectra of local Artinian rings,
(2) X, X', Y, Y' are of finite type over S, and
(3) $X \to X'$ (and hence $Y \to Y'$) is a closed immersion

the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

If A is an Artinian local ring with residue field k, then any morphism $\mathrm{Spec}(A) \to S$ is affine and of finite type if and only if the induced morphism $\mathrm{Spec}(k) \to S$ is of finite type, see Morphisms, Lemmas 29.11.13 and 29.16.2.

07WQ **Lemma 98.5.2.** Let $\mathcal{X}$ be an algebraic stack over a locally Noetherian base S. Then $\mathcal{X}$ satisfies (RS).

Proof. Immediate from the definitions and Lemma 98.4.1. □

07WR **Lemma 98.5.3.** Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS), then so does $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$.

Proof. This is formal. Let

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' = Y \amalg_X X' \end{tikzcd}$$

be a diagram as in Definition 98.5.1. We have to show that

$$(\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_{Y'} \longrightarrow (\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_Y \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_X} (\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_{X'}$$

is an equivalence. Using the definition of the 2-fibre product this becomes

07WS (98.5.3.1) $$\mathcal{X}_{Y'} \times_{\mathcal{Y}_{Y'}} \mathcal{Z}_{Y'} \longrightarrow (\mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y) \times_{(\mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X)} (\mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}).$$

We are given that each of the functors

$$\mathcal{X}_{Y'} \to \mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y, \quad \mathcal{Y}_{Y'} \to \mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X, \quad \mathcal{Z}_{Y'} \to \mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}$$

are equivalences. An object of the right hand side of (98.5.3.1) is a system

$$((x_Y, z_Y, \phi_Y), (x_{X'}, z_{X'}, \phi_{X'}), (\alpha, \beta)).$$

Then $(x_Y, x_{Y'}, \alpha)$ is isomorphic to the image of an object $x_{Y'}$ in $\mathcal{X}_{Y'}$ and $(z_Y, z_{Y'}, \beta)$ is isomorphic to the image of an object $z_{Y'}$ of $\mathcal{Z}_{Y'}$. The pair of morphisms $(\phi_Y, \phi_{X'})$ corresponds to a morphism ψ between the images of $x_{Y'}$ and $z_{Y'}$ in $\mathcal{Y}_{Y'}$. Then $(x_{Y'}, z_{Y'}, \psi)$ is an object of the left hand side of (98.5.3.1) mapping to the given object of the right hand side. This proves that (98.5.3.1) is essentially surjective. We omit the proof that it is fully faithful. $\square$

98.6. Deformation categories

07WT We match the notation introduced above with the notation from the chapter "Formal Deformation Theory".

07WU **Lemma 98.6.1.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS). For any field k of finite type over S and any object x_0 of $\mathcal{X}$ lying over k the predeformation category $p : \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{C}_\Lambda$ (98.3.0.2) is a deformation category, see Formal Deformation Theory, Definition 90.16.8.

Proof. Set $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$. Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. We have to show that the functor

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence, see Formal Deformation Theory, Lemma 90.16.4. Set $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A_2)$, $Y = \mathrm{Spec}(A_1)$ and $Y' = \mathrm{Spec}(A_1 \times_A A_2)$. Note that $Y' = Y \amalg_X X'$ in the category of schemes, see More on Morphisms, Lemma 37.14.3. We know that in the diagram of functors of fibre categories

$$\begin{tikzcd} \mathcal{X}_{Y'} \arrow[r] \arrow[d] & \mathcal{X}_Y \times_{\mathcal{X}_X} \mathcal{X}_{X'} \arrow[d] \\ \mathcal{X}_{\mathrm{Spec}(k)} \arrow[r, equal] & \mathcal{X}_{\mathrm{Spec}(k)} \end{tikzcd}$$

the top horizontal arrow is an equivalence by Definition 98.5.1. Since $\mathcal{F}(B)$ is the category of objects of $\mathcal{X}_{\mathrm{Spec}(B)}$ with an identification with x_0 over k we win. □

07WV Remark 98.6.2. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and x_0 an object of $\mathcal{X}$ over k. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be as in (98.3.0.2). If $\mathcal{F}$ is a deformation category, i.e., if $\mathcal{F}$ satisfies the Rim-Schlessinger condition (RS), then we see that $\mathcal{F}$ satisfies Schlessinger's conditions (S1) and (S2) by Formal Deformation Theory, Lemma 90.16.6. Let $\overline{\mathcal{F}}$ be the functor of isomorphism classes, see Formal Deformation Theory, Remarks 90.5.2 (10). Then $\overline{\mathcal{F}}$ satisfies (S1) and (S2) as well, see Formal Deformation Theory, Lemma 90.10.5. This holds in particular in the situation of Lemma 98.6.1.

98.7. Change of field

07WW This section is the analogue of Formal Deformation Theory, Section 90.29. As pointed out there, to discuss what happens under change of field we need to write $\mathcal{C}_{\Lambda,k}$ instead of $\mathcal{C}_\Lambda$. In the following lemma we use the notation $\mathcal{F}_{l/k}$ introduced in Formal Deformation Theory, Situation 90.29.1.

07WX Lemma 98.7.1. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let l/k be a finite extension. Let x_0 be an object of $\mathcal{F}$ lying over $\mathrm{Spec}(k)$. Denote $x_{l,0}$ the restriction of x_0 to $\mathrm{Spec}(l)$. Then there is a canonical functor

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$. If $\mathcal{X}$ satisfies (RS), then this functor is an equivalence.

Proof. Consider a factorization

$$\mathrm{Spec}(l) \to \mathrm{Spec}(B) \to S$$

as in (98.3.0.1). By definition we have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}(B) = \mathcal{F}_{\mathcal{X},k,x_0}(B \times_l k)$$

see Formal Deformation Theory, Situation 90.29.1. Thus an object of this is a morphism $x_0 \to x$ of $\mathcal{X}$ lying over the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(B \times_l k)$. Choosing pullback functor for $\mathcal{X}$ we can associate to $x_0 \to x$ the morphism $x_{l,0} \to x_B$ where x_B is the restriction of x to $\mathrm{Spec}(B)$ (via the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B \times_l k)$ coming from $B \times_l k \subset B$). This construction is functorial in B and compatible with morphisms.

Next, assume $\mathcal{X}$ satisfies (RS). Consider the diagrams

$$\begin{tikzcd} l & B \arrow[l] \\ k \arrow[u] & B \times_l k \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Spec}(l) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(B \times_l k) \end{tikzcd}$$

The diagram on the left is a fibre product of rings. The diagram on the right is a pushout in the category of schemes, see More on Morphisms, Lemma 37.14.3. These schemes are all of finite type over S (see remarks following Definition 98.5.1). Hence (RS) kicks in to give an equivalence of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(B \times_l k)} \longrightarrow \mathcal{X}_{\mathrm{Spec}(k)} \times_{\mathcal{X}_{\mathrm{Spec}(l)}} \mathcal{X}_{\mathrm{Spec}(B)}$$

This implies that the functor defined above gives an equivalence of fibre categories. Hence the functor is an equivalence on categories cofibred in groupoids by (the dual of) Categories, Lemma 4.35.9. □

98.8. Tangent spaces

07WY Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over k. In Formal Deformation Theory, Section 90.12 we have defined the tangent space

07WZ (98.8.0.1)
$$T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \begin{matrix} \text{isomorphism classes of morphisms} \\ x_0 \to x \text{ over } \operatorname{Spec}(k) \to \operatorname{Spec}(k[\epsilon]) \end{matrix} \right\}$$

of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. In Formal Deformation Theory, Section 90.19 we have defined

07X0 (98.8.0.2)
$$\operatorname{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \operatorname{Ker}\left(\operatorname{Aut}_{\operatorname{Spec}(k[\epsilon])}(x'_0) \to \operatorname{Aut}_{\operatorname{Spec}(k)}(x_0)\right)$$

where x'_0 is the pullback of x_0 to $\operatorname{Spec}(k[\epsilon])$. If $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS), then $T\mathcal{F}_{\mathcal{X},k,x_0}$ comes equipped with a natural k-vector space structure by Formal Deformation Theory, Lemma 90.12.2 (assumptions hold by Lemma 98.6.1 and Remark 98.6.2). Moreover, Formal Deformation Theory, Lemma 90.19.9 shows that $\operatorname{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ has a natural k-vector space structure such that addition agrees with composition of automorphisms. A natural condition is to ask these vector spaces to have finite dimension.

The following lemma tells us this is true if $\mathcal{X}$ is locally of finite type over S (see Morphisms of Stacks, Section 101.17).

07X1 **Lemma 98.8.1.** Let S be a locally Noetherian scheme. Assume

1. $\mathcal{X}$ is an algebraic stack,
2. U is a scheme locally of finite type over S, and
3. $(Sch/U)_{fppf} \to \mathcal{X}$ is a smooth surjective morphism.

Then, for any $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ as in Section 98.3 the tangent space $T\mathcal{F}$ and infinitesimal automorphism space $\operatorname{Inf}(\mathcal{F})$ have finite dimension over k.

Proof. Let us write $\mathcal{U} = (Sch/U)_{fppf}$. By our definition of algebraic stacks the 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence in particular the 2-fibre product
$$\mathcal{U}_{x_0} = (Sch/\operatorname{Spec}(k))_{fppf} \times_{\mathcal{X}} \mathcal{U}$$
is representable by an algebraic space U_{x_0} over $\operatorname{Spec}(k)$. Then $U_{x_0} \to \operatorname{Spec}(k)$ is smooth and surjective (in particular U_{x_0} is nonempty). By Spaces over Fields, Lemma 72.16.2 we can find a finite extension l/k and a point $\operatorname{Spec}(l) \to U_{x_0}$ over k. We have
$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} = \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$
by Lemma 98.7.1 and the fact that $\mathcal{X}$ satisfies (RS). Thus we see that
$$T\mathcal{F} \otimes_k l \cong T\mathcal{F}_{\mathcal{X},l,x_{l,0}} \quad \text{and} \quad \operatorname{Inf}(\mathcal{F}) \otimes_k l \cong \operatorname{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$$
by Formal Deformation Theory, Lemmas 90.29.3 and 90.29.4 (these are applicable by Lemmas 98.5.2 and 98.6.1 and Remark 98.6.2). Hence it suffices to prove that $T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$ and $\operatorname{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$ have finite dimension over l. Note that $x_{l,0}$ comes from a point u_0 of $\mathcal{U}$ over l.

We interrupt the flow of the argument to show that the lemma for infinitesimal automorphisms follows from the lemma for tangent spaces. Namely, let $\mathcal{R} = \mathcal{U} \times_\mathcal{X} \mathcal{U}$. Let r_0 be the l-valued point $(u_0, u_0, \text{id}_{x_0})$ of $\mathcal{R}$. Combining Lemma 98.3.3 and Formal Deformation Theory, Lemma 90.26.2 we see that

$$\text{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}}) \subset T\mathcal{F}_{\mathcal{R},l,r_0}$$

Note that $\mathcal{R}$ is an algebraic stack, see Algebraic Stacks, Lemma 94.14.2. Also, $\mathcal{R}$ is representable by an algebraic space R smooth over U (via either projection, see Algebraic Stacks, Lemma 94.16.2). Hence, choose an scheme U' and a surjective étale morphism $U' \to R$ we see that U' is smooth over U, hence locally of finite type over S. As $(Sch/U')_{fppf} \to \mathcal{R}$ is surjective and smooth, we have reduced the question to the case of tangent spaces.

The functor (98.3.1.1)

$$\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth by Lemma 98.3.2. The induced map on tangent spaces

$$T\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is l-linear (by Formal Deformation Theory, Lemma 90.12.4) and surjective (as smooth maps of predeformation categories induce surjective maps on tangent spaces by Formal Deformation Theory, Lemma 90.8.8). Hence it suffices to prove that the tangent space of the deformation space associated to the representable algebraic stack $\mathcal{U}$ at the point u_0 is finite dimensional. Let $\text{Spec}(R) \subset U$ be an affine open such that $u_0 : \text{Spec}(l) \to U$ factors through $\text{Spec}(R)$ and such that $\text{Spec}(R) \to S$ factors through $\text{Spec}(\Lambda) \subset S$. Let $\mathfrak{m}_R \subset R$ be the kernel of the Λ-algebra map $\varphi_0 : R \to l$ corresponding to u_0. Note that R, being of finite type over the Noetherian ring Λ, is a Noetherian ring. Hence $\mathfrak{m}_R = (f_1, \ldots, f_n)$ is a finitely generated ideal. We have

$$T\mathcal{F}_{\mathcal{U},l,u_0} = \{\varphi : R \to l[\epsilon] \mid \varphi \text{ is a } \Lambda\text{-algebra map and } \varphi \bmod \epsilon = \varphi_0\}$$

An element of the right hand side is determined by its values on $f_1, \ldots, f_n$ hence the dimension is at most n and we win. Some details omitted. $\square$

07X2 Lemma 98.8.2. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS). Let k be a field of finite type over S and let w_0 be an object of $\mathcal{W} = \mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ over k. Denote x_0, y_0, z_0 the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w_0. Then there is a 6-term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \text{Inf}(\mathcal{F}_{\mathcal{W},k,w_0}) \arrow[r] & \text{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) \oplus \text{Inf}(\mathcal{F}_{\mathcal{Z},k,z_0}) \arrow[r] & \text{Inf}(\mathcal{F}_{\mathcal{Y},k,y_0}) \arrow[dll] \\ & T\mathcal{F}_{\mathcal{W},k,w_0} \arrow[r] & T\mathcal{F}_{\mathcal{X},k,x_0} \oplus T\mathcal{F}_{\mathcal{Z},k,z_0} \arrow[r] & T\mathcal{F}_{\mathcal{Y},k,y_0} \end{tikzcd}$$

of k-vector spaces.

Proof. By Lemma 98.5.3 we see that $\mathcal{W}$ satisfies (RS) and hence the lemma makes sense. To see the lemma is true, apply Lemmas 98.3.3 and 98.6.1 and Formal Deformation Theory, Lemma 90.20.1. $\square$

98.9. Formal objects

07X3 In this section we transfer some of the notions already defined in the chapter "Formal Deformation Theory" to the current setting. In the following we will say "R is an S-algebra" to indicate that R is a ring endowed with a morphism of schemes $\mathrm{Spec}(R) \to S$.

07X4 **Definition 98.9.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ consists of a Noetherian complete local S-algebra R, objects ξ_n of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}_R^n)$, and morphisms $f_n : \xi_n \to \xi_{n+1}$ of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ such that $R/\mathfrak{m}$ is a field of finite type over S.
(2) A morphism of formal objects $a : \xi = (R, \xi_n, f_n) \to \eta = (T, \eta_n, g_n)$ is given by morphisms $a_n : \xi_n \to \eta_n$ such that for every n the diagram

$$\begin{tikzcd} \xi_n \arrow[r, "f_n"'] \arrow[d, "a_n"'] & \xi_{n+1} \arrow[d, "a_{n+1}"] \\ \eta_n \arrow[r, "g_n"] & \eta_{n+1} \end{tikzcd}$$

is commutative. Applying the functor p we obtain a compatible collection of morphisms $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(T/\mathfrak{m}_T^n)$ and hence a morphism $a_0 : \mathrm{Spec}(R) \to \mathrm{Spec}(T)$ over S. We say that a lies over a_0.

Thus we obtain a category of formal objects of $\mathcal{X}$.

0CXH **Remark 98.9.2.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. The formal object ξ defines a formal object ξ of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. This follows immediately from Definition 98.9.1 above, Formal Deformation Theory, Definition 90.7.1, and our construction of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ in Section 98.3.

If $F : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then F induces a functor between categories of formal objects as well.

07X5 **Lemma 98.9.3.** Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\eta = (R, \eta_n, g_n)$ be a formal object of $\mathcal{Y}$ and let ξ_1 be an object of $\mathcal{X}$ with $F(\xi_1) \cong \eta_1$. If F is formally smooth on objects (see Criteria for Representability, Section 97.6), then there exists a formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ such that $F(\xi) \cong \eta$.

Proof. Note that each of the morphisms $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ is a first order thickening of affine schemes over S. Hence the assumption on F means that we can successively lift ξ_1 to objects $\xi_2, \xi_3, \ldots$ of $\mathcal{X}$ endowed with compatible isomorphisms $\eta_n|_{\mathrm{Spec}(R/\mathfrak{m}^{n-1})} \cong \eta_{n-1}$ and $F(\eta_n) \cong \xi_n$. □

Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Suppose that x is an object of $\mathcal{X}$ over R, where R is a Noetherian complete local S-algebra with residue field of finite type over S. Then we can consider the system of restrictions $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ endowed with the natural morphisms $\xi_1 \to \xi_2 \to \ldots$ coming from transitivity of restriction. Thus

$\xi = (R, \xi_n, \xi_n \to \xi_{n+1})$ is a formal object of $\mathcal{X}$. This construction is functorial in the object x. Thus we obtain a functor

07X6 (98.9.3.1)
$$\left\{\begin{matrix} \text{objects } x \text{ of } \mathcal{X} \text{ such that } p(x) = \mathrm{Spec}(R) \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S \end{matrix}\right\} \longrightarrow \left\{\text{formal objects of } \mathcal{X}\right\}$$

To be precise the left hand side is the full subcategory of $\mathcal{X}$ consisting of objects as indicated and the right hand side is the category of formal objects of $\mathcal{X}$ as in Definition 98.9.1.

07X7 Definition 98.9.4. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. A formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ is called effective if it is in the essential image of the functor (98.9.3.1).

If the category fibred in groupoids is an algebraic stack, then every formal object is effective as follows from the next lemma.

07X8 Lemma 98.9.5. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be an algebraic stack over S. The functor (98.9.3.1) is an equivalence.

Proof. Case I: $\mathcal{X}$ is representable (by a scheme). Say $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X over S. Unwinding the definitions we have to prove the following: Given a Noetherian complete local S-algebra R with $R/\mathfrak{m}$ of finite type over S we have

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective. This follows from Formal Spaces, Lemma 87.33.2.

Case II. $\mathcal{X}$ is representable by an algebraic space. Say $\mathcal{X}$ is representable by X. Again we have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for R as above. This is Formal Spaces, Lemma 87.33.3.

Case III: General case of an algebraic stack. A general remark is that the left and right hand side of (98.9.3.1) are categories fibred in groupoids over the category of affine schemes over S which are spectra of Noetherian complete local rings with residue field of finite type over S. We will also see in the proof below that they form stacks for a certain topology on this category.

We first prove fully faithfulness. Let R be a Noetherian complete local S-algebra with $k = R/\mathfrak{m}$ of finite type over S. Let x, x' be objects of $\mathcal{X}$ over R. As $\mathcal{X}$ is an algebraic stack $Isom(x, x')$ is representable by an algebraic space I over $\mathrm{Spec}(R)$, see Algebraic Stacks, Lemma 94.10.11. Applying Case II to I over $\mathrm{Spec}(R)$ implies immediately that (98.9.3.1) is fully faithful on fibre categories over $\mathrm{Spec}(R)$. Hence the functor is fully faithful by Categories, Lemma 4.35.9.

Essential surjectivity. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$. Choose a scheme U over S and a surjective smooth morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$. For every n consider the fibre product

$$(Sch/\mathrm{Spec}(R/\mathfrak{m}^n))_{fppf} \times_{\xi_n, \mathcal{X}, f} (Sch/U)_{fppf}$$

By assumption this is representable by an algebraic space V_n surjective and smooth over $\mathrm{Spec}(R/\mathfrak{m}^n)$. The morphisms $f_n : \xi_n \to \xi_{n+1}$ induce cartesian squares

$$\begin{tikzcd} V_{n+1} \arrow[d] & V_n \arrow[l] \arrow[d] \\ \mathrm{Spec}(R/\mathfrak{m}^{n+1}) & \mathrm{Spec}(R/\mathfrak{m}^n) \arrow[l] \end{tikzcd}$$

of algebraic spaces. By Spaces over Fields, Lemma 72.16.2 we can find a finite separable extension k'/k and a point $v_1' : \mathrm{Spec}(k') \to V_1$ over k. Let $R \subset R'$ be the finite étale extension whose residue field extension is k'/k (exists and is unique by Algebra, Lemmas 10.153.7 and 10.153.9). By the infinitesimal lifting criterion of smoothness (see More on Morphisms of Spaces, Lemma 76.19.6) applied to $V_n \to \mathrm{Spec}(R/\mathfrak{m}^n)$ for $n = 2, 3, 4, \ldots$ we can successively find morphisms $v_n' : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \to V_n$ over $\mathrm{Spec}(R/\mathfrak{m}^n)$ fitting into commutative diagrams

$$\begin{tikzcd} \mathrm{Spec}(R'/(\mathfrak{m}')^{n+1}) \arrow[d, "v_{n+1}'"'] & \mathrm{Spec}(R'/(\mathfrak{m}')^n) \arrow[l] \arrow[d, "v_n'"] \\ V_{n+1} & V_n \arrow[l] \end{tikzcd}$$

Composing with the projection morphisms $V_n \to U$ we obtain a compatible system of morphisms $u_n' : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \to U$. By Case I the family (u_n') comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote x' the object of $\mathcal{X}$ over $\mathrm{Spec}(R')$ we get by applying the 1-morphism f to u'. By construction, there exists a morphism of formal objects

$$(98.9.3.1)(x') = (R', x'|_{\mathrm{Spec}(R'/(\mathfrak{m}')^n)}, \ldots) \longrightarrow (R, \xi_n, f_n)$$

lying over $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Denote $p_0, p_1 : \mathrm{Spec}(R' \otimes_R R') \to \mathrm{Spec}(R')$ the two projections. By the fully faithfulness shown above there exists a canonical isomorphism $\varphi : p_0^*x' \to p_1^*x'$ because we have such isomorphisms over $\mathrm{Spec}((R' \otimes_R R')/\mathfrak{m}^n(R' \otimes_R R'))$. We omit the proof that the isomorphism φ satisfies the cocycle condition (see Stacks, Definition 8.3.1). Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to an object x of $\mathcal{X}$ over $\mathrm{Spec}(R)$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. □

07X9 Lemma 98.9.6. Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If the functor (98.9.3.1) is an equivalence for $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, then it is an equivalence for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

Proof. The left and the right hand side of (98.9.3.1) for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ are simply the 2-fibre products of the left and the right hand side of (98.9.3.1) for $\mathcal{X}$, $\mathcal{Z}$ over $\mathcal{Y}$. Hence the result follows as taking 2-fibre products is compatible with equivalences of categories, see Categories, Lemma 4.31.7. □

98.10. Approximation

07XA A fundamental insight of Michael Artin is that you can approximate objects of a limit preserving stack. Namely, given an object x of the stack over a Noetherian complete local ring, you can find an object x_A over an algebraic ring which is “close to" x. Here an algebraic ring means a finite type S-algebra and close means adically close. In this section we present this in a simple, yet general form.

To formulate the result we need to pull together some definitions from different places in the Stacks project. First, in Criteria for Representability, Section 97.5 we introduced limit preserving on objects for 1-morphisms of categories fibred in groupoids over the category of schemes. In More on Algebra, Definition 15.50.1 we defined the notion of a G-ring. Let S be a locally Noetherian scheme. Let A be an S-algebra. We say that A is of finite type over S or is a finite type S-algebra if $\mathrm{Spec}(A) \to S$ is of finite type. In this case A is a Noetherian ring. Finally, given a ring A and ideal I we denote $\mathrm{Gr}_I(A) = \bigoplus I^n/I^{n+1}$.

07XB Lemma 98.10.1. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ where R is a Noetherian complete local ring with residue field k of finite type over S. Let $s \in S$ be the image of $\mathrm{Spec}(k) \to S$. Assume that (a) $\mathcal{O}_{S,s}$ is a G-ring and (b) p is limit preserving on objects. Then for every integer $N \geq 1$ there exist

(1) a finite type S-algebra A,
(2) a maximal ideal $\mathfrak{m}_A \subset A$,
(3) an object x_A of $\mathcal{X}$ over $\mathrm{Spec}(A)$,
(4) an S-isomorphism $R/\mathfrak{m}_R^N \cong A/\mathfrak{m}_A^N$,
(5) an isomorphism $x|_{\mathrm{Spec}(R/\mathfrak{m}_R^N)} \cong x_A|_{\mathrm{Spec}(A/\mathfrak{m}_A^N)}$ compatible with (4), and
(6) an isomorphism $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_A}(A)$ of graded k-algebras.

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra, see Morphisms, Lemma 29.16.1. We may and do replace S by $\mathrm{Spec}(\Lambda)$.

We may write R as a directed colimit $R = \mathrm{colim}\, C_j$ where each C_j is a finite type Λ-algebra (see Algebra, Lemma 10.127.2). By assumption (b) the object x is isomorphic to the restriction of an object over one of the C_j. Hence we may choose a finite type Λ-algebra C, a Λ-algebra map $C \to R$, and an object x_C of $\mathcal{X}$ over $\mathrm{Spec}(C)$ such that $x = x_C|_{\mathrm{Spec}(R)}$. The choice of C is a bookkeeping device and could be avoided. For later use, let us write $C = \Lambda[y_1, \ldots, y_u]/(f_1, \ldots, f_v)$ and we denote $\overline{a}_i \in R$ the image of y_i under the map $C \to R$. Set $\mathfrak{m}_C = C \cap \mathfrak{m}_R$.

Choose a Λ-algebra surjection $\Lambda[x_1, \ldots, x_s] \to k$ and denote $\mathfrak{m}'$ the kernel. By the universal property of polynomial rings we may lift this to a Λ-algebra map $\Lambda[x_1, \ldots, x_s] \to R$. We add some variables (i.e., we increase s a bit) mapping to generators of $\mathfrak{m}_R$. Having done this we see that $\Lambda[x_1, \ldots, x_s] \to R/\mathfrak{m}_R^2$ is surjective. Then we see that

07XC (98.10.1.1)
$$P = \Lambda[x_1, \ldots, x_s]^\wedge_{\mathfrak{m}'} \longrightarrow R$$

is a surjective map of Noetherian complete local rings, see for example Formal Deformation Theory, Lemma 90.4.2.

Choose lifts $a_i \in P$ of $\overline{a}_i$ we found above. Choose generators $b_1, \ldots, b_r \in P$ for the kernel of (98.10.1.1). Choose $c_{ji} \in P$ such that

$$f_j(a_1, \ldots, a_u) = \sum c_{ji} b_i$$

in P which is possible by the choices made so far. Choose generators

$$k_1, \ldots, k_t \in \mathrm{Ker}(P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P)$$

and write $k_i = (k_{i1}, \ldots, k_{ir})$ and $K = (k_{ij})$ so that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \to R \to 0$$

is an exact sequence of P-modules. In particular we have $\sum k_{ij} b_j = 0$. After possibly increasing N we may assume $N-1$ works in the Artin-Rees lemma for the first two maps of this exact sequence (see More on Algebra, Section 15.4 for terminology).

By assumption $\mathcal{O}_{S,s} = \Lambda_{\Lambda \cap \mathfrak{m}'}$ is a G-ring. Hence by More on Algebra, Proposition 15.50.10 the ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$ is a G-ring. Hence by Smoothing Ring Maps, Theorem 16.13.2 there exist an étale ring map

$$\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \to B,$$

a maximal ideal $\mathfrak{m}_B$ of B lying over $\mathfrak{m}'$, and elements $a'_i, b'_i, c'_{ij}, k'_{ij} \in B'$ such that

(1) $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m}_B)$ which implies that $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \subset B_{\mathfrak{m}_B} \subset P$ and P is identified with the completion of B at $\mathfrak{m}_B$, see remark preceding Smoothing Ring Maps, Theorem 16.13.2,
(2) $a_i - a'_i, b_i - b'_i, c_{ij} - c'_{ij}, k_{ij} - k'_{ij} \in (\mathfrak{m}')^N P$, and
(3) $f_j(a'_1, \ldots, a'_u) = \sum c'_{ji} b'_i$ and $\sum k'_{ij} b'_j = 0$.

Set $A = B/(b'_1, \ldots, b'_r)$ and denote $\mathfrak{m}_A$ the image of $\mathfrak{m}_B$ in A. (Note that A is essentially of finite type over Λ; at the end of the proof we will show how to obtain an A which is of finite type over Λ.) There is a ring map $C \to A$ sending $y_i \mapsto a'_i$ because the a'_i satisfy the desired equations modulo $(b'_1, \ldots, b'_r)$. Note that $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$ as quotients of $P = B^\wedge$ by property (2) above. Set $x_A = x_C|_{\mathrm{Spec}(A)}$. Since the maps

$$C \to A \to A/\mathfrak{m}_A^N \cong R/\mathfrak{m}_R^N \quad \text{and} \quad C \to R \to R/\mathfrak{m}_R^N$$

are equal we see that x_A and x agree modulo $\mathfrak{m}_R^N$ via the isomorphism $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$. At this point we have shown properties (1) – (5) of the statement of the lemma. To see (6) note that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \quad \text{and} \quad P^{\oplus t} \xrightarrow{K'} P^{\oplus r} \xrightarrow{(b'_1, \ldots, b'_r)} P$$

are two complexes of P-modules which are congruent modulo $(\mathfrak{m}')^N$ with the first one being exact. By our choice of N above we see from More on Algebra, Lemma 15.4.2 that $R = P/(b_1, \ldots, b_r)$ and $P/(b'_1, \ldots, b'_r) = B^\wedge/(b'_1, \ldots, b'_r) = A^\wedge$ have isomorphic associated graded algebras, which is what we wanted to show.

This last paragraph of the proof serves to clean up the issue that A is essentially of finite type over S and not yet of finite type. The construction above gives $A = B/(b'_1, \ldots, b'_r)$ and $\mathfrak{m}_A \subset A$ with B étale over $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$. Hence A is of finite type over the Noetherian ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$. Thus we can write $A = (A_0)_{\mathfrak{m}'}$ for some finite type $\Lambda[x_1, \ldots, x_n]$ algebra A_0. Then $A = \mathrm{colim}(A_0)_f$ where $f \in \Lambda[x_1, \ldots, x_n] \setminus \mathfrak{m}'$, see Algebra, Lemma 10.9.9. Because $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, we see that x_A comes from some object $x_{(A_0)_f}$ over $\mathrm{Spec}((A_0)_f)$ for an f as above. After replacing A by $(A_0)_f$ and x_A by $x_{(A_0)_f}$ and $\mathfrak{m}_A$ by $(A_0)_f \cap \mathfrak{m}_A$ the proof is finished. $\square$

98.11. Limit preserving

07XK The morphism $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, as defined in Criteria for Representability, Section 97.5, if the functor of the definition below is essentially surjective. However, the example in Examples, Section 110.53 shows that this isn't equivalent to being limit preserving.

07XL Definition 98.11.1. Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ is limit preserving if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have an equivalence

$$\operatorname{colim} \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T$$

of fibre categories.

We spell out what this means. First, given objects x, y of $\mathcal{X}$ over T_i we should have

$$\operatorname{Mor}_{\mathcal{X}_T}(x|_T, y|_T) = \operatorname{colim}_{i' \geq i} \operatorname{Mor}_{\mathcal{X}_{T_{i'}}}(x|_{T_{i'}}, y|_{T_{i'}})$$

and second every object of $\mathcal{X}_T$ is isomorphic to the restriction of an object over T_i for some i. Note that the first condition means that the presheaves $Isom_{\mathcal{X}}(x, y)$ (see Stacks, Definition 8.2.2) are limit preserving.

07XM Lemma 98.11.2. Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.

(1) If $\mathcal{X} \to (Sch/S)_{fppf}$ and $\mathcal{Z} \to (Sch/S)_{fppf}$ are limit preserving on objects and $\mathcal{Y}$ is limit preserving, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to (Sch/S)_{fppf}$ is limit preserving on objects.
(2) If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are limit preserving, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

Proof. This is formal. Proof of (1). Let $T = \lim_{i \in I} T_i$ be the directed limit of affine schemes T_i over S. We will prove that the functor $\operatorname{colim} \mathcal{X}_{T_i} \to \mathcal{X}_T$ is essentially surjective. Recall that an object of the fibre product over T is a quadruple (T, x, z, α) where x is an object of $\mathcal{X}$ lying over T, z is an object of $\mathcal{Z}$ lying over T, and $\alpha : p(x) \to q(z)$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By assumption on $\mathcal{X}$ and $\mathcal{Z}$ we can find an i and objects x_i and z_i over T_i such that $x_i|_T \cong T$ and $z_i|_T \cong z$. Then α corresponds to an isomorphism $p(x_i)|_T \to q(z_i)|_T$ which comes from an isomorphism $\alpha_{i'} : p(x_i)|_{T_{i'}} \to q(z_i)|_{T_{i'}}$ by our assumption on $\mathcal{Y}$. After replacing i by i', x_i by $x_i|_{T_{i'}}$, and z_i by $z_i|_{T_{i'}}$ we see that $(T_i, x_i, z_i, \alpha_i)$ is an object of the fibre product over T_i which restricts to an object isomorphic to (T, x, z, α) over T as desired.

We omit the arguments showing that $\operatorname{colim} \mathcal{X}_{T_i} \to \mathcal{X}_T$ is fully faithful in (2). □

07XN Lemma 98.11.3. Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Then the following are equivalent

(1) $\mathcal{X}$ is a stack in setoids and $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects,
(2) $\mathcal{X}$ is a stack in setoids and limit preserving,
(3) $\mathcal{X}$ is representable by an algebraic space locally of finite presentation.

Proof. Under each of the three assumptions $\mathcal{X}$ is representable by an algebraic space X over S, see Algebraic Stacks, Proposition 94.13.3. It is clear that (1) and (2) are equivalent as a functor between setoids is an equivalence if and only if it is surjective on isomorphism classes. Finally, (1) and (3) are equivalent by Limits of Spaces, Proposition 70.3.10. □

0CXI Lemma 98.11.4. Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and $\mathcal{X}$ is limit preserving. Then Δ is locally of finite type.

Proof. We apply Criteria for Representability, Lemma 97.5.6. Let V be an affine scheme V locally of finite presentation over S and let θ be an object of $\mathcal{X} \times \mathcal{X}$ over V. Let F_θ be an algebraic space representing $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \theta} (Sch/V)_{fppf}$ and let $f_\theta : F_\theta \to V$ be the canonical morphism (see Algebraic Stacks, Section 94.9). It suffices to show that $F_\theta \to V$ has the corresponding properties. By Lemmas 98.11.2 and 98.11.3 we see that $F_\theta \to S$ is locally of finite presentation. It follows that $F_\theta \to V$ is locally of finite type by Morphisms of Spaces, Lemma 67.23.6. $\square$

98.12. Versality

07XD In the previous section we explained how to approximate objects over complete local rings by algebraic objects. But in order to show that a stack $\mathcal{X}$ is an algebraic stack, we need to find smooth 1-morphisms from schemes towards $\mathcal{X}$. Since we are not going to assume a priori that $\mathcal{X}$ has a representable diagonal, we cannot even speak about smooth morphisms towards $\mathcal{X}$. Instead, borrowing terminology from deformation theory, we will introduce versal objects.

0CXJ Definition 98.12.1. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. We will say that ξ is versal if ξ as a formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$ (Remark 98.9.2) is versal in the sense of Formal Deformation Theory, Definition 90.8.9.

We briefly spell out what this means. With notation as in the definition, suppose given morphisms $\xi_1 = x_0 \to y \to z$ of $\mathcal{X}$ lying over closed immersions $\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ where A, B are Artinian local rings with residue field k. Suppose given an $n \geq 1$ and a commutative diagram

$$\begin{tikzcd} & y \arrow[ld] \\ \xi_n & \xi_1 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & \mathrm{Spec}(A) \arrow[ld] \\ \mathrm{Spec}(R/\mathfrak{m}^n) & \mathrm{Spec}(k) \arrow[l] \arrow[u] \end{tikzcd}$$

Versality means that for any data as above there exists an $m \geq n$ and a commutative diagram

$$\begin{tikzcd} & & z \arrow[lldd] \\ & & y \arrow[u] \arrow[ld] \\ \xi_m & \xi_n \arrow[l] & \xi_1 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & & \mathrm{Spec}(B) \arrow[lldd] \\ & & \mathrm{Spec}(A) \arrow[u] \arrow[ld] \\ \mathrm{Spec}(R/\mathfrak{m}^m) & \mathrm{Spec}(R/\mathfrak{m}^n) \arrow[l] & \mathrm{Spec}(k) \arrow[l] \arrow[u] \end{tikzcd}$$

Please compare with Formal Deformation Theory, Remark 90.8.10.

Let S be a locally Noetherian scheme. Let U be a scheme over S with structure morphism $U \to S$ locally of finite type. Let $u_0 \in U$ be a finite type point of U, see Morphisms, Definition 29.16.3. Set $k = \kappa(u_0)$. Note that the composition

$\mathrm{Spec}(k) \to S$ is also of finite type, see Morphisms, Lemma 29.15.3. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ which lies over U. Denote x_0 the pullback of x by u_0. By the 2-Yoneda lemma x corresponds to a 1-morphism

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X},$$

see Algebraic Stacks, Section 94.5. We obtain a morphism of predeformation categories

07XE (98.12.1.1) $$\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0},$$

over $\mathcal{C}_\Lambda$ see (98.3.1.1).

07XF Definition 98.12.2. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ lying over U. Let u_0 be finite type point of U. We say x is versal at u_0 if the morphism $\hat{x}$ (98.12.1.1) is smooth, see Formal Deformation Theory, Definition 90.8.1.

This definition matches our notion of versality for formal objects of $\mathcal{X}$.

0CXK Lemma 98.12.3. With notation as in Definition 98.12.2. Let $R = \mathcal{O}_{U,u_0}^\wedge$. Let ξ be the formal object of $\mathcal{X}$ over R associated to $x|_{\mathrm{Spec}(R)}$, see (98.9.3.1). Then

$$x \text{ is versal at } u_0 \Leftrightarrow \xi \text{ is versal}$$

Proof. Observe that $\mathcal{O}_{U,u_0}$ is a Noetherian local S-algebra with residue field k. Hence $R = \mathcal{O}_{U,u_0}^\wedge$ is an object of $\mathcal{C}_\Lambda^\wedge$, see Formal Deformation Theory, Definition 90.4.1. Recall that ξ is versal if $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth and x is versal at u_0 if $\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth. There is an identification of predeformation categories

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0},$$

see Formal Deformation Theory, Remark 90.7.12 for notation. Namely, given an Artinian local S-algebra A with residue field identified with k we have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda^\wedge}(R, A) = \{\varphi \in \mathrm{Mor}_S(\mathrm{Spec}(A), U) \mid \varphi|_{\mathrm{Spec}(k)} = u_0\}$$

Unwinding the definitions the reader verifies that the resulting map

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \xrightarrow{\hat{x}} \mathcal{F}_{\mathcal{X},k,x_0},$$

is equal to $\underline{\xi}$ and we see that the lemma is true. $\square$

Here is a sanity check.

0CXL Lemma 98.12.4. Let S be a locally Noetherian scheme. Let $f : U \to V$ be a morphism of schemes locally of finite type over S. Let $u_0 \in U$ be a finite type point. The following are equivalent

(1) f is smooth at u_0,
(2) f viewed as an object of $(Sch/V)_{fppf}$ over U is versal at u_0.

Proof. This is a restatement of More on Morphisms, Lemma 37.12.1. $\square$

It turns out that this notion is well behaved with respect to field extensions.

07XG **Lemma 98.12.5.** Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 98.12.2. Let l be a field and let $u_{l,0} : \mathrm{Spec}(l) \to U$ be a morphism with image u_0 such that $l/k = \kappa(u_0)$ is finite. Set $x_{l,0} = x_0|_{\mathrm{Spec}(l)}$. If $\mathcal{X}$ satisfies (RS) and x is versal at u_0, then

$$\mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth.

Proof. Note that $(Sch/U)_{fppf}$ satisfies (RS) by Lemma 98.5.2. Hence the functor of the lemma is the functor

$$(\mathcal{F}_{(Sch/U)_{fppf},k,u_0})_{l/k} \longrightarrow (\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}$$

associated to $\hat{x}$, see Lemma 98.7.1. Hence the lemma follows from Formal Deformation Theory, Lemma 90.29.5. □

The following lemma is another sanity check. It more or less signifies that if x is versal at u_0 as in Definition 98.12.2, then x viewed as a morphism from U to $\mathcal{X}$ is smooth whenever we make a base change by a scheme.

0CXM **Lemma 98.12.6.** Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 98.12.2. Assume

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(2) Δ is locally of finite type (for example if $\mathcal{X}$ is limit preserving), and
(3) $\mathcal{X}$ has (RS).

Let V be a scheme locally of finite type over S and let y be an object of $\mathcal{X}$ over V. Form the 2-fibre product

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] & (Sch/U)_{fppf} \arrow[d, "x"] \\ (Sch/V)_{fppf} \arrow[r, "y"] & \mathcal{X} \end{tikzcd}$$

Let Z be the algebraic space representing $\mathcal{Z}$ and let $z_0 \in |Z|$ be a finite type point lying over u_0. If x is versal at u_0, then the morphism $Z \to V$ is smooth at z_0.

Proof. (The parenthetical remark in the statement holds by Lemma 98.11.4.) Observe that Z exists by assumption (1) and Algebraic Stacks, Lemma 94.10.11. By assumption (2) we see that $Z \to V \times_S U$ is locally of finite type. Choose a scheme W, a closed point $w_0 \in W$, and an étale morphism $W \to Z$ mapping w_0 to z_0, see Morphisms of Spaces, Definition 67.25.2. Then W is locally of finite type over S and w_0 is a finite type point of W. Let $l = \kappa(z_0)$. Denote $z_{l,0}$, $v_{l,0}$, $u_{l,0}$, and $x_{l,0}$ the objects of $\mathcal{Z}$, $(Sch/V)_{fppf}$, $(Sch/U)_{fppf}$, and $\mathcal{X}$ over $\mathrm{Spec}(l)$ obtained by pullback to $\mathrm{Spec}(l) = w_0$. Consider

$$\begin{tikzcd} \mathcal{F}_{(Sch/W)_{fppf},l,w_0} \arrow[r] & \mathcal{F}_{\mathcal{Z},l,z_{l,0}} \arrow[r] \arrow[d] & \mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \arrow[d] \\ & \mathcal{F}_{(Sch/V)_{fppf},l,v_{l,0}} \arrow[r] & \mathcal{F}_{\mathcal{X},l,x_{l,0}} \end{tikzcd}$$

By Lemma 98.3.3 the square is a fibre product of predeformation categories. By Lemma 98.12.5 we see that the right vertical arrow is smooth. By Formal Deformation Theory, Lemma 90.8.7 the left vertical arrow is smooth. By Lemma 98.3.2 we see that the left horizontal arrow is smooth. We conclude that the map

$$\mathcal{F}_{(Sch/W)_{fppf},l,w_0} \to \mathcal{F}_{(Sch/V)_{fppf},l,v_{l,0}}$$

is smooth by Formal Deformation Theory, Lemma 90.8.7. Thus we conclude that $W \to V$ is smooth at w_0 by More on Morphisms, Lemma 37.12.1. This exactly means that $Z \to V$ is smooth at z_0 and the proof is complete. □

We restate the approximation result in terms of versal objects.

07XH Lemma 98.12.7. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$ with ξ_1 lying over $\mathrm{Spec}(k) \to S$ with image $s \in S$. Assume

(1) ξ is versal,
(2) ξ is effective,
(3) $\mathcal{O}_{S,s}$ is a G-ring, and
(4) $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects.

Then there exist a morphism of finite type $U \to S$, a finite type point $u_0 \in U$ with residue field k, and an object x of $\mathcal{X}$ over U such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}_{u_0}^n)} \cong \xi_n$.

Proof. Choose an object x_R of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ whose associated formal object is ξ. Let $N = 2$ and apply Lemma 98.10.1. We obtain $A, \mathfrak{m}_A, x_A, \ldots$. Let $\eta = (A^\wedge, \eta_n, g_n)$ be the formal object associated to $x_A|_{\mathrm{Spec}(A^\wedge)}$. We have a diagram

$$\begin{tikzcd} & \eta \arrow[d] \\ \xi \arrow[r] \arrow[ru, dotted] & \xi_2 = \eta_2 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A^\wedge \arrow[d] \\ R \arrow[r] \arrow[ru, dotted] & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{tikzcd}$$

The versality of ξ means exactly that we can find the dotted arrows in the diagrams, because we can successively find morphisms $\xi \to \eta_3$, $\xi \to \eta_4$, and so on by Formal Deformation Theory, Remark 90.8.10. The corresponding ring map $R \to A^\wedge$ is surjective by Formal Deformation Theory, Lemma 90.4.2. On the other hand, we have $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ for all n by construction. Hence $R/\mathfrak{m}_R^n$ and $A/\mathfrak{m}_A^n$ have the same (finite) length as Λ-modules by additivity of length and Formal Deformation Theory, Lemma 90.3.4. It follows that $R/\mathfrak{m}_R^n \to A/\mathfrak{m}_A^n$ is an isomorphism for all n, hence $R \to A^\wedge$ is an isomorphism. Thus η is isomorphic to a versal object, hence versal itself. By Lemma 98.12.3 we conclude that x_A is versal at the point u_0 of $U = \mathrm{Spec}(A)$ corresponding to $\mathfrak{m}_A$. □

07XI Example 98.12.8. In this example we show that the local ring $\mathcal{O}_{S,s}$ has to be a G-ring in order for the result of Lemma 98.12.7 to be true. Namely, let Λ be a Noetherian ring and let $\mathfrak{m}$ be a maximal ideal of Λ. Set $R = \Lambda_\mathfrak{m}^\wedge$. Let $\Lambda \to C \to R$ be a factorization with C of finite type over Λ. Set $S = \mathrm{Spec}(\Lambda)$, $U = S \setminus \{\mathfrak{m}\}$, and $S' = U \amalg \mathrm{Spec}(C)$. Consider the functor $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ defined by the rule

$$F(T) = \begin{cases} * & \text{if } T \to S \text{ factors through } S' \\ \emptyset & \text{else} \end{cases}$$

Let $\mathcal{X} = \mathcal{S}_F$ is the category fibred in sets associated to F, see Algebraic Stacks, Section 94.7. Then $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects and there exists an effective, versal formal object ξ over R. Hence if the conclusion of Lemma 98.12.7 holds for $\mathcal{X}$, then there exists a finite type ring map $\Lambda \to A$ and a maximal ideal $\mathfrak{m}_A$ lying over $\mathfrak{m}$ such that

(1) $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}_A)$,
(2) $\Lambda \to A$ and $\mathfrak{m}_A$ satisfy condition (4) of Algebra, Lemma 10.141.2, and
(3) there exists a Λ-algebra map $C \to A$.

Thus $\Lambda \to A$ is smooth at $\mathfrak{m}_A$ by the lemma cited. Slicing A we may assume that $\Lambda \to A$ is étale at $\mathfrak{m}_A$, see for example More on Morphisms, Lemma 37.38.5 or argue directly. Write $C = \Lambda[y_1, \ldots, y_n]/(f_1, \ldots, f_m)$. Then $C \to R$ corresponds to a solution in R of the system of equations $f_1 = \ldots = f_m = 0$, see Smoothing Ring Maps, Section 16.13. Thus if the conclusion of Lemma 98.12.7 holds for every $\mathcal{X}$ as above, then a system of equations which has a solution in R has a solution in the henselization of $\Lambda_\mathfrak{m}$. In other words, the approximation property holds for $\Lambda_\mathfrak{m}^h$. This implies that $\Lambda_\mathfrak{m}^h$ is a G-ring (insert future reference here; see also discussion in Smoothing Ring Maps, Section 16.1) which in turn implies that $\Lambda_\mathfrak{m}$ is a G-ring.

98.13. Openness of versality

07XP Next, we come to openness of versality.

07XQ Definition 98.13.1. Let S be a locally Noetherian scheme.

(1) Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies openness of versality if given a scheme U locally of finite type over S, an object x of $\mathcal{X}$ over U, and a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.
(2) Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f satisfies openness of versality if given a scheme U locally of finite type over S, an object y of $\mathcal{Y}$ over U, openness of versality holds for $(Sch/U)_{fppf} \times_\mathcal{Y} \mathcal{X}$.

Openness of versality is often the hardest to check. The following example shows that requiring this is necessary however.

07XR Example 98.13.2. Let k be a field and set $\Lambda = k[s, t]$. Consider the functor $F : \Lambda\text{-algebras} \longrightarrow \text{Sets}$ defined by the rule

$$F(A) = \begin{cases} * & \text{if there exist } f_1, \ldots, f_n \in A \text{ such that} \\ & \quad A = (s, t, f_1, \ldots, f_n) \text{ and } f_i s = 0\ \forall i \\ \emptyset & \quad\quad\quad\quad\quad \text{else} \end{cases}$$

Geometrically $F(A) = *$ means there exists a quasi-compact open neighbourhood W of $V(s, t) \subset \text{Spec}(A)$ such that $s|_W = 0$. Let $\mathcal{X} \subset (Sch/\text{Spec}(\Lambda))_{fppf}$ be the full subcategory consisting of schemes T which have an affine open covering $T = \bigcup \text{Spec}(A_j)$ with $F(A_j) = *$ for all j. Then $\mathcal{X}$ satisfies [0], [1], [2], [3], and [4] but not [5]. Namely, over $U = \text{Spec}(k[s, t]/(s))$ there exists an object x which is versal at $u_0 = (s, t)$ but not at any other point. Details omitted.

Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Consider the following property

07XS (98.13.2.1)
$$\begin{matrix}\text{for all fields } k \text{ of finite type over } S \text{ and all } x_0 \in \text{Ob}(\mathcal{X}_{\text{Spec}(k)}) \text{ the} \\ \text{map } \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{F}_{\mathcal{Y},k,f(x_0)} \text{ of predeformation categories is smooth}\end{matrix}$$

We formulate some lemmas around this concept. First we link it with (openness of) versality.

07XT Lemma 98.13.3. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ over U. Assume that x is versal at every finite type point of U and that $\mathcal{X}$ satisfies (RS). Then $x : (Sch/U)_{fppf} \to \mathcal{X}$ satisfies (98.13.2.1).

Proof. Let $\mathrm{Spec}(l) \to U$ be a morphism with l of finite type over S. Then the image $u_0 \in U$ is a finite type point of U and $l/\kappa(u_0)$ is a finite extension, see discussion in Morphisms, Section 29.16. Hence we see that $\mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \to \mathcal{F}_{\mathcal{X},l,x_{l,0}}$ is smooth by Lemma 98.12.5. □

07XU Lemma 98.13.4. Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f and g satisfy (98.13.2.1) so does $g \circ f$.

Proof. This follows formally from Formal Deformation Theory, Lemma 90.8.7. □

07XV Lemma 98.13.5. Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f satisfies (98.13.2.1) so does the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$.

Proof. Follows immediately from Lemma 98.3.3 and Formal Deformation Theory, Lemma 90.8.7. □

07XW Lemma 98.13.6. Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is formally smooth on objects, then f satisfies (98.13.2.1). If f is representable by algebraic spaces and smooth, then f satisfies (98.13.2.1).

Proof. A reformulation of Lemma 98.3.2. □

07XX Lemma 98.13.7. Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume

(1) f is representable by algebraic spaces,
(2) f satisfies (98.13.2.1),
(3) $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, and
(4) $\mathcal{Y}$ is limit preserving.

Then f is smooth.

Proof. The key ingredient of the proof is More on Morphisms, Lemma 37.12.1 which (almost) says that a morphism of schemes of finite type over S satisfying (98.13.2.1) is a smooth morphism. The other arguments of the proof are essentially bookkeeping.

Let V be a scheme over S and let y be an object of $\mathcal{Y}$ over V. Let Z be an algebraic space representing the 2-fibre product $\mathcal{Z} = \mathcal{X} \times_{f,\mathcal{X},y} (Sch/V)_{fppf}$. We have to show that the projection morphism $Z \to V$ is smooth, see Algebraic Stacks, Definition 94.10.1. In fact, it suffices to do this when V is an affine scheme locally of finite presentation over S, see Criteria for Representability, Lemma 97.5.6. Then $(Sch/V)_{fppf}$ is limit preserving by Lemma 98.11.3. Hence $Z \to S$ is locally of finite presentation by Lemmas 98.11.2 and 98.11.3. Choose a scheme W and a surjective étale morphism $W \to Z$. Then W is locally of finite presentation over S.

Since f satisfies (98.13.2.1) we see that so does $\mathcal{Z} \to (Sch/V)_{fppf}$, see Lemma 98.13.5. Next, we see that $(Sch/W)_{fppf} \to \mathcal{Z}$ satisfies (98.13.2.1) by Lemma 98.13.6. Thus the composition

$$(Sch/W)_{fppf} \to \mathcal{Z} \to (Sch/V)_{fppf}$$

satisfies (98.13.2.1) by Lemma 98.13.4. More on Morphisms, Lemma 37.12.1 shows that the composition $W \to Z \to V$ is smooth at every finite type point w_0 of W. Since the smooth locus is open we conclude that $W \to V$ is a smooth morphism of schemes by Morphisms, Lemma 29.16.7. Thus we conclude that $Z \to V$ is a smooth morphism of algebraic spaces by definition. □

The lemma below is how we will use openness of versality.

07XY Lemma 98.13.8. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let k be a finite type field over S and let x_0 be an object of $\mathcal{X}$ over $\mathrm{Spec}(k)$ with image $s \in S$. Assume

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ satisfies axioms [1], [2], [3] (see Section 98.14),
(3) every formal object of $\mathcal{X}$ is effective,
(4) openness of versality holds for $\mathcal{X}$, and
(5) $\mathcal{O}_{S,s}$ is a G-ring.

Then there exist a morphism of finite type $U \to S$ and an object x of $\mathcal{X}$ over U such that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth and such that there exists a finite type point $u_0 \in U$ whose residue field is k and such that $x|_{u_0} \cong x_0$.

Proof. By axiom [2], Lemma 98.6.1, and Remark 98.6.2 we see that $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (S1) and (S2). Since also the tangent space has finite dimension by axiom [3] we deduce from Formal Deformation Theory, Lemma 90.13.4 that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a versal formal object ξ. Assumption (3) says ξ is effective. By axiom [1] and Lemma 98.12.7 there exists a morphism of finite type $U \to S$, an object x of $\mathcal{X}$ over U, and a finite type point u_0 of U with residue field k such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(k)} \cong x_0$. By openness of versality we may shrink U and assume that x is versal at every finite type point of U. We claim that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth which proves the lemma. Namely, by Lemma 98.13.3 x satisfies (98.13.2.1) whereupon Lemma 98.13.7 finishes the proof. □

98.14. Axioms

07XJ Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Here are the axioms we will consider on $\mathcal{X}$.

[-1] a set theoretic condition[1] to be ignored by readers who are not interested in set theoretical issues,
[0] $\mathcal{X}$ is a stack in groupoids for the étale topology,

[1]The condition is the following: the supremum of all the cardinalities $|\mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})/\cong|$ and $|\mathrm{Arrows}(\mathcal{X}_{\mathrm{Spec}(k)})|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

[1] $\mathcal{X}$ is limit preserving,
[2] $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS),
[3] the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional for every k and x_0, see (98.8.0.1) and (98.8.0.2),
[4] the functor (98.9.3.1) is an equivalence,
[5] $\mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfy openness of versality.

98.15. Axioms for functors

07XZ Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ be a functor. Denote $\mathcal{X} = \mathcal{S}_F$ the category fibred in sets associated to F, see Algebraic Stacks, Section 94.7. In this section we provide a translation between the material above as it applies to $\mathcal{X}$, to statements about F.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)_{fppf}^{opp} \to \textit{Sets}$ be a functor. Let k be a field of finite type over S. Let $x_0 \in F(\text{Spec}(k))$. The associated predeformation category (98.3.0.2) corresponds to the functor

$$F_{k,x_0} : \mathcal{C}_\Lambda \longrightarrow \textit{Sets}, \quad A \longmapsto \{x \in F(\text{Spec}(A)) \mid x|_{\text{Spec}(k)} = x_0\}.$$

Recall that we do not distinguish between categories cofibred in sets over $\mathcal{C}_\Lambda$ and functor $\mathcal{C}_\Lambda \to \textit{Sets}$, see Formal Deformation Theory, Remarks 90.5.2 (11). Given a transformation of functors $a : F \to G$, setting $y_0 = a(x_0)$ we obtain a morphism

$$F_{k,x_0} \longrightarrow G_{k,y_0}$$

see (98.3.1.1). Lemma 98.3.2 tells us that if $a : F \to G$ is formally smooth (in the sense of More on Morphisms of Spaces, Definition 76.13.1), then $F_{k,x_0} \longrightarrow G_{k,y_0}$ is smooth as in Formal Deformation Theory, Remark 90.8.4.

Lemma 98.4.1 says that if $Y' = Y \amalg_X X'$ in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, then the map

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection, provided that F is an algebraic space. We say a general functor F satisfies the Rim-Schlessinger condition or we say F satisfies (RS) if given any pushout $Y' = Y \amalg_X X'$ where Y, X, X' are spectra of Artinian local rings of finite type over S, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. Thus every algebraic space satisfies (RS).

Lemma 98.6.1 says that given a functor F which satisfies (RS), then all F_{k,x_0} are deformation functors as in Formal Deformation Theory, Definition 90.16.8, i.e., they satisfy (RS) as in Formal Deformation Theory, Remark 90.16.5. In particular the tangent space

$$TF_{k,x_0} = \{x \in F(\text{Spec}(k[\epsilon])) \mid x|_{\text{Spec}(k)} = x_0\}$$

has the structure of a k-vector space by Formal Deformation Theory, Lemma 90.12.2.

Lemma 98.8.1 says that an algebraic space F locally of finite type over S gives rise to deformation functors F_{k,x_0} with finite dimensional tangent spaces TF_{k,x_0}.

A formal object[2] $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\mathrm{Spec}(R/\mathfrak{m}^n))$ such that $\xi_{n+1}|_{\mathrm{Spec}(R/\mathfrak{m}^n)} = \xi_n$. A formal object ξ defines a formal object ξ of $F_{R/\mathfrak{m},\xi_1}$. We say ξ is versal if and only if it is versal in the sense of Formal Deformation Theory, Definition 90.8.9. A formal object $\xi = (R, \xi_n)$ is called effective if there exists an $x \in F(\mathrm{Spec}(R))$ such that $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ for all $n \geq 1$. Lemma 98.9.5 says that if F is an algebraic space, then every formal object is effective.

Let U be a scheme locally of finite type over S and let $x \in F(U)$. Let $u_0 \in U$ be a finite type point. We say that x is versal at u_0 if and only if $\xi = (\mathcal{O}_{U,u_0}^\wedge, x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}_{u_0}^n)})$ is a versal formal object in the sense described above.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sch$ be a functor. Here are the axioms we will consider on F.

- [-1] a set theoretic condition[3] to be ignored by readers who are not interested in set theoretical issues,
- [0] F is a sheaf for the étale topology,
- [1] F is limit preserving,
- [2] F satisfies the Rim-Schlessinger condition (RS),
- [3] every tangent space TF_{k,x_0} is finite dimensional,
- [4] every formal object is effective,
- [5] F satisfies openness of versality.

Here limit preserving is the notion defined in Limits of Spaces, Definition 70.3.1 and openness of versality means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

98.16. Algebraic spaces

07Y0 The following is our first main result on algebraic spaces.

07Y1 Proposition 98.16.1. Let S be a locally Noetherian scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that

1. $\Delta : F \to F \times F$ is representable by algebraic spaces,
2. F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see Section 98.15), and
3. $\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.

Then F is an algebraic space.

Proof. Lemma 98.13.8 applies to F. Using this we choose, for every finite type field k over S and $x_0 \in F(\mathrm{Spec}(k))$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $U_{k,x_0} \to F$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$U = \coprod\nolimits_{k,x_0} U_{k,x_0} \longrightarrow F$$

[2]This is what Artin calls a formal deformation.

[3]The condition is the following: the supremum of all the cardinalities $|F(\mathrm{Spec}(k))|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

is smooth[4]. To finish the proof it suffices to show this map is surjective, see Bootstrap, Lemma 80.12.3 (this is where we use axiom [0]). By Criteria for Representability, Lemma 97.5.6 it suffices to show that $U \times_F V \to V$ is surjective for those $V \to F$ where V is an affine scheme locally of finite presentation over S. Since $U \times_F V \to V$ is smooth the image is open. Hence it suffices to show that the image of $U \times_F V \to V$ contains all finite type points of V, see Morphisms, Lemma 29.16.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote x_0 the composition $\mathrm{Spec}(k) \xrightarrow{v_0} V \to F$. Then $(u_{k,x_0}, v_0) : \mathrm{Spec}(k) \to U \times_F V$ is a point mapping to v_0 and we win. □

07Y2 Lemma 98.16.2. Let S be a locally Noetherian scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)_{fppf}^{opp} \to \mathrm{Sets}$. Assume that

(1) a is injective,
(2) F satisfies axioms [0], [1], [2], [4], and [5],
(3) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S,
(4) G is an algebraic space locally of finite type over S,

Then F is an algebraic space.

Proof. By Lemma 98.8.1 the functor G satisfies [3]. As $F \to G$ is injective, we conclude that F also satisfies [3]. Moreover, as $F \to G$ is injective, we see that given schemes U, V and morphisms $U \to F$ and $V \to F$, then $U \times_F V = U \times_G V$. Hence $\Delta : F \to F \times F$ is representable (by schemes) as this holds for G by assumption. Thus Proposition 98.16.1 applies[5]. □

98.17. Algebraic stacks

07Y3 Proposition 98.17.2 is our first main result on algebraic stacks.

07Y4 Lemma 98.17.1. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume that

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ satisfies axioms [-1], [0], [1], [2], [3] (see Section 98.14),
(3) every formal object of $\mathcal{X}$ is effective,
(4) $\mathcal{X}$ satisfies openness of versality, and
(5) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.

Then $\mathcal{X}$ is an algebraic stack.

Proof. Lemma 98.13.8 applies to $\mathcal{X}$. Using this we choose, for every finite type field k over S and every isomorphism class of object $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $(Sch/U_{k,x_0})_{fppf} \to \mathcal{X}$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$(Sch/U)_{fppf} \to \mathcal{X}, \quad \text{with} \quad U = \coprod\nolimits_{k,x_0} U_{k,x_0}$$

[4]Set theoretical remark: This coproduct is (isomorphic) to an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 3.9.9.

[5]The set theoretic condition [-1] holds for F as it holds for G. Details omitted.

is smooth[6]. To finish the proof it suffices to show this map is surjective, see Criteria for Representability, Lemma 97.19.1 (this is where we use axiom [0]). By Criteria for Representability, Lemma 97.5.6 it suffices to show that $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf} \to (Sch/V)_{fppf}$ is surjective for those $y : (Sch/V)_{fppf} \to \mathcal{X}$ where V is an affine scheme locally of finite presentation over S. By assumption (1) the fibre product $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf}$ is representable by an algebraic space W. Then $W \to V$ is smooth, hence the image is open. Hence it suffices to show that the image of $W \to V$ contains all finite type points of V, see Morphisms, Lemma 29.16.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote $x_0 = y|_{\mathrm{Spec}(k)}$ the pullback of y by v_0. Then (u_{k,x_0}, v_0) will give a morphism $\mathrm{Spec}(k) \to W$ whose composition with $W \to V$ is v_0 and we win. □

07Y5 Proposition 98.17.2. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume that

(1) $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ satisfies axioms [-1], [0], [1], [2], [3], [4], and [5] (see Section 98.14),
(3) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.

Then $\mathcal{X}$ is an algebraic stack.

Proof. We first prove that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. To do this it suffices to show that

$$\mathcal{Y} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (Sch/V)_{fppf}$$

is representable by an algebraic space for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V, see Criteria for Representability, Lemma 97.5.5[7]. Observe that $\mathcal{Y}$ is fibred in setoids (Stacks, Lemma 8.2.5) and let $Y : (Sch/S)_{fppf}^{opp} \to \mathrm{Sets}$, $T \mapsto \mathrm{Ob}(\mathcal{Y}_T)/\cong$ be the functor of isomorphism classes. We will apply Proposition 98.16.1 to see that Y is an algebraic space.

Note that $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ (and hence also $Y \to Y \times Y$) is representable by algebraic spaces by condition (1) and Criteria for Representability, Lemma 97.4.4. Observe that Y is a sheaf for the étale topology by Stacks, Lemmas 8.6.3 and 8.6.7, i.e., axiom [0] holds. Also Y is limit preserving by Lemma 98.11.2, i.e., we have [1]. Note that Y has (RS), i.e., axiom [2] holds, by Lemmas 98.5.2 and 98.5.3. Axiom [3] for Y follows from Lemmas 98.8.1 and 98.8.2. Axiom [4] follows from Lemmas 98.9.5 and 98.9.6. Axiom [5] for Y follows directly from openness of versality for $\Delta_{\mathcal{X}}$ which is part of axiom [5] for $\mathcal{X}$. Thus all the assumptions of Proposition 98.16.1 are satisfied and Y is an algebraic space.

At this point it follows from Lemma 98.17.1 that $\mathcal{X}$ is an algebraic stack. □

98.18. Strong Rim-Schlessinger

0CXN In the rest of this chapter the following strictly stronger version of the Rim-Schlessinger conditions will play an important role.

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[6]Set theoretical remark: This coproduct is (isomorphic to) an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 3.9.9.

[7]The set theoretic condition in Criteria for Representability, Lemma 97.5.5 will hold: the size of the algebraic space Y representing $\mathcal{Y}$ is suitably bounded. Namely, $Y \to S$ will be locally of finite type and Y will satisfy axiom [-1]. Details omitted.

07Y8 **Definition 98.18.1.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies condition (RS*) if given a fibre product diagram

$$\begin{tikzcd} B' \arrow[r] & B \\ A' = A \times_B B' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of S-algebras, with $B' \to B$ surjective with square zero kernel, the functor of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(A')} \longrightarrow \mathcal{X}_{\mathrm{Spec}(A)} \times_{\mathcal{X}_{\mathrm{Spec}(B)}} \mathcal{X}_{\mathrm{Spec}(B')}$$

is an equivalence of categories.

We make some observations: with $A \to B \leftarrow B'$ as in Definition 98.18.1

(1) we have $\mathrm{Spec}(A') = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(B')$ in the category of schemes, see More on Morphisms, Lemma 37.14.3, and
(2) if $\mathcal{X}$ is an algebraic stack, then $\mathcal{X}$ satisfies (RS*) by Lemma 98.18.2.

If S is locally Noetherian, then

(3) if A, B, B' are of finite type over S and B is finite over A, then A' is of finite type over S[8], and
(4) if $\mathcal{X}$ satisfies (RS*), then $\mathcal{X}$ satisfies (RS) because (RS) covers exactly those cases of (RS*) where A, B, B' are Artinian local.

0CXP **Lemma 98.18.2.** Let $\mathcal{X}$ be an algebraic stack over a base S. Then $\mathcal{X}$ satisfies (RS*).

Proof. This is implied by Lemma 98.4.1, see remarks following Definition 98.18.1. □

0CXQ **Lemma 98.18.3.** Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS*), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

Proof. The proof is exactly the same as the proof of Lemma 98.5.3. □

98.19. Versality and generalizations

0G2I We prove that versality is preserved under generalizations for stacks which have (RS*) and are limit preserving. We suggest skipping this section on a first reading.

0G2J **Lemma 98.19.1.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u \in U$ be a finite type point such that x is not versal at u. Then there exists a morphism $x \to y$ of $\mathcal{X}$ lying over $U \to T$ satisfying

(1) the morphism $U \to T$ is a first order thickening,
(2) we have a short exact sequence

$$0 \to \kappa(u) \to \mathcal{O}_T \to \mathcal{O}_U \to 0$$

[8]If $\mathrm{Spec}(A)$ maps into an affine open of S this follows from More on Algebra, Lemma 15.5.1. The general case follows using More on Algebra, Lemma 15.5.3.

(3) there does not exist a pair (W, α) consisting of an open neighbourhood $W \subset T$ of u and a morphism $\beta : y|_W \to x$ such that the composition

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to y} y|_W \xrightarrow{\beta} x$$

is the canonical morphism $x|_{U \cap W} \to x$.

Proof. Let $R = \mathcal{O}_{U,u}^\wedge$. Let $k = \kappa(u)$ be the residue field of R. Let ξ be the formal object of $\mathcal{X}$ over R associated to x. Since x is not versal at u, we see that ξ is not versal, see Lemma 98.12.3. By the discussion following Definition 98.12.1 this means we can find morphisms $\xi_1 \to x_A \to x_B$ of $\mathcal{X}$ lying over closed immersions $\text{Spec}(k) \to \text{Spec}(A) \to \text{Spec}(B)$ where A, B are Artinian local rings with residue field k, an $n \geq 1$ and a commutative diagram

$$\begin{tikzcd} & x_A \arrow[dl] \\ \xi_n & \xi_1 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & \text{Spec}(A) \arrow[dl] \\ \text{Spec}(R/\mathfrak{m}^n) & \text{Spec}(k) \arrow[l] \arrow[u] \end{tikzcd}$$

such that there does not exist an $m \geq n$ and a commutative diagram

$$\begin{tikzcd} & & x_B \arrow[dll] \\ & & x_A \arrow[u] \arrow[dl] \\ \xi_m & \xi_n \arrow[l] & \xi_1 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & & \text{Spec}(B) \arrow[dll] \\ & & \text{Spec}(A) \arrow[u] \arrow[dl] \\ \text{Spec}(R/\mathfrak{m}^m) & \text{Spec}(R/\mathfrak{m}^n) \arrow[l] & \text{Spec}(k) \arrow[l] \arrow[u] \end{tikzcd}$$

We may moreover assume that $B \to A$ is a small extension, i.e., that the kernel I of the surjection $B \to A$ is isomorphic to k as an A-module. This follows from Formal Deformation Theory, Remark 90.8.10. Then we simply define

$$T = U \amalg_{\text{Spec}(A)} \text{Spec}(B)$$

By property (RS*) we find y over T whose restriction to $\text{Spec}(B)$ is x_B and whose restriction to U is x (this gives the arrow $x \to y$ lying over $U \to T$). To finish the proof we verify conditions (1), (2), and (3).

By the construction of the pushout we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & B \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & \Gamma(T, \mathcal{O}_T) \arrow[r] \arrow[u] & \Gamma(U, \mathcal{O}_U) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. This immediately proves (1) and (2). To finish the proof we will argue by contradiction. Assume we have a pair (W, β) as in (3). Since $\text{Spec}(B) \to T$ factors through W we get the morphism

$$x_B \to y|_W \xrightarrow{\beta} x$$

Since B is Artinian local with residue field $k = \kappa(u)$ we see that $x_B \to x$ lies over a morphism $\text{Spec}(B) \to U$ which factors through $\text{Spec}(\mathcal{O}_{U,u}/\mathfrak{m}_u^m)$ for some $m \geq n$. In other words, $x_B \to x$ factors through ξ_m giving a map $x_B \to \xi_m$. The compatibility condition on the morphism α in condition (3) translates into the condition that

$$\begin{tikzcd} x_B \arrow[d] & x_A \arrow[l] \arrow[d] \\ \xi_m & \xi_n \arrow[l] \end{tikzcd}$$

is commutative. This gives the contradiction we were looking for. $\square$

0G2K **Lemma 98.19.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ has (RS*),
(3) $\mathcal{X}$ is limit preserving.

Let x be an object of $\mathcal{X}$ over a scheme U of finite type over S. Let $u \leadsto u_0$ be a specialization of finite type points of U such that x is versal at u_0. Then x is versal at u.

Proof. After shrinking U we may assume U is affine and U maps into an affine open $\mathrm{Spec}(\Lambda)$ of S. If x is not versal at u then we may pick $x \to y$ lying over $U \to T$ as in Lemma 98.19.1. Write $U = \mathrm{Spec}(R_0)$ and $T = \mathrm{Spec}(R)$. The morphism $U \to T$ corresponds to a surjective ring map $R \to R_0$ whose kernel is an ideal of square zero. By assumption (3) we get that y comes from an object x' over $U' = \mathrm{Spec}(R')$ for some finite type Λ-subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \to R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \to y \to x' \text{ lying over } U \to T \to U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x, \mathcal{X}, x'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 94.10.11. By construction of 2-fibre products, a V-valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : V \to U$, $a' : V \to U'$ and a morphism $\alpha : a^*x \to (a')^*x'$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[dr] \arrow[drr] \arrow[ddr] & & \\ & Z \arrow[r, "p'"'] \arrow[d, "p"] & U' \arrow[d] \\ & U \arrow[r] & S \end{tikzcd}$$

The morphism $i : U \to Z$ comes the isomorphism $x \to x'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 98.12.6 we see that $Z \to U'$ is smooth at z_0. After replacing U by an affine open neighbourhood of u_0, replacing U' by the corresponding open, and replacing Z by the intersection of the inverse images of these opens by p and p', we reach the situation where $Z \to U'$ is smooth along $i(U)$. Since $u \leadsto u_0$ the point u is in this open. Condition (3) of Lemma 98.19.1 is clearly preserved by shrinking U (all of the schemes U, T, U' have the same underlying topological space). Since $U \to U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \to Z$

such that $p' \circ i' = \mathrm{id}_{U'}$ and whose restriction to U is i (More on Morphisms of Spaces, Lemma 76.19.6). Pulling back the universal morphism $p^*x \to (p')^*x'$ by i' we obtain a morphism

$$x' \to x$$

lying over $p \circ i' : U' \to U$ such that the composition

$$x \to x' \to x$$

is the identity. Recall that we have $y \to x'$ lying over the morphism $T \to U'$. Composing we get a morphism $y \to x$ whose existence contradicts condition (3) of Lemma 98.19.1. This contradiction finishes the proof. □

98.20. Strong formal effectiveness

0CXR In this section we demonstrate how a strong version of effectiveness of formal objects implies openness of versality. The proof of [Bha16, Theorem 1.1] shows that quasi-compact and quasi-separated algebraic spaces satisfy the strong formal effectiveness discussed in Remark 98.20.2. In addition, the theory we develop is nonempty: we use it later to show openness of versality for the stack of coherent sheaves and for moduli of complexes, see Quot, Theorems 99.6.1 and 99.16.12.

0G2S **Lemma 98.20.1.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u_n \in U$, $n \geq 1$ be finite type points such that (a) there are no specializations $u_n \rightsquigarrow u_m$ for $n \neq m$, and (b) x is not versal at u_n for all n. Then there exist morphisms

$$x \to x_1 \to x_2 \to \ldots \quad \text{in } \mathcal{X} \text{ lying over} \quad U \to U_1 \to U_2 \to \ldots$$

over S such that

(1) for each n the morphism $U \to U_n$ is a first order thickening,
(2) for each n we have a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0$$

with $U_0 = U$ for $n = 1$,
(3) for each n there does not exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \to x$ such that the composition

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to x_n} x_n|_W \xrightarrow{\alpha} x$$

is the canonical morphism $x|_{U \cap W} \to x$.

Proof. Since there are no specializations among the points u_n (and in particular the u_n are pairwise distinct), for every n we can find an open $U' \subset U$ such that $u_n \in U'$ and $u_i \notin U'$ for $i = 1, \ldots, n-1$. By Lemma 98.19.1 for each $n \geq 1$ we can find

$$x \to y_n \quad \text{in } \mathcal{X} \text{ lying over} \quad U \to T_n$$

such that

(1) the morphism $U \to T_n$ is a first order thickening,
(2) we have a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{T_n} \to \mathcal{O}_U \to 0$$

(3) there does not exist a pair (W, α) consisting of an open neighbourhood $W \subset T_n$ of u_n and a morphism $\beta : y_n|_W \to x$ such that the composition

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to y_n} y_n|_W \xrightarrow{\beta} x$$

is the canonical morphism $x|_{U \cap W} \to x$.

Thus we can define inductively

$$U_1 = T_1, \quad U_{n+1} = U_n \amalg_U T_{n+1}$$

Setting $x_1 = y_1$ and using (RS*) we find inductively x_{n+1} over U_{n+1} restricting to x_n over U_n and y_{n+1} over T_{n+1}. Property (1) for $U \to U_n$ follows from the construction of the pushout in More on Morphisms, Lemma 37.14.3. Property (2) for U_n similarly follows from property (2) for T_n by the construction of the pushout. After shrinking to an open neighbourhood U' of u_n as discussed above, property (3) for (U_n, x_n) follows from property (3) for (T_n, y_n) simply because the corresponding open subschemes of T_n and U_n are isomorphic. Some details omitted. □

0CXT Remark 98.20.2 (Strong effectiveness). Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume we have

(1) an affine open $\text{Spec}(\Lambda) \subset S$,
(2) an inverse system (R_n) of Λ-algebras with surjective transition maps whose kernels are locally nilpotent,
(3) a system (ξ_n) of objects of $\mathcal{X}$ lying over the system $(\text{Spec}(R_n))$.

In this situation, set $R = \lim R_n$. We say that (ξ_n) is effective if there exists an object ξ of $\mathcal{X}$ over $\text{Spec}(R)$ whose restriction to $\text{Spec}(R_n)$ gives the system (ξ_n).

It is not the case that every algebraic stack $\mathcal{X}$ over S satisfies a strong effectiveness axiom of the form: every system (ξ_n) as in Remark 98.20.2 is effective. An example is given in Examples, Section 110.72.

0CXU Lemma 98.20.3. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ has (RS*),
(3) $\mathcal{X}$ is limit preserving,
(4) systems (ξ_n) as in Remark 98.20.2 where $\text{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$ are effective.

Then $\mathcal{X}$ satisfies openness of versality.

Proof. Choose a scheme U locally of finite type over S, a finite type point u_0 of U, and an object x of $\mathcal{X}$ over U such that x is versal at u_0. After shrinking U we may assume U is affine and U maps into an affine open $\text{Spec}(\Lambda)$ of S. Let $E \subset U$ be the set of finite type points u such that x is not versal at u. By Lemma 98.19.2 if $u \in E$ then u_0 is not a specialization of u. If openness of versality does not hold, then u_0 is in the closure $\overline{E}$ of E. By Properties, Lemma 28.5.13 we may choose a countable subset $E' \subset E$ with the same closure as E. By Properties, Lemma 28.5.12 we may assume there are no specializations among the points of E'. Observe that E' has to be (countably) infinite as u_0 isn't the specialization of any point of E' as pointed out above. Thus we can write $E' = \{u_1, u_2, u_3, \ldots\}$, there are no specializations among the u_i, and u_0 is in the closure of E'.

Choose $x \to x_1 \to x_2 \to \ldots$ lying over $U \to U_1 \to U_2 \to \ldots$ as in Lemma 98.20.1. Write $U_n = \mathrm{Spec}(R_n)$ and $U = \mathrm{Spec}(R_0)$. Set $R = \lim R_n$. Observe that $R \to R_0$ is surjective with kernel an ideal of square zero. By assumption (4) we get ξ over $\mathrm{Spec}(R)$ whose base change to R_n is x_n. By assumption (3) we get that ξ comes from an object ξ' over $U' = \mathrm{Spec}(R')$ for some finite type Λ-subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \to R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \to x_1 \to x_2 \to \ldots \to \xi' \text{ lying over } U \to U_1 \to U_2 \to \ldots \to U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x, \mathcal{X}, \xi'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 94.10.11. By construction of 2-fibre products, a T-valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : T \to U$, $a' : T \to U'$ and a morphism $\alpha : a^*x \to (a')^*\xi'$. We obtain a commutative diagram

$$\begin{tikzcd}
U \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& Z \arrow[r, "p'"] \arrow[d, "p"] & U' \arrow[d] \\
& U \arrow[r] & S
\end{tikzcd}$$

The morphism $i : U \to Z$ comes the isomorphism $x \to \xi'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 98.12.6 we see that $Z \to U'$ is smooth at z_0. After replacing U by an affine open neighbourhood of u_0, replacing U' by the corresponding open, and replacing Z by the intersection of the inverse images of these opens by p and p', we reach the situation where $Z \to U'$ is smooth along $i(U)$. Note that this also involves replacing u_n by a subsequence, namely by those indices such that u_n is in the open. Moreover, condition (3) of Lemma 98.20.1 is clearly preserved by shrinking U (all of the schemes U, U_n, U' have the same underlying topological space). Since $U \to U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \to Z$ such that $p' \circ i' = \mathrm{id}_{U'}$ and whose restriction to U is i (More on Morphisms of Spaces, Lemma 76.19.6). Pulling back the universal morphism $p^*x \to (p')^*\xi'$ by i' we obtain a morphism

$$\xi' \to x$$

lying over $p \circ i' : U' \to U$ such that the composition

$$x \to \xi' \to x$$

is the identity. Recall that we have $x_1 \to \xi'$ lying over the morphism $U_1 \to U'$. Composing we get a morphism $x_1 \to x$ whose existence contradicts condition (3) of Lemma 98.20.1. This contradiction finishes the proof. □

0CXV Remark 98.20.4. There is a way to deduce openness of versality of the diagonal of an category fibred in groupoids from a strong formal effectiveness axiom. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume

(1) $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ has (RS*),

(3) $\mathcal{X}$ is limit preserving,
(4) given an inverse system (R_n) of S-algebras as in Remark 98.20.2 where $\mathrm{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$ the functor
$$\mathcal{X}_{\mathrm{Spec}(\lim R_n)} \longrightarrow \lim_n \mathcal{X}_{\mathrm{Spec}(R_n)}$$
is fully faithful.

Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfies openness of versality. This follows by applying Lemma 98.20.3 to fibre products of the form $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (Sch/V)_{fppf}$ for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V. If we ever need this, we will change this remark into a lemma and provide a detailed proof.

98.21. Infinitesimal deformations

07Y6 In this section we discuss a generalization of the notion of the tangent space introduced in Section 98.8. To do this intelligently, we borrow some notation from Formal Deformation Theory, Sections 90.11, 90.17, and 90.19.

Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Given a homomorphism $A' \to A$ of S-algebras and an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we write $\mathrm{Lift}(x, A')$ for the category of lifts of x to $\mathrm{Spec}(A')$. An object of $\mathrm{Lift}(x, A')$ is a morphism $x \to x'$ of $\mathcal{X}$ lying over $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ and morphisms of $\mathrm{Lift}(x, A')$ are defined as commutative diagrams. The set of isomorphism classes of $\mathrm{Lift}(x, A')$ is denoted $\mathrm{Lift}(x, A')$. See Formal Deformation Theory, Definition 90.17.1 and Remark 90.17.2. If $A' \to A$ is surjective with locally nilpotent kernel we call an element x' of $\mathrm{Lift}(x, A')$ a (infinitesimal) deformation of x. In this case the group of infinitesimal automorphisms of x' over x is the kernel
$$\mathrm{Inf}(x'/x) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A')}}(x') \to \mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A)}}(x)\right)$$
Note that an element of $\mathrm{Inf}(x'/x)$ is the same thing as a lift of id_x over $\mathrm{Spec}(A')$ for (the category fibred in sets associated to) $Aut_{\mathcal{X}}(x')$. Compare with Formal Deformation Theory, Definition 90.19.1 and Formal Deformation Theory, Remark 90.19.8.

If M is an A-module we denote $A[M]$ the A-algebra whose underlying A-module is $A \oplus M$ and whose multiplication is given by $(a, m) \cdot (a', m') = (aa', am' + a'm)$. When $M = A$ this is the ring of dual numbers over A, which we denote $A[\epsilon]$ as is customary. There is an A-algebra map $A[M] \to A$. The pullback of x to $\mathrm{Spec}(A[M])$ is called the trivial deformation of x to $\mathrm{Spec}(A[M])$.

07Y7 **Lemma 98.21.1.** Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let
$$\begin{tikzcd} B' \arrow[r] & B \\ A' \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$
be a commutative diagram of S-algebras. Let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$, let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(B)$, and let $\phi : f(x)|_{\mathrm{Spec}(B)} \to y$ be a morphism of $\mathcal{Y}$ over $\mathrm{Spec}(B)$. Then there is a canonical functor
$$\mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(y, B')$$

of categories of lifts induced by f and ϕ. The construction is compatible with compositions of 1-morphisms of categories fibred in groupoids in an obvious manner.

Proof. This lemma proves itself. $\square$

Let S be a base scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We define a category whose objects are pairs $(x, A' \to A)$ where

(1) $A' \to A$ is a surjection of S-algebras whose kernel is an ideal of square zero,
(2) x is an object of $\mathcal{X}$ lying over $\mathrm{Spec}(A)$.

A morphism $(y, B' \to B) \to (x, A' \to A)$ is given by a commutative diagram

$$\begin{tikzcd} B' \arrow[r] & B \\ A' \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

of S-algebras together with a morphism $x|_{\mathrm{Spec}(B)} \to y$ over $\mathrm{Spec}(B)$. Let us call this the category of deformation situations.

07Y9 Lemma 98.21.2. Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra and let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$.

(1) There exists an A-linear functor $\mathrm{Inf}_x : \mathrm{Mod}_A \to \mathrm{Mod}_A$ such that given a deformation situation $(x, A' \to A)$ and a lift x' there is an isomorphism $\mathrm{Inf}_x(I) \to \mathrm{Inf}(x'/x)$ where $I = \mathrm{Ker}(A' \to A)$.
(2) There exists an A-linear functor $T_x : \mathrm{Mod}_A \to \mathrm{Mod}_A$ such that
  (a) given M in Mod_A there is a bijection $T_x(M) \to \mathrm{Lift}(x, A[M])$,
  (b) given a deformation situation $(x, A' \to A)$ there is an action
$$T_x(I) \times \mathrm{Lift}(x, A') \to \mathrm{Lift}(x, A')$$
  where $I = \mathrm{Ker}(A' \to A)$. It is simply transitive if $\mathrm{Lift}(x, A') \neq \emptyset$.

Proof. We define Inf_x as the functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{Sets}, \quad M \longrightarrow \mathrm{Inf}(x'_M/x) = \mathrm{Lift}(\mathrm{id}_x, A[M])$$

mapping M to the group of infinitesimal automorphisms of the trivial deformation x'_M of x to $\mathrm{Spec}(A[M])$ or equivalently the group of lifts of id_x in $Aut_{\mathcal{X}}(x'_M)$. We define T_x as the functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{Sets}, \quad M \longrightarrow \mathrm{Lift}(x, A[M])$$

of isomorphism classes of infintesimal deformations of x to $\mathrm{Spec}(A[M])$. We apply Formal Deformation Theory, Lemma 90.11.4 to Inf_x and T_x. This lemma is applicable, since (RS*) tells us that

$$\mathrm{Lift}(x, A[M \times N]) = \mathrm{Lift}(x, A[M]) \times \mathrm{Lift}(x, A[N])$$

as categories (and trivial deformations match up too).

Let $(x, A' \to A)$ be a deformation situation. Consider the ring map $g : A' \times_A A' \to A[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$. There is an isomorphism

$$A' \times_A A' \longrightarrow A' \times_A A[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections to A. Thus applying (RS*) twice we find equivalences of categories

$$\begin{aligned} \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') &= \mathrm{Lift}(x, A' \times_A A') \\ &= \mathrm{Lift}(x, A' \times_A A[I]) \\ &= \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A[I]) \end{aligned}$$

Using these maps and projection onto the last factor of the last product we see that we obtain "difference maps"

$$\mathrm{Inf}(x'/x) \times \mathrm{Inf}(x'/x) \longrightarrow \mathrm{Inf}_x(I) \quad \text{and} \quad \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow T_x(I)$$

These difference maps satisfy the transitivity rule "$(x'_1 - x'_2) + (x'_2 - x'_3) = x'_1 - x'_3$" because

$$\begin{tikzcd} A' \times_A A' \times_A A' \arrow[rr, "(a_1,a_2,a_3)\mapsto(g(a_1,a_2),g(a_2,a_3))"'] \arrow[drr, "(a_1,a_2,a_3)\mapsto g(a_1,a_3)"'] & & A[I] \times_A A[I] = A[I \times I] \arrow[d, "+"] \\ & & A[I] \end{tikzcd}$$

is commutative. Inverting the string of equivalences above we obtain an action which is free and transitive provided $\mathrm{Inf}(x'/x)$, resp. $\mathrm{Lift}(x, A')$ is nonempty. Note that $\mathrm{Inf}(x'/x)$ is always nonempty as it is a group. □

07YA Remark 98.21.3 (Functoriality). Assumptions and notation as in Lemma 98.21.2. Suppose $A \to B$ is a ring map and $y = x|_{\mathrm{Spec}(B)}$. Let $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$ and let $M \to N$ an A-linear map. Then there are canonical maps $\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N)$ and $T_x(M) \to T_y(N)$ simply because there is a pullback functor

$$\mathrm{Lift}(x, A[M]) \to \mathrm{Lift}(y, B[N])$$

coming from the ring map $A[M] \to B[N]$. Similarly, given a morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ we obtain a pullback functor $\mathrm{Lift}(x, A') \to \mathrm{Lift}(y, B')$. Since the construction of the action, the addition, and the scalar multiplication on Inf_x and T_x use only morphisms in the categories of lifts (see proof of Formal Deformation Theory, Lemma 90.11.4) we see that the constructions above are functorial. In other words we obtain A-linear maps

$$\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N) \quad \text{and} \quad T_x(M) \to T_y(N)$$

such that the diagrams

$$\begin{tikzcd} \mathrm{Inf}_y(J) \arrow[r] & \mathrm{Inf}(y'/y) \\ \mathrm{Inf}_x(I) \arrow[r] \arrow[u] & \mathrm{Inf}(x'/x) \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T_y(J) \times \mathrm{Lift}(y, B') \arrow[r] & \mathrm{Lift}(y, B') \\ T_x(I) \times \mathrm{Lift}(x, A') \arrow[r] \arrow[u] & \mathrm{Lift}(x, A') \arrow[u] \end{tikzcd}$$

commute. Here $I = \mathrm{Ker}(A' \to A)$, $J = \mathrm{Ker}(B' \to B)$, x' is a lift of x to A' (which may not always exist) and $y' = x'|_{\mathrm{Spec}(B')}$.

07YB Remark 98.21.4 (Automorphisms). Assumptions and notation as in Lemma 98.21.2. Let x', x'' be lifts of x to A'. Then we have a composition map

$$\mathrm{Inf}(x'/x) \times \mathrm{Mor}_{\mathrm{Lift}(x,A')}(x', x'') \times \mathrm{Inf}(x''/x) \longrightarrow \mathrm{Mor}_{\mathrm{Lift}(x,A')}(x', x'').$$

Since $\text{Lift}(x, A')$ is a groupoid, if $\text{Mor}_{\text{Lift}(x,A')}(x', x'')$ is nonempty, then this defines a simply transitive left action of $\text{Inf}(x'/x)$ on $\text{Mor}_{\text{Lift}(x,A')}(x', x'')$ and a simply transitive right action by $\text{Inf}(x''/x)$. Now the lemma says that $\text{Inf}(x'/x) = \text{Inf}_x(I) = \text{Inf}(x''/x)$. We claim that the two actions described above agree via these identifications. Namely, either $x' \not\cong x''$ in which the claim is clear, or $x' \cong x''$ and in that case we may assume that $x'' = x'$ in which case the result follows from the fact that $\text{Inf}(x'/x)$ is commutative. In particular, we obtain a well defined action

$$\text{Inf}_x(I) \times \text{Mor}_{\text{Lift}(x,A')}(x', x'') \longrightarrow \text{Mor}_{\text{Lift}(x,A')}(x', x'')$$

which is simply transitive as soon as $\text{Mor}_{\text{Lift}(x,A')}(x', x'')$ is nonempty.

07YE Remark 98.21.5. Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let A be an S-algebra. There is a notion of a short exact sequence

$$(x, A'_1 \to A) \to (x, A'_2 \to A) \to (x, A'_3 \to A)$$

of deformation situations: we ask the corresponding maps between the kernels $I_i = \text{Ker}(A'_i \to A)$ give a short exact sequence

$$0 \to I_3 \to I_2 \to I_1 \to 0$$

of A-modules. Note that in this case the map $A'_3 \to A'_1$ factors through A, hence there is a canonical isomorphism $A'_1 = A[I_1]$.

0DNN Lemma 98.21.6. Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS*). Let A be an S-algebra and let w be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over A. Denote x, y, z the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w. For any A-module M there is a 6-term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \text{Inf}_w(M) \arrow[r] & \text{Inf}_x(M) \oplus \text{Inf}_z(M) \arrow[r] & \text{Inf}_y(M) \arrow[lld] \\ & T_w(M) \arrow[r] & T_x(M) \oplus T_z(M) \arrow[r] & T_y(M) \end{tikzcd}$$

of A-modules.

Proof. By Lemma 98.18.3 we see that $\mathcal{W}$ satisfies (RS*) and hence $T_w(M)$ and $\text{Inf}_w(M)$ are defined. The horizontal arrows are defined using the functoriality of Lemma 98.21.1.

Definition of the "boundary" map $\delta : \text{Inf}_y(M) \to T_w(M)$. Choose isomorphisms $p(x) \to y$ and $y \to q(z)$ such that $w = (x, z, p(x) \to y \to q(z))$ in the description of the 2-fibre product of Categories, Lemma 4.35.7 and more precisely Categories, Lemma 4.32.3. Let x', y', z', w' denote the trivial deformation of x, y, z, w over $A[M]$. By pullback we get isomorphisms $y' \to p(x')$ and $q(z') \to y'$. An element $\alpha \in \text{Inf}_y(M)$ is the same thing as an automorphism $\alpha : y' \to y'$ over $A[M]$ which restricts to the identity on y over A. Thus setting

$$\delta(\alpha) = (x', z', p(x') \to y' \xrightarrow{\alpha} y' \to q(z'))$$

we obtain an object of $T_w(M)$. This is a map of A-modules by Formal Deformation Theory, Lemma 90.11.5.

The rest of the proof is exactly the same as the proof of Formal Deformation Theory, Lemma 90.20.1. $\square$

0D18 Remark 98.21.7 (Compatibility with previous tangent spaces). Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ has (RS*). Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over $\mathrm{Spec}(k)$. Then we have equalities of k-vector spaces

$$T\mathcal{F}_{\mathcal{X},k,x_0} = T_{x_0}(k) \quad \text{and} \quad \mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Inf}_{x_0}(k)$$

where the spaces on the left hand side of the equality signs are given in (98.8.0.1) and (98.8.0.2) and the spaces on the right hand side are given by Lemma 98.21.2.

07YC Remark 98.21.8 (Canonical element). Assumptions and notation as in Lemma 98.21.2. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(A) \to S$ corresponds to a ring map $\Lambda \to A$. Consider the ring map

$$A \longrightarrow A[\Omega_{A/\Lambda}], \quad a \longmapsto (a, \mathrm{d}_{A/\Lambda}(a))$$

Pulling back x along the corresponding morphism $\mathrm{Spec}(A[\Omega_{A/\Lambda}]) \to \mathrm{Spec}(A)$ we obtain a deformation x_{can} of x over $A[\Omega_{A/\Lambda}]$. We call this the canonical element

$$x_{can} \in T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}]).$$

Next, assume that Λ is Noetherian and $\Lambda \to A$ is of finite type. Let $k = \kappa(\mathfrak{p})$ be a residue field at a finite type point u_0 of $U = \mathrm{Spec}(A)$. Let $x_0 = x|_{u_0}$. By (RS*) and the fact that $A[k] = A \times_k k[k]$ the space $T_x(k)$ is the tangent space to the deformation functor $\mathcal{F}_{\mathcal{X},k,x_0}$. Via

$$T\mathcal{F}_{U,k,u_0} = \mathrm{Der}_\Lambda(A, k) = \mathrm{Hom}_A(\Omega_{A/\Lambda}, k)$$

(see Formal Deformation Theory, Example 90.11.11) and functoriality of T_x the canonical element produces the map on tangent spaces induced by the object x over U. Namely, $\theta \in T\mathcal{F}_{U,k,u_0}$ maps to $T_x(\theta)(x_{can})$ in $T_x(k) = T\mathcal{F}_{\mathcal{X},k,x_0}$.

07YD Remark 98.21.9 (Canonical automorphism). Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra such that $\mathrm{Spec}(A) \to S$ maps into an affine open and let x, y be objects of $\mathcal{X}$ over $\mathrm{Spec}(A)$. Further, let $A \to B$ be a ring map and let $\alpha : x|_{\mathrm{Spec}(B)} \to y|_{\mathrm{Spec}(B)}$ be a morphism of $\mathcal{X}$ over $\mathrm{Spec}(B)$. Consider the ring map

$$B \longrightarrow B[\Omega_{B/A}], \quad b \longmapsto (b, \mathrm{d}_{B/A}(b))$$

Pulling back α along the corresponding morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ we obtain a morphism α_{can} between the pullbacks of x and y over $B[\Omega_{B/A}]$. On the other hand, we can pullback α by the morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ corresponding to the injection of B into the first summand of $B[\Omega_{B/A}]$. By the discussion of Remark 98.21.4 we can take the difference

$$\varphi(x, y, \alpha) = \alpha_{can} - \alpha|_{\mathrm{Spec}(B[\Omega_{B/A}])} \in \mathrm{Inf}_{x|_{\mathrm{Spec}(B)}}(\Omega_{B/A}).$$

We will call this the canonical automorphism. It depends on all the ingredients A, x, y, $A \to B$ and α.

98.22. Obstruction theories

07YF In this section we describe what an obstruction theory is. Contrary to the spaces of infinitesimal deformations and infinitesimal automorphisms, an obstruction theory is an additional piece of data. The formulation is motivated by the results of Lemma 98.21.2 and Remark 98.21.3.

07YG **Definition 98.22.1.** Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. An *obstruction theory* is given by the following data

(1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ an A-linear functor
$$\mathcal{O}_x : \mathrm{Mod}_A \to \mathrm{Mod}_A$$
of obstruction modules,

(2) for (x, A) as in (1), a ring map $A \to B$, $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$, and an A-linear map $M \to N$ an induced A-linear map $\mathcal{O}_x(M) \to \mathcal{O}_y(N)$ where $y = x|_{\mathrm{Spec}(B)}$, and

(3) for every deformation situation $(x, A' \to A)$ an obstruction element $o_x(A') \in \mathcal{O}_x(I)$ where $I = \mathrm{Ker}(A' \to A)$.

These data are subject to the following conditions

(i) the functoriality maps turn the obstruction modules into a functor from the category of triples (x, A, M) to sets,

(ii) for every morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ the element $o_x(A')$ maps to $o_y(B')$, and

(iii) we have
$$\mathrm{Lift}(x, A') \neq \emptyset \Leftrightarrow o_x(A') = 0$$
for every deformation situation $(x, A' \to A)$.

This last condition explains the terminology. The module $\mathcal{O}_x(A')$ is called the *obstruction module*. The element $o_x(A')$ is the *obstruction*. Most obstruction theories have additional properties, and in order to make them useful additional conditions are needed. Moreover, this is just a sample definition, for example in the definition we could consider only deformation situations of finite type over S.

One of the main reasons for introducing obstruction theories is to check openness of versality. An example of this type of result is Lemma 98.22.2 below. The initial idea to do this is due to Artin, see the papers of Artin mentioned in the introduction. It has been taken up for example in the work by Flenner [Fle81], Hall [Hal17], Hall and Rydh [HR12], Olsson [Ols06a], Olsson and Starr [OS03], and Lieblich [Lie06a] (random order of references). Moreover, for particular categories fibred in groupoids, often authors develop a little bit of theory adapted to the problem at hand. We will develop this theory later (insert future reference here).

0CYF **Lemma 98.22.2.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume* This is [Hal17, Theorem 4.4]

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is representable by algebraic spaces,*
(2) $\mathcal{X}$ *has (RS*),*
(3) $\mathcal{X}$ *is limit preserving,*
(4) *there exists an obstruction theory*[9],
(5) *for an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ and A-modules M_n, $n \geq 1$ we have*
  (a) $T_x(\prod M_n) = \prod T_x(M_n)$,
  (b) $\mathcal{O}_x(\prod M_n) \to \prod \mathcal{O}_x(M_n)$ *is injective.*

[9]Analyzing the proof the reader sees that in fact it suffices to check the functoriality (ii) of obstruction classes in Definition 98.22.1 for maps $(y, B' \to B) \to (x, A' \to A)$ with $B = A$ and $y = x$.

Then $\mathcal{X}$ satisfies openness of versality.

Proof. We prove this by verifying condition (4) of Lemma 98.20.3. Let (ξ_n) and (R_n) be as in Remark 98.20.2 such that $\text{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$. Set $A = R_1$ and $x = \xi_1$. Denote $M_n = \text{Ker}(R_n \to R_1)$. Then M_n is an A-module. Set $R = \lim R_n$. Let

$$\tilde{R} = \{(r_1, r_2, r_3 \ldots) \in \prod R_n \text{ such that all have the same image in } A\}$$

Then $\tilde{R} \to A$ is surjective with kernel $M = \prod M_n$. There is a map $R \to \tilde{R}$ and a map $\tilde{R} \to A[M]$, $(r_1, r_2, r_3, \ldots) \mapsto (r_1, r_2 - r_1, r_3 - r_2, \ldots)$. Together these give a short exact sequence

$$(x, R \to A) \to (x, \tilde{R} \to A) \to (x, A[M])$$

of deformation situations, see Remark 98.21.5. The associated sequence of kernels $0 \to \lim M_n \to M \to M \to 0$ is the canonical sequence computing the limit of the system of modules (M_n).

Let $o_x(\tilde{R}) \in \mathcal{O}_x(M)$ be the obstruction element. Since we have the lifts ξ_n we see that $o_x(\tilde{R})$ maps to zero in $\mathcal{O}_x(M_n)$. By assumption (5)(b) we see that $o_x(\tilde{R}) = 0$. Choose a lift $\tilde{\xi}$ of x to $\text{Spec}(\tilde{R})$. Let $\tilde{\xi}_n$ be the restriction of $\tilde{\xi}$ to $\text{Spec}(R_n)$. There exists elements $t_n \in T_x(M_n)$ such that $t_n \cdot \tilde{\xi}_n = \xi_n$ by Lemma 98.21.2 part (2)(b). By assumption (5)(a) we can find $t \in T_x(M)$ mapping to t_n in $T_x(M_n)$. After replacing $\tilde{\xi}$ by $t \cdot \tilde{\xi}$ we find that $\tilde{\xi}$ restricts to ξ_n over $\text{Spec}(R_n)$ for all n. In particular, since ξ_{n+1} restricts to ξ_n over $\text{Spec}(R_n)$, the restriction $\overline{\xi}$ of $\tilde{\xi}$ to $\text{Spec}(A[M])$ has the property that it restricts to the trivial deformation over $\text{Spec}(A[M_n])$ for all n. Hence by assumption (5)(a) we find that $\overline{\xi}$ is the trivial deformation of x. By axiom (RS*) applied to $R = \tilde{R} \times_{A[M]} A$ this implies that $\tilde{\xi}$ is the pullback of a deformation ξ of x over R. This finishes the proof. $\square$

07YH Example 98.22.3. Let $S = \text{Spec}(\Lambda)$ for some Noetherian ring Λ. Let $W \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_W$-module flat over S. Consider the functor

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow \text{Sets}, \quad T/S \longrightarrow H^0(W_T, \mathcal{F}_T)$$

where $W_T = T \times_S W$ is the base change and $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ to W_T. If $T = \text{Spec}(A)$ we will write $W_T = W_A$, etc. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be the category fibred in groupoids associated to F. Then $\mathcal{X}$ has an obstruction theory. Namely,

(1) given A over Λ and $x \in H^0(W_A, \mathcal{F}_A)$ we set $\mathcal{O}_x(M) = H^1(W_A, \mathcal{F}_A \otimes_A M)$,
(2) given a deformation situation $(x, A' \to A)$ we let $o_x(A') \in \mathcal{O}_x(A)$ be the image of x under the boundary map

$$H^0(W_A, \mathcal{F}_A) \longrightarrow H^1(W_A, \mathcal{F}_A \otimes_A I)$$

coming from the short exact sequence of modules

$$0 \to \mathcal{F}_A \otimes_A I \to \mathcal{F}_{A'} \to \mathcal{F}_A \to 0.$$

We have omitted some details, in particular the construction of the short exact sequence above (it uses that W_A and $W_{A'}$ have the same underlying topological space) and the explanation for why flatness of $\mathcal{F}$ over S implies that the sequence above is short exact.

07YI Example 98.22.4 (Key example). Let $S = \operatorname{Spec}(\Lambda)$ for some Noetherian ring Λ. Say $\mathcal{X} = (Sch/X)_{fppf}$ with $X = \operatorname{Spec}(R)$ and $R = \Lambda[x_1, \ldots, x_n]/J$. The naive cotangent complex $NL_{R/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R\mathrm{d}x_i,$$

see Algebra, Lemma 10.134.2. Consider a deformation situation $(x, A' \to A)$. Denote I the kernel of $A' \to A$. The object x corresponds to $(a_1, \ldots, a_n)$ with $a_i \in A$ such that $f(a_1, \ldots, a_n) = 0$ in A for all $f \in J$. Set

$$\begin{aligned}\mathcal{O}_x(A') &= \operatorname{Hom}_R(J/J^2, I)/\operatorname{Hom}_R(R^{\oplus n}, I)\\ &= \operatorname{Ext}^1_R(NL_{R/\Lambda}, I)\\ &= \operatorname{Ext}^1_A(NL_{R/\Lambda} \otimes_R A, I).\end{aligned}$$

Choose lifts $a_i' \in A'$ of a_i in A. Then $o_x(A')$ is the class of the map $J/J^2 \to I$ defined by sending $f \in J$ to $f(a_1', \ldots, a_n') \in I$. We omit the verification that $o_x(A')$ is independent of choices. It is clear that if $o_x(A') = 0$ then the map lifts. Finally, functoriality is straightforward. Thus we obtain an obstruction theory. We observe that $o_x(A')$ can be described a bit more canonically as the composition

$$NL_{R/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'} = I[1]$$

in $D(A)$, see Algebra, Lemma 10.134.6 for the last identification.

98.23. Naive obstruction theories

07YJ The title of this section refers to the fact that we will use the naive cotangent complex in this section. Let $(x, A' \to A)$ be a deformation situation for a given category fibred in groupoids over a locally Noetherian scheme S. The key Example 98.22.4 suggests that any obstruction theory should be closely related to maps in $D(A)$ with target the naive cotangent complex of A. Working this out we find a criterion for versality in Lemma 98.23.3 which leads to a criterion for openness of versality in Lemma 98.23.4. We introduce a notion of a naive obstruction theory in Definition 98.23.5 to try to formalize the notion a bit further.

In the following we will use the naive cotangent complex as defined in Algebra, Section 10.134. In particular, if $A' \to A$ is a surjection of Λ-algebras with square zero kernel I, then there are maps

$$NL_{A'/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'}$$

whose composition is homotopy equivalent to zero (see Algebra, Remark 10.134.5). This doesn't form a distinguished triangle in general as we are using the naive cotangent complex and not the full one. There is a homotopy equivalence $NL_{A/A'} \to I[1]$ (the complex consisting of I placed in degree -1, see Algebra, Lemma 10.134.6). Finally, note that there is a canonical map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$.

07YK Lemma 98.23.1. Let $A \to k$ be a ring map with k a field. Let $E \in D^-(A)$. Then $\operatorname{Ext}^i_A(E, k) = \operatorname{Hom}_k(H^{-i}(E \otimes^{\mathbf{L}} k), k)$.

Proof. Omitted. Hint: Replace E by a bounded above complex of free A-modules and compute both sides. $\square$

07YL Lemma 98.23.2. Let $\Lambda \to A \to k$ be finite type ring maps of Noetherian rings with $k = \kappa(\mathfrak p)$ for some prime $\mathfrak p$ of A. Let $\xi : E \to NL_{A/\Lambda}$ be morphism of $D^-(A)$ such that $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is not surjective. Then there exists a surjection $A' \to A$ of Λ-algebras such that

(a) $I = \mathrm{Ker}(A' \to A)$ has square zero and is isomorphic to k as an A-module,
(b) $\Omega_{A'/\Lambda} \otimes k = \Omega_{A/\Lambda} \otimes k$, and
(c) $E \to NL_{A/A'}$ is zero.

Proof. Let $f \in A$, $f \notin \mathfrak p$. Suppose that $A'' \to A_f$ satisfies (a), (b), (c) for the induced map $E \otimes_A A_f \to NL_{A_f/\Lambda}$, see Algebra, Lemma 10.134.13. Then we can set $A' = A'' \times_{A_f} A$ and get a solution. Namely, it is clear that $A' \to A$ satisfies (a) because $\mathrm{Ker}(A' \to A) = \mathrm{Ker}(A'' \to A) = I$. Pick $f'' \in A''$ lifting f. Then the localization of A' at (f'', f) is isomorphic to A'' (for example by More on Algebra, Lemma 15.5.3). Thus (b) and (c) are clear for A' too. In this way we see that we may replace A by the localization A_f (finitely many times). In particular (after such a replacement) we may assume that $\mathfrak p$ is a maximal ideal of A, see Morphisms, Lemma 29.16.1.

Choose a presentation $A = \Lambda[x_1, \ldots, x_n]/J$. Then $NL_{A/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i,$$

see Algebra, Lemma 10.134.2. After localizing if necessary (using Nakayama's lemma) we can choose generators $f_1, \ldots, f_m$ of J such that $f_j \otimes 1$ form a basis for $J/J^2 \otimes_A k$. Moreover, after renumbering, we can assume that the images of $\mathrm{d}f_1, \ldots, \mathrm{d}f_r$ form a basis for the image of $J/J^2 \otimes k \to \bigoplus k\mathrm{d}x_i$ and that $\mathrm{d}f_{r+1}, \ldots, \mathrm{d}f_m$ map to zero in $\bigoplus k\mathrm{d}x_i$. With these choices the space

$$H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k) = H^{-1}(NL_{A/\Lambda} \otimes_A k)$$

has basis $f_{r+1} \otimes 1, \ldots, f_m \otimes 1$. Changing basis once again we may assume that the image of $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is contained in the k-span of $f_{r+1} \otimes 1, \ldots, f_{m-1} \otimes 1$. Set

$$A' = \Lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_{m-1}, \mathfrak p f_m)$$

By construction $A' \to A$ satisfies (a). Since $\mathrm{d}f_m$ maps to zero in $\bigoplus k\mathrm{d}x_i$ we see that (b) holds. Finally, by construction the induced map $E \to NL_{A/A'} = I[1]$ induces the zero map $H^{-1}(E \otimes_A^{\mathbf{L}} k) \to I \otimes_A k$. By Lemma 98.23.1 we see that the composition is zero. $\square$

The following lemma is our key technical result.

07YM Lemma 98.23.3. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume

(i) for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero, and
(ii) there is an isomorphism of functors $T_x(-) \to \mathrm{Ext}^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 98.21.8).

Let $u_0 \in U$ be a finite type point with residue field $k = \kappa(u_0)$. Consider the following statements

(1) x is versal at u_0, and
(2) $\xi : E \to NL_{A/\Lambda}$ induces a surjection $H^{-1}(E \otimes_A^{\mathbf{L}} k) \to H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ and an injection $H^0(E \otimes_A^{\mathbf{L}} k) \to H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$.

Then we always have (2) $\Rightarrow$ (1) and we have (1) $\Rightarrow$ (2) if u_0 is a closed point.

Proof. Let $\mathfrak p = \mathrm{Ker}(A \to k)$ be the prime corresponding to u_0.

Assume that x versal at u_0 and that u_0 is a closed point of U. If $H^{-1}(\xi \otimes_A^{\mathbf{L}} k)$ is not surjective, then let $A' \to A$ be an extension with kernel I as in Lemma 98.23.2. Because u_0 is a closed point, we see that I is a finite A-module, hence that A' is a finite type Λ-algebra (this fails if u_0 is not closed). In particular A' is Noetherian. By property (c) for A' and (i) for ξ we see that x lifts to an object x' over A'. Let $\mathfrak p' \subset A'$ be kernel of the surjective map to k. By Artin-Rees (Algebra, Lemma 10.51.2) there exists an $n > 1$ such that $(\mathfrak p')^n \cap I = 0$. Then we see that

$$B' = A'/(\mathfrak p')^n \longrightarrow A/\mathfrak p^n = B$$

is a small, essential extension of local Artinian rings, see Formal Deformation Theory, Lemma 90.3.12. On the other hand, as x is versal at u_0 and as $x'|_{\mathrm{Spec}(B')}$ is a lift of $x|_{\mathrm{Spec}(B)}$, there exists an integer $m \geq n$ and a map $q : A/\mathfrak p^m \to B'$ such that the composition $A/\mathfrak p^m \to B' \to B$ is the quotient map. Since the maximal ideal of B' has nth power equal to zero, this q factors through B which contradicts the fact that $B' \to B$ is an essential surjection. This contradiction shows that $H^{-1}(\xi \otimes_A^{\mathbf{L}} k)$ is surjective.

Assume that x versal at u_0. By Lemma 98.23.1 the map $H^0(\xi \otimes_A^{\mathbf{L}} k)$ is dual to the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$. Note that

$$\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) = \mathrm{Der}_\Lambda(A, k) \quad \text{and} \quad T_x(k) = \mathrm{Ext}^0_A(E, k)$$

Condition (ii) assures us the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$ sends a tangent vector θ to U at u_0 to the corresponding infinitesimal deformation of x_0, see Remark 98.21.8. Hence if x is versal, then this map is surjective, see Formal Deformation Theory, Lemma 90.13.2. Hence $H^0(\xi \otimes_A^{\mathbf{L}} k)$ is injective. This finishes the proof of (1) $\Rightarrow$ (2) in case u_0 is a closed point.

For the rest of the proof assume $H^{-1}(E \otimes_A^{\mathbf{L}} k) \to H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ is surjective and $H^0(E \otimes_A^{\mathbf{L}} k) \to H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ injective. Set $R = A_{\mathfrak p}^\wedge$ and let η be the formal object over R associated to $x|_{\mathrm{Spec}(R)}$. The map $d\underline{\eta}$ on tangent spaces is surjective because it is identified with the dual of the injective map $H^0(E \otimes_A^{\mathbf{L}} k) \to H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ (see previous paragraph). According to Formal Deformation Theory, Lemma 90.13.2 it suffices to prove the following: Let $C' \to C$ be a small extension of finite type Artinian local Λ-algebras with residue field k. Let $R \to C$ be a Λ-algebra map compatible with identifications of residue fields. Let $y = x|_{\mathrm{Spec}(C)}$ and let y' be a lift of y to C'. To show: we can lift the Λ-algebra map $R \to C$ to $R \to C'$.

Observe that it suffices to lift the Λ-algebra map $A \to C$. Let $I = \mathrm{Ker}(C' \to C)$. Note that I is a 1-dimensional k-vector space. The obstruction ob to lifting $A \to C$ is an element of $\mathrm{Ext}^1_A(NL_{A/\Lambda}, I)$, see Example 98.22.4. By Lemma 98.23.1 and our assumption the map ξ induces an injection

$$\mathrm{Ext}^1_A(NL_{A/\Lambda}, I) \longrightarrow \mathrm{Ext}^1_A(E, I)$$

By the construction of ob and (i) the image of ob in $\text{Ext}^1_A(E, I)$ is the obstruction to lifting x to $A \times_C C'$. By (RS*) the fact that y/C lifts to y'/C' implies that x lifts to $A \times_C C'$. Hence $ob = 0$ and we are done. □

The key lemma above allows us to conclude that we have openness of versality in some cases.

07YN Lemma 98.23.4. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \text{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\text{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume

(i) for every deformation situation $(x, A' \to A)$ we have: x lifts to $\text{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero,
(ii) there is an isomorphism of functors $T_x(-) \to \text{Ext}^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 98.21.8),
(iii) the cohomology groups of E are finite A-modules.

If x is versal at a closed point $u_0 \in U$, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

Proof. Let C be the cone of ξ so that we have a distinguished triangle

$$E \to NL_{A/\Lambda} \to C \to E[1]$$

in $D^-(A)$. By Lemma 98.23.3 the assumption that x is versal at u_0 implies that $H^{-1}(C \otimes^{\mathbf{L}} k) = 0$. By More on Algebra, Lemma 15.76.4 there exists an $f \in A$ not contained in the prime corresponding to u_0 such that $H^{-1}(C \otimes^{\mathbf{L}}_A M) = 0$ for any A_f-module M. Using Lemma 98.23.3 again we see that we have versality for all finite type points of the open $D(f) \subset U$. □

The technical lemmas above suggest the following definition.

07YP Definition 98.23.5. Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ satisfies (RS*). A naive obstruction theory is given by the following data

07YQ (1) for every S-algebra A such that $\text{Spec}(A) \to S$ maps into an affine open $\text{Spec}(\Lambda) \subset S$ and every object x of $\mathcal{X}$ over $\text{Spec}(A)$ we are given an object $E_x \in D^-(A)$ and a map $\xi_x : E \to NL_{A/\Lambda}$,
07YR (2) given (x, A) as in (1) there are transformations of functors

$$\text{Inf}_x(-) \to \text{Ext}^{-1}_A(E_x, -) \quad \text{and} \quad T_x(-) \to \text{Ext}^0_A(E_x, -)$$

07YS (3) for (x, A) as in (1) and a ring map $A \to B$ setting $y = x|_{\text{Spec}(B)}$ there is a functoriality map $E_x \to E_y$ in $D(A)$.

These data are subject to the following conditions

(i) in the situation of (3) the diagram

$$\begin{tikzcd} E_y \arrow[r, "\xi_y"'] & NL_{B/\Lambda} \\ E_x \arrow[u] \arrow[r, "\xi_x"] & NL_{A/\Lambda} \arrow[u] \end{tikzcd}$$

is commutative in $D(A)$,

(ii) given (x, A) as in (1) and $A \to B \to C$ setting $y = x|_{\mathrm{Spec}(B)}$ and $z = x|_{\mathrm{Spec}(C)}$ the composition of the functoriality maps $E_x \to E_y$ and $E_y \to E_z$ is the functoriality map $E_x \to E_z$,

(iii) the maps of (2) are isomorphisms compatible with the functoriality maps and the maps of Remark 98.21.3,

(iv) the composition $E_x \to NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ corresponds to the canonical element of $T_x(\Omega_{A/\Lambda}) = \mathrm{Ext}^0(E_x, \Omega_{A/\Lambda})$, see Remark 98.21.8,

(v) given a deformation situation $(x, A' \to A)$ with $I = \mathrm{Ker}(A' \to A)$ the composition $E_x \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero in

$$\mathrm{Hom}_A(E_x, NL_{A/\Lambda}) = \mathrm{Ext}^0_A(E_x, NL_{A/A'}) = \mathrm{Ext}^1_A(E_x, I)$$

if and only if x lifts to A'.

Thus we see in particular that we obtain an obstruction theory as in Section 98.22 by setting $\mathcal{O}_x(-) = \mathrm{Ext}^1_A(E_x, -)$.

07YT Lemma 98.23.6. Let S and $\mathcal{X}$ be as in Definition 98.23.5 and let $\mathcal{X}$ be endowed with a naive obstruction theory. Let $A \to B$ and $y \to x$ be as in (3). Let k be a B-algebra which is a field. Then the functoriality map $E_x \to E_y$ induces bijections

$$H^i(E_x \otimes_A^{\mathbf{L}} k) \to H^i(E_y \otimes_B^{\mathbf{L}} k)$$

for $i = 0, 1$.

Proof. Let $z = x|_{\mathrm{Spec}(k)}$. Then (RS*) implies that

$$\mathrm{Lift}(x, A[k]) = \mathrm{Lift}(z, k[k]) \quad \text{and} \quad \mathrm{Lift}(y, B[k]) = \mathrm{Lift}(z, k[k])$$

because $A[k] = A \times_k k[k]$ and $B[k] = B \times_k k[k]$. Hence the properties of a naive obstruction theory imply that the functoriality map $E_x \to E_y$ induces bijections $\mathrm{Ext}^i_A(E_x, k) \to \mathrm{Ext}^i_B(E_y, k)$ for $i = -1, 0$. By Lemma 98.23.1 our maps $H^i(E_x \otimes_A^{\mathbf{L}} k) \to H^i(E_y \otimes_B^{\mathbf{L}} k)$, $i = 0, 1$ induce isomorphisms on dual vector spaces hence are isomorphisms. $\square$

07YU Lemma 98.23.7. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}^{opp}$ be a category fibred in groupoids. Assume that $\mathcal{X}$ satisfies (RS*) and that $\mathcal{X}$ has a naive obstruction theory. Then openness of versality holds for $\mathcal{X}$ provided the complexes E_x of Definition 98.23.5 have finitely generated cohomology groups for pairs (A, x) where A is of finite type over S.

Proof. Let U be a scheme locally of finite type over S, let x be an object of $\mathcal{X}$ over U, and let u_0 be a finite type point of U such that x is versal at u_0. We may first shrink U to an affine scheme such that u_0 is a closed point and such that $U \to S$ maps into an affine open $\mathrm{Spec}(\Lambda)$. Say $U = \mathrm{Spec}(A)$. Let $\xi_x : E_x \to NL_{A/\Lambda}$ be the obstruction map. At this point we may apply Lemma 98.23.4 to conclude. $\square$

98.24. A dual notion

07YV Let $(x, A' \to A)$ be a deformation situation for a given category $\mathcal{X}$ fibred in groupoids over a locally Noetherian scheme S. Assume $\mathcal{X}$ has an obstruction theory, see Definition 98.22.1. In practice one often has a complex $K^\bullet$ of A-modules and isomorphisms of functors

$$\mathrm{Inf}_x(-) \to H^0(K^\bullet \otimes_A^{\mathbf{L}} -), \quad T_x(-) \to H^1(K^\bullet \otimes_A^{\mathbf{L}} -), \quad \mathcal{O}_x(-) \to H^2(K^\bullet \otimes_A^{\mathbf{L}} -)$$

In this section we formalize this a little bit and show how this leads to a verification of openness of versality in some cases.

07YW Example 98.24.1. Let $\Lambda, S, W, \mathcal{F}$ be as in Example 98.22.3. Assume that $W \to S$ is proper and $\mathcal{F}$ coherent. By Cohomology of Schemes, Remark 30.22.2 there exists a finite complex of finite projective Λ-modules $N^\bullet$ which universally computes the cohomology of $\mathcal{F}$. In particular the obstruction spaces from Example 98.22.3 are $\mathcal{O}_x(M) = H^1(N^\bullet \otimes_\Lambda M)$. Hence with $K^\bullet = N^\bullet \otimes_\Lambda A[-1]$ we see that $\mathcal{O}_x(M) = H^2(K^\bullet \otimes_A^{\mathbf{L}} M)$.

07YX Situation 98.24.2. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ has (RS*) so that we can speak of the functor $T_x(-)$, see Lemma 98.21.2. Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Assume we are given

(1) a complex of A-modules $K^\bullet$,
(2) a transformation of functors $T_x(-) \to H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$,
(3) for every deformation situation $(x, A' \to A)$ with kernel $I = \mathrm{Ker}(A' \to A)$ an element $o_x(A') \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I)$

satisfying the following (minimal) conditions

(i) the transformation $T_x(-) \to H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$ is an isomorphism,
(ii) given a morphism $(x, A'' \to A) \to (x, A' \to A)$ of deformation situations the element $o_x(A')$ maps to the element $o_x(A'')$ via the map $H^2(K^\bullet \otimes_A^{\mathbf{L}} I) \to H^2(K^\bullet \otimes_A^{\mathbf{L}} I')$ where $I' = \mathrm{Ker}(A'' \to A)$, and
(iii) x lifts to an object over $\mathrm{Spec}(A')$ if and only if $o_x(A') = 0$.

It is possible to incorporate infinitesimal automorphisms as well, but we refrain from doing so in order to get the sharpest possible result.

In Situation 98.24.2 an important role will be played by $K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda}$. Suppose we are given an element $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda})$. Then (1) for any surjection $A' \to A$ of Λ-algebras with kernel I of square zero the canonical map $NL_{A/\Lambda} \to NL_{A/A'} = I[1]$ sends ξ to an element $\xi_{A'} \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I)$ and (2) the map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ sends ξ to an element ξ_{can} of $H^1(K^\bullet \otimes_A^{\mathbf{L}} \Omega_{A/\Lambda})$.

07YY Lemma 98.24.3. In Situation 98.24.2. Assume furthermore that

(iv) given a short exact sequence of deformation situations as in Remark 98.21.5 and a lift $x'_2 \in \mathrm{Lift}(x, A'_2)$ then $o_x(A'_3) \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I_3)$ equals $\partial\theta$ where $\theta \in H^1(K^\bullet \otimes_A^{\mathbf{L}} I_1)$ is the element corresponding to $x'_2|_{\mathrm{Spec}(A'_1)}$ via $A'_1 = A[I_1]$ and the given map $T_x(-) \to H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$.

In this case there exists an element $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda})$ such that

(1) for every deformation situation $(x, A' \to A)$ we have $\xi_{A'} = o_x(A')$, and
(2) ξ_{can} matches the canonical element of Remark 98.21.8 via the given transformation $T_x(-) \to H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$.

Proof. Choose a $\alpha : \Lambda[x_1, \ldots, x_n] \to A$ with kernel J. Write $P = \Lambda[x_1, \ldots, x_n]$. In the rest of this proof we work with

$$NL(\alpha) = (J/J^2 \longrightarrow \bigoplus A\mathrm{d}x_i)$$

which is permissible by Algebra, Lemma 10.134.2 and More on Algebra, Lemma 15.58.2. Consider the element $o_x(P/J^2) \in H^2(K^\bullet \otimes_A^{\mathbf{L}} J/J^2)$ and consider the quotient

$$C = (P/J^2 \times \bigoplus A\mathrm{d}x_i)/(J/J^2)$$

where J/J^2 is embedded diagonally. Note that $C \to A$ is a surjection with kernel $\bigoplus A\mathrm{d}x_i$. Moreover there is a section $A \to C$ to $C \to A$ given by mapping the class of $f \in P$ to the class of $(f, \mathrm{d}f)$ in the pushout. For later use, denote x_C the pullback of x along the corresponding morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Thus we see that $o_x(C) = 0$. We conclude that $o_x(P/J^2)$ maps to zero in $H^2(K^\bullet \otimes_A^{\mathbf{L}} \bigoplus A\mathrm{d}x_i)$. It follows that there exists some element $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL(\alpha))$ mapping to $o_x(P/J^2)$.

Note that for any deformation situation $(x, A' \to A)$ there exists a Λ-algebra map $P/J^2 \to A'$ compatible with the augmentations to A. Hence the element ξ satisfies the first property of the lemma by construction and property (ii) of Situation 98.24.2.

Note that our choice of ξ was well defined up to the choice of an element of $H^1(K^\bullet \otimes_A^{\mathbf{L}} \bigoplus A\mathrm{d}x_i)$. We will show that after modifying ξ by an element of the aforementioned group we can arrange it so that the second assertion of the lemma is true. Let $C' \subset C$ be the image of P/J^2 under the Λ-algebra map $P/J^2 \to C$ (inclusion of first factor). Observe that $\mathrm{Ker}(C' \to A) = \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i)$. Set $\overline{C} = A[\Omega_{A/\Lambda}]$. The map $P/J^2 \times \bigoplus A\mathrm{d}x_i \to \overline{C}$, $(f, \sum f_i\mathrm{d}x_i) \mapsto (f \bmod J, \sum f_i\mathrm{d}x_i)$ factors through a surjective map $C \to \overline{C}$. Then

$$(x, \overline{C} \to A) \to (x, C \to A) \to (x, C' \to A)$$

is a short exact sequence of deformation situations. The associated splitting $\overline{C} = A[\Omega_{A/\Lambda}]$ (from Remark 98.21.5) equals the given splitting above. Moreover, the section $A \to C$ composed with the map $C \to \overline{C}$ is the map $(1, \mathrm{d}) : A \to A[\Omega_{A/\Lambda}]$ of Remark 98.21.8. Thus x_C restricts to the canonical element x_{can} of $T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}])$. By condition (iv) we conclude that $o_x(P/J^2)$ maps to ∂x_{can} in

$$H^1(K^\bullet \otimes_A^{\mathbf{L}} \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i))$$

By construction ξ maps to $o_x(P/J^2)$. It follows that x_{can} and ξ_{can} map to the same element in the displayed group which means (by the long exact cohomology sequence) that they differ by an element of $H^1(K^\bullet \otimes_A^{\mathbf{L}} \bigoplus A\mathrm{d}x_i)$ as desired. $\square$

07YZ **Lemma 98.24.4.** In Situation 98.24.2 assume that (iv) of Lemma 98.24.3 holds and that $K^\bullet$ is a perfect object of $D(A)$. In this case, if x is versal at a closed point $u_0 \in U$ then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

Proof. We may assume that $K^\bullet$ is a finite complex of finite projective A-modules. Thus the derived tensor product with $K^\bullet$ is the same as simply tensoring with $K^\bullet$. Let $E^\bullet$ be the dual perfect complex to $K^\bullet$, see More on Algebra, Lemma 15.74.15. (So $E^n = \mathrm{Hom}_A(K^{-n}, A)$ with differentials the transpose of the differentials of $K^\bullet$.) Let $E \in D^-(A)$ denote the object represented by the complex $E^\bullet[-1]$. Let $\xi \in H^1(\mathrm{Tot}(K^\bullet \otimes_A NL_{A/\Lambda}))$ be the element constructed in Lemma 98.24.3 and denote $\xi : E = E^\bullet[-1] \to NL_{A/\Lambda}$ the corresponding map (loc.cit.). We claim that the pair (E, ξ) satisfies all the assumptions of Lemma 98.23.4 which finishes the proof.

Namely, assumption (i) of Lemma 98.23.4 follows from conclusion (1) of Lemma 98.24.3 and the fact that $H^2(K^\bullet \otimes_A^{\mathbf{L}} -) = \text{Ext}^1(E, -)$ by loc.cit. Assumption (ii) of Lemma 98.23.4 follows from conclusion (2) of Lemma 98.24.3 and the fact that $H^1(K^\bullet \otimes_A^{\mathbf{L}} -) = \text{Ext}^0(E, -)$ by loc.cit. Assumption (iii) of Lemma 98.23.4 is clear. $\square$

98.25. Limit preserving functors on Noetherian schemes

0GE1 It is sometimes convenient to consider functors or stacks defined only on the full subcategory of (locally) Noetherian schemes. In this section we discuss this in the case of algebraic spaces.

Let S be a locally Noetherian scheme. Let us be a bit pedantic in order to line up our categories correctly; people who are ignoring set theoretical issues can just replace the sets of schemes we choose by the collection of all schemes in what follows. As in Topologies, Remark 34.11.1 we choose a category Sch_α of schemes containing S such that we obtain big sites $(Sch/S)_{Zar}$, $(Sch/S)_{étale}$, $(Sch/S)_{smooth}$, $(Sch/S)_{syntomic}$, and $(Sch/S)_{fppf}$ all with the same underlying category Sch_α/S. Denote

$$\text{Noetherian}_\alpha \subset Sch_\alpha$$

the full subcategory consisting of locally Noetherian schemes. This determines a full subcategory

$$\text{Noetherian}_\alpha/S \subset Sch_\alpha/S$$

For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have

(1) if $f : X \to Y$ is a morphism of Sch_α/S with Y in $\text{Noetherian}_\alpha/S$ and f locally of finite type, then X is in $\text{Noetherian}_\alpha/S$,
(2) for morphisms $f : X \to Y$ and $g : Z \to Y$ of $\text{Noetherian}_\alpha/S$ with f locally of finite type the fibre product $X \times_Y Z$ in $\text{Noetherian}_\alpha/S$ exists and agrees with the fibre product in Sch_α/S,
(3) if $\{X_i \to X\}_{i \in I}$ is a covering of $(Sch/S)_\tau$ and X is in $\text{Noetherian}_\alpha/S$, then the objects X_i are in $\text{Noetherian}_\alpha/S$
(4) the category $\text{Noetherian}_\alpha/S$ endowed with the set of coverings of $(Sch/S)_\tau$ whose objects are in $\text{Noetherian}_\alpha/S$ is a site we will denote $(\text{Noetherian}/S)_\tau$,
(5) the inclusion functor $(\text{Noetherian}/S)_\tau \to (Sch/S)_\tau$ is fully faithful, continuous, and cocontinuous.

By Sites, Lemmas 7.21.1 and 7.21.5 we obtain a morphism of topoi

$$g_\tau : Sh((\text{Noetherian}/S)_\tau) \longrightarrow Sh((Sch/S)_\tau)$$

whose pullback functor is the restriction of sheaves along the inclusion functor $(\text{Noetherian}/S)_\tau \to (Sch/S)_\tau$.

0GE2 Remark 98.25.1 (Warning). The site $(\text{Noetherian}/S)_\tau$ does not have fibre products. Hence we have to be careful in working with sheaves. For example, the continuous inclusion functor $(\text{Noetherian}/S)_\tau \to (Sch/S)_\tau$ does not define a morphism of sites. See Examples, Section 110.59 for an example in case $\tau = fppf$.

Let $F : (\text{Noetherian}/S)_\tau^{opp} \to \textit{Sets}$ be a functor. We say F is limit preserving if for any directed limit of affine schemes $X = \lim X_i$ of $(\text{Noetherian}/S)_\tau$ we have $F(X) = \text{colim}\, F(X_i)$.

0GE3 Lemma 98.25.2. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Restricting along the inclusion functor $(\text{Noetherian}/S)_\tau \to (Sch/S)_\tau$ defines an equivalence of categories between

(1) the category of limit preserving sheaves on $(Sch/S)_\tau$ and
(2) the category of limit preserving sheaves on $(\text{Noetherian}/S)_\tau$

Proof. Let $F : (\text{Noetherian}/S)_\tau^{opp} \to \text{Sets}$ be a functor which is both limit preserving and a sheaf. By Topologies, Lemmas 34.13.1 and 34.13.3 there exists a unique functor $F' : (Sch/S)_\tau^{opp} \to \text{Sets}$ which is limit preserving, a sheaf, and restricts to F. In fact, the construction of F' in Topologies, Lemma 34.13.1 is functorial in F and this construction is a quasi-inverse to restriction. Some details omitted. □

0GE4 Lemma 98.25.3. Let X be an object of $(\text{Noetherian}/S)_\tau$. If the functor of points $h_X : (\text{Noetherian}/S)_\tau^{opp} \to \text{Sets}$ is limit preserving, then X is locally of finite presentation over S.

Proof. Let $V \subset X$ be an affine open subscheme which maps into an affine open $U \subset S$. We may write $V = \lim V_i$ as a directed limit of affine schemes V_i of finite presentation over U, see Algebra, Lemma 10.127.2. By assumption, the arrow $V \to X$ factors as $V \to V_i \to X$ for some i. After increasing i we may assume $V_i \to X$ factors through V as the inverse image of $V \subset X$ in V_i eventually becomes equal to V_i by Limits, Lemma 32.4.11. Then the identity morphism $V \to V$ factors through V_i for some i in the category of schemes over U. Thus $V \to U$ is of finite presentation; the corresponding algebra fact is that if B is an A-algebra such that $\text{id} : B \to B$ factors through a finitely presented A-algebra, then B is of finite presentation over A (nice exercise). Hence X is locally of finite presentation over S. □

The following lemma has a variant for transformations representable by algebraic spaces.

0GE5 Lemma 98.25.4. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $F', G' : (Sch/S)_\tau^{opp} \to \text{Sets}$ be limit preserving and sheaves. Let $a' : F' \to G'$ be a transformation of functors. Denote $a : F \to G$ the restriction of $a' : F' \to G'$ to $(\text{Noetherian}/S)_\tau$. The following are equivalent

(1) a' is representable (as a transformation of functors, see Categories, Definition 4.6.4), and
(2) for every object V of $(\text{Noetherian}/S)_\tau$ and every map $V \to G$ the fibre product $F \times_G V : (\text{Noetherian}/S)_\tau^{opp} \to \text{Sets}$ is a representable functor, and
(3) same as in (2) but only for V affine finite type over S mapping into an affine open of S.

Proof. Assume (1). By Limits of Spaces, Lemma 70.3.4 the transformation a' is limit preserving[10]. Take $\xi : V \to G$ as in (2). Denote $V' = V$ but viewed as an object of $(Sch/S)_\tau$. Since G is the restriction of G' to $(\text{Noetherian}/S)_\tau$ we see that $\xi \in G(V)$ corresponds to $\xi' \in G'(V')$. By assumption $V' \times_{\xi', G'} F'$ is representable by a scheme U'. The morphism of schemes $U' \to V'$ corresponding to the projection $V' \times_{\xi', G'} F' \to V'$ is locally of finite presentation by Limits of Spaces, Lemma

[10] This makes sense even if $\tau \neq fppf$ as the underlying category of $(Sch/S)_\tau$ equals the underlying category of $(Sch/S)_{fppf}$ and the statement doesn't refer to the topology.

70.3.5 and Limits, Proposition 32.6.1. Hence U' is a locally Noetherian scheme and therefore U' is isomorphic to an object U of $(\text{Noetherian}/S)_\tau$. Then U represents $F \times_G V$ as desired.

The implication (2) $\Rightarrow$ (3) is immediate. Assume (3). We will prove (1). Let T be an object of $(Sch/S)_\tau$ and let $T \to G'$ be a morphism. We have to show the functor $F' \times_{G'} T$ is representable by a scheme X over T. Let $\mathcal{B}$ be the set of affine opens of T which map into an affine open of S. This is a basis for the topology of T. Below we will show that for $W \in \mathcal{B}$ the fibre product $F' \times_{G'} W$ is representable by a scheme X_W over W. If $W_1 \subset W_2$ in $\mathcal{B}$, then we obtain an isomorphism $X_{W_1} \to X_{W_2} \times_{W_2} W_1$ because both X_{W_1} and $X_{W_2} \times_{W_2} W_1$ represent the functor $F' \times_{G'} W_1$. These isomorphisms are canonical and satisfy the cocycle condition mentioned in Constructions, Lemma 27.2.1. Hence we can glue the schemes X_W to a scheme X over T. Compatibility of the glueing maps with the maps $X_W \to F'$ provide us with a map $X \to F'$. The resulting map $X \to F' \times_{G'} T$ is an isomorphism as we may check this locally on T (as source and target of this arrow are sheaves for the Zariski topology).

Let W be an affine scheme which maps into an affine open $U \subset S$. Let $W \to G'$ be a map. Still assuming (3) we have to show that $F' \times_{G'} W$ is representable by a scheme. We may write $W = \lim V_i'$ as a directed limit of affine schemes V_i' of finite presentation over U, see Algebra, Lemma 10.127.2. Since V_i' is of finite type over an Noetherian scheme, we see that V_i' is a Noetherian scheme. Denote $V_i = V_i'$ but viewed as an object of $(\text{Noetherian}/S)_\tau$. As G' is limit preserving can choose an i and a map $V_i' \to G'$ such that $W \to G'$ is the composition $W \to V_i' \to G'$. Since G is the restriction of G' to $(\text{Noetherian}/S)_\tau$ the morphism $V_i' \to G'$ is the same thing as a morphism $V_i \to G$ (see above). By assumption (3) the functor $F \times_G V_i$ is representable by an object X_i of $(\text{Noetherian}/S)_\tau$. The functor $F \times_G V_i$ is limit preserving as it is the restriction of $F' \times_{G'} V_i'$ and this functor is limit preserving by Limits of Spaces, Lemma 70.3.6, the assumption that F' and G' are limit preserving, and Limits, Remark 32.6.2 which tells us that the functor of points of V_i' is limit preserving. By Lemma 98.25.3 we conclude that X_i is locally of finite presentation over S. Denote $X_i' = X_i$ but viewed as an object of $(Sch/S)_\tau$. Then we see that $F' \times_{G'} V_i'$ and the functors of points $h_{X_i'}$ are both extensions of $h_{X_i} : (\text{Noetherian}/S)_\tau^{opp} \to \text{Sets}$ to limit preserving sheaves on $(Sch/S)_\tau$. By the equivalence of categories of Lemma 98.25.2 we deduce that X_i' represents $F' \times_{G'} V_i'$. Then finally

$$F' \times_{G'} W = F' \times_{G'} V_i' \times_{V_i'} W = X_i' \times_{V_i'} W$$

is representable as desired. $\square$

98.26. Algebraic spaces in the Noetherian setting

0GE6 Let S be a locally Noetherian scheme. Let $(\text{Noetherian}/S)_{\acute{e}tale} \subset (Sch/S)_{\acute{e}tale}$ denote the site studied in Section 98.25. Let $F : (\text{Noetherian}/S)_{\acute{e}tale}^{opp} \to \text{Sets}$ be a functor, i.e., F is a presheaf on $(\text{Noetherian}/S)_{\acute{e}tale}$. In this setting all the axioms [-1], [0], [1], [2], [3], [4], [5] of Section 98.15 make sense. We will review them one by one and make sure the reader knows exactly what we mean.

Axiom [-1]. This is a set theoretic condition to be ignored by readers who are not interested in set theoretic questions. It makes sense for F since it concerns the evaluation of F on spectra of fields of finite type over S which are objects of $(\text{Noetherian}/S)_{\acute{e}tale}$.

Axiom [0]. This is the axiom that F is a sheaf on $(\text{Noetherian}/S)_{\acute{e}tale}^{opp}$, i.e., satisfies the sheaf condition for étale coverings.

Axiom [1]. This is the axiom that F is limit preserving as defined in Section 98.25: for any directed limit of affine schemes $X = \lim X_i$ of $(\text{Noetherian}/S)_{\acute{e}tale}$ we have $F(X) = \text{colim}\, F(X_i)$.

Axiom [2]. This is the axiom that F satisfies the Rim-Schlessinger condition (RS). Looking at the definition of condition (RS) in Definition 98.5.1 and the discussion in Section 98.15 we see that this means: given any pushout $Y' = Y \amalg_X X'$ of schemes of finite type over S where Y, X, X' are spectra of Artinian local rings, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. This condition makes sense as the schemes X, X', Y, and Y' are in $(\text{Noetherian}/S)_{\acute{e}tale}$ since they are of finite type over S.

Axiom [3]. This is the axiom that every tangent space TF_{k,x_0} is finite dimensional. This makes sense as the tangent spaces TF_{k,x_0} are constructed from evaluations of F at $\text{Spec}(k)$ and $\text{Spec}(k[\epsilon])$ with k a field of finite type over S and hence are obtained by evaluating at objects of the category $(\text{Noetherian}/S)_{\acute{e}tale}$.

Axiom [4]. This is axiom that the every formal object is effective. Looking at the discussion in Sections 98.9 and 98.15 we see that this involves evaluating our functor at Noetherian schemes only and hence this condition makes sense for F.

Axiom [5]. This is the axiom stating that F satisfies openness of versality. Recall that this means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'. As before, verifying this only involves evaluating our functor at Noetherian schemes.

0GE7 Proposition 98.26.1. Let S be a locally Noetherian scheme. Let $F : (\text{Noetherian}/S)_{\acute{e}tale}^{opp} \to$ Sets be a functor. Assume that

(1) $\Delta : F \to F \times F$ is representable (as a transformation of functors, see Categories, Definition 4.6.4),
(2) F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see above), and
(3) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.

Then there exists a unique algebraic space $F' : (Sch/S)_{fppf}^{opp} \to$ Sets whose restriction to $(\text{Noetherian}/S)_{\acute{e}tale}$ is F (see proof for elucidation).

Proof. Recall that the sites $(Sch/S)_{fppf}$ and $(Sch/S)_{\acute{e}tale}$ have the same underlying category, see discussion in Section 98.25. Similarly the sites $(\text{Noetherian}/S)_{\acute{e}tale}$ and $(\text{Noetherian}/S)_{fppf}$ have the same underlying categories. By axioms [0] and [1] the functor F is a sheaf and limit preserving. Let $F' : (Sch/S)_{\acute{e}tale}^{opp} \to$ Sets be the unique extension of F which is a sheaf (for the étale topology) and which is limit preserving, see Lemma 98.25.2. Then F' satisfies axioms [0] and [1] as given in Section 98.15. By Lemma 98.25.4 we see that $\Delta' : F' \to F' \times F'$ is representable (by schemes). On the other hand, it is immediately clear that F' satisfies axioms [-1], [2], [3], [4], [5] of Section 98.15 as each of these involves only evaluating F' at objects of $(\text{Noetherian}/S)_{étale}$ and we've assumed the corresponding conditions for F. Whence F' is an algebraic space by Proposition 98.16.1. □

98.27. Artin's theorem on contractions

0GH7 In this section we will freely use the language of formal algebraic spaces, see Formal Spaces, Section 87.1. Artin's theorem on contractions is one of the two main theorems of Artin's paper [Art70]; the first one is his theorem on dilatations which we stated and proved in Algebraization of Formal Spaces, Section 88.29.

0GH8 Situation 98.27.1. Let S be a locally Noetherian scheme. Let X' be an algebraic space locally of finite type over S. Let $T' \subset |X'|$ be a closed subset. Let $U' \subset X'$ be the open subspace with $|U'| = |X'| \setminus T'$. Let W be a locally Noetherian formal algebraic space over S with W_{red} locally of finite type over S. Finally, we let

$$g : X'_{/T'} \longrightarrow W$$

be a formal modification, see Algebraization of Formal Spaces, Definition 88.24.1. Recall that $X'_{/T'}$ denotes the formal completion of X' along T', see Formal Spaces, Section 87.14.

In the situation above our goal is to prove that there exists a proper morphism $f : X' \to X$ of algebraic spaces over S, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ of formal algebraic spaces such that

1. T' is the inverse image of T by $|f| : |X'| \to |X|$,
2. $f : X' \to X$ maps U' isomorphically to an open subspace U of X, and
3. $g = a \circ f_{/T}$ where $f_{/T} : X'_{/T'} \to X_{/T}$ is the induced morphism.

Let us say that $(f : X' \to X, T, a)$ is a solution.

We will follow Artin's strategy by constructing a functor F on the category of locally Noetherian schemes over S, showing that F is an algebraic space using Proposition 98.26.1, and proving that setting $X = F$ works.

0GH9 Remark 98.27.2. In particular, we cannot prove that the desired result is true for every Situation 98.27.1 because we will need to assume the local rings of S are G-rings. If you can prove the result in general or if you have a counter example, please let us know at stacks.project@gmail.com.

In Situation 98.27.1 let V be a locally Noetherian scheme over S. The value of our functor F on V will be all triples

$$(Z, u' : V \setminus Z \to U', \hat{x} : V_{/Z} \to W)$$

satisfying the following conditions

1. $Z \subset V$ is a closed subset,
2. $u' : V \setminus Z \to U'$ is a morphism over S,
3. $\hat{x} : V_{/Z} \to W$ is an adic morphism of formal algebraic spaces over S,
4. u' and $\hat{x}$ are compatible (see below).

The compatibility condition is the following: pulling back the formal modification g we obtain a formal modification

$$X'_{/T'} \times_{g,W,\hat{x}} V_{/Z} \longrightarrow V_{/Z}$$

See Algebraization of Formal Spaces, Lemma 88.24.4. By the main theorem on dilatations (Algebraization of Formal Spaces, Theorem 88.29.1), there is a unique proper morphism $V' \to V$ of algebraic spaces which is an isomorphism over $V \setminus Z$ such that $V'_{/Z} \to V_{/Z}$ is isomorphic to the displayed arrow. In other words, for some morphism $\hat{x}' : V'_{/Z} \to X'_{/T'}$ we have a cartesian diagram

$$\begin{tikzcd} V'_{/Z} \arrow[r] \arrow[d, "\hat{x}'"'] & V_{/Z} \arrow[d, "\hat{x}"] \\ X'_{/T'} \arrow[r, "g"] & W \end{tikzcd}$$

of formal algebraic spaces. We will think of $V \setminus Z$ as an open subspace of V' without further mention. The compatibility condition is that there should be a morphism $x' : V' \to X'$ restricting to u' and $\hat{x}$ over $V \setminus Z \subset V'$ and $V'_{/Z}$. In other words, such that the diagram

$$\begin{tikzcd} V \setminus Z \arrow[r] \arrow[d, "u'"'] & V' \arrow[d, "x'"] & V'_{/Z} \arrow[l] \arrow[r] \arrow[d, "\hat{x}'"] & V_{/Z} \arrow[d, "\hat{x}"] \\ U' \arrow[r] & X' & X'_{/T'} \arrow[l] \arrow[r, "g"] & W \end{tikzcd}$$

is commutative. Observe that by Algebraization of Formal Spaces, Lemma 88.25.5 the morphism x' is unique if it exists. We will indicate this situation by saying "$V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$".

0GID Remark 98.27.3. In Situation 98.27.1 let V be a locally Noetherian scheme over S. Let $(Z, u', \hat{x})$ be a triple satisfying (1), (2), and (3) above. We want to explain a way to think about the compatibility condition (4). It will not be mathematically precise as we are going use a fictitious category An_S of analytic spaces over S and a fictitious analytification functor

$$\left\{\begin{matrix} \text{locally Noetherian formal} \\ \text{algebraic spaces over } S \end{matrix}\right\} \longrightarrow \text{An}_S, \qquad Y \longmapsto Y^{an}$$

For example if $Y = \text{Spf}(k[[t]])$ over $S = \text{Spec}(k)$, then Y^{an} should be thought of as an open unit disc. If $Y = \text{Spec}(k)$, then Y^{an} is a single point. The category An_S should have open and closed immersions and we should be able to take the open complement of a closed. Given Y the morphism $Y_{red} \to Y$ should induces a closed immersion $Y_{red}^{an} \to Y^{an}$. We set $Y^{rig} = Y^{an} \setminus Y^{an}_{red}$ equal to its open complement. If Y is an algebraic space and if $Z \subset Y$ is closed, then the morphism $Y_{/Z} \to Y$ should induce an open immersion $Y_{/Z}^{an} \to Y^{an}$ which in turn should induce an open immersion

$$can : (Y_{/Z})^{rig} \longrightarrow (Y \setminus Z)^{an}$$

Also, given a formal modification $g : Y' \to Y$ of locally Noetherian formal algebraic spaces, the induced morphism $g^{rig} : (Y')^{rig} \to Y^{rig}$ should be an isomorphism. Given An_S and the analytification functor, we can consider the requirement that

$$\begin{tikzcd} (V_{/Z})^{rig} \arrow[r, "can"'] \arrow[d, "(g^{rig})^{-1} \circ \hat{x}^{an}"'] & (V \setminus Z)^{an} \arrow[d, "(u')^{an}"] \\ (X'_{/T'})^{rig} \arrow[r, "can"] & (X' \setminus T')^{an} \end{tikzcd}$$

commutes. This makes sense as $g^{rig} : (X'_{T'})^{rig} \to W^{rig}$ is an isomorphism and $U' = X' \setminus T'$. Finally, under some assumptions of faithfulness of the analytification functor, this requirement will be equivalent to the compatibility condition formulated above. We hope this will motivate the reader to think of the compatibility of u' and $\hat{x}$ as the requirement that some maps be equal, rather than asking for the existence of a certain commutative diagram.

0GHA Lemma 98.27.4. In Situation 98.27.1 the rule F that sends a locally Noetherian scheme V over S to the set of triples $(Z, u', \hat{x})$ satisfying the compatibility condition and which sends a a morphism $\varphi : V_2 \to V_1$ of locally Noetherian schemes over S to the map

$$F(\varphi) : F(V_1) \longrightarrow F(V_2)$$

sending an element $(Z_1, u'_1, \hat{x}_1)$ of $F(V_1)$ to $(Z_2, u'_2, \hat{x}_2)$ in $F(V_2)$ given by

(1) $Z_2 \subset V_2$ is the inverse image of Z_1 by φ,
(2) u'_2 is the composition of u'_1 and $\varphi|_{V_2 \setminus Z_2} : V_2 \setminus Z_2 \to V_1 \setminus Z_1$,
(3) $\hat{x}_2$ is the composition of $\hat{x}_1$ and $\varphi_{/Z_2} : V_{2,/Z_2} \to V_{1,/Z_1}$

is a contravariant functor.

Proof. To see the compatibility condition between u'_2 and $\hat{x}_2$, let $V'_1 \to V_1$, $\hat{x}'_1$, and x'_1 witness the compatibility between u'_1 and $\hat{x}_1$. Set $V'_2 = V_2 \times_{V_1} V'_1$, set $\hat{x}'_2$ equal to the composition of $\hat{x}'_1$ and $V'_{2,/Z_2} \to V'_{1,/Z_1}$, and set x'_2 equal to the composition of x'_1 and $V'_2 \to V'_1$. Then $V'_2 \to V_2$, $\hat{x}'_2$, and x'_2 witness the compatibility between u'_2 and $\hat{x}_2$. We omit the detailed verification. □

0GHB Lemma 98.27.5. In Situation 98.27.1 if there exists a solution $(f : X' \to X, T, a)$ then there is a functorial bijection $F(V) = \mathrm{Mor}_S(V, X)$ on the category of locally Noetherian schemes V over S.

Proof. Let V be a locally Noetherian scheme over S. Let $x : V \to X$ be a morphism over S. Then we get an element $(Z, u', \hat{x})$ in $F(V)$ as follows

(1) $Z \subset V$ is the inverse image of T by x,
(2) $u' : V \setminus Z \to U' = U$ is the restriction of x to $V \setminus Z$,
(3) $\hat{x} : V_{/Z} \to W$ is the composition of $x_{/Z} : V_{/Z} \to X_{/T}$ with the isomorphism $a : X_{/T} \to W$.

This triple satisfies the compatibility condition because we can take $V' = V \times_{x,X} X'$, we can take $\hat{x}'$ the completion of the projection $x' : V' \to X'$.

Conversely, suppose given an element $(Z, u', \hat{x})$ of $F(V)$. We claim there is a unique morphism $x : V \to X$ compatible with u' and $\hat{x}$. Namely, let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. Then Algebraization of Formal Spaces, Proposition 88.26.1 is exactly the result we need to find a unique morphism $x : V \to X$ agreeing with $\hat{x}$ over $V_{/Z}$ and with x' over V' (and a fortiori agreeing with u' over $V \setminus Z$).

We omit the verification that the two constructions above define inverse bijections between their respective domains. □

0GHC Lemma 98.27.6. In Situation 98.27.1 if there exists an algebraic space X locally of finite type over S and a functorial bijection $F(V) = \mathrm{Mor}_S(V, X)$ on the category of locally Noetherian schemes V over S, then X is a solution.

Proof. We have to construct a proper morphism $f : X' \to X$, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ with properties (1), (2), (3) listed just below Situation 98.27.1.

The discussion in this proof is a bit pedantic because we want to carefully match the underlying categories. In this paragraph we explain how the adventurous reader can proceed less timidly. Namely, the reader may extend our definition of the functor F to all locally Noetherian algebraic spaces over S. Doing so the reader may then conclude that F and X agree as functors on the category of these algebraic spaces, i.e., X represents F. Then one considers the universal object $(T, u', \hat{x})$ in $F(X)$. Then the reader will find that for the triple $X'' \to X$, $\hat{x}'$, x' witnessing the compatibility between u' and $\hat{x}$ the morphism $x' : X'' \to X'$ is an isomorphism and this will produce $f : X' \to X$ by inverting x'. Finally, we already have $T \subset |X|$ and the reader may show that $\hat{x}$ is an isomorphism which can served as the last ingredient namely a.

Denote $h_X(-) = \mathrm{Mor}_S(-, X)$ the functor of points of X restricted to the category $(\text{Noetherian}/S)_{\acute{e}tale}$ of Section 98.25. By Limits of Spaces, Remark 70.3.11 the algebraic spaces X and X' are limit preserving. Hence so are the restrictions h_X and $h_{X'}$. To construct f it therefore suffices to construct a transformation $h_{X'} \to h_X = F$, see Lemma 98.25.2. Thus let $V \to S$ be an object of $(\text{Noetherian}/S)_{\acute{e}tale}$ and let $\tilde{x} : V \to X'$ be in $h_{X'}(V)$. Then we get an element $(Z, u', \hat{x})$ in $F(V)$ as follows

(1) $Z \subset V$ is the inverse image of T' by $\tilde{x}$,
(2) $u' : V \setminus Z \to U'$ is the restriction of $\tilde{x}$ to $V \setminus Z$,
(3) $\hat{x} : V_{/Z} \to W$ is the composition of $x_{/Z} : V_{/Z} \to X'_{/T'}$ with $g : X'_{/T'} \to W$.

This triple satisfies the compatibility condition: first we always obtain $V' \to V$ and $\hat{x}' : V'_{/Z'} \to X'_{/T'}$ for free (see discussion preceding Lemma 98.27.4). Then we just define $x' : V' \to X'$ to be the composition of $V' \to V$ and the morphism $\tilde{x} : V \to X'$. We omit the verification that this works.

If $\xi : V \to X$ is an étale morphism where V is a scheme, then we obtain $\xi = (Z, u', \hat{x}) \in F(V) = h_X(V) = X(V)$. Of course, if $\varphi : V' \to V$ is a further étale morphism of schemes, then $(Z, u', \hat{x})$ pulled back to $F(V')$ corresponds to $\xi \circ \varphi$. The closed subset $T \subset |X|$ is just defined as the closed subset such that $\xi : V \to X$ for $\xi = (Z, u', \hat{x})$ pulls T back to Z

Consider Noetherian schemes V over S and a morphism $\xi : V \to X$ corresponding to $(Z, u', \hat{x})$ as above. Then we see that $\xi(V)$ is set theoretically contained in T if and only if $V = Z$ (as topological spaces). Hence we see that $X_{/T}$ agrees with W as a functor. This produces the isomorphism $a : X_{/T} \to W$. (We've omitted a small detail here which is that for the locally Noetherian formal algebraic spaces $X_{/T}$ and W it suffices to check one gets an isomorphism after evaluating on locally Noetherian schemes over S.)

We omit the proof of conditions (1), (2), and (3). □

0GHD **Remark 98.27.7.** In Situation 98.27.1. Let V be a locally Noetherian scheme over S. Let $(Z_i, u'_i, \hat{x}_i) \in F(V)$ for $i = 1, 2$. Let $V'_i \to V$, $\hat{x}'_i$ and x'_i witness the compatibility between u'_i and $\hat{x}_i$ for $i = 1, 2$.

Set $V' = V_1' \times_V V_2'$. Let $E' \to V'$ denote the equalizer of the morphisms

$$V' \to V_1' \xrightarrow{x_1'} X' \quad \text{and} \quad V' \to V_2' \xrightarrow{x_2'} X'$$

Set $Z = Z_1 \cap Z_2$. Let $E_W \to V_{/Z}$ be the equalizer of the morphisms

$$V_{/Z} \to V_{/Z_1} \xrightarrow{\hat{x}_1} W \quad \text{and} \quad V_{/Z} \to V_{/Z_2} \xrightarrow{\hat{x}_2} W$$

Observe that $E' \to V$ is separated and locally of finite type and that E_W is a locally Noetherian formal algebraic space separated over V. The compatibilities between the various morphisms involved show that

(1) $\text{Im}(E' \to V) \cap (Z_1 \cup Z_2)$ is contained in $Z = Z_1 \cap Z_2$,
(2) the morphism $E' \times_V (V \setminus Z) \to V \setminus Z$ is a monomorphism and is equal to the equalizer of the restrictions of u_1' and u_2' to $V \setminus (Z_1 \cup Z_2)$,
(3) the morphism $E'_{/Z} \to V_{/Z}$ factors through E_W and the diagram

$$\begin{tikzcd} E'_{/Z} \arrow[r] \arrow[d] & X'_{/T'} \arrow[d, "g"] \\ E_W \arrow[r] & W \end{tikzcd}$$

is cartesian. In particular, the morphism $E'_{/Z} \to E_W$ is a formal modification as the base change of g,
(4) E', $(E' \to V)^{-1}Z$, and $E'_{/Z} \to E_W$ is a triple as in Situation 98.27.1 with base scheme the locally Noetherian scheme V,
(5) given a morphism $\varphi : A \to V$ of locally Noetherian schemes, the following are equivalent
   (a) $(Z_1, u_1', \hat{x}_1)$ and $(Z_2, u_2', \hat{x}_2)$ restrict to the same element of $F(A)$,
   (b) $A \setminus \varphi^{-1}(Z) \to V \setminus Z$ factors through $E' \times_V (V \setminus Z)$ and $A_{/\varphi^{-1}(Z)} \to V_{/Z}$ factors through E_W.

We conclude, using Lemmas 98.27.5 and 98.27.6, that if there is a solution $E \to V$ for the triple in (4), then E represents $F \times_{\Delta, F \times F} V$ on the category of locally Noetherian schemes over V.

0GHE Lemma 98.27.8. In Situation 98.27.1 assume given a closed subset $Z \subset S$ such that

(1) the inverse image of Z in X' is T',
(2) $U' \to S \setminus Z$ is a closed immersion,
(3) $W \to S_{/Z}$ is a closed immersion.

Then there exists a solution $(f : X' \to X, T, a)$ and moreover $X \to S$ is a closed immersion.

Proof. Suppose we have a closed subscheme $X \subset S$ such that $X \cap (S \setminus Z) = U'$ and $X_{/Z} = W$. Then X represents the functor F (some details omitted) and hence is a solution. To find X is clearly a local question on S. In this way we reduce to the case discussed in the next paragraph.

Assume $S = \text{Spec}(A)$ is affine. Let $I \subset A$ be the radical ideal cutting out Z. Write $I = (f_1, \ldots, f_r)$. By assumption we are given

(1) the closed immersion $U' \to S \setminus Z$ determines ideals $J_i \subset A[1/f_i]$ such that J_i and J_j generate the same ideal in $A[1/f_i f_j]$,

(2) the closed immersion $W \to S_{/Z}$ is the map $\text{Spf}(A^\wedge/J') \to \text{Spf}(A^\wedge)$ for some ideal $J' \subset A^\wedge$ in the I-adic completion $A^\wedge$ of A.

To finish the proof we need to find an ideal $J \subset A$ such that $J_i = J[1/f_i]$ and $J' = JA^\wedge$. By More on Algebra, Proposition 15.89.15 it suffices to show that J_i and J' generate the same ideal in $A^\wedge[1/f_i]$ for all i.

Recall that $A' = H^0(X', \mathcal{O})$ is a finite A-algebra whose formation commutes with flat base change (Cohomology of Spaces, Lemmas 69.20.3 and 69.11.2). Denote $J'' = \text{Ker}(A \to A')$[11]. We have $J_i = J''A[1/f_i]$ as follows from base change to the spectrum of $A[1/f_i]$. Observe that we have a commutative diagram

$$\begin{tikzcd} X' \arrow[d] & X'_{/T'} \times_{S_{/Z}} \text{Spf}(A^\wedge) \arrow[l] \arrow[d] \arrow[r, equal] & X'_{/T'} \times_W \text{Spf}(A^\wedge/J') \arrow[d] \\ \text{Spec}(A) & \text{Spf}(A^\wedge) \arrow[l] & \text{Spf}(A^\wedge/J') \arrow[l] \end{tikzcd}$$

The middle vertical arrow is the completion of the left vertical arrow along the obvious closed subsets. By the theorem on formal functions we have

$$(A')^\wedge = \Gamma(X' \times_S \text{Spec}(A^\wedge), \mathcal{O}) = \lim H^0(X' \times_S \text{Spec}(A/I^n), \mathcal{O})$$

See Cohomology of Spaces, Theorem 69.22.5. From the diagram we conclude that J' maps to zero in $(A')^\wedge$. Hence $J' \subset J''A^\wedge$. Consider the arrows

$$X'_{/T'} \to \text{Spf}(A^\wedge/J''A^\wedge) \to \text{Spf}(A^\wedge/J') = W$$

We know the composition g is a formal modification (in particular rig-étale and rig-surjective) and the second arrow is a closed immersion (in particular an adic monomorphism). Hence $X'_{/T'} \to \text{Spf}(A^\wedge/J''A^\wedge)$ is rig-surjective and rig-étale, see Algebraization of Formal Spaces, Lemmas 88.21.5 and 88.20.8. Applying Algebraization of Formal Spaces, Lemmas 88.21.14 and 88.21.6 we conclude that $\text{Spf}(A^\wedge/J''A^\wedge) \to W$ is rig-étale and rig-surjective. By Algebraization of Formal Spaces, Lemma 88.21.13 we conclude that $I^nJ''A^\wedge \subset J'$ for some $n > 0$. It follows that $J''A^\wedge[1/f_i] = J'A^\wedge[1/f_i]$ and we deduce $J_iA^\wedge[1/f_i] = J'A^\wedge[1/f_i]$ for all i as desired. □

0GHF Lemma 98.27.9. In Situation 98.27.1 assume $X' \to S$ and $W \to S$ are separated. Then the diagonal $\Delta : F \to F \times F$ is representable by closed immersions.

Proof. Combine Lemma 98.27.8 with the discussion in Remark 98.27.7. □

0GHG Lemma 98.27.10. In Situation 98.27.1 the functor F satisfies the sheaf property for all étale coverings of locally Noetherian schemes over S.

Proof. Omitted. Hint: morphisms may be defined étale locally. □

0GI5 Lemma 98.27.11. In Situation 98.27.1 the functor F is limit preserving: for any directed limit $V = \lim V_\lambda$ of Noetherian affine schemes over S we have $F(V) = \text{colim}\, F(V_\lambda)$.

Proof. This is an absurdly long proof. Much of it consists of standard arguments on limits and étale localization. We urge the reader to skip ahead to the last part of the proof where something interesting happens.

[11]Contrary to what the reader may expect, the ideals J and J'' won't agreee in general.

Let $V = \lim_{\lambda \in \Lambda} V_i$ be a directed limit of schemes over S with V and V_λ Noetherian and with affine transition morphisms. See Limits, Section 32.2 for material on limits of schemes. We will prove that $\text{colim}\, F(V_\lambda) \to F(V)$ is bijective.

Proof of injectivity: notation. Let $\lambda \in \Lambda$ and $\xi_{\lambda,1}, \xi_{\lambda,2} \in F(V_\lambda)$ be elements which restrict to the same element of $F(V)$. Write $\xi_{\lambda,1} = (Z_{\lambda,1}, u'_{\lambda,1}, \hat{x}_{\lambda,1})$ and $\xi_{\lambda,2} = (Z_{\lambda,2}, u'_{\lambda,2}, \hat{x}_{\lambda,2})$.

Proof of injectivity: agreement of $Z_{\lambda,i}$. Since $Z_{\lambda,1}$ and $Z_{\lambda,2}$ restrict to the same closed subset of V, we may after increasing i assume $Z_{\lambda,1} = Z_{\lambda,2}$, see Limits, Lemma 32.4.2 and Topology, Lemma 5.14.2. Let us denote the common value $Z_\lambda \subset V_\lambda$, for $\mu \geq \lambda$ denote $Z_\mu \subset V_\mu$ the inverse image in V_μ and and denote Z the inverse image in V. We will use below that $Z = \lim_{\mu \geq \lambda} Z_\mu$ as schemes if we view Z and Z_μ as reduced closed subschemes.

Proof of injectivity: agreement of $u'_{\lambda,i}$. Since U' is locally of finite type over S and since the restrictions of $u'_{\lambda,1}$ and $u'_{\lambda,2}$ to $V \setminus Z$ are the same, we may after increasing λ assume $u'_{\lambda,1} = u'_{\lambda,2}$, see Limits, Proposition 32.6.1. Let us denote the common value u'_λ and denote u' the restriction to $V \setminus Z$.

Proof of injectivity: restatement. At this point we have $\xi_{\lambda,1} = (Z_\lambda, u'_\lambda, \hat{x}_{\lambda,1})$ and $\xi_{\lambda,2} = (Z_\lambda, u'_\lambda, \hat{x}_{\lambda,2})$. The main problem we face in this part of the proof is to show that the morphisms $\hat{x}_{\lambda,1}$ and $\hat{x}_{\lambda,2}$ become the same after increasing λ.

Proof of injectivity: agreement of $\hat{x}_{\lambda,i}|_{Z_\lambda}$. Consider the morpisms $\hat{x}_{\lambda,1}|_{Z_\lambda}, \hat{x}_{\lambda,2}|_{Z_\lambda} : Z_\lambda \to W_{red}$. These morphisms restrict to the same morphism $Z \to W_{red}$. Since W_{red} is a scheme locally of finite type over S we see using Limits, Proposition 32.6.1 that after replacing λ by a bigger index we may assume $\hat{x}_{\lambda,1}|_{Z_\lambda} = \hat{x}_{\lambda,2}|_{Z_\lambda} : Z_\lambda \to W_{red}$.

Proof of injectivity: end. Next, we are going to apply the discussion in Remark 98.27.7 to V_λ and the two elements $\xi_{\lambda,1}, \xi_{\lambda,2} \in F(V_\lambda)$. This gives us

(1) $e_\lambda : E'_\lambda \to V_\lambda$ separated and locally of finite type,
(2) $e_\lambda^{-1}(V_\lambda \setminus Z_\lambda) \to V_\lambda \setminus Z_\lambda$ is an isomorphism,
(3) a monomorphism $E_{W,\lambda} \to V_{\lambda,/Z_\lambda}$ which is the equalizer of $\hat{x}_{\lambda,1}$ and $\hat{x}_{\lambda,2}$,
(4) a formal modification $E'_{\lambda,/Z_\lambda} \to E_{W,\lambda}$

Assertion (2) holds by assertion (2) in Remark 98.27.7 and the preparatory work we did above getting $u'_{\lambda,1} = u'_{\lambda,2} = u'_\lambda$. Since $Z_\lambda = (V_{\lambda,/Z_\lambda})_{red}$ factors through $E_{W,\lambda}$ because $\hat{x}_{\lambda,1}|_{Z_\lambda} = \hat{x}_{\lambda,2}|_{Z_\lambda}$ we see from Formal Spaces, Lemma 87.27.7 that $E_{W,\lambda} \to V_{\lambda,/Z_\lambda}$ is a closed immersion. Then we see from assertion (4) in Remark 98.27.7 and Lemma 98.27.8 applied to the triple E'_λ, $e_\lambda^{-1}(Z_\lambda)$, $E'_{\lambda,/Z_\lambda} \to E_{W,\lambda}$ over V_λ that there exists a closed immersion $E_\lambda \to V_\lambda$ which is a solution for this triple. Next we use assertion (5) in Remark 98.27.7 which combined with Lemma 98.27.5 says that E_λ is the "equalizer" of $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$. In particular, we see that $V \to V_\lambda$ factors through E_λ. Then using Limits, Proposition 32.6.1 once more we find $\mu \geq \lambda$ such that $V_\mu \to V_\lambda$ factors through E_λ and hence the pullbacks of $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$ to V_μ are the same as desired.

Proof of surjectivity: statement. Let $\xi = (Z, u', \hat{x})$ be an element of $F(V)$. We have to find a $\lambda \in \Lambda$ and an element $\xi_\lambda \in F(V_\lambda)$ restricting to ξ.

Proof of surjectivity: the question is étale local. By the unicity proved in the previous part of the proof and by the sheaf property of F in Lemma 98.27.10, the problem is local on V in the étale topology. More precisely, let $v \in V$. We claim it suffices to find an étale morphism $(\tilde{V}, \tilde{v}) \to (V, v)$ and some λ, some an étale morphism $\tilde{V}_\lambda \to V_\lambda$, and some element $\tilde{\xi}_\lambda \in F(\tilde{V}_\lambda)$ such that $\tilde{V} = \tilde{V}_\lambda \times_{V_\lambda} V$ and $\xi|_U = \tilde{\xi}_\lambda|_U$. We omit a detailed proof of this claim[12].

Proof of surjectivity: rephrasing the problem. Recall that any étale morphism $(\tilde{V}, \tilde{v}) \to (V, v)$ with $\tilde{V}$ affine is the base change of an étale morphism $\tilde{V}_\lambda \to V_\lambda$ with $\tilde{V}_\lambda$ affine for some λ, see for example Topologies, Lemma 34.13.2. Given $\tilde{V}_\lambda$ we have $\tilde{V} = \lim_{\mu \geq \lambda} \tilde{V}_\lambda \times_{V_\lambda} V_\mu$. Hence given $(\tilde{V}, \tilde{v}) \to (V, v)$ étale with $\tilde{V}$ affine, we may replace (V, v) by $(\tilde{V}, \tilde{v})$ and ξ by the restriction of ξ to $\tilde{V}$.

Proof of surjectivity: reduce to base being affine. In particular, suppose $\tilde{S} \subset S$ is an affine open subscheme such that $v \in V$ maps to a point of $\tilde{S}$. Then we may according to the previous paragraph, replace V by $\tilde{V} = \tilde{S} \times_S V$. Of course, if we do this, it suffices to solve the problem for the functor F restricted to the category of locally Noetherian schemes over $\tilde{S}$. This functor is of course the functor associated to the whole situation base changed to $\tilde{S}$. Thus we may and do assume $S = \text{Spec}(R)$ is a Noetherian affine scheme for the rest of the proof.

Proof of surjectivity: easy case. If $v \in V \setminus Z$, then we can take $\tilde{V} = V \setminus Z$. This descends to an open subscheme $\tilde{V}_\lambda \subset V_\lambda$ for some λ by Limits, Lemma 32.4.11. Next, after increasing λ we may assume there is a morphism $u'_\lambda : \tilde{V}_\lambda \to U'$ restricting to u'. Taking $\tilde{\xi}_\lambda = (\emptyset, u'_\lambda, \emptyset)$ gives the desired element of $F(\tilde{V}_\lambda)$.

Proof of surjectivity: hard case and reduction to affine W. The most difficult case comes from considering $v \in Z \subset V$. We claim that we can reduce this to the case where W is an affine formal scheme; we urge the reader to skip this argument[13]. Namely, we can choose an étale morphism $\tilde{W} \to W$ where $\tilde{W}$ is an affine formal algebraic space such that the image of v by $\hat{x} : V_{/Z} \to W$ is in the image of $\tilde{W} \to W$ (on reductions). Then the morphisms

$$p : \tilde{W} \times_{W,g} X'_{/T'} \longrightarrow X'_{/T'}$$

and

$$q : \tilde{W} \times_{W,\hat{x}} V_{/Z} \to V_{/Z}$$

are étale morphisms of locally Noetherian formal algebraic spaces. By (an easy case of) Algebraization of Formal Spaces, Theorem 88.27.4 there exists a morphism $\tilde{X}' \to X'$ of algebraic spaces which is locally of finite type, is an isomorphism over U', and such that $\tilde{X}'_{/T'} \to X'_{/T'}$ is isomorphic to p. By Algebraization of Formal Spaces, Lemma 88.28.5 the morphism $\tilde{X}' \to X'$ is étale. Denote $\tilde{T}' \subset |\tilde{X}'|$ the inverse image of T'. Denote $\tilde{U}' \subset \tilde{X}'$ the complementary open subspace. Denote $\tilde{g}' : \tilde{X}'_{/\tilde{T}'} \to \tilde{W}$ the formal modification which is the base change of g by $\tilde{W} \to W$. Then we see that

$$\tilde{X}',\ \tilde{T}',\ \tilde{U}',\ \tilde{W},\ \tilde{g} : \tilde{X}'_{/\tilde{T}'} \to \tilde{W}$$

[12]To prove this one assembles a collection of the morphisms $\tilde{V} \to V$ into a finite étale covering and shows that the corresponding morphisms $\tilde{V}_\lambda \to V_\lambda$ form an étale covering as well (after increasing λ). Next one uses the injectivity to see that the elements $\tilde{\xi}_\lambda$ glue (after increasing λ) and one uses the sheaf property for F to descend these elements to an element of $F(V_\lambda)$.

[13]Artin's approach to the proof of this lemma is to work around this and consequently he can avoid proving the injectivity first. Namely, Artin consistently works with a finite affine étale coverings of all spaces in sight keeping track of the maps between them during the proof. In hindsight that might be preferable to what we do here.

is another example of Situation 98.27.1. Denote $\tilde{F}$ the functor constructed from this triple. There is a transformation of functors

$$\tilde{F} \longrightarrow F$$

constructed using the morphisms $\tilde{X}' \to X'$ and $\tilde{W} \to W$ in the obvious manner; details omitted.

Proof of surjectivity: hard case and reduction to affine W, part 2. By the same theorem as used above, there exists a morphism $\tilde{V} \to V$ of algebraic spaces which is locally of finite type, is an isomorphism over $V \setminus Z$ and such that $\tilde{V}_{/Z} \to V_{/Z}$ is isomorphic to q. Denote $\tilde{Z} \subset \tilde{V}$ the inverse image of Z. By Algebraization of Formal Spaces, Lemmas 88.28.5 and 88.28.3 the morphism $\tilde{V} \to V$ is étale and separated. In particular $\tilde{V}$ is a (locally Noetherian) scheme, see for example Morphisms of Spaces, Proposition 67.50.2. We have the morphism u' which we may view as a morphism

$$\tilde{u}' : \tilde{V} \setminus \tilde{Z} \longrightarrow \tilde{U}'$$

where $\tilde{U}' \subset \tilde{X}'$ is the open mapping isomorphically to U'. We have a morphism

$$\tilde{\hat{x}} : \tilde{V}_{/\tilde{Z}} = \tilde{W} \times_{W,\hat{x}} V_{/Z} \longrightarrow \tilde{W}$$

Namely, here we just use the projection. Thus we have the triple

$$\tilde{\xi} = (\tilde{Z}, \tilde{u}', \tilde{\hat{x}}) \in \tilde{F}(\tilde{V})$$

We omit proving the compatibility condition; hints: if $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$, then one sets $\tilde{V}' = V' \times_V \tilde{V}$ which comes with morphsms $\tilde{\hat{x}}'$ and $\tilde{x}'$ and show this works. The image of $\tilde{\xi}$ under the transformation $\tilde{F} \to F$ is the restriction of ξ to $\tilde{V}$.

Proof of surjectivity: hard case and reduction to affine W, part 3. By our choice of $\tilde{W} \to W$, there is an affine open $\tilde{V}_{open} \subset \tilde{V}$ (we're running out of notation) whose image in V contains our chosen point $v \in V$. Now by the case studied in the next paragraph and the remarks made earlier, we can descend $\tilde{\xi}|_{\tilde{V}_{open}}$ to some element $\tilde{\xi}_\lambda$ of $\tilde{F}$ over $\tilde{V}_{\lambda,open}$ for some étale morphism $\tilde{V}_{\lambda,open} \to V_\lambda$ whose base change to V is $\tilde{V}_{open}$. Applying the transformation of functors $\tilde{F} \to F$ we obtain the element of $F(\tilde{V}_{\lambda,open})$ we were looking for. This reduces us to the case discussed in the next paragraph.

Proof of surjectivity: the case of an affine W. We have $v \in Z \subset V$ and W is an affine formal algebraic space. Recall that

$$\xi = (Z, u', \hat{x}) \in F(V)$$

We may still replace V by an étale neighbourhood of v. In particular we may and do assume V and V_λ are affine.

Proof of surjectivity: descending Z. We can find a λ and a closed subscheme $Z_\lambda \subset V_\lambda$ such that Z is the base change of Z_λ to V. See Limits, Lemma 32.10.1. Warning: we don't know (and in general it won't be true) that Z_λ is a reduced closed subscheme of V_λ. For $\mu \geq \lambda$ denote $Z_\mu \subset V_\mu$ the scheme theoretic inverse image in V_μ. We will use below that $Z = \lim_{\mu \geq \lambda} Z_\mu$ as schemes.

Proof of surjectivity: descending u'. Since U' is locally of finite type over S we may assume after increasing λ that there exists a morphism $u'_\lambda : V_\lambda \setminus Z_\lambda \to U'$ whose restriction to $V \setminus Z$ is u'. See Limits, Proposition 32.6.1. For $\mu \geq \lambda$ we will denote u'_μ the restriction of u'_λ to $V_\mu \setminus Z_\mu$.

Proof of surjectivity: descending a witness. Let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. Using the same references as above we may assume (after increasing λ) that there exists a morphism $V'_\lambda \to V_\lambda$ of finite type whose base change to V is $V' \to V$. After increasing λ we may assume $V'_\lambda \to V_\lambda$ is proper (Limits, Lemma 32.13.1). Next, we may assume $V'_\lambda \to V_\lambda$ is an isomorphism over $V_\lambda \setminus Z_\lambda$ (Limits, Lemma 32.8.11). Next, we may assume there is a morphism $x'_\lambda : V'_\lambda \to X'$ whose restriction to V' is x'. Increasing λ again we may assume x'_λ agrees with u'_λ over $V_\lambda \setminus Z_\lambda$. For $\mu \geq \lambda$ we denote V'_μ and x'_μ the base change of V'_λ and the restriction of x'_λ.

Proof of surjectivity: algebra. Write $W = \text{Spf}(B)$, $V = \text{Spec}(A)$, and for $\mu \geq \lambda$ write $V_\mu = \text{Spec}(A_\mu)$. Denote $I_\mu \subset A_\mu$ and $I \subset A$ the ideals cutting out Z_μ and Z. Then $I_\lambda A_\mu = I_\mu$ and $I_\lambda A = I$. The morphism $\hat{x}$ determines and is determined by a continuous ring map

$$(\hat{x})^\sharp : B \longrightarrow A^\wedge$$

where $A^\wedge$ is the I-adic completion of A. To finish the proof we have to show that this map descends to a map into $A_\mu^\wedge$ for some sufficiently large μ where $A_\mu^\wedge$ is the I_μ-adic completion of A_μ. This is a nontrivial fact; Artin writes in his paper [Art70]: "Since the data (3.5) involve I-adic completions, which do not commute with direct limits, the verification is somewhat delicate. It is an algebraic analogue of a convergence proof in analysis."

Proof of surjectivity: algebra, more rings. Let us denote

$$C_\mu = \Gamma(V'_\mu, \mathcal{O}) \quad \text{and} \quad C = \Gamma(V', \mathcal{O})$$

Observe that $A \to C$ and $A_\mu \to C_\mu$ are finite ring maps as $V' \to V$ and $V'_\mu \to V_\mu$ are proper morphisms, see Cohomology of Spaces, Lemma 69.20.3. Since $V = \lim V_\mu$ and $V' = \lim V'_\mu$ we have

$$A = \text{colim}\, A_\mu \quad \text{and} \quad C = \text{colim}\, C_\mu$$

by Limits, Lemma 32.4.7[14]. For an element $a \in I$, resp. $a \in I_\mu$ the maps $A_a \to C_a$, resp. $(A_\mu)_a \to (C_\mu)_a$ are isomorphisms by flat base change (Cohomology of Spaces, Lemma 69.11.2). Hence the kernel and cokernel of $A \to C$ is supported on $V(I)$ and similarly for $A_\mu \to C_\mu$. We conclude the kernel and cokernel of $A \to C$ are annihilated by a power of I and the kernel and cokernel of $A_\mu \to C_\mu$ are annihilated by a power of I_μ, see Algebra, Lemma 10.62.4.

Proof of surjectivity: algebra, more ring maps. Denote $Z_n \subset V$ the nth infinitesimal neighbourhood of Z and denote $Z_{\mu,n} \subset V_\mu$ the nth infinitesimal neighbourhoof of Z_μ. By the theorem on formal functions (Cohomology of Spaces, Theorem 69.22.5) we have

$$C^\wedge = \lim_n H^0(V' \times_V Z_n, \mathcal{O}) \quad \text{and} \quad C_\mu^\wedge = \lim_n H^0(V'_\mu \times_{V_\mu} Z_{\mu,n}, \mathcal{O})$$

where $C^\wedge$ and $C_\mu^\wedge$ are the completion with respect to I and I_μ. Combining the completion of the morphism $x'_\mu : V'_\mu \to X'$ with the morphism $g : X'_{/T'} \to W$ we obtain

$$g \circ x'_{\mu,/Z_\mu} : V'_{\mu,/Z_\mu} = \text{colim}\, V'_\mu \times_{V_\mu} Z_{\mu,n} \longrightarrow W$$

[14]We don't know that $C_\mu = C_\lambda \otimes_{A_\lambda} A_\mu$ as the various morphisms aren't flat.

and hence by the description of the completion $C_\mu^\wedge$ above we obtain a continuous ring homomorphism

$$(g \circ x'_{\mu,/Z_\mu})^\sharp : B \longrightarrow C_\mu^\wedge$$

The fact that $V' \to V$, $\hat{x}'$, x' witnesses the compatibility between u' and $\hat{x}$ implies the commutativity of the following diagram

$$\begin{tikzcd} C_\mu^\wedge \arrow[r] & C^\wedge \\ B \arrow[u, "(g\circ x'_{\mu,/Z_\mu})^\sharp"] \arrow[r, "(\hat{x})^\sharp"] & A^\wedge \arrow[u] \end{tikzcd}$$

Proof of surjectivity: more algebra arguments. Recall that the finite A-modules $\mathrm{Ker}(A \to C)$ and $\mathrm{Coker}(A \to C)$ are annihilated by a power of I and similarly the finite A_μ-modules $\mathrm{Ker}(A_\mu \to C_\mu)$ and $\mathrm{Coker}(A_\mu \to C_\mu)$ are annihilated by a power of I_μ. This implies that these modules are equal to their completions. Since I-adic completion on the category of finite A-modules is exact (see Algebra, Section 10.97) it follows that we have

$$\mathrm{Coker}(A^\wedge \to C^\wedge) = \mathrm{Coker}(A \to C)$$

and similarly for kernels and for the maps $A_\mu \to C_\mu$. Of course we also have

$$\mathrm{Ker}(A \to C) = \mathrm{colim}\,\mathrm{Ker}(A_\mu \to C_\mu) \quad \text{and} \quad \mathrm{Coker}(A \to C) = \mathrm{colim}\,\mathrm{Coker}(A_\mu \to C_\mu)$$

Recall that $S = \mathrm{Spec}(R)$ is affine. All of the ring maps above are R-algebra homomorphisms as all of the morphisms are morphisms over S. By Algebraization of Formal Spaces, Lemma 88.12.5 we see that B is topologically of finite type over R. Say B is topologically generated by $b_1, \ldots, b_n$. Pick some μ (for example λ) and consider the elements

$$\text{images of } (g \circ x'_{\mu,/Z_\mu})^\sharp(b_1), \ldots, (g \circ x'_{\mu,/Z_\mu})^\sharp(b_n) \text{ in } \mathrm{Coker}(A_\mu \to C_\mu)$$

The image of these elements in $\mathrm{Coker}(\alpha)$ are zero by the commutativity of the square above. Since $\mathrm{Coker}(A \to C) = \mathrm{colim}\,\mathrm{Coker}(A_\mu \to C_\mu)$ and these cokernels are equal to their completions we see that after increasing μ we may assume these images are all zero. This means that the continuous homomorphism $(g \circ x'_{\mu,/Z_\mu})^\sharp$ has image contained in $\mathrm{Im}(A_\mu \to C_\mu)$. Choose elements $a_{\mu,j} \in (A_\mu)^\wedge$ mapping to $(g \circ x'_{\mu,/Z_\mu})^\sharp(b_1)$ in $(C_\mu)^\wedge$. Then $a_{\mu,j} \in A_\mu^\wedge$ and $(\hat{x})^\sharp(b_j) \in A^\wedge$ map to the same element of $C^\wedge$ by the commutativity of the square above. Since $\mathrm{Ker}(A \to C) = \mathrm{colim}\,\mathrm{Ker}(A_\mu \to C_\mu)$ and these kernels are equal to their completions, we may after increasing μ adjust our choices of $a_{\mu,j}$ such that the image of $a_{\mu,j}$ in $A^\wedge$ is equal to $(\hat{x})^\sharp(b_j)$.

Proof of surjectivity: final algebra arguments. Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements. Let $J \subset R[x_1, \ldots, x_n]$ be the ideal consisting of those $h(x_1, \ldots, x_n)$ such that $h(b_1, \ldots, b_n) \in \mathfrak{b}$. Then we get a continuous surjection of topological R-algebras

$$\Phi : R[x_1, \ldots, x_n]^\wedge \longrightarrow B, \quad x_j \longmapsto b_j$$

where the completion on the left hand side is with respect to J. Since $R[x_1, \ldots, x_n]$ is Noetherian we can choose generators $h_1, \ldots, h_m$ for J. By the commutativity of the square above we see that $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ is an element of $A_\mu^\wedge$ whose image in $A^\wedge$ is contained in $IA^\wedge$. Namely, the ring map $(\hat{x})^\sharp$ is continuous and

$IA^\wedge$ is the ideal of topological nilpotent elements of $A^\wedge$ because $A^\wedge/IA^\wedge = A/I$ is reduced. (See Algebra, Section 10.97 for results on completion in Noetherian rings.) Since $A/I = \text{colim}\, A_\mu/I_\mu$ we conclude that after increasing μ we may assume $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ is in $I_\mu A_\mu^\wedge$. In particular the elements $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ of $A_\mu^\wedge$ are topologically nilpotent in $A_\mu^\wedge$. Thus we obtain a continuous R-algebra homomorphism

$$\Psi : R[x_1, \ldots, x_n]^\wedge \longrightarrow A_\mu^\wedge, \quad x_j \longmapsto a_{\mu,j}$$

In order to conclude what we want, we need to see if $\text{Ker}(\Phi)$ is annihilated by Ψ. This may not be true, but we can achieve this after increasing μ. Indeed, since $R[x_1, \ldots, x_n]^\wedge$ is Noetherian, we can choose generators $g_1, \ldots, g_l$ of the ideal $\text{Ker}(\Phi)$. Then we see that

$$\Psi(g_1), \ldots, \Psi(g_l) \in \text{Ker}(A_\mu^\wedge \to C_\mu^\wedge) = \text{Ker}(A_\mu \to C_\mu)$$

map to zero in $\text{Ker}(A \to C) = \text{colim}\,\text{Ker}(A_\mu \to C_\mu)$. Hence increasing μ as before we get the desired result.

Proof of surjectivity: mopping up. The continuous ring homomorphism $B \to (A_\mu)^\wedge$ constructed above determines a morphism $\hat{x}_\mu : V_{\mu,/Z_\mu} \to W$. The compatibility of $\hat{x}_\mu$ and u'_μ follows from the fact that the ring map $B \to (A_\mu)^\wedge$ is by construction compatible with the ring map $A_\mu \to C_\mu$. In fact, the compatibility will be witnessed by the proper morphism $V'_\mu \to V_\mu$ and the morphisms x'_μ and $\hat{x}'_\mu = x'_{\mu,/Z_\mu}$ we used in the construction. This finishes the proof. $\square$

0GI6 Lemma 98.27.12. In Situation 98.27.1 the functor F satisfies the Rim-Schlessinger condition (RS).

Proof. Recall that the condition only involves the evaluation $F(V)$ of the functor F on schemes V over S which are spectra of Artinian local rings and the restriction maps $F(V_2) \to F(V_1)$ for morphisms $V_1 \to V_2$ of schemes over S which are spectra of Artinian local rings. Thus let V/S be the spetruim of an Artinian local ring. If $\xi = (Z, u', \hat{x}) \in F(V)$ then either $Z = \emptyset$ or $Z = V$ (set theoretically). In the first case we see that $\hat{x}$ is a morphism from the empty formal algebraic space into W. In the second case we see that u' is a morphism from the empty scheme into X' and we see that $\hat{x} : V \to W$ is a morphism into W. We conclude that

$$F(V) = U'(V) \amalg W(V)$$

and moreover for $V_1 \to V_2$ as above the induced map $F(V_2) \to F(V_1)$ is compatible with this decomposition. Hence it suffices to prove that both U' and W satisfy the Rim-Schlessinger condition. For U' this follows from Lemma 98.5.2. To see that it is true for W, we write $W = \text{colim}\, W_n$ as in Formal Spaces, Lemma 87.20.11. Say $V = \text{Spec}(A)$ with $(A, \mathfrak{m})$ an Artinian local ring. Pick $n \geq 1$ such that $\mathfrak{m}^n = 0$. Then we have $W(V) = W_n(V)$. Hence we see that the Rim-Schlessinger condition for W follows from the Rim-Schlessinger condition for W_n for all n (which in turn follows from Lemma 98.5.2). $\square$

0GI7 Lemma 98.27.13. In Situation 98.27.1 the tangent spaces of the functor F are finite dimensional.

Proof. In the proof of Lemma 98.27.12 we have seen that $F(V) = U'(V) \amalg W(V)$ if V is the spectrum of an Artinian local ring. The tangent spaces are computed entirely from evaluations of F on such schemes over S. Hence it suffices to prove that the tangent spaces of the functors U' and W are finite dimensional. For U' this follows from Lemma 98.8.1. Write $W = \mathrm{colim}\, W_n$ as in the proof of Lemma 98.27.12. Then we see that the tangent spaces of W are equal to the tangent spaces of W_2, as to get at the tangent space we only need to evaluate W on spectra of Artinian local rings $(A, \mathfrak{m})$ with $\mathfrak{m}^2 = 0$. Then again we see that the tangent spaces of W_2 have finite dimension by Lemma 98.8.1. □

0GI8 Lemma 98.27.14. In Situation 98.27.1 assume $X' \to S$ is separated. Then every formal object for F is effective.

Proof. A formal object $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\mathrm{Spec}(R/\mathfrak{m}^n))$ for all n such that $\xi_{n+1}|_{\mathrm{Spec}(R/\mathfrak{m}^n)} = \xi_n$. By the discussion in the proof of Lemma 98.27.12 we see that either ξ is a formal object of U' or a formal object of W. In the first case we see that ξ is effective by Lemma 98.9.5. The second case is the interesting case. Set $V = \mathrm{Spec}(R)$. We will construct an element $(Z, u', \hat{x}) \in F(V)$ whose image in $F(\mathrm{Spec}(R/\mathfrak{m}^n))$ is ξ_n for all $n \geq 1$.

We may view the collection of elements ξ_n as a morphism

$$\xi : \mathrm{Spf}(R) \longrightarrow W$$

of locally Noetherian formal algebraic spaces over S. Observe that ξ is not an adic morphism in general. To fix this, let $I \subset R$ be the ideal corresponding to the formal closed subspace

$$\mathrm{Spf}(R) \times_{\xi, W} W_{red} \subset \mathrm{Spf}(R)$$

Note that $I \subset \mathfrak{m}_R$. Set $Z = V(I) \subset V = \mathrm{Spec}(R)$. Since R is $\mathfrak{m}_R$-adically complete it is a fortiori I-adically complete (Algebra, Lemma 10.96.8). Moreover, we claim that for each $n \geq 1$ the morphism

$$\xi|_{\mathrm{Spf}(R/I^n)} : \mathrm{Spf}(R/I^n) \longrightarrow W$$

actually comes from a morphism

$$\xi'_n : \mathrm{Spec}(R/I^n) \longrightarrow W$$

Namely, this follows from writing $W = \mathrm{colim}\, W_n$ as in the proof of Lemma 98.27.12, noticing that $\xi|_{\mathrm{Spf}(R/I^n)}$ maps into W_n, and applying Formal Spaces, Lemma 87.33.3 to algebraize this to a morphism $\mathrm{Spec}(R/I^n) \to W_n$ as desired. Let us denote $\mathrm{Spf}'(R) = V_{/Z}$ the formal spectrum of R endowed with the I-adic topology – equivalently the formal completion of V along Z. Using the morphisms ξ'_n we obtain an adic morphism

$$\hat{x} = (\xi'_n) : \mathrm{Spf}'(R) \longrightarrow W$$

of locally Noetherian formal algebraic spaces over S. Consider the base change

$$\mathrm{Spf}'(R) \times_{\hat{x}, W, g} X'_{/T'} \longrightarrow \mathrm{Spf}'(R)$$

This is a formal modification by Algebraization of Formal Spaces, Lemma 88.24.4. Hence by the main theorem on dilatations (Algebraization of Formal Spaces, Theorem 88.29.1) we obtain a proper morphism

$$V' \longrightarrow V = \mathrm{Spec}(R)$$

which is an isomorphism over $\text{Spec}(R) \setminus V(I)$ and whose completion recovers the formal modification above, in other words

$$V' \times_{\text{Spec}(R)} \text{Spec}(R/I^n) = \text{Spec}(R/I^n) \times_{\xi'_n, W, g} X'_{/T'}$$

This in particular tells us we have a compatible system of morphisms

$$V' \times_{\text{Spec}(R)} \text{Spec}(R/I^n) \longrightarrow X' \times_S \text{Spec}(R/I^n)$$

Hence by Grothendieck's algebraization theorem (in the form of More on Morphisms of Spaces, Lemma 76.43.3) we obtain a morphism

$$x' : V' \to X'$$

over S recovering the morphisms displayed above. Then finally setting $u' : V \setminus Z \to X'$ the restriction of x' to $V \setminus Z \subset V'$ gives the third component of our desired element $(Z, u', \hat{x}) \in F(V)$. $\square$

0GI9 Lemma 98.27.15. Let S be a locally Noetherian scheme. Let V be a scheme locally of finite type over S. Let $Z \subset V$ be closed. Let W be a locally Noetherian formal algebraic space over S such that W_{red} is locally of finite type over S. Let $g : V_{/Z} \to W$ be an adic morphism of formal algebraic spaces over S. Let $v \in V$ be a closed point such that g is versal at v (as in Section 98.15). Then after replacing V by an open neighbourhood of v the morphism g is smooth (see proof).

Proof. Since g is adic it is representable by algebraic spaces (Formal Spaces, Section 87.23). Thus by saying g is smooth we mean that g should be smooth in the sense of Bootstrap, Definition 80.4.1.

Write $W = \text{colim}\, W_n$ as in Formal Spaces, Lemma 87.20.11. Set $V_n = V_{/Z} \times_{\hat{x}, W} W_n$. Then V_n is a closed subscheme with underlying set Z. Smoothness of $V \to W$ is equivalent to the smoothness of all the morphisms $V_n \to W_n$ (this holds because any morphism $T \to W$ with T a quasi-compact scheme factors through W_n for some n). We know that the morphism $V_n \to W_n$ is smooth at v by Lemma 98.12.6[15]. Of course this means that given any n we can shrink V such that $V_n \to W_n$ is smooth. The problem is to find an open which works for all n at the same time.

The question is local on V, hence we may assume $S = \text{Spec}(R)$ and $V = \text{Spec}(A)$ are affine.

In this paragraph we reduce to the case where W is an affine formal algebraic space. Choose an affine formal scheme W' and an étale morphism $W' \to W$ such that the image of v in W_{red} is in the image of $W'_{red} \to W_{red}$. Then $V_{/Z} \times_{g,W} W' \to V_{/Z}$ is an adic étale morphism of formal algebraic spaces over S and $V_{/Z} \times_{g,W} W'$ is an affine fromal algebraic space. By Algebraization of Formal Spaces, Lemma 88.25.1 there exists an étale morphism $\varphi : V' \to V$ of affine schemes such that the completion of V' along $Z' = \varphi^{-1}(Z)$ is isomorphic to $V_{/Z} \times_{g,W} W'$ over $V_{/Z}$. Observe that v is the image of some $v' \in V'$. Since smoothness is preserved under base change we see that $V'_n \to W'_n$ is smooth for all n. In the next paragraph we show that after replacing V' by an open neighbourhood of v' the morphisms $V'_n \to W'_n$ are smooth for all n. Then, after we replace V by the open image of $V' \to V$, we obtain that $V_n \to W_n$ is smooth by étale descent of smoothness. Some details omitted.

[15]The lemma applies since the diagonal of W is representable by algebraic spaces and locally of finite type, see Formal Spaces, Lemma 87.15.5 and we have seen that W has (RS) in the proof of Lemma 98.27.12.

Assume $S = \text{Spec}(R)$, $V = \text{Spec}(A)$, $Z = V(I)$, and $W = \text{Spf}(B)$. Let v correspond to the maximal ideal $I \subset \mathfrak{m} \subset A$. We are given an adic continuous R-algebra homomorphism

$$B \longrightarrow A^\wedge$$

Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements (this is the maximal ideal of definition of the Noetherian adic topological ring B). Observe that $\mathfrak{b}A^\wedge$ and $IA^\wedge$ are both ideals of definition of the Noetherian adic ring $A^\wedge$. Also, $\mathfrak{m}A^\wedge$ is a maximal ideal of $A^\wedge$ containing both $\mathfrak{b}A^\wedge$ and $IA^\wedge$. We are given that

$$B_n = B/\mathfrak{b}^n \to A^\wedge/\mathfrak{b}^n A^\wedge = A_n$$

is smooth at $\mathfrak{m}$ for all n. By the discussion above we may and do assume that $B_1 \to A_1$ is a smooth ring map. Denote $\mathfrak{m}_1 \subset A_1$ the maximal ideal corresponing to $\mathfrak{m}$. Since smoothness implies flatness, we see that: for all $n \geq 1$ the map

$$\mathfrak{b}^n/\mathfrak{b}^{n+1} \otimes_{B_1} (A_1)_{\mathfrak{m}_1} \longrightarrow \left(\mathfrak{b}^n A^\wedge/\mathfrak{b}^{n+1} A^\wedge\right)_{\mathfrak{m}_1}$$

is an isomorphism (see Algebra, Lemma 10.99.9). Consider the Rees algebra

$$B' = \bigoplus\nolimits_{n \geq 0} \mathfrak{b}^n/\mathfrak{b}^{n+1}$$

which is a finite type graded algebra over the Noetherian ring B_1 and the Rees algebra

$$A' = \bigoplus\nolimits_{n \geq 0} \mathfrak{b}^n A^\wedge/\mathfrak{b}^{n+1} A^\wedge$$

which is a a finite type graded algebra over the Noetherian ring A_1. Consider the homomorphism of graded A_1-algebras

$$\Psi : B' \otimes_{B_1} A_1 \longrightarrow A'$$

By the above this map is an isomorphism after localizing at the maximal ideal $\mathfrak{m}_1$ of A_1. Hence $\text{Ker}(\Psi)$, resp. $\text{Coker}(\Psi)$ is a finite module over $B' \otimes_{B_1} A_1$, resp. A' whose localization at $\mathfrak{m}_1$ is zero. It follows that after replacing A_1 (and correspondingly A) by a principal localization we may assume Ψ is an isomorphism. (This is the key step of the proof.) Then working backwards we see that $B_n \to A_n$ is flat, see Algebra, Lemma 10.99.9. Hence $A_n \to B_n$ is smooth (as a flat ring map with smooth fibres, see Algebra, Lemma 10.137.17) and the proof is complete. □

0GIA Lemma 98.27.16. In Situation 98.27.1 the functor F satisfies openness of versality.

Proof. We have to show the following. Given a scheme V locally of finite type over S, given $\xi \in F(V)$, and given a finite type point $v_0 \in V$ such that ξ is versal at v_0, after replacing V by an open neighbourhood of v_0 we have that ξ is versal at every finite type point of V. Write $\xi = (Z, u', \hat{x})$.

First case: $v_0 \notin Z$. Then we can first replace V by $V \setminus Z$. Hence we see that $\xi = (\emptyset, u', \emptyset)$ and the morphism $u' : V \to X'$ is versal at v_0. By More on Morphisms of Spaces, Lemma 76.20.1 this means that $u' : V \to X'$ is smooth at v_0. Since the set of a points where a morphism is smooth is open, we can after shrinking V assume u' is smooth. Then the same lemma tells us that ξ is versal at every point as desired.

Second case: $v_0 \in Z$. Write $W = \text{colim}\, W_n$ as in Formal Spaces, Lemma 87.20.11. By Lemma 98.27.15 we may assume $\hat{x} : V_{/Z} \to W$ is a smooth morphism of formal algebraic spaces. It follows immediately that $\xi = (Z, u', \hat{x})$ is versal at all finite type points of Z. Let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and

$\hat{x}$. We see that $\hat{x}' : V'_{/Z} \to X'_{/T'}$ is smooth as a base change of $\hat{x}$. Since $\hat{x}'$ is the completion of $x' : V' \to X'$ this implies that $x' : V' \to X'$ is smooth at all points of $(V' \to V)^{-1}(Z) = |x'|^{-1}(T') \subset |V'|$ by the already used More on Morphisms of Spaces, Lemma 76.20.1. Since the set of smooth points of a morphism is open, we see that the closed set of points $B \subset |V'|$ where x' is not smooth does not meet $(V' \to V)^{-1}(Z)$. Since $V' \to V$ is proper and hence closed, we see that $(V' \to V)(B) \subset V$ is a closed subset not meeting Z. Hence after shrinking V we may assume $B = \emptyset$, i.e., x' is smooth. By the discussion in the previous paragraph this exactly means that ξ is versal at all finite type points of V not contained in Z and the proof is complete. □

Here is the final result.

0GIB Theorem 98.27.17. Let S be a locally Noetherian scheme such that $\mathcal{O}_{S,s}$ is a G-ring for all finite type points $s \in S$. Let X' be an algebraic space locally of finite type over S. Let $T' \subset |X'|$ be a closed subset. Let W be a locally Noetherian formal algebraic space over S with W_{red} locally of finite type over S. Finally, we let [Art70, Theorem 3.1]

$$g : X'_{/T'} \longrightarrow W$$

be a formal modification, see Algebraization of Formal Spaces, Definition 88.24.1. If X' and W are separated[16] over S, then there exists a proper morphism $f : X' \to X$ of algebraic spaces over S, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ of formal algebraic spaces such that

(1) T' is the inverse image of T by $|f| : |X'| \to |X|$,
(2) $f : X' \to X$ maps $X' \setminus T'$ isomorphically to $X \setminus T$, and
(3) $g = a \circ f_{/T}$ where $f_{/T} : X'_{/T'} \to X_{/T}$ is the induced morphism.

In other words, $(f : X' \to X, T, a)$ is a solution as defined earlier in this section.

Proof. Let F be the functor constructed using X', T', W, g in this section. By Lemma 98.27.6 it suffices to show that F corresponds to an algebraic space X locally of finite type over S. In order to do this, we will apply Proposition 98.26.1. Namely, by Lemma 98.27.9 the diagonal of F is representable by closed immersions and by Lemmas 98.27.10, 98.27.11, 98.27.12, 98.27.13, 98.27.14, and 98.27.16 we have axioms [0], [1], [2], [3], [4], and [5]. □

0GIC Remark 98.27.18. The proof of Theorem 98.27.17 uses that X' and W are separated over S in two places. First, the proof uses this in showing $\Delta : F \to F \times F$ is representable by algebraic spaces. This use of the assumption can be entirely avoided by proving that Δ is representable by applying the theorem in the separated case to the triples E', $(E' \to V)^{-1}Z$, and $E'_{/Z} \to E_W$ found in Remark 98.27.7 (this is the usual bootstrap procedure for the diagonal). Thus the proof of Lemma 98.27.14 is the only place in our proof of Theorem 98.27.17 where we really need to use that $X' \to S$ is separated. The reader checks that we use the assumption only to obtain the morphism $x' : V' \to X'$. The existence of x' can be shown, using results in the literature, if $X' \to S$ is quasi-separated, see More on Morphisms of Spaces, Remark 76.43.4. We conclude the theorem holds as stated with "separated" replaced by "quasi-separated". If we ever need this we will precisely state and carefully prove this here.

[16] See Remark 98.27.18.

98.28. Other chapters

CHAPTER 99

Quot and Hilbert Spaces

05X4 99.1. Introduction

05X5 As initially conceived, the purpose of this chapter was to write about Quot and Hilbert functors and to prove that these are algebraic spaces provided certain technical conditions are satisfied. This material, in the setting of schemes, is covered in Grothendieck's lectures in the séminair Bourbaki, see [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. For projective schemes the Quot and Hilbert schemes live inside Grassmannians of spaces of sections of suitable very ample invertible sheaves, and this provides a method of construction for these schemes. Our approach is different: we use Artin's axioms to prove Quot and Hilb are algebraic spaces.

Upon further consideration, it turned out to be more convenient for the development of theory in the Stacks project, to start the discussion with the stack $\mathcal{C}oh_{X/B}$ of coherent sheaves (with proper support over the base) as introduced in [Lie06b]. For us $f : X \to B$ is a morphism of algebraic spaces satisfying suitable technical conditions, although this can be generalized (see below). Given modules $\mathcal{F}$ and $\mathcal{G}$ on X, under suitably hypotheses, the functor $T/B \mapsto \mathrm{Hom}_{X_T}(\mathcal{F}_T, \mathcal{G}_T)$ is an algebraic space $\mathit{Hom}(\mathcal{F}, \mathcal{G})$ over B. See Section 99.3. The subfunctor $\mathit{Isom}(\mathcal{F}, \mathcal{G})$ of isomorphisms is shown to be an algebraic space in Section 99.4. This is used in the next sections to show the diagonal of the stack $\mathcal{C}oh_{X/B}$ is representable. We prove $\mathcal{C}oh_{X/B}$ is an algebraic stack in Section 99.5 when $X \to B$ is flat and in Section 99.6 in general. Please see the introduction of this section for pointers to the literature.

Having proved this, it is rather straightforward to prove that $\mathrm{Quot}_{\mathcal{F}/X/B}$, $\mathrm{Hilb}_{X/B}$, and $\mathrm{Pic}_{X/B}$ are algebraic spaces and that $\mathcal{P}ic_{X/B}$ is an algebraic stack. See Sections 99.8, 99.9, 99.11, and 99.10.

In the usual manner we deduce that the functor $\mathit{Mor}_B(Z, X)$ of relative morphisms is an algebraic space (under suitable hypotheses) in Section 99.12.

In Section 99.13 we prove that the stack in groupoids

$$\mathcal{S}paces'_{fp,flat,proper}$$

parametrizing flat families of proper algebraic spaces satisfies all of Artin's axioms (including openness of versality) except for formal effectiveness. We've chosen the very awkward notation for this stack intentionally, because the reader should be carefull in using its properties.

In Section 99.14 we prove that the stack $\mathcal{P}olarized$ parametrizing flat families of polarized proper algebraic spaces is an algebraic stack. Because of our work on flat families of proper algebraic spaces, this comes down to proving formal effectiveness for polarized schemes which is often known as Grothendieck's algebraization theorem.

In Section 99.15 we prove that the stack *Curves* parametrizing families of curves is algebraic.

In Section 99.16 we study moduli of complexes on a proper morphism and we obtain an algebraic stack $\mathit{Complexes}_{X/B}$. The idea of the statement and the proof are taken from [Lie06a].

What is not in this chapter? There is almost no discussion of the properties the resulting moduli spaces and moduli stacks possess (beyond their algebraicity); to read about this we refer to Moduli Stacks, Section 108.1. In most of the results discussed, we can generalize the constructions by considering a morphism $\mathcal{X} \to \mathcal{B}$ of algebraic stacks instead of a morphism $X \to B$ of algebraic space. We will discuss this (insert future reference here). In the case of Hilbert spaces there is a more general notion of "Hilbert stacks" which we will discuss in a separate chapter, see (insert future reference here).

99.2. Conventions

05X6 We have intentionally placed this chapter, as well as the chapters "Examples of Stacks", "Sheaves on Algebraic Stacks", "Criteria for Representability", and "Artin's Axioms" before the general development of the theory of algebraic stacks. The reason for this is that starting with the next chapter (see Properties of Stacks, Section 100.2) we will no longer distinguish between a scheme and the algebraic stack it gives rise to. Thus our language will become more flexible and easier for a human to parse, but also less precise. These first few chapters, including the initial chapter "Algebraic Stacks", lay the groundwork that later allow us to ignore some of the very technical distinctions between different ways of thinking about algebraic stacks. But especially in the chapters "Artin's Axioms" and "Criteria of Representability" we need to be very precise about what objects exactly we are working with, as we are trying to show that certain constructions produce algebraic stacks or algebraic spaces.

Unfortunately, this means that some of the notation, conventions and terminology is awkward and may seem backwards to the more experienced reader. We hope the reader will forgive us!

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

99.3. The Hom functor

08JS In this section we study the functor of homomorphisms defined below.

08JT Situation 99.3.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $\mathcal{F}_T$ and $\mathcal{G}_T$ the base changes of $\mathcal{F}$ and $\mathcal{G}$ to T, in other words, the pullbacks via the projection morphism $X_T = X \times_B T \to X$. We consider the functor

08JU (99.3.1.1) $$\mathit{Hom}(\mathcal{F}, \mathcal{G}) : (\mathit{Sch}/B)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T)$$

In Situation 99.3.1 we sometimes think of $\mathit{Hom}(\mathcal{F}, \mathcal{G})$ as a functor $(\mathit{Sch}/S)^{opp} \to$ Sets endowed with a morphism $\mathit{Hom}(\mathcal{F}, \mathcal{G}) \to B$. Namely, if T is a scheme over S, then an element of $\mathit{Hom}(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u), where h is a morphism $h : T \to B$ and $u : \mathcal{F}_T \to \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$-module map where $X_T = T \times_{h,B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T. In particular, when we say that $\mathit{Hom}(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(\mathit{Sch}/S)^{opp} \to$ Sets is an algebraic space.

08JV Lemma 99.3.2. In Situation 99.3.1 the functor $\mathit{Hom}(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = u_{T_i}$ and $\mathcal{G}_i = \mathcal{G}_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 73.9.3. Thus a family of maps $u_i : \mathcal{F}_i \to \mathcal{G}_i$ such that u_i and u_j restrict to the same map on $X_{T_i \times_T T_j}$ comes from a unique map $u : \mathcal{F}_T \to \mathcal{G}_T$ by descent (Descent on Spaces, Proposition 74.4.1). □

Sanity check: Hom sheaf plays the same role among algebraic spaces over S.

0D3S Lemma 99.3.3. In Situation 99.3.1. Let T be an algebraic space over S. We have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathit{Hom}(\mathcal{F}, \mathcal{G})) = \{(h, u) \mid h : T \to B, u : \mathcal{F}_T \to \mathcal{G}_T\}$$

where $\mathcal{F}_T, \mathcal{G}_T$ denote the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the algebraic space $X \times_{B,h} T$.

Proof. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $t, s : R \to U$.

Let $v : T \to \mathit{Hom}(\mathcal{F}, \mathcal{G})$ be a natural transformation. Then $v(p)$ corresponds to a pair (h_U, u_U) over U. As v is a transformation of functors we see that the pullbacks of (h_U, u_U) by s and t agree. Since $T = U/R$ (Spaces, Lemma 65.9.1), we obtain a morphism $h : T \to B$ such that $h_U = h \circ p$. Then $\mathcal{F}_U$ is the pullback of $\mathcal{F}_T$ to X_U and similarly for $\mathcal{G}_U$. Hence u_U descends to a $\mathcal{O}_{X_T}$-module map $u : \mathcal{F}_T \to \mathcal{G}_T$ by Descent on Spaces, Proposition 74.4.1.

Conversely, let (h, u) be a pair over T. Then we get a natural transformation $v : T \to \mathit{Hom}(\mathcal{F}, \mathcal{G})$ by sending a morphism $a : T' \to T$ where T' is a scheme to $(h \circ a, a^*u)$. We omit the verification that the construction of this and the previous paragraph are mutually inverse. □

08JW Remark 99.3.4. In Situation 99.3.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$, $\mathcal{G}'$ the pullback of $\mathcal{F}$, $\mathcal{G}$ to X'. Then we obtain a functor $\mathit{Hom}(\mathcal{F}', \mathcal{G}') : (\mathit{Sch}/B')^{opp} \to$ Sets associated to the base change $f' : X' \to B'$. For a scheme T over B' it is clear that we have

$$\mathit{Hom}(\mathcal{F}', \mathcal{G}')(T) = \mathit{Hom}(\mathcal{F}, \mathcal{G})(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. This trivial remark will occasionally be useful to change the base algebraic space.

08K3 Lemma 99.3.5. In Situation 99.3.1 let $\{X_i \to X\}_{i \in I}$ be an fppf covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. Similarly define $\mathcal{G}_i$ and $\mathcal{G}_{ijk}$. For every scheme T over B the diagram

$$\begin{tikzcd} Hom(\mathcal{F}, \mathcal{G})(T) \arrow[r] & \prod_i Hom(\mathcal{F}_i, \mathcal{G}_i)(T) \arrow[r, shift left, "\mathrm{pr}_0^*"] \arrow[r, shift right, "\mathrm{pr}_1^*"'] & \prod_{i,j,k} Hom(\mathcal{F}_{ijk}, \mathcal{G}_{ijk})(T) \end{tikzcd}$$

presents the first arrow as the equalizer of the other two.

Proof. Let $u_i : \mathcal{F}_{i,T} \to \mathcal{G}_{i,T}$ be an element in the equalizer of pr_0^* and pr_1^*. Since the base change of an fppf covering is an fppf covering (Topologies on Spaces, Lemma 73.7.3) we see that $\{X_{i,T} \to X_T\}_{i \in I}$ and $\{X_{ijk,T} \to X_{i,T} \times_{X_T} X_{j,T}\}$ are fppf coverings. Applying Descent on Spaces, Proposition 74.4.1 we first conclude that u_i and u_j restrict to the same morphism over $X_{i,T} \times_{X_T} X_{j,T}$, whereupon a second application shows that there is a unique morphism $u : \mathcal{F}_T \to \mathcal{G}_T$ restricting to u_i for each i. This finishes the proof. $\square$

08K4 Lemma 99.3.6. In Situation 99.3.1. If $\mathcal{F}$ is of finite presentation and f is quasi-compact and quasi-separated, then $Hom(\mathcal{F}, \mathcal{G})$ is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes. We have to show that

$$Hom(\mathcal{F}, \mathcal{G})(T) = \mathrm{colim}\, Hom(\mathcal{F}, \mathcal{G})(T_i)$$

Pick $0 \in I$. We may replace B by T_0, X by X_{T_0}, $\mathcal{F}$ by $\mathcal{F}_{T_0}$, $\mathcal{G}$ by $\mathcal{G}_{T_0}$, and I by $\{i \in I \mid i \geq 0\}$. See Remark 99.3.4. Thus we may assume $B = \mathrm{Spec}(R)$ is affine.

When B is affine, then X is quasi-compact and quasi-separated. Choose a surjective étale morphism $U \to X$ where U is an affine scheme (Properties of Spaces, Lemma 66.6.3). Since X is quasi-separated, the scheme $U \times_X U$ is quasi-compact and we may choose a surjective étale morphism $V \to U \times_X U$ where V is an affine scheme. Applying Lemma 99.3.5 we see that $Hom(\mathcal{F}, \mathcal{G})$ is the equalizer of two maps between

$$Hom(\mathcal{F}|_U, \mathcal{G}|_U) \quad \text{and} \quad Hom(\mathcal{F}|_V, \mathcal{G}|_V)$$

This reduces us to the case that X is affine.

In the affine case the statement of the lemma reduces to the following problem: Given a ring map $R \to A$, two A-modules M, N and a directed system of R-algebras $C = \mathrm{colim}\, C_i$. When is it true that the map

$$\mathrm{colim}\, \mathrm{Hom}_{A \otimes_R C_i}(M \otimes_R C_i, N \otimes_R C_i) \longrightarrow \mathrm{Hom}_{A \otimes_R C}(M \otimes_R C, N \otimes_R C)$$

is bijective? By Algebra, Lemma 10.127.5 this holds if $M \otimes_R C$ is of finite presentation over $A \otimes_R C$, i.e., when M is of finite presentation over A. $\square$

08K5 Lemma 99.3.7. Let S be a scheme. Let B be an algebraic space over S. Let $i : X' \to X$ be a closed immersion of algebraic spaces over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\mathcal{G}'$ be a quasi-coherent $\mathcal{O}_{X'}$-module. Then

$$Hom(\mathcal{F}, i_*\mathcal{G}') = Hom(i^*\mathcal{F}, \mathcal{G}')$$

as functors on (Sch/B).

Proof. Let $g : T \to B$ be a morphism where T is a scheme. Denote $i_T : X'_T \to X_T$ the base change of i. Denote $h : X_T \to X$ and $h' : X'_T \to X'$ the projections. Observe that $(h')^*i^*\mathcal{F} = i_T^*h^*\mathcal{F}$. As a closed immersion is affine (Morphisms of Spaces, Lemma 67.20.6) we have $h^*i_*\mathcal{G} = i_{T,*}(h')^*\mathcal{G}$ by Cohomology of Spaces, Lemma 69.11.1. Thus we have

$$\begin{aligned} \mathit{Hom}(\mathcal{F}, i_*\mathcal{G}')(T) &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, h^*i_*\mathcal{G}') \\ &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, i_{T,*}(h')^*\mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{O}_{X'_T}}(i_T^*h^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{O}_{X'_T}}((h')^*i^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= \mathit{Hom}(i^*\mathcal{F}, \mathcal{G}')(T) \end{aligned}$$

as desired. The middle equality follows from the adjointness of the functors $i_{T,*}$ and i_T^*. $\square$

08JX Lemma 99.3.8. Let S be a scheme. Let B be an algebraic space over S. Let K be a pseudo-coherent object of $D(\mathcal{O}_B)$.

(1) If for all $g : T \to B$ in (Sch/B) the cohomology sheaf $H^{-1}(Lg^*K)$ is zero, then the functor

$$(Sch/B)^{opp} \longrightarrow \mathrm{Sets}, \quad (g : T \to B) \longmapsto H^0(T, H^0(Lg^*K))$$

is an algebraic space affine and of finite presentation over B.

(2) If for all $g : T \to B$ in (Sch/B) the cohomology sheaves $H^i(Lg^*K)$ are zero for $i < 0$, then K is perfect, K locally has tor amplitude in $[0, b]$, and the functor

$$(Sch/B)^{opp} \longrightarrow \mathrm{Sets}, \quad (g : T \to B) \longmapsto H^0(T, Lg^*K)$$

is an algebraic space affine and of finite presentation over B.

Proof. Under the assumptions of (2) we have $H^0(T, Lg^*K) = H^0(T, H^0(Lg^*K))$. Let us prove that the rule $T \mapsto H^0(T, H^0(Lg^*K))$ satisfies the sheaf property for the fppf topology. To do this assume we have an fppf covering $\{h_i : T_i \to T\}$ of a scheme $g : T \to B$ over B. Set $g_i = g \circ h_i$. Note that since h_i is flat, we have $Lh_i^* = h_i^*$ and h_i^* commutes with taking cohomology. Hence

$$H^0(T_i, H^0(Lg_i^*K)) = H^0(T_i, H^0(h_i^*Lg^*K)) = H^0(T, h_i^*H^0(Lg^*K))$$

Similarly for the pullback to $T_i \times_T T_j$. Since Lg^*K is a pseudo-coherent complex on T (Cohomology on Sites, Lemma 21.45.3) the cohomology sheaf $\mathcal{F} = H^0(Lg^*K)$ is quasi-coherent (Derived Categories of Spaces, Lemma 75.13.6). Hence by Descent on Spaces, Proposition 74.4.1 we see that

$$H^0(T, \mathcal{F}) = \mathrm{Ker}(\prod H^0(T_i, h_i^*\mathcal{F}) \to \prod H^0(T_i \times_T T_j, (T_i \times_T T_j \to T)^*\mathcal{F}))$$

In this way we see that the rules in (1) and (2) satisfy the sheaf property for fppf coverings. This means we may apply Bootstrap, Lemma 80.11.2 to see it suffices to prove the representability étale locally on B. Moreover, we may check whether the end result is affine and of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 67.20.3 and 67.28.4. Hence we may assume that B is an affine scheme.

Assume $B = \mathrm{Spec}(A)$ is an affine scheme. By the results of Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, and 75.13.2 we deduce that in the rest of the proof we may think of K as a perfect object of the derived category of complexes of modules on B in the Zariski topology. By Derived Categories of Schemes, Lemmas 36.10.1, 36.3.5, and 36.10.2 we can find a pseudo-coherent complex $M^\bullet$ of A-modules such that K is the corresponding object of $D(\mathcal{O}_B)$. Our assumption on pullbacks implies that $M^\bullet \otimes_A^{\mathbf{L}} \kappa(\mathfrak p)$ has vanishing H^{-1} for all primes $\mathfrak p \subset A$. By More on Algebra, Lemma 15.76.4 we can write

$$M^\bullet = \tau_{\geq 0}M^\bullet \oplus \tau_{\leq -1}M^\bullet$$

with $\tau_{\geq 0}M^\bullet$ perfect with Tor amplitude in $[0, b]$ for some $b \geq 0$ (here we also have used More on Algebra, Lemmas 15.74.12 and 15.66.16). Note that in case (2) we also see that $\tau_{\leq -1}M^\bullet = 0$ in $D(A)$ whence $M^\bullet$ and K are perfect with tor amplitude in $[0, b]$. For any B-scheme $g : T \to B$ we have

$$H^0(T, H^0(Lg^*K)) = H^0(T, H^0(Lg^*\tau_{\geq 0}K))$$

(by the dual of Derived Categories, Lemma 13.16.1) hence we may replace K by $\tau_{\geq 0}K$ and correspondingly $M^\bullet$ by $\tau_{\geq 0}M^\bullet$. In other words, we may assume $M^\bullet$ has tor amplitude in $[0, b]$.

Assume $M^\bullet$ has tor amplitude in $[0, b]$. We may assume $M^\bullet$ is a bounded above complex of finite free A-modules (by our definition of pseudo-coherent complexes, see More on Algebra, Definition 15.64.1 and the discussion following the definition). By More on Algebra, Lemma 15.66.2 we see that $M = \mathrm{Coker}(M^{-1} \to M^0)$ is flat. By Algebra, Lemma 10.78.2 we see that M is finite locally free. Hence $M^\bullet$ is quasi-isomorphic to

$$M \to M^1 \to M^2 \to \ldots \to M^d \to 0 \ldots$$

Note that this is a K-flat complex (Cohomology, Lemma 20.26.9), hence derived pullback of K via a morphism $T \to B$ is computed by the complex

$$g^*\widetilde{M} \to g^*\widetilde{M^1} \to \ldots$$

Thus it suffices to show that the functor

$$(g : T \to B) \longmapsto \mathrm{Ker}(\Gamma(T, g^*\widetilde{M}) \to \Gamma(T, g^*(\widetilde{M^1}))$$

is representable by an affine scheme of finite presentation over B.

We may still replace B by the members of an affine open covering in order to prove this last statement. Hence we may assume that M is finite free (recall that M^1 is finite free to begin with). Write $M = A^{\oplus n}$ and $M^1 = A^{\oplus m}$. Let the map $M \to M^1$ be given by the $m \times n$ matrix (a_{ij}) with coefficients in A. Then $\widetilde{M} = \mathcal{O}_B^{\oplus n}$ and $\widetilde{M^1} = \mathcal{O}_B^{\oplus m}$. Thus the functor above is equal to the functor

$$(g : T \to B) \longmapsto \{(f_1, \ldots, f_n) \in \Gamma(T, \mathcal{O}_T) \mid \sum g^\sharp(a_{ij})f_i = 0,\ j = 1, \ldots, m\}$$

Clearly this is representable by the affine scheme

$$\mathrm{Spec}\left(A[x_1, \ldots, x_n]/(\sum a_{ij}x_i; j = 1, \ldots, m)\right)$$

and the lemma has been proved. $\square$

The functor $Hom(\mathcal{F}, \mathcal{G})$ is representable in a number of situations. All of our results will be based on the following basic case. The proof of this lemma as given below is in some sense the natural generalization to the proof of [DG67, III, Cor 7.7.8].

08JY Lemma 99.3.9. In Situation 99.3.1 assume that

(1) B is a Noetherian algebraic space,
(2) f is locally of finite type and quasi-separated,
(3) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and
(4) $\mathcal{G}$ is a finite type $\mathcal{O}_X$-module, flat over B, with support proper over B.

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space affine and of finite presentation over B.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to \mathcal{F}$ by a perfect complex P of the triple $(X, \mathcal{F}, -1)$, see Derived Categories of Spaces, Definition 75.14.1 and Theorem 75.14.7). Then the induced map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(P, \mathcal{G})$$

is an isomorphism because $P \to \mathcal{F}$ induces an isomorphism $H^0(P) \to \mathcal{F}$ and $H^i(P) = 0$ for $i > 0$. Moreover, for any morphism $g : T \to B$ denote $h : X_T = T \times_B X \to X$ the projection and set $P_T = Lh^*P$. Then it is equally true that

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T)$$

is an isomorphism, as $P_T = Lh^*P \to Lh^*\mathcal{F} \to \mathcal{F}_T$ induces an isomorphism $H^0(P_T) \to \mathcal{F}_T$ (because h^* is right exact and $H^i(P) = 0$ for $i > 0$). Thus it suffices to prove the result for the functor

$$T \longmapsto \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T).$$

By the Leray spectral sequence (see Cohomology on Sites, Remark 21.14.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T) = H^0(X_T, R\mathcal{H}om(P_T, \mathcal{G}_T)) = H^0(T, Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T))$$

where $f_T : X_T \to T$ is the base change of f. By Derived Categories of Spaces, Lemma 75.21.5 we have

$$Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T) = Lg^*Rf_*R\mathcal{H}om(P, \mathcal{G}).$$

By Derived Categories of Spaces, Lemma 75.22.3 the object $K = Rf_*R\mathcal{H}om(P, \mathcal{G})$ of $D(\mathcal{O}_B)$ is perfect. This means we can apply Lemma 99.3.8 as long as we can prove that the cohomology sheaf $H^i(Lg^*K)$ is 0 for all $i < 0$ and $g : T \to B$ as above. This is clear from the last displayed formula as the cohomology sheaves of $Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T)$ are zero in negative degrees due to the fact that $R\mathcal{H}om(P_T, \mathcal{G}_T)$ has vanishing cohomology sheaves in negative degrees as P_T is perfect with vanishing cohomology sheaves in positive degrees. $\square$

Here is a cheap consequence of Lemma 99.3.9.

08K6 Proposition 99.3.10. In Situation 99.3.1 assume that

(1) f is of finite presentation, and
(2) $\mathcal{G}$ is a finitely presented $\mathcal{O}_X$-module, flat over B, with support proper over B.

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $Hom(\mathcal{F}, \mathcal{G})$ is of finite presentation over B.

Proof. By Lemma 99.3.2 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for fppf coverings. This mean we may[1] apply Bootstrap, Lemma 80.11.1 to check the representability étale locally on B. Moreover, we may check whether the end result is affine or of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 67.20.3 and 67.28.4. Hence we may assume that B is an affine scheme.

Assume B is an affine scheme. As f is of finite presentation, it follows X is quasi-compact and quasi-separated. Thus we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of $\mathcal{O}_X$-modules of finite presentation (Limits of Spaces, Lemma 70.9.1). It is clear that

$$Hom(\mathcal{F}, \mathcal{G}) = \lim Hom(\mathcal{F}_i, \mathcal{G})$$

Hence if we can show that each $Hom(\mathcal{F}_i, \mathcal{G})$ is representable by an affine scheme, then we see that the same thing holds for $Hom(\mathcal{F}, \mathcal{G})$. Use the material in Limits, Section 32.2 and Limits of Spaces, Section 70.4. Thus we may assume that $\mathcal{F}$ is of finite presentation.

Say $B = \operatorname{Spec}(R)$. Write $R = \operatorname{colim} R_i$ with each R_i a finite type $\mathbf{Z}$-algebra. Set $B_i = \operatorname{Spec}(R_i)$. By the results of Limits of Spaces, Lemmas 70.7.1 and 70.7.2 we can find an i, a morphism of algebraic spaces $X_i \to B_i$, and finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{F}_i$ and $\mathcal{G}_i$ such that the base change of $(X_i, \mathcal{F}_i, \mathcal{G}_i)$ to B recovers $(X, \mathcal{F}, \mathcal{G})$. By Limits of Spaces, Lemma 70.6.12 we may, after increasing i, assume that $\mathcal{G}_i$ is flat over B_i. By Limits of Spaces, Lemma 70.12.3 we may similarly assume the scheme theoretic support of $\mathcal{G}_i$ is proper over B_i. At this point we can apply Lemma 99.3.9 to see that $H_i = Hom(\mathcal{F}_i, \mathcal{G}_i)$ is an algebraic space affine of finite presentation over B_i. Pulling back to B (using Remark 99.3.4) we see that $H_i \times_{B_i} B = Hom(\mathcal{F}, \mathcal{G})$ and we win. □

99.4. The Isom functor

08K7 In Situation 99.3.1 we can consider the subfunctor

$$Isom(\mathcal{F}, \mathcal{G}) \subset Hom(\mathcal{F}, \mathcal{G})$$

whose value on a scheme T over B is the set of invertible $\mathcal{O}_{X_T}$-homomorphisms $u : \mathcal{F}_T \to \mathcal{G}_T$.

We sometimes think of $Isom(\mathcal{F}, \mathcal{G})$ as a functor $(Sch/S)^{opp} \to$ Sets endowed with a morphism $Isom(\mathcal{F}, \mathcal{G}) \to B$. Namely, if T is a scheme over S, then an element of $Isom(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u), where h is a morphism $h : T \to B$ and $u : \mathcal{F}_T \to \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$-module isomorphism where $X_T = T \times_{h,B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T. In particular, when we say that $Isom(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to$ Sets is an algebraic space.

08K8 Lemma 99.4.1. In Situation 99.3.1 the functor $Isom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.

[1] We omit the verification of the set theoretical condition (3) of the referenced lemma.

Proof. We have already seen that $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property. Hence it remains to show the following: Given an fpqc covering $\{T_i \to T\}_{i \in I}$ of schemes over B and an $\mathcal{O}_{X_T}$-linear map $u : \mathcal{F}_T \to \mathcal{G}_T$ such that u_{T_i} is an isomorphism for all i, then u is an isomorphism. Since $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 73.9.3, this follows from Descent on Spaces, Proposition 74.4.1. □

Sanity check: *Isom* sheaf plays the same role among algebraic spaces over S.

0D3T Lemma 99.4.2. In Situation 99.3.1. Let T be an algebraic space over S. We have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Isom(\mathcal{F}, \mathcal{G})) = \{(h, u) \mid h : T \to B, u : \mathcal{F}_T \to \mathcal{G}_T \text{ isomorphism}\}$$

where $\mathcal{F}_T, \mathcal{G}_T$ denote the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the algebraic space $X \times_{B,h} T$.

Proof. Observe that the left and right hand side of the equality are subsets of the left and right hand side of the equality in Lemma 99.3.3. We omit the verification that these subsets correspond under the identification given in the proof of that lemma. □

08K9 Proposition 99.4.3. In Situation 99.3.1 assume that

(1) f is of finite presentation, and
(2) $\mathcal{F}$ and $\mathcal{G}$ are finitely presented $\mathcal{O}_X$-modules, flat over B, with support proper over B.

Then the functor $Isom(\mathcal{F}, \mathcal{G})$ is an algebraic space affine of finite presentation over B.

Proof. We will use the abbreviations $H = Hom(\mathcal{F}, \mathcal{G})$, $I = Hom(\mathcal{F}, \mathcal{F})$, $H' = Hom(\mathcal{G}, \mathcal{F})$, and $I' = Hom(\mathcal{G}, \mathcal{G})$. By Proposition 99.3.10 the functors H, I, H', I' are algebraic spaces and the morphisms $H \to B$, $I \to B$, $H' \to B$, and $I' \to B$ are affine and of finite presentation. The composition of maps gives a morphism

$$c : H' \times_B H \longrightarrow I \times_B I', \quad (u', u) \longmapsto (u \circ u', u' \circ u)$$

of algebraic spaces over B. Since $I \times_B I' \to B$ is separated, the section $\sigma : B \to I \times_B I'$ corresponding to $(\mathrm{id}_\mathcal{F}, \mathrm{id}_\mathcal{G})$ is a closed immersion (Morphisms of Spaces, Lemma 67.4.7). Moreover, σ is of finite presentation (Morphisms of Spaces, Lemma 67.28.9). Hence

$$Isom(\mathcal{F}, \mathcal{G}) = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

is an algebraic space affine of finite presentation over B as well. Some details omitted. □

99.5. The stack of coherent sheaves

08KA In this section we prove that the stack of coherent sheaves on X/B is algebraic under suitable hypotheses. This is a special case of [Lie06b, Theorem 2.1.1] which treats the case of the stack of coherent sheaves on an Artin stack over a base.

08KB Situation 99.5.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. We denote $\mathcal{C}oh_{X/B}$ the category whose objects are triples $(T, g, \mathcal{F})$ where

(1) T is a scheme over S,
(2) $g : T \to B$ is a morphism over S, and setting $X_T = T \times_{g,B} X$

(3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_T}$-module of finite presentation, flat over T, with support proper over T.

A morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ is given by a pair (h, φ) where

(1) $h : T \to T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
(2) $\varphi : (h')^*\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $\mathcal{O}_{X_T}$-modules where $h' : X_T \to X_{T'}$ is the base change of h.

Thus $\mathcal{C}oh_{X/B}$ is a category and the rule

$$p : \mathcal{C}oh_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (T, g, \mathcal{F}) \longmapsto T$$

is a functor. For a scheme T over S we denote $\mathcal{C}oh_{X/B,T}$ the fibre category of p over T. These fibre categories are groupoids.

08W5 Lemma 99.5.2. In Situation 99.5.1 the functor $p : \mathcal{C}oh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is fibred in groupoids.

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 4.35.1. Given an object $(T', g', \mathcal{F}')$ of $\mathcal{C}oh_{X/B}$ and a morphism $h : T \to T'$ of schemes over S we can set $g = h \circ g'$ and $\mathcal{F} = (h')^*\mathcal{F}'$ where $h' : X_T \to X_{T'}$ is the base change of h. Then it is clear that we obtain a morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ of $\mathcal{C}oh_{X/B}$ lying over h. This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, \mathcal{F}_1) \to (T, g, \mathcal{F}) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, \mathcal{F}_2) \to (T, g, \mathcal{F})$$

of $\mathcal{C}oh_{X/B}$ and a morphism $h : T_1 \to T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$(h')^*\mathcal{F}_2 \xrightarrow{(h')^*\varphi_2^{-1}} (h')^*(h_2)^*\mathcal{F} = (h_1)^*\mathcal{F} \xrightarrow{\varphi_1} \mathcal{F}_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, \mathcal{F}_1) \to (T_2, g_2, \mathcal{F}_2)$ that witnesses the truth of condition (2). □

08W6 Lemma 99.5.3. In Situation 99.5.1. Denote $\mathcal{X} = \mathcal{C}oh_{X/B}$. Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

Proof. Consider two objects $x = (T, g, \mathcal{F})$ and $y = (T, h, \mathcal{G})$ of $\mathcal{X}$ over a scheme T. We have to show that $Isom_{\mathcal{X}}(x, y)$ is an algebraic space over T, see Algebraic Stacks, Lemma 94.10.11. If for $a : T' \to T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ a = h \circ a$. Hence there is a transformation of presheaves

$$Isom_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, h)$$

Since the diagonal of B is representable (by schemes) this equalizer is a scheme. Thus we may replace T by this equalizer and the sheaves $\mathcal{F}$ and $\mathcal{G}$ by their pullbacks. Thus we may assume $g = h$. In this case we have $Isom_{\mathcal{X}}(x, y) = Isom(\mathcal{F}, \mathcal{G})$ and the result follows from Proposition 99.4.3. □

08KC Lemma 99.5.4. In Situation 99.5.1 the functor $p : \mathcal{C}oh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is a stack in groupoids.

Proof. To prove that $\mathcal{C}oh_{X/B}$ is a stack in groupoids, we have to show that the presheaves $Isom$ are sheaves and that descent data are effective. The statement on $Isom$ follows from Lemma 99.5.3, see Algebraic Stacks, Lemma 94.10.11. Let us prove the statement on descent data. Suppose that $\{a_i : T_i \to T\}$ is an fppf covering of schemes over S. Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \to T\}$ with values in $\mathcal{C}oh_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, \mathcal{F}_i)$. Denote $\text{pr}_0 : T_i \times_T T_j \to T_i$ and $\text{pr}_1 : T_i \times_T T_j \to T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} = \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \text{pr}_0 = g_j \circ \text{pr}_1$. Thus there exists a unique morphism $g : T \to B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 74.7.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i,B} X = T_i \times_{a_i,T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j. Observe that the pullback of $(T_i, g_i, \mathcal{F}_i)$ by $\text{pr}_0 : T_i \times_T T_j \to T_i$ is given by $(T_i \times_T T_j, g_i \circ \text{pr}_0, \text{pr}_i^*\mathcal{F}_i)$. Hence a descent datum for $\{T_i \to T\}$ in $\mathcal{C}oh_{X/B}$ is given by the objects $(T_i, g \circ a_i, \mathcal{F}_i)$ and for each pair i, j an isomorphism of $\mathcal{O}_{X_{ij}}$-modules

$$\varphi_{ij} : \text{pr}_i^*\mathcal{F}_i \longrightarrow \text{pr}_j^*\mathcal{F}_j$$

satisfying the cocycle condition over (the pullback of X to) $T_i \times_T T_j \times_T T_k$. Ok, and now we simply use that $\{X_i \to X_T\}$ is an fppf covering so that we can view $(\mathcal{F}_i, \varphi_{ij})$ as a descent datum for this covering. By Descent on Spaces, Proposition 74.4.1 this descent datum is effective and we obtain a quasi-coherent sheaf $\mathcal{F}$ over X_T restricting to $\mathcal{F}_i$ on X_i. By Morphisms of Spaces, Lemma 67.31.5 we see that $\mathcal{F}$ is flat over T and Descent on Spaces, Lemma 74.6.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. Finally, by Descent on Spaces, Lemma 74.11.19 we see that the scheme theoretic support of $\mathcal{F}$ is proper over T as we've assumed the scheme theoretic support of $\mathcal{F}_i$ is proper over T_i (note that taking scheme theoretic support commutes with flat base change by Morphisms of Spaces, Lemma 67.30.10). In this way we obtain our desired object over T. □

08LP Remark 99.5.5. In Situation 99.5.1 the rule $(T, g, \mathcal{F}) \mapsto (T, g)$ defines a 1-morphism

$$\mathcal{C}oh_{X/B} \longrightarrow \mathcal{S}_B$$

of stacks in groupoids (see Lemma 99.5.4, Algebraic Stacks, Section 94.7, and Examples of Stacks, Section 95.10). Let $B' \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{S}_{B'} \to \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $\mathcal{C}oh_{X'/B'} \to (Sch/S)_{fppf}$ associated to the base change $f' : X' \to B'$. In this situation the diagram

$$\begin{tikzcd} \mathcal{C}oh_{X'/B'} \arrow[r] \arrow[d] & \mathcal{C}oh_{X/B} \arrow[d] \\ \mathcal{S}_{B'} \arrow[r] & \mathcal{S}_B \end{tikzcd} \quad \begin{matrix}\text{or in}\\ \text{another}\\ \text{notation}\end{matrix} \quad \begin{tikzcd} \mathcal{C}oh_{X'/B'} \arrow[r] \arrow[d] & \mathcal{C}oh_{X/B} \arrow[d] \\ Sch/B' \arrow[r] & Sch/B \end{tikzcd}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

08KD Lemma 99.5.6. In Situation 99.5.1 assume that $B \to S$ is locally of finite presentation. Then $p : \mathcal{C}oh_{X/B} \to (Sch/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).

Proof. Write $B(T)$ for the discrete category whose objects are the S-morphisms $T \to B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S. Assigning to an object $(T, h, \mathcal{F})$ of $\mathcal{C}oh_{X/B,T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{tikzcd} \operatorname{colim} \mathcal{C}oh_{X/B,T_i} \arrow[r] \arrow[d] & \mathcal{C}oh_{X/B,T} \arrow[d] \\ \operatorname{colim} B(T_i) \arrow[r] & B(T) \end{tikzcd}$$

We have to show the top horizontal arrow is an equivalence. Since we have assumed that B is locally of finite presentation over S we see from Limits of Spaces, Remark 70.3.11 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B. Denote $g_i : T_i \to B$ and $g : T \to B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i,B} X$ and $X_T = T \times_{g,B} X$. Observe that $X_T = \operatorname{colim} X_i$ and that the algebraic spaces X_i and X_T are quasi-separated and quasi-compact (as they are of finite presentation over the affines T_i and T). By Limits of Spaces, Lemma 70.7.2 we see that

$$\operatorname{colim} \mathrm{FP}(X_i) = \mathrm{FP}(X_T).$$

where $\mathrm{FP}(W)$ is short hand for the category of finitely presented $\mathcal{O}_W$-modules. The results of Limits of Spaces, Lemmas 70.6.12 and 70.12.3 tell us the same thing is true if we replace $\mathrm{FP}(X_i)$ and $\mathrm{FP}(X_T)$ by the full subcategory of objects flat over T_i and T with scheme theoretic support proper over T_i and T. This proves the lemma. $\square$

08LQ **Lemma 99.5.7.** In Situation 99.5.1. Let

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

be a pushout in the category of schemes over S where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{C}oh_{X/B,Y'} \longrightarrow \mathcal{C}oh_{X/B,Y} \times_{\mathcal{C}oh_{X/B,Z}} \mathcal{C}oh_{X/B,Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 81.6.1. Thus using the commutative diagram

$$\begin{tikzcd} \mathcal{C}oh_{X/B,Y'} \arrow[r] \arrow[d] & \mathcal{C}oh_{X/B,Y} \times_{\mathcal{C}oh_{X/B,Z}} \mathcal{C}oh_{X/B,Z'} \arrow[d] \\ B(Y') \arrow[r] & B(Y) \times_{B(Z)} B(Z') \end{tikzcd}$$

we see that we may assume that Y' is a scheme over B'. By Remark 99.5.5 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$. In this case the statement follows from Pushouts of Spaces, Lemma 81.6.6. $\square$

08W7 **Lemma 99.5.8.** Let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

be a cartesian square of algebraic spaces where $T \to T'$ is a first order thickening. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module flat over T'. Set $\mathcal{F} = i^*\mathcal{F}'$. The following are equivalent

(1) $\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{X'}$-module of finite presentation,
(2) $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module of finite presentation,
(3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation,
(4) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation,

Proof. Recall that a finitely presented module is quasi-coherent hence the equivalence of (1) and (2) and (3) and (4). The equivalence of (2) and (4) is a special case of Deformation Theory, Lemma 91.11.3. □

08W8 **Lemma 99.5.9.** In Situation 99.5.1 assume that S is a locally Noetherian scheme and $B \to S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, \mathcal{G}_0)$ be an object of $\mathcal{X} = \mathcal{C}oh_{X/B}$ over k. Then the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ (Artin's Axioms, Section 98.8) are finite dimensional.

Proof. Observe that by Lemma 99.5.7 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 98.21. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 98.6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0,B} X$ and denote $\mathcal{X}_0 = \mathcal{C}oh_{X_0/k}$. In Remark 99.5.5 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(Sch/S)_{fppf}$. Thus by Artin's Axioms, Lemma 98.8.2 we reduce to proving that B, $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 98.8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k. Let $\mathrm{Spec}(k[\epsilon]) \to B$ be the composition of $g_0 : \mathrm{Spec}(k) \to B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ coming from the inclusion $k \to k[\epsilon]$. Set $X_0 = \mathrm{Spec}(k) \times_B X$ and $X_\epsilon = \mathrm{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. Unwinding the definitions and using Lemma 99.5.8 we see that $T\mathcal{F}_{\mathcal{X}_0,k,x_0}$ is the set of lifts of $\mathcal{G}_0$ to a flat module on X_ϵ. By Deformation Theory, Lemma 91.12.1 we conclude that

$$T\mathcal{F}_{\mathcal{X}_0,k,x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$-modules. Using Deformation Theory, Lemma 91.12.1 once more we see that

$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

These spaces are finite dimensional over k as $\mathcal{G}_0$ has support proper over $\mathrm{Spec}(k)$. Namely, X_0 is of finite presentation over $\mathrm{Spec}(k)$, hence Noetherian. Since $\mathcal{G}_0$ is of finite presentation it is a coherent $\mathcal{O}_{X_0}$-module. Thus we may apply Derived Categories of Spaces, Lemma 75.8.4 to conclude the desired finiteness. □

08W9 **Lemma 99.5.10.** In Situation 99.5.1 assume that S is a locally Noetherian scheme and that $f : X \to B$ is separated. Let $\mathcal{X} = \mathcal{C}oh_{X/B}$. Then the functor Artin's Axioms, Equation (98.9.3.1) is an equivalence.

Proof. Let A be an S-algebra which is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ whose residue field k is of finite type over S. We have to show that the category of objects over A is equivalent to the category of formal objects over A. Since we know this holds for the category $\mathcal{S}_B$ fibred in sets associated to B by Artin's Axioms, Lemma 98.9.5, it suffices to prove this for those objects lying over a given morphism $\mathrm{Spec}(A) \to B$.

Set $X_A = \mathrm{Spec}(A) \times_B X$ and $X_n = \mathrm{Spec}(A/\mathfrak{m}^n) \times_B X$. By Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 76.42.11) we see that the category of coherent modules $\mathcal{F}$ on X_A with support proper over $\mathrm{Spec}(A)$ is equivalent to the category of systems $(\mathcal{F}_n)$ of coherent modules $\mathcal{F}_n$ on X_n with support proper over $\mathrm{Spec}(A/\mathfrak{m}^n)$. The equivalence sends $\mathcal{F}$ to the system $(\mathcal{F} \otimes_A A/\mathfrak{m}^n)$. See discussion in More on Morphisms of Spaces, Remark 76.42.12. To finish the proof of the lemma, it suffices to show that $\mathcal{F}$ is flat over A if and only if all $\mathcal{F} \otimes_A A/\mathfrak{m}^n$ are flat over $A/\mathfrak{m}^n$. This follows from More on Morphisms of Spaces, Lemma 76.24.3. $\square$

08WA **Lemma 99.5.11.** In Situation 99.5.1 assume that S is a locally Noetherian scheme, $S = B$, and $f : X \to B$ is flat. Let $\mathcal{X} = \mathcal{C}oh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 98.13.1).

First proof. This proof is based on the criterion of Artin's Axioms, Lemma 98.24.4. Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 29.16.1) and $U = \mathrm{Spec}(A)$ with $U \to S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S. Let $\mathcal{F}$ be the coherent module on $X_A = \mathrm{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

According to Deformation Theory, Lemma 91.12.1 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

and given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$$

This uses that for any $A' \to A$ as above the base change $X_{A'} = \mathrm{Spec}(A') \times_B X$ is flat over A'. Moreover, the construction of the obstruction class is functorial in the surjection $A' \to A$ (for fixed A) by Deformation Theory, Lemma 91.12.3. Apply Derived Categories of Spaces, Lemma 75.23.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 98.24.2. Finally, condition (iv) of Artin's Axioms, Lemma 98.24.3 holds by Deformation Theory, Lemma 91.12.5. Thus Artin's Axioms, Lemma 98.24.4 does indeed apply and the lemma is proved. $\square$

Second proof. This proof is based on Artin's Axioms, Lemma 98.22.2. Conditions (1), (2), and (3) of that lemma correspond to Lemmas 99.5.3, 99.5.7, and 99.5.6.

We have constructed an obstruction theory in the chapter on deformation theory. Namely, given an S-algebra A and an object x of $\mathcal{C}oh_{X/B}$ over $\mathrm{Spec}(A)$ given by $\mathcal{F}$ on X_A we set $\mathcal{O}_x(M) = \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ and if $A' \to A$ is a surjection with kernel I, then as obstruction element we take the element

$$o_x(A') = o(\mathcal{F}, \mathcal{F} \otimes_A I, 1) \in \mathcal{O}_x(I) = \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$$

of Deformation Theory, Lemma 91.12.1. All properties of an obstruction theory as defined in Artin's Axioms, Definition 98.22.1 follow from this lemma except for functoriality of obstruction classes as formulated in condition (ii) of the definition. But as stated in the footnote to assumption (4) of Artin's Axioms, Lemma 98.22.2 it suffices to check functoriality of obstruction classes for a fixed A which follows from Deformation Theory, Lemma 91.12.3. Deformation Theory, Lemma 91.12.1 also tells us that $T_x(M) = \mathrm{Ext}^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for any A-module M.

To finish the proof it suffices to show that $T_x(\prod M_n) = \prod T_x(M_n)$ and $\mathcal{O}_x(\prod M_n) = \prod \mathcal{O}_x(M)$. Apply Derived Categories of Spaces, Lemma 75.23.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i = 1, 2$. A straightforward argument shows that

$$H^i(K \otimes_A^{\mathbf{L}} \prod M_n) = \prod H^i(K \otimes_A^{\mathbf{L}} M_n)$$

whenever K is a pseudo-coherent object of $D(A)$. In fact, this property (for all i) characterizes pseudo-coherent complexes, see More on Algebra, Lemma 15.65.5. □

08WC Theorem 99.5.12 (Algebraicity of the stack of coherent sheaves; flat case). Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation, separated, and flat[2]. Then $\mathcal{C}oh_{X/B}$ is an algebraic stack over S.

Proof. Set $\mathcal{X} = \mathcal{C}oh_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 99.5.4 and 99.5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = \mathcal{C}oh_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 99.5.5). By the material in Algebraic Stacks, Section 94.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 94.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth

[2] This assumption is not necessary. See Section 99.6.

morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \text{Spec}(\Lambda)$. Write $\Lambda = \text{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \text{Spec}(\Lambda_i)$ which is of finite presentation, separated, and flat and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1, 70.6.9, and 70.6.12. If we show that $\mathcal{C}oh_{X_i/\text{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 99.5.5 and Algebraic Stacks, Section 94.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \text{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 98.17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.50.12. Hence all local rings of S are G-rings. Thus (5) holds. By Lemma 99.5.11 we have that $\mathcal{X}$ satisfies openness of versality, hence (4) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 98.14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 99.5.4, 99.5.6, 99.5.7, 99.5.9. Condition (3) follows from Lemma 99.5.10. Finally, condition (1) is Lemma 99.5.3. This finishes the proof of the theorem. □

99.6. The stack of coherent sheaves in the non-flat case

08WB In Theorem 99.5.12 the assumption that $f : X \to B$ is flat is not necessary. In this section we give a different proof which avoids the flatness assumption and avoids checking openness of versality by using the results in Flatness on Spaces, Section 77.12 and Artin's Axioms, Section 98.20.

For a different approach to this problem the reader may wish to consult [Art69b] and follow the method discussed in the papers [OS03], [Lie06b], [Ols05], [HR13], [HR10], [Ryd11]. Some of these papers deal with the more general case of the stack of coherent sheaves on an algebraic stack over an algebraic stack and others deal with similar problems in the case of Hilbert stacks or Quot functors. Our strategy will be to show algebraicity of some cases of Hilbert stacks and Quot functors as a consequence of the algebraicity of the stack of coherent sheaves.

09DS Theorem 99.6.1 (Algebraicity of the stack of coherent sheaves; general case). Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation and separated. Then $\mathcal{C}oh_{X/B}$ is an algebraic stack over S.

Proof. Only the last step of the proof is different from the proof in the flat case, but we repeat all the arguments here to make sure everything works.

Set $\mathcal{X} = \mathcal{C}oh_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 99.5.4 and 99.5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = \mathcal{C}oh_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 99.5.5). By the material in

Algebraic Stacks, Section 94.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 94.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is separated and of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1 and 70.6.9. If we show that $\mathcal{C}oh_{X_i/\mathrm{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 99.5.5 and Algebraic Stacks, Section 94.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \mathrm{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 98.17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.50.12. Hence all local rings of S are G-rings. Thus (5) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 98.14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 99.5.4, 99.5.6, 99.5.7, 99.5.9. Condition (3) is Lemma 99.5.10. Condition (1) is Lemma 99.5.3.

It remains to show condition (4) which is openness of versality. To see this we will use Artin's Axioms, Lemma 98.20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving (see lemmas used above). Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 98.20.3. This is Flatness on Spaces, Theorem 77.12.8 and the proof is complete. □

99.7. The functor of quotients

082L In this section we discuss some generalities regarding the functor $Q_{\mathcal{F}/X/B}$ defined below. The notation $\mathrm{Quot}_{\mathcal{F}/X/B}$ is reserved for a subfunctor of $\mathrm{Q}_{\mathcal{F}/X/B}$. We urge the reader to skip this section on a first reading.

082M Situation 99.7.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_B T \to X$. Given such a T we set

$$\mathrm{Q}_{\mathcal{F}/X/B}(T) = \left\{ \begin{matrix} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a} \\ \text{quasi-coherent } \mathcal{O}_{X_T}\text{-module flat over } T \end{matrix} \right\}$$

We identify quotients if they have the same kernel. Suppose that $T' \to T$ is a morphism of schemes over B and $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$. Then the pullback $\mathcal{Q}' = (X_{T'} \to X_T)^*\mathcal{Q}$ is a quasi-coherent $\mathcal{O}_{X_{T'}}$-module flat over T' by Morphisms of Spaces, Lemma 67.31.3. Thus we obtain a functor

082N (99.7.1.1)
$$\mathrm{Q}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow \textit{Sets}$$

This is the functor of quotients of $\mathcal{F}/X/B$. We define a subfunctor

0CZL (99.7.1.2)
$$\mathrm{Q}^{fp}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow \textit{Sets}$$

which assigns to T the subset of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$ consisting of those quotients $\mathcal{F}_T \to \mathcal{Q}$ such that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. This is a subfunctor by Properties of Spaces, Section 66.30.

In Situation 99.7.1 we sometimes think of $\mathrm{Q}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to \textit{Sets}$ endowed with a morphism $\mathrm{Q}_{\mathcal{F}/X/S} \to B$. Namely, if T is a scheme over S, then an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$ is a pair $(h, \mathcal{Q})$ where h a morphism $h : T \to B$ and $\mathcal{Q}$ is a T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ of finite presentation on $X_T = X \times_{B,h} T$. In particular, when we say that $\mathrm{Q}_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to \textit{Sets}$ is an algebraic space. Similar remarks apply to $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$.

08IT **Remark 99.7.2.** In Situation 99.7.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. Thus we have the functor $Q_{\mathcal{F}'/X'/B'}$ on the category of schemes over B'. For a scheme T over B' it is clear that we have

$$Q_{\mathcal{F}'/X'/B'}(T) = Q_{\mathcal{F}/X/B}(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. Similar remarks apply to $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$. These trivial remarks will occasionally be useful to change the base algebraic space.

08IU **Remark 99.7.3.** Let S be a scheme, X an algebraic space over S, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\{f_i : X_i \to X\}_{i \in I}$ is an fpqc covering and for each $i, j \in I$ we are given an fpqc covering $\{X_{ijk} \to X_i \times_X X_j\}$. In this situation we have a bijection

$$\left\{ \begin{matrix} \text{quotients } \mathcal{F} \to \mathcal{Q} \text{ where} \\ \mathcal{Q} \text{ is a quasi-coherent} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{families of quotients } f_i^*\mathcal{F} \to \mathcal{Q}_i \text{ where} \\ \mathcal{Q}_i \text{ is quasi-coherent and } \mathcal{Q}_i \text{ and } \mathcal{Q}_j \\ \text{restrict to the same quotient on } X_{ijk} \end{matrix} \right\}$$

Namely, let $(f_i^*\mathcal{F} \to \mathcal{Q}_i)_{i \in I}$ be an element of the right hand side. Then since $\{X_{ijk} \to X_i \times_X X_j\}$ is an fpqc covering we see that the pullbacks of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ restrict to the same quotient of the pullback of $\mathcal{F}$ to $X_i \times_X X_j$ (by fully faithfulness in Descent on Spaces, Proposition 74.4.1). Hence we obtain a descent datum for quasi-coherent modules with respect to $\{X_i \to X\}_{i \in I}$. By Descent on Spaces, Proposition 74.4.1 we find a map of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{Q}$ whose restriction to X_i recovers the given maps $f_i^*\mathcal{F} \to \mathcal{Q}_i$. Since the family of morphisms $\{X_i \to X\}$ is jointly surjective and flat, for every point $x \in |X|$ there exists an i and a point $x_i \in |X_i|$ mapping to x. Note that the induced map on local rings $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is faithfully flat, see Morphisms of Spaces, Section 67.30. Thus we see that $\mathcal{F} \to \mathcal{Q}$ is surjective.

082P **Lemma 99.7.4.** *In Situation 99.7.1. The functors $\mathrm{Q}_{\mathcal{F}/X/B}$ and $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$ satisfy the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = \mathcal{F}_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T (Topologies on Spaces, Lemma 73.9.3) and that $X_{T_i \times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $\mathcal{F}_i \to \mathcal{Q}_i$ is a collection of elements of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i)$ such that $\mathcal{Q}_i$ and $\mathcal{Q}_{i'}$ restrict to the same element of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i \times_T T_{i'})$. By Remark 99.7.3 we obtain a surjective map of quasi-coherent $\mathcal{O}_{X_T}$-modules $\mathcal{F}_T \to \mathcal{Q}$ whose restriction to X_i recovers the given quotients. By Morphisms of Spaces, Lemma 67.31.5 we see that $\mathcal{Q}$ is flat over T. Finally, in the case of $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$, i.e., if $\mathcal{Q}_i$ are of finite presentation, then Descent on Spaces, Lemma 74.6.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. $\square$

Sanity check: $\mathrm{Q}_{\mathcal{F}/X/B}$, $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$ play the same role among algebraic spaces over S.

0D3U Lemma 99.7.5. In Situation 99.7.1. Let T be an algebraic space over S. We have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Q}_{\mathcal{F}/X/B}) = \left\{ \begin{matrix} (h, \mathcal{F}_T \to \mathcal{Q}) \text{ where } h : T \to B \text{ and} \\ \mathcal{Q} \text{ is quasi-coherent and flat over } T \end{matrix} \right\}$$

where $\mathcal{F}_T$ denotes the pullback of $\mathcal{F}$ to the algebraic space $X \times_{B,h} T$. Similarly, we have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Q}^{fp}_{\mathcal{F}/X/B}) = \left\{ \begin{matrix} (h, \mathcal{F}_T \to \mathcal{Q}) \text{ where } h : T \to B \text{ and} \\ \mathcal{Q} \text{ is of finite presentation and flat over } T \end{matrix} \right\}$$

Proof. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $t, s : R \to U$.

Let $v : T \to \mathrm{Q}_{\mathcal{F}/X/B}$ be a natural transformation. Then $v(p)$ corresponds to a pair $(h_U, \mathcal{F}_U \to \mathcal{Q}_U)$ over U. As v is a transformation of functors we see that the pullbacks of $(h_U, \mathcal{F}_U \to \mathcal{Q}_U)$ by s and t agree. Since $T = U/R$ (Spaces, Lemma 65.9.1), we obtain a morphism $h : T \to B$ such that $h_U = h \circ p$. By Descent on Spaces, Proposition 74.4.1 the quotient $\mathcal{Q}_U$ descends to a quotient $\mathcal{F}_T \to \mathcal{Q}$ over X_T. Since $U \to T$ is surjective and flat, it follows from Morphisms of Spaces, Lemma 67.31.5 that $\mathcal{Q}$ is flat over T.

Conversely, let $(h, \mathcal{F}_T \to \mathcal{Q})$ be a pair over T. Then we get a natural transformation $v : T \to \mathrm{Q}_{\mathcal{F}/X/B}$ by sending a morphism $a : T' \to T$ where T' is a scheme to $(h \circ a, \mathcal{F}_{T'} \to a^*\mathcal{Q})$. We omit the verification that the construction of this and the previous paragraph are mutually inverse.

In the case of $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$ we add: given a morphism $h : T \to B$, a quasi-coherent sheaf on X_T is of finite presentation as an $\mathcal{O}_{X_T}$-module if and only if the pullback to X_U is of finite presentation as an $\mathcal{O}_{X_U}$-module. This follows from the fact that $X_U \to X_T$ is surjective and étale and Descent on Spaces, Lemma 74.6.2. $\square$

08IV Lemma 99.7.6. In Situation 99.7.1 let $\{X_i \to X\}_{i \in I}$ be an fpqc covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fpqc covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. For every scheme T over B the diagram

$$Q_{\mathcal{F}/X/B}(T) \longrightarrow \prod\nolimits_i Q_{\mathcal{F}_i/X_i/B}(T) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod\nolimits_{i,j,k} Q_{\mathcal{F}_{ijk}/X_{ijk}/B}(T)$$

presents the first arrow as the equalizer of the other two. The same is true for the functor $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$.

Proof. Let $\mathcal{F}_{i,T} \to \mathcal{Q}_i$ be an element in the equalizer of pr_0^* and pr_1^*. By Remark 99.7.3 we obtain a surjection $\mathcal{F}_T \to \mathcal{Q}$ of quasi-coherent $\mathcal{O}_{X_T}$-modules whose restriction to $X_{i,T}$ recovers $\mathcal{F}_i \to \mathcal{Q}_i$. By Morphisms of Spaces, Lemma 67.31.5 we see that $\mathcal{Q}$ is flat over T as desired. In the case of the functor $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$, i.e., if $\mathcal{Q}_i$ is of finite presentation, then $\mathcal{Q}$ is of finite presentation too by Descent on Spaces, Lemma 74.6.2. $\square$

082Q **Lemma 99.7.7.** In Situation 99.7.1 assume also that (a) f is quasi-compact and quasi-separated and (b) $\mathcal{F}$ is of finite presentation. Then the functor $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$ is limit preserving in the following sense: If $T = \lim T_i$ is a directed limit of affine schemes over B, then $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}(T) = \text{colim}\, \mathrm{Q}^{fp}_{\mathcal{F}/X/B}(T_i)$.

Proof. Let $T = \lim T_i$ be as in the statement of the lemma. Choose $i_0 \in I$ and replace I by $\{i \in I \mid i \geq i_0\}$. We may set $B = S = T_{i_0}$ and we may replace X by X_{T_0} and $\mathcal{F}$ by the pullback to X_{T_0}. Then $X_T = \lim X_{T_i}$, see Limits of Spaces, Lemma 70.4.1. Let $\mathcal{F}_T \to \mathcal{Q}$ be an element of $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}(T)$. By Limits of Spaces, Lemma 70.7.2 there exists an i and a map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ of $\mathcal{O}_{X_{T_i}}$-modules of finite presentation whose pullback to X_T is the given quotient map.

We still have to check that, after possibly increasing i, the map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ is surjective and $\mathcal{Q}_i$ is flat over T_i. To do this, choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). We may check surjectivity and flatness over T_i after pulling back to the étale cover $U_{T_i} \to X_{T_i}$ (by definition). This reduces us to the case where $X = \text{Spec}(B_0)$ is an affine scheme of finite presentation over $B = S = T_0 = \text{Spec}(A_0)$. Writing $T_i = \text{Spec}(A_i)$, then $T = \text{Spec}(A)$ with $A = \text{colim}\, A_i$ we have reached the following algebra problem. Let $M_i \to N_i$ be a map of finitely presented $B_0 \otimes_{A_0} A_i$-modules such that $M_i \otimes_{A_i} A \to N_i \otimes_{A_i} A$ is surjective and $N_i \otimes_{A_i} A$ is flat over A. Show that for some $i' \geq i$ $M_i \otimes_{A_i} A_{i'} \to N_i \otimes_{A_i} A_{i'}$ is surjective and $N_i \otimes_{A_i} A_{i'}$ is flat over A. The first follows from Algebra, Lemma 10.127.5 and the second from Algebra, Lemma 10.168.1. $\square$

08IW **Lemma 99.7.8.** In Situation 99.7.1. Let

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

be a pushout in the category of schemes over B where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Then the natural map

$$Q_{\mathcal{F}/X/B}(Y') \longrightarrow Q_{\mathcal{F}/X/B}(Y) \times_{Q_{\mathcal{F}/X/B}(Z)} Q_{\mathcal{F}/X/B}(Z')$$

is bijective. If $X \to B$ is locally of finite presentation, then the same thing is true for $Q^{fp}_{\mathcal{F}/X/B}$.

Proof. Let us construct an inverse map. Namely, suppose we have $\mathcal{F}_Y \to \mathcal{A}$, $\mathcal{F}_{Z'} \to \mathcal{B}'$, and an isomorphism $\mathcal{A}|_{X_Z} \to \mathcal{B}'|_{X_Z}$ compatible with the given surjections. Then we apply Pushouts of Spaces, Lemma 81.6.6 to get a quasi-coherent module $\mathcal{A}'$ on $X_{Y'}$ flat over Y'. Since this sheaf is constructed as a fibre product (see proof of cited lemma) there is a canonical map $\mathcal{F}_{Y'} \to \mathcal{A}'$. That this map is surjective can be seen because it factors as

$$\begin{array}{c} \mathcal{F}_{Y'} \\ \downarrow \\ (X_Y \to X_{Y'})_*\mathcal{F}_Y \times_{(X_Z \to X_{Y'})_*\mathcal{F}_Z} (X_{Z'} \to X_{Y'})_*\mathcal{F}_{Z'} \\ \downarrow \\ \mathcal{A}' = (X_Y \to X_{Y'})_*\mathcal{A} \times_{(X_Z \to X_{Y'})_*\mathcal{A}|_{X_Z}} (X_{Z'} \to X_{Y'})_*\mathcal{B}' \end{array}$$

and the first arrow is surjective by More on Algebra, Lemma 15.6.5 and the second by More on Algebra, Lemma 15.6.6.

In the case of $Q^{fp}_{\mathcal{F}/X/B}$ all we have to show is that the construction above produces a finitely presented module. This is explained in More on Algebra, Remark 15.7.8 in the commutative algebra setting. The current case of modules over algebraic spaces follows from this by étale localization. □

0CZU Remark 99.7.9 (Obstructions for quotients). In Situation 99.7.1 assume that $\mathcal{F}$ is flat over B. Let $T \subset T'$ be an first order thickening of schemes over B with ideal sheaf $\mathcal{J}$. Then $X_T \subset X_{T'}$ is a first order thickening of algebraic spaces whose ideal sheaf $\mathcal{I}$ is a quotient of $f_T^*\mathcal{J}$. We will think of sheaves on $X_{T'}$, resp. T' as sheaves on X_T, resp. T using the fundamental equivalence described in More on Morphisms of Spaces, Section 76.9. Let

$$0 \to \mathcal{K} \to \mathcal{F}_T \to \mathcal{Q} \to 0$$

define an element x of $Q_{\mathcal{F}/X/B}(T)$. Since $\mathcal{F}_{T'}$ is flat over T' we have a short exact sequence

$$0 \to f_T^*\mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T \xrightarrow{i} \mathcal{F}_{T'} \xrightarrow{\pi} \mathcal{F}_T \to 0$$

and we have $f_T^*\mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T = \mathcal{I} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T$, see Deformation Theory, Lemma 91.11.2. Let us use the abbreviation $f_T^*\mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_T} \mathcal{J}$ for an $\mathcal{O}_{X_T}$-module $\mathcal{G}$. Since $\mathcal{Q}$ is flat over T, we obtain a short exact sequence

$$0 \to \mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J} \to \mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{J} \to \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J} \to\to 0$$

Combining the above we obtain an canonical extension

$$0 \to \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J} \to \pi^{-1}(\mathcal{K})/i(\mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J}) \to \mathcal{K} \to 0$$

of $\mathcal{O}_{X_T}$-modules. This defines a canonical class

$$o_x(T') \in \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(\mathcal{K}, \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J})$$

If $o_x(T')$ is zero, then we obtain a splitting of the short exact sequence defining it, in other words, we obtain a $\mathcal{O}_{X_{T'}}$-submodule $\mathcal{K}' \subset \pi^{-1}(\mathcal{K})$ sitting in a short exact sequence $0 \to \mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J} \to \mathcal{K}' \to \mathcal{K} \to 0$. Then it follows from the lemma reference above that $\mathcal{Q}' = \mathcal{F}_{T'}/\mathcal{K}'$ is a lift of x to an element of $Q_{\mathcal{F}/X/B}(T')$. Conversely, the reader sees that the existence of a lift implies that $o_x(T')$ is zero. Moreover, if $x \in Q^{fp}_{\mathcal{F}/X/B}(T)$, then automatically $x' \in Q^{fp}_{\mathcal{F}/X/B}(T')$ by Deformation Theory, Lemma 91.11.3. If we ever need this remark we will turn this remark into a lemma, precisely formulate the result and give a detailed proof (in fact, all of the above works in the setting of arbitrary ringed topoi).

0CZV Remark 99.7.10 (Deformations of quotients). In Situation 99.7.1 assume that $\mathcal{F}$ is flat over B. We continue the discussion of Remark 99.7.9. Assume $o_x(T') = 0$. Then we claim that the set of lifts $x' \in Q_{\mathcal{F}/X/B}(T')$ is a principal homogeneous space under the group

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{K}, \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J})$$

Namely, given any $\mathcal{F}_{T'} \to \mathcal{Q}'$ flat over T' lifting the quotient $\mathcal{Q}$ we obtain a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K} \otimes \mathcal{J} \arrow[r] \arrow[d] & \mathcal{F}_T \otimes \mathcal{J} \arrow[r] \arrow[d] & \mathcal{Q} \otimes \mathcal{J} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}' \arrow[r] \arrow[d] & \mathcal{F}_{T'} \arrow[r] \arrow[d] & \mathcal{Q}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{F}_T \arrow[r] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(to see this use the observations made in the previous remark). Given a map $\varphi : \mathcal{K} \to \mathcal{Q} \otimes \mathcal{J}$ we can consider the subsheaf $\mathcal{K}'_\varphi \subset \mathcal{F}_{T'}$ consisting of those local sections s whose image in $\mathcal{F}_T$ is a local section k of $\mathcal{K}$ and whose image in $\mathcal{Q}'$ is the local section $\varphi(k)$ of $\mathcal{Q} \otimes \mathcal{J}$. Then set $\mathcal{Q}'_\varphi = \mathcal{F}_{T'}/\mathcal{K}'_\varphi$. Conversely, any second lift of x corresponds to one of the qotients constructed in this manner. If we ever need this remark we will turn this remark into a lemma, precisely formulate the result and give a detailed proof (in fact, all of the above works in the setting of arbitrary ringed topoi).

99.8. The Quot functor

09TQ In this section we prove the Quot functor is an algebraic space.

09TR Situation 99.8.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Given such a T we set

$$\mathrm{Quot}_{\mathcal{F}/X/B}(T) = \left\{ \begin{matrix} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \\ \text{with support proper over } T \end{matrix} \right\}$$

By Derived Categories of Spaces, Lemma 75.7.8 this is a subfunctor of the functor $Q^{fp}_{\mathcal{F}/X/B}$ we discussed in Section 99.7. Thus we obtain a functor

09TS (99.8.1.1) $$\mathrm{Quot}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow \mathrm{Sets}$$

This is the Quot functor associated to $\mathcal{F}/X/B$.

In Situation 99.8.1 we sometimes think of $\mathrm{Quot}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to$ Sets endowed with a morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$. Namely, if T is a scheme over S, then an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ is a pair $(h, \mathcal{Q})$ where h is a morphism $h : T \to B$ and Q is a finitely presented, T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ on $X_T = X \times_{B,h} T$ with support proper over T. In particular, when we say that $\mathrm{Quot}_{\mathcal{F}/X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to$ Sets is an algebraic space.

09TT **Lemma 99.8.2.** In Situation 99.8.1. The functor $\mathrm{Quot}_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology.

Proof. In Lemma 99.7.4 we have seen that the functor $\mathrm{Q}^{fp}_{\mathcal{F}/X/S}$ is a sheaf. Recall that for a scheme T over S the subset $\mathrm{Quot}_{\mathcal{F}/X/S}(T) \subset \mathrm{Q}_{\mathcal{F}/X/S}(T)$ picks out those quotients whose support is proper over T. This defines a subsheaf by the result of Descent on Spaces, Lemma 74.11.19 combined with Morphisms of Spaces, Lemma 67.30.10 which shows that taking scheme theoretic support commutes with flat base change. $\square$

Sanity check: $\mathrm{Quot}_{\mathcal{F}/X/B}$ plays the same role among algebraic spaces over S.

0D3V **Lemma 99.8.3.** In Situation 99.8.1. Let T be an algebraic space over S. We have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Quot}_{\mathcal{F}/X/B}) = \left\{ \begin{array}{c} (h, \mathcal{F}_T \to \mathcal{Q}) \text{ where } h : T \to B \text{ and} \\ \mathcal{Q} \text{ is of finite presentation and} \\ \text{flat over } T \text{ with support proper over } T \end{array} \right\}$$

where $\mathcal{F}_T$ denotes the pullback of $\mathcal{F}$ to the algebraic space $X \times_{B,h} T$.

Proof. Observe that the left and right hand side of the equality are subsets of the left and right hand side of the second equality in Lemma 99.7.5. To see that these subsets correspond under the identification given in the proof of that lemma it suffices to show: given $h : T \to B$, a surjective étale morphism $U \to T$, a finite type quasi-coherent $\mathcal{O}_{X_T}$-module $\mathcal{Q}$ the following are equivalent

(1) the scheme theoretic support of $\mathcal{Q}$ is proper over T, and
(2) the scheme theoretic support of $(X_U \to X_T)^*\mathcal{Q}$ is proper over U.

This follows from Descent on Spaces, Lemma 74.11.19 combined with Morphisms of Spaces, Lemma 67.30.10 which shows that taking scheme theoretic support commutes with flat base change. $\square$

09TU **Proposition 99.8.4.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. If f is of finite presentation and separated, then $\mathrm{Quot}_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is locally of finite presentation.

Proof. By Lemma 99.8.2 we have that $\mathrm{Quot}_{\mathcal{F}/X/B}$ is a sheaf in the fppf topology. Let $\mathcal{Q}uot_{\mathcal{F}/X/B}$ be the stack in groupoids corresponding to $\mathrm{Quot}_{\mathcal{F}/X/S}$, see Algebraic Stacks, Section 94.7. By Algebraic Stacks, Proposition 94.13.3 it suffices to show that $\mathcal{Q}uot_{\mathcal{F}/X/B}$ is an algebraic stack. Consider the 1-morphism of stacks in groupoids

$$\mathcal{Q}uot_{\mathcal{F}/X/S} \longrightarrow \mathcal{C}oh_{X/B}$$

on $(Sch/S)_{fppf}$ which associates to the quotient $\mathcal{F}_T \to \mathcal{Q}$ the module $\mathcal{Q}$. By Theorem 99.6.1 we know that $\mathcal{C}oh_{X/B}$ is an algebraic stack. By Algebraic Stacks, Lemma 94.15.4 it suffices to show that this 1-morphism is representable by algebraic spaces.

Let T be a scheme over S and let the object $(h, \mathcal{G})$ of $\mathcal{C}oh_{X/B}$ over T correspond to a 1-morphism $\xi : (Sch/T)_{fppf} \to \mathcal{C}oh_{X/B}$. The 2-fibre product

$$\mathcal{Z} = (Sch/T)_{fppf} \times_{\xi, \mathcal{C}oh_{X/B}} \mathrm{Quot}_{\mathcal{F}/X/S}$$

is a stack in setoids, see Stacks, Lemma 8.6.7. The corresponding sheaf of sets (i.e., functor, see Stacks, Lemmas 8.6.7 and 8.6.2) assigns to a scheme T'/T the set of surjections $u : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$ of quasi-coherent modules on $X_{T'}$. Thus we see that $\mathcal{Z}$ is representable by an open subspace (by Flatness on Spaces, Lemma 77.9.3) of the algebraic space $Hom(\mathcal{F}_T, \mathcal{G})$ from Proposition 99.3.10. $\square$

0CZW Remark 99.8.5 (Quot via Artin's axioms). Let S be a Noetherian scheme all of whose local rings are G-rings. Let X be an algebraic space over S whose structure morphism $f : X \to S$ is of finite presentation and separated. Let $\mathcal{F}$ be a finitely presented quasi-coherent sheaf on X flat over S. In this remark we sketch how one can use Artin's axioms to prove that $\mathrm{Quot}_{\mathcal{F}/X/S}$ is an algebraic space locally of finite presentation over S and avoid using the algebraicity of the stack of coherent sheaves as was done in the proof of Proposition 99.8.4.

We check the conditions listed in Artin's Axioms, Proposition 98.16.1. Representability of the diagonal of $\mathrm{Quot}_{\mathcal{F}/X/S}$ can be seen as follows: suppose we have two quotients $\mathcal{F}_T \to \mathcal{Q}_i$, $i = 1, 2$. Denote $\mathcal{K}_1$ the kernel of the first one. Then we have to show that the locus of T over which $u : \mathcal{K}_1 \to \mathcal{Q}_2$ becomes zero is representable. This follows for example from Flatness on Spaces, Lemma 77.8.6 or from a discussion of the *Hom* sheaf earlier in this chapter. Axioms [0] (sheaf), [1] (limits), [2] (Rim-Schlessinger) follow from Lemmas 99.8.2, 99.7.7, and 99.7.8 (plus some extra work to deal with the properness condition). Axiom [3] (finite dimensionality of tangent spaces) follows from the description of the infinitesimal deformations in Remark 99.7.10 and finiteness of cohomology of coherent sheaves on proper algebraic spaces over fields (Cohomology of Spaces, Lemma 69.20.2). Axiom [4] (effectiveness of formal objects) follows from Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 76.42.11). As usual, the trickiest to verify is axiom [5] (openness of versality). One can for example use the obstruction theory described in Remark 99.7.9 and the description of deformations in Remark 99.7.10 to do this using the criterion in Artin's Axioms, Lemma 98.22.2. Please compare with the second proof of Lemma 99.5.11.

99.9. The Hilbert functor

0CZX In this section we prove the Hilb functor is an algebraic space.

0CZY Situation 99.9.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. For any scheme T over B we will denote X_T the base change of X to T. Given such a T we set

$$\mathrm{Hilb}_{X/B}(T) = \left\{ \begin{matrix} \text{closed subspaces } Z \subset X_T \text{ such that } Z \to T \\ \text{is of finite presentation, flat, and proper} \end{matrix} \right\}$$

Since base change preserves the required properties (Spaces, Lemma 65.12.3 and Morphisms of Spaces, Lemmas 67.28.3, 67.30.4, and 67.40.3) we obtain a functor

0CZZ (99.9.1.1) $$\mathrm{Hilb}_{X/B} : (Sch/B)^{opp} \longrightarrow \mathrm{Sets}$$

This is the Hilbert functor associated to X/B.

In Situation 99.9.1 we sometimes think of $\mathrm{Hilb}_{X/B}$ as a functor $(Sch/S)^{opp} \to \textit{Sets}$ endowed with a morphism $\mathrm{Hilb}_{X/S} \to B$. Namely, if T is a scheme over S, then an element of $\mathrm{Hilb}_{X/B}(T)$ is a pair (h, Z) where h is a morphism $h : T \to B$ and $Z \subset X_T = X \times_{B,h} T$ is a closed subscheme, flat, proper, and of finite presentation over T. In particular, when we say that $\mathrm{Hilb}_{X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to \textit{Sets}$ is an algebraic space.

Of course the Hilbert functor is just a special case of the Quot functor.

0D00 Lemma 99.9.2. In Situation 99.9.1 we have $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$.

Proof. Let T be a scheme over B. Given an element $Z \in \mathrm{Hilb}_{X/B}(T)$ we can consider the quotient $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$ where $i : Z \to X_T$ is the inclusion morphism. Note that $i_*\mathcal{O}_Z$ is quasi-coherent. Since $Z \to T$ and $X_T \to T$ are of finite presentation, we see that i is of finite presentation (Morphisms of Spaces, Lemma 67.28.9), hence $i_*\mathcal{O}_Z$ is an $\mathcal{O}_{X_T}$-module of finite presentation (Descent on Spaces, Lemma 74.6.7). Since $Z \to T$ is proper we see that $i_*\mathcal{O}_Z$ has support proper over T (as defined in Derived Categories of Spaces, Section 75.7). Since $\mathcal{O}_Z$ is flat over T and i is affine, we see that $i_*\mathcal{O}_Z$ is flat over T (small argument omitted). Hence $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$ is an element of $\mathrm{Quot}_{\mathcal{O}_X/X/B}(T)$.

Conversely, given an element $\mathcal{O}_{X_T} \to \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{O}_X/X/B}(T)$, we can consider the closed immersion $i : Z \to X_T$ corresponding to the quasi-coherent ideal sheaf $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X_T} \to \mathcal{Q})$ (Morphisms of Spaces, Lemma 67.13.1). By construction of Z we see that $\mathcal{Q} = i_*\mathcal{O}_Z$. Then we can read the arguments given above backwards to see that Z defines an element of $\mathrm{Hilb}_{X/B}(T)$. For example, $\mathcal{I}$ is quasi-coherent of finite type (Modules on Sites, Lemma 18.24.1) hence $i : Z \to X_T$ is of finite presentation (Morphisms of Spaces, Lemma 67.28.12) hence $Z \to T$ is of finite presentation (Morphisms of Spaces, Lemma 67.28.2). Properness of $Z \to T$ follows from the discussion in Derived Categories of Spaces, Section 75.7. Flatness of $Z \to T$ follows from flatness of $\mathcal{Q}$ over T.

We omit the (immediate) verification that the two constructions given above are mutually inverse. □

Sanity check: $\mathrm{Hilb}_{X/B}$ sheaf plays the same role among algebraic spaces over S.

0D3W Lemma 99.9.3. In Situation 99.9.1. Let T be an algebraic space over S. We have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Hilb}_{X/B}) = \left\{ \begin{matrix} (h, Z) \text{ where } h : T \to B,\ Z \subset X_T \\ \text{finite presentation, flat, proper over } T \end{matrix} \right\}$$

where $X_T = X \times_{B,h} T$.

Proof. By Lemma 99.9.2 we have $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Thus we can apply Lemma 99.8.3 to see that the left hand side is bijective with the set of surjections $\mathcal{O}_{X_T} \to \mathcal{Q}$ which are finitely presented, flat over T, and have support proper over T. Arguing exactly as in the proof of Lemma 99.9.2 we see that such quotients correspond exactly to the closed immersions $Z \to X_T$ such that $Z \to T$ is proper, flat, and of finite presentation. □

0D01 Proposition 99.9.4. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. If f is of finite presentation and separated, then $\mathrm{Hilb}_{X/B}$ is an algebraic space locally of finite presentation over B.

Proof. Immediate consequence of Lemma 99.9.2 and Proposition 99.8.4. □

99.10. The Picard stack

0D02 The Picard stack for a morphism of algebraic spaces was introduced in Examples of Stacks, Section 95.16. We will deduce it is an open substack of the stack of coherent sheaves (in good cases) from the following lemma.

0D03 Lemma 99.10.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S which is flat, of finite presentation, and proper. The natural map

$$\mathcal{P}ic_{X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

is representable by open immersions.

Proof. Observe that the map simply sends a triple $(T, g, \mathcal{L})$ as in Examples of Stacks, Section 95.16 to the same triple $(T, g, \mathcal{L})$ but where now we view this as a triple of the kind described in Situation 99.5.1. This works because the invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ is certainly a finitely presented $\mathcal{O}_{X_T}$-module, it is flat over T because $X_T \to T$ is flat, and the support is proper over T as $X_T \to T$ is proper (Morphisms of Spaces, Lemmas 67.30.4 and 67.40.3). Thus the statement makes sense.

Having said this, it is clear that the content of the lemma is the following: given an object $(T, g, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ there is an open subscheme $U \subset T$ such that for a morphism of schemes $T' \to T$ the following are equivalent

(a) $T' \to T$ factors through U,
(b) the pullback $\mathcal{F}_{T'}$ of $\mathcal{F}$ by $X_{T'} \to X_T$ is invertible.

Let $W \subset |X_T|$ be the set of points $x \in |X_T|$ such that $\mathcal{F}$ is locally free in a neighbourhood of x. By More on Morphisms of Spaces, Lemma 76.23.8. W is open and formation of W commutes with arbitrary base change. Clearly, if $T' \to T$ satisfies (b), then $|X_{T'}| \to |X_T|$ maps into W. Hence we may replace T by the open $T \setminus f_T(|X_T| \setminus W)$ in order to construct U. After doing so we reach the situation where $\mathcal{F}$ is finite locally free. In this case we get a disjoint union decomposition $X_T = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$ into open and closed subspaces such that the restriction of $\mathcal{F}$ is locally free of rank i on X_i. Then clearly

$$U = T \setminus f_T(|X_0| \cup |X_2| \cup |X_3| \cup \ldots)$$

works. (Note that if we assume that T is quasi-compact, then X_T is quasi-compact hence only a finite number of X_i are nonempty and so U is indeed open.) □

0D04 Proposition 99.10.2. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. If f is flat, of finite presentation, and proper, then $\mathcal{P}ic_{X/B}$ is an algebraic stack.

Proof. Immediate consequence of Lemma 99.10.1, Algebraic Stacks, Lemma 94.15.4 and either Theorem 99.5.12 or Theorem 99.6.1 □

99.11. The Picard functor

0D24 In this section we revisit the Picard functor discussed in Picard Schemes of Curves, Section 44.4. The discussion will be more general as we want to study the Picard functor of a morphism of algebraic spaces as in the section on the Picard stack, see Section 99.10.

Let S be a scheme and let X be an algebraic space over S. An invertible sheaf on X is an invertible $\mathcal{O}_X$-module on $X_{étale}$, see Modules on Sites, Definition 18.32.1. The group of isomorphism classes of invertible modules is denoted $\text{Pic}(X)$, see Modules on Sites, Definition 18.32.6. Given a morphism $f : X \to Y$ of algebraic spaces over S pullback defines a group homomorphism $\text{Pic}(Y) \to \text{Pic}(X)$. The assignment $X \rightsquigarrow \text{Pic}(X)$ is a contravariant functor from the category of schemes to the category of abelian groups. This functor is not representable, but it turns out that a relative variant of this construction sometimes is representable.

0D25 Situation 99.11.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. We define

$$\text{Pic}_{X/B} : (Sch/B)^{opp} \longrightarrow \text{Sets}$$

as the fppf sheafification of the functor which to a scheme T over B associates the group $\text{Pic}(X_T)$.

In Situation 99.11.1 we sometimes think of $\text{Pic}_{X/B}$ as a functor $(Sch/S)^{opp} \to \text{Sets}$ endowed with a morphism $\text{Pic}_{X/B} \to B$. In this point of view, we define $\text{Pic}_{X/B}$ to be the fppf sheafification of the functor

$$T/S \longmapsto \{(h, \mathcal{L}) \mid h : T \to B,\ \mathcal{L} \in \text{Pic}(X \times_{B,h} T)\}$$

In particular, when we say that $\text{Pic}_{X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to \text{Sets}$ is an algebraic space.

An often used remark is that if T is a scheme over B, then $\text{Pic}_{X_T/T}$ is the restriction of $\text{Pic}_{X/B}$ to $(Sch/T)_{fppf}$.

0D26 Lemma 99.11.2. In Situation 99.11.1 the functor $\text{Pic}_{X/B}$ is the sheafification of the functor $T \mapsto \text{Ob}(\mathcal{P}ic_{X/B,T})/\cong$.

Proof. Since the fibre category $\mathcal{P}ic_{X/B,T}$ of the Picard stack $\mathcal{P}ic_{X/B}$ over T is the category of invertible sheaves on X_T (see Section 99.10 and Examples of Stacks, Section 95.16) this is immediate from the definitions. $\square$

It turns out to be nontrivial to see what the value of $\text{Pic}_{X/B}$ is on schemes T over B. Here is a lemma that helps with this task.

0D27 Lemma 99.11.3. In Situation 99.11.1. If $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all schemes T over B, then

$$0 \to \text{Pic}(T) \to \text{Pic}(X_T) \to \text{Pic}_{X/B}(T)$$

is an exact sequence for all T.

Proof. We may replace B by T and X by X_T and assume that $B = T$ to simplify the notation. Let $\mathcal{N}$ be an invertible $\mathcal{O}_B$-module. If $f^*\mathcal{N} \cong \mathcal{O}_X$, then we see that $f_*f^*\mathcal{N} \cong f_*\mathcal{O}_X \cong \mathcal{O}_B$ by assumption. Since $\mathcal{N}$ is locally trivial, we see that the canonical map $\mathcal{N} \to f_*f^*\mathcal{N}$ is locally an isomorphism (because $\mathcal{O}_B \to f_*f^*\mathcal{O}_B$ is an isomorphism by assumption). Hence we conclude that $\mathcal{N} \to f_*f^*\mathcal{N} \to \mathcal{O}_B$ is an isomorphism and we see that $\mathcal{N}$ is trivial. This proves the first arrow is injective.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is in the kernel of $\text{Pic}(X) \to \text{Pic}_{X/B}(B)$. Then there exists an fppf covering $\{B_i \to B\}$ such that $\mathcal{L}$ pulls back to the trivial invertible sheaf on X_{B_i}. Choose a trivializing section s_i. Then $\text{pr}_0^*s_i$ and $\text{pr}_1^*s_j$ are both trivialising sections of $\mathcal{L}$ over $X_{B_i \times_B B_j}$ and hence differ by a multiplicative unit

$$f_{ij} \in \Gamma(X_{S_i \times_B B_j}, \mathcal{O}^*_{X_{B_i \times_B B_j}}) = \Gamma(B_i \times_B B_j, \mathcal{O}^*_{B_i \times_N B_j})$$

(equality by our assumption on pushforward of structure sheaves). Of course these elements satisfy the cocycle condition on $B_i \times_B B_j \times_B B_k$, hence they define a descent datum on invertible sheaves for the fppf covering $\{B_i \to B\}$. By Descent, Proposition 35.5.2 there is an invertible $\mathcal{O}_B$-module $\mathcal{N}$ with trivializations over B_i whose associated descent datum is $\{f_{ij}\}$. (The proposition applies because B is a scheme by the replacement performed at the start of the proof.) Then $f^*\mathcal{N} \cong \mathcal{L}$ as the functor from descent data to modules is fully faithful. $\square$

0D28 Lemma 99.11.4. In Situation 99.11.1 let $\sigma : B \to X$ be a section. Assume that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all T over B. Then

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/B}(T) \to 0$$

is a split exact sequence with splitting given by $\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T)$.

Proof. Denote $K(T) = \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$. Since σ is a section of f we see that $\mathrm{Pic}(X_T)$ is the direct sum of $\mathrm{Pic}(T)$ and $K(T)$. Thus by Lemma 99.11.3 we see that $K(T) \subset \mathrm{Pic}_{X/B}(T)$ for all T. Moreover, it is clear from the construction that $\mathrm{Pic}_{X/B}$ is the sheafification of the presheaf K. To finish the proof it suffices to show that K satisfies the sheaf condition for fppf coverings which we do in the next paragraph.

Let $\{T_i \to T\}$ be an fppf covering. Let $\mathcal{L}_i$ be elements of $K(T_i)$ which map to the same elements of $K(T_i \times_T T_j)$ for all i and j. Choose an isomorphism $\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^*\mathcal{L}_i$ for all i. Choose an isomorphism

$$\varphi_{ij} : \mathcal{L}_i|_{X_{T_i \times_T T_j}} \longrightarrow \mathcal{L}_j|_{X_{T_i \times_T T_j}}$$

If the map

$$\alpha_j|_{T_i \times_T T_j} \circ \sigma^*_{T_i \times_T T_j}\varphi_{ij} \circ \alpha_i|_{T_i \times_T T_j} : \mathcal{O}_{T_i \times_T T_j} \to \mathcal{O}_{T_i \times_T T_j}$$

is not equal to multiplication by 1 but some u_{ij}, then we can scale φ_{ij} by u_{ij}^{-1} to correct this. Having done this, consider the self map

$$\varphi_{ki}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{jk}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{ij}|_{X_{T_i \times_T T_j \times_T T_k}} \quad \text{on} \quad \mathcal{L}_i|_{X_{T_i \times_T T_j \times_T T_k}}$$

which is given by multiplication by some section f_{ijk} of the structure sheaf of $X_{T_i \times_T T_j \times_T T_k}$. By our choice of φ_{ij} we see that the pullback of this map by σ is equal to multiplication by 1. By our assumption on functions on X, we see that $f_{ijk} = 1$. Thus we obtain a descent datum for the fppf covering $\{X_{T_i} \to X\}$. By Descent on Spaces, Proposition 74.4.1 there is an invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ and an isomorphism $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ whose pullback to X_{T_i} recovers $(\mathcal{L}_i, \alpha_i)$ (small detail omitted). Thus $\mathcal{L}$ defines an object of $K(T)$ as desired. $\square$

In Situation 99.11.1 let $\sigma : B \to X$ be a section. We denote $\mathcal{P}ic_{X/B,\sigma}$ the category defined as follows:

(1) An object is a quadruple $(T, h, \mathcal{L}, \alpha)$, where $(T, h, \mathcal{L})$ is an object of $\mathcal{P}ic_{X/B}$ over T and $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ is an isomorphism.
(2) A morphism $(g, \varphi) : (T, h, \mathcal{L}, \alpha) \to (T', h', \mathcal{L}', \alpha')$ is given by a morphism of schemes $g : T \to T'$ with $h = h' \circ g$ and an isomorphism $\varphi : (g')^*\mathcal{L}' \to \mathcal{L}$ such that $\sigma_T^*\varphi \circ g^*\alpha' = \alpha$. Here $g' : X_{T'} \to X_T$ is the base change of g.

There is a natural faithful forgetful functor

$$\mathcal{P}ic_{X/B,\sigma} \longrightarrow \mathcal{P}ic_{X/B}$$

In this way we view $\mathcal{P}ic_{X/B,\sigma}$ as a category over $(Sch/S)_{fppf}$.

0D29 Lemma 99.11.5. In Situation 99.11.1 let $\sigma : B \to X$ be a section. Then $\mathcal{P}ic_{X/B,\sigma}$ as defined above is a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. We already know that $\mathcal{P}ic_{X/B}$ is a stack in groupoids over $(Sch/S)_{fppf}$ by Examples of Stacks, Lemma 95.16.1. Let us show descent for objects for $\mathcal{P}ic_{X/B,\sigma}$. Let $\{T_i \to T\}$ be an fppf covering and let $\xi_i = (T_i, h_i, \mathcal{L}_i, \alpha_i)$ be an object of $\mathcal{P}ic_{X/B,\sigma}$ lying over T_i, and let $\varphi_{ij} : \text{pr}_0^*\xi_i \to \text{pr}_1^*\xi_j$ be a descent datum. Applying the result for $\mathcal{P}ic_{X/B}$ we see that we may assume we have an object $(T, h, \mathcal{L})$ of $\mathcal{P}ic_{X/B}$ over T which pulls back to ξ_i for all i. Then we get

$$\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^*\mathcal{L}_i = (T_i \to T)^*\sigma_T^*\mathcal{L}$$

Since the maps φ_{ij} are compatible with the α_i we see that α_i and α_j pullback to the same map on $T_i \times_T T_j$. By descent of quasi-coherent sheaves (Descent, Proposition 35.5.2, we see that the α_i are the restriction of a single map $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ as desired. We omit the proof of descent for morphisms. $\square$

0D2A Lemma 99.11.6. In Situation 99.11.1 let $\sigma : B \to X$ be a section. The morphism $\mathcal{P}ic_{X/B,\sigma} \to \mathcal{P}ic_{X/B}$ is representable, surjective, and smooth.

Proof. Let T be a scheme and let $(Sch/T)_{fppf} \to \mathcal{P}ic_{X/B}$ be given by the object $\xi = (T, h, \mathcal{L})$ of $\mathcal{P}ic_{X/B}$ over T. We have to show that

$$(Sch/T)_{fppf} \times_{\xi, \mathcal{P}ic_{X/B}} \mathcal{P}ic_{X/B,\sigma}$$

is representable by a scheme V and that the corresponding morphism $V \to T$ is surjective and smooth. See Algebraic Stacks, Sections 94.6, 94.9, and 94.10. The forgetful functor $\mathcal{P}ic_{X/B,\sigma} \to \mathcal{P}ic_{X/B}$ is faithful on fibre categories and for T'/T the set of isomorphism classes is the set of isomorphisms

$$\alpha' : \mathcal{O}_{T'} \longrightarrow (T' \to T)^*\sigma_T^*\mathcal{L}$$

See Algebraic Stacks, Lemma 94.9.2. We know this functor is representable by an affine scheme U of finite presentation over T by Proposition 99.4.3 (applied to $\text{id} : T \to T$ and $\mathcal{O}_T$ and $\sigma^*\mathcal{L}$). Working Zariski locally on T we may assume that $\sigma_T^*\mathcal{L}$ is isomorphic to $\mathcal{O}_T$ and then we see that our functor is representable by $\mathbf{G}_m \times T$ over T. Hence $U \to T$ Zariski locally on T looks like the projection $\mathbf{G}_m \times T \to T$ which is indeed smooth and surjective. $\square$

0D2B Lemma 99.11.7. In Situation 99.11.1 let $\sigma : B \to X$ be a section. If $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all T over B, then $\mathcal{P}ic_{X/B,\sigma} \to (Sch/S)_{fppf}$ is fibred in setoids with set of isomorphism classes over T given by

$$\coprod\nolimits_{h : T \to B} \text{Ker}(\sigma_T^* : \text{Pic}(X \times_{B,h} T) \to \text{Pic}(T))$$

Proof. If $\xi = (T, h, \mathcal{L}, \alpha)$ is an object of $\mathcal{P}ic_{X/B,\sigma}$ over T, then an automorphism φ of ξ is given by multiplication with an invertible global section u of the structure sheaf of X_T such that moreover $\sigma_T^*u = 1$. Then $u = 1$ by our assumption that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. Hence $\mathcal{P}ic_{X/B,\sigma}$ is fibred in setoids over $(Sch/S)_{fppf}$. Given T and $h : T \to B$ the set of isomorphism classes of pairs $(\mathcal{L}, \alpha)$ is the same as the set of isomorphism classes of $\mathcal{L}$ with $\sigma_T^*\mathcal{L} \cong \mathcal{O}_T$ (isomorphism not specified).

This is clear because any two choices of α differ by a global unit on T and this is the same thing as a global unit on X_T. $\square$

0D2C Proposition 99.11.8. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that

(1) f is flat, of finite presentation, and proper, and
(2) $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all schemes T over B.

Then $\mathrm{Pic}_{X/B}$ is an algebraic space.

In the situation of the proposition the algebraic stack $\mathcal{P}ic_{X/B}$ is a gerbe over the algebraic space $\mathrm{Pic}_{X/B}$. After developing the general theory of gerbes, this provides a shorter proof of the proposition (but using more general theory).

Proof. There exists a surjective, flat, finitely presented morphism $B' \to B$ of algebraic spaces such that the base change $X' = X \times_B B'$ over B' has a section: namely, we can take $B' = X$. Observe that $\mathrm{Pic}_{X'/B'} = B' \times_B \mathrm{Pic}_{X/B}$. Hence $\mathrm{Pic}_{X'/B'} \to \mathrm{Pic}_{X/B}$ is representable by algebraic spaces, surjective, flat, and finitely presented. Hence, if we can show that $\mathrm{Pic}_{X'/B'}$ is an algebraic space, then it follows that $\mathrm{Pic}_{X/B}$ is an algebraic space by Bootstrap, Theorem 80.10.1. In this way we reduce to the case described in the next paragraph.

In addition to the assumptions of the proposition, assume that we have a section $\sigma : B \to X$. By Proposition 99.10.2 we see that $\mathcal{P}ic_{X/B}$ is an algebraic stack. By Lemma 99.11.6 and Algebraic Stacks, Lemma 94.15.4 we see that $\mathcal{P}ic_{X/B,\sigma}$ is an algebraic stack. By Lemma 99.11.7 and Algebraic Stacks, Lemma 94.8.2 we see that $T \mapsto \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$ is an algebraic space. By Lemma 99.11.4 this functor is the same as $\mathrm{Pic}_{X/B}$. $\square$

0D2D Lemma 99.11.9. With assumptions and notation as in Proposition 99.11.8. Then the diagonal $\mathrm{Pic}_{X/B} \to \mathrm{Pic}_{X/B} \times_B \mathrm{Pic}_{X/B}$ is representable by immersions. In other words, $\mathrm{Pic}_{X/B} \to B$ is locally separated.

Proof. Let T be a scheme over B and let $s, t \in \mathrm{Pic}_{X/B}(T)$. We want to show that there exists a locally closed subscheme $Z \subset T$ such that $s|_Z = t|_Z$ and such that a morphism $T' \to T$ factors through Z if and only if $s|_{T'} = t|_{T'}$.

We first reduce the general problem to the case where s and t come from invertible modules on X_T. We suggest the reader skip this step. Choose an fppf covering $\{T_i \to T\}_{i \in I}$ such that $s|_{T_i}$ and $t|_{T_i}$ come from $\mathrm{Pic}(X_{T_i})$ for all i. Suppose that we can show the result for all the pairs $s|_{T_i}, t|_{T_i}$. Then we obtain locally closed subschemes $Z_i \subset T_i$ with the desired universal property. It follows that Z_i and Z_j have the same scheme theoretic inverse image in $T_i \times_T T_j$. This determines a descend datum on Z_i/T_i. Since $Z_i \to T_i$ is locally quasi-finite, it follows from More on Morphisms, Lemma 37.57.1 that we obtain a locally quasi-finite morphism $Z \to T$ recovering $Z_i \to T_i$ by base change. Then $Z \to T$ is an immersion by Descent, Lemma 35.24.1. Finally, because $\mathrm{Pic}_{X/B}$ is an fppf sheaf, we conclude that $s|_Z = t|_Z$ and that Z satisfies the universal property mentioned above.

Assume s and t come from invertible modules $\mathcal{V}$, $\mathcal{W}$ on X_T. Set $\mathcal{L} = \mathcal{V} \otimes \mathcal{W}^{\otimes -1}$ We are looking for a locally closed subscheme Z of T such that $T' \to T$ factors through Z if and only if $\mathcal{L}_{X_{T'}}$ is the pullback of an invertible sheaf on T', see Lemma 99.11.3. Hence the existence of Z follows from More on Morphisms of Spaces, Lemma 76.53.1. $\square$

99.12. Relative morphisms

0D19 We continue the discussion from Criteria for Representability, Section 97.10. In that section, starting with a scheme S and morphisms of algebraic spaces $Z \to B$ and $X \to B$ over S we constructed a functor

$$Mor_B(Z, X) : (Sch/B)^{opp} \longrightarrow \text{Sets}, \quad T \longmapsto \{f : Z_T \to X_T\}$$

We sometimes think of $Mor_B(Z, X)$ as a functor $(Sch/S)^{opp} \to \text{Sets}$ endowed with a morphism $Mor_B(Z, X) \to B$. Namely, if T is a scheme over S, then an element of $Mor_B(Z, X)(T)$ is a pair (f, h) where h is a morphism $h : T \to B$ and $f : Z \times_{B,h} T \to X \times_{B,h} T$ is a morphism of algebraic spaces over T. In particular, when we say that $Mor_B(Z, X)$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to \text{Sets}$ is an algebraic space.

0D1A **Lemma 99.12.1.** Let S be a scheme. Consider morphisms of algebraic spaces $Z \to B$ and $X \to B$ over S. If $X \to B$ is separated and $Z \to B$ is of finite presentation, flat, and proper, then there is a natural injective transformation of functors

$$Mor_B(Z, X) \longrightarrow \text{Hilb}_{Z \times_B X/B}$$

which maps a morphism $f : Z_T \to X_T$ to its graph.

Proof. Given a scheme T over B and a morphism $f_T : Z_T \to X_T$ over T, the graph of f is the morphism $\Gamma_f = (\text{id}, f) : Z_T \to Z_T \times_T X_T = (Z \times_B X)_T$. Recall that being separated, flat, proper, or finite presentation are properties of morphisms of algebraic spaces which are stable under base change (Morphisms of Spaces, Lemmas 67.4.4, 67.30.4, 67.40.3, and 67.28.3). Hence Γ_f is a closed immersion by Morphisms of Spaces, Lemma 67.4.6. Moreover, $\Gamma_f(Z_T)$ is flat, proper, and of finite presentation over T. Thus $\Gamma_f(Z_T)$ defines an element of $\text{Hilb}_{Z \times_B X/B}(T)$. To show the transformation is injective it suffices to show that two morphisms with the same graph are the same. This is true because if $Y \subset (Z \times_B X)_T$ is the graph of a morphism f, then we can recover f by using the inverse of $\text{pr}_1|_Y : Y \to Z_T$ composed with $\text{pr}_2|_Y$. $\square$

0D1B **Lemma 99.12.2.** Assumption and notation as in Lemma 99.12.1. The transformation $Mor_B(Z, X) \longrightarrow \text{Hilb}_{Z \times_B X/B}$ is representable by open immersions.

Proof. Let T be a scheme over B and let $Y \subset (Z \times_B X)_T$ be an element of $\text{Hilb}_{Z \times_B X/B}(T)$. Then we see that Y is the graph of a morphism $Z_T \to X_T$ over T if and only if $k = \text{pr}_1|_Y : Y \to Z_T$ is an isomorphism. By More on Morphisms of Spaces, Lemma 76.49.6 there exists an open subscheme $V \subset T$ such that for any morphism of schemes $T' \to T$ we have $k_{T'} : Y_{T'} \to Z_{T'}$ is an isomorphism if and only if $T' \to T$ factors through V. This proves the lemma. $\square$

0D1C **Proposition 99.12.3.** Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Assume $X \to B$ is of finite presentation and separated and $Z \to B$ is of finite presentation, flat, and proper. Then $Mor_B(Z, X)$ is an algebraic space locally of finite presentation over B.

Proof. Immediate consequence of Lemma 99.12.2 and Proposition 99.9.4. $\square$

99.13. The stack of algebraic spaces

0D1D This section continuous the discussion started in Examples of Stacks, Sections 95.7, 95.8, and 95.12. Working over $\mathbf{Z}$, the discussion therein shows that we have a stack in groupoids

$$p'_{ft} : \mathcal{S}paces'_{ft} \longrightarrow Sch_{fppf}$$

parametrizing (nonflat) families of finite type algebraic spaces. More precisely, an object[3] of $\mathcal{S}paces'_{ft}$ is a finite type morphism $X \to S$ from an algebraic space X to a scheme S and a morphism $(X' \to S') \to (X \to S)$ is given by a pair (f, g) where $f : X' \to X$ is a morphism of algebraic spaces and $g : S' \to S$ is a morphism of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian in the category of algebraic spaces. The functor p'_{ft} sends $(X \to S)$ to S and sends (f, g) to g. We define a full subcategory

$$\mathcal{S}paces'_{fp,flat,proper} \subset \mathcal{S}paces'_{ft}$$

consisting of objects $X \to S$ of $\mathcal{S}paces'_{ft}$ such that $X \to S$ is of finite presentation, flat, and proper. We denote

$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow Sch_{fppf}$$

the restriction of the functor p'_{ft} to the indicated subcategory. We first review the results already obtained in the references listed above, and then we start adding further results.

0D1E **Lemma 99.13.1.** *The category $\mathcal{S}paces'_{ft}$ is fibred in groupoids over Sch_{fppf}. The same is true for $\mathcal{S}paces'_{fp,flat,proper}$.*

Proof. We have seen this in Examples of Stacks, Section 95.12 for the case of $\mathcal{S}paces'_{ft}$ and this easily implies the result for the other case. However, let us also prove this directly by checking conditions (1) and (2) of Categories, Definition 4.35.1.

Condition (1). Let $X \to S$ be an object of $\mathcal{S}paces'_{ft}$ and let $S' \to S$ be a morphism of schemes. Then we set $X' = S' \times_S X$. Note that $X' \to S'$ is of finite type by Morphisms of Spaces, Lemma 67.23.3. to obtain a morphism $(X' \to S') \to (X \to S)$ lying over $S' \to S$. Argue similarly for the other case using Morphisms of Spaces, Lemmas 67.28.3, 67.30.4, and 67.40.3.

Condition (2). Consider morphisms $(f, g) : (X' \to S') \to (X \to S)$ and $(a, b) : (Y \to T) \to (X \to S)$ of $\mathcal{S}paces'_{ft}$. Given a morphism $h : T \to S'$ with $g \circ h = b$ we have to show there is a unique morphism $(k, h) : (Y \to T) \to (X' \to S')$ of $\mathcal{S}paces'_{ft}$ such that $(f, g) \circ (k, h) = (a, b)$. This is clear from the fact that $X' = S' \times_S X$. The same therefore works for any full subcategory of $\mathcal{S}paces'_{ft}$ satisfying (1). $\square$

[3] We always perform a replacement as in Examples of Stacks, Lemma 95.8.2.

0D1F Lemma 99.13.2. The diagonal

$$\Delta : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow \mathcal{S}paces'_{fp,flat,proper} \times \mathcal{S}paces'_{fp,flat,proper}$$

is representable by algebraic spaces.

Proof. We will use criterion (2) of Algebraic Stacks, Lemma 94.10.11. Let S be a scheme and let X and Y be algebraic spaces of finite presentation over S, flat over S, and proper over S. We have to show that the functor

$$Isom_S(X,Y) : (Sch/S)_{fppf} \longrightarrow \text{Sets}, \quad T \longmapsto \{f : X_T \to Y_T \text{ isomorphism}\}$$

is an algebraic space. An elementary argument shows that $Isom_S(X,Y)$ sits in a fibre product

$$\begin{tikzcd} Isom_S(X,Y) \arrow[r] \arrow[d] & S \arrow[d, "(\mathrm{id},\mathrm{id})"] \\ Mor_S(X,Y) \times Mor_S(Y,X) \arrow[r] & Mor_S(X,X) \times Mor_S(Y,Y) \end{tikzcd}$$

The bottom arrow sends (φ, ψ) to $(\psi \circ \varphi, \varphi \circ \psi)$. By Proposition 99.12.3 the functors on the bottom row are algebraic spaces over S. Hence the result follows from the fact that the category of algebraic spaces over S has fibre products. $\square$

0D1G Lemma 99.13.3. The category $\mathcal{S}paces'_{ft}$ is a stack in groupoids over Sch_{fppf}. The same is true for $\mathcal{S}paces'_{fp,flat,proper}$.

Proof. The reason this lemma holds is the slogan: any fppf descent datum for algebraic spaces is effective, see Bootstrap, Section 80.11. More precisely, the lemma for $\mathcal{S}paces'_{ft}$ follows from Examples of Stacks, Lemma 95.8.1 as we saw in Examples of Stacks, Section 95.12. However, let us review the proof. We need to check conditions (1), (2), and (3) of Stacks, Definition 8.5.1.

Property (1) we have seen in Lemma 99.13.1.

Property (2) follows from Lemma 99.13.2 in the case of $\mathcal{S}paces'_{fp,flat,proper}$. In the case of $\mathcal{S}paces'_{ft}$ it follows from Examples of Stacks, Lemma 95.7.2 (and this is really the "correct" reference).

Condition (3) for $\mathcal{S}paces'_{ft}$ is checked as follows. Suppose given

(1) an fppf covering $\{U_i \to U\}_{i \in I}$ in Sch_{fppf},
(2) for each $i \in I$ an algebraic space X_i of finite type over U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$.

We have to show there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. This follows from Bootstrap, Lemma 80.11.3 part (2). By Descent on Spaces, Lemma 74.11.11 we see that $X \to U$ is of finite type. In the case of $\mathcal{S}paces'_{fp,flat,proper}$ one additionally uses Descent on Spaces, Lemma 74.11.12, 74.11.13, and 74.11.19 in the last step. $\square$

Sanity check: the stacks $\mathcal{S}paces'_{ft}$ and $\mathcal{S}paces'_{fp,flat,proper}$ play the same role among algebraic spaces.

0E93 Lemma 99.13.4. Let T be an algebraic space over $\mathbf{Z}$. Let $\mathcal{S}_T$ denote the corresponding algebraic stack (Algebraic Stacks, Sections 94.7, 94.8, and 94.13). We have an equivalence of categories

$$\left\{\begin{matrix}\text{morphisms of algebraic spaces}\\ X \to T \text{ of finite type}\end{matrix}\right\} \longrightarrow \mathrm{Mor}_{\mathrm{Cat}/Sch_{fppf}}(\mathcal{S}_T, \mathcal{S}paces'_{ft})$$

and an equivalence of categories

$$\left\{\begin{matrix}\text{morphisms of algebraic spaces } X \to T\\ \text{of finite presentation, flat, and proper}\end{matrix}\right\} \longrightarrow \mathrm{Mor}_{\mathrm{Cat}/Sch_{fppf}}(\mathcal{S}_T, \mathcal{S}paces'_{fp,flat,proper})$$

Proof. We are going to deduce this lemma from the fact that it holds for schemes (essentially by construction of the stacks) and the fact that fppf descent data for algebraic spaces over algerbaic spaces are effective. We strongly encourage the reader to skip the proof.

The construction from left to right in either arrow is straightforward: given $X \to T$ of finite type the functor $\mathcal{S}_T \to \mathcal{S}paces'_{ft}$ assigns to U/T the base change $X_U \to U$. We will explain how to construct a quasi-inverse.

If T is a scheme, then there is a quasi-inverse by the 2-Yoneda lemma, see Categories, Lemma 4.41.2. Let $p : U \to T$ be a surjective étale morphism where U is a scheme. Let $R = U \times_T U$ with projections $s, t : R \to U$. Observe that we obtain morphisms

$$\mathcal{S}_{U \times_T U \times_T U} \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; \mathcal{S}_R \; \substack{\longrightarrow\\ \longrightarrow} \; \mathcal{S}_U \longrightarrow \mathcal{S}_T$$

satisfying various compatibilities (on the nose).

Let $G : \mathcal{S}_T \to \mathcal{S}paces'_{ft}$ be a functor over Sch_{fppf}. The restriction of G to $\mathcal{S}_U$ via the map displayed above corresponds to a finite type morphism $X_U \to U$ of algebraic spaces via the 2-Yoneda lemma. Since $p \circ s = p \circ t$ we see that $R \times_{s,U} X_U$ and $R \times_{t,U} X_U$ both correspond to the restriction of G to $\mathcal{S}_R$. Thus we obtain a canonical isomorphism $\varphi : X_U \times_{U,t} R \to R \times_{s,U} X_U$ over R. This isomorphism satisfies the cocycle condition by the various compatibilities of the diagram given above. Thus a descent datum which is effective by Bootstrap, Lemma 80.11.3 part (2). In other words, we obtain an object $X \to T$ of the right hand side category. We omit checking the construction $G \rightsquigarrow X$ is functorial and that it is quasi-inverse to the other construction. In the case of $\mathcal{S}paces'_{fp,flat,proper}$ one additionally uses Descent on Spaces, Lemma 74.11.12, 74.11.13, and 74.11.19 in the last step to see that $X \to T$ is of finite presentation, flat, and proper. $\square$

0D1H Remark 99.13.5. Let B be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let $B\text{-Spaces}'_{ft}$ be the category consisting of pairs $(X \to S, h : S \to B)$ where $X \to S$ is an object of $\mathcal{S}paces'_{ft}$ and $h : S \to B$ is a morphism. A morphism $(X' \to S', h') \to (X \to S, h)$ in $B\text{-Spaces}'_{ft}$ is a morphism (f, g) in $\mathcal{S}paces'_{ft}$ such that $h \circ g = h'$. In this situation the diagram

$$\begin{tikzcd} B\text{-Spaces}'_{ft} \arrow[r] \arrow[d] & \mathcal{S}paces'_{ft} \arrow[d] \\ (Sch/B)_{fppf} \arrow[r] & Sch_{fppf} \end{tikzcd}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case $\mathcal{S}paces'_{ft}$ to the case of families over a given base algebraic space. Of course, a similar construction works for B-Spaces$'_{fp,flat,proper}$

0D1I Lemma 99.13.6. The stack $p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \to Sch_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).

Proof. Let $T = \lim T_i$ be the limits of a directed inverse system of affine schemes. By Limits of Spaces, Lemma 70.7.1 the category of algebraic spaces of finite presentation over T is the colimit of the categories of algebraic spaces of finite presentation over T_i. To finish the proof use that flatness and properness descends through the limit, see Limits of Spaces, Lemmas 70.6.12 and 70.6.13. $\square$

0D1J Lemma 99.13.7. Let

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

be a pushout in the category of schemes where $T \to T'$ is a thickening and $T \to S$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\begin{array}{c} \mathcal{S}paces'_{fp,flat,proper,S'} \\ \downarrow \\ \mathcal{S}paces'_{fp,flat,proper,S} \times_{\mathcal{S}paces'_{fp,flat,proper,T}} \mathcal{S}paces'_{fp,flat,proper,T'} \end{array}$$

is an equivalence.

Proof. The functor is an equivalence if we drop "proper" from the list of conditions and replace "of finite presentation" by "locally of finite presentation", see Pushouts of Spaces, Lemma 81.6.7. Thus it suffices to show that given a morphism $X' \to S'$ of an algebraic space to S' which is flat and locally of finite presentation, then $X' \to S'$ is proper if and only if $S \times_{S'} X' \to S$ and $T' \times_{S'} X' \to T'$ are proper. One implication follows from the fact that properness is preserved under base change (Morphisms of Spaces, Lemma 67.40.3) and the other from the fact that properness of $S \times_{S'} X' \to S$ implies properness of $X' \to S'$ by More on Morphisms of Spaces, Lemma 76.10.2. $\square$

0D1K Lemma 99.13.8. Let k be a field and let $x = (X \to \mathrm{Spec}(k))$ be an object of $\mathcal{X} = \mathcal{S}paces'_{fp,flat,proper}$ over $\mathrm{Spec}(k)$.

(1) If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 98.8) are finite dimensional, and
(2) in general the vector spaces $T_x(k)$ and $\mathrm{Inf}_x(k)$ (see Artin's Axioms, Section 98.21) are finite dimensional.

Proof. The discussion in Artin's Axioms, Section 98.8 only applies to fields of finite type over the base scheme $\mathrm{Spec}(\mathbf{Z})$. Our stack satisfies (RS*) by Lemma 99.13.7 and we may apply Artin's Axioms, Lemma 98.21.2 to get the vector spaces $T_x(k)$ and $\mathrm{Inf}_x(k)$ mentioned in (2). Moreover, in the finite type case these spaces agree with the ones mentioned in (1) by Artin's Axioms, Remark 98.21.7. With this out of the way we can start the proof. Observe that the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]) = \mathrm{Spec}(k[k])$ has conormal module k. Hence the formula in Deformation Theory, Lemma 91.14.2 describing infinitesimal deformations of X and infinitesimal automorphisms of X become

$$T_x(k) = \text{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) \quad \text{and} \quad \text{Inf}_x(k) = \text{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

By More on Morphisms of Spaces, Lemma 76.21.5 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1. By Derived Categories of Spaces, Lemma 75.8.4 the displayed Ext-groups are finite k-vector spaces and the proof is complete. □

Beware that openness of versality (as proved in the next lemma) is a bit strange because our stack does not satisfy formal effectiveness, see Examples, Section 110.70. Later we will apply the openness of versality to suitable substacks of $\mathit{Spaces}'_{fp,flat,proper}$ which do satisfy formal effectiveness to conclude that these stacks are algebraic.

0D3X Lemma 99.13.9. The stack in groupoids $\mathcal{X} = \mathit{Spaces}'_{fp,flat,proper}$ satisfies openness of versality over $\text{Spec}(\mathbf{Z})$. Similarly, after base change (Remark 99.13.5) openness of versality holds over any Noetherian base scheme S.

Proof. For the "usual" proof of this fact, please see the discussion in the remark following this proof. We will prove this using Artin's Axioms, Lemma 98.20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving, see Lemmas 99.13.2, 99.13.7, and 99.13.6. Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 98.20.3.

Let (R_n) be an inverse system of rings such that $R_n \to R_m$ is surjective with square zero kernel for all $n \geq m$. Let $X_n \to \text{Spec}(R_n)$ be a finitely presented, flat, proper morphism where X_n is an algebraic space and let $X_{n+1} \to X_n$ be a morphism over $\text{Spec}(R_{n+1})$ inducing an isomorphism $X_n = X_{n+1} \times_{\text{Spec}(R_{n+1})} \text{Spec}(R_n)$. We have to find a flat, proper, finitely presented morphism $X \to \text{Spec}(\lim R_n)$ whose source is an algebraic space such that X_n is the base change of X for all n.

Let $I_n = \text{Ker}(R_n \to R_1)$. We may think of $(X_1 \subset X_n) \to (\text{Spec}(R_1) \subset \text{Spec}(R_n))$ as a morphism of first order thickenings. (Please read some of the material on thickenings of algebraic spaces in More on Morphisms of Spaces, Section 76.9 before continuing.) The structure sheaf of X_n is an extension

$$0 \to \mathcal{O}_{X_1} \otimes_{R_1} I_n \to \mathcal{O}_{X_n} \to \mathcal{O}_{X_1} \to 0$$

over $0 \to I_n \to R_n \to R_1$, see More on Morphisms of Spaces, Lemma 76.18.1. Let's consider the extension

$$0 \to \lim \mathcal{O}_{X_1} \otimes_{R_1} I_n \to \lim \mathcal{O}_{X_n} \to \mathcal{O}_{X_1} \to 0$$

over $0 \to \lim I_n \to \lim R_n \to R_1 \to 0$. The displayed sequence is exact as the $R^1 \lim$ of the system of kernels is zero by Derived Categories of Spaces, Lemma 75.5.4. Observe that the map

$$\mathcal{O}_{X_1} \otimes_{R_1} \lim I_n \longrightarrow \lim \mathcal{O}_{X_1} \otimes_{R_1} I_n$$

induces an isomorphism upon applying the functor DQ_X, see Derived Categories of Spaces, Lemma 75.25.6. Hence we obtain a unique extension

$$0 \to \mathcal{O}_{X_1} \otimes_{R_1} \lim I_n \to \mathcal{O}' \to \mathcal{O}_{X_1} \to 0$$

over $0 \to \lim I_n \to \lim R_n \to R_1 \to 0$ by the equivalence of categories of Deformation Theory, Lemma 91.14.4. The sheaf $\mathcal{O}'$ determines a first order thickening of algebraic spaces $X_1 \subset X$ over $\mathrm{Spec}(R_1) \subset \mathrm{Spec}(\lim R_n)$ by More on Morphisms of Spaces, Lemma 76.9.7. Observe that $X \to \mathrm{Spec}(\lim R_n)$ is flat by the already used More on Morphisms of Spaces, Lemma 76.18.1. By More on Morphisms of Spaces, Lemma 76.18.3 we see that $X \to \mathrm{Spec}(\lim R_n)$ is proper and of finite presentation. This finishes the proof. □

0D1P Remark 99.13.10. Lemma 99.13.9 can also be shown using either Artin's Axioms, Lemma 98.24.4 (as in the first proof of Lemma 99.5.11), or using an obstruction theory as in Artin's Axioms, Lemma 98.22.2 (as in the second proof of Lemma 99.5.11). In both cases one uses the deformation and obstruction theory developed in Cotangent, Section 92.23 to translate the needed properties of deformations and obstructions into Ext-groups to which Derived Categories of Spaces, Lemma 75.23.3 can be applied. The second method (using an obstruction theory and therefore using the full cotangent complex) is perhaps the "standard" method used in most references.

99.14. The stack of polarized proper schemes

0D1L To study the stack of polarized proper schemes it suffices to work over $\mathbf{Z}$ as we can later pullback to any scheme or algebraic space we want (see Remark 99.14.5).

0D1M Situation 99.14.1. We define a category $\mathcal{P}olarized$ as follows. Objects are pairs $(X \to S, \mathcal{L})$ where

(1) $X \to S$ is a morphism of schemes which is proper, flat, and of finite presentation, and
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S (Morphisms, Definition 29.37.1).

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner (see Examples of Stacks, Sections 95.7 and 95.4). The forgetful functor

$$p : \mathcal{P}olarized \longrightarrow Sch_{fppf}, \quad (X \to S, \mathcal{L}) \longmapsto S$$

is how we view $\mathcal{P}olarized$ as a category over Sch_{fppf} (see Section 99.2 for notation).

In the previous section we have done a substantial amount of work on the stack $\mathcal{S}paces'_{fp,flat,proper}$ of finitely presented, flat, proper algebraic spaces. To use this material we consider the forgetful functor

0D3Y (99.14.1.1) $$\mathcal{P}olarized \longrightarrow \mathcal{S}paces'_{fp,flat,proper}, \quad (X \to S, \mathcal{L}) \longmapsto (X \to S)$$

This functor will be a useful tool in what follows. Observe that if $(X \to S)$ is in the essential image of (99.14.1.1), then X and S are schemes.

0D3Z Lemma 99.14.2. The category $\mathcal{P}olarized$ is fibred in groupoids over $\mathcal{S}paces'_{fp,flat,proper}$. The category $\mathcal{P}olarized$ is fibred in groupoids over Sch_{fppf}.

Proof. We check conditions (1) and (2) of Categories, Definition 4.35.1.

Condition (1). Let $(X \to S, \mathcal{L})$ be an object of $\mathcal{P}olarized$ and let $(X' \to S') \to (X \to S)$ be a morphism of $\mathcal{S}paces'_{fp,flat,proper}$. Then we let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to X'. Observe that X, S, S' are schemes, hence X' is a scheme as well (as the fibre product of schemes). Then $\mathcal{L}'$ is ample on X'/S' by Morphisms, Lemma 29.37.9. In this way we obtain a morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ lying over $(X' \to S') \to (X \to S)$.

Condition (2). Consider morphisms $(f, g, \varphi) : (X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ and $(a, b, \psi) : (Y \to T, \mathcal{N}) \to (X \to S, \mathcal{L})$ of $\mathcal{P}olarized$. Given a morphism $(k, h) : (Y \to T) \to (X' \to S')$ of $\mathcal{S}paces'_{fp,flat,proper}$ with $(f, g) \circ (k, h) = (a, b)$ we have to show there is a unique morphism $(k, h, \chi) : (Y \to T, \mathcal{N}) \to (X' \to S', \mathcal{L}')$ of $\mathcal{P}olarized$ such that $(f, g, \varphi) \circ (k, h, \chi) = (a, b, \psi)$. We can just take

$$\chi = \psi \circ (k^*\varphi)^{-1}$$

This proves condition (2). A composition of functors defining fibred categories defines a fibred category, see Categories, Lemma 4.33.12. This we see that $\mathcal{P}olarized$ is fibred in groupoids over Sch_{fppf} (strictly speaking we should check the fibre categories are groupoids and apply Categories, Lemma 4.35.2). $\square$

0D40 Lemma 99.14.3. The category $\mathcal{P}olarized$ is a stack in groupoids over $\mathcal{S}paces'_{fp,flat,proper}$ (endowed with the inherited topology, see Stacks, Definition 8.10.2). The category $\mathcal{P}olarized$ is a stack in groupoids over Sch_{fppf}.

Proof. We prove $\mathcal{P}olarized$ is a stack in groupoids over $\mathcal{S}paces'_{fp,flat,proper}$ by checking conditions (1), (2), and (3) of Stacks, Definition 8.5.1. We have already seen (1) in Lemma 99.14.2.

A covering of $\mathcal{S}paces'_{fp,flat,proper}$ comes about in the following manner: Let $X \to S$ be an object of $\mathcal{S}paces'_{fp,flat,proper}$. Suppose that $\{S_i \to S\}_{i \in I}$ is a covering of Sch_{fppf}. Set $X_i = S_i \times_S X$. Then $\{(X_i \to S_i) \to (X \to S)\}_{i \in I}$ is a covering of $\mathcal{S}paces'_{fp,flat,proper}$ and every covering of $\mathcal{S}paces'_{fp,flat,proper}$ is isomorphic to one of these. Set $S_{ij} = S_i \times_S S_j$ and $X_{ij} = S_{ij} \times_S X$ so that $(X_{ij} \to S_{ij}) = (X_i \to S_i) \times_{(X \to S)} (X_j \to S_j)$. Next, suppose that $\mathcal{L}, \mathcal{N}$ are ample invertible sheaves on X/S so that $(X \to S, \mathcal{L})$ and $(X \to S, \mathcal{N})$ are two objects of $\mathcal{P}olarized$ over the object $(X \to S)$. To check descent for morphisms, we assume we have morphisms $(\text{id}, \text{id}, \varphi_i)$ from $(X_i \to S_i, \mathcal{L}|_{X_i})$ to $(X_i \to S_i, \mathcal{N}|_{X_i})$ whose base changes to morphisms from $(X_{ij} \to S_{ij}, \mathcal{L}|_{X_{ij}})$ to $(X_{ij} \to S_{ij}, \mathcal{N}|_{X_{ij}})$ agree. Then $\varphi_i : \mathcal{L}|_{X_i} \to \mathcal{N}|_{X_i}$ are isomorphisms of invertible modules over X_i such that φ_i and φ_j restrict to the same isomorphisms over X_{ij}. By descent for quasi-coherent sheaves (Descent on Spaces, Proposition 74.4.1) we obtain a unique isomorphism $\varphi : \mathcal{L} \to \mathcal{N}$ whose restriction to X_i recovers φ_i.

Decent for objects is proved in exactly the same manner. Namely, suppose that $\{(X_i \to S_i) \to (X \to S)\}_{i \in I}$ is a covering of $\mathcal{S}paces'_{fp,flat,proper}$ as above. Suppose we have objects $(X_i \to S_i, \mathcal{L}_i)$ of $\mathcal{P}olarized$ lying over $(X_i \to S_i)$ and a descent datum

$$(\text{id}, \text{id}, \varphi_{ij}) : (X_{ij} \to S_{ij}, \mathcal{L}_i|_{X_{ij}}) \to (X_{ij} \to S_{ij}, \mathcal{L}_j|_{X_{ij}})$$

satisfying the obvious cocycle condition over $(X_{ijk} \to S_{ijk})$ for every triple of indices. Then by descent for quasi-coherent sheaves (Descent on Spaces, Proposition 74.4.1) we obtain a unique invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $\mathcal{L}|_{X_i} \to \mathcal{L}_i$ recovering the descent datum φ_{ij}. To show that $(X \to S, \mathcal{L})$ is an object of $\mathcal{P}olarized$ we have to prove that $\mathcal{L}$ is ample. This follows from Descent on Spaces, Lemma 74.13.1.

Since we already have seen that $\mathcal{S}paces'_{fp,flat,proper}$ is a stack in groupoids over Sch_{fppf} (Lemma 99.13.3) it now follows formally that $\mathcal{P}olarized$ is a stack in groupoids over Sch_{fppf}. See Stacks, Lemma 8.10.6. □

Sanity check: the stack $\mathcal{P}olarized$ plays the same role among algebraic spaces.

0E94 Lemma 99.14.4. Let T be an algebraic space over $\mathbf{Z}$. Let $\mathcal{S}_T$ denote the corresponding algebraic stack (Algebraic Stacks, Sections 94.7, 94.8, and 94.13). We have an equivalence of categories

$$\left\{\begin{matrix}(X \to T, \mathcal{L}) \text{ where } X \to T \text{ is a morphism} \\ \text{of algebraic spaces, is proper, flat, and of} \\ \text{finite presentation and } \mathcal{L} \text{ ample on } X/T\end{matrix}\right\} \longrightarrow \mathrm{Mor}_{\mathrm{Cat}/Sch_{fppf}}(\mathcal{S}_T, \mathcal{P}olarized)$$

Proof. Omitted. Hints: Argue exactly as in the proof of Lemma 99.13.4 and use Descent on Spaces, Proposition 74.4.1 to descent the invertible sheaf in the construction of the quasi-inverse functor. The relative ampleness property descends by Descent on Spaces, Lemma 74.13.1. □

0D1N Remark 99.14.5. Let B be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let B-Polarized be the category consisting of triples $(X \to S, \mathcal{L}, h : S \to B)$ where $(X \to S, \mathcal{L})$ is an object of $\mathcal{P}olarized$ and $h : S \to B$ is a morphism. A morphism $(X' \to S', \mathcal{L}', h') \to (X \to S, \mathcal{L}, h)$ in B-Polarized is a morphism (f, g, φ) in $\mathcal{P}olarized$ such that $h \circ g = h'$. In this situation the diagram

$$\begin{tikzcd} B\text{-Polarized} \arrow[r] \arrow[d] & \mathcal{P}olarized \arrow[d] \\ (Sch/B)_{fppf} \arrow[r] & Sch_{fppf} \end{tikzcd}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case $\mathcal{P}olarized$ to the case of families over a given base algebraic space.

0D41 Lemma 99.14.6. The functor (99.14.1.1) defines a 1-morphism

$$\mathcal{P}olarized \to \mathcal{S}paces'_{fp,flat,proper}$$

of stacks in groupoids over Sch_{fppf} which is algebraic in the sense of Criteria for Representability, Definition 97.8.1.

Proof. By Lemmas 99.13.3 and 99.14.3 the statement makes sense. To prove it, we choose a scheme S and an object $\xi = (X \to S)$ of $\mathcal{S}paces'_{fp,flat,proper}$ over S. We have to show that

$$\mathcal{X} = (Sch/S)_{fppf} \times_{\xi, \mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized$$

is an algebraic stack over S. Observe that an object of $\mathcal{X}$ is given by a pair $(T/S, \mathcal{L})$ where T is a scheme over S and $\mathcal{L}$ is an invertible $\mathcal{O}_{X_T}$-module which is ample on X_T/T. Morphisms are defined in the obvious manner. In particular, we see immediately that we have an inclusion

$$\mathcal{X} \subset \mathcal{P}ic_{X/S}$$

of categories over $(Sch/S)_{fppf}$, inducing equality on morphism sets. Since $\mathcal{P}ic_{X/S}$ is an algebraic stack by Proposition 99.10.2 it suffices to show that the inclusion above is representable by open immersions. This is exactly the content of Descent on Spaces, Lemma 74.13.2. □

0D42 Lemma 99.14.7. The diagonal

$$\Delta : \mathcal{P}olarized \longrightarrow \mathcal{P}olarized \times \mathcal{P}olarized$$

is representable by algebraic spaces.

Proof. This is a formal consequence of Lemmas 99.14.6 and 99.13.2. See Criteria for Representability, Lemma 97.8.4. □

0D43 Lemma 99.14.8. The stack in groupoids $\mathcal{P}olarized$ is limit preserving (Artin's Axioms, Definition 98.11.1).

Proof. Let I be a directed set and let $(A_i, \varphi_{ii'})$ be a system of rings over I. Set $S = \text{Spec}(A)$ and $S_i = \text{Spec}(A_i)$. We have to show that on fibre categories we have

$$\mathcal{P}olarized_S = \text{colim}\, \mathcal{P}olarized_{S_i}$$

We know that the category of schemes of finite presentation over S is the colimit of the category of schemes of finite presentation over S_i, see Limits, Lemma 32.10.1. Moreover, given $X_i \to S_i$ of finite presentation, with limit $X \to S$, then the category of invertible $\mathcal{O}_X$-modules $\mathcal{L}$ is the colimit of the categories of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$, see Limits, Lemma 32.10.2 and 32.10.3. If $X \to S$ is proper and flat, then for sufficiently large i the morphism $X_i \to S_i$ is proper and flat too, see Limits, Lemmas 32.13.1 and 32.8.7. Finally, if $\mathcal{L}$ is ample on X then $\mathcal{L}_i$ is ample on X_i for i sufficiently large, see Limits, Lemma 32.4.15. Putting everything together finishes the proof. □

0D44 Lemma 99.14.9. In Situation 99.5.1. Let

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

be a pushout in the category of schemes where $T \to T'$ is a thickening and $T \to S$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{P}olarized_{S'} \longrightarrow \mathcal{P}olarized_S \times_{\mathcal{P}olarized_T} \mathcal{P}olarized_{T'}$$

is an equivalence.

Proof. By More on Morphisms, Lemma 37.14.6 there is an equivalence

$$\text{flat-lfp}_{S'} \longrightarrow \text{flat-lfp}_S \times_{\text{flat-lfp}_T} \text{flat-lfp}_{T'}$$

where flat-lfp_S signifies the category of schemes flat and locally of finite presentation over S. Let X'/S' on the left hand side correspond to the triple $(X/S, Y'/T', \varphi)$

on the right hand side. Set $Y = T \times_{T'} Y'$ which is isomorphic with $T \times_S X$ via φ. Then More on Morphisms, Lemma 37.14.5 shows that we have an equivalence

$$\text{QCoh-flat}_{X'/S'} \longrightarrow \text{QCoh-flat}_{X/S} \times_{\text{QCoh-flat}_{Y/T}} \text{QCoh-flat}_{Y'/T'}$$

where $\text{QCoh-flat}_{X/S}$ signifies the category of quasi-coherent $\mathcal{O}_X$-modules flat over S. Since $X \to S$, $Y \to T$, $X' \to S'$, $Y' \to T'$ are flat, this will in particular apply to invertible modules to give an equivalence of categories

$$\text{Pic}(X') \longrightarrow \text{Pic}(X) \times_{\text{Pic}(Y)} \text{Pic}(Y')$$

where $\text{Pic}(X)$ signifies the category of invertible $\mathcal{O}_X$-modules. There is a small point here: one has to show that if an object $\mathcal{F}'$ of $\text{QCoh-flat}_{X'/S'}$ pulls back to invertible modules on X and Y', then $\mathcal{F}'$ is an invertible $\mathcal{O}_{X'}$-module. It follows from the cited lemma that $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module of finite presentation. By More on Morphisms, Lemma 37.16.7 it suffices to check the restriction of $\mathcal{F}'$ to fibres of $X' \to S'$ is invertible. But the fibres of $X' \to S'$ are the same as the fibres of $X \to S$ and hence these restrictions are invertible.

Having said the above we obtain an equivalence of categories if we drop the assumption (for the category of objects over S) that $X \to S$ be proper and the assumption that $\mathcal{L}$ be ample. Now it is clear that if $X' \to S'$ is proper, then $X \to S$ and $Y' \to T'$ are proper (Morphisms, Lemma 29.41.5). Conversely, if $X \to S$ and $Y' \to T'$ are proper, then $X' \to S'$ is proper by More on Morphisms, Lemma 37.3.3. Similarly, if $\mathcal{L}'$ is ample on X'/S', then $\mathcal{L}'|_X$ is ample on X/S and $\mathcal{L}'|_{Y'}$ is ample on Y'/T' (Morphisms, Lemma 29.37.9). Finally, if $\mathcal{L}'|_X$ is ample on X/S and $\mathcal{L}'|_{Y'}$ is ample on Y'/T', then $\mathcal{L}'$ is ample on X'/S' by More on Morphisms, Lemma 37.3.2. $\square$

0D4S Lemma 99.14.10. Let k be a field and let $x = (X \to \text{Spec}(k), \mathcal{L})$ be an object of $\mathcal{X} = \mathcal{P}olarized$ over $\text{Spec}(k)$.

(1) If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 98.8) are finite dimensional, and
(2) in general the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ (see Artin's Axioms, Section 98.21) are finite dimensional.

Proof. The discussion in Artin's Axioms, Section 98.8 only applies to fields of finite type over the base scheme $\text{Spec}(\mathbf{Z})$. Our stack satisfies (RS*) by Lemma 99.14.9 and we may apply Artin's Axioms, Lemma 98.21.2 to get the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ mentioned in (2). Moreover, in the finite type case these spaces agree with the ones mentioned in part (1) by Artin's Axioms, Remark 98.21.7. With this out of the way we can start the proof.

One proof is to use an argument as in the proof of Lemma 99.13.8; this would require us to develop a deformation theory for pairs consisting of a scheme and a quasi-coherent module. Another proof would be the use the result from Lemma 99.13.8, the algebraicity of $\mathcal{P}olarized \to \mathcal{S}paces'_{fp,flat,proper}$, and a computation of the deformation space of an invertible module. However, what we will do instead is to translate the question into a deformation question on graded k-algebras and deduce the result that way.

Let $\mathcal{C}_k$ be the category of Artinian local k-algebras A with residue field k. We get a predeformation category $p : \mathcal{F} \to \mathcal{C}_k$ from our object x of $\mathcal{X}$ over k, see Artin's Axioms, Section 98.3. Thus $\mathcal{F}(A)$ is the category of triples $(X_A, \mathcal{L}_A, \alpha)$, where $(X_A, \mathcal{L}_A)$ is an object of $\mathcal{P}olarized$ over A and α is an isomorphism $(X_A, \mathcal{L}_A) \times_{\text{Spec}(A)}$

$\mathrm{Spec}(k) \cong (X, \mathcal{L})$. On the other hand, let $q : \mathcal{G} \to \mathcal{C}_k$ be the category cofibred in groupoids defined in Deformation Problems, Example 93.7.1. Choose $d_0 \gg 0$ (we'll see below how large). Let P be the graded k-algebra

$$P = k \oplus \bigoplus\nolimits_{d \geq d_0} H^0(X, \mathcal{L}^{\otimes d})$$

Then $y = (k, P)$ is an object of $\mathcal{G}(k)$. Let $\mathcal{G}_y$ be the predeformation category of Formal Deformation Theory, Remark 90.6.4. Given $(X_A, \mathcal{F}_A, \alpha)$ as above we set

$$Q = A \oplus \bigoplus\nolimits_{d \geq d_0} H^0(X_A, \mathcal{L}_A^{\otimes d})$$

The isomorphism α induces a map $\beta : Q \to P$. By deformation theory of projective schemes (More on Morphisms, Lemma 37.10.6) we obtain a 1-morphism

$$\mathcal{F} \longrightarrow \mathcal{G}_y, \quad (X_A, \mathcal{F}_A, \alpha) \longmapsto (Q, \beta : Q \to P)$$

of categories cofibred in groupoids over $\mathcal{C}_k$. In fact, this functor is an equivalence with quasi-inverse given by $Q \mapsto \underline{\mathrm{Proj}}_A(Q)$. Namely, the scheme $X_A = \underline{\mathrm{Proj}}_A(Q)$ is flat over A by Divisors, Lemma 31.30.6. Set $\mathcal{L}_A = \mathcal{O}_{X_A}(1)$; this is flat over A by the same lemma. We get an isomorphism $(X_A, \mathcal{L}_A) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = (X, \mathcal{L})$ from β. Then we can deduce all the desired properties of the pair $(X_A, \mathcal{L}_A)$ from the corresponding properties of $(X, \mathcal{L})$ using the techniques in More on Morphisms, Sections 37.3 and 37.10. Some details omitted.

In conclusion, we see that $T\mathcal{F} = T\mathcal{G}_y = T_y\mathcal{G}$ and $\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_y(\mathcal{G})$. These vector spaces are finite dimensional by Deformation Problems, Lemma 93.7.3 and the proof is complete. ☐

0D4T Lemma 99.14.11 (Strong formal effectiveness for polarized schemes). Let (R_n) be an inverse system of rings with surjective transition maps whose kernels are locally nilpotent. Set $R = \lim R_n$. Set $S_n = \mathrm{Spec}(R_n)$ and $S = \mathrm{Spec}(R)$. Consider a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots \\
S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots
\end{tikzcd}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^*\mathcal{L}_{n+1} \to \mathcal{L}_n$ is an isomorphism. If

(1) $X_n \to S_n$ is proper, flat, of finite presentation, and
(2) $\mathcal{L}_1$ is ample on X_1

then there exists a morphism of schemes $X \to S$ proper, flat, and of finite presentation and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n.

Proof. Choose d_0 for $X_1 \to S_1$ and $\mathcal{L}_1$ as in More on Morphisms, Lemma 37.10.6. For any $n \geq 1$ set

$$A_n = R_n \oplus \bigoplus\nolimits_{d \geq d_0} H^0(X_n, \mathcal{L}_n^{\otimes d})$$

By the lemma each A_n is a finitely presented graded R_n-algebra whose homogeneous parts $(A_n)_d$ are finite projective R_n-modules such that $X_n = \mathrm{Proj}(A_n)$ and $\mathcal{L}_n = \mathcal{O}_{\mathrm{Proj}(A_n)}(1)$. The lemma also guarantees that the maps

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \ldots$$

induce isomorphisms $A_n = A_m \otimes_{R_m} R_n$ for $n \leq m$. We set

$$B = \bigoplus\nolimits_{d \geq 0} B_d \quad \text{with} \quad B_d = \lim\nolimits_n (A_n)_d$$

By More on Algebra, Lemma 15.13.3 we see that B_d is a finite projective R-module and that $B \otimes_R R_n = A_n$. Thus the scheme

$$X = \text{Proj}(B) \quad \text{and} \quad \mathcal{L} = \mathcal{O}_X(1)$$

is flat over S and $\mathcal{L}$ is a quasi-coherent $\mathcal{O}_X$-module flat over S, see Divisors, Lemma 31.30.6. Because formation of Proj commutes with base change (Constructions, Lemma 27.11.6) we obtain canonical isomorphisms

$$X \times_S S_n = X_n \quad \text{and} \quad \mathcal{L}|_{X_n} \cong \mathcal{L}_n$$

compatible with the transition maps of the system. Thus we may think of $X_1 \subset X$ as a closed subscheme. Below we will show that B is of finite presentation over R. By Divisors, Lemmas 31.30.4 and 31.30.7 this implies that $X \to S$ is of finite presentation and proper and that $\mathcal{L} = \mathcal{O}_X(1)$ is of finite presentation as an $\mathcal{O}_X$-module. Since the restriction of $\mathcal{L}$ to the base change $X_1 \to S_1$ is invertible, we see from More on Morphisms, Lemma 37.16.8 that $\mathcal{L}$ is invertible on an open neighbourhood of X_1 in X. Since $X \to S$ is closed and since $\text{Ker}(R \to R_1)$ is contained in the Jacobson radical (More on Algebra, Lemma 15.11.3) we see that any open neighbourhood of X_1 in X is equal to X. Thus $\mathcal{L}$ is invertible. Finally, the set of points in S where $\mathcal{L}$ is ample on the fibre is open in S (More on Morphisms, Lemma 37.50.3) and contains S_1 hence equals S. Thus $X \to S$ and $\mathcal{L}$ have all the properties required of them in the statement of the lemma.

We prove the claim above. Choose a presentation $A_1 = R_1[X_1, \ldots, X_s]/(F_1, \ldots, F_t)$ where X_i are variables having degrees d_i and F_j are homogeneous polynomials in X_i of degree e_j. Then we can choose a map

$$\Psi : R[X_1, \ldots, X_s] \longrightarrow B$$

lifting the map $R_1[X_1, \ldots, X_s] \to A_1$. Since each B_d is finite projective over R we conclude from Nakayama's lemma (Algebra, Lemma 10.20.1 using again that $\text{Ker}(R \to R_1)$ is contained in the Jacobson radical of R) that Ψ is surjective. Since $- \otimes_R R_1$ is right exact we can find $G_1, \ldots, G_t \in \text{Ker}(\Psi)$ mapping to $F_1, \ldots, F_t$ in $R_1[X_1, \ldots, X_s]$. Observe that $\text{Ker}(\Psi)_d$ is a finite projective R-module for all $d \geq 0$ as the kernel of the surjection $R[X_1, \ldots, X_s]_d \to B_d$ of finite projective R-modules. We conclude from Nakayama's lemma once more that $\text{Ker}(\Psi)$ is generated by $G_1, \ldots, G_t$. □

0D4U Lemma 99.14.12. Consider the stack $\mathcal{P}olarized$ over the base scheme $\text{Spec}(\mathbf{Z})$. Then every formal object is effective.

Proof. For definitions of the notions in the lemma, please see Artin's Axioms, Section 98.9. From the definitions we see the lemma follows immediately from the more general Lemma 99.14.11. □

0D4V Lemma 99.14.13. The stack in groupoids $\mathcal{P}olarized$ satisfies openness of versality over $\text{Spec}(\mathbf{Z})$. Similarly, after base change (Remark 99.14.5) openness of versality holds over any Noetherian base scheme S.

Proof. This follows from Artin's Axioms, Lemma 98.20.3 and Lemmas 99.14.7, 99.14.9, 99.14.8, and 99.14.11. For the "usual" proof of this fact, please see the discussion in the remark following this proof. $\square$

0D4W Remark 99.14.14. Lemma 99.14.13 can also be shown using an obstruction theory as in Artin's Axioms, Lemma 98.22.2 (as in the second proof of Lemma 99.5.11). To do this one has to generalize the deformation and obstruction theory developed in Cotangent, Section 92.23 to the case of pairs of algebraic spaces and quasi-coherent modules. Another possibility is to use that the 1-morphism $\mathcal{P}olarized \to \mathcal{S}paces'_{fp,flat,proper}$ is algebraic (Lemma 99.14.6) and the fact that we know openness of versality for the target (Lemma 99.13.9 and Remark 99.13.10).

0D4X Theorem 99.14.15 (Algebraicity of the stack of polarized schemes). The stack $\mathcal{P}olarized$ (Situation 99.14.1) is algebraic. In fact, for any algebraic space B the stack B-Polarized (Remark 99.14.5) is algebraic.

Proof. The absolute case follows from Artin's Axioms, Lemma 98.17.1 and Lemmas 99.14.7, 99.14.9, 99.14.8, 99.14.12, and 99.14.13. The case over B follows from this, the description of B-Polarized as a 2-fibre product in Remark 99.14.5, and the fact that algebraic stacks have 2-fibre products, see Algebraic Stacks, Lemma 94.14.3. $\square$

99.15. The stack of curves

0D4Y In this section we prove the stack of curves is algebraic. For a further discussion of moduli of curves, we refer the reader to Moduli of Curves, Section 109.1.

A curve in the Stacks project is a variety of dimension 1. However, when we speak of families of curves, we often allow the fibres to be reducible and/or nonreduced. In this section, the stack of curves will "parametrize proper schemes of dimension ≤ 1". However, it turns out that in order to get the correct notion of a family we need to allow the total space of our family to be an algebraic space. This leads to the following definition.

0D4Z Situation 99.15.1. We define a category $\mathcal{C}urves$ as follows:

(1) Objects are families of curves. More precisely, an object is a morphism $f : X \to S$ where the base S is a scheme, the total space X is an algebraic space, and f is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 67.33.2).
(2) A morphism $(X' \to S') \to (X \to S)$ between objects is given by a pair (f, g) where $f : X' \to X$ is a morphism of algebraic spaces and $g : S' \to S$ is a morphism of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian.

The forgetful functor

$$p : \mathcal{C}urves \longrightarrow Sch_{fppf}, \quad (X \to S) \longmapsto S$$

is how we view *Curves* as a category over Sch_{fppf} (see Section 99.2 for notation).

It follows from Spaces over Fields, Lemma 72.9.3 and more generally More on Morphisms of Spaces, Lemma 76.43.6 that if S is the spectrum of a field, or an Artinian local ring, or a Noetherian complete local ring, then for any family of curves $X \to S$ the total space X is a scheme. On the other hand, there are families of curves over $\mathbf{A}^1_k$ where the total space is not a scheme, see Examples, Section 110.66.

It is clear that

0D50 (99.15.1.1)
$$\mathcal{C}urves \subset \mathcal{S}paces'_{fp,flat,proper}$$

and that an object $X \to S$ of $\mathcal{S}paces'_{fp,flat,proper}$ is in $\mathcal{C}urves$ if and only if $X \to S$ has relative dimension ≤ 1. We will use this to verify Artin's axioms for $\mathcal{C}urves$.

0D51 **Lemma 99.15.2.** *The category $\mathcal{C}urves$ is fibred in groupoids over Sch_{fppf}.*

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.1) this reduces to the following statement: Given a morphism

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

in $\mathcal{S}paces'_{fp,flat,proper}$ (recall that this implies in particular the diagram is cartesian) if $X \to S$ has relative dimension ≤ 1, then $X' \to S'$ has relative dimension ≤ 1. This follows from Morphisms of Spaces, Lemma 67.34.3. $\square$

0D52 **Lemma 99.15.3.** *The category $\mathcal{C}urves$ is a stack in groupoids over Sch_{fppf}.*

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.3) this reduces to the following statement: Given an object $X \to S$ of $\mathcal{S}paces'_{fp,flat,proper}$ and an fppf covering $\{S_i \to S\}_{i \in I}$ the following are equivalent:

(1) $X \to S$ has relative dimension ≤ 1, and
(2) for each i the base change $X_i \to S_i$ has relative dimension ≤ 1.

This follows from Morphisms of Spaces, Lemma 67.34.3. $\square$

0D53 **Lemma 99.15.4.** *The diagonal*

$$\Delta : \mathcal{C}urves \longrightarrow \mathcal{C}urves \times \mathcal{C}urves$$

is representable by algebraic spaces.

Proof. This is immediate from the fully faithful embedding (99.15.1.1) and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.2). $\square$

0D54 **Remark 99.15.5.** Let B be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let B-$\mathcal{C}urves$ be the category consisting of pairs $(X \to S, h : S \to B)$ where $X \to S$ is an object of $\mathcal{C}urves$ and $h : S \to B$ is a morphism. A morphism $(X' \to S', h') \to (X \to S, h)$ in

B-*Curves* is a morphism (f, g) in *Curves* such that $h \circ g = h'$. In this situation the diagram

$$\begin{tikzcd} B\text{-}\mathit{Curves} \arrow[r] \arrow[d] & \mathit{Curves} \arrow[d] \\ (Sch/B)_{fppf} \arrow[r] & Sch_{fppf} \end{tikzcd}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case *Curves* to the case of families of curves over a given base algebraic space.

0D55 Lemma 99.15.6. The stack $\mathit{Curves} \to Sch_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathit{Spaces}'_{fp,flat,proper}$ (Lemma 99.13.6) this reduces to the following statement: Let $T = \lim T_i$ be the limits of a directed inverse system of affine schemes. Let $i \in I$ and let $X_i \to T_i$ be an object of $\mathit{Spaces}'_{fp,flat,proper}$ over T_i. Assume that $T \times_{T_i} X_i \to T$ has relative dimension ≤ 1. Then for some $i' \geq i$ the morphism $T_{i'} \times_{T_i} X_i \to T_i$ has relative dimension ≤ 1. This follows from Limits of Spaces, Lemma 70.6.14. $\square$

0D56 Lemma 99.15.7. Let

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

be a pushout in the category of schemes where $T \to T'$ is a thickening and $T \to S$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathit{Curves}_{S'} \longrightarrow \mathit{Curves}_S \times_{\mathit{Curves}_T} \mathit{Curves}_{T'}$$

is an equivalence.

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathit{Spaces}'_{fp,flat,proper}$ (Lemma 99.13.7) this reduces to the following statement: given a morphism $X' \to S'$ of an algebraic space to S' which is of finite presentation, flat, proper then $X' \to S'$ has relative dimension ≤ 1 if and only if $S \times_{S'} X' \to S$ and $T' \times_{S'} X' \to T'$ have relative dimension ≤ 1. One implication follows from the fact that having relative dimension ≤ 1 is preserved under base change (Morphisms of Spaces, Lemma 67.34.3). The other follows from the fact that having relative dimension ≤ 1 is checked on the fibres and that the fibres of $X' \to S'$ (over points of the scheme S') are the same as the fibres of $S \times_{S'} X' \to S$ since $S \to S'$ is a thickening by More on Morphisms, Lemma 37.14.3. $\square$

0D57 Lemma 99.15.8. Let k be a field and let $x = (X \to \text{Spec}(k))$ be an object of $\mathcal{X} = \mathit{Curves}$ over $\text{Spec}(k)$.

(1) If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 98.8) are finite dimensional, and
(2) in general the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ (see Artin's Axioms, Section 98.21) are finite dimensional.

Proof. This is immediate from the fully faithful embedding (99.15.1.1) and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.8). $\square$

0D58 **Lemma 99.15.9.** Consider the stack $\mathcal{C}urves$ over the base scheme $\mathrm{Spec}(\mathbf{Z})$. Then every formal object is effective.

Proof. For definitions of the notions in the lemma, please see Artin's Axioms, Section 98.9. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. Let $(X_n \to \mathrm{Spec}(A/\mathfrak{m}^n))$ be a formal object of $\mathcal{C}urves$ over A. By More on Morphisms of Spaces, Lemma 76.43.5 there exists a projective morphism $X \to \mathrm{Spec}(A)$ and a compatible system of ismomorphisms $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n) \cong X_n$. By More on Morphisms, Lemma 37.12.4 we see that $X \to \mathrm{Spec}(A)$ is flat. By More on Morphisms, Lemma 37.30.6 we see that $X \to \mathrm{Spec}(A)$ has relative dimension ≤ 1. This proves the lemma. $\square$

0D59 **Lemma 99.15.10.** The stack in groupoids $\mathcal{X} = \mathcal{C}urves$ satisfies openness of versality over $\mathrm{Spec}(\mathbf{Z})$. Similarly, after base change (Remark 99.15.5) openness of versality holds over any Noetherian base scheme S.

Proof. This is immediate from the fully faithful embedding (99.15.1.1) and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.9). $\square$

0D5A **Theorem 99.15.11** (Algebraicity of the stack of curves)**.** The stack $\mathcal{C}urves$ (Situation 99.15.1) is algebraic. In fact, for any algebraic space B the stack $B\text{-}\mathcal{C}urves$ (Remark 99.15.5) is algebraic. See [dJHS11, Proposition 3.3, page 8] and [Smy13, Appendix B by Jack Hall, Theorem B.1].

Proof. The absolute case follows from Artin's Axioms, Lemma 98.17.1 and Lemmas 99.15.4, 99.15.7, 99.15.6, 99.15.9, and 99.15.10. The case over B follows from this, the description of $B\text{-}\mathcal{C}urves$ as a 2-fibre product in Remark 99.15.5, and the fact that algebraic stacks have 2-fibre products, see Algebraic Stacks, Lemma 94.14.3. $\square$

0D5B **Lemma 99.15.12.** The 1-morphism (99.15.1.1)

$$\mathcal{C}urves \longrightarrow \mathcal{S}paces'_{fp,flat,proper}$$

is representable by open and closed immersions.

Proof. Since (99.15.1.1) is a fully faithful embedding of categories it suffices to show the following: given an object $X \to S$ of $\mathcal{S}paces'_{fp,flat,proper}$ there exists an open and closed subscheme $U \subset S$ such that a morphism $S' \to S$ factors through U if and only if the base change $X' \to S'$ of $X \to S$ has relative dimension ≤ 1. This follows immediately from More on Morphisms of Spaces, Lemma 76.31.5. $\square$

0D5C **Remark 99.15.13.** Consider the 2-fibre product

$$\begin{tikzcd} \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized \arrow[r] \arrow[d] & \mathcal{P}olarized \arrow[d] \\ \mathcal{C}urves \arrow[r] & \mathcal{S}paces'_{fp,flat,proper} \end{tikzcd}$$

This fibre product parametrized polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. It turns out that the left vertical arrow

$$\text{PolarizedCurves} \longrightarrow \mathcal{C}urves$$

is algebraic, smooth, and surjective. Namely, this 1-morphism is algebraic (as base change of the arrow in Lemma 99.14.6), every point is in the image, and there are no obstructions to deforming invertible sheaves on curves (see proof of Lemma 99.15.9). This gives another approach to the algebraicity of $\mathcal{C}urves$. Namely, by Lemma 99.15.12 we see that PolarizedCurves is an open and closed substack of the algebraic stack $\mathcal{P}olarized$ and any stack in groupoids which is the target of a smooth algebraic morphism from an algebraic stack is an algebraic stack.

99.16. Moduli of complexes on a proper morphism

0DLB The title and the material of this section are taken from [Lie06a]. Let S be a scheme and let $f : X \to B$ be a proper, flat, finitely presented morphism of algebraic spaces. We will prove that there is an algebraic stack

$$\mathcal{C}omplexes_{X/B}$$

parametrizing "families" of objects of D^b_{Coh} of the fibres with vanishing negative self-exts. More precisely a family is given by a relatively perfect object of the derived category of the total space; this somewhat technical notion is studied in More on Morphisms of Spaces, Section 76.52.

Already if X is a proper algebraic space over a field k we obtain a very interesting algebraic stack. Namely, there is an embedding

$$\mathcal{C}oh_{X/k} \longrightarrow \mathcal{C}omplexes_{X/k}$$

since for any $\mathcal{O}$-module $\mathcal{F}$ (on any ringed topos) we have $\text{Ext}^i_{\mathcal{O}}(\mathcal{F}, \mathcal{F}) = 0$ for $i < 0$. Although this certainly shows our stack is nonempty, the true motivation for the study of $\mathcal{C}omplexes_{X/k}$ is that there are often objects of the derived category $D^b_{\text{Coh}}(\mathcal{O}_X)$ with vanishing negative self-exts and nonvanishing cohomology sheaves in more than one degree. For example, X could be derived equivalent to another proper algebraic space Y over k, i.e., we have a k-linear equivalence

$$F : D^b_{\text{Coh}}(\mathcal{O}_Y) \longrightarrow D^b_{\text{Coh}}(\mathcal{O}_X)$$

There are cases where this happens and F is not given by an automorphism between X and Y; for example in the case of an abelian variety and its dual. In this situation F induces an isomorphism of algebraic stacks

$$\mathcal{C}omplexes_{Y/k} \longrightarrow \mathcal{C}omplexes_{X/k}$$

(insert future reference here) and in particular the stack of coherent sheaves on Y maps into the stack of complexes on X. Turning this around, if we can understand well enough the geometry of $\mathcal{C}omplexes_{X/k}$, then we can try to use this to study all possible derived equivalent Y.

0DLC **Lemma 99.16.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper, flat, and of finite presentation. Let $K, E \in D(\mathcal{O}_X)$. Assume K is pseudo-coherent and E is Y-perfect (More on Morphisms of Spaces, Definition 76.52.1). For a field k and a morphism $y : \text{Spec}(k) \to Y$ denote K_y, E_y the pullback to the fibre X_y.

(1) There is an open $W \subset Y$ characterized by the property

$$y \in |W| \Leftrightarrow \text{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y) = 0 \text{ for } i < 0.$$

(2) For any morphism $V \to Y$ factoring through W we have
$$\mathrm{Ext}^i_{\mathcal{O}_{X_V}}(K_V, E_V) = 0 \quad \text{for} \quad i < 0$$
where X_V is the base change of X and K_V and E_V are the derived pullbacks of K and E to X_V.

(3) The functor $V \mapsto \mathrm{Hom}_{\mathcal{O}_{X_V}}(K_V, E_V)$ is a sheaf on $(\textit{Spaces}/W)_{fppf}$ representable by an algebraic space affine and of finite presentation over W.

Proof. For any morphism $V \to Y$ the complex K_V is pseudo-coherent (Cohomology on Sites, Lemma 21.45.3) and E_V is V-perfect (More on Morphisms of Spaces, Lemma 76.52.6). Another observation is that given $y : \mathrm{Spec}(k) \to Y$ and a field extension k'/k with $y' : \mathrm{Spec}(k') \to Y$ the induced morphism, we have
$$\mathrm{Ext}^i_{\mathcal{O}_{X_{y'}}}(K_{y'}, E_{y'}) = \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y) \otimes_k k'$$
by Derived Categories of Schemes, Lemma 36.22.6. Thus the vanishing in (1) is really a property of the induced point $y \in |Y|$. We will use these two observations without further mention in the proof.

Assume first Y is an affine scheme. Then we may apply More on Morphisms of Spaces, Lemma 76.52.11 and find a pseudo-coherent $L \in D(\mathcal{O}_Y)$ which "universally computes" $Rf_*R\mathcal{H}om(K, E)$ in the sense described in that lemma. Unwinding the definitions, we obtain for a point $y \in Y$ the equality
$$\mathrm{Ext}^i_{\kappa(y)}(L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \kappa(y), \kappa(y)) = \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y)$$
We conclude that
$$H^i(L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \kappa(y)) = 0 \text{ for } i > 0 \Leftrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y) = 0 \text{ for } i < 0.$$
By Derived Categories of Schemes, Lemma 36.31.1 the set W of $y \in Y$ where this happens defines an open of Y. This open W then satisfies the requirement in (1) for all morphisms from spectra of fields, by the "universality" of L.

Let's go back to Y a general algebraic space. Choose an étale covering $\{V_i \to Y\}$ by affine schemes V_i. Then we see that the subset $W \subset |Y|$ pulls back to the corresponding subset $W_i \subset |V_i|$ for X_{V_i}, K_{V_i}, E_{V_i}. By the previous paragraph we find that W_i is open, hence W is open. This proves (1) in general. Moreover, parts (2) and (3) are entirely formulated in terms of the category $\textit{Spaces}/W$ and the restrictions X_W, K_W, E_W. This reduces us to the case $W = Y$.

Assume $W = Y$. We claim that for any algebraic space V over Y we have $Rf_{V,*}R\mathcal{H}om(K_V, E_V)$ has vanishing cohomology sheaves in degrees < 0. This will prove (2) because
$$\mathrm{Ext}^i_{\mathcal{O}_{X_V}}(K_V, E_V) = H^i(X_V, R\mathcal{H}om(K_V, E_V)) = H^i(V, Rf_{V,*}R\mathcal{H}om(K_V, E_V))$$
by Cohomology on Sites, Lemmas 21.35.1 and 21.20.5 and the vanishing of the cohomology sheaves implies the cohomology group H^i is zero for $i < 0$ by Derived Categories, Lemma 13.16.1.

To prove the claim, we may work étale locally on V. In particular, we may assume Y is affine and $W = Y$. Let $L \in D(\mathcal{O}_Y)$ be as in the second paragraph of the proof. For an algebraic space V over Y denote L_V the derived pullback of L to V. (An important feature we will use is that L "works" for all algebraic spaces V over Y and not just affine V.) As $W = Y$ we have $H^i(L) = 0$ for $i > 0$ (use More on

Algebra, Lemma 15.75.5 to go from fibres to stalks). Hence $H^i(L_V) = 0$ for $i > 0$. The property defining L is that

$$Rf_{V,*}R\mathcal{H}om(K_V, E_V) = R\mathcal{H}om(L_V, \mathcal{O}_V)$$

Since L_V sits in degrees ≤ 0, we conclude that $R\mathcal{H}om(L_V, \mathcal{O}_V)$ sits in degrees ≥ 0 thereby proving the claim. This finishes the proof of (2).

Assume $W = Y$ but make no assumptions on the algebraic space Y. Since we have (2), we see from Simplicial Spaces, Lemma 85.35.1 that the functor F given by $F(V) = \mathrm{Hom}_{\mathcal{O}_{X_V}}(K_V, E_V)$ is a sheaf[4] on $(\mathrm{Spaces}/Y)_{fppf}$. Thus to prove that F is an algebraic space and that $F \to Y$ is affine and of finite presentation, we may work étale locally on Y; see Bootstrap, Lemma 80.11.2 and Morphisms of Spaces, Lemmas 67.20.3 and 67.28.4. We conclude that it suffices to prove F is an affine algebraic space of finite presentation over Y when Y is an affine scheme. In this case we go back to our pseudo-coherent complex $L \in D(\mathcal{O}_Y)$. Since $H^i(L) = 0$ for $i > 0$, we can represent L by a complex of the form

$$\ldots \to \mathcal{O}_Y^{\oplus m_1} \to \mathcal{O}_Y^{\oplus m_0} \to 0 \to \ldots$$

with the last term in degree 0, see More on Algebra, Lemma 15.64.5. Combining the two displayed formulas earlier in the proof we find that

$$F(V) = \mathrm{Ker}(\mathrm{Hom}_V(\mathcal{O}_V^{\oplus m_0}, \mathcal{O}_V) \to \mathrm{Hom}_V(\mathcal{O}_V^{\oplus m_1}, \mathcal{O}_V))$$

In other words, there is a fibre product diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & Y \arrow[d, "0"] \\ \mathbf{A}^{m_0}_Y \arrow[r] & \mathbf{A}^{m_1}_Y \end{tikzcd}$$

which proves what we want. □

0DLD Lemma 99.16.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper, flat, and of finite presentation. Let $E \in D(\mathcal{O}_X)$. Assume

(1) E is S-perfect (More on Morphisms of Spaces, Definition 76.52.1), and
(2) for every point $s \in S$ we have
$$\mathrm{Ext}^i_{\mathcal{O}_{X_s}}(E_s, E_s) = 0 \quad \text{for} \quad i < 0$$
where E_s is the pullback to the fibre X_s.

Then

(a) (1) and (2) are preserved by arbitrary base change $V \to Y$,
(b) $\mathrm{Ext}^i_{\mathcal{O}_{X_V}}(E_V, E_V) = 0$ for $i < 0$ and all V over Y,
(c) $V \mapsto \mathrm{Hom}_{\mathcal{O}_{X_V}}(E_V, E_V)$ is representable by an algebraic space affine and of finite presentation over Y.

Here X_V is the base change of X and E_V is the derived pullback of E to X_V.

Proof. Immediate consequence of Lemma 99.16.1. □

[4]To check the sheaf property for a covering $\{V_i \to V\}_{i \in I}$ first consider the Čech fppf hypercovering $a : V_\bullet \to V$ with $V_n = \coprod_{i_0 \ldots i_n} V_{i_0} \times_V \ldots \times_V V_{i_n}$ and then set $U_\bullet = V_\bullet \times_{a,V} X_V$. Then $U_\bullet \to X_V$ is an fppf hypercovering to which we may apply Simplicial Spaces, Lemma 85.35.1.

0DLE Situation 99.16.3. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume f is proper, flat, and of finite presentation. We denote $\mathcal{C}omplexes_{X/B}$ the category whose objects are triples (T, g, E) where

(1) T is a scheme over S,
(2) $g : T \to B$ is a morphism over S, and setting $X_T = T \times_{g,B} X$
(3) E is an object of $D(\mathcal{O}_{X_T})$ satisfying conditions (1) and (2) of Lemma 99.16.2.

A morphism $(T, g, E) \to (T', g', E')$ is given by a pair (h, φ) where

(1) $h : T \to T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
(2) $\varphi : L(h')^*E' \to E$ is an isomorphism of $D(\mathcal{O}_{X_T})$ where $h' : X_T \to X_{T'}$ is the base change of h.

Thus $\mathcal{C}omplexes_{X/B}$ is a category and the rule

$$p : \mathcal{C}omplexes_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (T, g, E) \longmapsto T$$

is a functor. For a scheme T over S we denote $\mathcal{C}omplexes_{X/B,T}$ the fibre category of p over T. These fibre categories are groupoids.

0DLF Lemma 99.16.4. In Situation 99.16.3 the functor $p : \mathcal{C}omplexes_{X/B} \longrightarrow (Sch/S)_{fppf}$ is fibred in groupoids.

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 4.35.1. Given an object (T', g', E') of $\mathcal{C}omplexes_{X/B}$ and a morphism $h : T \to T'$ of schemes over S we can set $g = h \circ g'$ and $E = L(h')^*E'$ where $h' : X_T \to X_{T'}$ is the base change of h. Then it is clear that we obtain a morphism $(T, g, E) \to (T', g', E')$ of $\mathcal{C}omplexes_{X/B}$ lying over h. This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, E_1) \to (T, g, E) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, E_2) \to (T, g, E)$$

of $\mathcal{C}omplexes_{X/B}$ and a morphism $h : T_1 \to T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$L(h')^*E_2 \xrightarrow{L(h')^*\varphi_2^{-1}} L(h')^*L(h_2)^*E = L(h_1)^*E \xrightarrow{\varphi_1} E_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, E_1) \to (T_2, g_2, E_2)$ that witnesses the truth of condition (2). $\square$

0DLG Lemma 99.16.5. In Situation 99.16.3. Denote $\mathcal{X} = \mathcal{C}omplexes_{X/B}$. Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

Proof. Consider two objects $x = (T, g, E)$ and $y = (T, g', E')$ of $\mathcal{X}$ over a scheme T. We have to show that $Isom_{\mathcal{X}}(x, y)$ is an algebraic space over T, see Algebraic Stacks, Lemma 94.10.11. If for $h : T' \to T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ h = g' \circ h$. Hence there is a transformation of presheaves

$$Isom_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, g')$$

Since the diagonal of B is representable (by schemes) this equalizer is a scheme. Thus we may replace T by this equalizer and E and E' by their pullbacks. Thus we may assume $g = g'$.

Assume $g = g'$. After replacing B by T and X by X_T we arrive at the following problem. Given $E, E' \in D(\mathcal{O}_X)$ satisfying conditions (1), (2) of Lemma 99.16.2

we have to show that $Isom(E, E')$ is an algebraic space. Here $Isom(E, E')$ is the functor

$$(Sch/B)^{opp} \to \text{Sets}, \quad T \mapsto \{\varphi : E_T \to E'_T \text{ isomorphism in } D(\mathcal{O}_{X_T})\}$$

where E_T and E'_T are the derived pullbacks of E and E' to X_T. Now, let $W \subset B$, resp. $W' \subset B$ be the open subspace of B associated to E, E', resp. to E', E by Lemma 99.16.1. Clearly, if there exists an isomorphism $E_T \to E'_T$ as in the definition of $Isom(E, E')$, then we see that $T \to B$ factors into both W and W' (because we have condition (1) for E and E' and we'll obviously have $E_t \cong E'_t$ so no nonzero maps $E_t[i] \to E_t$ or $E'_t[i] \to E_t$ over the fibre X_t for $i > 0$. Thus we may replace B by the open $W \cap W'$. In this case the functor $H = \mathcal{H}om(E, E')$

$$(Sch/B)^{opp} \to \text{Sets}, \quad T \mapsto \text{Hom}_{\mathcal{O}_{X_T}}(E_T, E'_T)$$

is an algebraic space affine and of finite presentation over B by Lemma 99.16.1. The same is true for $H' = \mathcal{H}om(E', E)$, $I = \mathcal{H}om(E, E)$, and $I' = \mathcal{H}om(E', E')$. Therefore we can repeat the argument of the proof of Proposition 99.4.3 to see that

$$Isom(E, E') = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

for some morphisms c and σ. Thus $Isom(E, E')$ is an algebraic space. □

0DLH Lemma 99.16.6. In Situation 99.16.3 the functor $p : \mathcal{C}omplexes_{X/B} \longrightarrow (Sch/S)_{fppf}$ is a stack in groupoids.

Proof. To prove that $\mathcal{C}omplexes_{X/B}$ is a stack in groupoids, we have to show that the presheaves $Isom$ are sheaves and that descent data are effective. The statement on $Isom$ follows from Lemma 99.16.5, see Algebraic Stacks, Lemma 94.10.11. Let us prove the statement on descent data.

Suppose that $\{a_i : T_i \to T\}$ is an fppf covering of schemes over S. Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \to T\}$ with values in $\mathcal{C}omplexes_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, E_i)$. Denote $\text{pr}_0 : T_i \times_T T_j \to T_i$ and $\text{pr}_1 : T_i \times_T T_j \to T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} \cong \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \text{pr}_0 = g_j \circ \text{pr}_1$. Thus there exists a unique morphism $g : T \to B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 74.7.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i,B} X = T_i \times_{a_i,T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j. Observe that the pullback of (T_i, g_i, E_i) by $\text{pr}_0 : T_i \times_T T_j \to T_i$ is given by $(T_i \times_T T_j, g_i \circ \text{pr}_0, L\text{pr}_i^* E_i)$. Hence a descent datum for $\{T_i \to T\}$ in $\mathcal{C}omplexes_{X/B}$ is given by the objects $(T_i, g \circ a_i, E_i)$ and for each pair i, j an isomorphism in $D\mathcal{O}_{X_{ij}})$

$$\varphi_{ij} : L\text{pr}_i^* E_i \longrightarrow L\text{pr}_j^* E_j$$

satisfying the cocycle condition over the pullback of X to $T_i \times_T T_j \times_T T_k$. Using the vanishing of negative Exts provided by (b) of Lemma 99.16.2, we may apply Simplicial Spaces, Lemma 85.35.2 to obtain descent[5] for these complexes. In other words, we find there exists an object E in $D_{QCoh}(\mathcal{O}_{X_T})$ restricting to E_i on X_{T_i} compatible with φ_{ij}. Recall that being T-perfect signifies being pseudo-coherent

[5]To check this, first consider the Čech fppf hypercovering $a : T_\bullet \to T$ with $T_n = \coprod_{i_0 \ldots i_n} T_{i_0} \times_T \ldots \times_T T_{i_n}$ and then set $U_\bullet = T_\bullet \times_{a,T} X_T$. Then $U_\bullet \to X_T$ is an fppf hypercovering to which we may apply Simplicial Spaces, Lemma 85.35.2.

and having locally finite tor dimension over $f^{-1}\mathcal{O}_T$. Thus E is T-perfect by an application of More on Morphisms of Spaces, Lemmas 76.54.1 and 76.54.2. Finally, we have to check condition (2) from Lemma 99.16.2 for E. This immediately follows from the description of the open W in Lemma 99.16.1 and the fact that (2) holds for E_i on X_{T_i}/T_i. □

0DLI Remark 99.16.7. In Situation 99.16.3 the rule $(T, g, E) \mapsto (T, g)$ defines a 1-morphism

$$\mathcal{C}omplexes_{X/B} \longrightarrow \mathcal{S}_B$$

of stacks in groupoids (see Lemma 99.16.6, Algebraic Stacks, Section 94.7, and Examples of Stacks, Section 95.10). Let $B' \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{S}_{B'} \to \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $\mathcal{C}omplexes_{X'/B'} \to (Sch/S)_{fppf}$ associated to the base change $f' : X' \to B'$. In this situation the diagram

$$\begin{tikzcd} \mathcal{C}omplexes_{X'/B'} \arrow[r] \arrow[d] & \mathcal{C}omplexes_{X/B} \arrow[d] \\ \mathcal{S}_{B'} \arrow[r] & \mathcal{S}_B \end{tikzcd} \quad \text{or in another notation} \quad \begin{tikzcd} \mathcal{C}omplexes_{X'/B'} \arrow[r] \arrow[d] & \mathcal{C}omplexes_{X/B} \arrow[d] \\ Sch/B' \arrow[r] & Sch/B \end{tikzcd}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

0DLJ Lemma 99.16.8. In Situation 99.16.3 assume that $B \to S$ is locally of finite presentation. Then $p : \mathcal{C}omplexes_{X/B} \to (Sch/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).

Proof. Write $B(T)$ for the discrete category whose objects are the S-morphisms $T \to B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S. Assigning to an object (T, h, E) of $\mathcal{C}omplexes_{X/B,T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{tikzcd} \operatorname{colim} \mathcal{C}omplexes_{X/B,T_i} \arrow[r] \arrow[d] & \mathcal{C}omplexes_{X/B,T} \arrow[d] \\ \operatorname{colim} B(T_i) \arrow[r] & B(T) \end{tikzcd}$$

We have to show the top horizontal arrow is an equivalence. Since we have assume that B is locally of finite presentation over S we see from Limits of Spaces, Remark 70.3.11 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B. Denote $g_i : T_i \to B$ and $g : T \to B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i,B} X$ and $X_T = T \times_{g,B} X$. Observe that $X_T = \operatorname{colim} X_i$. By More on Morphisms of Spaces, Lemma 76.52.9 the category of T-perfect objects of $D(\mathcal{O}_{X_T})$ is the colimit of the categories of T_i-perfect objects of $D(\mathcal{O}_{X_{T_i}})$. Thus all we have to prove is that given an T_i-perfect object E_i of $D(\mathcal{O}_{X_{T_i}})$ such that the derived pullback E of E_i to X_T satisfies condition (2) of Lemma 99.16.2, then after increasing i we have that E_i satisfies condition (2) of Lemma 99.16.2. Let $W \subset |T_i|$ be the open constructed in Lemma 99.16.1 for E_i and E_i. By assumption on E we find that $T \to T_i$ factors through T. Hence there is an $i' \geq i$ such that $T_{i'} \to T_i$ factors through W, see Limits, Lemma 32.4.10 Then i' works by construction of W. □

0DLK **Lemma 99.16.9.** In Situation 99.16.3. Let

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

be a pushout in the category of schemes over S where $Z \to Z'$ is a finite order thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{C}omplexes_{X/B,Y'} \longrightarrow \mathcal{C}omplexes_{X/B,Y} \times_{\mathcal{C}omplexes_{X/B,Z}} \mathcal{C}omplexes_{X/B,Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 81.6.1. Thus using the commutative diagram

$$\begin{tikzcd} \mathcal{C}omplexes_{X/B,Y'} \arrow[r] \arrow[d] & \mathcal{C}omplexes_{X/B,Y} \times_{\mathcal{C}omplexes_{X/B,Z}} \mathcal{C}omplexes_{X/B,Z'} \arrow[d] \\ B(Y') \arrow[r] & B(Y) \times_{B(Z)} B(Z') \end{tikzcd}$$

we see that we may assume that Y' is a scheme over B'. By Remark 99.16.7 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$.

Assume $B = Y'$. We first prove fully faithfulness of our functor. To do this, let ξ_1, ξ_2 be two objects of $\mathcal{C}omplexes_{X/B}$ over Y'. Then we have to show that

$$\mathit{Isom}(\xi_1, \xi_2)(Y') \longrightarrow \mathit{Isom}(\xi_1, \xi_2)(Y) \times_{\mathit{Isom}(\xi_1,\xi_2)(Z)} \mathit{Isom}(\xi_1, \xi_2)(Z')$$

is bijective. However, we already know that $\mathit{Isom}(\xi_1, \xi_2)$ is an algebraic space over $B = Y'$. Thus this bijectivity follows from Artin's Axioms, Lemma 98.4.1 (or the aforementioned Pushouts of Spaces, Lemma 81.6.1).

Essential surjectivity. Let $(E_Y, E_{Z'}, \alpha)$ be a triple, where $E_Y \in D(\mathcal{O}_Y)$ and $E_{Z'} \in D(\mathcal{O}_{X_{Z'}})$ are objects such that $(Y, Y \to B, E_Y)$ is an object of $\mathcal{C}omplexes_{X/B}$ over Y, such that $(Z', Z' \to B, E_{Z'})$ is an object of $\mathcal{C}omplexes_{X/B}$ over Z', and $\alpha : L(X_Z \to X_Y)^*E_Y \to L(X_Z \to X_{Z'})^*E_{Z'}$ is an isomorphism in $D(\mathcal{O}_{Z'})$. That is to say

$$((Y, Y \to B, E_Y), (Z', Z' \to B, E_{Z'}), \alpha)$$

is an object of the target of the arrow of our lemma. Observe that the diagram

$$\begin{tikzcd} X_Z \arrow[r] \arrow[d] & X_{Z'} \arrow[d] \\ X_Y \arrow[r] & X_{Y'} \end{tikzcd}$$

is a pushout with $X_Z \to X_Y$ affine and $X_Z \to X_{Z'}$ a thickening (see Pushouts of Spaces, Lemma 81.6.7). Hence by Pushouts of Spaces, Lemma 81.8.1 we find an object $E_{Y'} \in D(\mathcal{O}_{X_{Y'}})$ together with isomorphisms $L(X_Y \to X_{Y'})^*E_{Y'} \to E_Y$ and

$L(X_{Z'} \to X_{Y'})^*E_{Y'} \to E_Z$ compatible with α. Clearly, if we show that $E_{Y'}$ is Y'-perfect, then we are done, because property (2) of Lemma 99.16.2 is a property on points (and Y and Y' have the same points). This follows from More on Morphisms of Spaces, Lemma 76.54.4. □

0DLL Lemma 99.16.10. In Situation 99.16.3 assume that S is a locally Noetherian scheme and $B \to S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, E_0)$ be an object of $\mathcal{X} = \mathcal{C}omplexes_{X/B}$ over k. Then the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ (Artin's Axioms, Section 98.8) are finite dimensional.

Proof. Observe that by Lemma 99.16.9 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 98.18. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 98.6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0,B} X$ and denote $\mathcal{X}_0 = \mathcal{C}omplexes_{X_0/k}$. In Remark 99.16.7 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(Sch/S)_{fppf}$. Thus by Artin's Axioms, Lemma 98.8.2 we reduce to proving that B, $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 98.8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k. Let $\mathrm{Spec}(k[\epsilon]) \to B$ be the composition of $g_0 : \mathrm{Spec}(k) \to B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ coming from the inclusion $k \to k[\epsilon]$. Set $X_0 = \mathrm{Spec}(k) \times_B X$ and $X_\epsilon = \mathrm{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. Observe that X_0 and X_ϵ give rise to canonically equivalent small étale topoi, see More on Morphisms of Spaces, Section 76.9. By More on Morphisms of Spaces, Lemma 76.54.4 we see that $T\mathcal{F}_{\mathcal{X}_0,k,x_0}$ is the set of isomorphism classes of lifts of E_0 to X_ϵ in the sense of Deformation Theory, Lemma 91.16.7. We conclude that

$$T\mathcal{F}_{\mathcal{X}_0,k,x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(E_0, E_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$-modules. Using Deformation Theory, Lemma 91.16.7 once more we see that there is a surjection

$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) \leftarrow \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(E_0, E_0)$$

of k-vector spaces. As E_0 is pseudo-coherent it lies in $D^-_{\mathrm{Coh}}(\mathcal{O}_{X_0})$ by Derived Categories of Spaces, Lemma 75.13.7. Since E_0 locally has finite tor dimension and X_0 is quasi-compact we see $E_0 \in D^b_{\mathrm{Coh}}(\mathcal{O}_{X_0})$. Thus the Exts above are finite dimensional k-vector spaces by Derived Categories of Spaces, Lemma 75.8.4. □

0DLM Lemma 99.16.11. In Situation 99.16.3 assume $B = S$ is locally Noetherian. Then strong formal effectiveness in the sense of Artin's Axioms, Remark 98.20.2 holds for $p : \mathcal{C}omplexes_{X/S} \to (Sch/S)_{fppf}$.

Proof. Let (R_n) be an inverse system of S-algebras with surjective transition maps whose kernels are locally nilpotent. Set $R = \lim R_n$. Let (ξ_n) be a system of objects of $\mathcal{C}omplexes_{X/B}$ lying over $(\mathrm{Spec}(R_n))$. We have to show (ξ_n) is effective, i.e., there exists an object ξ of $\mathcal{C}omplexes_{X/B}$ lying over $\mathrm{Spec}(R)$.

Write $X_R = \mathrm{Spec}(R) \times_S X$ and $X_n = \mathrm{Spec}(R_n) \times_S X$. Of course X_n is the base change of X_R by $R \to R_n$. Since $S = B$, we see that ξ_n corresponds simply to an R_n-perfect object $E_n \in D(\mathcal{O}_{X_n})$ satisfying condition (2) of Lemma 99.16.2. In particular E_n is pseudo-coherent. The isomorphisms $\xi_{n+1}|_{\mathrm{Spec}(R_n)} \cong \xi_n$ correspond to isomorphisms $L(X_n \to X_{n+1})^*E_{n+1} \to E_n$. Therefore by Flatness on Spaces, Theorem 77.13.6 we find a pseudo-coherent object E of $D(\mathcal{O}_{X_R})$ with E_n equal to the derived pullback of E for all n compatible with the transition isomorphisms.

Observe that $(R, \mathrm{Ker}(R \to R_1))$ is a henselian pair, see More on Algebra, Lemma 15.11.3. In particular, $\mathrm{Ker}(R \to R_1)$ is contained in the Jacobson radical of R. Then we may apply More on Morphisms of Spaces, Lemma 76.54.5 to see that E is R-perfect.

Finally, we have to check condition (2) of Lemma 99.16.2. By Lemma 99.16.1 the set of points t of $\mathrm{Spec}(R)$ where the negative self-exts of E_t vanish is an open. Since this condition is true in $V(\mathrm{Ker}(R \to R_1))$ and since $\mathrm{Ker}(R \to R_1)$ is contained in the Jacobson radical of R we conclude it holds for all points. $\square$

Theorem 99.16.12 (Algebraicity of moduli of complexes on a proper morphism).
0DLN Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. [Lie06a]
Assume that f is proper, flat, and of finite presentation. Then $\mathcal{C}omplexes_{X/B}$ is an algebraic stack over S.

Proof. Set $\mathcal{X} = \mathcal{C}omplexes_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 99.16.6 and 99.16.5). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = \mathcal{C}omplexes_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 99.16.7). By the material in Algebraic Stacks, Section 94.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 94.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is proper, flat, of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1, 70.6.12, and 70.6.13. If we show that $\mathcal{C}omplexes_{X_i/\mathrm{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 99.16.7 and Algebraic Stacks, Section 94.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \mathrm{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 98.17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.50.12. Hence all local rings of S are G-rings. Thus (5) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 98.14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 99.16.6, 99.16.8, 99.16.9, 99.16.10. Condition (3) follows from Lemma 99.16.11. Condition (1) is Lemma 99.16.5.

It remains to show condition (4) which is openness of versality. To see this we will use Artin's Axioms, Lemma 98.20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving (see lemmas used above). Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 98.20.3. This follows from Lemma 99.16.11 and the proof is complete. □

99.17. Other chapters

CHAPTER 100

Properties of Algebraic Stacks

04X8 100.1. Introduction

04X9 Please see Algebraic Stacks, Section 94.1 for a brief introduction to algebraic stacks, and please read some of that chapter for our foundations of algebraic stacks. The intent is that in that chapter we are careful to distinguish between schemes, algebraic spaces, algebraic stacks, and starting with this chapter we employ the customary abuse of language where all of these concepts are used interchangeably.

The goal of this chapter is to introduce some basic notions and properties of algebraic stacks. A fundamental reference for the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

100.2. Conventions and abuse of language

04XA We choose a big fppf site Sch_{fppf}. All schemes are contained in Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

We also fix a base scheme S, by the conventions above an element of Sch_{fppf}. The reader who is only interested in the absolute case can take $S = \text{Spec}(\mathbf{Z})$.

Here are our conventions regarding algebraic stacks:

(1) When we say algebraic stack we will mean an algebraic stacks over S, i.e., a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which satisfies the conditions of Algebraic Stacks, Definition 94.12.1.
(2) We will say $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks to indicate a 1-morphism of algebraic stacks over S, i.e., a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Algebraic Stacks, Definition 94.12.3.
(3) A 2-morphism $\alpha : f \to g$ will indicate a 2-morphism in the 2-category of algebraic stacks over S, see Algebraic Stacks, Definition 94.12.3.
(4) Given morphisms $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ of algebraic stacks we abusively call the 2-fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ the fibre product.
(5) We will write $\mathcal{X} \times_S \mathcal{Y}$ for the product of the algebraic stacks $\mathcal{X}$, $\mathcal{Y}$.
(6) We will often abuse notation and say two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic if they are equivalent in this 2-category.

Here are our conventions regarding algebraic spaces.

(1) If we say X is an algebraic space then we mean that X is an algebraic space over S, i.e., X is a presheaf on $(Sch/S)_{fppf}$ which satisfies the conditions of Spaces, Definition 65.6.1.
(2) A morphism of algebraic spaces $f : X \to Y$ is a morphism of algebraic spaces over S as defined in Spaces, Definition 65.6.3.

(3) We will not distinguish between an algebraic space X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 94.13.1.
(4) In particular, a morphism $f : X \to \mathcal{Y}$ from X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ is an algebraic space to indicate that $\mathcal{X}$ is representable by an algebraic space, see Algebraic Stacks, Definition 94.8.1.
(6) We will use the following notational convention: If we indicate an algebraic stack by a roman capital (such as $X, Y, Z, A, B, \ldots$) then it will be the case that its inertia stack is trivial, and hence it is an algebraic space, see Algebraic Stacks, Proposition 94.13.3.

Here are our conventions regarding schemes.

(1) If we say X is a scheme then we mean that X is a scheme over S, i.e., X is an object of $(Sch/S)_{fppf}$.
(2) By a morphism of schemes we mean a morphism of schemes over S.
(3) We will not distinguish between a scheme X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 94.13.1.
(4) In particular, a morphism $f : X \to \mathcal{Y}$ from a scheme X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ is a scheme to indicate that $\mathcal{X}$ is representable, see Algebraic Stacks, Section 94.4.

Here are our conventions regarding morphisms of algebraic stacks:

(1) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable, or representable by schemes if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is a scheme. See Algebraic Stacks, Section 94.6.
(2) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable by algebraic spaces if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. See Algebraic Stacks, Definition 94.9.1. In this case $Z \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space whenever $Z \to \mathcal{Y}$ is a morphism whose source is an algebraic space, see Algebraic Stacks, Lemma 94.9.8.
(3) We may abuse notation and say that a diagram of algebraic stacks commutes if the diagram is 2-commutative in the 2-category of algebraic stacks.

Note that every morphism $X \to \mathcal{Y}$ from an algebraic space to an algebraic stack is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.11. We will use this basic result without further mention.

100.3. Properties of morphisms representable by algebraic spaces

04XB We will study properties of (arbitrary) morphisms of algebraic stacks in its own chapter. For morphisms representable by algebraic spaces we know what it means to be surjective, smooth, or étale, etc. This applies in particular to morphisms $X \to \mathcal{Y}$ from algebraic spaces to algebraic stacks. In this section, we recall how this works, we list the properties to which this applies, and we prove a few easy lemmas.

Our first lemma says a morphism is representable by algebraic spaces if it is so after a base change by a flat, locally finitely presented, surjective morphism.

04ZP Lemma 100.3.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. The following are equivalent

(1) f is representable by algebraic spaces, and
(2) $W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.

Proof. The implication (1) $\Rightarrow$ (2) is Algebraic Stacks, Lemma 94.9.8. Conversely, let $W \to \mathcal{Y}$ be as in (2). To prove (1) it suffices to show that f is faithful on fibre categories, see Algebraic Stacks, Lemma 94.15.2. Assumption (2) implies in particular that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is faithful. Hence the faithfulness of f follows from Stacks, Lemma 8.6.9. $\square$

Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then we say f has property P if and only if for every scheme T and morphism $T \to \mathcal{Y}$ the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ has property P, see Algebraic Stacks, Definition 94.10.1.

It turns out that if $f : \mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces and has property P, then for any morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property P, see Algebraic Stacks, Lemmas 94.9.7 and 94.10.6. If the property P is preserved under compositions, then this holds also in the setting of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemmas 94.9.9 and 94.10.5. Moreover, in this case products $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ of morphisms representable by algebraic spaces having property $\mathcal{P}$ have property $\mathcal{P}$, see Algebraic Stacks, Lemma 94.10.8.

Finally, if we have two properties P, P' of morphisms of algebraic spaces which are fppf local on the target and preserved by arbitrary base change and if $P(f) \Rightarrow P'(f)$ for every morphism f, then the same implication holds for the corresponding property of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.9. We will use this without further mention in the following and in the following chapters.

The discussion above applies to each of the following properties of morphisms of algebraic spaces

(1) quasi-compact, see Morphisms of Spaces, Lemma 67.8.4 and Descent on Spaces, Lemma 74.11.1,
(2) quasi-separated, see Morphisms of Spaces, Lemma 67.4.4 and Descent on Spaces, Lemma 74.11.2,
(3) universally closed, see Morphisms of Spaces, Lemma 67.9.3 and Descent on Spaces, Lemma 74.11.3,
(4) universally open, see Morphisms of Spaces, Lemma 67.6.3 and Descent on Spaces, Lemma 74.11.4,
(5) universally submersive, see Morphisms of Spaces, Lemma 67.7.3 and Descent on Spaces, Lemma 74.11.5,
(6) universal homeomorphism, see Morphisms of Spaces, Lemma 67.53.4 and Descent on Spaces, Lemma 74.11.8,

(7) surjective, see Morphisms of Spaces, Lemma 67.5.5 and Descent on Spaces, Lemma 74.11.6,
(8) universally injective, see Morphisms of Spaces, Lemma 67.19.5 and Descent on Spaces, Lemma 74.11.7,
(9) locally of finite type, see Morphisms of Spaces, Lemma 67.23.3 and Descent on Spaces, Lemma 74.11.9,
(10) locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.3 and Descent on Spaces, Lemma 74.11.10,
(11) finite type, see Morphisms of Spaces, Lemma 67.23.3 and Descent on Spaces, Lemma 74.11.11,
(12) finite presentation, see Morphisms of Spaces, Lemma 67.28.3 and Descent on Spaces, Lemma 74.11.12,
(13) flat, see Morphisms of Spaces, Lemma 67.30.4 and Descent on Spaces, Lemma 74.11.13,
(14) open immersion, see Morphisms of Spaces, Section 67.12 and Descent on Spaces, Lemma 74.11.14,
(15) isomorphism, see Descent on Spaces, Lemma 74.11.15,
(16) affine, see Morphisms of Spaces, Lemma 67.20.5 and Descent on Spaces, Lemma 74.11.16,
(17) closed immersion, see Morphisms of Spaces, Section 67.12 and Descent on Spaces, Lemma 74.11.17,
(18) separated, see Morphisms of Spaces, Lemma 67.4.4 and Descent on Spaces, Lemma 74.11.18,
(19) proper, see Morphisms of Spaces, Lemma 67.40.3 and Descent on Spaces, Lemma 74.11.19,
(20) quasi-affine, see Morphisms of Spaces, Lemma 67.21.5 and Descent on Spaces, Lemma 74.11.20,
(21) integral, see Morphisms of Spaces, Lemma 67.45.5 and Descent on Spaces, Lemma 74.11.22,
(22) finite, see Morphisms of Spaces, Lemma 67.45.5 and Descent on Spaces, Lemma 74.11.23,
(23) (locally) quasi-finite, see Morphisms of Spaces, Lemma 67.27.4 and Descent on Spaces, Lemma 74.11.24,
(24) syntomic, see Morphisms of Spaces, Lemma 67.36.3 and Descent on Spaces, Lemma 74.11.25,
(25) smooth, see Morphisms of Spaces, Lemma 67.37.3 and Descent on Spaces, Lemma 74.11.26,
(26) unramified, see Morphisms of Spaces, Lemma 67.38.4 and Descent on Spaces, Lemma 74.11.27,
(27) étale, see Morphisms of Spaces, Lemma 67.39.4 and Descent on Spaces, Lemma 74.11.28,
(28) finite locally free, see Morphisms of Spaces, Lemma 67.46.5 and Descent on Spaces, Lemma 74.11.29,
(29) monomorphism, see Morphisms of Spaces, Lemma 67.10.5 and Descent on Spaces, Lemma 74.11.30,
(30) immersion, see Morphisms of Spaces, Section 67.12 and Descent on Spaces, Lemma 74.12.1,

(31) locally separated, see Morphisms of Spaces, Lemma 67.4.4 and Descent on Spaces, Lemma 74.12.2,

04XC Lemma 100.3.2. Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. The following are equivalent:

(1) f has P,
(2) for every algebraic space Z and morphism $Z \to \mathcal{Y}$ the morphism $Z \times_\mathcal{Y} \mathcal{X} \to Z$ has P.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). Let $Z \to \mathcal{Y}$ be as in (2). Choose a scheme U and a surjective étale morphism $U \to Z$. By assumption the morphism $U \times_\mathcal{Y} \mathcal{X} \to U$ has P. But the diagram

$$\begin{tikzcd} U \times_\mathcal{Y} \mathcal{X} \arrow[r] \arrow[d] & Z \times_\mathcal{Y} \mathcal{X} \arrow[d] \\ U \arrow[r] & Z \end{tikzcd}$$

is cartesian, hence the right vertical arrow has P as $\{U \to Z\}$ is an fppf covering. $\square$

The following lemma tells us it suffices to check P after a base change by a surjective, flat, locally finitely presented morphism.

04XD Lemma 100.3.3. Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. Set $V = W \times_\mathcal{Y} \mathcal{X}$. Then

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } V \to W \text{ has } P).$$

Proof. The implication from left to right follows from Lemma 100.3.2. Assume $V \to W$ has P. Let T be a scheme, and let $T \to \mathcal{Y}$ be a morphism. Consider the commutative diagram

$$\begin{tikzcd} T \times_\mathcal{Y} \mathcal{X} \arrow[d] & T \times_\mathcal{Y} V \arrow[l] \arrow[r] \arrow[d] & V \arrow[d] \\ T & T \times_\mathcal{Y} W \arrow[l] \arrow[r] & W \end{tikzcd}$$

of algebraic spaces. The squares are cartesian. The bottom left morphism is a surjective, flat morphism which is locally of finite presentation, hence $\{T \times_\mathcal{Y} V \to T\}$ is an fppf covering. Hence the fact that the right vertical arrow has property P implies that the left vertical arrow has property P. $\square$

06TY Lemma 100.3.4. Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_\mathcal{Y} \mathcal{X}$. Then

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By the discussion above the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V =$

$W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 100.3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. □

06M2 Lemma 100.3.5. Let P be a property of morphisms of algebraic spaces as above. Let $\tau \in \{\textit{étale}, \textit{smooth}, \textit{syntomic}, \textit{fppf}\}$. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks representable by algebraic spaces. Assume

(1) $\mathcal{X} \to \mathcal{Y}$ is surjective and étale, smooth, syntomic, or flat and locally of finite presentation,
(2) the composition has P, and
(3) P is local on the source in the τ topology.

Then $\mathcal{Y} \to \mathcal{Z}$ has property P.

Proof. Let Z be a scheme and let $Z \to \mathcal{Z}$ be a morphism. Set $X = \mathcal{X} \times_{\mathcal{Z}} Z$, $Y = \mathcal{Y} \times_{\mathcal{Z}} Z$. By (1) $\{X \to Y\}$ is a τ covering of algebraic spaces and by (2) $X \to Z$ has property P. By (3) this implies that $Y \to Z$ has property P and we win. □

04Y6 Lemma 100.3.6. Let $g : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Let $[U/R] \to \mathcal{X}$ be a presentation. Set $U' = U \times_{\mathcal{X}} \mathcal{X}'$, and $R' = R \times_{\mathcal{X}} \mathcal{X}'$. Then there exists a groupoid in algebraic spaces of the form (U', R', s', t', c'), a presentation $[U'/R'] \to \mathcal{X}'$, and the diagram

$$\begin{tikzcd} {[U'/R']} \arrow[r] \arrow[d, "{[\mathrm{pr}]}"'] & \mathcal{X}' \arrow[d, "g"] \\ {[U/R]} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is 2-commutative where the morphism [pr] comes from a morphism of groupoids $\mathrm{pr} : (U', R', s', t', c') \to (U, R, s, t, c)$.

Proof. Since $U \to \mathcal{Y}$ is surjective and smooth, see Algebraic Stacks, Lemma 94.17.2 the base change $U' \to \mathcal{X}'$ is also surjective and smooth. Hence, by Algebraic Stacks, Lemma 94.16.2 it suffices to show that $R' = U' \times_{\mathcal{X}'} U'$ in order to get a smooth groupoid (U', R', s', t', c') and a presentation $[U'/R'] \to \mathcal{X}'$. Using that $R = V \times_{\mathcal{Y}} V$ (see Groupoids in Spaces, Lemma 78.22.2) this follows from

$$R' = U \times_{\mathcal{X}} U \times_{\mathcal{X}} \mathcal{X}' = (U \times_{\mathcal{X}} \mathcal{X}') \times_{\mathcal{X}'} (U \times_{\mathcal{X}} \mathcal{X}')$$

see Categories, Lemmas 4.31.8 and 4.31.10. Clearly the projection morphisms $U' \to U$ and $R' \to R$ give the desired morphism of groupoids $\mathrm{pr} : (U', R', s', t', c') \to (U, R, s, t, c)$. Hence the morphism [pr] of quotient stacks by Groupoids in Spaces, Lemma 78.21.1.

We still have to show that the diagram 2-commutes. It is clear that the diagram

$$\begin{tikzcd} U' \arrow[r, "f'"'] \arrow[d, "\mathrm{pr}_U"'] & \mathcal{X}' \arrow[d, "g"] \\ U \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

2-commutes where $\mathrm{pr}_U : U' \to U$ is the projection. There is a canonical 2-arrow $\tau : f \circ t \to f \circ s$ in $\mathrm{Mor}(R, \mathcal{X})$ coming from $R = U \times_{\mathcal{X}} U$, $t = \mathrm{pr}_0$, and $s = \mathrm{pr}_1$. Using the isomorphism $R' \to U' \times_{\mathcal{X}'} U'$ we get similarly an isomorphism $\tau' : f' \circ t' \to f' \circ s'$.

Note that $g \circ f' \circ t' = f \circ t \circ \text{pr}_R$ and $g \circ f' \circ s' = f \circ s \circ \text{pr}_R$, where $\text{pr}_R : R' \to R$ is the projection. Thus it makes sense to ask if

04Y7 (100.3.6.1) $$\tau \star \text{id}_{\text{pr}_R} = \text{id}_g \star \tau'.$$

Now we make two claims: (1) if Equation (100.3.6.1) holds, then the diagram 2-commutes, and (2) Equation (100.3.6.1) holds. We omit the proof of both claims. Hints: part (1) follows from the construction of $f = f_{can}$ and $f' = f'_{can}$ in Algebraic Stacks, Lemma 94.16.1. Part (2) follows by carefully working through the definitions. □

04ZQ Remark 100.3.7. Let $\mathcal{Y}$ be an algebraic stack. Consider the following 2-category:

(1) An object is a morphism $f : \mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces,
(2) a 1-morphism $(g, \beta) : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ consists of a morphism $g : \mathcal{X}_1 \to \mathcal{X}_2$ and a 2-morphism $\beta : f_1 \to f_2 \circ g$, and
(3) a 2-morphism between $(g, \beta), (g', \beta') : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ is a 2-morphism $\alpha : g \to g'$ such that $(\text{id}_{f_2} \star \alpha) \circ \beta = \beta'$.

Let us denote this 2-category Spaces$/\mathcal{Y}$ by analogy with the notation of Topologies on Spaces, Section 73.2. Now we claim that in this 2-category the morphism categories

$$\text{Mor}_{\text{Spaces}/\mathcal{Y}}((f_1 : \mathcal{X}_1 \to \mathcal{Y}), (f_2 : \mathcal{X}_2 \to \mathcal{Y}))$$

are all setoids. Namely, a 2-morphism α is a rule which to each object x_1 of $\mathcal{X}_1$ assigns an isomorphism $\alpha_{x_1} : g(x_1) \longrightarrow g'(x_1)$ in the relevant fibre category of $\mathcal{X}_2$ such that the diagram

$$\begin{tikzcd} & f_2(x_1) \arrow[dl, "\beta_{x_1}"'] \arrow[dr, "\beta'_{x_1}"] & \\ f_2(g(x_1)) \arrow[rr, "f_2(\alpha_{x_1})"] & & f_2(g'(x_1)) \end{tikzcd}$$

commutes. But since f_2 is faithful (see Algebraic Stacks, Lemma 94.15.2) this means that if α_{x_1} exists, then it is unique! In other words the 2-category Spaces$/\mathcal{Y}$ is very close to being a category. Namely, if we replace 1-morphisms by isomorphism classes of 1-morphisms we obtain a category. We will often perform this replacement without further mention.

100.4. Points of algebraic stacks

04XE Let $\mathcal{X}$ be an algebraic stack. Let K, L be two fields and let $p : \text{Spec}(K) \to \mathcal{X}$ and $q : \text{Spec}(L) \to \mathcal{X}$ be morphisms. We say that p and q are equivalent if there exists a field Ω and a 2-commutative diagram

$$\begin{tikzcd} \text{Spec}(\Omega) \arrow[r] \arrow[d] & \text{Spec}(L) \arrow[d, "q"] \\ \text{Spec}(K) \arrow[r, "p"] & \mathcal{X}. \end{tikzcd}$$

04XF Lemma 100.4.1. The notion above does indeed define an equivalence relation on morphisms from spectra of fields into the algebraic stack $\mathcal{X}$.

Proof. It is clear that the relation is reflexive and symmetric. Hence we have to prove that it is transitive. This comes down to the following: Given a diagram

$$\begin{tikzcd} \mathrm{Spec}(\Omega) \arrow[r, "b"'] \arrow[d, "a"'] & \mathrm{Spec}(L) \arrow[d, "q"] & \mathrm{Spec}(\Omega') \arrow[l, "b'"] \arrow[d, "a'"] \\ \mathrm{Spec}(K) \arrow[r, "p"] & \mathcal{X} & \mathrm{Spec}(K') \arrow[l, "p'"'] \end{tikzcd}$$

with both squares 2-commutative we have to show that p is equivalent to p'. By the 2-Yoneda lemma (see Algebraic Stacks, Section 94.5) the morphisms p, p', and q are given by objects x, x', and y in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(K)$, $\mathrm{Spec}(K')$, and $\mathrm{Spec}(L)$. The 2-commutativity of the squares means that there are isomorphisms $\alpha : a^*x \to b^*y$ and $\alpha' : (a')^*x' \to (b')^*y$ in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(\Omega)$ and $\mathrm{Spec}(\Omega')$. Choose any field Ω'' and embeddings $\Omega \to \Omega''$ and $\Omega' \to \Omega''$ agreeing on L. Then we can extend the diagram above to

$$\begin{tikzcd} & \mathrm{Spec}(\Omega'') \arrow[dl, "c"'] \arrow[d, "q'"] \arrow[dr, "c'"] & \\ \mathrm{Spec}(\Omega) \arrow[r, "b"'] \arrow[d, "a"'] & \mathrm{Spec}(L) \arrow[d, "q"] & \mathrm{Spec}(\Omega') \arrow[l, "b'"] \arrow[d, "a'"] \\ \mathrm{Spec}(K) \arrow[r, "p"] & \mathcal{X} & \mathrm{Spec}(K') \arrow[l, "p'"'] \end{tikzcd}$$

with commutative triangles and

$$(q')^*(\alpha')^{-1} \circ (q')^*\alpha : (a \circ c)^*x \longrightarrow (a' \circ c')^*x'$$

is an isomorphism in the fibre category over $\mathrm{Spec}(\Omega'')$. Hence p is equivalent to p' as desired. ☐

04XG Definition 100.4.2. Let $\mathcal{X}$ be an algebraic stack. A point of $\mathcal{X}$ is an equivalence class of morphisms from spectra of fields into $\mathcal{X}$. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

This agrees with our definition of points of algebraic spaces, see Properties of Spaces, Definition 66.4.1. Moreover, for a scheme we recover the usual notion of points, see Properties of Spaces, Lemma 66.4.2. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks then there is an induced map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ which maps a representative $x : \mathrm{Spec}(K) \to \mathcal{X}$ to the representative $f \circ x : \mathrm{Spec}(K) \to \mathcal{Y}$. This is well defined: namely 2-isomorphic 1-morphisms remain 2-isomorphic after pre- or post-composing by a 1-morphism because you can horizontally pre- or post-compose by the identity of the given 1-morphism. This holds in any (strict) $(2,1)$-category. If

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

is a 2-commutative diagram of algebraic stacks, then the diagram of sets

$$\begin{tikzcd} |\mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Y}| \arrow[d] \\ |\mathcal{W}| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

is commutative. In particular, if $\mathcal{X} \to \mathcal{Y}$ is an equivalence then $|\mathcal{X}| \to |\mathcal{Y}|$ is a bijection.

04XH Lemma 100.4.3. Let

$$\begin{tikzcd} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Z} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. Then the map of sets of points

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{Y}|} |\mathcal{X}|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to \mathcal{X}$, $\mathrm{Spec}(L) \to \mathcal{Z}$, then the assumption that they agree as elements of $|\mathcal{Y}|$ means that there is a common extension M/K and M/L such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to \mathcal{X} \to \mathcal{Y}$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to \mathcal{Z} \to \mathcal{Y}$ are 2-isomorphic. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. $\square$

04XI Lemma 100.4.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:

(1) $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is surjective, and
(2) f is surjective (in the sense of Section 100.3).

Proof. Assume (1). Let $T \to \mathcal{Y}$ be a morphism whose source is a scheme. To prove (2) we have to show that the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is surjective. By Morphisms of Spaces, Definition 67.5.2 this means we have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is surjective. Applying Lemma 100.4.3 we see that this follows from (1).

Conversely, assume (2). Let $y : \mathrm{Spec}(K) \to \mathcal{Y}$ be a morphism from the spectrum of a field into $\mathcal{Y}$. By assumption the morphism $\mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} \to \mathrm{Spec}(K)$ of algebraic spaces is surjective. By Morphisms of Spaces, Definition 67.5.2 this means there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X}$ such that the left square of the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, equal] & \mathrm{Spec}(K) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

is commutative. This shows that $|X| \to |\mathcal{Y}|$ is surjective. $\square$

Here is a lemma explaining how to compute the set of points in terms of a presentation.

04XJ **Lemma 100.4.5.** Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X} = [U/R]$ be a presentation of $\mathcal{X}$, see Algebraic Stacks, Definition 94.16.5. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.

Proof. The assumption means that we have a smooth groupoid (U, R, s, t, c) in algebraic spaces, and an equivalence $f : [U/R] \to \mathcal{X}$. We may assume $\mathcal{X} = [U/R]$. The induced morphism $p : U \to \mathcal{X}$ is smooth and surjective, see Algebraic Stacks, Lemma 94.17.2. Hence $|U| \to |\mathcal{X}|$ is surjective by Lemma 100.4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 78.22.2. Hence Lemma 100.4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. $\square$

04XK **Remark 100.4.6.** The result of Lemma 100.4.5 can be generalized as follows. Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be a surjective morphism (which makes sense by Section 100.3). Let $R = U \times_{\mathcal{X}} U$, let (U, R, s, t, c) be the groupoid in algebraic spaces, and let $f_{can} : [U/R] \to \mathcal{X}$ be the canonical morphism as constructed in Algebraic Stacks, Lemma 94.16.1. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}| = |U|/|R|$. The proof of Lemma 100.4.5 works without change. (Of course in general $[U/R]$ is not an algebraic stack, and in general f_{can} is not an isomorphism.)

04XL **Lemma 100.4.7.** There exists a unique topology on the sets of points of algebraic stacks with the following properties:

(1) for every morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ the map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous, and
(2) for every morphism $U \to \mathcal{X}$ which is flat and locally of finite presentation with U an algebraic space the map of topological spaces $|U| \to |\mathcal{X}|$ is continuous and open.

Proof. Choose a morphism $p : U \to \mathcal{X}$ which is surjective, flat, and locally of finite presentation with U an algebraic space. Such exist by the definition of an algebraic stack, as a smooth morphism is flat and locally of finite presentation (see Morphisms of Spaces, Lemmas 67.37.5 and 67.37.7). We define a topology on $|\mathcal{X}|$ by the rule: $W \subset |\mathcal{X}|$ is open if and only if $|p|^{-1}(W)$ is open in $|U|$. To show that this is independent of the choice of p, let $p' : U' \to \mathcal{X}$ be another morphism which is surjective, flat, locally of finite presentation from an algebraic space to $\mathcal{X}$. Set $U'' = U \times_{\mathcal{X}} U'$ so that we have a 2-commutative diagram

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

As $U \to \mathcal{X}$ and $U' \to \mathcal{X}$ are surjective, flat, locally of finite presentation we see that $U'' \to U'$ and $U'' \to U$ are surjective, flat and locally of finite presentation, see Lemma 100.3.2. Hence the maps $|U''| \to |U'|$ and $|U''| \to |U|$ are continuous, open and surjective, see Morphisms of Spaces, Definition 67.5.2 and Lemma 67.30.6. This clearly implies that our definition is independent of the choice of $p : U \to \mathcal{X}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. By Algebraic Stacks, Lemma 94.15.1 we can find a 2-commutative diagram

$$\begin{tikzcd} U \arrow[r, "a"'] \arrow[d, "x"'] & V \arrow[d, "y"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with surjective smooth vertical arrows. Consider the associated commutative diagram

$$\begin{tikzcd} {|U|} \arrow[r, "|a|"'] \arrow[d, "|x|"'] & {|V|} \arrow[d, "|y|"] \\ {|\mathcal{X}|} \arrow[r, "|f|"] & {|\mathcal{Y}|} \end{tikzcd}$$

of sets. If $W \subset |\mathcal{Y}|$ is open, then by the definition above this means exactly that $|y|^{-1}(W)$ is open in $|V|$. Since $|a|$ is continuous we conclude that $|a|^{-1}|y|^{-1}(W) = |x|^{-1}|f|^{-1}(W)$ is open in $|W|$ which means by definition that $|f|^{-1}(W)$ is open in $|\mathcal{X}|$. Thus $|f|$ is continuous.

Finally, we have to show that if U is an algebraic space, and $U \to \mathcal{X}$ is flat and locally of finite presentation, then $|U| \to |\mathcal{X}|$ is open. Let $V \to \mathcal{X}$ be surjective, flat, and locally of finite presentation with V an algebraic space. Consider the commutative diagram

$$\begin{tikzcd} {|U \times_{\mathcal{X}} V|} \arrow[r, "e"'] \arrow[dr, "f"'] & {|U| \times_{|\mathcal{X}|} |V|} \arrow[r, "d"'] \arrow[d, "c"'] & {|V|} \arrow[d, "b"] \\ & {|U|} \arrow[r, "a"] & {|\mathcal{X}|} \end{tikzcd}$$

Now the morphism $U \times_{\mathcal{X}} V \to U$ is surjective, i.e, $f : |U \times_{\mathcal{X}} V| \to |U|$ is surjective. The left top horizontal arrow is surjective, see Lemma 100.4.3. The morphism $U \times_{\mathcal{X}} V \to V$ is flat and locally of finite presentation, hence $d \circ e : |U \times_{\mathcal{X}} V| \to |V|$ is open, see Morphisms of Spaces, Lemma 67.30.6. Pick $W \subset |U|$ open. The properties above imply that $b^{-1}(a(W)) = (d \circ e)(f^{-1}(W))$ is open, which by construction means that $a(W)$ is open as desired. $\square$

04Y8 Definition 100.4.8. Let $\mathcal{X}$ be an algebraic stack. The underlying topological space of $\mathcal{X}$ is the set of points $|\mathcal{X}|$ endowed with the topology constructed in Lemma 100.4.7.

This definition does not conflict with the already existing topology on $|\mathcal{X}|$ if $\mathcal{X}$ is an algebraic space.

04Y9 Lemma 100.4.9. Let $\mathcal{X}$ be an algebraic stack. Every point of $|\mathcal{X}|$ has a fundamental system of quasi-compact open neighbourhoods. In particular $|\mathcal{X}|$ is locally quasi-compact in the sense of Topology, Definition 5.13.1.

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |\mathcal{X}|$ of topological spaces. Namely, if $U \to \mathcal{X}$ is surjective and smooth, then Lemma 100.4.7 guarantees that $|U| \to |\mathcal{X}|$ is continuous, surjective, and open. $\square$

100.5. Surjective morphisms

04ZR Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. In Section 100.3 we have already defined what it means for f to be surjective. In Lemma 100.4.4 we have seen that this is equivalent to requiring $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ to be surjective. This clears the way for the following definition.

04ZS Definition 100.5.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is surjective if the map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ of associated topological spaces is surjective.

Here are some lemmas.

04ZT Lemma 100.5.2. The composition of surjective morphisms is surjective.

Proof. Omitted. ☐

04ZU Lemma 100.5.3. The base change of a surjective morphism is surjective.

Proof. Omitted. Hint: Use Lemma 100.4.3. ☐

06PM Lemma 100.5.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Y}' \to \mathcal{Y}$ be a surjective morphism of algebraic stacks. If the base change $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ of f is surjective, then f is surjective.

Proof. Immediate from Lemma 100.4.3. ☐

06PN Lemma 100.5.5. Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is surjective so is $\mathcal{Y} \to \mathcal{Z}$.

Proof. Immediate. ☐

100.6. Quasi-compact algebraic stacks

04YA The following definition is equivalent with the definition for algebraic spaces by Properties of Spaces, Lemma 66.5.2.

04YB Definition 100.6.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is quasi-compact if and only if $|\mathcal{X}|$ is quasi-compact.

04YC Lemma 100.6.2. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:

(1) $\mathcal{X}$ is quasi-compact,
(2) there exists a surjective smooth morphism $U \to \mathcal{X}$ with U an affine scheme,
(3) there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact scheme,
(4) there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact algebraic space, and
(5) there exists a surjective morphism $\mathcal{U} \to \mathcal{X}$ of algebraic stacks such that $\mathcal{U}$ is quasi-compact.

Proof. We will use Lemma 100.4.4. Suppose $\mathcal{U}$ and $\mathcal{U} \to \mathcal{X}$ are as in (5). Then since $|\mathcal{U}| \to |\mathcal{X}|$ is surjective and continuous we conclude that $|\mathcal{X}|$ is quasi-compact. Thus (5) implies (1). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate. Assume (1), i.e., $\mathcal{X}$ is quasi-compact, i.e., that $|\mathcal{X}|$ is quasi-compact. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then since $|U| \to |\mathcal{X}|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still smooth). Choose a finite affine open covering $U' = U_1 \cup \ldots \cup U_n$. Then $U_1 \amalg \ldots \amalg U_n \to \mathcal{X}$ is a surjective smooth morphism whose source is an affine scheme (Schemes, Lemma 26.6.8). Hence (2) holds. □

04YD Lemma 100.6.3. A finite disjoint union of quasi-compact algebraic stacks is a quasi-compact algebraic stack.

Proof. This is clear from the corresponding topological fact. □

100.7. Properties of algebraic stacks defined by properties of schemes

04YE Any smooth local property of schemes gives rise to a corresponding property of algebraic stacks via the following lemma. Note that a property of schemes which is smooth local is also étale local as any étale covering is also a smooth covering. Hence for a smooth local property P of schemes we know what it means to say that an algebraic space has P, see Properties of Spaces, Section 66.7.

04YF Lemma 100.7.1. Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology, see Descent, Definition 35.15.1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent

(1) for some scheme U and some surjective smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,
(2) for every scheme U and every smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,
(3) for some algebraic space U and some surjective smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$, and
(4) for every algebraic space U and every smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$.

If $\mathcal{X}$ is a scheme this is equivalent to $\mathcal{P}(U)$. If $\mathcal{X}$ is an algebraic space this is equivalent to X having property $\mathcal{P}$.

Proof. Let $U \to \mathcal{X}$ surjective and smooth with U an algebraic space. Let $V \to \mathcal{X}$ be a smooth morphism with V an algebraic space. Choose schemes U' and V' and surjective étale morphisms $U' \to U$ and $V' \to V$. Finally, choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2. Moreover, $W \to V'$ is surjective as $U' \to \mathcal{X}$ is surjective. Hence, we have

$$\mathcal{P}(U) \Leftrightarrow \mathcal{P}(U') \Rightarrow \mathcal{P}(W) \Rightarrow \mathcal{P}(V') \Leftrightarrow \mathcal{P}(V)$$

where the equivalences are by definition of property $\mathcal{P}$ for algebraic spaces, and the two implications come from Descent, Definition 35.15.1. This proves (3) $\Rightarrow$ (4).

The implications (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (2) are immediate. □

04YG Definition 100.7.2. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology. We say $\mathcal{X}$ has property $\mathcal{P}$ if any of the equivalent conditions of Lemma 100.7.1 hold.

04YH Remark 100.7.3. Here is a list of properties which are local for the smooth topology (keep in mind that the fpqc, fppf, and syntomic topologies are stronger than the smooth topology):

(1) locally Noetherian, see Descent, Lemma 35.16.1,

(2) Jacobson, see Descent, Lemma 35.16.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 35.17.1,
(4) Cohen-Macaulay, see Descent, Lemma 35.17.2,
(5) reduced, see Descent, Lemma 35.18.1,
(6) normal, see Descent, Lemma 35.18.2,
(7) locally Noetherian and (R_k), see Descent, Lemma 35.18.3,
(8) regular, see Descent, Lemma 35.18.4,
(9) Nagata, see Descent, Lemma 35.18.5.

Any smooth local property of germs of schemes gives rise to a corresponding property of algebraic stacks. Note that a property of germs which is smooth local is also étale local. Hence for a smooth local property of germs of schemes P we know what it means to say that an algebraic space X has property P at $x \in |X|$, see Properties of Spaces, Section 100.7.

04YI Lemma 100.7.4. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local, see Descent, Definition 35.21.1. The following are equivalent

(1) for any smooth morphism $U \to \mathcal{X}$ with U a scheme and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,
(2) for some smooth morphism $U \to \mathcal{X}$ with U a scheme and some $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,
(3) for any smooth morphism $U \to \mathcal{X}$ with U an algebraic space and $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u, and
(4) for some smooth morphism $U \to \mathcal{X}$ with U a an algebraic space and some $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u.

If $\mathcal{X}$ is representable, then this is equivalent to $\mathcal{P}(\mathcal{X}, x)$. If $\mathcal{X}$ is an algebraic space then this is equivalent to $\mathcal{X}$ having property $\mathcal{P}$ at x.

Proof. Let $a : U \to \mathcal{X}$ and $u \in |U|$ as in (3). Let $b : V \to \mathcal{X}$ be another smooth morphism with V an algebraic space and $v \in |V|$ with $b(v) = x$ also. Choose a scheme U', an étale morphism $U' \to U$ and $u' \in U'$ mapping to u. Choose a scheme V', an étale morphism $V' \to V$ and $v' \in V'$ mapping to v. By Lemma 100.4.3 there exists a point $\overline{w} \in |V' \times_{\mathcal{X}} U'|$ mapping to u' and v'. Choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. We may choose a $w \in |W|$ mapping to $\overline{w}$ (see Properties of Spaces, Lemma 66.4.4). Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2. Hence

$$\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u') \Leftrightarrow \mathcal{P}(W, w) \Leftrightarrow \mathcal{P}(V', v') \Leftrightarrow \mathcal{P}(V, v)$$

The outer two equivalences by Properties of Spaces, Definition 66.7.5 and the other two by what it means to be a smooth local property of germs of schemes. This proves (4) $\Rightarrow$ (3).

The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (1) are immediate. □

04YJ Definition 100.7.5. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. We say $\mathcal{X}$ has property $\mathcal{P}$ at x if any of the equivalent conditions of Lemma 100.7.4 holds.

100.8. Monomorphisms of algebraic stacks

04ZV We define a monomorphism of algebraic stacks in the following way. We will see in Lemma 100.8.4 that this is compatible with the corresponding 2-category theoretic notion.

04ZW **Definition 100.8.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *monomorphism* if it is representable by algebraic spaces and a monomorphism in the sense of Section 100.3.

First some basic lemmas.

04ZX **Lemma 100.8.2.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a monomorphism. Then $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X}$ is a monomorphism.*

Proof. This follows from the general discussion in Section 100.3. □

04ZY **Lemma 100.8.3.** *Compositions of monomorphisms of algebraic stacks are monomorphisms.*

Proof. This follows from the general discussion in Section 100.3 and Morphisms of Spaces, Lemma 67.10.4. □

04ZZ **Lemma 100.8.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *f is a monomorphism,*
(2) *f is fully faithful,*
(3) *the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is an equivalence, and*
(4) *there exists an algebraic space W and a surjective, flat morphism $W \to \mathcal{Y}$ which is locally of finite presentation such that $V = \mathcal{X} \times_\mathcal{Y} W$ is an algebraic space, and the morphism $V \to W$ is a monomorphism of algebraic spaces.*

Proof. The equivalence of (1) and (4) follows from the general discussion in Section 100.3 and in particular Lemmas 100.3.1 and 100.3.3.

The equivalence of (2) and (3) is Categories, Lemma 4.35.10.

Assume the equivalent conditions (2) and (3). Then f is representable by algebraic spaces according to Algebraic Stacks, Lemma 94.15.2. Moreover, the 2-Yoneda lemma combined with the fully faithfulness implies that for every scheme T the functor

$$\mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, \mathcal{Y})$$

is fully faithful. Hence given a morphism $y : T \to \mathcal{Y}$ there exists up to unique 2-isomorphism at most one morphism $x : T \to \mathcal{X}$ such that $y \cong f \circ x$. In particular, given a morphism of schemes $h : T' \to T$ there exists at most one lift $\tilde{h} : T' \to T \times_\mathcal{Y} \mathcal{X}$ of h. Thus $T \times_\mathcal{Y} \mathcal{X} \to T$ is a monomorphism of algebraic spaces, which proves that (1) holds.

Finally, assume that (1) holds. Then for any scheme T and morphism $y : T \to \mathcal{Y}$ the fibre product $T \times_\mathcal{Y} \mathcal{X}$ is an algebraic space, and $T \times_\mathcal{Y} \mathcal{X} \to T$ is a monomorphism. Hence there exists up to unique isomorphism exactly one pair (x, α) where $x : T \to \mathcal{X}$ is a morphism and $\alpha : f \circ x \to y$ is a 2-morphism. Applying the 2-Yoneda lemma this says exactly that f is fully faithful, i.e., that (2) holds. □

0500 **Lemma 100.8.5.** *A monomorphism of algebraic stacks induces an injective map of sets of points.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Suppose that $x_i : \mathrm{Spec}(K_i) \to \mathcal{X}$ be morphisms such that $f \circ x_1$ and $f \circ x_2$ define the same element of $|\mathcal{Y}|$. Applying the definition we find a common extension Ω with corresponding morphisms $c_i : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K_i)$ and a 2-isomorphism $\beta : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. As f is fully faithful, see Lemma 100.8.4, we can lift β to an isomorphism $\alpha : x_1 \circ c_1 \to x_1 \circ c_2$. Hence x_1 and x_2 define the same point of $|\mathcal{X}|$ as desired. □

0CBB Lemma 100.8.6. Let $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{X}'$ is a monomorphism then the canonical diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[d] \\ \mathcal{X}' \arrow[r] & \mathcal{X}' \times_{\mathcal{Y}} \mathcal{X}' \end{tikzcd}$$

is a fibre product square.

Proof. We have $\mathcal{X} = \mathcal{X} \times_{\mathcal{X}'} \mathcal{X}$ by Lemma 100.8.4. Thus the result by applying Categories, Lemma 4.31.13. □

100.9. Immersions of algebraic stacks

04YK Immersions of algebraic stacks are defined as follows.

04YL Definition 100.9.1. Immersions.

(1) A morphism of algebraic stacks is called an open immersion if it is representable, and an open immersion in the sense of Section 100.3.
(2) A morphism of algebraic stacks is called a closed immersion if it is representable, and a closed immersion in the sense of Section 100.3.
(3) A morphism of algebraic stacks is called an immersion if it is representable, and an immersion in the sense of Section 100.3.

This is not the most convenient way to think about immersions for us. For us it is a little bit more convenient to think of an immersion as a morphism of algebraic stacks which is representable by algebraic spaces and is an immersion in the sense of Section 100.3. Similarly for closed and open immersions. Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

0501 Lemma 100.9.2. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a (closed, resp. open) immersion. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a (closed, resp. open) immersion.

Proof. This follows from the general discussion in Section 100.3. □

0502 Lemma 100.9.3. Compositions of immersions of algebraic stacks are immersions. Similarly for closed immersions and open immersions.

Proof. This follows from the general discussion in Section 100.3 and Spaces, Lemma 65.12.2. □

0503 Lemma 100.9.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. let W be an algebraic space and let $W \to \mathcal{Y}$ be a surjective, flat morphism which is locally of finite presentation. The following are equivalent:

(1) f is an (open, resp. closed) immersion, and
(2) $V = W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $V \to W$ is an (open, resp. closed) immersion.

Proof. This follows from the general discussion in Section 100.3 and in particular Lemmas 100.3.1 and 100.3.3. $\square$

0504 Lemma 100.9.5. An immersion is a monomorphism.

Proof. See Morphisms of Spaces, Lemma 67.10.7. $\square$

0H20 Lemma 100.9.6. If $f : \mathcal{X} \to \mathcal{Y}$ is an immersion, then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism onto a locally closed subset. If f is a closed, resp. open immersion, then $|f|$ is closed, resp. open.

Proof. Omitted. $\square$

The following two lemmas explain how to think about immersions in terms of presentations.

0505 Lemma 100.9.7. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $i : \mathcal{Z} \to [U/R]$ be an immersion. Then there exists an R-invariant locally closed subspace $Z \subset U$ and a presentation $[Z/R_Z] \to \mathcal{Z}$ where R_Z is the restriction of R to Z such that

$$\begin{tikzcd} {[Z/R_Z]} \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dl, "i"] \\ & {[U/R]} & \end{tikzcd}$$

is 2-commutative. If i is a closed (resp. open) immersion then Z is a closed (resp. open) subspace of U.

Proof. By Lemma 100.3.6 we get a commutative diagram

$$\begin{tikzcd} {[U'/R']} \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dl] \\ & {[U/R]} & \end{tikzcd}$$

where $U' = \mathcal{Z} \times_{[U/R]} U$ and $R' = \mathcal{Z} \times_{[U/R]} R$. Since $\mathcal{Z} \to [U/R]$ is an immersion we see that $U' \to U$ is an immersion of algebraic spaces. Let $Z \subset U$ be the locally closed subspace such that $U' \to U$ factors through Z and induces an isomorphism $U' \to Z$. It is clear from the construction of R' that $R' = U' \times_{U,t} R = R \times_{s,U} U'$. This implies that $Z \cong U'$ is R-invariant and that the image of $R' \to R$ identifies R' with the restriction $R_Z = s^{-1}(Z) = t^{-1}(Z)$ of R to Z. Hence the lemma holds. $\square$

04YN Lemma 100.9.8. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $\mathcal{X} = [U/R]$ be the associated algebraic stack, see Algebraic Stacks, Theorem 94.17.3. Let $Z \subset U$ be an R-invariant locally closed subspace. Then

$$[Z/R_Z] \longrightarrow [U/R]$$

is an immersion of algebraic stacks, where R_Z is the restriction of R to Z. If $Z \subset U$ is open (resp. closed) then the morphism is an open (resp. closed) immersion of algebraic stacks.

Proof. Recall that by Groupoids in Spaces, Definition 78.18.1 (see also discussion following the definition) we have $R_Z = s^{-1}(Z) = t^{-1}(Z)$ as locally closed subspaces of R. Hence the two morphisms $R_Z \to Z$ are smooth as base changes of s and t. Hence $(Z, R_Z, s|_{R_Z}, t|_{R_Z}, c|_{R_Z \times_{s,Z,t} R_Z})$ is a smooth groupoid in algebraic spaces, and we see that $[Z/R_Z]$ is an algebraic stack, see Algebraic Stacks, Theorem 94.17.3. The assumptions of Groupoids in Spaces, Lemma 78.25.3 are all satisfied and it follows that we have a 2-fibre square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & {[Z/R_Z]} \arrow[d] \\ U \arrow[r] & {[U/R]} \end{tikzcd}$$

It follows from this and Lemma 100.3.1 that $[Z/R_Z] \to [U/R]$ is representable by algebraic spaces, whereupon it follows from Lemma 100.3.3 that the right vertical arrow is an immersion (resp. closed immersion, resp. open immersion) if and only if the left vertical arrow is. $\square$

We can define open, closed, and locally closed substacks as follows.

04YM Definition 100.9.9. Let $\mathcal{X}$ be an algebraic stack.

(1) An open substack of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an open immersion.
(2) A closed substack of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is a closed immersion.
(3) A locally closed substack of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an immersion.

This definition should be used with caution. Namely, if $f : \mathcal{X} \to \mathcal{Y}$ is an equivalence of algebraic stacks and $\mathcal{X}' \subset \mathcal{X}$ is an open substack, then it is not necessarily the case that the subcategory $f(\mathcal{X}')$ is an open substack of $\mathcal{Y}$. The problem is that it may not be a strictly full subcategory; but this is also the only problem. Here is a formal statement.

0506 Lemma 100.9.10. For any immersion $i : \mathcal{Z} \to \mathcal{X}$ there exists a unique locally closed substack $\mathcal{X}' \subset \mathcal{X}$ such that i factors as the composition of an equivalence $i' : \mathcal{Z} \to \mathcal{X}'$ followed by the inclusion morphism $\mathcal{X}' \to \mathcal{X}$. If i is a closed (resp. open) immersion, then $\mathcal{X}'$ is a closed (resp. open) substack of $\mathcal{X}$.

Proof. Omitted. $\square$

0507 Lemma 100.9.11. Let $[U/R] \to \mathcal{X}$ be a presentation of an algebraic stack. There is a canonical bijection

$$\text{locally closed substacks } \mathcal{Z} \text{ of } \mathcal{X} \longrightarrow R\text{-invariant locally closed subspaces } Z \text{ of } U$$

which sends $\mathcal{Z}$ to $U \times_{\mathcal{X}} \mathcal{Z}$. Moreover, a morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. Similarly for closed substacks and open substacks.

Proof. By Lemmas 100.9.7 and 100.9.8 we find that the map is a bijection. If $\mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ then of course the base change $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. Converse, suppose that $\mathcal{Y} \to \mathcal{X}$ is a morphism such that $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. We will show that for every scheme T and morphism $T \to \mathcal{Y}$, given by an object y of the fibre category of $\mathcal{Y}$ over T, the object y is in fact in the fibre category of $\mathcal{Z}$ over T. Namely, the fibre product $T \times_{\mathcal{X}} U$ is an algebraic space and $T \times_{\mathcal{X}} U \to T$ is a surjective smooth morphism. Hence there is an fppf covering $\{T_i \to T\}$ such that $T_i \to T$ factors through $T \times_{\mathcal{X}} U \to T$ for all i. Then $T_i \to \mathcal{X}$ factors through $\mathcal{Y} \times_{\mathcal{X}} U$ and hence through $Z \subset U$. Thus $y|_{T_i}$ is an object of $\mathcal{Z}$ (as Z is the fibre product of U with $\mathcal{Z}$ over $\mathcal{X}$). Since $\mathcal{Z}$ is a strictly full substack, we conclude that y is an object of $\mathcal{Z}$ as desired. □

06FJ Lemma 100.9.12. Let $\mathcal{X}$ be an algebraic stack. The rule $\mathcal{U} \mapsto |\mathcal{U}|$ defines an inclusion preserving bijection between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$.

Proof. Choose a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 94.16.2. By Lemma 100.9.11 we see that open substacks correspond to R-invariant open subschemes of U. On the other hand Lemmas 100.4.5 and 100.4.7 guarantee these correspond bijectively to open subsets of $|\mathcal{X}|$. □

05UP Lemma 100.9.13. Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and $U \to \mathcal{X}$ a surjective smooth morphism. For an open immersion $V \hookrightarrow U$, there exists an algebraic stack $\mathcal{Y}$, an open immersion $\mathcal{Y} \to \mathcal{X}$, and a surjective smooth morphism $V \to \mathcal{Y}$.

Proof. We define a category fibred in groupoids $\mathcal{Y}$ by letting the fiber category $\mathcal{Y}_T$ over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $y \in \mathrm{Ob}(\mathcal{X}_T)$ such that the projection morphism $V \times_{\mathcal{X},y} T \to T$ surjective. Now for any morphism $x : T \to \mathcal{X}$, the 2-fibred product $T \times_{x,\mathcal{X}} \mathcal{Y}$ has fiber category over T' consisting of triples $(f : T' \to T, y \in \mathcal{X}_{T'}, f^*x \simeq y)$ such that $V \times_{\mathcal{X},y} T' \to T'$ is surjective. Note that $T \times_{x,\mathcal{X}} \mathcal{Y}$ is fibered in setoids since $\mathcal{Y} \to \mathcal{X}$ is faithful (see Stacks, Lemma 8.6.7). Now the isomorphism $f^*x \simeq y$ gives the diagram

$$\begin{tikzcd} V \times_{\mathcal{X},y} T' \arrow[r] \arrow[d] & V \times_{\mathcal{X},x} T \arrow[r] \arrow[d] & V \arrow[d] \\ T' \arrow[r, "f"] & T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

where both squares are cartesian. The morphism $V \times_{\mathcal{X},x} T \to T$ is smooth by base change, and hence open. Let $T_0 \subset T$ be its image. From the cartesian squares we deduce that $V \times_{\mathcal{X},y} T' \to T'$ is surjective if and only if f lands in T_0. Therefore $T \times_{x,\mathcal{X}} \mathcal{Y}$ is representable by T_0, so the inclusion $\mathcal{Y} \to \mathcal{X}$ is an open immersion. By Algebraic Stacks, Lemma 94.15.5 we conclude that $\mathcal{Y}$ is an algebraic stack. Lastly if we denote the morphism $V \to \mathcal{X}$ by g, we have $V \times_{\mathcal{X}} V \to V$ is surjective (the diagonal gives a section). Hence g is in the image of $\mathcal{Y}_V \to \mathcal{X}_V$, i.e., we obtain a morphism $g' : V \to \mathcal{Y}$ fitting into the commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "g'"] & U \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Since $V \times_{g,\mathcal{X}} \mathcal{Y} \to V$ is a monomorphism, it is in fact an isomorphism since $(1, g')$ defines a section. Therefore $g' : V \to \mathcal{Y}$ is a smooth morphism, as it is the base change of the smooth morphism $g : V \to \mathcal{X}$. It is surjective by our construction of $\mathcal{Y}$ which finishes the proof of the lemma. □

05UQ **Lemma 100.9.14.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}_i \subset \mathcal{X}$ a collection of open substacks indexed by $i \in I$. Then there exists an open substack, which we denote $\bigcup_{i \in I} \mathcal{X}_i \subset \mathcal{X}$, such that the $\mathcal{X}_i$ are open substacks covering it.

Proof. We define a fibred subcategory $\mathcal{X}' = \bigcup_{i \in I} \mathcal{X}_i$ by letting the fiber category over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $x \in \mathrm{Ob}(\mathcal{X}_T)$ such that the morphism $\coprod_{i \in I}(\mathcal{X}_i \times_\mathcal{X} T) \to T$ is surjective. Let $x_i \in \mathrm{Ob}((\mathcal{X}_i)_T)$. Then $(x_i, 1)$ gives a section of $\mathcal{X}_i \times_\mathcal{X} T \to T$, so we have an isomorphism. Thus $\mathcal{X}_i \subset \mathcal{X}'$ is a full subcategory. Now let $x \in \mathrm{Ob}(\mathcal{X}_T)$. Then $\mathcal{X}_i \times_\mathcal{X} T$ is representable by an open subscheme $T_i \subset T$. The 2-fibred product $\mathcal{X}' \times_\mathcal{X} T$ has fiber over T' consisting of $(y \in \mathcal{X}_{T'}, f : T' \to T, f^*x \simeq y)$ such that $\coprod(\mathcal{X}_i \times_{\mathcal{X},y} T') \to T'$ is surjective. The isomorphism $f^*x \simeq y$ induces an isomorphism $\mathcal{X}_i \times_{\mathcal{X},y} T' \simeq T_i \times_T T'$. Then the $T_i \times_T T'$ cover T' if and only if f lands in $\bigcup T_i$. Therefore we have a diagram

$$\begin{tikzcd} T_i \arrow[r] \arrow[d] & \bigcup T_i \arrow[r] \arrow[d] & T \arrow[d] \\ \mathcal{X}_i \arrow[r] & \mathcal{X}' \arrow[r] & \mathcal{X} \end{tikzcd}$$

with both squares cartesian. By Algebraic Stacks, Lemma 94.15.5 we conclude that $\mathcal{X}' \subset \mathcal{X}$ is algebraic and an open substack. It is also clear from the cartesian squares above that the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}'$ which finishes the proof of the lemma. $\square$

05UR **Lemma 100.9.15.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}' \subset \mathcal{X}$ a quasi-compact open substack. Suppose that we have a collection of open substacks $\mathcal{X}_i \subset \mathcal{X}$ indexed by $i \in I$ such that $\mathcal{X}' \subset \bigcup_{i \in I} \mathcal{X}_i$, where we define the union as in Lemma 100.9.14. Then there exists a finite subset $I' \subset I$ such that $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$.

Proof. Since $\mathcal{X}$ is algebraic, there exists a scheme U with a surjective smooth morphism $U \to \mathcal{X}$. Let $U_i \subset U$ be the open subscheme representing $\mathcal{X}_i \times_\mathcal{X} U$ and $U' \subset U$ the open subscheme representing $\mathcal{X}' \times_\mathcal{X} U$. By hypothesis, $U' \subset \bigcup_{i \in I} U_i$. From the proof of Lemma 100.6.2, there is a quasi-compact open $V \subset U'$ such that $V \to \mathcal{X}'$ is a surjective smooth morphism. Therefore there exists a finite subset $I' \subset I$ such that $V \subset \bigcup_{i \in I'} U_i$. We claim that $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$. Take $x \in \mathrm{Ob}(\mathcal{X}'_T)$ for $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Since $\mathcal{X}' \to \mathcal{X}$ is a monomorphism, we have cartesian squares

$$\begin{tikzcd} V \times_\mathcal{X} T \arrow[r] \arrow[d] & T \arrow[r, equal] \arrow[d, "x"] & T \arrow[d, "x"] \\ V \arrow[r] & \mathcal{X}' \arrow[r] & \mathcal{X} \end{tikzcd}$$

By base change, $V \times_\mathcal{X} T \to T$ is surjective. Therefore $\bigcup_{i \in I'} U_i \times_\mathcal{X} T \to T$ is also surjective. Let $T_i \subset T$ be the open subscheme representing $\mathcal{X}_i \times_\mathcal{X} T$. By a formal argument, we have a Cartesian square

$$\begin{tikzcd} U_i \times_{\mathcal{X}_i} T_i \arrow[r] \arrow[d] & U \times_\mathcal{X} T \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$

where the vertical arrows are surjective by base change. Since $U_i \times_{\mathcal{X}_i} T_i \simeq U_i \times_{\mathcal{X}} T$, we find that $\bigcup_{i \in I'} T_i = T$. Hence x is an object of $(\bigcup_{i \in I'} \mathcal{X}_i)_T$ by definition of the union. Observe that the inclusion $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$ is automatically an open substack. □

05WE Lemma 100.9.16. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume

(1) $\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and
(2) each $\mathcal{X}_i$ is an algebraic space.

Then $\mathcal{X}$ is an algebraic space.

Proof. Apply Stacks, Lemma 8.6.10 to the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$ and the morphism $\mathrm{id} : \mathcal{X} \to \mathcal{X}$ to see that $\mathcal{X}$ is a stack in setoids. Hence $\mathcal{X}$ is an algebraic space, see Algebraic Stacks, Proposition 94.13.3. □

05WF Lemma 100.9.17. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume

(1) $\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and
(2) each $\mathcal{X}_i$ is a scheme

Then $\mathcal{X}$ is a scheme.

Proof. By Lemma 100.9.16 we see that $\mathcal{X}$ is an algebraic space. Since any algebraic space has a largest open subspace which is a scheme, see Properties of Spaces, Lemma 66.13.1 we see that $\mathcal{X}$ is a scheme. □

The following lemma is the analogue of More on Groupoids, Lemma 40.6.1.

06M3 Lemma 100.9.18. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of algebraic spaces. Assume

(1) $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are fppf local on the target and stable under arbitrary base change,
(2) smooth $\Rightarrow \mathcal{R}$,
(3) for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open subspace $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and
(4) for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Assume f has $\mathcal{Q}$. Then

(A) there exists a largest open substack $\mathcal{X}' \subset \mathcal{X}$ such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$, and
(B) if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}'$ is the largest open substack of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ over which the base change $\mathrm{id}_{\mathcal{Z}} \times f$ has property $\mathcal{P}$.

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ and let $f' : U \to V$ be the base change of f. The morphism of algebraic spaces $f' : U \to V$ has property $\mathcal{Q}$. Thus we obtain the open $W(\mathcal{P}, f') \subset U$ by assumption (3). Note that $U \times_{\mathcal{X}} U = (V \times_{\mathcal{Y}} V) \times_{\mathcal{Y}} \mathcal{X}$ hence the morphism $f'' : U \times_{\mathcal{X}} U \to V \times_{\mathcal{Y}} V$ is the base change of f via either projection $V \times_{\mathcal{Y}} V \to V$. By our choice of V these projections are smooth, hence have property $\mathcal{R}$ by (2). Thus by (4) we see that the inverse images of $W(\mathcal{P}, f')$ under the two projections $\mathrm{pr}_i : U \times_{\mathcal{X}} U \to U$ agree. In other words, $W(\mathcal{P}, f')$ is an R-invariant subspace of U

(where $R = U \times_\mathcal{X} U$). Let $\mathcal{X}'$ be the open substack of $\mathcal{X}$ corresponding to $W(\mathcal{P}, f)$ via Lemma 100.9.7. By construction $W(\mathcal{P}, f') = \mathcal{X}' \times_\mathcal{Y} V$ hence $f|_{\mathcal{X}'}$ has property $\mathcal{P}$ by Lemma 100.3.3. Also, $\mathcal{X}'$ is the largest open substack such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$ as the same maximality holds for $W(\mathcal{P}, f)$. This proves (A).

Finally, if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then we set $T = V \times_\mathcal{Y} \mathcal{Z}$ and we see that $T \to V$ is a morphism of algebraic spaces having property $\mathcal{R}$. Set $f'_T : T \times_V U \to T$ the base change of f'. By (4) again we see that $W(\mathcal{P}, f'_T)$ is the inverse image of $W(\mathcal{P}, f)$ in $T \times_V U$. This implies (B); some details omitted. □

06M4 Remark 100.9.19. Warning: Lemma 100.9.18 should be used with care. For example, it applies to $\mathcal{P}$ =“flat”, $\mathcal{Q}$ =“empty”, and $\mathcal{R}$ =“flat and locally of finite presentation”. But given a morphism of algebraic spaces $f : X \to Y$ the largest open subspace $W \subset X$ such that $f|_W$ is flat is not the set of points where f is flat!

06M5 Remark 100.9.20. Notwithstanding the warning in Remark 100.9.19 there are some cases where Lemma 100.9.18 can be used without causing ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

06M6 (1) $\mathcal{Q}$ =“locally of finite type”, $\mathcal{R} = \emptyset$, and $\mathcal{P}$ =“relative dimension $\leq d$”. See Morphisms of Spaces, Definition 67.33.2 and Morphisms of Spaces, Lemmas 67.34.4 and 67.34.3.
06M7 (2) $\mathcal{Q}$ =“locally of finite type”, $\mathcal{R} = \emptyset$, and $\mathcal{P}$ =“locally quasi-finite”. This is the case $d = 0$ of the previous item, see Morphisms of Spaces, Lemma 67.34.6. On the other hand, properties (3) and (4) are spelled out in Morphisms of Spaces, Lemma 67.34.7.
06M8 (3) $\mathcal{Q}$ =“locally of finite type”, $\mathcal{R} = \emptyset$, and $\mathcal{P}$ =“unramified”. This is Morphisms of Spaces, Lemma 67.38.10.
06M9 (4) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“flat”. See More on Morphisms of Spaces, Theorem 76.22.1 and Lemma 76.22.2. Note that here $W(\mathcal{P}, f)$ is always exactly the set of points where the morphism f is flat because we only consider this open when f has $\mathcal{Q}$ (see loc.cit.).
06MA (5) $\mathcal{Q}$ =“locally of finite presentation”, $\mathcal{R}$ =“flat and locally of finite presentation”, and $\mathcal{P}$ =“étale”. This follows on combining (3) and (4) because an unramified morphism which is flat and locally of finite presentation is étale, see Morphisms of Spaces, Lemma 67.39.12.
(6) Add more here as needed (compare with the longer list at More on Groupoids, Remark 40.6.3).

100.10. Reduced algebraic stacks

0508 We have already defined reduced algebraic stacks in Section 100.7.

0509 Lemma 100.10.1. Let $\mathcal{X}$ be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a closed subset. There exists a unique closed substack $\mathcal{Z} \subset \mathcal{X}$ with the following properties: (a) we have $|\mathcal{Z}| = T$, and (b) $\mathcal{Z}$ is reduced.

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism, where U is an algebraic space. Set $R = U \times_\mathcal{X} U$, so that there is a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 94.16.2. As usual we denote $s, t : R \to U$ the two smooth projection morphisms. By Lemma 100.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $|s|^{-1}(T') = |t|^{-1}(T')$. Let $Z \subset U$ be the reduced induced algebraic space structure on T', see Properties of Spaces, Definition 66.12.5. The fibre products $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are closed subspaces of R (Spaces, Lemma 65.12.3). The projections $Z \times_{U,t} R \to Z$ and $R \times_{s,U} Z \to Z$ are smooth by Morphisms of Spaces, Lemma 67.37.3. Thus as Z is reduced, it follows that $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are reduced, see Remark 100.7.3. Since

$$|Z \times_{U,t} R| = |t|^{-1}(T') = |s|^{-1}(T') = R \times_{s,U} Z$$

we conclude from the uniqueness in Properties of Spaces, Lemma 66.12.3 that $Z \times_{U,t} R = R \times_{s,U} Z$. Hence Z is an R-invariant closed subspace of U. By the correspondence of Lemma 100.9.11 we obtain a closed substack $\mathcal{Z} \subset \mathcal{X}$ with $Z = \mathcal{Z} \times_{\mathcal{X}} U$. Then $[Z/R_Z] \to \mathcal{Z}$ is a presentation (Lemma 100.9.7). Then $|\mathcal{Z}| = |Z|/|R_Z| = |T'|/\sim$ is the given closed subset T. We omit the proof of unicity. $\square$

050A Lemma 100.10.2. Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}' \subset \mathcal{X}$ is a closed substack, $\mathcal{X}$ is reduced and $|\mathcal{X}'| = |\mathcal{X}|$, then $\mathcal{X}' = \mathcal{X}$.

Proof. Choose a presentation $[U/R] \to \mathcal{X}$ with U a scheme. As $\mathcal{X}$ is reduced, we see that U is reduced (by definition of reduced algebraic stacks). By Lemma 100.9.11 $\mathcal{X}'$ corresponds to an R-invariant closed subscheme $Z \subset U$. But now $|Z| \subset |U|$ is the inverse image of $|\mathcal{X}'|$, and hence $|Z| = |U|$. Hence Z is a closed subscheme of U whose underlying sets of points agree. By Schemes, Lemma 26.12.7 the map $\mathrm{id}_U : U \to U$ factors through $Z \to U$, and hence $Z = U$, i.e., $\mathcal{X}' = \mathcal{X}$. $\square$

050B Lemma 100.10.3. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack Assume $\mathcal{Y}$ is reduced. A morphism $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$.

Proof. Assume $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$. Consider $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}$. There is an equivalence $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}'$ where $\mathcal{Y}'$ is a closed substack of $\mathcal{Y}$, see Lemmas 100.9.2 and 100.9.10. Using Lemmas 100.4.3, 100.8.5, and 100.9.5 we see that $|\mathcal{Y}'| = |\mathcal{Y}|$. Hence we have reduced the lemma to Lemma 100.10.2. $\square$

050C Definition 100.10.4. Let $\mathcal{X}$ be an algebraic stack. Let $Z \subset |\mathcal{X}|$ be a closed subset. An algebraic stack structure on Z is given by a closed substack $\mathcal{Z}$ of $\mathcal{X}$ with $|\mathcal{Z}|$ equal to Z. The reduced induced algebraic stack structure on Z is the one constructed in Lemma 100.10.1. The reduction $\mathcal{X}_{red}$ of $\mathcal{X}$ is the reduced induced algebraic stack structure on $|\mathcal{X}|$.

In fact we can use this to define the reduced induced algebraic stack structure on a locally closed subset.

06FK Remark 100.10.5. Let X be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|\mathcal{X}|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $\mathcal{U} \subset \mathcal{X}$ be the open substack of X with $|\mathcal{U}| = |\mathcal{X}| \setminus \partial T$, see Lemma 100.9.12. Let $\mathcal{Z}$ be the reduced closed substack of $\mathcal{U}$ with $|\mathcal{Z}| = T$ obtained by taking the reduced induced closed subspace structure, see Definition 100.10.4. By construction $\mathcal{Z} \to \mathcal{U}$ is a closed immersion of algebraic stacks and $\mathcal{U} \to \mathcal{X}$ is an open immersion, hence

$\mathcal{Z} \to \mathcal{X}$ is an immersion of algebraic stacks by Lemma 100.9.3. Note that $\mathcal{Z}$ is a reduced algebraic stack and that $|\mathcal{Z}| = T$ as subsets of $|X|$. We sometimes say $\mathcal{Z}$ is the reduced induced substack structure on T.

100.11. Residual gerbes

06ML In the Stacks project we would like to define the residual gerbe of an algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ to be a monomorphism of algebraic stacks $m_x : \mathcal{Z}_x \to \mathcal{X}$ where $\mathcal{Z}_x$ is a reduced algebraic stack having a unique point which is mapped by m_x to x. It turns out that there are many issues with this notion; existence is not clear in general and neither is uniqueness. We resolve the uniqueness issue by imposing a slightly stronger condition on the algebraic stacks $\mathcal{Z}_x$. We discuss this in more detail by working through a few simple lemmas regarding reduced algebraic stacks having a unique point.

06MM **Lemma 100.11.1.** Let $\mathcal{Z}$ be an algebraic stack. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to \mathcal{Z}$ where k' is a field is surjective and flat.

Proof. Consider the fibre square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ \mathrm{Spec}(k') \arrow[r] & \mathcal{Z} \end{tikzcd}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to \mathcal{Z}$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 67.31.5 (via the discussion in Section 100.3) that $\mathrm{Spec}(k') \to \mathcal{Z}$ is flat. It is clear that it is surjective as by assumption $|\mathcal{Z}|$ is a singleton. $\square$

06MN **Lemma 100.11.2.** Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent

(1) $\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton,
(2) there exists a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field, and
(3) there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.

Proof. Assume (1). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to \mathcal{Z}$ is flat (see Morphisms of Spaces, Lemma 67.30.3). It is also surjective as $|\mathcal{Z}|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence regular (see Spaces over Fields, Lemma 72.16.1) and in particular reduced. Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent on Spaces, Lemma 74.9.2). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite type by Morphisms of Spaces, Lemmas 67.23.2 and 67.37.6. It is also flat and surjective by Lemma 100.11.1. Hence (3) holds. $\square$

The following lemma singles out a slightly better class of singleton algebraic stacks than the preceding lemma.

06MP Lemma 100.11.3. Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent

(1) $\mathcal{Z}$ is reduced, locally Noetherian, and $|\mathcal{Z}|$ is a singleton, and
(2) there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.

Proof. Assume (2) holds. By Lemma 100.11.2 we see that $\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton. Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence locally Noetherian (see Morphisms of Spaces, Lemma 67.23.5). Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent on Spaces, Lemma 74.9.3). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 29.21.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 67.28.2 and 67.37.5. It is also flat and surjective by Lemma 100.11.1. Hence (2) holds. $\square$

06MQ Lemma 100.11.4. Let $\mathcal{Z}' \to \mathcal{Z}$ be a monomorphism of algebraic stacks. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then either $\mathcal{Z}'$ is empty or $\mathcal{Z}' \to \mathcal{Z}$ is an equivalence.

Proof. We may assume that $\mathcal{Z}'$ is nonempty. In this case the fibre product $T = \mathcal{Z}' \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is nonempty, see Lemma 100.4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 67.10.8. We conclude that $\mathrm{Spec}(k) \to \mathcal{Z}$ factors through $\mathcal{Z}'$. Suppose the morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ is given by the object ξ over $\mathrm{Spec}(k)$. We have just seen that ξ is isomorphic to an object ξ' of $\mathcal{Z}'$ over $\mathrm{Spec}(k)$. Since z is surjective, flat, and locally of finite presentation we see that every object of $\mathcal{Z}$ over any scheme is fppf locally isomorphic to a pullback of ξ, hence also to a pullback of ξ'. By descent of objects for stacks in groupoids this implies that $\mathcal{Z}' \to \mathcal{Z}$ is essentially surjective (as well as fully faithful, see Lemma 100.8.4). Hence we win. $\square$

06MR Lemma 100.11.5. Let $\mathcal{Z}$ be an algebraic stack. Assume $\mathcal{Z}$ satisfies the equivalent conditions of Lemma 100.11.2. Then there exists a unique strictly full subcategory $\mathcal{Z}' \subset \mathcal{Z}$ such that $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of

Lemma 100.11.3. The inclusion morphism $\mathcal{Z}' \to \mathcal{Z}$ is a monomorphism of algebraic stacks.

Proof. The last part is immediate from the first part and Lemma 100.8.4. Pick a field k and a morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ which is surjective, flat, and locally of finite type. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$. The projections $s, t : R \to U$ are locally of finite type. Since U is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 67.28.7). We see that $\mathcal{Z}' = [U/R]$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. By Algebraic Stacks, Lemma 94.16.1 we obtain a canonical morphism

$$f : \mathcal{Z}' \longrightarrow \mathcal{Z}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.15.2 and a monomorphism, see Lemma 100.8.4. By Criteria for Representability, Lemma 97.17.1 the morphism $U \to \mathcal{Z}'$ is surjective, flat, and locally of finite presentation. Hence $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 100.11.3. By Algebraic Stacks, Lemma 94.12.4 we may replace $\mathcal{Z}'$ by its essential image in $\mathcal{Z}$. Hence we have proved all the assertions of the lemma except for the uniqueness of $\mathcal{Z}' \subset \mathcal{Z}$. Suppose that $\mathcal{Z}'' \subset \mathcal{Z}$ is a second such algebraic stack. Then the projections

$$\mathcal{Z}' \longleftarrow \mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Z}'' \longrightarrow \mathcal{Z}''$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 100.4.3. Hence the two projections are isomorphisms by Lemma 100.11.4 and we win. $\square$

06MS Example 100.11.6. Here is an example where the morphism constructed in Lemma 100.11.5 isn't an isomorphism. This example shows that imposing that residual gerbes are locally Noetherian is necessary in Definition 100.11.8. In fact, the example is even an algebraic space! Let $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the absolute Galois group of $\mathbf{Q}$ with the pro-finite topology. Let

$$U = \mathrm{Spec}(\overline{\mathbf{Q}}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\overline{\mathbf{Q}}) = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times \mathrm{Spec}(\overline{\mathbf{Q}})$$

(we omit a precise explanation of the meaning of the last equal sign). Let G denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with the discrete topology viewed as a constant group scheme over $\mathrm{Spec}(\overline{\mathbf{Q}})$, see Groupoids, Example 39.5.6. Then G acts freely and transitively on U. Let $X = U/G$, see Spaces, Definition 65.14.4. Then X is a non-noetherian reduced algebraic space with exactly one point. Furthermore, X has a (locally) finite type point:

$$x : \mathrm{Spec}(\overline{\mathbf{Q}}) \longrightarrow U \longrightarrow X$$

Indeed, every point of U is actually closed! As X is an algebraic space over $\overline{\mathbf{Q}}$ it follows that x is a monomorphism. So x is the morphism constructed in Lemma 100.11.5 but x is not an isomorphism. In fact $\mathrm{Spec}(\overline{\mathbf{Q}}) \to X$ is the residual gerbe of X at x.

It will turn out later that under mild assumptions on the algebraic stack $\mathcal{X}$ the equivalent conditions of the following lemma are satisfied for every point $x \in |\mathcal{X}|$ (see Morphisms of Stacks, Section 101.31).

06MT Lemma 100.11.7. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent

(1) there exists an algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,
(2) there exists a reduced algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,
(3) there exists an algebraic stack $\mathcal{Z}$, a monomorphism $f : \mathcal{Z} \to \mathcal{X}$, and a surjective flat morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field such that $x = f(z)$.

Moreover, if these conditions hold, then there exists a unique strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ such that $\mathcal{Z}_x$ is a reduced, locally Noetherian algebraic stack and $|\mathcal{Z}_x|$ is a singleton which maps to x via the map $|\mathcal{Z}_x| \to |\mathcal{X}|$.

Proof. If $\mathcal{Z} \to \mathcal{X}$ is as in (1), then $\mathcal{Z}_{red} \to \mathcal{X}$ is as in (2). (See Section 100.10 for the notion of the reduction of an algebraic stack.) Hence (1) implies (2). It is immediate that (2) implies (1). The equivalence of (2) and (3) is immediate from Lemma 100.11.2.

At this point we've seen the equivalence of (1) – (3). Pick a monomorphism $f : \mathcal{Z} \to \mathcal{X}$ as in (2). Note that this implies that f is fully faithful, see Lemma 100.8.4. Denote $\mathcal{Z}' \subset \mathcal{X}$ the essential image of the functor f. Then $f : \mathcal{Z} \to \mathcal{Z}'$ is an equivalence and hence $\mathcal{Z}'$ is an algebraic stack, see Algebraic Stacks, Lemma 94.12.4. Apply Lemma 100.11.5 to get a strictly full subcategory $\mathcal{Z}_x \subset \mathcal{Z}'$ as in the statement of the lemma. This proves all the statements of the lemma except for uniqueness.

In order to prove the uniqueness suppose that $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}'_x \subset \mathcal{X}$ are two strictly full subcategories as in the statement of the lemma. Then the projections

$$\mathcal{Z}'_x \longleftarrow \mathcal{Z}'_x \times_{\mathcal{X}} \mathcal{Z}_x \longrightarrow \mathcal{Z}_x$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 100.4.3. Hence the two projections are isomorphisms by Lemma 100.11.4 and we win. ☐

Having explained the above we can now make the following definition.

06MU Definition 100.11.8. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

(1) We say the residual gerbe of $\mathcal{X}$ at x exists if the equivalent conditions (1), (2), and (3) of Lemma 100.11.7 hold.
(2) If the residual gerbe of $\mathcal{X}$ at x exists, then the residual gerbe of $\mathcal{X}$ at x[1] is the strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ constructed in Lemma 100.11.7.

In particular we know that $\mathcal{Z}_x$ (if it exists) is a locally Noetherian, reduced algebraic stack and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow \mathcal{Z}_x.$$

We will see in Morphisms of Stacks, Lemma 101.28.12 that $\mathcal{Z}_x$ is a gerbe. Existence of residual gerbes is discussed in Morphisms of Stacks, Section 101.31.

[1]This clashes with [LMB00] in spirit, but not in fact. Namely, in Chapter 11 they associate to any point on any quasi-separated algebraic stack a gerbe (not necessarily algebraic) which they call the residual gerbe. We will see in Morphisms of Stacks, Lemma 101.31.1 that on a quasi-separated algebraic stack every point has a residual gerbe in our sense which is then equivalent to theirs. For more information on this topic see [Ryd10, Appendix B].

0H21 **Example 100.11.9.** Let X be a scheme and let $x \in X$ be a point. Then the monomorphism $x \to X$ is the residual gerbe of X at x where we, as usual, identify x with the scheme $x = \mathrm{Spec}(\kappa(x))$. If X is an algebraic space and $x \in |X|$, then the residual gerbe at x (which is called the residual space) always exists, see Decent Spaces, Section 68.13.

The residual gerbe, if it exists, is a regular algebraic stack by the following lemma.

06MV **Lemma 100.11.10.** A reduced, locally Noetherian algebraic stack $\mathcal{Z}$ such that $|\mathcal{Z}|$ is a singleton is regular.

Proof. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective, flat, and locally of finite presentation (see Lemma 100.11.3). The algebraic space $T = W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is smooth over k in particular regular, see Spaces over Fields, Lemma 72.16.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent on Spaces, Lemma 74.9.4. By definition this means that $\mathcal{Z}$ is regular. $\square$

06MW **Lemma 100.11.11.** Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume that the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ exists. Let $f : \mathrm{Spec}(K) \to \mathcal{X}$ be a morphism where K is a field in the equivalence class of x. Then f factors through the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$.

Proof. Choose a field k and a surjective flat locally finite presentation morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. Set $T = \mathrm{Spec}(K) \times_{\mathcal{X}} \mathcal{Z}_x$. By Lemma 100.4.3 we see that T is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we see that $T \to \mathrm{Spec}(K)$ is a monomorphism. Hence by Morphisms of Spaces, Lemma 67.10.8 we see that $T = \mathrm{Spec}(K)$ which proves the lemma. $\square$

06MX **Lemma 100.11.12.** Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Let $\mathcal{Z}$ be an algebraic stack satisfying the equivalent conditions of Lemma 100.11.3 and let $\mathcal{Z} \to \mathcal{X}$ be a monomorphism such that the image of $|\mathcal{Z}| \to |\mathcal{X}|$ is x. Then the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z} \to \mathcal{X}$ factors as $\mathcal{Z} \to \mathcal{Z}_x \to \mathcal{X}$ where the first arrow is an equivalence.

Proof. Let $\mathcal{Z}_x \subset \mathcal{X}$ be the full subcategory corresponding to the essential image of the functor $\mathcal{Z} \to \mathcal{X}$. Then $\mathcal{Z} \to \mathcal{Z}_x$ is an equivalence, hence $\mathcal{Z}_x$ is an algebraic stack, see Algebraic Stacks, Lemma 94.12.4. Since $\mathcal{Z}_x$ inherits all the properties of $\mathcal{Z}$ from this equivalence it is clear from the uniqueness in Lemma 100.11.7 that $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x. $\square$

0DTH **Lemma 100.11.13.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. If the residual gerbes $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}_y \subset \mathcal{Y}$ of x and y exist, then f induces a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "f"'] & \mathcal{Z}_x \arrow[l] \arrow[d] \\ \mathcal{Y} & \mathcal{Z}_y \arrow[l] \end{tikzcd}$$

Proof. Choose a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. The morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ factors through $\mathcal{Z}_y$ by Lemma 100.11.11. Thus $\mathcal{Z}_x \times_{\mathcal{Y}} \mathcal{Z}_y$ is a nonempty substack of $\mathcal{Z}_x$ hence equal to $\mathcal{Z}_x$ by Lemma 100.11.4. $\square$

0DTI Lemma 100.11.14. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Assume the residual gerbes $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}_y \subset \mathcal{Y}$ of x and y exist and that there exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x such that

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathrm{Spec}(k)$$

is an isomorphism. Then $\mathcal{Z}_x \to \mathcal{Z}_y$ is an isomorphism.

Proof. Let k'/k be an extension of fields. Then

$$\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k') \times_{\mathcal{Y}} \mathrm{Spec}(k')$$

is the base change of the morphism in the lemma by the faithfully flat morphism $\mathrm{Spec}(k' \otimes k') \to \mathrm{Spec}(k \otimes k)$. Thus the property described in the lemma is independent of the choice of the morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x. Thus we may assume that $\mathrm{Spec}(k) \to \mathcal{Z}_x$ is surjective, flat, and locally of finite presentation. In this situation we have

$$\mathcal{Z}_x = [\mathrm{Spec}(k)/R]$$

with $R = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)$. See proof of Lemma 100.11.5. Since also $R = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathrm{Spec}(k)$ we conclude that the morphism $\mathcal{Z}_x \to \mathcal{Z}_y$ of Lemma 100.11.13 is fully faithful by Algebraic Stacks, Lemma 94.16.1. We conclude for example by Lemma 100.11.12. □

100.12. Dimension of a stack

0AFL We can define the dimension of an algebraic stack $\mathcal{X}$ at a point x, using the notion of dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1). In the following lemma the output may be ∞ either because $\mathcal{X}$ is not quasi-compact or because we run into the phenomenon described in Examples, Section 110.15.

0AFM Lemma 100.12.1. Let $\mathcal{X}$ be a locally Noetherian algebraic stack over a scheme S. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $[U/R] \to \mathcal{X}$ be a presentation (Algebraic Stacks, Definition 94.16.5) where U is a scheme. Let $u \in U$ be a point that maps to x. Let $e : U \to R$ be the "identity" map and let $s : R \to U$ be the "source" map, which is a smooth morphism of algebraic spaces. Let R_u be the fiber of $s : R \to U$ over u. The element

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u) \in \mathbf{Z} \cup \infty$$

is independent of the choice of presentation and the point u over x.

Proof. Since $R \to U$ is smooth, the scheme R_u is smooth over $\kappa(u)$ and hence has finite dimension. On the other hand, the scheme U is locally Noetherian, but this does not guarantee that $\dim_u(U)$ is finite. Thus the difference is an element of $\mathbf{Z} \cup \{\infty\}$.

Let $[U'/R'] \to \mathcal{X}$ and $u' \in U'$ be a second presentation where U' is a scheme and u' maps to x. Consider the algebraic space $P = U \times_{\mathcal{X}} U'$. By Lemma 100.4.3 there exists a $p \in |P|$ mapping to u and u'. Since $P \to U$ and $P \to U'$ are smooth we see that $\dim_p(P) = \dim_u(U) + \dim_p(P_u)$ and $\dim_p(P) = \dim_{u'}(U') + \dim_p(P_{u'})$, see Morphisms of Spaces, Lemma 67.37.10. Note that

$$R'_{u'} = \mathrm{Spec}(\kappa(u')) \times_{\mathcal{X}} U' \quad \text{and} \quad P_u = \mathrm{Spec}(\kappa(u)) \times_{\mathcal{X}} U'$$

Let us represent $p \in |P|$ by a morphism $\mathrm{Spec}(\Omega) \to P$. Since p maps to both u and u' it induces a 2-morphism between the compositions $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\kappa(u')) \to \mathcal{X}$ and $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\kappa(u)) \to \mathcal{X}$ which in turn defines an isomorphism

$$\mathrm{Spec}(\Omega) \times_{\mathrm{Spec}(\kappa(u'))} R'_{u'} \cong \mathrm{Spec}(\Omega) \times_{\mathrm{Spec}(\kappa(u))} P_u$$

as algebraic spaces over $\mathrm{Spec}(\Omega)$ mapping the Ω-rational point $(1, e'(u'))$ to $(1, p)$ (some details omitted). We conclude that

$$\dim_{e'(u')}(R'_{u'}) = \dim_p(P_u)$$

by Morphisms of Spaces, Lemma 67.34.3. By symmetry we have $\dim_{e(u)}(R_u) = \dim_p(P_{u'})$. Putting everything together we obtain the independence of choices. □

We can use the lemma above to make the following definition.

0AFN Definition 100.12.2. Let $\mathcal{X}$ be a locally Noetherian algebraic stack over a scheme S. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $[U/R] \to \mathcal{X}$ be a presentation (Algebraic Stacks, Definition 94.16.5) where U is a scheme and let $u \in U$ be a point that maps to x. We define the dimension of $\mathcal{X}$ at x to be the element $\dim_x(\mathcal{X}) \in \mathbf{Z} \cup \infty$ such that

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u).$$

with notation as in Lemma 100.12.1.

The dimension of a stack at a point agrees with the usual notion when $\mathcal{X}$ is a scheme (Topology, Definition 5.10.1), or more generally when $\mathcal{X}$ is a locally Noetherian algebraic space (Properties of Spaces, Definition 66.9.1).

0AFP Definition 100.12.3. Let S be a scheme. Let $\mathcal{X}$ be a locally Noetherian algebraic stack over S. The dimension $\dim(\mathcal{X})$ of $\mathcal{X}$ is defined to be

$$\dim(\mathcal{X}) = \sup_{x \in |\mathcal{X}|} \dim_x(\mathcal{X})$$

This definition of dimension agrees with the usual notion if $\mathcal{X}$ is a scheme (Properties, Lemma 28.10.2) or an algebraic space (Properties of Spaces, Definition 66.9.2).

0AFQ Remark 100.12.4. If $\mathcal{X}$ is a nonempty stack of finite type over a field, then $\dim(\mathcal{X})$ is an integer. For an arbitrary locally Noetherian algebraic stack $\mathcal{X}$, $\dim(\mathcal{X})$ is in $Z \cup \{\pm\infty\}$, and $\dim(\mathcal{X}) = -\infty$ if and only if $\mathcal{X}$ is empty.

0AFR Example 100.12.5. Let X be a scheme of finite type over a field k, and let G be a group scheme of finite type over k which acts on X. Then the dimension of the quotient stack $[X/G]$ is equal to $\dim(X) - \dim(G)$. In particular, the dimension of the classifying stack $BG = [\mathrm{Spec}(k)/G]$ is $-\dim(G)$. Thus the dimension of an algebraic stack can be a negative integer, in contrast to what happens for schemes or algebraic spaces.

100.13. Local irreducibility

0DQG We have defined the geometric number of branches of a scheme at a point in Properties, Section 28.15 and for an algebraic space at a point in Properties of Spaces, Section 66.23. Let $n \in \mathbf{N}$. For a local ring A set

$$P_n(A) = \text{the number of geometric branches of } A \text{ is } n$$

For a smooth ring map $A \to B$ and a prime ideal $\mathfrak{q}$ of B lying over $\mathfrak{p}$ of A we have

$$P_n(A_{\mathfrak{p}}) \Leftrightarrow P_n(B_{\mathfrak{q}})$$

by More on Algebra, Lemma 15.106.8. As in Properties of Spaces, Remark 66.7.6 we may use P_n to define an étale local property $\mathcal{P}_n$ of germs (U, u) of schemes by setting $\mathcal{P}_n(U, u) = P_n(\mathcal{O}_{U,u})$. The corresponding property $\mathcal{P}_n$ of an algebraic space X at a point x (see Properties of Spaces, Definition 66.7.5) is just the property "the number of geometric branches of X at x is n", see Properties of Spaces, Definition 66.23.4. Moreover, the property $\mathcal{P}_n$ is smooth local, see Descent, Definition 35.21.1. This follows either from the equivalence displayed above or More on Morphisms, Lemma 37.36.4. Thus Definition 100.7.5 applies and we obtain a notion for algebraic stacks at a point.

0DQH Definition 100.13.1. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

(1) The number of geometric branches of $\mathcal{X}$ at x is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 100.7.4 hold for $\mathcal{P}_n$ defined above, or else ∞.
(2) We say $\mathcal{X}$ is geometrically unibranch at x if the number of geometric branches of $\mathcal{X}$ at x is 1.

100.14. Finiteness conditions and points

0DTJ This section is the analogue of Decent Spaces, Section 68.4 for points of algebraic stacks.

0DTK Lemma 100.14.1. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent

(1) some morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-compact, and
(2) any morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-compact.

Proof. Let $\mathrm{Spec}(k) \to \mathcal{X}$ be in the equivalence class of x. Let k'/k be a field extension. Then we have to show that $\mathrm{Spec}(k) \to \mathcal{X}$ is quasi-compact if and only if $\mathrm{Spec}(k') \to \mathcal{X}$ is quasi-compact. This follows from Morphisms of Spaces, Lemma 67.8.6 and the principle of Algebraic Stacks, Lemma 94.10.9. □

Sometimes people say that a point $x \in |\mathcal{X}|$ satisfying the equivalent conditions of Lemma 100.14.1 is a "quasi-compact point".

100.15. Other chapters

Preliminaries

CHAPTER 101

Morphisms of Algebraic Stacks

04XM 101.1. Introduction

04XN In this chapter we introduce some types of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

The goal is to extend the definition of each of the types of morphisms of algebraic spaces to morphisms of algebraic stacks. Each case is slightly different and it seems best to treat them all separately.

For morphisms of algebraic stacks which are representable by algebraic spaces we have already defined a large number of types of morphisms, see Properties of Stacks, Section 100.3. For each corresponding case in this chapter we have to make sure the definition in the general case is compatible with the definition given there.

101.2. Conventions and abuse of language

04XP We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

101.3. Properties of diagonals

04XQ The diagonal of an algebraic stack is closely related to the *Isom*-sheaves, see Algebraic Stacks, Lemma 94.10.11. By the second defining property of an algebraic stack these *Isom*-sheaves are always algebraic spaces.

04XR Lemma 101.3.1. Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then the morphism $Isom_\mathcal{X}(x, y) \to T$ is locally of finite type.

Proof. By Algebraic Stacks, Lemma 94.16.2 we may assume that $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces. By Descent on Spaces, Lemma 74.11.9 it suffices to check the property fppf locally on T. Thus we may assume that x, y come from morphisms $x', y' : T \to U$. By Groupoids in Spaces, Lemma 78.22.1 we see that in this case $Isom_\mathcal{X}(x, y) = T \times_{(y', x'), U \times_S U} R$. Hence it suffices to prove that $R \to U \times_S U$ is locally of finite type. This follows from the fact that the composition $s : R \to U \times_S U \to U$ is smooth (hence locally of finite type, see Morphisms of Spaces, Lemmas 67.37.5 and 67.28.5) and Morphisms of Spaces, Lemma 67.23.6. $\square$

04YP Lemma 101.3.2. Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then

(1) $Isom_\mathcal{X}(y, y)$ is a group algebraic space over T, and
(2) $Isom_\mathcal{X}(x, y)$ is a pseudo torsor for $Isom_\mathcal{X}(y, y)$ over T.

Proof. See Groupoids in Spaces, Definitions 78.5.1 and 78.9.1. The lemma follows immediately from the fact that $\mathcal{X}$ is a stack in groupoids. □

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The diagonal of f is the morphism

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_\mathcal{Y} \mathcal{X}$$

Here are two properties that every diagonal morphism has.

04XS Lemma 101.3.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then

(1) Δ_f is representable by algebraic spaces, and
(2) Δ_f is locally of finite type.

Proof. Let T be a scheme and let $a : T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ be a morphism. By definition of the fibre product and the 2-Yoneda lemma the morphism a is given by a triple $a = (x, x', \alpha)$ where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By definition of an algebraic stack the sheaves $\mathit{Isom}_\mathcal{X}(x, x')$ and $\mathit{Isom}_\mathcal{Y}(f(x), f(x'))$ are algebraic spaces over T. In this language α defines a section of the morphism $\mathit{Isom}_\mathcal{Y}(f(x), f(x')) \to T$. A T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}, a} T$ for $T' \to T$ a scheme over T is the same thing as an isomorphism $x|_{T'} \to x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Thus we see that

04XT (101.3.3.1)
$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}, a} T \arrow[r] \arrow[d] & \mathit{Isom}_\mathcal{X}(x, x') \arrow[d] \\ T \arrow[r, "\alpha"] & \mathit{Isom}_\mathcal{Y}(f(x), f(x')) \end{tikzcd}$$

is a fibre square of sheaves over T. In particular we see that $\mathcal{X} \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}, a} T$ is an algebraic space which proves part (1) of the lemma.

To prove the second statement we have to show that the left vertical arrow of Diagram (101.3.3.1) is locally of finite type. By Lemma 101.3.1 the algebraic space $\mathit{Isom}_\mathcal{X}(x, x')$ and is locally of finite type over T. Hence the right vertical arrow of Diagram (101.3.3.1) is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. We conclude by Morphisms of Spaces, Lemma 67.23.3. □

04YQ Lemma 101.3.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then

(1) Δ_f is representable (by schemes),
(2) Δ_f is locally of finite type,
(3) Δ_f is a monomorphism,
(4) Δ_f is separated, and
(5) Δ_f is locally quasi-finite.

Proof. We have already seen in Lemma 101.3.3 that Δ_f is representable by algebraic spaces. Hence the statements (2) – (5) make sense, see Properties of Stacks, Section 100.3. Also Lemma 101.3.3 guarantees (2) holds. Let $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ be a morphism and contemplate Diagram (101.3.3.1). By Algebraic Stacks, Lemma 94.9.2 the right vertical arrow is injective as a map of sheaves, i.e., a monomorphism of algebraic spaces. Hence also the morphism $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X} \to T$ is a monomorphism. Thus (3) holds. We already know that $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X} \to T$ is locally of finite type. Thus Morphisms of Spaces, Lemma 67.27.10 allows us to conclude that $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X} \to T$ is locally quasi-finite and separated. This proves (4) and (5). Finally, Morphisms of Spaces, Proposition 67.50.2 implies that $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X}$ is a scheme which proves (1). □

04YS Lemma 101.3.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent

(1) f is separated,
(2) Δ_f is a closed immersion,
(3) Δ_f is proper, or
(4) Δ_f is universally closed.

Proof. The statements "f is separated", "Δ_f is a closed immersion", "Δ_f is universally closed", and "Δ_f is proper" refer to the notions defined in Properties of Stacks, Section 100.3. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_\mathcal{Y} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.31.14 and Properties of Stacks, Lemma 100.3.3 we see that for any property P (as in that lemma) we have: Δ_f has P if and only if $\Delta_{U/V} : U \to U \times_V U$ has P. Hence the equivalence of (2), (3) and (4) follows from Morphisms of Spaces, Lemma 67.40.9 applied to $U \to V$. Moreover, if (1) holds, then $U \to V$ is separated and we see that $\Delta_{U/V}$ is a closed immersion, i.e., (2) holds. Finally, assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_\mathcal{Y} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is separated. Using Categories, Lemma 4.31.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is a closed immersion, hence $T' \to T$ is separated as desired. □

04YT Lemma 101.3.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent

(1) f is quasi-separated,
(2) Δ_f is quasi-compact, or
(3) Δ_f is of finite type.

Proof. The statements "f is quasi-separated", "Δ_f is quasi-compact", and "Δ_f is of finite type" refer to the notions defined in Properties of Stacks, Section 100.3. Note that (2) and (3) are equivalent in view of the fact that Δ_f is locally of finite type by Lemma 101.3.4 (and Algebraic Stacks, Lemma 94.10.9). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_\mathcal{Y} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.31.14 and Properties of Stacks, Lemma 100.3.3 we see that we have: Δ_f is quasi-compact if and only if $\Delta_{U/V} : U \to U \times_V U$ is quasi-compact. If (1) holds, then $U \to V$ is quasi-separated and we see that $\Delta_{U/V}$ is quasi-compact, i.e., (2) holds. Assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_\mathcal{Y} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is quasi-separated. Using Categories, Lemma 4.31.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is quasi-compact, hence $T' \to T$ is quasi-separated as desired. □

04YU Lemma 101.3.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent

(1) f is locally separated, and
(2) Δ_f is an immersion.

Proof. The statements "f is locally separated", and "Δ_f is an immersion" refer to the notions defined in Properties of Stacks, Section 100.3. Proof omitted. Hint: Argue as in the proofs of Lemmas 101.3.5 and 101.3.6. $\square$

101.4. Separation axioms

04YV Let $\mathcal{X} = [U/R]$ be a presentation of an algebraic stack. Then the properties of the diagonal of $\mathcal{X}$ over S, are the properties of the morphism $j : R \to U \times_S U$. For example, if $\mathcal{X} = [S/G]$ for some smooth group G in algebraic spaces over S then j is the structure morphism $G \to S$. Hence the diagonal is not automatically separated itself (contrary to what happens in the case of schemes and algebraic spaces). To say that $[S/G]$ is quasi-separated over S should certainly imply that $G \to S$ is quasi-compact, but we hesitate to say that $[S/G]$ is quasi-separated over S without also requiring the morphism $G \to S$ to be quasi-separated. In other words, requiring the diagonal morphism to be quasi-compact does not really agree with our intuition for a "quasi-separated algebraic stack", and we should also require the diagonal itself to be quasi-separated.

What about "separated algebraic stacks"? We have seen in Morphisms of Spaces, Lemma 67.40.9 that an algebraic space is separated if and only if the diagonal is proper. This is the condition that is usually used to define separated algebraic stacks too. In the example $[S/G] \to S$ above this means that $G \to S$ is a proper group scheme. This means algebraic stacks of the form $[\mathrm{Spec}(k)/E]$ are proper over k where E is an elliptic curve over k (insert future reference here). In certain situations it may be more natural to assume the diagonal is finite.

04YW Definition 101.4.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is DM if Δ_f is unramified[1].
(2) We say f is quasi-DM if Δ_f is locally quasi-finite[2].
(3) We say f is separated if Δ_f is proper.
(4) We say f is quasi-separated if Δ_f is quasi-compact and quasi-separated.

In this definition we are using that Δ_f is representable by algebraic spaces and we are using Properties of Stacks, Section 100.3 to make sense out of imposing conditions on Δ_f. We note that these definitions do not conflict with the already existing notions if f is representable by algebraic spaces, see Lemmas 101.3.6 and 101.3.5. There is an interesting way to characterize these conditions by looking at higher diagonals, see Lemma 101.6.5.

050D Definition 101.4.2. Let $\mathcal{X}$ be an algebraic stack over the base scheme S. Denote $p : \mathcal{X} \to S$ the structure morphism.

(1) We say $\mathcal{X}$ is DM over S if $p : \mathcal{X} \to S$ is DM.
(2) We say $\mathcal{X}$ is quasi-DM over S if $p : \mathcal{X} \to S$ is quasi-DM.

---

[1]The letters DM stand for Deligne-Mumford. If f is DM then given any scheme T and any morphism $T \to \mathcal{Y}$ the fibre product $\mathcal{X}_T = \mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack over T whose diagonal is unramified, i.e., $\mathcal{X}_T$ is DM. This implies $\mathcal{X}_T$ is a Deligne-Mumford stack, see Theorem 101.21.6. In other words a DM morphism is one whose "fibres" are Deligne-Mumford stacks. This hopefully at least motivates the terminology.

[2]If f is quasi-DM, then the "fibres" $\mathcal{X}_T$ of $\mathcal{X} \to \mathcal{Y}$ are quasi-DM. An algebraic stack $\mathcal{X}$ is quasi-DM exactly if there exists a scheme U and a surjective flat morphism $U \to \mathcal{X}$ of finite presentation which is locally quasi-finite, see Theorem 101.21.3. Note the similarity to being Deligne-Mumford, which is defined in terms of having an étale covering by a scheme.

(3) We say $\mathcal{X}$ is separated over S if $p : \mathcal{X} \to S$ is separated.
(4) We say $\mathcal{X}$ is quasi-separated over S if $p : \mathcal{X} \to S$ is quasi-separated.
(5) We say $\mathcal{X}$ is DM if $\mathcal{X}$ is DM[3] over $\mathrm{Spec}(\mathbf{Z})$.
(6) We say $\mathcal{X}$ is quasi-DM if $\mathcal{X}$ is quasi-DM over $\mathrm{Spec}(\mathbf{Z})$.
(7) We say $\mathcal{X}$ is separated if $\mathcal{X}$ is separated over $\mathrm{Spec}(\mathbf{Z})$.
(8) We say $\mathcal{X}$ is quasi-separated if $\mathcal{X}$ is quasi-separated over $\mathrm{Spec}(\mathbf{Z})$.

In the last 4 definitions we view $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 94.19.2.

Thus in each case we have an absolute notion and a notion relative to our given base scheme (mention of which is usually suppressed by our abuse of notation introduced in Properties of Stacks, Section 100.2). We will see that (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (6) in Lemma 101.4.13. We spend some time proving some standard results on these notions.

050E Lemma 101.4.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) If f is separated, then f is quasi-separated.
(2) If f is DM, then f is quasi-DM.
(3) If f is representable by algebraic spaces, then f is DM.

Proof. To see (1) note that a proper morphism of algebraic spaces is quasi-compact and quasi-separated, see Morphisms of Spaces, Definition 67.40.1. To see (2) note that an unramified morphism of algebraic spaces is locally quasi-finite, see Morphisms of Spaces, Lemma 67.38.7. Finally (3) follows from Lemma 101.3.4. □

050F Lemma 101.4.4. All of the separation axioms listed in Definition 101.4.1 are stable under base change.

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Let $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ be the base change of f by $\mathcal{Y}' \to \mathcal{Y}$. Then $\Delta_{f'}$ is the base change of Δ_f by the morphism $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, see Categories, Lemma 4.31.14. By the results of Properties of Stacks, Section 100.3 each of the properties of the diagonal used in Definition 101.4.1 is stable under base change. Hence the lemma is true. □

06TZ Lemma 101.4.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has one of the separation properties of Definition 101.4.1 then so does f.

Proof. Denote $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ the base change. Then Δ_g is the base change of Δ_f by the morphism $q : W \times_{\mathcal{Y}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Since q is the base change of $W \to \mathcal{Y}$ we see that q is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Hence the result follows from Properties of Stacks, Lemma 100.3.4. □

050G Lemma 101.4.6. Let S be a scheme. The property of being quasi-DM over S, quasi-separated over S, or separated over S (see Definition 101.4.2) is stable under change of base scheme, see Algebraic Stacks, Definition 94.19.3.

Proof. Follows immediately from Lemma 101.4.4. □

[3] Theorem 101.21.6 shows that this is equivalent to $\mathcal{X}$ being a Deligne-Mumford stack.

050H **Lemma 101.4.7.** Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of algebraic stacks. Consider the induced morphism $i : \mathcal{X} \times_\mathcal{Z} \mathcal{Y} \to \mathcal{X} \times_\mathcal{T} \mathcal{Y}$. Then

(1) i is representable by algebraic spaces and locally of finite type,
(2) if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is quasi-separated, then i is quasi-separated,
(3) if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is separated, then i is separated,
(4) if $\mathcal{Z} \to \mathcal{T}$ is DM, then i is unramified,
(5) if $\mathcal{Z} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,
(6) if $\mathcal{Z} \to \mathcal{T}$ is separated, then i is proper, and
(7) if $\mathcal{Z} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.

Proof. The following diagram

$$\begin{tikzcd} \mathcal{X} \times_\mathcal{Z} \mathcal{Y} \arrow[r, "i"'] \arrow[d] & \mathcal{X} \times_\mathcal{T} \mathcal{Y} \arrow[d] \\ \mathcal{Z} \arrow[r, "\Delta_{\mathcal{Z}/\mathcal{T}}"] & \mathcal{Z} \times_\mathcal{T} \mathcal{Z} \end{tikzcd}$$

is a 2-fibre product diagram, see Categories, Lemma 4.31.13. Hence i is the base change of the diagonal morphism $\Delta_{\mathcal{Z}/\mathcal{T}}$. Thus the lemma follows from Lemma 101.3.3, and the material in Properties of Stacks, Section 100.3. □

050I **Lemma 101.4.8.** Let $\mathcal{T}$ be an algebraic stack. Let $g : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{T}$. Consider the graph $i : \mathcal{X} \to \mathcal{X} \times_\mathcal{T} \mathcal{Y}$ of g. Then

(1) i is representable by algebraic spaces and locally of finite type,
(2) if $\mathcal{Y} \to \mathcal{T}$ is DM, then i is unramified,
(3) if $\mathcal{Y} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,
(4) if $\mathcal{Y} \to \mathcal{T}$ is separated, then i is proper, and
(5) if $\mathcal{Y} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.

Proof. This is a special case of Lemma 101.4.7 applied to the morphism $\mathcal{X} = \mathcal{X} \times_\mathcal{Y} \mathcal{Y} \to \mathcal{X} \times_\mathcal{T} \mathcal{Y}$. □

050J **Lemma 101.4.9.** Let $f : \mathcal{X} \to \mathcal{T}$ be a morphism of algebraic stacks. Let $s : \mathcal{T} \to \mathcal{X}$ be a morphism such that $f \circ s$ is 2-isomorphic to $\text{id}_\mathcal{T}$. Then

(1) s is representable by algebraic spaces and locally of finite type,
(2) if f is DM, then s is unramified,
(3) if f is quasi-DM, then s is locally quasi-finite,
(4) if f is separated, then s is proper, and
(5) if f is quasi-separated, then s is quasi-compact and quasi-separated.

Proof. This is a special case of Lemma 101.4.8 applied to $g = s$ and $\mathcal{Y} = \mathcal{T}$ in which case $i : \mathcal{T} \to \mathcal{T} \times_\mathcal{T} \mathcal{X}$ is 2-isomorphic to s. □

050K **Lemma 101.4.10.** All of the separation axioms listed in Definition 101.4.1 are stable under composition of morphisms.

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks to which the axiom in question applies. The diagonal $\Delta_{\mathcal{X}/\mathcal{Z}}$ is the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_\mathcal{Y} \mathcal{X} \longrightarrow \mathcal{X} \times_\mathcal{Z} \mathcal{X}.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 101.4.7 above we see that the second arrow also has this property. Hence the lemma follows since the composition of morphisms which are representable by algebraic spaces with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see our general discussion in Properties of Stacks, Section 100.3 and Morphisms of Spaces, Lemmas 67.38.3, 67.27.3, 67.40.4, 67.8.5, and 67.4.8. $\square$

050L Lemma 101.4.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

(1) If $\mathcal{Y}$ is DM over S and f is DM, then $\mathcal{X}$ is DM over S.
(2) If $\mathcal{Y}$ is quasi-DM over S and f is quasi-DM, then $\mathcal{X}$ is quasi-DM over S.
(3) If $\mathcal{Y}$ is separated over S and f is separated, then $\mathcal{X}$ is separated over S.
(4) If $\mathcal{Y}$ is quasi-separated over S and f is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.
(5) If $\mathcal{Y}$ is DM and f is DM, then $\mathcal{X}$ is DM.
(6) If $\mathcal{Y}$ is quasi-DM and f is quasi-DM, then $\mathcal{X}$ is quasi-DM.
(7) If $\mathcal{Y}$ is separated and f is separated, then $\mathcal{X}$ is separated.
(8) If $\mathcal{Y}$ is quasi-separated and f is quasi-separated, then $\mathcal{X}$ is quasi-separated.

Proof. Parts (1), (2), (3), and (4) follow immediately from Lemma 101.4.10 and Definition 101.4.2. For (5), (6), (7), and (8) think of $\mathcal{X}$ and $\mathcal{Y}$ as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 101.4.10. Details omitted. $\square$

The following lemma is a bit different to the analogue for algebraic spaces. To compare take a look at Morphisms of Spaces, Lemma 67.4.10.

050M Lemma 101.4.12. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks.

(1) If $g \circ f$ is DM then so is f.
(2) If $g \circ f$ is quasi-DM then so is f.
(3) If $g \circ f$ is separated and Δ_g is separated, then f is separated.
(4) If $g \circ f$ is quasi-separated and Δ_g is quasi-separated, then f is quasi-separated.

Proof. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces, see Lemmas 101.3.3 and 101.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T.$$

If $g \circ f$ is DM (resp. quasi-DM), then the composition $A \to T$ is unramified (resp. locally quasi-finite). Hence (1) (resp. (2)) follows on applying Morphisms of Spaces, Lemma 67.38.11 (resp. Morphisms of Spaces, Lemma 67.27.8). This proves (1) and (2).

Proof of (4). Assume $g \circ f$ is quasi-separated and Δ_g is quasi-separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is quasi-separated, see Lemmas 101.3.3 and 101.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is quasi-separated. The composition $A \to T$ is quasi-compact and quasi-separated as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is quasi-separated by Morphisms of Spaces, Lemma 67.4.10. And $A \to B$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.9. Thus f is quasi-separated.

Proof of (3). Assume $g \circ f$ is separated and Δ_g is separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is separated, see Lemmas 101.3.3 and 101.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is separated. The composition $A \to T$ is proper as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is proper by Morphisms of Spaces, Lemma 67.40.6 which means that f is separated. □

050N Lemma 101.4.13. Let $\mathcal{X}$ be an algebraic stack over the base scheme S.

(1) $\mathcal{X}$ is DM $\Leftrightarrow$ $\mathcal{X}$ is DM over S.
(2) $\mathcal{X}$ is quasi-DM $\Leftrightarrow$ $\mathcal{X}$ is quasi-DM over S.
(3) If $\mathcal{X}$ is separated, then $\mathcal{X}$ is separated over S.
(4) If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

(5) If $\mathcal{X}$ is DM over S, then f is DM.
(6) If $\mathcal{X}$ is quasi-DM over S, then f is quasi-DM.
(7) If $\mathcal{X}$ is separated over S and $\Delta_{\mathcal{Y}/S}$ is separated, then f is separated.
(8) If $\mathcal{X}$ is quasi-separated over S and $\Delta_{\mathcal{Y}/S}$ is quasi-separated, then f is quasi-separated.

Proof. Parts (5), (6), (7), and (8) follow immediately from Lemma 101.4.12 and Spaces, Definition 65.13.2. To prove (3) and (4) think of X and Y as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 101.4.12. Similarly, to prove (1) and (2), think of $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ consider the morphisms

$$\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(\mathbf{Z})} \mathcal{X}$$

Both arrows are representable by algebraic spaces. The second arrow is unramified and locally quasi-finite as the base change of the immersion $\Delta_{S/\mathbf{Z}}$. Hence the composition is unramified (resp. locally quasi-finite) if and only if the first arrow is unramified (resp. locally quasi-finite), see Morphisms of Spaces, Lemmas 67.38.3 and 67.38.11 (resp. Morphisms of Spaces, Lemmas 67.27.3 and 67.27.8). □

06MB Lemma 101.4.14. Let $\mathcal{X}$ be an algebraic stack. Let W be an algebraic space, and let $f : W \to \mathcal{X}$ be a surjective, flat, locally finitely presented morphism.

(1) If f is unramified (i.e., étale, i.e., $\mathcal{X}$ is Deligne-Mumford), then $\mathcal{X}$ is DM.
(2) If f is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.

Proof. Note that if f is unramified, then it is étale by Morphisms of Spaces, Lemma 67.39.12. This explains the parenthetical remark in (1). Assume f is unramified (resp. locally quasi-finite). We have to show that $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (resp. locally quasi-finite). Note that $W \times W \to \mathcal{X} \times \mathcal{X}$ is also surjective, flat, and locally of finite presentation. Hence it suffices to show that

$$W \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X} = W \times_{\mathcal{X}} W \longrightarrow W \times W$$

is unramified (resp. locally quasi-finite), see Properties of Stacks, Lemma 100.3.3. By assumption the morphism $\text{pr}_i : W \times_{\mathcal{X}} W \to W$ is unramified (resp. locally quasi-finite). Hence the displayed arrow is unramified (resp. locally quasi-finite) by Morphisms of Spaces, Lemma 67.38.11 (resp. Morphisms of Spaces, Lemma 67.27.8). □

06MY Lemma 101.4.15. A monomorphism of algebraic stacks is separated and DM. The same is true for immersions of algebraic stacks.

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of algebraic stacks, then Δ_f is an isomorphism, see Properties of Stacks, Lemma 100.8.4. Since an isomorphism of algebraic spaces is proper and unramified we see that f is separated and DM. The second assertion follows from the first as an immersion is a monomorphism, see Properties of Stacks, Lemma 100.9.5. □

06MZ Lemma 101.4.16. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists. If $\mathcal{X}$ is DM, resp. quasi-DM, resp. separated, resp. quasi-separated, then so is $\mathcal{Z}_x$.

Proof. This is true because $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism hence DM and separated by Lemma 101.4.15. Apply Lemma 101.4.11 to conclude. □

101.5. Inertia stacks

050P The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 8.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.34.

050Q Lemma 101.5.1. Let $\mathcal{X}$ be an algebraic stack. Then the inertia stack $\mathcal{I}_{\mathcal{X}}$ is an algebraic stack as well. The morphism

$$\mathcal{I}_{\mathcal{X}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type. More generally, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then the relative inertia $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an algebraic stack and the morphism

$$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type.

Proof. By Categories, Lemma 4.34.1 there are equivalences

$$\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \quad \text{and} \quad \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, \Delta} \mathcal{X}$$

which shows that the inertia stacks are algebraic stacks. Let $T \to \mathcal{X}$ be a morphism given by the object x of the fibre category of $\mathcal{X}$ over T. Then we get a 2-fibre product square

$$\begin{tikzcd} Isom_{\mathcal{X}}(x, x) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

This follows immediately from the definition of $\mathcal{I}_\mathcal{X}$. Since $\mathit{Isom}_\mathcal{X}(x, x)$ is always an algebraic space locally of finite type over T (see Lemma 101.3.1) we conclude that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is representable by algebraic spaces and locally of finite type. Finally, for the relative inertia we get

$$\begin{tikzcd} \mathit{Isom}_\mathcal{X}(x, x) \arrow[d] & K \arrow[l] \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] \\ \mathit{Isom}_\mathcal{Y}(f(x), f(x)) & T \arrow[l, "e"'] \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

with both squares 2-fibre products. This follows from Categories, Lemma 4.34.3. The left vertical arrow is a morphism of algebraic spaces locally of finite type over T, and hence is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. Thus K is an algebraic space and $K \to T$ is locally of finite type. This proves the assertion on the relative inertia. $\square$

050R Remark 101.5.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In Properties of Stacks, Remark 100.3.7 we have seen that the 2-category of morphisms $\mathcal{Z} \to \mathcal{X}$ representable by algebraic spaces with target $\mathcal{X}$ forms a category. In this category the inertia stack of $\mathcal{X}/\mathcal{Y}$ is a group object. Recall that an object of $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is just a pair (x, α) where x is an object of $\mathcal{X}$ and α is an automorphism of x in the fibre category of $\mathcal{X}$ that x lives in with $f(\alpha) = \mathrm{id}$. The composition

$$c : \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \times_\mathcal{X} \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

is given by the rule on objects

$$((x, \alpha), (x', \alpha'), \beta) \mapsto (x, \alpha \circ \beta^{-1} \circ \alpha' \circ \beta)$$

which makes sense as $\beta : x \to x'$ is an isomorphism in the fibre category by our definition of fibre products. The neutral element $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is given by the functor $x \mapsto (x, \mathrm{id}_x)$. We omit the proof that the axioms of a group object hold.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ be its inertia stack. Let T be a scheme and let x be an object of $\mathcal{X}$ over T. Set $y = f(x)$. We have seen in the proof of Lemma 101.5.1 that for any scheme T and object x of $\mathcal{X}$ over T there is an exact sequence of sheaves of groups

0CPJ (101.5.2.1)
$$0 \to \mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x) \to \mathit{Isom}_\mathcal{X}(x, x) \to \mathit{Isom}_\mathcal{Y}(y, y)$$

The group structure on the second and third term is the one defined in Lemma 101.3.2 and the sequence gives a meaning to the first term. Also, there is a canonical cartesian square

$$\begin{tikzcd} \mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

In fact, the group structure on $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ discussed in Remark 101.5.2 induces the group structure on $\mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x)$. This allows us to define the sheaf $\mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x)$ also for morphisms from algebraic spaces to $\mathcal{X}$. We formalize this in the following definition.

06PP Definition 101.5.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let Z be an algebraic space.

(1) Let $x : Z \to \mathcal{X}$ be a morphism. We set

$$\mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x) = Z \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

We endow it with the structure of a group algebraic space over Z by pulling back the composition law discussed in Remark 101.5.2. We will sometimes refer to $\mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x)$ as the relative sheaf of automorphisms of x.

(2) Let $x_1, x_2 : Z \to \mathcal{X}$ be morphisms. Set $y_i = f \circ x_i$. Let $\alpha : y_1 \to y_2$ be a 2-morphism. Then α determines a morphism $\Delta^\alpha : Z \to Z \times_{y_1,\mathcal{Y},y_2} Z$ and we set

$$\mathit{Isom}^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) = (Z \times_{x_1,\mathcal{X},x_2} Z) \times_{Z \times_{y_1,\mathcal{Y},y_2} Z, \Delta^\alpha} Z.$$

We will sometimes refer to $\mathit{Isom}^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ as the relative sheaf of isomorphisms from x_1 to x_2.

If $\mathcal{Y} = \mathrm{Spec}(\mathbf{Z})$ or more generally when $\mathcal{Y}$ is an algebraic space, then we use the notation $\mathit{Isom}_{\mathcal{X}}(x, x)$ and $\mathit{Isom}_{\mathcal{X}}(x_1, x_2)$ and we use the terminology sheaf of automorphisms of x and sheaf of isomorphisms from x_1 to x_2.

0CPK Lemma 101.5.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let Z be an algebraic space and let $x_i : Z \to \mathcal{X}$, $i = 1, 2$ be morphisms. Then

(1) $\mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ is a group algebraic space over Z,
(2) there is an exact sequence of groups

$$0 \to \mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x_2, x_2) \to \mathit{Isom}_{\mathcal{X}}(x_2, x_2) \to \mathit{Isom}_{\mathcal{Y}}(f \circ x_2, f \circ x_2)$$

(3) there is a map of algebraic spaces $\mathit{Isom}_{\mathcal{X}}(x_1, x_2) \to \mathit{Isom}_{\mathcal{Y}}(f \circ x_1, f \circ x_2)$ such that for any 2-morphism $\alpha : f \circ x_1 \to f \circ x_2$ we obtain a cartesian diagram

$$\begin{tikzcd} \mathit{Isom}^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) \arrow[r] \arrow[d] & Z \arrow[d, "\alpha"] \\ \mathit{Isom}_{\mathcal{X}}(x_1, x_2) \arrow[r] & \mathit{Isom}_{\mathcal{Y}}(f \circ x_1, f \circ x_2) \end{tikzcd}$$

(4) for any 2-morphism $\alpha : f \circ x_1 \to f \circ x_2$ the algebraic space $\mathit{Isom}^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ is a pseudo torsor for $\mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ over Z.

Proof. Part (1) follows from Definition 101.5.3. Part (2) comes from the exact sequence (101.5.2.1) étale locally on Z. Part (3) can be seen by unwinding the definitions. Locally on Z in the étale topology part (4) reduces to part (2) of Lemma 101.3.2. □

06PQ Lemma 101.5.5. Let $\pi : \mathcal{X} \to \mathcal{Y}$ and $f : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Set $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$. Then both squares in the diagram

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'/\mathcal{Y}'} \arrow[r] \arrow[d, "\text{Categories, Equation (4.34.2.3)}"'] & \mathcal{X}' \arrow[r, "\pi'"] \arrow[d] & \mathcal{Y}' \arrow[d, "f"] \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathcal{X} \arrow[r, "\pi"] & \mathcal{Y} \end{tikzcd}$$

are fibre product squares.

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}'/\mathcal{Y}'}$ is defined as the category of pairs (x', α') where x' is an object of $\mathcal{X}'$ and α' is an automorphism of x' with $\pi'(\alpha') = \text{id}$, see Categories, Section 4.34. Suppose that x' lies over the scheme U and maps to the object x of $\mathcal{X}$. By the construction of the 2-fibre product in Categories, Lemma 4.32.3 we see that $x' = (U, x, y', \beta)$ where y' is an object of $\mathcal{Y}'$ over U and β is an isomorphism $\beta : \pi(x) \to f(y')$ in the fibre category of $\mathcal{Y}$ over U. By the very construction of the 2-fibre product the automorphism α' is a pair (α, γ) where α is an automorphism of x over U and γ is an automorphism of y' over U such that α and γ are compatible via β. The condition $\pi'(\alpha') = \text{id}$ signifies that $\gamma = \text{id}$ whereupon the condition that α, β, γ are compatible is exactly the condition $\pi(\alpha) = \text{id}$, i.e., means exactly that (x, α) is an object of $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. In this way we see that the left square is a fibre product square (some details omitted). $\square$

06R5 Lemma 101.5.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Then the diagram

$$\begin{tikzcd} \mathcal{I}_\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{I}_\mathcal{Y} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is a fibre product square.

Proof. This follows immediately from the fact that f is fully faithful (see Properties of Stacks, Lemma 100.8.4) and the definition of the inertia in Categories, Section 4.34. Namely, an object of $\mathcal{I}_\mathcal{X}$ over a scheme T is the same thing as a pair (x, α) consisting of an object x of $\mathcal{X}$ over T and a morphism $\alpha : x \to x$ in the fibre category of $\mathcal{X}$ over T. As f is fully faithful we see that α is the same thing as a morphism $\beta : f(x) \to f(x)$ in the fibre category of $\mathcal{Y}$ over T. Hence we can think of objects of $\mathcal{I}_\mathcal{X}$ over T as triples $((y, \beta), x, \gamma)$ where y is an object of $\mathcal{Y}$ over T, $\beta : y \to y$ in $\mathcal{Y}_T$ and $\gamma : y \to f(x)$ is an isomorphism over T, i.e., an object of $\mathcal{I}_\mathcal{Y} \times_\mathcal{Y} \mathcal{X}$ over T. $\square$

06PR Lemma 101.5.7. Let $\mathcal{X}$ be an algebraic stack. Let $[U/R] \to \mathcal{X}$ be a presentation. Let G/U be the stabilizer group algebraic space associated to the groupoid (U, R, s, t, c). Then

$$\begin{tikzcd} G \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{I}_\mathcal{X} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is a fibre product diagram.

Proof. Immediate from Groupoids in Spaces, Lemma 78.26.2. $\square$

101.6. Higher diagonals

04YX Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In this situation it makes sense to consider not only the diagonal

$$\Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$$

but also the diagonal of the diagonal, i.e., the morphism

$$\Delta_{\Delta_f} : \mathcal{X} \longrightarrow \mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{X})} \mathcal{X}$$

Because of this we sometimes use the following terminology. We denote $\Delta_{f,0} = f$ the zeroth diagonal, we denote $\Delta_{f,1} = \Delta_f$ the first diagonal, and we denote $\Delta_{f,2} = \Delta_{\Delta_f}$ the second diagonal. Note that $\Delta_{f,1}$ is representable by algebraic spaces and locally of finite type, see Lemma 101.3.3. Hence $\Delta_{f,2}$ is representable, a monomorphism, locally of finite type, separated, and locally quasi-finite, see Lemma 101.3.4.

We can describe the second diagonal using the relative inertia stack. Namely, the fibre product $\mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{X})} \mathcal{X}$ is equivalent to the relative inertia stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by Categories, Lemma 4.34.1. Moreover, via this identification the second diagonal becomes the neutral section

$$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

of the relative inertia stack. By analogy with what happens for groupoids in algebraic spaces (Groupoids in Spaces, Lemma 78.29.2) we have the following equivalences.

0CL0 Lemma 101.6.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) The following are equivalent
  (a) $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is separated,
  (b) $\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is separated, and
  (c) $\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a closed immersion.
(2) The following are equivalent
  (a) $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is quasi-separated,
  (b) $\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is quasi-separated, and
  (c) $\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a quasi-compact.
(3) The following are equivalent
  (a) $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is locally separated,
  (b) $\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is locally separated, and
  (c) $\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an immersion.
(4) The following are equivalent
  (a) $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is unramified,
  (b) f is DM.
(5) The following are equivalent
  (a) $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is locally quasi-finite,
  (b) f is quasi-DM.

Proof. Proof of (1), (2), and (3). Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X}$. Then $G = U \times_\mathcal{X} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an algebraic space over U (Lemma 101.5.1). In fact, G is a group algebraic space over U by the group law on relative inertia constructed in Remark 101.5.2. Moreover, $G \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is surjective and smooth as a base change of $U \to \mathcal{X}$. Finally, the base change of $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by $G \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is the identity $U \to G$ of G/U. Thus the equivalence of (a) and (c) follows from Groupoids in Spaces, Lemma 78.6.1. Since $\Delta_{f,2}$ is the diagonal of Δ_f we have (b) $\Leftrightarrow$ (c) by definition.

Proof of (4) and (5). Recall that (4)(b) means Δ_f is unramified and (5)(b) means that Δ_f is locally quasi-finite. Choose a scheme Z and a morphism $a : Z \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$. Then $a = (x_1, x_2, \alpha)$ where $x_i : Z \to \mathcal{X}$ and $\alpha : f \circ x_1 \to f \circ x_2$ is a 2-morphism.

Recall that

$$
\begin{tikzcd}
Isom^{\alpha}_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) \arrow[r] \arrow[d] & Z \arrow[d] \\
\mathcal{X} \arrow[r, "\Delta_f"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2) \arrow[r] \arrow[d] & Z \arrow[d, "x_2"] \\
\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathcal{X}
\end{tikzcd}
$$

are cartesian squares. By Lemma 101.5.4 the algebraic space $Isom^{\alpha}_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ is a pseudo torsor for $Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ over Z. Thus the equivalences in (4) and (5) follow from Groupoids in Spaces, Lemma 78.9.5. ☐

04YY Lemma 101.6.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:

(1) the morphism f is representable by algebraic spaces,
(2) the second diagonal of f is an isomorphism,
(3) the group stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is trivial over $\mathcal{X}$, and
(4) for a scheme T and a morphism $x : T \to \mathcal{X}$ the kernel of $Isom_{\mathcal{X}}(x, x) \to Isom_{\mathcal{Y}}(f(x), f(x))$ is trivial.

Proof. We first prove the equivalence of (1) and (2). Namely, f is representable by algebraic spaces if and only if f is faithful, see Algebraic Stacks, Lemma 94.15.2. On the other hand, f is faithful if and only if for every object x of $\mathcal{X}$ over a scheme T the functor f induces an injection $Isom_{\mathcal{X}}(x, x) \to Isom_{\mathcal{Y}}(f(x), f(x))$, which happens if and only if the kernel K is trivial, which happens if and only if $e : T \to K$ is an isomorphism for every $x : T \to \mathcal{X}$. Since $K = T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ as discussed above, this proves the equivalence of (1) and (2). To prove the equivalence of (2) and (3), by the discussion above, it suffices to note that a group stack is trivial if and only if its identity section is an isomorphism. Finally, the equivalence of (3) and (4) follows from the definitions: in the proof of Lemma 101.5.1 we have seen that the kernel in (4) corresponds to the fibre product $T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ over T. ☐

This lemma leads to the following hierarchy for morphisms of algebraic stacks.

0AHJ Lemma 101.6.3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is

(1) a monomorphism if and only if $\Delta_{f,1}$ is an isomorphism, and
(2) representable by algebraic spaces if and only if $\Delta_{f,1}$ is a monomorphism.

Moreover, the second diagonal $\Delta_{f,2}$ is always a monomorphism.

Proof. Recall from Properties of Stacks, Lemma 100.8.4 that a morphism of algebraic stacks is a monomorphism if and only if its diagonal is an isomorphism of stacks. Thus Lemma 101.6.2 can be rephrased as saying that a morphism is representable by algebraic spaces if the diagonal is a monomorphism. In particular, it shows that condition (3) of Lemma 101.3.4 is actually an if and only if, i.e., a morphism of algebraic stacks is representable by algebraic spaces if and only if its diagonal is a monomorphism. ☐

04YZ Lemma 101.6.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then

(1) $\Delta_{f,1}$ separated $\Leftrightarrow$ $\Delta_{f,2}$ closed immersion $\Leftrightarrow$ $\Delta_{f,2}$ proper $\Leftrightarrow$ $\Delta_{f,2}$ universally closed,
(2) $\Delta_{f,1}$ quasi-separated $\Leftrightarrow$ $\Delta_{f,2}$ finite type $\Leftrightarrow$ $\Delta_{f,2}$ quasi-compact, and
(3) $\Delta_{f,1}$ locally separated $\Leftrightarrow$ $\Delta_{f,2}$ immersion.

Proof. Follows from Lemmas 101.3.5, 101.3.6, and 101.3.7 applied to $\Delta_{f,1}$. □

The following lemma is kind of cute and it may suggest a generalization of these conditions to higher algebraic stacks.

04Z0 Lemma 101.6.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then

(1) f is separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are universally closed, and
(2) f is quasi-separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are quasi-compact.
(3) f is quasi-DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are locally quasi-finite.
(4) f is DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are unramified.

Proof. Proof of (1). Assume that $\Delta_{f,2}$ and $\Delta_{f,1}$ are universally closed. Then $\Delta_{f,1}$ is separated and universally closed by Lemma 101.6.4. By Morphisms of Spaces, Lemma 67.9.7 and Algebraic Stacks, Lemma 94.10.9 we see that $\Delta_{f,1}$ is quasi-compact. Hence it is quasi-compact, separated, universally closed and locally of finite type (by Lemma 101.3.3) so proper. This proves "$\Leftarrow$" of (1). The proof of the implication in the other direction is omitted.

Proof of (2). This follows immediately from Lemma 101.6.4.

Proof of (3). This follows from the fact that $\Delta_{f,2}$ is always locally quasi-finite by Lemma 101.3.4 applied to $\Delta_f = \Delta_{f,1}$.

Proof of (4). This follows from the fact that $\Delta_{f,2}$ is always unramified as Lemma 101.3.4 applied to $\Delta_f = \Delta_{f,1}$ shows that $\Delta_{f,2}$ is locally of finite type and a monomorphism. See More on Morphisms of Spaces, Lemma 76.14.8. □

0CPL Lemma 101.6.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated (resp. quasi-separated, resp. quasi-DM, resp. DM) morphism of algebraic stacks. Then

(1) given algebraic spaces T_i, $i = 1, 2$ and morphisms $x_i : T_i \to \mathcal{X}$, with $y_i = f \circ x_i$ the morphism
$$T_1 \times_{x_1,\mathcal{X},x_2} T_2 \longrightarrow T_1 \times_{y_1,\mathcal{Y},y_2} T_2$$
is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified),
(2) given an algebraic space T and morphisms $x_i : T \to \mathcal{X}$, $i = 1, 2$, with $y_i = f \circ x_i$ the morphism
$$Isom_{\mathcal{X}}(x_1, x_2) \longrightarrow Isom_{\mathcal{Y}}(y_1, y_2)$$
is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified).

Proof. Proof of (1). Observe that the diagram
$$\begin{tikzcd} T_1 \times_{x_1,\mathcal{X},x_2} T_2 \arrow[r] \arrow[d] & T_1 \times_{y_1,\mathcal{Y},y_2} T_2 \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$
is cartesian. Hence this follows from the fact that f is separated (resp. quasi-separated, resp. quasi-DM, resp. DM) if and only if the diagonal is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified).

Proof of (2). This is true because

$$Isom_{\mathcal{X}}(x_1, x_2) = (T \times_{x_1, \mathcal{X}, x_2} T) \times_{T \times T, \Delta_T} T$$

hence the morphism in (2) is a base change of the morphism in (1). □

101.7. Quasi-compact morphisms

050S Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be quasi-compact. Here is another characterization.

050T Lemma 101.7.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:

(1) f is quasi-compact (as in Properties of Stacks, Section 100.3), and
(2) for every quasi-compact algebraic stack $\mathcal{Z}$ and any morphism $\mathcal{Z} \to \mathcal{Y}$ the algebraic stack $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. By Properties of Stacks, Lemma 100.6.2 there exists a quasi-compact scheme U and a surjective smooth morphism $U \to \mathcal{Z}$. Since f is representable by algebraic spaces and quasi-compact we see by definition that $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and that $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact algebraic space. The morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective (as the base change of the smooth and surjective morphism $U \to \mathcal{Z}$). Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact by another application of Properties of Stacks, Lemma 100.6.2

Assume (2). Let $Z \to \mathcal{Y}$ be a morphism, where Z is a scheme. We have to show that the morphism of algebraic spaces $p : Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_{\mathcal{Y}} \mathcal{Z}$ and the algebraic space $U \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Morphisms of Spaces, Lemma 67.8.8. □

This motivates the following definition.

050U Definition 101.7.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is quasi-compact if for every quasi-compact algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ the fibre product $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

By Lemma 101.7.1 above this agrees with the already existing notion for morphisms of algebraic stacks representable by algebraic spaces. In particular this notion agrees with the notions already defined for morphisms between algebraic stacks and schemes.

050V Lemma 101.7.3. The base change of a quasi-compact morphism of algebraic stacks by any morphism of algebraic stacks is quasi-compact.

Proof. Omitted. □

050W Lemma 101.7.4. The composition of a pair of quasi-compact morphisms of algebraic stacks is quasi-compact.

Proof. Omitted. □

0CL1 Lemma 101.7.5. A closed immersion of algebraic stacks is quasi-compact.

Proof. This follows from the fact that immersions are always representable and the corresponding fact for closed immersion of algebraic spaces. ☐

050X Lemma 101.7.6. Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & \mathcal{Z} & \end{tikzcd}$$

be a 2-commutative diagram of morphisms of algebraic stacks. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $\mathcal{T}$ be a quasi-compact algebraic stack, and let $\mathcal{T} \to \mathcal{Z}$ be a morphism. By Properties of Stacks, Lemma 100.5.3 the morphism $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is surjective and by assumption $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X}$ is quasi-compact. Hence $\mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is quasi-compact by Properties of Stacks, Lemma 100.6.2. ☐

050Y Lemma 101.7.7. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.

Proof. This is true because f equals the composition $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first map is quasi-compact by Lemma 101.4.9 because it is a section of the quasi-separated morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ (a base change of g, see Lemma 101.4.4). The second map is quasi-compact as it is the base change of f, see Lemma 101.7.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 101.7.4. ☐

075S Lemma 101.7.8. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) If $\mathcal{X}$ is quasi-compact and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact.
(2) If $\mathcal{X}$ is quasi-compact and quasi-separated and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact and quasi-separated.
(3) A fibre product of quasi-compact and quasi-separated algebraic stacks is quasi-compact and quasi-separated.

Proof. Part (1) follows from Lemma 101.7.7. Part (2) follows from (1) and Lemma 101.4.12. For (3) let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be morphisms of quasi-compact and quasi-separated algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is quasi-compact and quasi-separated as a base change of $\mathcal{X} \to \mathcal{Y}$ using (2) and Lemmas 101.7.3 and 101.4.4. Hence $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact and quasi-separated as an algebraic stack quasi-compact and quasi-separated over $\mathcal{Z}$, see Lemmas 101.4.11 and 101.7.4. ☐

0CL2 Lemma 101.7.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Let $y \in |\mathcal{Y}|$ be a point in the closure of the image of $|f|$. There exists a valuation ring A with fraction field K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y.

Proof. Choose an affine scheme V and a point $v \in V$ and a smooth morphism $V \to \mathcal{Y}$ sending v to y. Consider the base change diagram

$$\begin{tikzcd} V \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d, "g"] & \mathcal{X} \arrow[d, "f"] \\ V \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Recall that $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V| \times_{|\mathcal{Y}|} |\mathcal{X}|$ is surjective (Properties of Stacks, Lemma 100.4.3). Because $|V| \to |\mathcal{Y}|$ is open (Properties of Stacks, Lemma 100.4.7) we conclude that v is in the closure of the image of $|g|$. Thus it suffices to prove the lemma for the quasi-compact morphism g (Lemma 101.7.3) which we do in the next paragraph.

Assume $\mathcal{Y} = Y$ is an affine scheme. Then $\mathcal{X}$ is quasi-compact as f is quasi-compact (Definition 101.7.2). Choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Then the image of $|f|$ is the image of $W \to Y$. By Morphisms, Lemma 29.6.5 we can choose a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y. Composing with $W \to \mathcal{X}$ we obtain a solution. $\square$

0DTL Lemma 101.7.10. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is quasi-compact, then f is quasi-compact.

Proof. Assume $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is quasi-compact. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism with $\mathcal{Z}$ a quasi-compact algebraic stack. Choose a scheme U and a surjective smooth morphism $U \to W \times_{\mathcal{Y}} \mathcal{Z}$. Since $U \to \mathcal{Z}$ is flat, surjective, and locally of finite presentation and $\mathcal{Z}$ is quasi-compact, we can find a quasi-compact open subscheme $U' \subset U$ such that $U' \to \mathcal{Z}$ is surjective. Then $U' \times_{\mathcal{Y}} \mathcal{X} = U' \times_W (W \times_{\mathcal{Y}} \mathcal{X})$ is quasi-compact by assumption and surjects onto $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact as desired. $\square$

101.8. Noetherian algebraic stacks

050Z We have already defined locally Noetherian algebraic stacks in Properties of Stacks, Section 100.7.

0510 Definition 101.8.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is Noetherian if $\mathcal{X}$ is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic stack $\mathcal{X}$ is not just quasi-compact and locally Noetherian, but also quasi-separated. In the language of Section 101.6 if we denote $p : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ the "absolute" structure morphism (i.e., the structure morphism of $\mathcal{X}$ viewed as an algebraic stack over $\mathbf{Z}$), then

$$\mathcal{X} \text{ Noetherian} \Leftrightarrow \mathcal{X} \text{ locally Noetherian and } \Delta_{p,0}, \Delta_{p,1}, \Delta_{p,2} \text{ quasi-compact.}$$

This will later mean that an algebraic stack of finite type over a Noetherian algebraic stack is not automatically Noetherian.

0CPM Lemma 101.8.2. Let $j : \mathcal{X} \to \mathcal{Y}$ be an immersion of algebraic stacks.

(1) If $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian and j is quasi-compact.
(2) If $\mathcal{Y}$ is Noetherian, then $\mathcal{X}$ is Noetherian.

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Then $U = \mathcal{X} \times_{\mathcal{Y}} V$ is a scheme and $V \to U$ is an immersion, see Properties of Stacks, Definition 100.9.1. Recall that $\mathcal{Y}$ is locally Noetherian if and only if V is locally Noetherian. In this case U is locally Noetherian too (Morphisms, Lemmas 29.15.5 and 29.15.6) and $U \to V$ is quasi-compact (Properties, Lemma 28.5.3). This shows that j is quasi-compact (Lemma 101.7.10) and that $\mathcal{X}$ is locally Noetherian. Finally, if $\mathcal{Y}$ is Noetherian, then we see from the above that $\mathcal{X}$ is quasi-compact and locally Noetherian. To finish the proof observe that j is separated and hence $\mathcal{X}$ is quasi-separated because $\mathcal{Y}$ is so by Lemma 101.4.11. $\square$

0DQI Lemma 101.8.3. Let $\mathcal{X}$ be an algebraic stack.

(1) If $\mathcal{X}$ is locally Noetherian then $|\mathcal{X}|$ is a locally Noetherian topological space.
(2) If $\mathcal{X}$ is quasi-compact and locally Noetherian, then $|\mathcal{X}|$ is a Noetherian topological space.

Proof. Assume $\mathcal{X}$ is locally Noetherian. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. As $\mathcal{X}$ is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 28.5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |\mathcal{X}|$ is open and surjective we conclude that $|\mathcal{X}|$ is locally Noetherian by Topology, Lemma 5.9.3. This proves (1). If $\mathcal{X}$ is quasi-compact and locally Noetherian, then $|\mathcal{X}|$ is quasi-compact and locally Noetherian. Hence $|\mathcal{X}|$ is Noetherian by Topology, Lemma 5.12.14. $\square$

0GVX Lemma 101.8.4. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then $|\mathcal{X}|$ is quasi-sober (Topology, Definition 5.8.6).

Proof. We have to prove that every irreducible closed subset $T \subset |\mathcal{X}|$ has a generic point. Choose an affine scheme U and a smooth morphism $f : U \to \mathcal{X}$ such that $f^{-1}(T) \subset |U|$ is nonempty. Since U is Noetherian, the closed subset $f^{-1}(T)$ has finitely many irreducible components (Topology, Lemma 5.9.2). Say $f^{-1}(T) = Z_1 \cup \ldots \cup Z_n$ is the decomposition into irreducible components. As f is open, the image of $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ contains a nonempty open subset of T. Since T is irreducible, this means that $f(f^{-1}(T))$ is dense. Since T is irreducible, it follows that $f(Z_i)$ is dense for some i. Then if $\xi_i \in Z_i$ is the generic point we see that $f(\xi_i)$ is a generic point of T. $\square$

101.9. Affine morphisms

0CHP Affine morphisms of algebraic stacks are defined as follows.

0CHQ Definition 101.9.1. A morphism of algebraic stacks is said to be affine if it is representable and affine in the sense of Properties of Stacks, Section 100.3.

For us it is a little bit more convenient to think of an affine morphism of algebraic stacks as a morphism of algebraic stacks which is representable by algebraic spaces and affine in the sense of Properties of Stacks, Section 100.3. (Recall that the default for "representable" in the Stacks project is representable by schemes.) Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

0CHR Lemma 101.9.2. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is an affine morphism of algebraic stacks.

Proof. This follows from the discussion in Properties of Stacks, Section 100.3. □

0CHS Lemma 101.9.3. Compositions of affine morphisms of algebraic stacks are affine.

Proof. This follows from the discussion in Properties of Stacks, Section 100.3 and Morphisms of Spaces, Lemma 67.20.4. □

0GQE Lemma 101.9.4. Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "a"'] & & \mathcal{Y} \arrow[dl, "b"] \\ & \mathcal{Z} & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic stacks. If a is affine and Δ_b is affine, then f is affine.

Proof. The base change $\text{pr}_2 : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ of a is affine by Lemma 101.9.2. The morphism $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is the base change of $\Delta_b : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ by the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ (see material in Categories, Section 4.31). Hence it is affine by Lemma 101.9.2. The composition $f = \text{pr}_2 \circ (1, f)$ of affine morphisms is affine by Lemma 101.9.3 and the proof is done. □

101.10. Integral and finite morphisms

0CHT Integral and finite morphisms of algebraic stacks are defined as follows.

0CHU Definition 101.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is integral if f is representable and integral in the sense of Properties of Stacks, Section 100.3.
(2) We say f is finite if f is representable and finite in the sense of Properties of Stacks, Section 100.3.

For us it is a little bit more convenient to think of an integral, resp. finite morphism of algebraic stacks as a morphism of algebraic stacks which is representable by algebraic spaces and integral, resp. finite in the sense of Properties of Stacks, Section 100.3. (Recall that the default for "representable" in the Stacks project is representable by schemes.) Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

0CHV Lemma 101.10.2. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be an integral (or finite) morphism of algebraic stacks. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is an integral (or finite) morphism of algebraic stacks.

Proof. This follows from the discussion in Properties of Stacks, Section 100.3. $\square$

0CHW Lemma 101.10.3. Compositions of integral, resp. finite morphisms of algebraic stacks are integral, resp. finite.

Proof. This follows from the discussion in Properties of Stacks, Section 100.3 and Morphisms of Spaces, Lemma 67.45.4. $\square$

101.11. Open morphisms

06U0 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally open. Here is another characterization.

06U1 Lemma 101.11.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent

(1) f is universally open (as in Properties of Stacks, Section 100.3), and
(2) for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is open.

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally open, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is open. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_\mathcal{Y} \mathcal{X}|} \arrow[r] \arrow[d] & {|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|} \arrow[d] \\ {|V|} \arrow[r] & {|\mathcal{Z}|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

06U2 Definition 101.11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is open if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is open.
(2) We say f is universally open if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces

$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$

is open, i.e., the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is open.

06U3 Lemma 101.11.3. The base change of a universally open morphism of algebraic stacks by any morphism of algebraic stacks is universally open.

Proof. This is immediate from the definition. $\square$

06U4 Lemma 101.11.4. The composition of a pair of (universally) open morphisms of algebraic stacks is (universally) open.

Proof. Omitted. $\square$

101.12. Submersive morphisms

06U5 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally submersive. Here is another characterization.

0CHX **Lemma 101.12.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent

(1) f is universally submersive (as in Properties of Stacks, Section 100.3), and
(2) for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is submersive.

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally submersive, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is submersive. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{tikzcd} |V \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \arrow[d] \\ |V| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is submersive it follows that the right vertical arrow is submersive. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

06U6 **Definition 101.12.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *submersive*[4] if the continuous map $|\mathcal{X}| \to |\mathcal{Y}|$ is submersive, see Topology, Definition 5.6.3.
(2) We say f is *universally submersive* if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \to \mathcal{Y}'$ is submersive.

We note that a submersive morphism is in particular surjective.

0CHY **Lemma 101.12.3.** The base change of a universally submersive morphism of algebraic stacks by any morphism of algebraic stacks is universally submersive.

Proof. This is immediate from the definition. $\square$

0CHZ **Lemma 101.12.4.** The composition of a pair of (universally) submersive morphisms of algebraic stacks is (universally) submersive.

Proof. Omitted. $\square$

[4] This is very different from the notion of a submersion of differential manifolds.

101.13. Universally closed morphisms

0511 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally closed. Here is another characterization.

0512 **Lemma 101.13.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent

1. f is universally closed (as in Properties of Stacks, Section 100.3), and
2. for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is closed.

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally closed, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is closed. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{tikzcd} |V \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \arrow[d] \\ |V| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

0513 **Definition 101.13.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

1. We say f is *closed* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is closed.
2. We say f is *universally closed* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$
is closed, i.e., the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is closed.

0514 **Lemma 101.13.3.** The base change of a universally closed morphism of algebraic stacks by any morphism of algebraic stacks is universally closed.

Proof. This is immediate from the definition. $\square$

0515 **Lemma 101.13.4.** The composition of a pair of (universally) closed morphisms of algebraic stacks is (universally) closed.

Proof. Omitted. $\square$

0CL3 **Lemma 101.13.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

1. f is universally closed,
2. for every scheme Z and every morphism $Z \to \mathcal{Y}$ the projection $|Z \times_\mathcal{Y} \mathcal{X}| \to |Z|$ is closed,

(3) for every affine scheme Z and every morphism $Z \to \mathcal{Y}$ the projection $|Z \times_\mathcal{Y} \mathcal{X}| \to |Z|$ is closed, and
(4) there exists an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$ such that $V \times_\mathcal{Y} \mathcal{X} \to V$ is a universally closed morphism of algebraic stacks.

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective smooth morphism $V \to \mathcal{Y}$. We are going to show that $V \times_\mathcal{Y} \mathcal{X} \to V$ is a universally closed morphism of algebraic stacks. Let $\mathcal{Z} \to V$ be a morphism from an algebraic stack to V. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where $W = \coprod W_i$ is a disjoint union of affine schemes. Then we have the following commutative diagram

$$\begin{tikzcd}
\coprod_i |W_i \times_\mathcal{Y} \mathcal{X}| \arrow[r, equal] \arrow[d] & |W \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \arrow[r, equal] \arrow[d] & |\mathcal{Z} \times_V (V \times_\mathcal{Y} \mathcal{X})| \arrow[dl] \\
\coprod |W_i| \arrow[r, equal] & |W| \arrow[r] & |\mathcal{Z}| &
\end{tikzcd}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Stacks, Lemma 100.4.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the diagram

$$\begin{tikzcd}
|(V \times_\mathcal{Y} \mathcal{Z}) \times_V (V \times_\mathcal{Y} \mathcal{X})| \arrow[r, equal] \arrow[dr] & |V \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |Z \times_\mathcal{Y} \mathcal{X}| \arrow[d] \\
& |V \times_\mathcal{Y} \mathcal{Z}| \arrow[r] & |\mathcal{Z}|
\end{tikzcd}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic stacks are surjective and smooth (see reference above). It follows that the right vertical arrow is closed. □

101.14. Universally injective morphisms

0CI0 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally injective. Here is another characterization.

0CI1 **Lemma 101.14.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent

(1) f is universally injective (as in Properties of Stacks, Section 100.3), and
(2) for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is injective.

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally injective, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is injective. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_{\mathcal{Y}} \mathcal{X}|} \arrow[r] \arrow[d] & {|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|} \arrow[d] \\ {|V|} \arrow[r] & {|\mathcal{Z}|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is injective it follows that the right vertical arrow is injective. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

0CI2 Definition 101.14.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is universally injective if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is injective.

0CI3 Lemma 101.14.3. The base change of a universally injective morphism of algebraic stacks by any morphism of algebraic stacks is universally injective.

Proof. This is immediate from the definition. □

0CI4 Lemma 101.14.4. The composition of a pair of universally injective morphisms of algebraic stacks is universally injective.

Proof. Omitted. □

0CPN Lemma 101.14.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) f is universally injective,
(2) $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, and
(3) for an algebraically closed field, for $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$, and for a 2-arrow $\beta : f \circ x_1 \to f \circ x_2$ there is a 2-arrow $\alpha : x_1 \to x_2$ with $\beta = \mathrm{id}_f \star \alpha$.

Proof. (1) $\Rightarrow$ (2). If f is universally injective, then the first projection $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{X}|$ is injective, which implies that $|\Delta|$ is surjective.

(2) $\Rightarrow$ (1). Assume Δ is surjective. Then any base change of Δ is surjective (see Properties of Stacks, Section 100.5). Since the diagonal of a base change of f is a base change of Δ, we see that it suffices to show that $|\mathcal{X}| \to |\mathcal{Y}|$ is injective. If not, then by Properties of Stacks, Lemma 100.4.3 we find that the first projection $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{X}|$ is not injective. Of course this means that $|\Delta|$ is not surjective.

(3) $\Rightarrow$ (2). Let $t \in |\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}|$. Then we can represent t by a morphism $t : \mathrm{Spec}(k) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ with k an algebraically closed field. By our construction of 2-fibre products we can represent t by (x_1, x_2, β) where $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$ and $\beta : f \circ x_1 \to f \circ x_2$ is a 2-morphism. Then (3) implies that there is a 2-morphism $\alpha : x_1 \to x_2$ mapping to β. This exactly means that $\Delta(x_1) = (x_1, x_1, \mathrm{id})$ is isomorphic to t. Hence (2) holds.

(2) $\Rightarrow$ (3). Let $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$ be morphisms with k an algebraically closed field. Let $\beta : f \circ x_1 \to f \circ x_2$ be a 2-morphism. As in the previous paragraph, we obtain a morphism $t = (x_1, x_2, \beta) : \mathrm{Spec}(k) \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$. By Lemma 101.3.3

$$T = \mathcal{X} \times_{\Delta, \mathcal{X} \times_\mathcal{Y} \mathcal{X}, t} \mathrm{Spec}(k)$$

is an algebraic space locally of finite type over $\mathrm{Spec}(k)$. Condition (2) implies that T is nonempty. Then since k is algebraically closed, there is a k-point in T. Unwinding the definitions this means there is a morphism $\alpha : x_1 \to x_2$ in $\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})$ such that $\beta = \mathrm{id}_f \star \alpha$. $\square$

0DTM Lemma 101.14.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a universally injective morphism of algebraic stacks. Let $y : \mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism where k is an algebraically closed field. If y is in the image of $|\mathcal{X}| \to |\mathcal{Y}|$, then there is a morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ with $y = f \circ x$.

Proof. We first remark this lemma is not a triviality, because the assumption that y is in the image of $|f|$ means only that we can lift y to a morphism into $\mathcal{X}$ after possibly replacing k by an extension field. To prove the lemma we may base change f by y, hence we may assume we have a nonempty algebraic stack $\mathcal{X}$ and a universally injective morphism $\mathcal{X} \to \mathrm{Spec}(k)$ and we want to find a k-valued point of $\mathcal{X}$. We may replace $\mathcal{X}$ by its reduction. We may choose a field k' and a surjective, flat, locally finite type morphism $\mathrm{Spec}(k') \to \mathcal{X}$, see Properties of Stacks, Lemma 100.11.2. Since $\mathcal{X} \to \mathrm{Spec}(k)$ is universally injective, we find that

$$\mathrm{Spec}(k') \times_\mathcal{X} \mathrm{Spec}(k') \to \mathrm{Spec}(k' \otimes_k k')$$

is surjective as the base change of the surjective morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$ (Lemma 101.14.5). Since k is algebraically closed $k' \otimes_k k'$ is a domain (Algebra, Lemma 10.49.4). Let $\xi \in \mathrm{Spec}(k') \times_\mathcal{X} \mathrm{Spec}(k')$ be a point mapping to the generic point of $\mathrm{Spec}(k' \otimes_k k')$. Let U be the reduced induced closed subscheme structure on the connected component of $\mathrm{Spec}(k') \times_\mathcal{X} \mathrm{Spec}(k')$ containing ξ. Then the two projections $U \to \mathrm{Spec}(k')$ are locally of finite type, as this was true for the projections $\mathrm{Spec}(k') \times_\mathcal{X} \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ as base changes of the morphism $\mathrm{Spec}(k') \to \mathcal{X}$. Applying Varieties, Proposition 33.31.1 we find that the integral closures of the two images of k' in $\Gamma(U, \mathcal{O}_U)$ are equal. Looking in $\kappa(\xi)$ means that any element of the form $\lambda \otimes 1$ is algebraically dependend on the subfield

$$1 \otimes k' \subset (\text{fraction field of } k' \otimes_k k') \subset \kappa(\xi).$$

Since k is algebraically closed, this is only possible if $k' = k$ and the proof is complete. $\square$

0DTN Lemma 101.14.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:

(1) f is universally injective,
(2) for every affine scheme Z and any morphism $Z \to \mathcal{Y}$ the morphism $Z \times_\mathcal{Y} \mathcal{X} \to Z$ is universally injective, and
(3) add more here.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2) holds. We will show that $\Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is surjective, which implies (1) by Lemma 101.14.5. Consider an affine scheme V and a smooth morphism $V \to \mathcal{Y}$. Since $g : V \times_\mathcal{Y} \mathcal{X} \to V$ is universally injective by (2), we see that Δ_g is surjective. However, Δ_g is the base change of Δ_f by the smooth morphism $V \to \mathcal{Y}$. Since the collection of these morphisms $V \to \mathcal{Y}$ are jointly surjective, we conclude Δ_f is surjective. $\square$

0DTP Lemma 101.14.8. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_\mathcal{Y} \mathcal{X} \to W$ is universally injective, then f is universally injective.

Proof. Observe that the diagonal Δ_g of the morphism $g : W \times_\mathcal{Y} \mathcal{X} \to W$ is the base change of Δ_f by $W \to \mathcal{Y}$. Hence if Δ_g is surjective, then so is Δ_f by Properties of Stacks, Lemma 100.3.3. Thus the lemma follows from the characterization (2) in Lemma 101.14.5. $\square$

101.15. Universal homeomorphisms

0CI5 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be a universal homeomorphism. Here is another characterization.

0CI6 Lemma 101.15.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent

1. (1) f is a universal homeomorphism (Properties of Stacks, Section 100.3), and
2. (2) for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is a homeomorphism.

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is a universal homeomorphism, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is a homeomorphism. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_\mathcal{Y} \mathcal{X}|} \arrow[r] \arrow[d] & {|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|} \arrow[d] \\ {|V|} \arrow[r] & {|\mathcal{Z}|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is a homeomorphism it follows that the right vertical arrow is a homeomorphism. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

0CI7 Definition 101.15.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a universal homeomorphism if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map of topological spaces

$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$

is a homeomorphism.

0CI8 Lemma 101.15.3. The base change of a universal homeomorphism of algebraic stacks by any morphism of algebraic stacks is a universal homeomorphism.

Proof. This is immediate from the definition. $\square$

0CI9 Lemma 101.15.4. The composition of a pair of universal homeomorphisms of algebraic stacks is a universal homeomorphism.

Proof. Omitted. □

0DTQ Lemma 101.15.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_\mathcal{Y} \mathcal{X} \to W$ is a universal homeomorphism, then f is a universal homeomorphism.

Proof. Assume $g : W \times_\mathcal{Y} \mathcal{X} \to W$ is a universal homeomorphism. Then g is universally injective, hence f is universally injective by Lemma 101.14.8. On the other hand, let $\mathcal{Z} \to \mathcal{Y}$ be a morphism with $\mathcal{Z}$ an algebraic stack. Choose a scheme U and a surjective smooth morphism $U \to W \times_\mathcal{Y} \mathcal{Z}$. Consider the diagram

$$\begin{tikzcd} W \times_\mathcal{Y} \mathcal{X} \arrow[d, "g"] & U \times_\mathcal{Y} \mathcal{X} \arrow[l] \arrow[r] \arrow[d] & \mathcal{Z} \times_\mathcal{Y} \mathcal{X} \arrow[d] \\ W & U \arrow[l] \arrow[r] & \mathcal{Z} \end{tikzcd}$$

The middle vertical arrow induces a homeomorphism on topological space by assumption on g. The morphism $U \to \mathcal{Z}$ and $U \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z} \times_\mathcal{Y} \mathcal{X}$ are surjective, flat, and locally of finite presentation hence induce open maps on topological spaces. We conclude that $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is open. Surjectivity is easy to prove; we omit the proof. □

101.16. Types of morphisms smooth local on source-and-target

06FL Given a property of morphisms of algebraic spaces which is smooth local on the source-and-target, see Descent on Spaces, Definition 74.20.1 we may use it to define a corresponding property of morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

06FM Lemma 101.16.1. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider commutative diagrams

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces and the vertical arrows are smooth. The following are equivalent

(1) for any diagram as above such that in addition $U \to \mathcal{X} \times_\mathcal{Y} V$ is smooth the morphism h has property $\mathcal{P}$, and
(2) for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 100.3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. Pick an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "h'"'] \arrow[d] & V' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{tikzcd} U \arrow[dd, "h"] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d] \arrow[dl] \arrow[dd, bend left, "{(h,h')}"] & U' \arrow[dd, "h'"] \\ & U \times_{\mathcal{Y}} V' \arrow[d] & \\ V & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that

$$\begin{tikzcd} U \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \times_{\mathcal{Y}} V' \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V' \end{tikzcd}$$

is cartesian, hence the left vertical arrow is smooth as U', V', a', b', h' is as in (1). Since $\mathcal{P}$ is smooth local on the target by Descent on Spaces, Lemma 74.20.2 part (2) we see that the base change $U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ has $\mathcal{P}$. Since $\mathcal{P}$ is smooth local on the source by Descent on Spaces, Lemma 74.20.2 part (1) we can precompose by the smooth morphism $U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ and conclude (h, h') has $\mathcal{P}$. Since $V \times_{\mathcal{Y}} V' \to V'$ is smooth we conclude $U \times_{\mathcal{X}} U' \to V'$ has $\mathcal{P}$ by Descent on Spaces, Lemma 74.20.2 part (3). Finally, since $U \times_X U' \to U'$ is surjective and smooth and $\mathcal{P}$ is smooth local on the source (same lemma) we conclude that h' has $\mathcal{P}$. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 74.20.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property

$\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} Z \times_{\mathcal{Y}} U \arrow[r] \arrow[d] & Z \times_{\mathcal{Y}} V \arrow[d] \\ Z \times_{\mathcal{Y}} \mathcal{X} \arrow[r] & Z \end{tikzcd}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is smooth and surjective and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 74.20.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. □

06FN Definition 101.16.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks has property $\mathcal{P}$ if the equivalent conditions of Lemma 101.16.1 hold.

06FP Remark 101.16.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under composition. Then the property of morphisms of algebraic stacks defined in Definition 101.16.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks having property $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Finally, choose an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

06FQ Remark 101.16.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under base change. Then the property of morphisms of algebraic stacks defined in Definition 101.16.2 is stable under base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks and assume f has property $\mathcal{P}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{tikzcd} V' \times_V U \arrow[r] \arrow[d] & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

we see that the left top horizontal arrow is smooth and surjective, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

06PS Remark 101.16.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are smooth local on the source-and-target. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 101.16.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

101.17. Morphisms of finite type

06FR The property "locally of finite type" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.23.3 and Descent on Spaces, Lemma 74.11.9. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

06FS **Definition 101.17.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f locally of finite type if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} =$ locally of finite type.
(2) We say f is of finite type if it is locally of finite type and quasi-compact.

06FT **Lemma 101.17.2.** *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.23.2. □

06FU **Lemma 101.17.3.** *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.23.3. □

06FV **Lemma 101.17.4.** *An immersion is locally of finite type.*

Proof. Combine Remark 101.16.5 with Morphisms of Spaces, Lemma 67.23.7. □

06R6 **Lemma 101.17.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian.*

Proof. Let

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a commutative diagram where U, V are schemes, $V \to \mathcal{Y}$ is surjective and smooth, and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and smooth. Then $U \to V$ is locally of finite type. If $\mathcal{Y}$ is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 29.15.6 we see that U is locally Noetherian, which means that $\mathcal{X}$ is locally Noetherian. □

The following two lemmas will be improved on later (after we have discussed morphisms of algebraic stacks which are locally of finite presentation).

06U7 **Lemma 101.17.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is locally of finite type, then f is locally of finite type.*

Proof. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_\mathcal{Y} U$, $V' = W \times_\mathcal{Y} V$, $\mathcal{X}' = W \times_\mathcal{Y} \mathcal{X}$, and $\mathcal{Y}' = W \times_\mathcal{Y} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finite type morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite type, we see that $U' \to V'$ is locally of finite type. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite type by Descent on Spaces, Lemma 74.11.9 and we win. □

06U8 Lemma 101.17.7. Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume $\mathcal{X} \to \mathcal{Z}$ is locally of finite type and that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite type.

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_\mathcal{Z} \mathcal{Y}$. Set $U = V \times_\mathcal{Y} \mathcal{X}$ which is an algebraic space. We know that $U \to V$ is surjective, flat, and locally of finite presentation and that $U \to W$ is locally of finite type. Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 74.16.2. □

06U9 Lemma 101.17.8. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f : \mathcal{X} \to \mathcal{Z}$ is locally of finite type, then $f : \mathcal{X} \to \mathcal{Y}$ is locally of finite type.

Proof. We can find a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

where U, V, W are schemes, the vertical arrow $W \to \mathcal{Z}$ is surjective and smooth, the arrow $V \to \mathcal{Y} \times_\mathcal{Z} W$ is surjective and smooth, and the arrow $U \to \mathcal{X} \times_\mathcal{Y} V$ is surjective and smooth. Then also $U \to \mathcal{X} \times_\mathcal{Z} V$ is surjective and smooth (as a composition of a surjective and smooth morphism with a base change of such). By definition we see that $U \to W$ is locally of finite type. Hence $U \to V$ is locally of finite type by Morphisms, Lemma 29.15.8 which in turn means (by definition) that $\mathcal{X} \to \mathcal{Y}$ is locally of finite type. □

101.18. Points of finite type

06FW Let $\mathcal{X}$ be an algebraic stack. A finite type point $x \in |\mathcal{X}|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

06FX Lemma 101.18.1. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:

(1) There exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type and represents x.

(2) There exists a scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ such that $\varphi(u) = x$.

Proof. Let $u \in U$ and $U \to \mathcal{X}$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to \mathcal{X}$ is representable and locally of finite type (by Morphisms of Spaces, Lemmas 67.39.8 and 67.28.5). Hence we see (1) holds by Lemma 101.17.2.

Conversely, assume $\mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type and represents x. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. By assumption $U \times_{\mathcal{X}} \mathrm{Spec}(k) \to U$ is a morphism of algebraic spaces which is locally of finite type. Pick a finite type point v of $U \times_{\mathcal{X}} \mathrm{Spec}(k)$ (there exists at least one, see Morphisms of Spaces, Lemma 67.25.3). By Morphisms of Spaces, Lemma 67.25.4 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 29.16.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

06FY Definition 101.18.2. Let $\mathcal{X}$ be an algebraic stack. We say a point $x \in |\mathcal{X}|$ is a finite type point[5] if the equivalent conditions of Lemma 101.18.1 are satisfied. We denote $\mathcal{X}_{\text{ft-pts}}$ the set of finite type points of $\mathcal{X}$.

We can describe the set of finite type points as follows.

06FZ Lemma 101.18.3. Let $\mathcal{X}$ be an algebraic stack. We have

$$\mathcal{X}_{\text{ft-pts}} = \bigcup\nolimits_{\varphi : U \to \mathcal{X} \text{ smooth}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes smooth over $\mathcal{X}$ or over all affine schemes smooth over $\mathcal{X}$.

Proof. Immediate from Lemma 101.18.1. □

06G0 Lemma 101.18.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type, then $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$.

Proof. Take $x \in \mathcal{X}_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$. Then $f \circ x$ is locally of finite type by Lemma 101.17.2. Hence $f(x) \in \mathcal{Y}_{\text{ft-pts}}$. □

06G1 Lemma 101.18.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and surjective, then $f(\mathcal{X}_{\text{ft-pts}}) = \mathcal{Y}_{\text{ft-pts}}$.

Proof. We have $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$ by Lemma 101.18.4. Let $y \in |\mathcal{Y}|$ be a finite type point. Represent y by a morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ which is locally of finite type. As f is surjective the algebraic stack $\mathcal{X}_k = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$ is nonempty, therefore has a finite type point $x \in |\mathcal{X}_k|$ by Lemma 101.18.3. Now $\mathcal{X}_k \to \mathcal{X}$ is a morphism which is locally of finite type as a base change of $\mathrm{Spec}(k) \to \mathcal{Y}$ (Lemma 101.17.3). Hence the image of x in $\mathcal{X}$ is a finite type point by Lemma 101.18.4 which maps to y by construction. □

06G2 Lemma 101.18.6. Let $\mathcal{X}$ be an algebraic stack. For any locally closed subset $T \subset |\mathcal{X}|$ we have

$$T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{\text{ft-pts}} \neq \emptyset.$$

In particular, for any closed subset $T \subset |\mathcal{X}|$ we see that $T \cap \mathcal{X}_{\text{ft-pts}}$ is dense in T.

[5]This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

Proof. Let $i : \mathcal{Z} \to \mathcal{X}$ be the reduced induced substack structure on T, see Properties of Stacks, Remark 100.10.5. An immersion is locally of finite type, see Lemma 101.17.4. Hence by Lemma 101.18.4 we see $\mathcal{Z}_{\text{ft-pts}} \subset \mathcal{X}_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with a smooth morphism towards $\mathcal{Z}$ has at least one closed point, hence $\mathcal{Z}$ has at least one finite type point by Lemma 101.18.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic stack. It tells us in particular that the residual gerbe of $\mathcal{X}$ at x exists whenever x is a finite type point!

06G3 Lemma 101.18.7. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:

(1) x is a finite type point,
(2) there exists an algebraic stack $\mathcal{Z}$ whose underlying topological space $|\mathcal{Z}|$ is a singleton, and a morphism $f : \mathcal{Z} \to \mathcal{X}$ which is locally of finite type such that $\{x\} = |f|(|\mathcal{Z}|)$, and
(3) the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type.

Proof. (All of the morphisms occurring in this paragraph are representable by algebraic spaces, hence the conventions and results of Properties of Stacks, Section 100.3 are applicable.) Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ with $\varphi(u) = x$, see Lemma 101.18.3. Set $u = \text{Spec}(\kappa(u))$ as usual. Set $R = u \times_\mathcal{X} u$ so that we obtain a groupoid in algebraic spaces (u, R, s, t, c), see Algebraic Stacks, Lemma 94.16.1. The projection morphisms $R \to u$ are the compositions

$$R = u \times_\mathcal{X} u \to u \times_\mathcal{X} U \to u \times_\mathcal{X} X = u$$

where the first arrow is of finite type (a base change of the closed immersion of schemes $u \to U$) and the second arrow is smooth (a base change of the smooth morphism $U \to \mathcal{X}$). Hence $s, t : R \to u$ are locally of finite type (as compositions, see Morphisms of Spaces, Lemma 67.23.2). Since u is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 67.28.7). We see that $\mathcal{Z} = [u/R]$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. By Algebraic Stacks, Lemma 94.16.1 we obtain a canonical morphism

$$f : \mathcal{Z} \longrightarrow \mathcal{X}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.15.2 and a monomorphism, see Properties of Stacks, Lemma 100.8.4. It follows that the residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ of $\mathcal{X}$ at x exists and that f factors through an equivalence $\mathcal{Z} \to \mathcal{Z}_x$, see Properties of Stacks, Lemma 100.11.12. By construction the diagram

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

is commutative. By Criteria for Representability, Lemma 97.17.1 the left vertical arrow is surjective, flat, and locally of finite presentation. Consider

$$\begin{tikzcd} u \times_{\mathcal{X}} U \arrow[r] \arrow[d] & \mathcal{Z} \times_{\mathcal{X}} U \arrow[r] \arrow[d] & U \arrow[d] \\ u \arrow[r] & \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

As $u \to \mathcal{X}$ is locally of finite type, we see that the base change $u \times_{\mathcal{X}} U \to U$ is locally of finite type. Moreover, $u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U$ is surjective, flat, and locally of finite presentation as a base change of $u \to \mathcal{Z}$. Thus $\{u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U\}$ is an fppf covering of algebraic spaces, and we conclude that $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is locally of finite type by Descent on Spaces, Lemma 74.16.1. By definition this means that f is locally of finite type (because the vertical arrow $\mathcal{Z} \times_{\mathcal{X}} U \to \mathcal{Z}$ is smooth as a base change of $U \to \mathcal{X}$ and surjective as $\mathcal{Z}$ has only one point). Since $\mathcal{Z} = \mathcal{Z}_x$ we see that (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of $\mathcal{X}$ by Lemma 101.18.4 and Lemma 101.18.6 to see that $\mathcal{Z}_{\text{ft-pts}}$ is nonempty, i.e., the unique point of $\mathcal{Z}$ is a finite type point of $\mathcal{Z}$. □

101.19. Automorphism groups

0DTR Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ correspond to $x : \mathrm{Spec}(k) \to \mathcal{X}$. In this situation we often use the phrase "let G_x/k be the automorphism group algebraic space of x". This just means that

$$G_x = \mathit{Isom}_{\mathcal{X}}(x, x) = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$$

is the group algebraic space of automorphism of x. This is a group algebraic space over $\mathrm{Spec}(k)$. If k'/k is an extension of fields then the automorphism group algebraic space of the induced morphism $x' : \mathrm{Spec}(k') \to \mathcal{X}$ is the base change of G_x to $\mathrm{Spec}(k')$.

0DTS **Lemma 101.19.1.** In the situation above G_x is a scheme if one of the following holds

1. (1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-separated
2. (2) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is locally separated,
3. (3) $\mathcal{X}$ is quasi-DM,
4. (4) $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is quasi-separated,
5. (5) $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is locally separated, or
6. (6) $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is locally quasi-finite.

Proof. Observe that (1) ⇒ (4), (2) ⇒ (5), and (3) ⇒ (6) by Lemma 101.6.1. In case (4) we see that G_x is a quasi-separated algebraic space and in case (5) we see that G_x is a locally separated algebraic space. In both cases G_x is a decent algebraic space (Decent Spaces, Section 68.6 and Lemma 68.15.2). Then G_x is separated by More on Groupoids in Spaces, Lemma 79.9.4 whereupon we conclude that G_x is a scheme by More on Groupoids in Spaces, Proposition 79.10.3. In case (6) we see that $G_x \to \mathrm{Spec}(k)$ is locally quasi-finite and hence G_x is a scheme by Spaces over Fields, Lemma 72.10.8. □

0DTT Lemma 101.19.2. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. Let P be a property of algebraic spaces over fields which is invariant under ground field extensions; for example $P(X/k) = X \to \mathrm{Spec}(k)$ is finite. The following are equivalent

(1) for some morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x the automorphism group algebraic space G_x/k has P, and
(2) for any morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x the automorphism group algebraic space G_x/k has P.

Proof. Omitted. $\square$

0DTU Remark 101.19.3. Let P be a property of algebraic spaces over fields which is invariant under ground field extensions. Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, we say the automorphism group of $\mathcal{X}$ at x has P if the equivalent conditions of Lemma 101.19.2 are satisfied. For example, we say the automorphism group of $\mathcal{X}$ at x is finite, if $G_x \to \mathrm{Spec}(k)$ is finite whenever $x : \mathrm{Spec}(k) \to \mathcal{X}$ is a representative of x. Similarly for smooth, proper, etc. (There is clearly an abuse of language going on here, but we believe it will not cause confusion or imprecision.)

0DTV Lemma 101.19.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent

(1) for some morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x setting $y = f \circ x$ the map $G_x \to G_y$ of automorphism group algebraic spaces is an isomorphism, and
(2) for any morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x setting $y = f \circ x$ the map $G_x \to G_y$ of automorphism group algebraic spaces is an isomorphism.

Proof. This comes down to the fact that being an isomorphism is fpqc local on the target, see Descent on Spaces, Lemma 74.11.15. Namely, suppose that k'/k is an extension of fields and denote $x' : \mathrm{Spec}(k') \to \mathcal{X}$ the composition and set $y' = f \circ x'$. Then the morphism $G_{x'} \to G_{y'}$ is the base change of $G_x \to G_y$ by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Hence $G_x \to G_y$ is an isomorphism if and only if $G_{x'} \to G_{y'}$ is an isomorphism. Thus we see that the property propagates through the equivalence class if it holds for one. $\square$

0DTW Remark 101.19.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. To indicate the equivalent conditions of Lemma 101.19.4 are satisfied for f and x in the literature the terminology f is stabilizer preserving at x or f is fixed-point reflecting at x is used. We prefer to say f induces an isomorphism between automorphism groups at x and $f(x)$.

101.20. Presentations and properties of algebraic stacks

0DTX Let (U, R, s, t, c) be a groupoid in algebraic spaces. If $s, t : R \to U$ are flat and locally of finite presentation, then the quotient stack $[U/R]$ is an algebraic stack, see Criteria for Representability, Theorem 97.17.2. In this section we study what properties of (U, R, s, t, c) imply for the algebraic stack $[U/R]$.

0DTY Lemma 101.20.1. Let (U, R, s, t, c) be a groupoid in algebraic spaces such that $s, t : R \to U$ are flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$ (see above).

(1) If $R \to U \times U$ is separated, then $\Delta_{\mathcal{X}}$ is separated.

(2) If U, R are separated, then $\Delta_\mathcal{X}$ is separated.
(3) If $R \to U \times U$ is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.
(4) If $s, t : R \to U$ are locally quasi-finite, then $\mathcal{X}$ is quasi-DM.
(5) If $R \to U \times U$ is proper, then $\mathcal{X}$ is separated.
(6) If $s, t : R \to U$ are proper and U is separated, then $\mathcal{X}$ is separated.
(7) Add more here.

Proof. Observe that the morphism $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 97.17.1. Hence the same is true for $U \times U \to \mathcal{X} \times \mathcal{X}$. We have the cartesian diagram

$$\begin{tikzcd} R = U \times_\mathcal{X} U \arrow[r] \arrow[d] & U \times U \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

(see Groupoids in Spaces, Lemma 78.22.2). Thus we see that $\Delta_\mathcal{X}$ has one of the properties listed in Properties of Stacks, Section 100.3 if and only if the morphism $R \to U \times U$ does, see Properties of Stacks, Lemma 100.3.3. This explains why (1), (3), and (5) are true. The condition in (2) implies $R \to U \times U$ is separated hence (2) follows from (1). The condition in (4) implies the condition in (3) hence (4) follows from (3). The condition in (6) implies the condition in (5) by Morphisms of Spaces, Lemma 67.40.6 hence (6) follows from (5). ☐

0DTZ Lemma 101.20.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces such that $s, t : R \to U$ are flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$ (see above). Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.

Proof. The induced morphism $p : U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Criteria for Representability, Lemma 97.17.1. Hence $|U| \to |\mathcal{X}|$ is surjective by Properties of Stacks, Lemma 100.4.4. Note that $R = U \times_\mathcal{X} U$, see Groupoids in Spaces, Lemma 78.22.2. Hence Properties of Stacks, Lemma 100.4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. ☐

101.21. Special presentations of algebraic stacks

06MC In this section we prove two important theorems. The first is the characterization of quasi-DM stacks $\mathcal{X}$ as the stacks of the form $\mathcal{X} = [U/R]$ with $s, t : R \to U$ locally quasi-finite (as well as flat and locally of finite presentation). The second is the statement that DM algebraic stacks are Deligne-Mumford.

The following lemma gives a criterion for when a “slice” of a presentation is still flat over the algebraic stack.

06MD Lemma 101.21.1. Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow \mathcal{X}$$

is flat and locally of finite presentation.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Choose an extension of fields k'/k and a morphism $w : \mathrm{Spec}(k') \to W$ such that $\mathrm{Spec}(k') \to W \to \mathcal{X}$ is 2-isomorphic to $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to \mathcal{X}$. This is possible as $W \to \mathcal{X}$ is surjective. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[d] & U \times_{\mathcal{X}} W \arrow[l, "\mathrm{pr}_0"'] \arrow[d] & F' \arrow[l, "p'"'] \arrow[d] \\ \mathcal{X} & W \arrow[l] & \mathrm{Spec}(k') \arrow[l] \end{tikzcd}$$

both of whose squares are cartesian. By our choice of w we see that $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Thus $F' \to F$ is surjective and we can choose a point $z' \in |F'|$ mapping to z. Since $F' \to F$ is flat we see that $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 67.30.8. Hence $f_1, \ldots, f_r$ map to a regular sequence in $\mathcal{O}_{F',\overline{z}'}$, see Algebra, Lemma 10.68.5. Note that $U \times_{\mathcal{X}} W \to W$ is a morphism of algebraic spaces which is flat and locally of finite presentation. Hence by More on Morphisms of Spaces, Lemma 76.28.1 we see that there exists an open subspace U' of $U \times_{\mathcal{X}} W$ containing $p(z')$ such that the intersection $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W)$ is flat and locally of finite presentation over W. Note that $\mathrm{pr}_0(U')$ is an open subspace of U containing $p(z)$ as pr_0 is smooth hence open. Now we see that $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to \mathcal{X}$ is flat and locally of finite presentation as the composition

$$U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to W \to \mathcal{X}.$$

Hence Properties of Stacks, Lemma 100.3.5 implies $\mathrm{pr}_0(U') \cap V(f_1, \ldots, f_r) \to \mathcal{X}$ is flat and locally of finite presentation as desired. $\square$

06ME Lemma 101.21.2. Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is locally of finite type. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is locally quasi-finite.

Proof. Since $f : U \to \mathcal{X}$ is locally of finite type there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is locally quasi-finite, see Properties of Stacks, Remark 100.9.20 (2). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is locally quasi-finite at z. This follows immediately from Morphisms of Spaces, Lemma 67.34.6. $\square$

A quasi-DM stack has a locally quasi-finite "covering" by a scheme.

06MF Theorem 101.21.3. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent

(1) $\mathcal{X}$ is quasi-DM, and
(2) there exists a scheme W and a surjective, flat, locally finitely presented, locally quasi-finite morphism $W \to \mathcal{X}$.

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 101.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

Let U be an affine scheme, $U \to \mathcal{X}$ a smooth morphism, and $u \in U$ a closed point which maps to x, see Lemma 101.18.1. Denote $u = \mathrm{Spec}(\kappa(u))$ as usual. Consider the following commutative diagram

$$\begin{tikzcd} u \arrow[d] & R \arrow[l] \arrow[d] \\ U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & u \arrow[l] \end{tikzcd}$$

with both squares fibre product squares, in particular $R = u \times_{\mathcal{X}} u$. In the proof of Lemma 101.18.7 we have seen that (u, R, s, t, c) is a groupoid in algebraic spaces with s, t locally of finite type. Let $G \to u$ be the stabilizer group algebraic space (see Groupoids in Spaces, Definition 78.16.2). Note that

$$G = R \times_{(u \times u)} u = (u \times_{\mathcal{X}} u) \times_{(u \times u)} u = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} u.$$

As $\mathcal{X}$ is quasi-DM we see that G is locally quasi-finite over u. By More on Groupoids in Spaces, Lemma 79.9.11 we have $\dim(R) = 0$.

Let $e : u \to R$ be the identity of the groupoid. Thus both compositions $u \to R \to u$ are equal to the identity morphism of u. Note that $R \subset F$ is a closed subspace as $u \subset U$ is a closed subscheme. Hence we can also think of e as a point of F. Consider the maps of étale local rings

$$\mathcal{O}_{U,u} \xrightarrow{p^\sharp} \mathcal{O}_{F,\overline{e}} \longrightarrow \mathcal{O}_{R,\overline{e}}$$

Note that $\mathcal{O}_{R,\overline{e}}$ has dimension 0 by the result of the first paragraph. On the other hand, the kernel of the second arrow is $p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}$ as R is cut out in F by $\mathfrak{m}_u$.

Thus we see that

$$\mathfrak{m}_{\overline{z}} = \sqrt{p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}}$$

On the other hand, as the morphism $U \to \mathcal{X}$ is smooth we see that $F \to u$ is a smooth morphism of algebraic spaces. This means that F is a regular algebraic space (Spaces over Fields, Lemma 72.16.1). Hence $\mathcal{O}_{F,\overline{e}}$ is a regular local ring (Properties of Spaces, Lemma 66.25.1). Note that a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.106.3). Let $d = \dim(\mathcal{O}_{F,\overline{e}})$. By Algebra, Lemma 10.104.10 we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ whose images $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{F,\overline{z}}$. By Lemma 101.21.1 after shrinking U we may assume that $Z = V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation. Note that by construction $F_Z = Z \times_\mathcal{X} u$ is a closed subspace of $F = U \times_\mathcal{X} u$, that e is a point of this closed subspace, and that

$$\dim(\mathcal{O}_{F_Z,\overline{e}}) = 0.$$

By Morphisms of Spaces, Lemma 67.34.1 it follows that $\dim_e(F_Z) = 0$ because the transcendence degree of e relative to u is zero. Hence it follows from Lemma 101.21.2 that after possibly shrinking U the morphism $Z \to \mathcal{X}$ is locally quasi-finite.

We conclude that for every finite type point x of $\mathcal{X}$ there exists a locally quasi-finite, flat, locally finitely presented morphism $f_x : Z_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x Z_x$ and $f = \coprod f_x$. Then f is flat, locally of finite presentation, and locally quasi-finite. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 100.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 101.18.6 (and Properties of Stacks, Lemma 100.4.4). □

06N0 Lemma 101.21.4. Let $\mathcal{Z}$ be a DM, locally Noetherian, reduced algebraic stack with $|\mathcal{Z}|$ a singleton. Then there exists a field k and a surjective étale morphism $\mathrm{Spec}(k) \to \mathcal{Z}$.

Proof. By Properties of Stacks, Lemma 100.11.3 there exists a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Set $U = \mathrm{Spec}(k)$ and $R = U \times_\mathcal{Z} U$ so we obtain a groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 94.9.2. Note that by Algebraic Stacks, Remark 94.16.3 we have an equivalence

$$f_{can} : [U/R] \longrightarrow \mathcal{Z}$$

The projections $s, t : R \to U$ are locally of finite presentation. As $\mathcal{Z}$ is DM we see that the stabilizer group algebraic space

$$G = U \times_{U \times U} R = U \times_{U \times U} (U \times_\mathcal{Z} U) = U \times_{\mathcal{Z} \times \mathcal{Z}, \Delta_\mathcal{Z}} \mathcal{Z}$$

is unramified over U. In particular $\dim(G) = 0$ and by More on Groupoids in Spaces, Lemma 79.9.11 we have $\dim(R) = 0$. This implies that R is a scheme, see Spaces over Fields, Lemma 72.9.1. By Varieties, Lemma 33.20.2 we see that R (and also G) is the disjoint union of spectra of Artinian local rings finite over k via either s or t. Let $P = \mathrm{Spec}(A) \subset R$ be the open and closed subscheme whose underlying point is the identity e of the groupoid scheme (U, R, s, t, c). As $s \circ e = t \circ e = \mathrm{id}_{\mathrm{Spec}(k)}$ we see that A is an Artinian local ring whose residue field is identified with k via either $s^\sharp : k \to A$ or $t^\sharp : k \to A$. Note that $s, t : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ are finite (by the lemma referenced above). Since $G \to \mathrm{Spec}(k)$ is unramified we see that

$$G \cap P = P \times_{U \times U} U = \mathrm{Spec}(A \otimes_{k \otimes k} k)$$

is unramified over k. On the other hand $A \otimes_{k \otimes k} k$ is local as a quotient of A and surjects onto k. We conclude that $A \otimes_{k \otimes k} k = k$. It follows that $P \to U \times U$ is universally injective (as P has only one point with residue field k), unramified (by the computation of the fibre over the unique image point above), and of finite type (because s, t are) hence a monomorphism (see Étale Morphisms, Lemma 41.7.1). Thus $s|_P, t|_P : P \to U$ define a finite flat equivalence relation. Thus we may apply Groupoids, Proposition 39.23.9 to conclude that U/P exists and is a scheme $\overline{U}$. Moreover, $U \to \overline{U}$ is finite locally free and $P = U \times_{\overline{U}} U$. In fact $\overline{U} = \mathrm{Spec}(k_0)$ where $k_0 \subset k$ is the ring of R-invariant functions. As k is a field it follows from the definition Groupoids, Equation (39.23.0.1) that k_0 is a field.

We claim that

06N1 (101.21.4.1)
$$\mathrm{Spec}(k_0) = \overline{U} = U/P \to [U/R] = \mathcal{Z}$$

is the desired surjective étale morphism. It follows from Properties of Stacks, Lemma 100.11.1 that this morphism is surjective. Thus it suffices to show that (101.21.4.1) is étale[6]. Instead of proving the étaleness directly we first apply Bootstrap, Lemma 80.9.1 to see that there exists a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ such that (U, R, s, t, c) is the restriction of $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ via the quotient morphism $U \to \overline{U}$. (We verified all the hypothesis of the lemma above except for the assertion that $j : R \to U \times U$ is separated and locally quasi-finite which follows from the fact that R is a separated scheme locally quasi-finite over k.) Since $U \to \overline{U}$ is finite locally free we see that $[U/R] \to [\overline{U}/\overline{R}]$ is an equivalence, see Groupoids in Spaces, Lemma 78.25.2.

Note that s, t are the base changes of the morphisms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 35.23.15, 35.23.11, and 35.23.24. Consider the commutative diagram

$$\begin{tikzcd}
U \times_{\overline{U}} U \arrow[r, equal] \arrow[rd] & P \arrow[r] \arrow[d] & R \arrow[d] \\
 & \overline{U} \arrow[r, "\overline{e}"] & \overline{R}
\end{tikzcd}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 35.23.16.

But of course, if $\overline{e}$ is an open immersion and $\overline{s}, \overline{t}$ are flat and locally of finite presentation then the morphisms $\overline{t}, \overline{s}$ are étale. For example you can see this by applying More on Groupoids, Lemma 40.4.1 which shows that $\Omega_{\overline{R}/\overline{U}} = 0$ implies that $\overline{s}, \overline{t} : \overline{R} \to \overline{U}$ is unramified (see Morphisms, Lemma 29.35.2), which in turn implies that $\overline{s}, \overline{t}$ are étale (see Morphisms, Lemma 29.36.16). Hence $\mathcal{Z} = [\overline{U}/\overline{R}]$ is an étale presentation of the algebraic stack $\mathcal{Z}$ and we conclude that $\overline{U} \to \mathcal{Z}$ is étale by Properties of Stacks, Lemma 100.3.3. □

[6] We urge the reader to find his/her own proof of this fact. In fact the argument has a lot in common with the final argument of the proof of Bootstrap, Theorem 80.10.1 hence probably should be isolated into its own lemma somewhere.

06N2 **Lemma 101.21.5.** Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $z \in |F|$ be such that $F \to \mathrm{Spec}(k)$ is unramified at z. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is étale.

Proof. Since $f : U \to \mathcal{X}$ is flat and locally of finite presentation there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is étale, see Properties of Stacks, Remark 100.9.20 (5). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is étale at z. Since it is flat and locally of finite presentation as a base change of $U \to \mathcal{X}$ and since $F \to \mathrm{Spec}(k)$ is unramified at z by assumption, this follows from Morphisms of Spaces, Lemma 67.39.12. ☐

A DM stack is a Deligne-Mumford stack.

06N3 **Theorem 101.21.6.** Let $\mathcal{X}$ be an algebraic stack. The following are equivalent

(1) $\mathcal{X}$ is DM,
(2) $\mathcal{X}$ is Deligne-Mumford, and
(3) there exists a scheme W and a surjective étale morphism $W \to \mathcal{X}$.

Proof. Recall that (3) is the definition of (2), see Algebraic Stacks, Definition 94.12.2. The implication (3) $\Rightarrow$ (1) is Lemma 101.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

By Lemma 101.18.7 the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type. By Lemma 101.4.16 the algebraic stack $\mathcal{Z}_x$ is DM. By Lemma 101.21.4 there exists a field k and a surjective étale morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}_x$. In particular the composition $x : \mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type (by Morphisms of Spaces, Lemmas 67.23.2 and 67.39.9).

Pick a scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of $|U| \to |\mathcal{X}|$. Consider the following fibre square

$$\begin{tikzcd} U \arrow[d] & F \arrow[l] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l, "x"] \end{tikzcd}$$

in other words $F = U \times_{\mathcal{X},x} \mathrm{Spec}(k)$. By Properties of Stacks, Lemma 100.4.3 we see that F is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we have

$$\mathrm{Spec}(k) \times_{z,\mathcal{Z}_x,z} \mathrm{Spec}(k) = \mathrm{Spec}(k) \times_{x,\mathcal{X},x} \mathrm{Spec}(k)$$

with étale projection maps to $\mathrm{Spec}(k)$ by construction of z. Since

$$F \times_U F = (\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)) \times_{\mathrm{Spec}(k)} F$$

we see that the projections maps $F \times_U F \to F$ are étale as well. It follows that $\Delta_{F/U} : F \to F \times_U F$ is étale (see Morphisms of Spaces, Lemma 67.39.11). By Morphisms of Spaces, Lemma 67.51.2 this implies that $\Delta_{F/U}$ is an open immersion, which finally implies by Morphisms of Spaces, Lemma 67.38.9 that $F \to U$ is unramified.

Pick a nonempty affine scheme V and an étale morphism $V \to F$. (This could be avoided by working directly with F, but it seems easier to explain what's going on by doing so.) Picture

$$\begin{tikzcd} U \arrow[d] & F \arrow[l] \arrow[d] & V \arrow[l] \arrow[dl] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l, "x"'] \end{tikzcd}$$

Then $V \to \mathrm{Spec}(k)$ is a smooth morphism of schemes and $V \to U$ is an unramified morphism of schemes (see Morphisms of Spaces, Lemmas 67.37.2 and 67.38.3). Pick a closed point $v \in V$ with $k \subset \kappa(v)$ finite separable, see Varieties, Lemma 33.25.6. Let $u \in U$ be the image point. The local ring $\mathcal{O}_{V,v}$ is regular (see Varieties, Lemma 33.25.3) and the local ring homomorphism

$$\varphi : \mathcal{O}_{U,u} \longrightarrow \mathcal{O}_{V,v}$$

coming from the morphism $V \to U$ is such that $\varphi(\mathfrak{m}_u)\mathcal{O}_{V,v} = \mathfrak{m}_v$, see Morphisms, Lemma 29.35.14. Hence we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ such that the images $\varphi(f_1), \ldots, \varphi(f_d)$ form a basis for $\mathfrak{m}_v/\mathfrak{m}_v^2$ over $\kappa(v)$. Since $\mathcal{O}_{V,v}$ is a regular local ring this implies that $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{V,v}$ (see Algebra, Lemma 10.106.3). After replacing U by an open neighbourhood of u we may assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$. After replacing U by a possibly even smaller open neighbourhood of u we may assume that $V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation, see Lemma 101.21.1. By construction

$$V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \longleftarrow V(f_1, \ldots, f_d) \times_U V$$

is étale and $V(f_1, \ldots, f_d) \times_U V$ is the closed subscheme $T \subset V$ cut out by $f_1|_V, \ldots, f_d|_V$. Hence by construction $v \in T$ and

$$\mathcal{O}_{T,v} = \mathcal{O}_{V,v}/(\varphi(f_1), \ldots, \varphi(f_d)) = \kappa(v)$$

a finite separable extension of k. It follows that $T \to \mathrm{Spec}(k)$ is unramified at v, see Morphisms, Lemma 29.35.14. By definition of an unramified morphism of algebraic spaces this means that $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is unramified at the image of v in $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k)$. Applying Lemma 101.21.5 we see that on shrinking U to yet another open neighbourhood of u the morphism $V(f_1, \ldots, f_d) \to \mathcal{X}$ is étale.

We conclude that for every finite type point x of $\mathcal{X}$ there exists an étale morphism $f_x : W_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x W_x$ and $f = \coprod f_x$. Then f is étale. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 100.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 101.18.6 (and Properties of Stacks, Lemma 100.4.4). $\square$

Here is a useful corollary which tells us that the "fibres" of a DM morphism of algebraic stacks are Deligne-Mumford.

0CIA Lemma 101.21.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks. Then

(1) For every DM algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ there exists a scheme and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.
(2) For every algebraic space Z and morphism $Z \to \mathcal{Y}$ there exists a scheme and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} Z$.

Proof. Proof of (1). As f is DM we see that the base change $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is DM by Lemma 101.4.4. Since $\mathcal{Z}$ is DM this implies that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is DM by Lemma 101.4.11. Hence there exists a scheme U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$, see Theorem 101.21.6. Part (2) is a special case of (1) since an algebraic space (when viewed as an algebraic stack) is DM by Lemma 101.4.3. $\square$

101.22. The Deligne-Mumford locus

0DSL Every algebraic stack has a largest open substack which is a Deligne-Mumford stack; this is more or less clear but we also write out the proof below. Of course this substack may be empty, for example if $X = [\mathrm{Spec}(\mathbf{Z})/\mathbf{G}_{m,\mathbf{Z}}]$. Below we will characterize the points of the DM locus.

0DSM Lemma 101.22.1. Let $\mathcal{X}$ be an algebraic stack. There exist open substacks

$$\mathcal{X}'' \subset \mathcal{X}' \subset \mathcal{X}$$

such that $\mathcal{X}''$ is DM, $\mathcal{X}'$ is quasi-DM, and such that these are the largest open substacks with these properties.

Proof. All we are really saying here is that if $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{X}$ are open substacks which are DM, then the open substack $\mathcal{W} \subset \mathcal{X}$ with $|\mathcal{W}| = |\mathcal{U}| \cup |\mathcal{V}|$ is DM as well. (Similarly for quasi-DM.) Although this is a cheat, let us use Theorem 101.21.6 to prove this. By that theorem we can choose schemes U and V and surjective étale morphisms $U \to \mathcal{U}$ and $V \to \mathcal{V}$. Then of course $U \amalg V \to \mathcal{W}$ is surjective and étale. The quasi-DM case is proven by exactly the same method using Theorem 101.21.3. $\square$

0DSN Lemma 101.22.2. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ correspond to $x : \mathrm{Spec}(k) \to \mathcal{X}$. Let G_x/k be the automorphism group algebraic space of x. Then

(1) x is in the DM locus of $\mathcal{X}$ if and only if $G_x \to \mathrm{Spec}(k)$ is unramified, and
(2) x is in the quasi-DM locus of $\mathcal{X}$ if and only if $G_x \to \mathrm{Spec}(k)$ is locally quasi-finite.

Proof. Proof of (2). Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Consider the fibre product

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Recall that G is the automorphism group algebraic space of $U \to \mathcal{X}$. By Groupoids in Spaces, Lemma 78.6.3 there is a maximal open subscheme $U' \subset U$ such that $G_{U'} \to U'$ is locally quasi-finite. Moreover, formation of U' commutes with arbitrary base change. In particular the two inverse images of U' in $R = U \times_{\mathcal{X}} U$ are the same open subspace of R (since after all the two maps $R \to \mathcal{X}$ are isomorphic and hence have isomorphic automorphism group spaces). Hence U' is the inverse image of an open substack $\mathcal{X}' \subset \mathcal{X}$ by Properties of Stacks, Lemma 100.9.11 and we have a cartesian diagram

$$\begin{tikzcd} G_{U'} \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}'} \arrow[d] \\ U' \arrow[r] & \mathcal{X}' \end{tikzcd}$$

Thus the morphism $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is locally quasi-finite and we conclude that $\mathcal{X}'$ is quasi-DM by Lemma 101.6.1 part (5). On the other hand, if $\mathcal{W} \subset \mathcal{X}$ is an open substack which is quasi-DM, then the inverse image $W \subset U$ of $\mathcal{W}$ must be contained in U' by our construction of U' since $\mathcal{I}_\mathcal{W} = \mathcal{W} \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is locally quasi-finite over $\mathcal{W}$. Thus $\mathcal{X}'$ is the quasi-DM locus. Finally, choose a field extension K/k and a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d, "x"] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Then we find an isomorphism $G_x \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \cong G \times_U \mathrm{Spec}(K)$ of group algebraic spaces over K. Hence G_x is locally quasi-finite over k if and only if $\mathrm{Spec}(K) \to U$ maps into U' (use the commutation of formation of U' and Groupoids in Spaces, Lemma 78.6.3 applied to $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ and G_x to see this). This finishes the proof of (2). The proof of (1) is exactly the same. $\square$

101.23. Locally quasi-finite morphisms

06PT The property "locally quasi-finite" of morphisms of algebraic spaces is not smooth local on the source-and-target so we cannot use the material in Section 101.16 to define locally quasi-finite morphisms of algebraic stacks. We do already know what it means for a morphism of algebraic stacks representable by algebraic spaces to be locally quasi-finite, see Properties of Stacks, Section 100.3. To find a condition suitable for general morphisms we make the following observation.

06UA **Lemma 101.23.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces. The following are equivalent*

1. *f is locally quasi-finite (as in Properties of Stacks, Section 100.3), and*
2. *f is locally of finite type and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathrm{Spec}(k) \times_\mathcal{Y} \mathcal{X}|$ is discrete.*

Proof. Assume (1). In this case the morphism of algebraic spaces $\mathcal{X}_k \to \mathrm{Spec}(k)$ is locally quasi-finite as a base change of f. Hence $|\mathcal{X}_k|$ is discrete by Morphisms of Spaces, Lemma 67.27.5. Conversely, assume (2). Pick a surjective smooth morphism $V \to \mathcal{Y}$ where V is a scheme. It suffices to show that the morphism of algebraic spaces $V \times_\mathcal{Y} \mathcal{X} \to V$ is locally quasi-finite, see Properties of Stacks, Lemma 100.3.3. The morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ is locally of finite type by assumption. For any morphism $\mathrm{Spec}(k) \to V$ where k is a field

$$\mathrm{Spec}(k) \times_V (V \times_\mathcal{Y} \mathcal{X}) = \mathrm{Spec}(k) \times_\mathcal{Y} \mathcal{X}$$

has a discrete space of points by assumption. Hence we conclude that $V \times_\mathcal{Y} \mathcal{X} \to V$ is locally quasi-finite by Morphisms of Spaces, Lemma 67.27.5. $\square$

A morphism of algebraic stacks which is representable by algebraic spaces is quasi-DM, see Lemma 101.4.3. Combined with the lemma above we see that the following definition does not conflict with the already existing notion in the case of morphisms representable by algebraic spaces.

06PU Definition 101.23.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is locally quasi-finite if f is quasi-DM, locally of finite type, and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathcal{X}_k|$ is discrete.

The condition that f be quasi-DM is natural. For example, let k be a field and consider the morphism $\pi : [\mathrm{Spec}(k)/\mathbf{G}_m] \to \mathrm{Spec}(k)$ which has singleton fibres and is locally of finite type. As we will see later this morphism is smooth of relative dimension -1, and we'd like our locally quasi-finite morphisms to have relative dimension 0. Also, note that the section $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/\mathbf{G}_m]$ does not have discrete fibres, hence is not locally quasi-finite, and we'd like to have the following permanence property for locally quasi-finite morphisms: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of algebraic stacks locally quasi-finite over the algebraic stack $\mathcal{Y}$, then f is locally quasi-finite (in fact something a bit stronger holds, see Lemma 101.23.8).

Another justification for the definition above is Lemma 101.23.7 below which characterizes being locally quasi-finite in terms of the existence of suitable "presentations" or "coverings" of $\mathcal{X}$ and $\mathcal{Y}$.

06UB Lemma 101.23.3. A base change of a locally quasi-finite morphism is locally quasi-finite.

Proof. We have seen this for quasi-DM morphisms in Lemma 101.4.4 and for locally finite type morphisms in Lemma 101.17.3. It is immediate that the condition on fibres is inherited by a base change. $\square$

06UC Lemma 101.23.4. Let $\mathcal{X} \to \mathrm{Spec}(k)$ be a locally quasi-finite morphism where $\mathcal{X}$ is an algebraic stack and k is a field. Let $f : V \to \mathcal{X}$ be a locally quasi-finite morphism where V is a scheme. Then $V \to \mathrm{Spec}(k)$ is locally quasi-finite.

Proof. By Lemma 101.17.2 we see that $V \to \mathrm{Spec}(k)$ is locally of finite type. Assume, to get a contradiction, that $V \to \mathrm{Spec}(k)$ is not locally quasi-finite. Then there exists a nontrivial specialization $v \rightsquigarrow v'$ of points of V, see Morphisms, Lemma 29.20.6. In particular $\mathrm{trdeg}_k(\kappa(v)) > \mathrm{trdeg}_k(\kappa(v'))$, see Morphisms, Lemma 29.28.7. Because $|\mathcal{X}|$ is discrete we see that $|f|(v) = |f|(v')$. Consider $R = V \times_\mathcal{X} V$. Then R is an algebraic space and the projections $s, t : R \to V$ are locally quasi-finite as base changes of $V \to \mathcal{X}$ (which is representable by algebraic spaces so this follows from the discussion in Properties of Stacks, Section 100.3). By Properties of Stacks, Lemma 100.4.3 we see that there exists an $r \in |R|$ such that $s(r) = v$ and $t(r) = v'$. By Morphisms of Spaces, Lemma 67.33.3 we see that the transcendence degree of v/k is equal to the transcendence degree of r/k is equal to the transcendence degree of v'/k. This contradiction proves the lemma. $\square$

06UD Lemma 101.23.5. A composition of a locally quasi-finite morphisms is locally quasi-finite.

Proof. We have seen this for quasi-DM morphisms in Lemma 101.4.10 and for locally finite type morphisms in Lemma 101.17.2. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be locally quasi-finite. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be a morphism. It suffices to show that $|\mathcal{X}_k|$ is discrete. By Lemma 101.23.3 the morphisms $\mathcal{X}_k \to \mathcal{Y}_k$ and $\mathcal{Y}_k \to \mathrm{Spec}(k)$ are locally quasi-finite. In particular we see that $\mathcal{Y}_k$ is a quasi-DM algebraic stack, see Lemma 101.4.13. By Theorem 101.21.3 we can find a scheme V and a surjective, flat, locally finitely presented, locally quasi-finite morphism $V \to \mathcal{Y}_k$. By Lemma 101.23.4 we see that V is locally quasi-finite over k, in particular $|V|$ is discrete. The morphism $V \times_{\mathcal{Y}_k} \mathcal{X}_k \to \mathcal{X}_k$ is surjective, flat, and locally of finite presentation hence $|V \times_{\mathcal{Y}_k} \mathcal{X}_k| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to show that $|V \times_{\mathcal{Y}_k} \mathcal{X}_k|$ is discrete. Note that V is a disjoint union of spectra of Artinian local k-algebras A_i with residue fields k_i, see Varieties, Lemma 33.20.2. Thus it suffices to show that each

$$|\mathrm{Spec}(A_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}} \mathcal{X}|$$

is discrete, which follows from the assumption that $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite. □

Before we characterize locally quasi-finite morphisms in terms of coverings we do it for quasi-DM morphisms.

06UE Lemma 101.23.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) f is quasi-DM,
(2) for any morphism $V \to \mathcal{Y}$ with V an algebraic space there exists a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space, and
(3) there exist algebraic spaces U, V and a morphism $V \to \mathcal{Y}$ which is surjective, flat, and locally of finite presentation, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite.

Proof. The implication (2) ⇒ (3) is immediate.

Assume (1) and let $V \to \mathcal{Y}$ be as in (2). Then $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 101.4.4. By Lemma 101.4.3 the algebraic space V is DM, hence quasi-DM. Thus $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM by Lemma 101.4.11. Hence we may apply Theorem 101.21.3 to get the morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ as in (2).

Assume (3). Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (3). To prove that f is quasi-DM it suffices to show that $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 101.4.5. By Lemma 101.4.14 we see that $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM. Hence $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM by Lemma 101.4.13 and (1) holds. This finishes the proof of the lemma. □

06UF Lemma 101.23.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) f is locally quasi-finite,
(2) f is quasi-DM and for any morphism $V \to \mathcal{Y}$ with V an algebraic space and any locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space the morphism $U \to V$ is locally quasi-finite,
(3) for any morphism $V \to \mathcal{Y}$ from an algebraic space V there exists a surjective, flat, locally finitely presented, and locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space such that $U \to V$ is locally quasi-finite,

(4) there exists algebraic spaces U, V, a surjective, flat, and locally of finite presentation morphism $V \to \mathcal{Y}$, and a morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite such that $U \to V$ is locally quasi-finite.

Proof. Assume (1). Then f is quasi-DM by assumption. Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_\mathcal{Y} V$ be as in (2). By Lemma 101.23.5 the composition $U \to \mathcal{X} \times_\mathcal{Y} V \to V$ is locally quasi-finite. Thus (1) implies (2).

Assume (2). Let $V \to \mathcal{Y}$ be as in (3). By Lemma 101.23.6 we can find an algebraic space U and a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_\mathcal{Y} V$. By (2) the composition $U \to V$ is locally quasi-finite. Thus (2) implies (3).

It is immediate that (3) implies (4).

Assume (4). We will prove (1) holds, which finishes the proof. By Lemma 101.23.6 we see that f is quasi-DM. To prove that f is locally of finite type it suffices to prove that $g : \mathcal{X} \times_\mathcal{Y} V \to V$ is locally of finite type, see Lemma 101.17.6. Then it suffices to check that g precomposed with $h : U \to \mathcal{X} \times_\mathcal{Y} V$ is locally of finite type, see Lemma 101.17.7. Since $g \circ h : U \to V$ was assumed to be locally quasi-finite this holds, hence f is locally of finite type. Finally, let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism. Then $V \times_\mathcal{Y} \mathrm{Spec}(k)$ is a nonempty algebraic space which is locally of finite presentation over k. Hence we can find a finite extension k'/k and a morphism $\mathrm{Spec}(k') \to V$ such that

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

commutes (details omitted). Then $\mathcal{X}_{k'} \to \mathcal{X}_k$ is representable (by schemes), surjective, and finite locally free. In particular $|\mathcal{X}_{k'}| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to prove that $|\mathcal{X}_{k'}|$ is discrete. Since

$$U \times_V \mathrm{Spec}(k') = U \times_{\mathcal{X} \times_\mathcal{Y} V} \mathcal{X}_{k'}$$

we see that $U \times_V \mathrm{Spec}(k') \to \mathcal{X}_{k'}$ is surjective, flat, and locally of finite presentation (as a base change of $U \to \mathcal{X} \times_\mathcal{Y} V$). Hence $|U \times_V \mathrm{Spec}(k')| \to |\mathcal{X}_{k'}|$ is surjective and open. Thus it suffices to show that $|U \times_V \mathrm{Spec}(k')|$ is discrete. This follows from the fact that $U \to V$ is locally quasi-finite (either by our definition above or from the original definition for morphisms of algebraic spaces, via Morphisms of Spaces, Lemma 67.27.5). $\square$

06UG Lemma 101.23.8. Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume that $\mathcal{X} \to \mathcal{Z}$ is locally quasi-finite and $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Then $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite.

Proof. Write $\mathcal{X} \to \mathcal{Y}$ as the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_\mathcal{Z} \mathcal{Y} \longrightarrow \mathcal{Y}$$

The second arrow is locally quasi-finite as a base change of $\mathcal{X} \to \mathcal{Z}$, see Lemma 101.23.3. The first arrow is locally quasi-finite by Lemma 101.4.8 as $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Hence $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite by Lemma 101.23.5. $\square$

101.24. Quasi-finite morphisms

0G2L We have defined "locally quasi-finite" morphisms of algebraic stacks in Section 101.23 and "quasi-compact" morphisms of algebraic stacks in Section 101.7. Since a morphism of algebraic spaces is by definition quasi-finite if and only if it is both locally quasi-finite and quasi-compact (Morphisms of Spaces, Definition 67.27.1), we may define what it means for a morphism of algebraic stacks to be quasi-finite as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

0G2M **Definition 101.24.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is quasi-finite if f is locally quasi-finite (Definition 101.23.2) and quasi-compact (Definition 101.7.2). [Ryd08]

0G2N **Lemma 101.24.2.** The composition of quasi-finite morphisms is quasi-finite.

Proof. Combine Lemmas 101.23.5 and 101.7.4. □

0G2P **Lemma 101.24.3.** A base change of a quasi-finite morphism is quasi-finite.

Proof. Combine Lemmas 101.23.3 and 101.7.3. □

0G2Q **Lemma 101.24.4.** Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-finite and g is quasi-separated and quasi-DM then f is quasi-finite.

Proof. Combine Lemmas 101.23.8 and 101.7.7. □

101.25. Flat morphisms

06PV The property "being flat" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.30.4 and Descent on Spaces, Lemma 74.11.13. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be flat as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

06PW **Definition 101.25.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is flat if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} = $ flat.

06PX **Lemma 101.25.2.** The composition of flat morphisms is flat.

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.30.3. □

06PY **Lemma 101.25.3.** A base change of a flat morphism is flat.

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.30.4. □

06PZ **Lemma 101.25.4.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective flat morphism of algebraic stacks. If the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is flat, then f is flat.

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective and flat (Morphisms of Spaces, Lemma 67.37.7) hence $W \to \mathcal{Y}$ is surjective and flat (by Properties of Stacks, Lemma 100.5.2 and Lemma 101.25.2). Since the base change of $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is a flat morphism (Lemma 101.25.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is flat. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of flat morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is flat, we see that $U' \to V'$ is flat. Then, since $V' \to V$ is surjective as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is flat by Morphisms of Spaces, Lemma 67.31.3 (2) and we win. $\square$

06Q0 Lemma 101.25.5. Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is surjective and flat, then $\mathcal{Y} \to \mathcal{Z}$ is flat.

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and that $U \to W$ is flat. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Morphisms of Spaces, Lemma 67.31.5. $\square$

0DN5 Lemma 101.25.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Let $\mathrm{Spec}(A) \to \mathcal{Y}$ be a morphism where A is a valuation ring. If the closed point of $\mathrm{Spec}(A)$ maps to a point of $|\mathcal{Y}|$ in the image of $|\mathcal{X}| \to |\mathcal{Y}|$, then there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(A') \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where $A \to A'$ is an extension of valuation rings (More on Algebra, Definition 15.123.1).

Proof. The base change $\mathcal{X}_A \to \mathrm{Spec}(A)$ is flat (Lemma 101.25.3) and the closed point of $\mathrm{Spec}(A)$ is in the image of $|\mathcal{X}_A| \to |\mathrm{Spec}(A)|$ (Properties of Stacks, Lemma 100.4.3). Thus we may assume $\mathcal{Y} = \mathrm{Spec}(A)$. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. Then we can apply Morphisms of Spaces, Lemma 67.42.4 to the morphism $U \to \mathrm{Spec}(A)$ to conclude. $\square$

101.26. Flat at a point

0CIB We still have to develop the general machinery needed to say what it means for a morphism of algebraic stacks to have a given property at a point. For the moment the following lemma is sufficient.

0CIC Lemma 101.26.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$. Consider commutative diagrams

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \quad \text{with points} \quad \begin{tikzcd} u \in |U| \arrow[d, mapsto] \\ x \in |\mathcal{X}| \end{tikzcd}$$

where U and V are algebraic spaces, b is flat, and $(a, h) : U \to \mathcal{X} \times_{\mathcal{Y}} V$ is flat. The following are equivalent

(1) h is flat at u for one diagram as above,
(2) h is flat at u for every diagram as above.

Proof. Suppose we are given a second diagram U', V', u', a', b', h' as in the lemma. Then we can consider

$$\begin{tikzcd} U & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d] & U' \arrow[d] \\ V & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V' \arrow[from=1-1, to=2-1] \end{tikzcd}$$

By Properties of Stacks, Lemma 100.4.3 there is a point $u'' \in |U \times_{\mathcal{X}} U'|$ mapping to u and u'. If h is flat at u, then the base change $U \times_V (V \times_{\mathcal{Y}} V') \to V \times_{\mathcal{Y}} V'$ is flat at any point over u, see Morphisms of Spaces, Lemma 67.31.3. On the other hand, the morphism

$$U \times_{\mathcal{X}} U' \to U \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V') = U \times_{\mathcal{Y}} V' = U \times_V (V \times_{\mathcal{Y}} V')$$

is flat as a base change of (a', h'), see Lemma 101.25.3. Composing and using Morphisms of Spaces, Lemma 67.31.4 we conclude that $U \times_{\mathcal{X}} U' \to V \times_{\mathcal{Y}} V'$ is flat at u''. Then we can use composition by the flat map $V \times_{\mathcal{Y}} V' \to V'$ to conclude that $U \times_{\mathcal{X}} U' \to V'$ is flat at u''. Finally, since $U \times_{\mathcal{X}} U' \to U'$ is flat at u'' and u'' maps to u' we conclude that $U' \to V'$ is flat at u' by Morphisms of Spaces, Lemma 67.31.5. □

0CID Definition 101.26.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$. We say f is flat at x if the equivalent conditions of Lemma 101.26.1 hold.

101.27. Morphisms of finite presentation

06Q1 The property "locally of finite presentation" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.28.3 and Descent on Spaces, Lemma 74.11.10. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic stacks to be locally of finite presentation as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

06Q2 Definition 101.27.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f locally of finite presentation if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P}$ = locally of finite presentation.
(2) We say f is of finite presentation if it is locally of finite presentation, quasi-compact, and quasi-separated.

Note that a morphism of finite presentation is not just a quasi-compact morphism which is locally of finite presentation.

06Q3 Lemma 101.27.2. The composition of finitely presented morphisms is of finite presentation. The same holds for morphisms which are locally of finite presentation.

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.28.2. □

06Q4 **Lemma 101.27.3.** A base change of a finitely presented morphism is of finite presentation. The same holds for morphisms which are locally of finite presentation.

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.28.3. □

06Q5 **Lemma 101.27.4.** A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.

Proof. Combine Remark 101.16.5 with Morphisms of Spaces, Lemma 67.28.5. □

0DQJ **Lemma 101.27.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) If $\mathcal{Y}$ is locally Noetherian and f locally of finite type then f is locally of finite presentation.
(2) If $\mathcal{Y}$ is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.

Proof. Assume $f : \mathcal{X} \to \mathcal{Y}$ locally of finite type and $\mathcal{Y}$ locally Noetherian. This means there exists a diagram as in Lemma 101.16.1 with h locally of finite type and surjective vertical arrow a. By Morphisms of Spaces, Lemma 67.28.7 h is locally of finite presentation. Hence $\mathcal{X} \to \mathcal{Y}$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

06Q6 **Lemma 101.27.6.** Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. The lemma follows upon applying Morphisms of Spaces, Lemma 67.28.9 to the morphisms $U \to V \to W$. □

0CMG **Lemma 101.27.7.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks with diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.

Proof. Note that Δ is a morphism over $\mathcal{X}$ (via the second projection). If f is locally of finite type, then $\mathcal{X}$ is of finite presentation over $\mathcal{X}$ and $\text{pr}_2 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is locally of finite type by Lemma 101.17.3. Thus the first statement holds by Lemma 101.27.6. The second statement follows from the first and the definitions (because f being quasi-separated means by definition that Δ_f is quasi-compact and quasi-separated). □

06Q7 **Lemma 101.27.8.** An open immersion is locally of finite presentation.

Proof. In view of Properties of Stacks, Definition 100.9.1 this follows from Morphisms of Spaces, Lemma 67.28.11. □

0CPP **Lemma 101.27.9.** Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

For the meaning of this statement see Properties of Stacks, Section 100.3.

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By Properties of Stacks, Lemma 100.5.2 and Lemmas 101.25.2 and 101.27.2 the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V = W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Properties of Stacks, Lemma 100.3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. $\square$

0DN6 **Lemma 101.27.10.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and fppf local on the target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. Assume $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Observe that $W \times_{\mathcal{Z}} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} = W \times_{\mathcal{Y}} \mathcal{X}$. Thus by the very definition of what it means for $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ to have $\mathcal{P}$ (see Definition 101.16.2 and Lemma 101.16.1) we see that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has $\mathcal{P}$. On the other hand, $W \to \mathcal{Z}$ is surjective, flat, and locally of finite presentation (Morphisms of Spaces, Lemmas 67.37.7 and 67.37.5) hence $W \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation (by Properties of Stacks, Lemma 100.5.2 and Lemmas 101.25.2 and 101.27.2). Thus we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ has $\mathcal{P}$. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by Lemma 101.16.1 used in the definition of $\mathcal{X}' \to \mathcal{Y}' = W$ having $\mathcal{P}$ we see that $U' \to V'$ has $\mathcal{P}$. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is local in the fppf topology on the target. $\square$

06Q8 **Lemma 101.27.11.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is locally of finite presentation, then f is locally of finite presentation.

Proof. The property "locally of finite presentation" satisfies the conditions of Lemma 101.27.10. Smooth local on the source-and-target we have seen in the introduction to this section and fppf local on the target is Descent on Spaces, Lemma 74.11.10. $\square$

06Q9 **Lemma 101.27.12.** Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is locally of finite presentation and $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite presentation.

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and locally of finite presentation and that $U \to W$ is locally of finite presentation. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 74.16.1. $\square$

06QA Lemma 101.27.13. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then for every scheme U and object y of $\mathcal{Y}$ over U there exists an fppf covering $\{U_i \to U\}$ and objects x_i of $\mathcal{X}$ over U_i such that $f(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.

Proof. We may think of y as a morphism $U \to \mathcal{Y}$. By Properties of Stacks, Lemma 100.5.3 and Lemmas 101.27.3 and 101.25.3 we see that $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is surjective, flat, and locally of finite presentation. Let V be a scheme and let $V \to \mathcal{X} \times_{\mathcal{Y}} U$ smooth and surjective. Then $V \to \mathcal{X} \times_{\mathcal{Y}} U$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 67.37.7 and 67.37.5). Hence also $V \to U$ is surjective, flat, and locally of finite presentation, see Properties of Stacks, Lemma 100.5.2 and Lemmas 101.27.2, and 101.25.2. Hence $\{V \to U\}$ is the desired fppf covering and $x : V \to \mathcal{X}$ is the desired object. $\square$

07AN Lemma 101.27.14. Let $f_j : \mathcal{X}_j \to \mathcal{X}$, $j \in J$ be a family of morphisms of algebraic stacks which are each flat and locally of finite presentation and which are jointly surjective, i.e., $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Then for every scheme U and object x of $\mathcal{X}$ over U there exists an fppf covering $\{U_i \to U\}_{i \in I}$, a map $a : I \to J$, and objects x_i of $\mathcal{X}_{a(i)}$ over U_i such that $f_{a(i)}(x_i) \cong y|_{U_i}$ in $\mathcal{X}_{U_i}$.

Proof. Apply Lemma 101.27.13 to the morphism $\coprod_{j \in J} \mathcal{X}_j \to \mathcal{X}$. (There is a slight set theoretic issue here – due to our setup of things – which we ignore.) To finish, note that a morphism $x_i : U_i \to \coprod_{j \in J} \mathcal{X}_j$ is given by a disjoint union decomposition $U_i = \coprod U_{i,j}$ and morphisms $U_{i,j} \to \mathcal{X}_j$. Then the fppf covering $\{U_{i,j} \to U\}$ and the morphisms $U_{i,j} \to \mathcal{X}_j$ do the job. $\square$

06R7 Lemma 101.27.15. Let $f : \mathcal{X} \to \mathcal{Y}$ be flat and locally of finite presentation. Then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is open.

Proof. Choose a scheme V and a smooth surjective morphism $V \to \mathcal{Y}$. Choose a scheme U and a smooth surjective morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. By assumption the morphism of schemes $U \to V$ is flat and locally of finite presentation. Hence $U \to V$ is open by Morphisms, Lemma 29.25.10. By construction of the topology on $|\mathcal{Y}|$ the map $|V| \to |\mathcal{Y}|$ is open. The map $|U| \to |\mathcal{X}|$ is surjective. The result follows from these facts by elementary topology. $\square$

0DQK Lemma 101.27.16. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is quasi-compact, then f is quasi-compact.

Proof. We have to show that given $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'$ quasi-compact, we have $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact. Denote $\mathcal{Z}' = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Y}'$. Then $|\mathcal{Z}'| \to |\mathcal{Y}'|$ is open, see Lemma 101.27.15. Hence we can find a quasi-compact open substack $\mathcal{W} \subset \mathcal{Z}'$ mapping onto $\mathcal{Y}'$. Because $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is quasi-compact, we know that

$$\mathcal{W} \times_{\mathcal{Z}} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} = \mathcal{W} \times_{\mathcal{Y}} \mathcal{X}$$

is quasi-compact. And the map $\mathcal{W} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, hence we win. Some details omitted. $\square$

0CPQ **Lemma 101.27.17.** Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be composable morphisms of algebraic stacks with composition $h = g \circ f : \mathcal{X} \to \mathcal{Z}$. If f is surjective, flat, locally of finite presentation, and universally injective and if h is separated, then g is separated.

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"'] \arrow[rd] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times_{\mathcal{Z}} \mathcal{X} \arrow[d] \\ & \mathcal{Y} \arrow[r] & \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y} \end{tikzcd}$$

The square is cartesian. We have to show the bottom horizontal arrow is proper. We already know that it is representable by algebraic spaces and locally of finite type (Lemma 101.3.3). Since the right vertical arrow is surjective, flat, and locally of finite presentation it suffices to show the top right horizontal arrow is proper (Lemma 101.27.9). Since h is separated, the composition of the top horizontal arrows is proper.

Since f is universally injective Δ is surjective (Lemma 101.14.5). Since the composition of Δ with the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is the identity, we see that Δ is universally closed. By Morphisms of Spaces, Lemma 67.9.8 we conclude that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is separated as $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is separated. Here we use that implications between properties of morphisms of algebraic spaces can be transferred to the same implications between properties of morphisms of algebraic stacks representable by algebraic spaces; this is discussed in Properties of Stacks, Section 100.3. Finally, we use the same principle to conlude that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is proper from Morphisms of Spaces, Lemma 67.40.7. $\square$

101.28. Gerbes

06QB An important type of algebraic stack are the stacks of the form $[B/G]$ where B is an algebraic space and G is a flat and locally finitely presented group algebraic space over B (acting trivially on B), see Criteria for Representability, Lemma 97.18.3. It turns out that an algebraic stack is a gerbe when it locally in the fppf topology is of this form, see Lemma 101.28.7. In this section we briefly discuss this notion and the corresponding relative notion.

06QC **Definition 101.28.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ as stacks in groupoids over $(Sch/S)_{fppf}$, see Stacks, Definition 8.11.4. We say an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X.

The condition that $\mathcal{X}$ be a gerbe over $\mathcal{Y}$ is defined purely in terms of the topology and category theory underlying the given algebraic stacks; but as we will see later this condition has geometric consequences. For example it implies that $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, see Lemma 101.28.8. The absolute notion is trickier to parse, because it may not be at first clear that X is well determined. Actually, it is.

06QD Lemma 101.28.2. Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe, then the sheafification of the presheaf

$$(Sch/S)_{fppf}^{opp} \to \text{Sets}, \quad U \mapsto \text{Ob}(\mathcal{X}_U)/\cong$$

is an algebraic space and $\mathcal{X}$ is a gerbe over it.

Proof. (In this proof the abuse of language introduced in Section 101.2 really pays off.) Choose a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X. It suffices to prove that X is the sheafification of the presheaf $\mathcal{F}$ displayed in the lemma. It is clear that there is a map $c : \mathcal{F} \to X$. We will use Stacks, Lemma 8.11.3 properties (2)(a) and (2)(b) to see that the map $c^\# : \mathcal{F}^\# \to X$ is surjective and injective, hence an isomorphism, see Sites, Lemma 7.11.2. Surjective: Let T be a scheme and let $f : T \to X$. By property (2)(a) there exists an fppf covering $\{h_i : T_i \to T\}$ and morphisms $x_i : T_i \to \mathcal{X}$ such that $f \circ h_i$ corresponds to $\pi \circ x_i$. Hence we see that $f|_{T_i}$ is in the image of c. Injective: Let T be a scheme and let $x, y : T \to \mathcal{X}$ be morphisms such that $c \circ x = c \circ y$. By (2)(b) we can find a covering $\{T_i \to T\}$ and morphisms $x|_{T_i} \to y|_{T_i}$ in the fibre category $\mathcal{X}_{T_i}$. Hence the restrictions $x|_{T_i}, y|_{T_i}$ are equal in $\mathcal{F}(T_i)$. This proves that x, y give the same section of $\mathcal{F}^\#$ over T as desired. $\square$

06QE Lemma 101.28.3. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. Immediate from the definitions and Stacks, Lemma 8.11.5. $\square$

06R8 Lemma 101.28.4. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.

Proof. Immediate from Stacks, Lemma 8.11.6. $\square$

06QF Lemma 101.28.5. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. If $\mathcal{Y}' \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. Follows immediately from Lemma 101.27.13 and Stacks, Lemma 8.11.7. $\square$

06QG Lemma 101.28.6. Let $\pi : \mathcal{X} \to U$ be a morphism from an algebraic stack to an algebraic space and let $x : U \to \mathcal{X}$ be a section of π. Set $G = \mathit{Isom}_\mathcal{X}(x, x)$, see Definition 101.5.3. If $\mathcal{X}$ is a gerbe over U, then

(1) there is a canonical equivalence of stacks in groupoids

$$x_{can} : [U/G] \longrightarrow \mathcal{X}.$$

where $[U/G]$ is the quotient stack for the trivial action of G on U,

(2) $G \to U$ is flat and locally of finite presentation, and
(3) $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation.

Proof. Set $R = U \times_{x,\mathcal{X},x} U$. The morphism $R \to U \times U$ factors through the diagonal $\Delta_U : U \to U \times U$ as it factors through $U \times_U U = U$. Hence $R = G$ because

$$\begin{aligned} G &= Isom_{\mathcal{X}}(x, x) \\ &= U \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}} \\ &= U \times_{x,\mathcal{X}} (\mathcal{X} \times_{\Delta,\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}) \\ &= (U \times_{x,\mathcal{X},x} U) \times_{U\times U,\Delta_U} U \\ &= R \times_{U\times U,\Delta_U} U \\ &= R \end{aligned}$$

for the fourth equality use Categories, Lemma 4.31.12. Let $t, s : R \to U$ be the projections. The composition law $c : R \times_{s,U,t} R \to R$ constructed on R in Algebraic Stacks, Lemma 94.16.1 agrees with the group law on G (proof omitted). Thus Algebraic Stacks, Lemma 94.16.1 shows we obtain a canonical fully faithful 1-morphism

$$x_{can} : [U/G] \longrightarrow \mathcal{X}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. To see that it is an equivalence it suffices to show that it is essentially surjective. To do this it suffices to show that any object of $\mathcal{X}$ over a scheme T comes fppf locally from x via a morphism $T \to U$, see Stacks, Lemma 8.4.8. However, this follows the condition that π turns $\mathcal{X}$ into a gerbe over U, see property (2)(a) of Stacks, Lemma 8.11.3.

By Criteria for Representability, Lemma 97.18.3 we conclude that $G \to U$ is flat and locally of finite presentation. Finally, $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 97.17.1. □

06QH Lemma 101.28.7. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and
(2) there exists an algebraic space U, a group algebraic space G flat and locally of finite presentation over U, and a surjective, flat, and locally finitely presented morphism $U \to \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} U \cong [U/G]$ over U.

Proof. Assume (2). By Lemma 101.28.5 to prove (1) it suffices to show that $[U/G]$ is a gerbe over U. This is immediate from Groupoids in Spaces, Lemma 78.27.2.

Assume (1). Any base change of π is a gerbe, see Lemma 101.28.3. As a first step we choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Thus we may assume that $\pi : \mathcal{X} \to V$ is a gerbe over a scheme. This means that there exists an fppf covering $\{V_i \to V\}$ such that the fibre category $\mathcal{X}_{V_i}$ is nonempty, see Stacks, Lemma 8.11.3 (2)(a). Note that $U = \coprod V_i \to V$ is surjective, flat, and locally of finite presentation. Hence we may replace V by U and assume that $\pi : \mathcal{X} \to U$ is a gerbe over a scheme U and that there exists an object x of $\mathcal{X}$ over U. By Lemma 101.28.6 we see that $\mathcal{X} = [U/G]$ over U for some flat and locally finitely presented group algebraic space G over U. □

06QI Lemma 101.28.8. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective, flat, and locally of finite presentation.

Proof. By Properties of Stacks, Lemma 100.5.4 and Lemmas 101.25.4 and 101.27.11 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 101.28.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U. Then $U \to [U/G]$ is surjective, flat, and locally of finite presentation, see Lemma 101.28.6. This implies that π is surjective, flat, and locally of finite presentation by Properties of Stacks, Lemma 100.5.5 and Lemmas 101.25.5 and 101.27.12. □

06QJ Proposition 101.28.9. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent

(1) $\mathcal{X}$ is a gerbe, and
(2) $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation.

Proof. Assume (1). Choose a morphism $\mathcal{X} \to X$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $X' \to X$ be a surjective, flat, locally finitely presented morphism and set $\mathcal{X}' = X' \times_X \mathcal{X}$. Note that $\mathcal{X}'$ is a gerbe over X' by Lemma 101.28.3. Then both squares in

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathcal{I}_\mathcal{X} \arrow[r] & \mathcal{X} \arrow[r] & X \end{tikzcd}$$

are fibre product squares, see Lemma 101.5.5. Hence to prove $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation it suffices to do so after such a base change by Lemmas 101.25.4 and 101.27.11. Thus we can apply Lemma 101.28.7 to assume that $\mathcal{X} = [U/G]$. By Lemma 101.28.6 we see G is flat and locally of finite presentation over U and that $x : U \to [U/G]$ is surjective, flat, and locally of finite presentation. Moreover, the pullback of $\mathcal{I}_\mathcal{X}$ by x is G and we conclude that (2) holds by descent again, i.e., by Lemmas 101.25.4 and 101.27.11.

Conversely, assume (2). Choose a smooth presentation $\mathcal{X} = [U/R]$, see Algebraic Stacks, Section 94.16. Denote $G \to U$ the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Groupoids in Spaces, Definition 78.16.2. By Lemma 101.5.7 we see that $G \to U$ is flat and locally of finite presentation as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemmas 101.25.3 and 101.27.3. Consider the following action

$$a : G \times_{U,t} R \to R, \quad (g, r) \mapsto c(g, r)$$

of G on R. This action is free on T-valued points for any scheme T as R is a groupoid. Hence $R' = R/G$ is an algebraic space and the quotient morphism $\pi : R \to R'$ is surjective, flat, and locally of finite presentation by Bootstrap, Lemma 80.11.7. The projections $s, t : R \to U$ are G-invariant, hence we obtain morphisms $s', t' : R' \to U$ such that $s = s' \circ \pi$ and $t = t' \circ \pi$. Since $s, t : R \to U$ are flat and locally of finite presentation we conclude that s', t' are flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 67.31.5 and Descent on Spaces, Lemma 74.16.1. Consider the morphism

$$j' = (t', s') : R' \longrightarrow U \times U.$$

We claim this is a monomorphism. Namely, suppose that T is a scheme and that $a, b : T \to R'$ are morphisms which have the same image in $U \times U$. By definition of the quotient $R' = R/G$ there exists an fppf covering $\{h_j : T_j \to T\}$ such that $a \circ h_j = \pi \circ a_j$ and $b \circ h_j = \pi \circ b_j$ for some morphisms $a_j, b_j : T_j \to R$. Since

a_j, b_j have the same image in $U \times U$ we see that $g_j = c(a_j, i(b_j))$ is a T_j-valued point of G such that $c(g_j, b_j) = a_j$. In other words, a_j and b_j have the same image in R' and the claim is proved. Since $j : R \to U \times U$ is a pre-equivalence relation (see Groupoids in Spaces, Lemma 78.11.2) and $R \to R'$ is surjective (as a map of sheaves) we see that $j' : R' \to U \times U$ is an equivalence relation. Hence Bootstrap, Theorem 80.10.1 shows that $X = U/R'$ is an algebraic space. Finally, we claim that the morphism

$$\mathcal{X} = [U/R] \longrightarrow X = U/R'$$

turns $\mathcal{X}$ into a gerbe over X. This follows from Groupoids in Spaces, Lemma 78.27.1 as $R \to R'$ is surjective, flat, and locally of finite presentation (if needed use Bootstrap, Lemma 80.4.6 to see this implies the required hypothesis). $\square$

0CPR Lemma 101.28.10. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which makes $\mathcal{X}$ a gerbe over $\mathcal{Y}$. Then

(1) $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is flat and locally of finite presentation,
(2) $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, flat, and locally of finite presentation,
(3) given algebraic spaces T_i, $i = 1, 2$ and morphisms $x_i : T_i \to \mathcal{X}$, with $y_i = f \circ x_i$ the morphism

$$T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \longrightarrow T_1 \times_{y_1, \mathcal{Y}, y_2} T_2$$

is surjective, flat, and locally of finite presentation,
(4) given an algebraic space T and morphisms $x_i : T \to \mathcal{X}$, $i = 1, 2$, with $y_i = f \circ x_i$ the morphism

$$\mathit{Isom}_{\mathcal{X}}(x_1, x_2) \longrightarrow \mathit{Isom}_{\mathcal{Y}}(y_1, y_2)$$

is surjective, flat, and locally of finite presentation.

Proof. Proof of (1). Choose a scheme Y and a surjective smooth morphism $Y \to \mathcal{Y}$. Set $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} Y$. By Lemma 101.5.5 we obtain cartesian squares

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

By Lemmas 101.25.4 and 101.27.11 it suffices to prove that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is flat and locally of finite presentation. This follows from Proposition 101.28.9 (because $\mathcal{X}'$ is a gerbe over Y by Lemma 101.28.3).

Proof of (2). With notation as above, note that we may assume that $\mathcal{X}' = [Y/G]$ for some group algebraic space G flat and locally of finite presentation over Y, see Lemma 101.28.7. The base change of the morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over $\mathcal{Y}$ by the morphism $Y \to \mathcal{Y}$ is the morphism $\Delta' : \mathcal{X}' \to \mathcal{X}' \times_Y \mathcal{X}'$. Hence it suffices to show that Δ' is surjective, flat, and locally of finite presentation (see Lemmas 101.25.4 and 101.27.11). In other words, we have to show that

$$[Y/G] \longrightarrow [Y/G \times_Y G]$$

is surjective, flat, and locally of finite presentation. This is true because the base change by the surjective, flat, locally finitely presented morphism $Y \to [Y/G \times_Y G]$ is the morphism $G \to Y$.

Proof of (3). Observe that the diagram

$$\begin{tikzcd} T_1 \times_{x_1,\mathcal{X},x_2} T_2 \arrow[r] \arrow[d] & T_1 \times_{y_1,\mathcal{Y},y_2} T_2 \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

is cartesian. Hence (3) follows from (2).

Proof of (4). This is true because

$$Isom_{\mathcal{X}}(x_1, x_2) = (T \times_{x_1,\mathcal{X},x_2} T) \times_{T\times T,\Delta_T} T$$

hence the morphism in (4) is a base change of the morphism in (3). □

0CPS Proposition 101.28.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and
(2) $f : \mathcal{X} \to \mathcal{Y}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are surjective, flat, and locally of finite presentation.

Proof. The implication (1) ⇒ (2) follows from Lemmas 101.28.8 and 101.28.10.

Assume (2). It suffices to prove (1) for the base change of f by a surjective, flat, and locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$, see Lemma 101.28.5 (note that the base change of the diagonal of f is the diagonal of the base change). Thus we may assume $\mathcal{Y}$ is a scheme Y. In this case $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is a base change of Δ and we conclude that $\mathcal{X}$ is a gerbe by Proposition 101.28.9. We still have to show that $\mathcal{X}$ is a gerbe over Y. Let $\mathcal{X} \to X$ be the morphism of Lemma 101.28.2 turning $\mathcal{X}$ into a gerbe over the algebraic space X classifying isomorphism classes of objects of $\mathcal{X}$. It is clear that $f : \mathcal{X} \to Y$ factors as $\mathcal{X} \to X \to Y$. Since f is surjective, flat, and locally of finite presentation, we conclude that $X \to Y$ is surjective as a map of fppf sheaves (for example use Lemma 101.27.13). On the other hand, $X \to Y$ is injective too: for any scheme T and any two T-valued points x_1, x_2 of X which map to the same point of Y, we can first fppf locally on T lift x_1, x_2 to objects ξ_1, ξ_2 of $\mathcal{X}$ over T and second deduce that ξ_1 and ξ_2 are fppf locally isomorphic by our assumption that $\Delta : \mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ is surjective, flat, and locally of finite presentation. Whence $x_1 = x_2$ by construction of X. Thus $X = Y$ and the proof is complete. □

At this point we have developed enough machinery to prove that residual gerbes (when they exist) are gerbes.

06QK Lemma 101.28.12. Let $\mathcal{Z}$ be a reduced, locally Noetherian algebraic stack such that $|\mathcal{Z}|$ is a singleton. Then $\mathcal{Z}$ is a gerbe over a reduced, locally Noetherian algebraic space Z with $|Z|$ a singleton.

Proof. By Properties of Stacks, Lemma 100.11.3 there exists a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field. Then $\mathcal{I}_{\mathcal{Z}} \times_{\mathcal{Z}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is representable by algebraic spaces and locally of finite type (as a base change of $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$, see Lemmas 101.5.1 and 101.17.3). Therefore it is locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.7. Of course it is also flat as k is a field. Hence we may apply Lemmas 101.25.4 and 101.27.11 to see that $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$ is flat and locally of finite presentation. We conclude that $\mathcal{Z}$

is a gerbe by Proposition 101.28.9. Let $\pi : \mathcal{Z} \to Z$ be a morphism to an algebraic space such that $\mathcal{Z}$ is a gerbe over Z. Then π is surjective, flat, and locally of finite presentation by Lemma 101.28.8. Hence $\mathrm{Spec}(k) \to Z$ is surjective, flat, and locally of finite presentation as a composition, see Properties of Stacks, Lemma 100.5.2 and Lemmas 101.25.2 and 101.27.2. Hence by Properties of Stacks, Lemma 100.11.3 we see that $|Z|$ is a singleton and that Z is locally Noetherian and reduced. $\square$

06R9 Lemma 101.28.13. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ then f is a universal homeomorphism.

Proof. By Lemma 101.28.3 the assumption on f is preserved under base change. Hence it suffices to show that the map $|\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism of topological spaces. Let k be a field and let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(k)$. By Stacks, Lemma 8.11.3 property (2)(a) there exists an fppf covering $\{T_i \to \mathrm{Spec}(k)\}$ and objects x_i of $\mathcal{X}$ over T_i with $f(x_i) \cong y|_{T_i}$. Choose an i such that $T_i \neq \emptyset$. Choose a morphism $\mathrm{Spec}(K) \to T_i$ for some field K. Then $k \subset K$ and $x_i|_K$ is an object of $\mathcal{X}$ lying over $y|_K$. Thus we see that $|\mathcal{Y}| \to |\mathcal{X}|$. is surjective. The map $|\mathcal{Y}| \to |\mathcal{X}|$ is also injective. Namely, if x, x' are objects of $\mathcal{X}$ over $\mathrm{Spec}(k)$ whose images $f(x), f(x')$ become isomorphic (over an extension) in $\mathcal{Y}$, then Stacks, Lemma 8.11.3 property (2)(b) guarantees the existence of an extension of k over which x and x' become isomorphic (details omitted). Hence $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous and bijective and it suffices to show that it is also open. This follows from Lemmas 101.28.8 and 101.27.15. $\square$

0DQL Lemma 101.28.14. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks such that $\mathcal{X}$ is a gerbe over $\mathcal{Y}$. If $\Delta_{\mathcal{X}}$ is quasi-compact, so is $\Delta_{\mathcal{Y}}$.

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times \mathcal{X} \arrow[d] \\ & \mathcal{Y} \arrow[r] & \mathcal{Y} \times \mathcal{Y} \end{tikzcd}$$

By Proposition 101.28.11 we find that the arrow on the top left is surjective. Since the composition of the top horizontal arrows is quasi-compact, we conclude that the top right arrow is quasi-compact by Lemma 101.7.6. The square is cartesian and the right vertical arrow is surjective, flat, and locally of finite presentation. Thus we conclude by Lemma 101.27.16. $\square$

The following lemma tells us that residual gerbes exist for all points on any algebraic stack which is a gerbe.

06RA Lemma 101.28.15. Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe then for every $x \in |\mathcal{X}|$ the residual gerbe of $\mathcal{X}$ at x exists.

Proof. Let $\pi : \mathcal{X} \to X$ be a morphism from $\mathcal{X}$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $Z_x \to X$ be the residual space of X at x, see Decent Spaces, Definition 68.13.6. Let $\mathcal{Z} = \mathcal{X} \times_X Z_x$. By Lemma 101.28.3 the algebraic stack $\mathcal{Z}$ is a gerbe over Z_x. Hence $|\mathcal{Z}| = |Z_x|$ (Lemma 101.28.13) is a singleton. Since $\mathcal{Z} \to Z_x$ is locally of finite presentation as a base change of π (see Lemmas 101.28.8 and 101.27.3) we see that $\mathcal{Z}$ is locally Noetherian, see Lemma 101.17.5. Thus the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and is equal to

$\mathcal{Z}_x = \mathcal{Z}_{red}$ the reduction of the algebraic stack $\mathcal{Z}$. Namely, we have seen above that $|\mathcal{Z}_{red}|$ is a singleton mapping to $x \in |\mathcal{X}|$, it is reduced by construction, and it is locally Noetherian (as the reduction of a locally Noetherian algebraic stack is locally Noetherian, details omitted). □

101.29. Stratification by gerbes

06RB The goal of this section is to show that many algebraic stacks $\mathcal{X}$ have a "stratification" by locally closed substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is a gerbe. This shows that in some sense gerbes are the building blocks out of which any algebraic stack is constructed. Note that by stratification we only mean that

$$|\mathcal{X}| = \bigcup\nolimits_i |\mathcal{X}_i|$$

is a stratification of the topological space associated to $\mathcal{X}$ and nothing more (in this section). Hence it is harmless to replace $\mathcal{X}$ by its reduction (see Properties of Stacks, Section 100.10) in order to study this stratification.

The following proposition tells us there is (almost always) a dense open substack of the reduction of $\mathcal{X}$

06RC **Proposition 101.29.1.** Let $\mathcal{X}$ be a reduced algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe.

Proof. According to Proposition 101.28.9 it is enough to find a dense open substack $\mathcal{U}$ such that $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation. Note that $\mathcal{I}_\mathcal{U} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{U}$, see Lemma 101.5.5.

Choose a presentation $\mathcal{X} = [U/R]$. Let $G \to U$ be the stabilizer group algebraic space of the groupoid R. By Lemma 101.5.7 we see that $G \to U$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ hence quasi-compact (by assumption) and locally of finite type (by Lemma 101.5.1). Let $W \subset U$ be the largest open (possibly empty) subscheme such that the restriction $G_W \to W$ is flat and locally of finite presentation (we omit the proof that W exists; hint: use that the properties are local). By Morphisms of Spaces, Proposition 67.32.1 we see that $W \subset U$ is dense. Note that $W \subset U$ is R-invariant by More on Groupoids in Spaces, Lemma 79.6.2. Hence W corresponds to an open substack $\mathcal{U} \subset \mathcal{X}$ by Properties of Stacks, Lemma 100.9.11. Since $|U| \to |\mathcal{X}|$ is open and $|W| \subset |U|$ is dense we conclude that $\mathcal{U}$ is dense in $\mathcal{X}$. Finally, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation because the base change by the surjective smooth morphism $W \to \mathcal{U}$ is the morphism $G_W \to W$ which is flat and locally of finite presentation by construction. See Lemmas 101.25.4 and 101.27.11. □

The above proposition immediately implies that any point has a residual gerbe on an algebraic stack with quasi-compact inertia, as we will show in Lemma 101.31.1. It turns out that there doesn't always exist a finite stratification by gerbes. Here is an example.

06RE **Example 101.29.2.** Let k be a field. Take $U = \text{Spec}(k[x_0, x_1, x_2, \ldots])$ and let $\mathbf{G}_m$ act by $t(x_0, x_1, x_2, \ldots) = (tx_0, t^p x_1, t^{p^2} x_2, \ldots)$ where p is a prime number. Let $\mathcal{X} = [U/\mathbf{G}_m]$. This is an algebraic stack. There is a stratification of $\mathcal{X}$ by strata

(1) $\mathcal{X}_0$ is where x_0 is not zero,
(2) $\mathcal{X}_1$ is where x_0 is zero but x_1 is not zero,

(3) $\mathcal{X}_2$ is where x_0, x_1 are zero, but x_2 is not zero,
(4) and so on, and
(5) $\mathcal{X}_\infty$ is where all the x_i are zero.

Each stratum is a gerbe over a scheme with group μ_{p^i} for $\mathcal{X}_i$ and $\mathbf{G}_m$ for $\mathcal{X}_\infty$. The strata are reduced locally closed substacks. There is no coarser stratification with the same properties.

Nonetheless, using transfinite induction we can use Proposition 101.29.1 find possibly infinite stratifications by gerbes...!

06RF Lemma 101.29.3. Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a well-ordered index set I and for every $i \in I$ a reduced locally closed substack $\mathcal{U}_i \subset \mathcal{X}$ such that

(1) each $\mathcal{U}_i$ is a gerbe,
(2) we have $|\mathcal{X}| = \bigcup_{i \in I} |\mathcal{U}_i|$,
(3) $T_i = |\mathcal{X}| \setminus \bigcup_{i' < i} |\mathcal{U}_{i'}|$ is closed in $|\mathcal{X}|$ for all $i \in I$, and
(4) $|\mathcal{U}_i|$ is open in T_i.

We can moreover arrange it so that either (a) $|\mathcal{U}_i| \subset T_i$ is dense, or (b) $\mathcal{U}_i$ is quasi-compact. In case (a), if we choose $\mathcal{U}_i$ as large as possible (see proof for details), then the stratification is canonical.

Proof. Let $T \subset |\mathcal{X}|$ be a nonempty closed subset. We are going to find (resp. choose) for every such T a reduced locally closed substack $\mathcal{U}(T) \subset \mathcal{X}$ with $|\mathcal{U}(T)| \subset T$ open dense (resp. nonempty quasi-compact). Namely, by Properties of Stacks, Lemma 100.10.1 there exists a unique reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 101.5.6. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 101.7.3. Therefore Proposition 101.29.1 implies there exists a dense maximal (see proof proposition) open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. In case (a) we set $\mathcal{U}(T) = \mathcal{U}$ (this is canonical) and in case (b) we simply choose a nonempty quasi-compact open $\mathcal{U}(T) \subset \mathcal{U}$, see Properties of Stacks, Lemma 100.4.9 (we can do this for all T simultaneously by the axiom of choice).

Using transfinite recursion we construct for every ordinal α a closed subset $T_\alpha \subset |\mathcal{X}|$. For $\alpha = 0$ we set $T_0 = |\mathcal{X}|$. Given T_α set

$$T_{\alpha+1} = T_\alpha \setminus |\mathcal{U}(T_\alpha)|.$$

If β is a limit ordinal we set

$$T_\beta = \bigcap\nolimits_{\alpha < \beta} T_\alpha.$$

We claim that $T_\alpha = \emptyset$ for all α large enough. Namely, assume that $T_\alpha \neq \emptyset$ for all α. Then we obtain an injective map from the class of ordinals into the set of subsets of $|\mathcal{X}|$ which is a contradiction.

The claim implies the lemma. Namely, let

$$I = \{\alpha \mid \mathcal{U}_\alpha \neq \emptyset\}.$$

This is a well-ordered set by the claim. For $i = \alpha \in I$ we set $\mathcal{U}_i = \mathcal{U}_\alpha$. So $\mathcal{U}_i$ is a reduced locally closed substack and a gerbe, i.e., (1) holds. By construction $T_i = T\alpha$ if $i = \alpha \in I$, hence (3) holds. Also, (4) and (a) or (b) hold by our choice of $\mathcal{U}(T)$ as well. Finally, to see (2) let $x \in |\mathcal{X}|$. There exists a smallest ordinal β with $x \notin T_\beta$ (because the ordinals are well-ordered). In this case β has to be a successor ordinal by the definition of T_β for limit ordinals. Hence $\beta = \alpha + 1$ and $x \in |\mathcal{U}(T_\alpha)|$ and we win. $\square$

06RG Remark 101.29.4. We can wonder about the order type of the canonical stratifications which occur as output of the stratifications of type (a) constructed in Lemma 101.29.3. A natural guess is that the well-ordered set I has cardinality at most $\aleph_0$. We have no idea if this is true or false. If you do please email stacks.project@gmail.com.

101.30. The topological space of an algebraic stack

0DQM In this section we apply the previous results to the topological space $|\mathcal{X}|$ associated to an algebraic stack.

0DQN Lemma 101.30.1. Let $\mathcal{X}$ be a quasi-compact algebraic stack whose diagonal Δ is quasi-compact. Then $|\mathcal{X}|$ is a spectral topological space.

Proof. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$, see Properties of Stacks, Lemma 100.6.2. Then $|U| \to |\mathcal{X}|$ is continuous, open, and surjective, see Properties of Stacks, Lemma 100.4.7. Hence the quasi-compact opens of $|\mathcal{X}|$ form a basis for the topology. For $W_1, W_2 \subset |\mathcal{X}|$ quasi-compact open, we may choose a quasi-compact opens V_1, V_2 of U mapping to W_1 and W_2. Since Δ is quasi-compact, we see that

$$V_1 \times_{\mathcal{X}} V_2 = (V_1 \times V_2) \times_{\mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

is quasi-compact. Then image of $|V_1 \times_{\mathcal{X}} V_2|$ in $|\mathcal{X}|$ is $W_1 \cap W_2$ by Properties of Stacks, Lemma 100.4.3. Thus $W_1 \cap W_2$ is quasi-compact. To finish the proof, it suffices to show that $|\mathcal{X}|$ is sober, see Topology, Definition 5.23.1.

Let $T \subset |\mathcal{X}|$ be an irreducible closed subset. We have to show T has a unique generic point. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced closed substack corresponding to T, see Properties of Stacks, Definition 100.10.4. Since $\mathcal{Z} \to \mathcal{X}$ is a closed immersion, we see that $\Delta_{\mathcal{Z}}$ is quasi-compact: first show that $\mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact as the composition of $\mathcal{Z} \to \mathcal{X}$ with Δ, then write $\mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ as the composition of $\Delta_{\mathcal{Z}}$ and $\mathcal{Z} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ and use Lemma 101.7.7 and the fact that $\mathcal{Z} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ is separated. Thus we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is quasi-compact, Δ is quasi-compact, $\mathcal{X}$ is reduced, and $|\mathcal{X}|$ irreducible. We have to show $|\mathcal{X}|$ has a unique generic point. Since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is a base change of Δ, we see that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is quasi-compact (Lemma 101.7.3). Thus there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe by Proposition 101.29.1. In other words, $|\mathcal{U}| \subset |\mathcal{X}|$ is open dense. Thus we may assume that $\mathcal{X}$ is a gerbe. Say $\mathcal{X} \to X$ turns $\mathcal{X}$ into a gerbe over the algebraic space X. Then $|\mathcal{X}| \cong |X|$ by Lemma 101.28.13. In particular, X is quasi-compact. By Lemma 101.28.14 we see that X has quasi-compact diagonal, i.e., X is a quasi-separated algebraic space. Then $|X|$ is spectral by Properties of Spaces, Lemma 66.15.2 which implies what we want is true. $\square$

0DQP Lemma 101.30.2. Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then $|\mathcal{X}|$ is a spectral topological space.

Proof. This is a special case of Lemma 101.30.1. $\square$

0DQQ **Lemma 101.30.3.** Let $\mathcal{X}$ be an algebraic stack whose diagonal is quasi-compact (for example if $\mathcal{X}$ is quasi-separated). Then there is an open covering $|\mathcal{X}| = \bigcup U_i$ with U_i spectral. In particular $|\mathcal{X}|$ is a sober topological space.

Proof. Immediate consequence of Lemma 101.30.1. $\square$

101.31. Existence of residual gerbes

06UH The definition of a residual gerbe of a point on an algebraic stack is Properties of Stacks, Definition 100.11.8. We have already shown that residual gerbes exist for finite type points (Lemma 101.18.7) and for any point of a gerbe (Lemma 101.28.15). In this section we prove that residual gerbes exist on many algebraic stacks. First, here is the promised application of Proposition 101.29.1.

06RD **Lemma 101.31.1.** Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 100.10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 101.5.6. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 101.7.3. Therefore Proposition 101.29.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$ because $\{x\} \subset T$ is a dense subset too. Hence a residual gerbe $\mathcal{Z}_x \subset \mathcal{U}$ of $\mathcal{U}$ at x exists by Lemma 101.28.15. It is immediate from the definitions that $\mathcal{Z}_x \to \mathcal{X}$ is a residual gerbe of $\mathcal{X}$ at x. $\square$

If the stack is quasi-DM then residual gerbes exist too. In particular, residual gerbes always exist for Deligne-Mumford stacks.

06UI **Lemma 101.31.2.** Let $\mathcal{X}$ be a quasi-DM algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.

Proof. Choose a scheme U and a surjective, flat, locally finite presented, and locally quasi-finite morphism $U \to \mathcal{X}$, see Theorem 101.21.3. Set $R = U \times_\mathcal{X} U$. The projections $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite as base changes of the morphism $U \to \mathcal{X}$. There is a canonical morphism $[U/R] \to \mathcal{X}$ (see Algebraic Stacks, Lemma 94.16.1) which is an equivalence because $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Remark 94.16.3. Thus we may assume that $\mathcal{X} = [U/R]$ where (U, R, s, t, c) is a groupoid in algebraic spaces such that $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite. Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \mathrm{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_\mathcal{X} U' = R \times_{(U \times U)} (U' \times U')$$

Because $U' \to U$ is a monomorphism we see that both projections $s', t' : R' \to U'$ factor as a monomorphism followed by a locally quasi-finite morphism. Hence, as U' is a disjoint union of spectra of fields, using Spaces over Fields, Lemma 72.10.9 we conclude that the morphisms $s', t' : R' \to U'$ are locally quasi-finite. Again since U' is a disjoint union of spectra of fields, the morphisms s', t' are also flat. Finally, s', t' locally quasi-finite implies s', t' locally of finite type, hence s', t' locally of finite presentation (because U' is a disjoint union of spectra of fields in particular locally Noetherian, so that Morphisms of Spaces, Lemma 67.28.7 applies). Hence $\mathcal{Z} = [U'/R']$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. As R' is the restriction of R by $U' \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 78.25.1 and Properties of Stacks, Lemma 100.8.4. Since $\mathcal{Z} \to \mathcal{X}$ is a monomorphism we see that $|\mathcal{Z}| \to |\mathcal{X}|$ is injective, see Properties of Stacks, Lemma 100.8.5. By Properties of Stacks, Lemma 100.4.3 we see that

$$|U'| = |\mathcal{Z} \times_{\mathcal{X}} U'| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{X}|} |U'|$$

is surjective which implies (by our choice of U') that $|\mathcal{Z}| \to |\mathcal{X}|$ has image $\{x\}$. We conclude that $|\mathcal{Z}|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., $\mathcal{Z}$ is reduced and locally Noetherian. This means that the essential image of $\mathcal{Z} \to \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x, see Properties of Stacks, Lemma 100.11.12. $\square$

0H22 **Lemma 101.31.3.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.

Proof. Choose an affine scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of the open continuous map $|U| \to |\mathcal{X}|$. We may and do replace $\mathcal{X}$ with the open substack corresponding to the image of $|U| \to |\mathcal{X}|$, see Properties of Stacks, Lemma 100.9.12. Thus we may assume $\mathcal{X} = [U/R]$ for a smooth groupoid (U, R, s, t, c) in algebraic spaces where U is a Noetherian affine scheme, see Algebraic Stacks, Section 94.16.

Let $E \subset |U|$ be the inverse image of $\{x\} \subset |\mathcal{X}|$. Of course $E \neq \emptyset$. Since $|U|$ is a Noetherian topological space, we can choose an element $u \in E$ such that $\overline{\{u\}} \cap E = \{u\}$. As usual, we think of $u = \mathrm{Spec}(\kappa(u))$ as the spectrum of its residue field. Let us write

$$F = u \times_{U,t} R = u \times_{\mathcal{X}} U \quad \text{and} \quad R' = (u \times u) \times_{(U \times U),(t,s)} R = u \times_{\mathcal{X}} u$$

Furthermore, denote $Z = \overline{\{u\}} \subset U$ with the reduced induced scheme structure. Denote $p : F \to U$ the morphism induced by the second projection (using $s : R \to U$ in the first fibre product description of F). Then E is the set theoretic image of p. The morphism $R' \to F$ is a monomorphism which factors through the inverse image $p^{-1}(Z)$ of Z. This inverse image $p^{-1}(Z) \subset F$ is a closed subscheme and the restriction $p|_{p^{-1}(Z)} : p^{-1}(Z) \to Z$ has image set theoretically contained in $\{u\} \subset Z$ by our careful choice of $u \in E$ above. Since $u = \lim W$ where the limit is over the nonempty affine open subschemes of the irreducible reduced scheme Z, we conclude that the morphism $p|_{p^{-1}(Z)} : p^{-1}(Z) \to Z$ factors through the morphism $u \to Z$. Clearly this implies that $R' = p^{-1}(Z)$. In particular the morphism $t' : R' \to u$ is locally of finite presentation as the composition of the closed immersion $p^{-1}(Z) \to F$ of locally Noetherian algebraic spaces with the smooth morphism $\mathrm{pr}_1 : F \to u$; use Morphisms of Spaces, Lemmas 67.23.5, 67.28.12, and 67.28.2. Hence the restriction (u, R', s', t', c') of (U, R, s, t, c) by $u \to U$ is a groupoid in algebraic spaces where s' and t' are flat and locally of finite presentation. Therefore $\mathcal{Z} = [u/R']$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. As R' is the restriction of R by $u \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 78.25.1 and Properties of Stacks, Lemma 100.8.4. Then $\mathcal{Z}$ is (isomorphic to) the residual gerbe by the material in Properties of Stacks, Section 100.11. $\square$

101.32. Étale local structure

0DU0 In this section we start discussing what we can say about the étale local structure of an algebraic stack.

0DU1 **Lemma 101.32.1.** Let Y be an algebraic space. Let (U, R, s, t, c) be a groupoid in algebraic spaces over Y. Assume $U \to Y$ is flat and locally of finite presentation and $R \to U \times_Y U$ an open immersion. Then $X = [U/R] = U/R$ is an algebraic space and $X \to Y$ is étale.

Proof. The quotient stack $[U/R]$ is an algebraic stacks by Criteria for Representability, Theorem 97.17.2. On the other hand, since $R \to U \times U$ is a monomorphism, it is an algebraic space (by our abuse of language and Algebraic Stacks, Proposition 94.13.3) and of course it is equal to the algebraic space U/R (of Bootstrap, Theorem 80.10.1). Since $U \to X$ is surjective, flat, and locally of finite presenation (Bootstrap, Lemma 80.11.6) we conclude that $X \to Y$ is flat and locally of finite presentation by Morphisms of Spaces, Lemma 67.31.5 and Descent on Spaces, Lemma 74.8.2. Finally, consider the cartesian diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \times_Y U \arrow[d] \\ X \arrow[r] & X \times_Y X \end{tikzcd}$$

Since the right vertical arrow is surjective, flat, and locally of finite presentation (small detail omitted), we find that $X \to X \times_Y X$ is an open immersion as the top horizonal arrow has this property by assumption (use Properties of Stacks, Lemma 100.3.3). Thus $X \to Y$ is unramified by Morphisms of Spaces, Lemma 67.38.9. We conclude by Morphisms of Spaces, Lemma 67.39.12. $\square$

0DU2 **Lemma 101.32.2.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $P \subset R$ be an open subspace such that $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over S. Then

$$[U/P] \longrightarrow [U/R]$$

is a morphism of algebraic stacks which is representable by algebraic spaces, surjective, and étale.

Proof. Since $P \subset R$ is open, we see that $s|_P$ and $t|_P$ are flat and locally of finite presentation. Thus $[U/R]$ and $[U/P]$ are algebraic stacks by Criteria for Representability, Theorem 97.17.2. To see that the morphism is representable by algebraic spaces, it suffices to show that $[U/P] \to [U/R]$ is faithful on fibre categories, see Algebraic Stacks, Lemma 94.15.2. This follows immediately from the fact that $P \to R$ is a monomorphism and the explicit description of quotient stacks given in Groupoids in Spaces, Lemma 78.24.1. Having said this, we know what it means for $[U/P] \to [U/R]$ to be surjective and étale by Algebraic Stacks, Definition 94.10.1. Surjectivity follows for example from Criteria for Representability, Lemma 97.7.3 and the description of objects of quotient stacks (see lemma cited above) over spectra of fields. It remains to prove that our morphism is étale.

To do this it suffices to show that $U \times_{[U/R]} [U/P] \to U$ is étale, see Properties of Stacks, Lemma 100.3.3. By Groupoids in Spaces, Lemma 78.21.2 the fibre product is equal to $[R/P \times_{s,U,t} R]$ with morphism to U induced by $s : R \to U$. The maps $s', t' : P \times_{s,U,t} R \to R$ are given by $s' : (p, r) \mapsto r$ and $t' : (p, r) \mapsto c(p, r)$. Since $P \subset R$ is open we conclude that $(t', s') : P \times_{s,U,t} R \to R \times_{s,U,s} R$ is an open immersion. Thus we may apply Lemma 101.32.1 to conclude. $\square$

0DU3 Lemma 101.32.3. Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Then there exists a morphism of algebraic stacks

$$\mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

(1) there exists a point $u \in |\mathcal{U}|$ mapping to x,
(2) $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,
(3) $\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.

Proof. (The parenthetical statement follows from the equivalences in Lemma 101.6.1). Choose an affine scheme U and a flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X}$ such that x is the image of some point $u \in U$. This is possible by Theorem 101.21.3 and the assumption that $\mathcal{X}$ is quasi-DM. Let (U, R, s, t, c) be the groupoid in algebraic spaces we obtain by setting $R = U \times_\mathcal{X} U$, see Algebraic Stacks, Lemma 94.16.1. Let $\mathcal{X}' \subset \mathcal{X}$ be the open substack corresponding to the open image of $|U| \to |\mathcal{X}|$ (Properties of Stacks, Lemmas 100.4.7 and 100.9.12). Clearly, we may replace $\mathcal{X}$ by the open substack $\mathcal{X}'$. Thus we may assume $U \to \mathcal{X}$ is surjective and then Algebraic Stacks, Remark 94.16.3 gives $\mathcal{X} = [U/R]$. Observe that $s, t : R \to U$ are flat, locally of finite presentation, and locally quasi-finite. Since $R = U \times U \times_{(\mathcal{X} \times \mathcal{X})} \mathcal{X}$ and since the diagonal of $\mathcal{X}$ is separated, we find that R is separated. Hence $s, t : R \to U$ are separated. It follows that R is a scheme by Morphisms of Spaces, Proposition 67.50.2 applied to $s : R \to U$.

Above we have verified all the assumptions of More on Groupoids in Spaces, Lemma 79.15.13 are satisfied for (U, R, s, t, c) and u. Hence we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is quasi-split over u. Note that $R' = U' \times_\mathcal{X} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a quasi-splitting over u (the point u is irrelevant from now on as can be seen from the footnote in More on Groupoids in Spaces, Definition 79.15.1). Let $P \subset R$ be a quasi-splitting of R over u. Apply Lemma 101.32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

has all the desired properties. $\square$

0DU4 Lemma 101.32.4. Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Assume the automorphism group of $\mathcal{X}$ at x is finite (Remark 101.19.3). Then there exists a morphism of algebraic stacks

$$g : \mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

(1) there exists a point $u \in |\mathcal{U}|$ mapping to x and g induces an isomorphism between automorphism groups at u and x (Remark 101.19.5),
(2) $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,
(3) $\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.

Proof. Observe that G_x is a group scheme by Lemma 101.19.1. The first part of the proof is exactly the same as the first part of the proof of Lemma 101.32.3. Thus we may assume $\mathcal{X} = [U/R]$ where (U, R, s, t, c) and $u \in U$ mapping to x satisfy all the assumptions of More on Groupoids in Spaces, Lemma 79.15.13. Our assumption on G_x implies that G_u is finite over u. Hence all the assumptions of More on Groupoids in Spaces, Lemma 79.15.12 are satisfied. Hence we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is split over u. Note that $R' = U' \times_\mathcal{X} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a splitting over u. Let $P \subset R$ be a splitting of R over u. Apply Lemma 101.32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

is representable by algebraic spaces and étale. By construction G_u is contained in P, hence this morphism defines an isomorphism on automorphism groups at u as desired. □

0DU5 Lemma 101.32.5. Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Assume x can be represented by a quasi-compact morphism $\mathrm{Spec}(k) \to \mathcal{X}$. Then there exists a morphism of algebraic stacks

$$g : \mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

(1) there exists a point $u \in |\mathcal{U}|$ mapping to x and g induces an isomorphism between the residual gerbes at u and x,
(2) $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,
(3) $\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.

Proof. The first part of the proof is exactly the same as the first part of the proof of Lemma 101.32.3. Thus we may assume $\mathcal{X} = [U/R]$ where (U, R, s, t, c) and $u \in U$ mapping to x satisfy all the assumptions of More on Groupoids in Spaces, Lemma 79.15.13. Observe that $u = \mathrm{Spec}(\kappa(u)) \to \mathcal{X}$ is quasi-compact, see Properties of Stacks, Lemma 100.14.1. Consider the cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & U \arrow[d] \\ u \arrow[r, "u"] & \mathcal{X} \end{tikzcd}$$

Since U is an affine scheme and $F \to U$ is quasi-compact, we see that F is quasi-compact. Since $U \to \mathcal{X}$ is locally quasi-finite, we see that $F \to u$ is locally quasi-finite. Hence $F \to u$ is quasi-finite and F is an affine scheme whose underlying topological space is finite discrete (Spaces over Fields, Lemma 72.10.8). Observe that we have a monomorphism $u \times_\mathcal{X} u \to F$. In particular the set $\{r \in R : s(r) =$

$u, t(r) = u\}$ which is the image of $|u \times_{\mathcal{X}} u| \to |R|$ is finite. we conclude that all the assumptions of More on Groupoids in Spaces, Lemma 79.15.11 hold.

Thus we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is strongly split over u'. Note that $R' = U' \times_{\mathcal{X}} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a strong splitting over u. Let $P \subset R$ be a strong splitting of R over u. Apply Lemma 101.32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

is representable by algebraic spaces and étale. Since $P \subset R$ is open and contains $\{r \in R : s(r) = u, t(r) = u\}$ by construction we see that $u \times_{\mathcal{U}} u \to u \times_{\mathcal{X}} u$ is an isomorphism. The statement on residual gerbes then follows from Properties of Stacks, Lemma 100.11.14 (we observe that the residual gerbes in question exist by Lemma 101.31.2). $\square$

101.33. Smooth morphisms

075T The property "being smooth" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.37.3 and Descent on Spaces, Lemma 74.11.26. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be smooth as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

075U **Definition 101.33.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *smooth* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} =$ smooth.

075V **Lemma 101.33.2.** *The composition of smooth morphisms is smooth.*

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.37.2. $\square$

075W **Lemma 101.33.3.** *A base change of a smooth morphism is smooth.*

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.37.3. $\square$

0DN7 **Lemma 101.33.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is smooth, then f is smooth.*

Proof. The property "smooth" satisfies the conditions of Lemma 101.27.10. Smooth local on the source-and-target we have seen in the introduction to this section and fppf local on the target is Descent on Spaces, Lemma 74.11.26. $\square$

0DNP **Lemma 101.33.5.** *A smooth morphism of algebraic stacks is locally of finite presentation.*

Proof. Omitted. $\square$

0DZR **Lemma 101.33.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. There is a maximal open substack $\mathcal{U} \subset \mathcal{X}$ such that $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$ is smooth. Moreover, formation of this open commutes with*

(1) *precomposing by smooth morphisms,*

(2) base change by morphisms which are flat and locally of finite presentation,
(3) base change by flat morphisms provided f is locally of finite presentation.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces, the vertical arrows are smooth, and $a : U \to \mathcal{X}$ surjective. There is a maximal open subspace $U' \subset U$ such that $h_{U'} : U' \to V$ is smooth, see Morphisms of Spaces, Lemma 67.37.9. Let $\mathcal{U} \subset \mathcal{X}$ be the open substack corresponding to the image of $|U'| \to |\mathcal{X}|$ (Properties of Stacks, Lemmas 100.4.7 and 100.9.12). By the equivalence in Lemma 101.16.1 we find that $f|_\mathcal{U} : \mathcal{U} \to \mathcal{Y}$ is smooth and that $\mathcal{U}$ is the largest open substack with this property.

Assertion (1) follows from the fact that being smooth is smooth local on the source (this property was used to even define smooth morphisms of algebraic stacks). Assertions (2) and (3) follow from the case of algebraic spaces, see Morphisms of Spaces, Lemma 67.37.9. ☐

0DLS Lemma 101.33.7. Let $X \to Y$ be a smooth morphism of algebraic spaces. Let G be a group algebraic space over Y which is flat and locally of finite presentation over Y. Let G act on X over Y. Then the quotient stack $[X/G]$ is smooth over Y.

This holds even if G is not smooth over S!

Proof. The quotient $[X/G]$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. The smoothness of $[X/G]$ over Y follows from the fact that smoothness descends under fppf coverings: Choose a surjective smooth morphism $U \to [X/G]$ where U is a scheme. Smoothness of $[X/G]$ over Y is equivalent to smoothness of U over Y. Observe that $U \times_{[X/G]} X$ is smooth over X and hence smooth over Y (because compositions of smooth morphisms are smooth). On the other hand, $U \times_{[X/G]} X \to U$ is locally of finite presentation, flat, and surjective (because it is the base change of $X \to [X/G]$ which has those properties for example by Criteria for Representability, Lemma 97.17.1). Therefore we may apply Descent on Spaces, Lemma 74.8.4. ☐

0DN8 Lemma 101.33.8. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective and smooth.

Proof. We have seen surjectivity in Lemma 101.28.8. By Lemma 101.33.4 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 101.28.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U with $G \to U$ flat and locally of finite presentation. Then we win by Lemma 101.33.7. ☐

101.34. Types of morphisms étale-smooth local on source-and-target

0CIE Given a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target, see Descent on Spaces, Definition 74.21.1 we may use it to define a corresponding property of DM morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

0CIF **Lemma 101.34.1.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks. Consider commutative diagrams

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces, $V \to \mathcal{Y}$ is smooth, and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is étale. The following are equivalent

(1) for any diagram as above the morphism h has property $\mathcal{P}$, and
(2) for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 100.3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. As f is DM there exists a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$, see Lemma 101.21.7. Thus we see that (2) holds. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "h'"'] \arrow[d] & V' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{tikzcd} U \arrow[d, "h"] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d, "{(h,h')}"] & U' \arrow[d, "h'"] \\ V & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that the squares

$$\begin{tikzcd} U \arrow[d] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[d] \\ V \times_{\mathcal{Y}} \mathcal{X} & V \times_{\mathcal{Y}} U' \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} U \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \times_{\mathcal{Y}} V' \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V' \end{tikzcd}$$

are cartesian, hence the vertical arrows are étale by our assumptions on U', V', a', b', h' and U, V, a, b, h. Since $\mathcal{P}$ is smooth local on the target by Descent on Spaces, Lemma 74.21.2 part (2) we see that the base change $t : U \times_\mathcal{Y} V' \to V \times_\mathcal{Y} V'$ of h has $\mathcal{P}$. Since $\mathcal{P}$ is étale local on the source by Descent on Spaces, Lemma 74.21.2 part (1) and $s : U \times_\mathcal{X} U' \to U \times_\mathcal{Y} V'$ is étale, we see the morphism $(h, h') = t \circ s$ has $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} U \times_\mathcal{X} U' \arrow[r, "{(h,h')}"'] \arrow[d] & V \times_\mathcal{Y} V' \arrow[d] \\ U' \arrow[r, "h'"] & V' \end{tikzcd}$$

The left vertical arrow is surjective, the right vertical arrow is smooth, and the induced morphism

$$U \times_\mathcal{X} U' \longrightarrow U' \times_{V'} (V \times_\mathcal{Y} V') = V \times_\mathcal{Y} U'$$

is étale as seen above. Hence by Descent on Spaces, Definition 74.21.1 part (3) we conclude that h' has $\mathcal{P}$. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 74.21.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_\mathcal{Y} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} Z \times_\mathcal{Y} U \arrow[r] \arrow[d] & Z \times_\mathcal{Y} V \arrow[d] \\ Z \times_\mathcal{Y} \mathcal{X} \arrow[r] & Z \end{tikzcd}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is surjective, the induced morphism

$$Z \times_\mathcal{Y} U \longrightarrow (Z \times_\mathcal{Y} \mathcal{X}) \times_Z (Z \times_\mathcal{Y} V)$$

is étale, and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 74.21.3 kicks in and shows that $Z \times_\mathcal{Y} \mathcal{X} \to Z$ has property $\mathcal{P}$. $\square$

0CIG Definition 101.34.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target. We say a DM morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks has property $\mathcal{P}$ if the equivalent conditions of Lemma 101.16.1 hold.

0CIH Remark 101.34.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target and stable under composition. Then the property of DM morphisms of algebraic stacks defined in Definition 101.34.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be DM morphisms of algebraic stacks having property $\mathcal{P}$. By Lemma 101.4.10 the composition $g \circ f$ is DM. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective étale morphism $V \to \mathcal{Y} \times_\mathcal{Z} W$ (Lemma 101.21.7). Choose an algebraic space U and a surjective étale morphism $U \to \mathcal{X} \times_\mathcal{Y} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

0CII Remark 101.34.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target and stable under base change. Then the property of DM morphisms of algebraic stacks defined in Definition 101.34.2 is stable under arbitrary base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks and $g : \mathcal{Y}' \to \mathcal{Y}$ be a morphism of algebraic stacks and assume f has property $\mathcal{P}$. Then the base change $\mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \to \mathcal{Y}'$ is a DM morphism by Lemma 101.4.4. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective étale morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ (Lemma 101.21.7). Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_\mathcal{Y} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{tikzcd} V' \times_V U \arrow[r] \arrow[d] & \mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

we see that the left top horizontal arrow is surjective and

$$V' \times_V U \to V' \times_\mathcal{Y} (\mathcal{Y}' \times_{\mathcal{Y}'} \mathcal{X}) = V' \times_V (\mathcal{X} \times_\mathcal{Y} V)$$

is étale as a base change of $U \to \mathcal{X} \times_\mathcal{Y} V$, whence by definition we see that the projection $\mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

0CIJ Remark 101.34.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are étale-smooth local on the source-and-target. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 101.34.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

101.35. Étale morphisms

0CIK An étale morphism of algebraic stacks should not be defined as a smooth morphism of relative dimension 0. Namely, the morphism

$$[\mathbf{A}^1_k/\mathbf{G}_{m,k}] \longrightarrow \mathrm{Spec}(k)$$

is smooth of relative dimension 0 for any choice of action of the group scheme $\mathbf{G}_{m,k}$ on $\mathbf{A}^1_k$. This does not correspond to our usual idea that étale morphisms should identify tangent spaces. The example above isn't quasi-finite, but the morphism

$$\mathcal{X} = [\mathrm{Spec}(k)/\mu_{p,k}] \longrightarrow \mathrm{Spec}(k)$$

is smooth and quasi-finite (Section 101.23). However, if the characteristic of k is $p > 0$, then we see that the representable morphism $\mathrm{Spec}(k) \to \mathcal{X}$ isn't étale as the base change $\mu_{p,k} = \mathrm{Spec}(k) \times_\mathcal{X} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a morphism from a nonreduced scheme to the spectrum of a field. Thus if we define an étale morphism as smooth and locally quasi-finite, then the analogue of Morphisms of Spaces, Lemma 67.39.11 would fail.

Instead, our approach will be to start with the requirements that "étaleness" should be a property preserved under base change and that if $\mathcal{X} \to X$ is an étale morphism from an algebraic stack to a scheme, then $\mathcal{X}$ should be Deligne-Mumford. In other words, we will require étale morphisms to be DM and we will use the material in Section 101.34 to define étale morphisms of algebraic stacks.

In Lemma 101.36.10 we will characterize étale morphisms of algebraic stacks as morphisms which are (a) locally of finite presentation, (b) flat, and (c) have étale diagonal.

The property "étale" of morphisms of algebraic spaces is étale-smooth local on the source-and-target, see Descent on Spaces, Remark 74.21.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.39.4 and Descent on Spaces, Lemma 74.11.28. Hence, by Lemma 101.34.1 above, we may define what it means for a morphism of algebraic spaces to be étale as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces because such a morphism is automatically DM by Lemma 101.4.3.

0CIL Definition 101.35.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is étale if f is DM and the equivalent conditions of Lemma 101.34.1 hold with $\mathcal{P} = \textit{étale}$.

We will use without further mention that this agrees with the already existing notion of étale morphisms in case f is representable by algebraic spaces or if $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces.

0CIM Lemma 101.35.2. The composition of étale morphisms is étale.

Proof. Combine Remark 101.34.3 with Morphisms of Spaces, Lemma 67.39.3. □

0CIN Lemma 101.35.3. A base change of an étale morphism is étale.

Proof. Combine Remark 101.34.4 with Morphisms of Spaces, Lemma 67.39.4. □

0CIP Lemma 101.35.4. An open immersion is étale.

Proof. Let $j : \mathcal{U} \to \mathcal{X}$ be an open immersion of algebraic stacks. Since j is representable, it is DM by Lemma 101.4.3. On the other hand, if $X \to \mathcal{X}$ is a smooth and surjective morphism where X is a scheme, then $U = \mathcal{U} \times_{\mathcal{X}} X$ is an open subscheme of X. Hence $U \to X$ is étale (Morphisms, Lemma 29.36.9) and we conclude that j is étale from the definition. □

0CIQ Lemma 101.35.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

1. f is étale,
2. f is DM and for any morphism $V \to \mathcal{Y}$ where V is an algebraic space and any étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ where U is an algebraic space, the morphism $U \to V$ is étale,
3. there exists some surjective, locally of finite presentation, and flat morphism $W \to \mathcal{Y}$ where W is an algebraic space and some surjective étale morphism $T \to W \times_{\mathcal{Y}} \mathcal{X}$ where T is an algebraic space such that the morphism $T \to W$ is étale.

Proof. Assume (1). Then f is DM and since being étale is preserved by base change, we see that (2) holds.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective étale morphism $U \to V \times_\mathcal{Y} \mathcal{X}$ (Lemma 101.21.7). Thus we see that (3) holds.

Assume $W \to \mathcal{Y}$ and $T \to W \times_\mathcal{Y} \mathcal{X}$ are as in (3). We first check f is DM. Namely, it suffices to check $W \times_\mathcal{Y} \mathcal{X} \to W$ is DM, see Lemma 101.4.5. By Lemma 101.4.12 it suffices to check $W \times_\mathcal{Y} \mathcal{X}$ is DM. This follows from the existence of $T \to W \times_\mathcal{Y} \mathcal{X}$ by (the easy direction of) Theorem 101.21.6.

Assume f is DM and $W \to \mathcal{Y}$ and $T \to W \times_\mathcal{Y} \mathcal{X}$ are as in (3). Let V be an algebraic space, let $V \to \mathcal{Y}$ be surjective smooth, let U be an algebraic space, and let $U \to V \times_\mathcal{Y} \mathcal{X}$ is surjective and étale (Lemma 101.21.7). We have to check that $U \to V$ is étale. It suffices to prove $U \times_\mathcal{Y} W \to V \times_\mathcal{Y} W$ is étale by Descent on Spaces, Lemma 74.11.28. We may replace $\mathcal{X}, \mathcal{Y}, W, T, U, V$ by $\mathcal{X} \times_\mathcal{Y} W, W, W, T, U \times_\mathcal{Y} W, V \times_\mathcal{Y} W$ (small detail omitted). Thus we may assume that $Y = \mathcal{Y}$ is an algebraic space, there exists an algebraic space T and a surjective étale morphism $T \to \mathcal{X}$ such that $T \to Y$ is étale, and U and V are as before. In this case we know that

$$U \to V \text{ is étale} \Leftrightarrow \mathcal{X} \to Y \text{ is étale} \Leftrightarrow T \to Y \text{ is étale}$$

by the equivalence of properties (1) and (2) of Lemma 101.34.1 and Definition 101.35.1. This finishes the proof. □

0CIR Lemma 101.35.6. Let $\mathcal{X}, \mathcal{Y}$ be algebraic stacks étale over an algebraic stack $\mathcal{Z}$. Any morphism $\mathcal{X} \to \mathcal{Y}$ over $\mathcal{Z}$ is étale.

Proof. The morphism $\mathcal{X} \to \mathcal{Y}$ is DM by Lemma 101.4.12. Let $W \to \mathcal{Z}$ be a surjective smooth morphism whose source is an algebraic space. Let $V \to \mathcal{Y} \times_\mathcal{Z} W$ be a surjective étale morphism whose source is an algebraic space (Lemma 101.21.7). Let $U \to \mathcal{X} \times_\mathcal{Y} V$ be a surjective étale morphism whose source is an algebraic space (Lemma 101.21.7). Then

$$U \longrightarrow \mathcal{X} \times_\mathcal{Z} W$$

is surjective étale as the composition of $U \to \mathcal{X} \times_\mathcal{Y} V$ and the base change of $V \to \mathcal{Y} \times_\mathcal{Z} W$ by $\mathcal{X} \times_\mathcal{Z} W \to \mathcal{Y} \times_\mathcal{Z} W$. Hence it suffices to show that $U \to W$ is étale. Since $U \to W$ and $V \to W$ are étale because $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are étale, this follows from the version of the lemma for algebraic spaces, namely Morphisms of Spaces, Lemma 67.39.11. □

101.36. Unramified morphisms

0CIS For a justification of our choice of definition of unramified morphisms we refer the reader to the discussion in the section on étale morphisms Section 101.35.

In Lemma 101.36.9 we will characterize unramified morphisms of algebraic stacks as morphisms which are locally of finite type and have étale diagonal.

The property "unramified" of morphisms of algebraic spaces is étale-smooth local on the source-and-target, see Descent on Spaces, Remark 74.21.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.38.4 and Descent on Spaces, Lemma 74.11.27. Hence, by Lemma 101.34.1 above, we may define what it means for a morphism of algebraic spaces to be unramified as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces because such a morphism is automatically DM by Lemma 101.4.3.

0CIT Definition 101.36.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is unramified if f is DM and the equivalent conditions of Lemma 101.34.1 hold with $\mathcal{P}$ ="unramified".

We will use without further mention that this agrees with the already existing notion of unramified morphisms in case f is representable by algebraic spaces or if $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces.

0CIU Lemma 101.36.2. The composition of unramified morphisms is unramified.

Proof. Combine Remark 101.34.3 with Morphisms of Spaces, Lemma 67.38.3. $\square$

0CIV Lemma 101.36.3. A base change of an unramified morphism is unramified.

Proof. Combine Remark 101.34.4 with Morphisms of Spaces, Lemma 67.38.4. $\square$

0CIW Lemma 101.36.4. An étale morphism is unramified.

Proof. Follows from Remark 101.34.5 and Morphisms of Spaces, Lemma 67.39.10. $\square$

0CIX Lemma 101.36.5. An immersion is unramified.

Proof. Let $j : \mathcal{Z} \to \mathcal{X}$ be an immersion of algebraic stacks. Since j is representable, it is DM by Lemma 101.4.3. On the other hand, if $X \to \mathcal{X}$ is a smooth and surjective morphism where X is a scheme, then $Z = \mathcal{Z} \times_{\mathcal{X}} X$ is a locally closed subscheme of X. Hence $Z \to X$ is unramified (Morphisms, Lemmas 29.35.7 and 29.35.8) and we conclude that j is unramified from the definition. $\square$

0CIY Lemma 101.36.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) f is unramified,
(2) f is DM and for any morphism $V \to \mathcal{Y}$ where V is an algebraic space and any étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ where U is an algebraic space, the morphism $U \to V$ is unramified,
(3) there exists some surjective, locally of finite presentation, and flat morphism $W \to \mathcal{Y}$ where W is an algebraic space and some surjective étale morphism $T \to W \times_{\mathcal{Y}} \mathcal{X}$ where T is an algebraic space such that the morphism $T \to W$ is unramified.

Proof. Assume (1). Then f is DM and since being unramified is preserved by base change, we see that (2) holds.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (Lemma 101.21.7). Thus we see that (3) holds.

Assume $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). We first check f is DM. Namely, it suffices to check $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is DM, see Lemma 101.4.5. By Lemma 101.4.12 it suffices to check $W \times_{\mathcal{Y}} \mathcal{X}$ is DM. This follows from the existence of $T \to W \times_{\mathcal{Y}} \mathcal{X}$ by (the easy direction of) Theorem 101.21.6.

Assume f is DM and $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). Let V be an algebraic space, let $V \to \mathcal{Y}$ be surjective smooth, let U be an algebraic space, and let $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and étale (Lemma 101.21.7). We have to check that $U \to V$ is unramified. It suffices to prove $U \times_{\mathcal{Y}} W \to V \times_{\mathcal{Y}} W$ is unramified by

Descent on Spaces, Lemma 74.11.27. We may replace $\mathcal{X}, \mathcal{Y}, W, T, U, V$ by $\mathcal{X} \times_\mathcal{Y} W, W, W, T, U \times_\mathcal{Y} W, V \times_\mathcal{Y} W$ (small detail omitted). Thus we may assume that $Y = \mathcal{Y}$ is an algebraic space, there exists an algebraic space T and a surjective étale morphism $T \to \mathcal{X}$ such that $T \to Y$ is unramified, and U and V are as before. In this case we know that

$$U \to V \text{ is unramified} \Leftrightarrow \mathcal{X} \to Y \text{ is unramified} \Leftrightarrow T \to Y \text{ is unramified}$$

by the equivalence of properties (1) and (2) of Lemma 101.34.1 and Definition 101.36.1. This finishes the proof. □

0H2Z Lemma 101.36.7. An unramified morphism of algebraic stacks is locally quasi-finite.

Proof. This follows from Lemma 101.36.6 (characterizing unramified morphisms), Lemma 101.23.7 (characterizing locally quasi-finite morphisms), and Morphisms of Spaces, Lemma 67.38.7 (the corresponding result for algebraic spaces). □

0CIZ Lemma 101.36.8. Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is unramified and $\mathcal{Y} \to \mathcal{Z}$ is DM, then $\mathcal{X} \to \mathcal{Y}$ is unramified.

Proof. Assume $\mathcal{X} \to \mathcal{Z}$ is unramified. By Lemma 101.4.12 the morphism $\mathcal{X} \to \mathcal{Y}$ is DM. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

with U, V, W algebraic spaces, with $W \to \mathcal{Z}$ surjective smooth, $V \to \mathcal{Y} \times_\mathcal{Z} W$ surjective étale, and $U \to \mathcal{X} \times_\mathcal{Y} V$ surjective étale (see Lemma 101.21.7). Then also $U \to \mathcal{X} \times_\mathcal{Z} W$ is surjective and étale. Hence we know that $U \to W$ is unramified and we have to show that $U \to V$ is unramified. This follows from Morphisms of Spaces, Lemma 67.38.11. □

0CJ0 Lemma 101.36.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) f is unramified, and
(2) f is locally of finite type and its diagonal is étale.

Proof. Assume f is unramified. Then f is DM hence we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ (Lemma 101.21.7). Since f is unramified the induced morphism $U \to V$ is unramified. Thus $U \to V$ is locally of finite type (Morphisms of Spaces, Lemma 67.38.6) and we conclude that f is locally of finite type. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is a morphism of algebraic stacks over $\mathcal{Y}$. The base change of Δ by the surjective smooth morphism $V \to \mathcal{Y}$ is the diagonal of the base change of f, i.e., of $\mathcal{X}_V = \mathcal{X} \times_\mathcal{Y} V \to V$. In other words, the diagram

$$\begin{tikzcd} \mathcal{X}_V \arrow[r] \arrow[d] & \mathcal{X}_V \times_V \mathcal{X}_V \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

is cartesian. Since the right vertical arrow is surjective and smooth it suffices to show that the top horizontal arrow is étale by Properties of Stacks, Lemma 100.3.4. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \times_V U \arrow[d] \\ \mathcal{X}_V \arrow[r] & \mathcal{X}_V \times_V \mathcal{X}_V \end{tikzcd}$$

All arrows are representable by algebraic spaces, the vertical arrows are étale, the left one is surjective, and the top horizontal arrow is an open immersion by Morphisms of Spaces, Lemma 67.38.9. This implies what we want: first we see that $U \to \mathcal{X}_V \times_V \mathcal{X}_V$ is étale as a composition of étale morphisms, and then we can use Properties of Stacks, Lemma 100.3.5 to see that $\mathcal{X}_V \to \mathcal{X}_V \times_V \mathcal{X}_V$ is étale because being étale (for morphisms of algebraic spaces) is local on the source in the étale topology (Descent on Spaces, Lemma 74.19.1).

Assume f is locally of finite type and that its diagonal is étale. Then f is DM by definition (as étale morphisms of algebraic spaces are unramified). As above this means we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 101.21.7). To finish the proof we have to show that $U \to V$ is unramified. We already know that $U \to V$ is locally of finite type. Arguing as above we find a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \times_V U \arrow[d] \\ \mathcal{X}_V \arrow[r] & \mathcal{X}_V \times_V \mathcal{X}_V \end{tikzcd}$$

where all arrows are representable by algebraic spaces, the vertical arrows are étale, and the lower horizontal one is étale as a base change of Δ. It follows that $U \to U \times_V U$ is étale for example by Lemma 101.35.6[7]. Thus $U \to U \times_V U$ is an étale monomorphism hence an open immersion (Morphisms of Spaces, Lemma 67.51.2). Then $U \to V$ is unramified by Morphisms of Spaces, Lemma 67.38.9. □

0CJ1 **Lemma 101.36.10.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent

(1) f is étale, and
(2) f is locally of finite presentation, flat, and unramified,
(3) f is locally of finite presentation, flat, and its diagonal is étale.

Proof. The equivalence of (2) and (3) follows immediately from Lemma 101.36.9. Thus in each case the morphism f is DM. Then we can choose Then we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 101.21.7). To finish the proof we have to show that $U \to V$ is étale if and only if it is locally of finite presentation, flat, and unramified. This follows from Morphisms of Spaces, Lemma 67.39.12 (and the more trivial Morphisms of Spaces, Lemmas 67.39.10, 67.39.8, and 67.39.7). □

[7]It is quite easy to deduce this directly from Morphisms of Spaces, Lemma 67.39.11.

101.37. Proper morphisms

0CL4 The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic stacks.

0CL5 **Definition 101.37.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *proper* if f is separated, finite type, and universally closed.

This does not conflict with the already existing notion of a proper morphism of algebraic spaces: a morphism of algebraic spaces is proper if and only if it is separated, finite type, and universally closed (Morphisms of Spaces, Definition 67.40.1) and we've already checked the compatibility of these notions in Lemma 101.3.5, Section 101.17, and Lemmas 101.13.1. Similarly, if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks which is representable by algebraic spaces then we have defined what it means for f to be proper in Properties of Stacks, Section 100.3. However, the discussion in that section shows that this is equivalent to requiring f to be separated, finite type, and universally closed and the same references as above give the compatibility.

0CL6 **Lemma 101.37.2.** *A base change of a proper morphism is proper.*

Proof. See Lemmas 101.4.4, 101.17.3, and 101.13.3. $\square$

0CL7 **Lemma 101.37.3.** *A composition of proper morphisms is proper.*

Proof. See Lemmas 101.4.10, 101.17.2, and 101.13.4. $\square$

0CL8 **Lemma 101.37.4.** *A closed immersion of algebraic stacks is a proper morphism of algebraic stacks.*

Proof. A closed immersion is by definition representable (Properties of Stacks, Definition 100.9.1). Hence this follows from the discussion in Properties of Stacks, Section 100.3 and the corresponding result for morphisms of algebraic spaces, see Morphisms of Spaces, Lemma 67.40.5. $\square$

0CPT **Lemma 101.37.5.** *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{Z} & \end{tikzcd}$$

of algebraic stacks.

1. *If $\mathcal{X} \to \mathcal{Z}$ is universally closed and $\mathcal{Y} \to \mathcal{Z}$ is separated, then the morphism $\mathcal{X} \to \mathcal{Y}$ is universally closed. In particular, the image of $|\mathcal{X}|$ in $|\mathcal{Y}|$ is closed.*
2. *If $\mathcal{X} \to \mathcal{Z}$ is proper and $\mathcal{Y} \to \mathcal{Z}$ is separated, then the morphism $\mathcal{X} \to \mathcal{Y}$ is proper.*

Proof. Assume $\mathcal{X} \to \mathcal{Z}$ is universally closed and $\mathcal{Y} \to \mathcal{Z}$ is separated. We factor the morphism as $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first morphism is proper (Lemma 101.4.8) hence universally closed. The projection $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ is the base change of a universally closed morphism and hence universally closed, see Lemma 101.13.3. Thus $\mathcal{X} \to \mathcal{Y}$ is universally closed as the composition of universally closed morphisms, see Lemma 101.13.4. This proves (1). To deduce (2) combine (1) with Lemmas 101.4.12, 101.7.7, and 101.17.8. $\square$

0CQK Lemma 101.37.6. Let $\mathcal{Z}$ be an algebraic stack. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{Z}$. If $\mathcal{X}$ is universally closed over $\mathcal{Z}$ and f is surjective then $\mathcal{Y}$ is universally closed over $\mathcal{Z}$. In particular, if also $\mathcal{Y}$ is separated and of finite type over $\mathcal{Z}$, then $\mathcal{Y}$ is proper over $\mathcal{Z}$.

Proof. Assume $\mathcal{X}$ is universally closed and f surjective. Denote $p : \mathcal{X} \to \mathcal{Z}$, $q : \mathcal{Y} \to \mathcal{Z}$ the structure morphisms. Let $\mathcal{Z}' \to \mathcal{Z}$ be a morphism of algebraic stacks. The base change $f' : \mathcal{X}' \to \mathcal{Y}'$ of f by $\mathcal{Z}' \to \mathcal{Z}$ is surjective (Properties of Stacks, Lemma 100.5.3) and the base change $p' : \mathcal{X}' \to \mathcal{Z}'$ of p is closed. If $T \subset |\mathcal{Y}'|$ is closed, then $(f')^{-1}(T) \subset |\mathcal{X}'|$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. $\square$

101.38. Scheme theoretic image

0CMH Here is the definition.

0CMI Definition 101.38.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The scheme theoretic image of f is the smallest closed substack $\mathcal{Z} \subset \mathcal{Y}$ through which f factors[8].

We often denote $f : \mathcal{X} \to \mathcal{Z}$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open substacks of $\mathcal{Y}$. However, if f is quasi-compact then the scheme theoretic image commutes with flat base change (Lemma 101.38.5).

0CMJ Lemma 101.38.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $g : \mathcal{W} \to \mathcal{X}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then the scheme theoretic image of f exists if and only if the scheme theoretic image of $f \circ g$ exists and if so then these scheme theoretic images are the same.

Proof. Assume $\mathcal{Z} \subset \mathcal{Y}$ is a closed substack and $f \circ g$ factors through $\mathcal{Z}$. To prove the lemma it suffices to show that f factors through $\mathcal{Z}$. Consider a scheme T and a morphism $T \to \mathcal{X}$ given by an object x of the fibre category of $\mathcal{X}$ over T. We will show that $f(x)$ is in fact in the fibre category of $\mathcal{Z}$ over T. Namely, the projection $T \times_{\mathcal{X}} \mathcal{W} \to T$ is a surjective, flat, locally finitely presented morphism. Hence there is an fppf covering $\{T_i \to T\}$ such that $T_i \to T$ factors through $T \times_{\mathcal{X}} \mathcal{W} \to T$ for all i. Then $T_i \to \mathcal{X}$ factors through $\mathcal{W}$ and hence $T_i \to \mathcal{Y}$ factors through $\mathcal{Z}$. Thus $x|_{T_i}$ is an object of $\mathcal{Z}$. Since $\mathcal{Z}$ is a strictly full substack, we conclude that x is an object of $\mathcal{Z}$ as desired. $\square$

0CPU Lemma 101.38.3. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Then the scheme theoretic image of f exists.

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. By Lemma 101.38.2 we may replace $\mathcal{Y}$ by V. Thus it suffices to show that if $X \to \mathcal{X}$ is a morphism from a scheme to an algebraic stack, then the scheme theoretic image exists. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$. We have $\mathcal{X} = [U/R]$ by Algebraic Stacks, Lemma 94.16.2. By Properties of Stacks, Lemma 100.9.11 the closed substacks $\mathcal{Z}$ of $\mathcal{X}$ are in 1-to-1 correspondence with R-invariant closed subschemes $Z \subset U$. Let $Z_1 \subset U$ be the scheme theoretic image of $X \times_{\mathcal{X}} U \to U$. Observe that $X \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if

[8]We will see in Lemma 101.38.3 that the scheme theoretic image always exists.

$X \times_{\mathcal{X}} U \to U$ factors through the corresponding R-invariant closed subscheme Z (details omitted; hint: this follows because $X \times_{\mathcal{X}} U \to X$ is surjective and smooth). Thus we have to show that there exists a smallest R-invariant closed subscheme $Z \subset U$ containing Z_1.

Let $\mathcal{I}_1 \subset \mathcal{O}_U$ be the quasi-coherent ideal sheaf corresponding to the closed subscheme $Z_1 \subset U$. Let Z_α, $\alpha \in A$ be the set of all R-invariant closed subschemes of U containing Z_1. For $\alpha \in A$, let $\mathcal{I}_\alpha \subset \mathcal{O}_U$ be the quasi-coherent ideal sheaf corresponding to the closed subscheme $Z_\alpha \subset U$. The containment $Z_1 \subset Z_\alpha$ means $\mathcal{I}_\alpha \subset \mathcal{I}_1$. The R-invariance of Z_α means that

$$s^{-1}\mathcal{I}_\alpha \cdot \mathcal{O}_R = t^{-1}\mathcal{I}_\alpha \cdot \mathcal{O}_R$$

as (quasi-coherent) ideal sheaves on (the algebraic space) R. Consider the image

$$\mathcal{I} = \operatorname{Im}\left(\bigoplus\nolimits_{\alpha \in A} \mathcal{I}_\alpha \to \mathcal{I}_1\right) = \operatorname{Im}\left(\bigoplus\nolimits_{\alpha \in A} \mathcal{I}_\alpha \to \mathcal{O}_X\right)$$

Since direct sums of quasi-coherent sheaves are quasi-coherent and since images of maps between quasi-coherent sheaves are quasi-coherent, we find that $\mathcal{I}$ is quasi-coherent. Since pull back is exact and commutes with direct sums we find

$$s^{-1}\mathcal{I} \cdot \mathcal{O}_R = t^{-1}\mathcal{I} \cdot \mathcal{O}_R$$

Hence $\mathcal{I}$ defines an R-invariant closed subscheme $Z \subset U$ which is contained in every Z_α and containes Z_1 as desired. □

0CPV Lemma 101.38.4. Let

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "f_1"'] \arrow[d] & \mathcal{Y}_1 \arrow[d] \\ \mathcal{X}_2 \arrow[r, "f_2"] & \mathcal{Y}_2 \end{tikzcd}$$

be a commutative diagram of algebraic stacks. Let $\mathcal{Z}_i \subset \mathcal{Y}_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ induces a morphism $\mathcal{Z}_1 \to \mathcal{Z}_2$ and a commutative diagram

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r] \arrow[d] & \mathcal{Z}_1 \arrow[r] \arrow[d] & \mathcal{Y}_1 \arrow[d] \\ \mathcal{X}_2 \arrow[r] & \mathcal{Z}_2 \arrow[r] & \mathcal{Y}_2 \end{tikzcd}$$

Proof. The scheme theoretic inverse image of $\mathcal{Z}_2$ in $\mathcal{Y}_1$ is a closed substack of $\mathcal{Y}_1$ through which f_1 factors. Hence $\mathcal{Z}_1$ is contained in this. This proves the lemma. □

0CMK Lemma 101.38.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Then formation of the scheme theoretic image commutes with flat base change.

Proof. Let $\mathcal{Y}' \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme V' and a surjective smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. We may and do assume that $V = \coprod_{i \in I} V_i$ is a disjoint union of affine schemes and that $V' = \coprod_{i \in I} \coprod_{j \in J_i} V_{i,j}$ is a disjoint union of affine schemes with each $V_{i,j}$ mapping into V_i. Let

(1) $\mathcal{Z} \subset \mathcal{Y}$ be the scheme theoretic image of f,
(2) $\mathcal{Z}' \subset \mathcal{Y}'$ be the scheme theoretic image of the base change of f by $\mathcal{Y}' \to \mathcal{Y}$,
(3) $Z \subset V$ be the scheme theoretic image of the base change of f by $V \to \mathcal{Y}$,

(4) $Z' \subset V'$ be the scheme theoretic image of the base change of f by $V' \to \mathcal{Y}$.

If we can show that (a) $Z = V \times_\mathcal{Y} \mathcal{Z}$, (b) $Z' = V' \times_{\mathcal{Y}'} \mathcal{Z}'$, and (c) $Z' = V' \times_V Z$ then the lemma follows: the inclusion $\mathcal{Z}' \to \mathcal{Z} \times_\mathcal{Y} \mathcal{Y}'$ (Lemma 101.38.4) has to be an isomorphism because after base change by the surjective smooth morphism $V' \to \mathcal{Y}'$ it is.

Proof of (a). Set $R = V \times_\mathcal{Y} V$. By Properties of Stacks, Lemma 100.9.11 the rule $\mathcal{Z} \mapsto \mathcal{Z} \times_\mathcal{Y} V$ defines a 1-to-1 correspondence between closed substacks of $\mathcal{Y}$ and R-invariant closed subspaces of V. Moreover, $f : \mathcal{X} \to \mathcal{Y}$ factors through $\mathcal{Z}$ if and only if the base change $g : \mathcal{X} \times_\mathcal{Y} V \to V$ factors through $\mathcal{Z} \times_\mathcal{Y} V$. We claim: the scheme theoretic image $Z \subset V$ of g is R-invariant. The claim implies (a) by what we just said.

For each i the morphism $\mathcal{X} \times_\mathcal{Y} V_i \to V_i$ is quasi-compact and hence $\mathcal{X} \times_\mathcal{Y} V_i$ is quasi-compact. Thus we can choose an affine scheme W_i and a surjective smooth morphism $W_i \to \mathcal{X} \times_\mathcal{Y} V_i$. Observe that $W = \coprod W_i$ is a scheme endowed with a smooth and surjective morphism $W \to \mathcal{X} \times_\mathcal{Y} V$ such that the composition $W \to V$ with g is quasi-compact. Let $Z \to V$ be the scheme theoretic image of $W \to V$, see Morphisms, Section 29.6 and Morphisms of Spaces, Section 67.16. It follows from Lemma 101.38.2 that $Z \subset V$ is the scheme theoretic image of g. To show that Z is R-invariant we claim that both

$$\mathrm{pr}_0^{-1}(Z), \mathrm{pr}_1^{-1}(Z) \subset R = V \times_\mathcal{Y} V$$

are the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} R \to R$. Namely, we first use Morphisms of Spaces, Lemma 67.30.12 to see that $\mathrm{pr}_0^{-1}(Z)$ is the scheme theoretic image of the composition

$$W \times_{V, \mathrm{pr}_0} R = W \times_\mathcal{Y} V \to \mathcal{X} \times_\mathcal{Y} R \to R$$

Since the first arrow here is surjective and smooth we see that $\mathrm{pr}_0^{-1}(Z)$ is the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} R \to R$. The same argument applies that $\mathrm{pr}_1^{-1}(Z)$. Hence Z is R-invariant.

Statement (b) is proved in exactly the same way as one proves (a).

Proof of (c). Let $Z_i \subset V_i$ be the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} V_i \to V_i$ and let $Z_{i,j} \subset V_{i,j}$ be the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} V_{i,j} \to V_{i,j}$. Clearly it suffices to show that the inverse image of Z_i in $V_{i,j}$ is $Z_{i,j}$. Above we've seen that Z_i is the scheme theoretic image of $W_i \to V_i$ and by the same token $Z_{i,j}$ is the scheme theoretic image of $W_i \times_{V_i} V_{i,j} \to V_{i,j}$. Hence the equality follows from the case of schemes (Morphisms, Lemma 29.25.16) and the fact that $V_{i,j} \to V_i$ is flat. $\square$

0CML Lemma 101.38.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Let $\mathcal{Z} \subset \mathcal{Y}$ be the scheme theoretic image of f. Then $|\mathcal{Z}|$ is the closure of the image of $|f|$.

Proof. Let $z \in |\mathcal{Z}|$ be a point. Choose an affine scheme V, a point $v \in V$, and a smooth morphism $V \to \mathcal{Y}$ mapping v to z. Then $\mathcal{X} \times_\mathcal{Y} V$ is a quasi-compact algebraic stack. Hence we can find an affine scheme W and a surjective smooth morphism $W \to \mathcal{X} \times_\mathcal{Y} V$. By Lemma 101.38.5 the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} V \to V$ is $Z = \mathcal{Z} \times_\mathcal{Y} V$. Hence the inverse image of $|\mathcal{Z}|$ in $|V|$ is $|Z|$ by Properties of Stacks, Lemma 100.4.3. By Lemma 101.38.2 Z is the scheme theoretic image of $W \to V$. By Morphisms of Spaces, Lemma 67.16.3 we see that the image of $|W| \to |Z|$ is dense. Hence the image of $|\mathcal{X} \times_\mathcal{Y} V| \to |Z|$ is dense.

Observe that $v \in Z$. Since $|V| \to |\mathcal{Y}|$ is open, a topology argument tells us that z is in the closure of the image of $|f|$ as desired. $\square$

0CPW Lemma 101.38.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces and separated. Let $\mathcal{V} \subset \mathcal{Y}$ be an open substack such that $\mathcal{V} \to \mathcal{Y}$ is quasi-compact. Let $s : \mathcal{V} \to \mathcal{X}$ be a morphism such that $f \circ s = \mathrm{id}_\mathcal{V}$. Let $\mathcal{Y}'$ be the scheme theoretic image of s. Then $\mathcal{Y}' \to \mathcal{Y}$ is an isomorphism over $\mathcal{V}$.

Proof. By Lemma 101.7.7 the morphism $s : \mathcal{V} \to \mathcal{Y}$ is quasi-compact. Hence the construction of the scheme theoretic image $\mathcal{Y}'$ of s commutes with flat base change by Lemma 101.38.5. Thus to prove the lemma we may assume $\mathcal{Y}$ is representable by an algebraic space and we reduce to the case of algebraic spaces which is Morphisms of Spaces, Lemma 67.16.7. $\square$

101.39. Valuative criteria

0CL9 We need to be careful when stating the valuative criterion. Namely, in the formulation we need to speak about commutative diagrams but we are working in a 2-category and we need to make sure the 2-morphisms compose correctly as well!

0CLA Definition 101.39.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

0CLB (101.39.1.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ur, dotted] & \mathcal{Y} \end{tikzcd}$$

where A is a valuation ring with field of fractions K. Let

$$\gamma : y \circ j \longrightarrow f \circ x$$

be a 2-morphism witnessing the 2-commutativity of the diagram. (Notation as in Categories, Sections 4.28 and 4.29.) Given (101.39.1.1) and γ a dotted arrow is a triple (a, α, β) consisting of a morphism $a : \mathrm{Spec}(A) \to \mathcal{X}$ and 2-arrows $\alpha : a \circ j \to x$, $\beta : y \to f \circ a$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_j)$, in other words such that

$$\begin{tikzcd} & f \circ a \circ j \arrow[dr, "\mathrm{id}_f \star \alpha"] & \\ y \circ j \arrow[ur, "\beta \star \mathrm{id}_j"] \arrow[rr, "\gamma"] & & f \circ x \end{tikzcd}$$

is commutative. A morphism of dotted arrows $(a, \alpha, \beta) \to (a', \alpha', \beta')$ is a 2-arrow $\theta : a \to a'$ such that $\alpha = \alpha' \circ (\theta \star \mathrm{id}_j)$ and $\beta' = (\mathrm{id}_f \star \theta) \circ \beta$.

The preceding definition is a special case of Categories, Definition 4.44.1. The category of dotted arrows depends on γ in general. If $\mathcal{Y}$ is representable by an algebraic space (or if automorphism groups of objects over fields are trivial), then of course there is at most one γ and one does not need to check the commutativity of the triangle. More generally, we have Lemma 101.39.3. The commutativity of the triangle is important in the proof of compatibility with base change, see proof of Lemma 101.39.4.

0CLC **Lemma 101.39.2.** In the situation of Definition 101.39.1 the category of dotted arrows is a groupoid. If Δ_f is separated, then it is a setoid.

Proof. Since 2-arrows are invertible it is clear that the category of dotted arrows is a groupoid. Given a dotted arrow (a, α, β) an automorphism of (a, α, β) is a 2-morphism $\theta : a \to a$ satisfying two conditions. The first condition $\beta = (\mathrm{id}_f \star \theta) \circ \beta$ signifies that θ defines a morphism $(a, \theta) : \mathrm{Spec}(A) \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. The second condition $\alpha = \alpha \circ (\theta \star \mathrm{id}_j)$ implies that the restriction of (a, θ) to $\mathrm{Spec}(K)$ is the identity. Picture

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] & \mathrm{Spec}(K) \arrow[l, "(a\circ j, \mathrm{id})"'] \arrow[d, "j"] \\ \mathcal{X} & \mathrm{Spec}(A) \arrow[l, "a"] \arrow[ul, "(a,\theta)"] \end{tikzcd}$$

In other words, if $G \to \mathrm{Spec}(A)$ is the group algebraic space we get by pulling back the relative inertia by a, then θ defines a point $\theta \in G(A)$ whose image in $G(K)$ is trivial. Certainly, if the identity $e : \mathrm{Spec}(A) \to G$ is a closed immersion, then this can happen only if θ is the identity. Looking at Lemma 101.6.1 we obtain the result we want. $\square$

0CLD **Lemma 101.39.3.** In Definition 101.39.1 assume $\mathcal{I}_\mathcal{Y} \to \mathcal{Y}$ is proper (for example if $\mathcal{Y}$ is separated or if $\mathcal{Y}$ is separated over an algebraic space). Then the category of dotted arrows is independent (up to noncanonical equivalence) of the choice of γ and the existence of a dotted arrow (for some and hence equivalently all γ) is equivalent to the existence of a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ur, "a"] & \mathcal{Y} \end{tikzcd}$$

with 2-commutative triangles (without checking the 2-morphisms compose correctly).

Proof. Let $\gamma, \gamma' : y \circ j \longrightarrow f \circ x$ be two 2-morphisms. Then $\gamma^{-1} \circ \gamma'$ is an automorphism of y over $\mathrm{Spec}(K)$. Hence if $\mathit{Isom}_\mathcal{Y}(y, y) \to \mathrm{Spec}(A)$ is proper, then by the valuative criterion of properness (Morphisms of Spaces, Lemma 67.44.1) we can find $\delta : y \to y$ whose restriction to $\mathrm{Spec}(K)$ is $\gamma^{-1} \circ \gamma'$. Then we can use δ to define an equivalence between the category of dotted arrows for γ to the category of dotted arrows for γ' by sending (a, α, β) to $(a, \alpha, \beta \circ \delta)$. The final statement is clear. $\square$

0CLE **Lemma 101.39.4.** Assume given a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x'"'] \arrow[d, "j"'] & \mathcal{X}' \arrow[r, "q"'] \arrow[d, "p"] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y'"] & \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

with the right square 2-cartesian. Choose a 2-arrow $\gamma' : y' \circ j \to p \circ x'$. Set $x = q \circ x'$, $y = g \circ y'$ and let $\gamma : y \circ j \to f \circ x$ be the composition of γ' with the 2-arrow implicit in the 2-commutativity of the right square. Then the category of dotted arrows for the left square and γ' is equivalent to the category of dotted arrows for the outer rectangle and γ.

Proof. (We do not know how to prove the analogue of this lemma if instead of the category of dotted arrows we look at the set of isomorphism classes of morphisms producing two 2-commutative triangles as in Lemma 101.39.3; in fact this analogue may very well be wrong.) First proof: this lemma is a special case of Categories, Lemma 4.44.2. Second proof: we are allowed to replace $\mathcal{X}'$ by the 2-fibre product $\mathcal{Y}' \times_\mathcal{Y} \mathcal{X}$ as described in Categories, Lemma 4.32.3. Then the object x' becomes the triple $(y' \circ j, x, \gamma)$. Then we can go from a dotted arrow (a, α, β) for the outer rectangle to a dotted arrow (a', α', β') for the left square by taking $a' = (y', a, \beta)$ and $\alpha' = (\mathrm{id}_{y' \circ j}, \alpha)$ and $\beta' = \mathrm{id}_{y'}$. Details omitted. $\square$

0CLF Lemma 101.39.5. Assume given a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[dd, "j"'] & \mathcal{X} \arrow[d, "f"] \\ & \mathcal{Y} \arrow[d, "g"] \\ \mathrm{Spec}(A) \arrow[r, "z"] & \mathcal{Z} \end{tikzcd}$$

Choose a 2-arrow $\gamma : z \circ j \to g \circ f \circ x$. Let $\mathcal{C}$ be the category of dotted arrows for the outer rectangle and γ. Let $\mathcal{C}'$ be the category of dotted arrows for the square

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "f \circ x"'] \arrow[d, "j"'] & \mathcal{Y} \arrow[d, "g"] \\ \mathrm{Spec}(A) \arrow[r, "z"] & \mathcal{Z} \end{tikzcd}$$

and γ. Then $\mathcal{C}$ is equivalent to a category $\mathcal{C}''$ which has the following property: there is a functor $\mathcal{C}'' \to \mathcal{C}'$ which turns $\mathcal{C}''$ into a category fibred in groupoids over $\mathcal{C}'$ and whose fibre categories are categories of dotted arrows for certain squares of the form

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

and some choices of $y \circ j \to f \circ x$.

Proof. This lemma is a special case of Categories, Lemma 4.44.3. $\square$

0CLG Definition 101.39.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f satisfies the uniqueness part of the valuative criterion if for every diagram (101.39.1.1) and γ as in Definition 101.39.1 the category of dotted arrows is either empty or a setoid with exactly one isomorphism class.

0CLH Lemma 101.39.7. The base change of a morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion by any morphism of algebraic stacks is a morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion.

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLI Lemma 101.39.8. The composition of morphisms of algebraic stacks which satisfy the uniqueness part of the valuative criterion is another morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion.

Proof. Follows from Lemma 101.39.5 and the definition. □

0CLJ Lemma 101.39.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent

(1) f satisfies the uniqueness part of the valuative criterion,
(2) for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_\mathcal{Y} T \to T$ satisfies the uniqueness part of the valuative criterion as a morphism of algebraic spaces.

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLK Definition 101.39.10. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f satisfies the existence part of the valuative criterion if for every diagram (101.39.1.1) and γ as in Definition 101.39.1 there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the outer rectangle of the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d, "j'"'] \arrow[rr, bend left, "x'"] & \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rr, bend right, "y'"'] & \mathrm{Spec}(A) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty.

We have already seen in the case of morphisms of algebraic spaces, that it is necessary to allow extensions of the fractions fields in order to get the correct notion of the valuative criterion. See Morphisms of Spaces, Example 67.41.6. Still, for morphisms between separated algebraic spaces, such an extension is not needed (Morphisms of Spaces, Lemma 67.41.5). However, for morphisms between algebraic stacks, an extension may be needed even if $\mathcal{X}$ and $\mathcal{Y}$ are both separated. For example consider the morphism of algebraic stacks

$$[\mathrm{Spec}(\mathbf{C})/G] \to \mathrm{Spec}(\mathbf{C})$$

over the base scheme $\mathrm{Spec}(\mathbf{C})$ where G is a group of order 2. Both source and target are separated algebraic stacks and the morphism is proper. Whence it satisfies the uniqueness and existence parts of the valuative criterion (see Lemma 101.43.1). But on the other hand, there is a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & {[\mathrm{Spec}(\mathbf{C})/G]} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(\mathbf{C}) \end{tikzcd}$$

where no dotted arrow exists with $A = \mathbf{C}[[t]]$ and $K = \mathbf{C}((t))$. Namely, the top horizontal arrow is given by a G-torsor over the spectrum of $K = \mathbf{C}((t))$. Since any G-torsor over the strictly henselian local ring $A = \mathbf{C}[[t]]$ is trivial, we see that if a dotted arrow always exists, then every G-torsor over K is trivial. This is not true because $G = \{+1, -1\}$ and by Kummer theory G-torsors over K are classified by $K^*/(K^*)^2$ which is nontrivial.

0CLL Lemma 101.39.11. The base change of a morphism of algebraic stacks which satisfies the existence part of the valuative criterion by any morphism of algebraic stacks is a morphism of algebraic stacks which satisfies the existence part of the valuative criterion.

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLM Lemma 101.39.12. The composition of morphisms of algebraic stacks which satisfy the existence part of the valuative criterion is another morphism of algebraic stacks which satisfies the existence part of the valuative criterion.

Proof. Follows from Lemma 101.39.5 and the definition. □

0CLN Lemma 101.39.13. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent

(1) f satisfies the existence part of the valuative criterion,
(2) for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ satisfies the existence part of the valuative criterion as a morphism of algebraic spaces.

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLP Lemma 101.39.14. A closed immersion of algebraic stacks satisfies both the existence and uniqueness part of the valuative criterion.

Proof. Omitted. Hint: reduce to the case of a closed immersion of schemes by Lemmas 101.39.9 and 101.39.13. □

101.40. Valuative criterion for second diagonal

0CLQ The converse statement has already been proved in Lemma 101.39.2. The criterion itself is the following.

0CLR Lemma 101.40.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If Δ_f is quasi-separated and if for every diagram (101.39.1.1) and choice of γ as in Definition 101.39.1 the category of dotted arrows is a setoid, then Δ_f is separated.

Proof. We are going to write out a detailed proof, but we strongly urge the reader to find their own proof, inspired by reading the argument given in the proof of Lemma 101.39.2.

Assume Δ_f is quasi-separated and for every diagram (101.39.1.1) and choice of γ as in Definition 101.39.1 the category of dotted arrows is a setoid. By Lemma 101.6.1 it suffices to show that $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a closed immersion. By Lemma 101.6.4 it in fact suffices to show that $e = \Delta_{f,2}$ is universally closed. Either of these lemmas tells us that $e = \Delta_{f,2}$ is quasi-compact by our assumption that Δ_f is quasi-separated.

In this paragraph we will show that e satisfies the existence part of the valuative criterion. Consider a 2-commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "e"] \\ \mathrm{Spec}(A) \arrow[r, "{(a,\theta)}"] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \end{tikzcd}$$

and let $\alpha : (a, \theta) \circ j \to e \circ x$ be any 2-morphism witnessing the 2-commutativity of the diagram (we use α instead of the letter γ used in Definition 101.39.1). Note that $f \circ \theta = \mathrm{id}$; we will use this below. Observe that $e \circ x = (x, \mathrm{id}_x)$ and $(a, \theta) \circ j = (a \circ j, \theta \star \mathrm{id}_j)$. Thus we see that α is a 2-arrow $\alpha : a \circ j \to x$ compatible with $\theta \star \mathrm{id}_j$ and id_x. Set $y = f \circ x$ and $\beta = \mathrm{id}_{f \circ a}$. Reading the arguments given in the proof of Lemma 101.39.2 backwards, we see that θ is an automorphism of the dotted arrow (a, α, β) with

$$\gamma : y \circ j \to f \circ x \quad \text{equal to} \quad \mathrm{id}_f \star \alpha : f \circ a \circ j \to f \circ x$$

On the other hand, id_a is an automorphism too, hence we conclude $\theta = \mathrm{id}_a$ from the assumption on f. Then we can take as dotted arrow for the displayed diagram above the morphism $a : \mathrm{Spec}(A) \to \mathcal{X}$ with 2-morphisms $(a, \mathrm{id}_a) \circ j \to (x, \mathrm{id}_x)$ given by α and $(a, \theta) \to e \circ a$ given by id_a.

By Lemma 101.39.11 any base change of e satisfies the existence part of the valuative criterion. Since e is representable by algebraic spaces, it suffices to show that e is universally closed after a base change by a morphism $I \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ which is surjective and smooth and with I an algebraic space (see Properties of Stacks, Section 100.3). This base change $e' : X' \to I'$ is a quasi-compact morphism of algebraic spaces which satisfies the existence part of the valuative criterion and hence is universally closed by Morphisms of Spaces, Lemma 67.42.1. $\square$

101.41. Valuative criterion for the diagonal

0CLS The result is Lemma 101.41.2. We first state and prove a formal helper lemma.

0E8L Lemma 101.41.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathrm{Spec}(A) \arrow[r, "{(a_1,a_2,\varphi)}"] \arrow[ur, dotted] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

where A is a valuation ring with field of fractions K. Let $\gamma : (a_1, a_2, \varphi) \circ j \longrightarrow \Delta_f \circ x$ be a 2-morphism witnessing the 2-commutativity of the diagram. Then

(1) Writing $\gamma = (\alpha_1, \alpha_2)$ with $\alpha_i : a_i \circ j \to x$ we obtain two dotted arrows $(a_1, \alpha_1, \mathrm{id})$ and (a_2, α_2, φ) in the diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "f \circ a_1"] \arrow[ur, dotted] & \mathcal{Y} \end{tikzcd}$$

(2) The category of dotted arrows for the original diagram and γ is a setoid whose set of isomorphism classes of objects equal to the set of morphisms $(a_1, \alpha_1, \mathrm{id}) \to (a_2, \alpha_2, \varphi)$ in the category of dotted arrows.

Proof. Since Δ_f is representable by algebraic spaces (hence the diagonal of Δ_f is separated), we see that the category of dotted arrows in the first commutative diagram of the lemma is a setoid by Lemma 101.39.2. All the other statements of the lemma are consequences of 2-diagramatic computations which we omit. $\square$

0CLT **Lemma 101.41.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is quasi-separated. If f satisfies the uniqueness part of the valuative criterion, then f is separated.

Proof. The assumption on f means that Δ_f is quasi-compact and quasi-separated (Definition 101.4.1). We have to show that Δ_f is proper. Lemma 101.40.1 says that Δ_f is separated. By Lemma 101.3.3 we know that Δ_f is locally of finite type. To finish the proof we have to show that Δ_f is universally closed. A formal argument (see Lemma 101.41.1) shows that the uniqueness part of the valuative criterion implies that we have the existence of a dotted arrow in any solid diagram like so:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

Using that this property is preserved by any base change we conclude that any base change by Δ_f by an algebraic space mapping into $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ has the existence part of the valuative criterion and we conclude is universally closed by the valuative criterion for morphisms of algebraic spaces, see Morphisms of Spaces, Lemma 67.42.1. $\square$

Here is a converse.

0CLU **Lemma 101.41.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is separated, then f satisfies the uniqueness part of the valuative criterion.

Proof. Since f is separated we see that all categories of dotted arrows are setoids by Lemma 101.39.2. Consider a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ur, dotted] & \mathcal{Y} \end{tikzcd}$$

and a 2-morphism $\gamma : y \circ j \to f \circ x$ as in Definition 101.39.1. Consider two objects (a, α, β) and (a', β', α') of the category of dotted arrows. To finish the proof we have to show these objects are isomorphic. The isomorphism

$$f \circ a \xrightarrow{\beta^{-1}} y \xrightarrow{\beta'} f \circ a'$$

means that $(a, a', \beta' \circ \beta^{-1})$ is a morphism $\mathrm{Spec}(A) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. On the other hand, α and α' define a 2-arrow

$$(a, a', \beta' \circ \beta^{-1}) \circ j = (a \circ j, a' \circ j, (\beta' \star \mathrm{id}_j) \circ (\beta \star \mathrm{id}_j)^{-1}) \xrightarrow{(\alpha, \alpha')} (x, x, \mathrm{id}) = \Delta_f \circ x$$

Here we use that both (a, α, β) and (a', α', β') are dotted arrows with respect to γ. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathrm{Spec}(A) \arrow[r, "(a, a', \beta' \circ \beta^{-1})"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

with 2-commutativity witnessed by (α, α'). Now Δ_f is representable by algebraic spaces (Lemma 101.3.3) and proper as f is separated. Hence by Lemma 101.39.13 and the valuative criterion for properness for algebraic spaces (Morphisms of Spaces, Lemma 67.44.1) we see that there exists a dotted arrow. Unwinding the construction, we see that this means (a, α, β) and (a', α', β') are isomorphic in the category of dotted arrows as desired. $\square$

101.42. Valuative criterion for universal closedness

0CLV Here is a statement.

0CLW **Lemma 101.42.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume

(1) f is quasi-compact, and
(2) f satisfies the existence part of the valuative criterion.

Then f is universally closed.

Proof. By Lemmas 101.7.3 and 101.39.11 properties (1) and (2) are preserved under any base change. By Lemma 101.13.5 we only have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is closed, whenever T is an affine scheme mapping into $\mathcal{Y}$. Hence it suffices to show: if $f : \mathcal{X} \to Y$ is a quasi-compact morphism from an algebraic stack to an affine scheme satisfying the existence part of the valuative criterion, then $|f|$ is closed. Let $T \subset |\mathcal{X}|$ be a closed subset. We have to show that $f(T)$ is closed to finish the proof.

Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced algebraic stack structure on T (Properties of Stacks, Definition 100.10.4). Then $i : \mathcal{Z} \to \mathcal{X}$ is a closed immersion and we have to show that the image of $|\mathcal{Z}| \to |Y|$ is closed. Since closed immersions are quasi-compact (Lemma 101.7.5) and satisfies the existence part of the valuative criterion (Lemma 101.39.14) and since compositions of quasi-compact morphisms are quasi-compact (Lemma 101.7.4) and since compositions preserve the property of satisfying the existence part of the valuative criterion (Lemma 101.39.12) we conclude that it suffices to show: if $f : \mathcal{X} \to Y$ is a quasi-compact morphism from an algebraic stack to an affine scheme satisfying the existence part of the valuative criterion, then $|f|(|\mathcal{X}|)$ is closed.

Since $\mathcal{X}$ is quasi-compact (being quasi-compact over the affine Y), we can choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$ (Properties of Stacks, Lemma 100.6.2). Suppose that $y \in Y$ is in the closure of the image of $U \to Y$ (in other words, in the closure of the image of $|f|$). Then by Morphisms, Lemma 29.6.5

we can find a valuation ring A with fraction field K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y. By assumption we get an extension K'/K and a valuation ring $A' \subset K'$ dominating A and the dotted arrow in the following diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rrru, dotted] & \mathrm{Spec}(A) \arrow[r] & Y \arrow[r, equal] & Y \end{tikzcd}$$

Thus y is in the image of $|f|$ and we win. $\square$

Here is a converse.

0CLX Lemma 101.42.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume

(1) f is quasi-separated, and
(2) f is universally closed.

Then f satisfies the existence part of the valuative criterion.

Proof. Consider a solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ru, dotted] & \mathcal{Y} \end{tikzcd}$$

where A is a valuation ring with field of fractions K and $\gamma : y \circ j \longrightarrow f \circ x$ as in Definition 101.39.1. By Lemma 101.39.4 in order to find a dotted arrow (after possibly replacing K by an extension and A by a valuation ring dominating it) we may replace $\mathcal{Y}$ by $\mathrm{Spec}(A)$ and $\mathcal{X}$ by $\mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$. Of course we use here that being quasi-separated and universally closed are preserved under base change. Thus we reduce to the case discussed in the next paragraph.

Consider a solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, equal] \arrow[ru, dotted] & \mathrm{Spec}(A) \end{tikzcd}$$

where A is a valuation ring with field of fractions K as in Definition 101.39.1. By Lemma 101.7.7 and the fact that f is quasi-separated we have that the morphism x is quasi-compact. Since f is universally closed, we have in particular that $|f|(\overline{\{x\}})$ is closed in $\mathrm{Spec}(A)$. Since this image contains the generic point of $\mathrm{Spec}(A)$ there exists a point $x' \in |\mathcal{X}|$ in the closure of x mapping to the closed point of $\text{Spec}(A)$. By Lemma 101.7.9 we can find a commutative diagram

$$\begin{tikzcd} \text{Spec}(K') \arrow[r] \arrow[d] & \text{Spec}(K) \arrow[d] \\ \text{Spec}(A') \arrow[r] & \mathcal{X} \end{tikzcd}$$

such that the closed point of $\text{Spec}(A')$ maps to $x' \in |\mathcal{X}|$. It follows that $\text{Spec}(A') \to \text{Spec}(A)$ maps the closed point to the closed point, i.e., A' dominates A and this finishes the proof. □

101.43. Valuative criterion for properness

0CLY Here is the statement.

0CLZ **Lemma 101.43.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is of finite type and quasi-separated. Then the following are equivalent*

1. *f is proper, and*
2. *f satisfies both the uniqueness and existence parts of the valuative criterion.*

Proof. A proper morphism is the same thing as a separated, finite type, and universally closed morphism. Thus this lemma follows from Lemmas 101.41.2, 101.41.3, 101.42.1, and 101.42.2. □

101.44. Local complete intersection morphisms

0CJ2 The property "being a local complete intersection morphism" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Lemma 74.20.4 and More on Morphisms of Spaces, Lemmas 76.48.9 and 76.48.10. By Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be a local complete intersection morphism as follows and it agrees with the already existing notion defined in More on Morphisms of Spaces, Section 76.48 when both source and target are algebraic spaces.

0CJ3 **Definition 101.44.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *local complete intersection morphism* or *Koszul* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P}$ = local complete intersection.

0CJ4 **Lemma 101.44.2.** *The composition of local complete intersection morphisms is a local complete intersection.*

Proof. Combine Remark 101.16.3 with More on Morphisms of Spaces, Lemma 76.48.5. □

0CJ5 **Lemma 101.44.3.** *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Argue exactly as in Remark 101.16.4 (but only for flat $\mathcal{Y}' \to \mathcal{Y}$) using More on Morphisms of Spaces, Lemma 76.48.4. □

0CJ6 **Lemma 101.44.4.** Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{Z} & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic stacks. Assume $\mathcal{Y} \to \mathcal{Z}$ is smooth and $\mathcal{X} \to \mathcal{Z}$ is a local complete intersection morphism. Then $f : \mathcal{X} \to \mathcal{Y}$ is a local complete intersection morphism.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose a scheme V and a surjective smooth morphism $V \to W \times_\mathcal{Z} \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. Then $U \to W$ is a local complete intersection morphism of schemes and $V \to W$ is a smooth morphism of schemes. By the result for schemes (More on Morphisms, Lemma 37.62.10) we conclude that $U \to V$ is a local complete intersection morphism. By definition this means that f is a local complete intersection morphism. □

101.45. Stabilizer preserving morphisms

0DU6 In the literature a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be stabilizer preserving or fixed-point reflecting if the induced morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism. Such a morphism induces an isomorphism between automorphism groups (Remark 101.19.5) in every point of $\mathcal{X}$. In this section we prove some simple lemmas around this concept.

0DU7 **Lemma 101.45.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, then f is representable by algebraic spaces.

Proof. Immediate from Lemma 101.6.2. □

0DU8 **Remark 101.45.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $U \to \mathcal{X}$ be a morphism whose source is an algebraic space. Let $G \to H$ be the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. If Δ_f is unramified, étale, etc, so is $G \to H$. This is true because

$$\begin{tikzcd} U \times_\mathcal{X} U \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta_f"] \\ U \times_\mathcal{Y} U \arrow[r] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

is cartesian and the morphism $G \to H$ is the base change of the left vertical arrow by the diagonal $U \to U \times U$. Compare with the proof of Lemma 101.6.6.

0DU9 **Lemma 101.45.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be an unramified morphism of algebraic stacks. The following are equivalent

(1) $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, and
(2) f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 101.19.5) for all $x \in |\mathcal{X}|$.

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Denote $G \to H$ the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. By Remark 101.45.2 and Lemma 101.36.9 the morphism $G \to H$ is étale. Condition (1) is equivalent to the condition that $G \to H$ is an isomorphism (this follows for example by applying

Properties of Stacks, Lemma 100.3.3). Condition (2) is equivalent to the condition that for every $u \in U$ the morphism $G_u \to H_u$ of fibres is an isomorphism. Thus (1) $\Rightarrow$ (2) is trivial. If (2) holds, then $G \to H$ is a surjective, universally injective, étale morphism of algebraic spaces. Such a morphism is an isomorphism by Morphisms of Spaces, Lemma 67.51.2. $\square$

0DUA Lemma 101.45.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume [Ryd07a, Proposition 3.5] and [Alp10, Proposition 2.5]

(1) f is representable by algebraic spaces and unramified, and
(2) $\mathcal{I}_\mathcal{Y} \to \mathcal{Y}$ is proper.

Then the set of $x \in |\mathcal{X}|$ such that f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 101.19.5) is open. Letting $\mathcal{U} \subset \mathcal{X}$ be the corresponding open substack, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism.

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Denote $G \to H$ the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. By Remark 101.45.2 and Lemma 101.36.9 the morphism $G \to H$ is étale. Since f is representable by algebraic spaces, we see that $G \to H$ is a monomorphism. Hence $G \to H$ is an open immersion, see Morphisms of Spaces, Lemma 67.51.2. By assumption $H \to U$ is proper.

With these preparations out of the way, we can prove the lemma as follows. The inverse image of the subset of $|\mathcal{X}|$ of the lemma is clearly the set of $u \in U$ such that $G_u \to H_u$ is an isomorphism (since after all G_u is an open sub group algebraic space of H_u). This is an open subset because the complement is the image of the closed subset $|H| \setminus |G|$ and $|H| \to |U|$ is closed. By Properties of Stacks, Lemma 100.9.12 we can consider the corresponding open substack $\mathcal{U}$ of $\mathcal{X}$. The final statement of the lemma follows from applying Lemma 101.45.3 to $\mathcal{U} \to \mathcal{Y}$. $\square$

0DUB Lemma 101.45.5. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a cartesian diagram of algebraic stacks.

(1) Let $x' \in |\mathcal{X}'|$ with image $x \in |\mathcal{X}|$. If f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 101.19.5), then f' induces an isomorphism between automorphism groups at x' and $f(x')$.
(2) If $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, then $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{I}_{\mathcal{Y}'}$ is an isomorphism.

Proof. Omitted. $\square$

0DUC Lemma 101.45.6. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a cartesian diagram of algebraic stacks. If f induces an isomorphism between automorphism groups at points (Remark 101.19.5), then

$$\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}') \longrightarrow \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{Y}') \times \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})$$

is injective on isomorphism classes for any field k.

Proof. We have to show that given (y', x) there is at most one x' mapping to it. By our construction of 2-fibre products, a morphism x' is given by a triple (x, y', α) where $\alpha : g \circ y' \to f \circ x$ is a 2-morphism. Now, suppose we have a second such triple (x, y', β). Then α and β differ by a k-valued point ϵ of the automorphism group algebraic space $G_{f(x)}$. Since f induces an isomorphism $G_x \to G_{f(x)}$ by assumption, this means we can lift ϵ to a k-valued point γ of G_x. Then $(\gamma, \mathrm{id}) : (x, y', \alpha) \to (x, y', \beta)$ is an isomorphism as desired. □

0DUD Lemma 101.45.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is étale, f induces an isomorphism between automorphism groups at points (Remark 101.19.5), and for every algebraically closed field k the functor

$$f : \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \longrightarrow \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{Y})$$

is an equivalence. Then f is an isomorphism.

Proof. By Lemma 101.14.5 we see that f is universally injective. Combining Lemmas 101.45.1 and 101.45.3 we see that f is representable by algebraic spaces. Hence f is an open immersion by Morphisms of Spaces, Lemma 67.51.2. To finish we remark that the condition in the lemma also guarantees that f is surjective. □

101.46. Normalization

0GMH This section is the analogue of Morphisms of Spaces, Section 67.49.

0GMI Lemma 101.46.1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent

(1) there is a surjective smooth morphism $U \to \mathcal{X}$ where U is a scheme such that every quasi-compact open of U has finitely many irreducible components,
(2) for every scheme U and every smooth morphism $U \to \mathcal{X}$ every quasi-compact open of U has finitely many irreducible components,
(3) for every algebraic space Y and smooth morphism $Y \to \mathcal{X}$ the space Y satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1, and
(4) for every quasi-compact algebraic stack $\mathcal{Y}$ smooth over $\mathcal{X}$ the space $|\mathcal{Y}|$ has finitely many irreducible components.

Proof. The equivalence of (1), (2), and (3) follow from Descent, Lemma 35.16.3, Properties of Stacks, Lemma 100.7.1, and Morphisms of Spaces, Lemma 67.49.1. It is also clear from these references that condition (4) implies condition (1). Conversely, assume the equivalent conditions (1), (2), and (3) hold and let $\mathcal{Y} \to \mathcal{X}$ be a smooth morphism of algebraic stacks with $\mathcal{Y}$ quasi-compact. Then we can choose an affine scheme V and a surjective smooth morphism $V \to \mathcal{Y}$ by Properties of Stacks, Lemma 100.6.2. Since V has finitely many irreducible components by (2) and since $|V| \to |\mathcal{Y}|$ is surjective and continuous, we conclude that $|\mathcal{Y}|$ has finitely many irreducible components by Topology, Lemma 5.8.5. □

0GMJ Lemma 101.46.2. Let $\mathcal{X}$ be an algebraic stack satisfying the equivalent conditions of Lemma 101.46.1. Then there exists an integral morphism of algebraic stacks

$$\mathcal{X}^\nu \longrightarrow \mathcal{X}$$

such that for every scheme U and smooth morphism $U \to \mathcal{X}$ the fibre product $\mathcal{X}^\nu \times_\mathcal{X} U$ is the normalization of U.

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. Set $R = U \times_\mathcal{X} U$. Recall that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces and a presentation $\mathcal{X} = [U/R]$ of $\mathcal{X}$, see Algebraic Stacks, Lemmas 94.16.1 and 94.16.2 and Definition 94.16.5. The assumption on $\mathcal{X}$ means that the normalization U^ν of U is defined, see Morphisms, Definition 29.54.1. By Morphisms of Spaces, Lemma 67.49.5 taking normalization commutes with smooth morphisms of algebraic spaces. Thus we see that the normalization R^ν of R is isomorphic to both $R \times_{s,U} U^\nu$ and $U^\nu \times_{U,t} R$. Thus we obtain two smooth morphisms $s^\nu : R^\nu \to U^\nu$ and $t^\nu : R^\nu \to U^\nu$ of algebraic spaces. A formal computation with fibre products shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the smooth morphism $c : R \times_{s,U,t} R \to R$ extends to c^ν as well. Similarly, the inverse $i : R \to R$ (an isomorphism) induces an isomorphism $i^\nu : R^\nu \to R^\nu$. Finally, the identity $e : U \to R$ lifts to $e^\nu : U^\nu \to R^\nu$ for example because e is a section of s and $R^\nu = R \times_{U,s} U^\nu$. We claim that $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu)$ is a smooth groupoid in algebraic spaces. To see this involves checking the axioms (1), (2)(a), (2)(b), (3)(a), and (3)(b) of Groupoids, Section 39.13 for $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu, e^\nu, i^\nu)$. For example, for (1) we have to see that the two morphisms $a, b : R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu \to R^\nu$ we obtain are the same. This holds because we know that the corresponding pair of morphisms $R \times_{s,U,t} R \times_{s,U,t} R \to R$ are the same and the morphisms a and b are the unique extensions of this morphism to the normalizations. Similarly for the other axioms.

Consider the algebraic stack $\mathcal{X}^\nu = [U^\nu / R^\nu]$ (Algebraic Stacks, Theorem 94.17.3). Since we have a morphism $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu) \to (U, R, s, t, c)$ of groupoids in algebraic spaces, we obtain a morphism $\nu : \mathcal{X}^\nu \to \mathcal{X}$ of algebraic stacks. Since $R^\nu = R \times_{s,U} U^\nu$ we see that $U^\nu = \mathcal{X}^\nu \times_\mathcal{X} U$ by Groupoids in Spaces, Lemma 78.25.3. In particular, as $U^\nu \to U$ is integral, we see that ν is integral. We omit the verification that the base change property stated in the lemma holds for every smooth morphism from a scheme to $\mathcal{X}$. $\square$

This leads us to the following definition.

0GMK Definition 101.46.3. Let $\mathcal{X}$ be an algebraic stack satisfying the equivalent conditions of Lemma 101.46.1. We define the normalization of $\mathcal{X}$ as the morphism

$$\nu : \mathcal{X}^\nu \longrightarrow \mathcal{X}$$

constructed in Lemma 101.46.2.

101.47. Points and specializations

0GVY This section is the analogue of Decent Spaces, Section 68.7.

0GVZ Lemma 101.47.1. Let $\mathcal{X}$ be an algebraic stack. Let $f : U \to \mathcal{X}$ be a smooth morphism where U is an algebraic space. Let $x' \rightsquigarrow x$ be a specialization of points of $|\mathcal{X}|$. Let $u \in |U|$ with $f(u) = x$. If $(\mathcal{X}, x')$ satisfy the equivalent conditions of Properties of Stacks, Lemma 100.14.1, then there exists a specialization $u' \rightsquigarrow u$ in $|U|$ with $f(u') = x'$.

Proof. Choose an étale morphism $(U_1, u_1) \to (U, u)$ where U_1 is an affine scheme. Then we may and do replace U by U_1. Thus we may assume U is an affine scheme.

Consider the algebraic space $R = U \times_{\mathcal{X}} U$ with smooth projections $t, s : R \to U$. Choose a point $w \in U$ mapping to x'; this is possible as $f : |U| \to |\mathcal{X}|$ is open. By our assumption on x' the fibre $F' = t^{-1}(w) = R \times_{t,U} w$ of $t : R \to U$ over w is a quasi-compact algebraic space. Choose an affine scheme T and a surjective étale morphism $T \to F'$. The fact that $x' \rightsquigarrow x$ means that u is in the closure of the image of the morphism

$$T \to F' \to R \xrightarrow{s} U$$

Namely, this image is the fibre of $|U| \to |\mathcal{X}'|$ over x'; if some $u \in V \subset |U|$ open is disjoint from this fibre, then $f(V)$ is an open neighbourhood of x not containing x'; contradiction. Thus by Morphisms, Lemma 29.6.5 we see that there exists $u' \in |U|$ in the fibre of $|U| \to |\mathcal{X}|$ over x' which specializes to u. $\square$

101.48. Decent algebraic stacks

0GW0 This section is the analogue of Decent Spaces, Section 68.6. In particular, the following definition is compatible with the notion of a decent algebraic space defined there.

0GW1 Definition 101.48.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is decent if for every $x \in |\mathcal{X}|$ the equivalent conditions of Properties of Stacks, Lemma 100.14.1 are satisfied.

Some people would rephrase this definition by saying that every point of $\mathcal{X}$ is quasi-compact. A slightly stronger condition would be to ask that any morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-separated as well as quasi-compact.

0GW2 Lemma 101.48.2. A quasi-separated algebraic stack $\mathcal{X}$ is decent. More generally, if $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, then $\mathcal{X}$ is decent.

Proof. Namely, if $\mathcal{X}$ is quasi-separated, then any morphism $f : T \to \mathcal{X}$ whose source is a quasi-compact scheme T, is quasi-compact, see Lemma 101.7.7. If Δ is on known to be quasi-compact, then one uses the description

$$T \times_{f,\mathcal{X},f'} T' = (T \times T') \times_{(f,f'),\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$$

to prove this. Details omitted. $\square$

0GW3 Lemma 101.48.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume Y is decent and f is representable (by schemes) or f is representable by algebraic spaces and quasi-separated. Then $\mathcal{X}$ is decent.

Proof. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Choose a morphism $y : \mathrm{Spec}(k) \to \mathcal{Y}$ in the equivalence class defining y. Set $\mathcal{X}_y = \mathrm{Spec}(k) \times_{y,\mathcal{Y}} \mathcal{X}$. Choose a point $x' \in |\mathcal{X}_y|$ mapping to x, see Properties of Stacks, Lemma 100.4.3. Choose a morphism $x' : \mathrm{Spec}(k') \to \mathcal{X}_y$ in the equivalence class of x'. Diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "x'"'] & \mathcal{X}_y \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \mathrm{Spec}(k) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

The morphism y is quasi-compact if $\mathcal{Y}$ is decent. Hence $\mathcal{X}_y \to \mathcal{X}$ is quasi-compact as a base change (Lemma 101.7.3). Thus to conclude it suffices to prove that x' is quasi-compact (Lemma 101.7.4). If f is representable, then $\mathcal{X}_y$ is a scheme and x' is quasi-compact. If f is representable by algebraic spaces and quasi-separated, then $\mathcal{X}_y$ is a quasi-separated algebraic space and hence decent (Decent Spaces, Lemma 68.17.2). □

0GW4 Lemma 101.48.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is quasi-compact and surjective and $\mathcal{X}$ is decent, then $\mathcal{Y}$ is decent.

Proof. Let $x : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism where k is a field and denote $y = f \circ x$. Since f is surjective, every point of $|\mathcal{Y}|$ arises in this manner, see Properties of Stacks, Lemma 100.4.4. Consider an affine scheme T and morphism $T \to \mathcal{Y}$. It suffices to show that $T \times_{\mathcal{Y},y} \mathrm{Spec}(k)$ is quasi-compact, see Lemma 101.7.10. We have

$$(T \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X},x} \mathrm{Spec}(k) = T \times_{\mathcal{Y},y} \mathrm{Spec}(k)$$

The morphism $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is quasi-compact hence $T \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact. Since x is a quasi-compact morphism as $\mathcal{X}$ is decent we see that the displayed fibre product is quasi-compact. □

0GW5 Lemma 101.48.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{X}$ is decent, then $\mathcal{Y}$ is decent.

Proof. Assume $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{X}$ is decent. Note that f is a universal homeomorphism by Lemma 101.28.13. Thus the lemma follows from Lemma 101.48.4. □

101.49. Points on decent stacks

0GW6 This section is the analogue of Decent Spaces, Section 68.12. We do not know whether or not the topological space associated to a decent algebraic stack is always sober; see Proposition 101.49.3 for a slightly weaker result.

0GW7 Lemma 101.49.1. Let $\mathcal{X}$ be a decent algebraic stack. Then $|\mathcal{X}|$ is Kolmogorov (see Topology, Definition 5.8.6).

Proof. Let $x_1, x_2 \in |\mathcal{X}|$ with $x_1 \leadsto x_2$ and $x_2 \leadsto x_1$. We have to show that $x_1 = x_2$. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced closed substack with $|\mathcal{Z}|$ equal to $\overline{\{x_1\}} = \overline{\{x_2\}}$. By Lemma 101.48.3 we see that $\mathcal{Z}$ is decent. After replacing $\mathcal{X}$ by $\mathcal{Z}$ we reduce to the case discussed in the next paragraph.

Assume $|\mathcal{X}|$ is irreducible with generic points x_1 and x_2. Pick an affine scheme U and $u_1, u_2 \in U$ and a smooth morphism $f : U \to \mathcal{X}$ such that $f(u_i) = x_i$. Then we find a third point $u_3 \in U$ which is the generic point of an irreducible component of U whose image $x_3 \in |\mathcal{X}|$ is also a generic point of $|\mathcal{X}|$. Namely, we can simply choose u_3 any generic point of an irreducible component passing through u_1 (or u_2 if you like). In the next paragraph we will show that $x_1 = x_3$ and $x_2 = x_3$ which will prove what we want.

By symmetry it suffices to prove that $x_1 = x_3$. Since x_1 is a generic point of $|\mathcal{X}|$ we have a specialization $x_1 \leadsto x_3$. By Lemma 101.47.1 we can find a specialization $u_1' \leadsto u_3$ in U (!) mapping to $x_1 \leadsto x_3$. However, u_3 is the generic point of an irreducible component and hence $u_1' = u_3$ as desired. □

0GW8 Lemma 101.49.2. Let $\mathcal{X}$ be a decent, locally Noetherian algebraic stack. Then $|\mathcal{X}|$ is a sober locally Noetherian topological space.

Proof. By Lemma 101.8.3 the topological space $|\mathcal{X}|$ is locally Noetherian. By Lemma 101.49.1 the topological space $|\mathcal{X}|$ is Kolmogorov. By Lemma 101.8.4 the topological space $|\mathcal{X}|$ is quasi-sober. This finishes the proof, see Topology, Definition 5.8.6. □

0GW9 Proposition 101.49.3. Let $\mathcal{X}$ be a decent algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then $|\mathcal{X}|$ is sober.

Proof. By Lemma 101.49.1 we know that $|\mathcal{X}|$ is Kolmogorov (in fact we will reprove this). Let $T \subset |\mathcal{X}|$ be an irreducible closed subset. We have to show T has a generic point. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced closed substack corresponding to T, see Properties of Stacks, Definition 100.10.4. Since $\mathcal{Z} \to \mathcal{X}$ is a closed immersion, we see that $\mathcal{Z}$ is a decent algebraic stack, see Lemma 101.48.3. Also, the morphism $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ (Lemma 101.5.6). Hence $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is quasi-compact (Lemma 101.7.3). Thus we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is decent, $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact, $\mathcal{X}$ is reduced, and $|\mathcal{X}|$ irreducible. We have to show $|\mathcal{X}|$ has a generic point. By Proposition 101.29.1. there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe. In other words, $|\mathcal{U}| \subset |\mathcal{X}|$ is open dense. Thus we may assume that $\mathcal{X}$ is a gerbe in addition to all the other properties. Say $\mathcal{X} \to X$ turns $\mathcal{X}$ into a gerbe over the algebraic space X. Then $|\mathcal{X}| \cong |X|$ by Lemma 101.28.13. In particular, X is quasi-compact and $|X|$ is irreducible. Also, by Lemma 101.48.5 we see that X is a decent algebraic space. Then $|\mathcal{X}| = |X|$ is sober by Decent Spaces, Proposition 68.12.4 and hence has a (unique) generic point. □

101.50. Integral algebraic stacks

0GWA This section is the analogue of Spaces over Fields, Section 72.4. Motivated by the considerations in that section and by the result of Proposition 101.49.3 we define an integral algebraic stack as follows (and it does not conflict with the already existing definitions of integral schemes and integral algebraic spaces).

0GWB Definition 101.50.1. We say an algebraic stack $\mathcal{X}$ is integral if it is reduced, decent, $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact, and $|\mathcal{X}|$ is irreducible.

Note that if $\mathcal{X}$ is quasi-separated, then for it to be integral, it suffices that $\mathcal{X}$ is reduced and that $|\mathcal{X}|$ is irreducible, see Lemma 101.50.3.

0GWC Lemma 101.50.2. Let $\mathcal{X}$ be an integral algebraic stack. Then

(1) $|\mathcal{X}|$ is sober, irreducible, and has a unique generic point,
(2) there exists an open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over an integral scheme U.

Proof. Proposition 101.49.3 tells us that $|\mathcal{X}|$ is sober. Of course it is also irreducible and hence has a unique generic point x (by the definition of sobriety). Proposition 101.29.1 shows the existence of a dense open $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over an algebraic space U. Then U is a decent algebraic space by Lemma 101.48.5 (and the fact that $\mathcal{U}$ is decent by Lemma 101.48.3). Since $|U| = |\mathcal{U}|$ we see that $|U|$ is irreducible. Finally, since $\mathcal{U}$ is reduced the morphism $\mathcal{U} \to U$ factors through U_{red}, see Properties of Stacks, Lemma 100.10.3. Now since $\mathcal{U} \to U$ is flat, locally of finite presentation, and surjective (Lemma 101.28.8), this implies that $U = U_{red}$, i.e., U is reduced (small detail omitted). It follows that U is an integral algebraic space, see Spaces over Fields, Definition 72.4.1. Then finally, we may replace U (and correspondingly $\mathcal{U}$) by an open subspace and assume that U is an integral scheme, see discussion in Spaces over Fields, Section 72.4. □

0GWD Lemma 101.50.3. Let $\mathcal{X}$ be an algebraic stack which is reduced and quasi-separated and whose associated topological space $|\mathcal{X}|$ is irreducible. Then $\mathcal{X}$ is integral.

Proof. If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is decent by Lemma 101.48.2. If $\mathcal{X}$ is quasi-separated, then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, hence $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact as the base change of Δ by Δ, see Lemma 101.7.3. Thus we see that all the hypotheses of Definition 101.50.1 hold (and we also see that we may replace "quasi-separated" by "$\Delta_\mathcal{X}$ is quasi-compact"). □

0GWE Lemma 101.50.4. Let $\mathcal{X}$ be a decent algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. There are canonical bijections between the following sets:

(1) the set of points of $\mathcal{X}$, i.e., $|\mathcal{X}|$,
(2) the set of irreducible closed subsets of $|\mathcal{X}|$,
(3) the set of integral closed substacks of $\mathcal{X}$.

The bijection from (1) to (2) sends x to $\overline{\{x\}}$. The bijection from (3) to (2) sends $\mathcal{Z}$ to $|\mathcal{Z}|$.

Proof. Our map defines a bijection between (1) and (2) as $|\mathcal{X}|$ is sober by Proposition 101.49.3. Given $T \subset |\mathcal{X}|$ closed and irreducible, there is a unique reduced closed substack $\mathcal{Z} \subset \mathcal{X}$ such that $|\mathcal{Z}| = T$, namely, $\mathcal{Z}$ is the reduced induced subspace structure on T, see Properties of Stacks, Definition 100.10.4. Then $\mathcal{Z}$ is an integral algebraic stack because it is decent (Lemma 101.48.3), the morphism $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is quasi-compact (as the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemma 101.5.6), $\mathcal{Z}$ is reduced, and $|\mathcal{Z}|$ is irreducible. □

101.51. Residual gerbes

0H23 This section is the continuation of Properties of Stacks, Section 100.11.

0H24 Lemma 101.51.1. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Assume the residual gerbe $\mathcal{Z}_y \subset \mathcal{Y}$ of $\mathcal{Y}$ at y exists and that $\mathcal{X}$ is a gerbe over $\mathcal{Y}$. Then $\mathcal{Z}_x = \mathcal{Z}_y \times_\mathcal{Y} \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x.

Proof. The morphism $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism as the base change of the monomorphism $\mathcal{Z}_y \to \mathcal{Y}$. The morphism π is a univeral homeomorphism by Lemma 101.28.13 and hence $|\mathcal{Z}_x| = \{x\}$. Finally, the morphism $\mathcal{Z}_x \to \mathcal{Z}_y$ is smooth as a base change of the smooth morphism π, see Lemma 101.33.8. Hence as $\mathcal{Z}_y$ is reduced and locally Noetherian, so is $\mathcal{Z}_x$ (details omitted). Thus $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x by Properties of Stacks, Definition 100.11.8. □

0H25 Lemma 101.51.2. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. Assume

(1) $\mathcal{X}$ is decent or locally Noetherian (or both),
(2) $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact,
(3) $|f|(|\mathcal{Y}|)$ is contained in $\{x\} \subset |\mathcal{X}|$, and
(4) $\mathcal{Y}$ is reduced.

Then f factors through the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x (whose existence is guaranteed by Lemma 101.31.1 or 101.31.3).

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 100.10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. By Properties of Stacks, Lemma 100.10.3 the morphism f factors through $\mathcal{X}'$. If $\mathcal{X}$ is decent, then by Lemma 101.48.3 the stack $\mathcal{X}'$ is decent. If $\mathcal{X}$ is locally Noetherian, then $\mathcal{X}'$ is locally Noetherian (details omitted). Note that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ by Lemma 101.5.6 we see that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact by Lemma 101.7.3. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is a generic point. By Proposition 101.29.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$. Repeating the arguments above we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X} \to X$ is a gerbe over the algebraic space X. If $\mathcal{X}$ is decent, then by Lemmas 101.28.13 and 101.48.4 the space X is decent. If $\mathcal{X}$ is locally Noetherian, then X is locally Noetherian by fppf descent (details omitted). Hence the corresponding result holds for X, see Decent Spaces, Lemma 68.13.10 or 68.13.9 (small detail omitted). Applying Lemma 101.51.1 we conclude that the result holds for $\mathcal{X}$ as well. $\square$

0H26 Remark 101.51.3. We do not know whether Lemma 101.51.2 holds if we only assume $\mathcal{X}$ is locally Noetherian, i.e., we drop the assumption on the inertia being quasi-compact. In this case, if x is a closed point, this is certainly true as follows from the following much simpler lemma.

0H27 Lemma 101.51.4. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ with residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ (Lemma 101.31.3). Then x is a closed point of $|\mathcal{X}|$ if and only if the morphism $\mathcal{Z}_x \to \mathcal{X}$ is a closed immersion.

Proof. If $\mathcal{Z}_x \to \mathcal{X}$ is a closed immersion, then x is a closed point of $|\mathcal{X}|$, see for example Lemma 101.37.4. Conversely, assume x is a closed point of $|\mathcal{X}|$. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced closed substack with $|Z| = \{x\}$ (Properties of Stacks, Lemma 100.10.1). Then $\mathcal{Z}$ is a locally Noetherian algebraic stack by Lemmas 101.17.4 and 101.17.5. Since also $\mathcal{Z}$ is reduced and $|\mathcal{Z}| = \{x\}$ it follows that $\mathcal{Z} = \mathcal{Z}_x$ is the residual gerbe by definition. $\square$

101.52. Other chapters

Preliminaries

CHAPTER 102

Limits of Algebraic Stacks

0CMM 102.1. Introduction

0CMN In this chapter we put material related to limits of algebraic stacks. Many results on limits of algebraic stacks and algebraic spaces have been obtained by David Rydh in [Ryd08].

102.2. Conventions

0CMP We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

102.3. Morphisms of finite presentation

0CMQ This section is the analogue of Limits of Spaces, Section 70.3. There we defined what it means for a transformation of functors on *Sch* to be limit preserving (we suggest looking at the characterization in Limits of Spaces, Lemma 70.3.2). In Criteria for Representability, Section 97.5 we defined the notion "limit preserving on objects". Recall that in Artin's Axioms, Section 98.11 we have defined what it means for a category fibred in groupoids over *Sch* to be limit preserving. Combining these we get the following notion.

0CMR Definition 102.3.1. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f is limit preserving if for every directed limit $U = \lim U_i$ of affine schemes over S the diagram

$$\begin{tikzcd} \operatorname{colim} \mathcal{X}_{U_i} \arrow[r] \arrow[d, "f"'] & \mathcal{X}_U \arrow[d, "f"] \\ \operatorname{colim} \mathcal{Y}_{U_i} \arrow[r] & \mathcal{Y}_U \end{tikzcd}$$

of fibre categories is 2-cartesian.

0CMS Lemma 102.3.2. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is limit preserving (Definition 102.3.1), then f is limit preserving on objects (Criteria for Representability, Section 97.5).

Proof. If for every directed limit $U = \lim U_i$ of affine schemes over U, the functor

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow (\operatorname{colim} \mathcal{Y}_{U_i}) \times_{\mathcal{Y}_U} \mathcal{X}_U$$

is essentially surjective, then f is limit preserving on objects. $\square$

0CMT Lemma 102.3.3. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. For each i we have

$$(\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{Z}_{U_i}$$

Filtered colimits commute with 2-fibre products of categories (details omitted) hence if p is limit preserving we get

$$\begin{aligned} \text{colim}(\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_{U_i} &= \text{colim}\, \mathcal{X}_{U_i} \times_{\text{colim}\, \mathcal{Y}_{U_i}} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i} \times_{\text{colim}\, \mathcal{Y}_{U_i}} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Z}_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \\ &= (\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \end{aligned}$$

as desired. □

0CMU Lemma 102.3.4. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving, then so is the composition $q \circ p$.

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. If p and q are limit preserving we get

$$\begin{aligned} \text{colim}\, \mathcal{X}_{U_i} &= \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Y}_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \end{aligned}$$

as desired. □

0CMV Lemma 102.3.5. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:

(1) p is limit preserving,
(2) p is limit preserving on objects, and
(3) p is locally of finite presentation (see Algebraic Stacks, Definition 94.10.1).

Proof. In Criteria for Representability, Lemma 97.5.3 we have seen that (2) and (3) are equivalent. Thus it suffices to show that (1) and (2) are equivalent. One direction we saw in Lemma 102.3.2. For the other direction, let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. We have to show that

$$\text{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i}$$

is an equivalence. Since we are assuming (2) we know that it is essentially surjective. Hence we need to prove it is fully faithful. Since p is faithful on fibre categories (Algebraic Stacks, Lemma 94.9.2) we see that the functor is faithful. Let x_i and x_i' be objects in the fibre category of $\mathcal{X}$ over U_i. The functor above sends x_i to $(x_i|_U, p(x_i), can)$ where can is the canonical isomorphism $p(x_i|_U) \to p(x_i)|_U$. Thus we assume given a morphism

$$(\alpha, \beta_i) : (x_i|_U, p(x_i), can) \longrightarrow (x_i'|_U, p(x_i'), can)$$

in the category of the right hand side of the first displayed arrow of this proof. Our task is to produce an $i' \geq i$ and a morphism $x_i|_{U_{i'}} \to x_i'|_{U_{i'}}$ which maps to $(\alpha, \beta_i|_{U_{i'}})$.

Set $y_i = p(x_i)$ and $y'_i = p(x'_i)$. By (Algebraic Stacks, Lemma 94.9.2) the functor

$$X_{y_i} : (Sch/U_i)^{opp} \to \text{Sets}, \quad V/U_i \mapsto \{(x, \phi) \mid x \in \text{Ob}(\mathcal{X}_V), \phi : f(x) \to y_i|V\}/\cong$$

is an algebraic space over U_i and the same is true for the analogously defined functor $X_{y'_i}$. Since (2) is equivalent to (3) we see that $X_{y'_i}$ is locally of finite presentation over U_i. Observe that (x_i, id) and (x'_i, id) define U_i-valued points of X_{y_i} and $X_{y'_i}$. There is a transformation of functors

$$\beta_i : X_{y_i} \to X_{y'_i}, \quad (x/V, \phi) \mapsto (x/V, \beta_i|_V \circ \phi)$$

in other words, this is a morphism of algebraic spaces over U_i. We claim that

$$\begin{tikzcd} U \arrow[rr] \arrow[d] & & U_i \arrow[d, "{(x'_i,\text{id})}"] \\ U_i \arrow[r, "{(x_i,\text{id})}"] & X_{y_i} \arrow[r, "\beta_i"] & X_{y'_i} \end{tikzcd}$$

commutes. Namely, this is equivalent to the condition that the pairs $(x_i|_U, \beta_i|_U)$ and $(x'_i|_U, \text{id})$ as in the definition of the functor $X_{y'_i}$ are isomorphic. And the morphism $\alpha : x_i|_U \to x'_i|_U$ exactly produces such an isomorphism. Arguing backwards the reader sees that if we can find an $i' \geq i$ such that the diagram

$$\begin{tikzcd} U_{i'} \arrow[rr] \arrow[d] & & U_i \arrow[d, "{(x'_i,\text{id})}"] \\ U_i \arrow[r, "{(x_i,\text{id})}"] & X_{y_i} \arrow[r, "\beta_i"] & X_{y'_i} \end{tikzcd}$$

commutes, then we obtain an isomorphism $x_i|_{U_{i'}} \to x'_i|_{U_{i'}}$ which is a solution to the problem posed in the preceding paragraph. However, the diagonal morphism

$$\Delta : X_{y'_i} \to X_{y'_i} \times_{U_i} X_{y'_i}$$

is locally of finite presentation (Morphisms of Spaces, Lemma 67.28.10) hence the fact that $U \to U_i$ equalizes the two morphisms to $X_{y'_i}$, means that for some $i' \geq i$ the morphism $U_{i'} \to U_i$ equalizes the two morphisms, see Limits of Spaces, Proposition 70.3.10. $\square$

0CMW Lemma 102.3.6. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent

(1) the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is limit preserving, and
(2) for every directed limit $U = \lim U_i$ of affine schemes over S the functor

$$\text{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i}$$

is fully faithful.

In particular, if p is limit preserving, then Δ is too.

Proof. Let $U = \lim U_i$ be a directed limit of affine schemes over S. We claim that the functor

$$\text{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i}$$

is fully faithful if and only if the functor

$$\text{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{X})_U} \text{colim} (\mathcal{X} \times_\mathcal{Y} \mathcal{X})_{U_i}$$

is an equivalence. This will prove the lemma. Since $(\mathcal{X} \times_\mathcal{Y} \mathcal{X})_U = \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{X}_U$ and $(\mathcal{X} \times_\mathcal{Y} \mathcal{X})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{X}_{U_i}$ this is a purely category theoretic assertion which we discuss in the next paragraph.

Let $\mathcal{I}$ be a filtered index category. Let $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ be systems of groupoids over $\mathcal{I}$. Let $p : (\mathcal{C}_i) \to (\mathcal{D}_i)$ be a map of systems of groupoids over $\mathcal{I}$. Suppose we have a functor $p : \mathcal{C} \to \mathcal{D}$ of groupoids and functors $f : \operatorname{colim} \mathcal{C}_i \to \mathcal{C}$ and $g : \operatorname{colim} \mathcal{D}_i \to \mathcal{D}$ fitting into a commutative diagram

$$\begin{tikzcd} \operatorname{colim} \mathcal{C}_i \arrow[r, "f"'] \arrow[d, "p"'] & \mathcal{C} \arrow[d, "p"] \\ \operatorname{colim} \mathcal{D}_i \arrow[r, "g"] & \mathcal{D} \end{tikzcd}$$

Then we claim that

$$A : \operatorname{colim} \mathcal{C}_i \longrightarrow \mathcal{C} \times_\mathcal{D} \operatorname{colim} \mathcal{D}_i$$

is fully faithful if and only if the functor

$$B : \operatorname{colim} \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_\mathcal{D} \mathcal{C}, f \times_g f} \operatorname{colim}(\mathcal{C}_i \times_{\mathcal{D}_i} \mathcal{C}_i)$$

is an equivalence. Set $\mathcal{C}' = \operatorname{colim} \mathcal{C}_i$ and $\mathcal{D}' = \operatorname{colim} \mathcal{D}_i$. Since 2-fibre products commute with filtered colimits we see that A and B become the functors

$$A' : \mathcal{C}' \to \mathcal{C} \times_\mathcal{D} \mathcal{D}' \quad \text{and} \quad B' : \mathcal{C}' \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_\mathcal{D} \mathcal{C}, f \times_g f} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}')$$

Thus it suffices to prove that if

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "f"'] \arrow[d, "p"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r, "g"] & \mathcal{D} \end{tikzcd}$$

is a commutative diagram of groupoids, then A' is fully faithful if and only if B' is an equivalence. This follows from Categories, Lemma 4.35.10 (with trivial, i.e., punctual, base category) because

$$\mathcal{C} \times_{\Delta, \mathcal{C} \times_\mathcal{D} \mathcal{C}, f \times_g f} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}') = \mathcal{C}' \times_{A', \mathcal{C} \times_\mathcal{D} \mathcal{D}', A'} \mathcal{C}'$$

This finishes the proof. □

0CMX Lemma 102.3.7. Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. If $\mathcal{X} \to S$ is locally of finite presentation, then $\mathcal{X}$ is limit preserving in the sense of Artin's Axioms, Definition 98.11.1 (equivalently: the morphism $\mathcal{X} \to S$ is limit preserving).

Proof. Choose a surjective smooth morphism $U \to \mathcal{X}$ for some scheme U. Then $U \to S$ is locally of finite presentation, see Morphisms of Stacks, Section 101.27. We can write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 94.16.2. Since U is locally of finite presentation over S it follows that the algebraic space R is locally of finite presentation over S. Recall that $[U/R]$ is the stack in groupoids over $(Sch/S)_{fppf}$ obtained by stackyfying the category fibred in groupoids whose fibre category over T is the groupoid $(U(T), R(T), s, t, c)$. Since U and R are limit preserving as functors (Limits of Spaces, Proposition 70.3.10) this category fibred in groupoids is limit preserving. Thus it suffices to show that fppf stackyfication preserves the property of being limit preserving. This is true (hint: use Topologies, Lemma 34.13.2). However, we give a direct proof below using that in this case we know what the stackyfication amounts to.

Let $T = \lim T_\lambda$ be a directed limit of affine schemes over S. We have to show that the functor

$$\mathrm{colim}[U/R]_{T_\lambda} \longrightarrow [U/R]_T$$

is an equivalence of categories. Let us show this functor is essentially surjective. Let $x \in \mathrm{Ob}([U/R]_T)$. In Groupoids in Spaces, Lemma 78.24.1 the reader finds a description of the category $[U/R]_T$. In particular x corresponds to an fppf covering $\{T_i \to T\}_{i \in I}$ and a $[U/R]$-descent datum (u_i, r_{ij}) relative to this covering. After refining this covering we may assume it is a standard fppf covering of the affine scheme T. By Topologies, Lemma 34.13.2 we may choose a λ and a standard fppf covering $\{T_{\lambda,i} \to T_\lambda\}_{i \in I}$ whose base change to T is equal to $\{T_i \to T\}_{i \in I}$. For each i, after increasing λ, we can find a $u_{\lambda,i} : T_{\lambda,i} \to U$ whose composition with $T_i \to T_{\lambda,i}$ is the given morphism u_i (this is where we use that U is limit preserving). Similarly, for each i, j, after increasing λ, we can find a $r_{\lambda,ij} : T_{\lambda,i} \times_{T_\lambda} T_{\lambda,j} \to R$ whose composition with $T_{ij} \to T_{\lambda,ij}$ is the given morphism r_{ij} (this is where we use that R is limit preserving). After increasing λ we can further assume that

$$s \circ r_{\lambda,ij} = u_{\lambda,i} \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{\lambda,ij} = u_{\lambda,j} \circ \mathrm{pr}_1,$$

and

$$c \circ (r_{\lambda,jk} \circ \mathrm{pr}_{12}, r_{\lambda,ij} \circ \mathrm{pr}_{01}) = r_{\lambda,ik} \circ \mathrm{pr}_{02}.$$

In other words, we may assume that $(u_{\lambda,i}, r_{\lambda,ij})$ is a $[U/R]$-descent datum relative to the covering $\{T_{\lambda,i} \to T_\lambda\}_{i \in I}$. Then we obtain a corresponding object of $[U/R]$ over T_λ whose pullback to T is isomorphic to x as desired. The proof of fully faithfulness works in exactly the same way using the description of morphisms in the fibre categories of $[U/T]$ given in Groupoids in Spaces, Lemma 78.24.1. □

0CMY **Proposition 102.3.8.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent This is a special case of [EG15, Lemma 2.3.15]

(1) f is limit preserving,
(2) f is limit preserving on objects, and
(3) f is locally of finite presentation.

Proof. Assume (3). Let $T = \lim T_i$ be a directed limit of affine schemes. Consider the functor

$$\mathrm{colim}\, \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T \times_{\mathcal{Y}_T} \mathrm{colim}\, \mathcal{Y}_{T_i}$$

Let (x, y_i, β) be an object on the right hand side, i.e., $x \in \mathrm{Ob}(\mathcal{X}_T)$, $y_i \in \mathrm{Ob}(\mathcal{Y}_{T_i})$, and $\beta : f(x) \to y_i|_T$ in $\mathcal{Y}_T$. Then we can consider (x, y_i, β) as an object of the algebraic stack $\mathcal{X}_{y_i} = \mathcal{X} \times_{\mathcal{Y}, y_i} T_i$ over T. Since $\mathcal{X}_{y_i} \to T_i$ is locally of finite presentation (as a base change of f) we see that it is limit preserving by Lemma 102.3.7. This means that (x, y_i, β) comes from an object over $T_{i'}$ for some $i' \geq i$ and unwinding the definitions we find that (x, y_i, β) is in the essential image of the displayed functor. In other words, the displayed functor is essentially surjective. Another formulation is that this means f is limit preserving on objects. Now we apply this to the diagonal Δ of f. Namely, by Morphisms of Stacks, Lemma 101.27.7 the morphism Δ is locally of finite presentation. Thus the argument above shows that Δ is limit preserving on objects. By Lemma 102.3.5 this implies that Δ is limit preserving. By Lemma 102.3.6 we conclude that the displayed functor above is fully faithful. Thus it is an equivalence (as we already proved essential surjectivity) and we conclude that (1) holds.

The implication (1) $\Rightarrow$ (2) is trivial. Assume (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. By Criteria for Representability, Lemma 97.5.1 the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is limit preserving on objects. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Since a smooth morphism is locally of finite presentation, we see that $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is limit preserving (first part of the proof). By Criteria for Representability, Lemma 97.5.2 we find that the composition $U \to V$ is limit preserving on objects. We conclude that $U \to V$ is locally of finite presentation, see Criteria for Representability, Lemma 97.5.3. This is exactly the condition that f is locally of finite presentation, see Morphisms of Stacks, Definition 101.27.1. □

102.4. Descending properties

0CPX This section is the analogue of Limits, Section 32.4.

0CPY Situation 102.4.1. Let $Y = \lim_{i \in I} Y_i$ be a limit of a directed system of algebraic spaces with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

0CPZ Lemma 102.4.2. In Situation 102.4.1 assume that $\mathcal{X}_0 \to Y_0$ is a morphism from algebraic stack to Y_0. Assume $\mathcal{X}_0$ is quasi-compact and quasi-separated. If $Y \times_{Y_0} \mathcal{X}_0 \to Y$ is separated, then $Y_i \times_{Y_0} \mathcal{X}_0 \to Y_i$ is separated for all sufficiently large $i \in I$.

Proof. Write $\mathcal{X} = Y \times_{Y_0} \mathcal{X}_0$ and $\mathcal{X}_i = Y_i \times_{Y_0} \mathcal{X}_0$. Choose an affine scheme U_0 and a surjective smooth morphism $U_0 \to \mathcal{X}_0$. Set $U = Y \times_{Y_0} U_0$ and $U_i = Y_i \times_{Y_0} U_0$. Then U and U_i are affine and $U \to \mathcal{X}$ and $U_i \to \mathcal{X}_i$ are smooth and surjective. Set $R_0 = U_0 \times_{\mathcal{X}_0} U_0$. Set $R = Y \times_{Y_0} R_0$ and $R_i = Y_i \times_{Y_0} R_0$. Then $R = U \times_{\mathcal{X}} U$ and $R_i = U_i \times_{\mathcal{X}_i} U_i$.

With this notation note that $\mathcal{X} \to Y$ is separated implies that $R \to U \times_Y U$ is proper as the base change of $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ by $U \times_Y U \to \mathcal{X} \times_Y \mathcal{X}$. Conversely, we see that $\mathcal{X}_i \to Y_i$ is separated if $R_i \to U_i \times_{Y_i} U_i$ is proper because $U_i \times_{Y_i} U_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is surjective and smooth, see Properties of Stacks, Lemma 100.3.3. Observe that $R_0 \to U_0 \times_{Y_0} U_0$ is locally of finite type and that R_0 is quasi-compact and quasi-separated. By Limits of Spaces, Lemma 70.6.13 we see that $R_i \to U_i \times_{Y_i} U_i$ is proper for large enough i which finishes the proof. □

102.5. Descending relative objects

0CN3 This section is the analogue of Limits of Spaces, Section 70.7.

0CN4 Lemma 102.5.1. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system of algebraic spaces over I. Assume

1. the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,
2. the spaces X_i are quasi-compact and quasi-separated.

Let $X = \lim X_i$. If $\mathcal{X}$ is an algebraic stack of finite presentation over X, then there exists an $i \in I$ and an algebraic stack $\mathcal{X}_i$ of finite presentation over X_i with $\mathcal{X} \cong \mathcal{X}_i \times_{X_i} X$ as algebraic stacks over X.

Proof. By Morphisms of Stacks, Definition 101.27.1 the morphism $\mathcal{X} \to X$ is quasi-compact, locally of finite presentation, and quasi-separated. Since X is quasi-compact and $\mathcal{X} \to X$ is quasi-compact, we see that $\mathcal{X}$ is quasi-compact (Morphisms of Stacks, Definition 101.7.2). Hence we can find an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$ (Properties of Stacks, Lemma 100.6.2). Set $R = U \times_{\mathcal{X}} U$. We obtain a smooth groupoid in algebraic spaces (U, R, s, t, c) over X such that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 94.16.2. Since $\mathcal{X} \to X$ is quasi-separated and X is quasi-separated we see that $\mathcal{X}$ is quasi-separated (Morphisms of Stacks, Lemma 101.4.10). Thus $R \to U \times U$ is quasi-compact and quasi-separated (Morphisms of Stacks, Lemma 101.4.7) and hence R is a quasi-separated and quasi-compact algebraic space. On the other hand $U \to X$ is locally of finite presentation and hence also $R \to X$ is locally of finite presentation (because $s : R \to U$ is smooth hence locally of finite presentation). Thus (U, R, s, t, c) is a groupoid object in the category of algebraic spaces which are of finite presentation over X. By Limits of Spaces, Lemma 70.7.1 there exists an i and a groupoid in algebraic spaces $(U_i, R_i, s_i, t_i, c_i)$ over X_i whose pullback to X is isomorphic to (U, R, s, t, c). After increasing i we may assume that s_i and t_i are smooth, see Limits of Spaces, Lemma 70.6.3. The quotient stack $\mathcal{X}_i = [U_i/R_i]$ is an algebraic stack (Algebraic Stacks, Theorem 94.17.3).

There is a morphism $[U/R] \to [U_i/R_i]$, see Groupoids in Spaces, Lemma 78.21.1. We claim that combined with the morphisms $[U/R] \to X$ and $[U_i/R_i] \to X_i$ (Groupoids in Spaces, Lemma 78.20.2) we obtain an isomorphism (i.e., equivalence)

$$[U/R] \longrightarrow [U_i/R_i] \times_{X_i} X$$

The corresponding map

$$[U/_pR] \longrightarrow [U_i/_pR_i] \times_{X_i} X$$

on the level of "presheaves of groupoids" as in Groupoids in Spaces, Equation (78.20.0.1) is an isomorphism. Thus the claim follows from the fact that stackification commutes with fibre products, see Stacks, Lemma 8.8.4. ☐

102.6. Finite type closed in finite presentation

0CQ0 This section is the analogue of Limits of Spaces, Section 70.11.

0CQ1 Lemma 102.6.1. Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume:

(1) f is of finite type and quasi-separated,
(2) Y is quasi-compact and quasi-separated.

Then there exists a morphism of finite presentation $f' : \mathcal{X}' \to Y$ and a closed immersion $\mathcal{X} \to \mathcal{X}'$ of algebraic stacks over Y.

Proof. Write $Y = \lim_{i \in I} Y_i$ as a limit of algebraic spaces over a directed set I with affine transition morphisms and with Y_i Noetherian, see Limits of Spaces, Proposition 70.8.1. We will use the material from Limits of Spaces, Section 70.23.

Choose a presentation $\mathcal{X} = [U/R]$. Denote (U, R, s, t, c, e, i) the corresponding groupoid in algebraic spaces over Y. We may and do assume U is affine. Then U, R, $R \times_{s,U,t} R$ are quasi-separated algebraic spaces of finite type over Y. We have two morpisms $s, t : R \to U$, three morphisms $c : R \times_{s,U,t} R \to R$, $\text{pr}_1 : R \times_{s,U,t} R \to R$, $\text{pr}_2 : R \times_{s,U,t} R \to R$, a morphism $e : U \to R$, and finally a morphism $i : R \to R$.

These morphisms satisfy a list of axioms which are detailed in Groupoids, Section 39.13.

According to Limits of Spaces, Remark 70.23.5 we can find an $i_0 \in I$ and inverse systems

(1) $(U_i)_{i \geq i_0}$,
(2) $(R_i)_{i \geq i_0}$,
(3) $(T_i)_{i \geq i_0}$

over $(Y_i)_{i \geq i_0}$ such that $U = \lim_{i \geq i_0} U_i$, $R = \lim_{i \geq i_0} R_i$, and $R \times_{s,U,t} R = \lim_{i \geq i_0} T_i$ and such that there exist morphisms of systems

(1) $(s_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (U_i)_{i \geq i_0}$,
(2) $(t_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (U_i)_{i \geq i_0}$,
(3) $(c_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(4) $(p_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(5) $(q_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(6) $(e_i)_{i \geq i_0} : (U_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(7) $(i_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$

with $s = \lim_{i \geq i_0} s_i$, $t = \lim_{i \geq i_0} t_i$, $c = \lim_{i \geq i_0} c_i$, $\text{pr}_1 = \lim_{i \geq i_0} p_i$, $\text{pr}_2 = \lim_{i \geq i_0} q_i$, $e = \lim_{i \geq i_0} e_i$, and $i = \lim_{i \geq i_0} i_i$. By Limits of Spaces, Lemma 70.23.7 we see that we may assume that s_i and t_i are smooth (this may require increasing i_0). By Limits of Spaces, Lemma 70.23.6 we may assume that the maps $R \to U \times_{U_i, s_i} R_i$ given by s and $R \to R_i$ and $R \to U \times_{U_i, t_i} R_i$ given by t and $R \to R_i$ are isomorphisms for all $i \geq i_0$. By Limits of Spaces, Lemma 70.23.9 we see that we may assume that the diagrams

$$\begin{tikzcd} T_i \arrow[r, "q_i"'] \arrow[d, "p_i"'] & R_i \arrow[d, "t_i"] \\ R_i \arrow[r, "s_i"] & U_i \end{tikzcd}$$

are cartesian. The uniqueness of Limits of Spaces, Lemma 70.23.4 then guarantees that for a sufficiently large i the relations between the morphisms s, t, c, e, i mentioned above are satisfied by s_i, t_i, c_i, e_i, i_i. Fix such an i.

It follows that $(U_i, R_i, s_i, t_i, c_i, e_i, i_i)$ is a smooth groupoid in algebraic spaces over Y_i. Hence $\mathcal{X}_i = [U_i/R_i]$ is an algebraic stack (Algebraic Stacks, Theorem 94.17.3). The morphism of groupoids

$$(U, R, s, t, c, e, i) \to (U_i, R_i, s_i, t_i, c_i, e_i, i_i)$$

over $Y \to Y_i$ determines a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}_i \arrow[d] \\ Y \arrow[r] & Y_i \end{tikzcd}$$

(Groupoids in Spaces, Lemma 78.21.1). We claim that the morphism $\mathcal{X} \to Y \times_{Y_i} \mathcal{X}_i$ is a closed immersion. The claim finishes the proof because the algebraic stack $\mathcal{X}_i \to Y_i$ is of finite presentation by construction. To prove the claim, note that the left diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U_i \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}_i \end{tikzcd} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & Y \times_{Y_i} U_i \arrow[d] \\ \mathcal{X} \arrow[r] & Y \times_{Y_i} \mathcal{X}_i \end{tikzcd}$$

is cartesian by Groupoids in Spaces, Lemma 78.25.3 and the results mentioned above. Hence the right commutative diagram is cartesian too. Then the desired result follows from the fact that $U \to Y \times_{Y_i} U_i$ is a closed immersion by construction of the inverse system (U_i) in Limits of Spaces, Lemma 70.23.3, the fact that $Y \times_{Y_i} U_i \to Y \times_{Y_i} \mathcal{X}_i$ is smooth and surjective, and Properties of Stacks, Lemma 100.9.4. $\square$

There is a version for separated algebraic stacks.

0CQ2 Lemma 102.6.2. Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume:

(1) f is of finite type and separated,
(2) Y is quasi-compact and quasi-separated.

Then there exists a separated morphism of finite presentation $f' : \mathcal{X}' \to Y$ and a closed immersion $\mathcal{X} \to \mathcal{X}'$ of algebraic stacks over Y.

Proof. First we use exactly the same procedure as in the proof of Lemma 102.6.1 (and we borrow its notation) to construct the embedding $\mathcal{X} \to \mathcal{X}'$ as a morphism $\mathcal{X} \to \mathcal{X}' = Y \times_{Y_i} \mathcal{X}_i$ with $\mathcal{X}_i = [U_i/R_i]$. Thus it is enough to show that $\mathcal{X}_i \to Y_i$ is separated for sufficiently large i. In other words, it is enough to show that $\mathcal{X}_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is proper for i sufficiently large. Since the morphism $U_i \times_{Y_i} U_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is surjective and smooth and since $R_i = \mathcal{X}_i \times_{\mathcal{X}_i \times_{Y_i} \mathcal{X}_i} U_i \times_{Y_i} U_i$ it is enough to show that the morphism $(s_i, t_i) : R_i \to U_i \times_{Y_i} U_i$ is proper for i sufficiently large, see Properties of Stacks, Lemma 100.3.3. We prove this in the next paragraph.

Observe that $U \times_Y U \to Y$ is quasi-separated and of finite type. Hence we can use the construction of Limits of Spaces, Remark 70.23.5 to find an $i_1 \in I$ and an inverse system $(V_i)_{i \geq i_1}$ with $U \times_Y U = \lim_{i \geq i_1} V_i$. By Limits of Spaces, Lemma 70.23.9 for i sufficiently large the functoriality of the construction applied to the projections $U \times_Y U \to U$ gives closed immersions

$$V_i \to U_i \times_{Y_i} U_i$$

(There is a small mismatch here because in truth we should replace Y_i by the scheme theoretic image of $Y \to Y_i$, but clearly this does not change the fibre product.) On the other hand, by Limits of Spaces, Lemma 70.23.8 the functoriality applied to the proper morphism $(s, t) : R \to U \times_Y U$ (here we use that $\mathcal{X}$ is separated) leads to morphisms $R_i \to V_i$ which are proper for large enough i. Composing these morphisms we obtain a proper morphisms $R_i \to U_i \times_{Y_i} U_i$ for all i large enough. The functoriality of the construction of Limits of Spaces, Remark 70.23.5 shows that this is the morphism is the same as (s_i, t_i) for large enough i and the proof is complete. $\square$

102.7. Universally closed morphisms

0H28 This section is the analogue of Limits of Spaces, Section 70.20.

0H29 Lemma 102.7.1. Let $g : Z \to Y$ be a morphism of affine schemes. Let $f : \mathcal{X} \to Y$ be a quasi-compact morphism of algebraic stacks. Let $z \in Z$ and let $T \subset |\mathcal{X} \times_Y Z|$ be a closed subset with $z \notin \mathrm{Im}(T \to |Z|)$. If $\mathcal{X}$ is quasi-compact, then there exist an open neighbourhood $V \subset Z$ of z, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

(1) Z' is an affine scheme of finite presentation over Y,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|\mathcal{X} \times_Y V|$ maps into T' via $|\mathcal{X} \times_Y V| \to |\mathcal{X} \times_Y Z'|$.

Proof. We will deduce this from the corresponding result for morphisms of schemes. Since $\mathcal{X}$ is quasi-compact, we may choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T. Then z is not in the image of T_W. By the schemes case (Limits, Lemma 32.14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

(1) Z' is an affine scheme of finite presentation over Y,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) $T_1 = T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \to |W \times_Y Z'|$.

The commutative diagram

$$\begin{tikzcd} W \times_Y Z \arrow[d] & W \times_Y V \arrow[l] \arrow[r, "a_1"'] \arrow[d, "c"'] & W \times_Y Z' \arrow[d, "q"] \\ \mathcal{X} \times_Y Z & \mathcal{X} \times_Y V \arrow[l] \arrow[r, "a_2"] & \mathcal{X} \times_Y Z' \end{tikzcd}$$

has cartesian squares and the vertical maps are surjective, smooth, and a fortiori open. Looking at the left hand square we see that $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |\mathcal{X} \times_Y V|$ by c. By Properties of Stacks, Lemma 100.4.3 we get $a_1(T_1) = q^{-1}(a_2(T_2))$. By Topology, Lemma 5.6.4 we get

$$q^{-1}\left(\overline{a_2(T_2)}\right) = \overline{q^{-1}(a_2(T_2))} = \overline{a_1(T_1)} \subset T'$$

As q is surjective the image of $\overline{a_2(T_2)} \to |Z'|$ does not contain z' since the same is true for T'. Thus we can take the diagram with Z', V, a, b above and the closed subset $\overline{a_2(T_2)} \subset |\mathcal{X} \times_Y Z'|$ as a solution to the problem posed by the lemma. $\square$

0H2A Lemma 102.7.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. The following are equivalent

(1) f is universally closed,

(2) for every morphism $Z \to \mathcal{Y}$ which is locally of finite presentation and where Z is an affine scheme the map $|\mathcal{X} \times_Y Z| \to |Z|$ is closed, and
(3) there exists a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$ such that $|\mathbf{A}^n \times (\mathcal{X} \times_{\mathcal{Y}} V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.

Proof. It is clear that (1) implies (2).

Assume (2). Choose a scheme V which is the disjoint union of affine schemes and a surjective smooth morphism $V \to \mathcal{Y}$. In order to show that f is universally closed, it suffices to show that the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ of f is universally closed, see Morphisms of Stacks, Lemma 101.13.5. Note that property (2) holds for this base change. Hence in order to prove that (2) implies (1) we may assume $Y = \mathcal{Y}$ is an affine scheme.

Assume (2) and assume $\mathcal{Y} = Y$ is an affine scheme. If f is not universally closed, then there exists an affine scheme Z over Y such that $|\mathcal{X} \times_Y Z| \to |Z|$ is not closed, see Morphisms of Stacks, Lemma 101.13.5. This means that there exists some closed subset $T \subset |\mathcal{X} \times_Y Z|$ such that $\mathrm{Im}(T \to |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 102.7.1. We find an open neighbourhood $V \subset Z$, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |\mathcal{X} \times_Y Z'|$ such that

(1) Z' is an affine scheme of finite presentation over Y,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|\mathcal{X} \times_Y V|$ maps into T' via $|\mathcal{X} \times_Y V| \to |\mathcal{X} \times_Y Z'|$.

We claim that z' is in the closure of $\mathrm{Im}(T' \to |Z'|)$. This implies that $|\mathcal{X} \times_Y Z'| \to |Z'|$ is not closed and this is absurd as we assumed (2), in other words, the claim shows that (2) implies (1). To see the claim is true we contemplate the following commutative diagram

$$\begin{tikzcd} \mathcal{X} \times_Y Z \arrow[d] & \mathcal{X} \times_Y V \arrow[l] \arrow[r] \arrow[d] & \mathcal{X} \times_Y Z' \arrow[d] \\ Z & V \arrow[l] \arrow[r, "a"] & Z' \end{tikzcd}$$

Let $T_V \subset |\mathcal{X} \times_Y V|$ be the inverse image of T. By Properties of Stacks, Lemma 100.4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \to |Z|$ and since $|V| \to |Z|$ is open, we see that z is in the closure of the image of $T_V \to |V|$. Since the image of T_V in $|\mathcal{X} \times_Y Z'|$ is contained in $|T'|$ it follows immediately that $z' = a(z)$ is in the closure of the image of T'.

It is clear that (1) implies (3). Let $V \to \mathcal{Y}$ be as in (3). If we can show that $\mathcal{X} \times_Y V \to V$ is universally closed, then f is universally closed by Morphisms of Stacks, Lemma 101.13.5. Thus it suffices to show that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (2) if f is a quasi-compact morphism of algebraic stacks, $\mathcal{Y} = Y$ is a scheme, and $|\mathbf{A}^n \times \mathcal{X}| \to |\mathbf{A}^n \times Y|$ is closed for all n. Let $Z \to Y$ be locally of finite presentation where Z is an affine scheme. We have to show the map $|\mathcal{X} \times_Y Z| \to |Z|$ is closed. Since Y is a scheme, Z is affine, and $Z \to Y$ is locally of finite presentation we can find an immersion $Z \to \mathbf{A}^n \times Y$, see Morphisms, Lemma 29.39.2. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{X} \times_Y Z \arrow[r] \arrow[d] & \mathbf{A}^n \times \mathcal{X} \arrow[d] \\ Z \arrow[r] & \mathbf{A}^n \times Y \end{tikzcd} \quad \text{inducing the cartesian square} \quad \begin{tikzcd} |\mathcal{X} \times_Y Z| \arrow[r] \arrow[d] & |\mathbf{A}^n \times \mathcal{X}| \arrow[d] \\ |Z| \arrow[r] & |\mathbf{A}^n \times Y| \end{tikzcd}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Stacks, Lemma 100.9.6). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. $\square$

102.8. Other chapters

CHAPTER 103

Cohomology of Algebraic Stacks

073P 103.1. Introduction

073Q In this chapter we write about cohomology of algebraic stacks. This means in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled "Cohomology of Schemes" and "Cohomology of Algebraic Spaces". The results in this chapter are different from those in [LMB00] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapter "Sheaves on Algebraic Stacks" in order to become familiar with the terminology introduced there, see Sheaves on Stacks, Section 96.1.

103.2. Conventions and abuse of language

073R We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

103.3. Notation

073S Different topologies. If we indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then the notation $\mathcal{X}_{Zar}, \mathcal{X}_{\acute{e}tale}, \mathcal{X}_{smooth}, \mathcal{X}_{syntomic}, \mathcal{X}_{fppf}$ indicates the site introduced in Sheaves on Stacks, Definition 96.4.1. (Think "big site".) Correspondingly the structure sheaf of $\mathcal{X}$ is a sheaf on $\mathcal{X}_{fppf}$. On the other hand, algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z, and in this case $X_{\acute{e}tale}$ indicates the small étale site of X (as defined in Topologies, Definition 34.4.8 or Properties of Spaces, Definition 66.18.1). It seems that the distinction should be clear enough.

The default topology is the fppf topology. Hence we will sometimes say "sheaf on $\mathcal{X}$" or "sheaf of $\mathcal{O}_\mathcal{X}$-modules" when we mean sheaf on $\mathcal{X}_{fppf}$ or object of $\textit{Mod}(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then the functors f_* and f^{-1} defined on presheaves preserves sheaves for any of the topologies mentioned above. In particular when we discuss the pushforward or pullback of a sheaf we don't have to mention which topology we are working with. The same isn't true when we compute cohomology groups and/or higher direct images. In this case we will always mention which topology we are working with.

Suppose that $f : X \to \mathcal{Y}$ is a morphism from an algebraic space X to an algebraic stack $\mathcal{Y}$. Let $\mathcal{G}$ be a sheaf on $\mathcal{Y}_\tau$ for some topology τ. In this case $f^{-1}\mathcal{G}$ is a sheaf for the τ topology on $\mathcal{S}_X$ (the algebraic stack associated to X) because (by our conventions) f really is a 1-morphism $f : \mathcal{S}_X \to \mathcal{Y}$. If $\tau = \textit{étale}$ or stronger, then we write $f^{-1}\mathcal{G}|_{X_{\acute{e}tale}}$ to denote the restriction to the étale site of X, see Sheaves on

Stacks, Section 96.22. If $\mathcal{G}$ is an $\mathcal{O}_\mathcal{X}$-module we sometimes write $f^*\mathcal{G}$ and $f^*\mathcal{G}|_{X_{\acute{e}tale}}$ instead.

103.4. Pullback of quasi-coherent modules

076W Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. It is a very general fact that quasi-coherent modules on ringed topoi are compatible with pullbacks. In particular the pullback f^* preserves quasi-coherent modules and we obtain a functor

$$f^* : QCoh(\mathcal{O}_\mathcal{Y}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}),$$

see Sheaves on Stacks, Lemma 96.11.2. In general this functor isn't exact, but if f is flat then it is.

076X Lemma 103.4.1. If $f : \mathcal{X} \to \mathcal{Y}$ is a flat morphism of algebraic stacks then $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ is an exact functor.

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. Then $U \to \mathcal{X}$ is still smooth and surjective as a composition of two such morphisms. From the commutative diagram

$$\begin{tikzcd} U \arrow[r, "f'"'] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd} QCoh(\mathcal{O}_U) & QCoh(\mathcal{O}_V) \arrow[l] \\ QCoh(\mathcal{O}_\mathcal{X}) \arrow[u] & QCoh(\mathcal{O}_\mathcal{Y}) \arrow[l] \arrow[u] \end{tikzcd}$$

of abelian categories. Our proof that the bottom two categories in this diagram are abelian showed that the vertical functors are faithful exact functors (see proof of Sheaves on Stacks, Lemma 96.15.1). Since f' is a flat morphism of schemes (by our definition of flat morphisms of algebraic stacks) we see that $(f')^*$ is an exact functor on quasi-coherent sheaves on V. Thus we win. $\square$

0GQF Lemma 103.4.2. Let $\mathcal{X}$ be an algebraic stack. Let I be a set and for $i \in I$ let $x_i : U_i \to \mathcal{X}$ be an object of $\mathcal{X}$. Assume that x_i is flat and $\coprod x_i : \coprod U_i \to \mathcal{X}$ is surjective. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an arrow of $QCoh(\mathcal{O}_\mathcal{X})$. Denote φ_i the restriction of φ to $(U_i)_{\acute{e}tale}$. Then φ is injective, resp. surjective, resp. an isomorphism if and only if each φ_i is so.

Proof. Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. We may and do think of x as an object of $\mathcal{X}$. This produces a presentation $\mathcal{X} = [U/R]$ for some groupoid in spaces (U, R, s, t, c) and correspondingly an equivalence

$$QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$$

See discussion in Sheaves on Stacks, Section 96.15. The structure of abelian category on the right hand is such that φ is injective, resp. surjective, resp. an isomorphism if and only if the restriction $\varphi|_{U_{\acute{e}tale}}$ is so, see Groupoids in Spaces, Lemma 78.12.6.

For each i we choose an étale covering $\{W_{i,j} \to V \times_{\mathcal{X}} U_i\}_{j \in J_i}$ by schemes. Denote $g_{i,j} : W_{i,j} \to V$ and $h_{i,j} : W_{i,j} \to U_i$ the obvious arrows. Each of the morphisms of schemes $g_{i,j} : W_{i,j} \to U$ is flat and they are jointly surjective. Similarly, for each fixed i the morphisms of schemes $h_{i,j} : W_{i,j} \to U_i$ are flat and jointly surjective. By Sheaves on Stacks, Lemma 96.12.2 the pullback by $(g_{i,j})_{small}$ of the restriction $\varphi|_{U_{étale}}$ is the restriction $\varphi|_{(W_{i,j})_{étale}}$ and the pullback by $(h_{i,j})_{small}$ of the restriction $\varphi|_{(U_i)_{étale}}$ is the restriction $\varphi|_{(W_{i,j})_{étale}}$. Pullback of quasi-coherent modules by a flat morphism of schemes is exact and pullback by a jointly surjective family of flat morphisms of schemes reflects injective, resp. surjective, resp. bijective maps of quasi-coherent modules (in fact this holds for all modules as we can check exactness at stalks). Thus we see

$$\varphi|_{U_{étale}} \text{ injective} \Leftrightarrow \varphi|_{(W_{i,j})_{étale}} \text{ injective for all } i, j \Leftrightarrow \varphi|_{(U_i)_{étale}} \text{ injective for all } i$$

This finishes the proof. □

103.5. Higher direct images of types of modules

076Y The following lemma is the basis for our understanding of higher direct images of certain types of sheaves of modules. There are two versions: one for the étale topology and one for the fppf topology.

076Z Lemma 103.5.1. Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_{\mathcal{X}}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ such that

1. $\mathcal{M}_{\mathcal{X}}$ is a weak Serre subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ (see Homology, Definition 12.10.1) for all algebraic stacks $\mathcal{X}$,
2. for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_{\mathcal{X}}$ into $\mathcal{M}_{\mathcal{Y}}$,
3. if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ is in $\mathcal{M}_{\mathcal{X}}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and
4. if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then R^if_* maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$.

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks R^if_* maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$. (Higher direct images computed in étale topology.)

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi-compact and quasi-separated morphism of algebraic stacks and let $\mathcal{F}$ be an object of $\mathcal{M}_{\mathcal{Y}}$. Choose a surjective smooth morphism $\mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ is representable by a scheme. By Sheaves on Stacks, Lemma 96.21.3 taking higher direct images commutes with base change. Assumption (2) shows that the pullback of $\mathcal{F}$ to $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}$ is in $\mathcal{M}_{\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}}$ because the projection $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is smooth as a base change of a smooth morphism. Hence (3) shows we may replace $\mathcal{Y} \to \mathcal{X}$ by the projection $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{U}$. In other words, we may assume that $\mathcal{X}$ is representable by a scheme. Using (3) once more, we see that the question is Zariski local on $\mathcal{X}$, hence we may assume that $\mathcal{X}$ is representable by an affine scheme. Since f is quasi-compact this implies that also $\mathcal{Y}$ is quasi-compact. Thus we may choose a surjective smooth morphism $g : \mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme.

In this situation we have the spectral sequence

$$E_2^{p,q} = R^q(f \circ g_p)_* g_p^* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

of Sheaves on Stacks, Proposition 96.21.1. Recall that this is a first quadrant spectral sequence hence we may use the last part of Homology, Lemma 12.25.3. Note that the morphisms

$$g_p : \mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{Y}$$

are smooth as compositions of base changes of the smooth morphism g. Thus the sheaves $g_p^*\mathcal{F}$ are in $\mathcal{M}_{\mathcal{V}_p}$ by (2). Hence it suffices to prove that the higher direct images of objects of $\mathcal{M}_{\mathcal{V}_p}$ under the morphisms

$$\mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{X}$$

are in $\mathcal{M}_\mathcal{X}$. The algebraic stacks $\mathcal{V}_p$ are quasi-compact and quasi-separated by Morphisms of Stacks, Lemma 101.7.8. Of course each $\mathcal{V}_p$ is representable by an algebraic space (the diagonal of the algebraic stack $\mathcal{Y}$ is representable by algebraic spaces). This reduces us to the case where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme.

In the situation where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme, we choose anew a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. Going through the argument above once again we once again reduce to the morphisms $\mathcal{V}_p \to \mathcal{X}$. But in the current situation the algebraic stacks $\mathcal{V}_p$ are representable by quasi-compact and quasi-separated schemes (because the diagonal of an algebraic space is representable by schemes).

Thus we may assume $\mathcal{Y}$ is representable by a scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by separated schemes (because the diagonal of a scheme is separated).

Thus we may assume $\mathcal{Y}$ is representable by a separated scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (yet again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by affine schemes (because the diagonal of a separated scheme is a closed immersion hence affine) and this case is handled by assumption (4). This finishes the proof. □

Here is the version for the fppf topology.

0770 Lemma 103.5.2. Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $\text{Mod}(\mathcal{O}_\mathcal{X})$ such that

(1) $\mathcal{O}_\mathcal{X}$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_\mathcal{X})$ for all algebraic stacks $\mathcal{X}$,
(2) for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,
(3) if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $\text{Mod}(\mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and
(4) if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in fppf topology.)

Proof. Identical to the proof of Lemma 103.5.1. □

103.6. Locally quasi-coherent modules

075X Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We can ask whether $\mathcal{F}$ is locally quasi-coherent, see Sheaves on Stacks, Definition 96.12.1. Briefly, this means $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module for the étale topology such that for any morphism $f : U \to \mathcal{X}$ the restriction $f^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent on $U_{étale}$. (The actual definition is slightly different, but equivalent.) A useful fact is that

$$\mathrm{LQCoh}(\mathcal{O}_\mathcal{X}) \subset \mathrm{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is a weak Serre subcategory, see Sheaves on Stacks, Lemma 96.12.4.

075Y Lemma 103.6.1. Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.

Proof. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 101.33.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 96.12.3. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective smooth morphism. Let x be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 96.19.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x, x_i live over schemes V, V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. The restriction $x_i^*\mathcal{F}|_{V_{i,étale}}$ is equal to the restriction of $f^*\mathcal{F}$ to $V_{i,étale}$, see Sheaves on Stacks, Lemma 96.9.3. Hence $x^*\mathcal{F}|_{V_{étale}}$ is a sheaf on the small étale site of V which is quasi-coherent when restricted to $V_{i,étale}$ for each i. This implies that it is quasi-coherent (as desired), for example by Properties of Spaces, Lemma 66.29.6. □

075Z Lemma 103.6.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$. Then $R^if_*\mathcal{F}$ (computed in the étale topology) is locally quasi-coherent on $\mathcal{Y}_{étale}$.

Proof. We will use Lemma 103.5.1 to prove this. We will check its assumptions (1) – (4). Parts (1) and (2) follows from Sheaves on Stacks, Lemma 96.12.4. Part (3) follows from Lemma 103.6.1. Thus it suffices to show (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. Choose any object y of $\mathcal{Y}$ lying over a scheme V. For clarity, denote $\mathcal{V} = (Sch/V)_{fppf}$ the algebraic stack corresponding to V. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "g"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{V} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

Thus $\mathcal{Z}$ is representable by the scheme $Z = V \times_Y X$ and f' is quasi-compact and separated (even affine). By Sheaves on Stacks, Lemma 96.22.3 we have

$$R^if_*\mathcal{F}|_{V_{étale}} = R^if'_{small,*}(g^*\mathcal{F}|_{Z_{étale}})$$

The right hand side is a quasi-coherent sheaf on $V_{étale}$ by Cohomology of Spaces, Lemma 69.3.1. This implies the left hand side is quasi-coherent which is what we had to prove. □

07AP **Lemma 103.6.3.** Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.

Proof. First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 96.19.7 and Lemma 96.19.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent (Sheaves on Stacks, Lemma 96.12.3). The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent by Lemma 103.6.2. (Note that a_* and b_* don't care about which topology is used to calculate them.) We conclude that $\mathcal{F}$ is locally quasi-coherent, see Sheaves on Stacks, Lemma 96.12.4.

We are going to reduce the proof of the general case the situation in the first paragraph. Let x be an object of $\mathcal{X}$ lying over the scheme U. We have to show that $\mathcal{F}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. It suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 101.27.14 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence $a_i^*\mathcal{F}$ is locally quasi-coherent on $(Sch/U_i)_{fppf}$. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent. Hence by the first paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent, which certainly implies that $\mathcal{F}|_{U_{étale}}$ is quasi-coherent. $\square$

103.7. Flat comparison maps

0760 Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ be an object of $\textit{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Given an object x of $\mathcal{X}$ lying over the scheme U the restriction $\mathcal{F}|_{U_{étale}}$ is the restriction of $x^{-1}\mathcal{F}$ to the small étale site of U, see Sheaves on Stacks, Definition 96.9.2. Next, let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a morphism of schemes $f : U \to U'$. Thus a 2-commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[rd, "x"'] & & U' \arrow[ld, "x'"] \\ & \mathcal{X} & \end{tikzcd}$$

Associated to φ we obtain a comparison map between restrictions

0761 (103.7.0.1) $$c_\varphi : f_{small}^*(\mathcal{F}|_{U'_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}}$$

see Sheaves on Stacks, Equation (96.9.4.1). In this situation we can consider the following property of $\mathcal{F}$.

0762 **Definition 103.7.1.** Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ in $\textit{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. We say $\mathcal{F}$ has the *flat base change property*[1] if and only if c_φ is an isomorphism whenever f is flat.

[1]This may be nonstandard notation.

Here is a lemma with some properties of this notion.

0764 Lemma 103.7.2. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{\acute{e}tale}$.

(1) If $\mathcal{F}$ has the flat base change property then for any morphism $g : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, the pullback $g^*\mathcal{F}$ does too.
(2) The full subcategory of $\textit{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ consisting of modules with the flat base change property is a weak Serre subcategory.
(3) Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ has the flat base change property then so does $\mathcal{F}$.
(4) The category of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{\acute{e}tale}$ with the flat base change property has colimits and they agree with colimits in $\textit{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$.
(5) Given $\mathcal{F}$ and $\mathcal{G}$ in $\textit{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ with the flat base change property then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ has the flat base change property.
(6) Given $\mathcal{F}$ and $\mathcal{G}$ in $\textit{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ having the flat base change property then the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ has the flat base change property.

Proof. Let $g : \mathcal{Y} \to \mathcal{X}$ be as in (1). Let y be an object of $\mathcal{Y}$ lying over a scheme V. By Sheaves on Stacks, Lemma 96.9.3 we have $(g^*\mathcal{F})|_{V_{\acute{e}tale}} = \mathcal{F}|_{V_{\acute{e}tale}}$. Moreover a comparison mapping for the sheaf $g^*\mathcal{F}$ on $\mathcal{Y}$ is a special case of a comparison map for the sheaf $\mathcal{F}$ on $\mathcal{X}$, see Sheaves on Stacks, Lemma 96.9.3. In this way (1) is clear.

Proof of (2). We use the characterization of weak Serre subcategories of Homology, Lemma 12.10.3. Kernels and cokernels of maps between sheaves having the flat base change property also have the flat base change property. This is clear because f_{small}^* is exact for a flat morphism of schemes and since the restriction functors $(-)|_{U_{\acute{e}tale}}$ are exact (because we are working in the étale topology). Finally, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\textit{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ and the outer two sheaves have the flat base change property then the middle one does as well, again because of the exactness of f_{small}^* and the restriction functors (and the 5 lemma).

Proof of (3). Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ has the flat base change property. By part (1), the definition of an algebraic stack, and the fact that compositions of smooth morphisms are smooth (see Morphisms of Stacks, Lemma 101.33.2) we may assume that each $\mathcal{X}_i$ is representable by a scheme. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a flat morphism $a : U \to U'$ of schemes. By Sheaves on Stacks, Lemma 96.19.10 there exists a jointly surjective family of étale morphisms $U'_i \to U'$ such that $U'_i \to U' \to \mathcal{X}$ factors through $\mathcal{X}_i$. Thus we obtain commutative diagrams

$$\begin{tikzcd} U_i = U \times_{U'} U'_i \arrow[r, "a_i"'] \arrow[d] & U'_i \arrow[r, "x'_i"'] \arrow[d] & \mathcal{X}_i \arrow[d, "f_i"] \\ U \arrow[r, "a"] & U' \arrow[r, "x'"] & \mathcal{X} \end{tikzcd}$$

Note that each a_i is a flat morphism of schemes as a base change of a. Denote $\psi_i : x_i \to x'_i$ the morphism of $\mathcal{X}_i$ lying over a_i with target x'_i. By assumption the comparison maps $c_{\psi_i} : (a_i)^*_{small}\big(f_i^*\mathcal{F}|_{(U'_i)_{\acute{e}tale}}\big) \to f_i^*\mathcal{F}|_{(U_i)_{\acute{e}tale}}$ is an isomorphism. Because the vertical arrows $U'_i \to U'$ and $U_i \to U$ are étale, the sheaves $f_i^*\mathcal{F}|_{(U'_i)_{\acute{e}tale}}$

and $f_i^*\mathcal{F}|_{(U_i)_{\acute{e}tale}}$ are the restrictions of $\mathcal{F}|_{U'_{\acute{e}tale}}$ and $\mathcal{F}|_{U_{\acute{e}tale}}$ and the map c_{ψ_i} is the restriction of c_φ to $(U_i)_{\acute{e}tale}$, see Sheaves on Stacks, Lemma 96.9.3. Since $\{U_i \to U\}$ is an étale covering, this implies that the comparison map c_φ is an isomorphism which is what we wanted to prove.

Proof of (4). Let $\mathcal{I} \to \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram and assume each $\mathcal{F}_i$ has the flat base change property. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over the flat morphism of schemes $f : U \to U'$. Recall that $\text{colim}_i \mathcal{F}_i$ is the sheafification of the presheaf colimit. As we are using the étale topology, it is clear that

$$(\text{colim}_i \mathcal{F}_i)|_{U_{\acute{e}tale}} = \text{colim}_i \mathcal{F}_i|_{U_{\acute{e}tale}}$$

and similarly for the restriction to $U'_{\acute{e}tale}$. Hence

$$\begin{aligned} f_{small}^*((\text{colim}_i \mathcal{F}_i)|_{U'_{\acute{e}tale}}) &= f_{small}^*(\text{colim}_i \mathcal{F}_i|_{U'_{\acute{e}tale}}) \\ &= \text{colim}_i f_{small}^*(\mathcal{F}_i|_{U'_{\acute{e}tale}}) \\ &\xrightarrow{\text{colim } c_\varphi} \text{colim}_i \mathcal{F}_i|_{U_{\acute{e}tale}} \\ &= (\text{colim}_i \mathcal{F}_i)|_{U_{\acute{e}tale}} \end{aligned}$$

For the second equality we used that f_{small}^* commutes with colimits (as a left adjoint). The arrow is an isomorphism as each $\mathcal{F}_i$ has the flat base change property. Thus the colimit has the flat base change property and (4) is true.

Part (5) holds because tensor products commute with pullbacks, see Modules on Sites, Lemma 18.26.2. Details omitted.

Let $\mathcal{F}$ and $\mathcal{G}$ be as in (6). Since $\mathcal{F}$ is quasi-coherent it has the flat base change property by Sheaves on Stacks, Lemma 96.12.2. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over the flat morphism of schemes $f : U \to U'$. As we are using the étale topology, we have

$$\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{\acute{e}tale}} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$$

and similarly for the restriction to $U'_{\acute{e}tale}$ (details omitted). Hence

$$\begin{aligned} f_{small}^*(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U'_{\acute{e}tale}}) &= f_{small}^*(\mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{F}|_{U'_{\acute{e}tale}}, \mathcal{G}|_{U'_{\acute{e}tale}})) \\ &= \mathcal{H}om_{\mathcal{O}_{U'}}(f_{small}^*(\mathcal{F}|_{U'_{\acute{e}tale}}), f_{small}^*(\mathcal{G}|_{U'_{\acute{e}tale}})) \\ &\xrightarrow{c_\varphi} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}}) \\ &= \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{\acute{e}tale}} \end{aligned}$$

Here the second equality is Modules on Sites, Lemma 18.31.4 which uses that $f : U \to U'$ is flat and hence the morphism of ringed sites f_{small} is flat too. The arrow is an isomorphism as both $\mathcal{F}$ and $\mathcal{G}$ have the flat base change property. Thus our $\mathcal{H}om$ has the flat base change property too as desired. □

0765 Lemma 103.7.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be an object of $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property. Then each $R^if_*\mathcal{F}$ (computed in the étale topology) has the flat base change property.

Proof. We will use Lemma 103.5.1 to prove this. For every algebraic stack $\mathcal{X}$ let $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ denote the full subcategory of $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ consisting of locally quasi-coherent sheaves with the flat base change property. Once we verify conditions (1) – (4) of Lemma 103.5.1 the lemma will follow. Properties (1), (2), and (3) follow from Sheaves on Stacks, Lemmas 96.12.3 and 96.12.4 and Lemmas 103.6.1 and 103.7.2. Thus it suffices to show part (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. In this case, suppose that $\psi : y \to y'$ is a morphism of $\mathcal{Y}$ lying over a flat morphism $b : V \to V'$ of schemes. For clarity denote $\mathcal{V} = (Sch/V)_{fppf}$ and $\mathcal{V}' = (Sch/V')_{fppf}$ the corresponding algebraic stacks. Consider the diagram of algebraic stacks

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "a"'] \arrow[d, "f''"'] & \mathcal{Z}' \arrow[r, "x'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{V} \arrow[r, "b"] & \mathcal{V}' \arrow[r, "y'"] & \mathcal{Y}
\end{tikzcd}$$

with both squares cartesian. As f is representable by schemes (and quasi-compact and separated – even affine) we see that $\mathcal{Z}$ and $\mathcal{Z}'$ are representable by schemes Z and Z' and in fact $Z = V \times_{V'} Z'$. Since $\mathcal{F}$ has the flat base change property we see that

$$a_{small}^*(\mathcal{F}|_{Z'_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{Z_{\acute{e}tale}}$$

is an isomorphism. Moreover,

$$R^i f_*\mathcal{F}|_{V'_{\acute{e}tale}} = R^i(f')_{small,*}(\mathcal{F}|_{Z'_{\acute{e}tale}})$$

and

$$R^i f_*\mathcal{F}|_{V_{\acute{e}tale}} = R^i(f'')_{small,*}(\mathcal{F}|_{Z_{\acute{e}tale}})$$

by Sheaves on Stacks, Lemma 96.22.3. Hence we see that the comparison map

$$c_\psi : b_{small}^*(R^i f_*\mathcal{F}|_{V'_{\acute{e}tale}}) \longrightarrow R^i f_*\mathcal{F}|_{V_{\acute{e}tale}}$$

is an isomorphism by Cohomology of Spaces, Lemma 69.11.2. Thus $R^i f_*\mathcal{F}$ has the flat base change property. Since $R^i f_*\mathcal{F}$ is locally quasi-coherent by Lemma 103.6.2 we win. ☐

103.8. Locally quasi-coherent modules with the flat base change property

0GQG Let $\mathcal{X}$ be an algebraic stack. We[2] will denote

$$\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \subset \mathrm{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$$

the full subcategory whose objects are étale $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ which are both locally quasi-coherent (Section 103.6) and have the flat base change property (Section 103.7). We have

$$QCoh(\mathcal{O}_\mathcal{X}) \subset \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$$

by Sheaves on Stacks, Lemma 96.12.2.

0771 Proposition 103.8.1. Summary of results on locally quasi-coherent modules having the flat base change property.

(1) Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}$ is in $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, then $\mathcal{F}$ is a sheaf for the fppf topology, i.e., it is an object of $\mathrm{Mod}(\mathcal{O}_\mathcal{X})$.
(2) The category $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ is a weak Serre subcategory of both $\mathrm{Mod}(\mathcal{O}_\mathcal{X})$ and $\mathrm{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$.
(3) Pullback f^* along any morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ induces a functor $f^* : \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y}) \to \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

[2]Apologies for the horrendous notation.

(4) If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of algebraic stacks and $\mathcal{F}$ is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, then
  (a) the total direct image $Rf_*\mathcal{F}$ and the higher direct images $R^if_*\mathcal{F}$ can be computed in either the étale or the fppf topology with the same result, and
  (b) each $R^if_*\mathcal{F}$ is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})$.

(5) The category $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ as well as in $\text{Mod}(\mathcal{O}_\mathcal{X})$.

(6) Given $\mathcal{F}$ and $\mathcal{G}$ in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

(7) Given $\mathcal{F}$ of finite presentation and $\mathcal{G}$ in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ then $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

Proof. Part (1) is Sheaves on Stacks, Lemma 96.23.1.

Part (2) for the embedding $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \subset \text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have seen in the proof of Lemma 103.7.3. Let us prove (2) for the embedding $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \subset \text{Mod}(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between objects of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. Since $\text{Ker}(\varphi)$ is the same whether computed in the étale or the fppf topology, we see that $\text{Ker}(\varphi)$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ by the étale case. On the other hand, the cokernel computed in the fppf topology is the fppf sheafification of the cokernel computed in the étale topology. However, this étale cokernel is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ hence an fppf sheaf by (1) and we see that the cokernel is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence in $\text{Mod}(\mathcal{O}_\mathcal{X})$ (i.e., using the fppf topology) with $\mathcal{F}_1$, $\mathcal{F}_2$ in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. In order to show that $\mathcal{F}_2$ is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ it suffices to show that the sequence is also exact in the étale topology. To do this it suffices to show that any element of $H^1_{fppf}(x, \mathcal{F}_1)$ becomes zero on the members of an étale covering of x (for any object x of $\mathcal{X}$). This is true because $H^1_{fppf}(x, \mathcal{F}_1) = H^1_{étale}(x, \mathcal{F}_1)$ by Sheaves on Stacks, Lemma 96.23.2 and because of locality of cohomology, see Cohomology on Sites, Lemma 21.7.3. This proves (2).

Part (3) follows from Lemma 103.7.2 and Sheaves on Stacks, Lemma 96.12.3.

Part (4)(b) for $R^if_*\mathcal{F}$ computed in the étale cohomology follows from Lemma 103.7.3. Whereupon part (4)(a) follows from Sheaves on Stacks, Lemma 96.23.2 combined with (1) above.

Part (5) for the étale topology follows from Sheaves on Stacks, Lemma 96.12.4 and Lemma 103.7.2. The fppf version then follows as the colimit in the étale topology is already an fppf sheaf by part (1).

Parts (6) and (7) follow from the corresponding parts of Lemma 103.7.2 and Sheaves on Stacks, Lemma 96.12.4. $\square$

07AQ Lemma 103.8.2. Let $\mathcal{X}$ be an algebraic stack.

(1) Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is in $\text{LQCoh}^{fpc}(\mathcal{O}_{\mathcal{X}_i})$, then $\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

(2) Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_{\mathcal{X}_i})$, then $\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

Proof. Part (1) follows from a combination of Lemmas 103.6.1 and 103.7.2. The proof of (2) is analogous to the proof of Lemma 103.6.3. Let $\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$.

First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 96.19.7 and Lemma 96.19.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent and has the flat base change property, see Proposition 103.8.1. The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent and have the flat base change property by Proposition 103.8.1. We conclude that $\mathcal{F}$ is locally quasi-coherent and has the flat base change property by Proposition 103.8.1.

Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By part (1) it suffices to show that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Again by part (1) it suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 101.27.14 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence the module $a_i^*\mathcal{F}$ on $(Sch/U_i)_{fppf}$ is locally quasi-coherent and has the flat base change property. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent and has the flat base change property. Hence by the previous paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property as desired. $\square$

0GQH Lemma 103.8.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is quasi-compact, quasi-separated, and representable by algebraic spaces. Let $\mathcal{F}$ be in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. Then for an object $y : V \to \mathcal{Y}$ of $\mathcal{Y}$ we have

$$(R^if_*\mathcal{F})|_{V_{étale}} = R^if'_{small,*}(\mathcal{F}|_{U_{étale}})$$

where $f' : U = V \times_\mathcal{Y} \mathcal{X} \to V$ is the base change of f.

Proof. By Sheaves on Stacks, Lemma 96.21.3 we can reduce to the case where $\mathcal{X}$ is represented by U and $\mathcal{Y}$ is represented by V. Of course this also uses that the pullback of $\mathcal{F}$ to U is in $\text{LQCoh}^{fbc}(\mathcal{O}_U)$ by Proposition 103.8.1. Then the result follows from Sheaves on Stacks, Lemma 96.22.2 and the fact that R^if_* may be computed in the étale topology by Proposition 103.8.1. $\square$

0GQI Lemma 103.8.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. The functor $f_* : \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})$ is exact and commutes with direct sums. The functors R^if_* for $i > 0$ vanish on $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

Proof. The functors exist by Proposition 103.8.1. By Lemma 103.8.3 this reduces to the case of an affine morphism of algebraic spaces taking higher direct images in the setting of quasi-coherent modules on algebraic spaces. By the discussion in Cohomology of Spaces, Section 69.3 we reduce to the case of an affine morphism of schemes. For affine morphisms of schemes we have the vanishing of higher direct images on quasi-coherent modules by Cohomology of Schemes, Lemma 30.2.3. The vanishing for R^1f_* implies exactness of f_*. Commuting with direct sums follows from Morphisms, Lemma 29.11.6 for example. □

103.9. Parasitic modules

0772 The following definition is compatible with Descent, Definition 35.12.1.

0773 Definition 103.9.1. Let $\mathcal{X}$ be an algebraic stack. A presheaf of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ is parasitic if we have $\mathcal{F}(x) = 0$ for any object x of $\mathcal{X}$ which lies over a scheme U such that the corresponding morphism $x : U \to \mathcal{X}$ is flat.

Here is a lemma with some properties of this notion.

0774 Lemma 103.9.2. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules.

(1) If $\mathcal{F}$ is parasitic and $g : \mathcal{Y} \to \mathcal{X}$ is a flat morphism of algebraic stacks, then $g^*\mathcal{F}$ is parasitic.
(2) For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have
   (a) the τ sheafification of a parasitic presheaf of modules is parasitic, and
   (b) the full subcategory of $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ consisting of parasitic modules is a Serre subcategory.
(3) Suppose $\mathcal{F}$ is a sheaf for the étale topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.
(4) Suppose $\mathcal{F}$ is a sheaf for the fppf topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.

Proof. To see part (1) let y be an object of $\mathcal{Y}$ which lies over a scheme V such that the corresponding morphism $y : V \to \mathcal{Y}$ is flat. Then $g(y) : V \to \mathcal{Y} \to \mathcal{X}$ is flat as a composition of flat morphisms (see Morphisms of Stacks, Lemma 101.25.2) hence $\mathcal{F}(g(y))$ is zero by assumption. Since $g^*\mathcal{F} = g^{-1}\mathcal{F}(y) = \mathcal{F}(g(y))$ we conclude $g^*\mathcal{F}$ is parasitic.

To see part (2)(a) note that if $\{x_i \to x\}$ is a τ-covering of $\mathcal{X}$, then each of the morphisms $x_i \to x$ lies over a flat morphism of schemes. Hence if x lies over a scheme U such that $x : U \to \mathcal{X}$ is flat, so do all of the objects x_i. Hence the presheaf $\mathcal{F}^+$ (see Sites, Section 7.10) is parasitic if the presheaf $\mathcal{F}$ is parasitic. This proves (2)(a) as the sheafification of $\mathcal{F}$ is $(\mathcal{F}^+)^+$.

Let $\mathcal{F}$ be a parasitic τ-module. It is immediate from the definitions that any submodule of $\mathcal{F}$ is parasitic. On the other hand, if $\mathcal{F}' \subset \mathcal{F}$ is a submodule, then it is equally clear that the presheaf $x \mapsto \mathcal{F}(x)/\mathcal{F}'(x)$ is parasitic. Hence the quotient $\mathcal{F}/\mathcal{F}'$ is a parasitic module by (2)(a). Finally, we have to show that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ parasitic, then $\mathcal{F}_2$ is parasitic. This follows immediately on evaluating on x lying over a scheme flat over $\mathcal{X}$. This proves (2)(b), see Homology, Lemma 12.10.2.

Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ is parasitic. Let x be an object of $\mathcal{X}$ which lies over a scheme U such that $x : U \to \mathcal{X}$ is flat. Consider a surjective smooth covering $W_i \to U \times_{x,\mathcal{X}} \mathcal{X}_i$. Denote $y_i : W_i \to \mathcal{X}_i$ the projection. It follows that $\{f_i(y_i) \to x\}$ is a covering for the smooth topology on $\mathcal{X}$. Since a composition of flat morphisms is flat we see that $f_i^*\mathcal{F}(y_i) = 0$. On the other hand, as we saw in the proof of (1), we have $f_i^*\mathcal{F}(y_i) = \mathcal{F}(f_i(y_i))$. Hence we see that for some smooth covering $\{x_i \to x\}_{i\in I}$ in $\mathcal{X}$ we have $\mathcal{F}(x_i) = 0$. This implies $\mathcal{F}(x) = 0$ because the smooth topology is the same as the étale topology, see More on Morphisms, Lemma 37.38.7. Namely, $\{x_i \to x\}_{i\in I}$ lies over a smooth covering $\{U_i \to U\}_{i\in I}$ of schemes. By the lemma just referenced there exists an étale covering $\{V_j \to U\}_{j\in J}$ which refines $\{U_i \to U\}_{i\in I}$. Denote $x'_j = x|_{V_j}$. Then $\{x'_j \to x\}$ is an étale covering in $\mathcal{X}$ refining $\{x_i \to x\}_{i\in I}$. This means the map $\mathcal{F}(x) \to \prod_{j\in J} \mathcal{F}(x'_j)$, which is injective as $\mathcal{F}$ is a sheaf in the étale topology, factors through $\mathcal{F}(x) \to \prod_{i\in I}\mathcal{F}(x_i)$ which is zero. Hence $\mathcal{F}(x) = 0$ as desired.

Proof of (4): omitted. Hint: similar, but simpler, than the proof of (3). □

Parasitic modules are preserved under absolutely any pushforward.

0775 Lemma 103.9.3. Let $\tau \in \{\acute{e}tale, fppf\}$. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a parasitic object of $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

(1) $H^i_\tau(\mathcal{X}, \mathcal{F}) = 0$ for all i.
(2) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then $R^if_*\mathcal{F}$ (computed in τ-topology) is a parasitic object of $\text{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$.

Proof. We first reduce (2) to (1). By Sheaves on Stacks, Lemma 96.21.2 we see that $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big(V \times_{y,\mathcal{Y}} \mathcal{X},\ \text{pr}^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V. By Lemma 103.9.2 it suffices to show that these cohomology groups are zero when $y : V \to \mathcal{Y}$ is flat. Note that $\text{pr} : V \times_{y,\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is flat as a base change of y. Hence by Lemma 103.9.2 we see that $\text{pr}^{-1}\mathcal{F}$ is parasitic. Thus it suffices to prove (1).

To see (1) we can use the spectral sequence of Sheaves on Stacks, Proposition 96.20.1 to reduce this to the case where $\mathcal{X}$ is an algebraic stack representable by an algebraic space. Note that in the spectral sequence each $f_p^{-1}\mathcal{F} = f_p^*\mathcal{F}$ is a parasitic module by Lemma 103.9.2 because the morphisms $f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ are flat. Reusing this spectral sequence one more time (as in the proof of Lemma 103.5.1) we reduce to the case where the algebraic stack $\mathcal{X}$ is representable by a scheme X. Then $H^i_\tau(\mathcal{X}, \mathcal{F}) = H^i((Sch/X)_\tau, \mathcal{F})$. In this case the vanishing follows easily from an argument with Čech coverings, see Descent, Lemma 35.12.2. □

The following lemma is one of the major reasons we care about parasitic modules. To understand the statement, recall that the functors $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$ aren't exact in general.

0776 Lemma 103.9.4. Let $\mathcal{X}$ be an algebraic stack. Let $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$ be maps in $QCoh(\mathcal{O}_\mathcal{X})$ with $\beta \circ \alpha = 0$. The following are equivalent:

(1) in the abelian category $QCoh(\mathcal{O}_\mathcal{X})$ the complex $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact at $\mathcal{G}$,

(2) $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ computed in either $\mathrm{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ or $\mathrm{Mod}(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ is parasitic.

Proof. We have $QCoh(\mathcal{O}_\mathcal{X}) \subset \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, see Section 103.8. Hence $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ computed in $\mathrm{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ or $\mathrm{Mod}(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ agree, see Proposition 103.8.1. From now on we will use the étale topology on $\mathcal{X}$.

Let $\mathcal{E}$ be the cohomology of $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ computed in the abelian category $QCoh(\mathcal{O}_\mathcal{X})$. Let $x : U \to \mathcal{X}$ be a flat morphism where U is a scheme. As we are using the étale topology, the restriction functor $\mathrm{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \to \mathrm{Mod}(U_{\acute{e}tale}, \mathcal{O}_U)$ is exact. On the other hand, by Lemma 103.4.1 and Sheaves on Stacks, Lemma 96.14.2 the restriction functor

$$QCoh(\mathcal{O}_\mathcal{X}) \xrightarrow{x^*} QCoh((Sch/U)_{\acute{e}tale}, \mathcal{O}) \xrightarrow{-|_{U_{\acute{e}tale}}} QCoh(U_{\acute{e}tale}, \mathcal{O}_U)$$

is exact too. We conclude that $\mathcal{E}|_{U_{\acute{e}tale}} = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha))|_{U_{\acute{e}tale}}$.

If (1) holds, then $\mathcal{E} = 0$ hence $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ restricts to zero on $U_{\acute{e}tale}$ for all U flat over $\mathcal{X}$ and this is the definition of a parasitic module. If (2) holds, then $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ restricts to zero on $U_{\acute{e}tale}$ for all U flat over $\mathcal{X}$ hence $\mathcal{E}$ restricts to zero on $U_{\acute{e}tale}$ for all U flat over $\mathcal{X}$. This certainly implies that the quasi-coherent module $\mathcal{E}$ is zero, for example apply Lemma 103.4.2 to the map $0 \to \mathcal{E}$. ☐

103.10. Quasi-coherent modules

0777 We have seen that the category of quasi-coherent modules on an algebraic stack is equivalent to the category of quasi-coherent modules on a presentation, see Sheaves on Stacks, Section 96.15. This fact is the basis for the following.

0778 Lemma 103.10.1. Let $\mathcal{X}$ be an algebraic stack. Let $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ be the category of locally quasi-coherent modules with the flat base change property, see Section 103.8. The inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ has a right adjoint

$$Q : \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$$

such that $Q \circ i$ is the identity functor.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Set $R = U \times_\mathcal{X} U$ so that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces with the property that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 94.16.2. We may and do replace $\mathcal{X}$ by $[U/R]$. By Sheaves on Stacks, Proposition 96.14.3 there is an equivalence

$$q_1 : QCoh(U, R, s, t, c) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

Let us construct a functor

$$q_2 : \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U, R, s, t, c)$$

by the following rule: if $\mathcal{F}$ is an object of $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ then we set

$$q_2(\mathcal{F}) = (f^*\mathcal{F}|_{U_{\acute{e}tale}}, \alpha)$$

where α is the isomorphism

$$t_{small}^*(f^*\mathcal{F}|_{U_{\acute{e}tale}}) \to t^*f^*\mathcal{F}|_{R_{\acute{e}tale}} \to s^*f^*\mathcal{F}|_{R_{\acute{e}tale}} \to s_{small}^*(f^*\mathcal{F}|_{U_{\acute{e}tale}})$$

where the outer two morphisms are the comparison maps. Note that $q_2(\mathcal{F})$ is quasi-coherent precisely because $\mathcal{F}$ is locally quasi-coherent and that we used (and needed) the flat base change property in the construction of the descent datum

α. We omit the verification that the cocycle condition (see Groupoids in Spaces, Definition 78.12.1) holds. Looking at the proof of Sheaves on Stacks, Proposition 96.14.3 we see that $q_2 \circ i$ is the quasi-inverse to q_1. We define $Q = q_1 \circ q_2$. Let $\mathcal{F}$ be an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ and let $\mathcal{G}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. We have

$$\begin{aligned}\text{Mor}_{\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(i(\mathcal{G}), \mathcal{F}) &= \text{Mor}_{QCoh(U,R,s,t,c)}(q_2(i(\mathcal{G})), q_2(\mathcal{F})) \\ &= \text{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(\mathcal{G}, Q(\mathcal{F}))\end{aligned}$$

where the first equality is Sheaves on Stacks, Lemma 96.14.4 and the second equality holds because $q_1 \circ i$ and q_2 are quasi-inverse equivalences of categories. The assertion $Q \circ i \cong \text{id}$ is a formal consequence of the fact that i is fully faithful. $\square$

0779 Lemma 103.10.2. Let $\mathcal{X}$ be an algebraic stack. Let $Q : \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$ be the functor constructed in Lemma 103.10.1.

(1) The kernel of Q is exactly the collection of parasitic objects of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.
(2) For any object $\mathcal{F}$ of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ both the kernel and the cokernel of the adjunction map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic.
(3) The functor Q is exact and commutes with all limits and colimits.

Proof. Write $\mathcal{X} = [U/R]$ as in the proof of Lemma 103.10.1. Let $\mathcal{F}$ be an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. It is clear from the proof of Lemma 103.10.1 that $\mathcal{F}$ is in the kernel of Q if and only if $\mathcal{F}|_{U_{étale}} = 0$. In particular, if $\mathcal{F}$ is parasitic then $\mathcal{F}$ is in the kernel. Next, let $x : V \to \mathcal{X}$ be a flat morphism, where V is a scheme. Set $W = V \times_\mathcal{X} U$ and consider the diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & V \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{étale}} \cong q_{small}^*\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of Q, then $\mathcal{F}|_{V_{étale}} = 0$ which completes the proof of (1).

Part (2) follows from the discussion above and the fact that the map $Q(\mathcal{F}) \to \mathcal{F}$ becomes an isomorphism after restricting to $U_{étale}$.

To see part (3) note that Q is left exact as a right adjoint. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Q(\mathcal{F}) \arrow[r] \arrow[d, "a"'] & Q(\mathcal{G}) \arrow[r] \arrow[d, "b"'] & Q(\mathcal{H}) \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0 \end{tikzcd}$$

Since the kernels and cokernels of a, b, and c are parasitic by part (2) and since the bottom row is a short exact sequence, we see that the top row as a complex of $\mathcal{O}_\mathcal{X}$-modules has parasitic cohomology sheaves (details omitted; this uses that the category of parasitic modules is a Serre subcategory of the category of all modules). By left exactness of Q we see that only exactness at $Q(\mathcal{H})$ is at issue. However, the cokernel $\mathcal{Q}$ of $Q(\mathcal{G}) \to Q(\mathcal{H}))$ may be computed either in $\text{Mod}(\mathcal{O}_\mathcal{X})$ or in $QCoh(\mathcal{O}_\mathcal{X})$

with the same result because the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ is a left adjoint and hence right exact. Hence $\mathcal{Q} = Q(\mathcal{Q})$ is both quasi-coherent and parasitic, whence 0 by part (1) as desired.

As a right adjoint Q commutes with all limits. Since Q is exact, to show that Q commutes with all colimits it suffices to show that Q commutes with direct sums, see Categories, Lemma 4.14.12. Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. To see that $Q(\bigoplus \mathcal{F}_i)$ is equal to $\bigoplus Q(\mathcal{F}_i)$ we look at the construction of Q in the proof of Lemma 103.10.1. This uses a presentation $\mathcal{X} = [U/R]$ where U is a scheme. Then $Q(\mathcal{F})$ is computed by first taking the pair $(\mathcal{F}|_{U_{\acute{e}tale}}, \alpha)$ in $QCoh(U, R, s, t, c)$ and then using the equivalence $QCoh(U, R, s, t, c) \cong QCoh(\mathcal{O}_\mathcal{X})$. Since the restriction functor $\text{Mod}(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_{U_{\acute{e}tale}})$, $\mathcal{F} \mapsto \mathcal{F}|_{U_{\acute{e}tale}}$ commutes with direct sums, the desired equality is clear. □

0GQJ Lemma 103.10.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Then $Q_\mathcal{X} \circ f^* = f^* \circ Q_\mathcal{Y}$ where $Q_\mathcal{X}$ and $Q_\mathcal{Y}$ are as in Lemma 103.10.1.

Proof. Observe that f^* preserves both $QCoh$ and LQCoh^{fbc}, see Sheaves on Stacks, Lemma 96.11.2 and Proposition 103.8.1. If $\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})$ then $Q_\mathcal{Y}(\mathcal{F}) \to \mathcal{F}$ has parasitic kernel and cokernel by Lemma 103.10.2. As f is flat we get that $f^*Q_\mathcal{Y}(\mathcal{F}) \to f^*\mathcal{F}$ has parasitic kernel and cokernel by Lemma 103.9.2. Thus the induced map $f^*Q_\mathcal{Y}(\mathcal{F}) \to Q_\mathcal{X}(f^*\mathcal{F})$ has parasitic kernel and cokernel and hence is an isomorphism for example by Lemma 103.9.4. □

0GQK Lemma 103.10.4. Let $\mathcal{X}$ be an algebraic stack. Let x be an object of $\mathcal{X}$ lying over the scheme U such that $x : U \to \mathcal{X}$ is flat. Then for $\mathcal{F}$ in $QCoh^{fbc}(\mathcal{O}_\mathcal{X})$ we have $Q(\mathcal{F})|_{U_{\acute{e}tale}} = \mathcal{F}|_{U_{\acute{e}tale}}$.

Proof. True because the kernel and cokernel of $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic, see Lemma 103.10.2. □

0GQL Remark 103.10.5. Let $\mathcal{X}$ be an algebraic stack. The category $QCoh(\mathcal{O}_\mathcal{X})$ is abelian, the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$ is right exact, but not exact in general, see Sheaves on Stacks, Lemma 96.15.1. We can use the functor Q from Lemmas 103.10.1 and 103.10.2 to understand this. Namely, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Then

(1) the cokernel $\text{Coker}(\varphi)$ computed in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is quasi-coherent and is the cokernel of φ in $QCoh(\mathcal{O}_\mathcal{X})$,
(2) the image $\text{Im}(\varphi)$ computed in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is quasi-coherent and is the image of φ in $QCoh(\mathcal{O}_\mathcal{X})$, and
(3) the kernel $\text{Ker}(\varphi)$ computed in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ by Proposition 103.8.1 and $Q(\text{Ker}(\varphi))$ is the kernel in $QCoh(\mathcal{O}_\mathcal{X})$.

This follows from the references given.

0GQM Remark 103.10.6. Let $\mathcal{X}$ be an algebraic stack. Given two quasi-coherent $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ and $\mathcal{G}$ the tensor product module $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ is quasi-coherent, see Sheaves on Stacks, Lemma 96.15.1 part (5). Similarly, given two locally quasi-coherent modules with the flat base change property, their tensor product has the same property, see Proposition 103.8.1. Thus the inclusion functors

$$QCoh(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{O}_\mathcal{X})$$

are functors of symmetric monoidal categories. What is more interesting is that the functor

$$Q : \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

is a functor of symmetric monoidal categories as well. Namely, given $\mathcal{F}$ and $\mathcal{G}$ in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ we obtain

$$\begin{tikzcd} Q(\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{X}} Q(\mathcal{G}) \arrow[rr] \arrow[rd] & & \mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G} \\ & Q(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) \arrow[ru] & \end{tikzcd}$$

where the south-west arrow comes from the universal property of the north-west arrow (and the fact already mentioned that the object in the upper left corner is quasi-coherent). If we restrict this diagram to $U_{étale}$ for $U \to \mathcal{X}$ flat, then all three arrows become isomorphisms (see Lemmas 103.10.1 and 103.10.2 and Definition 103.9.1). Hence $Q(\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{X}} Q(\mathcal{G}) \to Q(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G})$ is an isomorphism, see for example Lemma 103.4.2.

07B2 Remark 103.10.7. Let $\mathcal{X}$ be an algebraic stack. Let $\text{Parasitic}(\mathcal{O}_\mathcal{X}) \subset \text{Mod}(\mathcal{O}_\mathcal{X})$ denote the full subcategory consiting of parasitic modules. The results of Lemmas 103.10.1 and 103.10.2 imply that

$$QCoh(\mathcal{O}_\mathcal{X}) = \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) / \text{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$$

in words: the category of quasi-coherent modules is the category of locally quasi-coherent modules with the flat base change property divided out by the Serre subcategory consisting of parasitic objects. See Homology, Lemma 12.10.6. The existence of the inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ which is left adjoint to the quotient functor is a key feature of the situation. In Derived Categories of Stacks, Section 104.5 and especially Lemma 104.5.4 we prove that a similar result holds on the level of derived categories.

0GQN Lemma 103.10.8. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module of finite presentation and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. The internal homs $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ computed in $\text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $\text{Mod}(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. The quasi-coherent module $hom(\mathcal{F}, \mathcal{G}) = Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))$ has the following universal property

$$\text{Hom}_\mathcal{X}(\mathcal{H}, hom(\mathcal{F}, \mathcal{G})) = \text{Hom}_\mathcal{X}(\mathcal{H} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}, \mathcal{G})$$

for $\mathcal{H}$ in $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. The construction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in Modules on Sites, Section 18.27 depends only on $\mathcal{F}$ and $\mathcal{G}$ as presheaves of modules; the output $\mathcal{H}om$ is a sheaf for the fppf topology because $\mathcal{F}$ and $\mathcal{G}$ are assumed sheaves in the fppf topology, see Modules on Sites, Lemma 18.27.1. By Sheaves on Stacks, Lemma 96.12.4 we see that $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is locally quasi-coherent. By Lemma 103.7.2 we see that $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ has the flat base change property. Hence $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ and it makes sense to apply the functor Q of Lemma 103.10.1 to it. By the universal property of Q we have

$$\text{Hom}_\mathcal{X}(\mathcal{H}, Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))) = \text{Hom}_\mathcal{X}(\mathcal{H}, \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))$$

for $\mathcal{H}$ quasi-coherent, hence the displayed formula of the lemma follows from Modules on Sites, Lemma 18.27.6. $\square$

0GQP Lemma 103.10.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{Y}$-module of finite presentation and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then $f^*hom(\mathcal{F}, \mathcal{G}) = hom(f^*\mathcal{F}, f^*\mathcal{G})$ with notation as in Lemma 103.10.8.

Proof. We have $f^* \mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(f^*\mathcal{F}, f^*\mathcal{G})$ by Modules on Sites, Lemma 18.31.4. (Observe that this step is not where the flatness of f is used as the morphism of ringed topoi associated to f is always flat, see Sheaves on Stacks, Remark 96.6.3.) Then apply Lemma 103.10.3 (and here we do use flatness of f). $\square$

103.11. Pushforward of quasi-coherent modules

070A Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the pushforward

$$f_* : \text{Mod}(\mathcal{O}_\mathcal{X}) \longrightarrow \text{Mod}(\mathcal{O}_\mathcal{Y})$$

It turns out that this functor almost never preserves the subcategories of quasi-coherent sheaves. For example, consider the morphism of schemes

$$j : X = \mathbf{A}^2_k \setminus \{0\} \longrightarrow \mathbf{A}^2_k = Y.$$

Associated to this we have the corresponding morphism of algebraic stacks

$$f = j_{big} : \mathcal{X} = (Sch/X)_{fppf} \to (Sch/Y)_{fppf} = \mathcal{Y}$$

The pushforward $f_*\mathcal{O}_\mathcal{X}$ of the structure sheaf has global sections $k[x, y]$. Hence if $f_*\mathcal{O}_\mathcal{X}$ is quasi-coherent on $\mathcal{Y}$ then we would have $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. However, consider $T = \text{Spec}(k) \to \mathbf{A}^2_k = Y$ mapping to 0. Then $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = 0$ because $X \times_Y T = \emptyset$ whereas $\Gamma(T, \mathcal{O}_\mathcal{Y}) = k$. On the positive side, for any flat morphism $T \to Y$ we have the equality $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = \Gamma(T, \mathcal{O}_\mathcal{Y})$ as follows from Cohomology of Schemes, Lemma 30.5.2 using that j is quasi-compact and quasi-separated.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We work around the problem mentioned above using the following three observations:

(1) f_* does preserve locally quasi-coherent modules (Lemma 103.6.2),
(2) f_* transforms a quasi-coherent sheaf into a locally quasi-coherent sheaf whose flat comparison maps are isomorphisms (Lemma 103.7.3), and
(3) locally quasi-coherent $\mathcal{O}_\mathcal{Y}$-modules with the flat base change property give rise to quasi-coherent modules on a presentation of $\mathcal{Y}$ and hence quasi-coherent modules on $\mathcal{Y}$, see Sheaves on Stacks, Section 96.15.

Thus we obtain a functor

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which is a right adjoint to $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ such that moreover

$$\Gamma(y, f_*\mathcal{F}) = \Gamma(y, f_{QCoh,*}\mathcal{F})$$

for any $y \in \text{Ob}(\mathcal{Y})$ such that the associated 1-morphism $y : V \to \mathcal{Y}$ is flat, see Lemma 103.11.2. Moreover, a similar construction will produce functors $R^i f_{QCoh,*}$. However, these results will not be sufficient to produce a total direct image functor (of complexes with quasi-coherent cohomology sheaves).

077A Proposition 103.11.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ has a right adjoint

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which can be defined as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{f_*} \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

where the functors f_* and Q are as in Proposition 103.8.1 and Lemma 103.10.1. Moreover, if we define $R^if_{QCoh,*}$ as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{R^if_*} \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

then the sequence of functors $\{R^if_{QCoh,*}\}_{i\geq 0}$ forms a cohomological δ-functor.

Proof. This is a combination of the results mentioned in the statement. The adjointness can be shown as follows: Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then we have

$$\begin{aligned} \text{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(f^*\mathcal{G}, \mathcal{F}) &= \text{Mor}_{\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}) \\ &= \text{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, Q(f_*\mathcal{F})) \\ &= \text{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_{QCoh,*}\mathcal{F}) \end{aligned}$$

the first equality by adjointness of f_* and f^* (for arbitrary sheaves of modules). By Proposition 103.8.1 we see that $f_*\mathcal{F}$ is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})$ (and can be computed in either the fppf or étale topology) and we obtain the second equality by Lemma 103.10.1. The third equality is the definition of $f_{QCoh,*}$.

To see that $\{R^if_{QCoh,*}\}_{i\geq 0}$ is a cohomological δ-functor as defined in Homology, Definition 12.12.1 let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $QCoh(\mathcal{O}_\mathcal{X})$. This sequence may not be an exact sequence in $\text{Mod}(\mathcal{O}_\mathcal{X})$ but we know that it is up to parasitic modules, see Lemma 103.9.4. Thus we may break up the sequence into short exact sequences

$$\begin{aligned} 0 \to \mathcal{P}_1 \to \mathcal{F}_1 \to \mathcal{I}_2 \to 0 \\ 0 \to \mathcal{I}_2 \to \mathcal{F}_2 \to \mathcal{Q}_2 \to 0 \\ 0 \to \mathcal{P}_2 \to \mathcal{Q}_2 \to \mathcal{I}_3 \to 0 \\ 0 \to \mathcal{I}_3 \to \mathcal{F}_3 \to \mathcal{P}_3 \to 0 \end{aligned}$$

of $\text{Mod}(\mathcal{O}_\mathcal{X})$ with $\mathcal{P}_i$ parasitic. Note that each of the sheaves $\mathcal{P}_j$, $\mathcal{I}_j$, $\mathcal{Q}_j$ is an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, see Proposition 103.8.1. Applying R^if_* we obtain long exact sequences

$$\begin{aligned} 0 \to f_*\mathcal{P}_1 \to f_*\mathcal{F}_1 \to f_*\mathcal{I}_2 \to R^1f_*\mathcal{P}_1 \to \ldots \\ 0 \to f_*\mathcal{I}_2 \to f_*\mathcal{F}_2 \to f_*\mathcal{Q}_2 \to R^1f_*\mathcal{I}_2 \to \ldots \\ 0 \to f_*\mathcal{P}_2 \to f_*\mathcal{Q}_2 \to f_*\mathcal{I}_3 \to R^1f_*\mathcal{P}_2 \to \ldots \\ 0 \to f_*\mathcal{I}_3 \to f_*\mathcal{F}_3 \to f_*\mathcal{P}_3 \to R^1f_*\mathcal{I}_3 \to \ldots \end{aligned}$$

where are the terms are objects of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})$ by Proposition 103.8.1. By Lemma 103.9.3 the sheaves $R^if_*\mathcal{P}_j$ are parasitic, hence vanish on applying the functor Q, see Lemma 103.10.2. Since Q is exact the maps

$$Q(R^if_*\mathcal{F}_3) \cong Q(R^if_*\mathcal{I}_3) \cong Q(R^if_*\mathcal{Q}_2) \to Q(R^{i+1}f_*\mathcal{I}_2) \cong Q(R^{i+1}f_*\mathcal{F}_1)$$

can serve as the connecting map which turns the family of functors $\{R^if_{QCoh,*}\}_{i\geq 0}$ into a cohomological δ-functor. $\square$

0GQQ Lemma 103.11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $y : V \to \mathcal{Y}$ in $\mathrm{Ob}(\mathcal{Y})$ with y a flat morphism. Let $\mathcal{F}$ be in $QCoh(\mathcal{O}_\mathcal{X})$. Then $(f_*\mathcal{F})(y) = (f_{QCoh,*}\mathcal{F})(y)$ and $(R^if_*\mathcal{F})(y) = (R^if_{QCoh,*}\mathcal{F})(y)$ for all $i \in \mathbf{Z}$.

Proof. This follows from the construction of the functors $R^if_{QCoh,*}$ in Proposition 103.11.1, the definition of parasitic modules in Definition 103.9.1, and Lemma 103.10.2 part (2). □

0GQR Remark 103.11.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_\mathcal{X})$. Then there is a canonical commutative diagram

$$\begin{tikzcd} f_{QCoh,*}\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} f_{QCoh,*}\mathcal{G} \arrow[r] \arrow[d] & f_*\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} f_*\mathcal{G} \arrow[d, "c"] \\ f_{QCoh,*}(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) \arrow[r] & f_*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) \end{tikzcd}$$

The vertical arrow c on the right is the naive relative cup product (in degree 0), see Cohomology on Sites, Section 21.33. The source and target of c are in $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, see Proposition 103.8.1. Applying Q to c we obtain the left vertical arrow as Q commutes with tensor products, see Remark 103.10.6. This construction is functorial in $\mathcal{F}$ and $\mathcal{G}$.

0782 Lemma 103.11.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page

$$E_2^{p,q} = H^p(\mathcal{Y}, R^qf_{QCoh,*}\mathcal{F})$$

converging to $H^{p+q}(\mathcal{X}, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 21.14.5 the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{Y}, R^qf_*\mathcal{F})$$

converges to $H^{p+q}(\mathcal{X}, \mathcal{F})$. The kernel and cokernel of the adjunction map

$$R^qf_{QCoh,*}\mathcal{F} \longrightarrow R^qf_*\mathcal{F}$$

are parasitic modules on $\mathcal{Y}$ (Lemma 103.10.2) hence have vanishing cohomology (Lemma 103.9.3). It follows formally that $H^p(\mathcal{Y}, R^qf_{QCoh,*}\mathcal{F}) = H^p(\mathcal{Y}, R^qf_*\mathcal{F})$ and we win. □

0783 Lemma 103.11.5. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be quasi-compact and quasi-separated morphisms of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page

$$E_2^{p,q} = R^pg_{QCoh,*}(R^qf_{QCoh,*}\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$.

Proof. By Cohomology on Sites, Lemma 21.14.7 the Leray spectral sequence with

$$E_2^{p,q} = R^pg_*(R^qf_*\mathcal{F})$$

converges to $R^{p+q}(g \circ f)_*\mathcal{F}$. By the results of Proposition 103.8.1 all the terms of this spectral sequence are objects of $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Z})$. Applying the exact functor

$Q_\mathcal{Z} : \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Z}) \to QCoh(\mathcal{O}_\mathcal{Z})$ we obtain a spectral sequence in $QCoh(\mathcal{O}_\mathcal{Z})$ covering to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$. Hence the result follows if we can show that

$$Q_\mathcal{Z}(R^p g_*(R^q f_*\mathcal{F})) = Q_\mathcal{Z}(R^p g_*(Q_\mathcal{X}(R^q f_*\mathcal{F}))$$

This follows from the fact that the kernel and cokernel of the map

$$Q_\mathcal{X}(R^q f_*\mathcal{F}) \longrightarrow R^q f_*\mathcal{F}$$

are parasitic (Lemma 103.10.2) and that $R^p g_*$ transforms parasitic modules into parasitic modules (Lemma 103.9.3). □

To end this section we make explicit the spectral sequences associated to a smooth covering by a scheme. Please compare with Sheaves on Stacks, Sections 96.20 and 96.21.

0784 Proposition 103.11.6. Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. Then there is a spectral sequence

$$E_2^{p,q} = H^q(\mathcal{U}_p, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{F})$$

where f_p is the morphism $\mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ ($p + 1$ factors).

Proof. This is a special case of Sheaves on Stacks, Proposition 96.20.1. □

0785 Proposition 103.11.7. Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable morphisms of algebraic stacks. Assume that

(1) f is representable by algebraic spaces, surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated, and
(2) g is quasi-compact and quasi-separated.

If $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_{QCoh,*} f_p^*\mathcal{F} \Rightarrow R^{p+q} g_{QCoh,*}\mathcal{F}$$

in $QCoh(\mathcal{O}_\mathcal{Y})$.

Proof. Note that each of the morphisms $f_p : \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ is quasi-compact and quasi-separated, hence $g \circ f_p$ is quasi-compact and quasi-separated, hence the assertion makes sense (i.e., the functors $R^q(g \circ f_p)_{QCoh,*}$ are defined). There is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q} g_*\mathcal{F}$$

by Sheaves on Stacks, Proposition 96.21.1. Applying the exact functor $Q_\mathcal{Y} : \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{Y})$ gives the desired spectral sequence in $QCoh(\mathcal{O}_\mathcal{Y})$. □

103.12. Further remarks on quasi-coherent modules

0GQS In this section we collect some results that to help understand how to use quasi-coherent modules on algebraic stacks.

Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{U}$ is represented by the algebraic space U. Consider the functor

$$a : \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \longrightarrow \text{Mod}(U_{\acute{e}tale}, \mathcal{O}_U), \quad \mathcal{F} \longmapsto f^*\mathcal{F}|_{U_{\acute{e}tale}}$$

given by pullback (Sheaves on Stacks, Section 96.7) followed by restriction (Sheaves on Stacks, Section 96.10). Applying this functor to locally quasi-coherent modules we obtain a functor

$$b : \text{LQCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{\acute{e}tale}, \mathcal{O}_U)$$

See Sheaves on Stacks, Lemmas 96.12.3 and 96.14.1. We can further limit our functor to even smaller subcategories to obtain

$$c : \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{\acute{e}tale}, \mathcal{O}_U)$$

and

$$d : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{\acute{e}tale}, \mathcal{O}_U)$$

About these functors we can say the following:[3]

(1) The functor a is exact. Namely, pullback $f^* = f^{-1}$ is exact (Sheaves on Stacks, Section 96.7) and restriction to $U_{\acute{e}tale}$ is exact, see Sheaves on Stacks, Equation (96.10.2.1).
(2) The functor b is exact. Namely, by Sheaves on Stacks, Lemma 96.12.4 the inclusion $\text{LQCoh}(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ is exact.
(3) The functor c is exact. Namely, by Proposition 103.8.1 the inclusion functor $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ is exact.
(4) The functor d is right exact but not exact in general. Namely, by Sheaves on Stacks, Lemma 96.12.5 the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ is right exact. We omit giving an example showing non-exactness.
(5) If f is flat, then d is exact. This follows on combining Lemma 103.4.1 and Sheaves on Stacks, Lemma 96.14.2.
(6) If f is flat, then c kills parasitic objects. Namely, f^* preserves parasitic object by Lemma 103.9.2. Then for any scheme V étale over U and hence flat over $\mathcal{X}$ we see that $0 = f^*\mathcal{F}|_{V_{\acute{e}tale}} = c(\mathcal{F})|_{V_{\acute{e}tale}}$ by the compatibility of restriction with étale localization Sheaves on Stacks, Remark 96.10.2. Hence clearly $c(\mathcal{F}) = 0$.
(7) If f is flat, then $c = d \circ Q$. Namely, the kernel and cokernel of $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic by Lemma 103.10.2. Thus, since c is exact (3) and kills parasitic objects (6), we see that c applied to $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.
(8) The functors a, b, c, d commute with colimits and arbitrary direct sums. This is true for f^* and restriction as left adjoints and hence it holds for a. Then it follows for b, c, d by the references given above.
(9) The functors a, b, c, d commute with tensor products.
(10) If f is flat and surjective, $\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, and $c(\mathcal{F}) = 0$, then $\mathcal{F}$ is parasitic. Namely, by (7) we get $d(Q(\mathcal{F})) = 0$. We may assume U is a scheme by the compatibility of restriction with étale localization (see reference above). Then Lemma 103.4.2 applied to $0 \to Q(\mathcal{F})$ and the morphism $f : U \to \mathcal{X}$ shows that $Q(\mathcal{F}) = 0$. Thus $\mathcal{F}$ is parasitic by Lemma 103.10.2.
(11) If f is flat and surjective, then the functor d reflects exactness. More precisely, let $\mathcal{F}^\bullet$ be a complex in $QCoh(\mathcal{O}_\mathcal{X})$. Then $\mathcal{F}^\bullet$ is exact in $QCoh(\mathcal{O}_\mathcal{X})$ if and only if $d(\mathcal{F}^\bullet)$ is exact. Namely, we have seen one implication in (5). For the other, suppose that $H^i(d(\mathcal{F}^\bullet)) = 0$. Then $\mathcal{G} = H^i(\mathcal{F}^\bullet)$ is an

[3]We suggest working out why these statements are true on a napkin instead of following the references given.

object of $QCoh(\mathcal{O}_\mathcal{X})$ with $d(\mathcal{G}) = 0$. Hence $\mathcal{G}$ is both quasi-coherent and parasitic by (10), whence 0 for example by Remark 103.10.7.

0GQT (12) If f is flat, $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(QCoh(\mathcal{O}_\mathcal{X}))$, and $\mathcal{F}$ of finite presentation and let then we have

$$d(hom(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{O}_U}(d(\mathcal{F}), d(\mathcal{G}))$$

with notation as in Lemma 103.10.8. Perhaps the easiest way to see this is as follows

$$\begin{aligned} d(hom(\mathcal{F}, \mathcal{G})) &= d(Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))) \\ &= c(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})) \\ &= f^* \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{\acute{e}tale}} \\ &= \mathcal{H}om_{\mathcal{O}_\mathcal{U}}(f^*\mathcal{F}, f^*\mathcal{G})|_{U_{\acute{e}tale}} \\ &= \mathcal{H}om_{\mathcal{O}_U}(f^*\mathcal{F}|_{U_{\acute{e}tale}}, f^*\mathcal{G}|_{U_{\acute{e}tale}}) \end{aligned}$$

The first equality by construction of *hom*. The second equality by (7). The third equality by definition of c. The fourth equality by Modules on Sites, Lemma 18.31.4. The final equality by the same reference applied to the flat morphism of ringed topoi $i_U(U_{\acute{e}tale}, \mathcal{O}_U) \to (\mathcal{U}_{\acute{e}tale}, \mathcal{O}_\mathcal{U})$ of Sheaves on Stacks, Lemma 96.10.1.

(13) add more here.

103.13. Colimits and cohomology

0GQU The following lemma in particular applies to diagrams of quasi-coherent sheaves.

0GQV **Lemma 103.13.1.** Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then

$$\mathrm{colim}_i\, H^p(\mathcal{X}, \mathcal{F}_i) \longrightarrow H^p(\mathcal{X}, \mathrm{colim}_i\, \mathcal{F}_i)$$

is an isomorphism for every filtered diagram of abelian sheaves on $\mathcal{X}$. The same is true for abelian sheaves on $\mathcal{X}_{\acute{e}tale}$ taking cohomology in the étale topology.

Proof. Let $\tau = fppf$, resp. $\tau = \acute{e}tale$. The lemma follows from Cohomology on Sites, Lemma 21.16.2 applied to the site $\mathcal{X}_\tau$. In order to check the assumptions we use Cohomology on Sites, Remark 21.16.3. Namely, let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{X}_\tau)$ be the set of objects lying over affine schemes. In other words, an element of $\mathcal{B}$ is a morphism $x : U \to \mathcal{X}$ with U affine. We check each of the conditions (1) – (4) of the remark in turn:

(1) Since $\mathcal{X}$ is quasi-compact, there exists a surjetive and smooth morphism $x : U \to \mathcal{X}$ with U affine (Properties of Stacks, Lemma 100.6.2). Then $h_x^\# \to *$ is a surjective map of sheaves on $\mathcal{X}_\tau$.

(2) Since coverings in $\mathcal{X}_\tau$ are fppf, resp. étale coverings, we see that every covering of $U \in \mathcal{B}$ is refined by a finite affine fppf covering, see Topologies, Lemma 34.7.4, resp. Lemma 34.4.4.

(3) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. The product $h_x^\# \times h_{x'}^\#$ in $Sh(\mathcal{X}_\tau)$ is equal to the sheaf on $\mathcal{X}_\tau$ determined by the algebraic space $W = U \times_{x, \mathcal{X}, x'} U'$ over $\mathcal{X}$: for an object $y : V \to \mathcal{X}$ of $\mathcal{X}_\tau$ we have $(h_x^\# \times h_{x'}^\#)(y) = \{f : V \to W \mid y = x \circ \mathrm{pr}_1 \circ f = x' \circ \mathrm{pr}_2 \circ f\}$. The algebraic space W is quasi-compact because $\mathcal{X}$ is quasi-separated, see Morphisms of Stacks, Lemma 101.7.8 for example. Hence we can choose an affine scheme

U'' and a surjective étale morphism $U'' \to W$. Denote $x'' : U'' \to \mathcal{X}$ the composition of $U'' \to W$ and $W \to \mathcal{X}$. Then $h_{x''}^{\#} \to h_x^{\#} \times h_{x'}^{\#}$ is surjective as desired.

(4) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. Let $a, b : U \to U'$ be a morphism over $\mathcal{X}$, i.e., $a, b : x \to x'$ is a morphism in $\mathcal{X}_\tau$. Then the equalizer of h_a and h_b is represented by the equalizer of $a, b : U \to U'$ which is affine scheme over $\mathcal{X}$ and hence in $\mathcal{B}$.

This finished the proof. □

0GQW Lemma 103.13.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $\mathcal{X}$. Then for any $p \geq 0$ we have

$$R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i.$$

The same is true for abelian sheaves on $\mathcal{X}_{étale}$ taking higher direct images in the étale topology.

Proof. We will prove this for the fppf topology; the proof for the étale topology is the same. Recall that $R^i f_* \mathcal{F}$ is the sheaf on $\mathcal{Y}_{fppf}$ associated to the presheaf

$$(y : V \to \mathcal{Y}) \longmapsto H^i(V \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{F})$$

See Sheaves on Stacks, Lemma 96.21.2. Recall that the colimit is the sheaf associated to the presheaf colimit. When V is affine, the fibre product $V \times_\mathcal{Y} \mathcal{X}$ is quasi-compact and quasi-separated. Hence we can apply Lemma 103.13.1 to $H^p(V \times_\mathcal{Y} \mathcal{X}, -)$ where V is affine. Since every V has an fppf covering by affine objects this proves the lemma. Some details omitted. □

0GQX Lemma 103.13.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f_{QCoh,*}$ and the functors $R^i f_{QCoh,*}$ commute with direct sums and filtered colimits.

Proof. The functors f_* and $R^i f_*$ commute with direct sums and filtered colimits on all modules by Lemma 103.13.2. The lemma follows as $f_{QCoh,*} = Q \circ f_*$ and $R^i f_{QCoh,*} = Q \circ R^i f_*$ and Q commutes with all colimits, see Lemma 103.10.2. □

0GQY Lemma 103.13.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. The functors $R^i f_{QCoh,*}$, $i > 0$ vanish and the functor $f_{QCoh,*}$ is exact and commutes with direct sums and all colimits.

Proof. Since we have $R^i f_{QCoh,*} = Q \circ R^i f_*$ we obtain the vanishing from Lemma 103.8.4. The vanishing implies that $f_{QCoh,*}$ is exact as $\{R^i f_{QCoh,*}\}_{i \geq 0}$ form a δ-functor, see Proposition 103.11.1. Then $f_{QCoh,*}$ commutes with direct sums for example by Lemma 103.13.3. An exact functor which commutes with direct sums commutes with all colimits. □

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic stacks.

0GQZ Lemma 103.13.5. Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Let I be a directed set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of $\mathcal{O}_\mathcal{X}$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_\mathcal{X}$-module of finite presentation. Then we have

$$\operatorname{colim}_i \operatorname{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}_i) = \operatorname{Hom}_\mathcal{X}(\mathcal{G}, \operatorname{colim}_i \mathcal{F}_i).$$

In particular, $\mathrm{Hom}_{\mathcal{X}}(\mathcal{G}, -)$ commutes with filtered colimits in $QCoh(\mathcal{O}_{\mathcal{X}})$.

Proof. The displayed equality is a special case of Modules on Sites, Lemma 18.27.12. In order to apply it, we need to check the hypotheses of Sites, Lemma 7.17.8 part (4) for the site $\mathcal{X}_{fppf}$. In order to do this, we will check hypotheses (2)(a), (2)(b), (2)(c) of Sites, Remark 7.17.9. Namely, let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{X}_{fppf})$ be the set of objects lying over affine schemes. In other words, an element of $\mathcal{B}$ is a morphism $x : U \to \mathcal{X}$ with U affine. We check each of the conditions (2)(a), (2)(b), and (2)(c) of the remark in turn:

(1) Since $\mathcal{X}$ is quasi-compact, there exists a surjective and smooth morphism $x : U \to \mathcal{X}$ with U affine (Properties of Stacks, Lemma 100.6.2). Then $h_x^\# \to *$ is a surjective map of sheaves on $\mathcal{X}_{fppf}$.
(2) Since coverings in $\mathcal{X}_{fppf}$ are fppf coverings, we see that every covering of $U \in \mathcal{B}$ is refined by a finite affine fppf covering, see Topologies, Lemma 34.7.4.
(3) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. The product $h_x^\# \times h_{x'}^\#$ in $Sh(\mathcal{X}_{fppf})$ is equal to the sheaf on $\mathcal{X}_{fppf}$ determined by the algebraic space $W = U \times_{x, \mathcal{X}, x'} U'$ over $\mathcal{X}$: for an object $y : V \to \mathcal{X}$ of $\mathcal{X}_{fppf}$ we have $(h_x^\# \times h_{x'}^\#)(y) = \{f : V \to W \mid y = x \circ \mathrm{pr}_1 \circ f = x' \circ \mathrm{pr}_2 \circ f\}$. The algebraic space W is quasi-compact because $\mathcal{X}$ is quasi-separated, see Morphisms of Stacks, Lemma 101.7.8 for example. Hence we can choose an affine scheme U'' and a surjective étale morphism $U'' \to W$. Denote $x'' : U'' \to \mathcal{X}$ the composition of $U'' \to W$ and $W \to \mathcal{X}$. Then $h_{x''}^\# \to h_x^\# \times h_{x'}^\#$ is surjective as desired.

For the final statement, observe that the inclusion functor $QCoh(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$ commutes with colimits and that finitely presented modules are quasi-coherent. See Sheaves on Stacks, Lemma 96.15.1. □

103.14. The lisse-étale and the flat-fppf sites

0786 In the book [LMB00] many of the results above are proved using the lisse-étale site of an algebraic stack. We define this site here. In Examples, Section 110.58 we show that the lisse-étale site isn't functorial. We also define its analogue, the flat-fppf site, which is better suited to the development of algebraic stacks as given in the Stacks project (because we use the fppf topology as our base topology). Of course the flat-fppf site isn't functorial either.

0787 Definition 103.14.1. Let $\mathcal{X}$ be an algebraic stack.

(1) The lisse-étale site of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{lisse,étale}$[4] of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is smooth. A covering of $\mathcal{X}_{lisse,étale}$ is a family of morphisms $\{x_i \to x\}_{i \in I}$ of $\mathcal{X}_{lisse,étale}$ which forms a covering of $\mathcal{X}_{étale}$.
(2) The flat-fppf site of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{flat,fppf}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is flat. A covering of $\mathcal{X}_{flat,fppf}$ is a family of morphisms $\{x_i \to x\}_{i \in I}$ of $\mathcal{X}_{flat,fppf}$ which forms a covering of $\mathcal{X}_{fppf}$.

[4]In the literature the site is denoted Lis-ét($\mathcal{X}$) or Lis-Et($\mathcal{X}$) and the associated topos is denoted $\mathcal{X}_{\text{lis-ét}}$ or $\mathcal{X}_{\text{lis-et}}$. In the Stacks project our convention is to name the site and denote the corresponding topos by $Sh(\mathcal{C})$.

We denote $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$ the restriction of $\mathcal{O}_\mathcal{X}$ to the lisse-étale site and similarly for $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$. The relationship between the lisse-étale site and the étale site is as follows (we mainly stick to "topological" properties in this lemma).

0788 Lemma 103.14.2. Let $\mathcal{X}$ be an algebraic stack.

(1) The inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$ is fully faithful, continuous and cocontinuous. It follows that
  (a) there is a morphism of topoi
  $$g : Sh(\mathcal{X}_{lisse,étale}) \longrightarrow Sh(\mathcal{X}_{étale})$$
  with g^{-1} given by restriction,
  (b) the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,
  (c) the adjunction maps $g^{-1}g_* \to \text{id}$ and $\text{id} \to g^{-1}g_!^{Sh}$ are isomorphisms,
  (d) the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,
  (e) the adjunction map $\text{id} \to g^{-1}g_!$ is an isomorphism, and
  (f) we have $g^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.

(2) The inclusion functor $\mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ is fully faithful, continuous and cocontinuous. It follows that
  (a) there is a morphism of topoi
  $$g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$$
  with g^{-1} given by restriction,
  (b) the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,
  (c) the adjunction maps $g^{-1}g_* \to \text{id}$ and $\text{id} \to g^{-1}g_!^{Sh}$ are isomorphisms,
  (d) the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,
  (e) the adjunction map $\text{id} \to g^{-1}g_!$ is an isomorphism, and
  (f) we have $g^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.

Proof. In both cases it is immediate that the functor is fully faithful, continuous, and cocontinuous (see Sites, Definitions 7.13.1 and 7.20.1). Hence properties (a), (b), (c) follow from Sites, Lemmas 7.21.5 and 7.21.7. Parts (d), (e) follow from Modules on Sites, Lemmas 18.16.2 and 18.16.4. Part (f) is immediate. □

0GR0 Lemma 103.14.3. Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 103.14.2.

(1) For an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ we have
  (a) $H^p(\mathcal{X}_{étale}, \mathcal{F}) = H^p(\mathcal{X}_{lisse,étale}, g^{-1}\mathcal{F})$, and
  (b) $H^p(x, \mathcal{F}) = H^p(\mathcal{X}_{lisse,étale}/x, g^{-1}\mathcal{F})$ for any object x of $\mathcal{X}_{lisse,étale}$.
  The same holds for sheaves of modules.

(2) For an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{fppf}$ we have
  (a) $H^p(\mathcal{X}_{fppf}, \mathcal{F}) = H^p(\mathcal{X}_{flat,fppf}, g^{-1}\mathcal{F})$, and
  (b) $H^p(x, \mathcal{F}) = H^p(\mathcal{X}_{flat,fppf}/x, g^{-1}\mathcal{F})$ for any object x of $\mathcal{X}_{flat,fppf}$.
  The same holds for sheaves of modules.

Proof. Part (1)(a) follows from Sheaves on Stacks, Lemma 96.23.3 applied to the inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$. Part (1)(b) follows from part (1)(a). Namely, if x lies over the scheme U, then the site $\mathcal{X}_{étale}/x$ is equivalent to $(Sch/U)_{étale}$ and $\mathcal{X}_{lisse,étale}$ is equivalent to $U_{lisse,étale}$. Part (2) is proved in the same manner. □

0789 Lemma 103.14.4. Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 103.14.2.

(1) There exists a functor
$$g_! : \text{Mod}(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow \text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$
which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = \text{id}$.
(2) There exists a functor
$$g_! : \text{Mod}(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow \text{Mod}(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$$
which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = \text{id}$.

Proof. In both cases, the existence of the functor $g_!$ follows from Modules on Sites, Lemma 18.41.1. To see that $g_!$ agrees with the functor on abelian sheaves we will show the maps Modules on Sites, Equation (18.41.2.1) are isomorphisms.

Lisse-étale case. Let $x \in \text{Ob}(\mathcal{X}_{lisse,étale})$ lying over a scheme U with $x : U \to \mathcal{X}$ smooth. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{lisse,étale}/x \longrightarrow \mathcal{X}_{étale}/x$$
The right hand side is identified with $(Sch/U)_{étale}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is smooth. Thus Étale Cohomology, Lemma 59.49.2 applies.

Flat-fppf case. Let $x \in \text{Ob}(\mathcal{X}_{flat,fppf})$ lying over a scheme U with $x : U \to \mathcal{X}$ flat. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{flat,fppf}/x \longrightarrow \mathcal{X}_{fppf}/x$$
The right hand side is identified with $(Sch/U)_{fppf}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is flat. Thus Étale Cohomology, Lemma 59.49.2 applies.

In both cases the equality $g^*g_! = \text{id}$ follows from $g^* = g^{-1}$ and the equality for abelian sheaves in Lemma 103.14.2. □

078A Lemma 103.14.5. Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemmas 103.14.2 and 103.14.4.

(1) We have $g_!\mathcal{O}_{\mathcal{X}_{lisse,étale}} = \mathcal{O}_\mathcal{X}$.
(2) We have $g_!\mathcal{O}_{\mathcal{X}_{flat,fppf}} = \mathcal{O}_\mathcal{X}$.

Proof. In this proof we write $\mathcal{C} = \mathcal{X}_{étale}$ (resp. $\mathcal{C} = \mathcal{X}_{fppf}$) and we denote $\mathcal{C}' = \mathcal{X}_{lisse,étale}$ (resp. $\mathcal{C}' = \mathcal{X}_{flat,fppf}$). Then $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. In this proof we will think of objects V of $\mathcal{C}$ as schemes over $\mathcal{X}$ and objects U of $\mathcal{C}'$ as schemes smooth (resp. flat) over $\mathcal{X}$. Finally, we write $\mathcal{O} = \mathcal{O}_\mathcal{X}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). In the notation above we have $\mathcal{O}(V) = \Gamma(V, \mathcal{O}_V)$ and $\mathcal{O}'(U) = \Gamma(U, \mathcal{O}_U)$. Consider the $\mathcal{O}$-module homomorphism $g_!\mathcal{O}' \to \mathcal{O}$ adjoint to the identification $\mathcal{O}' = g^{-1}\mathcal{O}$.

Recall that $g_!\mathcal{O}'$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}'$ given by the rule
$$V \longmapsto \text{colim}_{V \to U} \mathcal{O}'(U)$$
where the colimit is taken in the category of abelian groups (Modules on Sites, Definition 18.16.1). Below we will use frequently that if
$$V \to U \to U'$$

are morphisms and if $f' \in \mathcal{O}'(U')$ restricts to $f \in \mathcal{O}'(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}' \to \mathcal{O}$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Let us prove that $g_!\mathcal{O}' \to \mathcal{O}$ is surjective. Let $h \in \mathcal{O}(V)$ for some object V of $\mathcal{C}$. It suffices to show that h is locally in the image. Choose an object U of $\mathcal{C}'$ corresponding to a surjective smooth morphism $U \to \mathcal{X}$. Since $U \times_\mathcal{X} V \to V$ is surjective smooth, after replacing V by the members of an étale covering of V we may assume there exists a morphism $V \to U$, see Topologies on Spaces, Lemma 73.4.4. Using h we obtain a morphism $V \to U \times \mathbf{A}^1$ such that writing $\mathbf{A}^1 = \text{Spec}(\mathbf{Z}[t])$ the element $t \in \mathcal{O}(U \times \mathbf{A}^1)$ pulls back to h. Since $U \times \mathbf{A}^1$ is an object of $\mathcal{C}'$ we see that $(V \to U \times \mathbf{A}^1, t)$ is an element of the colimit above which maps to $h \in \mathcal{O}(V)$ as desired.

Suppose that $s \in g_!\mathcal{O}'(V)$ is a section mapping to zero in $\mathcal{O}(V)$. To finish the proof we have to show that s is zero. After replacing V by the members of a covering we may assume s is an element of the colimit

$$\text{colim}_{V \to U} \mathcal{O}'(U)$$

Say $s = \sum(\varphi_i, s_i)$ is a finite sum with $\varphi_i : V \to U_i$, U_i smooth (resp. flat) over $\mathcal{X}$, and $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Choose a scheme W surjective étale over the algebraic space $U = U_1 \times_\mathcal{X} \ldots \times_\mathcal{X} U_n$. Note that W is still smooth (resp. flat) over $\mathcal{X}$, i.e., defines an object of $\mathcal{C}'$. The fibre product

$$V' = V \times_{(\varphi_1, \ldots, \varphi_n), U} W$$

is surjective étale over V, hence it suffices to show that s maps to zero in $g_!\mathcal{O}'(V')$. Note that the restriction $\sum(\varphi_i, s_i)|_{V'}$ corresponds to the sum of the pullbacks of the functions s_i to W. In other words, we have reduced to the case of (φ, s) where $\varphi : V \to U$ is a morphism with U in $\mathcal{C}'$ and $s \in \mathcal{O}'(U)$ restricts to zero in $\mathcal{O}(V)$. By the commutative diagram

$$\begin{tikzcd} V \arrow[rr, "(\varphi,0)"'] \arrow[drr, "\varphi"'] & & U \times \mathbf{A}^1 \\ & & U \arrow[u, "(\text{id},0)"'] \end{tikzcd}$$

we see that $((\varphi, 0) : V \to U \times \mathbf{A}^1, \text{pr}_2^*x)$ represents zero in the colimit above. Hence we may replace U by $U \times \mathbf{A}^1$, φ by $(\varphi, 0)$ and s by $\text{pr}_1^*s + \text{pr}_2^*x$. Thus we may assume that the vanishing locus $Z : s = 0$ in U of s is smooth (resp. flat) over $\mathcal{X}$. Then we see that $(V \to Z, 0)$ and (φ, s) have the same value in the colimit, i.e., we see that the element s is zero as desired. $\square$

The lisse-étale and the flat-fppf sites can be used to characterize parasitic modules as follows.

07AR Lemma 103.14.6. Let $\mathcal{X}$ be an algebraic stack.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module with the flat base change property on $\mathcal{X}_{étale}$. The following are equivalent
  (a) $\mathcal{F}$ is parasitic, and
  (b) $g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{lisse,étale}) \to Sh(\mathcal{X}_{étale})$ is as in Lemma 103.14.2.
(2) Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{fppf}$. The following are equivalent

(a) $\mathcal{F}$ is parasitic, and
(b) $g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ is as in Lemma 103.14.2.

Proof. Part (2) is immediate from the definitions (this is one of the advantages of the flat-fppf site over the lisse-étale site). The implication (1)(a) $\Rightarrow$ (1)(b) is immediate as well. To see (1)(b) $\Rightarrow$ (1)(a) let U be a scheme and let $x : U \to \mathcal{X}$ be a surjective smooth morphism. Then x is an object of the lisse-étale site of $\mathcal{X}$. Hence we see that (1)(b) implies that $\mathcal{F}|_{U_{étale}} = 0$. Let $V \to \mathcal{X}$ be an flat morphism where V is a scheme. Set $W = U \times_\mathcal{X} V$ and consider the diagram

$$\begin{tikzcd} W \arrow[r, "q"'] \arrow[d, "p"'] & V \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{étale}} \cong q_{small}^*\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of g^*, then $\mathcal{F}|_{V_{étale}} = 0$ as desired. $\square$

103.15. Functoriality of the lisse-étale and flat-fppf sites

0GR1 The lisse-étale site is functorial for smooth morphisms of algebraic stacks and the flat-fppf site is functorial for flat morphisms of algebraic stacks. We warn the reader that the lisse-étale and flat-fppf topoi are not functorial with respect to all morphisms of algebraic stacks, see Examples, Section 110.58.

07AT Lemma 103.15.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) If f is smooth, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ which gives a morphism of ringed topoi fitting into the following commutative diagram

$$\begin{tikzcd} Sh(\mathcal{X}_{lisse,étale}) \arrow[r, "g'"'] \arrow[d, "f'"'] & Sh(\mathcal{X}_{étale}) \arrow[d, "f"] \\ Sh(\mathcal{Y}_{lisse,étale}) \arrow[r, "g"] & Sh(\mathcal{Y}_{étale}) \end{tikzcd}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.

(2) If f is flat, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$ which gives a morphism of ringed topoi fitting into the following commutative diagram

$$\begin{tikzcd} Sh(\mathcal{X}_{flat,fppf}) \arrow[r, "g'"'] \arrow[d, "f'"'] & Sh(\mathcal{X}_{fppf}) \arrow[d, "f"] \\ Sh(\mathcal{Y}_{flat,fppf}) \arrow[r, "g"] & Sh(\mathcal{Y}_{fppf}) \end{tikzcd}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.

Proof. The initial statement comes from the fact that if $x \in \mathrm{Ob}(\mathcal{X})$ lies over a scheme U such that $x : U \to \mathcal{X}$ is smooth (resp. flat) and if f is smooth (resp. flat) then $f(x) : U \to \mathcal{Y}$ is smooth (resp. flat), see Morphisms of Stacks, Lemmas 101.33.2 and 101.25.2. The induced functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$) is continuous and cocontinuous by our definition of coverings in these categories. Finally, the commutativity of the diagram is a consequence of the fact that the horizontal morphisms are given by the inclusion functors (see Lemma 103.14.2) and Sites, Lemma 7.21.2.

To show that $f'_*(g')^{-1} = g^{-1}f_*$ let $\mathcal{F}$ be a sheaf on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$). There is a canonical pullback map

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

see Sites, Section 7.45. We claim this map is an isomorphism. To prove this pick an object y of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}f_*\mathcal{F}\right)(y) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

by Sheaves on Stacks, Equation (96.5.0.1). Let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects $z : W \to V \times_{y,\mathcal{Y}} \mathcal{X}$ such that the induced morphism $W \to \mathcal{X}$ is smooth (resp. flat). Denote

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \longrightarrow \mathcal{X}_{lisse,étale} \text{ (resp. } \mathcal{X}_{flat,fppf})$$

the restriction of the functor pr used in the formula above. Exactly the same argument that proves Sheaves on Stacks, Equation (96.5.0.1) shows that for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) we have

07AU (103.15.1.1) $$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

Since $(g')^{-1}$ is restriction we see that

$$\left(f'_*(g')^{-1}\mathcal{F}\right)(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'})$$

By Sheaves on Stacks, Lemma 96.23.3 we see that

$$\Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'}) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

Finally, the equality $g'_!(f')^{-1} = f^{-1}g_!$ follows formally from the equality $f'_*(g')^{-1} = g^{-1}f_*$ by the adjointness of f^{-1} and f_*, the adjointness of $g_!$ and g^{-1}, and their "primed" versions. $\square$

0GR2 Lemma 103.15.2. With assumptions and notation as in Lemma 103.15.1. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). Then

07AW (103.15.2.1) $$R^pf'_*\mathcal{H} = \text{sheaf associated to } y \longmapsto H^p((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{H})$$

Here y is an object of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) lying over the scheme V and the notation $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ and pr' are explained in the proof.

Proof. As in the proof of Lemma 103.15.1 let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects (x, φ) where x is an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) and $\varphi : f(x) \to y$ is a morphism in $\mathcal{Y}$. By Equation (103.15.1.1) we have

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

where pr' is the projection. For an object (x, φ) of $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ we can think of φ as a section of $(f')^{-1}h_y$ over x. Thus $(V \times_{\mathcal{Y}} \mathcal{X})'$ is the localization of the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) at the sheaf of sets $(f')^{-1}h_y$, see Sites, Lemma 7.30.3. The morphism

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{lisse,étale} \text{ (resp. } \mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{flat,fppf})$$

is the localization morphism. In particular, the pullback $(\mathrm{pr}')^{-1}$ preserves injective abelian sheaves, see Cohomology on Sites, Lemma 21.13.3.

Choose an injective resolution $\mathcal{H} \to \mathcal{I}^\bullet$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). By the formula for pushforward we see that $R^if'_*\mathcal{H}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{matrix} \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i+1}\Big) \end{matrix}$$

Since $(\mathrm{pr}')^{-1}$ is exact and preserves injectives the complex $(\mathrm{pr}')^{-1}\mathcal{I}^\bullet$ is an injective resolution of $(\mathrm{pr}')^{-1}\mathcal{H}$ and the proof is complete. □

0GR3 **Lemma 103.15.3.** With assumptions and notation as in Lemma 103.15.1 the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$).

Proof. Comparing the formula for $g^{-1}R^pf_*\mathcal{F}$ and $R^pf'_*(g')^{-1}\mathcal{F}$ given in Sheaves on Stacks, Lemma 96.21.2 and Lemma 103.15.2 we see that it suffices to show

$$H^p((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'}) = H^p_\tau(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

where $\tau =$ *étale* (resp. $\tau = fppf$). Here y is an object of $\mathcal{Y}$ lying over a scheme V such that the morphism $y : V \to \mathcal{Y}$ is smooth (resp. flat). This equality follows from Sheaves on Stacks, Lemma 96.23.3. Although we omit the verification of the assumptions of the lemma, we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition. □

103.16. Quasi-coherent modules and the lisse-étale and flat-fppf sites

07AY In this section we explain how to think of quasi-coherent modules on an algebraic stack in terms of its lisse-étale or flat-fppf site.

07AZ **Lemma 103.16.1.** Let $\mathcal{X}$ be an algebraic stack.

(1) Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.

(2) Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.

Proof. Proof of (1). We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 101.33.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still quasi-coherent, see Modules on Sites, Lemma 18.23.4. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a smooth surjective morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 96.19.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 18.23.3. By Sheaves on Stacks, Lemma 96.11.4 we see that $x^*\mathcal{F}$ is an fppf sheaf and since x was arbitrary we see that $\mathcal{F}$ is a sheaf in the fppf topology. Applying Sheaves on Stacks, Lemma 96.11.3 we see that $\mathcal{F}$ is quasi-coherent.

Proof of (2). This is proved using exactly the same argument, which we fully write out here. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that flat and locally finite presented morphisms are preserved by composition, see Morphisms of Stacks, Lemmas 101.25.2 and 101.27.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 96.11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective, flat, and locally finitely presented morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 96.19.10 there exists an fppf covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an fppf covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 18.23.3. By Sheaves on Stacks, Lemma 96.11.3 we see that $\mathcal{F}$ is quasi-coherent. ☐

We recall that we have defined the notion of a quasi-coherent module on any ringed topos in Modules on Sites, Section 18.23.

07B0 Lemma 103.16.2. Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 103.14.2.

(1) Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.
(2) Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 18.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i \in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 18.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 96.9.2 and the discussion following.) Since each f_i is smooth (resp. flat) by Lemma 103.15.1 we see that $f_i^{-1}g_!\mathcal{H} = g_{i,!}(f_i')^{-1}\mathcal{H}$.

Using Lemma 103.16.1 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation. Say we have

$$\bigoplus\nolimits_{j\in J}\mathcal{O}\longrightarrow\bigoplus\nolimits_{i\in I}\mathcal{O}\longrightarrow\mathcal{H}\longrightarrow 0$$

of $\mathcal{O}$-modules where $\mathcal{O}=\mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O}=\mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Since $g_!$ commutes with arbitrary colimits (as a left adjoint functor, see Lemma 103.14.4 and Categories, Lemma 4.24.5) we conclude that there exists an exact sequence

$$\bigoplus\nolimits_{j\in J}g_!\mathcal{O}\longrightarrow\bigoplus\nolimits_{i\in I}g_!\mathcal{O}\longrightarrow g_!\mathcal{H}\longrightarrow 0$$

Lemma 103.14.5 shows that $g_!\mathcal{O}=\mathcal{O}_\mathcal{X}$. In case (2) we are done. In case (1) we apply Sheaves on Stacks, Lemma 96.11.4 to conclude. $\square$

07B1 Lemma 103.16.3. Let $\mathcal{X}$ be an algebraic stack.

(1) With g as in Lemma 103.14.2 for the lisse-étale site we have
  (a) the functors g^{-1} and $g_!$ define mutually inverse functors
$$\begin{tikzcd} QCoh(\mathcal{O}_\mathcal{X}) \arrow[r, shift left, "g^{-1}"] & QCoh(\mathcal{X}_{lisse,étale},\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \arrow[l, shift left, "g_!"] \end{tikzcd}$$
  (b) if $\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and
  (c) $Q(\mathcal{F})=g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 103.10.1.

(2) With g as in Lemma 103.14.2 for the flat-fppf site we have
  (a) the functors g^{-1} and $g_!$ define mutually inverse functors
$$\begin{tikzcd} QCoh(\mathcal{O}_\mathcal{X}) \arrow[r, shift left, "g^{-1}"] & QCoh(\mathcal{X}_{flat,fppf},\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \arrow[l, shift left, "g_!"] \end{tikzcd}$$
  (b) if $\mathcal{F}$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ and
  (c) $Q(\mathcal{F})=g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 103.10.1.

Proof. Pullback by any morphism of ringed topoi preserves categories of quasi-coherent modules, see Modules on Sites, Lemma 18.23.4. Hence g^{-1} preserves the categories of quasi-coherent modules; here we use that $QCoh(\mathcal{O}_\mathcal{X})=QCoh(\mathcal{X}_{étale},\mathcal{O}_\mathcal{X})$ by Sheaves on Stacks, Lemma 96.11.4. The same is true for $g_!$ by Lemma 103.16.2. We know that $\mathcal{H}\to g^{-1}g_!\mathcal{H}$ is an isomorphism by Lemma 103.14.2. Conversely, if $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then the map $g_!g^{-1}\mathcal{F}\to\mathcal{F}$ is a map of quasi-coherent modules on $\mathcal{X}$ whose restriction to any scheme smooth over $\mathcal{X}$ is an isomorphism. Then the discussion in Sheaves on Stacks, Sections 96.14 and 96.15 (comparing with quasi-coherent modules on presentations) shows it is an isomorphism. This proves (1)(a) and (2)(a).

Let $\mathcal{F}$ be an object of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. By Lemma 103.10.2 the kernel and cokernel of the map $Q(\mathcal{F})\to\mathcal{F}$ are parasitic. Hence by Lemma 103.14.6 and since $g^*=g^{-1}$ is exact, we conclude $g^*Q(\mathcal{F})\to g^*\mathcal{F}$ is an isomorphism. Thus $g^*\mathcal{F}$ is quasi-coherent. This proves (1)(b) and (2)(b). Finally, (1)(c) and (2)(c) follow because $g_!g^*Q(\mathcal{F})\to Q(\mathcal{F})$ is an isomorphism by our arguments above. $\square$

07B4 Lemma 103.16.4. Let $\mathcal{X}$ be an algebraic stack.

(1) $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$.
(2) $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$.

Proof. We will verify conditions (1), (2), (3), (4) of Homology, Lemma 12.10.3.

Since 0 is a quasi-coherent module on any ringed site we see that (1) holds.

By definition $QCoh(\mathcal{O})$ is a strictly full subcategory $\text{Mod}(\mathcal{O})$, so (2) holds.

Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of quasi-coherent modules on $\mathcal{X}_{lisse,étale}$ or $\mathcal{X}_{flat,fppf}$. We have $g^*g_!\mathcal{F} = \mathcal{F}$ and similarly for $\mathcal{G}$ and φ, see Lemma 103.14.4. By Lemma 103.16.2 we see that $g_!\mathcal{F}$ and $g_!\mathcal{G}$ are quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. By Sheaves on Stacks, Lemma 96.15.1 we have that $\text{Coker}(g_!\varphi)$ is a quasi-coherent module on $\mathcal{X}$ (and the cokernel in the category of quasi-coherent modules on $\mathcal{X}$). Since g^* is exact (see Lemma 103.14.2) $g^* \text{Coker}(g_!\varphi) = \text{Coker}(g^*g_!\varphi) = \text{Coker}(\varphi)$ is quasi-coherent too (see Lemma 103.16.3). By Proposition 103.8.1 the kernel $\text{Ker}(g_!\varphi)$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. Since g^* is exact, we have $g^* \text{Ker}(g_!\varphi) = \text{Ker}(g^*g_!\varphi) = \text{Ker}(\varphi)$. Since g^* maps objects of $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ to quasi-coherent modules by Lemma 103.16.3 we conclude that $\text{Ker}(\varphi)$ is quasi-coherent as well. This proves (3).

Finally, suppose that
$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$
is an extension of $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-modules (resp. $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-modules) with $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent. To prove (4) and finish the proof we have to show that $\mathcal{E}$ is quasi-coherent on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). Let U be an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$; we think of U as a scheme smooth (resp. flat) over $\mathcal{X}$. We have to show that the restriction of $\mathcal{E}$ to $U_{lisse,étale}$ (resp. $= U_{flat,fppf}$) is quasi-coherent. Thus we may assume that $\mathcal{X} = U$ is a scheme. Because $\mathcal{G}$ is quasi-coherent on $U_{lisse,étale}$ (resp. $U_{flat,fppf}$), we may assume, after replacing U by the members of an étale (resp. fppf) covering, that $\mathcal{G}$ has a presentation
$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$
on $U_{lisse,étale}$ (resp. $U_{flat,fppf}$) where $\mathcal{O}$ is the structure sheaf on the site. We may also assume U is affine. Since $\mathcal{F}$ is quasi-coherent, we have
$$H^1(U_{lisse,étale}, \mathcal{F}) = 0, \quad \text{resp.} \quad H^1(U_{flat,fppf}, \mathcal{F}) = 0$$
Namely, $\mathcal{F}$ is the pullback of a quasi-coherent module $\mathcal{F}'$ on the big site of U (by Lemma 103.16.3), cohomology of $\mathcal{F}$ and $\mathcal{F}'$ agree (by Lemma 103.14.3), and we know that the cohomology of $\mathcal{F}'$ on the big site of the affine scheme U is zero (to get this in the current situation you have to combine Descent, Propositions 35.8.9 and 35.9.3 with Cohomology of Schemes, Lemma 30.2.2). Thus we can lift the map $\bigoplus_{i \in I} \mathcal{O} \to \mathcal{G}$ to $\mathcal{E}$. A diagram chase shows that we obtain an exact sequence
$$\bigoplus\nolimits_{j \in J} \mathcal{O} \to \mathcal{F} \oplus \bigoplus\nolimits_{i \in I} \mathcal{O} \to \mathcal{E} \to 0$$
By (3) proved above, we conclude that $\mathcal{E}$ is quasi-coherent as desired. □

103.17. Coherent sheaves on locally Noetherian stacks

0GR4 This section is the analogue of Cohomology of Spaces, Section 69.12. We have defined the notion of a coherent module on any ringed topos in Modules on Sites, Section 18.23. However, for any algebraic stack $\mathcal{X}$ the category of coherent $\mathcal{O}_\mathcal{X}$-modules is zero, essentially because the site $\mathcal{X}$ contains too many non-Noetherian objects (even if $\mathcal{X}$ is itself locally Noetherian). Instead, we will define coherent modules using the following lemma.

0GR5 Lemma 103.17.1. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module. The following are equivalent

(1) $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_\mathcal{X}$-module,
(2) $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module of finite presentation,
(3) $\mathcal{F}$ is quasi-coherent and for any morphism $f : U \to \mathcal{X}$ where U is a locally Noetherian algebraic space, the pullback $f^*\mathcal{F}|_{U_{\acute{e}tale}}$ is coherent, and
(4) $\mathcal{F}$ is quasi-coherent and there exists an algebraic space U and a morphism $f : U \to \mathcal{X}$ which is locally of finite type, flat, and surjective, such that the pullback $f^*\mathcal{F}|_{U_{\acute{e}tale}}$ is coherent.

Proof. Let $f : U \to \mathcal{X}$ be as in (4). Then U is locally Noetherian (Morphisms of Stacks, Lemma 101.17.5) and we see that the statement of the lemma makes sense. Additionally, f is locally of finite presentation by Morphisms of Stacks, Lemma 101.27.5. Let x be an object of $\mathcal{X}$ lying over the scheme V. In order to prove (2) we have to show that, after replacing V by the members of an fppf covering of V, the restriction $x^*\mathcal{F}$ has a global finite presentation on $\mathcal{X}/x \cong (Sch/V)_{fppf}$. The projection $W = U \times_\mathcal{X} V \to V$ is locally of finite presentation, flat, and surjective. Hence we may replace V by the members of an étale covering of W by schemes and assume we have a morphism $h : V \to U$ with $f \circ h = x$. Since $\mathcal{F}$ is quasi-coherent, we see that the restriction $x^*\mathcal{F}$ is the pullback of $h^*_{small}(f^*\mathcal{F})|_{U_{\acute{e}tale}}$ by π_V, see Sheaves on Stacks, Lemma 96.14.2. Since $f^*\mathcal{F}|_{U_{\acute{e}tale}}$ locally in the étale topology has a finite presentation by assumption, we conclude (4) $\Rightarrow$ (2).

Part (2) implies (1) for any ringed topos (immediate from the definition). The properties "finite type" and "quasi-coherent" are preserved under pullback by any morphism of ringed topoi, see Modules on Sites, Lemma 18.23.4. Hence (1) implies (3), see Cohomology of Spaces, Lemma 69.12.2. Finally, (3) trivially implies (4). □

0GR6 Definition 103.17.2. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. An $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ is called coherent if $\mathcal{F}$ satisfies one (and hence all) of the equivalent conditions of Lemma 103.17.1. The category of coherent $\mathcal{O}_\mathcal{X}$-modules is denote $\text{Coh}(\mathcal{O}_\mathcal{X})$.

0GR7 Lemma 103.17.3. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. The module $\mathcal{O}_\mathcal{X}$ is coherent, any invertible $\mathcal{O}_\mathcal{X}$-module is coherent, and more generally any finite locally free $\mathcal{O}_\mathcal{X}$-module is coherent.

Proof. Follows from the definition and Cohomology of Spaces, Lemma 69.12.2. □

0GR8 Lemma 103.17.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of locally Noetherian algebraic stacks. Then f^* sends coherent modules on $\mathcal{Y}$ to coherent modules on $\mathcal{X}$.

Proof. Immediate from the definition and the fact that pullback for any morphism of ringed topoi preserves finitely presented modules, see Modules on Sites, Lemma 18.23.4. □

0GR9 Lemma 103.17.5. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. The category of coherent $\mathcal{O}_\mathcal{X}$-modules is abelian. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of coherent $\mathcal{O}_\mathcal{X}$-modules, then

(1) the cokernel $\text{Coker}(\varphi)$ computed in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is a coherent $\mathcal{O}_\mathcal{X}$-module,
(2) the image $\text{Im}(\varphi)$ computed in $\text{Mod}(\mathcal{O}_\mathcal{X})$ is a coherent $\mathcal{O}_\mathcal{X}$-module, and
(3) the kernel $\text{Ker}(\varphi)$ computed in $\text{Mod}(\mathcal{O}_\mathcal{X})$ may not be coherent, but it is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ and $Q(\text{Ker}(\varphi))$ is coherent and is the kernel of φ in $\text{Coh}(\mathcal{O}_\mathcal{X})$.

The inclusion functor $\mathrm{Coh}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$ is exact.

Proof. The rules given for taking kernels, images, and cokernels in $\mathrm{Coh}(\mathcal{O}_\mathcal{X})$ agree with the prescription for quasi-coherent modules in Remark 103.10.5. Hence the lemma will follow if we can show that the quasi-coherent modules $\mathrm{Coker}(\varphi)$, $\mathrm{Im}(\varphi)$, and $Q(\mathrm{Ker}(\varphi))$ are coherent. By Lemma 103.17.1 it suffices to prove this after restricting to $U_{étale}$ for some surjective smooth morphism $f : U \to \mathcal{X}$. The functor $\mathcal{F} \mapsto f^*\mathcal{F}|_{U_{étale}}$ is exact. Hence $f^* \mathrm{Coker}(\varphi)$ and $f^* \mathrm{Im}(\varphi)$ are the cokernel and image of a map between coherent $\mathcal{O}_U$-modules hence coherent as desired. The functor $\mathcal{F} \mapsto f^*\mathcal{F}|_{U_{étale}}$ kills parasitic modules by Lemma 103.9.2. Hence $f^*Q(\mathrm{Ker}(\varphi))|_{U_{étale}} = f^* \mathrm{Ker}(\varphi)|_{U_{étale}}$ by part (2) of Lemma 103.10.2. Thus we conclude that $Q(\mathrm{Ker}(\varphi))$ is coherent in the same way. $\square$

0GRA Lemma 103.17.6. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $\mathrm{Mod}(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ coherent, then $\mathcal{F}_2$ is coherent.

Proof. By Sheaves on Stacks, Lemma 96.15.1 part (7) we see that $\mathcal{F}_2$ is quasi-coherent. Then we can check that $\mathcal{F}_2$ is coherent by restricting to $U_{étale}$ for some $U \to \mathcal{X}$ surjective and smooth. This follows from Cohomology of Spaces, Lemma 69.12.3. Some details omitted. $\square$

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. This does not hold for modules on a general ringed topos.

0GRB Lemma 103.17.7. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then $\mathrm{Coh}(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $QCoh(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. We have

(1) if $\mathcal{F}$ is coherent and φ surjective, then $\mathcal{G}$ is coherent,
(2) if $\mathcal{F}$ is coherent, then $\mathrm{Im}(\varphi)$ is coherent, and
(3) if $\mathcal{G}$ coherent and $\mathrm{Ker}(\varphi)$ parasitic, then $\mathcal{F}$ is coherent.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Then the functor $f^* : QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_U)$ is exact (Lemma 103.4.1) and moreover by definition $\mathrm{Coh}(\mathcal{O}_\mathcal{X})$ is the full subcategory of $QCoh(\mathcal{O}_\mathcal{X})$ consisting of objects $\mathcal{F}$ such that $f^*\mathcal{F}$ is in $\mathrm{Coh}(\mathcal{O}_U)$. The statement that $\mathrm{Coh}(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $QCoh(\mathcal{O}_\mathcal{X})$ follows immediately from this and the corresponding fact for U, see Cohomology of Spaces, Lemmas 69.12.3 and 69.12.4. We omit the proof of (1), (2), and (3). Hint: compare with the proof of Lemma 103.17.5. $\square$

Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be surjective, locally of finite presentation, and flat. Observe that U is locally Noetherian (Morphisms of Stacks, Lemma 101.17.5). Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ the isomorphism constructed in Algebraic Stacks, Lemma 94.16.1 and Remark 94.16.3. As in Sheaves on Stacks, Section 96.15 we obtain equivalences

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Sheaves on Stacks, Proposition 96.14.3. Recall that in Groupoids in Spaces, Section 78.13 we have defined the full subcategory

$$\mathrm{Coh}(U, R, s, t, c) \subset QCoh(U, R, s, t, c)$$

of coherent modules as those $(\mathcal{G}, \alpha)$ such that $\mathcal{G}$ is a coherent $\mathcal{O}_U$-module.

0GRC Lemma 103.17.8. In the situation discussed above, the equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$ sends coherent sheaves to coherent sheaves and vice versa, i.e., induces an equivalence $\text{Coh}(\mathcal{O}_\mathcal{X}) \cong \text{Coh}(U, R, s, t, c)$.

Proof. This is immediate from the definition of coherent $\mathcal{O}_\mathcal{X}$-modules. For bookkeeping purposes: the material above uses Morphisms of Stacks, Lemma 101.17.5, Algebraic Stacks, Lemma 94.16.1 and Remark 94.16.3, Sheaves on Stacks, Section 96.15, Sheaves on Stacks, Proposition 96.14.3, and Groupoids in Spaces, Section 78.13. □

0GRD Lemma 103.17.9. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent be $\mathcal{O}_\mathcal{X}$-modules. Then the internal hom $hom(\mathcal{F}, \mathcal{G})$ constructed in Lemma 103.10.8 is a coherent $\mathcal{O}_\mathcal{X}$-module.

Proof. Let $U \to \mathcal{X}$ be a smooth surjective morphism from a scheme. By item (12) in Section 103.12 we see that the restriction of $hom(\mathcal{F}, \mathcal{G})$ to U is the Hom sheaf of the restrictions. Hence this lemma follows from the case of algebraic spaces, see Cohomology of Spaces, Lemma 69.12.5. □

103.18. Coherent sheaves on Noetherian stacks

0GRE This section is the analogue of Cohomology of Spaces, Section 69.13.

0GRF Lemma 103.18.1. Let $\mathcal{X}$ be a Noetherian algebraic stack. Every quasi-coherent $\mathcal{O}_\mathcal{X}$-module is the filtered colimit of its coherent submodules.

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_\mathcal{X}$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_\mathcal{X}$-submodule which contains both of them, see Lemma 103.17.7. In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to \mathcal{X}$ a surjective smooth morphism (Properties of Stacks, Lemma 100.6.2). Set $R = U \times_\mathcal{X} U$ so that $\mathcal{X} = [U/R]$ as in Algebraic Stacks, Lemma 94.16.2. By Lemma 103.17.8 we have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $\text{Coh}(\mathcal{O}_X) = \text{Coh}(U, R, s, t, c)$. In this way we reduce to the problem of proving the corresponding thing for $QCoh(U, R, s, t, c)$. This is Groupoids in Spaces, Lemma 78.13.4; we check its assumptions in the next paragraph.

We urge the reader to skip the rest of the proof. The affine scheme U is Noetherian; this follows from our definition of $\mathcal{X}$ being locally Noetherian, see Properties of Stacks, Definition 100.7.2 and Remark 100.7.3. The projection morphisms $s, t : R \to U$ are smooth (see reference given above) and quasi-separated and quasi-compact (Morphisms of Stacks, Lemma 101.7.8). In particular, R is a quasi-compact and quasi-separated algebraic space smooth over U and hence Noetherian (Morphisms of Spaces, Lemma 67.28.6). □

103.19. Other chapters

Preliminaries

CHAPTER 104

Derived Categories of Stacks

08MW 104.1. Introduction

08MX In this chapter we write about derived categories associated to algebraic stacks. This means in particular derived categories of quasi-coherent sheaves, i.e., we prove analogues of the results on schemes (see Derived Categories of Schemes, Section 36.1) and algebraic spaces (see Derived Categories of Spaces, Section 75.1). The results in this chapter are different from those in [LMB00] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapters "Sheaves on Algebraic Stacks" and "Cohomology of Algebraic Stacks" where the terminology we use here is introduced.

104.2. Conventions, notation, and abuse of language

08MY We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2. We use notation as explained in Cohomology of Stacks, Section 103.3.

104.3. The lisse-étale and the flat-fppf sites

08MZ The section is the analogue of Cohomology of Stacks, Section 103.14 for derived categories.

07AS Lemma 104.3.1. Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemmas 103.14.2 and 103.14.4.

(1) The functor $g_! : \text{Ab}(\mathcal{X}_{lisse,étale}) \to \text{Ab}(\mathcal{X}_{étale})$ has a left derived functor
$$Lg_! : D(\mathcal{X}_{lisse,étale}) \longrightarrow D(\mathcal{X}_{étale})$$
which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = \text{id}$.

(2) The functor $g_! : \text{Mod}(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \to \text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ has a left derived functor
$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$
which is left adjoint to g^* and such that $g^*Lg_! = \text{id}$.

(3) The functor $g_! : \text{Ab}(\mathcal{X}_{flat,fppf}) \to \text{Ab}(\mathcal{X}_{fppf})$ has a left derived functor
$$Lg_! : D(\mathcal{X}_{flat,fppf}) \longrightarrow D(\mathcal{X}_{fppf})$$
which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = \text{id}$.

(4) The functor $g_! : \text{Mod}(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \to \text{Mod}(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ has a left derived functor
$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow D(\mathcal{O}_\mathcal{X})$$
which is left adjoint to g^* and such that $g^*Lg_! = \text{id}$.

Warning: It is not clear (a priori) that $Lg_!$ on modules agrees with $Lg_!$ on abelian sheaves, see Cohomology on Sites, Remark 21.37.3.

Proof. The existence of the functor $Lg_!$ and adjointness to g^* is Cohomology on Sites, Lemma 21.37.2. (For the case of abelian sheaves use the constant sheaf $\mathbf{Z}$ as the structure sheaves.) Moreover, it is computed on a complex $\mathcal{H}^\bullet$ by taking a suitable left resolution $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ and applying the functor $g_!$ to $\mathcal{K}^\bullet$. Since $g^{-1}g_!\mathcal{K}^\bullet = \mathcal{K}^\bullet$ by Cohomology of Stacks, Lemmas 103.14.4 and 103.14.2 we see that the final assertion holds in each case. $\square$

07AV Lemma 104.3.2. With assumptions and notation as in Cohomology of Stacks, Lemma 103.15.1. We have

$$g^{-1} \circ Rf_* = Rf'_* \circ (g')^{-1} \quad \text{and} \quad L(g')_! \circ (f')^{-1} = f^{-1} \circ Lg_!$$

on unbounded derived categories (both for the case of modules and for the case of abelian sheaves).

Proof. Let $\tau = \textit{étale}$ (resp. $\tau = fppf$). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}_\tau$. By Cohomology of Stacks, Lemma 103.15.3 the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism. The rest of the proof is formal. Since cohomology of abelian groups and sheaves of modules agree we also conclude that $g^{-1}Rf_*\mathcal{F} = Rf'_*(g')^{-1}\mathcal{F}$ when $\mathcal{F}$ is a sheaf of modules on $\mathcal{X}_\tau$.

Next we show that for $\mathcal{G}$ (either sheaf of modules or abelian groups) on $\mathcal{Y}_{lisse,\acute{e}tale}$ (resp. $\mathcal{Y}_{flat,fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G} \to f^{-1}Lg_!\mathcal{G}$$

is an isomorphism. To see this it is enough to prove for any injective sheaf $\mathcal{I}$ on $\mathcal{X}_\tau$ the induced map

$$\mathrm{Hom}(L(g')_!(f')^{-1}\mathcal{G}, \mathcal{I}[n]) \leftarrow \mathrm{Hom}(f^{-1}Lg_!\mathcal{G}, \mathcal{I}[n])$$

is an isomorphism for all $n \in \mathbf{Z}$. (Hom's taken in suitable derived categories.) By the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions this follows from the isomorphism $g^{-1}Rf_*\mathcal{I} \to Rf'_*(g')^{-1}\mathcal{I}$ proved above.

In the case of a bounded complex $\mathcal{G}^\bullet$ (of modules or abelian groups) on $\mathcal{Y}_{lisse,\acute{e}tale}$ (resp. $\mathcal{Y}_{fppf}$) the canonical map

07AX (104.3.2.1) $$L(g')_!(f')^{-1}\mathcal{G}^\bullet \to f^{-1}Lg_!\mathcal{G}^\bullet$$

is an isomorphism as follows from the case of a sheaf by the usual arguments involving truncations and the fact that the functors $L(g')_!(f')^{-1}$ and $f^{-1}Lg_!$ are exact functors of triangulated categories.

Suppose that $\mathcal{G}^\bullet$ is a bounded above complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,\acute{e}tale}$ (resp. $\mathcal{Y}_{fppf}$). The canonical map (104.3.2.1) is an isomorphism because we can use the stupid truncations $\sigma_{\geq -n}$ (see Homology, Section 12.15) to write $\mathcal{G}^\bullet$ as a colimit $\mathcal{G}^\bullet = \mathrm{colim}\, \mathcal{G}_n^\bullet$ of bounded complexes. This gives a distinguished triangle

$$\bigoplus\nolimits_{n \geq 1} \mathcal{G}_n^\bullet \to \bigoplus\nolimits_{n \geq 1} \mathcal{G}_n^\bullet \to \mathcal{G}^\bullet \to \ldots$$

and each of the functors $L(g')_!$, $(f')^{-1}$, f^{-1}, $Lg_!$ commutes with direct sums (of complexes).

If $\mathcal{G}^\bullet$ is an arbitrary complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) then we use the canonical truncations $\tau_{\leq n}$ (see Homology, Section 12.15) to write $\mathcal{G}^\bullet$ as a colimit of bounded above complexes and we repeat the argument of the paragraph above.

Finally, by the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions we conclude that the first identity of the lemma follows from the second in full generality. $\square$

07B3 Lemma 104.3.3. Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemma 103.14.2.

(1) Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.
(2) Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 18.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i \in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 18.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 96.9.2 and the discussion following.) After refining the covering we may assume each U_i is an affine scheme. Since each f_i is smooth (resp. flat) by Lemma 104.3.2 we see that $f_i^{-1}Lg_!\mathcal{H} = Lg_{i,!}(f_i')^{-1}\mathcal{H}$. Using Cohomology of Stacks, Lemma 103.8.2 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation and where $\mathcal{X} = (Sch/X)_{fppf}$ for some affine scheme $X = \mathrm{Spec}(A)$.

Say our presentation looks like

$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Note that the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) has a final object, namely X/X which is quasi-compact (see Cohomology on Sites, Section 21.16). Hence we have

$$\Gamma(\bigoplus\nolimits_{i \in I} \mathcal{O}) = \bigoplus\nolimits_{i \in I} A$$

by Sites, Lemma 7.17.7. Hence the map in the presentation corresponds to a similar presentation

$$\bigoplus\nolimits_{j \in J} A \longrightarrow \bigoplus\nolimits_{i \in I} A \longrightarrow M \longrightarrow 0$$

of an A-module M. Moreover, $\mathcal{H}$ is equal to the restriction to the lisse-étale (resp. flat-fppf) site of the quasi-coherent sheaf M^a associated to M. Choose a resolution

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

by free A-modules. The complex

$$\ldots \mathcal{O} \otimes_A F_2 \to \mathcal{O} \otimes_A F_1 \to \mathcal{O} \otimes_A F_0 \to \mathcal{H} \to 0$$

is a resolution of $\mathcal{H}$ by free $\mathcal{O}$-modules because for each object U/X of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) the structure morphism $U \to X$ is flat. Hence by construction the value of $Lg_!\mathcal{H}$ is

$$\ldots \to \mathcal{O}_\mathcal{X} \otimes_A F_2 \to \mathcal{O}_\mathcal{X} \otimes_A F_1 \to \mathcal{O}_\mathcal{X} \otimes_A F_0 \to 0 \to \ldots$$

Since this is a complex of quasi-coherent modules on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) it follows from Cohomology of Stacks, Proposition 103.8.1 that $H^p(Lg_!\mathcal{H})$ is quasi-coherent. □

104.4. Cohomology and the lisse-étale and flat-fppf sites

0H0Y We have already seen that cohomology of a sheaf on an algebraic stack $\mathcal{X}$ can be computed on flat-fppf site. In this section we prove the same is true for (possibly) unbounded objects of the direct category of $\mathcal{X}$.

0H0Z **Lemma 104.4.1.** Let $\mathcal{X}$ be an algebraic stack. We have $Lg_!\mathbf{Z} = \mathbf{Z}$ for either $Lg_!$ as in Lemma 104.3.1 part (1) or $Lg_!$ as in Lemma 104.3.1 part (3).

Proof. We prove this for the comparison between the flat-fppf site with the fppf site; the case of the lisse-étale site is exactly the same. We have to show that $H^i(Lg_!\mathbf{Z})$ is 0 for $i \neq 0$ and that the canonical map $H^0(Lg_!\mathbf{Z}) \to \mathbf{Z}$ is an isomorphism. Let $f : \mathcal{U} \to \mathcal{X}$ be a surjective, flat morphism where $\mathcal{U}$ is a scheme such that f is also locally of finite presentation. (For example, pick a presentation $U \to \mathcal{X}$ and let $\mathcal{U}$ be the algebraic stack corresponding to U.) By Sheaves on Stacks, Lemmas 96.19.6 and 96.19.10 it suffices to show that the pullback $f^{-1}H^i(Lg_!\mathbf{Z})$ is 0 for $i \neq 0$ and that the pullback $H^0(Lg_!\mathbf{Z}) \to f^{-1}\mathbf{Z}$ is an isomorphism. By Lemma 104.3.2 we find $f^{-1}Lg_!\mathbf{Z} = L(g')_!\mathbf{Z}$ where $g' : Sh(\mathcal{U}_{flat,fppf}) \to Sh(\mathcal{U}_{fppf})$ is the corresponding comparision morphism for $\mathcal{U}$. This reduces us to the case studied in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X. In this case the category $\mathcal{X}_{flat,fppf}$ has a final object e, namely X/X, and moreover the functor $u : \mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ sends e to the final object. Since $\mathbf{Z}$ is the free abelian sheaf on the final object (provided the final object exists) we find that $Lg_!\mathbf{Z} = \mathbf{Z}$ by the very construction of $Lg_!$ in Cohomology on Sites, Lemma 21.37.2. □

0H10 **Lemma 104.4.2.** Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 104.3.1.

(1) For K in $D(\mathcal{X}_{étale})$ we have
   (a) $R\Gamma(\mathcal{X}_{étale}, K) = R\Gamma(\mathcal{X}_{lisse,étale}, g^{-1}K)$, and
   (b) $R\Gamma(x, K) = R\Gamma(\mathcal{X}_{lisse,étale}/x, g^{-1}K)$ for any object x of $\mathcal{X}_{lisse,étale}$.
(2) For K in $D(\mathcal{X}_{fppf})$ we have
   (a) $R\Gamma(\mathcal{X}_{fppf}, K) = R\Gamma(\mathcal{X}_{flat,fppf}, g^{-1}K)$, and
   (b) $H^p(x, K) = R\Gamma(\mathcal{X}_{flat,fppf}/x, g^{-1}K)$ for any object x of $\mathcal{X}_{flat,fppf}$.

In both cases, the same holds for modules, since we have $g^{-1} = g^*$ and there is no difference in computing cohomology by Cohomology on Sites, Lemma 21.20.7.

Proof. We prove this for the comparison between the flat-fppf site with the fppf site; the case of the lisse-étale site is exactly the same. By Lemma 104.4.1 we have

$Lg_!\mathbf{Z} = \mathbf{Z}$. Then we obtain

$$\begin{aligned} R\Gamma(\mathcal{X}_{fppf}, K) &= R\operatorname{Hom}(\mathbf{Z}, K) \\ &= R\operatorname{Hom}(Lg_!\mathbf{Z}, K) \\ &= R\operatorname{Hom}(\mathbf{Z}, g^{-1}K) \\ &= R\Gamma(\mathcal{X}_{lisse,étale}, g^{-1}K) \end{aligned}$$

This proves (1)(a). Part (1)(b) follows from part (1)(a). Namely, if x lies over the scheme U, then the site $\mathcal{X}_{étale}/x$ is equivalent to $(Sch/U)_{étale}$ and $\mathcal{X}_{lisse,étale}$ is equivalent to $U_{lisse,étale}$. □

104.5. Derived categories of quasi-coherent modules

07B5 Let $\mathcal{X}$ be an algebraic stack. As the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to \operatorname{Mod}(\mathcal{O}_\mathcal{X})$ isn't exact, we cannot define $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as the full subcategory of $D(\mathcal{O}_\mathcal{X})$ consisting of complexes with quasi-coherent cohomology sheaves. Instead we define the derived category of quasi-coherent modules as a quotient by analogy with Cohomology of Stacks, Remark 103.10.7.

Recall that $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \subset \operatorname{Mod}(\mathcal{O}_\mathcal{X})$ denotes the full subcategory of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property, see Cohomology of Stacks, Section 103.8. We will abbreviate

$$D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) = D_{\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X})$$

From Derived Categories, Lemma 13.17.1 and Cohomology of Stacks, Proposition 103.8.1 part (2) we deduce that $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$.

Let $\mathrm{Parasitic}(\mathcal{O}_\mathcal{X}) \subset \operatorname{Mod}(\mathcal{O}_\mathcal{X})$ denote the full subcategory of parasitic $\mathcal{O}_\mathcal{X}$-modules, see Cohomology of Stacks, Section 103.9. Let us abbreviate

$$D_{\mathrm{Parasitic}}(\mathcal{O}_\mathcal{X}) = D_{\mathrm{Parasitic}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X})$$

As before this is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$ since $\mathrm{Parasitic}(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $\operatorname{Mod}(\mathcal{O}_\mathcal{X})$, see Cohomology of Stacks, Lemma 103.9.2.

The intersection of the weak Serre subcategories $\mathrm{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ of $\operatorname{Mod}(\mathcal{O}_\mathcal{X})$ is another one. Let us similarly abbreviate

$$\begin{aligned} D_{\mathrm{Parasitic}\cap\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) &= D_{\mathrm{Parasitic}(\mathcal{O}_\mathcal{X})\cap\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X}) \\ &= D_{\mathrm{Parasitic}}(\mathcal{O}_\mathcal{X}) \cap D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \end{aligned}$$

As before this is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$. Hence a fortiori it is a strictly full, saturated triangulated subcategory of $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$.

07B6 **Definition 104.5.1.** Let $\mathcal{X}$ be an algebraic stack. With notation as above we define the derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves as the Verdier quotient[1]

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) = D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) / D_{\mathrm{Parasitic}\cap\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$$

[1]This definition is different from the one in the literature, see [Ols07b, 6.3], but it agrees with that definition by Lemma 104.5.3.

The Verdier quotient is defined in Derived Categories, Section 13.6. A morphism $a : E \to E'$ of $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ becomes an isomorphism in $D_{QCoh}(\mathcal{O}_\mathcal{X})$ if and only if the cone $C(a)$ has parasitic cohomology sheaves, see Derived Categories, Lemma 13.6.10.

Consider the functors

$$D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \xrightarrow{H^i} \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{X})$$

Note that Q annihilates the subcategory $\mathrm{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, see Cohomology of Stacks, Lemma 103.10.2. By Derived Categories, Lemma 13.6.8 we obtain a cohomological functor

07B7 (104.5.1.1)
$$H^i : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

Moreover, note that $E \in D_{QCoh}(\mathcal{O}_\mathcal{X})$ is zero if and only if $H^i(E) = 0$ for all $i \in \mathbf{Z}$ since the kernel of Q is exactly equal to $\mathrm{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Lemma 103.10.2.

Note that the categories $\mathrm{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ and $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ are also weak Serre subcategories of the abelian category $\mathrm{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ of modules in the étale topology, see Cohomology of Stacks, Proposition 103.8.1 and Lemma 103.9.2. Hence the statement of the following lemma makes sense.

07B8 **Lemma 104.5.2.** Let $\mathcal{X}$ be an algebraic stack. Abbreviate $\mathcal{P}_\mathcal{X} = \mathrm{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$. The comparison morphism $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ induces a commutative diagram

$$\begin{tikzcd}
D_{\mathrm{Parasitic} \cap \mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D(\mathcal{O}_\mathcal{X}) \\
D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[r] \arrow[u, "\epsilon^*"] & D_{\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[r] \arrow[u, "\epsilon^*"] & D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[u, "\epsilon^*"]
\end{tikzcd}$$

Moreover, the left two vertical arrows are equivalences of triangulated categories, hence we also obtain an equivalence

$$D_{\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) / D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

Proof. Since ϵ^* is exact it is clear that we obtain a diagram as in the statement of the lemma. We will show the middle vertical arrow is an equivalence by applying Cohomology on Sites, Lemma 21.29.1 to the following situation: $\mathcal{C} = \mathcal{X}$, $\tau = fppf$, $\tau' = étale$, $\mathcal{O} = \mathcal{O}_\mathcal{X}$, $\mathcal{A} = \mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$, and $\mathcal{B}$ is the set of objects of $\mathcal{X}$ lying over affine schemes. To see the lemma applies we have to check conditions (1), (2), (3), (4). Conditions (1) and (2) are clear from the discussion above (explicitly this follows from Cohomology of Stacks, Proposition 103.8.1). Condition (3) holds because every scheme has a Zariski open covering by affines. Condition (4) follows from Descent, Lemma 35.12.4.

We omit the verification that the equivalence of categories $\epsilon^* : D_{\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ induces an equivalence of the subcategories of complexes with parasitic cohomology sheaves. $\square$

Let $\mathcal{X}$ be an algebraic stack. By Cohomology of Stacks, Lemma 103.16.4 the category of quasi-coherent modules $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ forms a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and the category of quasi-coherent modules $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ forms a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Thus we can consider

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) = D_{QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})}(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \subset D(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

and similarly

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) = D_{QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \subset D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

As above these are strictly full, saturated triangulated subcategories. It turns out that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ is equivalent to either of these.

07B9 Lemma 104.5.3. Let $\mathcal{X}$ be an algebraic stack. Set $\mathcal{P}_\mathcal{X} = \text{Parasitic}(\mathcal{O}_\mathcal{X}) \cap \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$.

(1) Let $\mathcal{F}^\bullet$ be an object of $D_{\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. With g as in Cohomology of Stacks, Lemma 103.14.2 for the lisse-étale site we have
  (a) $g^*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$,
  (b) $g^*\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$,
  (c) $Lg_!\mathcal{H}^\bullet$ is in $D_{\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$, and
  (d) the functors g^* and $Lg_!$ define mutually inverse functors

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_\mathcal{X}) \arrow[r, shift left, "g^*"] & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \arrow[l, shift left, "Lg_!"] \end{tikzcd}$$

(2) Let $\mathcal{F}^\bullet$ be an object of $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. With g as in Cohomology of Stacks, Lemma 103.14.2 for the flat-fppf site we have
  (a) $g^*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$,
  (b) $g^*\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,
  (c) $Lg_!\mathcal{H}^\bullet$ is in $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$, and
  (d) the functors g^* and $Lg_!$ define mutually inverse functors

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_\mathcal{X}) \arrow[r, shift left, "g^*"] & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \arrow[l, shift left, "Lg_!"] \end{tikzcd}$$

Proof. The functor $g^* = g^{-1}$ is exact, hence (1)(a), (2)(a), (1)(b), and (2)(b) follow from Cohomology of Stacks, Lemmas 103.16.3 and 103.14.6.

Proof of (1)(c) and (2)(c). The construction of $Lg_!$ in Lemma 104.3.1 (via Cohomology on Sites, Lemma 21.37.2 which in turn uses Derived Categories, Proposition 13.29.2) shows that $Lg_!$ on any object $\mathcal{H}^\bullet$ of $D(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is computed as

$$Lg_!\mathcal{H}^\bullet = \text{colim}\, g_!\mathcal{K}_n^\bullet = g_!\, \text{colim}\, \mathcal{K}_n^\bullet$$

(termwise colimits) where the quasi-isomorphism $\text{colim}\, \mathcal{K}_n^\bullet \to \mathcal{H}^\bullet$ induces quasi-isomorphisms $\mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{H}^\bullet$. Since the inclusion functors

$$\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \subset \text{Mod}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \quad \text{and} \quad \text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X}) \subset \text{Mod}(\mathcal{O}_\mathcal{X})$$

are compatible with filtered colimits we see that it suffices to prove (c) on bounded above complexes $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. In this case to show that $H^n(Lg_!\mathcal{H}^\bullet)$ is in $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ we can argue by induction on the integer m such that $\mathcal{H}^i = 0$ for $i > m$. If $m < n$, then $H^n(Lg_!\mathcal{H}^\bullet) = 0$ and the result holds. In general consider the distinguished triangle

$$\tau_{\leq m-1}\mathcal{H}^\bullet \to \mathcal{H}^\bullet \to H^m(\mathcal{H}^\bullet)[-m] \to \ldots$$

(Derived Categories, Remark 13.12.4) and apply the functor $Lg_!$. Since $\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ is a weak Serre subcategory of the module category it suffices to prove (c) for two out of three. We have the result for $Lg_!\tau_{\leq m-1}\mathcal{H}^\bullet$ by induction and we have the result for $Lg_!H^m(\mathcal{H}^\bullet)[-m]$ by Lemma 104.3.3. Whence (c) holds.

Let us prove (2)(d). By (2)(a) and (2)(b) the functor $g^{-1} = g^*$ induces a functor

$$c : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

see Derived Categories, Lemma 13.6.8. Thus we have the following diagram of triangulated categories

$$\begin{tikzcd} D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[rr, "q"] \arrow[dr, shift left, "g^{-1}"] & & D_{QCoh}(\mathcal{O}_\mathcal{X}) \arrow[dl, "c"] \\ & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \arrow[ul, shift left, "Lg_!"] & \end{tikzcd}$$

where q is the quotient functor, the inner triangle is commutative, and $g^{-1}Lg_! = \text{id}$. For any object of E of $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ the map $a : Lg_!g^{-1}E \to E$ maps to a quasi-isomorphism in $D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Hence the cone on a maps to zero under g^{-1} and by (2)(b) we see that $q(a)$ is an isomorphism. Thus $q \circ Lg_!$ is a quasi-inverse to c.

In the case of the lisse-étale site exactly the same argument as above proves that

$$D_{\text{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$$

is equivalent to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$. Applying the last equivalence of Lemma 104.5.2 finishes the proof. $\square$

The following lemma tells us that the quotient functor $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{X})$ has a left adjoint. See Remark 104.5.5.

07BA **Lemma 104.5.4.** Let $\mathcal{X}$ be an algebraic stack. Let E be an object of $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. There exists a canonical distinguished triangle

$$E' \to E \to P \to E'[1]$$

in $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ such that P is in $D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ and

$$\text{Hom}_{D(\mathcal{O}_\mathcal{X})}(E', P') = 0$$

for all P' in $D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$.

Proof. Consider the morphism of ringed topoi $g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$ studied in Cohomology of Stacks, Section 103.14. Set $E' = Lg_!g^*E$ and let P be the cone on the adjunction map $E' \to E$, see Lemma 104.3.1 part (4). By Lemma 104.5.3 parts (2)(a) and (2)(c) we have that E' is in $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. Hence also P is in $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. The map $g^*E' \to g^*E$ is an isomorphism as $g^*Lg_! = \text{id}$ by Lemma 104.3.1 part (4). Hence $g^*P = 0$ and whence P is an object of $D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ by Lemma 104.5.3 part (2)(b). Finally, for P' in $D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ we have

$$\text{Hom}(E', P') = \text{Hom}(Lg_!g^*E, P') = \text{Hom}(g^*E, g^*P') = 0$$

as $g^*P' = 0$ by Lemma 104.5.3 part (2)(b). The distinguished triangle $E' \to E \to P \to E'[1]$ is canonical (more precisely unique up to isomorphism of triangles induces the identity on E) by the discussion in Derived Categories, Section 13.40. $\square$

0H11 Remark 104.5.5. The result of Lemma 104.5.4 tells us that

$$D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \subset D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$$

is a left admissible subcategory, see Derived Categories, Section 13.40. In particular, if $\mathcal{A} \subset D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ denotes its left orthogonal, then Derived Categories, Proposition 13.40.10 implies that $\mathcal{A}$ is right admissible in $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ and that the composition

$$\mathcal{A} \longrightarrow D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is an equivalence. This means that we can view $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as a strictly full saturated triangulated subcategory of $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ and also of $D(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

104.6. Derived pushforward of quasi-coherent modules

07BB As a first application of the material above we construct the derived pushforward. In Examples, Section 110.60 the reader can find an example of a quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks such that the direct image functor Rf_* does not induce a functor $D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$. Thus restricting to bounded below complexes is necessary.

07BC Proposition 104.6.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor Rf_* induces a commutative diagram

$$\begin{array}{ccccc} D^+_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D^+_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\ D^+_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D^+_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D(\mathcal{O}_\mathcal{Y}) \end{array}$$

and hence induces a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_\mathcal{Y})$$

on quotient categories. Moreover, the functor R^if_{QCoh} of Cohomology of Stacks, Proposition 103.11.1 are equal to $H^i \circ Rf_{QCoh,*}$ with H^i as in (104.5.1.1).

Proof. We have to show that Rf_*E is an object of $D^+_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y})$ for E in $D^+_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. This follows from Cohomology of Stacks, Proposition 103.8.1 and the spectral sequence $R^if_*H^j(E) \Rightarrow R^{i+j}f_*E$. The case of parasitic modules works the same way using Cohomology of Stacks, Lemma 103.9.3. The final statement is clear from the definition of H^i in (104.5.1.1). □

104.7. Derived pullback of quasi-coherent modules

07BD Derived pullback of complexes with quasi-coherent cohomology sheaves exists in general.

07BE Proposition 104.7.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The exact functor f^* induces a commutative diagram

$$\begin{array}{ccc} D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D(\mathcal{O}_\mathcal{Y}) \end{array}$$

The composition

$$D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y}) \xrightarrow{f^*} D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X}) \xrightarrow{q_\mathcal{X}} D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is left derivable with respect to the localization $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and we may define Lf^*_{QCoh} as its left derived functor

$$Lf^*_{QCoh} : D_{QCoh}(\mathcal{O}_\mathcal{Y}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

(see Derived Categories, Definitions 13.14.2 and 13.14.9). If f is quasi-compact and quasi-separated, then Lf^*_{QCoh} and $Rf_{QCoh,*}$ satisfy the following adjointness:

$$\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) = \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)$$

for $A \in D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and $B \in D^+_{QCoh}(\mathcal{O}_\mathcal{X})$.

Proof. To prove the first statement, we have to show that f^*E is an object of $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ for E in $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y})$. Since $f^* = f^{-1}$ is exact this follows immediately from the fact that f^* maps $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{Y})$ into $\mathrm{LQCoh}^{fbc}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Proposition 103.8.1.

Set $\mathcal{D} = D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y})$. Let S be the collection of morphisms in $\mathcal{D}$ whose cone is an object of $D_{\mathrm{Parasitic} \cap \mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y})$. Set $\mathcal{D}' = D_{QCoh}(\mathcal{O}_\mathcal{X})$. Set $F = q_\mathcal{X} \circ f^* : \mathcal{D} \to \mathcal{D}'$. Then $\mathcal{D}, S, \mathcal{D}', F$ are as in Derived Categories, Situation 13.14.1 and Definition 13.14.2. Let us prove that $LF(E)$ is defined for any object E of $\mathcal{D}$. Namely, consider the triangle

$$E' \to E \to P \to E'[1]$$

constructed in Lemma 104.5.4. Note that $s : E' \to E$ is an element of S. We claim that E' computes LF. Namely, suppose that $s' : E'' \to E$ is another element of S, i.e., fits into a triangle $E'' \to E \to P' \to E''[1]$ with P' in $D_{\mathrm{Parasitic} \cap \mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y})$. By Lemma 104.5.4 (and its proof) we see that $E' \to E$ factors through $E'' \to E$. Thus we see that $E' \to E$ is cofinal in the system S/E. Hence it is clear that E' computes LF.

To see the final statement, write $B = q_\mathcal{X}(H)$ and $A = q_\mathcal{Y}(E)$. Choose $E' \to E$ as above. We will use on the one hand that $Rf_{QCoh,*}(B) = q_\mathcal{Y}(Rf_*H)$ and on the other that $Lf^*_{QCoh}(A) = q_\mathcal{X}(f^*E')$.

$$\begin{aligned} \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(q_\mathcal{X}(f^*E'), q_\mathcal{X}(H)) \\ &= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(f^*E', H') \\ &= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H') \\ &= \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H) \\ &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B) \end{aligned}$$

Here the colimit is over morphisms $s : H \to H'$ in $D^+_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ whose cone $P(s)$ is an object of $D^+_{\mathrm{Parasitic} \cap \mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. The first equality we've seen above. The second equality holds by construction of the Verdier quotient. The third equality holds by Cohomology on Sites, Lemma 21.19.1. Since $Rf_*P(s)$ is an object of $D^+_{\mathrm{Parasitic} \cap \mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{Y})$ by Proposition 104.6.1 we see that $\mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*P(s)) = 0$. Thus the fourth equality holds. The final equality holds by construction of E'. $\square$

104.8. Quasi-coherent objects in the derived category

0H12 This section is the continuation of Sheaves on Stacks, Section 96.26. Let $\mathcal{X}$ be an algebraic stack. In that section we defined a triangulated category

$$QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O})$$

and we proved that if $\mathcal{X}$ is representable by an algebraic space X then $QC(\mathcal{X})$ is equivalent to $D_{QCoh}(\mathcal{O}_X)$. It turns out that we have developed just enough theory to prove the same thing is true for any algebraic stack.

0H13 **Lemma 104.8.1.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{X}_{fppf})$ whose cohomology sheaves are parasitic. Then $R\Gamma(x, K) = 0$ for all objects x of $\mathcal{X}$ lying over a scheme U such that $U \to \mathcal{X}$ is flat.*

Proof. Denote $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ the morphism of topoi discussed in Section 104.3. Let x be an object of $\mathcal{X}$ lying over a scheme U such that $U \to \mathcal{X}$ is flat, i.e., x is an object of $\mathcal{X}_{flat,fppf}$. By Lemma 104.4.2 part (2)(b) we have $R\Gamma(x, K) = R\Gamma(\mathcal{X}_{flat,fppf}/x, g^{-1}K)$. However, our assumption means that the cohomology sheaves of the object $g^{-1}K$ of $D(\mathcal{X}_{flat,fppf})$ are zero, see Cohomology of Stacks, Definition 103.9.1. Hence $g^{-1}K = 0$ and we win. $\square$

0H14 **Lemma 104.8.2.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{X}_{fppf})$ such that $R\Gamma(x, K) = 0$ for all objects x of $\mathcal{X}$ lying over an affine scheme U such that $U \to \mathcal{X}$ is flat. Then $H^i(\mathcal{X}, K) = 0$ for all i.*

Proof. Denote $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ the morphism of topoi discussed in Section 104.3. By Lemma 104.4.2 part (2)(b) our assumption means that $g^{-1}K$ has vanishing cohomology over every object of $\mathcal{X}_{flat,fppf}$ which lies over an affine scheme. Since every object x of $\mathcal{X}_{flat,fppf}$ has a covering by such objects, we conclude that $g^{-1}K$ has vanishing cohomology sheaves, i.e., we conclude $g^{-1}K = 0$. Then of course $R\Gamma(\mathcal{X}_{flat,fppf}, g^{-1}K) = 0$ which in turn implies what we want by Lemma 104.4.2 part (2)(a). $\square$

0H15 **Lemma 104.8.3.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Then $Lg_!K$ satisfies the following property: for any morphism $x \to x'$ of $\mathcal{X}_{affine}$ the map*

$$R\Gamma(x', Lg_!K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, Lg_!K)$$

is a quasi-isomorphism.

Proof. By Lemma 104.5.3 part (2)(c) the object $Lg_!K$ is in $D_{\mathrm{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. It follows readily from this that the map displayed in the lemma is an isomorphism if $\mathcal{O}(x') \to \mathcal{O}(x)$ is a flat ring map; we omit the details.

In this paragraph we argue that the question is local for the étale topology. Let $x \to x'$ be a general morphism of $\mathcal{X}_{affine}$. Let $\{x'_i \to x'\}$ be a covering in $\mathcal{X}_{affine,étale}$. Set $x_i = x \times_{x'} x'_i$ so that $\{x_i \to x\}$ is a covering of $\mathcal{X}_{affine,étale}$ too. Then $\mathcal{O}(x') \to \prod \mathcal{O}(x'_i)$ is a faithfully flat étale ring map and

$$\prod \mathcal{O}(x_i) = \mathcal{O}(x) \otimes_{\mathcal{O}(x')} \left(\prod \mathcal{O}(x'_i)\right)$$

Thus a simple algebra argument we omit shows that it suffices to prove the result in the statement of the lemma holds for each of the morphisms $x_i \to x'_i$ in $\mathcal{X}_{affine}$. In other words, the problem is local in the étale topology.

Choose a scheme X and a surjective smooth morphism $f : X \to \mathcal{X}$. We may view f as an object of $\mathcal{X}$ (by our abuse of notation) and then $(Sch/X)_{fppf} = \mathcal{X}/f$, see Sheaves on Stacks, Section 96.9. By Sheaves on Stacks, Lemma 96.19.10 for example, there exist an étale covering $\{x'_i \to x'\}$ such that $x'_i : U'_i = p(x'_i) \to \mathcal{X}$ factors through f. By the result of the previous paragraph, we may assume that $x \to x'$ is a morphism which is the image of a morphism $U \to U'$ of $(\text{Aff}/X)_{fppf}$ by the functor $(Sch/X)_{fppf} \to \mathcal{X}$. At this point we see use that the restriction to $(Sch/X)_{fppf}$ of $Lg_!K$ is equal to $f^*Lg_!K = L(g')_!(f')^*K$ by Lemma 104.3.2. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ and $x \to x'$ corresponds to the morphism of affine schemes $U \to U'$. We may still work étale (or Zariski) locally on U' and hence we may assume $U' \to X$ factors through some affine open of X. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ where $X = \text{Spec}(R)$ is an affine scheme and $x \to x'$ corresponds to the morphism of affine schemes $U \to U'$. Let $M^\bullet$ be a complex of R-modules representing $R\Gamma(X, K)$. By the construction in More on Algebra, Lemma 15.59.10 we may assume $M^\bullet = \text{colim}\, P_n^\bullet$ where each $P_n^\bullet$ is a bounded above complex of free R-modules. Details omitted; see also More on Algebra, Remark 15.59.11. Consider the complex of modules $M^\bullet_{flat,fppf}$ on $X_{flat,fppf} = (Sch/X)_{flat,fppf}$ given by the rule

$$U \longmapsto \Gamma(U, M^\bullet \otimes_R \mathcal{O}_U)$$

This is a complex of sheaves by the discussion in Descent, Section 35.8. There is a canonical map $M^\bullet_{flat,fppf} \to K$ which by our initial remarks of the proof produces an isomorphism on sections over the affine objects of $X_{flat,fppf}$. Since every object of $X_{flat,fppf}$ has a covering by affine objects we see that $M^\bullet_{flat,fppf}$ agrees with K.

Let $M^\bullet_{fppf}$ be the complex of modules on X_{fppf} given by the same formula as displayed above. Recall that $Lg_!\mathcal{O} = g_!\mathcal{O} = \mathcal{O}$. Since $Lg_!$ is the left derived functor of $g_!$ we conclude that $Lg_!P^\bullet_{n,flat,fppf} = P^\bullet_{n,fppf}$. Since the functor $Lg_!$ commutes with homotopy colimits (or by its construction in Cohomology on Sites, Lemma 21.37.2) and since $M^\bullet = \text{colim}\, P_n^\bullet$ we conclude that $Lg_!M^\bullet_{flat,fppf} = M^\bullet_{fppf}$. Say $U = \text{Spec}(A)$, $U' = \text{Spec}(A')$ and $U \to U'$ corresponds to the ring map $A' \to A$. From the above we see that

$$R\Gamma(U, Lg_!K) = M^\bullet \otimes_R A \quad \text{and} \quad R\Gamma(U', Lg_!K) = M^\bullet \otimes_R A'$$

Since $M^\bullet$ is a K-flat complex of R-modules, by transitivity of tensor product it follows that

$$R\Gamma(U', Lg_!K) \otimes_{A'}^{\mathbf{L}} A \longrightarrow R\Gamma(U, Lg_!K)$$

is a quasi-isomorphism as desired. □

0H16 Proposition 104.8.4. Let $\mathcal{X}$ be an algebraic stack. Then $QC(\mathcal{X})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_\mathcal{X})$.

Proof. By Sheaves on Stacks, Lemma 96.26.6 pullback by the comparison morphism $\epsilon : \mathcal{X}_{affine,fppf} \to \mathcal{X}_{affine}$ identifies $QC(\mathcal{X})$ with a full subcategory $Q_\mathcal{X} \subset D(\mathcal{X}_{affine,fppf}, \mathcal{O})$. Using the equivalence of ringed topoi in Sheaves on Stacks, Equation (96.24.3.1) we may and do view $Q_\mathcal{X}$ as a full subcategory of $D(\mathcal{X}_{fppf}, \mathcal{O})$.

Similarly by Lemma 104.5.4 and Remark 104.5.5 we find that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ may be viewed as the left orthogonal $\mathcal{A}$ of the left admissible subcategory $D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$ of $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$.

To finish we will show that $Q_\mathcal{X}$ is equal to $\mathcal{A}$ as subcategories of $D(\mathcal{X}_{fppf}, \mathcal{O})$.

Step 1: $Q_\mathcal{X}$ is contained in $D_{\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. An object K of $Q_\mathcal{X}$ is characterized by the property that K, viewed as an object of $D(\mathcal{X}_{affine,fppf}, \mathcal{O})$ satisfies $R\epsilon_*K$ is an object of $QC(\mathcal{X}_{affine}, \mathcal{O})$. This in turn means exactly that for all morphisms $x \to x'$ of $\mathcal{X}_{affine}$ the map

$$R\Gamma(x', K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, K)$$

is an isomorphism, see footnote in statement of Cohomology on Sites, Lemma 21.43.12. Now, if $x' \to x$ lies over a flat morphism of affine schemes, then this means that

$$H^i(x', K) \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \cong H^i(x, K)$$

This clearly means that $H^i(K)$ is a sheaf for the étale topology (Sheaves on Stacks, Lemma 96.25.1) and that it has the flat base change property (small detail omitted).

Step 2: $Q_\mathcal{X}$ is contained in $\mathcal{A}$. To see this it suffices to show that for K in $Q_\mathcal{X}$ we have $\text{Hom}(K, P) = 0$ for all P in $D_{\text{Parasitic}\cap\text{LQCoh}^{fbc}}(\mathcal{O}_\mathcal{X})$. Consider the object

$$H = R\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(K, P)$$

Let x be an object of $\mathcal{X}$ which lies over an affine scheme $U = p(x)$. By Cohomology on Sites, Lemma 21.35.1 we have the first equality in

$$R\Gamma(x, H) = R\,\text{Hom}_{\mathcal{O}_\mathcal{X}}(K|_{\mathcal{X}/x}, P|_{\mathcal{X}/x}) = R\,\text{Hom}_{\mathcal{O}}(K|_{\mathcal{X}_{affine}/x}, P|_{\mathcal{X}_{affine}/x})$$

The second equality stems from the fact that the topos of the site $\mathcal{X}/x$ is equivalent to the topos of the site $\mathcal{X}_{affine}/x$, see Sheaves on Stacks, Equation (96.24.3.1). We may write $K = \epsilon^*N$ for some N in $QC(\mathcal{O})$. Then by Cohomology on Sites, Lemma 21.43.13 we see that

$$R\Gamma(x, H) = R\,\text{Hom}_{D(\mathcal{O}(x))}(R\Gamma(x, N), R\Gamma(x, P))$$

By Lemma 104.8.1 we see that $R\Gamma(x, P) = 0$ if $U \to \mathcal{X}$ is flat and hence $R\Gamma(x, H) = 0$ under the same hypothesis. By Lemma 104.8.2 we conclude that $R\Gamma(\mathcal{X}, H) = 0$ and therefore $\text{Hom}(K, P) = 0$.

Step 3: $\mathcal{A}$ is contained in $Q_\mathcal{X}$. Let K be an object of $\mathcal{A}$ and let $x \to x'$ be a morphism of $\mathcal{X}_{affine}$. We have to show that

$$R\Gamma(x', K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, K)$$

is a quasi-isomorphism, see footnote in statement of Cohomology on Sites, Lemma 21.43.12. By the proof of Lemma 104.5.4 and the discussion in Remark 104.5.5 we see that $\mathcal{A}$ is the image of the restriction of $Lg_!$ to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Thus we may assume $K = Lg_!M$ for some M in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Then the desired equality follow from Lemma 104.8.3. ☐

104.9. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology

CHAPTER 105

Introducing Algebraic Stacks

072H 105.1. Why read this?

072I We give an informal introduction to algebraic stacks. The goal is to quickly introduce a simple language which you can use to think about local and global properties of your favorite moduli problem. Having done this it should be possible to ask yourself well-posed questions about moduli problems and to start solving them, whilst assuming a general theory exists. If you end up with an interesting result, you can go back to the general theory in the other parts of the stacks project and fill in the gaps as needed.

The point of view we take here is close to the point of view taken in [KM85] and [Mum65].

105.2. Preliminary

072J Let S be a scheme. An elliptic curve over S is a triple $(E, f, 0)$ where E is a scheme and $f : E \to S$ and $0 : S \to E$ are morphisms of schemes such that

1. $f : E \to S$ is proper, smooth of relative dimension 1,
2. for every $s \in S$ the fibre E_s is a connected curve of genus 1, i.e., $H^0(E_s, \mathcal{O})$ and $H^1(E_s, \mathcal{O})$ both are 1-dimensional $\kappa(s)$-vector spaces, and
3. 0 is a section of f.

Given elliptic curves $(E, f, 0)/S$ and $(E', f', 0')/S'$ a morphism of elliptic curves over $a : S \to S'$ is a morphism $\alpha : E \to E'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d, "f"] & E' \arrow[d, "f'"'] \\ S \arrow[r, "a"] \arrow[u, bend left, "0"] & S' \arrow[u, bend right, "0'"'] \end{tikzcd}$$

is commutative and the inner square is cartesian, in other words the morphism α induces an isomorphism $E \to S \times_{S'} E'$. We are going to define the stack of elliptic curves $\mathcal{M}_{1,1}$. In the rest of the Stacks project we work out the method introduced in Deligne and Mumford's paper [DM69] which consists in presenting $\mathcal{M}_{1,1}$ as a category endowed with a functor

$$p : \mathcal{M}_{1,1} \longrightarrow Sch, \quad (E, f, 0)/S \longmapsto S$$

This means you work with fibred categories over the categories of schemes, topologies, stacks fibred in groupoids, coverings, etc, etc. In this chapter we throw all of that out of the window and we think about it a bit differently – probably closer to how the initiators of the theory started thinking about it themselves.

105.3. The moduli stack of elliptic curves

072K Here is what we are going to do:

(1) Start with your favorite category of schemes *Sch*.
(2) Add a new symbol $\mathcal{M}_{1,1}$.
(3) A morphism $S \to \mathcal{M}_{1,1}$ is an elliptic curve $(E, f, 0)$ over S.
(4) A diagram

$$\begin{tikzcd} S \arrow[rr, "a"'] \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',F',0')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

is commutative if and only if there exists a morphism $\alpha : E \to E'$ of elliptic curves over $a : S \to S'$. We say α witnesses the commutativity of the diagram.

(5) Note that commutative diagrams glue as follows

$$\begin{tikzcd} S \arrow[r, "a"'] \arrow[dr, "{(E,f,0)}"'] & S' \arrow[r, "a'"'] \arrow[d, "{(E',F',0')}"'] & S'' \arrow[dl, "{(E'',F'',0'')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

because $\alpha' \circ \alpha$ witnesses the commutativity of the outer triangle if α and α' witness the commutativity of the left and right triangles.

(6) The composition

$$S \xrightarrow{a} S' \xrightarrow{(E',f',0')} \mathcal{M}_{1,1}$$

is given by $(E' \times_{S'} S, f' \times_{S'} S, 0' \times_{S'} S)$.

At the end of this procedure we have enlarged the category *Sch* of schemes with exactly one object...

Except that we haven't defined what a morphism from $\mathcal{M}_{1,1}$ to a scheme T is. The answer is that it is the weakest possible notion such that compositions make sense. Thus a morphism $F : \mathcal{M}_{1,1} \to T$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates a morphism $F(E, f, 0) : S \to T$ such that given any commutative diagram

$$\begin{tikzcd} S \arrow[rr, "a"'] \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',F',0')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

the diagram

$$\begin{tikzcd} S \arrow[rr, "a"'] \arrow[dr, "{F(E,f,0)}"'] & & S' \arrow[dl, "{F(E',F',0')}"] \\ & T & \end{tikzcd}$$

is commutative also. An example is the j-invariant

$$j : \mathcal{M}_{1,1} \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$$

which you may have heard of. Aha, so now we're done...

Except, no we're not! We still have to define a notion of morphisms $\mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$. This we do in exactly the same way as before, i.e., a morphism $F : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates another elliptic curve $F(E, f, 0)$ preserving commutativity of diagrams as above. However, since I don't know of a nontrivial example of such a functor, I'll just define the set of morphisms from $\mathcal{M}_{1,1}$ to itself to consist of the identity for now.

I hope you see how to add other objects to this enlarged category. Somehow it seems intuitively clear that given any "well-behaved" moduli problem we can perform the construction above and add an object to our category. In fact, much of modern day algebraic geometry takes place in such a universe where *Sch* is enlarged with countably many (explicitly constructed) moduli stacks.

You may object that the category we obtain isn't a category because there is a "vagueness" about when diagrams commute and which combinations of diagrams continue to commute as we have to produce a witness to the commutativity. However, it turns out that this, the idea of having witnesses to commutativity, is a valid approach to 2-categories! Thus we stick with it.

105.4. Fibre products

072L The question we pose here is what should be the fibre product

$$\begin{tikzcd} & ? \arrow[dl, dotted] \arrow[dr, dotted] & \\ S \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',f',0')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

The answer: A morphism from a scheme T into ? should be a triple (a, a', α) where $a : T \to S$, $a' : T \to S'$ are morphisms of schemes and where $\alpha : E \times_{S,a} T \to E' \times_{S',a'} T$ is an isomorphism of elliptic curves over T. This makes sense because of our definition of composition and commutative diagrams earlier in the discussion.

072M Lemma 105.4.1 (Key fact). The functor $Sch^{opp} \to \text{Sets}$, $T \mapsto \{(a, a', \alpha) \text{ as above}\}$ is representable by a scheme $S \times_{\mathcal{M}_{1,1}} S'$.

Proof. Idea of proof. Relate this functor to

$$Isom_{S\times S'}(E \times S', S \times E')$$

and use Grothendieck's theory of Hilbert schemes. □

072N Remark 105.4.2. We have the formula $S \times_{\mathcal{M}_{1,1}} S' = (S \times S') \times_{\mathcal{M}_{1,1}\times\mathcal{M}_{1,1}} \mathcal{M}_{1,1}$. Hence the key fact is a property of the diagonal $\Delta_{\mathcal{M}_{1,1}}$ of $\mathcal{M}_{1,1}$.

In any case the key fact allows us to make the following definition.

072P Definition 105.4.3. We say a morphism $S \to \mathcal{M}_{1,1}$ is smooth if for every morphism $S' \to \mathcal{M}_{1,1}$ the projection morphism

$$S \times_{\mathcal{M}_{1,1}} S' \longrightarrow S'$$

is smooth.

Note that this is compatible with the notion of a smooth morphism of schemes as the base change of a smooth morphism is smooth. Moreover, it is clear how to extend this definition to other properties of morphisms into $\mathcal{M}_{1,1}$ (or your own favorite moduli stack). In particular we will use it below for surjective morphisms.

105.5. The definition

072Q We'll formulate it as a definition and not as a result since we expect the reader to try out other cases (not just the stack $\mathcal{M}_{1,1}$ and not just *Sch* the category of all schemes).

072R Definition 105.5.1. We say $\mathcal{M}_{1,1}$ is an algebraic stack if and only if

(1) We have descent for objects for the étale topology on *Sch*.
(2) The key fact holds.
(3) there exists a surjective and smooth morphism $S \to \mathcal{M}_{1,1}$.

The first condition is a "sheaf property". We're going to spell it out since there is a technical point we should make. Suppose given a scheme S and an étale covering $\{S_i \to S\}$ and morphisms $e_i : S_i \to \mathcal{M}_{1,1}$ such that the diagrams

$$\begin{tikzcd} S_i \times_S S_j \arrow[rr, "\mathrm{id}"'] \arrow[dr, "e_i \circ \mathrm{pr}_1"'] & & S_i \times_S S_j \arrow[dl, "e_j \circ \mathrm{pr}_2"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

commute. The sheaf condition does not guarantee the existence of a morphism $e : S \to \mathcal{M}_{1,1}$ in this situation. Namely, we need to pick witnesses α_{ij} for the diagrams above and require that

$$\mathrm{pr}_{02}^*\alpha_{ik} = \mathrm{pr}_{12}^*\alpha_{jk} \circ \mathrm{pr}_{01}^*\alpha_{ij}$$

as witnesses over $S_i \times_S S_j \times_S S_k$. I think it is clear what this means... If not, then I'm afraid you'll have to read some of the material on categories fibred in groupoids, etc. In any case, the displayed equation is often called the cocycle condition. A more precise statement of the "sheaf property" is: given $\{S_i \to S\}$, $e_i : S_i \to \mathcal{M}_{1,1}$ and witnesses α_{ij} satisfying the cocycle condition, there exists a unique (up to unique isomorphism) $e : S \to \mathcal{M}_{1,1}$ with $e_i \cong e|_{S_i}$ recovering the α_{ij}.

As you can see even formulating a precise statement takes a bit of work. The proof of this "sheaf property" relies on a fundamental technique in algebraic geometry, namely descent theory. My suggestion is to initially simply accept the "sheaf property" holds, and see what it implies in practice. In fact, a certain amount of mental agility is required to boil the "sheaf property" down to a manageable statement that you can fit on a napkin. Perhaps the simplest variant which is already a bit interesting is the following: Suppose we have a finite Galois extension L/K of fields with Galois group $G = \mathrm{Gal}(L/K)$. Set $T = \mathrm{Spec}(L)$ and $S = \mathrm{Spec}(K)$. Then $\{T \to S\}$ is an étale covering. Let $(E, f, 0)$ be an elliptic curve over L. (Yes, this just means that $E \subset \mathbf{P}^2_L$ is given by a Weierstrass equation and 0 is the usual point at infinity.) Denote $E_\sigma = E \times_{T,\mathrm{Spec}(\sigma)} T$ the base change. (Yes, this corresponds to applying σ to the coefficients of the Weierstrass equation, or is it σ^{-1}?) Now, suppose moreover that for every $\sigma \in G$ we are given an isomorphism

$$\alpha_\sigma : E \longrightarrow E_\sigma$$

over T. The cocycle condition above means in this situation that

$$(\alpha_\tau)^\sigma \circ \alpha_\sigma = \alpha_{\tau\sigma}$$

for $\sigma, \tau \in G$. If you've ever done any group cohomology then this should be familiar. Anyway, the "glueing" condition on $\mathcal{M}_{1,1}$ says that if you have a solution to this set of equations, then there exists an elliptic curve E' over S such that $E \cong E' \times_S T$ (it says a little bit more because it also tells you how to recover the α_σ).

Challenge: Can you prove this entirely using only elliptic curves defined in terms of Weierstrass equations?

105.6. A smooth cover

072S The last thing we have to do is find a smooth cover of $\mathcal{M}_{1,1}$. In fact, in some sense the existence of a smooth cover implies[1] the key fact! In the case of elliptic curves we use the Weierstrass equation to construct one.

Set

$$W = \text{Spec}(\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta])$$

where $\Delta \in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$ is a certain polynomial (see below). Set

$$\mathbf{P}^2_W \supset E_W : zy^2 + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

Denote $f_W : E_W \to W$ the projection. Finally, denote $0_W : W \to E_W$ the section of f_W given by $(0 : 1 : 0)$. It turns out that there is a degree 12 homogeneous polynomial Δ in a_i where $\deg(a_i) = i$ such that $E_W \to W$ is smooth. You can find it explicitly by computing partials of the Weierstrass equation – of course you can also look it up. You can also use pari/gp to compute it for you. Here it is

$$\begin{aligned}\Delta = & -a_6a_1^6 + a_4a_3a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 + \\ & (8a_4a_3a_2 + (a_3^3 + 36a_6a_3))a_1^3 + \\ & ((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2a_2 + (-30a_4a_3^2 + 72a_6a_4))a_1^2 + \\ & (16a_4a_3a_2^2 + (36a_3^3 + 144a_6a_3)a_2 - 96a_4^2a_3)a_1 + \\ & (-16a_3^2 - 64a_6)a_2^3 + 16a_4^2a_2^2 + (72a_4a_3^2 + 288a_6a_4)a_2 + \\ & -27a_3^4 - 216a_6a_3^2 - 64a_4^3 - 432a_6^2\end{aligned}$$

You may recognize the last two terms from the case $y^2 = x^3 + Ax + B$ having discriminant $-64A^3 - 432B^2 = -16(4A^3 + 27B^2)$.

072T Lemma 105.6.1. The morphism $W \xrightarrow{(E_W, f_W, 0_W)} \mathcal{M}_{1,1}$ is smooth and surjective.

Proof. Surjectivity follows from the fact that every elliptic curve over a field has a Weierstrass equation. We give a rough sketch of one way to prove smoothness. Consider the sub group scheme

$$H = \left\{ \begin{pmatrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix} \middle| \begin{array}{c} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{array} \right\} \subset \text{GL}_{3,\mathbf{Z}}$$

[1] This is a bit of a cheat because in checking the smoothness you have to prove something close to the key fact – after all smoothness is defined in terms of fibre products. The advantage is that you only have to prove the existence of these fibre products in the case that on one side you have the morphism that you are trying to show provides the smooth cover.

There is an action $H \times W \to W$ of H on the Weierstrass scheme W. To find the equations for this action write out what a coordinate change given by a matrix in H does to the general Weierstrass equation. Then it turns out the following statements hold

(1) any elliptic curve $(E, f, 0)/S$ has Zariski locally on S a Weierstrass equation,
(2) any two Weierstrass equations for $(E, f, 0)$ differ (Zariski locally) by an element of H.

Considering the fibre product $S \times_{\mathcal{M}_{1,1}} W = \mathit{Isom}_{S\times W}(E \times W, S \times E_W)$ we conclude that this means that the morphism $W \to \mathcal{M}_{1,1}$ is an H-torsor. Since $H \to \mathrm{Spec}(\mathbf{Z})$ is smooth, and since torsors over smooth group schemes are smooth we win. $\square$

072U Remark 105.6.2. The argument sketched above actually shows that $\mathcal{M}_{1,1} = [W/H]$ is a global quotient stack. It is true about 50% of the time that an argument proving a moduli stack is algebraic will show that it is a global quotient stack.

105.7. Properties of algebraic stacks

072V Ok, so now we know that $\mathcal{M}_{1,1}$ is an algebraic stack. What can we do with this? Well, it isn't so much the fact that it is an algebraic stack that helps us here, but more the point of view that properties of $\mathcal{M}_{1,1}$ should be encoded in the properties of morphisms $S \to \mathcal{M}_{1,1}$, i.e., in families of elliptic curves. We list some examples

Local properties:

$$\mathcal{M}_{1,1} \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth} \Leftrightarrow W \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth}$$

Idea. Local properties of an algebraic stack are encoded in the local properties of its smooth cover.

Global properties:

$$\mathcal{M}_{1,1} \text{ is quasi-compact} \Leftarrow W \text{ is quasi-compact}$$
$$\mathcal{M}_{1,1} \text{ is irreducible} \Leftarrow W \text{ is irreducible}$$

Idea. Some global properties of an algebraic stack can be read off from the corresponding property of a suitable[2] smooth cover.

Quasi-coherent sheaves:

$$\mathit{QCoh}(\mathcal{O}_{\mathcal{M}_{1,1}}) = H\text{-equivariant quasi-coherent modules on } W$$

Idea. On the one hand a quasi-coherent module on $\mathcal{M}_{1,1}$ should correspond to a quasi-coherent sheaf $\mathcal{F}_{S,e}$ on S for each morphism $e : S \to \mathcal{M}_{1,1}$. In particular for the morphism $(E_W, f_W, 0_W) : W \to \mathcal{M}_{1,1}$. Since this morphism is H-equivariant we see the quasi-coherent module $\mathcal{F}_W$ we obtain is H-equivariant. Conversely, given an H-equivariant module we can recover the sheaves $\mathcal{F}_{S,e}$ by descent theory starting with the observation that $S \times_{e,\mathcal{M}_{1,1}} W$ is an H-torsor.

Picard group:

$$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Pic}_H(W) = \mathbf{Z}/12\mathbf{Z}$$

[2] I suppose that it is possible an irreducible algebraic stack exists which doesn't have an irreducible smooth cover – but if so it is going to be quite nasty!

Idea. We have seen the first equality above. Note that $\text{Pic}(W) = 0$ because the ring $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta]$ has trivial class group. There is an exact sequence

$$\mathbf{Z}\Delta \to \text{Pic}_H(\mathbf{A}^5_{\mathbf{Z}}) \to \text{Pic}_H(W) \to 0$$

The middle group equals $\text{Hom}(H, \mathbf{G}_m) = \mathbf{Z}$. The image Δ is 12 because Δ has degree 12. This argument is roughly correct, see [FO10].

Étale cohomology: Let Λ be a ring. There is a first quadrant spectral sequence converging to $H^{p+q}_{\acute{e}tale}(\mathcal{M}_{1,1}, \Lambda)$ with E_2-page

$$E_2^{p,q} = H^q_{\acute{e}tale}(W \times H \times \ldots \times H, \Lambda) \quad (p \text{ factors } H)$$

Idea. Note that

$$W \times_{\mathcal{M}_{1,1}} W \times_{\mathcal{M}_{1,1}} \ldots \times_{\mathcal{M}_{1,1}} W = W \times H \times \ldots \times H$$

because $W \to \mathcal{M}_{1,1}$ is a H-torsor. The spectral sequence is the Čech-to-cohomology spectral sequence for the smooth cover $\{W \to \mathcal{M}_{1,1}\}$. For example we see that $H^0_{\acute{e}tale}(\mathcal{M}_{1,1}, \Lambda) = \Lambda$ because W is connected, and $H^1_{\acute{e}tale}(\mathcal{M}_{1,1}, \Lambda) = 0$ because $H^1_{\acute{e}tale}(W, \Lambda) = 0$ (of course this requires a proof). Of course, the smooth covering $W \to \mathcal{M}_{1,1}$ may not be "optimal" for the computation of étale cohomology.

105.8. Other chapters

CHAPTER 106

More on Morphisms of Stacks

0BPK 106.1. Introduction

0BPL In this chapter we continue our study of properties of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

106.2. Conventions and abuse of language

0BPM We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

106.3. Thickenings

0BPN The following terminology may not be completely standard, but it is convenient. If $\mathcal{Y}$ is a closed substack of an algebraic stack $\mathcal{X}$, then the morphism $\mathcal{Y} \to \mathcal{X}$ is representable.

0BPP Definition 106.3.1. Thickenings.

(1) We say an algebraic stack $\mathcal{X}'$ is a thickening of an algebraic stack $\mathcal{X}$ if $\mathcal{X}$ is a closed substack of $\mathcal{X}'$ and the associated topological spaces are equal.
(2) Given two thickenings $\mathcal{X} \subset \mathcal{X}'$ and $\mathcal{Y} \subset \mathcal{Y}'$ a morphism of thickenings is a morphism $f' : \mathcal{X}' \to \mathcal{Y}'$ of algebraic stacks such that $f'|_\mathcal{X}$ factors through the closed substack $\mathcal{Y}$. In this situation we set $f = f'|_\mathcal{X} : \mathcal{X} \to \mathcal{Y}$ and we say that $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings.
(3) Let $\mathcal{Z}$ be an algebraic stack. We similarly define thickenings over $\mathcal{Z}$ and morphisms of thickenings over $\mathcal{Z}$. This means that the algebraic stacks $\mathcal{X}'$ and $\mathcal{Y}'$ are endowed with a structure morphism to $\mathcal{Z}$ and that f' fits into a suitable 2-commutative diagram of algebraic stacks.

Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Let U' be a scheme and let $U' \to \mathcal{X}'$ be a surjective smooth morphism. Setting $U = \mathcal{X} \times_{\mathcal{X}'} U'$ we obtain a morphism of thickenings

$$(U \subset U') \longrightarrow (\mathcal{X} \subset \mathcal{X}')$$

and $U \to \mathcal{X}$ is a surjective smooth morphism. We can often deduce properties of the thickening $\mathcal{X} \subset \mathcal{X}'$ from the corresponding properties of the thickening $U \subset U'$. Sometimes, by abuse of language, we say that a morphism $\mathcal{X} \to \mathcal{X}'$ is a thickening if it is a closed immersion inducing a bijection $|\mathcal{X}| \to |\mathcal{X}'|$.

0CJ7 Lemma 106.3.2. Let $i : \mathcal{X} \to \mathcal{X}'$ be a morphism of algebraic stacks. The following are equivalent

(1) i is a thickening of algebraic stacks (abuse of language as above), and

(2) i is representable by algebraic spaces and is a thickening in the sense of Properties of Stacks, Section 100.3.

In this case i is a closed immersion and a universal homeomorphism.

Proof. By More on Morphisms of Spaces, Lemmas 76.9.10 and 76.9.8 the property P that a morphism of algebraic spaces is a (first order) thickening is fpqc local on the base and stable under base change. Thus the discussion in Properties of Stacks, Section 100.3 indeed applies. Having said this the equivalence of (1) and (2) follows from the fact that $P = P_1 + P_2$ where P_1 is the property of being a closed immersion and P_2 is the property of being surjective. (Strictly speaking, the reader should also consult More on Morphisms of Spaces, Definition 76.9.1, Properties of Stacks, Definition 100.9.1 and the discussion following, Morphisms of Spaces, Lemma 67.5.1, Properties of Stacks, Section 100.5 to see that all the concepts all match up.) The final assertion is clear from the foregoing. $\square$

We will use the lemma without further mention. Using the same references More on Morphisms of Spaces, Lemmas 76.9.10 and 76.9.8 as used in the lemma, allows us to define a first order thickening as follows.

0BPQ Definition 106.3.3. We say an algebraic stack $\mathcal{X}'$ is a first order thickening of an algebraic stack $\mathcal{X}$ if $\mathcal{X}$ is a closed substack of $\mathcal{X}'$ and $\mathcal{X} \to \mathcal{X}'$ is a first order thickening in the sense of Properties of Stacks, Section 100.3.

If $(U \subset U') \to (\mathcal{X} \subset \mathcal{X}')$ is a smooth cover by a scheme as above, then this simply means that $U \subset U'$ is a first order thickening. Next we formulate the obligatory lemmas.

0BPR Lemma 106.3.4. Let $\mathcal{Y} \subset \mathcal{Y}'$ be a thickening of algebraic stacks. Let $\mathcal{X}' \to \mathcal{Y}'$ be a morphism of algebraic stacks and set $\mathcal{X} = \mathcal{Y} \times_{\mathcal{Y}'} \mathcal{X}'$. Then $(\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings. If $\mathcal{Y} \subset \mathcal{Y}'$ is a first order thickening, then $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening.

Proof. See discussion above, Properties of Stacks, Section 100.3, and More on Morphisms of Spaces, Lemma 76.9.8. $\square$

0BPS Lemma 106.3.5. If $\mathcal{X} \subset \mathcal{X}'$ and $\mathcal{X}' \subset \mathcal{X}''$ are thickenings of algebraic stacks, then so is $\mathcal{X} \subset \mathcal{X}''$.

Proof. See discussion above, Properties of Stacks, Section 100.3, and More on Morphisms of Spaces, Lemma 76.9.9 $\square$

0BPT Example 106.3.6. Let $\mathcal{X}'$ be an algebraic stack. Then $\mathcal{X}'$ is a thickening of the reduction $\mathcal{X}'_{red}$, see Properties of Stacks, Definition 100.10.4. Moreover, if $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, then $\mathcal{X}'_{red} = \mathcal{X}_{red} \subset \mathcal{X}$. In other words, $\mathcal{X} = \mathcal{X}'_{red}$ if and only if $\mathcal{X}$ is a reduced algebraic stack.

0BPU Lemma 106.3.7. Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ is a thickening and the canonical diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"'] \arrow[d] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[d] \\ \mathcal{X}' \arrow[r, "\Delta'"] & \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \end{tikzcd}$$

is cartesian.

Proof. Since $\mathcal{X} \to \mathcal{Y}'$ factors through the closed substack $\mathcal{Y}$ we see that $\mathcal{X} \times_\mathcal{Y} \mathcal{X} = \mathcal{X} \times_{\mathcal{Y}'} \mathcal{X}$. Hence $\mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ is isomorphic to the composition

$$\mathcal{X} \times_{\mathcal{Y}'} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$$

both of which are thickenings as base changes of thickenings (Lemma 106.3.4). Hence so is the composition (Lemma 106.3.5). Since $\mathcal{X} \to \mathcal{X}'$ is a monomorphism, the final statement of the lemma follows from Properties of Stacks, Lemma 100.8.6 applied to $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}'$. □

0BPV Lemma 106.3.8. Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Let $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ and $\Delta' : \mathcal{X}' \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ be the corresponding diagonal morphisms. Then each property from the following list is satisfied by Δ if and only if it is satisfied by Δ': (a) representable by schemes, (b) affine, (c) surjective, (d) quasi-compact, (e) universally closed, (f) integral, (g) quasi-separated, (h) separated, (i) universally injective, (j) universally open, (k) locally quasi-finite, (l) finite, (m) unramified, (n) monomorphism, (o) immersion, (p) closed immersion, and (q) proper.

Proof. Observe that

$$(\Delta, \Delta') : (\mathcal{X} \subset \mathcal{X}') \longrightarrow (\mathcal{X} \times_\mathcal{Y} \mathcal{X} \subset \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}')$$

is a morphism of thickenings (Lemma 106.3.7). Moreover Δ and Δ' are representable by algebraic spaces by Morphisms of Stacks, Lemma 101.3.3. Hence, via the discussion in Properties of Stacks, Section 100.3 the lemma follows for cases (a), (b), (c), (d), (e), (f), (g), (h), (i), and (j) by using More on Morphisms of Spaces, Lemma 76.10.1.

Lemma 106.3.7 tells us that $\mathcal{X} = (\mathcal{X} \times_\mathcal{Y} \mathcal{X}) \times_{(\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}')} \mathcal{X}'$. Moreover, Δ and Δ' are locally of finite type by the aforementioned Morphisms of Stacks, Lemma 101.3.3. Hence the result for cases (k), (l), (m), (n), (o), (p), and (q) by using More on Morphisms of Spaces, Lemma 76.10.3. □

As a consequence we obtain the following pleasing result.

0BPW Lemma 106.3.9. Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Then [Con07a, Theorem 2.2.5]

(1) $\mathcal{X}$ is an algebraic space if and only if $\mathcal{X}'$ is an algebraic space,
(2) $\mathcal{X}$ is a scheme if and only if $\mathcal{X}'$ is a scheme,
(3) $\mathcal{X}$ is DM if and only if $\mathcal{X}'$ is DM,
(4) $\mathcal{X}$ is quasi-DM if and only if $\mathcal{X}'$ is quasi-DM,
(5) $\mathcal{X}$ is separated if and only if $\mathcal{X}'$ is separated,
(6) $\mathcal{X}$ is quasi-separated if and only if $\mathcal{X}'$ is quasi-separated, and
(7) add more here.

Proof. In each case we reduce to a question about the diagonal and then we use Lemma 106.3.8 applied to the morphism of thickenings

$$(\mathcal{X} \subset \mathcal{X}') \to (\mathrm{Spec}(\mathbf{Z}) \subset \mathrm{Spec}(\mathbf{Z}))$$

We do this after viewing $\mathcal{X} \subset \mathcal{X}'$ as a thickening of algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 94.19.2.

Case (1). An algebraic stack is an algebraic space if and only if its diagonal is a monomorphism, see Morphisms of Stacks, Lemma 101.6.3 (this also follows immediately from Algebraic Stacks, Proposition 94.13.3).

Case (2). By (1) we may assume that $\mathcal{X}$ and $\mathcal{X}'$ are algebraic spaces and then we can use More on Morphisms of Spaces, Lemma 76.9.5.

Case (3) – (6). Each of these cases corresponds to a condition on the diagonal, see Morphisms of Stacks, Definitions 101.4.1 and 101.4.2. □

106.4. Morphisms of thickenings

0CJ8 If $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings of algebraic stacks, then often properties of the morphism f are inherited by f'. There are several variants.

0CJ9 **Lemma 106.4.1.** *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Then*

1. *f is an affine morphism if and only if f' is an affine morphism,*
2. *f is a surjective morphism if and only if f' is a surjective morphism,*
3. *f is quasi-compact if and only if f' quasi-compact,*
4. *f is universally closed if and only if f' is universally closed,*
5. *f is integral if and only if f' is integral,*
6. *f is universally injective if and only if f' is universally injective,*
7. *f is universally open if and only if f' is universally open,*
8. *f is quasi-DM if and only if f' is quasi-DM,*
9. *f is DM if and only if f' is DM,*
10. *f is (quasi-)separated if and only if f' is (quasi-)separated,*
11. *f is representable if and only if f' is representable,*
12. *f is representable by algebraic spaces if and only if f' is representable by algebraic spaces,*
13. *add more here.*

Proof. By Lemma 106.3.2 the morphisms $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{Y} \to \mathcal{Y}'$ are universal homeomorphisms. Thus any condition on $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is equivalent with the corresponding condition on $|f'| : |\mathcal{X}'| \to |\mathcal{Y}'|$ and the same is true after arbitrary base change by a morphism $\mathcal{Z}' \to \mathcal{Y}'$. This proves that (2), (3), (4), (6), (7) hold.

In cases (8), (9), (10), (12) we can translate the conditions on f and f' into conditions on the diagonals Δ and Δ' as in Lemma 106.3.8. See Morphisms of Stacks, Definition 101.4.1 and Lemma 101.6.3. Hence these cases follow from Lemma 106.3.8.

Proof of (11). If f' is representable, then so is f, because for a scheme T and a morphism $T \to \mathcal{Y}$ we have $\mathcal{X} \times_{\mathcal{Y}} T = \mathcal{X} \times_{\mathcal{X}'} (\mathcal{X}' \times_{\mathcal{Y}'} T)$ and $\mathcal{X} \to \mathcal{X}'$ is a closed immersion (hence representable). Conversely, assume f is representable, and let $T' \to \mathcal{Y}'$ be a morphism where T' is a scheme. Then

$$\mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{Y}'} T') = \mathcal{X} \times_{\mathcal{X}'} (\mathcal{X}' \times_{\mathcal{Y}'} T') \to \mathcal{X}' \times_{\mathcal{Y}'} T'$$

is a thickening (by Lemma 106.3.4) and the source is a scheme. Hence the target is a scheme by Lemma 106.3.9.

In cases (1) and (5) if either f or f' has the stated property, then both f and f' are representable by (11). In this case choose an algebraic space V' and a surjective smooth morphism $V' \to \mathcal{Y}'$. Set $V = \mathcal{Y} \times_{\mathcal{Y}'} V'$, $U' = \mathcal{X}' \times_{\mathcal{Y}'} V'$, and $U = \mathcal{X} \times_{\mathcal{Y}'} V'$. Then the desired results follow from the corresponding results for the morphism $(U \subset U') \to (V \subset V')$ of thickenings of algebraic spaces via the principle of

Properties of Stacks, Lemma 100.3.3. See More on Morphisms of Spaces, Lemma 76.10.1 for the corresponding results in the case of algebraic spaces. $\square$

106.5. Infinitesimal deformations of algebraic stacks

0CJA This section is the analogue of More on Morphisms of Spaces, Section 76.18.

0CJB **Lemma 106.5.1.** *Consider a commutative diagram*

$$\begin{tikzcd} (\mathcal{X} \subset \mathcal{X}') \arrow[rr, "{(f,f')}"'] \arrow[rd] & & (\mathcal{Y} \subset \mathcal{Y}') \arrow[ld] \\ & (\mathcal{B} \subset \mathcal{B}') & \end{tikzcd}$$

of thickenings of algebraic stacks. Assume

1. *$\mathcal{Y}' \to \mathcal{B}'$ is locally of finite type,*
2. *$\mathcal{X}' \to \mathcal{B}'$ is flat and locally of finite presentation,*
3. *f is flat, and*
4. *$\mathcal{X} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{X}'$ and $\mathcal{Y} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{Y}'$.*

Then f' is flat and for all $y' \in |\mathcal{Y}'|$ in the image of $|f'|$ the morphism $\mathcal{Y}' \to \mathcal{B}'$ is flat at y'.

Proof. Choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective smooth morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[rd] & & (V \subset V') \arrow[ld] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 76.18.4. The statement about flatness of $\mathcal{Y}'/\mathcal{B}'$ at points in the image of $|f'|$ follows in the same manner. $\square$

0CJC **Lemma 106.5.2.** *Consider a commutative diagram*

$$\begin{tikzcd} (\mathcal{X} \subset \mathcal{X}') \arrow[rr, "{(f,f')}"'] \arrow[rd] & & (\mathcal{Y} \subset \mathcal{Y}') \arrow[ld] \\ & (\mathcal{B} \subset \mathcal{B}') & \end{tikzcd}$$

of thickenings of algebraic stacks. Assume $\mathcal{Y}' \to \mathcal{B}'$ locally of finite type, $\mathcal{X}' \to \mathcal{B}'$ flat and locally of finite presentation, $\mathcal{X} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{X}'$, and $\mathcal{Y} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{Y}'$. Then

0CJD (1) *f is flat if and only if f' is flat,*
0CJE (2) *f is an isomorphism if and only if f' is an isomorphism,*
0CJF (3) *f is an open immersion if and only if f' is an open immersion,*
0CJG (4) *f is a monomorphism if and only if f' is a monomorphism,*
0CJH (5) *f is locally quasi-finite if and only if f' is locally quasi-finite,*

0CJI (6) f is syntomic if and only if f' is syntomic,
0CJJ (7) f is smooth if and only if f' is smooth,
0CJK (8) f is unramified if and only if f' is unramified,
0CJL (9) f is étale if and only if f' is étale,
0CJM (10) f is finite if and only if f' is finite, and
(11) add more here.

Proof. In case (1) this follows from Lemma 106.5.1.

In cases (6), (7) this can be proved by the method used in the proof of Lemma 106.5.1. Namely, choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective smooth morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then the property for f, resp. f' is equivalent to the property for of $W' \to V'$, resp. $W \to V$. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[rd] & & (V \subset V') \arrow[ld] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 76.18.5.

In cases (8) and (9) we first see that the assumption for f or f' implies that both f and f' are DM morphisms of algebraic stacks, see Lemma 106.4.1. Then we can choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective étale(!) morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then $W \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective étale as well. Hence the property for f, resp. f' is equivalent to the property for of $W' \to V'$, resp. $W \to V$. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[rd] & & (V \subset V') \arrow[ld] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 76.18.5.

In cases (2), (3), (4), (10) we first conclude by Lemma 106.4.1 that f and f' are representable by algebraic spaces. Thus we may choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$, an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$, and then $W' = V' \times_{\mathcal{Y}'} \mathcal{X}'$ will be an algebraic space. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then $W = V \times_{\mathcal{Y}} \mathcal{X}$ as well. Then we have to see that $W' \to V'$ is an isomorphism, resp. an open immersion, resp. a monomorphism, resp. finite, if and only if $W \to V$ has the same property. See Properties of Stacks, Lemma 100.3.3. Thus we conclude by applying the results for algebraic spaces as above.

In the case (5) we first observe that f and f' are locally of finite type by Morphisms of Stacks, Lemma 101.17.8. On the other hand, the morphism f is quasi-DM if and only if f' is by Lemma 106.4.1. The last thing to check to see if f or f' is locally quasi-finite (Morphisms of Stacks, Definition 101.23.2) is a condition on underlying topological spaces which holds for f if and only if it holds for f' by the discussion in the first paragraph of the proof. □

106.6. Lifting affines

0CJN Consider a solid diagram

$$\begin{tikzcd} W \arrow[r, dotted] \arrow[d] & W' \arrow[d, dotted] \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

where $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, W is an affine scheme and $W \to \mathcal{X}$ is smooth. The question we address in this section is whether we can find W' and the dotted arrows so that the square is cartesian and $W' \to \mathcal{X}'$ is smooth. We do not know the answer in general, but if $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening we will prove the answer is yes.

To study this problem we introduce the following category.

0CJP Remark 106.6.1 (Category of lifts). Consider a diagram

$$\begin{tikzcd} W \arrow[d, "x"'] & \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

where $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, W is an algebraic space, and $W \to \mathcal{X}$ is smooth. We will construct a category $\mathcal{C}$ and a functor

$$p : \mathcal{C} \longrightarrow W_{spaces,étale}$$

(see Properties of Spaces, Definition 66.18.2 for notation) as follows. An object of $\mathcal{C}$ will be a system $(U, U', a, i, x', \alpha)$ which forms a commutative diagram

0CJQ (106.6.1.1)
$$\begin{tikzcd} U \arrow[r, "i"'] \arrow[d, "a"'] & U' \arrow[dd, "x'"] \\ W \arrow[d, "x"'] & \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

with commutativity witnessed by the 2-morphism $\alpha : x \circ a \to x' \circ i$ such that U and U' are algebraic spaces, $a : U \to W$ is étale, $x' : U' \to \mathcal{X}'$ is smooth, and such that $U = \mathcal{X} \times_{\mathcal{X}'} U'$. In particular $U \subset U'$ is a thickening. A morphism

$$(U, U', a, i, x', \alpha) \to (V, V', b, j, y', \beta)$$

is given by (f, f', γ) where $f : U \to V$ is a morphism over W, $f' : U' \to V'$ is a morphism whose restriction to U gives f, and $\gamma : x' \circ f' \to y'$ is a 2-morphism witnessing the commutativity in right triangle of the diagram below

0CJR (106.6.1.2)
$$
\begin{tikzcd}
& V \arrow[rr, "j"] \arrow[dl, "f"'] \arrow[ddl, "b"] & & V' \arrow[dl, "f'"'] \arrow[dddl, "y'"] \\
U \arrow[rr, "i"] \arrow[d, "a"'] & & U' \arrow[dd, "x'"'] & \\
W \arrow[d, "x"'] & & & \\
\mathcal{X} \arrow[rr] & & \mathcal{X}' &
\end{tikzcd}
$$

Finally, we require that γ is compatible with α and β: in the calculus of 2-categories of Categories, Sections 4.28 and 4.29 this reads

$$\beta = (\gamma \star \mathrm{id}_j) \circ (\alpha \star \mathrm{id}_f)$$

(more succinctly: $\beta = j^*\gamma \circ f^*\alpha$). Another formulation is that objects are commutative diagrams (106.6.1.1) with some additional properties and morphisms are commutative diagrams (106.6.1.2) in the category Spaces/$\mathcal{X}'$ introduced in Properties of Stacks, Remark 100.3.7. This makes it clear that $\mathcal{C}$ is a category and that the rule $p : \mathcal{C} \to W_{spaces,étale}$ sending $(U, U', a, i, x', \alpha)$ to $a : U \to W$ is a functor.

0CJS Lemma 106.6.2. For any morphism (106.6.1.2) the map $f' : V' \to U'$ is étale.

Proof. Namely $f : V \to U$ is étale as a morphism in $W_{spaces,étale}$ and we can apply Lemma 106.5.2 because $U' \to \mathcal{X}'$ and $V' \to \mathcal{X}'$ are smooth and $U = \mathcal{X} \times_{\mathcal{X}'} U'$ and $V = \mathcal{X} \times_{\mathcal{X}'} V'$. □

0CJT Lemma 106.6.3. The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 is fibred in groupoids.

Proof. We claim the fibre categories of p are groupoids. If (f, f', γ') as in (106.6.1.2) is a morphism such that $f : U \to V$ is an isomorphism, then f' is an isomorphism by Lemma 106.5.2 and hence (f, f', γ') is an isomorphism.

Consider a morphism $f : V \to U$ in $W_{spaces,étale}$ and an object $\xi = (U, U', a, i, x', \alpha)$ of $\mathcal{C}$ over U. We are going to construct the "pullback" $f^*\xi$ over V. Namely, set $b = a \circ f$. Let $f' : V' \to U'$ be the étale morphism whose restriction to V is f (More on Morphisms of Spaces, Lemma 76.8.2). Denote $j : V \to V'$ the corresponding thickening. Let $y' = x' \circ f'$ and $\gamma = \mathrm{id} : x' \circ f' \to y'$. Set

$$\beta = \alpha \star \mathrm{id}_f : x \circ b = x \circ a \circ f \to x' \circ i \circ f = x' \circ f' \circ j = y' \circ j$$

It is clear that $(f, f', \gamma) : (V, V', b, j, y', \beta) \to (U, U', a, i, x', \alpha)$ is a morphism as in (106.6.1.2). The morphisms (f, f', γ) so constructed are strongly cartesian (Categories, Definition 4.33.1). We omit the detailed proof, but essentially the reason is that given a morphism $(g, g', \epsilon) : (Y, Y', c, k, z', \delta) \to (U, U', a, i, x', \alpha)$ in $\mathcal{C}$ such that g factors as $g = f \circ h$ for some $h : Y \to V$, then we get a unique factorization $g' = f' \circ h'$ from More on Morphisms of Spaces, Lemma 76.8.2 and after that one can produce the necessary ζ such that $(h, h', \zeta) : (Y, Y', c, k, z', \delta) \to (V, V', b, j, y', \beta)$ is a morphism of $\mathcal{C}$ with $(g, g', \epsilon) = (f, f', \gamma) \circ (h, h', \zeta)$.

Therefore $p : \mathcal{C} \to W_{étale}$ is a fibred category (Categories, Definition 4.33.5). Combined with the fact that the fibre categories are groupoids seen above we conclude that $p : \mathcal{C} \to W_{étale}$ is fibred in groupoids by Categories, Lemma 4.35.2. $\square$

0CJU **Lemma 106.6.4.** The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 is a stack in groupoids.

Proof. By Lemma 106.6.3 we see the first condition of Stacks, Definition 8.5.1 holds. As is customary we check descent of objects and we leave it to the reader to check descent of morphisms. Thus suppose we have $a : U \to W$ in $W_{spaces,étale}$, a covering $\{U_k \to U\}_{k\in K}$ in $W_{spaces,étale}$, objects $\xi_k = (U_k, U'_k, a_k, i_k, x'_k, \alpha_k)$ of $\mathcal{C}$ over U_k, and morphisms

$$\varphi_{kk'} = (f_{kk'}, f'_{kk'}, \gamma_{kk'}) : \xi_k|_{U_k \times_U U_{k'}} \to \xi_{k'}|_{U_k \times_U U_{k'}}$$

between restrictions satisfying the cocycle condition. In order to prove effectivity we may first refine the covering. Hence we may assume each U_k is a scheme (even an affine scheme if you like). Let us write

$$\xi_k|_{U_k \times_U U_{k'}} = (U_k \times_U U_{k'}, U'_{kk'}, a_{kk'}, x'_{kk'}, \alpha_{kk'})$$

Then we get an étale (by Lemma 106.6.2) morphism $s_{kk'} : U'_{kk'} \to U'_k$ as the second component of the morphism $\xi_k|_{U_k \times_U U_{k'}} \to \xi_k$ of $\mathcal{C}$. Similarly we obtain an étale morphism $t_{kk'} : U'_{kk'} \to U'_{k'}$ by looking at the second component of the composition

$$\xi_k|_{U_k \times_U U_{k'}} \xrightarrow{\varphi_{kk'}} \xi_{k'}|_{U_k \times_U U_{k'}} \to \xi_{k'}$$

We claim that

$$j : \coprod\nolimits_{(k,k') \in K \times K} U'_{kk'} \xrightarrow{(\coprod s_{kk'}, \coprod t_{kk'})} (\coprod\nolimits_{k \in K} U'_k) \times (\coprod\nolimits_{k \in K} U'_k)$$

is an étale equivalence relation. First, we have already seen that the components s, t of the displayed morphism are étale. The base change of the morphism j by $(\coprod U_k) \times (\coprod U_k) \to (\coprod U'_k) \times (\coprod U'_k)$ is a monomorphism because it is the map

$$\coprod\nolimits_{(k,k') \in K \times K} U_k \times_U U_{k'} \longrightarrow (\coprod\nolimits_{k \in K} U_k) \times (\coprod\nolimits_{k \in K} U_k)$$

Hence j is a monomorphism by More on Morphisms, Lemma 37.3.4. Finally, symmetry of the relation j comes from the fact that $\varphi_{kk'}^{-1}$ is the "flip" of $\varphi_{k'k}$ (see Stacks, Remarks 8.3.2) and transitivity comes from the cocycle condition (details omitted). Thus the quotient of $\coprod U'_k$ by j is an algebraic space U' (Spaces, Theorem 65.10.5). Above we have already shown that there is a thickening $i : U \to U'$ as we saw that the restriction of j on $\coprod U_k$ gives $(\coprod U_k) \times_U (\coprod U_k)$. Finally, if we temporarily view the 1-morphisms $x'_k : U'_k \to \mathcal{X}'$ as objects of the stack $\mathcal{X}'$ over U'_k then we see that these come endowed with a descent datum with respect to the étale covering $\{U'_k \to U'\}$ given by the third component $\gamma_{kk'}$ of the morphisms $\varphi_{kk'}$ in $\mathcal{C}$. Since $\mathcal{X}'$ is a stack this descent datum is effective and translating back we obtain a 1-morphism $x' : U' \to \mathcal{X}'$ such that the compositions $U'_k \to U' \to \mathcal{X}'$ come equipped with isomorphisms to x'_k compatible with $\gamma_{kk'}$. This means that the morphisms $\alpha_k : x \circ a_k \to x'_k \circ i_k$ glue to a morphism $\alpha : x \circ a \to x' \circ i$. Then $\xi = (U, U', a, i, x', \alpha)$ is the desired object over U. $\square$

0CJV Lemma 106.6.5. Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{X}$ be a smooth morphism. There exists an étale covering $\{W_i \to W\}_{i \in I}$ and for each i a cartesian diagram

$$\begin{tikzcd} W_i \arrow[r] \arrow[d] & W_i' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

with $W_i' \to \mathcal{X}'$ smooth.

Proof. Choose a scheme U' and a surjective smooth morphism $U' \to \mathcal{X}'$. As usual we set $U = \mathcal{X} \times_{\mathcal{X}'} U'$. Then $U \to \mathcal{X}$ is a surjective smooth morphism. Therefore the base change

$$V = W \times_{\mathcal{X}} U \longrightarrow W$$

is a surjective smooth morphism of algebraic spaces. By Topologies on Spaces, Lemma 73.4.4 we can find an étale covering $\{W_i \to W\}$ such that $W_i \to W$ factors through $V \to W$. After covering W_i by affines (Properties of Spaces, Lemma 66.6.1) we may assume each W_i is affine. We may and do replace W by W_i which reduces us to the situation discussed in the next paragraph.

Assume W is affine and the given morphism $W \to \mathcal{X}$ factors through U. Picture

$$W \xrightarrow{i} U \to \mathcal{X}$$

Since W and U are smooth over $\mathcal{X}$ we see that i is locally of finite type (Morphisms of Stacks, Lemma 101.17.8). After replacing U by $\mathbf{A}^n_U$ we may assume that i is an immersion, see Morphisms, Lemma 29.39.2. By Morphisms of Stacks, Lemma 101.44.4 the morphism i is a local complete intersection. Hence i is a Koszul-regular immersion (as defined in Divisors, Definition 31.21.1) by More on Morphisms, Lemma 37.62.3.

We may still replace W by an affine open covering. For every point $w \in W$ we can choose an affine open $U'_w \subset U'$ such that if $U_w \subset U$ is the corresponding affine open, then $w \in i^{-1}(U_w)$ and $i^{-1}(U_w) \to U_w$ is a closed immersion cut out by a Koszul-regular sequence $f_1, \ldots, f_r \in \Gamma(U_w, \mathcal{O}_{U_w})$. This follows from the definition of Koszul-regular immersions and Divisors, Lemma 31.20.7. Set $W_w = i^{-1}(U_w)$; this is an affine open neighbourhood of $w \in W$. Choose lifts $f_1', \ldots, f_r' \in \Gamma(U'_w, \mathcal{O}_{U'_w})$ of $f_1, \ldots, f_r$. This is possible as $U_w \to U'_w$ is a closed immersion of affine schemes. Let $W'_w \subset U'_w$ be the closed subscheme cut out by $f_1', \ldots, f_r'$. We claim that $W'_w \to \mathcal{X}'$ is smooth. The claim finishes the proof as $W_w = \mathcal{X} \times_{\mathcal{X}'} W'_w$ by construction.

To check the claim it suffices to check that the base change $W'_w \times_{\mathcal{X}'} X' \to X'$ is smooth for every affine scheme X' smooth over $\mathcal{X}'$. Choose an étale morphism

$$Y' \to U'_w \times_{\mathcal{X}'} X'$$

with Y' affine. Because $U'_w \times_{\mathcal{X}'} X'$ is covered by the images of such morphisms, it is enough to show that the closed subscheme Z' of Y' cut out by $f_1', \ldots, f_r'$ is smooth over X'. Picture

$$\begin{tikzcd} & Z' \arrow[r] \arrow[d] & Y' \arrow[d] & \\ & W'_w \times_{\mathcal{X}'} X' \arrow[r] \arrow[d] & U'_w \times_{\mathcal{X}'} X' \arrow[r] \arrow[d] & X' \\ W'_w = V(f'_1, \ldots, f'_r) \arrow[rr] & & U'_w & \end{tikzcd}$$

Set $X = \mathcal{X} \times_{\mathcal{X}'} X'$, $Y = X \times_{X'} Y' = \mathcal{X} \times_{\mathcal{X}'} Y'$, and $Z = Y \times_{Y'} Z' = X \times_{X'} Z' = \mathcal{X} \times_{\mathcal{X}'} Z'$. Then $(Z \subset Z') \to (Y \subset Y') \subset (X \subset X')$ are (cartesian) morphisms of thickenings of affine schemes and we are given that $Z \to X$ and $Y' \to X'$ are smooth. Finally, the sequence of functions $f'_1, \ldots, f'_r$ map to a Koszul-regular sequence in $\Gamma(Y', \mathcal{O}_{Y'})$ by More on Algebra, Lemma 15.30.5 because $Y' \to U'_w$ is smooth and hence flat. By More on Algebra, Lemma 15.31.6 (and the fact that Koszul-regular sequences are quasi-regular sequences by More on Algebra, Lemmas 15.30.2, 15.30.3, and 15.30.6) we conclude that $Z' \to X'$ is smooth as desired. □

0CJW Lemma 106.6.6. Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Consider a commutative diagram

$$\begin{tikzcd} W'' \arrow[d, "x''"'] & W \arrow[l] \arrow[r] \arrow[d, "x"] & W' \arrow[d, "x'"] \\ \mathcal{X}' & \mathcal{X} \arrow[l] \arrow[r] & \mathcal{X}' \end{tikzcd}$$

with cartesian squares where W', W, W'' are algebraic spaces and the vertical arrows are smooth. Then there exist

(1) an étale covering $\{f'_k : W'_k \to W'\}_{k \in K}$,
(2) étale morphisms $f''_k : W'_k \to W''$, and
(3) 2-morphisms $\gamma_k : x'' \circ f''_k \to x' \circ f'_k$

such that (a) $(f'_k)^{-1}(W) = (f''_k)^{-1}(W)$, (b) $f'_k|_{(f'_k)^{-1}(W)} = f''_k|_{(f''_k)^{-1}(W)}$, and (c) pulling back γ_k to the closed subscheme of (a) agrees with the 2-morphism given by the commutativity of the initial diagram over W.

Proof. Denote $i : W \to W'$ and $i'' : W \to W''$ the given thickenings. The commutativity of the diagram in the statement of the lemma means there is a 2-morphism $\delta : x' \circ i' \to x'' \circ i''$ This is the 2-morphism referred to in part (c) of the statement. Consider the algebraic space

$$I' = W' \times_{x', \mathcal{X}', x''} W''$$

with projections $p' : I' \to W'$ and $q' : I' \to W''$. Observe that there is a "universal" 2-morphism $\gamma : x' \circ p' \to x'' \circ q'$ (we will use this later). The choice of δ defines a morphism

$$\begin{tikzcd} W \arrow[r, "\delta"'] & I' \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\ W' & & W'' \end{tikzcd}$$

such that the compositions $W \to I' \to W'$ and $W \to I' \to W''$ are $i : W \to W'$ and $i' : W \to W''$. Since x'' is smooth, the morphism $p' : I' \to W'$ is smooth as a base change of x''.

Suppose we can find an étale covering $\{f'_k : W'_k \to W'\}$ and morphisms $\delta_k : W'_k \to I'$ such that the restriction of δ_k to $W_k = (f'_k)^{-1}$ is equal to $\delta \circ f_k$ where $f_k = f'_k|_{W_k}$. Picture

$$\begin{tikzcd} W_k \arrow[r, "f_k"] \arrow[d] & W \arrow[r, "\delta"] & I' \arrow[d, "p'"] \\ W'_k \arrow[rru, "\delta_k"] \arrow[rr, "f'_k"] & & W' \end{tikzcd}$$

In other words, we want to be able to extend the given section $\delta : W \to I'$ of p' to a section over W' after possibly replacing W' by an étale covering.

If this is true, then we can set $f''_k = q' \circ \delta_k$ and $\gamma_k = \gamma \star \mathrm{id}_{\delta_k}$ (more succinctly $\gamma_k = \delta_k^*\gamma$). Namely, the only thing left to show at this is that the morphism f''_k is étale. By construction the morphism $x' \circ p'$ is 2-isomorphic to $x'' \circ q'$. Hence $x'' \circ f''_k$ is 2-isomorphic to $x' \circ f'_k$. We conclude that the composition

$$W'_k \xrightarrow{f''_k} W'' \xrightarrow{x''} \mathcal{X}'$$

is smooth because $x' \circ f'_k$ is so. As f_k is étale we conclude f''_k is étale by Lemma 106.5.2.

If the thickening is a first order thickening, then we can choose any étale covering $\{W'_k \to W'\}$ with W'_k affine. Namely, since p' is smooth we see that p' is formally smooth by the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 76.19.6). As W_k is affine and as $W_k \to W'_k$ is a first order thickening (as a base change of $\mathcal{X} \to \mathcal{X}'$, see Lemma 106.3.4) we get δ_k as desired.

In the general case the existence of the covering and the morphisms δ_k follows from More on Morphisms of Spaces, Lemma 76.19.7. □

0CJX Lemma 106.6.7. The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 is a gerbe.

Proof. In Lemma 106.6.4 we have seen that it is a stack in groupoids. Thus it remains to check conditions (2) and (3) of Stacks, Definition 8.11.1. Condition (2) follows from Lemma 106.6.5. Condition (3) follows from Lemma 106.6.6. □

0CKG Lemma 106.6.8. In Remark 106.6.1 assume $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening. Then

(1) the automorphism sheaves of objects of the gerbe $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 are abelian, and
(2) the sheaf of groups $\mathcal{G}$ constructed in Stacks, Lemma 8.11.8 is a quasi-coherent $\mathcal{O}_W$-module.

Proof. We will prove both statements at the same time. Namely, given an object $\xi = (U, U', a, i, x', \alpha)$ we will endow $Aut(\xi)$ with the structure of a quasi-coherent $\mathcal{O}_U$-module on $U_{spaces,étale}$ and we will show that this structure is compatible with pullbacks. This will be sufficient by glueing of sheaves (Sites, Section 7.26) and the construction of $\mathcal{G}$ in the proof of Stacks, Lemma 8.11.8 as the glueing of the automorphism sheaves $Aut(\xi)$ and the fact that it suffices to check a module is quasi-coherent after going to an étale covering (Properties of Spaces, Lemma 66.29.6).

We will describe the sheaf $Aut(\xi)$ using the same method as used in the proof of Lemma 106.6.6. Consider the algebraic space

$$I' = U' \times_{x', \mathcal{X}', x'} U'$$

with projections $p' : I' \to U'$ and $q' : I' \to U'$. Over I' there is a universal 2-morphism $\gamma : x' \circ p' \to x' \circ q'$. The identity $x' \to x'$ defines a diagonal morphism

$$\begin{tikzcd} U' \arrow[r, "\Delta'"'] & I' \arrow[dl, "p'"] \arrow[dr, "q'"] & \\ U' & & U' \end{tikzcd}$$

such that the compositions $U' \to I' \to U'$ and $U' \to I' \to U'$ are the identity morphisms. We will denote the base change of U', I', p', q', Δ' to $\mathcal{X}$ by U, I, p, q, Δ. Since $W' \to \mathcal{X}'$ is smooth, we see that $p' : I' \to U'$ is smooth as a base change.

A section of $Aut(\xi)$ over U is a morphism $\delta' : U' \to I'$ such that $\delta'|_U = \Delta$ and such that $p' \circ \delta' = \mathrm{id}_{U'}$. To be explicit, $(\mathrm{id}_U, q' \circ \delta', (\delta')^*\gamma) : \xi \to \xi$ is a formula for the corresponding automorphism. More generally, if $f : V \to U$ is an étale morphism, then there is a thickening $j : V \to V'$ and an étale morphism $f' : V' \to U'$ whose restriction to V is f and $f^*\xi$ corresponds to $(V, V', a \circ f, j, x' \circ f', f^*\alpha)$, see proof of Lemma 106.6.3. a section of $Aut(\xi)$ over V is a morphism $\delta' : V' \to I'$ such that $\delta'|_V = \Delta \circ f$ and $p' \circ \delta' = f'$[1].

We conclude that $Aut(\xi)$ as a sheaf of sets agrees with the sheaf defined in More on Morphisms of Spaces, Remark 76.17.7 for the thickenings $(U \subset U')$ and $(I \subset I')$ over $(U \subset U')$ via $\mathrm{id}_{U'}$ and p'. The diagonal Δ' is a section of this sheaf and by acting on this section using More on Morphisms of Spaces, Lemma 76.17.5 we get an isomorphism

0CKH (106.6.8.1)
$$\mathcal{H}om_{\mathcal{O}_U}(\Delta^*\Omega_{I/U}, \mathcal{C}_{U/U'}) \longrightarrow Aut(\xi)$$

on $U_{spaces,étale}$. There three things left to check

1. the construction of (106.6.8.1) commutes with étale localization,
2. $\mathcal{H}om_{\mathcal{O}_U}(\Delta^*\Omega_{I/U}, \mathcal{C}_{U/U'})$ is a quasi-coherent module on U,
3. the composition in $Aut(\xi)$ corresponds to addition of sections in this quasi-coherent module.

---

[1]A formula for the corresponding automorphism is $(\mathrm{id}_V, h', (\delta')^*\gamma)$. Here $h' : V' \to V'$ is the unique (iso)morphism such that $h'|_V = \mathrm{id}_V$ and such that

$$\begin{tikzcd} V' \arrow[r, "h'"'] \arrow[dr, "q' \circ \delta'"'] & V' \arrow[d, "f'"] \\ & U' \end{tikzcd}$$

commutes. Uniqueness and existence of h' by topological invariance of the étale site, see More on Morphisms of Spaces, Theorem 76.8.1. The reader may feel we should instead look at morphisms $\delta'' : V' \to V' \times_{\mathcal{X}'} V'$ with $\delta'' \circ j = \Delta_{V'/\mathcal{X}'}$ and $\mathrm{pr}_1 \circ \delta'' = \mathrm{id}_{V'}$. This would be fine too: as $V' \times_{\mathcal{X}'} V' \to I'$ is étale, the same topological invariance tells us that sending δ'' to $\delta' = (V' \times_{\mathcal{X}'} V' \to I') \circ \delta''$ is a bijection between the two sets of morphisms.

We will check these in order.

To see (1) we have to show that if $f : V \to U$ is étale, then (106.6.8.1) constructed using ξ over U, restricts to the map (106.6.8.1)

$$\mathcal{H}om_{\mathcal{O}_V}(\Delta_V^*\Omega_{V\times_{\mathcal{X}}V/V}, \mathcal{C}_{V/V'}) \to Aut(\xi|_V)$$

constructed using $\xi|_V$ over V on $V_{spaces,étale}$. This follows from the discussion in the footnote above and More on Morphisms of Spaces, Lemma 76.17.8.

Proof of (2). Since p' is smooth, the morphism $I \to U$ is smooth, and hence the relative module of differentials $\Omega_{I/U}$ is finite locally free (More on Morphisms of Spaces, Lemma 76.7.16). On the other hand, $\mathcal{C}_{U/U'}$ is quasi-coherent (More on Morphisms of Spaces, Definition 76.5.1). By Properties of Spaces, Lemma 66.29.7 we conclude.

Proof of (3). There exists a morphism $c' : I'\times_{p',U',q'}I' \to I'$ such that (U', I', p', q', c') is a groupoid in algebraic spaces with identity Δ'. See Algebraic Stacks, Lemma 94.16.1 for example. Composition in $Aut(\xi)$ is induced by the morphism c' as follows. Suppose we have two morphisms

$$\delta'_1, \delta'_2 : U' \longrightarrow I'$$

corresponding to sections of $Aut(\xi)$ over U as above, in other words, we have $\delta'_i|U = \Delta_U$ and $p' \circ \delta'_i = \mathrm{id}_{U'}$. Then the composition in $Aut(\xi)$ is

$$\delta'_1 \circ \delta'_2 = c'(\delta'_1 \circ q' \circ \delta'_2, \delta'_2)$$

We omit the detailed verification[2]. Thus we are in the situation described in More on Groupoids in Spaces, Section 79.5 and the desired result follows from More on Groupoids in Spaces, Lemma 79.5.2. $\square$

0CKI Proposition 106.6.9 (Emerton). Let $\mathcal{X} \subset \mathcal{X}'$ be a first order thickening of algebraic stacks. Let W be an affine scheme and let $W \to \mathcal{X}$ be a smooth morphism. Then there exists a cartesian diagram

Email of Matthew Emerton dated April 27, 2016.

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

with $W' \to \mathcal{X}'$ smooth and W' affine.

Proof. Consider the category $p : \mathcal{C} \to W_{spaces,étale}$ introduced in Remark 106.6.1. The proposition states that there exists an object of $\mathcal{C}$ lying over W. Namely, if we have such an object $(W, W', a, i, y', \alpha)$ then $W = \mathcal{X} \times_{\mathcal{X}'} W'$. Hence $W \to W'$ is a thickening of algebraic spaces so W' is affine by More on Morphisms of Spaces, Lemma 76.9.5 and More on Morphisms, Lemma 37.2.3.

Lemma 106.6.7 tells us $\mathcal{C}$ is a gerbe over $W_{spaces,étale}$. This means we can étale locally find a solution and these local solutions are étale locally isomorphic; this part does not require the assumption that the thickening is first order. By Lemma 106.6.8 the automorphism sheaves of objects of our gerbe are abelian and fit together to form a quasi-coherent module $\mathcal{G}$ on $W_{spaces,étale}$. We will verify conditions (1) and (2) of Cohomology on Sites, Lemma 21.11.1 to conclude the existence of an

[2]The reader can see immediately that it is necessary to precompose δ'_1 by $q' \circ \delta'_2$ to get a well defined U'-valued point of the fibre product $I' \times_{p',U',q'} I'$.

object of $\mathcal{C}$ lying over W. Condition (1) is true: the étale coverings $\{W_i \to W\}$ with each W_i affine are cofinal in the collection of all coverings. For such a covering W_i and $W_i \times_W W_j$ are affine and $H^1(W_i, \mathcal{G})$ and $H^1(W_i \times_W W_j, \mathcal{G})$ are zero: the cohomology of a quasi-coherent module over an affine algebraic space is zero for example by Cohomology of Spaces, Proposition 69.7.2. Finally, condition (2) is that $H^2(W, \mathcal{G}) = 0$ for our quasi-coherent sheaf $\mathcal{G}$ which again follows from Cohomology of Spaces, Proposition 69.7.2. This finishes the proof. □

106.7. Infinitesimal deformations

0DNQ We continue the discussion from Artin's Axioms, Section 98.21.

0DNR Lemma 106.7.1. Let $\mathcal{X}$ be an algebraic stack over a scheme S. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation. Let $A \to B$ be a flat S-algebra homomorphism. Let x be an object of $\mathcal{X}$ over A and set $y = x|_B$. Then $\text{Inf}_x(M) \otimes_A B = \text{Inf}_y(M \otimes_A B)$.

Proof. Recall that $\text{Inf}_x(M)$ is the set of automorphisms of the trivial deformation of x to $A[M]$ which induce the identity automorphism of x over A. The trivial deformation is the pullback of x to $\text{Spec}(A[M])$ via $\text{Spec}(A[M]) \to \text{Spec}(A)$. Let $G \to \text{Spec}(A)$ be the automorphism group algebraic space of x (this exists because $\mathcal{X}$ is an algebraic space). Let $e : \text{Spec}(A) \to G$ be the neutral element. The discussion in More on Morphisms of Spaces, Section 76.17 gives

$$\text{Inf}_x(M) = \text{Hom}_A(e^*\Omega_{G/A}, M)$$

By the same token

$$\text{Inf}_y(M \otimes_A B) = \text{Hom}_B(e_B^*\Omega_{G_B/B}, M \otimes_A B)$$

Since $G \to \text{Spec}(A)$ is locally of finite presentation by assumption, we see that $\Omega_{G/A}$ is locally of finite presentation, see More on Morphisms of Spaces, Lemma 76.7.15. Hence $e^*\Omega_{G/A}$ is a finitely presented A-module. Moreover, $\Omega_{G_B/B}$ is the pullback of $\Omega_{G/A}$ by More on Morphisms of Spaces, Lemma 76.7.12. Therefore $e_B^*\Omega_{G_B/B} = e^*\Omega_{G/A} \otimes_A B$. we conclude by More on Algebra, Lemma 15.65.4. □

0DNS Lemma 106.7.2. Let $\mathcal{X}$ be an algebraic stack over a base scheme S. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation. Let $(A' \to A, x)$ be a deformation situation. Then the functor

$$F : B' \longmapsto \{\text{lifts of } x|_{B' \otimes_{A'} A} \text{ to } B'\}/\text{isomorphisms}$$

is a sheaf on the site $(\text{Aff}/\text{Spec}(A'))_{fppf}$ of Topologies, Definition 34.7.8.

Proof. Let $\{T'_i \to T'\}_{i=1,\ldots,n}$ be a standard fppf covering of affine schemes over A'. Write $T' = \text{Spec}(B')$. As usual denote

$$T'_{i_0 \ldots i_p} = T'_{i_0} \times_{T'} \ldots \times_{T'} T'_{i_p} = \text{Spec}(B'_{i_0 \ldots i_p})$$

where the ring is a suitable tensor product. Set $B = B' \otimes_{A'} A$ and $B_{i_0 \ldots i_p} = B'_{i_0 \ldots i_p} \otimes_{A'} A$. Denote $y = x|_B$ and $y_{i_0 \ldots i_p} = x|_{B_{i_0 \ldots i_p}}$. Let $\gamma_i \in F(B'_i)$ such that γ_{i_0} and γ_{i_1} map to the same element of $F(B'_{i_0 i_1})$. We have to find a unique $\gamma \in F(B')$ mapping to γ_i in $F(B'_i)$.

Choose an actual object y'_i of $\text{Lift}(y_i, B'_i)$ in the isomorphism class γ_i. Choose isomorphisms $\varphi_{i_0 i_1} : y'_{i_0}|_{B'_{i_0 i_1}} \to y'_{i_1}|_{B'_{i_0 i_1}}$ in the category $\text{Lift}(y_{i_0 i_1}, B'_{i_0 i_1})$. If the maps $\varphi_{i_0i_1}$ satisfy the cocycle condition, then we obtain our object γ because $\mathcal{X}$ is a stack in the fppf topology. The cocycle condition is that the composition

$$y'_{i_0}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_0i_1}|_{B'_{i_0i_1i_2}}} y'_{i_1}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_1i_2}|_{B'_{i_0i_1i_2}}} y'_{i_2}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_2i_0}|_{B'_{i_0i_1i_2}}} y'_{i_0}|_{B'_{i_0i_1i_2}}$$

is the identity. If not, then these maps give elements

$$\delta_{i_0i_1i_2} \in \mathrm{Inf}_{y_{i_0i_1i_2}}(J_{i_0i_1i_2}) = \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1i_2}$$

Here $J = \mathrm{Ker}(B' \to B)$ and $J_{i_0\ldots i_p} = \mathrm{Ker}(B'_{i_0\ldots i_p} \to B_{i_0\ldots i_p})$. The equality in the displayed equation holds by Lemma 106.7.1 applied to $B' \to B'_{i_0\ldots i_p}$ and y and $y_{i_0\ldots i_p}$, the flatness of the maps $B' \to B'_{i_0\ldots i_p}$ which also guarantees that $J_{i_0\ldots i_p} = J \otimes_{B'} B'_{i_0\ldots i_p}$. A computation (omitted) shows that $\delta_{i_0i_1i_2}$ gives a 2-cocycle in the Čech complex

$$\prod \mathrm{Inf}_y(J) \otimes_B B_{i_0} \to \prod \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1} \to \prod \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1i_2} \to \ldots$$

By Descent, Lemma 35.9.2 this complex is acyclic in positive degrees and has $H^0 = \mathrm{Inf}_y(J)$. Since $\mathrm{Inf}_{y_{i_0i_1}}(J_{i_0i_1})$ acts on morphisms (Artin's Axioms, Remark 98.21.4) this means we can modify our choice of $\varphi_{i_0i_1}$ to get to the case where $\delta_{i_0i_1i_2} = 0$.

Uniqueness. We still have to show there is at most one γ restricting to γ_i for all i. Suppose we have objects y', z' of $\mathrm{Lift}(y, B')$ and isomorphisms $\psi_i : y'|_{B'_i} \to z'|_{B'_i}$ in $\mathrm{Lift}(y_i, B'_i)$. Then we can consider

$$\psi_{i_1}^{-1} \circ \psi_{i_0} \in \mathrm{Inf}_{y_{i_0i_1}}(J_{i_0i_1}) = \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1}$$

Arguing as before, the obstruction to finding an isomorphism between y' and z' over B' is an element in the H^1 of the Čech complex displayed above which is zero. □

0DNT Lemma 106.7.3. Let $\mathcal{X}$ be an algebraic stack over a scheme S whose structure morphism $\mathcal{X} \to S$ is locally of finite presentation. Let $A \to B$ be a flat S-algebra homomorphism. Let x be an object of $\mathcal{X}$ over A. Then $T_x(M) \otimes_A B = T_y(M \otimes_A B)$.

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. We first reduce the lemma to the case where x lifts to U. Recall that $T_x(M)$ is the set of isomorphism classes of lifts of x to $A[M]$. Therefore Lemma 106.7.2[3] says that the rule

$$A_1 \mapsto T_{x|_{A_1}}(M \otimes_A A_1)$$

is a sheaf on the small étale site of $\mathrm{Spec}(A)$; the tensor product is needed to make $A[M] \to A_1[M \otimes_A A_1]$ a flat ring map. We may choose a faithfully flat étale ring map $A \to A_1$ such that $x|_{A_1}$ lifts to a morphism $u_1 : \mathrm{Spec}(A_1) \to U$, see for example Sheaves on Stacks, Lemma 96.19.10. Write $A_2 = A_1 \otimes_A A_1$ and set $B_1 = B \otimes_A A_1$ and $B_2 = B \otimes_A A_2$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & T_y(M \otimes_A B) \arrow[r] & T_{y|_{B_1}}(M \otimes_A B_1) \arrow[r] & T_{y|_{B_2}}(M \otimes_A B_2) \\ 0 \arrow[r] & T_x(M) \arrow[r] \arrow[u] & T_{x|_{A_1}}(M \otimes_A A_1) \arrow[r] \arrow[u] & T_{x|_{A_2}}(M \otimes_A A_2) \arrow[u] \end{tikzcd}$$

[3]This lemma applies: $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is locally of finite presentation by Morphisms of Stacks, Lemma 101.27.6 and the assumption that $\mathcal{X} \to S$ is locally of finite presentation. Therefore $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation as a base change of Δ.

The rows are exact by the sheaf condition. We have $M \otimes_A B_i = (M \otimes_A A_i) \otimes_{A_i} B_i$. Thus if we prove the result for the middle and right vertical arrow, then the result follows. This reduces us to the case discussed in the next paragraph.

Assume that x is the image of a morphism $u : \mathrm{Spec}(A) \to U$. Observe that $T_u(M) \to T_x(M)$ is surjective since $U \to \mathcal{X}$ is smooth and representable by algebraic spaces, see Criteria for Representability, Lemma 97.6.3 (see discussion preceding it for explanation) and More on Morphisms of Spaces, Lemma 76.19.6. Set $R = U \times_{\mathcal{X}} U$. Recall that we obtain a groupoid (U, R, s, t, c, e, i) in algebraic spaces with $\mathcal{X} = [U/R]$. By Artin's Axioms, Lemma 98.21.6 we have an exact sequence

$$T_{e \circ u}(M) \to T_u(M) \oplus T_u(M) \to T_x(M) \to 0$$

where the zero on the right was shown above. A similar sequence holds for the base change to B. Thus the result we want follows if we can prove the result of the lemma for $T_u(M)$ and $T_{e \circ u}(M)$. This reduces us to the case discussed in the next paragraph.

Assume that $\mathcal{X} = X$ is an algebraic space locally of finite presentation over S. Then we have

$$T_x(M) = \mathrm{Hom}_A(x^*\Omega_{X/S}, M)$$

by the discussion in More on Morphisms of Spaces, Section 76.17. By the same token

$$T_y(M \otimes_A B) = \mathrm{Hom}_B(y^*\Omega_{X/S}, M \otimes_A B)$$

Since $X \to S$ is locally of finite presentation, we see that $\Omega_{X/S}$ is locally of finite presentation, see More on Morphisms of Spaces, Lemma 76.7.15. Hence $x^*\Omega_{X/S}$ is a finitely presented A-module. Clearly, we have $y^*\Omega_{X/S} = x^*\Omega_{X/S} \otimes_A B$. we conclude by More on Algebra, Lemma 15.65.4. □

0DNU Lemma 106.7.4. Let $\mathcal{X}$ be an algebraic stack over a scheme S whose structure morphism $\mathcal{X} \to S$ is locally of finite presentation. Let $(A' \to A, x)$ be a deformation situation. If there exists a faithfully flat finitely presented A'-algebra B' and an object y' of $\mathcal{X}$ over B' lifting $x|_{B' \otimes_{A'} A}$, then there exists an object x' over A' lifting x.

Proof. Let $I = \mathrm{Ker}(A' \to A)$. Set $B'_1 = B' \otimes_{A'} B'$ and $B'_2 = B' \otimes_{A'} B' \otimes_{A'} B'$. Let $J = IB'$, $J_1 = IB'_1$, $J_2 = IB'_2$ and $B = B'/J$, $B_1 = B'_1/J_1$, $B_2 = B'_2/J_2$. Set $y = x|_B$, $y_1 = x|_{B_1}$, $y_2 = x|_{B_2}$. Let F be the fppf sheaf of Lemma 106.7.2 (which applies, see footnote in the proof of Lemma 106.7.3). Thus we have an equalizer diagram

$$F(A') \longrightarrow F(B') \rightrightarrows F(B'_1)$$

On the other hand, we have $F(B') = \mathrm{Lift}(y, B')$, $F(B'_1) = \mathrm{Lift}(y_1, B'_1)$, and $F(B'_2) = \mathrm{Lift}(y_2, B'_2)$ in the terminology from Artin's Axioms, Section 98.21. These sets are nonempty and are (canonically) principal homogeneous spaces for $T_y(J)$, $T_{y_1}(J_1)$, $T_{y_2}(J_2)$, see Artin's Axioms, Lemma 98.21.2. Thus the difference of the two images of y' in $F(B'_1)$ is an element

$$\delta_1 \in T_{y_1}(J_1) = T_x(I) \otimes_A B_1$$

The equality in the displayed equation holds by Lemma 106.7.3 applied to $A' \to B'_1$ and x and y_1, the flatness of $A' \to B'_1$ which also guarantees that $J_1 = I \otimes_{A'} B'_1$.

We have similar equalities for B' and B'_2. A computation (omitted) shows that δ_1 gives a 1-cocycle in the Čech complex

$$T_x(I) \otimes_A B \to T_x(I) \otimes_A B_1 \to T_x(I) \otimes_A B_2 \to \ldots$$

By Descent, Lemma 35.9.2 this complex is acyclic in positive degrees and has $H^0 = T_x(I)$. Thus we may choose an element in $T_x(I) \otimes_A B = T_y(J)$ whose boundary is δ_1. Replacing y' by the result of this element acting on it, we find a new choice y' with $\delta_1 = 0$. Thus y' maps to the same element under the two maps $F(B') \to F(B'_1)$ and we obtain an element o $F(A')$ by the sheaf condition. $\square$

106.8. Formally smooth morphisms

0DNV In this section we introduce the notion of a formally smooth morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks. Such a morphism is characterized by the property that T-valued points of $\mathcal{X}$ lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 106.8.7. It turns out that this criterion is often easier to use than the Jacobian criterion.

0DNW Definition 106.8.1. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be formally smooth if it is formally smooth on objects as a 1-morphism in categories fibred in groupoids as explained in Criteria for Representability, Section 97.6.

We translate the condition of the definition into the language we are currently using (see Properties of Stacks, Section 100.2). Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

0DNX (106.8.1.1)
$$\begin{tikzcd} T \arrow[r, "x"] \arrow[d, "i"] & \mathcal{X} \arrow[d, "f"] \\ T' \arrow[r, "y"] \arrow[ru, dotted] & \mathcal{Y} \end{tikzcd}$$

where $i : T \to T'$ is a first order thickening of affine schemes. Let

$$\gamma : y \circ i \longrightarrow f \circ x$$

be a 2-morphism witnessing the 2-commutativity of the diagram. (Notation as in Categories, Sections 4.28 and 4.29.) Given (106.8.1.1) and γ a dotted arrow is a triple (x', α, β) consisting of a morphism $x' : T' \to \mathcal{X}$ and 2-arrows $\alpha : x' \circ i \to x$, $\beta : y \to f \circ x'$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_i)$, in other words such that

$$\begin{tikzcd} & f \circ x' \circ i \arrow[rd, "\mathrm{id}_f \star \alpha"] & \\ y \circ i \arrow[ru, "\beta \star \mathrm{id}_i"] \arrow[rr, "\gamma"] & & f \circ x \end{tikzcd}$$

is commutative. A morphism of dotted arrows $(x'_1, \alpha_1, \beta_1) \to (x'_2, \alpha_2, \beta_2)$ is a 2-arrow $\theta : x'_1 \to x'_2$ such that $\alpha_1 = \alpha_2 \circ (\theta \star \mathrm{id}_i)$ and $\beta_2 = (\mathrm{id}_f \star \theta) \circ \beta_1$.

The category of dotted arrows just described is a special case of Categories, Definition 4.44.1.

0DNY Lemma 106.8.2. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is formally smooth (Definition 106.8.1) if and only if for every diagram (106.8.1.1) and γ the category of dotted arrows is nonempty.

Proof. Translation between different languages omitted. □

0H1I **Lemma 106.8.3.** The base change of a formally smooth morphism of algebraic stacks by any morphism of algebraic stacks is formally smooth.

Proof. Follows from Categories, Lemma 4.44.2 and the definition. □

0H1J **Lemma 106.8.4.** The composition of formally smooth morphisms of algebraic stacks is formally smooth.

Proof. Follows from Categories, Lemma 4.44.3 and the definition. □

0H1K **Lemma 106.8.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent

1. f is formally smooth,
2. for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is formally smooth as a morphism of algebraic spaces.

Proof. Follows from Categories, Lemma 4.44.2 and the definition. □

0DNZ **Lemma 106.8.6.** Let $T \to T'$ be a first order thickening of affine schemes. Let $\mathcal{X}'$ be an algebraic stack over T' whose structure morphism $\mathcal{X}' \to T'$ is smooth. Let $x : T \to \mathcal{X}'$ be a morphism over T'. Then there exists a morphsm $x' : T' \to \mathcal{X}'$ over T' with $x'|_T = x$.

Proof. We may apply the result of Lemma 106.7.4. Thus it suffices to construct a smooth surjective morphism $W' \to T'$ with W' affine such that $x|_{T \times_{W'} T'}$ lifts to W'. (We urge the reader to find their own proof of this fact using the analogous result for algebraic spaces already established.) We choose a scheme U' and a surjective smooth morphism $U' \to \mathcal{X}'$. Observe that $U' \to T'$ is smooth and that the projection $T \times_{\mathcal{X}'} U' \to T$ is surjective smooth. Choose an affine scheme W and an étale morphism $W \to T \times_{\mathcal{X}'} U'$ such that $W \to T$ is surjective. Then $W \to T$ is a smooth morphism of affine schemes. After replacing W by a disjoint union of principal affine opens, we may assume there exists a smooth morphism of affines $W' \to T'$ such that $W = T \times_{T'} W'$, see Algebra, Lemma 10.137.20. By More on Morphisms of Spaces, Lemma 76.19.6 we can find a morphism $W' \to U'$ over T' lifting the given morphism $W \to U'$. This finishes the proof. □

The following lemma is the main result of this section. It implies, combined with Limits of Stacks, Proposition 102.3.8, that we can recognize whether a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is smooth in terms of "simple" properties of the 1-morphism of stacks in groupoids $\mathcal{X} \to \mathcal{Y}$.

0DP0 **Lemma 106.8.7** (Infinitesimal lifting criterion)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:

1. The morphism f is smooth.
2. The morphism f is locally of finite presentation and formally smooth.

Proof. Assume f is smooth. Then f is locally of finite presentation by Morphisms of Stacks, Lemma 101.33.5. Hence it suffices given a diagram (106.8.1.1) and a $\gamma : y \circ i \to f \circ x$ to find a dotted arrow (see Lemma 106.8.2). Forming fibre products we obtain

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T' \arrow[r] & T' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Thus we see it is sufficient to find a dotted arrow in the left square. Since $T' \times_{\mathcal{Y}} \mathcal{X} \to T'$ is smooth (Morphisms of Stacks, Lemma 101.33.3) existence of a dotted arrow in the left square is guaranteed by Lemma 106.8.6.

Conversely, suppose that f is locally of finite presentation and formally smooth. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then $a : U \to \mathcal{X}$ and $b : U \to \mathcal{Y}$ are representable by algebraic spaces and locally of finite presentation (use Morphisms of Stacks, Lemma 101.27.2 and the fact seen above that a smooth morphism is locally of finite presentation). We will apply the general principle of Algebraic Stacks, Lemma 94.10.9 with as input the equivalence of More on Morphisms of Spaces, Lemma 76.19.6 and simultaneously use the translation of Criteria for Representability, Lemma 97.6.3. We first apply this to a to see that a is formally smooth on objects. Next, we use that f is formally smooth on objects by assumption (see Lemma 106.8.2) and Criteria for Representability, Lemma 97.6.2 to see that $b = f \circ a$ is formally smooth on objects. Then we apply the principle once more to conclude that b is smooth. This means that f is smooth by the definition of smoothness for morphisms of algebraic stacks and the proof is complete. □

106.9. Blowing up and flatness

0CQ3 This section quickly discusses what you can deduce from More on Morphisms of Spaces, Sections 76.38 and 76.39 for algebraic stacks over algebraic spaces.

0CQ4 Lemma 106.9.1. Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Let $V \subset Y$ be an open subspace. Assume

(1) Y is quasi-compact and quasi-separated,
(2) f is of finite type and quasi-separated,
(3) V is quasi-compact, and
(4) $\mathcal{X}_V$ is flat and locally of finite presentation over V.

Then there exists a V-admissible blowup $Y' \to Y$ and a closed substack $\mathcal{X}' \subset \mathcal{X}_{Y'}$ with $\mathcal{X}'_V = \mathcal{X}_V$ such that $\mathcal{X}' \to Y'$ is flat and of finite presentation.

Proof. Observe that $\mathcal{X}$ is quasi-compact. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Let $R = U \times_{\mathcal{X}} U$ so that we obtain a groupoid (U, R, s, t, c) in algebraic spaces over Y with $\mathcal{X} = [U/R]$ (Algebraic Stacks, Lemma 94.16.2). We may apply More on Morphisms of Spaces, Lemma 76.39.1 to $U \to Y$ and the open $V \subset Y$. Thus we obtain a V-admissible blowup $Y' \to Y$ such that the strict transform $U' \subset U_{Y'}$ is flat and of finite presentation over Y'. Let $R' \subset R_{Y'}$ be the strict transform of R. Since s and t are smooth (and in particular flat) it follows from Divisors on Spaces, Lemma 71.18.4 that we have cartesian diagrams

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & R_{Y'} \arrow[d, "s_{Y'}"] \\ U' \arrow[r] & U_{Y'} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R' \arrow[r] \arrow[d] & R_{Y'} \arrow[d, "t_{Y'}"] \\ U' \arrow[r] & U_{Y'} \end{tikzcd}$$

In other words, U' is an $R_{Y'}$-invariant closed subspace of $U_{Y'}$. Thus U' defines a closed substack $\mathcal{X}' \subset \mathcal{X}_{Y'}$ by Properties of Stacks, Lemma 100.9.11. The morphism $\mathcal{X}' \to Y'$ is flat and locally of finite presentation because this is true for $U' \to Y'$. On the other hand, we already know $\mathcal{X}' \to Y'$ is quasi-compact and quasi-separated (by our assumptions on f and because this is true for closed immersions). This finishes the proof. □

106.10. Chow's lemma for algebraic stacks

0CQ5 In this section we discuss Chow's lemma for algebraic stacks.

0CQ6 **Lemma 106.10.1.** Let Y be a quasi-compact and quasi-separated algebraic space. Let $V \subset Y$ be a quasi-compact open. Let $f : \mathcal{X} \to V$ be surjective, flat, and locally of finite presentation. Then there exists a finite surjective morphism $g : Y' \to Y$ such that $V' = g^{-1}(V) \to Y$ factors Zariski locally through f.

Proof. We first prove this when Y is a scheme. We may choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then $\{U \to V\}$ is an fppf covering of schemes. By More on Morphisms, Lemma 37.48.6 there exists a finite surjective morphism $V' \to V$ such that $V' \to V$ factors Zariski locally through U. By More on Morphisms, Lemma 37.48.4 we can find a finite surjective morphism $Y' \to Y$ whose restriction to V is $V' \to V$ as desired.

If Y is an algebraic space, then we see the lemma is true by first doing a finite base change by a finite surjective morphism $Y' \to Y$ where Y' is a scheme. See Limits of Spaces, Proposition 70.16.1. □

0CQ7 **Lemma 106.10.2.** Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Let $V \subset Y$ be an open subspace. Assume

(1) f is separated and of finite type,
(2) Y is quasi-compact and quasi-separated,
(3) V is quasi-compact, and
(4) $\mathcal{X}_V$ is a gerbe over V.

Then there exists a commutative diagram

$$\begin{tikzcd} \overline{Z} \arrow[dr, "\overline{g}"'] & Z \arrow[l, "j"'] \arrow[r, "h"] \arrow[d, "g"] & \mathcal{X} \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

with j an open immersion, $\overline{g}$ and h proper, and such that $|V|$ is contained in the image of $|g|$.

Proof. Suppose we have a commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and a quasi-compact open $V' \subset Y'$, such that $Y' \to Y$ is a proper morphism of algebraic spaces, $\mathcal{X}' \to \mathcal{X}$ is a proper morphism of algebraic stacks, $V' \subset Y'$ maps surjectively onto V, and $\mathcal{X}'_{V'}$ is a gerbe over V'. Then it suffices to prove the lemma for the pair $(f' : \mathcal{X}' \to Y', V')$. Some details omitted.

Overall strategy of the proof. We will reduce to the case where the image of f is open and f has a section over this open by repeatedly applying the above remark. Each step is straightforward, but there are quite a few of them which makes the proof a bit involved.

Using Limits of Spaces, Proposition 70.16.1 we reduce to the case where Y is a scheme. (Let $Y' \to Y$ be a finite surjective morphism where Y' is a scheme. Set $\mathcal{X}' = \mathcal{X}_{Y'}$ and apply the initial remark of the proof.)

Using Lemma 106.9.1 (and Morphisms of Stacks, Lemma 101.28.8 to see that a gerbe is flat and locally of finite presentation) we reduce to the case where f is flat and of finite presentation.

Since f is flat and locally of finite presentation, we see that the image of $|f|$ is an open $W \subset Y$. Since $\mathcal{X}$ is quasi-compact (as f is of finite type and Y is quasi-compact) we see that W is quasi-compact. By Lemma 106.10.1 we can find a finite surjective morphism $g : Y' \to Y$ such that $g^{-1}(W) \to Y$ factors Zariski locally through $\mathcal{X} \to Y$. After replacing Y by Y' and $\mathcal{X}$ by $\mathcal{X} \times_Y Y'$ we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \text{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, and (c) W is the image of $|f|$. We will use induction on n. The base case is $n = 0$: this implies $V = \emptyset$ and in this case we can take $\overline{Z} = \emptyset$. If $n > 0$, then for $i = 1, \ldots, n$ consider the reduced closed subschemes Y_i with underlying topological space $Y \setminus W_i$. Consider the finite morphism

$$Y' = Y \amalg \coprod\nolimits_{i=1,\ldots,n} Y_i \longrightarrow Y$$

and the quasi-compact open

$$V' = (W_1 \cap \ldots \cap W_n \cap V) \amalg \coprod_{i=1,\ldots,n} (V \cap Y_i).$$

By the initial remark of the proof, if we can prove the lemma for the pairs

$$(\mathcal{X} \to Y, W_1 \cap \ldots \cap W_n \cap V) \quad \text{and} \quad (\mathcal{X} \times_Y Y_i \to Y_i, V \cap Y_i), \quad i = 1, \ldots, n$$

then the result is true. Here we use the settheoretic equality $V = (W_1 \cap \ldots \cap W_n \cap V) \cup \bigcup_{i=1,\ldots n} (V \cap Y_i)$. The induction hypothesis applies to the second type of pairs above. Hence we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \text{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, and (d) $V \subset W_1 \cap \ldots \cap W_n$. The morphisms

$$T_{ij} = \mathit{Isom}_{\mathcal{X}}(x_i|_{W_i \cap W_j \cap V}, x_j|_{W_i \cap W_j \cap V}) \longrightarrow W_i \cap W_j \cap V$$

are surjective, flat, and locally of finite presentation (Morphisms of Stacks, Lemma 101.28.10). We apply Lemma 106.10.1 to each quasi-compact open $W_i \cap W_j \cap V$ and the morphisms $T_{ij} \to W_i \cap W_j \cap V$ to get finite surjective morphisms $Y'_{ij} \to Y$. After replacing Y by the fibre product of all of the Y'_{ij} over Y we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \text{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (e) x_i and x_j are Zariski locally isomorphic over $W_i \cap W_j \cap V$. Let $y \in V$ be arbitrary. Suppose that we can find a quasi-compact open neighbourhood $y \in V_y \subset V$ such that the lemma is true for the pair $(\mathcal{X} \to Y, V_y)$, say with solution $\overline{Z}_y, Z_y, \overline{g}_y, g_y, h_y$. Since V is quasi-compact, we can find a finite number $y_1, \ldots, y_m$ such that $V = V_{y_1} \cup \ldots \cup V_{y_m}$. Then it follows that setting

$$\overline{Z} = \coprod \overline{Z}_{y_j}, \quad Z = \coprod Z_{y_j}, \quad \overline{g} = \coprod \overline{g}_{y_j}, \quad g = \coprod g_{y_j}, \quad h = \coprod h_{y_j}$$

is a solution for the lemma. Given y by condition (e) we can choose a quasi-compact open neighbourhood $y \in V_y \subset V$ and isomorphisms $\varphi_i : x_1|_{V_y} \to x_i|_{V_y}$ for $i = 2, \ldots, n$. Set $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. This leads us to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \text{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (f) there are isomorphisms $\varphi_{ij} : x_i|_V \to x_j|_V$ satisfying $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. The morphisms

$$I_{ij} = \mathit{Isom}_{\mathcal{X}}(x_i|_{W_i \cap W_j}, x_j|_{W_i \cap W_j}) \longrightarrow W_i \cap W_j$$

are proper because f is separated (Morphisms of Stacks, Lemma 101.6.6). Observe that φ_{ij} defines a section $V \to I_{ij}$ of $I_{ij} \to W_i \cap W_j$ over V. By More on Morphisms of Spaces, Lemma 76.39.6 we can find V-admissible blowups $p_{ij} : Y_{ij} \to Y$ such that s_{ij} extends to $p_{ij}^{-1}(W_i \cap W_j)$. After replacing Y by the fibre product of all the Y_{ij} over Y we get to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \text{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (g) there are isomorphisms $\varphi_{ij} : x_i|_{W_i \cap W_j} \to x_j|_{W_i \cap W_j}$ satisfying

$$\varphi_{jk}|_V \circ \varphi_{ij}|_V = \varphi_{ik}|_V.$$

After replacing Y by another V-admissible blowup if necessary we may assume that V is dense and scheme theoretically dense in Y and hence in any open subspace of Y containing V. After such a replacement we conclude that

$$\varphi_{jk}|_{W_i \cap W_j \cap W_k} \circ \varphi_{ij}|_{W_i \cap W_j \cap W_k} = \varphi_{ik}|_{W_i \cap W_j \cap W_k}$$

by appealing to Morphisms of Spaces, Lemma 67.17.8 and the fact that $I_{ik} \to W_i \cap W_j$ is proper (hence separated). Of course this means that (x_i, φ_{ij}) is a desent datum and we obtain a morphism $x : W \to \mathcal{X}$ agreeing with x_i over W_i because $\mathcal{X}$ is a stack. Since x is a section of the separated morphism $\mathcal{X} \to W$ we see that x is proper (Morphisms of Stacks, Lemma 101.4.9). Thus the lemma now holds with $\overline{Z} = Y$, $Z = W$, $\overline{g} = \text{id}_Y$, $g = \text{id}_W$, $h = x$. $\square$

0CQ8 Theorem 106.10.3 (Chow's lemma). Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume

(1) Y is quasi-compact and quasi-separated,
(2) f is separated of finite type.

This is a result due to Ofer Gabber, see [Ols05, Theorem 1.1]

Then there exists a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[dr] & X \arrow[l] \arrow[r] \arrow[d] & \overline{X} \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X \to \mathcal{X}$ is proper surjective, $X \to \overline{X}$ is an open immersion, and $\overline{X} \to Y$ is proper morphism of algebraic spaces.

Proof. The rough idea is to use that $\mathcal{X}$ has a dense open which is a gerbe (Morphisms of Stacks, Proposition 101.29.1) and appeal to Lemma 106.10.2. The reason this does not work is that the open may not be quasi-compact and one runs into technical problems. Thus we first do a (standard) reduction to the Noetherian case.

First we choose a closed immersion $\mathcal{X} \to \mathcal{X}'$ where $\mathcal{X}'$ is an algebraic stack separated and of finite type over Y. See Limits of Stacks, Lemma 102.6.2. Clearly it suffices to prove the theorem for $\mathcal{X}'$, hence we may assume $\mathcal{X} \to Y$ is separated and of finite presentation.

Assume $\mathcal{X} \to Y$ is separated and of finite presentation. By Limits of Spaces, Proposition 70.8.1 we can write $Y = \lim Y_i$ as the directed limit of a system of Noetherian algebraic spaces with affine transition morphisms. By Limits of Stacks, Lemma 102.5.1 there is an i and a morphism $\mathcal{X}_i \to Y_i$ of finite presentation from an algebraic stack to Y_i such that $\mathcal{X} = Y \times_{Y_i} \mathcal{X}_i$. After increasing i we may assume that $\mathcal{X}_i \to Y_i$ is separated, see Limits of Stacks, Lemma 102.4.2. Then it suffices to prove the theorem for $\mathcal{X}_i \to Y_i$. This reduces us to the case discussed in the next paragraph.

Assume Y is Noetherian. We may replace $\mathcal{X}$ by its reduction (Properties of Stacks, Definition 100.10.4). This reduces us to the case discussed in the next paragraph.

Assume Y is Noetherian and $\mathcal{X}$ is reduced. Since $\mathcal{X} \to Y$ is separated and Y quasi-separated, we see that $\mathcal{X}$ is quasi-separated as an algebraic stack. Hence the inertia $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Thus by Morphisms of Stacks, Proposition 101.29.1 there exists a dense open substack $\mathcal{V} \subset \mathcal{X}$ which is a gerbe. Let $\mathcal{V} \to V$ be the morphism which expresses $\mathcal{V}$ as a gerbe over the algebraic space V. See Morphisms of Stacks, Lemma 101.28.2 for a construction of $\mathcal{V} \to V$. This construction in particular shows that the morphism $\mathcal{V} \to Y$ factors as $\mathcal{V} \to V \to Y$. Picture

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

Since the morphism $\mathcal{V} \to V$ is surjective, flat, and of finite presentation (Morphisms of Stacks, Lemma 101.28.8) and since $\mathcal{V} \to Y$ is locally of finite presentation, it follows that $V \to Y$ is locally of finite presentation (Morphisms of Stacks, Lemma 101.27.12). Note that $\mathcal{V} \to V$ is a universal homeomorphism (Morphisms of Stacks, Lemma 101.28.13). Since $\mathcal{V}$ is quasi-compact (see Morphisms of Stacks, Lemma 101.8.2) we see that V is quasi-compact. Finally, since $\mathcal{V} \to Y$ is separated the same is true for $V \to Y$ by Morphisms of Stacks, Lemma 101.27.17 applied to $\mathcal{V} \to V \to Y$ (whose assumptions are satisfied as we've already seen).

All of the above means that the assumptions of Limits of Spaces, Lemma 70.13.3 apply to the morphism $V \to Y$. Thus we can find a dense open subspace $V' \subset V$ and an immersion $V' \to \mathbf{P}^n_Y$ over Y. Clearly we may replace V by V' and $\mathcal{V}$ by the inverse image of V' in $\mathcal{V}$ (recall that $|\mathcal{V}| = |V|$ as we've seen above). Thus we may assume we have a diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr] \arrow[d] & & \mathcal{X} \arrow[d] \\ V \arrow[r] & \mathbf{P}^n_Y \arrow[r] & Y \end{tikzcd}$$

where the arrow $V \to \mathbf{P}^n_Y$ is an immersion. Let $\mathcal{X}'$ be the scheme theoretic image of the morphism

$$j : \mathcal{V} \longrightarrow \mathbf{P}^n_Y \times_Y \mathcal{X}$$

and let Y' be the scheme theoretic image of the morphism $V \to \mathbf{P}^n_Y$. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & \mathbf{P}^n_Y \times_Y \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V \arrow[r] & Y' \arrow[r] & \mathbf{P}^n_Y \arrow[r] & Y \end{tikzcd}$$

(See Morphisms of Stacks, Lemma 101.38.4). We claim that $\mathcal{V} = V \times_{Y'} \mathcal{X}'$ and that Lemma 106.10.2 applies to the morphism $\mathcal{X}' \to Y'$ and the open subspace $V \subset Y'$. If the claim is true, then we obtain

$$\begin{tikzcd} \overline{X} \arrow[dr, "\overline{g}"'] & X \arrow[l] \arrow[r, "h"] \arrow[d, "g"] & \mathcal{X}' \arrow[dl, "f"] \\ & Y' & \end{tikzcd}$$

with $X \to \overline{X}$ an open immersion, $\overline{g}$ and h proper, and such that $|V|$ is contained in the image of $|g|$. Then the composition $X \to \mathcal{X}' \to \mathcal{X}$ is proper (as a composition of proper morphisms) and its image contains $|\mathcal{V}|$, hence this composition is surjective. As well, $\overline{X} \to Y' \to Y$ is proper as a composition of proper morphisms.

The last step is to prove the claim. Observe that $\mathcal{X}' \to Y'$ is separated and of finite type, that Y' is quasi-compact and quasi-separated, and that V is quasi-compact (we omit checking all the details completely). Next, we observe that $b : \mathcal{X}' \to \mathcal{X}$ is an isomorphism over $\mathcal{V}$ by Morphisms of Stacks, Lemma 101.38.7. In particular $\mathcal{V}$ is identified with an open substack of $\mathcal{X}'$. The morphism j is quasi-compact (source is quasi-compact and target is quasi-separated), so formation of the scheme theoretic image of j commutes with flat base change by Morphisms of Stacks, Lemma 101.38.5. In particular we see that $V \times_{Y'} \mathcal{X}'$ is the scheme theoretic image of $\mathcal{V} \to V \times_{Y'} \mathcal{X}'$. However, by Morphisms of Stacks, Lemma 101.37.5 the image of $|\mathcal{V}| \to |V \times_{Y'} \mathcal{X}'|$ is closed (use that $\mathcal{V} \to V$ is a universal homeomorphism as we've seen above and hence is universally closed). Also the image is dense (combine what we just said with Morphisms of Stacks, Lemma 101.38.6) we conclude $|\mathcal{V}| = |V \times_{Y'} \mathcal{X}'|$. Thus $\mathcal{V} \to V \times_{Y'} \mathcal{X}'$ is an isomorphism and the proof of the claim is complete. $\square$

106.11. Noetherian valuative criterion

0CQL In this section we will discuss (refined) valuative criteria for morphisms of algebraic stacks using only discrete valuation rings in the Noetherian setting. There are many different variants and we will add more here over time as needed.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks (or algebraic spaces or schemes). A refined valuative criterion is one where we are given a morphism $\mathcal{U} \to \mathcal{X}$ (with some properties) and we only look at existence or uniqueness of dotted arrows in solid diagrams of the form

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{U} \arrow[r] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dotted] & & \mathcal{Y} \end{tikzcd}$$

We use this terminology below to describe the results we have obtained sofar.

Non-Noetherian valuative criteria for morphisms of algebraic stacks

1. Morphisms of Stacks, Section 101.40 (for separatedness of the diagonal),
2. Morphisms of Stacks, Section 101.41 (for separatedness),
3. Morphisms of Stacks, Section 101.42 (for universal closedness),
4. Morphisms of Stacks, Section 101.43 (for properness).

For algebraic spaces we have the following valuative criteria

1. Morphisms of Spaces, Section 67.42 (for universal closedness),
2. Morphisms of Spaces, Lemma 67.42.5 (refined for universal closedness)
3. Morphisms of Spaces, Section 67.43 (for separatedness),
4. Morphisms of Spaces, Section 67.44 (for properness),
5. Decent Spaces, Section 68.16 (for universal closedness for decent spaces),
6. Decent Spaces, Lemma 68.17.11 (for universal closedness for decent morphisms between algebraic spaces),
7. Cohomology of Spaces, Section 69.19 contains Noetherian valuative criteria
   (a) Cohomology of Spaces, Lemma 69.19.1 (for separatedness using discrete valuation rings),
   (b) Cohomology of Spaces, Lemma 69.19.2 (for properness using discrete valuation rings),
   (c) Cohomology of Spaces, Remark 69.19.3 (discusses how to reduce to complete discrete valuation rings),
8. Limits of Spaces, Section 70.21 discussing Noetherian valuative criteria
   (a) Limits of Spaces, Lemma 70.21.2 (for separatedness using discrete valuation rings and generic points)
   (b) Limits of Spaces, Lemma 70.21.3 (for properness using discrete valuation rings and generic points)
   (c) Limits of Spaces, Lemma 70.21.4 (for universal closedness using discrete valuation rings).
9. Limits of Spaces, Section 70.22 discussing refined Noetherian valuative criteria
   (a) Limits of Spaces, Lemmas 70.22.1 and 70.22.3 (refined for properness using discrete valuation rings),

(b) Limits of Spaces, Lemma 70.22.2 (refined for separatedness using discrete valuation rings),

For schemes we have the following valuative criteria

(1) Schemes, Section 26.20 (for universal closedness)
(2) Schemes, Section 26.22 (for separatedness),
(3) Morphisms, Section 29.42 (for properness)
(4) Morphisms, Lemma 29.42.2 (refined for universal closedness),
(5) Limits, Section 32.15 discussing Noetherian valuative criteria
 (a) Limits, Lemma 32.15.2 (for separatedness using discrete valuation rings and generic points)
 (b) Limits, Lemma 32.15.3 (for properness using discrete valuation rings and generic points)
 (c) Limits, Lemma 32.15.4 (for universal closedness using discrete valuation rings).
(6) Limits, Section 32.16 discussing refined Noetherian valuative criteria
 (a) Limits, Lemmas 32.16.1 and 32.16.3 (refined for properness using discrete valuation rings),
 (b) Limits, Lemma 32.16.2 (refined for separatedness using discrete valuation rings),
(7) Limits, Section 32.17 discussing valuative criteria over a Noetherian base where one can get discrete valuation rings essentially of finite type over the base.

This ends our list of previous results.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0H2B **Lemma 106.11.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f finite type and $\mathcal{Y}$ locally Noetherian. Let $y \in |\mathcal{Y}|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y.

Proof. Choose an affine scheme V, a point $v \in V$ and a smooth morphism $V \to \mathcal{Y}$ mapping v to y. The map $|V| \to |\mathcal{Y}|$ is open and by Properties of Stacks, Lemma 100.4.3 the image of $|\mathcal{X} \times_\mathcal{Y} V| \to |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|\mathcal{X} \times_\mathcal{Y} V| \to |V|$. If we prove the lemma for $\mathcal{X} \times_\mathcal{Y} V \to V$ and the point v, then the lemma follows for f and y. In this way we reduce to the situation described in the next paragraph.

Assume we have $f : \mathcal{X} \to Y$ and $y \in |Y|$ as in the lemma where Y is a Noetherian affine scheme. Since f is quasi-compact, we conclude that $\mathcal{X}$ is quasi-compact. Hence we can choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Then the image of $|f|$ is the same as the image of $|W| \to |Y|$. In this way we reduce to the case of schemes which is Limits, Lemma 32.15.1. $\square$

0E80 **Lemma 106.11.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume

(1) $\mathcal{Y}$ is locally Noetherian,
(2) f is locally of finite type and quasi-separated,
(3) for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ru, dashed] & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field and any 2-arrow $\gamma : y \circ j \to f \circ x$ the category of dotted arrows (Morphisms of Stacks, Definition 101.39.1) is either empty or a setoid with exactly one isomorphism class.

Then f is separated.

Proof. To prove that f is separated we have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is proper. We already know that Δ is representable by algebraic spaces, locally of finite type (Morphisms of Stacks, Lemma 101.3.3) and quasi-compact and quasi-separated (by definition of f being quasi-separated). Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$. Set

$$V = \mathcal{X} \times_{\Delta, \mathcal{X} \times_\mathcal{Y} \mathcal{X}} U$$

It suffices to show that the morphism of algebraic spaces $V \to U$ is proper (Properties of Stacks, Lemma 100.3.3). Observe that U is locally Noetherian (use Morphisms of Stacks, Lemma 101.17.5 and the fact that $U \to \mathcal{Y}$ is locally of finite type) and $V \to U$ is of finite type and quasi-separated (as the base change of Δ and properties of Δ listed above). Applying Cohomology of Spaces, Lemma 69.19.2 it suffices to show: Given a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "v"'] \arrow[d, "j"'] & V \arrow[r] \arrow[d, "g"] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[r, "u"] \arrow[ru, dashed] \arrow[rru, dotted] & U \arrow[r] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field, there is a unique dashed arrow making the diagram commute. By Morphisms of Stacks, Lemma 101.39.4 the categories of dashed and dotted arrows are equivalent. Assumption (3) implies there is a unique dotted arrow up to isomorphism, see Morphisms of Stacks, Lemma 101.41.1. We conclude there is a unique dashed arrow as desired. $\square$

0CQM **Lemma 106.11.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{U} \to \mathcal{X}$ be morphisms of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian, that f and h are of finite type, that f is separated, and that the image of $|h| : |\mathcal{U}| \to |\mathcal{X}|$ is dense in $|\mathcal{X}|$. If given any 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "u"'] \arrow[d, "j"'] & \mathcal{U} \arrow[r, "h"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr, "y"] & & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ h \circ u$ there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d, "j'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r, "y'"] \arrow[ru, dotted, "x'"] & \mathcal{Y} \end{tikzcd}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty (Morphisms of Stacks, Definition 101.39.1), then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \to \mathcal{Y}$ be a smooth morphism where V is an affine scheme. By Properties of Stacks, Lemma 100.4.3 the image I of $|\mathcal{U} \times_\mathcal{Y} V| \to |\mathcal{X} \times_\mathcal{Y} V|$ is the inverse image of the image of $|h|$. Since $|\mathcal{X} \times_\mathcal{Y} V| \to |\mathcal{X}|$ is open (Morphisms of Stacks, Lemma 101.27.15) we conclude that I is dense in $|\mathcal{X} \times_\mathcal{Y} V|$. Also since the category of dotted arrows behaves well with respect to base change (Morphisms of Stacks, Lemma 101.39.4) the assumption on existence of dotted arrows (after extension) is inherited by the morphisms $\mathcal{U} \times_\mathcal{Y} V \to \mathcal{X} \times_\mathcal{Y} V \to V$. Therefore the assumptions of the lemma are satisfied for the morphisms $\mathcal{U} \times_\mathcal{Y} V \to \mathcal{X} \times_\mathcal{Y} V \to V$. Hence we may assume $\mathcal{Y}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ is an affine scheme. (From now on we no longer have to worry about the 2-arrows γ and γ', see Morphisms of Stacks, Lemma 101.39.3.) Then $\mathcal{U}$ is quasi-compact. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{U}$. Then we may and do replace $\mathcal{U}$ by U. Thus we may assume that $\mathcal{U}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ and $\mathcal{U} = U$ are affine schemes. By Chow's lemma (Theorem 106.10.3) we can choose a surjective proper morphism $X \to \mathcal{X}$ where X is an algebraic space. We will use below that $X \to Y$ is separated as a composition of separated morphisms. Consider the algebraic space $W = X \times_\mathcal{X} U$. The projection morphism $W \to X$ is of finite type. We may replace X by the scheme theoretic image of $W \to X$ and hence we may assume that the image of $|W|$ in $|X|$ is dense in $|X|$ (here we use that the image of $|h|$ is dense in $|\mathcal{X}|$, so after this replacement, the morphism $X \to \mathcal{X}$ is still surjective). We claim that for every solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & W \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dotted] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute. First, it is enough to prove there exists a dotted arrow after replacing K by an extension and A by a valuation ring in this extension dominating A, see Morphisms of Spaces, Lemma 67.41.4. By the assumption of the lemma we get an extension K'/K and a valuation ring $A' \subset K'$ dominating A and an arrow $\mathrm{Spec}(A') \to \mathcal{X}$ lifting the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to Y$

and compatible with the composition $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \to W \to X$. Because $X \to \mathcal{X}$ is proper, we can use the valuative criterion of properness (Morphisms of Stacks, Lemma 101.43.1) to find an extension K''/K' and a valuation ring $A'' \subset K''$ dominating A' and a morphism $\mathrm{Spec}(A'') \to X$ lifting the composition $\mathrm{Spec}(A'') \to \mathrm{Spec}(A') \to \mathcal{X}$ and compatible with the composition $\mathrm{Spec}(K'') \to \mathrm{Spec}(K') \to \mathrm{Spec}(K) \to X$. Then K''/K and $A'' \subset K''$ and the morphism $\mathrm{Spec}(A'') \to X$ is a solution to the problem posed above and the claim holds.

The claim implies the morphism $X \to Y$ is proper by the case of the lemma for algebraic spaces (Limits of Spaces, Lemma 70.22.1). By Morphisms of Stacks, Lemma 101.37.6 we conclude that $\mathcal{X} \to Y$ is proper as desired. □

0E95 Lemma 106.11.4. Let $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{U} \to \mathcal{X}$ be morphisms of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |\mathcal{U}| \to |\mathcal{X}|$ is dense in $|\mathcal{X}|$. If given any 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "u"'] \arrow[d, "j"'] & \mathcal{U} \arrow[r, "h"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr, "y"] \arrow[urr, dotted] & & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ h \circ u$, the category of dotted arrows is either empty or a setoid with exactly one isomorphism class, then f is separated.

Proof. We have to prove Δ is a proper morphism. Assume first that Δ is separated. Then we may apply Lemma 106.11.3 to the morphisms $\mathcal{U} \to \mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Observe that Δ is quasi-compact as f is quasi-separated. Of course Δ is locally of finite type (true for any diagonal morphism, see Morphisms of Stacks, Lemma 101.3.3). Finally, suppose given a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "u"'] \arrow[d, "j"'] & \mathcal{U} \arrow[r, "h"'] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rr, "y"] \arrow[urr, dotted] & & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to \Delta \circ h \circ u$. By Morphisms of Stacks, Lemma 101.41.1 and the assumption in the lemma we find there exists a unique dotted arrow. This proves the last assumption of Lemma 106.11.3 holds and the result follows.

In the general case, it suffices to prove Δ is separated since then we'll be back in the previous case. In fact, we claim that the assumptions of the lemma hold for

$$\mathcal{U} \to \mathcal{X} \quad \text{and} \quad \Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Namely, since Δ is representable by algebraic spaces, the category of dotted arrows for a diagram as in the previous paragraph is a setoid (see for example Morphisms of Stacks, Lemma 101.39.2). The argument in the preceding paragraph shows these categories are either empty or have one isomorphism class. Thus Δ is separated. □

0H2C Lemma 106.11.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian and that f is of finite type. If given any 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ x$ there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d, "j'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r, "y'"] \arrow[ru, dotted, "x'"] & \mathcal{Y} \end{tikzcd}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty (Morphisms of Stacks, Definition 101.39.1), then f is universally closed.

Proof. Let $V \to \mathcal{Y}$ be a smooth morphism where V is an affine scheme. The category of dotted arrows behaves well with respect to base change (Morphisms of Stacks, Lemma 101.39.4). Hence the assumption on existence of dotted arrows (after extension) is inherited by the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$. Therefore the assumptions of the lemma are satisfied for the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$. Hence we may assume $\mathcal{Y}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ is a Noetherian affine scheme. (From now on we no longer have to worry about the 2-arrows γ and γ', see Morphisms of Stacks, Lemma 101.39.3.) To prove that f is universally closed it suffices to show that $|\mathcal{X} \times \mathbf{A}^n| \to |Y \times \mathbf{A}^n|$ is closed for all n by Limits of Stacks, Lemma 102.7.2. Since the assumption in the lemma is inherited by the product morphism $\mathcal{X} \times \mathbf{A}^n \to Y \times \mathbf{A}^n$ (details omitted) we reduce to proving that $|\mathcal{X}| \to |Y|$ is closed.

Assume Y is a Noetherian affine scheme. Let $T \subset |\mathcal{X}|$ be a closed subset. We have to show that the image of T in $|Y|$ is closed. We may replace $\mathcal{X}$ by the reduced induced closed subspace structure on T; we omit the verification that property on the existence of dotted arrows is preserved by this replacement. Thus we reduce to proving that the image of $|\mathcal{X}| \to |Y|$ is closed.

Let $y \in |Y|$ be a point in the closure of the image of $|\mathcal{X}| \to |Y|$. By Lemma 106.11.1 we may choose a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. It follows immediately from the assumption in the lemma that y is in the image of $|\mathcal{X}| \to |Y|$ and the proof is complete. $\square$

106.12. Moduli spaces

0DUF This section discusses morphisms $f : \mathcal{X} \to Y$ from algebraic stacks to algebraic spaces. Under suitable hypotheses Y is called a moduli space for $\mathcal{X}$. If $\mathcal{X} = [U/R]$ is a presentation, then we obtain an R-invariant morphism $U \to Y$ and (under suitable hypotheses) Y is a quotient of the groupoid (U, R, s, t, c). A discussion of the different types of quotients can be found starting with Quotients of Groupoids, Section 83.1.

0DUG **Definition 106.12.1.** Let $\mathcal{X}$ be an algebraic stack. Let $f : \mathcal{X} \to Y$ be a morphism to an algebraic space Y.

(1) We say f is a *categorical moduli space* if any morphism $\mathcal{X} \to W$ to an algebraic space W factors uniquely through f.
(2) We say f is a *uniform categorical moduli space* if for any flat morphism $Y' \to Y$ of algebraic spaces the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space.

Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces.

(3) We say f is a *categorical moduli space in $\mathcal{C}$* if $Y \in \mathrm{Ob}(\mathcal{C})$ and any morphism $\mathcal{X} \to W$ with $W \in \mathrm{Ob}(\mathcal{C})$ factors uniquely through f.
(4) We say is a *uniform categorical moduli space in $\mathcal{C}$* if $Y \in \mathrm{Ob}(\mathcal{C})$ and for every flat morphism $Y' \to Y$ in $\mathcal{C}$ the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space in $\mathcal{C}$.

By the Yoneda lemma a categorical moduli space, if it exists, is unique. Let us match this with the language introduced for quotients.

0DUH **Lemma 106.12.2.** *Let (U, R, s, t, c) be a groupoid in algebraic spaces with $s, t : R \to U$ flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$. Given an algebraic space Y there is a 1-to-1 correspondence between morphisms $f : \mathcal{X} \to Y$ and R-invariant morphisms $\phi : U \to Y$.*

Proof. Criteria for Representability, Theorem 97.17.2 tells us $\mathcal{X}$ is an algebraic stack. Given a morphism $f : \mathcal{X} \to Y$ we let $\phi : U \to Y$ be the composition $U \to \mathcal{X} \to Y$. Since $R = U \times_{\mathcal{X}} U$ (Groupoids in Spaces, Lemma 78.22.2) it is immediate that ϕ is R-invariant. Conversely, if $\phi : U \to Y$ is an R-invariant morphism towards an algebraic space, we obtain a morphism $f : \mathcal{X} \to Y$ by Groupoids in Spaces, Lemma 78.23.2. You can also construct f from ϕ using the explicit description of the quotient stack in Groupoids in Spaces, Lemma 78.24.1. $\square$

0DUI **Lemma 106.12.3.** *With assumption and notation as in Lemma 106.12.2. Then f is a (uniform) categorical moduli space if and only if ϕ is a (uniform) categorical quotient. Similarly for moduli spaces in a full subcategory.*

Proof. It is immediate from the 1-to-1 correspondence established in Lemma 106.12.2 that f is a categorical moduli space if and only if ϕ is a categorical quotient (Quotients of Groupoids, Definition 83.4.1). If $Y' \to Y$ is a morphism, then $U' = Y' \times_Y U \to Y' \times_Y \mathcal{X} = \mathcal{X}'$ is a surjective, flat, locally finitely presented morphism as a base change of $U \to \mathcal{X}$ (Criteria for Representability, Lemma 97.17.1). And $R' = Y' \times_Y R$ is equal to $U' \times_{\mathcal{X}'} U'$ by transitivity of fibre products. Hence $\mathcal{X}' = [U'/R']$, see Algebraic Stacks, Remark 94.16.3. Thus the base change of our situation to Y' is another situation as in the statement of the lemma. From this it immediately follows that f is a uniform categorical moduli space if and only if ϕ is a uniform categorical quotient. $\square$

0DUJ Lemma 106.12.4. Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. If for every affine scheme Y' and flat morphism $Y' \to Y$ the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space, then f is a uniform categorical moduli space.

Proof. Choose an étale covering $\{Y_i \to Y\}$ where Y_i is an affine scheme. For each i and j choose a affine open covering $Y_i \times_Y Y_j = \bigcup Y_{ijk}$. Set $\mathcal{X}_i = Y_i \times_Y \mathcal{X}$ and $\mathcal{X}_{ijk} = Y_{ijk} \times_Y \mathcal{X}$. Let $g : \mathcal{X} \to W$ be a morphism towards an algebraic space. Then we consider the diagram

$$\begin{tikzcd}
\mathcal{X}_i \arrow[r] \arrow[d] & \mathcal{X} \arrow[r, "g"] \arrow[d] & W \\
Y_i \arrow[r] \arrow[rru, dotted] & Y &
\end{tikzcd}$$

The assumption that $\mathcal{X}_i \to Y_i$ is a categorical moduli space, produces a unique dotted arrow $h_i : Y_i \to W$. The assumption that $\mathcal{X}_{ijk} \to Y_{ijk}$ is a categorical moduli space, implies the restriction of h_i and h_j to Y_{ijk} are equal. Hence h_i and h_j agree on $Y_i \times_Y Y_j$. Since $Y = \coprod Y_i / \coprod Y_i \times_Y Y_j$ (by Spaces, Section 65.9) we conclude that there is a unique morphism $Y \to W$ through which g factors. Thus f is a categorical moduli space. The same argument applies after a flat base change, hence f is a uniform categorical moduli space. $\square$

106.13. The Keel-Mori theorem

0DUK In this section we start discussing the theorem of Keel and Mori in the setting of algebraic stacks. For a discussion of the literature, please see Guide to Literature, Subsection 112.5.2.

0DUL Definition 106.13.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is well-nigh affine if there exists an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$.

We give this property a somewhat ridiculous name because we do not intend to use it too much.

0DUM Lemma 106.13.2. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent

(1) $\mathcal{X}$ is well-nigh affine, and
(2) there exists a groupoid scheme (U, R, s, t, c) with U and R affine and $s, t : R \to U$ finite locally free such that $\mathcal{X} = [U/R]$.

If true then $\mathcal{X}$ is quasi-compact, quasi-DM, and separated.

Proof. Assume $\mathcal{X}$ is well-nigh affine. Choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$. Then we obtain a groupoid (U, R, s, t, c) in algebraic spaces and an isomorphism $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 94.16.1 and Remark 94.16.3. Since $s, t : R \to U$ are flat, finite, and finitely presented morphisms (as base changes of $U \to \mathcal{X}$) we see that s, t are finite locally free (Morphisms, Lemma 29.48.2). This implies that R is affine (as finite morphisms are affine) and hence (2) holds.

Suppose that we have a groupoid scheme (U, R, s, t, c) with U and R are affine and $s, t : R \to U$ finite locally free. Set $\mathcal{X} = [U/R]$. Then $\mathcal{X}$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2 (strictly speaking we don't need this here, but it can't be stressed enough that this is true). The morphism $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 97.17.1. Thus we can check whether $U \to \mathcal{X}$ is finite by checking whether the projection $U \times_{\mathcal{X}} U \to U$ has this property, see Properties of Stacks, Lemma 100.3.3. Since $U \times_{\mathcal{X}} U = R$ by Groupoids in Spaces, Lemma 78.22.2 we see that this is true. Thus $\mathcal{X}$ is well-nigh affine.

Proof of the final statement. We see that $\mathcal{X}$ is quasi-compact by Properties of Stacks, Lemma 100.6.2. We see that $\mathcal{X} = [U/R]$ is quasi-DM and separated by Morphisms of Stacks, Lemma 101.20.1. □

0DUN Lemma 106.13.3. Let the algebraic stack $\mathcal{X}$ be well-nigh affine.

(1) If $\mathcal{X}$ is an algebraic space, then it is affine.
(2) If $\mathcal{X}' \to \mathcal{X}$ is an affine morphism of algebraic stacks, then $\mathcal{X}'$ is well-nigh affine.

Proof. Part (1) follows from immediately from Limits of Spaces, Lemma 70.15.1. However, this is overkill, since (1) also follows from Lemma 106.13.2 combined with Groupoids, Proposition 39.23.9.

To prove (2) we choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Then $U' = \mathcal{X}' \times_{\mathcal{X}} U$ admits an affine morphism to U (Morphisms of Stacks, Lemma 101.9.2). Therefore U' is an affine scheme. Of course $U' \to \mathcal{X}'$ is surjective, flat, finite, and finitely presented as a base change of $U \to \mathcal{X}$. □

0DUP Lemma 106.13.4. Let the algebraic stack $\mathcal{X}$ be well-nigh affine. There exists a uniform categorical moduli space

$$f : \mathcal{X} \longrightarrow M$$

in the category of affine schemes. Moreover f is separated, quasi-compact, and a universal homeomorphism.

Proof. Write $\mathcal{X} = [U/R]$ with (U, R, s, t, c) as in Lemma 106.13.2. Let C be the ring of R-invariant functions on U, see Groupoids, Section 39.23. We set $M = \mathrm{Spec}(C)$. The R-invariant morphism $U \to M$ corresponds to a morphism $f : \mathcal{X} \to M$ by Lemma 106.12.2. The characterization of morphisms into affine schemes given in Schemes, Lemma 26.6.4 immediately guarantees that $\phi : U \to M$ is a categorical quotient in the category of affine schemes. Hence f is a categorical moduli space in the category of affine schemes (Lemma 106.12.3).

Since $\mathcal{X}$ is separated by Lemma 106.13.2 we find that f is separated by Morphisms of Stacks, Lemma 101.4.12.

Since $U \to \mathcal{X}$ is surjective and since $U \to M$ is quasi-compact, we see that f is quasi-compact by Morphisms of Stacks, Lemma 101.7.6.

By Groupoids, Lemma 39.23.4 the composition

$$U \to \mathcal{X} \to M$$

is an integral morphism of affine schemes. In particular, it is universally closed (Morphisms, Lemma 29.44.7). Since $U \to \mathcal{X}$ is surjective, it follows that $\mathcal{X} \to M$ is universally closed (Morphisms of Stacks, Lemma 101.37.6). To conclude that $\mathcal{X} \to M$ is a universal homeomorphism, it is enough to show that it is universally bijective, i.e., surjective and universally injective.

We have $|\mathcal{X}| = |U|/|R|$ by Morphisms of Stacks, Lemma 101.20.2. Thus $|f|$ is surjective and even bijective by Groupoids, Lemma 39.23.6.

Let $C \to C'$ be a ring map. Let (U', R', s', t', c') be the base change of (U, R, s, t, c) by $M' = \mathrm{Spec}(C') \to M$. Setting $\mathcal{X}' = [U'/R']$, we observe that $M' \times_M \mathcal{X} = \mathcal{X}'$ by Quotients of Groupoids, Lemma 83.3.6. Let C^1 be the ring of R'-invariant functions on U'. Set $M^1 = \mathrm{Spec}(C^1)$ and consider the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"] & \mathcal{X} \arrow[dd, "f"] \\ M^1 \arrow[d] & \\ M' \arrow[r] & M \end{tikzcd}$$

By Groupoids, Lemma 39.23.5 and Algebra, Lemma 10.46.11 the morphism $M^1 \to M'$ is a homeomorphism. On the other hand, the previous paragraph applied to (U', R', s', t', c') shows that $|f'|$ is bijective. We conclude that f induces a bijection on points after any base change by an affine scheme. Thus f is universally injective by Morphisms of Stacks, Lemma 101.14.7.

Finally, we still have to show that f is a uniform moduli space in the category of affine schemes. This follows from the discussion above and the fact that if the ring map $C \to C'$ is flat, then $C' \to C^1$ is an isomorphism by Groupoids, Lemma 39.23.5. □

0DUQ Lemma 106.13.5. Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{X}'$ and $\mathcal{X}$ are well-nigh affine, h is étale, and h induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5). Then there exists a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

where $M' \to M$ is étale and the vertical arrows are the moduli spaces constructed in Lemma 106.13.4.

Proof. Observe that h is representable by algebraic spaces by Morphisms of Stacks, Lemmas 101.45.3 and 101.45.1. Choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Then $U' = \mathcal{X}' \times_\mathcal{X} U$ is an algebraic space with a finite (in particular affine) morphism $U' \to \mathcal{X}'$. By Lemma 106.13.3 we conclude that U' is affine. Setting $R = U \times_\mathcal{X} U$ and $R' = U' \times_{\mathcal{X}'} U'$ we obtain groupoids (U, R, s, t, c) and (U', R', s', t', c') such that $\mathcal{X} = [U/R]$ and $\mathcal{X}' = [U'/R']$, see proof of Lemma 106.13.2. we see that the diagrams

$$\begin{tikzcd} R' \arrow[r, "f"] \arrow[d, "s'"'] & R \arrow[d, "s"] \\ U' \arrow[r, "f"] & U \end{tikzcd} \quad \begin{tikzcd} R' \arrow[r, "f"] \arrow[d, "t'"'] & R \arrow[d, "t"] \\ U' \arrow[r, "f"] & U \end{tikzcd} \quad \begin{tikzcd} G' \arrow[r, "f"] \arrow[d] & G \arrow[d] \\ U' \arrow[r, "f"] & U \end{tikzcd}$$

are cartesian where G and G' are the stabilizer group schemes. This follows for the first two by transitivity of fibre products and for the last one this follows because it is the pullback of the isomorphism $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$ (by the already used Morphisms of Stacks, Lemma 101.45.3). Recall that M, resp. M' was constructed in Lemma 106.13.4 as the spectrum of the ring of R-invariant functions on U, resp. the ring of R'-invariant functions on U'. Thus $M' \to M$ is étale and $U' = M' \times_M U$ by Groupoids, Lemma 39.23.7. It follows that $R' = M' \times_M U$, in other words the groupoid (U', R', s', t', c') is the base change of (U, R, s, t, c) by $M' \to M$. This implies that the diagram in the lemma is cartesian by Quotients of Groupoids, Lemma 83.3.6. ☐

0DUR Lemma 106.13.6. Let the algebraic stack $\mathcal{X}$ be well-nigh affine. The morphism

$$f : \mathcal{X} \longrightarrow M$$

of Lemma 106.13.4 is a uniform categorical moduli space.

Proof. We already know that M is a uniform categorical moduli space in the category of affine schemes. By Lemma 106.12.4 it suffices to show that the base change $f' : M' \times_M \mathcal{X} \to M'$ is a categorical moduli space for any flat morphism $M' \to M$ of affine schemes. Observe that $\mathcal{X}' = M' \times_M \mathcal{X}$ is well-nigh affine by Lemma 106.13.3. This after replacing $\mathcal{X}$ by $\mathcal{X}'$ and M by M', we reduce to proving f is a categorical moduli space.

Let $g : \mathcal{X} \to Y$ be a morphism where Y is an algebraic space. We have to show that $g = h \circ f$ for a unique morphism $h : M \to Y$.

Uniqueness. Suppose we have two morphisms $h_i : M \to Y$ with $g = h_1 \circ f = h_2 \circ f$. Let $M' \subset M$ be the equalizer of h_1 and h_2. Then $M' \to M$ is a monomorphism and $f : \mathcal{X} \to M$ factors through M'. Thus $M' \to M$ is a universal homeomorphism. We conclude M' is affine (Morphisms, Lemma 29.45.5). But then since $f : \mathcal{X} \to M$ is a categorical moduli space in the category of affine schemes, we see $M' = M$.

Existence. Below we will show that for every $p \in M$ there exists a cartesian square

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

with $M' \to M$ an étale morphism of affines and p in the image such that the composition $\mathcal{X}' \to \mathcal{X} \to Y$ factors through M'. This means we can construct the map $h : M \to Y$ étale locally on M. Since Y is a sheaf for the étale topology and by the uniqueness shown above, this is enough (small detail omitted).

Let $y \in |Y|$ be the image of p. Let $(V, v) \to (Y, y)$ be an étale morphism with V affine. Consider $\mathcal{X}' = V \times_Y \mathcal{X}$. Observe that $\mathcal{X}' \to \mathcal{X}$ is separated and étale as the base change of $V \to Y$. Moreover, $\mathcal{X}' \to \mathcal{X}$ induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5) as this is true for $V \to Y$, see Morphisms of Stacks, Lemma 101.45.5. Choose a presentation $\mathcal{X} = [U/R]$ as in Lemma 106.13.2. Set $U' = \mathcal{X}' \times_\mathcal{X} U = V \times_Y U$ and choose $u' \in U'$ mapping to p and v (possible by Properties of Spaces, Lemma 66.4.3). Since $U' \to U$ is separated and étale we see that every finite set of points of U' is contained in an affine open, see More on Morphisms, Lemma 37.45.1. On the other hand, the morphism $U' \to \mathcal{X}'$ is surjective, finite, flat, and locally of finite presentation. Setting $R' = U' \times_{\mathcal{X}'} U'$ we see that $s', t' : R' \to U'$ are finite locally free. By Groupoids, Lemma 39.24.1 there exists an R'-invariant affine open subscheme $U'' \subset U'$ containing u'. Let $\mathcal{X}'' \subset \mathcal{X}'$ be the corresponding open substack. Then $\mathcal{X}''$ is well-nigh affine. By Lemma 106.13.5 we obtain a cartesian square

$$\begin{tikzcd} \mathcal{X}'' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M'' \arrow[r] & M \end{tikzcd}$$

with $M'' \to M$ étale. Since $\mathcal{X}'' \to M''$ is a categorical moduli space in the category of affine schemes we obtain a morphism $M'' \to V$ such that the composition $\mathcal{X}'' \to \mathcal{X}' \to V$ is equal to the composition $\mathcal{X}'' \to M'' \to V$. This proves our claim and finishes the proof. $\square$

0DUS Lemma 106.13.7. Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{X}$ is well-nigh affine, h is étale, h is separated, and h induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5). Then there exists a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

where $M' \to M$ is a separated étale morphism of schemes and $\mathcal{X} \to M$ is the moduli space constructed in Lemma 106.13.4.

Proof. Choose an affine scheme U and a surjective, flat, finite, and locally finitely presented morphism $U \to \mathcal{X}$. Since h is representable by algebraic spaces (Morphisms of Stacks, Lemmas 101.45.3 and 101.45.1) we see that $U' = \mathcal{X}' \times_\mathcal{X} U$ is an algebraic space. Since $U' \to U$ is separated and étale, we see that U' is a scheme and that every finite set of points of U' is contained in an affine open, see Morphisms of Spaces, Lemma 67.51.1 and More on Morphisms, Lemma 37.45.1. Setting $R' = U' \times_{\mathcal{X}'} U'$ we see that $s', t' : R' \to U'$ are finite locally free. By Groupoids, Lemma 39.24.1 there exists an open covering $U' = \bigcup U'_i$ by R'-invariant affine open subschemes $U'_i \subset U'$. Let $\mathcal{X}'_i \subset \mathcal{X}'$ be the corresponding open substacks. These are well-nigh affine as $U'_i \to \mathcal{X}'_i$ is surjective, flat, finite and of finite presentation. By Lemma 106.13.5 we obtain cartesian diagrams

$$\begin{tikzcd} \mathcal{X}'_i \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M'_i \arrow[r] & M \end{tikzcd}$$

with $M'_i \to M$ an étale morphism of affine schemes and vertical arrows as in Lemma 106.13.4. Observe that $\mathcal{X}'_{ij} = \mathcal{X}'_i \cap \mathcal{X}'_j$ is an open subspace of $\mathcal{X}'_i$ and $\mathcal{X}'_j$. Hence we get corresponding open subschemes $V_{ij} \subset M'_i$ and $V_{ji} \subset M'_j$. By the result of Lemma 106.13.6 we see that both $\mathcal{X}'_{ij} \to V_{ij}$ and $\mathcal{X}'_{ji} \to V_{ji}$ are categorical moduli spaces! Thus we get a unique isomorphism $\varphi_{ij} : V_{ij} \to V_{ji}$ such that

$$\begin{tikzcd} \mathcal{X}'_i \arrow[d] & & \mathcal{X}'_i \cap \mathcal{X}'_j \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] & & \mathcal{X}'_j \arrow[d] \\ M'_i & V_{ij} \arrow[l] \arrow[rr, "\varphi_{ij}"] & & V_{ji} \arrow[r] & M'_j \end{tikzcd}$$

is commutative. These isomorphisms satisfy the cocyclce condition of Schemes, Section 26.14 by a computation (and another application of the previous lemma) which we omit. Thus we can glue the affine schemes in to scheme M', see Schemes, Lemma 26.14.1. Let us identify the M'_i with their image in M'. We claim there is a morphism $\mathcal{X}' \to M'$ fitting into cartesian diagrams

$$\begin{tikzcd} \mathcal{X}'_i \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ M'_i \arrow[r] & M' \end{tikzcd}$$

This is clear from the description of the morphisms into the glued scheme M' in Schemes, Lemma 26.14.1 and the fact that to give a morphism $\mathcal{X}' \to M'$ is the same thing as given a morphism $T \to M'$ for any morphism $T \to \mathcal{X}'$. Similarly, there is a morphism $M' \to M$ restricting to the given morphisms $M'_i \to M$ on M'_i. The morphism $M' \to M$ is étale (being étale on the members of an étale covering) and the fibre product property holds as it can be checked on members of the (affine) open covering $M' = \bigcup M'_i$. Finally, $M' \to M$ is separated because the composition $U' \to \mathcal{X}' \to M'$ is surjective and universally closed and we can apply Morphisms, Lemma 29.41.11. $\square$

0DUE Lemma 106.13.8. Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Then there exist a set I and for $i \in I$ a morphism of algebraic stacks

$$g_i : \mathcal{X}_i \longrightarrow \mathcal{X}$$

with the following properties

(1) $|\mathcal{X}| = \bigcup |g_i|(|\mathcal{X}_i|)$,
(2) $\mathcal{X}_i$ is well-nigh affine,
(3) $\mathcal{I}_{\mathcal{X}_i} \to \mathcal{X}_i \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism, and
(4) $g_i : \mathcal{X}_i \to \mathcal{X}$ is representable by algebraic spaces, separated, and étale,

Proof. For any $x \in |\mathcal{X}|$ we can choose $g : \mathcal{U} \to \mathcal{X}, \mathcal{U} = [U/R]$, and u as in Morphisms of Stacks, Lemma 101.32.4. Then by Morphisms of Stacks, Lemma 101.45.4 we see that there exists an open substack $\mathcal{U}' \subset \mathcal{U}$ containing u such that $\mathcal{I}_{\mathcal{U}'} \to \mathcal{U}' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism. Let $U' \subset U$ be the R-invariant open corresponding to the open substack $\mathcal{U}'$. Let $u' \in U'$ be a point of U' mapping to u. Observe that $t(s^{-1}(\{u'\}))$ is finite as $s : R \to U$ is finite. By Properties, Lemma 28.29.5 and Groupoids, Lemma 39.24.1 we can find an R-invariant affine open $U'' \subset U'$ containing u'. Let R'' be the restriction of R to U''. Then $\mathcal{U}'' = [U''/R'']$ is an open substack of $\mathcal{U}'$ containing u, is well-nigh affine, $\mathcal{I}_{\mathcal{U}''} \to \mathcal{U}'' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism, and $\mathcal{U}'' \to \mathcal{X}$

and is representable by algebraic spaces and étale. Finally, $\mathcal{U}'' \to \mathcal{X}$ is separated as $\mathcal{U}''$ is separated (Lemma 106.13.2) the diagonal of $\mathcal{X}$ is separated (Morphisms of Stacks, Lemma 101.6.1) and separatedness follows from Morphisms of Stacks, Lemma 101.4.12. Since the point $x \in |\mathcal{X}|$ is arbitrary the proof is complete. $\square$

0DUT Theorem 106.13.9 (Keel-Mori). Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Then there exists a uniform categorical moduli space

$$f : \mathcal{X} \longrightarrow M$$

and f is separated, quasi-compact, and a universal homeomorphism.

Proof. We choose a set I[4] and for $i \in I$ a morphism of algebraic stacks $g_i : \mathcal{X}_i \to \mathcal{X}$ as in Lemma 106.13.8; we will use all of the properties listed in this lemma without further mention. Let

$$f_i : \mathcal{X}_i \to M_i$$

be as in Lemma 106.13.4. Consider the stacks

$$\mathcal{X}_{ij} = \mathcal{X}_i \times_{g_i, \mathcal{X}, g_j} \mathcal{X}_j$$

for $i, j \in I$. The projections $\mathcal{X}_{ij} \to \mathcal{X}_i$ and $\mathcal{X}_{ij} \to \mathcal{X}_j$ are separated by Morphisms of Stacks, Lemma 101.4.4, étale by Morphisms of Stacks, Lemma 101.35.3, and induce isomorphisms on automorphism groups (as in Morphisms of Stacks, Remark 101.19.5) by Morphisms of Stacks, Lemma 101.45.5. Thus we may apply Lemma 106.13.7 to find a commutative diagram

$$\begin{tikzcd} \mathcal{X}_i \arrow[d, "f_i"] & \mathcal{X}_{ij} \arrow[l] \arrow[r] \arrow[d, "f_{ij}"] & \mathcal{X}_j \arrow[d, "f_j"] \\ M_i & M_{ij} \arrow[l] \arrow[r] & M_j \end{tikzcd}$$

with cartesian squares where $M_{ij} \to M_i$ and $M_{ij} \to M_j$ are separated étale morphisms of schemes; here we also use that f_i is a uniform categorical quotient by Lemma 106.13.6. Claim:

$$\coprod M_{ij} \longrightarrow \coprod M_i \times \coprod M_i$$

is an étale equivalence relation.

Proof of the claim. Set $R = \coprod M_{ij}$ and $U = \coprod M_i$. We have already seen that $t : R \to U$ and $s : R \to U$ are étale. Let us construct a morphism $c : R \times_{s,U,t} R \to R$ compatible with $\text{pr}_{13} : U \times U \times U \to U \times U$. Namely, for $i, j, k \in I$ we consider

$$\mathcal{X}_{ijk} = \mathcal{X}_i \times_{g_i, \mathcal{X}, g_j} \mathcal{X}_j \times_{g_j, \mathcal{X}, g_k} \mathcal{X}_k = \mathcal{X}_{ij} \times_{\mathcal{X}_j} \mathcal{X}_{jk}$$

Arguing exactly as in the previous paragraph, we find that $M_{ijk} = M_{ij} \times_{M_j} M_{jk}$ is a categorical moduli space for $\mathcal{X}_{ijk}$. In particular, there is a canonical morphism $M_{ijk} = M_{ij} \times_{M_j} M_{jk} \to M_{ik}$ coming from the projection $\mathcal{X}_{ijk} \to \mathcal{X}_{ik}$. Putting these morphisms together we obtain the morphism c. In a similar fashion we construct a morphism $e : U \to R$ compatible with $\Delta : U \to U \times U$ and $i : R \to R$ compatible with the flip $U \times U \to U \times U$. Let k be an algebraically closed field. Then

$$\text{Mor}(\text{Spec}(k), \mathcal{X}_i) \to \text{Mor}(\text{Spec}(k), M_i) = M_i(k)$$

[4]The reader who is still keeping track of set theoretic issues should make sure I is not too large.

is bijective on isomorphism classes and the same remains true after any base change by a morphism $M' \to M$. This follows from our choice of f_i and Morphisms of Stacks, Lemmas 101.14.5 and 101.14.6. By construction of 2-fibred products the diagram

$$\begin{tikzcd} \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_{ij}) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_j) \arrow[d] \\ \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_i) \arrow[r] & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \end{tikzcd}$$

is a fibre product of categories. By our choice of g_i the functors in this diagram induce bijections on automorphism groups. It follows that this diagram induces a fibre product diagram on sets of isomorphism classes! Thus we see that

$$R(k) = U(k) \times_{|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|} U(k)$$

where $|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|$ denotes the set of isomorphism classes. In particular, for any algebraically closed field k the map on k-valued point is an equivalence relation. We conclude the claim holds by Groupoids, Lemma 39.3.5.

Let $M = U/R$ be the algebraic space which is the quotient of the above étale equivalence relation, see Spaces, Theorem 65.10.5. There is a canonical morphism $f : \mathcal{X} \to M$ fitting into commutative diagrams

0DUU (106.13.9.1)
$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "g_i"'] \arrow[d, "f_i"'] & \mathcal{X} \arrow[d, "f"] \\ M_i \arrow[r] & M \end{tikzcd}$$

Namely, such a morphism f is given by a functor

$$f : \mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, M)$$

for any scheme T compatible with base change. Let $a : T \to \mathcal{X}$ be an object of the left hand side. We obtain an étale covering $\{T_i \to T\}$ with $T_i = \mathcal{X}_i \times_\mathcal{X} T$ and morphisms $a_i : T_i \to \mathcal{X}_i$. Then we get $b_i = f_i \circ a_i : T_i \to M_i$. Since $T_i \times_T T_j = \mathcal{X}_{ij} \times_\mathcal{X} T$ we moreover get a morphism $a_{ij} : T_i \times_T T_j \to \mathcal{X}_{ij}$. Setting $b_{ij} = f_{ij} \circ a_{ij}$ we find that $b_i \times b_j$ factors through the monomorphism $M_{ij} \to M_i \times M_j$. Hence the morphisms

$$T_i \xrightarrow{b_i} M_i \to M$$

agree on $T_i \times_T T_j$. As M is a sheaf for the étale topology, we see that these morphisms glue to a unique morphism $b = f(a) : T \to M$. We omit the verification that this construction is compatible with base change and we omit the verification that the diagrams (106.13.9.1) commute.

Claim: the diagrams (106.13.9.1) are cartesian. To see this we study the induced morphism

$$h_i : \mathcal{X}_i \longrightarrow M_i \times_M \mathcal{X}$$

This is a morphism of stacks étale over $\mathcal{X}$ and hence h_i is étale (Morphisms of Stacks, Lemma 101.35.6). Since g_i is separated, we see h_i is separated (use Morphisms of Stacks, Lemma 101.4.12 and the fact seen above that the diagonal of $\mathcal{X}$ is separated). The morphism h_i induces isomorphisms on automorphism groups

(Morphisms of Stacks, Remark 101.19.5) as this is true for g_i. For an algebraically closed field k the diagram

$$\begin{tikzcd} \mathrm{Mor}(\mathrm{Spec}(k), M_i \times_M \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \arrow[d] \\ M_i(k) \arrow[r] & M(k) \end{tikzcd}$$

is a catesian diagram of categories and the top arrow induces bijections on automorphism groups. On the other hand, we have

$$M(k) = U(k)/R(k) = U(k)/U(k) \times_{|\mathrm{Mor}(\mathrm{Spec}(k),\mathcal{X})|} U(k) = |\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|$$

by what we said above. Thus the right vertical arrow in the cartesian diagram above is a bijection on isomorphism classes. We conclude that $|\mathrm{Mor}(\mathrm{Spec}(k), M_i \times_M \mathcal{X})| \to M_i(k)$ is bijective. Review: h_i is a separated, étale, induces isomorphisms on automorphism groups (as in Morphisms of Stacks, Remark 101.19.5), and induces an equivalence on fibre categories over algebraically closed fields. Hence it is an isomorphism by Morphisms of Stacks, Lemma 101.45.7.

From the claim we get in particular the following: we have a surjective étale morphism $U \to M$ such that the base change of f is separated, quasi-compact, and a universal homeomorphism. It follows that f is separated, quasi-compact, and a universal homeomorphism. See Morphisms of Stacks, Lemma 101.4.5, 101.7.10, and 101.15.5

To finish the proof we have to show that $f : \mathcal{X} \to M$ is a uniform categorical moduli space. To prove this it suffices to show that given a flat morphism $M' \to M$ of algebraic spaces, the base change

$$M' \times_M \mathcal{X} \longrightarrow M'$$

is a categorical moduli space. Thus we consider a morphism

$$\theta : M' \times_M \mathcal{X} \longrightarrow E$$

where E is an algebraic space. For each i we know that f_i is a uniform categorical moduli space. Hence we obtain

$$\begin{tikzcd} M' \times_M \mathcal{X}_i \arrow[r] \arrow[d] & M' \times_M \mathcal{X} \arrow[d, "\theta"] \\ M' \times_M M_i \arrow[r, "\psi_i"] & E \end{tikzcd}$$

Since $\{M' \times_M M_i \to M'\}$ is an étale covering, to obtain the desired morphism $\psi : M' \to E$ it suffices to show that ψ_i and ψ_j agree over $M' \times_M M_i \times_M M_j = M' \times_M M_{ij}$. This follows easily from the fact that $f_{ij} : \mathcal{X}_{ij} = \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_j \to M_{ij}$ is a uniform categorical quotient; details omitted. Then finally one shows that ψ fits into the commutative diagram

$$\begin{tikzcd} M' \times_M \mathcal{X} \arrow[d] \arrow[dr, "\theta"] & \\ M' \arrow[r, "\psi"] & E \end{tikzcd}$$

because "$\{M' \times_M \mathcal{X}_i \to M' \times_M \mathcal{X}\}$ is an étale covering" and the morphisms ψ_i fit into the corresponding commutative diagrams by construction. This finishes the proof of the Keel-Mori theorem. $\square$

The following lemma emphasizes the étale local nature of the construction.

0DUV **Lemma 106.13.10.** Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume

(1) $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,
(2) h is étale, separated, and induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5).

Then there exists a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

where $M' \to M$ is a separated étale morphism of algebraic spaces and the vertical arrows are the moduli spaces constructed in Theorem 106.13.9.

Proof. By Morphisms of Stacks, Lemma 101.45.3 we see that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is finite as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$. Let $f' : \mathcal{X}' \to M'$ and $f : \mathcal{X} \to M$ be as in Theorem 106.13.9. We obtain a commutative diagram as in the lemma because f' is categorical moduli space. Choose I and $g_i' : \mathcal{X}_i' \to \mathcal{X}'$ as in Lemma 106.13.8. Observe that $g_i = h \circ g_i'$ is étale, separated, and induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5). Let $f_i' : \mathcal{X}_i' \to M_i'$ be as in Lemma 106.13.4. In the proof of Theorem 106.13.9 we have seen that the diagrams

$$\begin{tikzcd} \mathcal{X}_i' \arrow[r, "g_i'"'] \arrow[d, "f_i'"'] & \mathcal{X}' \arrow[d, "f'"] \\ M_i' \arrow[r] & M' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{X}_i' \arrow[r, "g_i"'] \arrow[d, "f_i'"'] & \mathcal{X} \arrow[d, "f"] \\ M_i' \arrow[r] & M \end{tikzcd}$$

are cartesian and that $M_i' \to M'$ and $M_i' \to M$ are étale (this also follows directly from the properties of the morphisms g_i', g_i, f', f_i', f listed sofar by arguing in exactly the same way). This first implies that $M' \to M$ is étale and second that the diagram in the lemma is cartesian. We still need to show that $M' \to M$ is separated. To do this we contemplate the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X}' \times_\mathcal{X} \mathcal{X}' \arrow[d] \\ M' \arrow[r] & M' \times_M M' \end{tikzcd}$$

The top horizontal arrow is universally closed as $\mathcal{X}' \to \mathcal{X}$ is separated. The vertical arrows are as in Theorem 106.13.9 (as flat base changes of $\mathcal{X} \to M$) hence universal homeomorphisms. Thus the lower horizontal arrow is universally closed. This (combined with it being an étale monomorphism of algebraic spaces) proves it is a closed immersion as desired. $\square$

106.14. Properties of moduli spaces

0DUW Once the existence of a moduli space has been proven, it becomes possible (and is usually straightforward) to esthablish properties of these moduli spaces.

0DUX **Lemma 106.14.1.** *Let $p : \mathcal{X} \to Y$ be a morphism of an algebraic stack to an algebraic space. Assume*

1. *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
2. *Y is locally Noetherian, and*
3. *p is locally of finite type.*

Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 106.13.9. Then $M \to Y$ is locally of finite type.

Proof. Since f is a uniform categorical moduli space we obtain the morphism $M \to Y$. It suffices to check that $M \to Y$ is locally of finite type étale locally on M and Y. Since f is a uniform categorical moduli space, we may first replace Y by an affine scheme étale over Y. Next, we may choose I and $g_i : \mathcal{X}_i \to \mathcal{X}$ as in Lemma 106.13.8. Then by Lemma 106.13.10 we reduce to the case $\mathcal{X} = \mathcal{X}_i$. In other words, we may assume $\mathcal{X}$ is well-nigh affine. In this case we have $Y = \mathrm{Spec}(A_0)$, we have $\mathcal{X} = [U/R]$ with $U = \mathrm{Spec}(A)$ and $M = \mathrm{Spec}(C)$ where $C \subset A$ is the set of R-invariant functions on U. See Lemmas 106.13.2 and 106.13.4. Then A_0 is Noetherian and $A_0 \to A$ is of finite type. Moreover A is integral over C by Groupoids, Lemma 39.23.4, hence finite over C (being of finite type over A_0). Thus we may finally apply Algebra, Lemma 10.51.7 to conclude. $\square$

0DUY **Lemma 106.14.2.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 106.13.9.*

1. *If $\mathcal{X}$ is quasi-separated, then M is quasi-separated.*
2. *If $\mathcal{X}$ is separated, then M is separated.*
3. *Add more here, for example relative versions of the above.*

Proof. To prove this consider the following diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta_\mathcal{X}"'] \arrow[d, "f"'] & \mathcal{X} \times \mathcal{X} \arrow[d, "f \times f"] \\ M \arrow[r, "\Delta_M"] & M \times M \end{tikzcd}$$

Since f is a universal homeomorphism, we see that $f \times f$ is a universal homeomorphism.

If $\mathcal{X}$ is separated, then $\Delta_\mathcal{X}$ is proper, hence $\Delta_\mathcal{X}$ is universally closed, hence Δ_M is universally closed, hence M is separated by Morphisms of Spaces, Lemma 67.40.9.

If $\mathcal{X}$ is quasi-separated, then $\Delta_\mathcal{X}$ is quasi-compact, hence Δ_M is quasi-compact, hence M is quasi-separated. $\square$

0DUZ **Lemma 106.14.3.** *Let $p : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume*

1. *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
2. *p is proper, and*
3. *Y is locally Noetherian.*

Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 106.13.9. Then $M \to Y$ is proper.

Proof. By Lemma 106.14.1 we see that $M \to Y$ is locally of finite type. By Lemma 106.14.2 we see that $M \to Y$ is separated. Of course $M \to Y$ is quasi-compact and universally closed as these are topological properties and $\mathcal{X} \to Y$ has these properties and $\mathcal{X} \to M$ is a universal homeomorphism. $\square$

106.15. Stacks and fpqc coverings

0GRG Certain algebraic stacks satisfy fpqc descent. The analogue of this section for algebraic spaces is Properties of Spaces, Section 66.17.

0GRH Proposition 106.15.1. Let $\mathcal{X}$ be an algebraic stack with quasi-affine[5] diagonal. Then $\mathcal{X}$ satisfies descent for fpqc coverings. Proposition 3.3.6 of "Intro to Algebraic Stacks" by Anatoly Preygel.

Proof. Our conventions are that $\mathcal{X}$ is a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ over the category of schemes over a base scheme S endowed with the fppf topology. The statement means the following: given an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of schemes over S the functor

$$\mathcal{X}_U \longrightarrow DD(\mathcal{U})$$

is an equivalence. Here on the left we have the category of objects of $\mathcal{X}$ over U and on the right we have the category of descent data in $\mathcal{X}$ relative to $\mathcal{U}$. See discussion in Stacks, Section 8.3.

Fully faithfulness. Suppose we have two objects x, y of $\mathcal{X}$ over U. Then $I = \mathit{Isom}(x, y)$ is an algebraic space over U. Hence a collection of sections of I over U_i whose restrictions to $U_i \times_U U_j$ agree, come from a unique section over U by the analogue of the proposition for algebraic spaces, see Properties of Spaces, Proposition 66.17.1. Thus our functor is fully faithful.

Essential surjectivity. Here we are given objects x_i over U_i and isomorphisms $\varphi_{ij} : \mathrm{pr}_0^* x_i \to \mathrm{pr}_1^* x_j$ over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$.

Let W be an affine scheme and let $W \to \mathcal{X}$ be a morphism. For each i we can form

$$W_i = U_i \times_{x_i, \mathcal{X}} W$$

The projection $W_i \to U_i$ is quasi-affine as the diagonal of $\mathcal{X}$ is quasi-affine. For each pair $i, j \in I$ the isomorphism φ_{ij} induces an isomorphism

$$W_i \times_U U_j = (U_i \times_U U_j) \times_{x_i \circ \mathrm{pr}_0, \mathcal{X}} W \to (U_i \times_U U_j) \times_{x_j \circ \mathrm{pr}_1, \mathcal{X}} W = U_i \times_U W_j$$

Moreover, these isomorphisms satisfy the cocycle condition over $U_i \times_U U_j \times_U U_k$. In other words, these isomorphisms define a descent datum on the schemes W_i/U_i relative to $\mathcal{U}$. By Descent, Lemma 35.38.1 we see that this descent datum is effective[6]. We conclude that there exists a quasi-affine morphism $W' \to U$ and a commutative diagram

$$\begin{tikzcd} W' \arrow[d] & W_i \arrow[l] \arrow[r] \arrow[d] & W \arrow[d] \\ U & U_i \arrow[l] \arrow[r, "x_i"] & \mathcal{X} \end{tikzcd}$$

[5]It suffices to assume ind-quasi-affine.

[6]Or use More on Groupoids, Lemma 40.15.3 in the case of ind-quasi-affine diagonal.

whose squares are cartesian. Since $\{W_i \to W'\}_{i \in I}$ is the base change of $\mathcal{U}$ by $W' \to U$ we conclude that it is an fpqc covering. Since W satisfies the sheaf condition for fpqc coverings, we obtain a unique morphism $W' \to W$ such that $W_i \to W' \to W$ is the given morphism $W_i \to W$. In other words, we have the commutative diagrams

$$\begin{tikzcd} W_i \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & W \arrow[d] \\ U_i \arrow[r] \arrow[rr, bend right, "x_i"'] & U & \mathcal{X} \end{tikzcd}$$

compatible with the isomorphisms φ_{ij} and whose square and rectangle are cartesian.

Choose a collection of affine schemes W_α, $\alpha \in A$ and smooth morphisms $W_\alpha \to \mathcal{X}$ such that $\coprod W_\alpha \to \mathcal{X}$ is surjective. By the procedure of the preceding paragraph we produce a diagram

$$\begin{tikzcd} W_{\alpha,i} \arrow[r] \arrow[d] & W'_\alpha \arrow[r] \arrow[d] & W_\alpha \arrow[d] \\ U_i \arrow[r] \arrow[rr, bend right, "x_i"'] & U & \mathcal{X} \end{tikzcd}$$

for each α. Then the morphisms $W'_\alpha \to U$ are smooth and jointly surjective.

Denote x_α the object of $\mathcal{X}$ over W'_α corresponding to $W'_\alpha \to W_\alpha \to \mathcal{X}$. Since $\mathcal{X}$ is an fppf stack and since $\{W'_\alpha \to U\}$ is an fppf covering, it suffices to show that there are isomorphisms $\text{pr}_0^*x_\alpha \to \text{pr}_1^*x_\beta$ over $W'_\alpha \times_U W'_\beta$ satisfying the cocycle condition. However, after pulling back to $W_{\alpha,i}$ we do have such isomorphisms over $W_{\alpha,i} \times_{U_i} W_{\beta,i} = U_i \times_U (W'_\alpha \times_U W'_\beta)$ since the pullback of x_α to $W_{\alpha,i}$ is isomorphic to the pullback of x_i to $W_{\alpha,i}$. Since $\{U_i \times_U (W'_\alpha \times_U W'_\beta) \to W'_\alpha \times_U W'_\beta\}_{i \in I}$ is an fpqc covering and by the aforementioned compatibility of the diagrams above with φ_{ij} these isomorphisms descend to $W'_\alpha \times_U W'_\beta$ and the proof is complete. $\square$

106.16. Tensor functors

0GRI Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of Noetherian algebraic stacks. The pullback functor

$$f^* : \text{Coh}(\mathcal{O}_\mathcal{X}) \longrightarrow \text{Coh}(\mathcal{O}_\mathcal{Y})$$

is a right exact tensor functor: it is additive, right exact, and commutes with tensor products of coherent modules. We can ask to what extent any right exact tensor functor $F : \text{Coh}(\mathcal{O}_\mathcal{X}) \to \text{Coh}(\mathcal{O}_\mathcal{Y})$ comes from a morphism $f : \mathcal{Y} \to \mathcal{X}$. The reader may consult [HR19] for a very general result of this nature. The aim of this section is to give a short proof of Theorem 106.16.8 as an introduction to these ideas.

We begin with some lemmas.

0GRJ **Lemma 106.16.1.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Noetherian algebraic stacks. Any right exact tensor functor $F : \text{Coh}(\mathcal{O}_\mathcal{X}) \to \text{Coh}(\mathcal{O}_\mathcal{Y})$ extends uniquely to a right exact tensor functor $F : QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$ commuting with all colimits.

Proof. The existence and uniqueness of the extension is a general fact, see Categories, Lemma 4.26.2. To see that the lemma applies observe that coherent modules on locally Noetherian algebraic stacks are by definition modules of finite presentation, see Cohomology of Stacks, Definition 103.17.2. Hence a coherent module on $\mathcal{X}$ is a categorically compact object of $QCoh(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Lemma 103.13.5. Finally, every quasi-coherent module is a filtered colimit of its coherent submodules by Cohomology of Stacks, Lemma 103.18.1.

Since F is additive, also the extension of F is additive (details omitted). Since F is a tensor functor and since colimits of modules commute with taking tensor products, also the extension of F is a tensor functor (details omitted).

In this paragraph we show the extension commutes with arbitrary direct sums. If $\mathcal{F} = \bigoplus_{j \in J} \mathcal{H}_j$ with $\mathcal{H}_j$ quasi-coherent, then $\mathcal{F} = \mathrm{colim}_{J' \subset J \text{ finite}} \bigoplus_{j \in J'} \mathcal{H}_j$. Denoting the extension of F also by F we obtain

$$\begin{aligned} F(\mathcal{F}) &= \mathrm{colim}_{J' \subset J \text{ finite}} F(\bigoplus\nolimits_{j \in J'} \mathcal{H}_j) \\ &= \mathrm{colim}_{J' \subset J \text{ finite}} \bigoplus\nolimits_{j \in J'} F(\mathcal{H}_j) \\ &= \bigoplus\nolimits_{j \in J} F(\mathcal{H}_j) \end{aligned}$$

Thus F commutes with arbitrary direct sums.

In this paragraph we show that the extension is right exact. Suppose $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Then we write $\mathcal{F}' = \bigcup \mathcal{F}'_i$ as the union of its coherent submodules (see reference given above). Denote $\mathcal{F}''_i \subset \mathcal{F}''$ the image of $\mathcal{F}'_i$ and denote $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \mathrm{Ker}(\mathcal{F}'_i \to \mathcal{F}''_i)$. Then it is clear that $\mathcal{F} = \bigcup \mathcal{F}_i$ and $\mathcal{F}'' = \bigcup \mathcal{F}''_i$ and that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{F}'_i \to \mathcal{F}''_i \to 0$$

Since the extension commutes with filtered colimits we have $F(\mathcal{F}) = \mathrm{colim}_{i \in I} F(\mathcal{F}_i)$, $F(\mathcal{F}') = \mathrm{colim}_{i \in I} F(\mathcal{F}'_i)$, and $F(\mathcal{F}'') = \mathrm{colim}_{i \in I} F(\mathcal{F}''_i)$. Since filtered colimits of sheaves of modules is exact we conclude that the extension of F is right exact.

The proof is finished as a right exact functor which commutes with all coproducts commutes with all colimits, see Categories, Lemma 4.14.12. ☐

0GRK Lemma 106.16.2. Let $\mathcal{X}$ be an algebraic stack with affine diagonal. Let B be a ring. Let $F : QCoh(\mathcal{O}_\mathcal{X}) \to \mathrm{Mod}_B$ be a right exact tensor functor which commutes with direct sums. Let $g : U \to \mathcal{X}$ be a morphism with $U = \mathrm{Spec}(A)$ affine. Then

(1) $C = F(g_{QCoh,*}\mathcal{O}_U)$ is a commutative B-algebra and
(2) there is a ring map $A \to C$

such that $F \circ g_{QCoh,*} : \mathrm{Mod}_A \to \mathrm{Mod}_B$ sends M to $M \otimes_A C$ seen as B-module.

Proof. We note that g is quasi-compact and quasi-separated, see Morphisms of Stacks, Lemma 101.7.8. In Cohomology of Stacks, Proposition 103.11.1 we have constructed the functor $g_{QCoh,*} : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_\mathcal{X})$. By Cohomology of Stacks, Remarks 103.11.3 and 103.10.6 we obtain a multiplication

$$\mu : g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \longrightarrow g_{QCoh,*}\mathcal{O}_U$$

which turns $g_{QCoh,*}\mathcal{O}_U$ into a commutative $\mathcal{O}_\mathcal{X}$-algebra. Hence $C = F(g_{QCoh,*}\mathcal{O}_U)$ is a commutative algebra object in Mod_B, in other words, C is a commutative B-algebra. Observe that we have a map $\kappa : A \to \mathrm{End}_{\mathcal{O}_\mathcal{X}}(g_{QCoh,*}\mathcal{O}_U)$ such that for any $a \in A$ the diagram

$$\begin{tikzcd} g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \arrow[r, "\mu"'] \arrow[d, "\kappa(r)\otimes 1"'] & g_{QCoh,*}\mathcal{O}_U \arrow[d, "\kappa(r)"] \\ g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \arrow[r, "\mu"] & g_{QCoh,*}\mathcal{O}_U \end{tikzcd}$$

commutes. It follows that we get a map $\kappa' = F(\kappa) : A \to \mathrm{End}_B(C)$ such that $\kappa'(a)(c)c' = \kappa'(a)(cc')$ and of course this means that $a \mapsto \kappa'(a)(1)$ is a ring map $A \to C$.

The morphism $g : U \to \mathcal{X}$ is affine, see Morphisms of Stacks, Lemma 101.9.4. Hence $g_{QCoh,*}$ is exact and commutes with direct sums by Cohomology of Stacks, Lemma 103.13.4. Thus $F \circ g_{QCoh,*} : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is a right exact functor which commutes with direct sums and which sends A to C. By Functors and Morphisms, Lemma 56.3.1 we see that the functor $F \circ g_{QCoh,*}$ sends an A-module M to $M \otimes_A C$ viewed as a B-module. $\square$

0GRL Lemma 106.16.3. Notation as in Lemma 106.16.2. Assume $\mathcal{X}$ is Noetherian and g is surjective and flat. Then $B \to C$ is universally injective.

Proof. Consider the natural map $1 : \mathcal{O}_\mathcal{X} \to g_{QCoh,*}\mathcal{O}_U$ in $QCoh(\mathcal{O}_\mathcal{X})$. Pulling back to U and using adjunction we find that the composition

$$\mathcal{O}_U = g^*\mathcal{O}_\mathcal{X} \xrightarrow{g^*1} g^*g_{QCoh,*}\mathcal{O}_U \to \mathcal{O}_U$$

is the identity in $QCoh(\mathcal{O}_U)$. Write $g_{QCoh,*}\mathcal{O}_U = \mathrm{colim}\, \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_\mathcal{X}$-modules, see Cohomology of Stacks, Lemma 103.18.1. For i large enough the map $1 : \mathcal{O}_\mathcal{X} \to g_{QCoh,*}\mathcal{O}_U$ factors through $\mathcal{F}_i$, see Cohomology of Stacks, Lemma 103.13.5. Say $s : \mathcal{O}_\mathcal{X} \to \mathcal{F}_i$ is the factorization. Then

$$\mathcal{O}_U \xrightarrow{g^*s} g^*\mathcal{F}_i \to g^*g_{QCoh,*}\mathcal{O}_U \to \mathcal{O}_U$$

is the identity. In other words, we see that s becomes the inclusion of a direct summand upon pullback to U. Set $\mathcal{F}_i^\vee = hom(\mathcal{F}_i, \mathcal{O}_\mathcal{X})$ with notation as in Cohomology of Stacks, Lemma 103.10.8. In particular there is an evaluation map $ev : \mathcal{F}_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_i^\vee \to \mathcal{O}_\mathcal{X}$. Evaluation at s defines a map $s^\vee : \mathcal{F}_i^\vee \to \mathcal{O}_\mathcal{X}$. Dual to the statement about s we see that $g^*(s^\vee)$ is surjective, see see Cohomology of Stacks, Section 103.12 for compatibility of hom and $\otimes$ with restriction to U. Since g is surjective and flat, we conclude that $s^\vee$ is surjective (see locus citatus). Since F is right exact, we conclude that $F(\mathcal{F}_i^\vee) \to F(\mathcal{O}_\mathcal{X}) = B$ is surjective. Choose $\lambda \in F(\mathcal{F}_i^\vee)$ mapping to $1 \in B$. Denote $e = F(s)(1) \in F(\mathcal{F}_i)$ the image of 1 by the map $F(s) : B = F(\mathcal{O}_\mathcal{X}) \to F(\mathcal{F}_i)$. Then the map

$$F(ev) : F(\mathcal{F}_i) \otimes_B F(\mathcal{F}_i^\vee) = F(\mathcal{F}_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_i^\vee) \longrightarrow F(\mathcal{O}_\mathcal{X}) = B$$

sends $e \otimes \lambda$ to 1 by construction. Hence the map $B \to F(\mathcal{F}_i)$, $b \mapsto be$ is universally injective because we have the one-sided inverse $F(\mathcal{F}_i) \to B$, $\xi \mapsto F(ev)(\xi \otimes \lambda)$. Since this is true for all i large enough we conclude. $\square$

0GRM Lemma 106.16.4. Let $B \to C$ be a ring map. If

(1) the coprojections $C \to C \otimes_B C$ are flat and
(2) $B \to C$ is universally injective,

then $B \to C$ is faithfully flat.

Proof. The map $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective as $B \to C$ is universally injective. Thus it suffices to show that $B \to C$ is flat which follows from Descent, Theorem 35.4.25. $\square$

The following very simple version of Künneth should become obsoleted when we write a section on Künneth theorems for cohomology of quasi-coherent modues on algebraic stacks.

0GRN Lemma 106.16.5. Let $a : \mathcal{Y} \to \mathcal{X}$ and $b : \mathcal{Z} \to \mathcal{X}$ be representable by schemes, quasi-compact, quasi-separated, and flat. Then $a_{QCoh,*}\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_{QCoh,*}\mathcal{O}_\mathcal{Z} = f_{QCoh,*}\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$ where $f : \mathcal{Y} \times_\mathcal{X} \mathcal{Z} \to \mathcal{X}$ is the obvious morphism.

Proof. We abbreviate $\mathcal{P} = \mathcal{Y} \times_\mathcal{X} \mathcal{Z}$. Since $a \circ \mathrm{pr}_1 = f$ and $b \circ \mathrm{pr}_2 = f$ we obtain maps $a_*\mathcal{O}_\mathcal{Y} \to f_*\mathcal{O}_\mathcal{P}$ and $b_*\mathcal{O}_\mathcal{Z} \to f_*\mathcal{O}_\mathcal{P}$ (using relative pullback maps, see Sites, Section 7.45). Hence we obtain a relative cup product

$$\mu : a_*\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_*\mathcal{O}_\mathcal{Z} \longrightarrow f_*\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$$

Applying Q and its compatibility with tensor products (Cohomology of Stacks, Remark 103.10.6) we obtain an arrow $Q(\mu) : a_{QCoh,*}\mathcal{O}_\mathcal{Y}\otimes_{\mathcal{O}_\mathcal{X}} b_{QCoh,*}\mathcal{O}_\mathcal{Z} \to f_{QCoh,*}\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$ in $QCoh(\mathcal{O}_\mathcal{X})$. Next, choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. It suffices to prove the restriction of $Q(\mu)$ to $U_{étale}$ is an isomorphism, see Cohomology of Stacks, Section 103.12. In turn, by the material in the same section, it suffices to prove the restriction of μ to $U_{étale}$ is an isomorphism (this uses that the source and target of μ are locally quasi-coherent modules with the base change property). Moreover, we may compute pushforwards in the étale topology, see Cohomology of Stacks, Proposition 103.8.1. Then finally, we see that $a_*\mathcal{O}_\mathcal{Y}|_{U_{étale}} = (V \to U)_{small,*}\mathcal{O}_V$ where $V = U \times_\mathcal{X} \mathcal{Y}$. Similarly for b_* and f_*. Thus the result follows from the Künneth formula for flat, quasi-compact, quasi-separated morphisms of schemes, see Derived Categories of Schemes, Lemma 36.23.1. $\square$

0GRP Lemma 106.16.6. Let $\mathcal{X}$ be an algebraic stack with affine diagonal. Let B be a ring. Let $f_i : \mathrm{Spec}(B) \to \mathcal{X}$, $i = 1, 2$ be two morphisms. Let $t : f_1^* \to f_2^*$ be an isomorphism of the tensor functors $f_i^* : QCoh(\mathcal{O}_\mathcal{X}) \to \mathrm{Mod}_B$. Then there is a 2-arrow $f_1 \to f_2$ inducing t.

Proof. Choose an affine scheme $U = \mathrm{Spec}(A)$ and a surjective smooth morphism $g : U \to \mathcal{X}$, see Properties of Stacks, Lemma 100.6.2. Since the diagonal of $\mathcal{X}$ is affine, we see that $U_i = \mathrm{Spec}(B) \times_{f_i,\mathcal{X},g} U$ is affine. Say $U_i = \mathrm{Spec}(C_i)$. Then C_i is the B-algebra endowed with ring map $A \to C_i$ constructed in Lemma 106.16.2 using the functor $F = f_i^*$. Therefore t induces an isomorphism $C_1 \to C_2$ of B-algebras, compatible with the ring maps $A \to C_1$ and $A \to C_2$. In other words, we have a commutative diagrams

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & U \arrow[d, "g"] \\ \mathrm{Spec}(B) \arrow[r, "f_i"] & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} & U_2 \arrow[dl] \arrow[d, "\cong"] \arrow[dr] & \\ \mathrm{Spec}(B) & U_1 \arrow[l] \arrow[r] & U \end{tikzcd}$$

This already shows that the objects f_1 and f_2 of $\mathcal{X}$ over $\mathrm{Spec}(B)$ become isomorphic after the smooth covering $\{U_1 \to \mathrm{Spec}(B)\}$. To show that this descends to an isomorphism of f_1 and f_2 over $\mathrm{Spec}(B)$, we have to show that our isomorphism (which comes from the commutative diagrams above) is compatible with the descent data over $U_1 \times_{\mathrm{Spec}(B)} U_1$. For this we observe that $U \times_\mathcal{X} U$ is affine too, that we have the morphism $g' : U \times_\mathcal{X} U \to \mathcal{X}$, and that

$$U_i \times_{\mathrm{Spec}(B)} U_i = \mathrm{Spec}(B) \times_{f_i, \mathcal{X}, g'} (U \times_\mathcal{X} U)$$

It follows that the isomorphism $C_1 \otimes_B C_1 \to C_2 \otimes_B C_2$ coming from the isomorphism $C_1 \to C_2$ is compatible with the morphisms $U_i \times_{\mathrm{Spec}(B)} U_i \to U \times_\mathcal{X} U$. Some details omitted. $\square$

0GRQ Lemma 106.16.7. Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal. Let B be a ring. Let $F : QCoh(\mathcal{O}_\mathcal{X}) \to \mathrm{Mod}_B$ be a right exact tensor functor which commutes with direct sums. Then F comes from a unique morphism $\mathrm{Spec}(B) \to \mathcal{X}$.

Proof. Choose a surjective smooth morphism $g : U \to \mathcal{X}$ with $U = \mathrm{Spec}(A)$ affine, see Properties of Stacks, Lemma 100.6.2. Apply Lemma 106.16.2 to get the finite type commutative B-algebra $C = F(g_{QCoh,*}\mathcal{O}_U)$ and the ring map $A \to C$. By Lemma 106.16.3 the ring map $B \to C$ is universally injective. Consider the algebra

$$C \otimes_B C = F(g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U)$$

Since g is flat, quasi-compact, and quasi-separated by Lemma 106.16.5 we have the first equality in

$$g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U = f_{QCoh,*}\mathcal{O}_{U \times_\mathcal{X} U} = g_{QCoh,*}(\mathrm{pr}_{2,*}\mathcal{O}_{U \times_\mathcal{X} U})$$

where $f : U \times_\mathcal{X} U \to \mathcal{X}$ is the obvious morphism and $\mathrm{pr}_2 : U \times_\mathcal{X} U \to U$ is the second projection. The second equality follows from Cohomology of Stacks, Lemma 103.11.5 and $f = g \circ \mathrm{pr}_2$. Since the diagonal of $\mathcal{X}$ is affine, we see that $U \times_\mathcal{X} U = \mathrm{Spec}(R)$ is affine. Let us use $\mathrm{pr}_2 : A \to R$ to view R as an A-algebra. All in all we obtain

$$C \otimes_B C = F(g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U) = F(g_{QCoh,*}(\mathrm{pr}_{2,*}\mathcal{O}_{U \times_\mathcal{X} U})) = R \otimes_A C$$

where the final equality follows from the final statement of Lemma 106.16.2. Since $A \to R$ is flat (because pr_2 is flat as a base change of $U \to \mathcal{X}$), we conclude that $C \otimes_B C$ is flat over C. By Lemma 106.16.4 we conclude that $B \to C$ is faithfully flat.

We claim there is a solid commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(C \otimes_B C) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & U \times_\mathcal{X} U \arrow[d, shift right] \arrow[d, shift left] \\
\mathrm{Spec}(C) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(B) \arrow[r, dotted] & \mathcal{X}
\end{tikzcd}$$

The arrow $\mathrm{Spec}(C) \to U = \mathrm{Spec}(A)$ comes from the ring map $A \to C$ in the statement of Lemma 106.16.2. The arrow $\mathrm{Spec}(C \otimes_B C) \to U \times_\mathcal{X} U$ simlarly comes from the ring map $R \to C \otimes_B C$. To verify the top square commutes use Lemma 106.16.6; details omitted. We conclude we get the dotted arrow $\mathrm{Spec}(B) \to \mathcal{X}$ by Proposition 106.15.1.

The statement that F is the functor corresponding to pullback by the dotted arrow is also clear from this and the corresponding statement in Lemma 106.16.2. Details omitted. $\square$

For a ring B let us denote Mod_B^{fg} the category of finitely generated B-modules (AKA finite B-modules).

0GRR Theorem 106.16.8. Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal. Let B be a Noetherian ring. Let $F : \text{Coh}(\mathcal{O}_\mathcal{X}) \to \text{Mod}_B^{fg}$ be a right exact tensor functor. Then F comes from a unique morphism $\text{Spec}(B) \to \mathcal{X}$.

Proof. By Lemma 106.16.1 we can extend F uniquely to a right exact tensor functor $F : QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}_B$ commuting with all direct susms. Then we can apply Lemma 106.16.7. $\square$

106.17. Other chapters

CHAPTER 107

The Geometry of Algebraic Stacks

0DQR 107.1. Introduction

0DQS This chapter discusses a few geometric properties of algebraic stacks. The initial versions of Sections 107.3 and 107.5 were written by Matthew Emerton and Toby Gee and can be found in their original form in [EG17].

107.2. Versal rings

0DQT In this section we elucidate the relationship between deformation rings and local rings on algebraic stacks of finite type over a locally Noetherian base.

0DQU Situation 107.2.1. Here $\mathcal{X}$ is an algebraic stack locally of finite type over a locally Noetherian scheme S.

Here is the definition.

0DQV Definition 107.2.2. In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. A versal ring to $\mathcal{X}$ at x_0 is a complete Noetherian local S-algebra A with residue field k such that there exists a versal formal object (A, ξ_n, f_n) as in Artin's Axioms, Definition 98.12.1 with $\xi_1 \cong x_0$ (a 2-isomorphism).

We want to prove that versal rings exist and are unique up to smooth factors. To do this, we will use the predeformation categories of Artin's Axioms, Section 98.3. These are always deformation categories in our situation.

0DQW Lemma 107.2.3. In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Then $\mathcal{F}_{\mathcal{X},k,x_0}$ is a deformation category and $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional k-vector spaces.

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \to S$ factors through it. By Artin's Axioms, Section 98.3 we obtain a predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ over the category $\mathcal{C}_\Lambda$. (As pointed out in locus citatus this category only depends on the morphism $\mathrm{Spec}(k) \to S$ and not on the choice of Λ.) By Artin's Axioms, Lemmas 98.6.1 and 98.5.2 $\mathcal{F}_{\mathcal{X},k,x_0}$ is actually a deformation category. By Artin's Axioms, Lemma 98.8.1 we find that $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional k-vector spaces. ☐

0DQX Lemma 107.2.4. In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Then a versal ring to $\mathcal{X}$ at x_0 exists. Given a pair A, A' of these, then $A \cong A'[[t_1, \ldots, t_r]]$ or $A' \cong A[[t_1, \ldots, t_r]]$ as S-algebras for some r.

Proof. By Lemma 107.2.3 and Formal Deformation Theory, Lemma 90.13.4 (note that the assumptions of this lemma hold by Formal Deformation Theory, Lemmas

90.16.6 and Definition 90.16.8). By the uniquness result of Formal Deformation Theory, Lemma 90.14.5 there exists a "minimal" versal ring A of $\mathcal{X}$ at x_0 such that any other versal ring of $\mathcal{X}$ at x_0 is isomorphic to $A[[t_1, \ldots, t_r]]$ for some r. This clearly implies the second statement. $\square$

0DQY Lemma 107.2.5. In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let l/k be a finite extension of fields and denote $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ the induced morphism. Given a versal ring A to $\mathcal{X}$ at x_0 there exists a versal ring A' to $\mathcal{X}$ at $x_{l,0}$ such that there is a S-algebra map $A \to A'$ which induces the given field extension l/k and is formally smooth in the $\mathfrak{m}_{A'}$-adic topology.

Proof. Follows immediately from Artin's Axioms, Lemma 98.7.1 and Formal Deformation Theory, Lemma 90.29.6. (We also use that $\mathcal{X}$ satisfies (RS) by Artin's Axioms, Lemma 98.5.2.) $\square$

0DQZ Lemma 107.2.6. In Situation 107.2.1 let $x : U \to \mathcal{X}$ be a morphism where U is a scheme locally of finite type over S. Let $u_0 \in U$ be a finite type point. Set $k = \kappa(u_0)$ and denote $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ the induced map. The following are equivalent

(1) x is versal at u_0 (Artin's Axioms, Definition 98.12.2),
(2) $\hat{x} : \mathcal{F}_{U,k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth,
(3) the formal object associated to $x|_{\mathrm{Spec}(\mathcal{O}^\wedge_{U,u_0})}$ is versal, and
(4) there is an open neighbourhood $U' \subset U$ of x such that $x|_{U'} : U' \to \mathcal{X}$ is smooth.

Moreover, in this case the completion $\mathcal{O}^\wedge_{U,u_0}$ is a versal ring to $\mathcal{X}$ at x_0.

Proof. Since $U \to S$ is locally of finite type (as a composition of such morphisms), we see that $\mathrm{Spec}(k) \to S$ is of finite type (again as a composition). Thus the statement makes sense. The equivalence of (1) and (2) is the definition of x being versal at u_0. The equivalence of (1) and (3) is Artin's Axioms, Lemma 98.12.3. Thus (1), (2), and (3) are equivalent.

If $x|_{U'}$ is smooth, then the functor $\hat{x} : \mathcal{F}_{U,k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth by Artin's Axioms, Lemma 98.3.2. Thus (4) implies (1), (2), and (3). For the converse, assume x is versal at u_0. Choose a surjective smooth morphism $y : V \to \mathcal{X}$ where V is a scheme. Set $Z = V \times_{\mathcal{X}} U$ and pick a finite type point $z_0 \in |Z|$ lying over u_0 (this is possible by Morphisms of Spaces, Lemma 67.25.5). By Artin's Axioms, Lemma 98.12.6 the morphism $Z \to V$ is smooth at z_0. By definition we can find an open neighbourhood $W \subset Z$ of z_0 such that $W \to V$ is smooth. Since $Z \to U$ is open, let $U' \subset U$ be the image of W. Then we see that $U' \to \mathcal{X}$ is smooth by our definition of smooth morphisms of stacks.

The final statement follows from the definitions as $\mathcal{O}^\wedge_{U,u_0}$ prorepresents $\mathcal{F}_{U,k,u_0}$. $\square$

0DZS Lemma 107.2.7. In Situation 107.2.1. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism such that $\mathrm{Spec}(k) \to S$ is of finite type with image s. Let A be a versal ring to $\mathcal{X}$ at x_0. The following are equivalent

(1) x_0 is in the smooth locus of $\mathcal{X} \to S$ (Morphisms of Stacks, Lemma 101.33.6),
(2) $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology, and
(3) $\mathcal{F}_{\mathcal{X},k,x_0}$ is unobstructed.

Proof. The equivalence of (2) and (3) follows immediately from Formal Deformation Theory, Lemma 90.9.4.

Note that $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology if and only if $\mathcal{O}_{S,s} \to A' = A[[t_1, \ldots, t_r]]$ is formally smooth in the $\mathfrak{m}_{A'}$-adic topology. Hence (2) does not depend on the choice of our versal ring by Lemma 107.2.4. Next, let l/k be a finite extension and choose $A \to A'$ as in Lemma 107.2.5. If $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology, then $\mathcal{O}_{S,s} \to A'$ is formally smooth in the $\mathfrak{m}_{A'}$-adic topology, see More on Algebra, Lemma 15.37.7. Conversely, if $\mathcal{O}_{S,s} \to A'$ is formally smooth in the $\mathfrak{m}_{A'}$-adic topology, then $\mathcal{O}_{S,s}^\wedge \to A'$ and $A \to A'$ are regular (More on Algebra, Proposition 15.49.2) and hence $\mathcal{O}_{S,s}^\wedge \to A$ is regular (More on Algebra, Lemma 15.41.7), hence $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology (same lemma as before). Thus the equivalence of (2) and (1) holds for k and x_0 if and only if it holds for l and $x_{0,l}$.

Choose a scheme U and a smooth morphism $U \to \mathcal{X}$ such that $\mathrm{Spec}(k) \times_{\mathcal{X}} U$ is nonempty. Choose a finite extension l/k and a point $w_0 : \mathrm{Spec}(l) \to \mathrm{Spec}(k) \times_{\mathcal{X}} U$. Let $u_0 \in U$ be the image of w_0. We may apply the above to l/k and to $l/\kappa(u_0)$ to see that we can reduce to u_0. Thus we may assume $A = \mathcal{O}_{U,u_0}^\wedge$, see Lemma 107.2.6. Observe that x_0 is in the smooth locus of $\mathcal{X} \to S$ if and only if u_0 is in the smooth locus of $U \to S$, see for example Morphisms of Stacks, Lemma 101.33.6. Thus the equivalence of (1) and (2) follows from More on Algebra, Lemma 15.38.6. □

We recall a consequence of Artin approximation.

0DR0 Lemma 107.2.8. In Situation 107.2.1. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism such that $\mathrm{Spec}(k) \to S$ is of finite type with image s. Let A be a versal ring to $\mathcal{X}$ at x_0. If $\mathcal{O}_{S,s}$ is a G-ring, then we may find a smooth morphism $U \to \mathcal{X}$ whose source is a scheme and a point $u_0 \in U$ with residue field k, such that

(1) $\mathrm{Spec}(k) \to U \to \mathcal{X}$ coincides with the given morphism x_0,
(2) there is an isomorphism $\mathcal{O}_{U,u_0}^\wedge \cong A$.

Proof. Let (ξ_n, f_n) be the versal formal object over A. By Artin's Axioms, Lemma 98.9.5 we know that $\xi = (A, \xi_n, f_n)$ is effective. By assumption $\mathcal{X}$ is locally of finite presentation over S (use Morphisms of Stacks, Lemma 101.27.5), and hence limit preserving by Limits of Stacks, Proposition 102.3.8. Thus Artin approximation as in Artin's Axioms, Lemma 98.12.7 shows that we may find a morphism $U \to \mathcal{X}$ with source a finite type S-scheme, containing a point $u_0 \in U$ of residue field k satisfying (1) and (2) such that $U \to \mathcal{X}$ is versal at u_0. By Lemma 107.2.6 after shrinking U we may assume $U \to \mathcal{X}$ is smooth. □

0DR1 Remark 107.2.9 (Upgrading versal rings). In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let A be a versal ring to $\mathcal{X}$ at x_0. By Artin's Axioms, Lemma 98.9.5 our versal formal object in fact comes from a morphism

$$\mathrm{Spec}(A) \longrightarrow \mathcal{X}$$

over S. Moreover, the results above each can be upgraded to be compatible with this morphism. Here is a list:

(1) in Lemma 107.2.4 the isomorphism $A \cong A'[[t_1, \ldots, t_r]]$ or $A' \cong A[[t_1, \ldots, t_r]]$ may be chosen compatible with these morphisms,

(2) in Lemma 107.2.5 the homomorphism $A \to A'$ may be chosen compatible with these morphisms,
(3) in Lemma 107.2.6 the morphism $\mathrm{Spec}(\mathcal{O}_{U,u_0}^\wedge) \to \mathcal{X}$ is the composition of the canonical map $\mathrm{Spec}(\mathcal{O}_{U,u_0}^\wedge) \to U$ and the given map $U \to \mathcal{X}$,
(4) in Lemma 107.2.8 the isomorphism $\mathcal{O}_{U,u_0}^\wedge \cong A$ may be chosen so $\mathrm{Spec}(A) \to \mathcal{X}$ corresponds to the canonical map in the item above.

In each case the statement follows from the fact that our maps are compatible with versal formal elements; we note however that the implied diagrams are 2-commutative only up to a (noncanonical) choice of a 2-arrow. Still, this means that the implied map $A' \to A$ or $A \to A'$ in (1) is well defined up to formal homotopy, see Formal Deformation Theory, Lemma 90.28.3.

0DR2 Lemma 107.2.10. In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let A be a versal ring to $\mathcal{X}$ at x_0. Then the morphism $\mathrm{Spec}(A) \to \mathcal{X}$ of Remark 107.2.9 is flat.

Proof. If the local ring of S at the image point is a G-ring, then this follows immediately from Lemma 107.2.8 and the fact that the map from a Noetherian local ring to its completion is flat. In general we prove it as follows.

Step I. If A and A' are two versal rings to $\mathcal{X}$ at x_0, then the result is true for A if and only if it is true for A'. Namely, after possible swapping A and A', we may assume there is a formally smooth map $\varphi : A \to A'$ such that the composition

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to \mathcal{X}$$

is the morphism $\mathrm{Spec}(A') \to \mathcal{X}$, see Lemma 107.2.4 and Remark 107.2.9. Since $A \to A'$ is faithfully flat we obtain the equivalence from Morphisms of Stacks, Lemmas 101.25.2 and 101.25.5.

Step II. Let l/k be a finite extension of fields. Let $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ be the induced morphism. Let A be a versal ring to $\mathcal{X}$ at x_0 and let $A \to A'$ be as in Lemma 107.2.5. Then again the composition

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to \mathcal{X}$$

is the morphism $\mathrm{Spec}(A') \to \mathcal{X}$, see Remark 107.2.9. Arguing as before and using step I to see choice of versal rings is irrelevant, we see that the lemma holds for x_0 if and only if it holds for $x_{l,0}$.

Step III. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then we can choose a finite type point z_0 on $Z = U \times_{\mathcal{X}} x_0$ (this is a nonempty algebraic space). Let $u_0 \in U$ be the image of z_0 in U. Choose a scheme and a surjective étale map $W \to Z$ such that z_0 is the image of a closed point $w_0 \in W$ (see Morphisms of Spaces, Section 67.25). Since $W \to \mathrm{Spec}(k)$ and $W \to U$ are of finite type, we see that $\kappa(w_0)/k$ and $\kappa(w_0)/\kappa(u_0)$ are finite extensions of fields (see Morphisms, Section 29.16). Applying Step II twice we may replace x_0 by $u_0 \to U \to \mathcal{X}$. Then we see our morphism is the composition

$$\mathrm{Spec}(\mathcal{O}_{U,u_0}^\wedge) \to U \to \mathcal{X}$$

The first arrow is flat because completion of Noetherian local rings are flat (Algebra, Lemma 10.97.2) and the second arrow is flat as a smooth morphism is flat. The composition is flat as composition preserves flatness. □

0DR3 Remark 107.2.11. In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. By Lemma 107.2.3 and Formal Deformation Theory, Theorem 90.26.4 we know that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Unwinding the definitions, this means we can choose

(1) a Noetherian complete local Λ-algebra A with residue field k and a versal formal object ξ of $\mathcal{F}_{\mathcal{X},k,x_0}$ over A,
(2) a Noetherian complete local Λ-algebra B with residue field k and an isomorphism

$$\underline{B}|_{\mathcal{C}_\Lambda} \longrightarrow \underline{A}|_{\mathcal{C}_\Lambda} \times_{\underline{\xi}, \mathcal{F}_{\mathcal{X},k,x_0}, \underline{\xi}} \underline{A}|_{\mathcal{C}_\Lambda}$$

The projections correspond to formally smooth maps $t : A \to B$ and $s : A \to B$ (because ξ is versal). There is a map $c : B \to B\widehat{\otimes}_{s,A,t}B$ which turns (A, B, s, t, c) into a cogroupoid in the category of Noetherian complete local Λ-algebras with residue field k (on prorepresentable functors this map is constructed in Formal Deformation Theory, Lemma 90.25.2). Finally, the cited theorem tells us that ξ induces an equivalence

$$[\underline{A}|_{\mathcal{C}_\Lambda}/\underline{B}|_{\mathcal{C}_\Lambda}] \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0}$$

of groupoids cofibred over $\mathcal{C}_\Lambda$. In fact, we also get an equivalence

$$[\underline{A}/\underline{B}] \longrightarrow \widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$$

of groupoids cofibred over the completed category $\widehat{\mathcal{C}}_\Lambda$ (see discussion in Formal Deformation Theory, Section 90.22 as to why this works). Of course A is a versal ring to $\mathcal{X}$ at x_0.

107.3. Multiplicities of components of algebraic stacks

0DR4 If X is a locally Noetherian scheme, then we may write X (thought of simply as a topological space) as a union of irreducible components, say $X = \bigcup T_i$. Each irreducible component is the closure of a unique generic point ξ_i, and the local ring $\mathcal{O}_{X,\xi_i}$ is a local Artin ring. We may define the multiplicity of X along T_i or the multiplicity of T_i in X by

$$m_{T_i,X} = \mathrm{length}_{\mathcal{O}_{X,\xi_i}} \mathcal{O}_{X,\xi_i}$$

In other words, it is the length of the local Artinian ring. Please compare with Chow Homology, Section 42.9.

Our goal here is to generalise this definition to locally Noetherian algebraic stacks. If $\mathcal{X}$ is a stack, then its topological space $|\mathcal{X}|$ (see Properties of Stacks, Definition 100.4.8) is locally Noetherian (Morphisms of Stacks, Lemma 101.8.3). The irreducible components of $|\mathcal{X}|$ are sometimes referred to as the irreducible components of $\mathcal{X}$. If $\mathcal{X}$ is quasi-separated, then $|\mathcal{X}|$ is sober (Morphisms of Stacks, Lemma 101.30.3), but it need not be in the non-quasi-separated case. Consider for example the non-quasi-separated algebraic space $X = \mathbf{A}^1_{\mathbf{C}}/\mathbf{Z}$. Furthermore, there is no structure sheaf on $|\mathcal{X}|$ whose stalks can be used to define multiplicities.

0DR5 Lemma 107.3.1. Let $f : U \to \mathcal{X}$ be a smooth morphism from a scheme to a locally Noetherian algebraic stack. The closure of the image of any irreducible component of $|U|$ is an irreducible component of $|\mathcal{X}|$. If $U \to \mathcal{X}$ is surjective, then all irreducible components of $|\mathcal{X}|$ are obtained in this way.

Proof. The map $|U| \to |\mathcal{X}|$ is continuous and open by Properties of Stacks, Lemma 100.4.7. Let $T \subset |U|$ be an irreducible component. Since U is locally Noetherian, we can find a nonempty affine open $W \subset U$ contained in T. Then $f(T) \subset |\mathcal{X}|$ is irreducible and contains the nonempty open subset $f(W)$. Thus the closure of $f(T)$ is irreducible and contains a nonempty open. It follows that this closure is an irreducible component.

Assume $U \to \mathcal{X}$ is surjective and let $Z \subset |\mathcal{X}|$ be an irreducible component. Choose a Noetherian open subset V of $|\mathcal{X}|$ meeting Z. After removing the other irreducible components from V we may assume that $V \subset Z$. Take an irreducible component of the nonempty open $f^{-1}(V) \subset |U|$ and let $T \subset |U|$ be its closure. This is an irreducible component of $|U|$ and the closure of $f(T)$ must agree with Z by our choice of T. $\square$

The preceding lemma applies in particular in the case of smooth morphisms between locally Noetherian schemes. This particular case is implicitly invoked in the statement of the following lemma.

0DR6 **Lemma 107.3.2.** Let $U \to X$ be a smooth morphism of locally Noetherian schemes. Let T' is an irreducible component of U. Let T be the irreducible component of X obtained as the closure of the image of T'. Then $m_{T',U} = m_{T,X}$.

Proof. Write ξ' for the generic point of T', and ξ for the generic point of T. Let $A = \mathcal{O}_{X,\xi}$ and $B = \mathcal{O}_{U,\xi'}$. We need to show that $\text{length}_A A = \text{length}_B B$. Since $A \to B$ is a flat local homomorphism of rings (since smooth morphisms are flat), we have

$$\text{length}_A(A)\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B)$$

by Algebra, Lemma 10.52.13. Thus it suffices to show $\mathfrak{m}_A B = \mathfrak{m}_B$, or equivalently, that $B/\mathfrak{m}_A B$ is reduced. Since $U \to X$ is smooth, so is its base change $U_\xi \to \operatorname{Spec} \kappa(\xi)$. As U_ξ is a smooth scheme over a field, it is reduced, and thus so its local ring at any point (Varieties, Lemma 33.25.4). In particular,

$$B/\mathfrak{m}_A B = \mathcal{O}_{U,\xi'}/\mathfrak{m}_{X,\xi}\mathcal{O}_{U,\xi'} = \mathcal{O}_{U_\xi,\xi'}$$

is reduced, as required. $\square$

Using this result, we may show that there exists a good notion of multiplicity by looking smooth locally.

0DR7 **Lemma 107.3.3.** Let $U_1 \to \mathcal{X}$ and $U_2 \to \mathcal{X}$ be two smooth morphisms from schemes to a locally Noetherian algebraic stack $\mathcal{X}$. Let T_1' and T_2' be irreducible components of $|U_1|$ and $|U_2|$ respectively. Assume the closures of the images of T_1' and T_2' are the same irreducible component T of $|\mathcal{X}|$. Then $m_{T_1',U_1} = m_{T_2',U_2}$.

Proof. Let V_1 and V_2 be dense subsets of T_1' and T_2', respectively, that are open in U_1 and U_2 respectively (see proof of Lemma 107.3.1). The images of $|V_1|$ and $|V_2|$ in $|\mathcal{X}|$ are non-empty open subsets of the irreducible subset T, and therefore have non-empty intersection. By Properties of Stacks, Lemma 100.4.3, the map $|V_1 \times_\mathcal{X} V_2| \to |V_1| \times_{|\mathcal{X}|} |V_2|$ is surjective. Consequently $V_1 \times_\mathcal{X} V_2$ is a non-empty algebraic space; we may therefore choose an étale surjection $V \to V_1 \times_\mathcal{X} V_2$ whose source is a (non-empty) scheme. If we let T' be any irreducible component of V, then Lemma 107.3.1 shows that the closure of the image of T' in U_1 (respectively U_2) is equal to T_1' (respectively T_2').

Applying Lemma 107.3.2 twice we find that

$$m_{T'_1,U_1} = m_{T',V} = m_{T'_2,U_2},$$

as required. □

At this point we have done enough work to show the following definition makes sense.

0DR8 Definition 107.3.4. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $T \subset |\mathcal{X}|$ be an irreducible component. The multiplicity of T in $\mathcal{X}$ is defined as $m_{T,\mathcal{X}} = m_{T',U}$ where $f : U \to \mathcal{X}$ is a smooth morphism from a scheme and $T' \subset |U|$ is an irreducible component with $f(T') \subset T$.

This is independent of the choice of $f : U \to \mathcal{X}$ and the choice of the irreducible component T' mapping to T by Lemmas 107.3.1 and 107.3.3.

As a closing remark, we note that it is sometimes convenient to think of an irreducible component of $\mathcal{X}$ as a closed substack. To this end, if $T \subset |\mathcal{X}|$ is an irreducible component, then we may consider the unique reduced closed substack $\mathcal{T} \subset \mathcal{X}$ with $|\mathcal{T}| = T$, see Properties of Stacks, Definition 100.10.4. If $\mathcal{X}$ is quasi-separated, then an irreducible component is an integral stack; see Morphisms of Stacks, Section 101.50 for further discussion.

107.4. Formal branches and multiplicities

0DR9 It will be convenient to have a comparison between the notion of multiplicity of an irreducible component given by Definition 107.3.4 and the related notion of multiplicities of irreducible components of (the spectra of) versal rings of $\mathcal{X}$ at finite type points.

In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let A, A' be versal rings to $\mathcal{X}$ at x_0. After possibly swapping A and A', we know there is a formally smooth[1] map $\varphi : A \to A'$ compatible with versal formal objects, see Lemma 107.2.4 and Remark 107.2.9. Moreover, φ is well defined up to formal homotopy, see Formal Deformation Theory, Lemma 90.28.3. In particular, we find that $\varphi(\mathfrak{p})A'$ is a well defined ideal of A' by Formal Deformation Theory, Lemma 90.28.4. Since $A \to A'$ is formally smooth, in fact $\varphi(\mathfrak{p})A'$ is a minimal prime of A' and every minimal prime of A' is of this form for a unique minimal prime $\mathfrak{p} \subset A$ (all of this is easy to prove by writing A' as a power series ring over A). Therefore, recalling that minimal primes correspond to irreducible components, the following definition makes sense.

0DRA Definition 107.4.1. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ is a morphism where k is a field of finite type over S. The formal branches of $\mathcal{X}$ through x_0 is the set of irreducible components of $\mathrm{Spec}(A)$ for any choice of versal ring to $\mathcal{X}$ at x_0 identified for different choices of A by the procedure described above.

Suppose in the situation of Definition 107.4.1 we are given a finite extension l/k. Set $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ equal to the composition of $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ with x_0. Let $A \to A'$ be as in Lemma 107.2.5. Since $A \to A'$ is faithfully flat, the morphism

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A)$$

[1] In the sense that A' becomes isomorphic to a power series ring over A.

sends (generic points of) irreducible components to (generic points of) irreducible components. This will be a surjective map, but in general this map will not be a bijection. In other words, we obtain a surjective map

$$\text{formal branches of } \mathcal{X} \text{ through } x_{l,0} \longrightarrow \text{formal branches of } \mathcal{X} \text{ through } x_0$$

It turns out that if l/k is purely inseparable, then the map is injective as well (we'll add a precise statement and proof here if we ever need this).

0DRB Lemma 107.4.2. In the situation of Definition 107.4.1 there is a canonical surjection from the set of formal branches of $\mathcal{X}$ through x_0 to the set of irreducible components of $|\mathcal{X}|$ containing x_0 in $|\mathcal{X}|$.

Proof. Let A be as in Definition 107.4.1 and let $\text{Spec}(A) \to \mathcal{X}$ be as in Remark 107.2.9. We claim that the generic point of an irreducible component of $\text{Spec}(A)$ maps to a generic point of an irreducible component of $|\mathcal{X}|$. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Consider the diagram

$$\begin{tikzcd} \text{Spec}(A) \times_{\mathcal{X}} U \arrow[r, "q"'] \arrow[d, "p"'] & U \arrow[d, "f"] \\ \text{Spec}(A) \arrow[r, "j"] & \mathcal{X} \end{tikzcd}$$

By Lemma 107.2.10 we see that j is flat. Hence q is flat. On the other hand, f is surjective smooth hence p is surjective smooth. This implies that any generic point $\eta \in \text{Spec}(A)$ of an irreducible component is the image of a codimension 0 point η' of the algebraic space $\text{Spec}(A) \times_{\mathcal{X}} U$ (see Properties of Spaces, Section 66.11 for notation and use going down on étale local rings). Since q is flat, $q(\eta')$ is a codimension 0 point of U (same argument). Since U is a scheme, $q(\eta')$ is the generic point of an irreducible component of U. Thus the closure of the image of $q(\eta')$ in $|\mathcal{X}|$ is an irreducible component by Lemma 107.3.1 as claimed.

Clearly the claim provides a mechanism for defining the desired map. To see that it is surjective, we choose $u_0 \in U$ mapping to x_0 in $|\mathcal{X}|$. Choose an affine open $U' \subset U$ neighbourhood of u_0. After shrinking U' we may assume every irreducible component of U' passes through u_0. Then we may replace $\mathcal{X}$ by the open substack corresponding to the image of $|U'| \to |\mathcal{X}|$. Thus we may assume U is affine has a point u_0 mapping to $x_0 \in |\mathcal{X}|$ and every irreducible component of U passes through u_0. By Properties of Stacks, Lemma 100.4.3 there is a point $t \in |\text{Spec}(A) \times_{\mathcal{X}} U|$ mapping to the closed point of $\text{Spec}(A)$ and to u_0. Using going down for the flat local ring homomorphisms

$$A \longrightarrow \mathcal{O}_{\text{Spec}(A) \times_{\mathcal{X}} U, \bar{t}} \longleftarrow \mathcal{O}_{U,u_0}$$

we see that every minimal prime of $\mathcal{O}_{U,u_0}$ is the image of a minimal prime of the local ring in the middle and such a minimal prime maps to a minimal prime of A. This proves the surjectivity. Some details omitted. $\square$

Let A be a Noetherian complete local ring. Then the irreducible components of $\text{Spec}(A)$ have multiplicities, see introduction to Section 107.3. If $A' = A[[t_1, \ldots, t_r]]$, then the morphism $\text{Spec}(A') \to \text{Spec}(A)$ induces a bijection on irreducible components preserving multiplicities (we omit the easy proof). This and the discussion preceding Definition 107.4.1 mean that the following definition makes sense.

0DRC Definition 107.4.3. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ is a morphism where k is a field of finite type over S. The multiplicity of a formal branch of $\mathcal{X}$ through x_0 is the multiplicity of the corresponding irreducible component of $\mathrm{Spec}(A)$ for any choice of versal ring to $\mathcal{X}$ at x_0 (see discussion above).

0DRD Lemma 107.4.4. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ is a morphism where k is a field of finite type over S with image $s \in S$. If $\mathcal{O}_{S,s}$ is a G-ring, then the map of Lemma 107.4.2 preserves multiplicities.

Proof. By Lemma 107.2.8 we may assume there is a smooth morphism $U \to \mathcal{X}$ where U is a scheme and a k-valued point u_0 of U such that $\mathcal{O}_{U,u_0}^\wedge$ is a versal ring to $\mathcal{X}$ at x_0. By construction of our map in the proof of Lemma 107.4.2 (which simplifies greatly because $A = \mathcal{O}_{U,u_0}^\wedge$) we find that it suffices to show: the multiplicity of an irreducible component of U passing through u_0 is the same as the multiplicity of any irreducible component of $\mathrm{Spec}(\mathcal{O}_{U,u_0}^\wedge)$ mapping into it.

Translated into commutative algebra we find the following: Let $C = \mathcal{O}_{U,u_0}$. This is essentially of finite type over $\mathcal{O}_{S,s}$ and hence is a G-ring (More on Algebra, Proposition 15.50.10). Then $A = C^\wedge$. Therefore $C \to A$ is a regular ring map. Let $\mathfrak{q} \subset C$ be a minimal prime and let $\mathfrak{p} \subset A$ be a minimal prime lying over $\mathfrak{q}$. Then

$$R = C_\mathfrak{p} \longrightarrow A_\mathfrak{p} = R'$$

is a regular ring map of Artinian local rings. For such a ring map it is always the case that

$$\mathrm{length}_R R = \mathrm{length}_{R'} R'$$

This is what we have to show because the left hand side is the multiplicity of our component on U and the right hand side is the multiplicity of our component on $\mathrm{Spec}(A)$. To see the equality, first we use that

$$\mathrm{length}_R(R)\mathrm{length}_{R'}(R'/\mathfrak{m}_R R') = \mathrm{length}_{R'}(R')$$

by Algebra, Lemma 10.52.13. Thus it suffices to show $\mathfrak{m}_R R' = \mathfrak{m}_{R'}$, which is a consequence of being a regular homomorphism of zero dimensional local rings. □

107.5. Dimension theory of algebraic stacks

0DRE The main results on the dimension theory of algebraic stacks in the literature that we are aware of are those of [Oss15], which makes a study of the notions of codimension and relative dimension. We make a more detailed examination of the notion of the dimension of an algebraic stack at a point, and prove various results relating the dimension of the fibres of a morphism at a point in the source to the dimension of its source and target. We also prove a result (Lemma 107.6.4 below) which allow us (under suitable hypotheses) to compute the dimension of an algebraic stack at a point in terms of a versal ring.

While we haven't always tried to optimise our results, we have largely tried to avoid making unnecessary hypotheses. However, in some of our results, in which we compare certain properties of an algebraic stack to the properties of a versal ring to this stack at a point, we have restricted our attention to the case of algebraic stacks that are locally finitely presented over a locally Noetherian scheme base, all of whose local rings are G-rings. This gives us the convenience of having Artin approximation available to compare the geometry of the versal ring to the geometry of the stack itself. However, this restrictive hypothesis may not be necessary for the truth of all of the various statements that we prove. Since it is satisfied in the applications that we have in mind, though, we have been content to make it when it helps.

If X is a scheme, then we define the dimension $\dim(X)$ of X to be the Krull dimension of the topological space underlying X, while if x is a point of X, then we define the dimension $\dim_x(X)$ of X at x to be the minimum of the dimensions of the open subsets U of X containing x, see Properties, Definition 28.10.1. One has the relation $\dim(X) = \sup_{x \in X} \dim_x(X)$, see Properties, Lemma 28.10.2. If X is locally Noetherian, then $\dim_x(X)$ coincides with the supremum of the dimensions at x of the irreducible components of X passing through x.

If X is an algebraic space and $x \in |X|$, then we define $\dim_x X = \dim_u U$, where U is any scheme admitting an étale surjection $U \to X$, and $u \in U$ is any point lying over x, see Properties of Spaces, Definition 66.9.1. We set $\dim(X) = \sup_{x \in |X|} \dim_x(X)$, see Properties of Spaces, Definition 66.9.2.

0DRF Remark 107.5.1. In general, the dimension of the algebraic space X at a point x may not coincide with the dimension of the underlying topological space $|X|$ at x. E.g. if k is a field of characteristic zero and $X = \mathbf{A}^1_k/\mathbf{Z}$, then X has dimension 1 (the dimension of $\mathbf{A}^1_k$) at each of its points, while $|X|$ has the indiscrete topology, and hence is of Krull dimension zero. On the other hand, in Algebraic Spaces, Example 65.14.9 there is given an example of an algebraic space which is of dimension 0 at each of its points, while $|X|$ is irreducible of Krull dimension 1, and admits a generic point (so that the dimension of $|X|$ at any of its points is 1); see also the discussion of this example in Properties of Spaces, Section 66.9.

On the other hand, if X is a decent algebraic space, in the sense of Decent Spaces, Definition 68.6.1 (in particular, if X is quasi-separated; see Decent Spaces, Section 68.6) then in fact the dimension of X at x does coincide with the dimension of $|X|$ at x; see Decent Spaces, Lemma 68.12.5.

In order to define the dimension of an algebraic stack, it will be useful to first have the notion of the relative dimension, at a point in the source, of a morphism whose source is an algebraic space, and whose target is an algebraic stack. The definition is slightly involved, just because (unlike in the case of schemes) the points of an algebraic stack, or an algebraic space, are not describable as morphisms from the spectrum of a field, but only as equivalence classes of such.

0DRG Definition 107.5.2. If $f : T \to \mathcal{X}$ is a locally of finite type morphism from an algebraic space to an algebraic stack, and if $t \in |T|$ is a point with image $x \in |\mathcal{X}|$, then we define the relative dimension of f at t, denoted $\dim_t(T_x)$, as follows: choose a morphism $\mathrm{Spec}\, k \to \mathcal{X}$, with source the spectrum of a field, which represents x, and choose a point $t' \in |T \times_{\mathcal{X}} \mathrm{Spec}\, k|$ mapping to t under the projection to $|T|$ (such a point t' exists, by Properties of Stacks, Lemma 100.4.3); then

$$\dim_t(T_x) = \dim_{t'}(T \times_{\mathcal{X}} \mathrm{Spec}\, k).$$

Note that since T is an algebraic space and $\mathcal{X}$ is an algebraic stack, the fibre product $T \times_{\mathcal{X}} \mathrm{Spec}\, k$ is an algebraic space, and so the quantity on the right hand side of this proposed definition is in fact defined (see discussion above).

0DRH Remark 107.5.3. (1) One easily verifies (for example, by using the invariance of the relative dimension of locally of finite type morphisms of schemes under base-change; see for example Morphisms, Lemma 29.28.3) that $\dim_t(T_x)$ is well-defined, independently of the choices used to compute it.

(2) In the case that $\mathcal{X}$ is also an algebraic space, it is straightforward to confirm that this definition agrees with the definition of relative dimension given in Morphisms of Spaces, Definition 67.33.1.

We next recall the following lemma, on which our study of the dimension of a locally Noetherian algebraic stack is founded.

0DRI Lemma 107.5.4. If $f : U \to X$ is a smooth morphism of locally Noetherian algebraic spaces, and if $u \in |U|$ with image $x \in |X|$, then

$$\dim_u(U) = \dim_x(X) + \dim_u(U_x)$$

where $\dim_u(U_x)$ is defined via Definition 107.5.2.

Proof. See Morphisms of Spaces, Lemma 67.37.10 noting that the definition of $\dim_u(U_x)$ used here coincides with the definition used there, by Remark 107.5.3 (2). □

0DRJ Lemma 107.5.5. If $\mathcal{X}$ is a locally Noetherian algebraic stack and $x \in |\mathcal{X}|$. Let $U \to \mathcal{X}$ be a smooth morphism from an algebraic space to $\mathcal{X}$, let u be any point of $|U|$ mapping to x. Then we have

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_u(U_x)$$

where the relative dimension $\dim_u(U_x)$ is defined by Definition 107.5.2 and the dimension of $\mathcal{X}$ at x is as in Properties of Stacks, Definition 100.12.2.

Proof. Lemma 107.5.4 can be used to verify that the right hand side $\dim_u(U) + \dim_u(U_x)$ is independent of the choice of the smooth morphism $U \to \mathcal{X}$ and $u \in |U|$. We omit the details. In particular, we may assume U is a scheme. In this case we can compute $\dim_u(U_x)$ by choosing the representative of x to be the composite $\mathrm{Spec}\,\kappa(u) \to U \to \mathcal{X}$, where the first morphism is the canonical one with image $u \in U$. Then, if we write $R = U \times_{\mathcal{X}} U$, and let $e : U \to R$ denote the diagonal morphism, the invariance of relative dimension under base-change shows that $\dim_u(U_x) = \dim_{e(u)}(R_u)$. Thus we see that the right hand side is equal to $\dim_u(U) - \dim_{e(u)}(R_u) = \dim_x(\mathcal{X})$ as desired. □

0DRK Remark 107.5.6. For Deligne–Mumford stacks which are suitably decent (e.g. quasi-separated), it will again be the case that $\dim_x(\mathcal{X})$ coincides with the topologically defined quantity $\dim_x |\mathcal{X}|$. However, for more general Artin stacks, this will typically not be the case. For example, if $\mathcal{X} = [\mathbf{A}^1/\mathbf{G}_m]$ (over some field, with the quotient being taken with respect to the usual multiplication action of $\mathbf{G}_m$ on $\mathbf{A}^1$), then $|\mathcal{X}|$ has two points, one the specialisation of the other (corresponding to the two orbits of $\mathbf{G}_m$ on $\mathbf{A}^1$), and hence is of dimension 1 as a topological space; but $\dim_x(\mathcal{X}) = 0$ for both points $x \in |\mathcal{X}|$. (An even more extreme example is given by the classifying space $[\mathrm{Spec}\,k/\mathbf{G}_m]$, whose dimension at its unique point is equal to -1.)

We can now extend Definition 107.5.2 to the context of (locally finite type) morphisms between (locally Noetherian) algebraic stacks.

0DRL Definition 107.5.7. If $f : \mathcal{T} \to \mathcal{X}$ is a locally of finite type morphism between locally Noetherian algebraic stacks, and if $t \in |\mathcal{T}|$ is a point with image $x \in |\mathcal{X}|$, then we define the relative dimension of f at t, denoted $\dim_t(\mathcal{T}_x)$, as follows: choose a morphism $\mathrm{Spec}\, k \to \mathcal{X}$, with source the spectrum of a field, which represents x, and choose a point $t' \in |\mathcal{T} \times_{\mathcal{X}} \mathrm{Spec}\, k|$ mapping to t under the projection to $|\mathcal{T}|$ (such a point t' exists, by Properties of Stacks, Lemma 100.4.3; then

$$\dim_t(\mathcal{T}_x) = \dim_{t'}(\mathcal{T} \times_{\mathcal{X}} \mathrm{Spec}\, k).$$

Note that since $\mathcal{T}$ is an algebraic stack and $\mathcal{X}$ is an algebraic stack, the fibre product $\mathcal{T} \times_{\mathcal{X}} \mathrm{Spec}\, k$ is an algebraic stack, which is locally Noetherian by Morphisms of Stacks, Lemma 101.17.5. Thus the quantity on the right side of this proposed definition is defined by Properties of Stacks, Definition 100.12.2.

0DRM Remark 107.5.8. Standard manipulations show that $\dim_t(\mathcal{T}_x)$ is well-defined, independently of the choices made to compute it.

We now establish some basic properties of relative dimension, which are obvious generalisations of the corresponding statements in the case of morphisms of schemes.

0DRN Lemma 107.5.9. Suppose given a Cartesian square of morphisms of locally Noetherian stacks

$$\begin{tikzcd} \mathcal{T}' \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\ \mathcal{X}' \arrow[r] & \mathcal{X} \end{tikzcd}$$

in which the vertical morphisms are locally of finite type. If $t' \in |\mathcal{T}'|$, with images t, x', and x in $|\mathcal{T}|$, $|\mathcal{X}'|$, and $|\mathcal{X}|$ respectively, then $\dim_{t'}(\mathcal{T}'_{x'}) = \dim_t(\mathcal{T}_x)$.

Proof. Both sides can (by definition) be computed as the dimension of the same fibre product. $\square$

0DRP Lemma 107.5.10. If $f : \mathcal{U} \to \mathcal{X}$ is a smooth morphism of locally Noetherian algebraic stacks, and if $u \in |\mathcal{U}|$ with image $x \in |\mathcal{X}|$, then

$$\dim_u(\mathcal{U}) = \dim_x(\mathcal{X}) + \dim_u(\mathcal{U}_x).$$

Proof. Choose a smooth surjective morphism $V \to \mathcal{U}$ whose source is a scheme, and let $v \in |V|$ be a point mapping to u. Then the composite $V \to \mathcal{U} \to \mathcal{X}$ is also smooth, and by Lemma 107.5.4 we have $\dim_x(\mathcal{X}) = \dim_v(V) - \dim_v(V_x)$, while $\dim_u(\mathcal{U}) = \dim_v(V) - \dim_v(V_u)$. Thus

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_v(V_x) - \dim_v(V_u).$$

Choose a representative $\mathrm{Spec}\, k \to \mathcal{X}$ of x and choose a point $v' \in |V \times_{\mathcal{X}} \mathrm{Spec}\, k|$ lying over v, with image u' in $|\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}\, k|$; then by definition $\dim_u(\mathcal{U}_x) = \dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}\, k)$, and $\dim_v(V_x) = \dim_{v'}(V \times_{\mathcal{X}} \mathrm{Spec}\, k)$.

Now $V \times_{\mathcal{X}} \mathrm{Spec}\, k \to \mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}\, k$ is a smooth surjective morphism (being the base-change of such a morphism) whose source is an algebraic space (since V and $\mathrm{Spec}\, k$ are schemes, and $\mathcal{X}$ is an algebraic stack). Thus, again by definition, we have

$$\begin{aligned} \dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}\, k) &= \dim_{v'}(V \times_{\mathcal{X}} \mathrm{Spec}\, k) - \dim_{v'}(V \times_{\mathcal{X}} \mathrm{Spec}\, k)_{u'}) \\ &= \dim_v(V_x) - \dim_{v'}((V \times_{\mathcal{X}} \mathrm{Spec}\, k)_{u'}). \end{aligned}$$

Now $V \times_{\mathcal{X}} \operatorname{Spec} k \cong V \times_{\mathcal{U}} (\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k)$, and so Lemma 107.5.9 shows that $\dim_{v'}((V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}) = \dim_v(V_u)$. Putting everything together, we find that

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_u(\mathcal{U}_x),$$

as required. □

0DRQ Lemma 107.5.11. Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of algebraic stacks.

(1) The function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is upper semi-continuous on $|\mathcal{T}|$.
(2) If f is smooth, then the function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is locally constant on $|\mathcal{T}|$.

Proof. Suppose to begin with that $\mathcal{T}$ is a scheme T, let $U \to \mathcal{X}$ be a smooth surjective morphism whose source is a scheme, and let $T' = T \times_{\mathcal{X}} U$. Let $f' : T' \to U$ be the pull-back of f over U, and let $g : T' \to T$ be the projection.

Lemma 107.5.9 shows that $\dim_{t'}(T'_{f'(t')}) = \dim_{g(t')}(T_{f(g(t'))})$, for $t' \in T'$, while, since g is smooth and surjective (being the base-change of a smooth surjective morphism) the map induced by g on underlying topological spaces is continuous and open (by Properties of Spaces, Lemma 66.4.6), and surjective. Thus it suffices to note that part (1) for the morphism f' follows from Morphisms of Spaces, Lemma 67.34.4, and part (2) from either of Morphisms, Lemma 29.29.4 or Morphisms, Lemma 29.34.12 (each of which gives the result for schemes, from which the analogous results for algebraic spaces can be deduced exactly as in Morphisms of Spaces, Lemma 67.34.4.

Now return to the general case, and choose a smooth surjective morphism $h : V \to \mathcal{T}$ whose source is a scheme. If $v \in V$, then, essentially by definition, we have

$$\dim_{h(v)}(\mathcal{T}_{f(h(v))}) = \dim_v(V_{f(h(v))}) - \dim_v(V_{h(v)}).$$

Since V is a scheme, we have proved that the first of the terms on the right hand side of this equality is upper semi-continuous (and even locally constant if f is smooth), while the second term is in fact locally constant. Thus their difference is upper semi-continuous (and locally constant if f is smooth), and hence the function $\dim_{h(v)}(\mathcal{T}_{f(h(v))})$ is upper semi-continuous on $|V|$ (and locally constant if f is smooth). Since the morphism $|V| \to |\mathcal{T}|$ is open and surjective, the lemma follows. □

Before continuing with our development, we prove two lemmas related to the dimension theory of schemes.

To put the first lemma in context, we note that if X is a finite dimensional scheme, then since $\dim X$ is defined to equal the supremum of the dimensions $\dim_x X$, there exists a point $x \in X$ such that $\dim_x X = \dim X$. The following lemma shows that we may furthermore take the point x to be of finite type.

0DRR Lemma 107.5.12. If X is a finite dimensional scheme, then there exists a closed (and hence finite type) point $x \in X$ such that $\dim_x X = \dim X$.

Proof. Let $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X, say

0DRS (107.5.12.1) $$Z_0 \supset Z_1 \supset \ldots \supset Z_d.$$

The subset Z_d is a minimal irreducible closed subset of X, and thus any point of Z_d is a generic point of Z_d. Since the underlying topological space of the scheme X is sober, we conclude that Z_d is a singleton, consisting of a single closed point $x \in X$. If U is any neighbourhood of x, then the chain

$$U \cap Z_0 \supset U \cap Z_1 \supset \ldots \supset U \cap Z_d = Z_d = \{x\}$$

is then a strictly descending chain of irreducible closed subsets of U, showing that $\dim U \geq d$. Thus we find that $\dim_x X \geq d$. The other inequality being obvious, the lemma is proved. □

The next lemma shows that $\dim_x X$ is a constant function on an irreducible scheme satisfying some mild additional hypotheses.

0DRT Lemma 107.5.13. If X is an irreducible, Jacobson, catenary, and locally Noetherian scheme of finite dimension, then $\dim U = \dim X$ for every non-empty open subset U of X. Equivalently, $\dim_x X$ is a constant function on X.

Proof. The equivalence of the two claims follows directly from the definitions. Suppose, then, that $U \subset X$ is a non-empty open subset. Certainly $\dim U \leq \dim X$, and we have to show that $\dim U \geq \dim X$. Write $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X, say

$$X = Z_0 \supset Z_1 \supset \ldots \supset Z_d.$$

Since X is Jacobson, the minimal irreducible closed subset Z_d is equal to $\{x\}$ for some closed point x.

If $x \in U$, then

$$U = U \cap Z_0 \supset U \cap Z_1 \supset \ldots \supset U \cap Z_d = \{x\}$$

is a strictly decreasing chain of irreducible closed subsets of U, and so we conclude that $\dim U \geq d$, as required. Thus we may suppose that $x \notin U$.

Consider the flat morphism $\mathrm{Spec}\,\mathcal{O}_{X,x} \to X$. The non-empty (and hence dense) open subset U of X pulls back to an open subset $V \subset \mathrm{Spec}\,\mathcal{O}_{X,x}$. Replacing U by a non-empty quasi-compact, and hence Noetherian, open subset, we may assume that the inclusion $U \to X$ is a quasi-compact morphism. Since the formation of scheme-theoretic images of quasi-compact morphisms commutes with flat base-change Morphisms, Lemma 29.25.16 we see that V is dense in $\mathrm{Spec}\,\mathcal{O}_{X,x}$, and so in particular non-empty, and of course $x \notin V$. (Here we use x also to denote the closed point of $\mathrm{Spec}\,\mathcal{O}_{X,x}$, since its image is equal to the given point $x \in X$.) Now $\mathrm{Spec}\,\mathcal{O}_{X,x} \setminus \{x\}$ is Jacobson Properties, Lemma 28.6.4 and hence V contains a closed point z of $\mathrm{Spec}\,\mathcal{O}_{X,x} \setminus \{x\}$. The closure in X of the image of z is then an irreducible closed subset Z of X containing x, whose intersection with U is non-empty, and for which there is no irreducible closed subset properly contained in Z and properly containing $\{x\}$ (because pull-back to $\mathrm{Spec}\,\mathcal{O}_{X,x}$ induces a bijection between irreducible closed subsets of X containing x and irreducible closed subsets of $\mathrm{Spec}\,\mathcal{O}_{X,x}$). Since $U \cap Z$ is a non-empty closed subset of U, it contains a point u that is closed in X (since X is Jacobson), and since $U \cap Z$ is a non-empty (and hence dense) open subset of the irreducible set Z (which contains a point not lying in U, namely x), the inclusion $\{u\} \subset U \cap Z$ is proper.

As X is catenary, the chain

$$X = Z_0 \supset Z \supset \{x\} = Z_d$$

can be refined to a chain of length $d+1$, which must then be of the form

$$X = Z_0 \supset W_1 \supset \ldots \supset W_{d-1} = Z \supset \{x\} = Z_d.$$

Since $U \cap Z$ is non-empty, we then find that

$$U = U \cap Z_0 \supset U \cap W_1 \supset \ldots \supset U \cap W_{d-1} = U \cap Z \supset \{u\}$$

is a strictly decreasing chain of irreducible closed subsets of U of length $d+1$, showing that $\dim U \geq d$, as required. □

We will prove a stack-theoretic analogue of Lemma 107.5.13 in Lemma 107.5.17 below, but before doing so, we have to introduce an additional definition, necessitated by the fact that the notion of a scheme being catenary is not an étale local one (see the example of Algebra, Remark 10.164.8 which makes it difficult to define what it means for an algebraic space or algebraic stack to be catenary (see the discussion of [Oss15, page 3]). For certain aspects of dimension theory, the following definition seems to provide a good substitute for the missing notion of a catenary algebraic stack.

0DRU Definition 107.5.14. We say that a locally Noetherian algebraic stack $\mathcal{X}$ is pseudo-catenary if there exists a smooth and surjective morphism $U \to \mathcal{X}$ whose source is a universally catenary scheme.

0DRV Example 107.5.15. If $\mathcal{X}$ is locally of finite type over a universally catenary locally Noetherian scheme S, and $U \to \mathcal{X}$ is a smooth surjective morphism whose source is a scheme, then the composite $U \to \mathcal{X} \to S$ is locally of finite type, and so U is universally catenary Morphisms, Lemma 29.17.2. Thus $\mathcal{X}$ is pseudo-catenary.

The following lemma shows that the property of being pseudo-catenary passes through finite-type morphisms.

0DRW Lemma 107.5.16. If $\mathcal{X}$ is a pseudo-catenary locally Noetherian algebraic stack, and if $\mathcal{Y} \to \mathcal{X}$ is a locally of finite type morphism, then there exists a smooth surjective morphism $V \to \mathcal{Y}$ whose source is a universally catenary scheme; thus $\mathcal{Y}$ is again pseudo-catenary.

Proof. By assumption we may find a smooth surjective morphism $U \to \mathcal{X}$ whose source is a universally catenary scheme. The base-change $U \times_{\mathcal{X}} \mathcal{Y}$ is then an algebraic stack; let $V \to U \times_{\mathcal{X}} \mathcal{Y}$ be a smooth surjective morphism whose source is a scheme. The composite $V \to U \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is then smooth and surjective (being a composite of smooth and surjective morphisms), while the morphism $V \to U \times_{\mathcal{X}} \mathcal{Y} \to U$ is locally of finite type (being a composite of morphisms that are locally finite type). Since U is universally catenary, we see that V is universally catenary (by Morphisms, Lemma 29.17.2), as claimed. □

We now study the behaviour of the function $\dim_x(\mathcal{X})$ on $|\mathcal{X}|$ (for some locally Noetherian stack $\mathcal{X}$) with respect to the irreducible components of $|\mathcal{X}|$, as well as various related topics.

0DRX Lemma 107.5.17. If $\mathcal{X}$ is a Jacobson, pseudo-catenary, and locally Noetherian algebraic stack for which $|\mathcal{X}|$ is irreducible, then $\dim_x(\mathcal{X})$ is a constant function on $|\mathcal{X}|$.

Proof. It suffices to show that $\dim_x(\mathcal{X})$ is locally constant on $|\mathcal{X}|$, since it will then necessarily be constant (as $|\mathcal{X}|$ is connected, being irreducible). Since $\mathcal{X}$ is pseudo-catenary, we may find a smooth surjective morphism $U \to \mathcal{X}$ with U being a universally catenary scheme. If $\{U_i\}$ is an cover of U by quasi-compact open subschemes, we may replace U by $\coprod U_i$,, and it suffices to show that the function $u \mapsto \dim_{f(u)}(\mathcal{X})$ is locally constant on U_i. Since we check this for one U_i at a time, we now drop the subscript, and write simply U rather than U_i. Since U is quasi-compact, it is the union of a finite number of irreducible components, say $T_1 \cup \ldots \cup T_n$. Note that each T_i is Jacobson, catenary, and locally Noetherian, being a closed subscheme of the Jacobson, catenary, and locally Noetherian scheme U.

By Lemma 107.5.4, we have $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$. Lemma 107.5.11 (2) shows that the second term in the right hand expression is locally constant on U, as f is smooth, and hence we must show that $\dim_u(U)$ is locally constant on U. Since $\dim_u(U)$ is the maximum of the dimensions $\dim_u T_i$, as T_i ranges over the components of U containing u, it suffices to show that if a point u lies on two distinct components, say T_i and T_j (with $i \neq j$), then $\dim_u T_i = \dim_u T_j$, and then to note that $t \mapsto \dim_t T$ is a constant function on an irreducible Jacobson, catenary, and locally Noetherian scheme T (as follows from Lemma 107.5.13).

Let $V = T_i \setminus (\bigcup_{i' \neq i} T_{i'})$ and $W = T_j \setminus (\bigcup_{i' \neq j} T_{i'})$. Then each of V and W is a non-empty open subset of U, and so each has non-empty open image in $|\mathcal{X}|$. As $|\mathcal{X}|$ is irreducible, these two non-empty open subsets of $|\mathcal{X}|$ have a non-empty intersection. Let x be a point lying in this intersection, and let $v \in V$ and $w \in W$ be points mapping to x. We then find that

$$\dim T_i = \dim V = \dim_v(U) = \dim_x(\mathcal{X}) + \dim_v(U_x)$$

and similarly that

$$\dim T_j = \dim W = \dim_w(U) = \dim_x(\mathcal{X}) + \dim_w(U_x).$$

Since $u \mapsto \dim_u(U_{f(u)})$ is locally constant on U, and since $T_i \cup T_j$ is connected (being the union of two irreducible, hence connected, sets that have non-empty intersection), we see that $\dim_v(U_x) = \dim_w(U_x)$, and hence, comparing the preceding two equations, that $\dim T_i = \dim T_j$, as required. $\square$

0DRY Lemma 107.5.18. If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of locally Noetherian algebraic stacks, and if $z \in |\mathcal{Z}|$ has image $x \in |\mathcal{X}|$, then $\dim_z(\mathcal{Z}) \leq \dim_x(\mathcal{X})$.

Proof. Choose a smooth surjective morphism $U \to \mathcal{X}$ whose source is a scheme; the base-changed morphism $V = U \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$ is then also smooth and surjective, and the projection $V \to U$ is a closed immersion. If $v \in |V|$ maps to $z \in |\mathcal{Z}|$, and if we let u denote the image of v in $|U|$, then clearly $\dim_v(V) \leq \dim_u(U)$, while $\dim_v(V_z) = \dim_u(U_x)$, by Lemma 107.5.9. Thus

$$\dim_z(\mathcal{Z}) = \dim_v(V) - \dim_v(V_z) \leq \dim_u(U) - \dim_u(U_x) = \dim_x(\mathcal{X}),$$

as claimed. $\square$

0DRZ Lemma 107.5.19. If $\mathcal{X}$ is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then $\dim_x(\mathcal{X}) = \sup_{\mathcal{T}}\{\dim_x(\mathcal{T})\}$, where $\mathcal{T}$ runs over all the irreducible components of $|\mathcal{X}|$ passing through x (endowed with their induced reduced structure).

Proof. Lemma 107.5.18 shows that $\dim_x(\mathcal{T}) \leq \dim_x(\mathcal{X})$ for each irreducible component $\mathcal{T}$ passing through the point x. Thus to prove the lemma, it suffices to show that

0DS0 (107.5.19.1) $$\dim_x(\mathcal{X}) \leq \sup_{\mathcal{T}}\{\dim_x(\mathcal{T})\}.$$

Let $U \to \mathcal{X}$ be a smooth cover by a scheme. If T is an irreducible component of U then we let $\mathcal{T}$ denote the closure of its image in $\mathcal{X}$, which is an irreducible component of $\mathcal{X}$. Let $u \in U$ be a point mapping to x. Then we have $\dim_x(\mathcal{X}) = \dim_u U - \dim_u U_x = \sup_T \dim_u T - \dim_u U_x$, where the supremum is over the irreducible components of U passing through u. Choose a component T for which the supremum is achieved, and note that $\dim_x(\mathcal{T}) = \dim_u T - \dim_u T_x$. The desired inequality (107.5.19.1) now follows from the evident inequality $\dim_u T_x \leq \dim_u U_x$. (Note that if $\operatorname{Spec} k \to \mathcal{X}$ is a representative of x, then $T \times_{\mathcal{X}} \operatorname{Spec} k$ is a closed subspace of $U \times_{\mathcal{X}} \operatorname{Spec} k$.) □

0DS1 **Lemma 107.5.20.** *If $\mathcal{X}$ is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then for any open substack $\mathcal{V}$ of $\mathcal{X}$ containing x, there is a finite type point $x_0 \in |\mathcal{V}|$ such that $\dim_{x_0}(\mathcal{X}) = \dim_x(\mathcal{V})$.*

Proof. Choose a smooth surjective morphism $f : U \to \mathcal{X}$ whose source is a scheme, and consider the function $u \mapsto \dim_{f(u)}(\mathcal{X})$; since the morphism $|U| \to |\mathcal{X}|$ induced by f is open (as f is smooth) as well as surjective (by assumption), and takes finite type points to finite type points (by the very definition of the finite type points of $|\mathcal{X}|$), it suffices to show that for any $u \in U$, and any open neighbourhood of u, there is a finite type point u_0 in this neighbourhood such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. Since, with this reformulation of the problem, the surjectivity of f is no longer required, we may replace U by the open neighbourhood of the point u in question, and thus reduce to the problem of showing that for each $u \in U$, there is a finite type point $u_0 \in U$ such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. By Lemma 107.5.4 $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$, while $\dim_{f(u_0)}(\mathcal{X}) = \dim_{u_0}(U) - \dim_{u_0}(U_{f(u_0)})$. Since f is smooth, the expression $\dim_{u_0}(U_{f(u_0)})$ is locally constant as u_0 varies over U (by Lemma 107.5.11 (2)), and so shrinking U further around u if necessary, we may assume it is constant. Thus the problem becomes to show that we may find a finite type point $u_0 \in U$ for which $\dim_{u_0}(U) = \dim_u(U)$. Since by definition $\dim_u U$ is the minimum of the dimensions $\dim V$, as V ranges over the open neighbourhoods V of u in U, we may shrink U down further around u so that $\dim_u U = \dim U$. The existence of desired point u_0 then follows from Lemma 107.5.12. □

0DS2 **Lemma 107.5.21.** *Let $\mathcal{T} \hookrightarrow \mathcal{X}$ be a locally of finite type monomorphism of algebraic stacks, with $\mathcal{X}$ (and thus also $\mathcal{T}$) being Jacobson, pseudo-catenary, and locally Noetherian. Suppose further that $\mathcal{T}$ is irreducible of some (finite) dimension d, and that $\mathcal{X}$ is reduced and of dimension less than or equal to d. Then there is a non-empty open substack $\mathcal{V}$ of $\mathcal{T}$ such that the induced monomorphism $\mathcal{V} \hookrightarrow \mathcal{X}$ is an open immersion which identifies $\mathcal{V}$ with an open subset of an irreducible component of $\mathcal{X}$.*

Proof. Choose a smooth surjective morphism $f : U \to \mathcal{X}$ with source a scheme, necessarily reduced since $\mathcal{X}$ is, and write $U' = \mathcal{T} \times_{\mathcal{X}} U$. The base-changed morphism $U' \to U$ is a monomorphism of algebraic spaces, locally of finite type, and thus representable Morphisms of Spaces, Lemma 67.51.1 and 67.27.10; since U is a scheme, so is U'. The projection $f' : U' \to \mathcal{T}$ is again a smooth surjection. Let $u' \in U'$, with image $u \in U$. Lemma 107.5.9 shows that $\dim_{u'}(U'_{f(u')}) = \dim_u(U_{f(u)})$, while $\dim_{f'(u')}(\mathcal{T}) = d \geq \dim_{f(u)}(\mathcal{X})$ by Lemma 107.5.17 and our assumptions on $\mathcal{T}$ and $\mathcal{X}$. Thus we see that

(107.5.21.1)

0DS3 $$\dim_{u'}(U') = \dim_{u'}(U'_{f(u')}) + \dim_{f'(u')}(\mathcal{T}) \geq \dim_u(U_{f(u)}) + \dim_{f(u)}(\mathcal{X}) = \dim_u(U).$$

Since $U' \to U$ is a monomorphism, locally of finite type, it is in particular unramified, and so by the étale local structure of unramified morphisms Étale Morphisms, Lemma 41.17.3, we may find a commutative diagram

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d] \\ U' \arrow[r] & U \end{tikzcd}$$

in which the scheme V' is non-empty, the vertical arrows are étale, and the upper horizontal arrow is a closed immersion. Replacing V by a quasi-compact open subset whose image has non-empty intersection with the image of U', and replacing V' by the preimage of V, we may further assume that V (and thus V') is quasi-compact. Since V is also locally Noetherian, it is thus Noetherian, and so is the union of finitely many irreducible components.

Since étale morphisms preserve pointwise dimension Descent, Lemma 35.21.2 we deduce from (107.5.21.1) that for any point $v' \in V'$, with image $v \in V$, we have $\dim_{v'}(V') \geq \dim_v(V)$. In particular, the image of V' can't be contained in the intersection of two distinct irreducible components of V, and so we may find at least one irreducible open subset of V which has non-empty intersection with V'; replacing V by this subset, we may assume that V is integral (being both reduced and irreducible). From the preceding inequality on dimensions, we conclude that the closed immersion $V' \hookrightarrow V$ is in fact an isomorphism. If we let W denote the image of V' in U', then W is a non-empty open subset of U' (as étale morphisms are open), and the induced monomorphism $W \to U$ is étale (since it is so étale locally on the source, i.e. after pulling back to V'), and hence is an open immersion (being an étale monomorphism). Thus, if we let $\mathcal{V}$ denote the image of W in $\mathcal{T}$, then $\mathcal{V}$ is a dense (equivalently, non-empty) open substack of $\mathcal{T}$, whose image is dense in an irreducible component of $\mathcal{X}$. Finally, we note that the morphism is $\mathcal{V} \to \mathcal{X}$ is smooth (since its composite with the smooth morphism $W \to \mathcal{V}$ is smooth), and also a monomorphism, and thus is an open immersion. □

0DS4 **Lemma 107.5.22.** Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of $\mathcal{T}$. Then for all $t \in |T|$, we have that $\dim_t(\mathcal{T}_{f(t)}) \geq \dim \mathcal{T} - \dim \mathcal{Z}$, and there is a non-empty (equivalently, dense) open subset of $|\mathcal{T}|$ over which equality holds.

Proof. Replacing $\mathcal{X}$ by $\mathcal{Z}$, we may and do assume that f is scheme theoretically dominant, and also that $\mathcal{X}$ is irreducible. By the upper semi-continuity of fibre dimensions (Lemma 107.5.11 (1)), it suffices to prove that the equality $\dim_t(\mathcal{T}_{f(t)}) = \dim \mathcal{T} - \dim \mathcal{Z}$ holds for t lying in some non-empty open substack of $\mathcal{T}$. For this reason, in the argument we are always free to replace $\mathcal{T}$ by a non-empty open substack.

Let $T' \to \mathcal{T}$ be a smooth surjective morphism whose source is a scheme, and let T be a non-empty quasi-compact open subset of T'. Since $\mathcal{Y}$ is quasi-separated, we find that $T \to \mathcal{Y}$ is quasi-compact (by Morphisms of Stacks, Lemma 101.7.7, applied to the morphisms $T \to \mathcal{Y} \to \operatorname{Spec} \mathbf{Z}$). Thus, if we replace $\mathcal{T}$ by the image of T in $\mathcal{T}$, then we may assume (appealing to Morphisms of Stacks, Lemma 101.7.6 that the morphism $f : \mathcal{T} \to \mathcal{X}$ is quasi-compact.

If we choose a smooth surjection $U \to \mathcal{X}$ with U a scheme, then Lemma 107.3.1 ensures that we may find an irreducible open subset V of U such that $V \to \mathcal{X}$ is smooth and scheme-theoretically dominant. Since scheme-theoretic dominance for quasi-compact morphisms is preserved by flat base-change, the base-change $\mathcal{T} \times_{\mathcal{X}} V \to V$ of the scheme-theoretically dominant morphism f is again scheme-theoretically dominant. We let Z denote a scheme admitting a smooth surjection onto this fibre product; then $Z \to \mathcal{T} \times_{\mathcal{X}} V \to V$ is again scheme-theoretically dominant. Thus we may find an irreducible component C of Z which scheme-theoretically dominates V. Since the composite $Z \to \mathcal{T} \times_{\mathcal{X}} V \to \mathcal{T}$ is smooth, and since $\mathcal{T}$ is irreducible, Lemma 107.3.1 shows that any irreducible component of the source has dense image in $|\mathcal{T}|$. We now replace C by a non-empty open subset W which is disjoint from every other irreducible component of Z, and then replace $\mathcal{T}$ and $\mathcal{X}$ by the images of W and V (and apply Lemma 107.5.17 to see that this doesn't change the dimension of either $\mathcal{T}$ or $\mathcal{X}$). If we let $\mathcal{W}$ denote the image of the morphism $W \to \mathcal{T} \times_{\mathcal{X}} V$, then $\mathcal{W}$ is open in $\mathcal{T} \times_{\mathcal{X}} V$ (since the morphism $W \to \mathcal{T} \times_{\mathcal{X}} V$ is smooth), and is irreducible (being the image of an irreducible scheme). Thus we end up with a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[dr] & \mathcal{W} \arrow[r] \arrow[d] & V \arrow[d] \\ & \mathcal{T} \arrow[r] & \mathcal{X} \end{tikzcd}$$

in which W and V are schemes, the vertical arrows are smooth and surjective, the diagonal arrows and the left-hand upper horizontal arroware smooth, and the induced morphism $\mathcal{W} \to \mathcal{T} \times_{\mathcal{X}} V$ is an open immersion. Using this diagram, together with the definitions of the various dimensions involved in the statement of the lemma, we will reduce our verification of the lemma to the case of schemes, where it is known.

Fix $w \in |W|$ with image $w' \in |\mathcal{W}|$, image $t \in |\mathcal{T}|$, image v in $|V|$, and image x in $|\mathcal{X}|$. Essentially by definition (using the fact that $\mathcal{W}$ is open in $\mathcal{T} \times_{\mathcal{X}} V$, and that the fibre of a base-change is the base-change of the fibre), we obtain the equalities

$$\dim_v V_x = \dim_{w'} \mathcal{W}_t$$

and

$$\dim_t \mathcal{T}_x = \dim_{w'} \mathcal{W}_v.$$

By Lemma 107.5.4 (the diagonal arrow and right-hand vertical arrow in our diagram realise W and V as smooth covers by schemes of the stacks $\mathcal{T}$ and $\mathcal{X}$), we find that

$$\dim_t \mathcal{T} = \dim_w W - \dim_w W_t$$

and

$$\dim_x \mathcal{X} = \dim_v V - \dim_v V_x.$$

Combining the equalities, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_{w'} \mathcal{W}_v - \dim_w W + \dim_w W_t + \dim_v V - \dim_{w'} \mathcal{W}_t$$

Since $W \to \mathcal{W}$ is a smooth surjection, the same is true if we base-change over the morphism $\operatorname{Spec} \kappa(v) \to V$ (thinking of $W \to \mathcal{W}$ as a morphism over V), and from this smooth morphism we obtain the first of the following two equalities

$$\dim_w W_v - \dim_{w'} \mathcal{W}_v = \dim_w (W_v)_{w'} = \dim_w W_{w'};$$

the second equality follows via a direct comparison of the two fibres involved. Similarly, if we think of $W \to \mathcal{W}$ as a morphism of schemes over $\mathcal{T}$, and base-change over some representative of the point $t \in |\mathcal{T}|$, we obtain the equalities

$$\dim_w W_t - \dim_{w'} \mathcal{W}_t = \dim_w (W_t)_{w'} = \dim_w W_{w'}.$$

Putting everything together, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_w W_v - \dim_w W + \dim_v V.$$

Our goal is to show that the left-hand side of this equality vanishes for a non-empty open subset of t. As w varies over a non-empty open subset of W, its image $t \in |\mathcal{T}|$ varies over a non-empty open subset of $|\mathcal{T}|$ (as $W \to \mathcal{T}$ is smooth).

We are therefore reduced to showing that if $W \to V$ is a scheme-theoretically dominant morphism of irreducible locally Noetherian schemes that is locally of finite type, then there is a non-empty open subset of points $w \in W$ such that $\dim_w W_v = \dim_w W - \dim_v V$ (where v denotes the image of w in V). This is a standard fact, whose proof we recall for the convenience of the reader.

We may replace W and V by their underlying reduced subschemes without altering the validity (or not) of this equation, and thus we may assume that they are in fact integral schemes. Since $\dim_w W_v$ is locally constant on W, replacing W by a non-empty open subset if necessary, we may assume that $\dim_w W_v$ is constant, say equal to d. Choosing this open subset to be affine, we may also assume that the morphism $W \to V$ is in fact of finite type. Replacing V by a non-empty open subset if necessary (and then pulling back W over this open subset; the resulting pull-back is non-empty, since the flat base-change of a quasi-compact and scheme-theoretically dominant morphism remains scheme-theoretically dominant), we may furthermore assume that W is flat over V. The morphism $W \to V$ is thus of relative dimension d in the sense of Morphisms, Definition 29.29.1 and it follows from Morphisms, Lemma 29.29.6 that $\dim_w(W) = \dim_v(V) + d$, as required. $\square$

0DS5 Remark 107.5.23. We note that in the context of the preceding lemma, it need not be that $\dim \mathcal{T} \geq \dim \mathcal{Z}$; this does not contradict the inequality in the statement of the lemma, because the fibres of the morphism f are again algebraic stacks, and so may have negative dimension. This is illustrated by taking k to be a field, and applying the lemma to the morphism $[\operatorname{Spec} k/\mathbf{G}_m] \to \operatorname{Spec} k$.

If the morphism f in the statement of the lemma is assumed to be quasi-DM (in the sense of Morphisms of Stacks, Definition 101.4.1; e.g. morphisms that are representable by algebraic spaces are quasi-DM), then the fibres of the morphism over points of the target are quasi-DM algebraic stacks, and hence are of non-negative dimension. In this case, the lemma implies that indeed $\dim \mathcal{T} \geq \dim \mathcal{Z}$. In fact, we obtain the following more general result.

0DS6 Lemma 107.5.24. Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks which is quasi-DM, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of $\mathcal{T}$. Then $\dim \mathcal{Z} \leq \dim \mathcal{T}$, and furthermore, exactly one of the following two conditions holds:

(1) for every finite type point $t \in |T|$, we have $\dim_t(\mathcal{T}_{f(t)}) > 0$, in which case $\dim \mathcal{Z} < \dim \mathcal{T}$; or
(2) $\mathcal{T}$ and $\mathcal{Z}$ are of the same dimension.

Proof. As was observed in the preceding remark, the dimension of a quasi-DM stack is always non-negative, from which we conclude that $\dim_t \mathcal{T}_{f(t)} \geq 0$ for all $t \in |\mathcal{T}|$, with the equality

$$\dim_t \mathcal{T}_{f(t)} = \dim_t \mathcal{T} - \dim_{f(t)} \mathcal{Z}$$

holding for a dense open subset of points $t \in |\mathcal{T}|$. $\square$

107.6. The dimension of the local ring

0DS7 An algebraic stack doesn't really have local rings in the usual sense, but we can define the dimension of the local ring as follows.

0DS8 Lemma 107.6.1. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $U \to \mathcal{X}$ be a smooth morphism and let $u \in U$. Then

$$\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R_u,e(\overline{u})}) = 2\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R,e(\overline{u})})$$

Here $R = U \times_\mathcal{X} U$ with projections $s, t : R \to U$ and diagonal $e : U \to R$ and R_u is the fibre of $s : R \to U$ over u.

Proof. This is true because $s : \mathcal{O}_{U,\overline{u}} \to \mathcal{O}_{R,e(\overline{u})}$ is a flat local homomorphism of Noetherian local rings and hence

$$\dim(\mathcal{O}_{R,e(\overline{u})}) = \dim(\mathcal{O}_{U,\overline{u}}) + \dim(\mathcal{O}_{R_u,e(\overline{u})})$$

by Algebra, Lemma 10.112.7. $\square$

0DS9 Lemma 107.6.2. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point Morphisms of Stacks, Definition 101.18.2). Let $d \in \mathbf{Z}$. The following are equivalent

(1) there exists a scheme U, a smooth morphism $U \to \mathcal{X}$, and a finite type point $u \in U$ mapping to x such that $2\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R,e(\overline{u})}) = d$, and
(2) for any scheme U, a smooth morphism $U \to \mathcal{X}$, and finite type point $u \in U$ mapping to x we have $2\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R,e(\overline{u})}) = d$.

Here $R = U \times_\mathcal{X} U$ with projections $s, t : R \to U$ and diagonal $e : U \to R$ and R_u is the fibre of $s : R \to U$ over u.

Proof. Suppose we have two smooth neighbourhoods (U, u) and (U', u') of x with u and u' finite type points. After shrinking U and U' we may assume that u and u' are closed points (by definition of finite type points). Then we choose a surjective étale morphism $W \to U \times_\mathcal{X} U'$. Let W_u be the fibre of $W \to U$ over u and let $W_{u'}$ be the fibre of $W \to U'$ over u'. Since u and u' map to the same point of $|\mathcal{X}|$ we see that $W_u \cap W_{u'}$ is nonempty. Hence we may choose a closed point $w \in W$ mapping to both u and u'. This reduces us to the discussion in the next paragraph.

Assume $(U', u') \to (U, u)$ is a smooth morphism of smooth neighbourhoods of x with u and u' closed points. Goal: prove the invariant defined for (U, u) is the same as the invariant defined for (U', u'). To see this observe that $\mathcal{O}_{U,u} \to \mathcal{O}_{U',u'}$ is a flat local homomorphism of Noetherian local rings and hence

$$\dim(\mathcal{O}_{U',\overline{u}'}) = \dim(\mathcal{O}_{U,\overline{u}}) + \dim(\mathcal{O}_{U'_u,\overline{u}'})$$

by Algebra, Lemma 10.112.7. (We omit working through all the steps to relate properties of local rings and their strict henselizations, see More on Algebra, Section 15.45). On the other hand we have

$$R' = U' \times_{U,t} R \times_{s,U} U'$$

Thus we see that

$$\dim(\mathcal{O}_{R',e(\overline{u}')}) = \dim(\mathcal{O}_{R,e(\overline{u})}) + \dim(\mathcal{O}_{U'_u \times_u U'_u,(\overline{u}',\overline{u}')})$$

To prove the lemma it suffices to show that

$$\dim(\mathcal{O}_{U'_u \times_u U'_u,(\overline{u}',\overline{u}')}) = 2\dim(\mathcal{O}_{U'_u,\overline{u}'})$$

Observe that this isn't always true (example: if U'_u is a curve and u' is the generic point of this curve). However, we know that u' is a closed point of the algebraic space U'_u locally of finite type over u. In this case the equality holds because, first $\dim_{(u',u')}(U'_u \times_u U'_u) = 2\dim_{u'}(U'_u)$ by Varieties, Lemma 33.20.5 and second the agreement of dimension with dimension of local rings in closed points of locally algebraic schemes, see Varieties, Lemma 33.20.3. We omit the translation of these results for schemes into the language of algebraic spaces. □

0DSA Definition 107.6.3. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point. The dimension of the local ring of $\mathcal{X}$ at x is $d \in \mathbf{Z}$ if the equivalent conditions of Lemma 107.6.2 are satisfied.

To be sure, this is motivated by Lemma 107.6.1 and Properties of Stacks, Definition 100.12.2. We close this section by establishing a formula allowing us to compute $\dim_x(\mathcal{X})$ in terms of properties of the versal ring to $\mathcal{X}$ at x.

0DSB Lemma 107.6.4. Suppose that $\mathcal{X}$ is an algebraic stack, locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism where k is a field of finite type over S. Represent $\mathcal{F}_{\mathcal{X},k,x_0}$ as in Remark 107.2.11 by a cogroupoid (A, B, s, t, c) of Noetherian complete local S-algebras with residue field k. Then

$$\text{the dimension of the local ring of } \mathcal{X} \text{ at } x_0 = 2\dim A - \dim B$$

Proof. Let $s \in S$ be the image of x_0. If $\mathcal{O}_{S,s}$ is a G-ring (a condition that is almost always satisfied in practice), then we can prove the lemma as follows. By Lemma 107.2.8, we may find a smooth morphism $U \to \mathcal{X}$, whose source is a scheme, containing a point $u_0 \in U$ of residue field k, such that induced morphism $\mathrm{Spec}(k) \to U \to \mathcal{X}$ coincides with x_0 and such that $A = \mathcal{O}^\wedge_{U,u_0}$. Write $R = U \times_{\mathcal{X}} U$. Then we may identify $\mathcal{O}^\wedge_{R,e(u_0)}$ with B. Hence the equality follows from the definitions.

In the rest of this proof we explain how to prove the lemma in general, but we urge the reader to skip this.

First let us show that the right hand side is independent of the choice of (A, B, s, t, c). Namely, suppose that (A', B', s', t', c') is a second choice. Since A and A' are versal rings to $\mathcal{X}$ at x_0, we can choose, after possibly switching A and A', a formally smooth map $A \to A'$ compatible with the given versal formal objects ξ and ξ' over A and A'. Recall that $\widehat{\mathcal{C}}_\Lambda$ has coproducts and that these are given by completed tensor product over Λ, see Formal Deformation Theory, Lemma 90.4.4. Then B prorepresents the functor of isomorphisms between the two pushforwards of ξ to $A\widehat{\otimes}_\Lambda A$. Similarly for B'. We conclude that

$$B' = B \otimes_{(A\widehat{\otimes}_\Lambda A)} (A'\widehat{\otimes}_\Lambda A')$$

It is straightforward to see that

$$A\widehat{\otimes}_\Lambda A \longrightarrow A\widehat{\otimes}_\Lambda A' \longrightarrow A'\widehat{\otimes}_\Lambda A'$$

is formally smooth of relative dimension equal to 2 times the relative dimension of the formally smooth map $A \to A'$. (This follows from general principles, but also because in this particular case A' is a power series ring over A in r variables.) Hence $B \to B'$ is formally smooth of relative dimension $2(\dim(A') - \dim(A))$ as desired.

Next, let l/k be a finite extension. let $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ be the induced point. We claim that the right hand side of the formula is the same for x_0 as it is for $x_{l,0}$. This can be shown by choosing $A \to A'$ as in Lemma 107.2.5 and arguing exactly as in the preceding paragraph. We omit the details.

Finally, arguing as in the proof of Lemma 107.2.10 we can use the compatibilities in the previous two paragraphs to reduce to the case (discussed in the first paragraph) where A is the complete local ring of U at u_0 for some scheme smooth over $\mathcal{X}$ and finite type point u_0. Details omitted. □

107.7. Other chapters

Part 8

Topics in Moduli Theory

CHAPTER 108

Moduli Stacks

0DLT 108.1. Introduction

0DLU In this chapter we verify basic properties of moduli spaces and moduli stacks such as Hom, Isom, $\mathcal{C}oh_{X/B}$, $\mathrm{Quot}_{\mathcal{F}/X/B}$, $\mathrm{Hilb}_{X/B}$, $\mathcal{P}ic_{X/B}$, $\mathrm{Pic}_{X/B}$, $\mathit{Mor}_B(Z, X)$, $\mathcal{S}paces'_{fp,flat,proper}$, $\mathcal{P}olarized$, and $\mathcal{C}omplexes_{X/B}$. We have already shown these algebraic spaces or algebraic stacks under suitable hypotheses, see Quot, Sections 99.3, 99.4, 99.5, 99.6, 99.8, 99.9, 99.10, 99.11, 99.12, 99.13, 99.14, and 99.16. The stack of curves, denoted Curves and introduced in Quot, Section 99.15, is discussed in the chapter on moduli of curves, see Moduli of Curves, Section 109.3.

In some sense this chapter is following the footsteps of Grothendieck's lectures [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d].

108.2. Conventions and abuse of language

0DLV We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

108.3. Properties of Hom and Isom

0DLW Let $f : X \to B$ be a morphism of algebraic spaces which is of finite presentation. Assume $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is of finite presentation, flat over B with support proper over B, then the functor $\mathit{Hom}(\mathcal{F}, \mathcal{G})$ defined by

$$T/B \longmapsto \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T)$$

is an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $\mathit{Hom}(\mathcal{F}, \mathcal{G}) \to B$ is of finite presentation. See Quot, Proposition 99.3.10.

If both $\mathcal{F}$ and $\mathcal{G}$ are of finite presentation, flat over B with support proper over B, then the subfunctor

$$\mathit{Isom}(\mathcal{F}, \mathcal{G}) \subset \mathit{Hom}(\mathcal{F}, \mathcal{G})$$

is an algebraic space affine of finite presentation over B. See Quot, Proposition 99.4.3.

108.4. Properties of the stack of coherent sheaves

0DLX Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Then the stack $\mathcal{C}oh_{X/B}$ parametrizing flat families of coherent modules with proper support is algebraic. See Quot, Theorem 99.6.1.

0DLY Lemma 108.4.1. The diagonal of $\mathcal{C}oh_{X/B}$ over B is affine and of finite presentation.

Proof. The representability of the diagonal by algebraic spaces was shown in Quot, Lemma 99.5.3. From the proof we find that we have to show $Isom(\mathcal{F}, \mathcal{G}) \to T$ is affine and of finite presentation for a pair of finitely presented $\mathcal{O}_{X_T}$-modules $\mathcal{F}$, $\mathcal{G}$ flat over T with support proper over T. This was discussed in Section 108.3. $\square$

0DLZ Lemma 108.4.2. The morphism $\mathcal{C}oh_{X/B} \to B$ is quasi-separated and locally of finite presentation.

Proof. To check $\mathcal{C}oh_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 108.4.1. To prove that $\mathcal{C}oh_{X/B} \to B$ is locally of finite presentation, we have to show that $\mathcal{C}oh_{X/B} \to B$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This follows from Quot, Lemma 99.5.6 (small detail omitted). $\square$

0DM0 Lemma 108.4.3. Assume $X \to B$ is proper as well as of finite presentation. Then $\mathcal{C}oh_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Stacks, Definition 101.39.10).

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K and an algebraic space X proper over R and a coherent $\mathcal{O}_{X_K}$-module $\mathcal{F}_K$, show there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ flat over R whose generic fibre is $\mathcal{F}_K$. Observe that by Flatness on Spaces, Theorem 77.4.5 any finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ flat over R is of finite presentation. Denote $j : X_K \to X$ the embedding of the generic fibre. As a base change of the affine morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ the morphism j is affine. Thus $j_*\mathcal{F}_K$ is quasi-coherent. Write

$$j_*\mathcal{F}_K = \mathrm{colim}\, \mathcal{F}_i$$

as a filtered colimit of its finite type quasi-coherent $\mathcal{O}_X$-submodules, see Limits of Spaces, Lemma 70.9.2. Since $j_*\mathcal{F}_K$ is a sheaf of K-vector spaces over X, it is flat over $\mathrm{Spec}(R)$. Thus each $\mathcal{F}_i$ is flat over R as flatness over a valuation ring is the same as being torsion free (More on Algebra, Lemma 15.22.10) and torsion freeness is inherited by submodules. Finally, we have to show that the map $j^*\mathcal{F}_i \to \mathcal{F}_K$ is an isomorphism for some i. Since $j^*j_*\mathcal{F}_K = \mathcal{F}_K$ (small detail omitted) and since j^* is exact, we see that $j^*\mathcal{F}_i \to \mathcal{F}_K$ is injective for all i. Since j^* commutes with colimits, we have $\mathcal{F}_K = j^*j_*\mathcal{F}_K = \mathrm{colim}\, j^*\mathcal{F}_i$. Since $\mathcal{F}_K$ is coherent (i.e., finitely presented), there is an i such that $j^*\mathcal{F}_i$ contains all the (finitely many) generators over an affine étale cover of X. Thus we get surjectivity of $j^*\mathcal{F}_i \to \mathcal{F}_K$ for i large enough. $\square$

0DN9 Lemma 108.4.4. Let B be an algebraic space. Let $\pi : X \to Y$ be a quasi-finite morphism of algebraic spaces which are separated and of finite presentation over B. Then π_* induces a morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$.

Proof. Let $(T \to B, \mathcal{F})$ be an object of $\mathcal{C}oh_{X/B}$. We claim

(a) $(T \to B, \pi_{T,*}\mathcal{F})$ is an object of $\mathcal{C}oh_{Y/B}$ and
(b) for $T' \to T$ we have $\pi_{T',*}(X_{T'} \to X_T)^*\mathcal{F} = (Y_{T'} \to Y_T)^*\pi_{T,*}\mathcal{F}$.

Part (b) guarantees that this construction defines a functor $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ as desired.

Let $i : Z \to X_T$ be the closed subspace cut out by the zeroth fitting ideal of $\mathcal{F}$ (Divisors on Spaces, Section 71.5). Then $Z \to B$ is proper by assumption (see Derived Categories of Spaces, Section 75.7). On the other hand i is of finite presentation (Divisors on Spaces, Lemma 71.5.2 and Morphisms of Spaces, Lemma 67.28.12). There exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type with $i_*\mathcal{G} = \mathcal{F}$ (Divisors on Spaces, Lemma 71.5.3). In fact $\mathcal{G}$ is of finite presentation as an $\mathcal{O}_Z$-module by Descent on Spaces, Lemma 74.6.7. Observe that $\mathcal{G}$ is flat over B, for example because the stalks of $\mathcal{G}$ and $\mathcal{F}$ agree (Morphisms of Spaces, Lemma 67.13.6). Observe that $\pi_T \circ i : Z \to Y_T$ is quasi-finite as a composition of quasi-finite morphisms and that $\pi_{T,*}\mathcal{F} = (\pi_T \circ i)_*\mathcal{G})$. Since i is affine, formation of i_* commutes with base change (Cohomology of Spaces, Lemma 69.11.1). Therefore we may replace B by T, X by Z, $\mathcal{F}$ by $\mathcal{G}$, and Y by Y_T to reduce to the case discussed in the next paragraph.

Assume that $X \to B$ is proper. Then π is proper by Morphisms of Spaces, Lemma 67.40.6 and hence finite by More on Morphisms of Spaces, Lemma 76.35.1. Since a finite morphism is affine we see that (b) holds by Cohomology of Spaces, Lemma 69.11.1. On the other hand, π is of finite presentation by Morphisms of Spaces, Lemma 67.28.9. Thus $\pi_{T,*}\mathcal{F}$ is of finite presentation by Descent on Spaces, Lemma 74.6.7. Finally, $\pi_{T,*}\mathcal{F}$ is flat over B for example by looking at stalks using Cohomology of Spaces, Lemma 69.4.2. □

0DNA Lemma 108.4.5. Let B be an algebraic space. Let $\pi : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Then the morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ of Lemma 108.4.4 is an open immersion.

Proof. Omitted. Hint: If $\mathcal{F}$ is an object of $\mathcal{C}oh_{Y/B}$ over T and for $t \in T$ we have $\text{Supp}(\mathcal{F}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. □

0DNB Lemma 108.4.6. Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Then the morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ of Lemma 108.4.4 is a closed immersion.

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the sheaf of ideals cutting out X as a closed subspace of Y. Recall that π_* induces an equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathcal{O}_Y$-modules annihilated by $\mathcal{I}$, see Morphisms of Spaces, Lemma 67.14.1. The same, mutatis mutandis, is true after base by $T \to B$ with $\mathcal{I}$ replaced by the ideal sheaf $\mathcal{I}_T = \text{Im}((Y_T \to Y)^*\mathcal{I} \to \mathcal{O}_{Y_T})$. Analyzing the proof of Lemma 108.4.4 we find that the essential image of $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ is exactly the objects $\xi = (T \to B, \mathcal{F})$ where $\mathcal{F}$ is annihilated by $\mathcal{I}_T$. In other words, ξ is in the essential image if and only if the multiplication map

$$\mathcal{F} \otimes_{\mathcal{O}_{Y_T}} (Y_T \to Y)^*\mathcal{I} \longrightarrow \mathcal{F}$$

is zero and similarly after any further base change $T' \to T$. Note that

$$(Y_{T'} \to Y_T)^*(\mathcal{F} \otimes_{\mathcal{O}_{Y_T}} (Y_T \to Y)^*\mathcal{I}) = (Y_{T'} \to Y_T)^*\mathcal{F} \otimes_{\mathcal{O}_{Y_{T'}}} (Y_{T'} \to Y)^*\mathcal{I})$$

Hence the vanishing of the multiplication map on T' is representable by a closed subspace of T by Flatness on Spaces, Lemma 77.8.6. □

0DNC Situation 108.4.7 (Numerical invariants). Let $f : X \to B$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Given an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ denote $E_{i,T}$ the derived pullback of E_i to X_T. The object

$$K_i = Rf_{T,*}(E_{i,T} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_T}} \mathcal{F})$$

of $D(\mathcal{O}_T)$ is perfect and its formation commutes with base change, see Derived Categories of Spaces, Lemma 75.25.1. Thus the function

$$\chi_i : |T| \longrightarrow \mathbf{Z}, \quad \chi_i(t) = \chi(X_t, E_{i,t} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_t}} \mathcal{F}_t) = \chi(K_i \otimes^{\mathbf{L}}_{\mathcal{O}_T} \kappa(t))$$

is locally constant by Derived Categories of Spaces, Lemma 75.26.3. Let $P : I \to \mathbf{Z}$ be a map. Consider the substack

$$\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$$

consisting of flat families of coherent sheaves with proper support whose numerical invariants agree with P. More precisely, an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ is in $\mathcal{C}oh^P_{X/B}$ if and only if $\chi_i(t) = P(i)$ for all $i \in I$ and $t \in T$.

0DND Lemma 108.4.8. In Situation 108.4.7 the stack $\mathcal{C}oh^P_{X/B}$ is algebraic and

$$\mathcal{C}oh^P_{X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

is a flat closed immersion. If I is finite or B is locally Noetherian, then $\mathcal{C}oh^P_{X/B}$ is an open and closed substack of $\mathcal{C}oh_{X/B}$.

Proof. This is immediately clear if I is finite, because the functions $t \mapsto \chi_i(t)$ are locally constant. If I is infinite, then we write

$$I = \bigcup\nolimits_{I' \subset I \text{ finite}} I'$$

and we denote $P' = P|_{I'}$. Then we have

$$\mathcal{C}oh^P_{X/B} = \bigcap\nolimits_{I' \subset I \text{ finite}} \mathcal{C}oh^{P'}_{X/B}$$

Therefore, $\mathcal{C}oh^P_{X/B}$ is always an algebraic stack and the morphism $\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$ is always a flat closed immersion, but it may no longer be an open substack. (We leave it to the reader to make examples). However, if B is locally Noetherian, then so is $\mathcal{C}oh_{X/B}$ by Lemma 108.4.2 and Morphisms of Stacks, Lemma 101.17.5. Hence if $U \to \mathcal{C}oh_{X/B}$ is a smooth surjective morphism where U is a locally Noetherian scheme, then the inverse images of the open and closed substacks $\mathcal{C}oh^{P'}_{X/B}$ have an open intersection in U (because connected components of locally Noetherian topological spaces are open). Thus the result in this case. □

0DNE Lemma 108.4.9. Let $f : X \to B$ be as in the introduction to this section. Let $E_1, \ldots, E_r \in D(\mathcal{O}_X)$ be perfect. Let $I = \mathbf{Z}^{\oplus r}$ and consider the map

$$I \longrightarrow D(\mathcal{O}_X), \quad (n_1, \ldots, n_r) \longmapsto E_1^{\otimes n_1} \otimes \ldots \otimes E_r^{\otimes n_r}$$

Let $P : I \to \mathbf{Z}$ be a map. Then $\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$ as defined in Situation 108.4.7 is an open and closed substack.

Proof. We may work étale locally on B, hence we may assume that B is affine. In this case we may perform absolute Noetherian reduction; we suggest the reader skip the proof. Namely, say $B = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is separated and of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1 and 70.6.9. Then after increasing i we may assume there exist perfect objects $E_{1,i}, \ldots, E_{r,i}$ in $D(\mathcal{O}_{X_i})$

whose derived pullback to X are isomorphic to $E_1, \ldots, E_r$, see Derived Categories of Spaces, Lemma 75.24.3. Clearly we have a cartesian square

$$\begin{tikzcd}
\mathcal{C}oh^P_{X/B} \arrow[r] \arrow[d] & \mathcal{C}oh_{X/B} \arrow[d] \\
\mathcal{C}oh^P_{X_i/\operatorname{Spec}(\Lambda_i)} \arrow[r] & \mathcal{C}oh_{X_i/\operatorname{Spec}(\Lambda_i)}
\end{tikzcd}$$

and hence we may appeal to Lemma 108.4.8 to finish the proof. □

0DNF Example 108.4.10 (Coherent sheaves with fixed Hilbert polynomial). Let $f : X \to B$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $P : \mathbf{Z} \to \mathbf{Z}$ be a numerical polynomial. Then we can consider the open and closed algebraic substack

$$\mathcal{C}oh^P_{X/B} = \mathcal{C}oh^{P,\mathcal{L}}_{X/B} \subset \mathcal{C}oh_{X/B}$$

consisting of flat families of coherent sheaves with proper support whose numerical invariants agree with P: an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ lies in $\mathcal{C}oh^P_{X/B}$ if and only if

$$P(n) = \chi(X_t, \mathcal{F}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}_t^{\otimes n})$$

for all $n \in \mathbf{Z}$ and $t \in T$. Of course this is a special case of Situation 108.4.7 where $I = \mathbf{Z} \to D(\mathcal{O}_X)$ is given by $n \mapsto \mathcal{L}^{\otimes n}$. It follows from Lemma 108.4.9 that this is an open and closed substack. Since the functions $n \mapsto \chi(X_t, \mathcal{F}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}_t^{\otimes n})$ are always numerical polynomials (Spaces over Fields, Lemma 72.18.1) we conclude that

$$\mathcal{C}oh_{X/B} = \coprod\nolimits_{P \text{ numerical polynomial}} \mathcal{C}oh^P_{X/B}$$

is a disjoint union decomposition.

108.5. Properties of Quot

0DM1 Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathrm{Quot}_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is locally of finite presentation. See Quot, Proposition 99.8.4.

0DM2 Lemma 108.5.1. The diagonal of $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then the diagonal is a closed immersion of finite presentation.

Proof. Suppose we have a scheme T/B and two quotients $\mathcal{F}_T \to \mathcal{Q}_i$, $i = 1, 2$ corresponding to T-valued points of $\mathrm{Quot}_{\mathcal{F}/X/B}$ over B. Denote $\mathcal{K}_1$ the kernel of the first one and set $u : \mathcal{K}_1 \to \mathcal{Q}_2$ the composition. By Flatness on Spaces, Lemma 77.8.6 there is a closed subspace of T such that $T' \to T$ factors through it if and only if the pullback $u_{T'}$ is zero. This proves the diagonal is a closed immersion. Moreover, if $\mathcal{F}$ is of finite type, then $\mathcal{K}_1$ is of finite type (Modules on Sites, Lemma 18.24.1) and we see that the diagonal is of finite presentation by the same lemma. □

0DM3 Lemma 108.5.2. The morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated. If $\mathcal{F}$ is of finite presentation, then it is also locally of finite presentation.

Proof. To check $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 108.5.1. The second statement is part of Quot, Proposition 99.8.4. $\square$

0DM4 Lemma 108.5.3. Assume $X \to B$ is proper as well as of finite presentation and $\mathcal{F}$ quasi-coherent of finite type. Then $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Spaces, Definition 67.41.1).

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K, an algebraic space X proper over R, a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, and a coherent quotient $\mathcal{F}_K \to \mathcal{Q}_K$, show there exists a quotient $\mathcal{F} \to \mathcal{Q}$ where $\mathcal{Q}$ is a finitely presented $\mathcal{O}_X$-module flat over R whose generic fibre is $\mathcal{Q}_K$. Observe that by Flatness on Spaces, Theorem 77.4.5 any finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ flat over R is of finite presentation. We first solve the existence of $\mathcal{Q}$ affine locally.

Affine locally we arrive at the following problem: let $R \to A$ be a finitely presented ring map, let M be a finite A-module, let $\varphi : M_K \to N_K$ be an A_K-quotient module. Then we may consider

$$L = \{x \in M \mid \varphi(x \otimes 1) = 0\}$$

The $M \to M/L$ is an A-module quotient which is torsion free as an R-module. Hence it is flat as an R-module (More on Algebra, Lemma 15.22.10). Since M is finite as an A-module so is L and we conclude that L is of finite presentation as an A-module (by the reference above). Clearly M/L is the unqiue such quotient with $(M/L)_K = N_K$.

The uniqueness in the construction of the previous paragraph guarantees these quotients glue and give the desired $\mathcal{Q}$. Here is a bit more detail. Choose a surjective étale morphism $U \to X$ where U is an affine scheme. Use the above construction to construct a quotient $\mathcal{F}|_U \to \mathcal{Q}_U$ which is quasi-coherent, is flat over R, and recovers $\mathcal{Q}_K|U$ on the generic fibre. Since X is separated, we see that $U \times_X U$ is an affine scheme étale over X as well. Then $\mathcal{F}|_{U \times_X U} \to \mathrm{pr}_1^*\mathcal{Q}_U$ and $\mathcal{F}|_{U \times_X U} \to \mathrm{pr}_2^*\mathcal{Q}_U$ agree as quotients by the uniquess in the construction. Hence we may descend $\mathcal{F}|_U \to \mathcal{Q}_U$ to a surjection $\mathcal{F} \to \mathcal{Q}$ as desired (Properties of Spaces, Proposition 66.32.1). $\square$

0DP1 Lemma 108.5.4. Let B be an algebraic space. Let $\pi : X \to Y$ be an affine quasi-finite morphism of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then π_* induces a morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$.

Proof. Set $\mathcal{G} = \pi_*\mathcal{F}$. Since π is affine we see that for any scheme T over B we have $\mathcal{G}_T = \pi_{T,*}\mathcal{F}_T$ by Cohomology of Spaces, Lemma 69.11.1. Moreover π_T is affine, hence $\pi_{T,*}$ is exact and transforms quotients into quotients. Observe that a quasi-coherent quotient $\mathcal{F}_T \to \mathcal{Q}$ defines a point of $\mathrm{Quot}_{X/B}$ if and only if $\mathcal{Q}$ defines an object of $\mathcal{C}oh_{X/B}$ over T (similarly for $\mathcal{G}$ and Y). Since we've seen in Lemma 108.4.4 that π_* induces a morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ we see that if $\mathcal{F}_T \to \mathcal{Q}$ is in $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then $\mathcal{G}_T \to \pi_{T,*}\mathcal{Q}$ is in $\mathrm{Quot}_{\mathcal{G}/Y/B}(T)$. $\square$

0DP2 Lemma 108.5.5. Let B be an algebraic space. Let $\pi : X \to Y$ be an affine open immersion of algebraic spaces which are separated and of finite presentation over

B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$ of Lemma 108.5.4 is an open immersion.

Proof. Omitted. Hint: If $(\pi_*\mathcal{F})_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}(T)$ and for $t \in T$ we have $\mathrm{Supp}(\mathcal{Q}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. $\square$

0DP3 Lemma 108.5.6. Let B be an algebraic space. Let $j : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module and set $\mathcal{F} = j^*\mathcal{G}$. Then there is an open immersion

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{G}/Y/B}$$

of algebraic spaces over B.

Proof. If $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ then we can consider $\mathcal{G}_T \to j_{T,*}\mathcal{F}_T \to j_{T,*}\mathcal{Q}$. Looking at stalks one finds that this is surjective. By Lemma 108.4.4 we see that $j_{T,*}\mathcal{Q}$ is finitely presented, flat over B with support proper over B. Thus we obtain a T-valued point of $\mathrm{Quot}_{\mathcal{G}/Y/B}$. This defines the morphism of the lemma. We omit the proof that this is an open immersion. Hint: If $\mathcal{G}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{G}/Y/B}(T)$ and for $t \in T$ we have $\mathrm{Supp}(\mathcal{Q}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. $\square$

0DP4 Lemma 108.5.7. Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$ of Lemma 108.5.4 is an isomorphism.

Proof. For every scheme T over B the morphism $\pi_T : X_T \to Y_T$ is a closed immersion. Then $\pi_{T,*}$ is an equivalence of categories between $QCoh(\mathcal{O}_{X_T})$ and the full subcategory of $QCoh(\mathcal{O}_{Y_T})$ whose objects are those quasi-coherent modules annihilated by the ideal sheaf of X_T, see Morphisms of Spaces, Lemma 67.14.1. Since a qotient of $(\pi_*\mathcal{F})_T$ is annihilated by this ideal we obtain the bijectivity of the map $\mathrm{Quot}_{\mathcal{F}/X/B}(T) \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}(T)$ for all T as desired. $\square$

0DP5 Lemma 108.5.8. Let $X \to B$ be as in the introduction to this section. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical closed immersion $\mathrm{Quot}_{\mathcal{G}/X/B} \to \mathrm{Quot}_{\mathcal{F}/X/B}$.

Proof. Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to \mathcal{G})$. By right exactness of pullbacks we find that $\mathcal{K}_T \to \mathcal{F}_T \to \mathcal{G}_T \to 0$ is an exact sequecnce for all schemes T over B. In particular, a quotient of $\mathcal{G}_T$ determines a quotient of $\mathcal{F}_T$ and we obtain our transformation of functors $\mathrm{Quot}_{\mathcal{G}/X/B} \to \mathrm{Quot}_{\mathcal{F}/X/B}$. This transformation is a closed immersion by Flatness on Spaces, Lemma 77.8.6. Namely, given an element $\mathcal{F}_T \to \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then we see that the pull back to T'/T is in the image of the transformation if and only if $\mathcal{K}_{T'} \to \mathcal{Q}_{T'}$ is zero. $\square$

0DP6 Remark 108.5.9 (Numerical invariants). Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that we have a morphism

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

which sends the element $\mathcal{F}_T \to \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ to the object $\mathcal{Q}$ of $\mathcal{C}oh_{X/B}$ over T, see proof of Quot, Proposition 99.8.4. Hence we can form the fibre product diagram

$$\begin{tikzcd} \mathrm{Quot}^P_{\mathcal{F}/X/B} \arrow[r] \arrow[d] & \mathcal{C}oh^P_{X/B} \arrow[d] \\ \mathrm{Quot}_{\mathcal{F}/X/B} \arrow[r] & \mathcal{C}oh_{X/B} \end{tikzcd}$$

This is the defining diagram for the algebraic space in the upper left corner. The left vertical arrow is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$ and $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_X$-module $\mathcal{L}$ (in the last case we sometimes use the notation $\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/B}$). See Situation 108.4.7 and Lemmas 108.4.8 and 108.4.9 and Example 108.4.10.

0DP7 Lemma 108.5.10. Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then tensoring with $\mathcal{L}$ defines an isomorphism

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}$$

Given a numerical polynomial $P(t)$, then setting $P'(t) = P(t+1)$ this map induces an isomorphism $\mathrm{Quot}^P_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}^{P'}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}$ of open and closed substacks.

Proof. Set $\mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$. Observe that $\mathcal{G}_T = \mathcal{F}_T \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$. If $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then we send it to the element $\mathcal{G}_T \to \mathcal{Q} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$ of $\mathrm{Quot}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}(T)$. This is compatible with pullbacks and hence defines a transformation of functors as desired. Since there is an obvious inverse transformation, it is an isomorphism. We omit the proof of the final statement. $\square$

0DP8 Lemma 108.5.11. Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

$$\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/B} = \mathrm{Quot}^{P',\mathcal{L}^{\otimes n}}_{\mathcal{F}/X/B}$$

where $P'(t) = P(nt)$.

Proof. Follows immediately after unwinding all the definitions. $\square$

108.6. Boundedness for Quot

0DP9 Contrary to what happens classically, we already know the Quot functor is an algebraic space, but we don't know that it is ever represented by a finite type algebraic space.

0DPA Lemma 108.6.1. Let $n \geq 0$, $r \geq 1$, $P \in \mathbf{Q}[t]$. The algebraic space

$$X = \mathrm{Quot}^P_{\mathcal{O}^{\oplus r}_{\mathbf{P}^n_{\mathbf{Z}}}/\mathbf{P}^n_{\mathbf{Z}}/\mathbf{Z}}$$

parametrizing quotients of $\mathcal{O}^{\oplus r}_{\mathbf{P}^n_{\mathbf{Z}}}$ with Hilbert polynomial P is proper over $\mathrm{Spec}(\mathbf{Z})$.

Proof. We already know that $X \to \mathrm{Spec}(\mathbf{Z})$ is separated and locally of finite presentation (Lemma 108.5.2). We also know that $X \to \mathrm{Spec}(\mathbf{Z})$ satisfies the existence part of the valuative criterion, see Lemma 108.5.3. By the valuative criterion for properness, it suffices to prove our Quot space is quasi-compact, see Morphisms of Spaces, Lemma 67.44.1. Thus it suffices to find a quasi-compact scheme T and a surjective morphism $T \to X$. Let m be the integer found in Varieties, Lemma 33.35.18. Let

$$N = r\binom{m+n}{n} - P(m)$$

We will write $\mathbf{P}^n$ for $\mathbf{P}^n_\mathbf{Z} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ and unadorned products will mean products over $\text{Spec}(\mathbf{Z})$. The idea of the proof is to construct a "universal" map

$$\Psi : \mathcal{O}_{T \times \mathbf{P}^n}(-m)^{\oplus N} \longrightarrow \mathcal{O}_{T \times \mathbf{P}^n}^{\oplus r}$$

over an affine scheme T and show that every point of X corresponds to a cokernel of this in some point of T.

Definition of T and Ψ. We take $T = \text{Spec}(A)$ where

$$A = \mathbf{Z}[a_{i,j,E}]$$

where $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, N\}$ and $E = (e_0, \ldots, e_n)$ runs through the multi-indices of total degree $|E| = \sum_{k=0,\ldots n} e_k = m$. Then we define Ψ to be the map whose (i,j) matrix entry is the map

$$\sum\nolimits_{E=(e_0,\ldots,e_n)} a_{i,j,E} T_0^{e_0} \ldots T_n^{e_n} : \mathcal{O}_{T \times \mathbf{P}^n}(-m) \longrightarrow \mathcal{O}_{T \times \mathbf{P}^n}$$

where the sum is over E as above (but i and j are fixed of course).

Consider the quotient $\mathcal{Q} = \text{Coker}(\Psi)$ on $T \times \mathbf{P}^n$. By More on Morphisms, Lemma 37.54.1 there exists a $t \geq 0$ and closed subschemes

$$T = T_0 \supset T_1 \supset \ldots \supset T_t = \emptyset$$

such that the pullback $\mathcal{Q}_p$ of $\mathcal{Q}$ to $(T_p \setminus T_{p+1}) \times \mathbf{P}^n$ is flat over $T_p \setminus T_{p+1}$. Observe that we have an exact sequence

$$\mathcal{O}_{(T_p \setminus T_{p+1}) \times \mathbf{P}^n}(-m)^{\oplus N} \to \mathcal{O}_{(T_p \setminus T_{p+1}) \times \mathbf{P}^n}^{\oplus r} \to \mathcal{Q}_p \to 0$$

by pulling back the exact sequence defining $\mathcal{Q} = \text{Coker}(\Psi)$. Therefore we obtain a morphism

$$\coprod (T_p \setminus T_{p+1}) \longrightarrow \text{Quot}_{\mathcal{O}^{\oplus r}/\mathbf{P}/\mathbf{Z}} \supset \text{Quot}^P_{\mathcal{O}^{\oplus r}/\mathbf{P}/\mathbf{Z}} = X$$

Since the left hand side is a Noetherian scheme and the inclusion on the right hand side is open, it suffices to show that any point of X is in the image of this morphism.

Let k be a field and let $x \in X(k)$. Then x corresponds to a surjection $\mathcal{O}_{\mathbf{P}^n_k}^{\oplus r} \to \mathcal{F}$ of coherent $\mathcal{O}_{\mathbf{P}^n_k}$-modules such that the Hilbert polynomial of $\mathcal{F}$ is P. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbf{P}^n_k}^{\oplus r} \to \mathcal{F} \to 0$$

By Varieties, Lemma 33.35.18 and our choice of m we see that $\mathcal{K}$ is m-regular. By Varieties, Lemma 33.35.12 we see that $\mathcal{K}(m)$ is globally generated. By Varieties, Lemma 33.35.10 and the definition of m-regularity we see that $H^i(\mathbf{P}^n_k, \mathcal{K}(m)) = 0$ for $i > 0$. Hence we see that

$$\dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m)) = \chi(\mathcal{K}(m)) = \chi(\mathcal{O}_{\mathbf{P}^n_k}(m)^{\oplus r}) - \chi(\mathcal{F}(m)) = N$$

by our choice of N. This gives a surjection

$$\mathcal{O}_{\mathbf{P}^n_k}^{\oplus N} \longrightarrow \mathcal{K}(m)$$

Twisting back down and using the short exact sequence above we see that $\mathcal{F}$ is the cokernel of a map

$$\Psi_x : \mathcal{O}_{\mathbf{P}^n_k}(-m)^{\oplus N} \to \mathcal{O}_{\mathbf{P}^n_k}^{\oplus r}$$

There is a unique ring map $\tau : A \to k$ such that the base change of Ψ by the corresponding morphism $t = \mathrm{Spec}(\tau) : \mathrm{Spec}(k) \to T$ is Ψ_x. This is true because the entries of the $N \times r$ matrix defining Ψ_x are homogeneous polynomials $\sum \lambda_{i,j,E} T_0^{e_0} \ldots T_n^{e_n}$ of degree m in $T_0, \ldots, T_n$ with coefficients $\lambda_{i,j,E} \in k$ and we can set $\tau(a_{i,j,E}) = \lambda_{i,j,E}$. Then $t \in T_p \setminus T_{p+1}$ for some p and the image of t under the morphism above is x as desired. $\square$

0DPB Lemma 108.6.2. Let B be an algebraic space. Let $X = B \times \mathbf{P}^n_{\mathbf{Z}}$. Let $\mathcal{L}$ be the pullback of $\mathcal{O}_{\mathbf{P}^n}(1)$ to X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.

Proof. The question is étale local over B, see Morphisms of Spaces, Lemma 67.40.2. Thus we may assume B is an affine scheme. In this case $\mathcal{L}$ is an ample invertible module on X (by Constructions, Lemma 27.10.6 and the definition of ample invertible modules in Properties, Definition 28.26.1). Thus we can find $r' \geq 0$ and $r \geq 0$ and a surjection

$$\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r'}$$

by Properties, Proposition 28.26.13. By Lemma 108.5.10 we may replace $\mathcal{F}$ by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r'}$ and $P(t)$ by $P(t+r')$. By Lemma 108.5.8 we obtain a closed immersion

$$\mathrm{Quot}^P_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}^P_{\mathcal{O}_X^{\oplus r}/X/B}$$

Since we've shown that $\mathrm{Quot}^P_{\mathcal{O}_X^{\oplus r}/X/B} \to B$ is proper in Lemma 108.6.1 we conclude. $\square$

0DPC Lemma 108.6.3. Let $f : X \to B$ be a proper morphism of finite presentation of algebraic spaces. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.

Proof. The question is étale local over B, see Morphisms of Spaces, Lemma 67.40.2. Thus we may assume B is an affine scheme. Then we can find a closed immersion $i : X \to \mathbf{P}^n_B$ such that $i^*\mathcal{O}_{\mathbf{P}^n_B}(1) \cong \mathcal{L}^{\otimes d}$ for some $d \geq 1$. See Morphisms, Lemma 29.39.3. Changing $\mathcal{L}$ into $\mathcal{L}^{\otimes d}$ and the numerical polynomial $P(t)$ into $P(dt)$ leaves $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ unaffected; some details omitted. Hence we may assume $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_B}(1)$. Then the isomorphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$ of Lemma 108.5.7 induces an isomorphism $\mathrm{Quot}^P_{\mathcal{F}/X/B} \cong \mathrm{Quot}^P_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$. Since $\mathrm{Quot}^P_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$ is proper over B by Lemma 108.6.2 we conclude. $\square$

0DPD Lemma 108.6.4. Let $f : X \to B$ be a separated morphism of finite presentation of algebraic spaces. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is separated of finite presentation over B.

Proof. We have already seen that $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated and locally of finite presentation, see Lemma 108.5.2. Thus it suffices to show that the open subspace $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ of Remark 108.5.9 is quasi-compact over B.

The question is étale local on B (Morphisms of Spaces, Lemma 67.8.8). Thus we may assume B is affine.

Assume $B = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Then we can find an i and a system $X_i, \mathcal{F}_i, \mathcal{L}_i$ as in the lemma over $B_i = \mathrm{Spec}(\Lambda_i)$ whose base change to B gives $X, \mathcal{F}, \mathcal{L}$. This follows from Limits of Spaces, Lemmas 70.7.1 (to find X_i), 70.7.2 (to find $\mathcal{F}_i$), 70.7.3 (to find $\mathcal{L}_i$), and 70.5.9 (to make X_i separated). Because

$$\mathrm{Quot}_{\mathcal{F}/X/B} = B \times_{B_i} \mathrm{Quot}_{\mathcal{F}_i/X_i/B_i}$$

and similarly for $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ we reduce to the case discussed in the next paragraph.

Assume B is affine and Noetherian. We may replace $\mathcal{L}$ by a positive power, see Lemma 108.5.11. Thus we may assume there exists an immersion $i : X \to \mathbf{P}^n_B$ such that $i^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{L}$. By Morphisms, Lemma 29.7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_B$ such that i factors through an open immersion $j : X \to X'$. By Properties, Lemma 28.22.5 there exists a finitely presented $\mathcal{O}_{X'}$-module $\mathcal{G}$ such that $j^*\mathcal{G} = \mathcal{F}$. Thus we obtain an open immersion

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{G}/X'/B}$$

by Lemma 108.5.6. Clearly this open immersion sends $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ into $\mathrm{Quot}^P_{\mathcal{G}/X'/B}$. Now $\mathrm{Quot}^P_{\mathcal{G}/X'/B}$ is proper over B by Lemma 108.6.3. Therefore it is Noetherian and since any open of a Noetherian algebraic space is quasi-compact we win. ☐

108.7. Properties of the Hilbert functor

0DM5 Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Then $\mathrm{Hilb}_{X/B}$ is an algebraic space locally of finite presentation over B. See Quot, Proposition 99.9.4.

0DM6 Lemma 108.7.1. The diagonal of $\mathrm{Hilb}_{X/B} \to B$ is a closed immersion of finite presentation.

Proof. In Quot, Lemma 99.9.2 we have seen that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Hence this follows from Lemma 108.5.1. ☐

0DM7 Lemma 108.7.2. The morphism $\mathrm{Hilb}_{X/B} \to B$ is separated and locally of finite presentation.

Proof. To check $\mathrm{Hilb}_{X/B} \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 108.7.1. The second statement is part of Quot, Proposition 99.9.4. ☐

0DM8 Lemma 108.7.3. Assume $X \to B$ is proper as well as of finite presentation. Then $\mathrm{Hilb}_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Spaces, Definition 67.41.1).

Proof. In Quot, Lemma 99.9.2 we have seen that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Hence this follows from Lemma 108.5.3. ☐

0DPE Lemma 108.7.4. Let B be an algebraic space. Let $\pi : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Then π induces an open immersion $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$.

Proof. Omitted. Hint: If $Z \subset X_T$ is a closed subscheme which is proper over T, then Z is also closed in Y_T. Thus we obtain the transformation $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$. If $Z \subset Y_T$ is an element of $\mathrm{Hilb}_{Y/B}(T)$ and for $t \in T$ we have $|Z_t| \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. □

0DPF Lemma 108.7.5. Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Then π induces a closed immersion $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$.

Proof. Since π is a closed immersion, it is immediate that given a closed subscheme $Z \subset X_T$, we can view Z as a closed subscheme of X_T. Thus we obtain the transformation $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$. This transformation is immediately seen to be a monomorphism. To prove that it is a closed immersion, you can use Lemma 108.5.8 for the map $\mathcal{O}_Y \to \mathcal{O}_X$ and the identifications $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, $\mathrm{Hilb}_{Y/B} = \mathrm{Quot}_{\mathcal{O}_Y/Y/B}$ of Quot, Lemma 99.9.2. □

0DPG Remark 108.7.6 (Numerical invariants). Let $f : X \to B$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 99.9.2. Thus we can define

$$\mathrm{Hilb}^P_{X/B} = \mathrm{Quot}^P_{\mathcal{O}_X/X/B}$$

where $\mathrm{Quot}^P_{\mathcal{O}_X/X/B}$ is as in Remark 108.5.9. The morphism

$$\mathrm{Hilb}^P_{X/B} \longrightarrow \mathrm{Hilb}_{X/B}$$

is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$ and $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_X$-module $\mathcal{L}$. In the last case we sometimes use the notation $\mathrm{Hilb}^{P,\mathcal{L}}_{X/B}$.

0DPH Lemma 108.7.7. Let $f : X \to B$ be a proper morphism of finite presentation of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Hilb}^P_{X/B}$ parametrizing closed subschemes having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.

Proof. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 99.9.2. Thus this lemma is an immediate consequence of Lemma 108.6.3. □

0DPI Lemma 108.7.8. Let $f : X \to B$ be a separated morphism of finite presentation of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Hilb}^P_{X/B}$ parametrizing closed subschemes having Hilbert polynomial P with respect to $\mathcal{L}$ is separated of finite presentation over B.

Proof. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 99.9.2. Thus this lemma is an immediate consequence of Lemma 108.6.4. □

108.8. Properties of the Picard stack

0DM9 Let $f : X \to B$ be a morphism of algebraic spaces which is flat, proper, and of finite presentation. Then the stack $\mathcal{P}ic_{X/B}$ parametrizing invertible sheaves on X/B is algebraic, see Quot, Proposition 99.10.2.

0DMA **Lemma 108.8.1.** *The diagonal of $\mathcal{P}ic_{X/B}$ over B is affine and of finite presentation.*

Proof. In Quot, Lemma 99.10.1 we have seen that $\mathcal{P}ic_{X/B}$ is an open substack of $\mathcal{C}oh_{X/B}$. Hence this follows from Lemma 108.4.1. $\square$

0DMB **Lemma 108.8.2.** *The morphism $\mathcal{P}ic_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. In Quot, Lemma 99.10.1 we have seen that $\mathcal{P}ic_{X/B}$ is an open substack of $\mathcal{C}oh_{X/B}$. Hence this follows from Lemma 108.4.2. $\square$

0DNG **Lemma 108.8.3.** *Assume $X \to B$ is smooth in addition to being proper. Then $\mathcal{P}ic_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Stacks, Definition 101.39.10).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K and an algebraic space X proper and smooth over R and an invertible $\mathcal{O}_{X_K}$-module $\mathcal{L}_K$, show there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ whose generic fibre is $\mathcal{L}_K$. Observe that X_K is Noetherian, separated, and regular (use Morphisms of Spaces, Lemma 67.28.6 and Spaces over Fields, Lemma 72.16.1). Thus we can write $\mathcal{L}_K$ as the difference in the Picard group of $\mathcal{O}_{X_K}(D_K)$ and $\mathcal{O}_{X_K}(D'_K)$ for two effective Cartier divisors D_K, D'_K in X_K, see Divisors on Spaces, Lemma 71.8.4. Finally, we know that D_K and D'_K are restrictions of effective Cartier divisors $D, D' \subset X$, see Divisors on Spaces, Lemma 71.8.5. $\square$

0DNH **Lemma 108.8.4.** *Assume $f_{T,*}\mathcal{O}_{X_T} \cong \mathcal{O}_T$ for all schemes T over B. Then the inertia stack of $\mathcal{P}ic_{X/B}$ is equal to $\mathbf{G}_m \times \mathcal{P}ic_{X/B}$.*

Proof. This is explained in Examples of Stacks, Example 95.17.2. $\square$

0DPJ **Lemma 108.8.5.** *Assume $f : X \to B$ has relative dimension ≤ 1 in addition to the other assumptions in this section. Then $\mathcal{P}ic_{X/B} \to B$ is smooth.*

Proof. We already know that $\mathcal{P}ic_{X/B} \to B$ is locally of finite presentation, see Lemma 108.8.2. Thus it suffices to show that $\mathcal{P}ic_{X/B} \to B$ is formally smooth, see More on Morphisms of Stacks, Lemma 106.8.7. Taking base change, this immediately reduces to the following problem: given a first order thickening $T \subset T'$ of affine schemes, given $X' \to T'$ proper, flat, of finite presentation and of relative dimension ≤ 1, and for $X = T \times_{T'} X'$ given an invertible $\mathcal{O}_X$-module $\mathcal{L}$, prove that there exists an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ whose restriction to X is $\mathcal{L}$. Since $T \subset T'$ is a first order thickening, the same is true for $X \subset X'$, see More on Morphisms of Spaces, Lemma 76.9.8. By More on Morphisms of Spaces, Lemma 76.11.1 we see that it suffices to show $H^2(X, \mathcal{I}) = 0$ where $\mathcal{I}$ is the quasi-coherent ideal cutting out X in X'. Denote $f : X \to T$ the structure morphism. By Cohomology of Spaces, Lemma 69.22.9 we see that $R^pf_*\mathcal{I} = 0$ for $p > 1$. Hence we get the desired vanishing by Cohomology of Spaces, Lemma 69.3.2 (here we finally use that T is affine). $\square$

108.9. Properties of the Picard functor

0DMD Let $f : X \to B$ be a morphism of algebraic spaces which is flat, proper, and of finite presentation such that moreover for every T/B the canonical map

$$\mathcal{O}_T \longrightarrow f_{T,*}\mathcal{O}_{X_T}$$

is an isomorphism. Then the Picard functor $\mathrm{Pic}_{X/B}$ is an algebraic space, see Quot, Proposition 99.11.8. There is a closed relationship with the Picard stack.

0DME Lemma 108.9.1. The morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ turns the Picard stack into a gerbe over the Picard functor.

Proof. The definition of $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ being a gerbe is given in Morphisms of Stacks, Definition 101.28.1, which in turn refers to Stacks, Definition 8.11.4. To prove it, we will check conditions (2)(a) and (2)(b) of Stacks, Lemma 8.11.3. This follows immediately from Quot, Lemma 99.11.2; here is a detailed explanation.

Condition (2)(a). Suppose that $\xi \in \mathrm{Pic}_{X/B}(U)$ for some scheme U over B. Since $\mathrm{Pic}_{X/B}$ is the fppf sheafification of the rule $T \mapsto \mathrm{Pic}(X_T)$ on schemes over B (Quot, Situation 99.11.1), we see that there exists an fppf covering $\{U_i \to U\}$ such that $\xi|_{U_i}$ corresponds to some invertible module $\mathcal{L}_i$ on X_{U_i}. Then $(U_i \to B, \mathcal{L}_i)$ is an object of $\mathcal{P}ic_{X/B}$ over U_i mapping to $\xi|_{U_i}$.

Condition (2)(b). Suppose that U is a scheme over B and $\mathcal{L}, \mathcal{N}$ are invertible modules on X_U which map to the same element of $\mathrm{Pic}_{X/B}(U)$. Then there exists an fppf covering $\{U_i \to U\}$ such that $\mathcal{L}|_{X_{U_i}}$ is isomorphic to $\mathcal{N}|_{X_{U_i}}$. Thus we find isomorphisms between $(U \to B, \mathcal{L})|_{U_i} \to (U \to B, \mathcal{N})|_{U_i}$ as desired. $\square$

0DMF Lemma 108.9.2. The diagonal of $\mathrm{Pic}_{X/B}$ over B is a quasi-compact immersion.

Proof. The diagonal is an immersion by Quot, Lemma 99.11.9. To finish we show that the diagonal is quasi-compact. The diagonal of $\mathcal{P}ic_{X/B}$ is quasi-compact by Lemma 108.8.1 and $\mathcal{P}ic_{X/B}$ is a gerbe over $\mathrm{Pic}_{X/B}$ by Lemma 108.9.1. We conclude by Morphisms of Stacks, Lemma 101.28.14. $\square$

0DNI Lemma 108.9.3. The morphism $\mathrm{Pic}_{X/B} \to B$ is quasi-separated and locally of finite presentation.

Proof. To check $\mathrm{Pic}_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact. This is immediate from Lemma 108.9.2. Since the morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ is surjective, flat, and locally of finite presentation (by Lemma 108.9.1 and Morphisms of Stacks, Lemma 101.28.8) it suffices to prove that $\mathcal{P}ic_{X/B} \to B$ is locally of finite presentation, see Morphisms of Stacks, Lemma 101.27.12. This follows from Lemma 108.8.2. $\square$

0DNJ Lemma 108.9.4. Assume the geometric fibres of $X \to B$ are integral in addition to the other assumptions in this section. Then $\mathrm{Pic}_{X/B} \to B$ is separated.

Proof. Since $\mathrm{Pic}_{X/B} \to B$ is quasi-separated, it suffices to check the uniqueness part of the valuative criterion, see Morphisms of Spaces, Lemma 67.43.2. This immediately reduces to the following problem: given

(1) a valuation ring R with fraction field K,
(2) an algebraic space X proper and flat over R with integral geometric fibre,
(3) an element $a \in \mathrm{Pic}_{X/R}(R)$ with $a|_{\mathrm{Spec}(K)} = 0$,

then we have to prove $a = 0$. Applying Morphisms of Stacks, Lemma 101.25.6 to the surjective flat morphism $\mathcal{P}ic_{X/R} \to \mathrm{Pic}_{X/R}$ (surjective and flat by Lemma 108.9.1 and Morphisms of Stacks, Lemma 101.28.8) after replacing R by an extension we may assume a is given by an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Since $a|_{\mathrm{Spec}(K)} = 0$ we find $\mathcal{L}_K \cong \mathcal{O}_{X_K}$ by Quot, Lemma 99.11.3.

Denote $f : X \to \mathrm{Spec}(R)$ the structure morphism. Let $\eta, 0 \in \mathrm{Spec}(R)$ be the generic and closed point. Consider the perfect complexes $K = Rf_*\mathcal{L}$ and $M = Rf_*(\mathcal{L}^{\otimes -1})$ on $\mathrm{Spec}(R)$, see Derived Categories of Spaces, Lemma 75.25.4. Consider the functions $\beta_{K,i}, \beta_{M,i} : \mathrm{Spec}(R) \to \mathbf{Z}$ of Derived Categories of Spaces, Lemma 75.26.1 associated to K and M. Since the formation of K amd M commutes with base change (see lemma cited above) we find $\beta_{K,0}(\eta) = \beta_{M,0}(\beta) = 1$ by Spaces over Fields, Lemma 72.14.3 and our assumption on the fibres of f. By upper semi-continuity we find $\beta_{K,0}(0) \geq 1$ and $\beta_{M,0} \geq 1$. By Spaces over Fields, Lemma 72.14.4 we conclude that the restriction of $\mathcal{L}$ to the special fibre X_0 is trivial. In turn this gives $\beta_{K,0}(0) = \beta_{M,0} = 1$ as above. Then by More on Algebra, Lemma 15.75.5 we can represent K by a complex of the form

$$\ldots \to 0 \to R \to R^{\oplus \beta_{K,1}(0)} \to R^{\oplus \beta_{K,2}(0)} \to \ldots$$

Now $R \to R^{\oplus \beta_{K,1}(0)}$ is zero because $\beta_{K,0}(\eta) = 1$. In other words $K = R \oplus \tau_{\geq 1}(K)$ in $D(R)$ where $\tau_{\geq 1}(K)$ has tor amplitude in $[1, b]$ for some $b \in \mathbf{Z}$. Hence there is a global section $s \in H^0(X, \mathcal{L})$ whose restriction s_0 to X_0 is nonvanishing (again because formation of K commutes with base change). Then $s : \mathcal{O}_X \to \mathcal{L}$ is a map of invertible sheaves whose restriction to X_0 is an isomorphism and hence is an isomorphism as desired. □

0DPK Lemma 108.9.5. Assume $f : X \to B$ has relative dimension ≤ 1 in addition to the other assumptions in this section. Then $\mathrm{Pic}_{X/B} \to B$ is smooth.

Proof. By Lemma 108.8.5 we know that $\mathcal{P}ic_{X/B} \to B$ is smooth. The morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ is surjective and smooth by combining Lemma 108.9.1 with Morphisms of Stacks, Lemma 101.33.8. Thus if U is a scheme and $U \to \mathcal{P}ic_{X/B}$ is surjective and smooth, then $U \to \mathrm{Pic}_{X/B}$ is surjective and smooth and $U \to B$ is surjective and smooth (because these properties are preserved by composition). Thus $\mathrm{Pic}_{X/B} \to B$ is smooth for example by Descent on Spaces, Lemma 74.8.3. □

108.10. Properties of relative morphisms

0DPL Let B be an algebraic space. Let X and Y be algebraic spaces over B such that $Y \to B$ is flat, proper, and of finite presentation and $X \to B$ is separated and of finite presentation. Then the functor $Mor_B(Y, X)$ of relative morphisms is an algebraic space locally of finite presentation over B. See Quot, Proposition 99.12.3.

0DPM Lemma 108.10.1. The diagonal of $Mor_B(Y, X) \to B$ is a closed immersion of finite presentation.

Proof. There is an open immersion $Mor_B(Y, X) \to \mathrm{Hilb}_{Y \times_B X/B}$, see Quot, Lemma 99.12.2. Thus the lemma follows from Lemma 108.7.1. □

0DPN Lemma 108.10.2. The morphism $Mor_B(Y, X) \to B$ is separated and locally of finite presentation.

Proof. To check $\mathit{Mor}_B(Y, X) \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 108.10.1. The second statement is part of Quot, Proposition 99.12.3. □

0DPP Lemma 108.10.3. With B, X, Y as in the introduction of this section, in addition assume $X \to B$ is proper. Then the subfunctor $\mathit{Isom}_B(Y, X) \subset \mathit{Mor}_B(Y, X)$ of isomorphisms is an open subspace.

Proof. Follows immediately from More on Morphisms of Spaces, Lemma 76.49.6. □

0DPQ Remark 108.10.4 (Numerical invariants). Let B, X, Y be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_{Y \times_B X})$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that

$$\mathit{Mor}_B(Y, X) \subset \mathrm{Hilb}_{Y \times_B X/B}$$

is an open subspace, see Quot, Lemma 99.12.2. Thus we can define

$$\mathit{Mor}^P_B(Y, X) = \mathit{Mor}_B(Y, X) \cap \mathrm{Hilb}^P_{Y \times_B X/B}$$

where $\mathrm{Hilb}^P_{Y \times_B X/B}$ is as in Remark 108.7.6. The morphism

$$\mathit{Mor}^P_B(Y, X) \longrightarrow \mathit{Mor}_B(Y, X)$$

is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$, $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_{Y \times_B X}$-module $\mathcal{L}$. In the last case we sometimes use the notation $\mathit{Mor}^{P,\mathcal{L}}_B(Y, X)$.

0DPR Lemma 108.10.5. With B, X, Y as in the introduction of this section, let $\mathcal{L}$ be ample on X/B and let $\mathcal{N}$ be ample on Y/B. See Divisors on Spaces, Definition 71.14.1. Let P be a numerical polynomial. Then

$$\mathit{Mor}^{P,\mathcal{M}}_B(Y, X) \longrightarrow B$$

is separated and of finite presentation where $\mathcal{M} = \mathrm{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_B X}} \mathrm{pr}_2^*\mathcal{L}$.

Proof. By Lemma 108.10.2 the morphism $\mathit{Mor}_B(Y, X) \to B$ is separated and locally of finite presentation. Thus it suffices to show that the open and closed subspace $\mathit{Mor}^{P,\mathcal{M}}_B(Y, X)$ of Remark 108.10.4 is quasi-compact over B.

The question is étale local on B (Morphisms of Spaces, Lemma 67.8.8). Thus we may assume B is affine.

Assume $B = \mathrm{Spec}(\Lambda)$. Note that X and Y are schemes and that $\mathcal{L}$ and $\mathcal{N}$ are ample invertible sheaves on X and Y (this follows immediately from the definitions). Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Then we can find an i and a system $X_i, Y_i, \mathcal{L}_i, \mathcal{N}_i$ as in the lemma over $B_i = \mathrm{Spec}(\Lambda_i)$ whose base change to B gives $X, Y, \mathcal{L}, \mathcal{N}$. This follows from Limits, Lemmas 32.10.1 (to find X_i, Y_i), 32.10.3 (to find $\mathcal{L}_i$, $\mathcal{N}_i$), 32.8.6 (to make $X_i \to B_i$ separated), 32.13.1 (to make $Y_i \to B_i$ proper), and 32.4.15 (to make $\mathcal{L}_i$, $\mathcal{N}_i$ ample). Because

$$\mathit{Mor}_B(Y, X) = B \times_{B_i} \mathit{Mor}_{B_i}(Y_i, X_i)$$

and similarly for $\mathit{Mor}^P_B(Y, X)$ we reduce to the case discussed in the next paragraph.

Assume B is a Noetherian affine scheme. By Properties, Lemma 28.26.15 we see that $\mathcal{M}$ is ample. By Lemma 108.7.8 we see that $\mathrm{Hilb}^{P,\mathcal{M}}_{Y \times_B X/B}$ is of finite presentation over B and hence Noetherian. By construction

$$\mathit{Mor}^{P,\mathcal{M}}_B(Y, X) = \mathit{Mor}_B(Y, X) \cap \mathrm{Hilb}^{P,\mathcal{M}}_{Y \times_B X/B}$$

is an open subspace of $\mathrm{Hilb}^{P,\mathcal{M}}_{Y \times_B X/B}$ and hence quasi-compact (as an open of a Noetherian algebraic space is quasi-compact). □

108.11. Properties of the stack of polarized proper schemes

0DPS In this section we discuss properties of the moduli stack

$$\mathcal{P}\mathit{olarized} \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

whose category of sections over a scheme S is the category of proper, flat, finitely presented scheme over S endowed with a relatively ample invertible sheaf. This is an algebraic stack by Quot, Theorem 99.14.15.

0DPT Lemma 108.11.1. The diagonal of $\mathcal{P}\mathit{olarized}$ is separated and of finite presentation.

Proof. Recall that $\mathcal{P}\mathit{olarized}$ is a limit preserving algebraic stack, see Quot, Lemma 99.14.8. By Limits of Stacks, Lemma 102.3.6 this implies that $\Delta : \mathcal{P}\mathit{olarized} \to \mathcal{P}\mathit{olarized} \times \mathcal{P}\mathit{olarized}$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 102.3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given an affine scheme U and two objects $\upsilon = (Y, \mathcal{N})$ and $\chi = (X, \mathcal{L})$ of $\mathcal{P}\mathit{olarized}$ over U, the algebraic space

$$\mathit{Isom}_{\mathcal{P}\mathit{olarized}}(\upsilon, \chi)$$

is separated. The rule which to an isomorphism $\upsilon_T \to \chi_T$ assigns the underlying isomorphism $Y_T \to X_T$ defines a morphism

$$\mathit{Isom}_{\mathcal{P}\mathit{olarized}}(\upsilon, \chi) \longrightarrow \mathit{Isom}_U(Y, X)$$

Since we have seen in Lemmas 108.10.2 and 108.10.3 that the target is a separated algebraic space, it suffices to prove that this morphism is separated. Given an isomorphism $f : Y_T \to X_T$ over some scheme T/U, then clearly

$$\mathit{Isom}_{\mathcal{P}\mathit{olarized}}(\upsilon, \chi) \times_{\mathit{Isom}_U(Y,X),[f]} T = \mathit{Isom}(\mathcal{N}_T, f^*\mathcal{L}_T)$$

Here $[f] : T \to \mathit{Isom}_U(Y, X)$ indicates the T-valued point corresponding to f and $\mathit{Isom}(\mathcal{N}_T, f^*\mathcal{L}_T)$ is the algebraic space discussed in Section 108.3. Since this algebraic space is affine over U, the claim implies Δ is separated.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffice to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{P}\mathit{olarized} \times \mathcal{P}\mathit{olarized}$ is quasi-compact (see for example Properties of Stacks, Lemma 100.3.3). We can assume $U = \coprod U_i$ is a disjoint union of affine opens. Since $\mathcal{P}\mathit{olarized}$ is limit preserving (see above), we see that $\mathcal{P}\mathit{olarized} \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 102.3.8 and Morphisms of Stacks, Lemmas 101.27.2 and 101.33.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $\upsilon = (Y, \mathcal{N})$ and $\chi = (X, \mathcal{L})$ of $\mathcal{P}olarized$ over U, we show the algebraic space

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let P be the Hilbert polynomial $n \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$, see Varieties, Lemma 33.45.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 36.32.2) we see that we get the same Hilbert polynomial in every point of U. Set $\mathcal{M} = \text{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \text{pr}_2^*\mathcal{L}$ on $Y \times_U X$. Given $(f, \varphi) \in Isom_{\mathcal{P}olarized}(\upsilon, \chi)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\chi(Y_t, (\text{id} \times f)^*\mathcal{M}^{\otimes n}) = \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) = \chi(Y_t, \mathcal{N}_t^{\otimes 2n}) = P(2n)$$

where in the middle equality we use the isomorphism $\varphi : f^*\mathcal{L}_T \to \mathcal{N}_T$. Setting $P'(t) = P(2t)$ we find that the morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X)$$

(see earlier) has image contained in the intersection

$$Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $Mor_U(Y, X)$ (see Lemma 108.10.3 and Remark 108.10.4). Now $Mor_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Lemma 108.10.5. Thus the intersection is a Noetherian algebraic space too. Since the morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

is affine (see above) we conclude. □

0DPU Lemma 108.11.2. The morphism $\mathcal{P}olarized \to \text{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.

Proof. To check $\mathcal{P}olarized \to \text{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 108.11.1. To prove that $\mathcal{P}olarized \to \text{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that $\mathcal{P}olarized$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This is Quot, Lemma 99.14.8. □

0E96 Lemma 108.11.3. Let $n \geq 1$ be an integer and let P be a numerical polynomial. Let

$$T \subset |\mathcal{P}olarized|$$

be a subset with the following property: for every $\xi \in T$ there exists a field k and an object $(X, \mathcal{L})$ of $\mathcal{P}olarized$ over k representing ξ such that

(1) the Hilbert polynomial of $\mathcal{L}$ on X is P, and
(2) there exists a closed immersion $i : X \to \mathbf{P}^n_k$ such that $i^*\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}$.

Then T is a Noetherian topological space, in particular quasi-compact.

Proof. Observe that $|\mathcal{P}olarized|$ is a locally Noetherian topological space, see Morphisms of Stacks, Lemma 101.8.3 (this also uses that $\text{Spec}(\mathbf{Z})$ is Noetherian and hence $\mathcal{P}olarized$ is a locally Noetherian algebraic stack by Lemma 108.11.2 and Morphisms of Stacks, Lemma 101.17.5). Thus any quasi-compact subset of $|\mathcal{P}olarized|$

is a Noetherian topological space and any subset of such is also Noetherian, see Topology, Lemmas 5.9.4 and 5.9.2. Thus all we have to do is a find a quasi-compact subset containing T.

By Lemma 108.7.7 the algebraic space

$$H = \mathrm{Hilb}^{P, \mathcal{O}(1)}_{\mathbf{P}^n_{\mathbf{Z}}/\mathrm{Spec}(\mathbf{Z})}$$

is proper over $\mathrm{Spec}(\mathbf{Z})$. By Quot, Lemma 99.9.3[1] the identity morphism of H corresponds to a closed subspace

$$Z \subset \mathbf{P}^n_H$$

which is proper, flat, and of finite presentation over H and such that the restriction $\mathcal{N} = \mathcal{O}(1)|_Z$ is relatively ample on Z/H and has Hilbert polynomial P on the fibres of $Z \to H$. In particular, the pair $(Z \to H, \mathcal{N})$ defines a morphism

$$H \longrightarrow \mathcal{P}olarized$$

which sends a morphism of schemes $U \to H$ to the classifying morphism of the family $(Z_U \to U, \mathcal{N}_U)$, see Quot, Lemma 99.14.4. Since H is a Noetherian algebraic space (as it is proper over $\mathbf{Z}$)) we see that $|H|$ is Noetherian and hence quasi-compact. The map

$$|H| \longrightarrow |\mathcal{P}olarized|$$

is continuous, hence the image is quasi-compact. Thus it suffices to prove T is contained in the image of $|H| \to |\mathcal{P}olarized|$. However, assumptions (1) and (2) exactly express the fact that this is the case: any choice of a closed immersion $i : X \to \mathbf{P}^n_k$ with $i^*\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}$ we get a k-valued point of H by the moduli interpretation of H. This finishes the proof of the lemma. □

108.12. Properties of moduli of complexes on a proper morphism

0DPV Let $f : X \to B$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Then the stack $\mathcal{C}omplexes_{X/B}$ parametrizing relatively perfect complexes with vanishing negative self-exts is algebraic. See Quot, Theorem 99.16.12.

0DPW **Lemma 108.12.1.** *The diagonal of $\mathcal{C}omplexes_{X/B}$ over B is affine and of finite presentation.*

Proof. The representability of the diagonal by algebraic spaces was shown in Quot, Lemma 99.16.5. From the proof we find that we have to show: given a scheme T over B and objects $E, E' \in D(\mathcal{O}_{X_T})$ such that (T, E) and (T, E') are objects of the fibre category of $\mathcal{C}omplexes_{X/B}$ over T, then $Isom(E, E') \to T$ is affine and of finite presentation. Here $Isom(E, E')$ is the functor

$$(Sch/T)^{opp} \to \mathrm{Sets}, \quad T' \mapsto \{\varphi : E_{T'} \to E'_{T'} \text{ isomorphism in } D(\mathcal{O}_{X_{T'}})\}$$

where $E_{T'}$ and $E'_{T'}$ are the derived pullbacks of E and E' to $X_{T'}$. Consider the functor $H = \mathcal{H}om(E, E')$ defined by the rule

$$(Sch/T)^{opp} \to \mathrm{Sets}, \quad T' \mapsto \mathrm{Hom}_{\mathcal{O}_{X_{T'}}}(E_T, E'_T)$$

[1]We will see later (insert future reference here) that H is a scheme and hence the use of this lemma and Quot, Lemma 99.14.4 isn't necessary.

By Quot, Lemma 99.16.1 this is an algebraic space affine and of finite presentation over T. The same is true for $H' = \mathcal{H}om(E', E)$, $I = \mathcal{H}om(E, E)$, and $I' = \mathcal{H}om(E', E')$. Therefore we see that

$$Isom(E, E') = (H' \times_T H) \times_{c, I \times_T I', \sigma} T$$

where $c(\varphi', \varphi) = (\varphi \circ \varphi', \varphi' \circ \varphi)$ and $\sigma = (\mathrm{id}, \mathrm{id})$ (compare with the proof of Quot, Proposition 99.4.3). Thus $Isom(E, E')$ is affine over T as a fibre product of schemes affine over T. Similarly, $Isom(E, E')$ is of finite presentation over T. $\square$

0DPX Lemma 108.12.2. The morphism $\mathcal{C}omplexes_{X/B} \to B$ is quasi-separated and locally of finite presentation.

Proof. To check $\mathcal{C}omplexes_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 108.12.1. To prove that $\mathcal{C}omplexes_{X/B} \to B$ is locally of finite presentation, we have to show that $\mathcal{C}omplexes_{X/B} \to B$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This follows from Quot, Lemma 99.16.8 (small detail omitted). $\square$

108.13. Other chapters

CHAPTER 109

Moduli of Curves

0DMG 109.1. Introduction

0DMH In this chapter we discuss some of the familiar moduli stacks of curves. A reference is the celebrated article of Deligne and Mumford, see [DM69].

109.2. Conventions and abuse of language

0DMI We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

109.3. The stack of curves

0DMJ This section is the continuation of Quot, Section 99.15. Let $\mathcal{C}urves$ be the stack whose category of sections over a scheme S is the category of families of curves over S. It is somewhat important to keep in mind that a family of curves is a morphism $f : X \to S$ where X is an algebraic space (!) and f is flat, proper, of finite presentation and of relative dimension ≤ 1. We already know that $\mathcal{C}urves$ is an algebraic stack over $\mathbf{Z}$, see Quot, Theorem 99.15.11. If we did not allow algebraic spaces in the definition of our stack, then this theorem would be false.

Often base change is denoted by a subscript, but we cannot use this notation for $\mathcal{C}urves$ because $\mathcal{C}urves_S$ is our notation for the fibre category over S. This is why in Quot, Remark 99.15.5 we used $B\text{-}\mathcal{C}urves$ for the base change

$$B\text{-}\mathcal{C}urves = \mathcal{C}urves \times B$$

to the algebraic space B. The product on the right is over the final object, i.e., over $\mathrm{Spec}(\mathbf{Z})$. The object on the left is the stack classifying families of curves on the category of schemes over B. In particular, if k is a field, then

$$k\text{-}\mathcal{C}urves = \mathcal{C}urves \times \mathrm{Spec}(k)$$

is the moduli stack classifying families of curves on the category of schemes over k. Before we continue, here is a sanity check.

0DMK **Lemma 109.3.1.** *Let $T \to B$ be a morphism of algebraic spaces. The category*

$$\mathrm{Mor}_B(T, B\text{-}\mathcal{C}urves) = \mathrm{Mor}(T, \mathcal{C}urves)$$

is the category of families of curves over T.

Proof. A family of curves over T is a morphism $f : X \to T$ of algebraic spaces, which is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 67.33.2). This is exactly the same as the definition in Quot, Situation 99.15.1 except that T the base is allowed to be an algebraic space. Our default base category for algebraic stacks/spaces is the category of schemes, hence the lemma does not follow immediately from the definitions. Having said this, we encourage the reader to skip the proof.

By the product description of B-*Curves* given above, it suffices to prove the lemma in the absolute case. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $s, t : R \to U$.

Let $v : T \to \mathcal{C}urves$ be a morphism. Then $v \circ p$ corresponds to a family of curves $X_U \to U$. The canonical 2-morphism $v \circ p \circ t \to v \circ p \circ s$ is an isomorphism $\varphi : X_U \times_{U,s} R \to X_U \times_{U,t} R$. This isomorphism satisfies the cocycle condition on $R \times_{s,t} R$. By Bootstrap, Lemma 80.11.3 we obtain a morphism of algebraic spaces $X \to T$ whose pullback to U is equal to X_U compatible with φ. Since $\{U \to T\}$ is an étale covering, we see that $X \to T$ is flat, proper, of finite presentation by Descent on Spaces, Lemmas 74.11.13, 74.11.19, and 74.11.12. Also $X \to T$ has relative dimension ≤ 1 because this is an étale local property. Hence $X \to T$ is a family of curves over T.

Conversely, let $X \to T$ be a family of curves. Then the base change X_U determines a morphism $w : U \to \mathcal{C}urves$ and the canonical isomorphism $X_U \times_{U,s} R \to X_U \times_{U,t} R$ determines a 2-arrow $w \circ s \to w \circ t$ satisfying the cocycle condition. Thus a morphism $v : T = [U/R] \to \mathcal{C}urves$ by the universal property of the quotient $[U/R]$, see Groupoids in Spaces, Lemma 78.23.2. (Actually, it is much easier in this case to go back to before we introduced our abuse of language and direct construct the functor $Sch/T \to \mathcal{C}urves$ which "is" the morphism $T \to \mathcal{C}urves$.)

We omit the verification that the constructions given above extend to morphisms between objects and are mutually quasi-inverse. $\square$

109.4. The stack of polarized curves

0DPY In this section we work out some of the material discussed in Quot, Remark 99.15.13. Consider the 2-fibre product

$$\begin{tikzcd} \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized \arrow[r] \arrow[d] & \mathcal{P}olarized \arrow[d] \\ \mathcal{C}urves \arrow[r] & \mathcal{S}paces'_{fp,flat,proper} \end{tikzcd}$$

We denote this 2-fibre product by

$$\text{PolarizedCurves} = \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized$$

This fibre product parametrizes polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. More precisely, an object of PolarizedCurves is a pair $(X \to S, \mathcal{L})$ where

1. (1) $X \to S$ is a morphism of schemes which is proper, flat, of finite presentation, and has relative dimension ≤ 1, and
2. (2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S.

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects of PolarizedCurves is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner.

0DPZ Lemma 109.4.1. The morphism PolarizedCurves $\to \mathcal{P}olarized$ is an open and closed immersion.

Proof. This is true because the 1-morphism $\mathcal{C}urves \to \mathcal{S}paces'_{fp,flat,proper}$ is representable by open and closed immersions, see Quot, Lemma 99.15.12. □

0DQ0 Lemma 109.4.2. The morphism PolarizedCurves $\to \mathcal{C}urves$ is smooth and surjective.

Proof. Surjective. Given a field k and a proper algebraic space X over k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Spaces over Fields, Lemma 72.9.3 the algebraic space X is a scheme. Hence X is a proper scheme of dimension ≤ 1 over k. By Varieties, Lemma 33.43.4 we see that X is H-projective over κ. In particular, there exists an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X. Then $(X, \mathcal{L})$ is an object of PolarizedCurves over k which maps to X.

Smooth. Let $X \to S$ be an object of $\mathcal{C}urves$, i.e., a morphism $S \to \mathcal{C}urves$. It is clear that

$$\text{PolarizedCurves} \times_{\mathcal{C}urves} S \subset \mathcal{P}ic_{X/S}$$

is the substack of objects $(T/S, \mathcal{L}/X_T)$ such that $\mathcal{L}$ is ample on X_T/T. This is an open substack by Descent on Spaces, Lemma 74.13.2. Since $\mathcal{P}ic_{X/S} \to S$ is smooth by Moduli Stacks, Lemma 108.8.5 we win. □

0E6F Lemma 109.4.3. Let $X \to S$ be a family of curves. Then there exists an étale covering $\{S_i \to S\}$ such that $X_i = X \times_S S_i$ is a scheme. We may even assume X_i is H-projective over S_i.

Proof. This is an immediate corollary of Lemma 109.4.2. Namely, unwinding the definitions, this lemma gives there is a surjective smooth morphism $S' \to S$ such that $X' = X \times_S S'$ comes endowed with an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ which is ample on X'/S'. Then we can refine the smooth covering $\{S' \to S\}$ by an étale covering $\{S_i \to S\}$, see More on Morphisms, Lemma 37.38.7. After replacing S_i by a suitable open covering we may assume $X_i \to S_i$ is H-projective, see Morphisms, Lemmas 29.43.6 and 29.43.4 (this is also discussed in detail in More on Morphisms, Section 37.50). □

109.5. Properties of the stack of curves

0DSP The following lemma isn't true for moduli of surfaces, see Remark 109.5.2.

0DSQ Lemma 109.5.1. The diagonal of $\mathcal{C}urves$ is separated and of finite presentation.

Proof. Recall that $\mathcal{C}urves$ is a limit preserving algebraic stack, see Quot, Lemma 99.15.6. By Limits of Stacks, Lemma 102.3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 102.3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given a scheme U and two objects $Y \to U$ and $X \to U$ of $\mathcal{C}urves$ over U, the algebraic space

$$\mathit{Isom}_U(Y, X)$$

is separated. This we have seen in Moduli Stacks, Lemmas 108.10.2 and 108.10.3 that the target is a separated algebraic space.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffices to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{C}urves \times \mathcal{C}urves$ is quasi-compact (see for example Properties of Stacks, Lemma 100.3.3). We choose $U = \coprod U_i$ to be a disjoint union of affine opens with a surjective smooth morphism

$$U \longrightarrow \text{PolarizedCurves} \times \text{PolarizedCurves}$$

Then $U \to \mathcal{C}urves \times \mathcal{C}urves$ will be surjective and smooth since PolarizedCurves $\to$ $\mathcal{C}urves$ is surjective and smooth by Lemma 109.4.2. Since PolarizedCurves is limit preserving (by Artin's Axioms, Lemma 98.11.2 and Quot, Lemmas 99.15.6, 99.14.8, and 99.13.6), we see that PolarizedCurves $\to \text{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \text{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 102.3.8 and Morphisms of Stacks, Lemmas 101.27.2 and 101.33.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $(Y, \mathcal{N})$ and $(X, \mathcal{L})$ of PolarizedCurves over U, we show the algebraic space

$$\mathit{Isom}_U(Y, X)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let Q, P be the Hilbert polynomials of these families, i.e.,

$$Q(n) = \chi(Y_u, \mathcal{N}_u^{\otimes n}) \quad \text{and} \quad P(n) = \chi(X_u, \mathcal{L}_u^{\otimes n})$$

see Varieties, Lemma 33.45.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ and $u \mapsto \chi(X_u, \mathcal{L}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 36.32.2) we see that we get the same Hilbert polynomial in every point of U. Set

$$\mathcal{M} = \text{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \text{pr}_2^*\mathcal{L}$$

on $Y \times_U X$. Given $(f, \varphi) \in \mathit{Isom}_U(Y, X)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\begin{aligned} \chi(Y_t, (\text{id} \times f)^*\mathcal{M}^{\otimes n}) &= \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) \\ &= n\deg(\mathcal{N}_t) + n\deg(f_t^*\mathcal{L}_t) + \chi(Y_t, \mathcal{O}_{Y_t}) \\ &= Q(n) + n\deg(\mathcal{L}_t) \\ &= Q(n) + P(n) - P(0) \end{aligned}$$

by Riemann-Roch for proper curves, more precisely by Varieties, Definition 33.44.1 and Lemma 33.44.7 and the fact that f_t is an isomorphism. Setting $P'(t) = Q(t) + P(t) - P(0)$ we find

$$Isom_U(Y, X) = Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $Mor_U(Y, X)$, see Moduli Stacks, Lemma 108.10.3 and Remark 108.10.4. Now $Mor_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Moduli Stacks, Lemma 108.10.5. Thus the intersection is a Noetherian algebraic space too and the proof is finished. $\square$

0DSR Remark 109.5.2. The boundedness argument in the proof of Lemma 109.5.1 does not work for moduli of surfaces and in fact, the result is wrong, for example because K3 surfaces over fields can have infinite discrete automorphism groups. The "reason" the argument does not work is that on a projective surface S over a field, given ample invertible sheaves $\mathcal{N}$ and $\mathcal{L}$ with Hilbert polynomials Q and P, there is no a priori bound on the Hilbert polynomial of $\mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{L}$. In terms of intersection theory, if H_1, H_2 are ample effective Cartier divisors on S, then there is no (upper) bound on the intersection number $H_1 \cdot H_2$ in terms of $H_1 \cdot H_1$ and $H_2 \cdot H_2$.

0DSS Lemma 109.5.3. The morphism $\mathcal{C}urves \to \text{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.

Proof. To check $\mathcal{C}urves \to \text{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 109.5.1. To prove that $\mathcal{C}urves \to \text{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that $\mathcal{C}urves$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This is Quot, Lemma 99.15.6. $\square$

109.6. Open substacks of the stack of curves

0E0E Below we will often characterize an open substack of $\mathcal{C}urves$ by a propery P of morphisms of algebraic spaces. To see that P defines an open substack it suffices to check

(o) given a family of curves $f : X \to S$ there exists a largest open subscheme $S' \subset S$ such that $f|_{f^{-1}(S')} : f^{-1}(S') \to S'$ has P and such that formation of S' commutes with arbitrary base change.

Namely, suppose (o) holds. Choose a scheme U and a surjective smooth morphism $m : U \to \mathcal{C}urves$. Let $R = U \times_{\mathcal{C}urves} U$ and denote $t, s : R \to U$ the projections. Recall that $\mathcal{C}urves = [U/R]$ is a presentation, see Algebraic Stacks, Lemma 94.16.2 and Definition 94.16.5. By construction of $\mathcal{C}urves$ as the stack of curves, the morphism m is the classifying morphism for a family of curves $C \to U$. The 2-commutativity of the diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d] \\ U \arrow[r] & \mathcal{C}urves \end{tikzcd}$$

implies that $C \times_{U,s} R \cong C \times_{U,t} R$ (isomorphism of families of curves over R). Let $W \subset U$ be the largest open subscheme such that $f|_{f^{-1}(W)} : f^{-1}(W) \to W$ has P as in (o). Since formation of W commutes with base change according to (o) and by the isomorphism above we find that $s^{-1}(W) = t^{-1}(W)$. Thus $W \subset U$ corresponds to an open substack

$$\mathcal{C}urves^P \subset \mathcal{C}urves$$

according to Properties of Stacks, Lemma 100.9.8.

Continuing with the setup of the previous paragrpah, we claim the open substack $\mathcal{C}urves^P$ has the following two universal properties:

(1) given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
   (b) the morphism $X \to S$ has P,
(2) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
   (b) the morphism $X \to \mathrm{Spec}(k)$ has P.

This follows by considering the 2-fibre product

$$\begin{tikzcd} T \arrow[r, "p"'] \arrow[d, "q"'] & U \arrow[d] \\ S \arrow[r] & \mathcal{C}urves \end{tikzcd}$$

Observe that $T \to S$ is surjective and smooth as the base change of $U \to \mathcal{C}urves$. Thus the open $S' \subset S$ given by (o) is determined by its inverse image in T. However, by the invariance under base change of these opens in (o) and because $X \times_S T \cong C \times_U T$ by the 2-commutativity, we find $q^{-1}(S') = p^{-1}(W)$ as opens of T. This immediately implies (1). Part (2) is a special case of (1).

Given two properties P and Q of morphisms of algebraic spaces, supposing we already have established $\mathcal{C}urves^Q$ is an open substack of $\mathcal{C}urves$, then we can use exactly the same method to prove openness of $\mathcal{C}urves^{Q,P} \subset \mathcal{C}urves^Q$. We omit a precise explanation.

109.7. Curves with finite reduced automorphism groups

0DST Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Lemma 109.5.1 the automorphism group algebraic space $\mathit{Aut}(X)$ is finite type and separated over k. In particular, $\mathit{Aut}(X)$ is a group scheme, see More on Groupoids in Spaces, Lemma 79.10.2. If the characteristic of k is zero, then $\mathit{Aut}(X)$ is reduced and even smooth over k (Groupoids, Lemma 39.8.2). However, in general $\mathit{Aut}(X)$ is not reduced, even if X is geometrically reduced.

0DSU Example 109.7.1 (Non-reduced automorphism group). Let k be an algebraically closed field of characteristic 2. Set $Y = Z = \mathbf{P}^1_k$. Choose three pairwise distinct k-valued points a, b, c in $\mathbf{A}^1_k$. Thinking of $\mathbf{A}^1_k \subset \mathbf{P}^1_k = Y = Z$ as an open subschemes, we get a closed immersion

$$T = \mathrm{Spec}(k[t]/(t-a)^2) \amalg \mathrm{Spec}(k[t]/(t-b)^2) \amalg \mathrm{Spec}(k[t]/(t-c)^2) \longrightarrow \mathbf{P}^1_k$$

Let X be the pushout in the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Let $U \subset X$ be the affine open part which is the image of $\mathbf{A}^1_k \amalg \mathbf{A}^1_k$. Then we have an equalizer diagram

$$\mathcal{O}_X(U) \longrightarrow k[t] \times k[t] \rightrightarrows k[t]/(t-a)^2 \times k[t]/(t-b)^2 \times k[t]/(t-c)^2$$

Over the dual numbers $A = k[\epsilon]$ we have a nontrivial automorphism of this equalizer diagram sending t to $t + \epsilon$. We leave it to the reader to see that this automorphism extends to an automorphism of X over A. On the other hand, the reader easily shows that the automorphism group of X over k is finite. Thus $Aut(X)$ must be non-reduced.

Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. If $Aut(X)$ is geometrically reduced, then it need not be the case that it has dimension 0, even if X is smooth and geometrically connected.

0DSV Example 109.7.2 (Smooth positive dimensional automorphism group). Let k be an algebraically closed field. If X is a smooth genus 0, resp. 1 curve, then the automorphism group has dimension 3, resp. 1. Namely, in the genus 0 case we have $X \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 53.10.4. Since

$$Aut(\mathbf{P}^1_k) = \text{PGL}_{2,k}$$

as functors we see that the dimension is 3. On the other hand, if the genus of X is 1, then we see that the map $X = \underline{\text{Hilb}}^1_{X/k} \to \underline{\text{Pic}}^1_{X/k}$ is an isomorphism, see Picard Schemes of Curves, Lemma 44.6.7 and Algebraic Curves, Theorem 53.2.6. Thus X has the structure of an abelian variety (since $\underline{\text{Pic}}^1_{X/k} \cong \underline{\text{Pic}}^0_{X/k}$). In particular the (co)tangent bundle of X are trivial (Groupoids, Lemma 39.6.3). We conclude that $\dim_k H^0(X, T_X) = 1$ hence $\dim Aut(X) \leq 1$. On the other hand, the translations (viewing X as a group scheme) provide a 1-dimensional piece of $\text{Aut}(X)$ and we conlude its dimension is indeed 1.

It turns out that there is an open substack of $\mathcal{C}urves$ parametrizing curves whose automorphism group is geometrically reduced and finite. Here is a precise statement.

0DSW Lemma 109.7.3. There exist an open substack $\mathcal{C}urves^{DM} \subset \mathcal{C}urves$ with the following properties

(1) $\mathcal{C}urves^{DM} \subset \mathcal{C}urves$ is the maximal open substack which is DM,
(2) given a family of curves $X \to S$ the following are equivalent
  (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$,
  (b) the group algebraic space $Aut_S(X)$ is unramified over S,
(3) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
  (a) the classifying morphism $\text{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$,
  (b) $Aut(X)$ is geometrically reduced over k and has dimension 0,
  (c) $Aut(X) \to \text{Spec}(k)$ is unramified.

Proof. The existence of an open substack with property (1) is Morphisms of Stacks, Lemma 101.22.1. The points of this open substack are characterized by (3)(c) by Morphisms of Stacks, Lemma 101.22.2. The equivalence of (3)(b) and (3)(c) is the statement that an algebraic space G which is locally of finite type, geometrically reduced, and of dimension 0 over a field k, is unramified over k. First, G is a scheme by Spaces over Fields, Lemma 72.9.1. Then we can take an affine open in G and observe that it will be proper over k and apply Varieties, Lemma 33.9.3. Minor details omitted.

Part (2) is true because (3) holds. Namely, the morphism $Aut_S(X) \to S$ is locally of finite type. Thus we can check whether $Aut_S(X) \to S$ is unramified at all points of $Aut_S(X)$ by checking on fibres at points of the scheme S, see Morphisms of Spaces, Lemma 67.38.10. But after base change to a point of S we fall back into the equivalence of (3)(a) and (3)(c). □

0E6G Lemma 109.7.4. Let X be a proper scheme over a field k of dimension ≤ 1. Then properties (3)(a), (b), (c) are also equivalent to $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.

Proof. In the discussion above we have seen that $G = Aut(X)$ is a group scheme over $\mathrm{Spec}(k)$ which is finite type and separated; this uses Lemma 109.5.1 and More on Groupoids in Spaces, Lemma 79.10.2. Then G is unramified over k if and only if $\Omega_{G/k} = 0$ (Morphisms, Lemma 29.35.2). By Groupoids, Lemma 39.6.3 the vanishing holds if $T_{G/k,e} = 0$, where $T_{G/k,e}$ is the tangent space to G at the identity element $e \in G(k)$, see Varieties, Definition 33.16.3 and the formula in Varieties, Lemma 33.16.4. Since $\kappa(e) = k$ the tangent space is defined in terms of morphisms $\alpha : \mathrm{Spec}(k[\epsilon]) \to G = Aut(X)$ whose restriction to $\mathrm{Spec}(k)$ is e. It follows that it suffices to show any automorphism

$$\alpha : X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \longrightarrow X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$$

over $\mathrm{Spec}(k[\epsilon])$ whose restriction to $\mathrm{Spec}(k)$ is id_X. Such automorphisms are called infinitesimal automorphisms.

The infinitesimal automorphisms of X correspond 1-to-1 with derivations of $\mathcal{O}_X$ over k. This follows from More on Morphisms, Lemmas 37.9.1 and 37.9.2 (we only need the first one as we don't care about the reverse direction; also, please look at More on Morphisms, Remark 37.9.7 for an elucidation). For a different argument proving this equality we refer the reader to Deformation Problems, Lemma 93.9.3. □

109.8. Cohen-Macaulay curves

0E0H There is an open substack of $\mathcal{C}urves$ parametrizing the Cohen-Macaulay "curves".

0E0I Lemma 109.8.1. There exist an open substack $\mathcal{C}urves^{CM} \subset \mathcal{C}urves$ such that

(1) given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM}$,
   (b) the morphism $X \to S$ is Cohen-Macaulay,
(2) given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM}$,
   (b) X is Cohen-Macaulay.

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 76.26.7 the set

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

0E1F Lemma 109.8.2. There exist an open substack $\mathcal{C}urves^{CM,1} \subset \mathcal{C}urves$ such that

(1) given a family of curves $X \to S$ the following are equivalent
 (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM,1}$,
 (b) the morphism $X \to S$ is Cohen-Macaulay and has relative dimension 1 (Morphisms of Spaces, Definition 67.33.2),
(2) given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
 (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM,1}$,
 (b) X is Cohen-Macaulay and X is equidimensional of dimension 1.

Proof. By Lemma 109.8.1 it is clear that we have $\mathcal{C}urves^{CM,1} \subset \mathcal{C}urves^{CM}$ if it exists. Let $f : X \to S$ be a family of curves such that f is a Cohen-Macaulay morphism. By More on Morphisms of Spaces, Lemma 76.26.8 we have a decomposition

$$X = X_0 \amalg X_1$$

by open and closed subspaces such that $X_0 \to S$ has relative dimension 0 and $X_1 \to S$ has relative dimension 1. Since f is proper the subset

$$S' = S \setminus f(|X_0|)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay and has relative dimension 1. Moreover, formation of S' commutes with arbitrary base change because this is true for the decomposition above (as relative dimension behaves well with respect to base change, see Morphisms of Spaces, Lemma 67.34.3). Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

109.9. Curves of a given genus

0E6H The convention in the Stacks project is that the genus g of a proper 1-dimensional scheme X over a field k is defined only if $H^0(X, \mathcal{O}_X) = k$. In this case $g = \dim_k H^1(X, \mathcal{O}_X)$. See Algebraic Curves, Section 53.8. The conditions needed to define the genus define an open substack which is then a disjoint union of open substacks, one for each genus.

0E6I Lemma 109.9.1. There exist an open substack $\mathcal{C}urves^{h0,1} \subset \mathcal{C}urves$ such that

(1) given a family of curves $f : X \to S$ the following are equivalent
 (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$,
 (b) $f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after arbitrary base change, and the fibres of f have dimension 1,
(2) given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
 (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$,
 (b) $H^0(X, \mathcal{O}_X) = k$ and $\dim(X) = 1$.

Proof. Given a family of curves $X \to S$ the set of $s \in S$ where $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ is open in S by Derived Categories of Spaces, Lemma 75.26.2. Also, the set of points in S where the fibre has dimension 1 is open by More on Morphisms of Spaces, Lemma 76.31.5. Moreover, if $f : X \to S$ is a family of curves all of whose fibres have dimension 1 (and in particular f is surjective), then condition (1)(b) is equivalent to $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for every $s \in S$, see Derived Categories of Spaces, Lemma 75.26.7. Thus we see that the lemma follows from the general discussion in Section 109.6. $\square$

0E6J Lemma 109.9.2. We have $\mathcal{C}urves^{h0,1} \subset \mathcal{C}urves^{CM,1}$ as open substacks of $\mathcal{C}urves$.

Proof. See Algebraic Curves, Lemma 53.6.1 and Lemmas 109.9.1 and 109.8.2. $\square$

0E1J Lemma 109.9.3. Let $f : X \to S$ be a family of curves such that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$, i.e., the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$ (Lemma 109.9.1). Then

(1) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds universally,
(2) $R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module,
(3) for any morphism $h : S' \to S$ if $f' : X' \to S'$ is the base change, then $h^*(R^1f_*\mathcal{O}_X) = R^1f'_*\mathcal{O}_{X'}$.

Proof. We apply Derived Categories of Spaces, Lemma 75.26.7. This proves part (1). It also implies that locally on S we can write $Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$ where P is perfect of tor amplitude in $[1, \infty)$. Recall that formation of $Rf_*\mathcal{O}_X$ commutes with arbitrary base change (Derived Categories of Spaces, Lemma 75.25.4). Thus for $s \in S$ we have

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition 20.20.7. Then $P = H^1(P)[-1]$ and $H^1(P)$ is finite locally free for example by More on Algebra, Lemma 15.75.6. Since everything is compatible with base change we also see that (3) holds. $\square$

0E6K Lemma 109.9.4. There is a decomposition into open and closed substacks

$$\mathcal{C}urves^{h0,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g$$

where each $\mathcal{C}urves_g$ is characterized as follows:

(1) given a family of curves $f : X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g$,
   (b) $f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after arbitrary base change, the fibres of f have dimension 1, and $R^1f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,
(2) given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g$,
   (b) $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and the genus of X is g.

Proof. We already have the existence of $\mathcal{C}urves^{h0,1}$ as an open substack of $\mathcal{C}urves$ characterized by the conditions of the lemma not involving R^1f_* or H^1, see Lemma 109.9.1. The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 109.6 and Lemma 109.9.3. This proves the characterization in (1). The characterization in (2) follows from the definition of the genus in Algebraic Curves, Definition 53.8.1. $\square$

109.10. Geometrically reduced curves

0E0F There is an open substack of *Curves* parametrizing the geometrically reduced "curves".

0E0G **Lemma 109.10.1.** *There exist an open substack $\mathit{Curves}^{geomred} \subset \mathit{Curves}$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   1. *the classifying morphism $S \to \mathit{Curves}$ factors through $\mathit{Curves}^{geomred}$,*
   2. *the fibres of the morphism $X \to S$ are geometrically reduced (More on Morphisms of Spaces, Definition 76.29.2),*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   1. *the classifying morphism $\mathrm{Spec}(k) \to \mathit{Curves}$ factors through $\mathit{Curves}^{geomred}$,*
   2. *X is geometrically reduced over k.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 76.29.6 the set

$$E = \{s \in S : \text{the fibre of } X \to S \text{ at } s \text{ is geometrically reduced}\}$$

is open in S. Formation of this open commutes with arbitrary base change by More on Morphisms of Spaces, Lemma 76.29.3. Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

0E1G **Lemma 109.10.2.** *We have $\mathit{Curves}^{geomred} \subset \mathit{Curves}^{CM}$ as open substacks of Curves.*

Proof. This is true because a reduced Noetherian scheme of dimension ≤ 1 is Cohen-Macaulay. See Algebra, Lemma 10.157.3. □

109.11. Geometrically reduced and connected curves

0E1H There is an open substack of *Curves* parametrizing the geometrically reduced and connected "curves". We will get rid of 0-dimensional objects right away.

0E1I **Lemma 109.11.1.** *There exist an open substack $\mathit{Curves}^{grc,1} \subset \mathit{Curves}$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   1. *the classifying morphism $S \to \mathit{Curves}$ factors through $\mathit{Curves}^{grc,1}$,*
   2. *the geometric fibres of the morphism $X \to S$ are reduced, connected, and have dimension 1,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   1. *the classifying morphism $\mathrm{Spec}(k) \to \mathit{Curves}$ factors through $\mathit{Curves}^{grc,1}$,*
   2. *X is geometrically reduced, geometrically connected, and has dimension 1.*

Proof. By Lemmas 109.10.1, 109.10.2, 109.8.1, and 109.8.2 it is clear that we have

$$\mathit{Curves}^{grc,1} \subset \mathit{Curves}^{geomred} \cap \mathit{Curves}^{CM,1}$$

if it exists. Let $f : X \to S$ be a family of curves such that f is Cohen-Macaulay, has geometrically reduced fibres, and has relative dimension 1. By More on Morphisms of Spaces, Lemma 76.36.9 in the Stein factorization

$$X \to T \to S$$

the morphism $T \to S$ is étale. This implies that there is an open and closed subscheme $S' \subset S$ such that $X \times_S S' \to S'$ has geometrically connected fibres (in the decomposition of Morphisms, Lemma 29.48.5 for the finite locally free morphism $T \to S$ this corresponds to S_1). Formation of this open commutes with arbitrary base change because the number of connected components of geometric fibres is invariant under base change (it is also true that the Stein factorization commutes with base change in our particular case but we don't need this to conclude). Thus we get the open substack with the desired properties by the method discussed in Section 109.6. $\square$

0E6L Lemma 109.11.2. We have $\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves^{h0,1}$ as open substacks of $\mathcal{C}urves$. In particular, given a family of curves $f : X \to S$ whose geometric fibres are reduced, connected and of dimension 1, then $R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module whose formation commutes with arbitrary base change.

Proof. This follows from Varieties, Lemma 33.9.3 and Lemmas 109.9.1 and 109.11.1. The final statement follows from Lemma 109.9.3. $\square$

0E1K Lemma 109.11.3. There is a decomposition into open and closed substacks

$$\mathcal{C}urves^{grc,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g^{grc,1}$$

where each $\mathcal{C}urves_g^{grc,1}$ is characterized as follows:

1. given a family of curves $f : X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{grc,1}$,
   (b) the geometric fibres of the morphism $f : X \to S$ are reduced, connected, of dimension 1 and $R^1f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,
2. given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{grc,1}$,
   (b) X is geometrically reduced, geometrically connected, has dimension 1, and has genus g.

Proof. First proof: set $\mathcal{C}urves_g^{grc,1} = \mathcal{C}urves^{grc,1} \cap \mathcal{C}urves_g$ and combine Lemmas 109.11.2 and 109.9.4. Second proof: The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 109.6 and Lemma 109.11.2. This proves the characterization in (1). The characterization in (2) follows as well since the genus of a geometrically reduced and connected proper 1-dimensional scheme X/k is defined (Algebraic Curves, Definition 53.8.1 and Varieties, Lemma 33.9.3) and is equal to $\dim_k H^1(X, \mathcal{O}_X)$. $\square$

109.12. Gorenstein curves

0E1L There is an open substack of $\mathcal{C}urves$ parametrizing the Gorenstein "curves".

0E1M Lemma 109.12.1. There exist an open substack $\mathcal{C}urves^{Gorenstein} \subset \mathcal{C}urves$ such that

1. given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,
   (b) the morphism $X \to S$ is Gorenstein,
2. given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,
   (b) X is Gorenstein.

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 76.27.7 the set

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Gorenstein. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 109.6. $\square$

0E6M Lemma 109.12.2. There exist an open substack $\mathcal{C}urves^{Gorenstein,1} \subset \mathcal{C}urves$ such that

(1) given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein,1}$,
   (b) the morphism $X \to S$ is Gorenstein and has relative dimension 1 (Morphisms of Spaces, Definition 67.33.2),
(2) given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein,1}$,
   (b) X is Gorenstein and X is equidimensional of dimension 1.

Proof. Recall that a Gorenstein scheme is Cohen-Macaulay (Duality for Schemes, Lemma 48.24.2) and that a Gorenstein morphism is a Cohen-Macaulay morphism (Duality for Schemes, Lemma 48.25.4. Thus we can set $\mathcal{C}urves^{Gorenstein,1}$ equal to the intersection of $\mathcal{C}urves^{Gorenstein}$ and $\mathcal{C}urves^{CM,1}$ inside of $\mathcal{C}urves$ and use Lemmas 109.12.1 and 109.8.2. $\square$

109.13. Local complete intersection curves

0E0J There is an open substack of $\mathcal{C}urves$ parametrizing the local complete intersection "curves".

0DZV Lemma 109.13.1. There exist an open substack $\mathcal{C}urves^{lci} \subset \mathcal{C}urves$ such that

(1) given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{lci}$,
   (b) $X \to S$ is a local complete intersection morphism, and
   (c) $X \to S$ is a syntomic morphism.
(2) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{lci}$,
   (b) X is a local complete intersection over k.

Proof. Recall that being a syntomic morphism is the same as being flat and a local complete intersection morphism, see More on Morphisms of Spaces, Lemma 76.48.6. Thus (1)(b) is equivalent to (1)(c). In Section 109.6 we have seen it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is a local complete intersection morphism and such that formation of S' commutes with arbitrary base change. This follows from the more general More on Morphisms of Spaces, Lemma 76.49.7. $\square$

109.14. Curves with isolated singularities

0E0K We can look at the open substack of Curves parametrizing "curves" with only a finite number of singular points (these may correspond to 0-dimensional components in our setup).

0DZW **Lemma 109.14.1.** There exist an open substack $\mathit{Curves}^+ \subset \mathit{Curves}$ such that

1. given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathit{Curves}$ factors through Curves^+,
   (b) the singular locus of $X \to S$ endowed with any/some closed subspace structure is finite over S.
2. given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathit{Curves}$ factors through Curves^+,
   (b) $X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.

Proof. To prove the lemma it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that the fibre of $S' \times_S X \to S'$ have property (2). (Formation of the open will automatically commute with base change.) By definition the locus $T \subset |X|$ of points where $X \to S$ is not smooth is closed. Let $Z \subset X$ be the closed subspace given by the reduced induced algebraic space structure on T (Properties of Spaces, Definition 66.12.5). Now if $s \in S$ is a point where Z_s is finite, then there is an open neighbourhood $U_s \subset S$ of s such that $Z \cap f^{-1}(U_s) \to U_s$ is finite, see More on Morphisms of Spaces, Lemma 76.35.2. This proves the lemma. $\square$

109.15. The smooth locus of the stack of curves

0DZT The morphism

$$\mathit{Curves} \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

is smooth over a maximal open substack, see Morphisms of Stacks, Lemma 101.33.6. We want to give a criterion for when a curve is in this locus. We will do this using a bit of deformation theory.

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Choose a Cohen ring Λ for k, see Algebra, Lemma 10.160.6. Then we are in the situation described in Deformation Problems, Example 93.9.1 and Lemma 93.9.2. Thus we obtain a deformation category $\mathcal{D}ef_X$ on the category $\mathcal{C}_\Lambda$ of Artinian local Λ-algebras with residue field k.

0DZU **Lemma 109.15.1.** In the situation above the following are equivalent

1. the classifying morphism $\mathrm{Spec}(k) \to \mathit{Curves}$ factors through the open where $\mathit{Curves} \to \mathrm{Spec}(\mathbf{Z})$ is smooth,
2. the deformation category $\mathcal{D}ef_X$ is unobstructed.

Proof. Since $\mathit{Curves} \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Lemma 109.5.3) formation of the open substack where $\mathit{Curves} \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is smooth commutes with flat base change (Morphisms of Stacks, Lemma 101.33.6). Since the Cohen ring Λ is flat over $\mathbf{Z}$, we may work over Λ. In other words, we are trying to prove that

$$\Lambda\text{-}\mathit{Curves} \longrightarrow \mathrm{Spec}(\Lambda)$$

is smooth in an open neighbourhood of the point $x_0 : \mathrm{Spec}(k) \to \Lambda\text{-}\mathcal{C}urves$ defined by X/k if and only if $\mathcal{D}ef_X$ is unobstructed.

The lemma now follows from Geometry of Stacks, Lemma 107.2.7 and the equality

$$\mathcal{D}ef_X = \mathcal{F}_{\Lambda\text{-}\mathcal{C}urves,k,x_0}$$

This equality is not completely trivial to esthablish. Namely, on the left hand side we have the deformation category classifying all flat deformations $Y \to \mathrm{Spec}(A)$ of X as a scheme over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. On the right hand side we have the deformation category classifying all flat morphisms $Y \to \mathrm{Spec}(A)$ with special fibre X where Y is an algebraic space and $Y \to \mathrm{Spec}(A)$ is proper, of finite presentation, and of relative dimension ≤ 1. Since A is Artinian, we find that Y is a scheme for example by Spaces over Fields, Lemma 72.9.3. Thus it remains to show: a flat deformation $Y \to \mathrm{Spec}(A)$ of X as a scheme over an Artinian local ring A with residue field k is proper, of finite presentation, and of relative dimension ≤ 1. Relative dimension is defined in terms of fibres and hence holds automatically for Y/A since it holds for X/k. The morphism $Y \to \mathrm{Spec}(A)$ is proper and locally of finite presentation as this is true for $X \to \mathrm{Spec}(k)$, see More on Morphisms, Lemma 37.10.3. □

Here is a "large" open of the stack of curves which is contained in the smooth locus.

0DZX Lemma 109.15.2. The open substack

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+} \subset \mathcal{C}urves$$

has the following properties

(1) $\mathcal{C}urves^{lci+} \to \mathrm{Spec}(\mathbf{Z})$ is smooth,
(2) given a family of curves $X \to S$ the following are equivalent
    (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{lci+}$,
    (b) $X \to S$ is a local complete intersection morphism and the singular locus of $X \to S$ endowed with any/some closed subspace structure is finite over S,
(3) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
    (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{lci+}$,
    (b) X is a local complete intersection over k and $X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.

Proof. If we can show that there is an open substack $\mathcal{C}urves^{lci+}$ whose points are characterized by (2), then we see that (1) holds by combining Lemma 109.15.1 with Deformation Problems, Lemma 93.16.4. Since

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+}$$

inside $\mathcal{C}urves$, we conclude by Lemmas 109.13.1 and 109.14.1. □

109.16. Smooth curves

0DZY In this section we study open substacks of $\mathcal{C}urves$ parametrizing smooth "curves".

0DZZ Lemma 109.16.1. There exist an open substacks

$$\mathcal{C}urves^{smooth,1} \subset \mathcal{C}urves^{smooth} \subset \mathcal{C}urves$$

such that

(1) given a family of curves $f : X \to S$ the following are equivalent

(a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$, resp. $\mathcal{C}urves^{smooth,1}$,
(b) f is smooth, resp. smooth of relative dimension 1,

(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
(a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$, resp. $\mathcal{C}urves^{smooth,1}$,
(b) X is smooth over k, resp. X is smooth over k and X is equidimensional of dimension 1.

Proof. To prove the statements regarding $\mathcal{C}urves^{smooth}$ it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is smooth and such that the formation of this open commutes with base change. We know that there is a maximal open $U \subset X$ such that $U \to S$ is smooth and that formation of U commutes with arbitrary base change, see Morphisms of Spaces, Lemma 67.37.9. If $T = |X| \setminus |U|$ then $f(T)$ is closed in S as f is proper. Setting $S' = S \setminus f(T)$ we obtain the desired open.

Let $f : X \to S$ be a family of curves with f smooth. Then the fibres X_s are smooth over $\kappa(s)$ and hence Cohen-Macaulay (for example you can see this using Algebra, Lemmas 10.137.5 and 10.135.3). Thus we see that we may set

$$\mathcal{C}urves^{smooth,1} = \mathcal{C}urves^{smooth} \cap \mathcal{C}urves^{CM,1}$$

and the desired equivalences follow from what we've already shown for $\mathcal{C}urves^{smooth}$ and Lemma 109.8.2. $\square$

0E1N Lemma 109.16.2. The morphism $\mathcal{C}urves^{smooth} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.

Proof. Follows immediately from the observation that $\mathcal{C}urves^{smooth} \subset \mathcal{C}urves^{lci+}$ and Lemma 109.15.2. $\square$

0E81 Lemma 109.16.3. There exist an open substack $\mathcal{C}urves^{smooth,h0} \subset \mathcal{C}urves$ such that

(1) given a family of curves $f : X \to S$ the following are equivalent
(a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$,
(b) $f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after any base change, and f is smooth of relative dimension 1,

(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
(a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth,h0}$,
(b) X is smooth, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$,
(c) X is smooth, $\dim(X) = 1$, and X is geometrically connected,
(d) X is smooth, $\dim(X) = 1$, and X is geometrically integral, and
(e) $X_{\overline{k}}$ is a smooth curve.

Proof. If we set

$$\mathcal{C}urves^{smooth,h0} = \mathcal{C}urves^{smooth} \cap \mathcal{C}urves^{h0,1}$$

then we see that (1) holds by Lemmas 109.9.1 and 109.16.1. In fact, this also gives the equivalence of (2)(a) and (2)(b). To finish the proof we have to show that (2)(b) is equivalent to each of (2)(c), (2)(d), and (2)(e).

A smooth scheme over a field is geometrically normal (Varieties, Lemma 33.25.4), smoothness is preserved under base change (Morphisms, Lemma 29.34.5), and being smooth is fpqc local on the target (Descent, Lemma 35.23.27). Keeping this in mind, the equivalence of (2)(b), (2)(c), 2(d), and (2)(e) follows from Varieties, Lemma 33.10.7. $\square$

0E82 Definition 109.16.4. We denote $\mathcal{M}$ and we name it the moduli stack of smooth proper curves the algebraic stack $\mathcal{C}urves^{smooth,h0}$ parametrizing families of curves introduced in Lemma 109.16.3. For $g \geq 0$ we denote $\mathcal{M}_g$ and we name it the moduli stack of smooth proper curves of genus g the algebraic stack introduced in Lemma 109.16.5. [DM69]

Here is the obligatory lemma.

0E83 Lemma 109.16.5. There is a decomposition into open and closed substacks

$$\mathcal{M} = \coprod\nolimits_{g \geq 0} \mathcal{M}_g$$

where each $\mathcal{M}_g$ is characterized as follows:

(1) given a family of curves $f : X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{M}_g$,
   (b) $X \to S$ is smooth, $f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after any base change, and $R^1 f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{M}_g$,
   (b) X is smooth, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and X has genus g,
   (c) X is smooth, $\dim(X) = 1$, X is geometrically connected, and X has genus g,
   (d) X is smooth, $\dim(X) = 1$, X is geometrically integral, and X has genus g, and
   (e) $X_{\overline{k}}$ is a smooth curve of genus g.

Proof. Combine Lemmas 109.16.3 and 109.9.4. You can also use Lemma 109.11.3 instead. $\square$

0E84 Lemma 109.16.6. The morphisms $\mathcal{M} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{M}_g \to \mathrm{Spec}(\mathbf{Z})$ are smooth.

Proof. Since $\mathcal{M}$ is an open substack of $\mathcal{C}urves^{lci+}$ this follows from Lemma 109.15.2. $\square$

109.17. Density of smooth curves

0E85 The title of this section is misleading as we don't claim $\mathcal{C}urves^{smooth}$ is dense in $\mathcal{C}urves$. In fact, this is false as was shown by Mumford in [Mum75]. However, we will see that the smooth "curves" are dense in a large open.

0E86 Lemma 109.17.1. The inclusion

$$|\mathcal{C}urves^{smooth}| \subset |\mathcal{C}urves^{lci+}|$$

is that of an open dense subset.

Proof. By the very construction of the topology on $|\mathcal{C}urves^{lci+}|$ in Properties of Stacks, Section 100.4 we find that $|\mathcal{C}urves^{smooth}|$ is an open subset. Let $\xi \in |\mathcal{C}urves^{lci+}|$ be a point. Then there exists a field k and a scheme X over k with X proper over k, with $\dim(X) \leq 1$, with X a local complete intersection over k, and with X is smooth over k except at finitely many points, such that ξ is the equivalence class of the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves^{lci+}$ determined by X. See Lemma 109.15.2. By Deformation Problems, Lemma 93.17.6 there exists a flat projective morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X. Consider the classifying morphism

$$\mathrm{Spec}(k[[t]]) \longrightarrow \mathcal{C}urves^{lci+}$$

determined by Y. The image of the closed point is ξ and the image of the generic point is in $|\mathcal{C}urves^{smooth}|$. Since the generic point specializes to the closed point in $|\mathrm{Spec}(k[[t]])|$ we conclude that ξ is in the closure of $|\mathcal{C}urves^{smooth}|$ as desired. □

109.18. Nodal curves

0DSX In algebraic geometry a special role is played by nodal curves. We suggest the reader take a brief look at some of the discussion in Algebraic Curves, Sections 53.19 and 53.20 and More on Morphisms of Spaces, Section 76.55.

0DSY Lemma 109.18.1. There exist an open substack $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves$ such that

1. given a family of curves $f : X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,
   (b) f is at-worst-nodal of relative dimension 1,
2. given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,
   (b) the singularities of X are at-worst-nodal and X is equidimensional of dimension 1.

Proof. In fact, it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is at-worst-nodal of relative dimension 1 and such that formation of S' commutes with arbitrary base change. By More on Morphisms of Spaces, Lemma 76.55.4 there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with base change. Hence we can take

$$S' = S \setminus |f|(|X| \setminus |X'|)$$

This is open because a proper morphism is universally closed by definition. □

0E00 Lemma 109.18.2. The morphism $\mathcal{C}urves^{nodal} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.

Proof. Follows immediately from the observation that $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves^{lci+}$ and Lemma 109.15.2. □

109.19. The relative dualizing sheaf

0E6N This section serves mainly to introduce notation in the case of families of curves. Most of the work has already been done in the chapter on duality.

Let $f : X \to S$ be a family of curves. There exists an object $\omega^\bullet_{X/S}$ in $D_{QCoh}(\mathcal{O}_X)$, called the relative dualizing complex, having the following property: for every base change diagram

$$\begin{tikzcd} X_U \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ U \arrow[r, "g"] & S \end{tikzcd}$$

with $U = \mathrm{Spec}(A)$ affine the complex $\omega_{X_U/U}^\bullet = L(g')^*\omega_{X/S}^\bullet$ represents the functor

$$D_{QCoh}(\mathcal{O}_{X_U}) \longrightarrow \mathrm{Mod}_A, \quad K \longmapsto \mathrm{Hom}_U(Rf_*K, \mathcal{O}_U)$$

More precisely, let $(\omega_{X/S}^\bullet, \tau)$ be the relative dualizing complex of the family as defined in Duality for Spaces, Definition 86.9.1. Existence is shown in Duality for Spaces, Lemma 86.9.5. Moreover, formation of $(\omega_{X/S}^\bullet, \tau)$ commutes with arbitrary base change (essentially by definition; a precise reference is Duality for Spaces, Lemma 86.9.6). From now on we will identify the base change of $\omega_{X/S}^\bullet$ with the relative dualizing complex of the base changed family without further mention.

Let $\{S_i \to S\}$ be an étale covering with S_i affine such that $X_i = X \times_S S_i$ is a scheme, see Lemma 109.4.3. By Duality for Spaces, Lemma 86.10.1 we find that $\omega_{X_i/S_i}^\bullet$ agrees with the relative dualizing complex for the proper, flat, and finitely presented morphism $f_i : X_i \to S_i$ of schemes discussed in Duality for Schemes, Remark 48.12.5. Thus to prove a property of $\omega_{X/S}^\bullet$ which is étale local, we may assume $X \to S$ is a morphism of schemes and use the theory developped in the chapter on duality for schemes. More generally, for any base change of X which is a scheme, the relative dualizing complex agrees with the relative dualizing complex of Duality for Schemes, Remark 48.12.5. From now on we will use this identification without further mention.

In particular, let $\mathrm{Spec}(k) \to S$ be a morphism where k is a field. Denote X_k the base change (this is a scheme by Spaces over Fields, Lemma 72.9.3). Then $\omega_{X_k/k}^\bullet$ is isomorphic to the complex $\omega_{X_k}^\bullet$ of Algebraic Curves, Lemma 53.4.1 (both represent the same functor and so we can use the Yoneda lemma, but really this holds because of the remarks above). We conclude that the cohomology sheaves $H^i(\omega_{X_k/k}^\bullet)$ are nonzero only for $i = 0, -1$. If X_k is Cohen-Macaulay and equidimensional of dimension 1, then we only have H^{-1} and if X_k is in addition Gorenstein, then $H^{-1}(\omega_{X_k/k})$ is invertible, see Algebraic Curves, Lemmas 53.4.2 and 53.5.2.

0E6P Lemma 109.19.1. Let $X \to S$ be a family of curves with Cohen-Macaulay fibres equidimensional of dimension 1 (Lemma 109.8.2). Then $\omega_{X/S}^\bullet = \omega_{X/S}[1]$ where $\omega_{X/S}$ is a pseudo-coherent $\mathcal{O}_X$-module flat over S whose formation commutes with arbitrary base change.

Proof. We urge the reader to deduce this directly from the discussion above of what happens after base change to a field. Our proof will use a somewhat cumbersome reduction to the Noetherian schemes case.

Once we show $\omega_{X/S}^\bullet = \omega_{X/S}[1]$ with $\omega_{X/S}$ flat over S, the statement on base change will follow as we already know that formation of $\omega_{X/S}^\bullet$ commutes with arbitrary base change. Moreover, the pseudo-coherence will be automatic as $\omega_{X/S}^\bullet$ is pseudo-coherent by definition. Vanishing of the other cohomology sheaves and flatness may be checked étale locally. Thus we may assume $f : X \to S$ is a morphism of schemes with S affine (see discussion above). Write $S = \lim S_i$ as a cofiltered limit of affine schemes S_i of finite type over $\mathbf{Z}$. Since $\mathit{Curves}^{CM,1}$ is locally of finite presentation over $\mathbf{Z}$ (as an open substack of Curves, see Lemmas 109.8.2 and 109.5.3), we can find an i and a family of curves $X_i \to S_i$ whose pullback is $X \to S$ (Limits of Stacks, Lemma 102.3.5). After increasing i if necessary we may assume X_i is a scheme, see Limits of Spaces, Lemma 70.5.11. Since formation of $\omega_{X/S}^\bullet$ commutes with arbitrary base change, we may replace S by S_i. Doing so we may and do assume S_i is Noetherian. Then f is clearly a Cohen-Macaulay morphism (More on Morphisms, Definition 37.22.1) by our assumption on the fibres. Also then $\omega_{X/S}^\bullet = f^!\mathcal{O}_S$ by the very construction of $f^!$ in Duality for Schemes, Section 48.16. Thus the lemma by Duality for Schemes, Lemma 48.23.3. □

0E6Q Definition 109.19.2. Let $f : X \to S$ be a family of curves with Cohen-Macaulay fibres equidimensional of dimension 1 (Lemma 109.8.2). Then the $\mathcal{O}_X$-module

$$\omega_{X/S} = H^{-1}(\omega_{X/S}^\bullet)$$

studied in Lemma 109.19.1 is called the relative dualizing sheaf of f.

In the situation of Definition 109.19.2 the relative dualizing sheaf $\omega_{X/S}$ has the following property (which moreover characterizes it locally on S): for every base change diagram

$$\begin{tikzcd} X_U \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ U \arrow[r, "g"] & S \end{tikzcd}$$

with $U = \mathrm{Spec}(A)$ affine the module $\omega_{X_U/U} = (g')^*\omega_{X/S}$ represents the functor

$$QCoh(\mathcal{O}_{X_U}) \longrightarrow \mathrm{Mod}_A, \quad \mathcal{F} \longmapsto \mathrm{Hom}_A(H^1(X, \mathcal{F}), A)$$

This follows immediately from the corresponding property of the relative dualizing complex given above. In particular, if $A = k$ is a field, then we recover the dualizing module of X_k as introduced and studied in Algebraic Curves, Lemmas 53.4.1, 53.4.2, and 53.5.2.

0E6R Lemma 109.19.3. Let $X \to S$ be a family of curves with Gorenstein fibres equidimensional of dimension 1 (Lemma 109.12.2). Then the relative dualizing sheaf $\omega_{X/S}$ is an invertible $\mathcal{O}_X$-module whose formation commutes with arbitrary base change.

Proof. This is true because the pullback of the relative dualizing module to a fibre is invertible by the discussion above. Alternatively, you can argue exactly as in the proof of Lemma 109.19.1 and deduce the result from Duality for Schemes, Lemma 48.25.10. □

109.20. Prestable curves

0E6S The following definition is equivalent to what appears to be the generally accepted notion of a prestable family of curves.

0E6T Definition 109.20.1. Let $f : X \to S$ be a family of curves. We say f is a prestable family of curves if

(1) f is at-worst-nodal of relative dimension 1, and

(2) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change[1].

Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is prestable[2] in the sense of the definition. Unwinding the definitions we see the following are equivalent

(1) X is prestable,
(2) the singularities of X are at-worst-nodal, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$,
(3) $X_{\overline{k}}$ is connected and it is smooth over $\overline{k}$ apart from a finite number of nodes (Algebraic Curves, Definition 53.16.2).

This shows that our definition agrees with most definitions one finds in the literature.

0E6U Lemma 109.20.2. There exist an open substack $\mathcal{C}urves^{prestable} \subset \mathcal{C}urves$ such that

(1) given a family of curves $f : X \to S$ the following are equivalent
 (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{prestable}$,
 (b) $X \to S$ is a prestable family of curves,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
 (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{prestable}$,
 (b) the singularities of X are at-worst-nodal, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$.

Proof. Given a family of curves $X \to S$ we see that it is prestable if and only if the classifying morphism factors both through $\mathcal{C}urves^{nodal}$ and $\mathcal{C}urves^{h0,1}$. An alternative is to use $\mathcal{C}urves^{grc,1}$ (since a nodal curve is geometrically reduced hence has H^0 equal to the ground field if and only if it is connected). In a formula

$$\mathcal{C}urves^{prestable} = \mathcal{C}urves^{nodal} \cap \mathcal{C}urves^{h0,1} = \mathcal{C}urves^{nodal} \cap \mathcal{C}urves^{grc,1}$$

Thus the lemma follows from Lemmas 109.9.1 and 109.18.1. □

For each genus $g \geq 0$ we have the algebraic stack classifying the prestable curves of genus g. In fact, from now on we will say that $X \to S$ is a prestable family of curves of genus g if and only if the classifying morphism $S \to \mathcal{C}urves$ factors through the open substack $\mathcal{C}urves_g^{prestable}$ of Lemma 109.20.3.

0E6V Lemma 109.20.3. There is a decomposition into open and closed substacks

$$\mathcal{C}urves^{prestable} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g^{prestable}$$

where each $\mathcal{C}urves_g^{prestable}$ is characterized as follows:

(1) given a family of curves $f : X \to S$ the following are equivalent
 (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{prestable}$,
 (b) $X \to S$ is a prestable family of curves and $R^1f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent

[1] In fact, it suffices to require $f_*\mathcal{O}_X = \mathcal{O}_S$ because the Stein factorization of f is étale in this case, see More on Morphisms of Spaces, Lemma 76.36.9. The condition may also be replaced by asking the geometric fibres to be connected, see Lemma 109.11.2.

[2] We can't use the term "prestable curve" here because curve implies irreducible. See discussion in Algebraic Curves, Section 53.20.

(a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{prestable}$,
(b) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and the genus of X is g.

Proof. Since we have seen that $\mathcal{C}urves^{prestable}$ is contained in $\mathcal{C}urves^{h0,1}$, this follows from Lemmas 109.20.2 and 109.9.4. $\square$

0E6W Lemma 109.20.4. The morphisms $\mathcal{C}urves^{prestable} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{C}urves_g^{prestable} \to \mathrm{Spec}(\mathbf{Z})$ are smooth.

Proof. Since $\mathcal{C}urves^{prestable}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 109.18.2. $\square$

109.21. Semistable curves

0E6X The following lemma will help us understand families of semistable curves.

0E6Y Lemma 109.21.1. Let $f : X \to S$ be a prestable family of curves of genus $g \geq 1$. Let $s \in S$ be a point of the base scheme. Let $m \geq 2$. The following are equivalent

(1) X_s does not have a rational tail (Algebraic Curves, Example 53.22.1), and
(2) $f^*f_*\omega_{X/S}^{\otimes m} \to \omega_{X/S}^{\otimes m}$, is surjective over $f^{-1}(U)$ for some $s \in U \subset S$ open.

Proof. Assume (2). Using the material in Section 109.19 we conclude that $\omega_{X_s}^{\otimes m}$ is globally generated. However, if $C \subset X_s$ is a rational tail, then $\deg(\omega_{X_s}|_C) < 0$ by Algebraic Curves, Lemma 53.22.2 hence $H^0(C, \omega_{X_s}|_C) = 0$ by Varieties, Lemma 33.44.12 which contradicts the fact that it is globally generated. This proves (1).

Assume (1). First assume that $g \geq 2$. Assumption (1) implies $\omega_{X_s}^{\otimes m}$ is globally generated, see Algebraic Curves, Lemma 53.22.6. Moreover, we have

$$\mathrm{Hom}_{\kappa(s)}(H^1(X_s, \omega_{X_s}^{\otimes m}), \kappa(s)) = H^0(X_s, \omega_{X_s}^{\otimes 1-m})$$

by duality, see Algebraic Curves, Lemma 53.4.2. Since $\omega_{X_s}^{\otimes m}$ is globally generated we find that the restriction to each irreducible component has nonegative degree. Hence the restriction of $\omega_{X_s}^{\otimes 1-m}$ to each irreducible component has nonpositive degree. Since $\deg(\omega_{X_s}^{\otimes 1-m}) = (1-m)(2g-2) < 0$ by Riemann-Roch (Algebraic Curves, Lemma 53.5.2) we conclude that the H^0 is zero by Varieties, Lemma 33.44.13. By cohomology and base change we conclude that

$$E = Rf_*\omega_{X/S}^{\otimes m}$$

is a perfect complex whose formation commutes with arbitrary base change (Derived Categories of Spaces, Lemma 75.25.4). The vanishing proved above tells us that $E \otimes^{\mathbf{L}} \kappa(s)$ is equal to $H^0(X_s, \omega_{X_s}^{\otimes m})$ placed in degree 0. After shrinking S we find $E = f_*\omega_{X/S}^{\otimes m}$ is a locally free $\mathcal{O}_S$-module placed in degree 0 (and its formation commutes with arbitrary base change as we've already said), see Derived Categories of Spaces, Lemma 75.26.5. The map $f^*f_*\omega_{X/S}^{\otimes m} \to \omega_{X/S}^{\otimes m}$ is surjective after restricting to X_s. Thus it is surjective in an open neighbourhood of X_s. Since f is proper, this open neighbourhood contains $f^{-1}(U)$ for some open neighbourhood U of s in S.

Assume (1) and $g = 1$. By Algebraic Curves, Lemma 53.22.6 the assumption (1) means that ω_{X_s} is isomorphic to $\mathcal{O}_{X_s}$. If we can show that after shrinking S the invertible sheaf $\omega_{X/S}$ because trivial, then we are done. We may assume S is affine. After shrinking S further, we can write

$$Rf_*\mathcal{O}_X = (\mathcal{O}_S \xrightarrow{0} \mathcal{O}_S)$$

sitting in degrees 0 and 1 compatibly with further base change, see Lemma 109.9.3. By duality this means that

$$Rf_*\omega_{X/S} = (\mathcal{O}_S \xrightarrow{0} \mathcal{O}_S)$$

sitting in degrees 0 and 1[3]. In particular we obtain an isomorphism $\mathcal{O}_S \to f_*\omega_{X/S}$ which is compatible with base change since formation of $Rf_*\omega_{X/S}$ is compatible with base change (see reference given above). By adjointness, we get a global section $\sigma \in \Gamma(X, \omega_{X/S})$. The restriction of this section to the fibre X_s is nonzero (a basis element in fact) and as ω_{X_s} is trivial on the fibres, this section is nowhere zero on X_s. Thus it nowhere zero in an open neighbourhood of X_s. Since f is proper, this open neighbourhood contains $f^{-1}(U)$ for some open neighbourhood U of s in S. □

Motivated by Lemma 109.21.1 we make the following definition.

0E6Z Definition 109.21.2. Let $f : X \to S$ be a family of curves. We say f is a semistable family of curves if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 1 and does not have a rational tail for all $s \in S$.

In particular, a prestable family of curves of genus 0 is never semistable. Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \text{Spec}(k)$ is a family of curves and hence we can ask whether or not it is semistable. Unwinding the definitions we see the following are equivalent

(1) X is semistable,
(2) X is prestable, has genus ≥ 1, and does not have a rational tail,
(3) $X_{\overline{k}}$ is connected, is smooth over $\overline{k}$ apart from a finite number of nodes, has genus ≥ 1, and has no irreducible component isomorphic to $\mathbf{P}^1_{\overline{k}}$ which meets the rest of $X_{\overline{k}}$ in only one point.

To see the equivalence of (2) and (3) use that X has no rational tails if and only if $X_{\overline{k}}$ has no rational tails by Algebraic Curves, Lemma 53.22.6. This shows that our definition agrees with most definitions one finds in the literature.

0E70 Lemma 109.21.3. There exist an open substack $\mathcal{C}urves^{semistable} \subset \mathcal{C}urves$ such that

(1) given a family of curves $f : X \to S$ the following are equivalent
  (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{semistable}$,
  (b) $X \to S$ is a semistable family of curves,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
  (a) the classifying morphism $\text{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{semistable}$,
  (b) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 1, and X has no rational tails,

[3] Use that $Rf_*\omega^\bullet_{X/S} = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X.\omega^\bullet_{X/S}) = R\mathcal{H}om_{\mathcal{O}_S}(Rf_*\mathcal{O}_X, \mathcal{O}_S)$ by Duality for Spaces, Lemma 86.3.3 and Remark 86.3.5 and then that $\omega^\bullet_{X/S} = \omega_{X/S}[1]$ by our definitions in Section 109.19.

(c) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and $\omega_{X_s}^{\otimes m}$ is globally generated for $m \geq 2$.

Proof. The equivalence of (2)(b) and (2)(c) is Algebraic Curves, Lemma 53.22.6. In the rest of the proof we will work with (2)(b) in accordance with Definition 109.21.2.

By the discussion in Section 109.6 it suffices to look at families $f : X \to S$ of prestable curves. By Lemma 109.21.1 we obtain the desired openness of the locus in question. Formation of this open commutes with arbitrary base change, because the (non)existence of rational tails is insensitive to ground field extensions by Algebraic Curves, Lemma 53.22.6. □

0E71 Lemma 109.21.4. There is a decomposition into open and closed substacks

$$\mathcal{C}urves^{semistable} = \coprod\nolimits_{g \geq 1} \mathcal{C}urves_g^{semistable}$$

where each $\mathcal{C}urves_g^{semistable}$ is characterized as follows:

(1) given a family of curves $f : X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{semistable}$,
   (b) $X \to S$ is a semistable family of curves and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{semistable}$,
   (b) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and X has no rational tail,
   (c) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and $\omega_{X_s}^{\otimes m}$ is globally generated for $m \geq 2$.

Proof. Combine Lemmas 109.21.3 and 109.20.3. □

0E72 Lemma 109.21.5. The morphisms $\mathcal{C}urves^{semistable} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{C}urves_g^{semistable} \to \mathrm{Spec}(\mathbf{Z})$ are smooth.

Proof. Since $\mathcal{C}urves^{semistable}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 109.18.2. □

109.22. Stable curves

0E73 The following lemma will help us understand families of stable curves.

0E74 Lemma 109.22.1. Let $f : X \to S$ be a prestable family of curves of genus $g \geq 2$. Let $s \in S$ be a point of the base scheme. The following are equivalent

(1) X_s does not have a rational tail and does not have a rational bridge (Algebraic Curves, Examples 53.22.1 and 53.23.1), and
(2) $\omega_{X/S}$ is ample on $f^{-1}(U)$ for some $s \in U \subset S$ open.

Proof. Assume (2). Then ω_{X_s} is ample on X_s. By Algebraic Curves, Lemmas 53.22.2 and 53.23.2 we conclude that (1) holds (we also use the characterization of ample invertible sheaves in Varieties, Lemma 33.44.15).

Assume (1). Then ω_{X_s} is ample on X_s by Algebraic Curves, Lemmas 53.23.6. We conclude by Descent on Spaces, Lemma 74.13.2. □

Motivated by Lemma 109.22.1 we make the following definition.

0E75 Definition 109.22.2. Let $f : X \to S$ be a family of curves. We say f is a stable family of curves if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 2 and does not have a rational tails or bridges for all $s \in S$.

In particular, a prestable family of curves of genus 0 or 1 is never stable. Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is stable. Unwinding the definitions we see the following are equivalent

(1) X is stable,
(2) X is prestable, has genus ≥ 2, does not have a rational tail, and does not have a rational bridge,
(3) X is geometrically connected, is smooth over k apart from a finite number of nodes, and ω_X is ample.

To see the equivalence of (2) and (3) use Lemma 109.22.1 above. This shows that our definition agrees with most definitions one finds in the literature.

0E76 Lemma 109.22.3. There exist an open substack $\mathcal{C}urves^{stable} \subset \mathcal{C}urves$ such that

(1) given a family of curves $f : X \to S$ the following are equivalent
  (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{stable}$,
  (b) $X \to S$ is a stable family of curves,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
  (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{stable}$,
  (b) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 2, and X has no rational tails or bridges,
  (c) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and ω_{X_s} is ample.

Proof. By the discussion in Section 109.6 it suffices to look at families $f : X \to S$ of prestable curves. By Lemma 109.22.1 we obtain the desired openness of the locus in question. Formation of this open commutes with arbitrary base change, either because the (non)existence of rational tails or bridges is insensitive to ground field extensions by Algebraic Curves, Lemmas 53.22.6 and 53.23.6 or because ampleness is insenstive to base field extensions by Descent, Lemma 35.25.6. □

0E77 Definition 109.22.4. We denote $\overline{\mathcal{M}}$ and we name the moduli stack of stable curves the algebraic stack $\mathcal{C}urves^{stable}$ parametrizing stable families of curves introduced in Lemma 109.22.3. For $g \geq 2$ we denote $\overline{\mathcal{M}}_g$ and we name the moduli stack of stable curves of genus g the algebraic stack introduced in Lemma 109.22.5. [DM69]

Here is the obligatory lemma.

0E78 Lemma 109.22.5. There is a decomposition into open and closed substacks

$$\overline{\mathcal{M}} = \coprod\nolimits_{g \geq 2} \overline{\mathcal{M}}_g$$

where each $\overline{\mathcal{M}}_g$ is characterized as follows:

(1) given a family of curves $f : X \to S$ the following are equivalent

   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\overline{\mathcal{M}}_g$,
   (b) $X \to S$ is a stable family of curves and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,
(2) given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\overline{\mathcal{M}}_g$,
   (b) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and X has no rational tails or bridges.
   (c) the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and ω_{X_s} is ample.

Proof. Combine Lemmas 109.22.3 and 109.20.3. □

0E79 Lemma 109.22.6. The morphisms $\overline{\mathcal{M}} \to \mathrm{Spec}(\mathbf{Z})$ and $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ are smooth.

Proof. Since $\overline{\mathcal{M}}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 109.18.2. □

0E7A Lemma 109.22.7. The stacks $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ are open substacks of $\mathcal{C}urves^{DM}$. In particular, $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ are DM (Morphisms of Stacks, Definition 101.4.2) as well as Deligne-Mumford stacks (Algebraic Stacks, Definition 94.12.2).

Proof. Proof of the first assertion. Let X be a scheme proper over a field k whose singularities are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 2, and X has no rational tails or bridges. We have to show that the classifying morphism $\mathrm{Spec}(k) \to \overline{\mathcal{M}} \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$. We may first replace k by the algebraic closure (since we already know the relevant stacks are open substacks of the algebraic stack $\mathcal{C}urves$). By Lemmas 109.22.3, 109.7.3, and 109.7.4 it suffices to show that $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. This is proven in Algebraic Curves, Lemma 53.25.3.

Since $\mathcal{C}urves^{DM}$ is the maximal open substack of $\mathcal{C}urves$ which is DM, we see this is true also for the open substack $\overline{\mathcal{M}}$ of $\mathcal{C}urves^{DM}$. Finally, a DM algebraic stack is Deligne-Mumford by Morphisms of Stacks, Theorem 101.21.6. □

0E87 Lemma 109.22.8. Let $g \geq 2$. The inclusion

$$|\mathcal{M}_g| \subset |\overline{\mathcal{M}}_g|$$

is that of an open dense subset.

Proof. Since $\overline{\mathcal{M}}_g \subset \mathcal{C}urves^{lci+}$ is open and since $\mathcal{C}urves^{smooth} \cap \overline{\mathcal{M}}_g = \mathcal{M}_g$ this follows immediately from Lemma 109.17.1. □

109.23. Contraction morphisms

0E7B We urge the reader to familiarize themselves with Algebraic Curves, Sections 53.22, 53.23, and 53.24 before continuing here. The main result of this section is the existence of a "stabilization" morphism

$$\mathcal{C}urves_g^{prestable} \longrightarrow \overline{\mathcal{M}}_g$$

See Lemma 109.23.5. Loosely speaking, this morphism sends the moduli point of a nodal genus g curve to the moduli point of the associated stable curve constructed in Algebraic Curves, Lemma 53.24.2.

0E88 **Lemma 109.23.1.** Let S be a scheme and $s \in S$ a point. Let $f : X \to S$ and $g : Y \to S$ be families of curves. Let $c : X \to Y$ be a morphism over S. If $c_{s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_s}$ and $R^1c_{s,*}\mathcal{O}_{X_s} = 0$, then after replacing S by an open neighbourhood of s we have $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$ and this remains true after base change by any morphism $S' \to S$.

Proof. Let $(U, u) \to (S, s)$ be an étale neighbourhood such that $\mathcal{O}_{Y_U} = (X_U \to Y_U)_*\mathcal{O}_{X_U}$ and $R^1(X_U \to Y_U)_*\mathcal{O}_{X_U} = 0$ and the same is true after base change by $U' \to U$. Then we replace S by the open image of $U \to S$. Given $S' \to S$ we set $U' = U \times_S S'$ and we obtain étale coverings $\{U' \to S'\}$ and $\{Y_{U'} \to Y_{S'}\}$. Thus the truth of the statement for the base change of c by $S' \to S$ follows from the truth of the statement for the base change of $X_U \to Y_U$ by $U' \to U$. In other words, the question is local in the étale topology on S. Thus by Lemma 109.4.3 we may assume X and Y are schemes. By More on Morphisms, Lemma 37.72.7 there exists an open subscheme $V \subset Y$ containing Y_s such that $c_*\mathcal{O}_X|_V = \mathcal{O}_V$ and $R^1c_*\mathcal{O}_X|_V = 0$ and such that this remains true after any base change by $S' \to S$. Since $g : Y \to S$ is proper, we can find an open neighbourhood $U \subset S$ of s such that $g^{-1}(U) \subset V$. Then U works. □

0E89 **Lemma 109.23.2.** Let S be a scheme and $s \in S$ a point. Let $f : X \to S$ and $g_i : Y_i \to S$, $i = 1, 2$ be families of curves. Let $c_i : X \to Y_i$ be morphisms over S. Assume there is an isomorphism $Y_{1,s} \cong Y_{2,s}$ of fibres compatible with $c_{1,s}$ and $c_{2,s}$. If $c_{1,s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_{1,s}}$ and $R^1c_{1,s,*}\mathcal{O}_{X_s} = 0$, then there exist an open neighbourhood U of s and an isomorphism $Y_{1,U} \cong Y_{2,U}$ of families of curves over U compatible with the given isomorphism of fibres and with c_1 and c_2.

Proof. Recall that $\mathcal{O}_{S,s} = \text{colim}\, \mathcal{O}_S(U)$ where the colimit is over the system of affine neighbourhoods U of s. Thus the category of algebraic spaces of finite presentation over the local ring is the colimit of the categories of algebraic spaces of finite presentation over the affine neighbourhoods of s. See Limits of Spaces, Lemma 70.7.1. In this way we reduce to the case where S is the spectrum of a local ring and s is the closed point.

Assume $S = \text{Spec}(A)$ where A is a local ring and s is the closed point. Write $A = \text{colim}\, A_j$ with A_j local Noetherian (say essentially of finite type over $\mathbf{Z}$) and local transition homomorphisms. Set $S_j = \text{Spec}(A_j)$ with closed point s_j. We can find a j and families of curves $X_j \to S_j$, $Y_{j,i} \to S_j$, see Lemma 109.5.3 and Limits of Stacks, Lemma 102.3.5. After possibly increasing j we can find morphisms $c_{j,i} : X_j \to Y_{j,i}$ whose base change to s is c_i, see Limits of Spaces, Lemma 70.7.1. Since $\kappa(s) = \text{colim}\, \kappa(s_j)$ we can similarly assume there is an isomorphism $Y_{j,1,s_j} \cong Y_{j,2,s_j}$ compatible with $c_{j,1,s_j}$ and $c_{j,2,s_j}$. Finally, the assumptions $c_{1,s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_{1,s}}$ and $R^1c_{1,s,*}\mathcal{O}_{X_s} = 0$ are inherited by $c_{j,1,s_j}$ because $\{s_j \to s\}$ is an fpqc covering and $c_{1,s}$ is the base of $c_{j,1,s_j}$ by this covering (details omitted). In this way we reduce the lemma to the case discussed in the next paragraph.

Assume S is the spectrum of a Noetherian local ring Λ and s is the closed point. Consider the scheme theoretic image Z of

$$(c_1, c_2) : X \longrightarrow Y_1 \times_S Y_2$$

The statement of the lemma is equivalent to the assertion that Z maps isomorphically to Y_1 and Y_2 via the projection morphisms. Since taking the scheme theoretic image of this morphism commutes with flat base change (Morphisms of Spaces, Lemma 67.30.12, we may replace Λ by its completion (More on Algebra, Section 15.43).

Assume S is the spectrum of a complete Noetherian local ring Λ. Observe that X, Y_1, Y_2 are schemes in this case (More on Morphisms of Spaces, Lemma 76.43.6). Denote X_n, $Y_{1,n}$, $Y_{2,n}$ the base changes of X, Y_1, Y_2 to $\mathrm{Spec}(\Lambda/\mathfrak m^{n+1})$. Recall that the arrow

$$\mathcal{D}ef_{X_s \to Y_{2,s}} \cong \mathcal{D}ef_{X_s \to Y_{1,s}} \longrightarrow \mathcal{D}ef_{X_s}$$

is an equivalence, see Deformation Problems, Lemma 93.10.6. Thus there is an isomorphism of formal objects $(X_n \to Y_{1,n}) \cong (X_n \to Y_{2,n})$ of $\mathcal{D}ef_{X_s \to Y_{1,s}}$. Finally, by Grothendieck's algebraization theorem (Cohomology of Schemes, Lemma 30.28.3) this produces an isomorphism $Y_1 \to Y_2$ compatible with c_1 and c_2. □

0E7C Lemma 109.23.3. Let $f : X \to S$ be a family of curves. Let $s \in S$ be a point. Let $h_0 : X_s \to Y_0$ be a morphism to a proper scheme Y_0 over $\kappa(s)$ such that $h_{0,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_0}$ and $R^1h_{0,*}\mathcal{O}_{X_s} = 0$. Then there exist an elementary étale neighbourhood $(U, u) \to (S, s)$, a family of curves $Y \to U$, and a morphism $h : X_U \to Y$ over U whose fibre in u is isomorphic to h_0.

Proof. We first do some reductions; we urge the reader to skip ahead. The question is local on S, hence we may assume S is affine. Write $S = \lim S_i$ as a cofiltered limit of affine schemes S_i of finite type over $\mathbf{Z}$. For some i we can find a family of curves $X_i \to S_i$ whose base change is $X \to S$. This follows from Lemma 109.5.3 and Limits of Stacks, Lemma 102.3.5. Let $s_i \in S_i$ be the image of s. Observe that $\kappa(s) = \mathrm{colim}\, \kappa(s_i)$ and that X_s is a scheme (Spaces over Fields, Lemma 72.9.3). After increasing i we may assume there exists a morphism $h_{i,0} : X_{i,s_i} \to Y_i$ of finite type schemes over $\kappa(s_i)$ whose base change to $\kappa(s)$ is h_0, see Limits, Lemma 32.10.1. After increasing i we may assume Y_i is proper over $\kappa(s_i)$, see Limits, Lemma 32.13.1. Let $g_{i,0} : Y_0 \to Y_{i,0}$ be the projection. Observe that this is a faithfully flat morphism as the base change of $\mathrm{Spec}(\kappa(s)) \to \mathrm{Spec}(\kappa(s_i))$. By flat base change we have

$$h_{0,*}\mathcal{O}_{X_s} = g_{i,0}^*h_{i,0,*}\mathcal{O}_{X_{i,s_i}} \quad \text{and} \quad R^1h_{0,*}\mathcal{O}_{X_s} = g_{i,0}^*Rh_{i,0,*}\mathcal{O}_{X_{i,s_i}}$$

see Cohomology of Schemes, Lemma 30.5.2. By faithful flatness we see that $X_i \to S_i$, $s_i \in S_i$, and $X_{i,s_i} \to Y_i$ satisfies all the assumptions of the lemma. This reduces us to the case discussed in the next paragraph.

Assume S is affine of finite type over $\mathbf{Z}$. Let $\mathcal{O}^h_{S,s}$ be the henselization of the local ring of S at s. Observe that $\mathcal{O}^h_{S,s}$ is a G-ring by More on Algebra, Lemma 15.50.8 and Proposition 15.50.12. Suppose we can construct a family of curves $Y' \to \mathrm{Spec}(\mathcal{O}^h_{S,s})$ and a morphism

$$h' : X \times_S \mathrm{Spec}(\mathcal{O}^h_{S,s}) \longrightarrow Y'$$

over $\mathrm{Spec}(\mathcal{O}^h_{S,s})$ whose base change to the closed point is h_0. This will be enough. Namely, first we use that

$$\mathcal{O}^h_{S,s} = \mathrm{colim}_{(U,u)}\, \mathcal{O}_U(U)$$

where the colimit is over the filtered category of elementary étale neighbourhoods (More on Morphisms, Lemma 37.35.5). Next, we use again that given Y' we can descend it to $Y \to U$ for some U (see references given above). Then we use Limits, Lemma 32.10.1 to descend h' to some h. This reduces us to the case discussed in the next paragraph.

Assume $S = \text{Spec}(\Lambda)$ where $(\Lambda, \mathfrak{m}, \kappa)$ is a henselian Noetherian local G-ring and s is the closed point of S. Recall that the map

$$\mathcal{D}ef_{X_s \to Y_0} \to \mathcal{D}ef_{X_s}$$

is an equivalence, see Deformation Problems, Lemma 93.10.6. (This is the only important step in the proof; everything else is technique.) Denote $\Lambda^\wedge$ the $\mathfrak{m}$-adic completion. The pullbacks X_n of X to $\Lambda/\mathfrak{m}^{n+1}$ define a formal object ξ of $\mathcal{D}ef_{X_s}$ over $\Lambda^\wedge$. From the equivalence we obtain a formal object ξ' of $\mathcal{D}ef_{X_s \to Y_0}$ over $\Lambda^\wedge$. Thus we obtain a huge commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & X_s \arrow[d] \\
\ldots \arrow[r] & Y_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & Y_0 \arrow[d] \\
\ldots \arrow[r] & \text{Spec}(\Lambda/\mathfrak{m}^{n+1}) \arrow[r] & \text{Spec}(\Lambda/\mathfrak{m}^n) \arrow[r] & \ldots \arrow[r] & \text{Spec}(\kappa)
\end{tikzcd}$$

The formal object (Y_n) comes from a family of curves $Y' \to \text{Spec}(\Lambda^\wedge)$ by Quot, Lemma 99.15.9. By More on Morphisms of Spaces, Lemma 76.43.3 we get a morphism $h' : X_{\Lambda^\wedge} \to Y'$ inducing the given morphisms $X_n \to Y_n$ for all n and in particular the given morphism $X_s \to Y_0$.

To finish we do a standard algebraization/approximation argument. First, we observe that we can find a finitely generated Λ-subalgebra $\Lambda \subset A \subset \Lambda^\wedge$, a family of curves $Y'' \to \text{Spec}(A)$ and a morphism $h'' : X_A \to Y''$ over A whose base change to $\Lambda^\wedge$ is h'. This is true because $\Lambda^\wedge$ is the filtered colimit of these rings A and we can argue as before using that $\mathcal{C}urves$ is locally of finite presentation (which gives us Y'' over A by Limits of Stacks, Lemma 102.3.5) and using Limits of Spaces, Lemma 70.7.1 to descend h' to some h''. Then we can apply the approximation property for G-rings (in the form of Smoothing Ring Maps, Theorem 16.13.1) to find a map $A \to \Lambda$ which induces the same map $A \to \kappa$ as we obtain from $A \to \Lambda^\wedge$. Base changing h'' to Λ the proof is complete. □

0E8A Lemma 109.23.4. Let $f : X \to S$ be a prestable family of curves of genus $g \geq 2$. There is a factorization $X \to Y \to S$ of f where $g : Y \to S$ is a stable family of curves and $c : X \to Y$ has the following properties

(1) $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$ and this remains true after base change by any morphism $S' \to S$, and
(2) for any $s \in S$ the morphism $c_s : X_s \to Y_s$ is the contraction of rational tails and bridges discussed in Algebraic Curves, Section 53.24.

Moreover $c : X \to Y$ is unique up to unique isomorphism.

Proof. Let $s \in S$. Let $c_0 : X_s \to Y_0$ be the contraction of Algebraic Curves, Section 53.24 (more precisely Algebraic Curves, Lemma 53.24.2). By Lemma 109.23.3 there exists an elementary étale neighbourhood (U, u) and a morphism $c : X_U \to Y$ of families of curves over U which recovers c_0 as the fibre at u. Since ω_{Y_0} is ample, after possibly shrinking U, we see that $Y \to U$ is a stable family of genus g by the openness inherent in Lemmas 109.22.3 and 109.22.5. After possibly shrinking U once more, assertion (1) of the lemma for $c : X_U \to Y$ follows from Lemma 109.23.1. Moreover, part (2) holds by the uniqueness in Algebraic Curves, Lemma 53.24.2. We conclude that a morphism c as in the lemma exists étale locally on S. More precisely, there exists an étale covering $\{U_i \to S\}$ and morphisms $c_i : X_{U_i} \to Y_i$ over U_i where $Y_i \to U_i$ is a stable family of curves having properties (1) and (2) stated in the lemma.

To finish the proof it suffices to prove uniqueness of $c : X \to Y$ (up to unique isomorphism). Namely, once this is done, then we obtain isomorphisms

$$\varphi_{ij} : Y_i \times_{U_i} (U_i \times_S U_j) \longrightarrow Y_i \times_{U_j} (U_i \times_S U_j)$$

satisfying the cocycle condition (by uniqueness) over $U_i \times U_j \times U_k$. Since $\overline{\mathcal{M}_g}$ is an algebraic stack, we have effectiveness of descent data and we obtain $Y \to S$. The morphisms c_i descend to a morphism $c : X \to Y$ over S. Finally, properties (1) and (2) for c are immediate from properties (1) and (2) for c_i.

Finally, if $c_1 : X \to Y_i$, $i = 1, 2$ are two morphisms towards stably families of curves over S satisfying (1) and (2), then we obtain a morphism $Y_1 \to Y_2$ compatible with c_1 and c_2 at least locally on S by Lemma 109.23.3. We omit the verification that these morphisms are unique (hint: this follows from the fact that the scheme theoretic image of c_1 is Y_1). Hence these locally given morphisms glue and the proof is complete. □

0E8B Lemma 109.23.5. Let $g \geq 2$. There is a morphism of algebraic stacks over $\mathbf{Z}$

$$stabilization : \mathcal{C}urves_g^{prestable} \longrightarrow \overline{\mathcal{M}}_g$$

which sends a prestable family of curves $X \to S$ of genus g to the stable family $Y \to S$ asssociated to it in Lemma 109.23.4.

Proof. To see this is true, it suffices to check that the construction of Lemma 109.23.4 is compatible with base change (and isomorphisms but that's immediate), see the (abuse of) language for algebraic stacks introduced in Properties of Stacks, Section 100.2. To see this it suffices to check properties (1) and (2) of Lemma 109.23.4 are stable under base change. This is immediately clear for (1). For (2) this follows either from the fact that the contractions of Algebraic Curves, Lemmas 53.22.6 and 53.23.6 are stable under ground field extensions, or because the conditions characterizing the morphisms on fibres in Algebraic Curves, Lemma 53.24.2 are preserved under ground field extensions. □

109.24. Stable reduction theorem

0E8C In the chapter on semistable reduction we have proved the celebrated theorem on semistable reduction of curves. Let K be the fraction field of a discrete valuation ring R. Let C be a projective smooth curve over K with $K = H^0(C, \mathcal{O}_C)$. According to Semistable Reduction, Definition 55.14.6 we say C has semistable reduction if either there is a prestable family of curves over R with generic fibre C, or some (equivalently any) minimal regular model of C over R is prestable. In this section we show that for curves of genus $g \geq 2$ this is also equivalent to stable reduction.

0E8D Lemma 109.24.1. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $K = H^0(C, \mathcal{O}_C)$ having genus $g \geq 2$. The following are equivalent

(1) C has semistable reduction (Semistable Reduction, Definition 55.14.6), or
(2) there is a stable family of curves over R with generic fibre C.

Proof. Since a stable family of curves is also prestable, it is immediate that (2) implies (1). Conversely, given a prestable family of curves over R with generic fibre C, we can contract it to a stable family of curves by Lemma 109.23.4. Since the generic fibre already is stable, it does not get changed by this procedure and the proof is complete. □

The following lemma tells us the stable family of curves over R promised in Lemma 109.24.1 is unique up to unique isomorphism.

0E97 Lemma 109.24.2. Let R be a discrete valuation ring with fraction field K. Let C be a smooth proper curve over K with $K = H^0(C, \mathcal{O}_C)$ and genus g. If X and X' are models of C (Semistable Reduction, Section 55.8) and X and X' are stable families of genus g curves over R, then there exists an unique isomorphism $X \to X'$ of models.

Proof. Let Y be the minimal model for C. Recall that Y exists, is unique, and is at-worst-nodal of relative dimension 1 over R, see Semistable Reduction, Proposition 55.8.6 and Lemmas 55.10.1 and 55.14.5 (applies because we have X). There is a contraction morphism

$$Y \longrightarrow Z$$

such that Z is a stable family of curves of genus g over R (Lemma 109.23.4). We claim there is a unique isomorphism of models $X \to Z$. By symmetry the same is true for X' and this will finish the proof.

By Semistable Reduction, Lemma 55.14.3 there exists a sequence

$$X_m \to \ldots \to X_1 \to X_0 = X$$

such that $X_{i+1} \to X_i$ is the blowing up of a closed point x_i where X_i is singular, $X_i \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1, and X_m is regular. By Semistable Reduction, Lemma 55.8.5 there is a sequence

$$X_m = Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0 = Y$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind[4]. By Semistable Reduction, Lemma 55.14.4 each Y_i is at-worst-nodal of relative dimension 1 over R. To prove the claim it suffices to show that there is an isomorphism $X \to Z$ compatible with the morphisms $X_m \to X$ and $X_m = Y_n \to Y \to Z$. Let $s \in \mathrm{Spec}(R)$ be the closed point. By either Lemma 109.23.2 or Lemma 109.23.4 we reduce to proving that the morphisms $X_{m,s} \to X_s$ and $X_{m,s} \to Z_s$ are both equal to the canonical morphism of Algebraic Curves, Lemma 53.24.2.

For a morphism $c : U \to V$ of schemes over $\kappa(s)$ we say c has property (*) if $\dim(U_v) \leq 1$ for $v \in V$, $\mathcal{O}_V = c_*\mathcal{O}_U$, and $R^1c_*\mathcal{O}_U = 0$. This property is stable under composition. Since both X_s and Z_s are stable genus g curves over $\kappa(s)$, it suffices to show that each of the morphisms $Y_s \to Z_s$, $X_{i+1,s} \to X_{i,s}$, and $Y_{i+1,s} \to Y_{i,s}$, satisfy property (*), see Algebraic Curves, Lemma 53.24.2.

Property (*) holds for $Y_s \to Z_s$ by construction.

[4] In fact we have $X_m = Y$, i.e., X_m does not contain any exceptional curves of the first kind. We encourage the reader to think this through as it simplifies the proof somewhat.

The morphisms $c : X_{i+1,s} \to X_{i,s}$ are constructed and studied in the proof of Semistable Reduction, Lemma 55.14.3. It suffices to check (*) étale locally on $X_{i,s}$. Hence it suffices to check (*) for the base change of the morphism "$X_1 \to X_0$" in Semistable Reduction, Example 55.14.1 to $R/\pi R$. We leave the explicit calculation to the reader.

The morphism $c : Y_{i+1,s} \to Y_{i,s}$ is the restriction of the blow down of an exceptional curve $E \subset Y_{i+1}$ of the first kind, i.e., $b : Y_{i+1} \to Y_i$ is a contraction of E, i.e., b is a blowing up of a regular point on the surface Y_i (Resolution of Surfaces, Section 54.16). Then $\mathcal{O}_{Y_i} = b_*\mathcal{O}_{Y_{i+1}}$ and $R^1b_*\mathcal{O}_{Y_{i+1}} = 0$, see for example Resolution of Surfaces, Lemma 54.3.4. We conclude that $\mathcal{O}_{Y_{i,s}} = c_*\mathcal{O}_{Y_{i+1,s}}$ and $R^1c_*\mathcal{O}_{Y_{i+1,s}} = 0$ by More on Morphisms, Lemmas 37.72.1, 37.72.2, and 37.72.4 (only gives surjectivity of $\mathcal{O}_{Y_{i,s}} \to c_*\mathcal{O}_{Y_{i+1,s}}$ but injectivity follows easily from the fact that $Y_{i,s}$ is reduced and c changes things only over one closed point). This finishes the proof. □

From Lemma 109.24.1 and Semistable Reduction, Theorem 55.18.1 we immediately deduce the stable reduction theorem.

0E98 Theorem 109.24.3. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus $g \geq 2$. Then [DM69, Corollary 2.7]

1. there exists an extension of discrete valuation rings $R \subset R'$ inducing a finite separable extension of fraction fields K'/K and a stable family of curves $Y \to \text{Spec}(R')$ of genus g with $Y_{K'} \cong C_{K'}$ over K', and
2. there exists a finite separable extension L/K and a stable family of curves $Y \to \text{Spec}(A)$ of genus g where $A \subset L$ is the integral closure of R in L such that $Y_L \cong C_L$ over L.

Proof. Part (1) is an immediate consequence of Lemma 109.24.1 and Semistable Reduction, Theorem 55.18.1.

Proof of (2). Let L/K be the finite separable extension found in part (3) of Semistable Reduction, Theorem 55.18.1. Let $A \subset L$ be the integral closure of R. Recall that A is a Dedekind domain finite over R with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, see More on Algebra, Remark 15.111.6. Set $S = \text{Spec}(A)$, $S_i = \text{Spec}(A_{\mathfrak{m}_i})$, $U = \text{Spec}(L)$, and $U_i = S_i \setminus \{\mathfrak{m}_i\}$. Observe that $U \cong U_i$ for $i = 1, \ldots, n$. Set $X = C_L$ viewed as a scheme over the open subscheme U of S. By our choice of L and A and Lemma 109.24.1 we have stable families of curves $X_i \to S_i$ and isomorphisms $X \times_U U_i \cong X_i \times_{S_i} U_i$. By Limits of Spaces, Lemma 70.18.4 we can find a finitely presented morphism $Y \to S$ whose base change to S_i is isomorphic to X_i for $i = 1, \ldots, n$. Alternatively, you can use that $S = \bigcup_{i=1,\ldots,n} S_i$ is an open covering of S and $S_i \cap S_j = U$ for $i \neq j$ and use $n - 1$ applications of Limits of Spaces, Lemma 70.18.1 to get $Y \to S$ whose base change to S_i is isomorphic to X_i for $i = 1, \ldots, n$. Clearly $Y \to S$ is the stable family of curves we were looking for. □

109.25. Properties of the stack of stable curves

0E99 In this section we prove the basic structure result for $\overline{\mathcal{M}}_g$ for $g \geq 2$.

0E9A Lemma 109.25.1. Let $g \geq 2$. The stack $\overline{\mathcal{M}}_g$ is separated.

Proof. The statement means that the morphism $\overline{\mathcal{M}}_g \to \text{Spec}(\mathbf{Z})$ is separated. We will prove this using the refined Noetherian valuative criterion as stated in More on Morphisms of Stacks, Lemma 106.11.4

Since $\overline{\mathcal{M}}_g$ is an open substack of $\mathcal{C}urves$, we see $\overline{\mathcal{M}}_g \to \text{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation by Lemma 109.5.3. In particular the stack $\overline{\mathcal{M}}_g$ is locally Noetherian (Morphisms of Stacks, Lemma 101.17.5). By Lemma 109.22.8 the open immersion $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ has dense image. Also, $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ is quasi-compact (Morphisms of Stacks, Lemma 101.8.2), hence of finite type. Thus all the preliminary assumptions of More on Morphisms of Stacks, Lemma 106.11.4 are satisfied for the morphisms

$$\mathcal{M}_g \to \overline{\mathcal{M}}_g \quad \text{and} \quad \overline{\mathcal{M}}_g \to \text{Spec}(\mathbf{Z})$$

and it suffices to check the following: given any 2-commutative diagram

$$\begin{tikzcd} \text{Spec}(K) \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[r] & \overline{\mathcal{M}}_g \arrow[d] \\ \text{Spec}(R) \arrow[rr] \arrow[rru, dotted] & & \text{Spec}(\mathbf{Z}) \end{tikzcd}$$

where R is a discrete valuation ring with field of fractions K the category of dotted arrows is either empty or a setoid with exactly one isomorphism class. (Observe that we don't need to worry about 2-arrows too much, see Morphisms of Stacks, Lemma 101.39.3). Unwinding what this means using that $\mathcal{M}_g$, resp. $\overline{\mathcal{M}}_g$ are the algebraic stacks parametrizing smooth, resp. stable families of genus g curves, we find that what we have to prove is exactly the uniqueness result stated and proved in Lemma 109.24.2. □

0E9B Lemma 109.25.2. Let $g \geq 2$. The stack $\overline{\mathcal{M}}_g$ is quasi-compact.

Proof. We will use the notation from Section 109.4. Consider the subset

$$T \subset |\text{PolarizedCurves}|$$

of points ξ such that there exists a field k and a pair $(X, \mathcal{L})$ over k representing ξ with the following two properties

(1) X is a stable genus g curve, and
(2) $\mathcal{L} = \omega_X^{\otimes 3}$.

Clearly, under the continuous map

$$|\text{PolarizedCurves}| \longrightarrow |\mathcal{C}urves|$$

the image of the set T is exactly the open subset

$$|\overline{\mathcal{M}}_g| \subset |\mathcal{C}urves|$$

Thus it suffices to show that T is quasi-compact. By Lemma 109.4.1 we see that

$$|\text{PolarizedCurves}| \subset |\mathcal{P}olarized|$$

is an open and closed immersion. Thus it suffices to prove quasi-compactness of T as a subset of $|\mathcal{P}olarized|$. For this we use the criterion of Moduli Stacks, Lemma 108.11.3. First, we observe that for $(X, \mathcal{L})$ as above the Hilbert polynomial P is the function $P(t) = (6g-6)t + (1-g)$ by Riemann-Roch, see Algebraic Curves, Lemma 53.5.2. Next, we observe that $H^1(X, \mathcal{L}) = 0$ and $\mathcal{L}$ is very ample by Algebraic

Curves, Lemma 53.24.3. This means exactly that with $n = P(3) - 1$ there is a closed immersion

$$i : X \longrightarrow \mathbf{P}^n_k$$

such that $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ as desired. $\square$

Here is the main theorem of this section.

0E9C Theorem 109.25.3. Let $g \geq 2$. The algebraic stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack, proper and smooth over $\mathrm{Spec}(\mathbf{Z})$. Moreover, the locus $\mathcal{M}_g$ parametrizing smooth curves is a dense open substack.

Proof. Most of the properties mentioned in the statement have already been shown. Smoothness is Lemma 109.22.6. Deligne-Mumford is Lemma 109.22.7. Openness of $\mathcal{M}_g$ is Lemma 109.22.8. We know that $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is separated by Lemma 109.25.1 and we know that $\overline{\mathcal{M}}_g$ is quasi-compact by Lemma 109.25.2. Thus, to show that $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is proper and finish the proof, we may apply More on Morphisms of Stacks, Lemma 106.11.3 to the morphisms $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$. Thus it suffices to check the following: given any 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, "j"'] & \mathcal{M}_g \arrow[r] & \overline{\mathcal{M}}_g \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] & & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d, "j'"'] & \overline{\mathcal{M}}_g \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[ur, dotted] & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

is nonempty (Morphisms of Stacks, Definition 101.39.1). (Observe that we don't need to worry about 2-arrows too much, see Morphisms of Stacks, Lemma 101.39.3). Unwinding what this means using that $\mathcal{M}_g$, resp. $\overline{\mathcal{M}}_g$ are the algebraic stacks parametrizing smooth, resp. stable families of genus g curves, we find that what we have to prove is exactly the result contained in the stable reduction theorem, i.e., Theorem 109.24.3. $\square$

109.26. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules

Part 9

Miscellany

CHAPTER 110

Examples

026Z 110.1. Introduction

0270 This chapter will contain examples which illuminate the theory.

110.2. An empty limit

0AKK This example is due to Waterhouse, see [Wat72]. Let S be an uncountable set. For every finite subset $T \subset S$ consider the set M_T of injective maps $T \to \mathbf{N}$. For $T \subset T' \subset S$ finite the restriction $M_{T'} \to M_T$ is surjective. Thus we have an inverse system over the directed partially ordered set of finite subsets of S with surjective transition maps. But $\lim M_T = \emptyset$ as an element in the limit would define an injective map $S \to \mathbf{N}$.

110.3. A zero limit

0ANX Let $(S_i)_{i \in I}$ be a directed inverse system of nonempty sets with surjective transition maps and with $\lim S_i = \emptyset$, see Section 110.2. Let K be a field and set

$$V_i = \bigoplus\nolimits_{s \in S_i} K$$

Then the transition maps $V_i \to V_j$ are surjective for $i \geq j$. However, $\lim V_i = 0$. Namely, if $v = (v_i)$ is an element of the limit, then the support of v_i would be a finite subset $T_i \subset S_i$ with $\lim T_i \neq \emptyset$, see Categories, Lemma 4.21.7.

For each i consider the unique K-linear map $V_i \to K$ which sends each basis vector $s \in S_i$ to 1. Let $W_i \subset V_i$ be the kernel. Then

$$0 \to (W_i) \to (V_i) \to (K) \to 0$$

is a nonsplit short exact sequence of inverse systems of vector spaces over the directed set I. Hence W_i is a directed system of K-vector spaces with surjective transition maps, vanishing limit, and nonvanishing $R^1 \lim$.

110.4. Non-quasi-compact inverse limit of quasi-compact spaces

09ZJ Let $\mathbf{N}$ denote the set of natural numbers. For every integer n, let I_n denote the set of all natural numbers $> n$. Define T_n to be the unique topology on $\mathbf{N}$ with basis $\{1\}, \ldots, \{n\}, I_n$. Denote by X_n the topological space $(\mathbf{N}, T_n)$. For each $m < n$, the identity map,

$$f_{n,m} : X_n \longrightarrow X_m$$

is continuous. Obviously for $m < n < p$, the composition $f_{p,n} \circ f_{n,m}$ equals $f_{p,m}$. So $((X_n), (f_{n,m}))$ is a directed inverse system of quasi-compact topological spaces.

Let T be the discrete topology on $\mathbf{N}$, and let X be $(\mathbf{N}, T)$. Then for every integer n, the identity map,

$$f_n : X \longrightarrow X_n$$

is continuous. We claim that this is the inverse limit of the directed system above. Let (Y, S) be any topological space. For every integer n, let

$$g_n : (Y, S) \longrightarrow (\mathbf{N}, T_n)$$

be a continuous map. Assume that for every $m < n$ we have $f_{n,m} \circ g_n = g_m$, i.e., the system (g_n) is compatible with the directed system above. In particular, all of the set maps g_n are equal to a common set map

$$g : Y \longrightarrow \mathbf{N}.$$

Moreover, for every integer n, since $\{n\}$ is open in X_n, also $g^{-1}(\{n\}) = g_n^{-1}(\{n\})$ is open in Y. Therefore the set map g is continuous for the topology S on Y and the topology T on $\mathbf{N}$. Thus $(X, (f_n))$ is the inverse limit of the directed system above.

However, clearly X is not quasi-compact, since the infinite open covering by singleton sets has no inverse limit.

09ZK **Lemma 110.4.1.** There exists an inverse system of quasi-compact topological spaces over $\mathbf{N}$ whose limit is not quasi-compact.

Proof. See discussion above. □

110.5. The structure sheaf on the fibre product

0FLS Let X, Y, S, a, b, p, q, f be as in the introduction to Derived Categories of Schemes, Section 36.23. Picture:

$$\begin{tikzcd}
& X \times_S Y \arrow[ld, "p"] \arrow[rd, "q"] \arrow[dd, "f"] & \\
X \arrow[rd, "a"] & & Y \arrow[ld, "b"] \\
& S &
\end{tikzcd}$$

Then we have a canonical map

$$can : p^{-1}\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_{X \times_S Y}$$

which is not an isomorphism in general.

For example, let $S = \text{Spec}(\mathbf{R})$, $X = \text{Spec}(\mathbf{C})$, and $Y = \text{Spec}(\mathbf{C})$. Then $X \times_S Y = \text{Spec}(\mathbf{C}) \amalg \text{Spec}(\mathbf{C})$ is a discrete space with two points and the sheaves $p^{-1}\mathcal{O}_X$, $q^{-1}\mathcal{O}_Y$ and $f^{-1}\mathcal{O}_S$ are the constant sheaves with values $\mathbf{C}$, $\mathbf{C}$, and $\mathbf{R}$. Hence the source of *can* is the constant sheaf with value $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ on the discrete space with two points. Thus its global sections have dimension 8 as an $\mathbf{R}$-vector space whereas taking global sections of the target of *can* we obtain $\mathbf{C} \times \mathbf{C}$ which has dimension 4 as an $\mathbf{R}$-vector space.

Another example is the following. Let k be an algebraically closed field. Consider $S = \text{Spec}(k)$, $X = \mathbf{A}^1_k$, and $Y = \mathbf{A}^1_k$. Then for $U \subset X \times_S Y = \mathbf{A}^2_k$ nonempty open the images $p(U) \subset X = \mathbf{A}^1_k$ and $q(U) \subset \mathbf{A}^1_k$ are open and the reader can show that

$$\left(p^{-1}\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{O}_Y\right)(U) = \mathcal{O}_X(p(U)) \otimes_k \mathcal{O}_Y(q(U))$$

This is not equal to $\mathcal{O}_{X \times_S Y}(U)$ if U is the complement of an irreducible curve C in $X \times_S Y = \mathbf{A}^2_k$ such that both $p|_C$ and $q|_C$ are nonconstant.

Returning to the general case, let $z = (x, y, s, \mathfrak{p})$ be a point of $X \times_S Y$ as in Schemes, Lemma 26.17.5. Then on stalks at z the map can gives the map

$$can_z : \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X \times_S Y, z}$$

This is a flat ring homomorphism as the target is a localization of the source (details omitted; hint reduce to the case that X, Y, and S are affine). Observe that the source is in general not a local ring, and this gives another way to see that can is not an isomorphism in general.

More generally, suppose we have an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then there is a canonical map

$$\begin{aligned}
& p^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G} \\
& = p^{-1}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \otimes_{f^{-1}\mathcal{O}_S} q^{-1}(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{G}) \\
& = p^{-1}\mathcal{F} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{O}_Y \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{G} \\
& \xrightarrow{can} p^{-1}\mathcal{F} \otimes_{q^{-1}\mathcal{O}_X} \mathcal{O}_{X \times_S Y} \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{G} \\
& = p^{-1}\mathcal{F} \otimes_{q^{-1}\mathcal{O}_X} \mathcal{O}_{X \times_S Y} \otimes_{\mathcal{O}_{X \times_S Y}} \mathcal{O}_{X \times_S Y} \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{G} \\
& = p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}
\end{aligned}$$

which is rarely an isomorphism.

110.6. A nonintegral connected scheme whose local rings are domains

0568 We give an example of an affine scheme $X = \text{Spec}(A)$ which is connected, all of whose local rings are domains, but which is not integral. Connectedness of X means A has no nontrivial idempotents, see Algebra, Lemma 10.21.3. The local rings of X are domains if, whenever $fg = 0$ in A, every point of X has a neighborhood where either f or g vanishes. As long as A is not a domain, then X is not integral (Properties, Definition 28.3.1).

Roughly speaking, the construction is as follows: let X_0 be the cross (the union of coordinate axes) on the affine plane. Then let X_1 be the (reduced) full preimage of X_0 on the blowup of the plane (X_1 has three rational components forming a chain). Then blow up the resulting surface at the two singularities of X_1, and let X_2 be the reduced preimage of X_1 (which has five rational components), etc. Take X to be the inverse limit. The only problem with this construction is that blowups glue in a projective line, so X_1 is not affine. Let us correct this by glueing in an affine line instead (so our scheme will be an open subset in what was described above).

Here is a completely algebraic construction: For every $k \geq 0$, let A_k be the following ring: its elements are collections of polynomials $p_i \in \mathbf{C}[x]$ where $i = 0, \ldots, 2^k$ such that $p_i(1) = p_{i+1}(0)$. Set $X_k = \text{Spec}(A_k)$. Observe that X_k is a union of 2^k+1 affine lines that meet transversally in a chain. Define a ring homomorphism $A_k \to A_{k+1}$ by

$$(p_0, \ldots, p_{2^k}) \longmapsto (p_0, p_0(1), p_1, p_1(1), \ldots, p_{2^k}),$$

in other words, every other polynomial is constant. This identifies A_k with a subring of A_{k+1}. Let A be the direct limit of A_k (basically, their union). Set $X = \text{Spec}(A)$. For every k, we have a natural embedding $A_k \to A$, that is, a map $X \to X_k$. Each

A_k is connected but not integral; this implies that A is connected but not integral. It remains to show that the local rings of A are domains.

Take $f, g \in A$ with $fg = 0$ and $x \in X$. Let us construct a neighborhood of x on which one of f and g vanishes. Choose k such that $f, g \in A_{k-1}$ (note the $k-1$ index). Let y be the image of x in X_k. It suffices to prove that y has a neighborhood on which either f or g viewed as sections of $\mathcal{O}_{X_k}$ vanishes. If y is a smooth point of X_k, that is, it lies on only one of the 2^k+1 lines, this is obvious. We can therefore assume that y is one of the 2^k singular points, so two components of X_k pass through y. However, on one of these two components (the one with odd index), both f and g are constant, since they are pullbacks of functions on X_{k-1}. Since $fg = 0$ everywhere, either f or g (say, f) vanishes on the other component. This implies that f vanishes on both components, as required.

110.7. Noncomplete completion

05JA Let R be a ring and let $\mathfrak{m}$ be a maximal ideal. Consider the completion

$$R^\wedge = \lim R/\mathfrak{m}^n.$$

Note that $R^\wedge$ is a local ring with maximal ideal $\mathfrak{m}' = \text{Ker}(R^\wedge \to R/\mathfrak{m})$. Namely, if $x = (x_n) \in R^\wedge$ is not in $\mathfrak{m}'$, then $y = (x_n^{-1}) \in R^\wedge$ satisfies $xy = 1$, whence $R^\wedge$ is local by Algebra, Lemma 10.18.2. Now it is always true that $R^\wedge$ complete in its limit topology (see the discussion in More on Algebra, Section 15.36). But beyond that, we have the following questions:

(1) Is it true that $\mathfrak{m}R^\wedge = \mathfrak{m}'$?
(2) Is $R^\wedge$ viewed as an $R^\wedge$-module $\mathfrak{m}'$-adically complete?
(3) Is $R^\wedge$ viewed as an R-module $\mathfrak{m}$-adically complete?

It turns out that these questions all have a negative answer. The example below was taken from an unpublished note of Bart de Smit and Hendrik Lenstra. See also [Bou61, Exercise III.2.12] and [Yek11, Example 1.8]

Let k be a field, $R = k[x_1, x_2, x_3, \ldots]$, and $\mathfrak{m} = (x_1, x_2, x_3, \ldots)$. We will think of an element f of $R^\wedge$ as a (possibly) infinite sum

$$f = \sum a_I x^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\mathfrak{m}' \subset R^\wedge$ is the collection of f with zero constant term. In particular, the element

$$f = x_1 + x_2^2 + x_3^3 + \ldots$$

is in $\mathfrak{m}'$ but not in $\mathfrak{m}R^\wedge$ which shows that (1) is false in this example. However, if (1) is false, then (3) is necessarily false because $\mathfrak{m}' = \text{Ker}(R^\wedge \to R/\mathfrak{m})$ and we can apply Algebra, Lemma 10.96.5 with $n = 1$.

To finish we prove that $R^\wedge$ is not $\mathfrak{m}'$-adically complete. For $n \geq 1$ let $K_n = \text{Ker}(R^\wedge \to R/\mathfrak{m}^n)$. Then we have short exact sequences

$$0 \to K_n/(\mathfrak{m}')^n \to R^\wedge/(\mathfrak{m}')^n \to R/\mathfrak{m}^n \to 0$$

The projection map $R^\wedge \to R/\mathfrak{m}^{n+1}$ sends $(\mathfrak{m}')^n$ onto $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. It follows that $K_{n+1} \to K_n/(\mathfrak{m}')^n$ is surjective. Hence the inverse system $(K_n/(\mathfrak{m}')^n)$ has surjective transition maps and taking inverse limits we obtain an exact sequence

$$0 \to \lim K_n/(\mathfrak{m}')^n \to \lim R^\wedge/(\mathfrak{m}')^n \to \lim R/\mathfrak{m}^n \to 0$$

by Algebra, Lemma 10.87.1. Thus we see that $R^\wedge$ is complete with respect to $\mathfrak{m}'$ if and only if $K_n = (\mathfrak{m}')^n$ for all $n \geq 1$.

To show that $R^\wedge$ is not $\mathfrak{m}'$-adically complete in our example we show that $K_2 = \text{Ker}(R^\wedge \to R/\mathfrak{m}^2)$ is not equal to $(\mathfrak{m}')^2$. Note that an element of $(\mathfrak{m}')^2$ can be written as a finite sum

05JB (110.7.0.1) $$\sum\nolimits_{i=1,\ldots,t} f_i g_i$$

with $f_i, g_i \in R^\wedge$ having vanishing constant terms. To get an example we are going to choose an $z \in K_2$ of the form

$$z = z_1 + z_2 + z_3 + \ldots$$

with the following properties

(1) there exist sequences $1 < d_1 < d_2 < d_3 < \ldots$ and $0 < n_1 < n_2 < n_3 < \ldots$ such that $z_i \in k[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]$ homogeneous of degree d_i, and
(2) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ the element z_i cannot be written as a sum (110.7.0.1) with $t \leq i$.

Clearly this implies that z is not in $(\mathfrak{m}')^2$ because the image of the relation (110.7.0.1) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ for i large enough would produce a contradiction. Hence it suffices to prove that for all $t > 0$ there exists a $d \gg 0$ and an integer n such that we can find an homogeneous element $z \in k[x_1, \ldots, x_n]$ of degree d which cannot be written as a sum (110.7.0.1) for the given t in $k[[x_1, \ldots, x_n]]$. Take $n > 2t$ and any $d > 1$ prime to the characteristic of k and set $z = \sum_{i=1,\ldots,n} x_i^d$. Then the vanishing locus of the ideal

$$(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}) = (dx_1^{d-1}, \ldots, dx_n^{d-1})$$

consists of one point. On the other hand,

$$\frac{\partial(\sum_{i=1,\ldots,t} f_i g_i)}{\partial x_j} \in (f_1, \ldots, f_t, g_1, \ldots, g_t)$$

by the Leibniz rule and hence the vanishing locus of these derivatives contains at least

$$V(f_1, \ldots, f_t, g_1, \ldots, g_t) \subset \text{Spec}(k[[x_1, \ldots, x_n]]).$$

Hence this is a contradiction as the dimension of $V(f_1, \ldots, f_t, g_1, \ldots, g_t)$ is at least $n - 2t \geq 1$.

05JC **Lemma 110.7.1.** There exists a local ring R and a maximal ideal $\mathfrak{m}$ such that the completion $R^\wedge$ of R with respect to $\mathfrak{m}$ has the following properties

(1) $R^\wedge$ is local, but its maximal ideal is not equal to $\mathfrak{m}R^\wedge$,
(2) $R^\wedge$ is not a complete local ring, and
(3) $R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module.

Proof. This follows from the discussion above as (with $R = k[x_1, x_2, x_3, \ldots]$) the completion of the localization $R_\mathfrak{m}$ is equal to the completion of R. $\square$

110.8. Noncomplete quotient

05JD Let k be a field. Let

$$R = k[t, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, x]/(z_i t - x^i w_i, z_i w_j)$$

Note that in particular $z_i z_j t = 0$ in this ring. Any element f of R can be uniquely written as a finite sum

$$f = \sum\nolimits_{i=0,\ldots,d} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Moreover, if f is written in this way, then $f \in (x^n)$ if and only if $f_i = 0$ for $i < n$. So x is a nonzerodivisor and $\bigcap(x^n) = 0$. Let $R^\wedge$ be the completion of R with respect to the ideal (x). Note that $R^\wedge$ is (x)-adically complete, see Algebra, Lemma 10.96.3. By the above we see that an element of $R^\wedge$ can be uniquely written as an infinite sum

$$f = \sum\nolimits_{i=0}^{\infty} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Consider the element

$$f = \sum\nolimits_{i=1}^{\infty} x^i w_i = x w_1 + x^2 w_2 + x^3 w_3 + \ldots$$

i.e., we have $f_n = w_n$. Note that $f \in (t, x^n)$ for every n because $x^m w_m \in (t)$ for all m. We claim that $f \notin (t)$. To prove this assume that $tg = f$ where $g = \sum g_l x^l$ in canonical form as above. Since $t z_i z_j = 0$ we may as well assume that none of the g_l have terms involving the products $z_i z_j$. Examining the process to get tg in canonical form we see the following: Given any term cm of g_l where $c \in k$ and m is a monomial in t, z_i, w_j and we make the following replacement

(1) if the monomial m does not involve any z_i, then ctm is a term of f_l, and
(2) if the monomial m does involve a z_i then it is equal to $m = z_i$ and we see that cw_i is term of f_{l+i}.

Since g_0 is a polynomial only finitely many of the variables z_i occur in it. Pick n such that z_n does not occur in g_0. Then the rules above show that w_n does not occur in f_n which is a contradiction. It follows that $R^\wedge/(t)$ is not complete, see Algebra, Lemma 10.96.10.

05JE **Lemma 110.8.1.** There exists a ring R complete with respect to a principal ideal I and a principal ideal J such that R/J is not I-adically complete.

Proof. See discussion above. $\square$

110.9. Completion is not exact

05JF A quick example is the following. Suppose that $R = k[t]$. Let $P = K = \bigoplus_{n \in \mathbf{N}} R$ and $M = \bigoplus_{n \in \mathbf{N}} R/(t^n)$. Then there is a short exact sequence $0 \to K \to P \to M \to 0$ where the first map is given by multiplication by t^n on the nth summand. We claim that $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is not exact in the middle. Namely, $\xi = (t^2, t^3, t^4, \ldots) \in P^\wedge$ maps to zero in $M^\wedge$ but is not in the image of $K^\wedge \to P^\wedge$, because it would be the image of $(t, t, t, \ldots)$ which is not an element of $K^\wedge$.

A "smaller" example is the following. In the situation of Lemma 110.8.1 the short exact sequence $0 \to J \to R \to R/J \to 0$ does not remain exact after completion. Namely, if $f \in J$ is a generator, then $f : R \to J$ is surjective, hence $R \to J^\wedge$ is surjective, hence the image of $J^\wedge \to R$ is $(f) = J$ but the fact that R/J is noncomplete means that the kernel of the surjection $R \to (R/J)^\wedge$ is strictly bigger than J, see Algebra, Lemmas 10.96.1 and 10.96.10. By the same token the sequence $R \to R \to R/(f) \to 0$ does not remain exact on completion.

05JG Lemma 110.9.1. Completion is not an exact functor in general; it is not even right exact in general. This holds even when I is finitely generated on the category of finitely presented modules.

Proof. See discussion above. $\square$

110.10. The category of complete modules is not abelian

07JQ Let R be a ring and let $I \subset R$ be a finitely generated ideal. Consider the category $\mathcal{A}$ of I-adically complete R-modules, see Algebra, Definition 10.96.2. Let $\varphi : M \to N$ be a morphism of $\mathcal{A}$. The cokernel of φ in $\mathcal{A}$ is the completion $(\mathrm{Coker}(\varphi))^\wedge$ of the usual cokernel (as I is finitely generated this completion is complete, see Algebra, Lemma 10.96.3). Let $K = \mathrm{Ker}(\varphi)$. We claim that K is complete and hence is the kernel of φ in $\mathcal{A}$. Namely, let $K^\wedge$ be the completion. As M is complete we obtain a factorization

$$K \to K^\wedge \to M \xrightarrow{\varphi} N$$

Since φ is continuous for the I-adic topology, $K \to K^\wedge$ has dense image, and $K = \mathrm{Ker}(\varphi)$ we conclude that $K^\wedge$ maps into K. Thus $K^\wedge = K \oplus C$ and K is a direct summand of a complete module, hence complete.

We will give an example that shows that $\mathrm{Im} \neq \mathrm{Coim}$ in general. We take $R = \mathbf{Z}_p = \lim_n \mathbf{Z}/p^n\mathbf{Z}$ to be the ring of p-adic integers and we take $I = (p)$. Consider the map

$$\mathrm{diag}(1, p, p^2, \ldots) : \left(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p\right)^\wedge \longrightarrow \prod\nolimits_{n \geq 1} \mathbf{Z}_p$$

where the left hand side is the p-adic completion of the direct sum. Hence an element of the left hand side is a vector $(x_1, x_2, x_3, \ldots)$ with $x_i \in \mathbf{Z}_p$ with p-adic valuation $v_p(x_i) \to \infty$ as $i \to \infty$. This maps to $(x_1, px_2, p^2x_3, \ldots)$. Hence we see that $(1, p, p^2, \ldots)$ is in the closure of the image but not in the image. By our description of kernels and cokernels above it is clear that $\mathrm{Im} \neq \mathrm{Coim}$ for this map.

07JR Lemma 110.10.1. Let R be a ring and let $I \subset R$ be a finitely generated ideal. The category of I-adically complete R-modules has kernels and cokernels but is not abelian in general.

Proof. See above. $\square$

110.11. The category of derived complete modules

0ARC Please read More on Algebra, Section 15.92 before reading this section.

Let A be a ring, let I be an ideal of A, and denote $\mathcal{C}$ the category of derived complete modules as defined in More on Algebra, Definition 15.91.4.

Let T be a set and let M_t, $t \in T$ be a family of derived complete modules. We claim that in general $\bigoplus M_t$ is not a derived complete module. For a specific example, let $A = \mathbf{Z}_p$ and $I = (p)$ and consider $\bigoplus_{n \in \mathbf{N}} \mathbf{Z}_p$. The map from $\bigoplus_{n \in \mathbf{N}} \mathbf{Z}_p$ to its p-adic completion isn't surjective. This means that $\bigoplus_{n \in \mathbf{N}} \mathbf{Z}_p$ cannot be derived complete as this would imply otherwise, see More on Algebra, Lemma 15.91.3. Hence the inclusion functor $\mathcal{C} \to \text{Mod}_A$ does not commute with either direct sums or (filtered) colimits.

Assume I is finitely generated. By the discussion in More on Algebra, Section 15.92 the category $\mathcal{C}$ has arbitrary colimits. However, we claim that filtered colimits are not exact in the category $\mathcal{C}$. Namely, suppose that $A = \mathbf{Z}_p$ and $I = (p)$. One has inclusions $f_n : \mathbf{Z}_p/p\mathbf{Z}_p \to \mathbf{Z}_p/p^n\mathbf{Z}_p$ of p-adically complete A-modules given by multiplication by p^{n-1}. There are commutative diagrams

$$\begin{tikzcd} \mathbf{Z}_p/p\mathbf{Z}_p \arrow[r, "f_n"'] \arrow[d, "1"] & \mathbf{Z}_p/p^n\mathbf{Z}_p \arrow[d, "p"'] \\ \mathbf{Z}_p/p\mathbf{Z}_p \arrow[r, "f_{n+1}"] & \mathbf{Z}_p/p^{n+1}\mathbf{Z}_p \end{tikzcd}$$

We claim: the colimit of these inclusions in the category $\mathcal{C}$ gives the map $\mathbf{Z}_p/p\mathbf{Z}_p \to 0$. Namely, the colimit in Mod_A of the system on the right is $\mathbf{Q}_p/\mathbf{Z}_p$. Thus the colimit in $\mathcal{C}$ is

$$H^0((\mathbf{Q}_p/\mathbf{Z}_p)^\wedge) = H^0(\mathbf{Z}_p[1]) = 0$$

by More on Algebra, Section 15.92 where $^\wedge$ is derived completion. This proves our claim.

0ARD Lemma 110.11.1. Let A be a ring and let $I \subset A$ be an ideal. The category $\mathcal{C}$ of derived complete modules is abelian and the inclusion functor $F : \mathcal{C} \to \text{Mod}_A$ is exact and commutes with arbitrary limits. If I is finitely generated, then $\mathcal{C}$ has arbitrary direct sums and colimits, but F does not commute with these in general. Finally, filtered colimits are not exact in $\mathcal{C}$ in general, hence $\mathcal{C}$ is not a Grothendieck abelian category.

Proof. See More on Algebra, Lemma 15.92.1 and discussion above. $\square$

110.12. Nonflat completions

0AL8 The completion of a ring with respect to an ideal isn't always flat, contrary to the Noetherian case. We have seen two examples of this phenomenon in More on Algebra, Example 15.90.10. In this section we give two more examples.

0AL9 Lemma 110.12.1. Let R be a ring. Let M be an R-module which is countable. Then M is a finite R-module if and only if $M \otimes_R R^{\mathbf{N}} \to M^{\mathbf{N}}$ is surjective.

Proof. If M is a finite module, then the map is surjective by Algebra, Proposition 10.89.2. Conversely, assume the map is surjective. Let $m_1, m_2, m_3, \ldots$ be an enumeration of the elements of M. Let $\sum_{j=1,\ldots,m} x_j \otimes a_j$ be an element of the tensor product mapping to the element $(m_n) \in M^{\mathbf{N}}$. Then we see that $x_1, \ldots, x_m$ generate M over R as in the proof of Algebra, Proposition 10.89.2. $\square$

0ALA Lemma 110.12.2. Let R be a countable ring. Let M be a countable R-module. Then M is finitely presented if and only if the canonical map $M \otimes_R R^{\mathbf{N}} \to M^{\mathbf{N}}$ is an isomorphism.

Proof. If M is a finitely presented module, then the map is an isomorphism by Algebra, Proposition 10.89.3. Conversely, assume the map is an isomorphism. By Lemma 110.12.1 the module M is finite. Choose a surjection $R^{\oplus m} \to M$ with kernel K. Then K is countable as a submodule of $R^{\oplus m}$. Arguing as in the proof of Algebra, Proposition 10.89.3 we see that $K \otimes_R R^{\mathbf{N}} \to K^{\mathbf{N}}$ is surjective. Hence we conclude that K is a finite R-module by Lemma 110.12.1. Thus M is finitely presented. $\square$

0ALB **Lemma 110.12.3.** Let R be a countable ring. Then R is coherent if and only if $R^{\mathbf{N}}$ is a flat R-module.

Proof. If R is coherent, then $R^{\mathbf{N}}$ is a flat module by Algebra, Proposition 10.90.6. Assume $R^{\mathbf{N}}$ is flat. Let $I \subset R$ be a finitely generated ideal. To prove the lemma we show that I is finitely presented as an R-module. Namely, the map $I \otimes_R R^{\mathbf{N}} \to R^{\mathbf{N}}$ is injective as $R^{\mathbf{N}}$ is flat and its image is $I^{\mathbf{N}}$ by Lemma 110.12.1. Thus we conclude by Lemma 110.12.2. $\square$

Let R be a countable ring. Observe that $R[[x]]$ is isomorphic to $R^{\mathbf{N}}$ as an R-module. By Lemma 110.12.3 we see that $R \to R[[x]]$ is flat if and only if R is coherent. There are plenty of noncoherent countable rings, for example

$$R = k[y, z, a_1, b_1, a_2, b_2, a_3, b_3, \ldots]/(a_1y + b_1z, a_2y + b_2z, a_3y + b_3z, \ldots)$$

where k is a countable field. This ring is not coherent because the ideal (y, z) of R is not a finitely presented R-module. Note that $R[[x]]$ is the completion of $R[x]$ by the principal ideal (x).

0ALC **Lemma 110.12.4.** There exists a ring such that the completion $R[[x]]$ of $R[x]$ at (x) is not flat over R and a fortiori not flat over $R[x]$.

Proof. See discussion above. $\square$

It turns out there is a ring R such that $R[[x]]$ is flat over R, but $R[[x]]$ is not flat over $R[x]$. See this post by Badam Baplan. Namely, let R be a valuation ring. Then R is coherent (Algebra, Example 10.90.2) and hence $R[[x]]$ is flat over R by Algebra, Proposition 10.90.6. On the other hand, we have the following lemma.

0F1Y **Lemma 110.12.5.** Let R be a domain with fraction field K. If $R[[x]]$ is flat over $R[x]$, then R is normal if and only if R is completely normal (Algebra, Definition 10.37.3).

Proof. Suppose we have $\alpha \in K$ and a nonzero $r \in R$ such that $r\alpha^n \in R$ for all $n \geq 1$. Then we consider $f = \sum r\alpha^{n-1}x^n$ in $R[[x]]$. Write $\alpha = a/b$ for $a, b \in R$ with b nonzero. Then we see that $(ax - b)f = -rb$. It follows that rb is in the ideal $(ax - b)R[[x]]$. Let $S = \{h \in R[x] : h(0) = 1\}$. This is a multiplicative subset and flatness of $R[x] \to R[[x]]$ implies that $S^{-1}R[x] \to R[[x]]$ is faithfully flat (details omitted; hint: use Algebra, Lemma 10.39.16). Hence

$$S^{-1}R/(ax - b)S^{-1}R \to R[[x]]/(ax - b)R[[x]]$$

is injective. We conclude that $hrb = (ax - b)g$ for some $h \in S$ and $g \in R[x]$. Writing $h = 1 + h_1x + \ldots + h_dx^d$ shows that we obtain

$$1 + h_1x + \ldots + h_dx^d = (1/r)(\alpha x - 1)g$$

This factorization in $K[x]$ gives a corresponding factorization in $K[x^{-1}]$ which shows that α is the root of a monic polynomial with coefficients in R as desired. $\square$

0F1Z **Lemma 110.12.6.** If R is a valuation ring of dimension > 1, then $R[[x]]$ is flat over R but not flat over $R[x]$.

Proof. The arguments above show that this is true if we can show that R is not completely normal (valuation rings are normal, see Algebra, Lemma 10.50.3). Let $\mathfrak{p} \subset \mathfrak{m} \subset R$ be a chain of primes. Pick nonzero $x \in \mathfrak{p}$ and $y \in \mathfrak{m} \setminus \mathfrak{p}$. Then $xy^{-n} \in R$ for all $n \geq 1$ (if not then $y^n/x \in R$ which is absurd because $y \notin \mathfrak{p}$). Hence $1/y$ is almost integral over R but not in R. □

Next, we will construct an example where the completion of a localization is nonflat. To do this consider the ring

$$R = k[y, z, a_1, a_2, a_3, \ldots]/(ya_i, a_ia_j)$$

Denote $f \in R$ the residue class of z. We claim the ring map

0ALD (110.12.6.1) $$R[[x]] \longrightarrow R_f[[x]]$$

isn't flat. Let I be the kernel of $y : R[[x]] \to R[[x]]$. A typical element g of I looks like $g = \sum g_{n,m} a_m x^n$ where $g_{n,m} \in k[z]$ and for a given n only a finite number of nonzero $g_{n,m}$. Let J be the kernel of $y : R_f[[x]] \to R_f[[x]]$. We claim that $J \neq IR_f[[x]]$. Namely, if this were true then we would have

$$\sum z^{-n} a_n x^n = \sum\nolimits_{i=1,\ldots,m} h_i g_i$$

for some $m \geq 1$, $g_i \in I$, and $h_i \in R_f[[x]]$. Say $h_i = \bar{h}_i \bmod (y, a_1, a_2, a_3, \ldots)$ with $\bar{h}_i \in k[z, 1/z][[x]]$. Looking at the coefficient of a_n and using the description of the elements g_i above we would get

$$z^{-n} x^n = \sum \bar{h}_i \bar{g}_{i,n}$$

for some $\bar{g}_{i,n} \in k[z][[x]]$. This would mean that all $z^{-n}x^n$ are contained in the finite $k[z][[x]]$-module generated by the elements $\bar{h}_i$. Since $k[z][[x]]$ is Noetherian this implies that the $R[z][[x]]$-submodule of $k[z, 1/z][[x]]$ generated by $1, z^{-1}x, z^{-2}x^2, \ldots$ is finite. By Algebra, Lemma 10.36.2 we would conclude that $z^{-1}x$ is integral over $k[z][[x]]$ which is absurd. On the other hand, if (110.12.6.1) were flat, then we would get $J = IR_f[[x]]$ by tensoring the exact sequence $0 \to I \to R[[x]] \xrightarrow{y} R[[x]]$ with $R_f[[x]]$.

0ALE Lemma 110.12.7. There exists a ring A complete with respect to a principal ideal I and an element $f \in A$ such that the I-adic completion $A_f^\wedge$ of A_f is not flat over A.

Proof. Set $A = R[[x]]$ and $I = (x)$ and observe that $R_f[[x]]$ is the completion of $R[[x]]_f$. □

110.13. Nonabelian category of quasi-coherent modules

0ALF In Sheaves on Stacks, Section 96.11 we defined the category of quasi-coherent modules on a category fibred in groupoids over *Sch*. Although we show in Sheaves on Stacks, Section 96.15 that this category is abelian for algebraic stacks, in this section we show that this is not the case for formal algebraic spaces.

Namely, consider $\mathbf{Z}_p$ viewed as topological ring using the p-adic topology. Let $X = \text{Spf}(\mathbf{Z}_p)$, see Formal Spaces, Definition 87.9.9. Then X is a sheaf in sets on $(Sch/\mathbf{Z})_{fppf}$ and gives rise to a stack in setoids $\mathcal{X}$, see Stacks, Lemma 8.6.2. Thus the discussion of Sheaves on Stacks, Section 96.15 applies.

Let $\mathcal{F}$ be a quasi-coherent module on $\mathcal{X}$. Since $X = \operatorname{colim} \operatorname{Spec}(\mathbf{Z}/p^n\mathbf{Z})$ it is clear from Sheaves on Stacks, Lemma 96.12.2 that $\mathcal{F}$ is given by a sequence $(\mathcal{F}_n)$ where

(1) $\mathcal{F}_n$ is a quasi-coherent module on $\operatorname{Spec}(\mathbf{Z}/p^n\mathbf{Z})$, and
(2) the transition maps give isomorphisms $\mathcal{F}_n = \mathcal{F}_{n+1}/p^n\mathcal{F}_{n+1}$.

Converting into modules we see that $\mathcal{F}$ corresponds to a system (M_n) where each M_n is an abelian group annihilated by p^n and the transition maps induce isomorphisms $M_n = M_{n+1}/p^nM_{n+1}$. In this situation the module $M = \lim M_n$ is a p-adically complete module and $M_n = M/p^nM$, see Algebra, Lemma 10.98.2. We conclude that the category of quasi-coherent modules on X is equivalent to the category of p-adically complete abelian groups. This category is not abelian, see Section 110.10.

0ALG Lemma 110.13.1. The category of quasi-coherent[1] modules on a formal algebraic space X is not abelian in general, even if X is a Noetherian affine formal algebraic space.

Proof. See discussion above. $\square$

110.14. Regular sequences and base change

063Z We are going to construct a ring R with a regular sequence (x, y, z) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.

To construct our example we first construct a peculiar module E over the ring $k[x, y, z]$ where k is any field. Namely, E will be a push-out as in the following diagram

$$\begin{tikzcd} \frac{xk[x,y,z,y^{-1}]}{xyk[x,y,z]} \arrow[r] \arrow[d, "z/x"] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]} \arrow[r] \arrow[d] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \arrow[d] \\ \frac{k[x,y,z,y^{-1}]}{yzk[x,y,z]} \arrow[r] & E \arrow[r] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \end{tikzcd}$$

where the rows are short exact sequences (we dropped the outer zeros due to typesetting problems). Another way to describe E is as

$$E = \{(f, g) \mid f \in k[x, y, z, x^{-1}, y^{-1}], g \in k[x, y, z, y^{-1}]\}/\sim$$

where $(f, g) \sim (f', g')$ if and only if there exists a $h \in k[x, y, z, y^{-1}]$ such that

$$f = f' + xh \bmod yk[x, y, z, x^{-1}], \quad g = g' - zh \bmod yzk[x, y, z]$$

We claim: (a) $x : E \to E$ is injective, (b) $y : E/xE \to E/xE$ is injective, (c) $E/(x, y)E = 0$, (d) there exists a nonzero element $\delta \in E/zE$ such that $x\delta = y\delta = 0$.

To prove (a) suppose that (f, g) is a pair that gives rise to an element of E and that $(xf, xg) \sim 0$. Then there exists a $h \in k[x, y, z, y^{-1}]$ such that $xf + xh \in yk[x, y, z, x^{-1}]$ and $xg - zh \in yzk[x, y, z]$. We may assume that $h = \sum a_{i,j,k}x^iy^jz^k$ is a sum of monomials where only $j \leq 0$ occurs. Then $xg - zh \in yzk[x, y, z]$ implies that only $i > 0$ occurs, i.e., $h = xh'$ for some $h' \in k[x, y, z, y^{-1}]$. Then $(f, g) \sim (f + xh', g - zh')$ and we see that we may assume that $g = 0$ and $h = 0$. In

[1] With quasi-coherent modules as defined above. Due to how things are setup in the Stacks project, this is really the correct definition; as seen above our definition agrees with what one would naively have defined to be quasi-coherent modules on $\operatorname{Spf}(A)$, namely complete A-modules.

this case $xf \in yk[x, y, z, x^{-1}]$ implies $f \in yk[x, y, z, x^{-1}]$ and we see that $(f, g) \sim 0$. Thus $x : E \to E$ is injective.

Since multiplication by x is an isomorphism on $\frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]}$ we see that E/xE is isomorphic to

$$\frac{k[x, y, z, y^{-1}]}{yzk[x, y, z] + xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]} = \frac{k[x, y, z, y^{-1}]}{xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]}$$

and hence multiplication by y is an isomorphism on E/xE. This clearly implies (b) and (c).

Let $e \in E$ be the equivalence class of $(1, 0)$. Suppose that $e \in zE$. Then there exist $f \in k[x, y, z, x^{-1}, y^{-1}]$, $g \in k[x, y, z, y^{-1}]$, and $h \in k[x, y, z, y^{-1}]$ such that

$$1 + zf + xh \in yk[x, y, z, x^{-1}], \quad 0 + zg - zh \in yzk[x, y, z].$$

This is impossible: the monomial 1 cannot occur in zf, nor in xh. On the other hand, we have $ye = 0$ and $xe = (x, 0) \sim (0, -z) = z(0, -1)$. Hence setting δ equal to the congruence class of e in E/zE we obtain (d).

0640 Lemma 110.14.1. There exists a local ring R and a regular sequence x, y, z (in the maximal ideal) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.

Proof. Let $R = k[x, y, z] \oplus E$ where E is the module above considered as a square zero ideal. Then it is clear that x, y, z is a regular sequence in R, and that the element $\delta \in E/zE \subset R/zR$ gives an element with the desired properties. To get a local example we may localize R at the maximal ideal $\mathfrak{m} = (x, y, z, E)$. The sequence x, y, z remains a regular sequence (as localization is exact), and the element δ remains nonzero as it is supported at $\mathfrak{m}$. $\square$

0641 Lemma 110.14.2. There exists a local homomorphism of local rings $A \to B$ and a regular sequence x, y in the maximal ideal of B such that $B/(x, y)$ is flat over A, but such that the images $\overline{x}, \overline{y}$ of x, y in $B/\mathfrak{m}_A B$ do not form a regular sequence, nor even a Koszul-regular sequence.

Proof. Set $A = k[z]_{(z)}$ and let $B = (k[x, y, z] \oplus E)_{(x,y,z,E)}$. Since x, y, z is a regular sequence in B, see proof of Lemma 110.14.1, we see that x, y is a regular sequence in B and that $B/(x, y)$ is a torsion free A-module, hence flat. On the other hand, there exists a nonzero element $\delta \in B/\mathfrak{m}_A B = B/zB$ which is annihilated by $\overline{x}, \overline{y}$. Hence $H_2(K_\bullet(B/\mathfrak{m}_A B, \overline{x}, \overline{y})) \neq 0$. Thus $\overline{x}, \overline{y}$ is not Koszul-regular, in particular it is not a regular sequence, see More on Algebra, Lemma 15.30.2. $\square$

110.15. A Noetherian ring of infinite dimension

02JC A Noetherian local ring has finite dimension as we saw in Algebra, Proposition 10.60.9. But there exist Noetherian rings of infinite dimension. See [Nag62b, Appendix, Example 1].

Namely, let k be a field, and consider the ring

$$R = k[x_1, x_2, x_3, \ldots].$$

Let $\mathfrak{p}_i = (x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i-1})$ for $i = 1, 2, \ldots$ which are prime ideals of R. Let S be the multiplicative subset

$$S = \bigcap\nolimits_{i \geq 1} (R \setminus \mathfrak{p}_i).$$

Consider the ring $A = S^{-1}R$. We claim that

(1) The maximal ideals of the ring A are the ideals $\mathfrak{m}_i = \mathfrak{p}_i A$.
(2) We have $A_{\mathfrak{m}_i} = R_{\mathfrak{p}_i}$ which is a Noetherian local ring of dimension 2^i.
(3) The ring A is Noetherian.

Hence it is clear that this is the example we are looking for. Details omitted.

110.16. Local rings with nonreduced completion

02JD In Algebra, Example 10.119.5 we gave an example of a characteristic p Noetherian local domain R of dimension 1 whose completion is nonreduced. In this section we present the example of [FR70, Proposition 3.1] which gives a similar ring in characteristic zero.

Let $\mathbf{C}\{x\}$ be the ring of convergent power series over the field $\mathbf{C}$ of complex numbers. The ring of all power series $\mathbf{C}[[x]]$ is its completion. Let $K = \mathbf{C}\{x\}[1/x]$ be the field of convergent Laurent series. The K-module $\Omega_{K/\mathbf{C}}$ of algebraic differentials of K over $\mathbf{C}$ is an infinite dimensional K-vector space (proof omitted). We may choose $f_n \in x\mathbf{C}\{x\}$, $n \geq 1$ such that $\mathrm{d}x, \mathrm{d}f_1, \mathrm{d}f_2, \ldots$ are part of a basis of $\Omega_{K/\mathbf{C}}$. Thus we can find a $\mathbf{C}$-derivation

$$D : \mathbf{C}\{x\} \longrightarrow \mathbf{C}((x))$$

such that $D(x) = 0$ and $D(f_i) = x^{-n}$. Let

$$A = \{f \in \mathbf{C}\{x\} \mid D(f) \in \mathbf{C}[[x]]\}$$

We claim that

(1) $\mathbf{C}\{x\}$ is integral over A,
(2) A is a local domain,
(3) $\dim(A) = 1$,
(4) the maximal ideal of A is generated by x and xf_1,
(5) A is Noetherian, and
(6) the completion of A is equal to the ring of dual numbers over $\mathbf{C}[[x]]$.

Since the dual numbers are nonreduced the ring A gives the example.

Note that if $0 \neq f \in x\mathbf{C}\{x\}$ then we may write $D(f) = h/f^n$ for some $n \geq 0$ and $h \in \mathbf{C}[[x]]$. Hence $D(f^{n+1}/(n+1)) \in \mathbf{C}[[x]]$ and $D(f^{n+2}/(n+2)) \in \mathbf{C}[[x]]$. Thus we see $f^{n+1}, f^{n+2} \in A$! In particular we see (1) holds. We also conclude that the fraction field of A is equal to the fraction field of $\mathbf{C}\{x\}$. It also follows immediately that $A \cap x\mathbf{C}\{x\}$ is the set of nonunits of A, hence A is a local domain of dimension 1. If we can show (4) then it will follow that A is Noetherian (proof omitted). Suppose that $f \in A \cap x\mathbf{C}\{x\}$. Write $D(f) = h$, $h \in \mathbf{C}[[x]]$. Write $h = c + xh'$ with $c \in \mathbf{C}$, $h' \in \mathbf{C}[[x]]$. Then $D(f - cxf_1) = c + xh' - c = xh'$. On the other hand $f - cxf_1 = xg$ with $g \in \mathbf{C}\{x\}$, but by the computation above we have $D(g) = h' \in \mathbf{C}[[x]]$ and hence $g \in A$. Thus $f = cxf_1 + xg \in (x, xf_1)$ as desired.

Finally, why is the completion of A nonreduced? Denote $\hat{A}$ the completion of A. Of course this maps surjectively to the completion $\mathbf{C}[[x]]$ of $\mathbf{C}\{x\}$ because $x \in A$. Denote this map $\psi : \hat{A} \to \mathbf{C}[[x]]$. Above we saw that $\mathfrak{m}_A = (x, xf_1)$ and hence $D(\mathfrak{m}_A^n) \subset (x^{n-1})$ by an easy computation. Thus $D : A \to \mathbf{C}[[x]]$ is continuous and gives rise to a continuous derivation $\hat{D} : \hat{A} \to \mathbf{C}[[x]]$ over ψ. Hence we get a ring map

$$\psi + \epsilon\hat{D} : \hat{A} \longrightarrow \mathbf{C}[[x]][\epsilon].$$

Since $\hat{A}$ is a one dimensional Noetherian complete local ring, if we can show this arrow is surjective then it will follow that $\hat{A}$ is nonreduced. Actually the map is an isomorphism but we omit the verification of this. The subring $\mathbf{C}[x]_{(x)} \subset A$ gives rise to a map $i : \mathbf{C}[[x]] \to \hat{A}$ on completions such that $i \circ \psi = \text{id}$ and such that $D \circ i = 0$ (as $D(x) = 0$ by construction). Consider the elements $x^n f_n \in A$. We have

$$(\psi + \epsilon D)(x^n f_n) = x^n f_n + \epsilon$$

for all $n \geq 1$. Surjectivity easily follows from these remarks.

110.17. Another local ring with nonreduced completion

0GHH In this section we make an example of a Noetherian local domain of dimension 2 complete with respect to a principal ideal such that the recompletion of a localization is nonreduced.

Let p be a prime number. Let k be a field of characteristic p such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \begin{matrix} \sum a_{i,j} x^i y^j \in k[[x, y]] \text{ such that for all } n \geq 0 \text{ we have} \\ [k^p(a_{n,n}, a_{n,n+1}, a_{n+1,n}, a_{n,n+2}, a_{n+2,n}, \ldots) : k^p] < \infty \end{matrix} \right\}$$

As a set we have

$$k^p[[x, y]] \subset A \subset k[[x, y]]$$

Every element f of A can be uniquely written as a series

$$f = f_0 + f_1 xy + f_2 (xy)^2 + f_3 (xy)^3 + \ldots$$

with

$$f_n = a_{n,n} + a_{n,n+1} y + a_{n+1,n} x + a_{n,n+2} y^2 + a_{n+2,n} x^2 + \ldots$$

and the condition in the formula defining A means that the coefficients of f_n generate a finite extension of k^p. From this presentation it is clear that A is an $k^p[[x, y]]$-subalgebra of $k[[x, y]]$ complete with respect to the ideal xy. Moreover, we clearly have

$$A/xyA = C \times_k D$$

where $k^p[[x]] \subset C \subset k[[x]]$ and $k^p[[y]] \subset D \subset k[[y]]$ are the subrings of power series from Algebra, Example 10.119.5. Hence C and D are dvrs and we see that A/xyA is Noetherian. By Algebra, Lemma 10.97.5 we conclude that A is Noetherian. Since $\dim(k[[x, y]]) = 2$ using Algebra, Lemma 10.112.4 we conclude that $\dim(A) = 2$.

Let $f = \sum a_i x^i$ be a power series such that $k^p(a_0, a_1, a_2, \ldots)$ has infinite degree over k^p. Then $f \notin A$ but $f^p \in A$. We set

$$B = A[f] \subset k[[x, y]]$$

Since B is finite over A we see that B is Noetherian. Also, B is complete with respect to the ideal generated by xy, see Algebra, Lemma 10.97.1. In fact B is free over A with basis $1, f, f^2, \ldots, f^{p-1}$; we omit the proof.

We claim the ring

$$(B_y)^\wedge = (B[1/y])^\wedge = \lim B[1/y]/(xy)^n B[1/y] = \lim B[1/y]/x^n B[1/y]$$

is nonreduced. Namely, this ring is free over

$$(A_y)^\wedge = (A[1/y])^\wedge = \lim A[1/y]/(xy)^n A[1/y] = \lim A[1/y]/x^n A[1/y]$$

with basis $1, f, \ldots, f^{p-1}$. However, there is an element $g \in (A_y)^\wedge$ such that $f^p = g^p$. Namely, we can just take $g = \sum a_i x^i$ (the same expression as we used for f) which makes sense in $(A_y)^\wedge$. Hence we see that

$$(B_y)^\wedge = (A_y)^\wedge[f]/(f^p - g^p) \cong (A_y)^\wedge[\tau]/(\tau^p)$$

is nonreduced. In fact, this example shows slightly more. Namely, observe that $(A_y)^\wedge$ is a dvr with uniformizer x and residue field the fraction field of the dvr D given above. Hence we see that even

$$(B_y)^\wedge[1/(xy)] = ((B_y)^\wedge)_{xy}$$

is nonreduced. This produces an example of the following kind.

0GHI Lemma 110.17.1. There exists a local Noetherian 2-dimensional domain $(B, \mathfrak{m})$ complete with respect to a principal ideal $I = (b)$ and an element $f \in \mathfrak{m}$, $f \notin I$ such that the I-adic completion $C = (B_f)^\wedge$ of the principal localization B_f is nonreduced and even such that $C_b = C[1/b] = (B_f)^\wedge[1/b]$ is nonreduced.

Proof. See discussion above. □

110.18. A non catenary Noetherian local ring

02JE Even though there is a succesful dimension theory of Noetherian local rings there are non-catenary Noetherian local rings. An example may be found in [Nag62b, Appendix, Example 2]. In fact, we will present this example in the simplest case. Namely, we will construct a local Noetherian domain A of dimension 2 which is not universally catenary. (Note that A is automatically catenary, see Exercises, Exercise 111.18.3.) The existence of a Noetherian local ring which is not universally catenary implies the existence of a Noetherian local ring which is not catenary – and we spell this out at the end of this section in the particular example at hand.

Let k be a field, and consider the formal power series ring $k[[x]]$ in one variable over k. Let

$$z = \sum\nolimits_{i=1}^{\infty} a_i x^i$$

be a formal power series. We assume z as an element of the Laurent series field $k((x)) = k[[x]][1/x]$ is transcendental over $k(x)$. Put

$$z_j = x^{-j}(z - \sum\nolimits_{i=1,\ldots,j-1} a_i x^i) = \sum\nolimits_{i=j}^{\infty} a_i x^{i-j} \in k[[x]].$$

Note that $z = xz_1$. Let R be the subring of $k[[x]]$ generated by x, z and all of the z_j, in other words

$$R = k[x, z_1, z_2, z_3, \ldots] \subset k[[x]].$$

Consider the ideals $\mathfrak{m} = (x)$ and $\mathfrak{n} = (x-1, z_1, z_2, \ldots)$ of R.

We have $xz_{j+1} + a_j = z_j$. Hence $R/\mathfrak{m} = k$ and $\mathfrak{m}$ is a maximal ideal. Moreover, any element of R not in $\mathfrak{m}$ maps to a unit in $k[[x]]$ and hence $R_\mathfrak{m} \subset k[[x]]$. In fact it is easy to deduce that $R_\mathfrak{m}$ is a discrete valuation ring and residue field k.

We claim that

$$R/(x-1) = k[x, z_1, z_2, z_3, \ldots]/(x-1) \cong k[z].$$

Namely, the relation above implies that $z_{j+1} = z_j - a_j - (x-1)z_{j+1}$, and hence we may express the class of z_{j+1} in terms of z_j in the quotient $R/(x-1)$. Since the fraction field of R has transcendence degree 2 over k by construction we see that

z is transcendental over k in $R/(x-1)$, whence the desired isomorphism. Hence $\mathfrak{n} = (x-1, z)$ and is a maximal ideal. In fact the map

$$k[x, x^{-1}, z]_{(x-1,z)} \longrightarrow R_\mathfrak{n}$$

is an isomorphism (since x^{-1} is invertible in $R_\mathfrak{n}$ and since $z_{j+1} = x^{-1}z_j - a_j = \ldots = f_j(x, x^{-1}, z)$). This shows that $R_\mathfrak{n}$ is a regular local ring of dimension 2 and residue field k.

Let S be the multiplicative subset

$$S = (R \setminus \mathfrak{m}) \cap (R \setminus \mathfrak{n}) = R \setminus (\mathfrak{m} \cup \mathfrak{n})$$

and set $B = S^{-1}R$. We claim that

(1) The ring B is a k-algebra.
(2) The maximal ideals of the ring B are the two ideals $\mathfrak{m}B$ and $\mathfrak{n}B$.
(3) The residue field at these maximal ideals is k.
(4) We have $B_{\mathfrak{m}B} = R_\mathfrak{m}$ and $B_{\mathfrak{n}B} = R_\mathfrak{n}$ which are Noetherian regular local rings of dimensions 1 and 2.
(5) The ring B is Noetherian.

We omit the details of the verifications.

Whenever given a k-algebra B with the properties listed above we get an example as follows. Take $A = k + \text{rad}(B) \subset B$ with $\text{rad}(B) = \mathfrak{m}B \cap \mathfrak{n}B$ the Jacobson radical. It is easy to see that B is finite over A and hence A is Noetherian by Eakin's theorem (see [Eak68], or [Nag62b, Appendix A1], or insert future reference here). Also A is a local domain with the same fraction field as B and residue field k. Since the dimension of B is 2 we see that A has dimension 2 as well, by Algebra, Lemma 10.112.4.

If A were universally catenary then the dimension formula, Algebra, Lemma 10.113.1 would give $\dim(B_{\mathfrak{m}B}) = 2$ contradiction.

Note that B is generated by one element over A. Hence $B = A[x]/\mathfrak{p}$ for some prime $\mathfrak{p}$ of $A[x]$. Let $\mathfrak{m}' \subset A[x]$ be the maximal ideal corresponding to $\mathfrak{m}B$. Then on the one hand $\dim(A[x]_{\mathfrak{m}'}) = 3$ and on the other hand

$$(0) \subset \mathfrak{p}A[x]_{\mathfrak{m}'} \subset \mathfrak{m}'A[x]_{\mathfrak{m}'}$$

is a maximal chain of primes. Hence $A[x]_{\mathfrak{m}'}$ is an example of a non catenary Noetherian local ring.

110.19. Existence of bad local Noetherian rings

0AL7 Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. In [Lec86a] it was shown that A is the completion of a Noetherian local domain if $\text{depth}(A) \geq 1$ and A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring, or contains $\mathbf{Z}$ as a subring and A is torsion free as a $\mathbf{Z}$-module. This produces many examples of Noetherian local domains with "bizarre" properties.

Applying this for example to $A = \mathbf{C}[[x, y]]/(y^2)$ we find a Noetherian local domain whose completion is nonreduced. Please compare with Section 110.16.

In [LLPY01] conditions were found that characterize when A is the completion of a reduced local Noetherian ring.

In [Hei93] it was shown that A is the completion of a local Noetherian UFD R if $\text{depth}(A) \geq 2$ and A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring, or contains $\mathbf{Z}$ as a subring and A is torsion free as a $\mathbf{Z}$-module. In particular R is normal (Algebra, Lemma 10.120.11) hence the henselization of R is a normal domain too (More on Algebra, Lemma 15.45.6). Thus A as above is the completion of a henselian Noetherian local normal domain (because the completion of R and its henselization agree, see More on Algebra, Lemma 15.45.3).

Apply this to find a Noetherian local UFD R such that $R^\wedge \cong \mathbf{C}[[x, y, z, w]]/(wx, wy)$. Note that $\text{Spec}(R^\wedge)$ is the union of a regular 2-dimensional and a regular 3-dimensional component. The ring R cannot be universally catenary: Let

$$X \longrightarrow \text{Spec}(R)$$

be the blowing up of the maximal ideal. Then X is an integral scheme. There is a closed point $x \in X$ such that $\dim(\mathcal{O}_{X,x}) = 2$, namely, on the level of the complete local ring we pick x to lie on the strict transform of the 2-dimensional component and not on the strict transform of the 3-dimensional component. By Morphisms, Lemma 29.52.1 we see that R is not universally catenary. Please compare with Section 110.18.

The ring above is catenary (being a 3-dimensional local Noetherian UFD). However, in [Ogo80] the author constructs a normal local Noetherian domain R with $R^\wedge \cong \mathbf{C}[[x, y, z, w]]/(wx, wy)$ such that R is not catenary. See also [Hei82] and [Lec86b].

In [Hei94] it was shown that A is the completion of a local Noetherian ring R with an isolated singularity provided A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring or A has residue characteristic $p > 0$ and p cannot map to a nonzero zerodivisor in any proper localization of A. Here we say a Noetherian local ring R has an isolated singularity if $R_\mathfrak{p}$ is a regular local ring for all nonmaximal primes $\mathfrak{p} \subset R$.

The papers [Nis12] and [Nis16] contain long lists of "bad" Noetherian local rings with given completions. In particular it constructs an example of a 2-dimensional Nagata local normal domain whose completion is $\mathbf{C}[[x, y, z]]/(yz)$ and one whose completion is $\mathbf{C}[[x, y, z]]/(y^2 - z^3)$.

As an aside, in [Loe03] it was shown that A is the completion of an excellent Noetherian local domain if A is reduced, equidimensional, and no integer in A is a zero divisor. However, this doesn't lead to "bad" Noetherian local rings as we obtain excellent ones!

110.20. Dimension in Noetherian Jacobson rings

0EEH Let k be the algebraic closure of a finite field. Let $A = k[x, y]$ and $X = \text{Spec}(A)$. Let $C = V(x)$ be the y-axis (this could be any other 1-dimensional integral closed subscheme of X). Let $C_1, C_2, C_3, \ldots$ be an enumeration of the other integral closed subschemes of X of dimension 1. Let $p_1, p_2, p_3, \ldots$ be an enumeration of the closed points of C.

Claim: for every n there exists an irreducible closed $Z_n \subset X$ of dimension 1 such that

$$\{p_n\} = Z_n \cap (C \cup C_1 \cup \ldots \cup C_n)$$

set theoretically. To do this set $Y = C \cup C_1 \cup C_2 \cup \ldots \cup C_n$. This is a reduced affine algebraic scheme of dimension 1 over k. It is enough to find $f \in k[x, y]$ with

$V(f) \cap Y = \{p_n\}$ set theoretically because then we can take Z_n to be a suitable irreducible component of $V(f)$. Since the restriction map

$$k[x, y] \longrightarrow \Gamma(Y, \mathcal{O}_Y)$$

is surjective, it suffices to find a regular function g on Y whose zero set is $\{p_n\}$ set theoretically. To see this is possible, we choose an effective Cartier divisor $D \subset Y$ whose support is p_n (this is possible by Varieties, Lemma 33.38.3). Thus it suffices to show that $\mathcal{O}_X(ND) \cong \mathcal{O}_X$ for some $N > 0$. But the Picard group of an affine 1-dimensional algebraic scheme over the algebraic closure of a finite field is torsion (insert future reference here) and we conclude the claim is true.

Choose Z_n as above for all n. Since $k[x, y]$ is a UFD we may write $Z_n = V(f_n)$ for some irreducible element $f_n \in A$. Let $S \subset k[x, y]$ be the multiplicative subset generated by $f_1, f_2, f_3, \ldots$. Consider the Noetherian ring $B = S^{-1}A$.

Obviously, the ring map $A \to B$ identifies local rings and induces an injection $\text{Spec}(B) \to \text{Spec}(A)$. Moreover, looking at the curve C_1 we see that only the points of $C \cap C_1$ are removed when passing from $\text{Spec}(A)$ to $\text{Spec}(B)$. In particular, we see that $\text{Spec}(B)$ has an infinite number of maximal ideals corresponding to maximal ideals of A. On the other hand, xB is a maximal ideal because the spectrum of B/xB consists of a unique prime ideal as we removed all the closed points of $C = V(x)$ (but not the generic point). Finally, for $i \geq 1$ consider the curve C_i. Write $C_i = V(g_i)$ for $g_i \in A$ irreducible. If $C_i = Z_n$ for some n, then g_iB is the unit ideal. If not, then all but finitely many of the closed points of C_i survive the passage from A to B: namely, only the points of $(Z_1 \cup \ldots \cup Z_{i-1} \cup C) \cap C_i$ are removed from C_i.

The structure of the prime spectrum of B given above shows that B is Jacobson by Algebra, Lemma 10.61.4. The maximal ideals are the maximal ideals of A which are in $\text{Spec}(B)$ (and there an inifinitude of these) together with the maximal ideal xB. Thus we see that we have local rings of dimensions 1 and 2.

0EEI Lemma 110.20.1. There exists a Jacobson, universally catenary, Noetherian domain B with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ such that $\dim(B_{\mathfrak{m}_1}) = 1$ and $\dim(B_{\mathfrak{m}_2}) = 2$.

Proof. The construction of B is given above. We just point out that B is universally catenary by Algebra, Lemma 10.105.4 and Morphisms, Lemma 29.17.5. $\square$

110.21. Underlying space Noetherian not Noetherian

0G61 We give two examples to show that a scheme whose underlying topological space is Noetherian may not be a Noetherian scheme.

0G62 Example 110.21.1. Let k be a field, and let $A = k[x_1, x_2, x_3, \ldots]/(x_1^2, x_2^2, x_3^2, \ldots)$. Any prime ideal of A contains the nilpotents $x_1, x_2, x_3, \ldots$, so $\mathfrak{p} = (x_1, x_2, x_3, \ldots)$ is the only prime ideal of A. Therefore the underlying topological space of $\text{Spec}\, A$ is a single point and in particular is Noetherian. However $\mathfrak{p}$ is clearly not finitely generated.

0G63 Example 110.21.2. Let k be a field, and let $A \subseteq k[x, y]$ be the subring generated by k and the monomials $\{xy^i\}_{i \geq 0}$. The prime ideals of A that do not contain x are in one-to-one correspondence with the prime ideals of $A_x \cong k[x, x^{-1}, y]$. If $\mathfrak{p}$ is a prime ideal that does contain x, then it contains every xy^i, $i \geq 0$, because

$(xy^i)^2 = x(xy^{2i}) \in \mathfrak{p}$ and $\mathfrak{p}$ is radical. Consequently $\mathfrak{p} = (\{xy^i\}_{i \geq 0})$. Therefore the underlying topological space of $\operatorname{Spec} A$ is Noetherian, since it consists of the points of the Noetherian scheme $\operatorname{Spec}(A[x, x^{-1}, y])$ and the prime ideal $\mathfrak{p}$. But the ring A is non-Noetherian because $\mathfrak{p}$ is not finitely generated. Note that in this example, A also has the property of being a domain.

110.22. Non-quasi-affine variety with quasi-affine normalization

0271 The existence of an example of this kind is mentioned in [DG67, II Remark 6.6.13]. They refer to the fifth volume of EGA for such an example, but the fifth volume did not appear.

Let k be a field. Let $Y = \mathbf{A}^2_k \setminus \{(0,0)\}$. We are going to construct a finite surjective birational morphism $\pi : Y \longrightarrow X$ with X a variety over k such that X is not quasi-affine. Namely, consider the following curves in Y:

$$\begin{matrix} C_1 & : & x = 0 \\ C_2 & : & y = 0 \end{matrix}$$

Note that $C_1 \cap C_2 = \emptyset$. We choose the isomorphism $\varphi : C_1 \to C_2$, $(0, y) \mapsto (y^{-1}, 0)$. We claim there is a unique morphism $\pi : Y \to X$ as above such that

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\varphi"'] & Y \arrow[r, "\pi"] & X \end{tikzcd}$$

is a coequalizer diagram in the category of varieties (and even in the category of schemes). Accepting this for the moment let us show that such an X cannot be quasi-affine. Namely, it is clear that we would get

$$\Gamma(X, \mathcal{O}_X) = \{f \in k[x, y] \mid f(0, y) = f(y^{-1}, 0)\} = k \oplus (xy) \subset k[x, y].$$

In particular these functions do not separate the points $(1, 0)$ and $(-1, 0)$ whose images in X (we will see below) are distinct (if the characteristic of k is not 2).

To show that X exists consider the Zariski open $D(x + y) \subset Y$ of Y. This is the spectrum of the ring $k[x, y, 1/(x + y)]$ and the curves C_1, C_2 are completely contained in $D(x + y)$. Moreover the morphism

$$C_1 \amalg C_2 \longrightarrow D(x + y) \cap Y = \operatorname{Spec}(k[x, y, 1/(x + y)])$$

is a closed immersion. It follows from More on Algebra, Lemma 15.5.1 that the ring

$$A = \{f \in k[x, y, 1/(x + y)] \mid f(0, y) = f(y^{-1}, 0)\}$$

is of finite type over k. On the other hand we have the open $D(xy) \subset Y$ of Y which is disjoint from the curves C_1 and C_2. It is the spectrum of the ring

$$B = k[x, y, 1/xy].$$

Note that we have $A_{xy} \cong B_{x+y}$ (since A clearly contains the elements $xyP(x, y)$ any polynomial P and the element $xy/(x + y)$). The scheme X is obtained by glueing the affine schemes $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ using the isomorphism $A_{xy} \cong B_{x+y}$ and hence is clearly of finite type over k. To see that it is separated one has to show that the ring map $A \otimes_k B \to B_{x+y}$ is surjective. To see this use that $A \otimes_k B$ contains the element $xy/(x + y) \otimes 1/xy$ which maps to $1/(x + y)$. The morphism $Y \to X$ is given by the natural maps $D(x + y) \to \operatorname{Spec}(A)$ and $D(xy) \to \operatorname{Spec}(B)$. Since these are both finite we deduce that $Y \to X$ is finite as desired. We omit the verification that X is indeed the coequalizer of the displayed diagram above, however, see (insert future reference for pushouts in the category of schemes here). Note that the morphism $\pi : Y \to X$ does map the points $(1,0)$ and $(-1,0)$ to distinct points in X because the function $(x+y^3)/(x+y)^2 \in A$ has value $1/1$, resp. $-1/(-1)^2 = -1$ which are always distinct (unless the characteristic is 2 – please find your own points for characteristic 2). We summarize this discussion in the form of a lemma.

0272 **Lemma 110.22.1.** Let k be a field. There exists a variety X whose normalization is quasi-affine but which is itself not quasi-affine.

Proof. See discussion above and (insert future reference on normalization here). $\square$

110.23. Taking scheme theoretic images

0GIK Let k be a field. Let t be a variable. Let $Y = \mathrm{Spec}(k[t])$ and $X = \coprod_{n \geq 1} \mathrm{Spec}(k[t]/(t^n))$. Denote $f : X \to Y$ the morphism using the closed immersion $\mathrm{Spec}(k[t]/(t^n)) \to \mathrm{Spec}(k[t])$ for each $n \geq 1$. In this case we have

1. The scheme theoretic image (Morphisms, Definition 29.6.2) of f is Y. On the other hand, the image of f is the closed point $t = 0$ in Y. Thus the underlying closed subset of the scheme theoretic image of f is not equal to the closure of the image of f.
2. The formation of the scheme theoretic image does not commute with restriction to the open subscheme $V = \mathrm{Spec}(k[t, 1/t]) \subset Y$. Namely, the preimage of V in X is empty and hence the scheme theoretic image of $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is the empty scheme. This is not equal to $Y \cap V$.

110.24. Images of locally closed subsets

0GZL Chevalley's theorem says that the image of a constructible set by a finitely presented morphism of affine schemes is constructible, see Algebra, Theorem 10.29.10 and Morphisms, Section 29.22. We will see the same thing does not hold for images of locally closed subsets.

Let k be a field of characteristic 0. Consider the projection morphism

$$f : X = \mathrm{Spec}(k[t, x_1, x_2, \ldots, y_1, y_2, \ldots]) \longrightarrow \mathrm{Spec}(k[x_1, x_2, \ldots, y_1, y_2, \ldots]) = Y$$

This is a morphism of finite presentation. Let Z be the closed subset of X defined by

$$x_1(t-1) = 0, \quad x_2(t-1)(t-2) = 0, \quad x_3(t-1)(t-2)(t-3) = 0, \quad \ldots$$

Let $U = \bigcup_{j \geq 1} U_j$ be the open of X defined by

$$U_j = \text{points where } y_j(t-1)(t-2)...(t-j) \text{ is nonzero}$$

Then we have

$$f(Z \cap U_j) = \text{points where } x_1, \ldots, x_j \text{ are zero and } y_j \text{ is nonzero}$$

We claim that $B = f(Z \cap U) = \bigcup_{j \geq 1} f(Z \cap U_j)$ is not a finite union of locally closed subsets of Y.

Proof of the claim. Suppose that $B = A_1 \cup \cdots \cup A_m$ is a finite cover of B by locally closed subsets of Y. We will show by induction on n that $m \geq n$. The base case $n = 1$ is OK as B is nonempty. Assume $n > 1$ and that the induction hypothesis holds for $n-1$. Since the closure of B is $(x_1 = 0)$, one of the A_i must contain some nonempty open subset of $(x_1 = 0)$. Then A_i must be open in $(x_1 = 0)$. But any such open subset cannot contain a point with $y_1 = 0$; indeed, for points of B, $y_1 = 0$ forces $x_2 = 0$, and this shows B contains no neighborhood of (x, y) inside $(x_1 = 0)$. Therefore, the remaining $m-1$ elements restrict to a constructible cover of $B \cap (y_1 = 0)$. However, observe that the right shift map $x_i \mapsto x_{i+1}$, $y_i \mapsto y_{i+1}$ identifies B with $B \cap (y_1 = 0)$! Thus by induction hypothesis, we see that $m - 1 \geq n - 1$ and we conclde $m \geq n$. This finishes the proof of the induction step and thereby establishes the claim.

0GZM Lemma 110.24.1. There exists a morphism $f : X \to Y$ of finite presentation between affine schemes and a locally closed subset T of X such that $f(T)$ is not a finite union of locally closed subsets of Y.

Proof. See discussion above. □

110.25. A locally closed subscheme which is not open in closed

078B This is a copy of Morphisms, Example 29.3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

The morphism $Z \to X$ also gives an example of bad behaviour of scheme theoretic images of immersions. Namely, the arguments above show that the scheme theoretic image of the immersion $Z \to X$ is X. On the other hand, we see

(1) Z is not topologically dense in X, and
(2) the scheme theoretic image of $Z = Z \cap U \to U$ is just Z. This is not equal to $U \cap X = U$ and hence formation of the scheme theoretic image in this case does not commute with restrictions to opens.

110.26. Nonexistence of suitable opens

086G This section complements the results of Properties, Section 28.29.

Let k be a field and let $A = k[z_1, z_2, z_3, \ldots]/I$ where I is the ideal generated by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $S = \mathrm{Spec}(A)$. Let $s \in S$ be the closed point corresponding to the maximal ideal (z_i). We claim there is no quasi-compact open $V \subset S \setminus \{s\}$ which is dense in $S \setminus \{s\}$. Note that $S \setminus \{s\} = \bigcup D(z_i)$. Each $D(z_i)$ is open and irreducible with generic point η_i. We conclude that $\eta_i \in V$ for all i. However, a principal affine open of $S \setminus \{s\}$ is of the form $D(f)$ where

$f \in (z_1, z_2, \ldots)$. Then $f \in (z_1, \ldots, z_n)$ for some n and we see that $D(f)$ contains only finitely many of the points η_i. Thus V cannot be quasi-compact.

Let k be a field and let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products z_iz_j, $i \neq j$, $i, j \in \mathbf{N}$. Set $T = \text{Spec}(B)$. Consider the principal open $U = D(x)$. We claim there is no quasi-compact open $V \subset S$ such that $V \cap U = \emptyset$ and $V \cup U$ is dense in S. Let $t \in T$ be the closed point corresponding to the maximal ideal (x, z_i). The closure of U in T is $\overline{U} = U \cup \{t\}$. Hence $V \subset \bigcup_i D(z_i)$ is a quasi-compact open. By the arguments of the previous paragraph we see that V cannot be dense in $\bigcup D(z_i)$.

086H Lemma 110.26.1. Nonexistence quasi-compact opens of affines:

(1) There exist an affine scheme S and affine open $U \subset S$ such that there is no quasi-compact open $V \subset S$ with $U \cap V = \emptyset$ and $U \cup V$ dense in S.
(2) There exists an affine scheme S and a closed point $s \in S$ such that $S \setminus \{s\}$ does not contain a quasi-compact dense open.

Proof. See discussion above. □

Let X be the glueing of two copies of the affine scheme T (see above) along the affine open U. Thus there is a morphism $\pi : X \to T$ and $X = U_1 \cup U_2$ such that π maps U_i isomorphically to T and $U_1 \cap U_2$ isomorphically to U. Note that X is quasi-separated (by Schemes, Lemma 26.21.6) and quasi-compact. We claim there does not exist a separated, dense, quasi-compact open $W \subset X$. Namely, consider the two closed points $x_1 \in U_1$, $x_2 \in U_2$ mapping to the closed point $t \in T$ introduced above. Let $\tilde{\eta} \in U_1 \cap U_2$ be the generic point mapping to the (unique) generic point η of U. Note that $\tilde{\eta} \rightsquigarrow x_1$ and $\tilde{\eta} \rightsquigarrow x_2$ lying over the specialization $\eta \rightsquigarrow s$. Since $\pi|_W : W \to T$ is separated we conclude that we cannot have both x_1 and $x_2 \in W$ (by the valuative criterion of separatedness Schemes, Lemma 26.22.2). Say $x_1 \notin W$. Then $W \cap U_1$ is a quasi-compact (as X is quasi-separated) dense open of U_1 which does not contain x_1. Now observe that there exists an isomorphism $(T, t) \cong (S, s)$ of schemes (by sending x to z_1 and z_i to z_{i+1}). Hence by the first paragraph of this section we arrive at a contradiction.

086I Lemma 110.26.2. There exists a quasi-compact and quasi-separated scheme X which does not contain a separated quasi-compact dense open.

Proof. See discussion above. □

110.27. Nonexistence of quasi-compact dense open subscheme

087H Let X be a quasi-compact and quasi-separated algebraic space over a field k. We know that the schematic locus $X' \subset X$ is a dense open subspace, see Properties of Spaces, Proposition 66.13.3. In fact, this result holds when X is reasonable, see Decent Spaces, Proposition 68.10.1. A natural question is whether one can find a quasi-compact dense open subscheme of X. It turns out this is not possible in general.

Assume the characteristic of k is not 2. Let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products z_iz_j, $i \neq j$, $i, j \in \mathbf{N}$. Set $U = \text{Spec}(B)$. Denote $0 \in U$ the closed point all of whose coordinates are zero. Set

$$j : R = \Delta \amalg \Gamma \longrightarrow U \times_k U$$

where Δ is the image of the diagonal morphism of U over k and

$$\Gamma = \{((x, 0, 0, 0, \ldots), (-x, 0, 0, 0, \ldots)) \mid x \in \mathbf{A}^1_k, x \neq 0\}.$$

It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space (Spaces, Theorem 65.10.5). Note that j is not an immersion because $(0, 0) \in \Delta$ is in the closure of Γ. Hence X is not a scheme. On the other hand, X is quasi-separated as R is quasi-compact. Denote 0_X the image of the point $0 \in U$. We claim that $X \setminus \{0_X\}$ is a scheme, namely

$$X \setminus \{0_X\} = \mathrm{Spec}\left(k[x^2, x^{-2}]\right) \amalg \mathrm{Spec}\left(k[z_1, z_2, z_3, \ldots]/(z_i z_j)\right) \setminus \{0\}$$

(details omitted). On the other hand, we have seen in Section 110.26 that the scheme on the right hand side does not contain a quasi-compact dense open.

087I Lemma 110.27.1. There exists a quasi-compact and quasi-separated algebraic space which does not contain a quasi-compact dense open subscheme.

Proof. See discussion above. □

Using the construction of Spaces, Example 65.14.2 in the same manner as we used the construction of Spaces, Example 65.14.1 above, one obtains an example of a quasi-compact, quasi-separated, and locally separated algebraic space which does not contain a quasi-compact dense open subscheme.

110.28. Affines over algebraic spaces

088V Suppose that $f : Y \to X$ is a morphism of schemes with f locally of finite type and Y affine. Then there exists an immersion $Y \to \mathbf{A}^n_X$ of Y into affine n-space over X. See the slightly more general Morphisms, Lemma 29.39.2.

Now suppose that $f : Y \to X$ is a morphism of algebraic spaces with f locally of finite type and Y an affine scheme. Then it is not true in general that we can find an immersion of Y into affine n-space over X.

A first (nasty) counter example is $Y = \mathrm{Spec}(k)$ and $X = [\mathbf{A}^1_k/\mathbf{Z}]$ where k is a field of characteristic zero and $\mathbf{Z}$ acts on $\mathbf{A}^1_k$ by translation $(n, t) \mapsto t + n$. Namely, for any morphism $Y \to \mathbf{A}^n_X$ over X we can pullback to the covering $\mathbf{A}^1_k$ of X and we get an infinite disjoint union of $\mathbf{A}^1_k$'s mapping into $\mathbf{A}^{n+1}_k$ which is not an immersion.

A second counter example is $Y = \mathbf{A}^1_k \to X = \mathbf{A}^1_k/R$ with $R = \{(t, t)\} \amalg \{(t, -t), t \neq 0\}$. Namely, in this case the morphism $Y \to \mathbf{A}^n_X$ would be given by some regular functions $f_1, \ldots, f_n$ on Y and hence the fibre product of Y with the covering $\mathbf{A}^{n+1}_k \to \mathbf{A}^n_X$ would be the scheme

$$\{(f_1(t), \ldots, f_n(t), t)\} \amalg \{(f_1(t), \ldots, f_n(t), -t), t \neq 0\}$$

with obvious morphism to $\mathbf{A}^{n+1}_k$ which is not an immersion. Note that this gives a counter example with X quasi-separated.

088W Lemma 110.28.1. There exists a finite type morphism of algebraic spaces $Y \to X$ with Y affine and X quasi-separated, such that there does not exist an immersion $Y \to \mathbf{A}^n_X$ over X.

Proof. See discussion above. □

110.29. Pushforward of quasi-coherent modules

078C In Schemes, Lemma 26.24.1 we proved that f_* transforms quasi-coherent modules into quasi-coherent modules when f is quasi-compact and quasi-separated. Here are some examples to show that these conditions are both necessary.

Suppose that $Y = \mathrm{Spec}(A)$ is an affine scheme and that $X = \coprod_{n \in \mathbf{N}} Y$. We claim that $f_*\mathcal{O}_X$ is not quasi-coherent where $f : X \to Y$ is the obvious morphism. Namely, for $a \in A$ we have

$$f_*\mathcal{O}_X(D(a)) = \prod\nolimits_{n \in \mathbf{N}} A_a$$

Hence, in order for $f_*\mathcal{O}_X$ to be quasi-coherent we would need

$$\prod\nolimits_{n \in \mathbf{N}} A_a = \left(\prod\nolimits_{n \in \mathbf{N}} A\right)_a$$

for all $a \in A$. This isn't true in general, for example if $A = \mathbf{Z}$ and $a = 2$, then $(1, 1/2, 1/4, 1/8, \ldots)$ is an element of the left hand side which is not in the right hand side. Note that f is a non-quasi-compact separated morphism.

Let k be a field. Set

$$A = k[t, z, x_1, x_2, x_3, \ldots]/(tx_1z, t^2x_2^2z, t^3x_3^3z, \ldots)$$

Let $Y = \mathrm{Spec}(A)$. Let $V \subset Y$ be the open subscheme $V = D(x_1) \cup D(x_2) \cup \ldots$. Let X be two copies of Y glued along V. Let $f : X \to Y$ be the obvious morphism. Then we have an exact sequence

$$0 \to f_*\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{(1,-1)} j_*\mathcal{O}_V$$

where $j : V \to Y$ is the inclusion morphism. Since

$$A \longrightarrow \prod A_{x_n}$$

is injective (details omitted) we see that $\Gamma(Y, f_*\mathcal{O}_X) = A$. On the other hand, the kernel of the map

$$A_t \longrightarrow \prod A_{tx_n}$$

is nonzero because it contains the element z. Hence $\Gamma(D(t), f_*\mathcal{O}_X)$ is strictly bigger than A_t because it contains $(z, 0)$. Thus we see that $f_*\mathcal{O}_X$ is not quasi-coherent. Note that f is quasi-compact but non-quasi-separated.

078D **Lemma 110.29.1.** Schemes, Lemma 26.24.1 is sharp in the sense that one can neither drop the assumption of quasi-compactness nor the assumption of quasi-separatedness.

Proof. See discussion above. $\square$

110.30. A nonfinite module with finite free rank 1 stalks

065J Let $R = \mathbf{Q}[x]$. Set $M = \sum_{n \in \mathbf{N}} \frac{1}{x-n} R$ as a submodule of the fraction field of R. Then M is not finitely generated, but for every prime $\mathfrak{p}$ of R we have $M_\mathfrak{p} \cong R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module.

An example of a similar flavor is $R = \mathbf{Z}$ and $M = \sum_{p \text{ prime}} \frac{1}{p}\mathbf{Z} \subset \mathbf{Q}$, which equals the set of fractions $\frac{a}{b}$ with b nonzero and squarefree.

110.31. A noninvertible ideal invertible in stalks

0CBZ Let A be a domain and let $I \subset A$ be a nonzero ideal. Recall that when we say I is invertible, we mean that I is invertible as an A-module. We are going to make an example of this situation where I is not invertible, yet $I_\mathfrak{q} = (f) \subset A_\mathfrak{q}$ is a (nonzero) principal ideal for every prime ideal $\mathfrak{q} \subset A$. In the literature the property that $I_\mathfrak{q}$ is principal for all primes $\mathfrak{q}$ is sometimes expressed by saying "I is a locally principal ideal". We can't use this terminology as our "local" always means "local in the Zariski topology" (or whatever topology we are currently working with).

Let $R = \mathbf{Q}[x]$ and let $M = \sum \frac{1}{x-n} R$ be the module constructed in Section 110.30. Consider the ring[2]

$$A = \text{Sym}^*_R(M)$$

and the ideal $I = MA = \bigoplus_{d \geq 1} \text{Sym}^d_R(M)$. Since M is not finitely generated as an R-module we see that I cannot be generated by finitely many elements as an ideal in A. Since an invertible module is finitely generated, this means that I is not invertible. On the other hand, let $\mathfrak{p} \subset R$ be a prime ideal. By construction $M_\mathfrak{p} \cong R_\mathfrak{p}$. Hence

$$A_\mathfrak{p} = \text{Sym}^*_{R_\mathfrak{p}}(M_\mathfrak{p}) \cong \text{Sym}^*_{R_\mathfrak{p}}(R_\mathfrak{p}) = R_\mathfrak{p}[T]$$

as a graded $R_\mathfrak{p}$-algebra. It follows that $I_\mathfrak{p} \subset A_\mathfrak{p}$ is generated by the nonzerodivisor T. Thus certainly for any prime ideal $\mathfrak{q} \subset A$ we see that $I_\mathfrak{q}$ is generated by a single element.

0CC0 **Lemma 110.31.1.** There exists a domain A and a nonzero ideal $I \subset A$ such that $I_\mathfrak{q} \subset A_\mathfrak{q}$ is a principal ideal for all primes $\mathfrak{q} \subset A$ but I is not an invertible A-module.

Proof. See discussion above. $\square$

110.32. A finite flat module which is not projective

052H This is a copy of Algebra, Remark 10.78.4. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0 \ \forall x, |x| < \epsilon\}$.

The morphism $\text{Spec}(R/I) \to \text{Spec}(R)$ is also an example of a flat closed immersion which is not open.

05FY **Lemma 110.32.1.** Strange flat modules.

(1) There exists a ring R and a finite flat R-module M which is not projective.
(2) There exists a closed immersion which is flat but not open.

Proof. See discussion above. $\square$

110.33. A projective module which is not locally free

05WG We give two examples. One where the rank is between 0 and 1 and one where the rank is $\aleph_0$.

05WH **Lemma 110.33.1.** Let R be a ring. Let $I \subset R$ be an ideal generated by a countable collection of idempotents. Then I is projective as an R-module.

[2] The ring A is an example of a non-Noetherian domain whose local rings are Noetherian.

Proof. Say $I = (e_1, e_2, e_3, \ldots)$ with e_n an idempotent of R. After inductively replacing e_{n+1} by $e_n + (1 - e_n)e_{n+1}$ we may assume that $(e_1) \subset (e_2) \subset (e_3) \subset \ldots$ and hence $I = \bigcup_{n \geq 1}(e_n) = \text{colim}_n\, e_nR$. In this case

$$\text{Hom}_R(I, M) = \text{Hom}_R(\text{colim}_n\, e_nR, M) = \lim_n \text{Hom}_R(e_nR, M) = \lim_n e_nM$$

Note that the transition maps $e_{n+1}M \to e_nM$ are given by multiplication by e_n and are surjective. Hence by Algebra, Lemma 10.86.4 the functor $\text{Hom}_R(I, M)$ is exact, i.e., I is a projective R-module. $\square$

Suppose that $P \subset Q$ is an inclusion of R-modules with Q a finite R-module and P locally free, see Algebra, Definition 10.78.1. Suppose that Q can be generated by N elements as an R-module. Then it follows from Algebra, Lemma 10.15.7 that P is finite locally free (with the free parts having rank at most N). And in this case P is a finite R-module, see Algebra, Lemma 10.78.2.

Combining this with the above we see that a non-finitely-generated ideal which is generated by a countable collection of idempotents is projective but not locally free. An explicit example is $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$ and I the ideal generated by the idempotents

$$e_n = (1, 1, \ldots, 1, 0, \ldots)$$

where the sequence of 1's has length n.

05WJ **Lemma 110.33.2.** *There exists a ring R and an ideal I such that I is projective as an R-module but not locally free as an R-module.*

Proof. See above. $\square$

05WK **Lemma 110.33.3.** *Let K be a field. Let C_i, $i = 1, \ldots, n$ be smooth, projective, geometrically irreducible curves over K. Let $P_i \in C_i(K)$ be a rational point and let $Q_i \in C_i$ be a point such that $[\kappa(Q_i) : K] = 2$. Then $[P_1 \times \ldots \times P_n]$ is nonzero in $\text{CH}_0(U_1 \times_K \ldots \times_K U_n)$ where $U_i = C_i \setminus \{Q_i\}$.*

Proof. There is a degree map $\deg : \text{CH}_0(C_1 \times_K \ldots \times_K C_n) \to \mathbf{Z}$ Because each Q_i has degree 2 over K we see that any zero cycle supported on the "boundary"

$$C_1 \times_K \ldots \times_K C_n \setminus U_1 \times_K \ldots \times_K U_n$$

has degree divisible by 2. $\square$

We can construct another example of a projective but not locally free module using the lemma above as follows. Let C_n, $n = 1, 2, 3, \ldots$ be smooth, projective, geometrically irreducible curves over $\mathbf{Q}$ each with a pair of points $P_n, Q_n \in C_n$ such that $\kappa(P_n) = \mathbf{Q}$ and $\kappa(Q_n)$ is a quadratic extension of $\mathbf{Q}$. Set $U_n = C_n \setminus \{Q_n\}$; this is an affine curve. Let $\mathcal{L}_n$ be the inverse of the ideal sheaf of P_n on U_n. Note that $c_1(\mathcal{L}_n) = [P_n]$ in the group of zero cycles $\text{CH}_0(U_n)$. Set $A_n = \Gamma(U_n, \mathcal{O}_{U_n})$. Let $L_n = \Gamma(U_n, \mathcal{L}_n)$ which is a locally free module of rank 1 over A_n. Set

$$B_n = A_1 \otimes_{\mathbf{Q}} A_2 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

so that $\text{Spec}(B_n) = U_1 \times \ldots \times U_n$ all products over $\text{Spec}(\mathbf{Q})$. For $i \leq n$ we set

$$L_{n,i} = A_1 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} M_i \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

which is a locally free B_n-module of rank 1. Note that this is also the global sections of $\text{pr}_i^*\mathcal{L}_n$. Set

$$B_\infty = \text{colim}_n\, B_n \quad \text{and} \quad L_{\infty,i} = \text{colim}_n\, L_{n,i}$$

Finally, set

$$M = \bigoplus\nolimits_{i \geq 1} L_{\infty, i}.$$

This is a direct sum of finite locally free modules, hence projective. We claim that M is not locally free. Namely, suppose that $f \in B_\infty$ is a nonzero function such that M_f is free over $(B_\infty)_f$. Let $e_1, e_2, \ldots$ be a basis. Choose $n \geq 1$ such that $f \in B_n$. Choose $m \geq n + 1$ such that $e_1, \ldots, e_{n+1}$ are in

$$\bigoplus\nolimits_{1 \leq i \leq m} L_{m,i}.$$

Because the elements $e_1, \ldots, e_{n+1}$ are part of a basis after a faithfully flat base change we conclude that the Chern classes

$$c_i(\text{pr}_1^*\mathcal{L}_1 \oplus \ldots \oplus \text{pr}_m^*\mathcal{L}_m), \quad i = m, m-1, \ldots, m-n$$

are zero in the chow group of

$$D(f) \subset U_1 \times \ldots \times U_m$$

Since f is the pullback of a function on $U_1 \times \ldots \times U_n$ this implies in particular that

$$c_{m-n}(\mathcal{O}_W^{\oplus n} \oplus \text{pr}_1^*\mathcal{L}_{n+1} \oplus \ldots \oplus \text{pr}_{m-n}^*\mathcal{L}_m) = 0.$$

on the variety

$$W = (C_{n+1} \times \ldots \times C_m)_K$$

over the field $K = \mathbf{Q}(C_1 \times \ldots \times C_n)$. In other words the cycle

$$[(P_{n+1} \times \ldots \times P_m)_K]$$

is zero in the chow group of zero cycles on W. This contradicts Lemma 110.33.3 above because the points Q_i, $n + 1 \leq i \leq m$ induce corresponding points Q_i' on $(C_n)_K$ and as $K/\mathbf{Q}$ is geometrically irreducible we have $[\kappa(Q_i') : K] = 2$.

05WL **Lemma 110.33.4.** There exists a countable ring R and a projective module M which is a direct sum of countably many locally free rank 1 modules such that M is not locally free.

Proof. See above. $\square$

110.34. Zero dimensional local ring with nonzero flat ideal

05FZ In [Laz67] and [Laz69] there is an example of a zero dimensional local ring with a nonzero flat ideal. Here is the construction. Let k be a field. Let X_i, Y_i, $i \geq 1$ be variables. Take $R = k[X_i, Y_i]/(X_i - Y_iX_{i+1}, Y_i^2)$. Denote x_i, resp. y_i the image of X_i, resp. Y_i in this ring. Note that

$$x_i = y_ix_{i+1} = y_iy_{i+1}x_{i+2} = y_iy_{i+1}y_{i+2}x_{i+3} = \ldots$$

in this ring. The ring R has only one prime ideal, namely $\mathfrak{m} = (x_i, y_i)$. We claim that the ideal $I = (x_i)$ is flat as an R-module.

Note that the annihilator of x_i in R is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Consider the R-module M generated by elements e_i, $i \geq 1$ and relations $e_i = y_ie_{i+1}$. Then M is flat as it is the colimit $\text{colim}_i R$ of copies of R with transition maps

$$R \xrightarrow{y_1} R \xrightarrow{y_2} R \xrightarrow{y_3} \ldots$$

Note that the annihilator of e_i in M is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Since every element of M, resp. I can be written as fe_i, resp. hx_i for some $f, h \in R$ we see that the map $M \to I$, $e_i \to x_i$ is an isomorphism and I is flat.

05G0 **Lemma 110.34.1.** There exists a local ring R with a unique prime ideal and a nonzero ideal $I \subset R$ which is a flat R-module

Proof. See discussion above. $\square$

110.35. An epimorphism of zero-dimensional rings which is not surjective

06RH In [Laz68] and [Laz69] one can find the following example. Let k be a field. Consider the ring homomorphism

$$k[x_1, x_2, \ldots, z_1, z_2, \ldots]/(x_i^{4^i}, z_i^{4^i}) \longrightarrow k[x_1, x_2, \ldots, y_1, y_2, \ldots]/(x_i^{4^i}, y_i - x_{i+1}y_{i+1}^2)$$

which maps x_i to x_i and z_i to $x_i y_i$. Note that $y_i^{4^{i+1}}$ is zero in the right hand side but that y_1 is not zero (details omitted). This map is not surjective: we can think of the above as a map of $\mathbf{Z}$-graded algebras by setting $\deg(x_i) = -1$, $\deg(z_i) = 0$, and $\deg(y_i) = 1$ and then it is clear that y_1 is not in the image. Finally, the map is an epimorphism because

$$y_{i-1} \otimes 1 = x_i y_i^2 \otimes 1 = y_i \otimes x_i y_i = x_i y_i \otimes y_i = 1 \otimes x_i y_i^2.$$

hence the tensor product of the target over the source is isomorphic to the target.

06RI **Lemma 110.35.1.** There exists an epimorphism of local rings of dimension 0 which is not a surjection.

Proof. See discussion above. $\square$

110.36. Finite type, not finitely presented, flat at prime

05G1 Let k be a field. Consider the local ring $A_0 = k[x, y]_{(x,y)}$. Denote $\mathfrak{p}_{0,n} = (y + x^n + x^{2n+1})$. This is a prime ideal. Set

$$A = A_0[z_1, z_2, z_3, \ldots]/(z_n z_m, z_n(y + x^n + x^{2n+1}))$$

Note that $A \to A_0$ is a surjection whose kernel is an ideal of square zero. Hence A is also a local ring and the prime ideals of A are in one-to-one correspondence with the prime ideals of A_0. Denote $\mathfrak{p}_n$ the prime ideal of A corresponding to $\mathfrak{p}_{0,n}$. Observe that $\mathfrak{p}_n$ is the annihilator of z_n in A. Let

$$C = A[z]/(xz^2 + z + y)[\frac{1}{2zx + 1}]$$

Note that $A \to C$ is an étale ring map, see Algebra, Example 10.137.8. Let $\mathfrak{q} \subset C$ be the maximal ideal generated by x, y, z and all z_n. As $A \to C$ is flat we see that the annihilator of z_n in C is $\mathfrak{p}_n C$. We compute

$$\begin{aligned} C/\mathfrak{p}_n C &= A_0[z]/(xz^2 + z + y, y + x^n + x^{2n+1})[1/(2zx + 1)] \\ &= k[x]_{(x)}[z]/(xz^2 + z - x^n - x^{2n+1})[1/(2zx + 1)] \\ &= k[x]_{(x)}[z]/(z - x^n) \times k[x]_{(x)}[z]/(xz + x^{n+1} + 1)[1/(2zx + 1)] \\ &= k[x]_{(x)} \times k(x) \end{aligned}$$

because $(z - x^n)(xz + x^{n+1} + 1) = xz^2 + z - x^n - x^{2n+1}$. Hence we see that $\mathfrak{p}_n C = \mathfrak{r}_n \cap \mathfrak{q}_n$ with $\mathfrak{r}_n = \mathfrak{p}_n C + (z - x^n)C$ and $\mathfrak{q}_n = \mathfrak{p}_n C + (xz + x^{n+1} + 1)C$. Since $\mathfrak{q}_n + \mathfrak{r}_n = C$ we also get $\mathfrak{p}_n C = \mathfrak{r}_n \mathfrak{q}_n$. It follows that $\mathfrak{q}_n$ is the annihilator of $\xi_n = (z - x^n)z_n$. Observe that on the one hand $\mathfrak{r}_n \subset \mathfrak{q}$, and on the other hand

$\mathfrak{q}_n + \mathfrak{q} = C$. This follows for example because $\mathfrak{q}_n$ is a maximal ideal of C distinct from $\mathfrak{q}$. Similarly we have $\mathfrak{q}_n + \mathfrak{q}_m = C$ for $n \neq m$. At this point we let

$$B = \mathrm{Im}(C \longrightarrow C_{\mathfrak{q}})$$

We observe that the elements ξ_n map to zero in B as $xz + x^{n+1} + 1$ is not in $\mathfrak{q}$. Denote $\mathfrak{q}' \subset B$ the image of $\mathfrak{q}$. By construction B is a finite type A-algebra, with $B_{\mathfrak{q}'} \cong C_{\mathfrak{q}}$. In particular we see that $B_{\mathfrak{q}'}$ is flat over A.

We claim there does not exist an element $g' \in B$, $g' \notin \mathfrak{q}'$ such that $B_{g'}$ is of finite presentation over A. We sketch a proof of this claim. Choose an element $g \in C$ which maps to $g' \in B$. Consider the map $C_g \to B_{g'}$. By Algebra, Lemma 10.6.3 we see that B_g is finitely presented over A if and only if the kernel of $C_g \to B_{g'}$ is finitely generated. But the element $g \in C$ is not contained in $\mathfrak{q}$, hence maps to a nonzero element of $A_0[z]/(xz^2 + z + y)$. Hence g can only be contained in finitely many of the prime ideals $\mathfrak{q}_n$, because the primes $(y + x^n + x^{2n+1}, xz + x^{n+1} + 1)$ are an infinite collection of codimension 1 points of the 2-dimensional irreducible Noetherian space $\mathrm{Spec}(k[x, y, z]/(xz^2 + z + y))$. The map

$$\bigoplus\nolimits_{g \notin \mathfrak{q}_n} C/\mathfrak{q}_n \longrightarrow C_g, \quad (c_n) \longrightarrow \sum c_n \xi_n$$

is injective and its image is the kernel of $C_g \to B_{g'}$. We omit the proof of this statement. (Hint: Write $A = A_0 \oplus I$ as an A_0-module where I is the kernel of $A \to A_0$. Similarly, write $C = C_0 \oplus IC$. Write $IC = \bigoplus Cz_n \cong \bigoplus (C/\mathfrak{r}_n \oplus C/\mathfrak{q}_n)$ and study the effect of multiplication by g on the summands.) This concludes the sketch of the proof of the claim. This also proves that $B_{g'}$ is not flat over A for any g' as above. Namely, if it were flat, then the annihilator of the image of z_n in $B_{g'}$ would be $\mathfrak{p}_n B_{g'}$, and would not contain $z - x^n$.

As a consequence we can answer (negatively) a question posed in [GR71, Part I, Remarques (3.4.7) (v)]. Here is a precise statement.

05G2 **Lemma 110.36.1.** There exists a local ring A, a finite type ring map $A \to B$ and a prime $\mathfrak{q}$ lying over $\mathfrak{m}_A$ such that $B_{\mathfrak{q}}$ is flat over A, and for any element $g \in B$, $g \notin \mathfrak{q}$ the ring B_g is neither finitely presented over A nor flat over A.

Proof. See discussion above. $\square$

110.37. Finite type, flat and not of finite presentation

05LB In this section we give some examples of ring maps and morphisms which are of finite type and flat but not of finite presentation.

Let R be a ring which has an ideal I such that R/I is a finite flat module but not projective, see Section 110.32 for an explicit example. Note that this means that I is not finitely generated, see Algebra, Lemma 10.108.5. Note that $I = I^2$, see Algebra, Lemma 10.108.2. The base ring in our examples will be R and correspondingly the base scheme $S = \mathrm{Spec}(R)$.

Consider the ring map $R \to R \oplus R/I\epsilon$ where $\epsilon^2 = 0$ by convention. This is a finite, flat ring map which is not of finite presentation. All the fibre rings are complete intersections and geometrically irreducible.

Let $A = R[x, y]/(xy, ay; a \in I)$. Note that as an R-module we have $A = \bigoplus_{i \geq 0} Ry^i \oplus \bigoplus_{j > 0} R/Ix^j$. Hence $R \to A$ is a flat finite type ring map which is not of finite presentation. Each fibre ring is isomorphic to either $\kappa(\mathfrak{p})[x, y]/(xy)$ or $\kappa(\mathfrak{p})[x]$.

We can turn the previous example into a projective morphism by taking $B = R[X_0, X_1, X_2]/(X_1X_2, aX_2; a \in I)$. In this case $X = \text{Proj}(B) \to S$ is a proper flat morphism which is not of finite presentation such that for each $s \in S$ the fibre X_s is isomorphic either to $\mathbf{P}^1_s$ or to the closed subscheme of $\mathbf{P}^2_s$ defined by the vanishing of X_1X_2 (this is a projective nodal curve of arithmetic genus 0).

Let $M = R \oplus R \oplus R/I$. Set $B = \text{Sym}_R(M)$ the symmetric algebra on M. Set $X = \text{Proj}(B)$. Then $X \to S$ is a proper flat morphism, not of finite presentation such that for $s \in S$ the geometric fibre is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$. In particular these fibres are smooth and geometrically irreducible.

05LC Lemma 110.37.1. There exist examples of

(1) a flat finite type ring map with geometrically irreducible complete intersection fibre rings which is not of finite presentation,
(2) a flat finite type ring map with geometrically connected, geometrically reduced, dimension 1, complete intersection fibre rings which is not of finite presentation,
(3) a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or to the vanishing locus of X_1X_2 in $\mathbf{P}^2_s$ which is not of finite presentation, and
(4) a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$ which is not of finite presentation.

Proof. See discussion above. □

110.38. Topology of a finite type ring map

05JH Let $A \to B$ be a local map of local domains. If A is Noetherian, $A \to B$ is essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite then there exists a prime $\mathfrak{q} \subset B$, $\mathfrak{q} \neq \mathfrak{m}_B$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. See More on Morphisms, Lemma 37.52.6.

In this section we give an example that shows this result is false A is no longer Noetherian. Namely, let k be a field and set

$$A = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k, a_i \in k((y)) \text{ for } i \geq 1\}$$

and

$$C = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k[y], a_i \in k((y)) \text{ for } i \geq 1\}.$$

The inclusion $A \to C$ is of finite type as C is generated by y over A. We claim that A is a local ring with maximal ideal $\mathfrak{m} = \{a_1x + a_2x^2 + \ldots \in A\}$ and no prime ideals besides (0) and $\mathfrak{m}$. Namely, an element $f = a_0 + a_1x + a_2x^2 + \ldots$ of A is invertible as soon as $a_0 \neq 0$. If $\mathfrak{q} \subset A$ is a nonzero prime ideal, and $f = a_ix^i + \ldots \in \mathfrak{q}$, then using properties of power series one sees that for any $g \in k((y))$ the element $g^{i+1}x^{i+1} \in \mathfrak{q}$, i.e., $gx \in \mathfrak{q}$. This proves that $\mathfrak{q} = \mathfrak{m}$.

As to the spectrum of the ring C, arguing in the same way as above we see that any nonzero prime ideal contains the prime $\mathfrak{p} = \{a_1x + a_2x^2 + \ldots \in C\}$ which lies over $\mathfrak{m}$. Thus the only prime of C which lies over (0) is (0). Set $\mathfrak{m}_C = yC + \mathfrak{p}$ and $B = C_{\mathfrak{m}_C}$. Then $A \to B$ is the desired example.

05JI Lemma 110.38.1. There exists a local homomorphism $A \to B$ of local domains which is essentially of finite type and such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is finite such that for every prime $\mathfrak{q} \neq \mathfrak{m}_B$ of B the ring map $A \to B/\mathfrak{q}$ is not the localization of a quasi-finite ring map.

Proof. See the discussion above. $\square$

110.39. Pure not universally pure

05JJ Let k be a field. Let

$$R = k[[x, xy, xy^2, \ldots]] \subset k[[x, y]].$$

In other words, a power series $f \in k[[x, y]]$ is in R if and only if $f(0, y)$ is a constant. In particular $R[1/x] = k[[x, y]][1/x]$ and R/xR is a local ring with a maximal ideal whose square is zero. Denote $R[y] \subset k[[x, y]]$ the set of power series $f \in k[[x, y]]$ such that $f(0, y)$ is a polynomial in y. Then $R \to R[y]$ is a finite type but not finitely presented ring map which induces an isomorphism after inverting x. Also there is a surjection $R[y]/xR[y] \to k[y]$ whose kernel has square zero. Consider the finitely presented ring map $R \to S = R[t]/(xt - xy)$. Again $R[1/x] \to S[1/x]$ is an isomorphism and in this case $S/xS \cong (R/xR)[t]/(xy)$ maps onto $k[t]$ with nilpotent kernel. There is a surjection $S \to R[y]$, $t \longmapsto y$ which induces an isomorphism on inverting x and a surjection with nilpotent kernel modulo x. Hence the kernel of $S \to R[y]$ is locally nilpotent. In particular $S \to R[y]$ induces a universal homeomorphism on spectra.

First we claim that S is an S-module which is relatively pure over R. Since on inverting x we obtain an isomorphism we only need to check this at the maximal ideal $\mathfrak{m} \subset R$. Since R is complete with respect to its maximal ideal it is henselian hence we need only check that every prime $\mathfrak{p} \subset R$, $\mathfrak{p} \neq \mathfrak{m}$, the unique prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ satisfies $\mathfrak{m}S + \mathfrak{q} \neq S$. Since $\mathfrak{p} \neq \mathfrak{m}$ it corresponds to a unique prime ideal of $k[[x, y]][1/x]$. Hence either $\mathfrak{p} = (0)$ or $\mathfrak{p} = (f)$ for some irreducible element $f \in k[[x, y]]$ which is not associated to x (here we use that $k[[x, y]]$ is a UFD – insert future reference here). In the first case $\mathfrak{q} = (0)$ and the result is clear. In the second case we may multiply f by a unit so that $f \in R[y]$ (Weierstrass preparation; details omitted). Then it is easy to see that $R[y]/fR[y] \cong k[[x, y]]/(f)$ hence f defines a prime ideal of $R[y]$ and $\mathfrak{m}R[y] + fR[y] \neq R[y]$. Since $S \to R[y]$ induces a universal homeomorphism on spectra we deduce the desired result for S also.

Second we claim that S is not universally relatively pure over R. Namely, to see this it suffices to find a valuation ring $\mathcal{O}$ and a local ring map $R \to \mathcal{O}$ such that $\mathrm{Spec}(R[y] \otimes_R \mathcal{O}) \to \mathrm{Spec}(\mathcal{O})$ does not hit the closed point of $\mathrm{Spec}(\mathcal{O})$. Equivalently, we have to find $\varphi : R \to \mathcal{O}$ such that $\varphi(x) \neq 0$ and $v(\varphi(x)) > v(\varphi(xy))$ where v is the valuation of $\mathcal{O}$. (Because this means that the valuation of y is negative.) To do this consider the ring map

$$R \longrightarrow \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0 \in k[y^{-1}], a_i \in k((y))\}$$

defined in the obvious way. We can find a valuation ring $\mathcal{O}$ dominating the localization of the right hand side at the maximal ideal (y^{-1}, x) and we win.

05JK Lemma 110.39.1. There exists a morphism of affine schemes of finite presentation $X \to S$ and an $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation such that $\mathcal{F}$ is pure relative to S, but not universally pure relative to S.

Proof. See discussion above. $\square$

110.40. A formally smooth non-flat ring map

057V Let k be a field. Consider the k-algebra $k[\mathbf{Q}]$. This is the k-algebra with basis $x_\alpha, \alpha \in \mathbf{Q}$ and multiplication determined by $x_\alpha x_\beta = x_{\alpha+\beta}$. (In particular $x_0 = 1$.) Consider the k-algebra homomorphism

$$k[\mathbf{Q}] \longrightarrow k, \quad x_\alpha \longmapsto 1.$$

It is surjective with kernel J generated by the elements $x_\alpha - 1$. Let us compute J/J^2. Note that multiplication by x_α on J/J^2 is the identity map. Denote z_α the class of $x_\alpha - 1$ modulo J^2. These classes generate J/J^2. Since

$$(x_\alpha - 1)(x_\beta - 1) = x_{\alpha+\beta} - x_\alpha - x_\beta + 1 = (x_{\alpha+\beta} - 1) - (x_\alpha - 1) - (x_\beta - 1)$$

we see that $z_{\alpha+\beta} = z_\alpha + z_\beta$ in J/J^2. A general element of J/J^2 is of the form $\sum \lambda_\alpha z_\alpha$ with $\lambda_\alpha \in k$ (only finitely many nonzero). Note that if the characteristic of k is $p > 0$ then

$$0 = p z_{\alpha/p} = z_{\alpha/p} + \ldots + z_{\alpha/p} = z_\alpha$$

and we see that $J/J^2 = 0$. If the characteristic of k is zero, then

$$J/J^2 = \mathbf{Q} \otimes_{\mathbf{Z}} k \cong k$$

(details omitted) is not zero.

We claim that $k[\mathbf{Q}] \to k$ is a formally smooth ring map if the characteristic of k is positive. Namely, suppose given a solid commutative diagram

$$\begin{tikzcd} k \arrow[r] \arrow[rd, dotted] & A \\ k[\mathbf{Q}] \arrow[u] \arrow[r, "\varphi"] & A' \arrow[u] \end{tikzcd}$$

with $A' \to A$ a surjection whose kernel I has square zero. To show that $k[\mathbf{Q}] \to k$ is formally smooth we have to prove that φ factors through k. Since $\varphi(x_\alpha - 1)$ maps to zero in A we see that φ induces a map $\overline{\varphi} : J/J^2 \to I$ whose vanishing is the obstruction to the desired factorization. Since $J/J^2 = 0$ if the characteristic is $p > 0$ we get the result we want, i.e., $k[\mathbf{Q}] \to k$ is formally smooth in this case. Finally, this ring map is not flat, for example as the nonzerodivisor $x_2 - 1$ is mapped to zero.

057W **Lemma 110.40.1.** There exists a formally smooth ring map which is not flat.

Proof. See discussion above. $\square$

110.41. A formally étale non-flat ring map

060H In this section we give a counterexample to the final sentence in [DG67, 0, Example 19.10.3(i)] (this was not one of the items caught in their later errata lists). Consider $A \to A/J$ for a local ring A and a nonzero proper ideal J such that $J^2 = J$ (so J isn't finitely generated); the valuation ring of an algebraically closed non-archimedean field with J its maximal ideal is a source of such (A, J). These non-flat quotient maps are formally étale. Namely, suppose given a commutative diagram

$$\begin{tikzcd} A/J \arrow[r] & R/I \\ A \arrow[u] \arrow[r, "\varphi"] & R \arrow[u] \end{tikzcd}$$

where I is an ideal of the ring R with $I^2 = 0$. Then $A \to R$ factors uniquely through A/J because

$$\varphi(J) = \varphi(J^2) \subset (\varphi(J)A)^2 \subset I^2 = 0.$$

Hence this also provides a counterexample to the formally étale case of the "structure theorem" for locally finite type and formally étale morphisms in [DG67, IV, Theorem 18.4.6(i)] (but not a counterexample to part (ii), which is what people actually use in practice). The error in the proof of the latter is that the very last step of the proof is to invoke the incorrect [DG67, 0, Example 19.3.10(i)], which is how the counterexample just mentioned creeps in.

060I Lemma 110.41.1. There exist formally étale nonflat ring maps.

Proof. See discussion above. □

110.42. A formally étale ring map with nontrivial cotangent complex

06E5 Let k be a field. Consider the ring

$$R = k[\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}]/(x_1y_1, x_{nm}^m - x_n, y_{nm}^m - y_n)$$

Let A be the localization at the maximal ideal generated by all x_n, y_n and denote $J \subset A$ the maximal ideal. Set $B = A/J$. By construction $J^2 = J$ and hence $A \to B$ is formally étale (see Section 110.41). We claim that the element $x_1 \otimes y_1$ is a nonzero element in the kernel of

$$J \otimes_A J \longrightarrow J.$$

Namely, (A, J) is the colimit of the localizations (A_n, J_n) of the rings

$$R_n = k[x_n, y_n]/(x_n^n y_n^n)$$

at their corresponding maximal ideals. Then $x_1 \otimes y_1$ corresponds to the element $x_n^n \otimes y_n^n \in J_n \otimes_{A_n} J_n$ and is nonzero (by an explicit computation which we omit). Since $\otimes$ commutes with colimits we conclude. By [Ill72, III Section 3.3] we see that J is not weakly regular. Hence by [Ill72, III Proposition 3.3.3] we see that the cotangent complex $L_{B/A}$ is not zero. In fact, we can be more precise. We have $H_0(L_{B/A}) = \Omega_{B/A}$ and $H_1(L_{B/A}) = 0$ because $J/J^2 = 0$. But from the five-term exact sequence of Quillen's fundamental spectral sequence (see Cotangent, Remark 92.12.5 or [Rei, Corollary 8.2.6]) and the nonvanishing of $\text{Tor}_2^A(B, B) = \text{Ker}(J \otimes_A J \to J)$ we conclude that $H_2(L_{B/A})$ is nonzero.

06E6 Lemma 110.42.1. There exists a formally étale surjective ring map $A \to B$ with $L_{B/A}$ not equal to zero.

Proof. See discussion above. □

110.43. Flat and formally unramified is not formally étale

0G64 In More on Morphisms, Lemma 37.8.7 it is shown that an unramified flat morphism of schemes $X \to S$ is formally étale. The goal of this section is to give two examples that illustrate that we cannot replace 'unramified' by 'formally unramified'. The first example exploits special properties of perfect rings, while the second example shows the result fails even for maps of Noetherian regular rings.

0G65 Lemma 110.43.1. Let $A = \mathbf{F}_p[T]$ be the polynomial ring in one variable over $\mathbf{F}_p$. Let A_{perf} denote the perfect closure of A. Then $A \to A_{perf}$ is flat and formally unramified, but not formally étale.

Proof. Note that under the Frobenius map $F_A : A \to A$, the target copy of A is a free-module over the domain with basis $\{1, T, \ldots, T^{p-1}\}$. Thus, F_A is faithfully flat, and consequently, so is $A \to A_{perf}$ since it is a colimit of faithfully flat maps. Since A_{perf} is a perfect ring, the relative Frobenius $F_{A_{perf}/A}$ is a surjection. In other words, $A_{perf} = A[A_{perf}^p]$, which readily implies $\Omega_{A_{perf}/A} = 0$. Then $A \to A_{perf}$ is formally unramified by More on Morphisms, Lemma 37.6.7

It suffices to show that $A \to A_{perf}$ is not formally smooth. Note that since A is a smooth $\mathbf{F}_p$-algebra, the cotangent complex $L_{A/\mathbf{F}_P} \simeq \Omega_{A/\mathbf{F}_p}[0]$ is concentrated in degree 0, see Cotangent, Lemma 92.9.1. Moreover, $L_{A_{perf}/\mathbf{F}_p} = 0$ in $D(A_{perf})$ by Cotangent, Lemma 92.10.3. Consider the distinguished triangle of cotangent complexes

$$L_{A/\mathbf{F}_p} \otimes_A A_{perf} \to L_{A_{perf}/\mathbf{F}_p} \to L_{A_{perf}/A} \to (L_{A/\mathbf{F}_p} \otimes_A A_{perf})[1]$$

in $D(A_{perf})$, see Cotangent, Section 92.7. We find $L_{A_{perf}/A} = \Omega_{A/\mathbf{F}_p} \otimes_A A_{perf}[1]$, that is, $L_{A_{perf}/A}$ is equal to a free rank 1 A_{perf} module placed in degree -1. Thus $A \to A_{perf}$ is not formally smooth by More on Morphisms, Lemma 37.13.5 and Cotangent, Lemma 92.11.3. □

The next example also involves rings of prime characteristic, but is perhaps a little more surprising. The drawback is that it requires more knowledge of characteristic p phenomena than the previous example. Recall that we say a ring A of prime characteristic is F-finite if the Frobenius map on A is finite.

0G66 Lemma 110.43.2. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring of prime characteristic $p > 0$ such that $[\kappa : \kappa^p] < \infty$. Then the canonical map $A \to A^\wedge$ to the completion of A is flat and formally unramified. However, if A is regular but not excellent, then this map is not formally étale.

Proof. Flatness of the completion is Algebra, Lemma 10.97.2. To show that the map is formally unramified, it suffices to show that $\Omega_{A^\wedge/A} = 0$, see Algebra, Lemma 10.148.2.

We sketch a proof. Choose $x_1, \ldots, x_r \in A$ which map to a p-basis $\overline{x}_1, \ldots, \overline{x}_r$ of κ, i.e., such that κ is minimally generated by $\overline{x}_i$ over κ^p. Choose a minimal set of generators $y_1, \ldots, y_s$ of $\mathfrak{m}$. For each n the elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ generate $A/\mathfrak{m}^n$ over $(A/\mathfrak{m}^n)^p$ by Frobenius. Some details omitted. We conclude that $F : A^\wedge \to A^\wedge$ is finite. Hence $\Omega_{A^\wedge/A}$ is a finite $A^\wedge$-module. On the other hand, for any $a \in A^\wedge$ and n we can find $a_0 \in A$ such that $a - a_0 \in \mathfrak{m}^n A^\wedge$. We conclude that $\mathrm{d}(a) \in \bigcap \mathfrak{m}^n \Omega_{A^\wedge/A}$ which implies that $\mathrm{d}(a)$ is zero by Algebra, Lemma 10.51.4. Thus $\Omega_{A^\wedge/A} = 0$.

Suppose A is regular. Then, using the Cohen structure theorem $x_1, \ldots, x_r, y_1, \ldots, y_s$ is a p-basis for the ring $A^\wedge$, i.e., we have

$$A^\wedge = \bigoplus\nolimits_{I,J} (A^\wedge)^p x_1^{i_1} \ldots x_r^{i_r} y_1^{j_1} \ldots y_s^{j_s}$$

with $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$ and $0 \leq i_a, j_b \leq p-1$. Details omitted. In particular, we see that $\Omega_{A^\wedge}$ is a free $A^\wedge$-module with basis $\mathrm{d}(x_1), \ldots, \mathrm{d}(x_r), \mathrm{d}(y_1), \ldots, \mathrm{d}(y_s)$.

Now if $A \to A^\wedge$ is formally étale or even just formally smooth, then we see that $NL_{A^\wedge/A}$ has vanishing cohomology in degrees $-1, 0$ by Algebra, Proposition 10.138.8. It follows from the Jacobi-Zariski sequence (Algebra, Lemma 10.134.4) for the ring maps $\mathbf{F}_p \to A \to A^\wedge$ that we get an isomorphism $\Omega_A \otimes_A A^\wedge \cong \Omega_{A^\wedge}$. Hence we find that Ω_A is free on $\text{d}(x_1), \ldots, \text{d}(x_r), \text{d}(y_1), \ldots, \text{d}(y_s)$. Looking at fraction fields and using that A is normal we conclude that $a \in A$ is a pth power if and only if its image in $A^\wedge$ is a pth power (details omitted; use Algebra, Lemma 10.158.2). A second consequence is that the operators $\partial/\partial x_a$ and $\partial/\partial y_b$ are defined on A.

We will show that the above lead to the conclusion that A is finite over A^p with p-basis $x_1, \ldots, x_r, y_1, \ldots, y_s$. This will contradict the non-excellency of A by a result of Kunz, see [Kun76, Corollary 2.6]. Namely, say $a \in A$ and write

$$a = \sum\nolimits_{I,J} (a_{I,J})^p x_1^{i_1} \ldots x_r^{i_r} y_1^{j_1} \ldots y_s^{j_s}$$

with $a_{I,J} \in A^\wedge$. To finish the proof it suffices to show that $a_{I,J} \in A$. Applying the operator

$$(\partial/\partial x_1)^{p-1} \ldots (\partial/\partial x_r)^{p-1} (\partial/\partial y_1)^{p-1} \ldots (\partial/\partial y_s)^{p-1}$$

to both sides we conclude that $a_{I,J}^p \in A$ where $I = (p-1, \ldots, p-1)$ and $J = (p-1, \ldots, p-1)$. By our remark above, this also implies $a_{I,J} \in A$. After replacing a by $a' = a - a_{I,J}^p x^I y^J$ we can use a 1-order lower differential operators to get another coefficient $a_{I,J}$ to be in A. Etc. □

0G67 Remark 110.43.3. Non-excellent regular rings whose residue fields have a finite p-basis can be constructed even in the function field of $\mathbb{P}^2_k$, over a characteristic p field $k = \overline{k}$. See [DS18, §4.1].

The proof of Lemma 110.43.2 actually shows a little more.

0G68 Lemma 110.43.4. Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of characteristic $p > 0$. Suppose $[\kappa : \kappa^p] < \infty$. Then A is excellent if and only if $A \to A^\wedge$ is formally étale.

Proof. The backward implication follows from Lemma 110.43.2. For the forward implication, note that we already know from Lemma 110.43.2 that $A \to A^\wedge$ is formally unramified or equivalently that $\Omega_{A^\wedge/A}$ is zero. Thus, it suffices to show that the completion map is formally smooth when A is excellent. By Néron-Popescu desingularization $A \to A^\wedge$ can be written as a filtered colimit of smooth A-algebras (Smoothing Ring Maps, Theorem 16.12.1). Hence $NL_{A^\wedge/A}$ has vanishing cohomology in degree -1. Thus $A \to A^\wedge$ is formally smooth by Algebra, Proposition 10.138.8. □

110.44. Ideals generated by sets of idempotents and localization

04QK Let R be a ring. Consider the ring

$$B(R) = R[x_n; n \in \mathbf{Z}]/(x_n(x_n - 1), x_n x_m; n \neq m)$$

It is easy to show that every prime $\mathfrak{q} \subset B(R)$ is either of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n; n \in \mathbf{Z})$$

or of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n - 1) + (x_m; n \neq m, m \in \mathbf{Z}).$$

Hence we see that

$$\mathrm{Spec}(B(R)) = \mathrm{Spec}(R) \amalg \coprod\nolimits_{n \in \mathbf{Z}} \mathrm{Spec}(R)$$

where the topology is not just the disjoint union topology. It has the following properties: Each of the copies indexed by $n \in \mathbf{Z}$ is an open subscheme, namely it is the standard open $D(x_n)$. The "central" copy of $\mathrm{Spec}(R)$ is in the closure of the union of any infinitely many of the other copies of $\mathrm{Spec}(R)$. Note that this last copy of $\mathrm{Spec}(R)$ is cut out by the ideal $(x_n, n \in \mathbf{Z})$ which is generated by the idempotents x_n. Hence we see that if $\mathrm{Spec}(R)$ is connected, then the decomposition above is exactly the decomposition of $\mathrm{Spec}(B(R))$ into connected components.

Next, let $A = \mathbf{C}[x,y]/((y-x^2+1)(y+x^2-1))$. The spectrum of A consists of two irreducible components $C_1 = \mathrm{Spec}(A_1)$, $C_2 = \mathrm{Spec}(A_2)$ with $A_1 = \mathbf{C}[x,y]/(y-x^2+1)$ and $A_2 = \mathbf{C}[x,y]/(y+x^2-1)$. Note that these are parametrized by $(x,y) = (t, t^2-1)$ and $(x,y) = (t, -t^2+1)$ which meet in $P = (-1, 0)$ and $Q = (1, 0)$. We can make a twisted version of $B(A)$ where we glue $B(A_1)$ to $B(A_2)$ in the following way: Above P we let $x_n \in B(A_1) \otimes \kappa(P)$ correspond to $x_n \in B(A_2) \otimes \kappa(P)$, but above Q we let $x_n \in B(A_1) \otimes \kappa(Q)$ correspond to $x_{n+1} \in B(A_2) \otimes \kappa(Q)$. Let $B^{twist}(A)$ denote the resulting A-algebra. Details omitted. By construction $B^{twist}(A)$ is Zariski locally over A isomorphic to the untwisted version. Namely, this happens over both the principal open $\mathrm{Spec}(A) \setminus \{P\}$ and the principal open $\mathrm{Spec}(A) \setminus \{Q\}$. However, our choice of glueing produces enough "monodromy" such that $\mathrm{Spec}(B^{twist}(A))$ is connected (details omitted). Finally, there is a central copy of $\mathrm{Spec}(A) \to \mathrm{Spec}(B^{twist}(A))$ which gives a closed subscheme whose ideal is Zariski locally on $B^{twist}(A)$ cut out by ideals generated by idempotents, but not globally (as $B^{twist}(A)$ has no nontrivial idempotents).

04QL Lemma 110.44.1. There exists an affine scheme $X = \mathrm{Spec}(A)$ and a closed subscheme $T \subset X$ such that T is Zariski locally on X cut out by ideals generated by idempotents, but T is not cut out by an ideal generated by idempotents.

Proof. See above. □

110.45. A ring map which identifies local rings which is not ind-étale

09AN Note that the ring map $R \to B(R)$ constructed in Section 110.44 is a colimit of finite products of copies of R. Hence $R \to B(R)$ is ind-Zariski, see Pro-étale Cohomology, Definition 61.4.1. Next, consider the ring map $A \to B^{twist}(A)$ constructed in Section 110.44. Since this ring map is Zariski locally on $\mathrm{Spec}(A)$ isomorphic to an ind-Zariski ring map $R \to B(R)$ we conclude that it identifies local rings (see Pro-étale Cohomology, Lemma 61.4.6). The discussion in Section 110.44 shows there is a section $B^{twist}(A) \to A$ whose kernel is not generated by idempotents. Now, if $A \to B^{twist}(A)$ were ind-étale, i.e., $B^{twist}(A) = \mathrm{colim}\, A_i$ with $A \to A_i$ étale, then the kernel of $A_i \to A$ would be generated by an idempotent (Algebra, Lemmas 10.143.8 and 10.143.9). This would contradict the result mentioned above.

09AP Lemma 110.45.1. There is a ring map $A \to B$ which identifies local rings but which is not ind-étale. A fortiori it is not ind-Zariski.

Proof. See discussion above. □

110.46. Non flasque quasi-coherent sheaf associated to injective module

0273 For more examples of this type see [BGI71, Exposé II, Appendix I] where Illusie explains some examples due to Verdier.

Consider the affine scheme $X = \mathrm{Spec}(A)$ where

$$A = k[x, y, z_1, z_2, \ldots]/(x^n z_n)$$

is the ring from Properties, Example 28.25.2. Set $I = (x) \subset A$. Consider the quasi-compact open $U = D(x)$ of X. We have seen in loc. cit. that there is a section $s \in \mathcal{O}_X(U)$ which does not come from an A-module map $I^n \to A$ for any $n \geq 0$.

Let $\alpha : A \to J$ be the embedding of A into an injective A-module. Let $Q = J/\alpha(A)$ and denote $\beta : J \to Q$ the quotient map. We claim that the map

$$\Gamma(X, \widetilde{J}) \longrightarrow \Gamma(U, \widetilde{J})$$

is not surjective. Namely, we claim that $\alpha(s)$ is not in the image. To see this, we argue by contradiction. So assume that $x \in J$ is an element which restricts to $\alpha(s)$ over U. Then $\beta(x) \in Q$ is an element which restricts to 0 over U. Hence we know that $I^n\beta(x) = 0$ for some n, see Properties, Lemma 28.25.1. This implies that we get a morphism $\varphi : I^n \to A$, $h \mapsto \alpha^{-1}(hx)$. It is easy to see that this morphism φ gives rise to the section s via the map of Properties, Lemma 28.25.1 which is a contradiction.

0274 Lemma 110.46.1. There exists an affine scheme $X = \mathrm{Spec}(A)$ and an injective A-module J such that $\widetilde{J}$ is not a flasque sheaf on X. Even the restriction $\Gamma(X, \widetilde{J}) \to \Gamma(U, \widetilde{J})$ with U a standard open need not be surjective.

Proof. See above. □

In fact, we can use a similar construction to get an example of an injective module whose associated quasi-coherent sheaf has nonzero cohomology over a quasi-compact open. Namely, we start with the ring

$$A = k[x, y, w_1, u_1, w_2, u_2, \ldots]/(x^n w_n, y^n u_n, u_n^2, w_n^2)$$

where k is a field. Choose an injective map $A \to I$ where I is an injective A-module. We claim that the element $1/xy$ in $A_{xy} \subset I_{xy}$ is not in the image of $I_x \oplus I_y \to I_{xy}$. Arguing by contradiction, suppose that

$$\frac{1}{xy} = \frac{i}{x^n} + \frac{j}{y^n}$$

for some $n \geq 1$ and $i, j \in I$. Clearing denominators we obtain

$$(xy)^{n+m-1} = x^m y^{n+m} i + x^{n+m} y^m j$$

for some $m \geq 0$. Multiplying with $u_{n+m}w_{n+m}$ we see that $u_{n+m}w_{n+m}(xy)^{n+m-1} = 0$ in A which is the desired contradiction. Let $U = D(x) \cup D(y) \subset X = \mathrm{Spec}(A)$. For any A-module M we have an exact sequence

$$0 \to H^0(U, \widetilde{M}) \to M_x \oplus M_y \to M_{xy} \to H^1(U, \widetilde{M}) \to 0$$

by Mayer-Vietoris. We conclude that $H^1(U, \widetilde{I})$ is nonzero.

0CRZ Lemma 110.46.2. There exists an affine scheme $X = \mathrm{Spec}(A)$ whose underlying topological space is Noetherian and an injective A-module I such that $\widetilde{I}$ has non-vanishing H^1 on some quasi-compact open U of X.

Proof. See above. Note that $\mathrm{Spec}(A) = \mathrm{Spec}(k[x, y])$ as topological spaces. $\square$

110.47. A non-separated flat group scheme

06E7 Every group scheme over a field is separated, see Groupoids, Lemma 39.7.3. This is not true for group schemes over a base.

Let k be a field. Let $S = \mathrm{Spec}(k[x]) = \mathbf{A}^1_k$. Let G be the affine line with 0 doubled (see Schemes, Example 26.14.3) seen as a scheme over S. Thus a fibre of $G \to S$ is either a singleton or a set with two elements (one in U and one in V). Thus we can endow these fibres with the structure of a group (by letting the element in U be the zero of the group structure). More precisely, G has two opens U, V which map isomorphically to S such that $U \cap V$ is mapped isomorphically to $S \setminus \{0\}$. Then

$$G \times_S G = U \times_S U \cup V \times_S U \cup U \times_S V \cup V \times_S V$$

where each piece is isomorphic to S. Hence we can define a multiplication $m : G \times_S G \to G$ as the unique S-morphism which maps the first and the last piece into U and the two middle pieces into V. This matches the pointwise description given above. We omit the verification that this defines a group scheme structure.

06E8 **Lemma 110.47.1.** *There exists a flat group scheme of finite type over the affine line which is not separated.*

Proof. See the discussion above. $\square$

08IX **Lemma 110.47.2.** *There exists a flat group scheme of finite type over the infinite dimensional affine space which is not quasi-separated.*

Proof. The same construction as above can be carried out with the infinite dimensional affine space $S = \mathbf{A}^\infty_k = \mathrm{Spec}\, k[x_1, x_2, \ldots]$ as the base and the origin $0 \in S$ corresponding to the maximal ideal $(x_1, x_2, \ldots)$ as the closed point which is doubled in G. The resulting group scheme $G \to S$ is not quasi-separated as explained in Schemes, Example 26.21.4. $\square$

110.48. A non-flat group scheme with flat identity component

06RJ Let $X \to S$ be a monomorphism of schemes. Let $G = S \amalg X$. Let $m : G \times_S G \to G$ be the S-morphism

$$G \times_S G = X \times_S X \amalg X \amalg X \amalg S \longrightarrow G = X \amalg S$$

which maps the summands $X \times_S X$ and S into S and maps the summands X into X by the identity morphism. This defines a group law. To see this we have to show that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$ as maps $G \times_S G \times_S G \to G$. Decomposing $G \times_S G \times_S G$ into components as above, we see that we need to verify this for the restriction to each of the 8-pieces. Each piece is isomorphic to either S, X, $X \times_S X$, or $X \times_S X \times_S X$. Moreover, both maps map these pieces to S, X, S, X respectively. Having said this, the fact that $X \to S$ is a monomorphism implies that $X \times_S X \cong X$ and $X \times_S X \times_S X \cong X$ and that there is in each case exactly one S-morphism $S \to S$ or $X \to X$. Thus we see that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$. Thus taking $X \to S$ to be any nonflat monomorphism of schemes (e.g., a closed immersion) we get an example of a group scheme over a base S whose identity component is S (hence flat) but which is not flat.

06RK Lemma 110.48.1. There exists a group scheme G over a base S whose identity component is flat over S but which is not flat over S.

Proof. See discussion above. □

110.49. A non-separated group algebraic space over a field

06E9 Every group scheme over a field is separated, see Groupoids, Lemma 39.7.3. This is not true for group algebraic spaces over a field (but see end of this section for positive results).

Let k be a field of characteristic zero. Consider the algebraic space $G = \mathbf{A}^1_k/\mathbf{Z}$ from Spaces, Example 65.14.8. By construction G is the fppf sheaf associated to the presheaf

$$T \longmapsto \Gamma(T, \mathcal{O}_T)/\mathbf{Z}$$

on the category of schemes over k. The obvious addition rule on the presheaf induces an addition $m : G \times G \to G$ which turns G into a group algebraic space over $\text{Spec}(k)$. Note that G is not separated (and not even quasi-separated or locally separated). On the other hand $G \to \text{Spec}(k)$ is of finite type!

06EA Lemma 110.49.1. There exists a group algebraic space of finite type over a field which is not separated (and not even quasi-separated or locally separated).

Proof. See discussion above. □

Positive results: If the group algebraic space G is either quasi-separated, or locally separated, or more generally a decent algebraic space, then G is in fact separated, see More on Groupoids in Spaces, Lemma 79.9.4. Moreover, a finite type, separated group algebraic space over a field is in fact a scheme by More on Groupoids in Spaces, Lemma 79.10.2. The idea of the proof is that the schematic locus is open dense, see Properties of Spaces, Proposition 66.13.3 or Decent Spaces, Theorem 68.10.2. By translating this open we see that every point of G has an open neighbourhood which is a scheme.

110.50. Specializations between points in fibre étale morphism

06UJ If $f : X \to Y$ is an étale, or more generally a locally quasi-finite morphism of schemes, then there are no specializations between points of fibres, see Morphisms, Lemma 29.20.8. However, for morphisms of algebraic spaces this doesn't hold in general.

To give an example, let k be a field. Set

$$P = k[u, u^{-1}, y, \{x_n\}_{n \in \mathbf{Z}}].$$

Consider the action of $\mathbf{Z}$ on P by k-algebra maps generated by the automorphism τ given by the rules $\tau(u) = u$, $\tau(y) = uy$, and $\tau(x_n) = x_{n+1}$. For $d \geq 1$ set $I_d = ((1 - u^d)y, x_n - x_{n+d}, n \in \mathbf{Z})$. Then $V(I_d) \subset \text{Spec}(P)$ is the fix point locus of τ^d. Let $S \subset P$ be the multiplicative subset generated by y and all $1 - u^d$, $d \in \mathbf{N}$. Then we see that $\mathbf{Z}$ acts freely on $U = \text{Spec}(S^{-1}P)$. Let $X = U/\mathbf{Z}$ be the quotient algebraic space, see Spaces, Definition 65.14.4.

Consider the prime ideals $\mathfrak{p}_n = (x_n, x_{n+1}, \ldots)$ in $S^{-1}P$. Note that $\tau(\mathfrak{p}_n) = \mathfrak{p}_{n+1}$. Hence each of these define point $\xi_n \in U$ whose image in X is the same point x of X. Moreover we have the specializations

$$\ldots \rightsquigarrow \xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots$$

We conclude that $U \to X$ is an example of the promised type.

06UK Lemma 110.50.1. There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization of points $x \rightsquigarrow x'$ in $|X|$ with $f(x) = f(x')$ in $|Y|$.

Proof. See discussion above. $\square$

110.51. A torsor which is not an fppf torsor

04AF In Groupoids, Remark 39.11.5 we raise the question whether any G-torsor is a G-torsor for the fppf topology. In this section we show that this is not always the case.

Let k be a field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be the group scheme

$$G = (\mu_{2,k})^\infty = \mu_{2,k} \times_k \mu_{2,k} \times_k \mu_{2,k} \times_k \ldots = \lim_n (\mu_{2,k})^n$$

where $\mu_{2,k}$ is the group scheme of second roots of unity over $\mathrm{Spec}(k)$, see Groupoids, Example 39.5.2. As an inverse limit of affine schemes we see that G is an affine group scheme. In fact it is the spectrum of the ring $k[t_1, t_2, t_3, \ldots]/(t_i^2 - 1)$. The multiplication map $m : G \times_k G \to G$ is on the algebra level given by $t_i \mapsto t_i \otimes t_i$.

We claim that any G-torsor over k is of the form

$$P = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots]/(x_i^2 - a_i))$$

for certain $a_i \in k^*$ and with G-action $G \times_k P \to P$ given by $x_i \to t_i \otimes x_i$ on the algebra level. We omit the proof. Actually for the example we only need that P is a G-torsor which is clear since over $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ the scheme P becomes isomorphic to G in a G-equivariant manner. Note that P is trivial if and only if $k' = k$ since if P has a k-rational point then all of the a_i are squares.

We claim that P is an fppf torsor if and only if the field extension $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)/k$ is finite. If k' is finite over k, then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fppf covering which trivializes P and we see that P is indeed an fppf torsor. Conversely, suppose that P is an fppf G-torsor. This means that there exists an fppf covering $\{S_i \to \mathrm{Spec}(k)\}$ such that each P_{S_i} is trivial. Pick an i such that S_i is not empty. Let $s \in S_i$ be a closed point. By Varieties, Lemma 33.14.1 the field extension $\kappa(s)/k$ is finite, and by construction $P_{\kappa(s)}$ has a $\kappa(s)$-rational point. Thus we see that $k \subset k' \subset \kappa(s)$ and k' is finite over k.

To get an explicit example take $k = \mathbf{Q}$ and $a_i = i$ for example (or a_i is the ith prime if you like).

077B Lemma 110.51.1. Let S be a scheme. Let G be a group scheme over S. The stack G-Principal classifying principal homogeneous G-spaces (see Examples of Stacks, Subsection 95.14.5) and the stack G-Torsors classifying fppf G-torsors (see Examples of Stacks, Subsection 95.14.8) are not equivalent in general.

Proof. The discussion above shows that the functor G-Torsors $\to$ G-Principal isn't essentially surjective in general. $\square$

110.52. Stack with quasi-compact flat covering which is not algebraic

04AG In this section we briefly describe an example due to Brian Conrad. You can find the example online at this location. Our example is slightly different.

Let k be an algebraically closed field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be an affine group scheme. In Examples of Stacks, Lemma 95.15.4 we have given several different equivalent ways to view $\mathcal{X} = [\mathrm{Spec}(k)/G]$ as a stack in groupoids over $(Sch/\mathrm{Spec}(k))_{fppf}$. In particular $\mathcal{X}$ classifies fppf G-torsors. More precisely, a 1-morphism $T \to \mathcal{X}$ corresponds to an fppf G_T-torsor P over T and 2-arrows correspond to isomorphisms of torsors. It follows that the diagonal 1-morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$$

is representable and affine. Namely, given any pair of fppf G_T-torsors P_1, P_2 over a scheme T/k the scheme $Isom(P_1, P_2)$ is affine over T. The trivial G-torsor over $\mathrm{Spec}(k)$ defines a 1-morphism

$$f : \mathrm{Spec}(k) \longrightarrow \mathcal{X}.$$

We claim that this is a surjective 1-morphism. The reason is simply that by definition for any 1-morphism $T \to \mathcal{X}$ there exists a fppf covering $\{T_i \to T\}$ such that P_{T_i} is isomorphic to the trivial G_{T_i}-torsor. Hence the compositions $T_i \to T \to \mathcal{X}$ factor through f. Thus it is clear that the projection $T \times_{\mathcal{X}} \mathrm{Spec}(k) \to T$ is surjective (which is how we define the property that f is surjective, see Algebraic Stacks, Definition 94.10.1). In a similar way you show that f is quasi-compact and flat (details omitted). We also record here the observation that

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong G$$

as schemes over k.

Suppose there exists a surjective smooth morphism $p : U \to \mathcal{X}$ where U is a scheme. Consider the fibre product

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathcal{X} \end{tikzcd}$$

Then we see that W is a nonempty smooth scheme over k which hence has a k-point. This means that we can factor f through U. Hence we obtain

$$G \cong \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong (\mathrm{Spec}(k) \times_k \mathrm{Spec}(k)) \times_{(U \times_k U)} (U \times_{\mathcal{X}} U)$$

and since the projections $U \times_{\mathcal{X}} U \to U$ were assumed smooth we conclude that $U \times_{\mathcal{X}} U \to U \times_k U$ is locally of finite type, see Morphisms, Lemma 29.15.8. It follows that in this case G is locally of finite type over k. Altogether we have proved the following lemma (which can be significantly generalized).

04AH **Lemma 110.52.1.** Let k be a field. Let G be an affine group scheme over k. If the stack $[\mathrm{Spec}(k)/G]$ has a smooth covering by a scheme, then G is of finite type over k.

Proof. See discussion above. □

To get an explicit example as in the title of this section, take for example $G = (\mu_{2,k})^\infty$ the group scheme of Section 110.51, which is not locally of finite type over k. By the discussion above we see that $\mathcal{X} = [\mathrm{Spec}(k)/G]$ has properties (1) and (2) of Algebraic Stacks, Definition 94.12.1, but not property (3). Hence $\mathcal{X}$ is not an algebraic stack. On the other hand, there does exist a scheme U and a surjective, flat, quasi-compact morphism $U \to \mathcal{X}$, namely the morphism $f : \mathrm{Spec}(k) \to \mathcal{X}$ we studied above.

110.53. Limit preserving on objects, not limit preserving

07Z0 Let S be a nonempty scheme. Let $\mathcal{G}$ be an injective abelian sheaf on $(Sch/S)_{fppf}$. We obtain a stack in groupoids

$$\mathcal{G}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

over S, see Examples of Stacks, Lemma 95.14.2. This stack is limit preserving on objects over $(Sch/S)_{fppf}$ (see Criteria for Representability, Section 97.5) because every $\mathcal{G}$-torsor is trivial. On the other hand, $\mathcal{G}$-Torsors is in general not limit preserving (see Artin's Axioms, Definition 98.11.1) as $\mathcal{G}$ need not be limit preserving as a sheaf. For example, take any nonzero injective sheaf $\mathcal{I}$ and set $\mathcal{G} = \prod_{n \in \mathbf{Z}} \mathcal{I}$ to get an example.

07Z1 Lemma 110.53.1. Let S be a nonempty scheme. There exists a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ such that p is limit preserving on objects, but $\mathcal{X}$ is not limit preserving.

Proof. See discussion above. $\square$

110.54. A non-algebraic classifying stack

077C Let $S = \mathrm{Spec}(\mathbf{F}_p)$ and let μ_p denote the group scheme of pth roots of unity over S. In Groupoids in Spaces, Section 78.20 we have introduced the quotient stack $[S/\mu_p]$ and in Examples of Stacks, Section 95.15 we have shown $[S/\mu_p]$ is the classifying stack for fppf μ_p-torsors: Given a scheme T over S the category $\mathrm{Mor}_S(T, [S/\mu_p])$ is canonically equivalent to the category of fppf μ_p-torsors over T. Finally, in Criteria for Representability, Theorem 97.17.2 we have seen that $[S/\mu_p]$ is an algebraic stack.

Now we can ask the question: "How about the category fibred in groupoids $\mathcal{S}$ classifying étale μ_p-torsors?" (In other words $\mathcal{S}$ is a category over Sch/S whose fibre category over a scheme T is the category of étale μ_p-torsors over T.)

The first objection is that this isn't a stack for the fppf topology, because descent for objects isn't going to hold. For example the μ_p-torsor $\mathrm{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$ over $T = \mathrm{Spec}(\mathbf{F}_p(T))$ is fppf locally trivial, but not étale locally trivial.

A fix for this first problem is to work with the étale topology and in this case descent for objects does work. Indeed it is true that $\mathcal{S}$ is a stack in groupoids over $(Sch/S)_{étale}$. Moreover, it is also the case that the diagonal $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable (by schemes). This is true because given two μ_p-torsors (whether they be étale locally trivial or not) the sheaf of isomorphisms between them is representable by a scheme.

Thus we can finally ask if there exists a scheme U and a smooth and surjective 1-morphism $U \to \mathcal{S}$. We will show in two ways that this is impossible: by a direct argument (which we advise the reader to skip) and by an argument using a general result.

Direct argument (sketch): Note that the 1-morphism $\mathcal{S} \to \text{Spec}(\mathbf{F}_p)$ satisfies the infinitesimal lifting criterion for formal smoothness. This is true because given a first order infinitesimal thickening of schemes $T \to T'$ the kernel of $\mu_p(T') \to \mu_p(T)$ is isomorphic to the sections of the ideal sheaf of T in T', and hence $H^1_{\acute{e}tale}(T, \mu_p) = H^1_{\acute{e}tale}(T', \mu_p)$. Moreover, $\mathcal{S}$ is a limit preserving stack. Hence if $U \to \mathcal{S}$ is smooth, then $U \to \text{Spec}(\mathbf{F}_p)$ is limit preserving and satisfies the infinitesimal lifting criterion for formal smoothness. This implies that U is smooth over $\mathbf{F}_p$. In particular U is reduced, hence $H^1_{\acute{e}tale}(U, \mu_p) = 0$. Thus $U \to \mathcal{S}$ factors as $U \to \text{Spec}(\mathbf{F}_p) \to \mathcal{S}$ and the first arrow is smooth. By descent of smoothness, we see that $U \to \mathcal{S}$ being smooth would imply $\text{Spec}(\mathbf{F}_p) \to \mathcal{S}$ is smooth. However, this is not the case as $\text{Spec}(\mathbf{F}_p) \times_{\mathcal{S}} \text{Spec}(\mathbf{F}_p)$ is μ_p which is not smooth over $\text{Spec}(\mathbf{F}_p)$.

Structural argument: In Criteria for Representability, Section 97.19 we have seen that we can think of algebraic stacks as those stacks in groupoids for the étale topology with diagonal representable by algebraic spaces having a smooth covering. Hence if a smooth surjective $U \to \mathcal{S}$ exists then $\mathcal{S}$ is an algebraic stack, and in particular satisfies descent in the fppf topology. But we've seen above that $\mathcal{S}$ does not satisfies descent in the fppf topology.

Loosely speaking the arguments above show that the classifying stack in the étale topology for étale locally trivial torsors for a group scheme G over a base B is algebraic if and only if G is smooth over B. One of the advantages of working with the fppf topology is that it suffices to assume that $G \to B$ is flat and locally of finite presentation. In fact the quotient stack (for the fppf topology) $[B/G]$ is algebraic if and only if $G \to B$ is flat and locally of finite presentation, see Criteria for Representability, Lemma 97.18.3.

110.55. Sheaf with quasi-compact flat covering which is not algebraic

078E Consider the functor $F = (\mathbf{P}^1)^\infty$, i.e., for a scheme T the value $F(T)$ is the set of $f = (f_1, f_2, f_3, \ldots)$ where each $f_i : T \to \mathbf{P}^1$ is a morphism of schemes. Note that $\mathbf{P}^1$ satisfies the sheaf property for fpqc coverings, see Descent, Lemma 35.13.7. A product of sheaves is a sheaf, so F also satisfies the sheaf property for the fpqc topology. The diagonal of F is representable: if $f : T \to F$ and $g : S \to F$ are morphisms, then $T \times_F S$ is the scheme theoretic intersection of the closed subschemes $T \times_{f_i, \mathbf{P}^1, g_i} S$ inside the scheme $T \times S$. Consider the group scheme SL_2 which comes with a surjective smooth affine morphism $\text{SL}_2 \to \mathbf{P}^1$. Next, consider $U = (\text{SL}_2)^\infty$ with its canonical (product) morphism $U \to F$. Note that U is an affine scheme. We claim the morphism $U \to F$ is flat, surjective, and universally open. Namely, suppose $f : T \to F$ is a morphism. Then $Z = T \times_F U$ is the infinite fibre product of the schemes $Z_i = T \times_{f_i, \mathbf{P}^1} \text{SL}_2$ over T. Each of the morphisms $Z_i \to T$ is surjective smooth and affine which implies that

$$Z = Z_1 \times_T Z_2 \times_T Z_3 \times_T \ldots$$

is a scheme flat and affine over Z. A simple limit argument shows that $Z \to T$ is open as well.

On the other hand, we claim that F isn't an algebraic space. Namely, if F where an algebraic space it would be a quasi-compact and separated (by our description of fibre products over F) algebraic space. Hence cohomology of quasi-coherent sheaves would vanish above a certain cutoff (see Cohomology of Spaces, Proposition 69.7.2 and remarks preceding it). But clearly by taking the pullback of $\mathcal{O}(-2, -2, \ldots, -2)$ under the projection

$$(\mathbf{P}^1)^\infty \longrightarrow (\mathbf{P}^1)^n$$

(which has a section) we can obtain a quasi-coherent sheaf whose cohomology is nonzero in degree n. Altogether we obtain an answer to a question asked by Anton Geraschenko on mathoverflow.

078F Lemma 110.55.1. There exists a functor $F : Sch^{opp} \to$ Sets which satisfies the sheaf condition for the fpqc topology, has representable diagonal $\Delta : F \to F \times F$, and such that there exists a surjective, flat, universally open, quasi-compact morphism $U \to F$ where U is a scheme, but such that F is not an algebraic space.

Proof. See discussion above. $\square$

110.56. Sheaves and specializations

05LD In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 34.4.6. Moreover, a scheme will be an object of this site. Recall that if x, x' are points of a scheme X we say x is a specialization of x' or we write $x' \leadsto x$ if $x \in \overline{\{x'\}}$. This is true in particular if $x = x'$.

Consider the functor $F : Sch_{étale} \to \text{Ab}$ defined by the following rules:

$$F(X) = \prod\nolimits_{x \in X} \prod\nolimits_{x' \in X, x' \leadsto x, x' \neq x} \mathbf{Z}/2\mathbf{Z}$$

Given a scheme X we denote $|X|$ the underlying set of points. An element $a \in F(X)$ will be viewed as a map of sets $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$, $(x, x') \mapsto a(x, x')$ which is zero if $x = x'$ or if x is not a specialization of x'. Given a morphism of schemes $f : X \to Y$ we define

$$F(f) : F(Y) \longrightarrow F(X)$$

by the rule that for $b \in F(Y)$ we set

$$F(f)(b)(x, x') = \begin{cases} 0 & \text{if } x \text{ is not a specialization of } x' \\ b(f(x), f(x')) & \text{else.} \end{cases}$$

Note that this really does define an element of $F(X)$. We claim that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms then $F(f) \circ F(g) = F(g \circ f)$. Namely, let $c \in F(Z)$ and let $x' \leadsto x$ be a specialization of points in X, then

$$F(g \circ f)(x, x') = c(g(f(x)), g(f(x'))) = F(g)(F(f)(c))(x, x')$$

because $f(x') \leadsto f(x)$. (This also works if $f(x) = f(x')$.)

Let G be the sheafification of F in the étale topology.

I claim that if X is a scheme and $x' \leadsto x$ is a specialization and $x' \neq x$, then $G(X) \neq 0$. Namely, let $a \in F(X)$ be an element such that when we think of a as a function $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$ it is nonzero at (x, x'). Let $\{f_i : U_i \to X\}$ be an étale covering of X. Then we can pick an i and a point $u_i \in U_i$ with $f_i(u_i) = x$. Since generalizations lift along flat morphisms (see Morphisms, Lemma 29.25.9) we can find a specialization $u'_i \leadsto u_i$ with $f_i(u'_i) = x'$. By our construction above we see that $F(f_i)(a) \neq 0$. Hence a determines a nonzero element of $G(X)$.

Note that if $X = \mathrm{Spec}(k)$ where k is a field (or more generally a ring all of whose prime ideals are maximal), then $F(X) = 0$ and for every étale morphism $U \to X$ we have $F(U) = 0$ because there are no specializations between distinct points in fibres of an étale morphism. Hence $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 37.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 59.45.2. Since it is always the case that $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

As a variant we can consider the presheaf F_n which associates to a scheme X the collection of maps $a : |X|^{n+1} \to \mathbf{Z}/2\mathbf{Z}$ where $a(x_0, \ldots, x_n)$ is nonzero only if $x_n \rightsquigarrow \ldots \rightsquigarrow x_0$ is a sequence of specializations and $x_n \neq x_{n-1} \neq \ldots \neq x_0$. Let G_n be the sheaf associated to F_n. In exactly the same way as above one shows that G_n is nonzero if $\dim(X) \geq n$ and is zero if $\dim(X) < n$.

05LE Lemma 110.56.1. There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties

(1) $G(X) = 0$ whenever $\dim(X) < n$,
(2) $G(X)$ is not zero if $\dim(X) \geq n$, and
(3) if $X \subset X'$ is a thickening, then $G(X) = G(X')$.

Proof. See the discussion above. □

05LF Remark 110.56.2. Here are some remarks:

(1) The presheaves F and F_n are separated presheaves.
(2) It turns out that F, F_n are not sheaves.
(3) One can show that G, G_n is actually a sheaf for the fppf topology.

We will prove these results if we need them.

110.57. Sheaves and constructible functions

05LG In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 34.4.6. Moreover, a scheme will be an object of this site. In this section we say that a constructible partition of a scheme X is a locally finite disjoint union decomposition $X = \coprod_{i \in I} X_i$ such that each $X_i \subset X$ is a locally constructible subset of X. Locally finite means that for any quasi-compact open $U \subset X$ there are only finitely many $i \in I$ such that $X_i \cap U$ is not empty. Note that if $f : X \to Y$ is a morphism of schemes and $Y = \coprod Y_j$ is a constructible partition, then $X = \coprod f^{-1}(Y_j)$ is a constructible partition of X. Given a set S and a scheme X a constructible function $f : |X| \to S$ is a map such that $X = \coprod_{s \in S} f^{-1}(s)$ is a constructible partition of X. If G is an (abstract group) and $a, b : |X| \to G$ are constructible functions, then $ab : |X| \to G, x \mapsto a(x)b(x)$ is a constructible function too. The reason is that given any two constructible partitions there is a third one refining both.

Let A be any abelian group. For any scheme X we define

$$F(X) = \frac{\{a : |X| \to A \mid a \text{ is a constructible function}\}}{\{\text{locally constant functions } |X| \to A\}}$$

We think of an element a of $F(X)$ simply as a function well defined up to adding a locally constant one. Given a morphism of schemes $f : X \to Y$ and an element $b \in F(Y)$, then we define $F(f)(b) = b \circ f$. Thus F is a presheaf on $Sch_{étale}$.

Note that if $\{f_i : U_i \to X\}$ is an fppf covering, and $a \in F(X)$ is such that $F(f_i)(a) = 0$ in $F(U_i)$, then $a \circ f_i$ is a locally constant function for each i. This means in turn that a is a locally constant function as the morphisms f_i are open. Hence $a = 0$ in $F(X)$. Thus we see that F is a separated presheaf (in the fppf topology hence a fortiori in the étale topology).

Let G be the sheafification of F in the étale topology. Since F is separated, and since $F(X) \neq 0$ for example when X is the spectrum of a discrete valuation ring, we see that G is not zero.

Let $X = \mathrm{Spec}(k)$ where k is a field. Then any étale covering of X can be dominated by a covering $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ with k'/k a finite separable extension of fields. Since $F(\mathrm{Spec}(k')) = 0$ we see that $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 37.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 59.45.2. Since $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

The sheaf G is limit preserving, see Limits of Spaces, Definition 70.3.1. Namely, let R be a ring which is written as a directed colimit $R = \mathrm{colim}_i R_i$ of rings. Set $X = \mathrm{Spec}(R)$ and $X_i = \mathrm{Spec}(R_i)$, so that $X = \lim_i X_i$. Then $G(X) = \mathrm{colim}_i G(X_i)$. To prove this one first proves that a constructible partition of $\mathrm{Spec}(R)$ comes from a constructible partitions of some $\mathrm{Spec}(R_i)$. Hence the result for F. To get the result for the sheafification, use that any étale ring map $R \to R'$ comes from an étale ring map $R_i \to R'_i$ for some i. Details omitted.

05LH Lemma 110.57.1. There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties

1. $G(\mathrm{Spec}(k)) = 0$ whenever k is a field,
2. G is limit preserving,
3. if $X \subset X'$ is a thickening, then $G(X) = G(X')$, and
4. G is not zero.

Proof. See discussion above. □

110.58. The lisse-étale site is not functorial

07BF The lisse-étale site $X_{lisse,étale}$ of X is the category of schemes smooth over X endowed with (usual) étale coverings, see Cohomology of Stacks, Section 103.14. Let $f : X \to Y$ be a morphism of schemes. There is a functor

$$u : Y_{lisse,étale} \longrightarrow X_{lisse,étale}, \quad V/Y \longmapsto V \times_Y X$$

which is continuous. Hence we obtain an adjoint pair of functors

$$u^s : Sh(X_{lisse,étale}) \longrightarrow Sh(Y_{lisse,étale}), \quad u_s : Sh(Y_{lisse,étale}) \longrightarrow Sh(X_{lisse,étale}),$$

see Sites, Section 7.13. We claim that, in general, u does not define a morphism of sites, see Sites, Definition 7.14.1. In other words, we claim that u_s is not left exact in general. Note that representable presheaves are sheaves on lisse-étale sites.

Hence, by Sites, Lemma 7.13.5 we see that $u_s h_V = h_{V \times_Y X}$. Now consider two morphisms

$$\begin{tikzcd} V_1 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] \arrow[dr] & & V_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

of schemes V_1, V_2 smooth over Y. Now if u_s is left exact, then we would have

$$u_s \text{Equalizer}(h_a, h_b : h_{V_1} \to h_{V_2}) = \text{Equalizer}(h_{a\times 1}, h_{b \times 1} : h_{V_1 \times_Y X} \to h_{V_2 \times_Y X})$$

We will take the morphisms $a, b : V_1 \to V_2$ such that there exists no morphism from a scheme smooth over Y into $(a = b) \subset V_1$, i.e., such that the left hand side is the empty sheaf, but such that after base change to X the equalizer is nonempty and smooth over X. A silly example is to take $X = \text{Spec}(\mathbf{F}_p)$, $Y = \text{Spec}(\mathbf{Z})$ and $V_1 = V_2 = \mathbf{A}^1_{\mathbf{Z}}$ with morphisms $a(x) = x$ and $b(x) = x + p$. Note that the equalizer of a and b is the fibre of $\mathbf{A}^1_{\mathbf{Z}}$ over (p).

07BG Lemma 110.58.1. The lisse-étale site is not functorial, even for morphisms of schemes.

Proof. See discussion above. $\square$

110.59. Sheaves on the category of Noetherian schemes

0GE8 Let S be a locally Noetherian scheme. As in Artin's Axioms, Section 98.25 consider the inclusion functor

$$u : (\text{Noetherian}/S)_{fppf} \longrightarrow (Sch/S)_{fppf}$$

of the fppf site of locally Noetherian schemes over S into a big fppf site of S. As explained in the section referenced, this functor is continuous. Hence we obtain an adjoint pair of functors

$$u^s : Sh((Sch/S)_{fppf}) \longrightarrow Sh((\text{Noetherian}/S)_{fppf})$$

and

$$u_s : Sh((\text{Noetherian}/S)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

see Sites, Section 7.13. However, we claim that u in general does not define a morphism of sites, see Sites, Definition 7.14.1. In other words, we claim that the functor u_s is not left exact in general.

Let p be a prime number and set $S = \text{Spec}(\mathbf{F}_p)$. Consider the injective map of sheaves

$$a : \mathcal{F} \longrightarrow \mathcal{G}$$

on $(\text{Noetherian}/S)_{fppf}$ defined as follows: for U a locally Noetherian scheme over S we define

$$\mathcal{G}(U) = \Gamma(U, \mathcal{O}_U)^* = \text{Mor}_S(U, \mathbf{G}_{m,S})$$

and we take

$$\mathcal{F}(U) = \{f \in \mathcal{G}(U) \mid \text{fppf locally } f \text{ has arbitrary } p\text{-power roots}\}$$

A Noetherian $\mathbf{F}_p$-algebra A has a nilpotent nilradical $I \subset A$, the p-power roots of 1 in A are of the elements of the form $1 + a$, $a \in I$, and hence no-nontrivial p-power root of 1 has arbitrary p-power roots. We conclude that $\mathcal{F}(U)$ is a p-torsion free abelian group for any locally Noetherian scheme U; some details omitted. It follows that $p : \mathcal{F} \to \mathcal{F}$ is an injective map of abelian sheaves on $(\text{Noetherian}/S)_{fppf}$.

To get a contradiction, assume u_s is exact. Then $p : u_s\mathcal{F} \to u_s\mathcal{F}$ is injective too and we find that $(u_s\mathcal{F})(V)$ is a p-torsion free abelian group for any V over S. Since representable presheaves are sheaves on fppf sites, by Sites, Lemma 7.13.5, we see that $u_s\mathcal{G}$ is represented by $\mathbf{G}_{m,S}$. Using that $u_s\mathcal{F} \to u_s\mathcal{G}$ is injective, we find a p-torsion free subgroup

$$(u_s\mathcal{F})(V) \subset \Gamma(V, \mathcal{O}_V)^*$$

for every scheme V over S with the following property: for every morphism $V \to U$ of schemes over S with U locally Noetherian the subgroup

$$\mathcal{F}(U) \subset \Gamma(U, \mathcal{O}_U)^*$$

maps into the subgroup $(u_s\mathcal{F})(V)$ by the restriction mapping $\Gamma(U, \mathcal{O}_U)^* \to \Gamma(V, \mathcal{O}_V)^*$.

The actual contradiction now is obtained as follows: let $k = \bigcup_{n \geq 0} \mathbf{F}_p(t^{1/p^n})$ and set

$$B = k \otimes_{\mathbf{F}_p(t)} k$$

and $V = \text{Spec}(B)$. Since we have the two projection morphisms $V \to \text{Spec}(k)$ corresponding to the two coprojections $k \to B$ and since $\text{Spec}(k)$ is Noetherian, we conclude the subgroup

$$(u_s\mathcal{F})(V) \subset B^*$$

contains $k^* \otimes 1$ and $1 \otimes k^*$. This is a contradiction because

$$(t^{1/p} \otimes 1) \cdot (1 \otimes t^{-1/p}) = t^{1/p} \otimes t^{-1/p}$$

is a nontrivial p-torsion unit of B.

0GE9 Lemma 110.59.1. With $S = \text{Spec}(\mathbf{F}_p)$ the inclusion functor $(\text{Noetherian}/S)_{fppf} \to (Sch/S)_{fppf}$ does not define a morphism of sites.

Proof. See discussion above. □

110.60. Derived pushforward of quasi-coherent modules

07DC Let k be a field of characteristic $p > 0$. Let $S = \text{Spec}(k[x])$. Let $G = \mathbf{Z}/p\mathbf{Z}$ viewed either as an abstract group or as a constant group scheme over S. Consider the algebraic stack $\mathcal{X} = [S/G]$ where G acts trivially on S, see Examples of Stacks, Remark 95.15.5 and Criteria for Representability, Lemma 97.18.3. Consider the structure morphism

$$f : \mathcal{X} \longrightarrow S$$

This morphism is quasi-compact and quasi-separated. Hence we get a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_S),$$

see Derived Categories of Stacks, Proposition 104.6.1. Let's compute $Rf_{QCoh,*}\mathcal{O}_\mathcal{X}$. Since $D_{QCoh}(\mathcal{O}_S)$ is equivalent to the derived category of $k[x]$-modules (see Derived Categories of Schemes, Lemma 36.3.5) this is equivalent to computing $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. For this we can use the covering $S \to \mathcal{X}$ and the spectral sequence

$$H^q(S \times_\mathcal{X} \ldots \times_\mathcal{X} S, O) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

see Cohomology of Stacks, Proposition 103.11.6. Note that

$$S \times_\mathcal{X} \ldots \times_\mathcal{X} S = S \times G^p$$

which is affine. Thus the complex

$$k[x] \to \mathrm{Map}(G, k[x]) \to \mathrm{Map}(G^2, k[x]) \to \ldots$$

computes $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Here for $\varphi \in \mathrm{Map}(G^{p-1}, k[x])$ its differential is the map which sends $(g_1, \ldots, g_p)$ to

$$\varphi(g_2, \ldots, g_p) + \sum\nolimits_{i=1}^{p-1} (-1)^i \varphi(g_1, \ldots, g_i + g_{i+1}, \ldots, g_p) + (-1)^p \varphi(g_1, \ldots, g_{p-1}).$$

This is just the complex computing the group cohomology of G acting trivially on $k[x]$ (insert future reference here). The cohomology of the cyclic group G on $k[x]$ is exactly one copy of $k[x]$ in each cohomological degree ≥ 0 (insert future reference here). We conclude that

$$Rf_*\mathcal{O}_\mathcal{X} = \bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[-n]$$

Now, consider the complex

$$E = \bigoplus\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

This is an object of $D_{QCoh}(\mathcal{O}_\mathcal{X})$. We interrupt the discussion for a general result.

08IY Lemma 110.60.1. Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{O}_\mathcal{X})$ whose cohomology sheaves are locally quasi-coherent (Sheaves on Stacks, Definition 96.12.1) and satisfy the flat base change property (Cohomology of Stacks, Definition 103.7.1). Then there exists a distinguished triangle

$$K \to \prod\nolimits_{n \geq 0} \tau_{\geq -n} K \to \prod\nolimits_{n \geq 0} \tau_{\geq -n} K \to K[1]$$

in $D(\mathcal{O}_\mathcal{X})$. In other words, K is the derived limit of its canonical truncations.

Proof. Recall that we work on the "big fppf site" $\mathcal{X}_{fppf}$ of $\mathcal{X}$ (by our conventions for sheaves of $\mathcal{O}_\mathcal{X}$-modules in the chapters Sheaves on Stacks and Cohomology on Stacks). Let $\mathcal{B}$ be the set of objects x of $\mathcal{X}_{fppf}$ which lie over an affine scheme U. Combining Sheaves on Stacks, Lemmas 96.23.2, 96.16.1, Descent, Lemma 35.12.4, and Cohomology of Schemes, Lemma 30.2.2 we see that $H^p(x, \mathcal{F}) = 0$ if $\mathcal{F}$ is locally quasi-coherent and $x \in \mathcal{B}$. Now the claim follows from Cohomology on Sites, Lemma 21.23.10 with $d = 0$. □

08IZ Lemma 110.60.2. Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}_n$ is a collection of locally quasi-coherent sheaves with the flat base change property on $\mathcal{X}$, then $\oplus_n \mathcal{F}_n[n] \to \prod_n \mathcal{F}_n[n]$ is an isomorphism in $D(\mathcal{O}_\mathcal{X})$.

Proof. This is true because by Lemma 110.60.1 we see that the direct sum is isomorphic to the product. □

We continue our discussion. Since a quasi-coherent module is locally quasi-coherent and satisfies the flat base change property (Sheaves on Stacks, Lemma 96.12.2) we get

$$E = \prod\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

Since cohomology commutes with limits we see that

$$Rf_*E = \prod\nolimits_{m \geq 0} \left(\bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[m-n] \right)$$

Note that this complex is not an object of $D_{QCoh}(\mathcal{O}_S)$ because the cohomology sheaf in degree 0 is an infinite product of copies of $\mathcal{O}_S$ which is not even a locally quasi-coherent $\mathcal{O}_S$-module.

07DD **Lemma 110.60.3.** A quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks need not induce a functor $Rf_* : D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$.

Proof. See discussion above. ☐

110.61. A big abelian category

07JS The purpose of this section is to give an example of a "big" abelian category $\mathcal{A}$ and objects M, N such that the collection of isomorphism classes of extensions $\mathrm{Ext}_\mathcal{A}(M, N)$ is not a set. The example is due to Freyd, see [Fre64, page 131, Exercise A].

We define $\mathcal{A}$ as follows. An object of $\mathcal{A}$ consists of a triple (M, α, f) where M is an abelian group and α is an ordinal and $f : \alpha \to \mathrm{End}(M)$ is a map. A morphism $(M, \alpha, f) \to (M', \alpha', f')$ is given by a homomorphism of abelian groups $\varphi : M \to M'$ such that for any ordinal β we have

$$\varphi \circ f(\beta) = f'(\beta) \circ \varphi$$

Here the rule is that we set $f(\beta) = 0$ if β is not in α and similarly we set $f'(\beta)$ equal to zero if β is not an element of α'. We omit the verification that the category so defined is abelian.

Consider the object $Z = (\mathbf{Z}, \emptyset, f)$, i.e., all the operators are zero. The observation is that computed in $\mathcal{A}$ the group $\mathrm{Ext}^1_\mathcal{A}(Z, Z)$ is a proper class and not a set. Namely, for each ordinal α we can find an extension $(M, \alpha{+}1, f)$ of Z by Z whose underlying group is $M = \mathbf{Z} \oplus \mathbf{Z}$ and where the value of f is always zero except for

$$f(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This clearly produces a proper class of isomorphism classes of extensions. In particular, the derived category of $\mathcal{A}$ has proper classes for its collections of morphism, see Derived Categories, Lemma 13.27.6. This means that some care has to be exercised when defining Verdier quotients of triangulated categories.

07JT **Lemma 110.61.1.** There exists a "big" abelian category $\mathcal{A}$ whose Ext-groups are proper classes.

Proof. See discussion above. ☐

110.62. Weakly associated points and scheme theoretic density

084J Let k be a field. Let $R = k[z, x_i, y_i]/(z^2, zx_iy_i)$ where i runs over the elements of $\mathbf{N}$. Note that $R = R_0 \oplus M_0$ where $R_0 = k[x_i, y_i]$ is a subring and M_0 is an ideal of square zero with $M_0 \cong R_0/(x_iy_i)$ as R_0-module. The prime $\mathfrak{p} = (z, x_i)$ is weakly associated to R as an R-module (Algebra, Definition 10.66.1). Indeed, the element z in $R_\mathfrak{p}$ is nonzero but annihilated by $\mathfrak{p}R_\mathfrak{p}$. On the other hand, consider the open subscheme

$$U = \bigcup D(x_i) \subset \mathrm{Spec}(R) = S$$

We claim that $U \subset S$ is scheme theoretically dense (Morphisms, Definition 29.7.1). To prove this it suffices to show that $\mathcal{O}_S \to j_*\mathcal{O}_U$ is injective where $j : U \to S$ is the inclusion morphism, see Morphisms, Lemma 29.7.5. Translated back into algebra, we have to show that for all $g \in R$ the map

$$R_g \longrightarrow \prod R_{x_i g}$$

is injective. Write $g = g_0 + m_0$ with $g_0 \in R_0$ and $m_0 \in M_0$. Then $R_g = R_{g_0}$ (details omitted). Hence we may assume $g \in R_0$. We may also assume g is not zero. Now $R_g = (R_0)_g \oplus (M_0)_g$. Since R_0 is a domain, the map $(R_0)_g \to \prod (R_0)_{x_i g}$ is injective. If $g \in (x_i y_i)$ then $(M_0)_g = 0$ and there is nothing to prove. If $g \notin (x_i y_i)$ then, since $(x_i y_i)$ is a radical ideal of R_0, we have to show that $M_0 \to \prod (M_0)_{x_i g}$ is injective. The kernel of $R_0 \to M_0 \to (M_0)_{x_n}$ is $(x_i y_i, y_n)$. Since $(x_i y_i, y_n)$ is a radical ideal, if $g \notin (x_i y_i, y_n)$ then the kernel of $R_0 \to M_0 \to (M_0)_{x_n g}$ is $(x_i y_i, y_n)$. As $g \notin (x_i y_i, y_n)$ for all $n \gg 0$ we conclude that the kernel is contained in $\bigcap_{n \gg 0} (x_i y_i, y_n) = (x_i y_i)$ as desired.

Second example due to Ofer Gabber. Let k be a field and let R, resp. R' be the ring of functions $\mathbf{N} \to k$, resp. the ring of eventually constant functions $\mathbf{N} \to k$. Then $\text{Spec}(R)$, resp. $\text{Spec}(R')$ is the Stone-Čech compactification[3] $\beta\mathbf{N}$, resp. the one point compactification[4] $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$. All points are weakly associated since all primes are minimal in the rings R and R'.

084K Lemma 110.62.1. There exists a reduced scheme X and a schematically dense open $U \subset X$ such that some weakly associated point $x \in X$ is not in U.

Proof. In the first example we have $\mathfrak{p} \notin U$ by construction. In Gabber's examples the schemes $\text{Spec}(R)$ or $\text{Spec}(R')$ are reduced. $\square$

110.63. Example of non-additivity of traces

087J Let k be a field and let $R = k[\epsilon]$ be the ring of dual numbers over k. In other words, $R = k[x]/(x^2)$ and ϵ is the congruence class of x in R. Consider the short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & R \arrow[r, "1"'] \arrow[d, "\epsilon"] & R \arrow[d] \\ R \arrow[r, "1"] & R \arrow[r] & 0 \end{tikzcd}$$

Here the columns are the complexes, the first row is placed in degree 0, and the second row in degree 1. Denote the first complex (i.e., the left column) by $A^\bullet$, the second by $B^\bullet$ and the third $C^\bullet$. We claim that the diagram

087K (110.63.0.1)
$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d, "1+\epsilon"'] & B^\bullet \arrow[r] \arrow[d, "1"'] & C^\bullet \arrow[d, "1"'] \\ A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \end{tikzcd}$$

[3] Every element $f \in R$ is of the form ue where u is a unit and e is an idempotent. Then Algebra, Lemma 10.26.5 shows $\text{Spec}(R)$ is Hausdorff. On the other hand, $\mathbf{N}$ with the discrete topology can be viewed as a dense open subset. Given a set map $\mathbf{N} \to X$ to a Hausdorff, quasi-compact topological space X, we obtain a ring map $\mathcal{C}^0(X; k) \to R$ where $\mathcal{C}^0(X; k)$ is the k-algebra of locally constant maps $X \to k$. This gives $\text{Spec}(R) \to \text{Spec}(\mathcal{C}^0(X; k)) = X$ proving the universal property.

[4] Here one argues that there is really only one extra maximal ideal in R'.

commutes in $K(R)$, i.e., is a diagram of complexes commuting up to homotopy. Namely, the square on the right commutes and the one on the left is off by the homotopy $1 : A^1 \to B^0$. On the other hand,

$$\mathrm{Tr}_{A^\bullet}(1+\epsilon) + \mathrm{Tr}_{C^\bullet}(1) \neq \mathrm{Tr}_{B^\bullet}(1).$$

087L Lemma 110.63.1. There exists a ring R, a distinguished triangle $(K, L, M, \alpha, \beta, \gamma)$ in the homotopy category $K(R)$, and an endomorphism (a, b, c) of this distinguished triangle, such that K, L, M are perfect complexes and $\mathrm{Tr}_K(a) + \mathrm{Tr}_M(c) \neq \mathrm{Tr}_L(b)$.

Proof. Consider the example above. The map $\gamma : C^\bullet \to A^\bullet[1]$ is given by multiplication by ϵ in degree 0, see Derived Categories, Definition 13.10.1. Hence it is also true that

$$\begin{tikzcd} C^\bullet \arrow[r, "\gamma"'] \arrow[d] & A^\bullet[1] \arrow[d] \\ C^\bullet \arrow[r, "\gamma"] & A^\bullet[1] \end{tikzcd}$$

commutes in $K(R)$ as $\epsilon(1+\epsilon) = \epsilon$. Thus we indeed have a morphism of distinguished triangles. $\square$

110.64. Being projective is not local on the base

08J0 In the chapter on descent we have seen that many properties of morphisms are local on the base, even in the fpqc topology. See Descent, Sections 35.22, 35.23, and 35.24. This is not true for projectivity of morphisms.

08J1 Lemma 110.64.1. The properties

$\mathcal{P}(f)$ =“f is projective”, and
$\mathcal{P}(f)$ =“f is quasi-projective”

are not Zariski local on the base. A fortiori, they are not fpqc local on the base.

Proof. Following Hironaka [Har77, Example B.3.4.1], we define a proper morphism of smooth complex 3-folds $f : V_Y \to Y$ which is Zariski-locally projective, but not projective. Since f is proper and not projective, it is also not quasi-projective.

Let Y be projective 3-space over the complex numbers $\mathbf{C}$. Let C and D be smooth conics in Y such that the closed subscheme $C \cap D$ is reduced and consists of two complex points P and Q. (For example, let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$, $D = \{[x, y, z, w] : xy = w^2, z = 0\}$, $P = [1, 0, 0, 0]$, and $Q = [0, 1, 0, 0]$.) On $Y - Q$, first blow up the curve C, and then blow up the strict transform of the curve D (Divisors, Definition 31.33.1). On $Y - P$, first blow up the curve D, and then blow up the strict transform of the curve C. Over $Y - P - Q$, the two varieties we have constructed are canonically isomorphic, and so we can glue them over $Y - P - Q$. The result is a smooth proper 3-fold V_Y over $\mathbf{C}$. The morphism $f : V_Y \to Y$ is proper and Zariski-locally projective (since it is a blowup over $Y - P$ and over $Y - Q$), by Divisors, Lemma 31.32.13. We will show that V_Y is not projective over $\mathbf{C}$. That will imply that f is not projective.

To do this, let L be the inverse image in V_Y of a complex point of $C - P - Q$, and M the inverse image of a complex point of $D - P - Q$. Then L and M are isomorphic to the projective line $\mathbf{P}^1_{\mathbf{C}}$. Next, let E be the inverse image in V_Y of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to

$\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0, and we have rational equivalences of cycles $L \sim L_0 + M_0$ and $M \sim M_0$ on E (using that C and D are isomorphic to $\mathbf{P}^1$). Note the asymmetry resulting from the order in which we blew up the two curves. Near Q, the opposite happens. So the inverse image of Q is the union of two lines L_0' and M_0', and we have rational equivalences $L \sim L_0'$ and $M \sim L_0' + M_0'$ on E. Combining these equivalences, we find that $L_0 + M_0' \sim 0$ on E and hence on V_Y. If V_Y were projective over $\mathbf{C}$, it would have an ample line bundle H, which would have degree > 0 on all curves in V_Y. In particular H would have positive degree on $L_0 + M_0'$, contradicting that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 42.20.3 and Lemma 42.28.2). So V_Y is not projective over $\mathbf{C}$. □

In different terminology, Hironaka's 3-fold V_Y is a small resolution of the blowup Y' of Y along the reduced subscheme $C \cup D$; here Y' has two node singularities. If we define Z by blowing up Y along C and then along the strict transform of D, then Z is a smooth projective 3-fold, and the non-projective 3-fold V_Y differs from Z by a "flop" over $Y - P$.

110.65. Non-effective descent data for projective schemes

08KE In the chapter on descent we have seen that descent data for schemes relative to an fpqc morphism are effective for several classes of morphisms. In particular, affine morphisms and more generally quasi-affine morphisms satisfy descent for fpqc coverings (Descent, Lemma 35.38.1). This is not true for projective morphisms.

08KF **Lemma 110.65.1.** *There is an etale covering $X \to S$ of schemes and a descent datum $(V/X, \varphi)$ relative to $X \to S$ such that $V \to X$ is projective, but the descent datum is not effective in the category of schemes.*

Proof. We imitate Hironaka's example of a smooth separated complex algebraic space of dimension 3 which is not a scheme [Har77, Example B.3.4.2].

Consider the action of the group $G = \mathbf{Z}/2 = \{1, g\}$ on projective 3-space $\mathbf{P}^3$ over the complex numbers by

$$g[x, y, z, w] = [y, x, w, z].$$

The action is free outside the two disjoint lines $L_1 = \{[x, x, z, z]\}$ and $L_2 = \{[x, -x, z, -z]\}$ in $\mathbf{P}^3$. Let $Y = \mathbf{P}^3 - (L_1 \cup L_2)$. There is a smooth quasi-projective scheme $S = Y/G$ over $\mathbf{C}$ such that $Y \to S$ is a G-torsor (Groupoids, Definition 39.11.3). Explicitly, we can define S as the image of the open subset Y in $\mathbf{P}^3$ under the morphism

$$\begin{aligned} \mathbf{P}^3 &\to \text{Proj } \mathbf{C}[x, y, z, w]^G \\ &= \text{Proj } \mathbf{C}[u_0, u_1, v_0, v_1, v_2]/(v_0 v_1 = v_2^2), \end{aligned}$$

where $u_0 = x + y$, $u_1 = z + w$, $v_0 = (x - y)^2$, $v_1 = (z - w)^2$, and $v_2 = (x - y)(z - w)$, and the ring is graded with u_0, u_1 in degree 1 and v_0, v_1, v_2 in degree 2.

Let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$ and $D = \{[x, y, z, w] : xy = w^2, z = 0\}$. These are smooth conic curves in $\mathbf{P}^3$, contained in the G-invariant open subset Y, with $g(C) = D$. Also, $C \cap D$ consists of the two points $P := [1, 0, 0, 0]$ and $Q := [0, 1, 0, 0]$, and these two points are switched by the action of G.

Let $V_Y \to Y$ be the scheme which over $Y - P$ is defined by blowing up D and then the strict transform of C, and over $Y - Q$ is defined by blowing up C and then the strict transform of D. (This is the same construction as in the proof of Lemma 110.64.1, except that Y here denotes an open subset of $\mathbf{P}^3$ rather than all of $\mathbf{P}^3$.) Then the action of G on Y lifts to an action of G on V_Y, which switches the inverse images of $Y - P$ and $Y - Q$. This action of G on V_Y gives a descent datum $(V_Y/Y, \varphi_Y)$ on V_Y relative to the G-torsor $Y \to S$. The morphism $V_Y \to Y$ is proper but not projective, as shown in the proof of Lemma 110.64.1.

Let X be the disjoint union of the open subsets $Y - P$ and $Y - Q$; then we have surjective etale morphisms $X \to Y \to S$. Let V be the pullback of $V_Y \to Y$ to X; then the morphism $V \to X$ is projective, since $V_Y \to Y$ is a blowup over each of the open subsets $Y - P$ and $Y - Q$. Moreover, the descent datum $(V_Y/Y, \varphi_Y)$ pulls back to a descent datum $(V/X, \varphi)$ relative to the etale covering $X \to S$.

Suppose that this descent datum is effective in the category of schemes. That is, there is a scheme $U \to S$ which pulls back to the morphism $V \to X$ together with its descent datum. Then U would be the quotient of V_Y by its G-action.

$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \arrow[d] \\ V_Y \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & S \end{tikzcd}$$

Let E be the inverse image of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0. It follows that the inverse image of $Q = g(P)$ in E is the union of two lines $L_0' = g(M_0)$ and $M_0' = g(L_0)$. As shown in the proof of Lemma 110.64.1, we have a rational equivalence $L_0 + M_0' = L_0 + g(L_0) \sim 0$ on E.

By descent of closed subschemes, there is a curve $L_1 \subset U$ (isomorphic to $\mathbf{P}^1$) whose inverse image in V_Y is $L_0 \cup g(L_0)$. (Use Descent, Lemma 35.37.1, noting that a closed immersion is an affine morphism.) Let R be a complex point of L_1. Since we assumed that U is a scheme, we can choose a function f in the local ring $O_{U,R}$ that vanishes at R but not on the whole curve L_1. Let D_{loc} be an irreducible component of the closed subset $\{f = 0\}$ in $\operatorname{Spec} O_{U,R}$; then D_{loc} has codimension 1. The closure of D_{loc} in U is an irreducible divisor D_U in U which contains the point R but not the whole curve L_1. The inverse image of D_U in V_Y is an effective divisor D which intersects $L_0 \cup g(L_0)$ but does not contain either curve L_0 or $g(L_0)$.

Since the complex 3-fold V_Y is smooth, $O(D)$ is a line bundle on V_Y. We use here that a regular local ring is factorial, or in other words is a UFD, see More on Algebra, Lemma 15.121.2. The restriction of $O(D)$ to the proper surface $E \subset V_Y$ is a line bundle which has positive degree on the 1-cycle $L_0 + g(L_0)$, by our information on D. Since $L_0 + g(L_0) \sim 0$ on E, this contradicts that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 42.20.3 and Lemma 42.28.2). Therefore the descent datum $(V/X, \varphi)$ is in fact not effective; that is, U does not exist as a scheme. $\square$

In this example, the descent datum is effective in the category of algebraic spaces. More precisely, U exists as a smooth separated algebraic space of dimension 3 over $\mathbf{C}$, for example by Algebraic Spaces, Lemma 65.14.3. Hironaka's 3-fold U is a small resolution of the blowup S' of the smooth quasi-projective 3-fold S along the irreducible nodal curve $(C \cup D)/G$; the 3-fold S' has a node singularity. The other small resolution of S' (differing from U by a "flop") is again an algebraic space which is not a scheme.

110.66. A family of curves whose total space is not a scheme

0D5D In Quot, Section 99.15 we define a family of curves over a scheme S to be a proper, flat, finitely presented morphism of relative dimension ≤ 1 from an algebraic space X to S. If S is the spectrum of a complete Noetherian local ring, then X is a scheme, see More on Morphisms of Spaces, Lemma 76.43.6. In this section we show this is not true in general.

Let k be a field. We start with a proper flat morphism

$$Y \longrightarrow \mathbf{A}^1_k$$

and a point $y \in Y(k)$ lying over $0 \in \mathbf{A}^1_k(k)$ with the following properties

(1) the fibre Y_0 is a smooth geometrically irreducible curve over k,
(2) for any proper closed subscheme $T \subset Y$ dominating $\mathbf{A}^1_k$ the intersection $T \cap Y_0$ contains at least one point distinct from y.

Given such a surface we construct our example as follows.

$$\begin{tikzcd} Y \arrow[dr] & Z \arrow[l] \arrow[r] \arrow[d] & X \arrow[dl] \\ & \mathbf{A}^1_k & \end{tikzcd}$$

Here $Z \to Y$ is the blowup of Y in y. Let $E \subset Z$ be the exceptional divisor and let $C \subset Z$ be the strict transform of Y_0. We have $Z_0 = E \cup C$ scheme theoretically (to see this use that Y is smooth at y and moreover $Y \to \mathbf{A}^1_k$ is smooth at y). By Artin's results ([Art70]; use Semistable Reduction, Lemma 55.9.7 to see that the normal bundle of C is negative) we can blow down the curve C in Z to obtain an algebraic space X as in the diagram. Let $x \in X(k)$ be the image of C.

We claim that X is not a scheme. Namely, if it were a scheme, then there would be an affine open neighbourhood $U \subset X$ of x. Set $T = X \setminus U$. Then T dominates $\mathbf{A}^1_k$ (as the fibres of $X \to \mathbf{A}^1_k$ are proper of dimension 1 and the fibres of $U \to \mathbf{A}^1_k$ are affine hence different). Let $T' \subset Z$ be the closed subscheme mapping isomorphically to T (as $x \notin T$). Then the image of T' in X contradicts condition (2) above (as $T' \cap Z_0$ is contained in the exceptional divisor E of the blowing up $Z \to Y$).

To finish the discussion we need to construct our Y. We will assume the characteristic of k is not 3. Write $\mathbf{A}^1_k = \mathrm{Spec}(k[t])$ and take

$$Y \quad : \quad T_0^3 + T_1^3 + T_2^3 - tT_0T_1T_2 = 0$$

in $\mathbf{P}^2_{k[t]}$. The fibre of this for $t = 0$ is a smooth projective genus 1 curve. On the affine piece $V_+(T_0)$ we get the affine equation

$$1 + x^3 + y^3 - txy = 0$$

which defines a smooth surface over k. Since the same is true on the other affine pieces by symmetry we see that Y is a smooth surface. Finally, we see from the affine equation also that the fraction field is $k(x, y)$ hence Y is a rational surface. Now the Picard group of a rational surface is finitely generated (insert future reference here). Hence in order to choose $y \in Y_0(k)$ with property (2) it suffices to choose y such that

0DYB (110.66.0.1) $$\mathcal{O}_{Y_0}(ny) \notin \mathrm{Im}(\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_0)) \text{ for all } n > 0$$

Namely, the sum of the 1-dimensional irreducible components of a T contradicting (2) would give an effective Cartier divisor intersection Y_0 in the divisor ny for some $n \geq 1$ and we would conclude that $\mathcal{O}_{Y_0}(ny)$ is in the image of the restriction map. Observe that since Y_0 has genus ≥ 1 the map

$$Y_0(k) \to \mathrm{Pic}(Y_0), \quad y \mapsto \mathcal{O}_{Y_0}(y)$$

is injective. Now if k is an uncountable algebraically closed field, then using the countability of $\mathrm{Pic}(Y)$ and the remark just made, we can find a $y \in Y_0(k)$ satisfying (110.66.0.1) and hence (2).

0D5E **Lemma 110.66.1.** There exists a field k and a family of curves $X \to \mathbf{A}^1_k$ such that X is not a scheme.

Proof. See discussion above. $\square$

110.67. Derived base change

08J2 Let $R \to R'$ be a ring map. In More on Algebra, Section 15.60 we construct a derived base change functor $- \otimes_R^{\mathbf{L}} R' : D(R) \to D(R')$. Next, let $R \to A$ be a second ring map. Picture

$$\begin{tikzcd} A \arrow[r] & A \otimes_R R' \arrow[r, equal] & A' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[ur] & \end{tikzcd}$$

Given an A-module M the tensor product $M \otimes_R R'$ is a $A \otimes_R R'$-module, i.e., an A'-module. For the ring map $A \to A'$ there is a derived functor

$$- \otimes_A^{\mathbf{L}} A' : D(A) \longrightarrow D(A')$$

but this functor does not agree with $- \otimes_R^{\mathbf{L}} R'$ in general. More precisely, for $K \in D(A)$ the canonical map

$$K \otimes_R^{\mathbf{L}} R' \longrightarrow K \otimes_A^{\mathbf{L}} A'$$

in $D(R')$ constructed in More on Algebra, Equation (15.61.0.1) isn't an isomorphism in general. Thus one may wonder if there exists a "derived base change functor" $T : D(A) \to D(A')$, i.e., a functor such that $T(K)$ maps to $K \otimes_R^{\mathbf{L}} R'$ in $D(R')$. In this section we show it does not exist in general.

Let k be a field. Set $R = k[x, y]$. Set $R' = R/(xy)$ and $A = R/(x^2)$. The object $A \otimes_R^{\mathbf{L}} R'$ in $D(R')$ is represented by

$$x^2 : R' \longrightarrow R'$$

and we have $H^0(A \otimes_R^{\mathbf{L}} R') = A \otimes_R R'$. We claim that there does not exist an object E of $D(A \otimes_R R')$ mapping to $A \otimes_R^{\mathbf{L}} R'$ in $D(R')$. Namely, for such an E the module $H^0(E)$ would be free, hence E would decompose as $H^0(E)[0] \oplus H^{-1}(E)[1]$. But it is easy to see that $A \otimes_R^{\mathbf{L}} R'$ is not isomorphic to the sum of its cohomology groups in $D(R')$.

08J3 Lemma 110.67.1. Let $R \to R'$ and $R \to A$ be ring maps. In general there does not exist a functor $T : D(A) \to D(A \otimes_R R')$ of triangulated categories such that an A-module M gives an object $T(M)$ of $D(A \otimes_R R')$ which maps to $M \otimes_R^{\mathbf{L}} R'$ under the map $D(A \otimes_R R') \to D(R')$.

Proof. See discussion above. □

110.68. An interesting compact object

09R4 Let R be a ring. Let (A, d) be a differential graded R-algebra. If $A = R$, then we know that every compact object of $D(A, \text{d}) = D(R)$ is represented by a finite complex of finite projective modules. In other words, compact objects are perfect, see More on Algebra, Proposition 15.78.3. The analogue in the language of differential graded modules would be the question: "Is every compact object of $D(A, \text{d})$ represented by a differential graded A-module P which is finite and graded projective?"

For general differential graded algebras, this is not true. Namely, let k be a field of characteristic 2 (so we don't have to worry about signs). Let $A = k[x, y]/(y^2)$ with

(1) x of degree 0
(2) y of degree -1,
(3) $\text{d}(x) = 0$, and
(4) $\text{d}(y) = x^2 + x$.

Then $x : A \to A$ is a projector in $K(A, \text{d})$. Hence we see that

$$A = \text{Ker}(x) \oplus \text{Im}(1 - x)$$

in $K(A, \text{d})$, see Differential Graded Algebra, Lemma 22.5.4 and Derived Categories, Lemma 13.4.14. It is clear that A is a compact object of $D(A, \text{d})$. Then $\text{Ker}(x)$ is a compact object of $D(A, \text{d})$ as follows from Derived Categories, Lemma 13.37.2.

Next, suppose that M is a differential graded (right) A-module representing $\text{Ker}(x)$ and suppose that M is finite and projective as a graded A-module. Because every finite graded projective module over $k[x, y]/(y^2)$ is graded free, we see that M is finite free as a graded $k[x, y]/(y^2)$-module (i.e., when we forget the differential). We set $N = M/M(x^2 + x)$. Consider the exact sequence

$$0 \to M \xrightarrow{x^2+x} M \to N \to 0$$

Since $x^2 + x$ is of degree 0, in the center of A, and $\text{d}(x^2 + x) = 0$ we see that this is a short exact sequence of differential graded A-modules. Moreover, as $\text{d}(y) = x^2 + x$ we see that the differential on N is linear. The maps

$$H^{-1}(N) \to H^0(M) \quad \text{and} \quad H^0(M) \to H^0(N)$$

are isomorphisms as $H^*(M) = H^0(M) = k$ since $M \cong \text{Ker}(x)$ in $D(A, \text{d})$. A computation of the boundary map shows that $H^*(N) = k[x, y]/(x, y^2)$ as a graded module; we omit the details. Since N is a free $k[x, y]/(y^2, x^2 + x)$-module we have a resolution

$$\ldots \to N[2] \xrightarrow{y} N[1] \xrightarrow{y} N \to N/Ny \to 0$$

compatible with differentials. Since N is bounded and since $H^0(N) = k[x, y]/(x, y^2)$ it follows from Homology, Lemma 12.25.3 that $H^0(N/Ny) = k[x]/(x)$. But as

N/Ny is a finite complex of free $k[x]/(x^2+x) = k \times k$-modules, we see that its cohomology has to have even dimension, a contradiction.

09R5 Lemma 110.68.1. There exists a differential graded algebra (A, d) and a compact object E of $D(A, \mathrm{d})$ such that E cannot be represented by a finite and graded projective differential graded A-module.

Proof. See discussion above. $\square$

110.69. Two differential graded categories

09R6 In this section we construct two differential graded categories satisfying axioms (A), (B), and (C) as in Differential Graded Algebra, Situation 22.27.2 whose objects do not come with a $\mathbf{Z}$-grading.

Example I. Let X be a topological space. Denote $\underline{\mathbf{Z}}$ the constant sheaf with value $\mathbf{Z}$. Let A be an $\underline{\mathbf{Z}}$-torsor. In this setting we say a sheaf of abelian groups $\mathcal{F}$ is A-graded if given a local section $a \in A(U)$ there is a projector $p_a : \mathcal{F}|_U \to \mathcal{F}|_U$ such that whenever we have a local isomorphism $\underline{\mathbf{Z}}|_U \to A|_U$ then $\mathcal{F}|_U = \bigoplus_{n \in \mathbf{Z}} p_n(\mathcal{F})$. Another way to say this is that locally on X the abelian sheaf $\mathcal{F}$ has a $\mathbf{Z}$-grading, but on overlaps the different choices of gradings differ by a shift in degree given by the transition functions for the torsor A. We say that a pair $(\mathcal{F}, \mathrm{d})$ is an A-graded complex of abelian sheaves, if $\mathcal{F}$ is an A-graded abelian sheaf and $\mathrm{d} : \mathcal{F} \to \mathcal{F}$ is a differential, i.e., $\mathrm{d}^2 = 0$ such that $p_{a+1} \circ \mathrm{d} = \mathrm{d} \circ p_a$ for every local section a of A. In other words, $\mathrm{d}(p_a(\mathcal{F}))$ is contained in $p_{a+1}(\mathcal{F})$.

Next, consider the category $\mathcal{A}$ with

(1) objects are A-graded complexes of abelian sheaves, and
(2) for objects $(\mathcal{F}, \mathrm{d})$, $(\mathcal{G}, \mathrm{d})$ we set
$$\mathrm{Hom}_{\mathcal{A}}((\mathcal{F}, \mathrm{d}), (\mathcal{G}, \mathrm{d})) = \bigoplus \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$$
where $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ is the group of maps of abelian sheaves f such that $f(p_a(\mathcal{F})) \subset p_{a+n}(\mathcal{G})$ for all local sections a of A. As differential we take $\mathrm{d}(f) = \mathrm{d} \circ f - (-1)^n f \circ \mathrm{d}$, see Differential Graded Algebra, Example 22.26.6.

We omit the verification that this is indeed a differential graded category satisfying (A), (B), and (C). All the properties may be verified locally on X where one just recovers the differential graded category of complexes of abelian sheaves. Thus we obtain a triangulated category $K(\mathcal{A})$.

Twisted derived category of X. Observe that given an object $(\mathcal{F}, \mathrm{d})$ of $\mathcal{A}$, there is a well defined A-graded cohomology sheaf $H(\mathcal{F}, \mathrm{d})$. Hence it is clear what is meant by a quasi-isomorphism in $K(\mathcal{A})$. We can invert quasi-isomorphisms to obtain the derived category $D(\mathcal{A})$ of complexes of A-graded sheaves. If A is the trivial torsor, then $D(\mathcal{A})$ is equal to $D(X)$, but for nonzero torsors, one obtains a kind of twisted derived category of X.

Example II. Let C be a smooth curve over a perfect field k of characteristic 2. Then $\Omega_{C/k}$ comes endowed with a canonical square root. Namely, we can write $\Omega_{C/k} = \mathcal{L}^{\otimes 2}$ such that for every local function f on C the section $\mathrm{d}(f)$ is equal to $s^{\otimes 2}$ for some local section s of $\mathcal{L}$. The "reason" is that
$$\mathrm{d}(a_0 + a_1 t + \ldots + a_d t^d) = (\textstyle\sum_{i \text{ odd}} a_i^{1/2} t^{(i-1)/2})^2 \mathrm{d}t$$

(insert future reference here). This in particular determines a canonical connection

$$\nabla_{can} : \Omega_{C/k} \longrightarrow \Omega_{C/k} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$

whose 2-curvature is zero (namely, the unique connection such that the squares have derivative equal to zero). Observe that the category of vector bundles with connections is a tensor category, hence we also obtain canonical connections ∇_{can} on the invertible sheaves $\Omega_{C/k}^{\otimes n}$ for all $n \in \mathbf{Z}$.

Let $\mathcal{A}$ be the category with

(1) objects are pairs $(\mathcal{F}, \nabla)$ consisting of a finite locally free sheaf $\mathcal{F}$ endowed with a connection
$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$
whose 2-curvature is zero, and

(2) morphisms between $(\mathcal{F}, \nabla_\mathcal{F})$ and $(\mathcal{G}, \nabla_\mathcal{G})$ are given by
$$\mathrm{Hom}_\mathcal{A}((\mathcal{F}, \nabla_\mathcal{F}), (\mathcal{G}, \nabla_\mathcal{G})) = \bigoplus \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_C} \Omega_{C/k}^{\otimes n})$$
For an element $f : \mathcal{F} \to \mathcal{G} \otimes \Omega_{C/k}^{\otimes n}$ of degree n we set
$$\mathrm{d}(f) = \nabla_{\mathcal{G} \otimes \Omega_{C/k}^{\otimes n}} \circ f + f \circ \nabla_\mathcal{F}$$
with suitable identifications.

We omit the verification that this forms a differential graded category with properties (A), (B), (C). Thus we obtain a triangulated homotopy category $K(\mathcal{A})$.

If $C = \mathbf{P}^1_k$, then $K(\mathcal{A})$ is the zero category. However, if C is a smooth proper curve of genus > 1, then $K(\mathcal{A})$ is not zero. Namely, suppose that $\mathcal{N}$ is an invertible sheaf of degree $0 \leq d < g - 1$ with a nonzero section σ. Then set $(\mathcal{F}, \nabla_\mathcal{F}) = (\mathcal{O}_C, \mathrm{d})$ and $(\mathcal{G}, \nabla_\mathcal{G}) = (\mathcal{N}^{\otimes 2}, \nabla_{can})$. We see that

$$\mathrm{Hom}^n_\mathcal{A}((\mathcal{F}, \nabla_\mathcal{F}), (\mathcal{G}, \nabla_\mathcal{G})) = \begin{cases} 0 & \text{if} \quad n < 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2}) & \text{if} \quad n = 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}) & \text{if} \quad n = 1 \end{cases}$$

The first 0 because the degree of $\mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}^{\otimes -1}$ is negative by the condition $d < g - 1$. Now, the section $\sigma^{\otimes 2}$ has derivative equal zero, hence the homomorphism group

$$\mathrm{Hom}_{K(\mathcal{A})}((\mathcal{F}, \nabla_\mathcal{F}), (\mathcal{G}, \nabla_\mathcal{G}))$$

is nonzero.

110.70. The stack of proper algebraic spaces is not algebraic

0D1Q In Quot, Section 99.13 we introduced and studied the stack in groupoids

$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow Sch_{fppf}$$

the stack whose category of sections over a scheme S is the category of flat, proper, finitely presented algebraic spaces over S. We proved that this satisfies many of Artin's axioms. In this section we why this stack is not algebraic by showing that formal effectiveness fails in general.

The canonical example uses that the universal deformation space of an abelian variety of dimension g has g^2 formal parameters whereas any effective formal deformation can be defined over a complete local ring of dimension $\leq g(g+1)/2$. Our example will be constructed by writing down a suitable non-effective deformation of a K3 surface. We will only sketch the argument and not give all the details.

Let $k = \mathbf{C}$ be the field of complex numbers. Let $X \subset \mathbf{P}^3_k$ be a smooth degree 4 surface over k. We have $\omega_X \cong \Omega^2_{X/k} \cong \mathcal{O}_X$. Finally, we have $\dim_k H^0(X, T_{X/k}) = 0$, $\dim_k H^1(X, T_{X/k}) = 20$, and $\dim_k H^2(X, T_{X/k}) = 0$. Since $L_{X/k} = \Omega_{X/k}$ because X is smooth over k, and since $\text{Ext}^i_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = H^i(X, T_{X/k})$, and because we have Cotangent, Lemma 92.23.1 we find that there is a universal deformation of X over

$$k[[x_1, \ldots, x_{20}]]$$

Suppose that this universal deformation is effective (as in Artin's Axioms, Section 98.9). Then we would get a flat, proper morphism

$$f : Y \longrightarrow \text{Spec}(k[[x_1, \ldots, x_{20}]])$$

where Y is an algebraic space recovering the universal deformation. This is impossible for the following reason. Since Y is separated we can find an affine open subscheme $V \subset Y$. Since the special fibre X of Y is smooth, we see that f is smooth. Hence Y is regular being smooth over regular and it follows that the complement D of V in Y is an effective Cartier divisor. Then $\mathcal{O}_Y(D)$ is a nontrivial element of $\text{Pic}(Y)$ (to prove this you show that the complement of a nonempty affine open in a proper smooth algebraic space over a field is always a nontrivial in the Picard group and you apply this to the generic fibre of f). Finally, to get a contradiction, we show that $\text{Pic}(Y) = 0$. Namely, the map $\text{Pic}(Y) \to \text{Pic}(X)$ is injective, because $H^1(X, \mathcal{O}_X) = 0$ (hence all deformations of $\mathcal{O}_X$ to $Y \times \text{Spec}(k[[x_i]]/\mathfrak{m}^n)$ are trivial) and Grothendieck's existence theorem (which says that coherent modules giving rise to the same sheaves on thickenings are isomorphic). If X is general enough, then $\text{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_X(1)$. Hence it suffices to show that $\mathcal{O}_X(n)$, $n > 0$ does not deform to the first order neighbourhood[5]. Consider the cup-product

$$H^1(X, \Omega_{X/k}) \times H^1(X, T_{X/k}) \longrightarrow H^2(X, \mathcal{O}_X)$$

This is a nondegenerate pairing by coherent duality. A computation shows that the Chern class $c_1(\mathcal{O}_X(n)) \in H^1(X, \Omega_{X/k})$ in Hodge cohomology is nonzero. Hence there is a first order deformation whose cup product with $c_1(\mathcal{O}_X(n))$ is nonzero. Then finally, one shows this cup product is the obstruction class to lifting.

0D1R Lemma 110.70.1. The stack in groupoids

$$p'_{fp,flat,proper} : \mathit{Spaces}'_{fp,flat,proper} \longrightarrow \mathit{Sch}_{fppf}$$

whose category of sections over a scheme S is the category of flat, proper, finitely presented algebraic spaces over S (see Quot, Section 99.13) is not an algebraic stack.

Proof. If it was an algebraic stack, then every formal object would be effective, see Artin's Axioms, Lemma 98.9.5. The discussion above show this is not the case after base change to $\text{Spec}(\mathbf{C})$. Hence the conclusion. $\square$

[5]This argument works as long as the map $c_1 : \text{Pic}(X) \to H^1(X, \Omega_{X/k})$ is injective, which is true for k any field of characteristic zero and any smooth hypersurface X of degree 4 in $\mathbf{P}^3_k$.

110.71. An example of a non-algebraic Hom-stack

0AF8 Let $\mathcal{Y}, \mathcal{Z}$ be algebraic stacks over a scheme S. The Hom-stack $\underline{\mathrm{Mor}}_S(\mathcal{Y}, \mathcal{Z})$ is the stack in groupoids over S whose category of sections over a scheme T is given by the category

$$\mathrm{Mor}_T(\mathcal{Y} \times_S T, \mathcal{Z} \times_S T)$$

whose objects are 1-morphisms and whose morphisms are 2-morphisms. We omit the proof this is indeed a stack in groupoids over $(Sch/S)_{fppf}$ (insert future reference here). Of course, in general the Hom-stack will not be algebraic. In this section we give an example where it is not true and where $\mathcal{Y}$ is representable by a proper flat scheme over S and $\mathcal{Z}$ is smooth and proper over S.

Let k be an algebraically closed field which is not the algebraic closure of a finite field. Let $S = \mathrm{Spec}(k[[t]])$ and $S_n = \mathrm{Spec}(k[t]/(t^n)) \subset S$. Let $f : X \to S$ be a map satisfying the following

(1) f is projective and flat, and its fibres are geometrically connected curves,
(2) the fibre $X_0 = X \times_S S_0$ is a nodal curve with smooth irreducible components whose dual graph has a loop consisting of rational curves,
(3) X is a regular scheme.

To make such a surface X we can take for example

$$X \quad : \quad T_0T_1T_2 - t(T_0^3 + T_1^3 + T_2^3) = 0$$

in $\mathbf{P}^2_{k[[t]]}$. Let A_0 be a non-zero abelian variety over k for example an elliptic curve. Let $A = A_0 \times_{\mathrm{Spec}(k)} S$ be the constant abelian scheme over S associated to A_0. We will show that the stack $\mathcal{X} = \underline{\mathrm{Mor}}_S(X, [S/A]))$ is not algebraic.

Recall that $[S/A]$ is on the one hand the quotient stack of A acting trivially on S and on the other hand equal to the stack classifying fppf A-torsors, see Examples of Stacks, Proposition 95.15.3. Observe that $[S/A] = [\mathrm{Spec}(k)/A_0] \times_{\mathrm{Spec}(k)} S$. This allows us to describe the fibre category over a scheme T as follows

$$\begin{aligned} \mathcal{X}_T &= \underline{\mathrm{Mor}}_S(X, [S/A])_T \\ &= \mathrm{Mor}_T(X \times_S T, [S/A] \times_S T) \\ &= \mathrm{Mor}_S(X \times_S T, [S/A]) \\ &= \mathrm{Mor}_{\mathrm{Spec}(k)}(X \times_S T, [\mathrm{Spec}(k)/A_0]) \end{aligned}$$

for any S-scheme T. In other words, the groupoid $\mathcal{X}_T$ is the groupoid of fppf A_0-torsors on $X \times_S T$. Before we discuss why $\mathcal{X}$ is not an algebraic stack, we need a few lemmas.

0AF9 **Lemma 110.71.1.** Let W be a two dimensional regular integral Noetherian scheme with function field K. Let $G \to W$ be an abelian scheme. Then the map $H^1_{fppf}(W, G) \to H^1_{fppf}(\mathrm{Spec}(K), G)$ is injective.

Sketch of proof. Let $P \to W$ be an fppf G-torsor which is trivial in the generic point. Then we have a morphism $\mathrm{Spec}(K) \to P$ over W and we can take its scheme theoretic image $Z \subset P$. Since $P \to W$ is proper (as a torsor for a proper group algebraic space over W) we see that $Z \to W$ is a proper birational morphism. By Spaces over Fields, Lemma 72.3.2 the morphism $Z \to W$ is finite away from finitely many closed points of W. By (insert future reference on resolving indeterminacies of morphisms by blowing quadratic transformations for surfaces) the irreducible components of the geometric fibres of $Z \to W$ are rational curves. By More on Groupoids in Spaces, Lemma 79.11.3 there are no nonconstant morphisms from rational curves to group schemes or torsors over such. Hence $Z \to W$ is finite, whence Z is a scheme and $Z \to W$ is an isomorphism by Morphisms, Lemma 29.54.8. In other words, the torsor P is trivial. □

0AFA Lemma 110.71.2. Let G be a smooth commutative group algebraic space over a field K. Then $H^1_{fppf}(\mathrm{Spec}(K), G)$ is torsion.

Proof. Every G-torsor P over $\mathrm{Spec}(K)$ is smooth over K as a form of G. Hence P has a point over a finite separable extension L/K. Say $[L : K] = n$. Let $[n](P)$ denote the G-torsor whose class is n times the class of P in $H^1_{fppf}(\mathrm{Spec}(K), G)$. There is a canonical morphism

$$P \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} P \to [n](P)$$

of algebraic spaces over K. This morphism is symmetric as G is abelian. Hence it factors through the quotient

$$(P \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} P)/S_n$$

On the other hand, the morphism $\mathrm{Spec}(L) \to P$ defines a morphism

$$(\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))/S_n \longrightarrow (P \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} P)/S_n$$

and the reader can verify that the scheme on the left has a K-rational point. Thus we see that $[n](P)$ is the trivial torsor. □

To prove $\mathcal{X} = \underline{\mathrm{Mor}}_S(X, [S/A])$ is not an algebraic stack, by Artin's Axioms, Lemma 98.9.5, it is enough to show the following.

0AFB Lemma 110.71.3. The canonical map $\mathcal{X}(S) \to \lim \mathcal{X}(S_n)$ is not essentially surjective.

Sketch of proof. Unwinding definitions, it is enough to check that $H^1(X, A_0) \to \lim H^1(X_n, A_0)$ is not surjective. As X is regular and projective, by Lemmas 110.71.2 and 110.71.1 each A_0-torsor over X is torsion. In particular, the group $H^1(X, A_0)$ is torsion. It is thus enough to show: (a) the group $H^1(X_0, A_0)$ is non-torsion, and (b) the maps $H^1(X_{n+1}, A_0) \to H^1(X_n, A_0)$ are surjective for all n.

Ad (a). One constructs a nontorsion A_0-torsor P_0 on the nodal curve X_0 by glueing trivial A_0-torsors on each component of X_0 using non-torsion points on A_0 as the isomorphisms over the nodes. More precisely, let $x \in X_0$ be a node which occurs in a loop consisting of rational curves. Let $X_0' \to X_0$ be the normalization of X_0 in $X_0 \setminus \{x\}$. Let $x', x'' \in X_0'$ be the two points mapping to x_0. Then we take $A_0 \times_{\mathrm{Spec}(k)} X_0'$ and we identify $A_0 \times x'$ with $A_0 \times \{x''\}$ using translation $A_0 \to A_0$ by a nontorsion point $a_0 \in A_0(k)$ (there is such a nontorsion point as k is algebraically closed and not the algebraic closure of a finite field – this is actually not trivial to prove). One can show that the glueing is an algebraic space (in fact one can show it is a scheme) and that it is an nontorsion A_0-torsor over X_0. The reason that it is nontorsion is that if $[n](P_0)$ has a section, then that section produces a morphism $s : X_0' \to A_0$ such that $[n](a_0) = s(x') - s(x'')$ in the group law on $A_0(k)$. However, since the irreducible components of the loop are rational to section s is constant on them (More on Groupoids in Spaces, Lemma 79.11.3). Hence $s(x') = s(x'')$ and we obtain a contradiction.

Ad (b). Deformation theory shows that the obstruction to deforming an A_0-torsor $P_n \to X_n$ to an A_0-torsor $P_{n+1} \to X_{n+1}$ lies in $H^2(X_0, \omega)$ for a suitable vector bundle ω on X_0. The latter vanishes as X_0 is a curve, proving the claim. □

0AFC Proposition 110.71.4. The stack $\mathcal{X} = \underline{\mathrm{Mor}}_S(X, [S/A])$ is not algebraic.

Proof. See discussion above. □

0AFD Remark 110.71.5. Proposition 110.71.4 contradicts [Aok06b, Theorem 1.1]. The problem is the non-effectivity of formal objects for $\underline{\mathrm{Mor}}_S(X, [S/A])$. The same problem is mentioned in the Erratum [Aok06a] to [Aok06b]. Unfortunately, the Erratum goes on to assert that $\underline{\mathrm{Mor}}_S(\mathcal{Y}, \mathcal{Z})$ is algebraic if $\mathcal{Z}$ is separated, which also contradicts Proposition 110.71.4 as $[S/A]$ is separated.

110.72. An algebraic stack not satisfying strong formal effectiveness

0CXW This is [Bha16, Example 4.12]. Let k be an algebraically closed field. Let A be an abelian variety over k. Assume that $A(k)$ is not torsion (this always holds if k is not the algebraic closure of a finite field). Let $\mathcal{X} = [\mathrm{Spec}(k)/A]$. We claim there exists an ideal $I \subset k[x, y]$ such that

$$\mathcal{X}_{\mathrm{Spec}(k[x,y]^\wedge)} \longrightarrow \lim \mathcal{X}_{\mathrm{Spec}(k[x,y]/I^n)}$$

is not essentially surjective. Namely, let I be the ideal generated by $xy(x + y - 1)$. Then $X_0 = V(I)$ consists of three copies of $\mathbf{A}^1_k$ glued into a triangle at three points. Hence we can make an infinite order torsor P_0 for A over X_0 by taking the trivial torsor over the irreducible components of X_0 and glueing using translation by nontorsion points. Exactly as in the proof of Lemma 110.71.3 we can lift P_0 to a torsor P_n over $X_n = \mathrm{Spec}(k[x, y]/I^n)$. Since $k[x, y]^\wedge$ is a two dimensional regular domain we see that any torsor P for A over $\mathrm{Spec}(k[x, y]^\wedge)$ is torsion (Lemmas 110.71.1 and 110.71.2). Hence the system of torsors is not in the image of the displayed functor.

0CXX Lemma 110.72.1. Let k be an algebraically closed field which is not the closure of a finite field. Let A be an abelian variety over k. Let $\mathcal{X} = [\mathrm{Spec}(k)/A]$. There exists an inverse system of k-algebras R_n with surjective transition maps whose kernels are locally nilpotent and a system (ξ_n) of $\mathcal{X}$ lying over the system $(\mathrm{Spec}(R_n))$ such that this system is not effective in the sense of Artin's Axioms, Remark 98.20.2.

Proof. See discussion above. □

110.73. A counter example to Grothendieck's existence theorem

0ARE Let k be a field and let $A = k[[t]]$. Let X be the glueing of $U = \mathrm{Spec}(A[x])$ and $V = \mathrm{Spec}(A[y])$ by the identification

$$U \setminus \{0_U\} \longrightarrow V \setminus \{0_V\}$$

sending x to y where $0_U \in U$ and $O_V \in V$ are the points corresponding to the maximal ideals (x, t) and (y, t). Set $A_n = A/(t^n)$ and set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$. Let $\mathcal{F}_n$ be the coherent sheaf on X_n corresponding to the $A_n[x]$-module $A_n[x]/(x) \cong A_n$ and the $A_n[y]$ module 0 with obvious glueing. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals generate by t. Then $(\mathcal{F}_n)$ is an object of the category $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ defined in Cohomology of Schemes, Section 30.27. On the other hand, this object is not in the image of the functor Cohomology of Schemes, Equation (30.27.0.1).

Namely, if it where there would be a finite $A[x]$-module M, a finite $A[y]$-module N and an isomorphism $M[1/t] \cong N[1/t]$ such that $M/t^nM \cong A_n[x]/(x)$ and $N/t^nN = 0$ for all n. It is easy to see that this is impossible.

0ARF Lemma 110.73.1. Counter examples to algebraization of coherent sheaves.

(1) Grothendieck's existence theorem as stated in Cohomology of Schemes, Theorem 30.27.1 is false if we drop the assumption that $X \to \text{Spec}(A)$ is separated.
(2) The stack of coherent sheaves $\mathcal{C}oh_{X/B}$ of Quot, Theorems 99.6.1 and 99.5.12 is in general not algebraic if we drop the assumption that $X \to S$ is separated
(3) The functor $\text{Quot}_{\mathcal{F}/X/B}$ of Quot, Proposition 99.8.4 is not an algebraic space in general if we drop the assumption that $X \to B$ is separated.

Proof. Part (1) we saw above. This shows that $\text{Coh}_{X/A}$ fails axiom [4] of Artin's Axioms, Section 98.14. Hence it cannot be an algebraic stack by Artin's Axioms, Lemma 98.9.5. In this way we see that (2) is true. To see (3), note that there are compatible surjections $\mathcal{O}_{X_n} \to \mathcal{F}_n$ for all n. Thus we see that $\text{Quot}_{\mathcal{O}_X/X/A}$ fails axiom [4] and we see that (3) is true as before. □

110.74. Affine formal algebraic spaces

0ANY Let K be a field and let $(V_i)_{i \in I}$ be a directed inverse system of nonzero vector spaces over K with surjective transition maps and with $\lim V_i = 0$, see Section 110.3. Let $R_i = K \oplus V_i$ as K-algebra where V_i is an ideal of square zero. Then R_i is an inverse system of K-algebras with surjective transition maps with nilpotent kernels and with $\lim R_i = K$. The affine formal algebraic space $X = \text{colim}\,\text{Spec}(R_i)$ is an example of an affine formal algebraic space which is not McQuillan.

0CBC Lemma 110.74.1. There exists an affine formal algebraic space which is not McQuillan.

Proof. See discussion above. □

Let $0 \to W_i \to V_i \to K \to 0$ be a system of exact sequences as in Section 110.3. Let $A_i = K[V_i]/(ww'; w, w' \in W_i)$. Then there is a compatible system of surjections $A_i \to K[t]$ with nilpotent kernels and the transition maps $A_i \to A_j$ are surjective with nilpotent kernels as well. Recall that V_i is free over K with basis given by $s \in S_i$. Then, if the characteristic of K is zero, the degree d part of A_i is free over K with basis given by s^d, $s \in S_i$ each of which map to t^d. Hence the inverse system of the degree d parts of the A_i is isomorphic to the inverse system of the vector spaces V_i. As $\lim V_i = 0$ we conclude that $\lim A_i = K$, at least when the characteristic of K is zero. This gives an example of an affine formal algebraic space whose "regular functions" do not separate points.

0CBD Lemma 110.74.2. There exists an affine formal algebraic space X whose regular functions do not separate points, in the following sense: If we write $X = \text{colim}\, X_\lambda$ as in Formal Spaces, Definition 87.9.1 then $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ is a field, but X_{red} has infinitely many points.

Proof. See discussion above. □

Let K, I, and (V_i) be as above. Consider systems

$$\Phi = (\Lambda, J_i \subset \Lambda, (M_i) \to (V_i))$$

where Λ is an augmented K-algebra, $J_i \subset \Lambda$ for $i \in I$ is an ideal of square zero, $(M_i) \to (V_i)$ is a map of inverse systems of K-vector spaces such that $M_i \to V_i$ is surjective for each i, such that M_i has a Λ-module structure, such that the transition maps $M_i \to M_j$, $i > j$ are Λ-linear, and such that $J_j M_i \subset \mathrm{Ker}(M_i \to M_j)$ for $i > j$. Claim: There exists a system as above such that $M_j = M_i/J_j M_i$ for all $i > j$.

If the claim is true, then we obtain a representable morphism

$$\mathrm{colim}_{i \in I} \mathrm{Spec}(\Lambda/J_i \oplus M_i) \longrightarrow \mathrm{Spf}(\lim \Lambda/J_i)$$

of affine formal algebraic spaces whose source is not McQuillan but the target is. Here $\Lambda/J_i \oplus M_i$ has the usual Λ/J_i-algebra structure where M_i is an ideal of square zero. Representability translates exactly into the condition that $M_i/J_j M_i = M_j$ for $i > j$. The source of the morphism is not McQuillan as the projections $\lim_{i \in I} M_i \to M_i$ are not be surjective. This is true because the maps $\lim V_i \to V_i$ are not surjective and we have the surjection $M_i \to V_i$. Some details omitted.

Proof of the claim. First, note that there exists at least one system, namely

$$\Phi_0 = (K, J_i = (0), (V_i) \xrightarrow{\mathrm{id}} (V_i))$$

Given a system Φ we will prove there exists a morphism of systems $\Phi \to \Phi'$ (morphisms of systems defined in the obvious manner) such that $\mathrm{Ker}(M_i/J_j M_i \to M_j)$ maps to zero in $M'_i/J'_j M'_i$. Once this is done we can do the usual trick of setting $\Phi_n = (\Phi_{n-1})'$ inductively for $n \geq 1$ and taking $\Phi = \mathrm{colim}\, \Phi_n$ to get a system with the desired properties. Details omitted.

Construction of Φ' given Φ. Consider the set U of triples $u = (i, j, \xi)$ where $i > j$ and $\xi \in \mathrm{Ker}(M_i \to M_j)$. We will let $s, t : U \to I$ denote the maps $s(i, j, \xi) = i$ and $t(i, j, \xi) = j$. Then we set $\xi_u \in M_{s(u)}$ the third component of u. We take

$$\Lambda' = \Lambda[x_u; u \in U]/(x_u x_{u'}; u, u' \in U)$$

with augmentation $\Lambda' \to K$ given by the augmentation of Λ and sending x_u to zero. We take $J'_k = J_k \Lambda' + (x_{u,\ t(u) \geq k})$. We set

$$M'_i = M_i \oplus \bigoplus\nolimits_{s(u) \geq i} K\epsilon_{i,u}$$

As transition maps $M'_i \to M'_j$ for $i > j$ we use the given map $M_i \to M_j$ and we send $\epsilon_{i,u}$ to $\epsilon_{j,u}$. The map $M'_i \to V_i$ induces the given map $M_i \to V_i$ and sends $\epsilon_{i,u}$ to zero. Finally, we let Λ' act on M'_i as follows: for $\lambda \in \Lambda$ we act by the Λ-module structure on M_i and via the augmentation $\Lambda \to K$ on $\epsilon_{i,u}$. The element x_u acts as 0 on M_i for all i. Finally, we define

$$x_u \epsilon_{i,u} = \text{image of } \xi_u \text{ in } M_i$$

and we set all other products $x_{u'}\epsilon_{i,u}$ equal to zero. The displayed formula makes sense because $s(u) \geq i$ and $\xi_u \in M_{s(u)}$. The main things the check are $J'_j M'_i \subset M'_i$ maps to zero in M'_j for $i > j$ and that $\mathrm{Ker}(M_i \to M_j)$ maps to zero in $M'_i/J_j M'_i$. The reason for the last fact is that $\xi = x_{(i,j,\xi)}\epsilon_{i,(i,j,\xi)} \in J'_j M'_i$ for any $\xi \in \mathrm{Ker}(M_i \to M_j)$. We omit the details.

0CBE **Lemma 110.74.3.** *There exists a representable morphism $f : X \to Y$ of affine formal algebraic spaces with Y McQuillan, but X not McQuillan.*

Proof. See discussion above. □

110.75. Flat maps are not directed limits of finitely presented flat maps

0ATE The goal of this section is to give an example of a flat ring map which is not a filtered colimit of flat and finitely presented ring maps. In [Gab96] it is shown that if A is a nonexcellent local ring of dimension 1 and residue characteristic zero, then the (flat) ring map $A \to A^\wedge$ to its completion is not a filtered colimit of finite type flat ring maps. The example in this section will have a source which is an excellent ring. We encourage the reader to submit other examples; please email stacks.project@gmail.com if you have one.

For the construction, fix a prime p, and let $A = \mathbf{F}_p[x_1, \ldots, x_n]$. Choose an absolute integral closure A^+ of A, i.e., A^+ is the integral closure of A in an algebraic closure of its fraction field. In [HH92, §6.7] it is shown that $A \to A^+$ is flat.

We claim that the A-algebra A^+ is not a filtered colimit of finitely presented flat A-algebras if $n \geq 3$.

We sketch the argument in the case $n = 3$, and we leave the generalization to higher n to the reader. It is enough to prove the analogous statement for the map $R \to R^+$, where R is the strict henselization of A at the origin and R^+ is its absolute integral closure. Observe that R is a henselian regular local ring whose residue field k is an algebraic closure of $\mathbf{F}_p$.

Choose an ordinary abelian surface X over k and a very ample line bundle L on X. The section ring $\Gamma_*(X, L) = \bigoplus_n H^0(X, L^n)$ is the coordinate ring of the affine cone over X with respect to L. It is a normal ring for L sufficiently positive. Let S denote the henselization of $\Gamma_*(X, L)$ at vertex of the cone. Then S is a henselian Noetherian normal domain of dimension 3. We obtain a finite injective map $R \to S$ as the henselization of a Noether normalization for the finite type k-algebra $\Gamma_*(X, L)$. As R^+ is an absolute integral closure of R, we can also fix an embedding $S \to R^+$. Thus R^+ is also the absolute integral closure of S. To show R^+ is not a filtered colimit of flat R-algebras, it suffices to show:

(1) If there exists a factorization $S \to P \to R^+$ with P flat and finite type over R, then there exists a factorization $S \to T \to R^+$ with T finite flat over R.
(2) For any factorization $S \to T \to R^+$ with $S \to T$ finite, the ring T is not R-flat.

Indeed, since S is finitely presented over R, if one could write $R^+ = \mathrm{colim}_i P_i$ as a filtered colimit of finitely presented flat R-algebras P_i, then $S \to R^+$ would factor as $S \to P_i \to R^+$ for $i \gg 0$, which contradicts the above pair of assertions. Assertion (1) follows from the fact that R is henselian and a slicing argument, see More on Morphisms, Lemma 37.23.5. Part (2) was proven in [Bha12]; for the convenience of the reader, we recall the argument.

Let $U \subset \mathrm{Spec}(S)$ be the punctured spectrum, so there are natural maps $X \leftarrow U \subset \mathrm{Spec}(S)$. The first map gives an identification $H^1(U, \mathcal{O}_U) \simeq H^1(X, \mathcal{O}_X)$. By passing to the Witt vectors of the perfection and using the Artin-Schreier sequence[6],

[6] Here we use that S is a strictly henselian local ring of characteristic p and hence $S \to S$, $f \mapsto f^p - f$ is surjective. Also S is a normal domain and hence $\Gamma(U, \mathcal{O}_U) = S$. Thus $H^1_{\acute{e}tale}(U, \mathbf{Z}/p)$ is the kernel of the map $H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$ induced by $f \mapsto f^p - f$.

this gives an identification $H^1_{\acute{e}tale}(U, \mathbf{Z}_p) \simeq H^1_{\acute{e}tale}(X, \mathbf{Z}_p)$. In particular, this group is a finite free $\mathbf{Z}_p$-module of rank 2 (since X is ordinary). To get a contradiction assume there exists an R-flat T as in (2) above. Let $V \subset \mathrm{Spec}(T)$ denote the preimage of U, and write $f : V \to U$ for the induced finite surjective map. Since U is normal, there is a trace map $f_*\mathbf{Z}_p \to \mathbf{Z}_p$ on $U_{\acute{e}tale}$ whose composition with the pullback $\mathbf{Z}_p \to f_*\mathbf{Z}_p$ is multiplication by $d = \deg(f)$. Passing to cohomology, and using that $H^1_{\acute{e}tale}(U, \mathbf{Z}_p)$ is nontorsion, then shows that $H^1_{\acute{e}tale}(V, \mathbf{Z}_p)$ is nonzero. Since $H^1_{\acute{e}tale}(V, \mathbf{Z}_p) \simeq \lim H^1_{\acute{e}tale}(V, \mathbf{Z}/p^n)$ as there is no $R^1 \lim$ interference, the group $H^1(V_{\acute{e}tale}, \mathbf{Z}/p)$ must be non-zero. Since T is R-flat we have $\Gamma(V, \mathcal{O}_V) = T$ which is strictly henselian and the Artin-Schreier sequence shows $H^1(V, \mathcal{O}_V) \neq 0$. This is equivalent to $H^2_{\mathfrak m}(T) \neq 0$, where $\mathfrak m \subset R$ is the maximal ideal. Thus, we obtain a contradiction since T is finite flat (i.e., finite free) as an R-module and $H^2_{\mathfrak m}(R) = 0$. This contradiction proves (2).

0ATF Lemma 110.75.1. There exists a commutative ring A and a flat A-algebra B which cannot be written as a filtered colimit of finitely presented flat A-algebras. In fact, we may either choose A to be a finite type $\mathbf{F}_p$-algebra or a 1-dimensional Noetherian local ring with residue field of characteristic 0.

Proof. See discussion above. □

110.76. The category of modules modulo torsion modules

0B0J The category of torsion groups is a Serre subcategory (Homology, Definition 12.10.1) of the category of all abelian groups. More generally, for any ring A, the category of torsion A-modules is a Serre subcategory of the category of all A-modules, see More on Algebra, Section 15.53. If A is a domain, then the quotient category (Homology, Lemma 12.10.6) is equivalent to the category of vector spaces over the fraction field. This follows from the following more general proposition.

0EA5 Proposition 110.76.1. Let A be a ring. Let S be a multiplicative subset of A. Let Mod_A denote the category of A-modules and $\mathcal{T}$ its Serre subcategory of modules for which any element is annihilated by some element of S. Then there is a canonical equivalence $\mathrm{Mod}_A/\mathcal{T} \to \mathrm{Mod}_{S^{-1}A}$.

Proof. The functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ given by $M \mapsto M \otimes_A S^{-1}A$ is exact (by Algebra, Proposition 10.9.12) and maps modules in $\mathcal{T}$ to zero. Thus, by the universal property given in Homology, Lemma 12.10.6, the functor descends to a functor $\mathrm{Mod}_A/\mathcal{T} \to \mathrm{Mod}_{S^{-1}A}$.

Conversely, any A-module M with $M \otimes_A S^{-1}A = 0$ is an object of $\mathcal{T}$, since $M \otimes_A S^{-1}A \cong S^{-1}M$ (Algebra, Lemma 10.12.15). Thus Homology, Lemma 12.10.7 shows that the functor $\mathrm{Mod}_A/\mathcal{T} \to \mathrm{Mod}_{S^{-1}A}$ is faithful.

Furthermore, this embedding is essentially surjective: a preimage to an $S^{-1}A$-module N is N_A, that is N regarded as an A-module, since the canonical map $N_A \otimes_A S^{-1}A \to N$ which maps $x \otimes a/s$ to $(a/s) \cdot x$ is an isomorphism of $S^{-1}A$-modules. □

0B0K Proposition 110.76.2. Let A be a ring. Let $Q(A)$ denote its total quotient ring (as in Algebra, Example 10.9.8). Let Mod_A denote the category of A-modules and $\mathcal{T}$ its Serre subcategory of torsion modules. Let $\mathrm{Mod}_{Q(A)}$ denote the category of $Q(A)$-modules. Then there is a canonical equivalence $\mathrm{Mod}_A/\mathcal{T} \to \mathrm{Mod}_{Q(A)}$.

Proof. Follows immediately from applying Proposition 110.76.1 to the multiplicative subset $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$, since a module is a torsion module if and only if all of its elements are each annihilated by some element of S. $\square$

0B0L Proposition 110.76.3. Let A be a Noetherian integral domain. Let K denote its field of fractions. Let Mod_A^{fg} denote the category of finitely generated A-modules and $\mathcal{T}^{fg}$ its Serre subcategory of finitely generated torsion modules. Then $\text{Mod}_A^{fg}/\mathcal{T}^{fg}$ is canonically equivalent to the category of finite dimensional K-vector spaces.

Proof. The equivalence given in Proposition 110.76.2 restricts along the embedding $\text{Mod}_A^{fg}/\mathcal{T}^{fg} \to \text{Mod}_A/\mathcal{T}$ to an equivalence $\text{Mod}_A^{fg}/\mathcal{T}^{fg} \to \text{Vect}_K^{fd}$. The Noetherian assumption guarantees that Mod_A^{fg} is an abelian category (see More on Algebra, Section 15.53) and that the canonical functor $\text{Mod}_A^{fg}/\mathcal{T}^{fg} \to \text{Mod}_A/\mathcal{T}$ is full (else torsion submodules of finitely generated modules might not be objects of $\mathcal{T}^{fg}$). $\square$

0B0M Proposition 110.76.4. The quotient of the category of abelian groups modulo its Serre subcategory of torsion groups is the category of $\mathbf{Q}$-vector spaces.

Proof. The claim follows directly from Proposition 110.76.2. $\square$

110.77. Different colimit topologies

0B2Y This example is [TSH98, Example 1.2, page 553]. Let $G_n = \mathbf{Q} \times \mathbf{R}^n$, $n \geq 1$ seen as a topological group for addition endowed with the usual (Euclidean) topology. Consider the closed embeddings $G_n \to G_{n+1}$ mapping $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, 0)$. We claim that $G = \text{colim}\, G_n$ endowed with the topology

$$U \subset G \text{ open} \Leftrightarrow G_n \cap U \text{ open } \forall n$$

is not a topological group.

To see this we consider the set

$$U = \{(x_0, x_1, x_2, \ldots) \text{ such that } |x_j| < |\cos(jx_0)| \text{ for } j > 0\}$$

Using that jx_0 is never an integral multiple of $\pi/2$ as π is not rational it is easy to show that $U \cap G_n$ is open. Since $0 \in U$, if the topology above made G into a topological group, then there would be an open neighbourhood $V \subset G$ of 0 such that $V + V \subset U$. Then, for every $j \geq 0$ there would exist $\epsilon_j > 0$ such that $(0, \ldots, 0, x_j, 0, \ldots) \in V$ for $|x_j| < \epsilon_j$. Since $V + V \subset U$ we would have

$$(x_0, 0, \ldots, 0, x_j, 0, \ldots) \in U$$

for $|x_0| < \epsilon_0$ and $|x_j| < \epsilon_j$. However, if we take j large enough such that $j\epsilon_0 > \pi/2$, then we can choose $x_0 \in \mathbf{Q}$ such that $|\cos(jx_0)|$ is smaller than ϵ_j, hence there exists an x_j with $|\cos(jx_0)| < |x_j| < \epsilon_j$. This contradiction proves the claim.

0B2Z Lemma 110.77.1. There exists a system $G_1 \to G_2 \to G_3 \to \ldots$ of (abelian) topological groups such that $\text{colim}\, G_n$ taken in the category of topological spaces is different from $\text{colim}\, G_n$ taken in the category of topological groups.

Proof. See discussion above. $\square$

110.78. Universally submersive but not V covering

0EU8 Let A be a valuation ring. Let $\mathfrak{p} \subset A$ be a prime ideal which is neither the minimal prime nor the maximal ideal. (A good case to keep in mind is when A has three prime ideals and $\mathfrak{p}$ is the one in the "middle".) Consider the morphism of affine schemes

$$\mathrm{Spec}(A_\mathfrak{p}) \amalg \mathrm{Spec}(A/\mathfrak{p}) \longrightarrow \mathrm{Spec}(A)$$

We claim this is universally submersive. In order to prove this, let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism of affine schemes given by the ring map $A \to B$. Then we have to show that

$$\mathrm{Spec}(B_\mathfrak{p}) \amalg \mathrm{Spec}(B/\mathfrak{p}B) \to \mathrm{Spec}(B)$$

is submersive. First of all it is surjective. Next, suppose that $T \subset \mathrm{Spec}(B)$ is a subset such that $T_1 = \mathrm{Spec}(B_\mathfrak{p}) \cap T$ and $T_2 = \mathrm{Spec}(B/\mathfrak{p}B) \cap T$ are closed. Then we see that T is the image of the spectrum of a B-algebra because both T_1 and T_2 are spectra of B-algebras. Hence to show that T is closed it suffices to show that T is stable under specialization, see Algebra, Lemma 10.41.5. To see this, suppose that $p \rightsquigarrow q$ is a specialization of points in $\mathrm{Spec}(B)$ with $p \in T$. Let A' be a valuation ring and let $\mathrm{Spec}(A') \to \mathrm{Spec}(B)$ be a morphism such that the generic point η of $\mathrm{Spec}(A')$ maps to p and the closed point s of $\mathrm{Spec}(A')$ maps to q, see Schemes, Lemma 26.20.4. Observe that the image of the composition $\gamma : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is exactly the set of points $\xi \in \mathrm{Spec}(A)$ with $\gamma(\eta) \rightsquigarrow \xi \rightsquigarrow \gamma(s)$ (details omitted). If $\mathfrak{p} \not\in \mathrm{Im}(\gamma)$, then we see that either both $p, q \in \mathrm{Spec}(B_\mathfrak{p})$ or both $p, q \in \mathrm{Spec}(B/\mathfrak{p}B)$. In this case the fact that T_1, resp. T_2 is closed implies that $q \in T_1$, resp. $q \in T_2$ and hence $q \in T$. Finally, suppose $\mathfrak{p} \in \mathrm{Im}(\gamma)$, say $\mathfrak{p} = \gamma(r)$. Then we have specializations $p \rightsquigarrow r$ and $r \rightsquigarrow q$. In this case $p, r \in \mathrm{Spec}(B_\mathfrak{p})$ and $r, q \in \mathrm{Spec}(B/\mathfrak{p}B)$. Then we fist conclude $r \in T_1 \subset T$, then $r \in T_2$ as r maps to $\mathfrak{p}$, and then $q \in T_2 \subset T$ as desired.

On the other hand, we claim that the singleton family

$$\{\mathrm{Spec}(A_\mathfrak{p}) \amalg \mathrm{Spec}(A/\mathfrak{p}) \longrightarrow \mathrm{Spec}(A)\}$$

is not a V covering. See Topologies, Definition 34.10.7. Namely, if it where a V covering, there would be an extension of valuation ring $A \subset B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(A_\mathfrak{p}) \amalg \mathrm{Spec}(A/\mathfrak{p})$. This would imply $\mathrm{Spec}(A')$ is disconnected which is absurd.

0EU9 **Lemma 110.78.1.** There exists a morphism $X \to Y$ of affine schemes which is universally submersive such that $\{X \to Y\}$ is not a V covering.

Proof. See discussion above. $\square$

110.79. The spectrum of the integers is not quasi-compact

0EUE Of course the title of this section doesn't refer to the spectrum of the integers as a topological space, because any spectrum is quasi-compact as a topological space (Algebra, Lemma 10.17.10). No, it refers to the spectrum of the integers in the canonical topology on the category of schemes, and the definition of a quasi-compact object in a site (Sites, Definition 7.17.1).

Let U be a nonprincipal ultrafilter on the set P of prime numbers. For a subset $T \subset P$ we denote $T^c = P \setminus T$ the complement. For $A \in U$ let $S_A \subset \mathbf{Z}$ be the multiplicative subset generated by $p \in A$. Set

$$\mathbf{Z}_A = S_A^{-1}\mathbf{Z}$$

Observe that $\text{Spec}(\mathbf{Z}_A) = \{(0)\} \cup A^c \subset \text{Spec}(\mathbf{Z})$ if we think of P as the set of closed points of $\text{Spec}(\mathbf{Z})$. If $A, B \in U$, then $A \cap B \in U$ and $A \cup B \in U$ and we have

$$\mathbf{Z}_{A\cap B} = \mathbf{Z}_A \times_{\mathbf{Z}_{A\cup B}} \mathbf{Z}_B$$

(fibre product of rings). In particular, for any integer n and elements $A_1, \ldots, A_n \in U$ the morphisms

$$\text{Spec}(\mathbf{Z}_{A_1}) \amalg \ldots \amalg \text{Spec}(\mathbf{Z}_{A_n}) \longrightarrow \text{Spec}(\mathbf{Z})$$

factors through $\text{Spec}(\mathbf{Z}[1/p])$ for some p (namely for any $p \in A_1 \cap \ldots \cap A_n$). We conclude that the family of flat morphisms $\{\text{Spec}(\mathbf{Z}_A) \to \text{Spec}(\mathbf{Z})\}_{A\in U}$ is jointly surjective, but no finite subset is.

For a $\mathbf{Z}$-module M we set

$$M_A = S_A^{-1}M = M \otimes_{\mathbf{Z}} \mathbf{Z}_A$$

Claim I: for every $\mathbf{Z}$-module M we have

$$M = \text{Equalizer}\left(\prod\nolimits_{A\in U} M_A \rightrightarrows \prod\nolimits_{A,B\in U} M_{A\cup B} \right)$$

First, assume M is torsion free. Then $M_A \subset M_P$ for all $A \in U$. Hence we see that we have to prove

$$M = \bigcap\nolimits_{A\in U} M_A \text{ inside } M_P = M \otimes \mathbf{Q}$$

Namely, since U is nonprincipal, for any prime p we have $\{p\}^c \in U$. Also, $M_{\{p\}^c} = M_{(p)}$ is equal to the localization at the prime (p). Thus the above is clear because already $M_{(2)} \cap M_{(3)} = M$. Next, assume M is torsion. Then we have

$$M = \bigoplus\nolimits_{p\in P} M[p^\infty]$$

and correspondingly we have

$$M_A = \bigoplus\nolimits_{p\not\in A} M[p^\infty]$$

because we are localizing at the primes in A. Suppose that $(x_A) \in \prod M_A$ is in the equalizer. Denote $x_p = x_{\{p\}^c} \in M[p^\infty]$. Then the equalizer property says

$$x_A = (x_p)_{p\not\in A}$$

and in particular it says that x_p is zero for all but a finite number of $p \not\in A$. To finish the proof in the torsion case it suffices to show that x_p is zero for all but a finite number of primes p. If not write $\{p \in P \mid x_p \neq 0\} = T \amalg T'$ as the disjoint union of two infinite sets. Then either $T \not\in U$ or $T' \not\in U$ because U is an ultrafilter (namely if both T, T' are in U then U contains $T \cap T' = \emptyset$ which is not allowed). Say $T \not\in U$. Then $T = A^c$ and this contradicts the finiteness mentioned above. Finally, suppose that M is a general module. Then we look at the short exact sequence

$$0 \to M_{tors} \to M \to M/M_{tors} \to 0$$

and we look at the following large diagram

$$\begin{tikzcd} M_{tors} \arrow[r] \arrow[d] & \prod_{A \in U} M_{tors, A} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{A, B \in U} M_{tors, A \cup B} \arrow[d] \\ M \arrow[r] \arrow[d] & \prod_{A \in U} M_A \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{A, B \in U} M_{A \cup B} \arrow[d] \\ M/M_{tors} \arrow[r] & \prod_{A \in U} (M/M_{tors})_A \arrow[r, shift left] \arrow[r, shift right] & \prod_{A, B \in U} (M/M_{tors})_{A \cup B} \end{tikzcd}$$

Doing a diagram chase using exactness of the columns and the result for the torsion module M_{tors} and the torsion free module M/M_{tors} proving Claim I for M. This gives an example of the phenomenon in the following lemma.

0EUF **Lemma 110.79.1.** There exists a ring A and an infinite family of flat ring maps $\{A \to A_i\}_{i \in I}$ such that for every A-module M

$$M = \text{Equalizer}\left(\prod_{i \in I} M \otimes_A A_i \rightrightarrows \prod_{i, j \in I} M \otimes_A A_i \otimes_A A_j \right)$$

but there is no finite subfamily where the same thing is true.

Proof. See discussion above. $\square$

We continue working with our nonprincipal ultrafilter U on the set P of prime numbers. Let R be a ring. Denote $R_A = S_A^{-1}R = R \otimes \mathbf{Z}_A$ for $A \in U$. Claim II: given closed subsets $T_A \subset \text{Spec}(R_A)$, $A \in U$ such that

$$(\text{Spec}(R_{A \cup B}) \to \text{Spec}(R_A))^{-1}T_A = (\text{Spec}(R_{A \cup B}) \to \text{Spec}(R_B))^{-1}T_B$$

for all $A, B \in U$, there is a closed subset $T \subset \text{Spec}(R)$ with $T_A = (\text{Spec}(R_A) \to \text{Spec}(R))^{-1}(T)$ for all $A \in U$. Let $I_A \subset R_A$ for $A \in U$ be the radical ideal cutting out T_A. Then the glueing condition implies $S_{A \cup B}^{-1} I_A = S_{A \cup B}^{-1} I_B$ in $R_{A \cup B}$ for all $A, B \in U$ (because localization preserves being a radical ideal). Let $I' \subset R$ be the set of elements mapping into $I_P \subset R_P = R \otimes \mathbf{Q}$. Then we see for $A \in U$ that

1. $I_A \subset I'_A = S_A^{-1} I'$, and
2. $M_A = I'_A / I_A$ is a torsion module.

Of course we obtain canonical identifications $S_{A \cup B}^{-1} M_A = S_{A \cup B}^{-1} M_B$ for $A, B \in U$. Decomposing the torsion modules M_A into their p-primary components, the reader easily shows that there exist p-power torsion R-modules M_p such that

$$M_A = \bigoplus\nolimits_{p \notin A} M_p$$

compatible with the canonical identifications given above. Setting $M = \bigoplus_{p \in P} M_p$ we find canonical isomorphisms $M_A = S_A^{-1} M$ compatible with the above canonical identifications. Then we get a canonical map

$$I' \longrightarrow M$$

of R-modules wich recovers the map $I_A \to M_A$ for all $A \in U$. This is true by all the compatibilities mentioned above and the claim proved previously that M is the equalizer of the two maps from $\prod_{A \in U} M_A$ to $\prod_{A, B \in U} M_{A \cup B}$. Let $I = \text{Ker}(I' \to M)$. Then I is an ideal and $T = V(I)$ is a closed subset which recovers the closed subsets T_A for all $A \in U$. This proves Claim II.

0EUG Lemma 110.79.2. The scheme $\text{Spec}(\mathbf{Z})$ is not quasi-compact in the canonical topology on the category of schemes.

Proof. With notation as above consider the family of morphisms

$$\mathcal{W} = \{\text{Spec}(\mathbf{Z}_A) \to \text{Spec}(\mathbf{Z})\}_{A \in U}$$

By Descent, Lemma 35.13.5 and the two claims proved above this is a universal effective epimorphism. In any category with fibre products, the universal effective epimorphisms give $\mathcal{C}$ the structure of a site (modulo some set theoretical issues which are easy to fix) defining the canonical topology. Thus $\mathcal{W}$ is a covering for the canonical topology. On the other hand, we have seen above that any finite subfamily

$$\{\text{Spec}(\mathbf{Z}_{A_i}) \to \text{Spec}(\mathbf{Z})\}_{i=1,\ldots,n}, \quad n \in \mathbf{N}, A_1, \ldots, A_n \in U$$

factors through $\text{Spec}(\mathbf{Z}[1/p])$ for some p. Hence this finite family cannot be a universal effective epimorphism and more generally no universal effective epimorphism $\{g_j : T_j \to \text{Spec}(\mathbf{Z})\}$ can refine $\{\text{Spec}(\mathbf{Z}_{A_i}) \to \text{Spec}(\mathbf{Z})\}_{i=1,\ldots,n}$. By Sites, Definition 7.17.1 this means that $\text{Spec}(\mathbf{Z})$ is not quasi-compact in the canonical topology. To see that our notion of quasi-compactness agrees with the usual topos theoretic definition, see Sites, Lemma 7.17.3. □

110.80. Other chapters

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CHAPTER 111

Exercises

0275 ## 111.1. Algebra

0276 This first section just contains some assorted questions.

02CG Exercise 111.1.1. Let A be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X - 1)$. Show that

$$A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}.$$

02CH Exercise 111.1.2. Find an example of a non Noetherian ring R such that every finitely generated ideal of R is finitely presented as an R-module. (A ring is said to be coherent if the last property holds.)

02CI Exercise 111.1.3. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring. For any finite A-module M define $r(M)$ to be the minimum number of generators of M as an A-module. This number equals $\dim_k M/\mathfrak{m}M = \dim_k M \otimes_A k$ by NAK.

(1) Show that $r(M \otimes_A N) = r(M)r(N)$.
(2) Let $I \subset A$ be an ideal with $r(I) > 1$. Show that $r(I^2) < r(I)^2$.
(3) Conclude that if every ideal in A is a flat module, then A is a PID (or a field).

02CJ Exercise 111.1.4. Let k be a field. Show that the following pairs of k-algebras are not isomorphic:

(1) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
(2) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
(3) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

02CK Remark 111.1.5. Of course the idea of this exercise is to find a simple argument in each case rather than applying a “big” theorem. Nonetheless it is good to be guided by general principles.

02CL Exercise 111.1.6. Algebra. (Silly and should be easy.)

(1) Give an example of a ring A and a nonsplit short exact sequence of A-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0.$$
(2) Give an example of a nonsplit sequence of A-modules as above and a faithfully flat $A \to B$ such that
$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0.$$
is split as a sequence of B-modules.

02CM Exercise 111.1.7. Suppose that k is a field having a primitive nth root of unity ζ. This means that $\zeta^n = 1$, but $\zeta^m \neq 1$ for $0 < m < n$.

(1) Show that the characteristic of k is prime to n.
(2) Suppose that $a \in k$ is an element of k which is not an dth power in k for any divisor d of n for $n \geq d > 1$. Show that $k[x]/(x^n - a)$ is a field. (Hint: Consider a splitting field for $x^n - a$ and use Galois theory.)

02CN Exercise 111.1.8. Let $\nu : k[x] \setminus \{0\} \to \mathbf{Z}$ be a map with the following properties: $\nu(fg) = \nu(f) + \nu(g)$ whenever f, g not zero, and $\nu(f+g) \geq min(\nu(f), \nu(g))$ whenever f, g, $f+g$ are not zero, and $\nu(c) = 0$ for all $c \in k^*$.

(1) Show that if f, g, and $f+g$ are nonzero and $\nu(f) \neq \nu(g)$ then we have equality $\nu(f+g) = min(\nu(f), \nu(g))$.
(2) Show that if $f = \sum a_i x^i$, $f \neq 0$, then $\nu(f) \geq min(\{i\nu(x)\}_{a_i \neq 0})$. When does equality hold?
(3) Show that if ν attains a negative value then $\nu(f) = -n \deg(f)$ for some $n \in \mathbf{N}$.
(4) Suppose $\nu(x) \geq 0$. Show that $\{f \mid f = 0, \text{ } or \text{ } \nu(f) > 0\}$ is a prime ideal of $k[x]$.
(5) Describe all possible ν.

Let A be a ring. An idempotent is an element $e \in A$ such that $e^2 = e$. The elements 1 and 0 are always idempotent. A nontrivial idempotent is an idempotent which is not equal to zero. Two idempotents $e, e' \in A$ are called orthogonal if $ee' = 0$.

078G Exercise 111.1.9. Let A be a ring. Show that A is a product of two nonzero rings if and only if A has a nontrivial idempotent.

078H Exercise 111.1.10. Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Show that the map $A \to A/I$ induces a bijection on idempotents. (Hint: It may be easier to prove this when I is nilpotent. Do this first. Then use "absolute Noetherian reduction" to reduce to the nilpotent case.)

111.2. Colimits

0277

078I Definition 111.2.1. A directed set is a nonempty set I endowed with a preorder $\leq$ such that given any pair $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. A system of rings over I is given by a ring A_i for each $i \in I$ and a map of rings $\varphi_{ij} : A_i \to A_j$ whenever $i \leq j$ such that the composition $A_i \to A_j \to A_k$ is equal to $A_i \to A_k$ whenever $i \leq j \leq k$.

One similarly defines systems of groups, modules over a fixed ring, vector spaces over a field, etc.

078J Exercise 111.2.2. Let I be a directed set and let (A_i, φ_{ij}) be a system of rings over I. Show that there exists a ring A and maps $\varphi_i : A_i \to A$ such that $\varphi_j \circ \varphi_{ij} = \varphi_i$ for all $i \leq j$ with the following universal property: Given any ring B and maps $\psi_i : A_i \to B$ such that $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j$, then there exists a unique ring map $\psi : A \to B$ such that $\psi_i = \psi \circ \varphi_i$.

078K Definition 111.2.3. The ring A constructed in Exercise 111.2.2 is called the colimit of the system. Notation $\operatorname{colim} A_i$.

078L Exercise 111.2.4. Let $(I, \geq)$ be a directed set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Prove that there is a bijection

$$\operatorname{Spec}(A) = \{(\mathfrak{p}_i)_{i \in I} \mid \mathfrak{p}_i \subset A_i \text{ and } \mathfrak{p}_i = \varphi_{ij}^{-1}(\mathfrak{p}_j) \ \forall i \leq j\} \subset \prod\nolimits_{i \in I} \operatorname{Spec}(A_i)$$

The set on the right hand side of the equality is the limit of the sets $\mathrm{Spec}(A_i)$. Notation $\lim \mathrm{Spec}(A_i)$.

078M Exercise 111.2.5. Let $(I, \geq)$ be a directed set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Suppose that $\mathrm{Spec}(A_j) \to \mathrm{Spec}(A_i)$ is surjective for all $i \leq j$. Show that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ is surjective for all i. (Hint: You can try to use Tychonoff, but there is also a basically trivial direct algebraic proof based on Algebra, Lemma 10.17.9.)

078N Exercise 111.2.6. Let $A \subset B$ be an integral ring extension. Prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Use the exercises above, the fact that this holds for a finite ring extension (proved in the lectures), and by proving that $B = \mathrm{colim}\, B_i$ is a directed colimit of finite extensions $A \subset B_i$.

02CO Exercise 111.2.7. Let $(I, \geq)$ be a directed set. Let A be a ring and let $(N_i, \varphi_{i,i'})$ be a directed system of A-modules indexed by I. Suppose that M is another A-module. Prove that

$$\mathrm{colim}_{i \in I}\, M \otimes_A N_i \cong M \otimes_A \Big(\mathrm{colim}_{i \in I}\, N_i \Big).$$

0278 Definition 111.2.8. A module M over R is said to be of finite presentation over R if it is isomorphic to the cokernel of a map of finite free modules $R^{\oplus n} \to R^{\oplus m}$.

02CP Exercise 111.2.9. Prove that any module over any ring is

(1) the colimit of its finitely generated submodules, and
(2) in some way a colimit of finitely presented modules.

111.3. Additive and abelian categories

057X

057Y Exercise 111.3.1. Let k be a field. Let $\mathcal{C}$ be the category of filtered vector spaces over k, see Homology, Definition 12.19.1 for the definition of a filtered object of any category.

(1) Show that this is an additive category (explain carefuly what the direct sum of two objects is).
(2) Let $f : (V, F) \to (W, F)$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (explain precisely what the kernel and cokernel of f are).
(3) Give an example of a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

057Z Exercise 111.3.2. Let R be a Noetherian domain. Let $\mathcal{C}$ be the category of finitely generated torsion free R-modules.

(1) Show that this is an additive category.
(2) Let $f : N \to M$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (make sure you define precisely what the kernel and cokernel of f are).
(3) Give an example of a Noetherian domain R and a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

0580 Exercise 111.3.3. Give an example of a category which is additive and has kernels and cokernels but which is not as in Exercises 111.3.1 and 111.3.2.

111.4. Tensor product

0CYG Tensor products are introduced in Algebra, Section 10.12. Let R be a ring. Let Mod_R be the category of R-modules. We will say that a functor $F : \text{Mod}_R \to \text{Mod}_R$

(1) is additive if $F : \text{Hom}_R(M, N) \to \text{Hom}_R(F(M), F(N))$ is a homomorphism of abelian groups for any R-modules M, N, see Homology, Definition 12.3.1.
(2) R-linear if $F : \text{Hom}_R(M, N) \to \text{Hom}_R(F(M), F(N))$ is R-linear for any R-modules M, N,
(3) right exact if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $F(M_1) \to F(M_2) \to F(M_3) \to 0$ is exact,
(4) left exact if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $0 \to F(M_1) \to F(M_2) \to F(M_3)$ is exact,
(5) commutes with direct sums, if given a set I and R-modules M_i the maps $F(M_i) \to F(\bigoplus M_i)$ induce an isomorphism $\bigoplus F(M_i) = F(\bigoplus M_i)$.

0CYH Exercise 111.4.1. Let R be a ring. With notation as above.

(1) Give an example of a ring R and an additive functor $F : \text{Mod}_R \to \text{Mod}_R$ which is not R-linear.
(2) Let N be an R-module. Show that the functor $F(M) = M \otimes_R N$ is R-linear, right exact, and commutes with direct sums,
(3) Conversely, show that any functor $F : \text{Mod}_R \to \text{Mod}_R$ which is R-linear, right exact, and commutes with direct sums is of the form $F(M) = M \otimes_R N$ for some R-module N.
(4) Show that if in (3) we drop the assumption that F commutes with direct sums, then the conclusion no longer holds.

111.5. Flat ring maps

0279

02CQ Exercise 111.5.1. Let S be a multiplicative subset of the ring A.

(1) For an A-module M show that $S^{-1}M = S^{-1}A \otimes_A M$.
(2) Show that $S^{-1}A$ is flat over A.

02CR Exercise 111.5.2. Find an injection $M_1 \to M_2$ of A-modules such that $M_1 \otimes N \to M_2 \otimes N$ is not injective in the following cases:

(1) $A = k[x, y]$ and $N = (x, y) \subset A$. (Here and below k is a field.)
(2) $A = k[x, y]$ and $N = A/(x, y)$.

02CS Exercise 111.5.3. Give an example of a ring A and a finite A-module M which is a flat but not a projective A-module.

02CT Remark 111.5.4. If M is of finite presentation and flat over A, then M is projective over A. Thus your example will have to involve a ring A which is not Noetherian. I know of an example where A is the ring of $\mathcal{C}^\infty$-functions on $\mathbf{R}$.

02CU Exercise 111.5.5. Find a flat but not free module over $\mathbf{Z}_{(2)}$.

02CV Exercise 111.5.6. Flat deformations.

(1) Suppose that k is a field and $k[\epsilon]$ is the ring of dual numbers $k[\epsilon] = k[x]/(x^2)$ and $\epsilon = \bar{x}$. Show that for any k-algebra A there is a flat $k[\epsilon]$-algebra B such that A is isomorphic to $B/\epsilon B$.

(2) Suppose that $k = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and
$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^p, x_2^p, x_3^p, x_4^p, x_5^p, x_6^p).$$
Show that there exists a flat $\mathbf{Z}/p^2\mathbf{Z}$-algebra B such that B/pB is isomorphic to A. (So here p plays the role of ϵ.)

(3) Now let $p = 2$ and consider the same question for $k = \mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ and
$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2 + x_3x_4 + x_5x_6).$$
However, in this case show that there does not exist a flat $\mathbf{Z}/4\mathbf{Z}$-algebra B such that $B/2B$ is isomorphic to A. (Find the trick! The same example works in arbitrary characteristic $p > 0$, except that the computation is more difficult.)

02CW Exercise 111.5.7. Let $(A, \mathfrak{m}, k)$ be a local ring and let k'/k be a finite field extension. Show there exists a flat, local map of local rings $A \to B$ such that $\mathfrak{m}_B = \mathfrak{m}B$ and $B/\mathfrak{m}B$ is isomorphic to k' as k-algebra. (Hint: first do the case where $k \subset k'$ is generated by a single element.)

02CX Remark 111.5.8. The same result holds for arbitrary field extensions K/k.

111.6. The Spectrum of a ring

027A

02CY Exercise 111.6.1. Compute $\mathrm{Spec}(\mathbf{Z})$ as a set and describe its topology.

02CZ Exercise 111.6.2. Let A be any ring. For $f \in A$ we define $D(f) := \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}$. Prove that the open subsets $D(f)$ form a basis of the topology of $\mathrm{Spec}(A)$.

02D0 Exercise 111.6.3. Prove that the map $I \mapsto V(I)$ defines a natural bijection
$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset \mathrm{Spec}(A) \text{ closed}\}$$

027B Definition 111.6.4. A topological space X is called quasi-compact if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $X = U_{i_1} \cup \ldots U_{i_n}$.

02D1 Exercise 111.6.5. Prove that $\mathrm{Spec}(A)$ is quasi-compact for any ring A.

027C Definition 111.6.6. A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ there is an open subset of X containing one but not the other. We say that X is Hausdorff if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

02D2 Exercise 111.6.7. Show that $\mathrm{Spec}(A)$ is not Hausdorff in general. Prove that $\mathrm{Spec}(A)$ is T_0. Give an example of a topological space X that is not T_0.

02D3 Remark 111.6.8. Usually the word compact is reserved for quasi-compact and Hausdorff spaces.

027D Definition 111.6.9. A topological space X is called irreducible if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called irreducible if it is an irreducible topological space with the topology induced from X. This definition implies T is irreducible if and only if the closure $\bar{T}$ of T in X is irreducible.

02D4 Exercise 111.6.10. Prove that Spec(A) is irreducible if and only if $Nil(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A.

02D5 Exercise 111.6.11. Prove that a closed subset $T \subset \text{Spec}(A)$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

027E Definition 111.6.12. A point x of an irreducible topological space X is called a generic point of X if X is equal to the closure of the subset $\{x\}$.

02D6 Exercise 111.6.13. Show that in a T_0 space X every irreducible closed subset has at most one generic point.

02D7 Exercise 111.6.14. Prove that in Spec(A) every irreducible closed subset does have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of Spec(A) with the set of irreducible closed subsets of X.

02D8 Exercise 111.6.15. Give an example to show that an irreducible subset of Spec($\mathbf{Z}$) does not necessarily have a generic point.

027F Definition 111.6.16. A topological space X is called Noetherian if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of X stabilizes. (It is called Artinian if any increasing sequence of closed subsets stabilizes.)

02D9 Exercise 111.6.17. Show that if the ring A is Noetherian then the topological space Spec(A) is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

027G Definition 111.6.18. A maximal irreducible subset $T \subset X$ is called an irreducible component of the space X. Such an irreducible component of X is automatically a closed subset of X.

02DA Exercise 111.6.19. Prove that any irreducible subset of X is contained in an irreducible component of X.

02DB Exercise 111.6.20. Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbf{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets. This is not terribly interesting.)

02DC Exercise 111.6.21. Show that irreducible components of Spec(A) correspond to minimal primes of A.

027H Definition 111.6.22. A point $x \in X$ is called closed if $\overline{\{x\}} = \{x\}$. Let x, y be points of X. We say that x is a specialization of y, or that y is a generalization of x if $x \in \overline{\{y\}}$.

02DD Exercise 111.6.23. Show that closed points of Spec(A) correspond to maximal ideals of A.

02DE Exercise 111.6.24. Show that $\mathfrak{p}$ is a generalization of $\mathfrak{q}$ in Spec(A) if and only if $\mathfrak{p} \subset \mathfrak{q}$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space X is called a generic point if it is a generic points of one of the irreducible components of X.)

02DF Exercise 111.6.25. Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

027I Definition 111.6.26. A topological space X is called connected if it is nonempty and not the union of two nonempty disjoint open subsets. A connected component of X is a maximal connected subset. Any point of X is contained in a connected component of X and any connected component of X is closed in X. (But in general a connected component need not be open in X.)

02DG Exercise 111.6.27. Let A be a nonzero ring. Show that $\text{Spec}(A)$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C.

02DH Exercise 111.6.28. Let T be a connected component of $\text{Spec}(A)$. Prove that T is stable under generalization. Prove that T is an open subset of $\text{Spec}(A)$ if A is Noetherian. (Remark: This is wrong when A is an infinite product of copies of $\mathbf{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

02DI Exercise 111.6.29. Compute $\text{Spec}(k[x])$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when k is algebraically closed but also when k is not.)

02DJ Exercise 111.6.30. Compute $\text{Spec}(k[x, y])$, where k is algebraically closed. [Hint: use the morphism $\varphi : \text{Spec}(k[x, y]) \to \text{Spec}(k[x])$; if $\varphi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think algebraic geometers call this affine 2-space?)

02DK Exercise 111.6.31. Compute $\text{Spec}(\mathbf{Z}[y])$. [Hint: as above.] (Affine 1-space over $\mathbf{Z}$.)

111.7. Localization

0766

0767 Exercise 111.7.1. Let A be a ring. Let $S \subset A$ be a multiplicative subset. Let M be an A-module. Let $N \subset S^{-1}M$ be an $S^{-1}A$-submodule. Show that there exists an A-submodule $N' \subset M$ such that $N = S^{-1}N'$. (This useful result applies in particular to ideals of $S^{-1}A$.)

0768 Exercise 111.7.2. Let A be a ring. Let M be an A-module. Let $m \in M$.

(1) Show that $I = \{a \in A \mid am = 0\}$ is an ideal of A.
(2) For a prime $\mathfrak{p}$ of A show that the image of m in $M_\mathfrak{p}$ is zero if and only if $I \not\subset \mathfrak{p}$.
(3) Show that m is zero if and only if the image of m is zero in $M_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A.
(4) Show that m is zero if and only if the image of m is zero in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A.
(5) Show that $M = 0$ if and only if $M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$.

0769 Exercise 111.7.3. Find a pair (A, f) where A is a domain with three or more pairwise distinct primes and $f \in A$ is an element such that the principal localization $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is a field.

076A Exercise 111.7.4. Let A be a ring. Let M be a finite A-module. Let $S \subset A$ be a multiplicative set. Assume that $S^{-1}M = 0$. Show that there exists an $f \in S$ such that the principal localization $M_f = \{1, f, f^2, \ldots\}^{-1}M$ is zero.

076B Exercise 111.7.5. Give an example of a triple (A, I, S) where A is a ring, $0 \neq I \neq A$ is a proper nonzero ideal, and $S \subset A$ is a multiplicative subset such that $A/I \cong S^{-1}A$ as A-algebras.

111.8. Nakayama's Lemma

076C

076D Exercise 111.8.1. Let A be a ring. Let I be an ideal of A. Let M be an A-module. Let $x_1, \ldots, x_n \in M$. Assume that

(1) M/IM is generated by $x_1, \ldots, x_n$,
(2) M is a finite A-module,
(3) I is contained in every maximal ideal of A.

Show that $x_1, \ldots, x_n$ generate M. (Suggested solution: Reduce to a localization at a maximal ideal of A using Exercise 111.7.2 and exactness of localization. Then reduce to the statement of Nakayama's lemma in the lectures by looking at the quotient of M by the submodule generated by $x_1, \ldots, x_n$.)

111.9. Length

027J

076E Definition 111.9.1. Let A be a ring. Let M be an A-module. The length of M as an R-module is

$$\text{length}_A(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words, the supremum of the lengths of chains of submodules.

076F Exercise 111.9.2. Show that a module M over a ring A has length 1 if and only if it is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

076G Exercise 111.9.3. Compute the length of the following modules over the following rings. Briefly(!) explain your answer. (Please feel free to use additivity of the length function in short exact sequences, see Algebra, Lemma 10.52.3).

(1) The length of $\mathbf{Z}/120\mathbf{Z}$ over $\mathbf{Z}$.
(2) The length of $\mathbf{C}[x]/(x^{100} + x + 1)$ over $\mathbf{C}[x]$.
(3) The length of $\mathbf{R}[x]/(x^4 + 2x^2 + 1)$ over $\mathbf{R}[x]$.

02DL Exercise 111.9.4. Let $A = k[x, y]_{(x,y)}$ be the local ring of the affine plane at the origin. Make any assumption you like about the field k. Suppose that $f = x^3 + x^2y^2 + y^{100}$ and $g = y^3 - x^{999}$. What is the length of $A/(f, g)$ as an A-module? (Possible way to proceed: think about the ideal that f and g generate in quotients of the form $A/\mathfrak{m}_A^n = k[x, y]/(x, y)^n$ for varying n. Try to find n such that $A/(f, g) + \mathfrak{m}_A^n \cong A/(f, g) + \mathfrak{m}_A^{n+1}$ and use NAK.)

111.10. Associated primes

0CR7 Associated primes are discussed in Algebra, Section 10.63

0CR8 Exercise 111.10.1. Compute the set of associated primes for each of the following modules.

(1) $R = k[x, y]$ and $M = R/(xy(x + y))$,
(2) $R = \mathbf{Z}[x]$ and $M = R/(300x + 75)$, and
(3) $R = k[x, y, z]$ and $M = R/(x^3, x^2y, xz)$.

Here as usual k is a field.

0CR9 Exercise 111.10.2. Give an example of a Noetherian ring R and a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}$ is not the only associated prime of $R/\mathfrak{p}^2$.

0CRA Exercise 111.10.3. Let R be a Noetherian ring with incomparable prime ideals $\mathfrak{p}$, $\mathfrak{q}$, i.e., $\mathfrak{p} \not\subset \mathfrak{q}$ and $\mathfrak{q} \not\subset \mathfrak{p}$.

(1) Show that for $N = R/(\mathfrak{p} \cap \mathfrak{q})$ we have $\text{Ass}(N) = \{\mathfrak{p}, \mathfrak{q}\}$.
(2) Show by an example that the module $M = R/\mathfrak{p}\mathfrak{q}$ can have an associated prime not equal to $\mathfrak{p}$ or $\mathfrak{q}$.

111.11. Ext groups

0CRB Ext groups are defined in Algebra, Section 10.71.

0CRC Exercise 111.11.1. Compute all the Ext groups $\text{Ext}^i(M, N)$ of the given modules in the category of $\mathbf{Z}$-modules (also known as the category of abelian groups).

(1) $M = \mathbf{Z}$ and $N = \mathbf{Z}$,
(2) $M = \mathbf{Z}/4\mathbf{Z}$ and $N = \mathbf{Z}/8\mathbf{Z}$,
(3) $M = \mathbf{Q}$ and $N = \mathbf{Z}/2\mathbf{Z}$, and
(4) $M = \mathbf{Z}/2\mathbf{Z}$ and $N = \mathbf{Q}/\mathbf{Z}$.

0CRD Exercise 111.11.2. Let $R = k[x, y]$ where k is a field.

(1) Show by hand that the Koszul complex
$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^{\oplus 2} \xrightarrow{(x,y)} R \xrightarrow{f \mapsto f(0,0)} k \to 0$$
is exact.
(2) Compute $\text{Ext}^i_R(k, k)$ where $k = R/(x, y)$ as an R-module.

0CRE Exercise 111.11.3. Give an example of a Noetherian ring R and finite modules M, N such that $\text{Ext}^i_R(M, N)$ is nonzero for all $i \geq 0$.

0CRF Exercise 111.11.4. Give an example of a ring R and ideal I such that $\text{Ext}^1_R(R/I, R/I)$ is not a finite R-module. (We know this cannot happen if R is Noetherian by Algebra, Lemma 10.71.9.)

111.12. Depth

0CS0 Depth is defined in Algebra, Section 10.72 and further studied in Dualizing Complexes, Section 47.11.

0CS1 Exercise 111.12.1. Let R be a ring, $I \subset R$ an ideal, and M an R-module. Compute $\text{depth}_I(M)$ in the following cases.

(1) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}$,
(2) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}/(300)$,
(3) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}/(7)$,
(4) $R = k[x, y, z]/(x^2 + y^2 + z^2)$, $I = (x, y, z)$, $M = R$,
(5) $R = k[x, y, z, w]/(xz, xw, yz, yw)$, $I = (x, y, z, w)$, $M = R$.

Here k is a field. In the last two cases feel free to localize at the maximal ideal I.

0CS2 Exercise 111.12.2. Give an example of a Noetherian local ring $(R, \mathfrak{m}, \kappa)$ of depth ≥ 1 and a prime ideal $\mathfrak{p}$ such that

(1) $\text{depth}_\mathfrak{m}(R) \geq 1$,
(2) $\text{depth}_\mathfrak{p}(R_\mathfrak{p}) = 0$, and
(3) $\dim(R_\mathfrak{p}) \geq 1$.

If we don't ask for (3) then the exercise is too easy. Why?

0CS3 Exercise 111.12.3. Let $(R, \mathfrak{m})$ be a local Noetherian domain. Let M be a finite R-module.

(1) If M is torsion free, show that M has depth at least 1 over R.
(2) Give an example with depth equal to 1.

0CS4 Exercise 111.12.4. For every $m \geq n \geq 0$ give an example of a Noetherian local ring R with $\dim(R) = m$ and $\text{depth}(R) = n$.

0CSZ Exercise 111.12.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite R-module. Show that there exists a canonical short exact sequence

$$0 \to K \to M \to Q \to 0$$

such that the following are true

(1) $\text{depth}(Q) \geq 1$,
(2) K is zero or $\text{Supp}(K) = \{\mathfrak{m}\}$, and
(3) $\text{length}_R(K) < \infty$.

Hint: using the Noetherian property show that there exists a maximal submodule K as in (2) and then show that $Q = M/K$ satisfies (1) and K satisfies (3).

0CT0 Exercise 111.12.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite R-module of depth ≥ 2. Let $N \subset M$ be a nonzero submodule.

(1) Show that $\text{depth}(N) \geq 1$.
(2) Show that $\text{depth}(N) = 1$ if and only if the quotient module M/N has $\text{depth}(M/N) = 0$.
(3) Show there exists a submodule $N' \subset M$ with $N \subset N'$ of finite colength, i.e., $\text{length}_R(N'/N) < \infty$, such that N' has depth ≥ 2. Hint: Apply Exercise 111.12.5 to M/N and choose N' to be the inverse image of K.

0CT1 Exercise 111.12.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Assume that R is reduced, i.e., R has no nonzero nilpotent elements. Assume moreover that R has two distinct minimal primes $\mathfrak{p}$ and $\mathfrak{q}$.

(1) Show that the sequence of R-modules
$$0 \to R \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/\mathfrak{p} + \mathfrak{q} \to 0$$
is exact (check at all the spots). The maps are $x \mapsto (x \bmod \mathfrak{p}, x \bmod \mathfrak{q})$ and $(y \bmod \mathfrak{p}, z \bmod \mathfrak{q}) \mapsto (y - z \bmod \mathfrak{p} + \mathfrak{q})$.
(2) Show that if $\text{depth}(R) \geq 2$, then $\dim(R/\mathfrak{p} + \mathfrak{q}) \geq 1$.
(3) Show that if $\text{depth}(R) \geq 2$, then $U = \text{Spec}(R) \setminus \{\mathfrak{m}\}$ is a connected topological space.

This proves a very special case of Hartshorne's connectedness theorem which says that the punctured spectrum U of a local Noetherian ring of depth ≥ 2 is connected.

0CT2 Exercise 111.12.8. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $x, y \in \mathfrak{m}$ be a regular sequence of length 2. For any $n \geq 2$ show that there do not exist $a, b \in R$ with

$$x^{n-1}y^{n-1} = ax^n + by^n$$

Suggestion: First try for $n = 2$ to see how to argue. Remark: There is a vast generalization of this result called the monomial conjecture.

111.13. Cohen-Macaulay modules and rings

0CT3 Cohen-Macaulay modules are studied in Algebra, Section 10.103 and Cohen-Macaulay rings are studied in Algebra, Section 10.104.

0CT4 Exercise 111.13.1. In the following cases, please answer yes or no. No explanation or proof necessary.

(1) Let p be a prime number. Is the local ring $\mathbf{Z}_{(p)}$ a Cohen-Macaulay local ring?
(2) Let p be a prime number. Is the local ring $\mathbf{Z}_{(p)}$ a regular local ring?
(3) Let k be a field. Is the local ring $k[x]_{(x)}$ a Cohen-Macaulay local ring?
(4) Let k be a field. Is the local ring $k[x]_{(x)}$ a regular local ring?
(5) Let k be a field. Is the local ring $(k[x,y]/(y^2-x^3))_{(x,y)} = k[x,y]_{(x,y)}/(y^2-x^3)$ a Cohen-Macaulay local ring?
(6) Let k be a field. Is the local ring $(k[x,y]/(y^2,xy))_{(x,y)} = k[x,y]_{(x,y)}/(y^2,xy)$ a Cohen-Macaulay local ring?

111.14. Singularities

027K

02DM Exercise 111.14.1. Let k be any field. Suppose that $A = k[[x,y]]/(f)$ and $B = k[[u,v]]/(g)$, where $f = xy$ and $g = uv + \delta$ with $\delta \in (u,v)^3$. Show that A and B are isomorphic rings.

02DN Remark 111.14.2. A singularity on a curve over a field k is called an ordinary double point if the complete local ring of the curve at the point is of the form $k'[[x,y]]/(f)$, where (a) k' is a finite separable extension of k, (b) the initial term of f has degree two, i.e., it looks like $q = ax^2 + bxy + cy^2$ for some $a,b,c \in k'$ not all zero, and (c) q is a nondegenerate quadratic form over k' (in char 2 this means that b is not zero). In general there is one isomorphism class of such rings for each isomorphism class of pairs (k', q).

0D1S Exercise 111.14.3. Let R be a ring. Let $n \geq 1$. Let A, B be $n \times n$ matrices with coefficients in R such that $AB = f1_{n\times n}$ for some nonzerodivisor f in R. Set $S = R/(f)$. Show that

$$\ldots \to S^{\oplus n} \xrightarrow{B} S^{\oplus n} \xrightarrow{A} S^{\oplus n} \xrightarrow{B} S^{\oplus n} \to \ldots$$

is exact.

111.15. Constructible sets

0FJ3 Let k be an algebraically closed field, for example the field $\mathbf{C}$ of complex numbers. Let $n \geq 0$. A polynomial $f \in k[x_1, \ldots, x_n]$ gives a function $f : k^n \to k$ by evaluation. A subset $Z \subset k^n$ is called an algebraic set if it is the common vanishing set of a collection of polynomials.

0FJ4 Exercise 111.15.1. Prove that an algebraic set can always be written as the zero locus of finitely many polynomials.

With notation as above a subset $E \subset k^n$ is called constructible if it is a finite union of sets of the form $Z \cap \{f \neq 0\}$ where f is a polynomial.

0FJ5 Exercise 111.15.2. Show the following

(1) the complement of a constructible set is a constructible set,
(2) a finite union of constructible sets is a constructible set,
(3) a finite intersection of constructible sets is a constructible set, and
(4) any constructible set E can be written as a finite disjoint union $E = \coprod E_i$ with each E_i of the form $Z \cap \{f \neq 0\}$ where Z is an algebraic set and f is a polynomial.

0FJ6 Exercise 111.15.3. Let R be a ring. Let $f = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in R[x]$. (As usual this notation means $a_0, \ldots, a_d \in R$.) Let $g \in R[x]$. Prove that we can find $N \geq 0$ and $r, q \in R[x]$ such that

$$a_d^N g = qf + r$$

with $\deg(r) < d$, i.e., for some $c_i \in R$ we have $r = c_0 + c_1 x + \ldots + c_{d-1}x^{d-1}$.

111.16. Hilbert Nullstellensatz

027L

02DO Exercise 111.16.1. A silly argument using the complex numbers! Let $\mathbf{C}$ be the complex number field. Let V be a vector space over $\mathbf{C}$. The spectrum of a linear operator $T : V \to V$ is the set of complex numbers $\lambda \in \mathbf{C}$ such that the operator $T - \lambda \text{id}_V$ is not invertible.

(1) Show that $\mathbf{C}(X)$ has uncountable dimension over $\mathbf{C}$.
(2) Show that any linear operator on V has a nonempty spectrum if the dimension of V is finite or countable.
(3) Show that if a finitely generated $\mathbf{C}$-algebra R is a field, then the map $\mathbf{C} \to R$ is an isomorphism.
(4) Show that any maximal ideal $\mathfrak{m}$ of $\mathbf{C}[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in \mathbf{C}$.

027M Remark 111.16.2. Let k be a field. Then for every integer $n \in \mathbf{N}$ and every maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k. This will be shown later in the course. Of course (please check this) it implies a similar statement for maximal ideals of finitely generated k-algebras. The exercise above proves it in the case $k = \mathbf{C}$.

02DP Exercise 111.16.3. Let k be a field. Please use Remark 111.16.2.

(1) Let R be a k-algebra. Suppose that $\dim_k R < \infty$ and that R is a domain. Show that R is a field.
(2) Suppose that R is a finitely generated k-algebra, and $f \in R$ not nilpotent. Show that there exists a maximal ideal $\mathfrak{m} \subset R$ with $f \notin \mathfrak{m}$.
(3) Show by an example that this statement fails when R is not of finite type over a field.
(4) Show that any radical ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it.

02DQ Remark 111.16.4. This is the Hilbert Nullstellensatz. Namely it says that the closed subsets of $\text{Spec}(k[x_1, \ldots, x_n])$ (which correspond to radical ideals by a previous exercise) are determined by the closed points contained in them.

02DR Exercise 111.16.5. Let $A = \mathbf{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Let I be the ideal of A generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Find the irreducible components of the closed subset $V(I)$ of $\mathrm{Spec}(A)$. (I mean describe them and give equations for each of them. You do not have to prove that the equations you write down define prime ideals.) Hints:

(1) You may use the Hilbert Nullstellensatz, and it suffices to find irreducible locally closed subsets which cover the set of closed points of $V(I)$.
(2) There are two easy components.
(3) An image of an irreducible set under a continuous map is irreducible.

111.17. Dimension

02LT

076H Exercise 111.17.1. Construct a ring A with finitely many prime ideals having dimension > 1.

076I Exercise 111.17.2. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that $\mathbf{C}[x, y]/(f)$ has dimension 1.

02LU Exercise 111.17.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $n \geq 1$. Let $\mathfrak{m}' = (\mathfrak{m}, x_1, \ldots, x_n)$ in the polynomial ring $R[x_1, \ldots, x_n]$. Show that

$$\dim(R[x_1, \ldots, x_n]_{\mathfrak{m}'}) = \dim(R) + n.$$

111.18. Catenary rings

027N

027O Definition 111.18.1. A Noetherian ring A is said to be catenary if for any triple of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ we have

$$ht(\mathfrak{p}_3/\mathfrak{p}_1) = ht(\mathfrak{p}_3/\mathfrak{p}_2) + ht(\mathfrak{p}_2/\mathfrak{p}_1).$$

Here $ht(\mathfrak{p}/\mathfrak{q})$ means the height of $\mathfrak{p}/\mathfrak{q}$ in the ring $A/\mathfrak{q}$. In a formula

$$ht(\mathfrak{p}/\mathfrak{q}) = \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) = \dim((A/\mathfrak{q})_\mathfrak{p}) = \dim((A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}})$$

A topological space X is catenary, if given $T \subset T' \subset X$ with T and T' closed and irreducible, then there exists a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_n = T'$$

and every such chain has the same (finite) length.

0CT5 Exercise 111.18.2. Show that the notion of catenary defined in Algebra, Definition 10.105.1 agrees with the notion of Definition 111.18.1 for Noetherian rings.

02DS Exercise 111.18.3. Show that a Noetherian local domain of dimension 2 is catenary.

077D Exercise 111.18.4. Let k be a field. Show that a finite type k-algebra is catenary.

0CT6 Exercise 111.18.5. Give an example of a finite, sober, catenary topological space X which does not have a dimension function $\delta : X \to \mathbf{Z}$. Here $\delta : X \to \mathbf{Z}$ is a dimension function if for $x, y \in X$ we have

(1) $x \rightsquigarrow y$ and $x \neq y$ implies $\delta(x) > \delta(y)$,

(2) $x \rightsquigarrow y$ and $\delta(x) \geq \delta(y) + 2$ implies there exists a $z \in X$ with $x \rightsquigarrow z \rightsquigarrow y$ and $\delta(x) > \delta(z) > \delta(y)$.

Describe your space clearly and succintly explain why there cannot be a dimension function.

111.19. Fraction fields

027P

02DT Exercise 111.19.1. Consider the domain

$$\mathbf{Q}[r, s, t]/(s^2 - (r-1)(r-2)(r-3), t^2 - (r+1)(r+2)(r+3)).$$

Find a domain of the form $\mathbf{Q}[x, y]/(f)$ with isomorphic field of fractions.

111.20. Transcendence degree

077E

077F Exercise 111.20.1. Let $K'/K/k$ be field extensions with K' algebraic over K. Prove that $\text{trdeg}_k(K) = \text{trdeg}_k(K')$. (Hint: Show that if $x_1, \ldots, x_d \in K$ are algebraically independent over k and $d < \text{trdeg}_k(K')$ then $k(x_1, \ldots, x_d) \subset K$ cannot be algebraic.)

0CVP Exercise 111.20.2. Let k be a field. Let K/k be a finitely generated extension of transcendence degree d. If $V, W \subset K$ are finite dimensional k-subvector spaces denote

$$VW = \{f \in K \mid f = \sum\nolimits_{i=1,\ldots,n} v_i w_i \text{ for some } n \text{ and } v_i \in V, w_i \in W\}$$

This is a finite dimensional k-subvector space. Set $V^2 = VV$, $V^3 = VV^2$, etc.

(1) Show you can find $V \subset K$ and $\epsilon > 0$ such that $\dim V^n \geq \epsilon n^d$ for all $n \geq 1$.
(2) Conversely, show that for every finite dimensional $V \subset K$ there exists a $C > 0$ such that $\dim V^n \leq C n^d$ for all $n \geq 1$. (One possible way to proceed: First do this for subvector spaces of $k[x_1, \ldots, x_d]$. Then do this for subvector spaces of $k(x_1, \ldots, x_d)$. Finally, if $K/k(x_1, \ldots, x_d)$ is a finite extension choose a basis of K over $k(x_1, \ldots, x_d)$ and argue using expansion in terms of this basis.)
(3) Conclude that you can redefine the transcendence degree in terms of growth of powers of finite dimensional subvector spaces of K.

This is related to Gelfand-Kirillov dimension of (noncommutative) algebras over k.

111.21. Dimension of fibres

0CVQ Some questions related to the dimension formula, see Algebra, Section 10.113.

0CVR Exercise 111.21.1. Let k be your favorite algebraically closed field. Below $k[x]$ and $k[x, y]$ denote the polynomial rings.

(1) For every integer $n \geq 0$ find a finite type extension $k[x] \subset A$ of domains such that the spectrum of A/xA has exactly n irreducible components.
(2) Make an example of a finite type extension $k[x] \subset A$ of domains such that the spectrum of $A/(x - \alpha)A$ is nonempty and reducible for every $\alpha \in k$.

(3) Make an example of a finite type extension $k[x, y] \subset A$ of domains such that the spectrum of $A/(x - \alpha, y - \beta)A$ is irreducible[1] for all $(\alpha, \beta) \in k^2 \setminus \{(0, 0)\}$ and the spectrum of $A/(x, y)A$ is nonempty and reducible.

0CVS Exercise 111.21.2. Let k be your favorite algebraically closed field. Let $n \geq 1$. Let $k[x_1, \ldots, x_n]$ be the polynomial ring. Set $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $k[x_1, \ldots, x_n] \subset A$ be a finite type extension of domains. Set $d = \dim(A)$.

(1) Show that $d - 1 \geq \dim(A/\mathfrak{m}A) \geq d - n$ if $A/\mathfrak{m}A \neq 0$.
(2) Show by example that every value can occur.
(3) Show by example that $\mathrm{Spec}(A/\mathfrak{m}A)$ can have irreducible components of different dimensions.

111.22. Finite locally free modules

027Q

027R Definition 111.22.1. Let A be a ring. Recall that a finite locally free A-module M is a module such that for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that M_f is a finite free A_f-module. We say M is an invertible module if M is finite locally free of rank 1, i.e., for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f \cong A_f$ as an A_f-module.

078P Exercise 111.22.2. Prove that the tensor product of finite locally free modules is finite locally free. Prove that the tensor product of two invertible modules is invertible.

078Q Definition 111.22.3. Let A be a ring. The class group of A, sometimes called the Picard group of A is the set $\mathrm{Pic}(A)$ of isomorphism classes of invertible A-modules endowed with a group operation defined by tensor product (see Exercise 111.22.2).

Note that the class group of A is trivial exactly when every invertible module is isomorphic to a free module of rank 1.

078R Exercise 111.22.4. Show that the class groups of the following rings are trivial

(1) a polynomial ring $A = k[x]$ where k is a field,
(2) the integers $A = \mathbf{Z}$,
(3) a polynomial ring $A = k[x, y]$ where k is a field, and
(4) the quotient $k[x, y]/(xy)$ where k is a field.

078S Exercise 111.22.5. Show that the class group of the ring $A = k[x, y]/(y^2 - f(x))$ where k is a field of characteristic not 2 and where $f(x) = (x - t_1) \ldots (x - t_n)$ with $t_1, \ldots, t_n \in k$ distinct and $n \geq 3$ an odd integer is not trivial. (Hint: Show that the ideal $(y, x - t_1)$ defines a nontrivial element of $\mathrm{Pic}(A)$.)

02DU Exercise 111.22.6. Let A be a ring.

(1) Suppose that M is a finite locally free A-module, and suppose that $\varphi : M \to M$ is an endomorphism. Define/construct the trace and determinant of φ and prove that your construction is "functorial in the triple (A, M, φ)".
(2) Show that if M, N are finite locally free A-modules, and if $\varphi : M \to N$ and $\psi : N \to M$ then $\mathrm{Trace}(\varphi \circ \psi) = \mathrm{Trace}(\psi \circ \varphi)$ and $\det(\varphi \circ \psi) = \det(\psi \circ \varphi)$.
(3) In case M is finite locally free show that Trace defines an A-linear map $\mathrm{End}_A(M) \to A$ and det defines a multiplicative map $\mathrm{End}_A(M) \to A$.

[1] Recall that irreducible implies nonempty.

02DV Exercise 111.22.7. Now suppose that B is an A-algebra which is finite locally free as an A-module, in other words B is a finite locally free A-algebra.

(1) Define $\text{Trace}_{B/A}$ and $\text{Norm}_{B/A}$ using Trace and det from Exercise 111.22.6.
(2) Let $b \in B$ and let $\pi : \text{Spec}(B) \to \text{Spec}(A)$ be the induced morphism. Show that $\pi(V(b)) = V(\text{Norm}_{B/A}(b))$. (Recall that $V(f) = \{\mathfrak{p} \mid f \in \mathfrak{p}\}$.)
(3) (Base change.) Suppose that $i : A \to A'$ is a ring map. Set $B' = B \otimes_A A'$. Indicate why $i(\text{Norm}_{B/A}(b))$ equals $\text{Norm}_{B'/A'}(b \otimes 1)$.
(4) Compute $\text{Norm}_{B/A}(b)$ when $B = A \times A \times A \times \ldots \times A$ and $b = (a_1, \ldots, a_n)$.
(5) Compute the norm of $y - y^3$ under the finite flat map $\mathbf{Q}[x] \to \mathbf{Q}[y]$, $x \to y^n$. (Hint: use the "base change" $A = \mathbf{Q}[x] \subset A' = \mathbf{Q}(\zeta_n)(x^{1/n})$.)

111.23. Glueing

027S

02DW Exercise 111.23.1. Suppose that A is a ring and M is an A-module. Let f_i, $i \in I$ be a collection of elements of A such that

$$\text{Spec}(A) = \bigcup D(f_i).$$

(1) Show that if M_{f_i} is a finite A_{f_i}-module, then M is a finite A-module.
(2) Show that if M_{f_i} is a flat A_{f_i}-module, then M is a flat A-module. (This is kind of silly if you think about it right.)

02DX Remark 111.23.2. In algebraic geometric language this means that the property of "being finitely generated" or "being flat" is local for the Zariski topology (in a suitable sense). You can also show this for the property "being of finite presentation".

078T Exercise 111.23.3. Suppose that $A \to B$ is a ring map. Let $f_i \in A$, $i \in I$ and $g_j \in B$, $j \in J$ be collections of elements such that

$$\text{Spec}(A) = \bigcup D(f_i) \quad \text{and} \quad \text{Spec}(B) = \bigcup D(g_j).$$

Show that if $A_{f_i} \to B_{f_i g_j}$ is of finite type for all i, j then $A \to B$ is of finite type.

111.24. Going up and going down

027T

027U Definition 111.24.1. Let $\phi : A \to B$ be a homomorphism of rings. We say that the going-up theorem holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in \text{Spec}(B)$ lying over $\mathfrak{p}$, there exists $P' \in \text{Spec}(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the going-down theorem holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in \text{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $P \in \text{Spec}(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

02DY Exercise 111.24.2. In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)

(1) k is a field, $A = k$, $B = k[x]$.
(2) k is a field, $A = k[x]$, $B = k[x, y]$.
(3) $A = \mathbf{Z}$, $B = \mathbf{Z}[1/11]$.
(4) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.

(5) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(2+i)]$.
(6) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(14+7i)]$.
(7) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy-1)]/(y^2-y)$.

0FKE Exercise 111.24.3. Let A be a ring. Let $B = A[x]$ be the polynomial algebra in one variable over A. Let $f = a_0 + a_1x + \ldots + a_rx^r \in B = A[x]$. Prove carefully that the image of $D(f)$ in $\mathrm{Spec}(A)$ is equal to $D(a_0) \cup \ldots \cup D(a_r)$.

02DZ Exercise 111.24.4. Let k be an algebraically closed field. Compute the image in $\mathrm{Spec}(k[x, y])$ of the following maps:

(1) $\mathrm{Spec}(k[x, yx^{-1}]) \to \mathrm{Spec}(k[x, y])$, where $k[x, y] \subset k[x, yx^{-1}] \subset k[x, y, x^{-1}]$. (Hint: To avoid confusion, give the element yx^{-1} another name.)
(2) $\mathrm{Spec}(k[x, y, a, b]/(ax - by - 1)) \to \mathrm{Spec}(k[x, y])$.
(3) $\mathrm{Spec}(k[t, 1/(t-1)]) \to \mathrm{Spec}(k[x, y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(4) $k = \mathbf{C}$ (complex numbers), $\mathrm{Spec}(k[s, t]/(s^3 + t^3 - 1)) \to \mathrm{Spec}(k[x, y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

02E0 Remark 111.24.5. Finding the image as above usually is done by using elimination theory.

111.25. Fitting ideals

027V

02E1 Exercise 111.25.1. Let R be a ring and let M be a finite R-module. Choose a presentation

$$\bigoplus\nolimits_{j \in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Note that the map $R^{\oplus n} \to M$ is given by a sequence of elements $x_1, \ldots, x_n$ of M. The elements x_i are generators of M. The map $\bigoplus_{j \in J} R \to R^{\oplus n}$ is given by a $n \times J$ matrix A with coefficients in R. In other words, $A = (a_{ij})_{i=1,\ldots,n, j \in J}$. The columns $(a_{1j}, \ldots, a_{nj})$, $j \in J$ of A are said to be the relations. Any vector $(r_i) \in R^{\oplus n}$ such that $\sum r_i x_i = 0$ is a linear combination of the columns of A. Of course any finite R-module has a lot of different presentations.

(1) Show that the ideal generated by the $(n-k) \times (n-k)$ minors of A is independent of the choice of the presentation. This ideal is the kth Fitting ideal of M. Notation $Fit_k(M)$.
(2) Show that $Fit_0(M) \subset Fit_1(M) \subset Fit_2(M) \subset \ldots$. (Hint: Use that a determinant can be computed by expanding along a column.)
(3) Show that the following are equivalent:
    (a) $Fit_{r-1}(M) = (0)$ and $Fit_r(M) = R$, and
    (b) M is locally free of rank r.

111.26. Hilbert functions

027W

027X Definition 111.26.1. A numerical polynomial is a polynomial $f(x) \in \mathbf{Q}[x]$ such that $f(n) \in \mathbf{Z}$ for every integer n.

027Y Definition 111.26.2. A graded module M over a ring A is an A-module M endowed with a direct sum decomposition $\bigoplus_{n \in \mathbf{Z}} M_n$ into A-submodules. We will say that M is locally finite if all of the M_n are finite A-modules. Suppose that A is a Noetherian ring and that φ is a Euler-Poincaré function on finite A-modules. This means that for every finitely generated A-module M we are given an integer $\varphi(M) \in \mathbf{Z}$ and for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have $\varphi(M) = \varphi(M') + \varphi(M'')$. The Hilbert function of a locally finite graded module M (with respect to φ) is the function $\chi_\varphi(M, n) = \varphi(M_n)$. We say that M has a Hilbert polynomial if there is some numerical polynomial P_φ such that $\chi_\varphi(M, n) = P_\varphi(n)$ for all sufficiently large integers n.

027Z Definition 111.26.3. A graded A-algebra is a graded A-module $B = \bigoplus_{n \geq 0} B_n$ together with an A-bilinear map

$$B \times B \longrightarrow B, \ (b, b') \longmapsto bb'$$

that turns B into an A-algebra so that $B_n \cdot B_m \subset B_{n+m}$. Finally, a graded module M over a graded A-algebra B is given by a graded A-module M together with a (compatible) B-module structure such that $B_n \cdot M_d \subset M_{n+d}$. Now you can define homomorphisms of graded modules/rings, graded submodules, graded ideals, exact sequences of graded modules, etc, etc.

02E2 Exercise 111.26.4. Let $A = k$ a field. What are all possible Euler-Poincaré functions on finite A-modules in this case?

02E3 Exercise 111.26.5. Let $A = \mathbf{Z}$. What are all possible Euler-Poincaré functions on finite A-modules in this case?

02E4 Exercise 111.26.6. Let $A = k[x, y]/(xy)$ with k algebraically closed. What are all possible Euler-Poincaré functions on finite A-modules in this case?

02E5 Exercise 111.26.7. Suppose that A is Noetherian. Show that the kernel of a map of locally finite graded A-modules is locally finite.

02E6 Exercise 111.26.8. Let k be a field and let $A = k$ and $B = k[x, y]$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

02E7 Exercise 111.26.9. Let k be a field and let $A = k$ and $B = k[x, y]/(x^2, xy)$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

02E8 Exercise 111.26.10. Let k be a field and let $A = k$. Let $\varphi(M) = \dim_k(M)$. Fix $d \in \mathbf{N}$. Consider the graded A-algebra $B = k[x, y, z]/(x^d + y^d + z^d)$, where x, y, z each have degree 1. Compute the Hilbert function of B. Is there a Hilbert polynomial in this case?

111.27. Proj of a ring

0280

0281 Definition 111.27.1. Let R be a graded ring. A homogeneous ideal is simply an ideal $I \subset R$ which is also a graded submodule of R. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous pieces in R then $f_i \in I$ for each i.

0282 Definition 111.27.2. We define the homogeneous spectrum $\text{Proj}(R)$ of the graded ring R to be the set of homogeneous, prime ideals $\mathfrak{p}$ of R such that $R_+ \not\subset \mathfrak{p}$. Note that $\text{Proj}(R)$ is a subset of $\text{Spec}(R)$ and hence has a natural induced topology.

0283 Definition 111.27.3. Let $R = \oplus_{d \geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R_{(f)}$ to be the subring of R_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. Furthermore, we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj}(R) | f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset R$ we define $V_+(I) = V(I) \cap \text{Proj}(R)$.

02E9 Exercise 111.27.4. On the topology on $\text{Proj}(R)$. With definitions and notation as above prove the following statements.

(1) Show that $D_+(f)$ is open in $\text{Proj}(R)$.
(2) Show that $D_+(ff') = D_+(f) \cap D_+(f')$.
(3) Let $g = g_0 + \ldots + g_m$ be an element of R with $g_i \in R_i$. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(g_i)$, $i \geq 1$ and $D(g_0) \cap \text{Proj}(R)$. No proof necessary.
(4) Let $g \in R_0$ be a homogeneous element of degree 0. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(f_\alpha)$ for a suitable family $f_\alpha \in R$ of homogeneous elements of positive degree.
(5) Show that the collection $\{D_+(f)\}$ of opens forms a basis for the topology of $\text{Proj}(R)$.
078U (6) Show that there is a canonical bijection $D_+(f) \to \text{Spec}(R_{(f)})$. (Hint: Imitate the proof for Spec but at some point thrown in the radical of an ideal.)
(7) Show that the map from (6) is a homeomorphism.
(8) Give an example of an R such that $\text{Proj}(R)$ is not quasi-compact. No proof necessary.
(9) Show that any closed subset $T \subset \text{Proj}(R)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset R$.

02EA Remark 111.27.5. There is a continuous map $\text{Proj}(R) \longrightarrow \text{Spec}(R_0)$.

02EB Exercise 111.27.6. If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\text{Proj}(R) \to \text{Spec}(A)$ is a bijection and in fact a homeomorphism.

02EC Exercise 111.27.7. Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \text{Proj}(R) \to \text{Spec}(A)$. Set $U = \text{Spec}(A) - V(I)$. Show that

$$b : b^{-1}(U) \longrightarrow U$$

is a homeomorphism. Thus we may think of U as an open subset of $\text{Proj}(R)$. Let $Z \subset \text{Spec}(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \notin V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \in U$, in other words $Z \cap U \neq \emptyset$. We define the strict transform Z' of Z to be the closure of the unique point ξ' lying above ξ. Another way to say this is that Z' is the closure in $\text{Proj}(R)$ of the locally closed subset $Z \cap U \subset U \subset \text{Proj}(R)$.

02ED Exercise 111.27.8. Blowing up: Part II. Let $A = k[x, y]$ where k is a field, and let $I = (x, y)$. Let R be the blowup algebra for A and I.

(1) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{y\})$ are disjoint.

(2) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are not disjoint.
(3) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ in the blowup along J are disjoint.

02EE Exercise 111.27.9. Let R be a graded ring.

(1) Show that $\text{Proj}(R)$ is empty if $R_n = (0)$ for all $n >> 0$.
(2) Show that $\text{Proj}(R)$ is an irreducible topological space if R is a domain and R_+ is not zero. (Recall that the empty topological space is not irreducible.)

02EF Exercise 111.27.10. Blowing up: Part III. Consider A, I and U, Z as in the definition of strict transform. Let $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $\bar{A} = A/\mathfrak{p}$ and let $\bar{I}$ be the image of I in $\bar{A}$.

(1) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
(2) Show that the ring map above induces a bijective map from $\text{Proj}(\bar{R})$ onto the strict transform Z' of Z. (This is not so easy. Hint: Use 5(b) above.)
(3) Conclude that the strict transform $Z' = V_+(P)$ where $P \subset R$ is the homogeneous ideal defined by $P_d = I^d \cap \mathfrak{p}$.
(4) Suppose that $Z_1 = V(\mathfrak{p})$ and $Z_2 = V(\mathfrak{q})$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = \mathfrak{p} + \mathfrak{q}$ separates the strict transforms of Z_1 and Z_2, i.e., $Z_1' \cap Z_2' = \emptyset$. (Hint: Consider the homogeneous ideal P and Q from part (c) and consider $V(P + Q)$.)

111.28. Cohen-Macaulay rings of dimension 1

0284

0285 Definition 111.28.1. A Noetherian local ring A is said to be Cohen-Macaulay of dimension d if it has dimension d and there exists a system of parameters $x_1, \ldots, x_d$ for A such that x_i is a nonzerodivisor in $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

02EG Exercise 111.28.2. Cohen-Macaulay rings of dimension 1. Part I: Theory.

(1) Let $(A, \mathfrak{m})$ be a local Noetherian with $\dim A = 1$. Show that if $x \in \mathfrak{m}$ is not a zerodivisor then
    (a) $\dim A/xA = 0$, in other words A/xA is Artinian, in other words $\{x\}$ is a system of parameters for A.
    (b) A is has no embedded prime.
(2) Conversely, let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension 1. Show that if A has no embedded prime then there exists a nonzerodivisor in $\mathfrak{m}$.

02EH Exercise 111.28.3. Cohen-Macaulay rings of dimension 1. Part II: Examples.

(1) Let A be the local ring at (x, y) of $k[x, y]/(x^2, xy)$.
    (a) Show that A has dimension 1.
    (b) Prove that every element of $\mathfrak{m} \subset A$ is a zerodivisor.
    (c) Find $z \in \mathfrak{m}$ such that $\dim A/zA = 0$ (no proof required).
(2) Let A be the local ring at (x, y) of $k[x, y]/(x^2)$. Find a nonzerodivisor in $\mathfrak{m}$ (no proof required).

02EI Exercise 111.28.4. Local rings of embedding dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Show that the function $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is either constant with value 1, or its values are

$$1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots$$

02EJ Exercise 111.28.5. Regular local rings of dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension 1. Recall that this means that A has dimension 1 and embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Let $x \in \mathfrak{m}$ be any element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is not zero.

(1) Show that for every element y of $\mathfrak{m}$ there exists an integer n such that y can be written as $y = ux^n$ with $u \in A^*$ a unit.
(2) Show that x is a nonzerodivisor in A.
(3) Conclude that A is a domain.

02EK Exercise 111.28.6. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with associated graded $Gr_{\mathfrak{m}}(A)$.

(1) Suppose that $x \in \mathfrak{m}^d$ maps to a nonzerodivisor $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$. Show that x is a nonzerodivisor.
(2) Suppose the depth of A is at least 1. Namely, suppose that there exists a nonzerodivisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$ which is a nonzerodivisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map of multiplication by $\bar{x}$

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$$

is injective. Then show that x is a nonzerodivisor.

02EL Exercise 111.28.7. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogeneous ideal I.

(1) Show that there exists a homogeneous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.
(2) Show that $f(n) \leq n + 1$.
(3) Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.
(4) Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.
(5) Show that $f(n) = \deg(F)$ for all $n >> 0$.

02EM Exercise 111.28.8. Cohen-Macaulay rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macaulay of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notations: f, F, $x, y \in \mathfrak{m}$, I as in Ex. 6 above. Please use any results from the problems above.

(1) Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha\bar{x} + \beta\bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with f.
  (a) Show that z is a nonzerodivisor on A.
  (b) Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d}\mathfrak{m}^{d-1}$ for all sufficiently large n. (Hint: First show $z^{n+1-d}\mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)

(2) What condition on k guarantees the existence of such a z? (No proof required; it's too easy.)
Now we are going to assume there exists a z as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).

(3) Now show that $\mathfrak{m}^\ell = z^{\ell-d+1}\mathfrak{m}^{d-1}$ for all $\ell \geq d$.

(4) Conclude that $I = (F)$.

(5) Conclude that the function f has values

$$2, 3, 4, \ldots, d-1, d, d, d, d, d, d, d, \ldots$$

02EN Remark 111.28.9. This suggests that a local Noetherian Cohen-Macaulay ring of dimension 1 and embedding dimension 2 is of the form B/FB, where B is a 2-dimensional regular local ring. This is more or less true (under suitable "niceness" properties of the ring).

111.29. Infinitely many primes

0286 A section with a collection of strange questions on rings where infinitely many primes are not invertible.

02EO Exercise 111.29.1. Give an example of a finite type $\mathbf{Z}$-algebra R with the following two properties:

(1) There is no ring map $R \to \mathbf{Q}$.
(2) For every prime p there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} \cong \mathbf{F}_p$.

02EP Exercise 111.29.2. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. Give an example of an $f \in \mathbf{Z}[x, u]$ such that the following two properties hold:

(1) There exist infinitely many p such that f_p does not have a zero in $\mathbf{F}_p$.
(2) For all $p >> 0$ the polynomial f_p either has a linear or a quadratic factor.

02EQ Exercise 111.29.3. For $f \in \mathbf{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \bmod p \in \mathbf{F}_p[x, y]$. Give an "interesting" example of an f such that f_p is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^p y = xy(x^{p-1} - y^{p-1})$ is "uninteresting"; any f depending only on x, u is "uninteresting", etc.

02ER Remark 111.29.4. Let $h \in \mathbf{Z}[y]$ be a monic polynomial of degree d. Then:

(1) The map $A = \mathbf{Z}[x] \to B = \mathbf{Z}[y]$, $x \mapsto h$ is finite locally free of rank d.
(2) For all primes p the map $A_p = \mathbf{F}_p[x] \to B_p = \mathbf{F}_p[y]$, $y \mapsto h(y) \bmod p$ is finite locally free of rank d.

02ES Exercise 111.29.5. Let h, A, B, A_p, B_p be as in the remark. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. For $g \in \mathbf{Z}[y, v]$ we define $g_p(y) = g(y, y^p) \bmod p \in \mathbf{F}_p[y]$.

(1) Give an example of a h and g such that there does not exist a f with the property
$$f_p = Norm_{B_p/A_p}(g_p).$$
(2) Show that for any choice of h and g as above there exists a nonzero f such that for all p we have
$$Norm_{B_p/A_p}(g_p) \quad \text{divides} \quad f_p.$$
If you want you can restrict to the case $h = y^n$, even with $n = 2$, but it is true in general.
(3) Discuss the relevance of this to Exercises 6 and 7 of the previous set.

02ET Exercise 111.29.6. Unsolved problems. They may be really hard or they may be easy. I don't know.

(1) Is there any $f \in \mathbf{Z}[x, u]$ such that f_p is irreducible for an infinite number of p? (Hint: Yes, this happens for $f(x, u) = u - x - 1$ and also for $f(x, u) = u^2 - x^2 + 1$.)
(2) Let $f \in \mathbf{Z}[x, u]$ nonzero, and suppose $\deg_x(f_p) = dp + d'$ for all large p. (In other words $\deg_u(f) = d$ and the coefficient c of u^d in f has $\deg_x(c) = d'$.) Suppose we can write $d = d_1 + d_2$ and $d' = d'_1 + d'_2$ with $d_1, d_2 > 0$ and $d'_1, d'_2 \geq 0$ such that for all sufficiently large p there exists a factorization
$$f_p = f_{1,p} f_{2,p}$$
with $\deg_x(f_{1,p}) = d_1 p + d'_1$. Is it true that f comes about via a norm construction as in Exercise 4? (More precisely, are there a h and g such that $Norm_{B_p/A_p}(g_p)$ divides f_p for all $p >> 0$.)
(3) Analogous question to the one in (b) but now with $f \in \mathbf{Z}[x_1, x_2, u_1, u_2]$ irreducible and just assuming that $f_p(x_1, x_2) = f(x_1, x_2, x_1^p, x_2^p) \bmod p$ factors for all $p >> 0$.

111.30. Filtered derived category

0287 In order to do the exercises in this section, please read the material in Homology, Section 12.19. We will say A is a filtered object of $\mathcal{A}$, to mean that A comes endowed with a filtration F which we omit from the notation.

0288 Exercise 111.30.1. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite and that each $\text{gr}^p(I)$ is an injective object of $\mathcal{A}$. Show that there exists an isomorphism $I \cong \bigoplus \text{gr}^p(I)$ with filtration $F^p(I)$ corresponding to $\bigoplus_{p' \geq p} \text{gr}^p(I)$.

0289 Exercise 111.30.2. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite. Show the following are equivalent:

(1) For any solid diagram
$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[d] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$
of filtered objects with (i) the filtrations on A and B are finite, and (ii) $\text{gr}(\alpha)$ injective the dotted arrow exists making the diagram commute.
(2) Each $\text{gr}^p I$ is injective.

Note that given a morphism $\alpha : A \to B$ of filtered objects with finite filtrations to say that $\mathrm{gr}(\alpha)$ injective is the same thing as saying that α is a strict monomorphism in the category $\mathrm{Fil}(\mathcal{A})$. Namely, being a monomorphism means $\mathrm{Ker}(\alpha) = 0$ and strict means that this also implies $\mathrm{Ker}(\mathrm{gr}(\alpha)) = 0$. See Homology, Lemma 12.19.13. (We only use the term "injective" for a morphism in an abelian category, although it makes sense in any additive category having kernels.) The exercises above justifies the following definition.

028A Definition 111.30.3. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume the filtration on I is finite. We say I is filtered injective if each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$.

We make the following definition to avoid having to keep saying "with a finite filtration" everywhere.

028B Definition 111.30.4. Let $\mathcal{A}$ be an abelian category. We denote $\mathrm{Fil}^f(\mathcal{A})$ the full subcategory of $\mathrm{Fil}(\mathcal{A})$ whose objects consist of those $A \in \mathrm{Ob}(\mathrm{Fil}(\mathcal{A}))$ whose filtration is finite.

028C Exercise 111.30.5. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show that there exists a strict monomorphism $\alpha : A \to I$ of A into a filtered injective object I of $\mathrm{Fil}^f(\mathcal{A})$.

028D Definition 111.30.6. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathrm{Fil}(\mathcal{A})$. We say that α is a filtered quasi-isomorphism if for each $p \in \mathbf{Z}$ the morphism $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$ is a quasi-isomorphism.

028E Definition 111.30.7. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathrm{Fil}^f(\mathcal{A})$. We say that $K^\bullet$ is filtered acyclic if for each $p \in \mathbf{Z}$ the complex $\mathrm{gr}^p(K^\bullet)$ is acyclic.

028F Exercise 111.30.8. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of bounded below complexes of $\mathrm{Fil}^f(\mathcal{A})$. (Note the superscript f.) Show that the following are equivalent:

(1) α is a filtered quasi-isomorphism,
(2) for each $p \in \mathbf{Z}$ the map $\alpha : F^pK^\bullet \to F^pL^\bullet$ is a quasi-isomorphism,
(3) for each $p \in \mathbf{Z}$ the map $\alpha : K^\bullet/F^pK^\bullet \to L^\bullet/F^pL^\bullet$ is a quasi-isomorphism, and
(4) the cone of α (see Derived Categories, Definition 13.9.1) is a filtered acyclic complex.

Moreover, show that if α is a filtered quasi-isomorphism then α is also a usual quasi-isomorphism.

028G Exercise 111.30.9. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a complex $I^\bullet$ of $\mathrm{Fil}^f(\mathcal{A})$, and a morphism $A[0] \to I^\bullet$ such that

(1) each I^p is filtered injective,
(2) $I^p = 0$ for $p < 0$, and
(3) $A[0] \to I^\bullet$ is a filtered quasi-isomorphism.

028H Exercise 111.30.10. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let $K^\bullet$ be a bounded below complex of objects of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ a complex of $\mathrm{Fil}^f(\mathcal{A})$ having filtered injective terms I^n, and bounded below. In fact, we may choose α such that each α^n is a strict monomorphism.

028I Exercise 111.30.11. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

of complexes of $\text{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ are bounded below and assume each I^n is a filtered injective object. Also assume that α is a filtered quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.
(2) If α is a strict monomorphism in every degree then we can find a β which makes the diagram commute.

028J Exercise 111.30.12. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$, $K^\bullet$ be complexes of $\text{Fil}^f(\mathcal{A})$. Assume

(1) $K^\bullet$ bounded below and filtered acyclic, and
(2) $I^\bullet$ bounded below and consisting of filtered injective objects.

Then any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.

028K Exercise 111.30.13. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"] & L^\bullet \arrow[dl, dashed, "\beta_i"] \\ I^\bullet & \end{tikzcd}$$

of complexes of $\text{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ bounded below and each I^n a filtered injective object. Also assume α a filtered quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

111.31. Regular functions

078V

0E9D Exercise 111.31.1. Consider the affine curve X given by the equation $t^2 = s^5 + 8$ in $\mathbf{C}^2$ with coordinates s, t. Let $x \in X$ be the point with coordinates $(1, 3)$. Let $U = X \setminus \{x\}$. Prove that there is a regular function on U which is not the restriction of a regular function on $\mathbf{C}^2$, i.e., is not the restriction of a polynomial in s and t to U.

0E9E Exercise 111.31.2. Let $n \geq 2$. Let $E \subset \mathbf{C}^n$ be a finite subset. Show that any regular function on $\mathbf{C}^n \setminus E$ is a polynomial.

0E9F Exercise 111.31.3. Let $X \subset \mathbf{C}^n$ be an affine variety. Let us say X is a cone if $x = (a_1, \ldots, a_n) \in X$ and $\lambda \in \mathbf{C}$ implies $(\lambda a_1, \ldots, \lambda a_n) \in X$. Of course, if $\mathfrak{p} \subset \mathbf{C}[x_1, \ldots, x_n]$ is a prime ideal generated by homogeneous polynomials in $x_1, \ldots, x_n$, then the affine variety $X = V(\mathfrak{p}) \subset \mathbf{C}^n$ is a cone. Show that conversely the prime ideal $\mathfrak{p} \subset \mathbf{C}[x_1, \ldots, x_n]$ corresponding to a cone can be generated by homogeneous polynomials in $x_1, \ldots, x_n$.

0E9G Exercise 111.31.4. Give an example of an affine variety $X \subset \mathbf{C}^n$ which is a cone (see Exercise 111.31.3) and a regular function f on $U = X \setminus \{(0, \ldots, 0)\}$ which is not the restriction of a polynomial function on $\mathbf{C}^n$.

078W Exercise 111.31.5. In this exercise we try to see what happens with regular functions over non-algebraically closed fields. Let k be a field. Let $Z \subset k^n$ be a Zariski locally closed subset, i.e., there exist ideals $I \subset J \subset k[x_1, \ldots, x_n]$ such that

$$Z = \{a \in k^n \mid f(a) = 0 \ \forall \ f \in I, \ \exists \ g \in J, \ g(a) \neq 0\}.$$

A function $\varphi : Z \to k$ is said to be regular if for every $z \in Z$ there exists a Zariski open neighbourhood $z \in U \subset Z$ and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that $g(u) \neq 0$ for all $u \in U$ and such that $\varphi(u) = f(u)/g(u)$ for all $u \in U$.

(1) If $k = \bar{k}$ and $Z = k^n$ show that regular functions are given by polynomials. (Only do this if you haven't seen this argument before.)
(2) If k is finite show that (a) every function φ is regular, (b) the ring of regular functions is finite dimensional over k. (If you like you can take $Z = k^n$ and even $n = 1$.)
(3) If $k = \mathbf{R}$ give an example of a regular function on $Z = \mathbf{R}$ which is not given by a polynomial.
(4) If $k = \mathbf{Q}_p$ give an example of a regular function on $Z = \mathbf{Q}_p$ which is not given by a polynomial.

111.32. Sheaves

028L A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an monomorphism if for every pair of morphisms $a, b : W \to X$ we have $f \circ a = f \circ b \Rightarrow a = b$. A monomorphism in the category of sets is an injective map of sets.

078X Exercise 111.32.1. Carefully prove that a map of sheaves of sets is an monomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are injective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an isomorphism if there exists a morphism $g : Y \to X$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. An isomorphism in the category of sets is a bijective map of sets.

078Y Exercise 111.32.2. Carefully prove that a map of sheaves of sets is an isomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are bijective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an epimorphism if for every pair of morphisms $a, b : Y \to Z$ we have $a \circ f = b \circ f \Rightarrow a = b$. An epimorphism in the category of sets is a surjective map of sets.

02EU Exercise 111.32.3. Carefully prove that a map of sheaves of sets is an epimorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are surjective.

02EV Exercise 111.32.4. Let $f : X \to Y$ be a map of topological spaces. Prove pushforward f_* and pullback f^{-1} for sheaves of sets form an adjoint pair of functors.

02EW Exercise 111.32.5. Let $j : U \to X$ be an open immersion. Show that

(1) Pullback $j^{-1} : Sh(X) \to Sh(U)$ has a left adjoint $j_! : Sh(U) \to Sh(X)$ called extension by the empty set.

(2) Characterize the stalks of $j_!(\mathcal{G})$ for $\mathcal{G} \in Sh(U)$.
(3) Pullback $j^{-1} : \text{Ab}(X) \to \text{Ab}(U)$ has a left adjoint $j_! : \text{Ab}(U) \to \text{Ab}(X)$ called extension by zero.
(4) Characterize the stalks of $j_!(\mathcal{G})$ for $\mathcal{G} \in \text{Ab}(U)$.

Observe that extension by zero differs from extension by the empty set!

028M Exercise 111.32.6. Let $X = \mathbf{R}$ with the usual topology. Let $\mathcal{O}_X = \underline{\mathbf{Z}/2\mathbf{Z}}_X$. Let $i : Z = \{0\} \to X$ be the inclusion and let $\mathcal{O}_Z = \underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Prove the following (the first three follow from the definitions but if you are not clear on the definitions you should elucidate them):

(1) $i_*\mathcal{O}_Z$ is a skyscraper sheaf.
(2) There is a canonical surjective map from $\underline{\mathbf{Z}/2\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Denote the kernel $\mathcal{I} \subset \mathcal{O}_X$.
(3) $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$.
(4) The sheaf $\mathcal{I}$ on X cannot be locally generated by sections (as in Modules, Definition 17.8.1.)

028N Exercise 111.32.7. Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Show that $\mathcal{F}$ is the quotient of a (possibly very large) direct sum of sheaves all of whose terms are of the form

$$j_!(\underline{\mathbf{Z}}_U)$$

where $U \subset X$ is open and $\underline{\mathbf{Z}}_U$ denotes the constant sheaf with value $\mathbf{Z}$ on U.

02EX Remark 111.32.8. Let X be a topological space. In the category of abelian sheaves the direct sum of a family of sheaves $\{\mathcal{F}_i\}_{i\in I}$ is the sheaf associated to the presheaf $U \mapsto \oplus \mathcal{F}_i(U)$. Consequently the stalk of the direct sum at a point x is the direct sum of the stalks of the $\mathcal{F}_i$ at x.

078Z Exercise 111.32.9. Let X be a topological space. Suppose we are given a collection of abelian groups A_x indexed by $x \in X$. Show that the rule $U \mapsto \prod_{x\in U} A_x$ with obvious restriction mappings defines a sheaf $\mathcal{G}$ of abelian groups. Show, by an example, that usually it is not the case that $\mathcal{G}_x = A_x$ for $x \in X$.

0790 Exercise 111.32.10. Let X, A_x, $\mathcal{G}$ be as in Exercise 111.32.9. Let $\mathcal{B}$ be a basis for the topology of X, see Topology, Definition 5.5.1. For $U \in \mathcal{B}$ let A_U be a subgroup $A_U \subset \mathcal{G}(U) = \prod_{x\in U} A_x$. Assume that for $U \subset V$ with $U, V \in \mathcal{B}$ the restriction maps A_V into A_U. For $U \subset X$ open set

$$\mathcal{F}(U) = \left\{ (s_x)_{x\in U} \middle| \begin{array}{l} \text{for every } x \text{ in } U \text{ there exists } V \in \mathcal{B} \\ x \in V \subset U \text{ such that } (s_y)_{y\in V} \in A_V \end{array} \right\}$$

Show that $\mathcal{F}$ defines a sheaf of abelian groups on X. Show, by an example, that it is usually not the case that $\mathcal{F}(U) = A_U$ for $U \in \mathcal{B}$.

0E9H Exercise 111.32.11. Give an example of a topological space X and a functor

$$F : Sh(X) \longrightarrow \text{Sets}$$

which is exact (commutes with finite products and equalizers and commutes with finite coproducts and coequalizers, see Categories, Section 4.23), but there is no point $x \in X$ such that F is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x$.

111.33. Schemes

028O Let *LRS* be the category of locally ringed spaces. An affine scheme is an object in *LRS* isomorphic in *LRS* to a pair of the form $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$. A scheme is an object $(X, \mathcal{O}_X)$ of *LRS* such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

028P Exercise 111.33.1. Find a 1-point locally ringed space which is not a scheme.

028Q Exercise 111.33.2. Suppose that X is a scheme whose underlying topological space has 2 points. Show that X is an affine scheme.

03KB Exercise 111.33.3. Suppose that X is a scheme whose underlying topological space is a finite discrete set. Show that X is an affine scheme.

028R Exercise 111.33.4. Show that there exists a non-affine scheme having three points.

028S Exercise 111.33.5. Suppose that X is a nonempty quasi-compact scheme. Show that X has a closed point.

02EY Remark 111.33.6. When $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open subset then $(U, \mathcal{O}_X|_U)$ is a ringed space. Notation: $\mathcal{O}_U = \mathcal{O}_X|_U$. There is a canonical morphisms of ringed spaces

$$j : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X).$$

If $(X, \mathcal{O}_X)$ is a locally ringed space, so is $(U, \mathcal{O}_U)$ and j is a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme so is $(U, \mathcal{O}_U)$ and j is a morphism of schemes. We say that $(U, \mathcal{O}_U)$ is an open subscheme of $(X, \mathcal{O}_X)$ and that j is an open immersion. More generally, any morphism $j' : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ that is isomorphic to a morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ as above is called an open immersion.

028T Exercise 111.33.7. Give an example of an affine scheme $(X, \mathcal{O}_X)$ and an open $U \subset X$ such that $(U, \mathcal{O}_X|U)$ is not an affine scheme.

028U Exercise 111.33.8. Given an example of a pair of affine schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, an open subscheme $(U, \mathcal{O}_X|_U)$ of X and a morphism of schemes $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ that does not extend to a morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

028V Exercise 111.33.9. (This is pretty hard.) Given an example of a scheme X, and open subscheme $U \subset X$ and a closed subscheme $Z \subset U$ such that Z does not extend to a closed subscheme of X.

028W Exercise 111.33.10. Give an example of a scheme X, a field K, and a morphism of ringed spaces $\text{Spec}(K) \to X$ which is NOT a morphism of schemes.

028X Exercise 111.33.11. Do all the exercises in [Har77, Chapter II], Sections 1 and 2... Just kidding!

028Y Definition 111.33.12. A scheme X is called integral if X is nonempty and for every nonempty affine open $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$ is a domain.

028Z Exercise 111.33.13. Give an example of a morphism of integral schemes $f : X \to Y$ such that the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$, but f is not a closed immersion.

0290 Exercise 111.33.14. Give an example of a fibre product $X \times_S Y$ such that X and Y are affine but $X \times_S Y$ is not.

02EZ Remark 111.33.15. It turns out this cannot happen with S separated. Do you know why?

0291 Exercise 111.33.16. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over $\mathbf{Q}$ such that $\text{Spec}(\mathbf{C}) \times_{\text{Spec}(\mathbf{Q})} V$ is not integral.

0292 Exercise 111.33.17. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over a field k such that $\text{Spec}(k') \times_{\text{Spec}(k)} V$ is not reduced for some finite field extension k'/k.

02F0 Remark 111.33.18. If your scheme is affine then dimension is the same as the Krull dimension of the underlying ring. So you can use last semesters results to compute dimension.

111.34. Morphisms

0293 An important question is, given a morphism $\pi : X \to S$, whether the morphism has a section or a rational section. Here are some example exercises.

0294 Exercise 111.34.1. Consider the morphism of schemes

$$\pi : X = \text{Spec}(\mathbf{C}[x, t, 1/xt]) \longrightarrow S = \text{Spec}(\mathbf{C}[t]).$$

(1) Show there does not exist a morphism $\sigma : S \to X$ such that $\pi \circ \sigma = \text{id}_S$.
(2) Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \text{id}_U$.

0295 Exercise 111.34.2. Consider the morphism of schemes

$$\pi : X = \text{Spec}(\mathbf{C}[x, t]/(x^2 + t)) \longrightarrow S = \text{Spec}(\mathbf{C}[t]).$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \text{id}_U$.

0296 Exercise 111.34.3. Let $A, B, C \in \mathbf{C}[t]$ be nonzero polynomials. Consider the morphism of schemes

$$\pi : X = \text{Spec}(\mathbf{C}[x, y, t]/(A + Bx^2 + Cy^2)) \longrightarrow S = \text{Spec}(\mathbf{C}[t]).$$

Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \text{id}_U$. (Hint: Symbolically, write $x = X/Z$, $y = Y/Z$ for some $X, Y, Z \in \mathbf{C}[t]$ of degree $\leq d$ for some d, and work out the condition that this solves the equation. Then show, using dimension theory, that if $d >> 0$ you can find nonzero X, Y, Z solving the equation.)

02F1 Remark 111.34.4. Exercise 111.34.3 is a special case of "Tsen's theorem". Exercise 111.34.5 shows that the method is limited to low degree equations (conics when the base and fibre have dimension 1).

0297 Exercise 111.34.5. Consider the morphism of schemes

$$\pi : X = \text{Spec}(\mathbf{C}[x, y, t]/(1 + tx^3 + t^2y^3)) \longrightarrow S = \text{Spec}(\mathbf{C}[t])$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \text{id}_U$.

0298 Exercise 111.34.6. Consider the schemes

$$X = \mathrm{Spec}(\mathbf{C}[\{x_i\}_{i=1}^8, s, t]/(1+sx_1^3+s^2x_2^3+tx_3^3+stx_4^3+s^2tx_5^3+t^2x_6^3+st^2x_7^3+s^2t^2x_8^3))$$

and

$$S = \mathrm{Spec}(\mathbf{C}[s, t])$$

and the morphism of schemes

$$\pi : X \longrightarrow S$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

0299 Exercise 111.34.7. (For the number theorists.) Give an example of a closed subscheme

$$Z \subset \mathrm{Spec}\left(\mathbf{Z}[x, \frac{1}{x(x-1)(2x-1)}]\right)$$

such that the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is finite and surjective.

029A Exercise 111.34.8. If you do not like number theory, you can try the variant where you look at

$$\mathrm{Spec}\left(\mathbf{F}_p[t, x, \frac{1}{x(x-t)(tx-1)}]\right) \longrightarrow \mathrm{Spec}(\mathbf{F}_p[t])$$

and you try to find a closed subscheme of the top scheme which maps finite surjectively to the bottom one. (There is a theoretical reason for having a finite ground field here; although it may not be necessary in this particular case.)

02F2 Remark 111.34.9. The interpretation of the results of Exercise 111.34.7 and 111.34.8 is that given the morphism $X \to S$ all of whose fibres are nonempty, there exists a finite surjective morphism $S' \to S$ such that the base change $X_{S'} \to S'$ does have a section. This is not a general fact, but it holds if the base is the spectrum of a dedekind ring with finite residue fields at closed points, and the morphism $X \to S$ is flat with geometrically irreducible generic fibre. See Exercise 111.34.10 below for an example where it doesn't work.

029B Exercise 111.34.10. Prove there exist a $f \in \mathbf{C}[x, t]$ which is not divisible by $t - \alpha$ for any $\alpha \in \mathbf{C}$ such that there does not exist any $Z \subset \mathrm{Spec}(\mathbf{C}[x, t, 1/f])$ which maps finite surjectively to $\mathrm{Spec}(\mathbf{C}[t])$. (I think that $f(x, t) = (xt - 2)(x - t + 3)$ works. To show any candidate has the required property is not so easy I think.)

0EG6 Exercise 111.34.11. Let $A \to B$ be a finite type ring map. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ factors through a closed immersion $\mathrm{Spec}(B) \to \mathbf{P}^n_A$ for some n. Prove that $A \to B$ is a finite ring map, i.e., that B is finite as an A-module. Hint: if A is Noetherian (please just assume this) you can argue using that $H^i(Z, \mathcal{O}_Z)$ for $i \in \mathbf{Z}$ is a finite A-module for every closed subscheme $Z \subset \mathbf{P}^n_A$.

0EG7 Exercise 111.34.12. Let k be an algebraically closed field. Let $f : X \to Y$ be a morphism of projective varieties such that $f^{-1}(\{y\})$ is finite for every closed point $y \in Y$. Prove that f is finite as a morphism of schemes. Hints: (a) being finite is a local property, (b) try to reduce to Exercise 111.34.11, and (c) use a closed immersion $X \to \mathbf{P}^n_k$ to get a closed immersion $X \to \mathbf{P}^n_Y$ over Y.

111.35. Tangent Spaces

029C

029D Definition 111.35.1. For any ring R we denote $R[\epsilon]$ the ring of dual numbers. As an R-module it is free with basis $1, \epsilon$. The ring structure comes from setting $\epsilon^2 = 0$.

029E Exercise 111.35.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point, let $s = f(x)$. Consider the solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa(x)) \arrow[r] \arrow[rr, bend left] \arrow[dr] & \mathrm{Spec}(\kappa(x)[\epsilon]) \arrow[r, dotted] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

with the curved arrow being the canonical morphism of $\mathrm{Spec}(\kappa(x))$ into X. If $\kappa(x) = \kappa(s)$ show that the set of dotted arrows which make the diagram commute are in one to one correspondence with the set of linear maps

$$\mathrm{Hom}_{\kappa(x)}(\frac{\mathfrak m_x}{\mathfrak m_x^2 + \mathfrak m_s\mathcal{O}_{X,x}}, \kappa(x))$$

In other words: describe such a bijection. (This works more generally if $\kappa(x) \supset \kappa(s)$ is a separable algebraic extension.)

029F Definition 111.35.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. We dub the set of dotted arrows of Exercise 111.35.2 the tangent space of X over S and we denote it $T_{X/S,x}$. An element of this space is called a tangent vector of X/S at x.

029G Exercise 111.35.4. For any field K prove that the diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Spec}(K[\epsilon_1]) \arrow[d] \\ \mathrm{Spec}(K[\epsilon_2]) \arrow[r] & \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \end{tikzcd}$$

is a pushout diagram in the category of schemes. (Here $\epsilon_i^2 = 0$ as before.)

029H Exercise 111.35.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Define addition of tangent vectors, using Exercise 111.35.4 and a suitable morphism

$$\mathrm{Spec}(K[\epsilon]) \longrightarrow \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)).$$

Similarly, define scalar multiplication of tangent vectors (this is easier). Show that $T_{X/S,x}$ becomes a $\kappa(x)$-vector space with your constructions.

029I Exercise 111.35.6. Let k be a field. Consider the structure morphism $f : X = \mathbf{A}^1_k \to \mathrm{Spec}(k) = S$.

(1) Let $x \in X$ be a closed point. What is the dimension of $T_{X/S,x}$?
(2) Let $\eta \in X$ be the generic point. What is the dimension of $T_{X/S,\eta}$?
(3) Consider now X as a scheme over $\mathrm{Spec}(\mathbf{Z})$. What are the dimensions of $T_{X/\mathbf{Z},x}$ and $T_{X/\mathbf{Z},\eta}$?

02F3 Remark 111.35.7. Exercise 111.35.6 explains why it is necessary to consider the tangent space of X over S to get a good notion.

029J Exercise 111.35.8. Consider the morphism of schemes

$$f : X = \mathrm{Spec}(\mathbf{F}_p(t)) \longrightarrow \mathrm{Spec}(\mathbf{F}_p(t^p)) = S$$

Compute the tangent space of X/S at the unique point of X. Isn't that weird? What do you think happens if you take the morphism of schemes corresponding to $\mathbf{F}_p[t^p] \to \mathbf{F}_p[t]$?

029K Exercise 111.35.9. Let k be a field. Compute the tangent space of X/k at the point $x = (0,0)$ where $X = \mathrm{Spec}(k[x,y]/(x^2 - y^3))$.

029L Exercise 111.35.10. Let $f : X \to Y$ be a morphism of schemes over S. Let $x \in X$ be a point. Set $y = f(x)$. Assume that the natural map $\kappa(y) \to \kappa(x)$ is bijective. Show, using the definition, that f induces a natural linear map

$$\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}.$$

Match it with what happens on local rings via Exercise 111.35.2 in case $\kappa(x) = \kappa(s)$.

029M Exercise 111.35.11. Let k be an algebraically closed field. Let

$$\begin{array}{ccc} f : \mathbf{A}^n_k & \longrightarrow & \mathbf{A}^m_k \\ (x_1, \ldots, x_n) & \longmapsto & (f_1(x_i), \ldots, f_m(x_i)) \end{array}$$

be a morphism of schemes over k. This is given by m polynomials $f_1, \ldots, f_m$ in n variables. Consider the matrix

$$A = \left(\frac{\partial f_j}{\partial x_i}\right)$$

Let $x \in \mathbf{A}^n_k$ be a closed point. Set $y = f(x)$. Show that the map on tangent spaces $T_{\mathbf{A}^n_k/k,x} \to T_{\mathbf{A}^m_k/k,y}$ is given by the value of the matrix A at the point x.

111.36. Quasi-coherent Sheaves

029N

029O Definition 111.36.1. Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is quasi-coherent if for every affine open $\mathrm{Spec}(R) = U \subset X$ the restriction $\mathcal{F}|_U$ is of the form $\widetilde{M}$ for some R-module M.

It is enough to check this conditions on the members of an affine open covering of X. See Schemes, Section 26.24 for more results.

029P Definition 111.36.2. Let X be a topological space. Let $x, x' \in X$. We say x is a specialization of x' if and only if $x \in \overline{\{x'\}}$.

029Q Exercise 111.36.3. Let X be a scheme. Let $x, x' \in X$. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Suppose that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$. Show that $\mathcal{F}_x \neq 0$.

029R Exercise 111.36.4. Find an example of a scheme X, points $x, x' \in X$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$ and $\mathcal{F}_x = 0$.

029S Definition 111.36.5. A scheme X is called locally Noetherian if and only if for every point $x \in X$ there exists an affine open $\mathrm{Spec}(R) = U \subset X$ such that R is Noetherian. A scheme is Noetherian if it is locally Noetherian and quasi-compact.

If X is locally Noetherian then any affine open of X is the spectrum of a Noetherian ring, see Properties, Lemma 28.5.2.

029T Definition 111.36.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is coherent if for every point $x \in X$ there exists an affine open $\mathrm{Spec}(R) = U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\widetilde{M}$ for some finite R-module M.

029U Exercise 111.36.7. Let $X = \mathrm{Spec}(R)$ be an affine scheme.

(1) Let $f \in R$. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_{D(f)}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.
(2) Let $I \subset R$ be an ideal. Let $i : Z \to X$ be the closed subscheme of X corresponding to I. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules on the closed subscheme Z. Show that $\mathcal{G} = i^*\mathcal{F}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. (Why is this silly?)
(3) Assume that R is Noetherian. Let $f \in R$. Let $\mathcal{G}$ be a coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_{D(f)}$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

029V Remark 111.36.8. If $U \to X$ is a quasi-compact immersion then any quasi-coherent sheaf on U is the restriction of a quasi-coherent sheaf on X. If X is a Noetherian scheme, and $U \subset X$ is open, then any coherent sheaf on U is the restriction of a coherent sheaf on X. Of course the exercise above is easier, and shouldn't use these general facts.

111.37. Proj and projective schemes

029W

029X Exercise 111.37.1. Give examples of graded rings S such that

(1) $\mathrm{Proj}(S)$ is affine and nonempty, and
(2) $\mathrm{Proj}(S)$ is integral, nonempty but not isomorphic to $\mathbf{P}^n_A$ for any $n \geq 0$, any ring A.

029Y Exercise 111.37.2. Give an example of a nonconstant morphism of schemes $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^5_{\mathbf{C}}$ over $\mathrm{Spec}(\mathbf{C})$.

029Z Exercise 111.37.3. Give an example of an isomorphism of schemes

$$\mathbf{P}^1_{\mathbf{C}} \to \mathrm{Proj}(\mathbf{C}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))$$

02A0 Exercise 111.37.4. Give an example of a morphism of schemes $f : X \to \mathbf{A}^1_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[T])$ such that the (scheme theoretic) fibre X_t of f over $t \in \mathbf{A}^1_{\mathbf{C}}$ is (a) isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when t is a closed point not equal to 0, and (b) not isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when $t = 0$. We will call X_0 the special fibre of the morphism. This can be done in many, many ways. Try to give examples that satisfy (each of) the following additional restraints (unless it isn't possible):

(1) Can you do it with special fibre projective?
(2) Can you do it with special fibre irreducible and projective?
(3) Can you do it with special fibre integral and projective?
(4) Can you do it with special fibre smooth and projective?
(5) Can you do it with f a flat morphism? This just means that for every affine open $\mathrm{Spec}(A) \subset X$ the induced ring map $\mathbf{C}[t] \to A$ is flat, which in this case means that any nonzero polynomial in t is a nonzerodivisor on A.

(6) Can you do it with f a flat and projective morphism?
(7) Can you do it with f flat, projective and special fibre reduced?
(8) Can you do it with f flat, projective and special fibre irreducible?
(9) Can you do it with f flat, projective and special fibre integral?

What do you think happens when you replace $\mathbf{P}^1_{\mathbf{C}}$ with another variety over $\mathbf{C}$? (This can get very hard depending on which of the variants above you ask for.)

02A1 Exercise 111.37.5. Let $n \geq 1$ be any positive integer. Give an example of a surjective morphism $X \to \mathbf{P}^n_{\mathbf{C}}$ with X affine.

02A2 Exercise 111.37.6. Maps of Proj. Let R and S be graded rings. Suppose we have a ring map

$$\psi : R \to S$$

and an integer $e \geq 1$ such that $\psi(R_d) \subset S_{de}$ for all $d \geq 0$. (By our conventions this is not a homomorphism of graded rings, unless $e = 1$.)

(1) For which elements $\mathfrak{p} \in \text{Proj}(S)$ is there a well-defined corresponding point in $\text{Proj}(R)$? In other words, find a suitable open $U \subset \text{Proj}(S)$ such that ψ defines a continuous map $r_\psi : U \to \text{Proj}(R)$.
(2) Give an example where $U \neq \text{Proj}(S)$.
(3) Give an example where $U = \text{Proj}(S)$.
(4) (Do not write this down.) Convince yourself that the continuous map $U \to \text{Proj}(R)$ comes canonically with a map on sheaves so that r_ψ is a morphism of schemes:
$$\text{Proj}(S) \supset U \longrightarrow \text{Proj}(R).$$
(5) What can you say about this map if $R = \bigoplus_{d \geq 0} S_{de}$ (as a graded ring with S_e, S_{2e}, etc in degree 1, 2, etc) and ψ is the inclusion mapping?

Notation. Let R be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \text{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on X such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D_+(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (=elements homogeneous of degree n in $R_f = R[1/f]$). See [Har77, page 116+]. Note that there are natural maps

$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

02A3 Exercise 111.37.7. Pathologies in Proj. Give examples of R as above such that

(1) $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.
(2) $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is NOT an isomorphism.

02A4 Exercise 111.37.8. Let S be a graded ring. Let $X = \text{Proj}(S)$. Show that any finite set of points of X is contained in a standard affine open.

02A5 Exercise 111.37.9. Let S be a graded ring. Let $X = \text{Proj}(S)$. Let $Z, Z' \subset X$ be two closed subschemes. Let $\varphi : Z \to Z'$ be an isomorphism. Assume $Z \cap Z' = \emptyset$. Show that for any $z \in Z$ there exists an affine open $U \subset X$ such that $z \in U$, $\varphi(z) \in U$ and $\varphi(Z \cap U) = Z' \cap U$. (Hint: Use Exercise 111.37.8 and something akin to Schemes, Lemma 26.11.5.)

111.38. Morphisms from the projective line

0DJ0 In this section we study morphisms from $\mathbf{P}^1$ to projective schemes.

0DJ1 Exercise 111.38.1. Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Let X be a finite type scheme over k. Show that for any morphism

$$\varphi : \mathrm{Spec}(k(t)) \longrightarrow X$$

over k, there exist a nonzero $f \in k[t]$ and a morphism $\psi : \mathrm{Spec}(k[t, 1/f]) \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t, 1/f]) \longrightarrow X$$

0DJ2 Exercise 111.38.2. Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Show that for any morphism

$$\varphi : \mathrm{Spec}(k(t)) \longrightarrow \mathbf{P}^n_k$$

over k, there exists a morphism $\psi : \mathrm{Spec}(k[t]) \to \mathbf{P}^n_k$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t]) \longrightarrow \mathbf{P}^n_k$$

Hint: the image of φ is in a standard open $D_+(T_i)$ for some i; then show that you can "clear denominators".

0DJ3 Exercise 111.38.3. Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Let X be a projective scheme over k. Show that for any morphism

$$\varphi : \mathrm{Spec}(k(t)) \longrightarrow X$$

over k, there exists a morphism $\psi : \mathrm{Spec}(k[t]) \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t]) \longrightarrow X$$

Hint: use Exercise 111.38.2.

0DJ4 Exercise 111.38.4. Let k be a field. Let X be a projective scheme over k. Let K be the function field of $\mathbf{P}^1_k$ (see hint below). Show that for any morphism

$$\varphi : \mathrm{Spec}(K) \longrightarrow X$$

over k, there exists a morphism $\psi : \mathbf{P}^1_k \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathbf{P}^1_k \longrightarrow X$$

Hint: use Exercise 111.38.3 for each of the two pieces of the affine open covering $\mathbf{P}^1_k = D_+(T_0) \cup D_+(T_1)$, use that $D_+(T_0)$ is the spectrum of a polynomial ring and that K is the fraction field of this polynomial ring.

111.39. Morphisms from surfaces to curves

02A6

02A7 Exercise 111.39.1. Let R be a ring. Let $R \to k$ be a map from R to a field. Let $n \geq 0$. Show that

$$\mathrm{Mor}_{\mathrm{Spec}(R)}(\mathrm{Spec}(k), \mathbf{P}^n_R) = (k^{n+1} \setminus \{0\})/k^*$$

where k^* acts via scalar multiplication on k^{n+1}. From now on we denote $(x_0 : \ldots : x_n)$ the morphism $\mathrm{Spec}(k) \to \mathbf{P}^n_k$ corresponding to the equivalence class of the element $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$.

02A8 Exercise 111.39.2. Let k be a field. Let $Z \subset \mathbf{P}^2_k$ be an irreducible and reduced closed subscheme. Show that either (a) Z is a closed point, or (b) there exists an homogeneous irreducible $F \in k[X_0, X_1, X_2]$ of degree > 0 such that $Z = V_+(F)$, or (c) $Z = \mathbf{P}^2_k$. (Hint: Look on a standard affine open.)

02A9 Exercise 111.39.3. Let k be a field. Let $Z_1, Z_2 \subset \mathbf{P}^2_k$ be irreducible closed subschemes of the form $V_+(F)$ for some homogeneous irreducible $F_i \in k[X_0, X_1, X_2]$ of degree > 0. Show that $Z_1 \cap Z_2$ is not empty. (Hint: Use dimension theory to estimate the dimension of the local ring of $k[X_0, X_1, X_2]/(F_1, F_2)$ at 0.)

02AA Exercise 111.39.4. Show there does not exist a nonconstant morphism of schemes $\mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ over $\mathrm{Spec}(\mathbf{C})$. Here a constant morphism is one whose image is a single point. (Hint: If the morphism is not constant consider the fibres over 0 and ∞ and argue that they have to meet to get a contradiction.)

02AB Exercise 111.39.5. Let k be a field. Suppose that $X \subset \mathbf{P}^3_k$ is a closed subscheme given by a single homogeneous equation $F \in k[X_0, X_1, X_2, X_3]$. In other words,

$$X = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F)) \subset \mathbf{P}^3_k$$

as explained in the course. Assume that

$$F = X_0 G + X_1 H$$

for some homogeneous polynomials $G, H \in k[X_0, X_1, X_2, X_3]$ of positive degree. Show that if X_0, X_1, G, H have no common zeros then there exists a nonconstant morphism

$$X \longrightarrow \mathbf{P}^1_k$$

of schemes over $\mathrm{Spec}(k)$ which on field points (see Exercise 111.39.1) looks like $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1)$ whenever x_0 or x_1 is not zero.

111.40. Invertible sheaves

02AC

02AD Definition 111.40.1. Let X be a locally ringed space. An invertible $\mathcal{O}_X$-module on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that every point has an open neighbourhood $U \subset X$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$ as $\mathcal{O}_U$-module. We say that $\mathcal{L}$ is trivial if it is isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module.

02AE Exercise 111.40.2. General facts.

(1) Show that an invertible $\mathcal{O}_X$-module on a scheme X is quasi-coherent.
(2) Suppose $X \to Y$ is a morphism of locally ringed spaces, and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Show that $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$ module.

02AF Exercise 111.40.3. Algebra.

(1) Show that an invertible $\mathcal{O}_X$-module on an affine scheme $\mathrm{Spec}(A)$ corresponds to an A-module M which is (i) finite, (ii) projective, (iii) locally free of rank 1, and hence (iv) flat, and (v) finitely presented. (Feel free to quote things from last semesters course; or from algebra books.)
(2) Suppose that A is a domain and that M is a module as in (a). Show that M is isomorphic as an A-module to an ideal $I \subset A$ such that $IA_{\mathfrak{p}}$ is principal for every prime $\mathfrak{p}$.

02AG Definition 111.40.4. Let R be a ring. An invertible module M is an R-module M such that $\widetilde{M}$ is an invertible sheaf on the spectrum of R. We say M is trivial if $M \cong R$ as an R-module.

In other words, M is invertible if and only if it satisfies all of the following conditions: it is flat, of finite presentation, projective, and locally free of rank 1. (Of course it suffices for it to be locally free of rank 1).

02AH Exercise 111.40.5. Simple examples.

02AI (1) Let k be a field. Let $A = k[x]$. Show that $X = \mathrm{Spec}(A)$ has only trivial invertible $\mathcal{O}_X$-modules. In other words, show that every invertible A-module is free of rank 1.

02AJ (2) Let A be the ring

$$A = \{f \in k[x] \mid f(0) = f(1)\}.$$

Show there exists a nontrivial invertible A-module, unless $k = \mathbf{F}_2$. (Hint: Think about $\mathrm{Spec}(A)$ as identifying 0 and 1 in $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$.)

02AK (3) Same question as in (2) for the ring $A = k[x^2, x^3] \subset k[x]$ (except now $k = \mathbf{F}_2$ works as well).

02AL Exercise 111.40.6. Higher dimensions.

(1) Prove that every invertible sheaf on two dimensional affine space is trivial. More precisely, let $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$ where k is a field. Show that every invertible sheaf on $\mathbf{A}^2_k$ is trivial. (Hint: One way to do this is to consider the corresponding module M, to look at $M \otimes_{k[x,y]} k(x)[y]$, and then use Exercise 111.40.5 (1) to find a generator for this; then you still have to think. Another way to is to use Exercise 111.40.3 and use what we know about ideals of the polynomial ring: primes of height one are generated by an irreducible polynomial; then you still have to think.)

(2) Prove that every invertible sheaf on any open subscheme of two dimensional affine space is trivial. More precisely, let $U \subset \mathbf{A}^2_k$ be an open subscheme where k is a field. Show that every invertible sheaf on U is trivial. Hint: Show that every invertible sheaf on U extends to one on $\mathbf{A}^2_k$. Not easy; but you can find it in [Har77].

(3) Find an example of a nontrivial invertible sheaf on a punctured cone over a field. More precisely, let k be a field and let $C = \mathrm{Spec}(k[x, y, z]/(xy - z^2))$. Let $U = C \setminus \{(x, y, z)\}$. Find a nontrivial invertible sheaf on U. Hint: It may be easier to compute the group of isomorphism classes of invertible sheaves on U than to just find one. Note that U is covered by the opens $\mathrm{Spec}(k[x, y, z, 1/x]/(xy - z^2))$ and $\mathrm{Spec}(k[x, y, z, 1/y]/(xy - z^2))$ which are "easy" to deal with.

02AM Definition 111.40.7. Let X be a locally ringed space. The Picard group of X is the set $\mathrm{Pic}(X)$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with addition given by tensor product. See Modules, Definition 17.25.9. For a ring R we set $\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Spec}(R))$.

02AN Exercise 111.40.8. Let R be a ring.

(1) Show that if R is a Noetherian normal domain, then $\mathrm{Pic}(R) = \mathrm{Pic}(R[t])$. [Hint: There is a map $R[t] \to R$, $t \mapsto 0$ which is a left inverse to the map $R \to R[t]$. Hence it suffices to show that any invertible $R[t]$-module

M such that $M/tM \cong R$ is free of rank 1. Let K be the fraction field of R. Pick a trivialization $K[t] \to M \otimes_{R[t]} K[t]$ which is possible by Exercise 111.40.5 (1). Adjust it so it agrees with the trivialization of M/tM above. Show that it is in fact a trivialization of M over $R[t]$ (this is where normality comes in).]

(2) Let k be a field. Show that $\mathrm{Pic}(k[x^2, x^3, t]) \neq \mathrm{Pic}(k[x^2, x^3])$.

111.41. Čech Cohomology

02AO

02F4 Exercise 111.41.1. Čech cohomology. Here k is a field.

(1) Let X be a scheme with an open covering $\mathcal{U} : X = U_1 \cup U_2$, with $U_1 = \mathrm{Spec}(k[x])$, $U_2 = \mathrm{Spec}(k[y])$ with $U_1 \cap U_2 = \mathrm{Spec}(k[z, 1/z])$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We've seen in the lectures that such an X exists; it is the affine line with zero doubled.) Compute $\check{H}^1(\mathcal{U}, \mathcal{O})$; eg. give a basis for it as a k-vectorspace.

(2) For each element in $\check{H}^1(\mathcal{U}, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules
$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$
such that the boundary $\delta(1) \in \check{H}^1(\mathcal{U}, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. See also below. It is also OK to show abstractly such a thing has to exist.)

02AP Definition 111.41.2. (Definition of delta.) Suppose that
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
is a short exact sequence of abelian sheaves on any topological space X. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to \check{H}^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that for each i there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of τ to U_i. Then consider the assignment
$$(i_0, i_1) \longmapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$
This is clearly a 1-coboundary in the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

111.42. Cohomology

0D8P

0D8Q Exercise 111.42.1. Let $X = \mathbf{R}$ with the usual Euclidean topology. Using only formal properties of cohomology (functoriality and the long exact cohomology sequence) show that there exists a sheaf $\mathcal{F}$ on X with nonzero $H^1(X, \mathcal{F})$.

0D8R Exercise 111.42.2. Let $X = U \cup V$ be a topological space written as the union of two opens. Then we have a long exact Mayer-Vietoris sequence
$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$
What property of injective sheaves is essential for the construction of the Mayer-Vietoris long exact sequence? Why does it hold?

0D8S Exercise 111.42.3. Let X be a topological space.

(1) Show that $H^i(X, \mathcal{F})$ is zero for $i > 0$ if X has 2 or fewer points.
(2) What if X has 3 points?

0D8T Exercise 111.42.4. Let X be the spectrum of a local ring. Show that $H^i(X, \mathcal{F})$ is zero for $i > 0$ and any sheaf of abelian groups $\mathcal{F}$.

0D8U Exercise 111.42.5. Let $f : X \to Y$ be an affine morphism of schemes. Prove that $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Feel free to impose some further conditions on X and Y and use the agreement of Čech cohomology with cohomology for quasi-coherent sheaves and affine open coverings of separated schemes.

0D8V Exercise 111.42.6. Let A be a ring. Let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Let $\mathbf{A}^{n+1}_A = \text{Spec}(A[T_0, \ldots, T_n])$ and let

$$U = \bigcup\nolimits_{i=0,\ldots,n} D(T_i) \subset \mathbf{A}^{n+1}_A$$

be the complement of the image of the closed immersion $0 : \text{Spec}(A) \to \mathbf{A}^{n+1}_A$. Construct an affine surjective morphism

$$f : U \longrightarrow \mathbf{P}^n_A$$

and prove that $f_*\mathcal{O}_U = \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{\mathbf{P}^n_A}(d)$. More generally, show that for a graded $A[T_0, \ldots, T_n]$-module M one has

$$f_*(\widetilde{M}|_U) = \bigoplus\nolimits_{d \in \mathbf{Z}} \widetilde{M(d)}$$

where on the left hand side we have the quasi-coherent sheaf $\widetilde{M}$ associated to M on $\mathbf{A}^{n+1}_A$ and on the right we have the quasi-coherent sheaves $\widetilde{M(d)}$ associated to the graded module $M(d)$.

0D8W Exercise 111.42.7. Let A be a ring and let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Carefully compute the cohomology of the Serre twists $\mathcal{O}_{\mathbf{P}^n_A}(d)$ of the structure sheaf on $\mathbf{P}^n_A$. Feel free to use Čech cohomology and the agreement of Čech cohomology with cohomology for quasi-coherent sheaves and affine open coverings of separated schemes.

0D8X Exercise 111.42.8. Let A be a ring and let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Let $F \in A[T_0, \ldots, T_n]$ be homogeneous of degree d. Let $X \subset \mathbf{P}^n_A$ be the closed subscheme corresponding to the graded ideal (F) of $A[T_0, \ldots, T_n]$. What can you say about $H^i(X, \mathcal{O}_X)$?

0D8Y Exercise 111.42.9. Let R be a ring such that for any left exact functor $F : \text{Mod}_R \to \text{Ab}$ we have $R^iF = 0$ for $i > 0$. Show that R is a finite product of fields.

111.43. More cohomology

0DAI

0DAJ Exercise 111.43.1. Let k be a field. Let $X \subset \mathbf{P}^n_k$ be the "coordinate cross". Namely, let X be defined by the homogeneous equations

$$T_iT_j = 0 \text{ for } i > j > 0$$

where as usual we write $\mathbf{P}^n_k = \text{Proj}(k[T_0, \ldots, T_n])$. In other words, X is the closed subscheme corresponding to the quotient $k[T_0, \ldots, T_n]/(T_iT_j; i > j > 0)$ of the polynomial ring. Compute $H^i(X, \mathcal{O}_X)$ for all i. Hint: use Čech cohomology.

0DAK Exercise 111.43.2. Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \text{Spec}(A)$ be the complement of $V(I)$. For any A-module M write down a complex of A-modules (in terms of A, $f_1, \ldots, f_t$, M) whose cohomology groups give $H^n(U, \widetilde{M})$.

0DAL Exercise 111.43.3. Let k be a field. Let $U \subset \mathbf{A}^d_k$ be the complement of the closed point 0 of $\mathbf{A}^d_k$. Compute $H^n(U, \mathcal{O}_U)$ for all n.

0DAM Exercise 111.43.4. Let k be a field. Find explicitly a scheme X projective over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ and $\dim_k H^1(X, \mathcal{O}_X) = 100$.

0DAN Exercise 111.43.5. Let $f : X \to Y$ be a finite locally free morphism of degree 2. Assume that X and Y are integral schemes and that 2 is invertible in the structure sheaf of Y, i.e., $2 \in \Gamma(Y, \mathcal{O}_Y)$ is invertible. Show that the $\mathcal{O}_Y$-module map

$$f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

has a left inverse, i.e., there is an $\mathcal{O}_Y$-module map $\tau : f_*\mathcal{O}_X \to \mathcal{O}_Y$ with $\tau \circ f^\sharp = \text{id}$. Conclude that $H^n(Y, \mathcal{O}_Y) \to H^n(X, \mathcal{O}_X)$ is injective[2].

0DAP Exercise 111.43.6. Let X be a scheme (or a locally ringed space). The rule $U \mapsto \mathcal{O}_X(U)^*$ defines a sheaf of groups denoted $\mathcal{O}^*_X$. Briefly explain why the Picard group of X (Definition 111.40.7) is equal to $H^1(X, \mathcal{O}^*_X)$.

0DAQ Exercise 111.43.7. Give an example of an affine scheme X with nontrivial $\text{Pic}(X)$. Conclude using Exercise 111.43.6 that $H^1(X, -)$ is not the zero functor for any such X.

0DAR Exercise 111.43.8. Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \text{Spec}(A)$ be the complement of $V(I)$. Given a quasi-coherent $\mathcal{O}_{\text{Spec}(A)}$-module $\mathcal{F}$ and $\xi \in H^p(U, \mathcal{F})$ with $p > 0$, show that there exists $n > 0$ such that $f_i^n\xi = 0$ for $i = 1, \ldots, t$. Hint: One possible way to proceed is to use the complex you found in Exercise 111.43.2.

0DAS Exercise 111.43.9. Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \text{Spec}(A)$ be the complement of $V(I)$. Let M be an A-module whose I-torsion is zero, i.e., $0 = \text{Ker}((f_1, \ldots, f_t) : M \to M^{\oplus t})$. Show that there is a canonical isomorphism

$$H^0(U, \widetilde{M}) = \text{colim}\, \text{Hom}_A(I^n, M).$$

Warning: this is not trivial.

0DAT Exercise 111.43.10. Let A be a Noetherian ring. Let I be an ideal of A. Let M be an A-module. Let $M[I^\infty]$ be the set of I-power torsion elements defined by

$$M[I^\infty] = \{x \in M \mid \text{ there exists an } n \geq 1 \text{ such that } I^nx = 0\}$$

Set $M' = M/M[I^\infty]$. Then the I-power torsion of M' is zero. Show that

$$\text{colim}\, \text{Hom}_A(I^n, M) = \text{colim}\, \text{Hom}_A(I^n, M').$$

[2]There does exist a finite locally free morphism $X \to Y$ between integral schemes of degree 2 where the map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ is not injective.

Warning: this is not trivial. Hints: (1) try to reduce to M finite, (2) show any element of $\mathrm{Ext}^1_A(I^n, N)$ maps to zero in $\mathrm{Ext}^1_A(I^{n+m}, N)$ for some $m > 0$ if $N = M[I^\infty]$ and M finite, (3) show the same thing as in (2) for $\mathrm{Hom}_A(I^n, N)$, (3) consider the long exact sequence

$$0 \to \mathrm{Hom}_A(I^n, M[I^\infty]) \to \mathrm{Hom}_A(I^n, M) \to \mathrm{Hom}_A(I^n, M') \to \mathrm{Ext}^1_A(I^n, M[I^\infty])$$

for M finite and compare with the sequence for I^{n+m} to conclude.

111.44. Cohomology revisited

0DB3

0DB4 Exercise 111.44.1. Make an example of a field k, a curve X over k, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a cohomology class $\xi \in H^1(X, \mathcal{L})$ with the following property: for every surjective finite morphism $\pi : Y \to X$ of schemes the element ξ pulls back to a nonzero element of $H^1(Y, \pi^*\mathcal{L})$. Hint: construct X, k, $\mathcal{L}$ such that there is a short exact sequence $0 \to \mathcal{L} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ where $Z \subset X$ is a closed subscheme consisting of more than 1 closed point. Then look at what happens when you pullback.

0DB5 Exercise 111.44.2. Let k be an algebraically closed field. Let X be a projective 1-dimensional scheme. Suppose that X contains a cycle of curves, i.e., suppose there exist an $n \geq 2$ and pairwise distinct 1-dimensional integral closed subschemes $C_1, \ldots, C_n$ and pairwise distinct closed points $x_1, \ldots, x_n \in X$ such that $x_n \in C_n \cap C_1$ and $x_i \in C_i \cap C_{i+1}$ for $i = 1, \ldots, n-1$. Prove that $H^1(X, \mathcal{O}_X)$ is nonzero. Hint: Let $\mathcal{F}$ be the image of the map $\mathcal{O}_X \to \bigoplus \mathcal{O}_{C_i}$, and show $H^1(X, \mathcal{F})$ is nonzero using that $\kappa(x_i) = k$ and $H^0(C_i, \mathcal{O}_{C_i}) = k$. Also use that $H^2(X, -) = 0$ by Grothendieck's theorem.

0DB6 Exercise 111.44.3. Let X be a projective surface over an algebraically closed field k. Prove there exists a proper closed subscheme $Z \subset X$ such that $H^1(Z, \mathcal{O}_Z)$ is nonzero. Hint: Use Exercise 111.44.2.

0DB7 Exercise 111.44.4. Let X be a projective surface over an algebraically closed field k. Show that for every $n \geq 0$ there exists a proper closed subscheme $Z \subset X$ such that $\dim_k H^1(Z, \mathcal{O}_Z) > n$. Only explain how to do this by modifying the arguments in Exercise 111.44.3 and 111.44.2; don't give all the details.

0DB8 Exercise 111.44.5. Let X be a projective surface over an algebraically closed field k. Prove there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $H^2(X, \mathcal{F})$ is nonzero. Hint: Use the result of Exercise 111.44.4 and a cleverly chosen exact sequence.

0DB9 Exercise 111.44.6. Let X and Y be schemes over a field k (feel free to assume X and Y are nice, for example qcqs or projective over k). Set $X \times Y = X \times_{\mathrm{Spec}(k)} Y$ with projections $p : X \times Y \to X$ and $q : X \times Y \to Y$. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ prove that

$$H^n(X \times Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G}) = \bigoplus\nolimits_{a+b=n} H^a(X, \mathcal{F}) \otimes_k H^b(Y, \mathcal{G})$$

or just show that this holds when one takes dimensions. Extra points for "clean" solutions.

0DBA Exercise 111.44.7. Let k be a field. Let $X = \mathbf{P}|^1 \times \mathbf{P}^1$ be the product of the projective line over k with itself with projections $p : X \to \mathbf{P}^1_k$ and $q : X \to \mathbf{P}^1_k$. Let

$$\mathcal{O}(a, b) = p^*\mathcal{O}_{\mathbf{P}^1_k}(a) \otimes_{\mathcal{O}_X} q^*\mathcal{O}_{\mathbf{P}^1_k}(b)$$

Compute the dimensions of $H^i(X, \mathcal{O}(a, b))$ for all i, a, b. Hint: Use Exercise 111.44.6.

111.45. Cohomology and Hilbert polynomials

0DCD

0DCE Situation 111.45.1. Let k be a field. Let $X = \mathbf{P}^n_k$ be n-dimensional projective space. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Recall that

$$\chi(X, \mathcal{F}) = \sum\nolimits_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathcal{F})$$

Recall that the Hilbert polynomial of $\mathcal{F}$ is the function

$$t \longmapsto \chi(X, \mathcal{F}(t))$$

We also recall that $\mathcal{F}(t) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$ where $\mathcal{O}_X(t)$ is the tth twist of the structure sheaf as in Constructions, Definition 27.10.1. In Varieties, Subsection 33.35.13 we have proved the Hilbert polynomial is a polynomial in t.

0DCF Exercise 111.45.2. In Situation 111.45.1.

(1) If $P(t)$ is the Hilbert polynomial of $\mathcal{F}$, what is the Hilbert polynomial of $\mathcal{F}(-13)$.
(2) If P_i is the Hilbert polynomial of $\mathcal{F}_i$, what is the Hilbert polynomial of $\mathcal{F}_1 \oplus \mathcal{F}_2$.
(3) If P_i is the Hilbert polynomial of $\mathcal{F}_i$ and $\mathcal{F}$ is the kernel of a surjective map $\mathcal{F}_1 \to \mathcal{F}_2$, what is the Hilbert polynomial of $\mathcal{F}$?

0DCG Exercise 111.45.3. In Situation 111.45.1 assume $n \geq 1$. Find a coherent sheaf whose Hilbert polynomial is $t - 101$.

0DCH Exercise 111.45.4. In Situation 111.45.1 assume $n \geq 2$. Find a coherent sheaf whose Hilbert polynomial is $t^2/2 + t/2 - 1$. (This is a bit tricky; it suffices if you just show there is such a coherent sheaf.)

0DCI Exercise 111.45.5. In Situation 111.45.1 assume $n \geq 2$ and k algebraically closed. Let $C \subset X$ be an integral closed subscheme of dimension 1. In other words, C is a projective curve. Let $dt + e$ be the Hilbert polynomial of $\mathcal{O}_C$ viewed as a coherent sheaf on X.

(1) Give an upper bound on e. (Hints: Use that $\mathcal{O}_C(t)$ only has cohomology in degrees 0 and 1 and study $H^0(C, \mathcal{O}_C)$.)
(2) Pick a global section s of $\mathcal{O}_X(1)$ which intersects C transversally, i.e., such that there are pairwise distinct closed points $c_1, \ldots, c_r \in C$ and a short exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{O}_C(1) \to \bigoplus\nolimits_{i=1,\ldots,r} k_{c_i} \to 0$$

where k_{c_i} is the skyscraper sheaf with value k in c_i. (Such an s exists; please just use this.) Show that $r = d$. (Hint: twist the sequence and see what you get.)
(3) Twisting the short exact sequence gives a k-linear map $\varphi_t : \Gamma(C, \mathcal{O}_C(t)) \to \bigoplus_{i=1,\ldots,d} k$ for any t. Show that if this map is surjective for $t \geq d - 1$.

(4) Give a lower bound on e in terms of d. (Hint: show that $H^1(C, \mathcal{O}_C(d-2)) = 0$ using the result of (3) and use vanishing.)

0DCJ Exercise 111.45.6. In Situation 111.45.1 assume $n = 2$. Let $s_1, s_2, s_3 \in \Gamma(X, \mathcal{O}_X(2))$ be three quadric equations. Consider the coherent sheaf

$$\mathcal{F} = \text{Coker}\left(\mathcal{O}_X(-2)^{\oplus 3} \xrightarrow{s_1, s_2, s_3} \mathcal{O}_X\right)$$

List the possible Hilbert polynomials of such $\mathcal{F}$. (Try to visualize intersections of quadrics in the projective plane.)

111.46. Curves

0EG8

0EG9 Exercise 111.46.1. Let k be an algebraically closed field. Let X be a projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ be global sections of $\mathcal{L}$. Prove there is a natural closed subscheme

$$Z \subset \mathbf{P}^n \times X$$

such that the closed point $((\lambda_0 : \ldots : \lambda_n), x)$ is in Z if and only if the section $\lambda_0 s_0 + \ldots + \lambda_n s_n$ vanishes at x. (Hint: describe Z affine locally.)

0EGA Exercise 111.46.2. Let k be an algebraically closed field. Let X be a smooth curve over k. Let $r \geq 1$. Show that the closed subset

$$D \subset X \times X^r = X^{r+1}$$

whose closed points are the tuples $(x, x_1, \ldots, x_r)$ with $x = x_i$ for some i, has an invertible ideal sheaf. In other words, show that D is an effective Cartier divisor. Hints: reduce to $r = 1$ and use that X is a smooth curves to say something about the diagonal (look in books for this).

0EGB Exercise 111.46.3. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let T be a scheme of finite type over k and let

$$D_1 \subset X \times T \quad \text{and} \quad D_2 \subset X \times T$$

be two effective Cartier divisors such that for $t \in T$ the fibres $D_{i,t} \subset X_t$ are not dense (i.e., do not contain the generic point of the curve X_t). Prove that there is a canonical closed subscheme $Z \subset T$ such that a closed point $t \in T$ is in Z if and only if for the scheme theoretic fibres $D_{1,t}$, $D_{2,t}$ of D_1, D_2 we have

$$D_{1,t} \subset D_{2,t}$$

as closed subschemes of X_t. Hints: Show that, possibly after shrinking T, you may assume $T = \text{Spec}(A)$ is affine and there is an affine open $U \subset X$ such that $D_i \subset U \times T$. Then show that $M_1 = \Gamma(D_1, \mathcal{O}_{D_1})$ is a finite locally free A-module (here you will need some nontrivial algebra — ask your friends). After shrinking T you may assume M_1 is a free A-module. Then look at

$$\Gamma(U \times T, \mathcal{I}_{D_2}) \to M_1 = A^{\oplus N}$$

and you define Z as the closed subscheme cut out by the ideal generated by coefficients of vectors in the image of this map. Explain why this works (this will require perhaps a bit more commutative algebra).

0EGC Exercise 111.46.4. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ be global sections of $\mathcal{L}$. Let $r \geq 1$. Prove there is a natural closed subscheme

$$Z \subset \mathbf{P}^n \times X \times \ldots \times X = \mathbf{P}^n \times X^r$$

such that the closed point $((\lambda_0 : \ldots : \lambda_n), x_1, \ldots, x_r)$ is in Z if and only if the section $s_\lambda = \lambda_0 s_0 + \ldots + \lambda_n s_n$ vanishes on the divisor $D = x_1 + \ldots + x_r$, i.e., the section s_λ is in $\mathcal{L}(-D)$. Hint: explain how this follows by combining then results of Exercises 111.46.2 and 111.46.3.

0EGD Exercise 111.46.5. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that there is a natural closed subset

$$Z \subset X^r$$

such that a closed point $(x_1, \ldots, x_r)$ of X^r is in Z if and only if $\mathcal{L}(-x_1 - \ldots - x_r)$ has a nonzero global section. Hint: use Exercise 111.46.4.

0EGE Exercise 111.46.6. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $r \geq s$ be integers. Show that there is a natural closed subset

$$Z \subset X^r \times X^s$$

such that a closed point $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ of $X^r \times X^s$ is in Z if and only if $x_1 + \ldots + x_r - y_1 - \ldots - y_s$ is linearly equivalent to an effective divisor. Hint: Choose an auxilliary invertible module $\mathcal{L}$ of very high degree so that $\mathcal{L}(-D)$ has a nonvanshing section for any effective divisor D of degree r. Then use the result of Exercise 111.46.5 twice.

0EGR Exercise 111.46.7. Choose your favorite algebraically closed field k. As best as you can determine all possible $\mathfrak{g}^r_d$ that can exist on some curve of genus 7. While doing this also try to

(1) determine in which cases the $\mathfrak{g}^r_d$ is base point free, and
(2) determine in which cases the $\mathfrak{g}^r_d$ gives a closed embedding in $\mathbf{P}^r$.

Do the same thing if you assume your curve is "general" (make up your own notion of general – this may be easier than the question above). Do the same thing if you assume your curve is hyperelliptic. Do the same thing if you assume your curve is trigonal (and not hyperelliptic). Etc.

111.47. Moduli

0EGN In this section we consider some naive approaches to moduli of algebraic geometric objects.

Let k be an algebraically closed field. Suppose that M is a moduli problem over k. We won't define exactly what this means here, but in each exercise it should be clear what we mean. To understand the following it suffices to know what the objects of M over k are, what the isomorphisms between objects of M over k are, and what the families of object of M over a variety are. Then we say the number of moduli of M is $d \geq 0$ if the following are true

(1) there is a finite number of families $X_i \to V_i$, $i = 1, \ldots, n$ such that every object of M over k is isomorphic to a fibre of one of these and such that $\max \dim(V_i) = d$, and

(2) there is no way to do this with a smaller d.

This is really just a very rough approximation of better notions in the literature.

0EGP Exercise 111.47.1. Let k be an algebraically closed field. Let $d \geq 1$ and $n \geq 1$. Let us say the moduli of hypersurfaces of degree d in P^n is given by

(1) an object is a hypersurface $X \subset \mathbf{P}^n_k$ of degree d,
(2) an isomorphism between two objects $X \subset \mathbf{P}^n_k$ and $Y \subset \mathbf{P}^n_k$ is an element $g \in \mathrm{PGL}_n(k)$ such that $g(X) = Y$, and
(3) a family of hypersurfaces over a variety V is a closed subscheme $X \subset \mathbf{P}^n_V$ such that for all $v \in V$ the scheme theoretic fibre X_v of $X \to V$ is a hypersurfaces in $\mathbf{P}^n_v$.

Compute (if you can – these get progressively harder)

(1) the number of moduli when $n = 1$ and d arbitrary,
(2) the number of moduli when $n = 2$ and $d = 1$,
(3) the number of moduli when $n = 2$ and $d = 2$,
(4) the number of moduli when $n \geq 1$ and $d = 2$,
(5) the number of moduli when $n = 2$ and $d = 3$,
(6) the number of moduli when $n = 3$ and $d = 3$, and
(7) the number of moduli when $n = 2$ and $d = 4$.

0EGQ Exercise 111.47.2. Let k be an algebraically closed field. Let $g \geq 2$. Let us say the moduli of hyperelliptic curves of genus g is given by

(1) an object is a smooth projective hyperelliptic curve X of genus g,
(2) an isomorphism between two objects X and Y is an isomorphism $X \to Y$ of schemes over k, and
(3) a family of hyperelliptic curves of genus g over a variety V is a proper flat[3] morphism $X \to Y$ such that all scheme theoretic fibres of $X \to V$ are smooth projective hyperelliptic curves of genus g.

Show that the number of moduli is $2g - 1$.

111.48. Global Exts

0DD0

0DD1 Exercise 111.48.1. Let k be a field. Let $X = \mathbf{P}^3_k$. Let $L \subset X$ and $P \subset X$ be a line and a plane, viewed as closed subschemes cut out by 1, resp., 2 linear equations. Compute the dimensions of

$$\mathrm{Ext}^i_X(\mathcal{O}_L, \mathcal{O}_P)$$

for all i. Make sure to do both the case where L is contained in P and the case where L is not contained in P.

0DD2 Exercise 111.48.2. Let k be a field. Let $X = \mathbf{P}^n_k$. Let $Z \subset X$ be a closed k-rational point viewed as a closed subscheme. For example the point with homogeneous coordinates $(1 : 0 : \ldots : 0)$. Compute the dimensions of

$$\mathrm{Ext}^i_X(\mathcal{O}_Z, \mathcal{O}_Z)$$

for all i.

[3]You can drop this assumption without changing the answer to the question.

0DD3 Exercise 111.48.3. Let X be a ringed space. Define cup-product maps

$$\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{H}) \times \mathrm{Ext}^j_X(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^{i+j}_X(\mathcal{F}, \mathcal{H})$$

for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$. (Hint: this is a super general thing.)

0DD4 Exercise 111.48.4. Let X be a ringed space. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. Prove the following statements

(1) $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$, and
(2) $\mathrm{Ext}^i_X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) = \mathrm{Ext}^i_X(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$.

Here $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Conclude that

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$$

0DD5 Exercise 111.48.5. Let X be a ringed space. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Construct a trace map

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E}) \to H^i(X, \mathcal{O}_X)$$

for all i. Generalize to a trace map

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \to H^i(X, \mathcal{F})$$

for any $\mathcal{O}_X$-module $\mathcal{F}$.

0DD6 Exercise 111.48.6. Let k be a field. Let $X = \mathbf{P}^d_k$. Set $\omega_{X/k} = \mathcal{O}_X(-d-1)$. Prove that for finite locally free modules $\mathcal{E}$, $\mathcal{F}$ the cup product on Ext combined with the trace map on Ext

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/k}) \times \mathrm{Ext}^{d-i}_X(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^d_X(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/k}) \to H^d(X, \omega_{X/k}) = k$$

produces a nondegenerate pairing. Hint: you can either reprove duality in this setting or you can reduce to cohomology of sheaves and apply the Serre duality theorem as proved in the lectures.

111.49. Divisors

02AQ We collect all relevant definitions here in one spot for convenience.

02AR Definition 111.49.1. Throughout, let S be any scheme and let X be a Noetherian, integral scheme.

(1) A Weil divisor on X is a formal linear combination $\Sigma n_i[Z_i]$ of prime divisors Z_i with integer coefficients.
(2) A prime divisor is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. We will use the notation $\mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ when $\xi \in Z \subset X$ is as above. Note that $\mathcal{O}_{X,Z} \subset K(X)$ is a subring of the function field of X.
(3) The Weil divisor associated to a rational function $f \in K(X)^*$ is the sum $\Sigma v_Z(f)[Z]$. Here $v_Z(f)$ is defined as follows
    (a) If $f \in \mathcal{O}^*_{X,Z}$ then $v_Z(f) = 0$.
    (b) If $f \in \mathcal{O}_{X,Z}$ then

$$v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f)).$$

    (c) If $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{X,Z}$ then

$$v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

(4) An effective Cartier divisor on a scheme S is a closed subscheme $D \subset S$ such that every point $d \in D$ has an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ in S so that $D \cap U = \mathrm{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.

(5) The Weil divisor $[D]$ associated to an effective Cartier divisor $D \subset X$ of our Noetherian integral scheme X is defined as the sum $\Sigma v_Z(D)[Z]$ where $v_Z(D)$ is defined as follows

  (a) If the generic point ξ of Z is not in D then $v_Z(D) = 0$.

  (b) If the generic point ξ of Z is in D then

$$v_Z(D) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f))$$

  where $f \in \mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ is the nonzerodivisor which defines D in an affine neighbourhood of ξ (as in (4) above).

(6) Let S be a scheme. The sheaf of total quotient rings $\mathcal{K}_S$ is the sheaf of $\mathcal{O}_S$-algebras which is the sheafification of the pre-sheaf $\mathcal{K}'$ defined as follows. For $U \subset S$ open we set $\mathcal{K}'(U) = S_U^{-1}\mathcal{O}_S(U)$ where $S_U \subset \mathcal{O}_S(U)$ is the multiplicative subset consisting of sections $f \in \mathcal{O}_S(U)$ such that the germ of f in $\mathcal{O}_{S,u}$ is a nonzerodivisor for every $u \in U$. In particular the elements of S_U are all nonzerodivisors. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{K}_S$, and we get a short exact sequence

$$0 \to \mathcal{O}_S^* \to \mathcal{K}_S^* \to \mathcal{K}_S^*/\mathcal{O}_S^* \to 0.$$

(7) A Cartier divisor on a scheme S is a global section of the quotient sheaf $\mathcal{K}_S^*/\mathcal{O}_S^*$.

(8) The Weil divisor associated to a Cartier divisor $\tau \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ over our Noetherian integral scheme X is the sum $\Sigma v_Z(\tau)[Z]$ where $v_Z(\tau)$ is defined as by the following recipe

  (a) If the germ of τ at the generic point ξ of Z is zero – in other words the image of τ in the stalk $(\mathcal{K}^*/\mathcal{O}^*)_\xi$ is "zero" – then $v_Z(\tau) = 0$.

  (b) Find an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ so that $\tau|_U$ is the image of a section $f \in \mathcal{K}(U)$ and moreover $f = a/b$ with $a, b \in A$. Then we set

$$v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

02F5 Remarks 111.49.2. Here are some trivial remarks.

(1) On a Noetherian integral scheme X the sheaf $\mathcal{K}_X$ is constant with value the function field $K(X)$.

(2) To make sense out of the definitions above one needs to show that

$$\mathrm{length}_{\mathcal{O}}(\mathcal{O}/(ab)) = \mathrm{length}_{\mathcal{O}}(\mathcal{O}/(a)) + \mathrm{length}_{\mathcal{O}}(\mathcal{O}/(b))$$

for any pair (a, b) of nonzero elements of a Noetherian 1-dimensional local domain $\mathcal{O}$. This will be done in the lectures.

02F6 Exercise 111.49.3. (On any scheme.) Describe how to assign a Cartier divisor to an effective Cartier divisor.

02F7 Exercise 111.49.4. (On an integral scheme.) Describe how to assign a Cartier divisor D to a rational function f such that the Weil divisor associated to D and to f agree. (This is silly.)

02F8 Exercise 111.49.5. Give an example of a Weil divisor on a variety which is not the Weil divisor associated to any Cartier divisor.

02F9 Exercise 111.49.6. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor but such that nD is the Weil divisor associated to a Cartier divisor for some $n > 1$.

02FA Exercise 111.49.7. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor and such that nD is NOT the Weil divisor associated to a Cartier divisor for any $n > 1$. (Hint: Consider a cone, for example $X : xy - zw = 0$ in $\mathbf{A}^4_k$. Try to show that $D = [x = 0, z = 0]$ works.)

02FB Exercise 111.49.8. On a separated scheme X of finite type over a field: Give an example of a Cartier divisor which is not the difference of two effective Cartier divisors. Hint: Find some X which does not have any nonempty effective Cartier divisors for example the scheme constructed in [Har77, III Exercise 5.9]. There is even an example with X a variety – namely the variety of Exercise 111.49.9.

02AS Exercise 111.49.9. Example of a nonprojective proper variety. Let k be a field. Let $L \subset \mathbf{P}^3_k$ be a line and let $C \subset \mathbf{P}^3_k$ be a nonsingular conic. Assume that $C \cap L = \emptyset$. Choose an isomorphism $\varphi : L \to C$. Let X be the k-variety obtained by glueing C to L via φ. In other words there is a surjective proper birational morphism

$$\pi : \mathbf{P}^3_k \longrightarrow X$$

and an open $U \subset X$ such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism, $\pi^{-1}(U) = \mathbf{P}^3_k \setminus (L \cup C)$ and such that $\pi|_L = \pi|_C \circ \varphi$. (These conditions do not yet uniquely define X. In order to do this you need to specify the structure sheaf of X along points of $Z = X \setminus U$.) Show X exists, is a proper variety, but is not projective. (Hint: For existence use the result of Exercise 111.37.9. For non-projectivity use that $\text{Pic}(\mathbf{P}^3_k) = \mathbf{Z}$ to show that X cannot have an ample invertible sheaf.)

111.50. Differentials

02AT Definitions and results. Kähler differentials.

(1) Let $R \to A$ be a ring map. The module of Kähler differentials of A over R is denoted $\Omega_{A/R}$. It is generated by the elements $\text{d}a$, $a \in A$ subject to the relations:

$$\text{d}(a_1 + a_2) = \text{d}a_1 + \text{d}a_2, \quad \text{d}(a_1 a_2) = a_1 \text{d}a_2 + a_2 \text{d}a_1, \quad \text{d}r = 0$$

The canonical universal R-derivation $\text{d} : A \to \Omega_{A/R}$ maps $a \mapsto \text{d}a$.

(2) Consider the short exact sequence

$$0 \to I \to A \otimes_R A \to A \to 0$$

which defines the ideal I. There is a canonical derivation $\text{d} : A \to I/I^2$ which maps a to the class of $a \otimes 1 - 1 \otimes a$. This is another presentation of the module of derivations of A over R, in other words

$$(I/I^2, \text{d}) \cong (\Omega_{A/R}, \text{d}).$$

(3) For multiplicative subsets $S_R \subset R$ and $S_A \subset A$ such that S_R maps into S_A we have

$$\Omega_{S_A^{-1}A/S_R^{-1}R} = S_A^{-1}\Omega_{A/R}.$$

(4) If A is a finitely presented R-algebra then $\Omega_{A/R}$ is a finitely presented A-module. Hence in this case the fitting ideals of $\Omega_{A/R}$ are defined.

(5) Let $f : X \to S$ be a morphism of schemes. There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega_{X/S}$ and a $\mathcal{O}_S$-linear derivation

$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$

such that for any affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ we have

$$\Gamma(\mathrm{Spec}(A), \Omega_{X/S}) = \Omega_{A/R}$$

compatibly with d.

02FC Exercise 111.50.1. Let $k[\epsilon]$ be the ring of dual numbers over the field k, i.e., $\epsilon^2 = 0$.

(1) Consider the ring map

$$R = k[\epsilon] \to A = k[x, \epsilon]/(\epsilon x)$$

Show that the Fitting ideals of $\Omega_{A/R}$ are (starting with the zeroth Fitting ideal)

$$(\epsilon), A, A, \ldots$$

(2) Consider the map $R = k[t] \to A = k[x, y, t]/(x(y - t)(y - 1), x(x - t))$. Show that the Fitting ideals of $\Omega_{A/R}$ in A are (assume characteristic k is zero for simplicity)

$$x(2x - t)(2y - t - 1)A,\ (x, y, t) \cap (x, y - 1, t),\ A,\ A, \ldots$$

So the 0-the Fitting ideal is cut out by a single element of A, the 1st Fitting ideal defines two closed points of $\mathrm{Spec}(A)$, and the others are all trivial.

(3) Consider the map $R = k[t] \to A = k[x, y, t]/(xy - t^n)$. Compute the Fitting ideals of $\Omega_{A/R}$.

02FD Remark 111.50.2. The kth Fitting ideal of $\Omega_{X/S}$ is commonly used to define the singular scheme of the morphism $X \to S$ when X has relative dimension k over S. But as part (a) shows, you have to be careful doing this when your family does not have "constant" fibre dimension, e.g., when it is not flat. As part (b) shows, flatness doesn't guarantee it works either (and yes this is a flat family). In "good cases" – such as in (c) – for families of curves you expect the 0-th Fitting ideal to be zero and the 1st Fitting ideal to define (scheme-theoretically) the singular locus.

02FE Exercise 111.50.3. Suppose that R is a ring and

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

Note that we are assuming that A is presented by the same number of equations as variables. Thus the matrix of partial derivatives

$$(\partial f_i/\partial x_j)$$

is $n \times n$, i.e., a square matrix. Assume that its determinant is invertible as an element in A. Note that this is exactly the condition that says that $\Omega_{A/R} = (0)$ in this case of n-generators and n relations. Let $\pi : B' \to B$ be a surjection of R-algebras whose kernel J has square zero (as an ideal in B'). Let $\varphi : A \to B$ be a homomorphism of R-algebras. Show there exists a unique homomorphism of R-algebras $\varphi' : A \to B'$ such that $\varphi = \pi \circ \varphi'$.

02FF Exercise 111.50.4. Find a generalization of the result of Exercise 111.50.3 to the case where $A = R[x, y]/(f)$.

0D1T Exercise 111.50.5. Let k be a field, let $f_1, \ldots, f_c \in k[x_1, \ldots, x_n]$, and let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Assume that $f_j(0, \ldots, 0) = 0$. This means that $\mathfrak{m} = (x_1, \ldots, x_n)A$ is a maximal ideal. Prove that the local ring $A_\mathfrak{m}$ is regular if the rank of the matrix

$$(\partial f_j/\partial x_i)|_{(x_1,\ldots,x_n)=(0,\ldots,0)}$$

is c. What is the dimension of $A_\mathfrak{m}$ in this case? Show that the converse is false by giving an example where $A_\mathfrak{m}$ is regular but the rank is less than c; what is the dimension of $A_\mathfrak{m}$ in your example?

111.51. Schemes, Final Exam, Fall 2007

02AU These were the questions in the final exam of a course on Schemes, in the Spring of 2007 at Columbia University.

02FG Exercise 111.51.1 (Definitions). Provide definitions of the following concepts.

(1) X is a scheme
(2) the morphism of schemes $f : X \to Y$ is finite
(3) the morphisms of schemes $f : X \to Y$ is of finite type
(4) the scheme X is Noetherian
(5) the $\mathcal{O}_X$-module $\mathcal{L}$ on the scheme X is invertible
(6) the genus of a nonsingular projective curve over an algebraically closed field

02FH Exercise 111.51.2. Let $X = \text{Spec}(\mathbf{Z}[x, y])$, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\mathcal{F}$ is zero when restricted to the standard affine open $D(x)$.

(1) Show that every global section s of $\mathcal{F}$ is killed by some power of x, i.e., $x^n s = 0$ for some $n \in \mathbf{N}$.
(2) Do you think the same is true if we do not assume that $\mathcal{F}$ is quasi-coherent?

02FI Exercise 111.51.3. Suppose that $X \to \text{Spec}(R)$ is a proper morphism and that R is a discrete valuation ring with residue field k. Suppose that $X \times_{\text{Spec}(R)} \text{Spec}(k)$ is the empty scheme. Show that X is the empty scheme.

02FJ Exercise 111.51.4. Consider the projective[4] variety

$$\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}^1_\mathbf{C} \times_{\text{Spec}(\mathbf{C})} \mathbf{P}^1_\mathbf{C}$$

over the field of complex numbers $\mathbf{C}$. It is covered by four affine pieces, corresponding to pairs of standard affine pieces of $\mathbf{P}^1_\mathbf{C}$. For example, suppose we use homogeneous coordinates X_0, X_1 on the first factor and Y_0, Y_1 on the second. Set $x = X_1/X_0$, and $y = Y_1/Y_0$. Then the 4 affine open pieces are the spectra of the rings

$$\mathbf{C}[x, y], \quad \mathbf{C}[x^{-1}, y], \quad \mathbf{C}[x, y^{-1}], \quad \mathbf{C}[x^{-1}, y^{-1}].$$

Let $X \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the closed subscheme which is the closure of the closed subset of the first affine piece given by the equation

$$y^3(x^4 + 1) = x^4 - 1.$$

(1) Show that X is contained in the union of the first and the last of the 4 affine open pieces.
(2) Show that X is a nonsingular projective curve.

[4] The projective embedding is $((X_0, X_1), (Y_0, Y_1)) \mapsto (X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1)$ in other words $(x, y) \mapsto (1, y, x, xy)$.

(3) Consider the morphism $pr_2 : X \to \mathbf{P}^1$ (projection onto the first factor). On the first affine piece it is the map $(x, y) \mapsto x$. Briefly explain why it has degree 3.
(4) Compute the ramification points and ramification indices for the map $pr_2 : X \to \mathbf{P}^1$.
(5) Compute the genus of X.

02FK Exercise 111.51.5. Let $X \to \mathrm{Spec}(\mathbf{Z})$ be a morphism of finite type. Suppose that there is an infinite number of primes p such that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{F}_p)$ is not empty.

(1) Show that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{Q})$ is not empty.
(2) Do you think the same is true if we replace the condition "finite type" by the condition "locally of finite type"?

111.52. Schemes, Final Exam, Spring 2009

02AV These were the questions in the final exam of a course on Schemes, in the Spring of 2009 at Columbia University.

02AW Exercise 111.52.1. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $x \in X$ be a point. Assume that $\mathrm{Supp}(\mathcal{F}) = \{x\}$.

(1) Show that x is a closed point of X.
(2) Show that $H^0(X, \mathcal{F})$ is not zero.
(3) Show that $\mathcal{F}$ is generated by global sections.
(4) Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

02AX Remark 111.52.2. Let k be a field. Let $\mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Any invertible sheaf on $\mathbf{P}^2_k$ is isomorphic to $\mathcal{O}_{\mathbf{P}^2_k}(n)$ for some $n \in \mathbf{Z}$. Recall that

$$\Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(n)) = k[X_0, X_1, X_2]_n$$

is the degree n part of the polynomial ring. For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^2_k$ set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^2_k}} \mathcal{O}_{\mathbf{P}^2_k}(n)$ as usual.

02AY Exercise 111.52.3. Let k be a field. Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^2_k$, i.e., a finite locally free $\mathcal{O}_{\mathbf{P}^2_k}$-module. We say $\mathcal{E}$ is split if $\mathcal{E}$ is isomorphic to a direct sum invertible $\mathcal{O}_{\mathbf{P}^2_k}$-modules.

(1) Show that $\mathcal{E}$ is split if and only if $\mathcal{E}(n)$ is split.
(2) Show that if $\mathcal{E}$ is split then $H^1(\mathbf{P}^2_k, \mathcal{E}(n)) = 0$ for all $n \in \mathbf{Z}$.
(3) Let

$$\varphi : \mathcal{O}_{\mathbf{P}^2_k} \longrightarrow \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1)$$

be given by linear forms $L_0, L_1, L_2 \in \Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1))$. Assume $L_i \neq 0$ for some i. What is the condition on L_0, L_1, L_2 such that the cokernel of φ is a vector bundle? Why?
(4) Given an example of such a φ.
(5) Show that $\mathrm{Coker}(\varphi)$ is not split (if it is a vector bundle).

02AZ Remark 111.52.4. Freely use the following facts on dimension theory (and add more if you need more).

(1) The dimension of a scheme is the supremum of the length of chains of irreducible closed subsets.

(2) The dimension of a finite type scheme over a field is the maximum of the dimensions of its affine opens.
(3) The dimension of a Noetherian scheme is the maximum of the dimensions of its irreducible components.
(4) The dimension of an affine scheme coincides with the dimension of the corresponding ring.
(5) Let k be a field and let A be a finite type k-algebra. If A is a domain, and $x \neq 0$, then $\dim(A) = \dim(A/xA) + 1$.

02B0 Exercise 111.52.5. Let k be a field. Let X be a projective, reduced scheme over k. Let $f : X \to \mathbf{P}^1_k$ be a morphism of schemes over k. Assume there exists an integer $d \geq 0$ such that for every point $t \in \mathbf{P}^1_k$ the fibre $X_t = f^{-1}(t)$ is irreducible of dimension d. (Recall that an irreducible space is not empty.)

(1) Show that $\dim(X) = d + 1$.
(2) Let $X_0 \subset X$ be an irreducible component of X of dimension $d+1$. Prove that for every $t \in \mathbf{P}^1_k$ the fibre $X_{0,t}$ has dimension d.
(3) What can you conclude about X_t and $X_{0,t}$ from the above?
(4) Show that X is irreducible.

02B1 Remark 111.52.6. Given a projective scheme X over a field k and a coherent sheaf $\mathcal{F}$ on X we set

$$\chi(X, \mathcal{F}) = \sum\nolimits_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

02B2 Exercise 111.52.7. Let k be a field. Write $\mathbf{P}^3_k = \text{Proj}(k[X_0, X_1, X_2, X_3])$. Let $C \subset \mathbf{P}^3_k$ be a type $(5, 6)$ complete intersection curve. This means that there exist $F \in k[X_0, X_1, X_2, X_3]_5$ and $G \in k[X_0, X_1, X_2, X_3]_6$ such that

$$C = \text{Proj}(k[X_0, X_1, X_2, X_3]/(F, G))$$

is a variety of dimension 1. (Variety implies reduced and irreducible, but feel free to assume C is nonsingular if you like.) Let $i : C \to \mathbf{P}^3_k$ be the corresponding closed immersion. Being a complete intersection also implies that

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3_k}(-11) \xrightarrow{\begin{pmatrix} -G \\ F \end{pmatrix}} \mathcal{O}_{\mathbf{P}^3_k}(-5) \oplus \mathcal{O}_{\mathbf{P}^3_k}(-6) \xrightarrow{(F,G)} \mathcal{O}_{\mathbf{P}^3_k} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

is an exact sequence of sheaves. Please use these facts to:

(1) compute $\chi(C, i^*\mathcal{O}_{\mathbf{P}^3_k}(n))$ for any $n \in \mathbf{Z}$, and
(2) compute the dimension of $H^1(C, \mathcal{O}_C)$.

02B3 Exercise 111.52.8. Let k be a field. Consider the rings

$$A = k[x, y]/(xy)$$
$$B = k[u, v]/(uv)$$
$$C = k[t, t^{-1}] \times k[s, s^{-1}]$$

and the k-algebra maps

$$\begin{array}{lll} A \longrightarrow C, & x \mapsto (t, 0), & y \mapsto (0, s) \\ B \longrightarrow C, & u \mapsto (t^{-1}, 0), & v \mapsto (0, s^{-1}) \end{array}$$

It is a true fact that these maps induce isomorphisms $A_{x+y} \to C$ and $B_{u+v} \to C$. Hence the maps $A \to C$ and $B \to C$ identify $\text{Spec}(C)$ with open subsets of $\text{Spec}(A)$

and Spec(B). Let X be the scheme obtained by glueing Spec(A) and Spec(B) along Spec(C):

$$X = \text{Spec}(A) \amalg_{\text{Spec}(C)} \text{Spec}(B).$$

As we saw in the course such a scheme exists and there are affine opens $\text{Spec}(A) \subset X$ and $\text{Spec}(B) \subset X$ whose overlap is exactly Spec(C) identified with an open of each of these using the maps above.

(1) Why is X separated?
(2) Why is X of finite type over k?
(3) Compute $H^1(X, \mathcal{O}_X)$, or what is its dimension?
(4) What is a more geometric way to describe X?

111.53. Schemes, Final Exam, Fall 2010

069Q These were the questions in the final exam of a course on Schemes, in the Fall of 2010 at Columbia University.

069R Exercise 111.53.1 (Definitions). Provide definitions of the following concepts.

(1) a separated scheme,
(2) a quasi-compact morphism of schemes,
(3) an affine morphism of schemes,
(4) a multiplicative subset of a ring,
(5) a Noetherian scheme,
(6) a variety.

069S Exercise 111.53.2. Prime avoidance.

(1) Let A be a ring. Let $I \subset A$ be an ideal and let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be prime ideals such that $I \not\subset \mathfrak{q}_i$. Show that $I \not\subset \mathfrak{q}_1 \cup \mathfrak{q}_2$.
(2) What is a geometric interpretation of (1)?
(3) Let $X = \text{Proj}(S)$ for some graded ring S. Let $x_1, x_2 \in X$. Show that there exists a standard open $D_+(F)$ which contains both x_1 and x_2.

069T Exercise 111.53.3. Why is a composition of affine morphisms affine?

069U Exercise 111.53.4 (Examples). Give examples of the following:

(1) A reducible projective scheme over a field k.
(2) A scheme with 100 points.
(3) A non-affine morphism of schemes.

069V Exercise 111.53.5. Chevalley's theorem and the Hilbert Nullstellensatz.

(1) Let $\mathfrak{p} \subset \mathbf{Z}[x_1, \ldots, x_n]$ be a maximal ideal. What does Chevalley's theorem imply about $\mathfrak{p} \cap \mathbf{Z}$?
(2) In turn, what does the Hilbert Nullstellensatz imply about $\kappa(\mathfrak{p})$?

069W Exercise 111.53.6. Let A be a ring. Let $S = A[X]$ as a graded A-algebra where X has degree 1. Show that $\text{Proj}(S) \cong \text{Spec}(A)$ as schemes over A.

069X Exercise 111.53.7. Let $A \to B$ be a finite ring map. Show that Spec(B) is a H-projective scheme over Spec(A).

069Y Exercise 111.53.8. Give an example of a scheme X over a field k such that X is irreducible and such that for some finite extension k'/k the base change $X_{k'} = X \times_{\text{Spec}(k)} \text{Spec}(k')$ is connected but reducible.

111.54. Schemes, Final Exam, Spring 2011

069Z These were the questions in the final exam of a course on Schemes, in the Spring of 2011 at Columbia University.

06A0 Exercise 111.54.1 (Definitions). Provide definitions of the italicized concepts.

(1) a separated scheme,
(2) a universally closed morphism of schemes,
(3) A dominates B for local rings A, B contained in a common field,
(4) the dimension of a scheme X,
(5) the codimension of an irreducible closed subscheme Y of a scheme X,

06A1 Exercise 111.54.2 (Results). State something formally equivalent to the fact discussed in the course.

(1) The valuative criterion of properness for a morphism $X \to Y$ of varieties for example.
(2) The relationship between $\dim(X)$ and the function field $k(X)$ of X for a variety X over a field k.
(3) Fill in the blank: The category of nonsingular projective curves over k and nonconstant morphisms is anti-equivalent to ....
(4) Noether normalization.
(5) Jacobian criterion.

06A2 Exercise 111.54.3. Let k be a field. Let $F \in k[X_0, X_1, X_2]$ be a homogeneous form of degree d. Assume that $C = V_+(F) \subset \mathbf{P}^2_k$ is a smooth curve over k. Denote $i : C \to \mathbf{P}^2_k$ the corresponding closed immersion.

(1) Show that there is a short exact sequence
$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_C \to 0$$
of coherent sheaves on $\mathbf{P}^2_k$: tell me what the maps are and briefly why it is exact.
(2) Conclude that $H^0(C, \mathcal{O}_C) = k$.
(3) Compute the genus of C.
(4) Assume now that $P = (0 : 0 : 1)$ is not on C. Prove that $\pi : C \to \mathbf{P}^1_k$ given by $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$ has degree d.
(5) Assume k is algebraically closed, assume all ramification indices (the "e_i") are 1 or 2, and assume the characteristic of k is not equal to 2. How many ramification points does $\pi : C \to \mathbf{P}^1_k$ have?
(6) In terms of F, what do you think is a set of equations of the set of ramification points of π?
(7) Can you guess K_C?

06A3 Exercise 111.54.4. Let k be a field. Let X be a "triangle" over k, i.e., you get X by glueing three copies of $\mathbf{A}^1_k$ to each other by identifying 0 on the first copy to 1 on the second copy, 0 on the second copy to 1 on the third copy, and 0 on the third copy to 1 on the first copy. It turns out that X is isomorphic to $\text{Spec}(k[x, y]/(xy(x+y+1)))$; feel free to use this. Compute the Picard group of X.

06A4 Exercise 111.54.5. Let k be a field. Let $\pi : X \to Y$ be a finite birational morphism of curves with X a projective nonsingular curve over k. It follows from the material in the course that Y is a proper curve and that π is the normalization morphism of Y. We have also seen in the course that there exists a dense open $V \subset Y$ such that $U = \pi^{-1}(V)$ is a dense open in X and $\pi : U \to V$ is an isomorphism.

(1) Show that there exists an effective Cartier divisor $D \subset X$ such that $D \subset U$ and such that $\mathcal{O}_X(D)$ is ample on X.
(2) Let D be as in (1). Show that $E = \pi(D)$ is an effective Cartier divisor on Y.
(3) Briefly indicate why
  (a) the map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ has a coherent cokernel Q which is supported in $Y \setminus V$, and
  (b) for every n there is a corresponding map $\mathcal{O}_Y(nE) \to \pi_*\mathcal{O}_X(nD)$ whose cokernel is isomorphic to Q.
(4) Show that $\dim_k H^0(X, \mathcal{O}_X(nD)) - \dim_k H^0(Y, \mathcal{O}_Y(nE))$ is bounded (by what?) and conclude that the invertible sheaf $\mathcal{O}_Y(nE)$ has lots of sections for large n (why?).

111.55. Schemes, Final Exam, Fall 2011

07DE These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2011 at Columbia University.

07DF **Exercise 111.55.1** (Definitions). Provide definitions of the italicized concepts.

(1) a Noetherian ring,
(2) a Noetherian scheme,
(3) a finite ring homomorphism,
(4) a finite morphism of schemes,
(5) the dimension of a ring.

07DG **Exercise 111.55.2** (Results). State something formally equivalent to the fact discussed in the course.

(1) Zariski's Main Theorem.
(2) Noether normalization.
(3) Chinese remainder theorem.
(4) Going up for finite ring maps.

07DH **Exercise 111.55.3.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring whose residue field has characteristic not 2. Suppose that $\mathfrak{m}$ is generated by three elements x, y, z and that $x^2 + y^2 + z^2 = 0$ in A.

(1) What are the possible values of $\dim(A)$?
(2) Give an example to show that each value is possible.
(3) Show that A is a domain if $\dim(A) = 2$. (Hint: look at $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.)

07DI **Exercise 111.55.4.** Let A be a ring. Let $S \subset T \subset A$ be multiplicative subsets. Assume that
$$\{\mathfrak{q} \mid \mathfrak{q} \cap S = \emptyset\} = \{\mathfrak{q} \mid \mathfrak{q} \cap T = \emptyset\}.$$
Show that $S^{-1}A \to T^{-1}A$ is an isomorphism.

07DJ **Exercise 111.55.5.** Let k be an algebraically closed field. Let
$$V_0 = \{A \in \text{Mat}(3 \times 3, k) \mid \text{rank}(A) = 1\} \subset \text{Mat}(3 \times 3, k) = k^9.$$

(1) Show that V_0 is the set of closed points of a (Zariski) locally closed subset $V \subset \mathbf{A}^9_k$.

(2) Is V irreducible?
(3) What is $\dim(V)$?

07DK Exercise 111.55.6. Prove that the ideal (x^2, xy, y^2) in $\mathbf{C}[x, y]$ cannot be generated by 2 elements.

07DL Exercise 111.55.7. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that for some $\alpha, \beta \in \mathbf{C}$ the $\mathbf{C}$-algebra map

$$\mathbf{C}[t] \longrightarrow \mathbf{C}[x, y]/(f), \quad t \longmapsto \alpha x + \beta y$$

is finite.

07DM Exercise 111.55.8. Show that given finitely many points $p_1, \ldots, p_n \in \mathbf{C}^2$ the scheme $\mathbf{A}^2_{\mathbf{C}} \setminus \{p_1, \ldots, p_n\}$ is a union of two affine opens.

07DN Exercise 111.55.9. Show that there exists a surjective morphism of schemes $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$. (Surjective just means surjective on underlying sets of points.)

07DP Exercise 111.55.10. Let k be an algebraically closed field. Let $A \subset B$ be an extension of domains which are both finite type k-algebras. Prove that the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains a nonempty open subset of $\mathrm{Spec}(A)$ using the following steps:

(1) Prove it if $A \to B$ is also finite.
(2) Prove it in case the fraction field of B is a finite extension of the fraction field of A.
(3) Reduce the statement to the previous case.

111.56. Schemes, Final Exam, Fall 2013

09TV These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2013 at Columbia University.

09TW Exercise 111.56.1 (Definitions). Provide definitions of the italicized concepts.

(1) a radical ideal of a ring,
(2) a finite type ring homomorphism,
(3) a differential a la Weil,
(4) a scheme.

09TX Exercise 111.56.2 (Results). State something formally equivalent to the fact discussed in the course.

(1) result on hilbert polynomials of graded modules.
(2) dimension of a Noetherian local ring $(R, \mathfrak{m})$ and $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.
(3) Riemann-Roch.
(4) Clifford's theorem.
(5) Chevalley's theorem.

09TY Exercise 111.56.3. Let $A \to B$ be a ring map. Let $S \subset A$ be a multiplicative subset. Assume that $A \to B$ is of finite type and $S^{-1}A \to S^{-1}B$ is surjective. Show that there exists an $f \in S$ such that $A_f \to B_f$ is surjective.

09TZ Exercise 111.56.4. Give an example of an injective local homomorphism $A \to B$ of local rings, such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not surjective.

09U0 Situation 111.56.5 (Notation plane curve). Let k be an algebraically closed field. Let $F(X_0, X_1, X_2) \in k[X_0, X_1, X_2]$ be an irreducible polynomial homogeneous of degree d. We let

$$D = V(F) \subset \mathbf{P}^2$$

be the projective plane curve given by the vanishing of F. Set $x = X_1/X_0$ and $y = X_2/X_0$ and $f(x, y) = X_0^{-d}F(X_0, X_1, X_2) = F(1, x, y)$. We denote K the fraction field of the domain $k[x, y]/(f)$. We let C be the abstract curve corresponding to K. Recall (from the lectures) that there is a surjective map $C \to D$ which is bijective over the nonsingular locus of D and an isomorphism if D is nonsingular. Set $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. Finally, we denote $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ the element of $\Omega_{K/k}$ discussed in the lectures. Denote K_C the divisor of zeros and poles of ω.

09U1 Exercise 111.56.6. In Situation 111.56.5 assume $d \geq 3$ and that the curve D has exactly one singular point, namely $P = (1 : 0 : 0)$. Assume further that we have the expansion

$$f(x, y) = xy + h.o.t$$

around $P = (0, 0)$. Then C has two points v and w lying over P characterized by

$$v(x) = 1, v(y) > 1 \quad \text{and} \quad w(x) > 1, w(y) = 1$$

(1) Show that the element $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ of $\Omega_{K/k}$ has a first order pole at both v and w. (The behaviour of ω at nonsingular points is as discussed in the lectures.)
(2) In the lectures we have shown that ω vanishes to order $d-3$ at the divisor $X_0 = 0$ pulled back to C under the map $C \to D$. Combined with the information of (1) what is the degree of the divisor of zeros and poles of ω on C?
(3) What is the genus of the curve C?

09U2 Exercise 111.56.7. In Situation 111.56.5 assume $d = 5$ and that the curve $C = D$ is nonsingular. In the lectures we have shown that the genus of C is 6 and that the linear system K_C is given by

$$L(K_C) = \{h\omega \mid h \in k[x, y],\ \deg(h) \leq 2\}$$

where deg indicates total degree[5]. Let $P_1, P_2, P_3, P_4, P_5 \in D$ be pairwise distinct points lying in the affine open $X_0 \neq 0$. We denote $\sum P_i = P_1 + P_2 + P_3 + P_4 + P_5$ the corresponding divisor of C.

(1) Describe $L(K_C - \sum P_i)$ in terms of polynomials.
(2) What are the possibilities for $l(\sum P_i)$?

09U3 Exercise 111.56.8. Write down an F as in Situation 111.56.5 with $d = 100$ such that the genus of C is 0.

09U4 Exercise 111.56.9. Let k be an algebraically closed field. Let K/k be finitely generated field extension of transcendence degree 1. Let C be the abstract curve corresponding to K. Let $V \subset K$ be a g^r_d and let $\Phi : C \to \mathbf{P}^r$ be the corresponding morphism. Show that the image of C is contained in a quadric[6] if V is a complete

[5] We get ≤ 2 because $d - 3 = 5 - 3 = 2$.

[6] A quadric is a degree 2 hypersurface, i.e., the zero set in $\mathbf{P}^r$ of a degree 2 homogeneous polynomial.

linear system and d is large enough relative to the genus of C. (Extra credit: good bound on the degree needed.)

09U5 Exercise 111.56.10. Notation as in Situation 111.56.5. Let $U \subset \mathbf{P}^2_k$ be the open subscheme whose complement is D. Describe the k-algebra $A = \mathcal{O}_{\mathbf{P}^2_k}(U)$. Give an upper bound for the number of generators of A as a k-algebra.

111.57. Schemes, Final Exam, Spring 2014

0AAL These were the questions in the final exam of a course on Schemes, in the Spring of 2014 at Columbia University.

0AAM Exercise 111.57.1 (Definitions). Let $(X, \mathcal{O}_X)$ be a scheme. Provide definitions of the italicized concepts.

(1) the local ring of X at a point x,
(2) a quasi-coherent sheaf of $\mathcal{O}_X$-modules,
(3) a coherent sheaf of $\mathcal{O}_X$-modules (please assume X is locally Noetherian,
(4) an affine open of X,
(5) a finite morphism of schemes $X \to Y$.

0AAN Exercise 111.57.2 (Theorems). Precisely state a nontrivial fact discussed in the lectures related to each item.

(1) on birational invariance of pluri-genera of varieties,
(2) being an affine morphism is a local property,
(3) the topology of a scheme theoretic fibre of a morphism, and
(4) valuative criterion of properness.

0AAP Exercise 111.57.3. Let $X = \mathbf{A}^2_{\mathbf{C}}$ where $\mathbf{C}$ is the field of complex numbers. A line will mean a closed subscheme of X defined by one linear equation $ax+by+c = 0$ for some $a, b, c \in \mathbf{C}$ with $(a, b) \neq (0, 0)$. A curve will mean an irreducible (so nonempty) closed subscheme $C \subset X$ of dimension 1. A quadric will mean a curve defined by one quadratic equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ for some $a, b, c, d, e, f \in \mathbf{C}$ and $(a, b, c) \neq (0, 0, 0)$.

(1) Find a curve C such that every line has nonempty intersection with C.
(2) Find a curve C such that every line and every quadric has nonempty intersection with C.
(3) Show that for every curve C there exists another curve such that $C \cap C' = \emptyset$.

0AAQ Exercise 111.57.4. Let k be a field. Let $b : X \to \mathbf{A}^2_k$ be the blow up of the affine plane in the origin. In other words, if $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$, then $X = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n)$ where $\mathfrak{m} = (x, y) \subset k[x, y]$. Prove the following statements

(1) the scheme theoretic fibre E of b over the origin is isomorphic to $\mathbf{P}^1_k$,
(2) E is an effective Cartier divisor on X,
(3) the restriction of $\mathcal{O}_X(-E)$ to E is a line bundle of degree 1.

(Recall that $\mathcal{O}_X(-E)$ is the ideal sheaf of E in X.)

0AAR Exercise 111.57.5. Let k be a field. Let X be a projective variety over k. Show there exists an affine variety U over k and a surjective morphism of varieties $U \to X$.

0AAS Exercise 111.57.6. Let k be a field of characteristic $p > 0$ different from $2, 3$. Consider the closed subscheme X of $\mathbf{P}^n_k$ defined by

$$\sum\nolimits_{i=0,\ldots,n} X_i = 0, \quad \sum\nolimits_{i=0,\ldots,n} X_i^2 = 0, \quad \sum\nolimits_{i=0,\ldots,n} X_i^3 = 0$$

For which pairs (n, p) is this variety singular?

111.58. Commutative Algebra, Final Exam, Fall 2016

0D5F These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2016 at Columbia University.

0D5G Exercise 111.58.1 (Definitions). Let R be a ring. Provide definitions of the italicized concepts.

(1) the local ring of R at a prime $\mathfrak{p}$,
(2) a finite R-module,
(3) a finitely presented R-module,
(4) R is regular,
(5) R is catenary,
(6) R is Cohen-Macaulay.

0D5H Exercise 111.58.2 (Theorems). Precisely state a nontrivial fact discussed in the lectures related to each item.

(1) regular rings,
(2) associated primes of Cohen-Macaulay modules,
(3) dimension of a finite type domain over a field, and
(4) Chevalley's theorem.

0D5I Exercise 111.58.3. Let $A \to B$ be a ring map such that

(1) A is local with maximal ideal $\mathfrak{m}$,
(2) $A \to B$ is a finite[7] ring map,
(3) $A \to B$ is injective (we think of A as a subring of B).

Show that there is a prime ideal $\mathfrak{q} \subset B$ with $\mathfrak{m} = A \cap \mathfrak{q}$.

0D5J Exercise 111.58.4. Let k be a field. Let $R = k[x, y, z, w]$. Consider the ideal $I = (xy, xz, xw)$. What are the irreducible components of $V(I) \subset \mathrm{Spec}(R)$ and what are their dimensions?

0D5K Exercise 111.58.5. Let k be a field. Let $A = k[x, x^{-1}]$ and $B = k[y]$. Show that any k-algebra map $\varphi : A \to B$ maps x to a constant.

0D5L Exercise 111.58.6. Consider the ring $R = \mathbf{Z}[x, y]/(xy - 7)$. Prove that R is regular.

Given a Noetherian local ring $(R, \mathfrak{m}, \kappa)$ for $n \geq 0$ we let $\varphi_R(n) = \dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1})$.

0D5M Exercise 111.58.7. Does there exist a Noetherian local ring R with $\varphi_R(n) = n + 1$ for all $n \geq 0$?

0D5N Exercise 111.58.8. Let R be a Noetherian local ring. Suppose that $\varphi_R(0) = 1$, $\varphi_R(1) = 3$, $\varphi_R(2) = 5$. Show that $\varphi_R(3) \leq 7$.

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[7]Recall that this means B is finite as an A-module.

111.59. Schemes, Final Exam, Spring 2017

0DSZ These were the questions in the final exam of a course on schemes, in the Spring of 2017 at Columbia University.

0DT0 Exercise 111.59.1 (Definitions). Let $f : X \to Y$ be a morphism of schemes. Provide brief definitions of the italicized concepts.

(1) the scheme theoretic fibre of f at $y \in Y$,
(2) f is a finite morphism,
(3) a quasi-coherent $\mathcal{O}_X$-module,
(4) X is variety,
(5) f is a smooth morphism,
(6) f is a proper morphism.

0DT1 Exercise 111.59.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item.

(1) pushforward of quasi-coherent sheaves,
(2) cohomology of coherent sheaves on projective varieties,
(3) Serre duality for a projective scheme over a field, and
(4) Riemann-Hurwitz.

0DT2 Exercise 111.59.3. Let k be an algebraically closed field. Let $\ell > 100$ be a prime number different from the characteristic of k. Let X be the nonsingular projective model of the affine curve given by the equation

$$y^\ell = x(x-1)^3$$

in $\mathbf{A}^2_k$. Answer the following questions:

(1) What is the genus of X?
(2) Give an upper bound for the gonality[8] of X.

0DT3 Exercise 111.59.4. Let k be an algebraically closed field. Let X be a reduced, projective scheme over k all of whose irreducible components have the same dimension 1. Let $\omega_{X/k}$ be the relative dualizing module. Show that if $\dim_k H^1(X, \omega_{X/k}) > 1$, then X is disconnected.

0DT4 Exercise 111.59.5. Give an example of a scheme X and a nontrivial invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that both $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are nonzero.

0DT5 Exercise 111.59.6. Let k be an algebraically closed field. Let $g \geq 3$. Let X and X' be smooth projective curves over k of genus g and $g+1$. Let $Y \subset X \times X'$ be a curve such that the projections $Y \to X$ and $Y \to X'$ are nonconstant. Prove that the nonsingular projective model of Y has genus $\geq 2g + 1$.

0DT6 Exercise 111.59.7. Let k be a finite field. Let $g > 1$. Sketch a proof of the following: there are only a finite number of isomorphism classes of smooth projective curves over k of genus g. (You will get credit for even just trying to answer this.)

111.60. Commutative Algebra, Final Exam, Fall 2017

0EEJ These were the questions in the final exam of a course on commutative algebra, in the Fall of 2017 at Columbia University.

[8]The gonality is the smallest degree of a nonconstant morphism from X to $\mathbf{P}^1_k$.

0EEK Exercise 111.60.1 (Definitions). Provide brief definitions of the italicized concepts.

(1) the left adjoint of a functor $F : \mathcal{A} \to \mathcal{B}$,
(2) the transcendence degree of an extension L/K of fields,
(3) a regular function on a classical affine variety $X \subset k^n$,
(4) a sheaf on a topological space,
(5) a local ring, and
(6) a morphism of schemes $f : X \to Y$ being affine.

0EEL Exercise 111.60.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Yoneda lemma,
(2) Mayer-Vietoris,
(3) dimension and cohomology,
(4) Hilbert polynomial, and
(5) duality for projective space.

0EEM Exercise 111.60.3. Let k be an algebraically closed field. Consider the closed subset X of k^5 with Zariski topology and coordinates x_1, x_2, x_3, x_4, x_5 given by the equations

$$x_1^2 - x_4 = 0, \quad x_2^5 - x_5 = 0, \quad x_3^2 + x_3 + x_4 + x_5 = 0$$

What is the dimension of X and why?

0EEN Exercise 111.60.4. Let k be a field. Let $X = \mathbf{P}^1_k$ be the projective space of dimension 1 over k. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For $d \in \mathbf{Z}$ denote $\mathcal{E}(d) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ the dth Serre twist of $\mathcal{E}$ and $h^i(X, \mathcal{E}(d)) = \dim_k H^i(X, \mathcal{E}(d))$.

(1) Why is there no $\mathcal{E}$ with $h^0(X, \mathcal{E}) = 5$ and $h^0(X, \mathcal{E}(1)) = 4$?
(2) Why is there no $\mathcal{E}$ with $h^1(X, \mathcal{E}(1)) = 5$ and $h^1(X, \mathcal{E}) = 4$?
(3) For which $a \in \mathbf{Z}$ can there exist a vector bundle $\mathcal{E}$ on X with

$$\begin{array}{ll} h^0(X, \mathcal{E}) \quad = 1 & h^1(X, \mathcal{E}) \quad = 1 \\ h^0(X, \mathcal{E}(1)) = 2 & h^1(X, \mathcal{E}(1)) = 0 \\ h^0(X, \mathcal{E}(2)) = 4 & h^1(X, \mathcal{E}(2)) = a \end{array}$$

Partial answers are welcomed and encouraged.

0EEP Exercise 111.60.5. Let X be a topological space which is the union $X = Y \cup Z$ of two closed subsets Y and Z whose intersection is denoted $W = Y \cap Z$. Denote $i : Y \to X$, $j : Z \to X$, and $k : W \to X$ the inclusion maps.

(1) Show that there is a short exact sequence of sheaves

$$0 \to \underline{\mathbf{Z}}_X \to i_*(\underline{\mathbf{Z}}_Y) \oplus j_*(\underline{\mathbf{Z}}_Z) \to k_*(\underline{\mathbf{Z}}_W) \to 0$$

where $\underline{\mathbf{Z}}_X$ denotes the constant sheaf with value $\mathbf{Z}$ on X, etc.
(2) What can you conclude about the relationship between the cohomology groups of X, Y, Z, W with $\mathbf{Z}$-coefficients?

0EEQ Exercise 111.60.6. Let k be a field. Let $A = k[x_1, x_2, x_3, \ldots]$ be the polynomial ring in infinitely many variables. Denote $\mathfrak{m}$ the maximal ideal of A generated by all the variables. Let $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

(1) Show $H^1(U, \mathcal{O}_U) = 0$. Hint: Čech cohomology computation.
(2) What is your guess for $H^i(U, \mathcal{O}_U)$ for $i \geq 1$?

0EER Exercise 111.60.7. Let A be a local ring. Let $a \in A$ be a nonzerodivisor. Let $I, J \subset A$ be ideals such that $IJ = (a)$. Show that the ideal I is principal, i.e., generated by one element (which will turn out to be a nonzerodivisor).

111.61. Schemes, Final Exam, Spring 2018

0ELF These were the questions in the final exam of a course on schemes, in the Spring of 2018 at Columbia University.

0ELG Exercise 111.61.1 (Definitions). Provide brief definitions of the italicized concepts. Let k be an algebraically closed field. Let X be a projective curve over k.

(1) a smooth algebra over k,
(2) the degree of an invertible $\mathcal{O}_X$-module on X,
(3) the genus of X,
(4) the Weil divisor class group of X,
(5) X is hyperelliptic, and
(6) the intersection number of two curves on a smooth projective surface over k.

0ELH Exercise 111.61.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Riemann-Hurwitz theorem,
(2) Clifford's theorem,
(3) factorization of maps between smooth projective surfaces,
(4) Hodge index theorem, and
(5) Riemann hypothesis for curves over finite fields.

0ELI Exercise 111.61.3. Let k be an algebraically closed field. Let $X \subset \mathbf{P}^3_k$ be a smooth curve of degree d and genus ≥ 2. Assume X is not contained in a plane and that there is a line ℓ in $\mathbf{P}^3_k$ meeting X in $d-2$ points. Show that X is hyperelliptic.

0ELJ Exercise 111.61.4. Let k be an algebraically closed field. Let X be a projective curve with pairwise distinct singular points $p_1, \ldots, p_n$. Explain why the genus of the normalization of X is at most $-n + \dim_k H^1(X, \mathcal{O}_X)$.

0ELK Exercise 111.61.5. Let k be a field. Let $X = \text{Spec}(k[x, y])$ be affine 2 space. Let

$$I = (x^3, x^2y, xy^2, y^3) \subset k[x, y].$$

Let $Y \subset X$ be the closed subscheme corresponding to I. Let $b : X' \to X$ be the blowing up of the ideal (x, y), i.e., the blow up of affine space at the origin.

(1) Show that the scheme theoretic inverse image $b^{-1}Y \subset X'$ is an effective Cartier divisor.
(2) Given an example of an ideal $J \subset k[x, y]$ with $I \subset J \subset (x, y)$ such that if $Z \subset X$ is the closed subscheme corresponding to J, then the scheme theoretic inverse image $b^{-1}Z$ is not an effective Cartier divisor.

0ELL Exercise 111.61.6. Let k be an algebraically closed field. Consider the following types of surfaces

(1) $S = C_1 \times C_2$ where C_1 and C_2 are smooth projective curves,
(2) $S = C_1 \times C_2$ where C_1 and C_2 are smooth projective curves and the genus of C_1 is > 0,

(3) $S \subset \mathbf{P}^3_k$ is a hypersurface of degree 4, and
(4) $S \subset \mathbf{P}^3_k$ is a smooth hypersurface of degree 4.

For each type briefly indicate why or why not the class of surfaces of this type contains rational surfaces.

0ELM Exercise 111.61.7. Let k be an algebraically closed field. Let $S \subset \mathbf{P}^3_k$ be a smooth hypersurface of degree d. Assume that S contains a line ℓ. What is the self square of ℓ viewed as a divisor on S?

111.62. Commutative Algebra, Final Exam, Fall 2019

0FWJ These were the questions in the final exam of a course on commutative algebra, in the Fall of 2019 at Columbia University.

0FWK Exercise 111.62.1 (Definitions). Provide brief definitions of the italicized concepts.

(1) a constructible subset of a Noetherian topological space,
(2) the localization of an R-module M at a prime $\mathfrak{p}$,
(3) the length of a module over a Noetherian local ring $(A, \mathfrak{m}, \kappa)$,
(4) a projective module over a ring R, and
(5) a Cohen-Macaulay module over a Noetherian local ring $(A, \mathfrak{m}, \kappa)$.

0FWL Exercise 111.62.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) images of constructible sets,
(2) Hilbert Nullstellensatz,
(3) dimension of finite type algebras over fields,
(4) Noether normalization, and
(5) regular local rings.

For a ring R and an ideal $I \subset R$ recall that $V(I)$ denotes the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $I \subset \mathfrak{p}$.

0FWM Exercise 111.62.3 (Making primes). Construct infinitely many distinct prime ideals $\mathfrak{p} \subset \mathbf{C}[x, y]$ such that $V(\mathfrak{p})$ contains (x, y) and $(x - 1, y - 1)$.

0FWN Exercise 111.62.4 (No prime). Let $R = \mathbf{C}[x, y, z]/(xy)$. Argue briefly there does not exist a prime ideal $\mathfrak{p} \subset R$ such that $V(\mathfrak{p})$ contains $(x, y-1, z-5)$ and $(x-1, y, z-7)$.

0FWP Exercise 111.62.5 (Frobenius). Let p be a prime number (you may assume $p = 2$ to simplify the formulas). Let R be a ring such that $p = 0$ in R.

(1) Show that the map $F : R \to R$, $x \mapsto x^p$ is a ring homomorphism.
(2) Show that $\mathrm{Spec}(F) : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is the identity map.

Recall that a specialization $x \rightsquigarrow y$ of points of a topological space simply means y is in the closure of x. We say $x \rightsquigarrow y$ is an immediate specialization if there does not exist a z different from x and y such that $x \rightsquigarrow z$ and $z \rightsquigarrow y$.

0FWQ Exercise 111.62.6 (Dimension). Suppose we have a sober topological space X containing 5 distinct points x, y, z, u, v having the following specializations

$$\begin{tikzcd} x \arrow[r] \arrow[d] & u & v \arrow[l] \arrow[dl] \\ y \arrow[r] & z & \end{tikzcd}$$

What is the minimal dimension such an X can have? If X is the spectrum of a finite type algebra over a field and $x \rightsquigarrow u$ is an immediate specialization, what can you say about the specialization $v \rightsquigarrow z$?

0FWR Exercise 111.62.7 (Tor computation). Let $R = \mathbf{C}[x, y, z]$. Let $M = R/(x, z)$ and $N = R/(y, z)$. For which $i \in \mathbf{Z}$ is $\text{Tor}_i^R(M, N)$ nonzero?

0FWS Exercise 111.62.8. Let $A \to B$ be a flat local homomorphism of local Noetherian rings. Show that if A has depth k, then B has depth at least k.

111.63. Algebraic Geometry, Final Exam, Spring 2020

0G12 These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2020 at Columbia University.

0G13 Exercise 111.63.1 (Definitions). Provide brief definitions of the italicized concepts.

(1) a scheme,
(2) a morphism of schemes,
(3) a quasi-coherent module on a scheme,
(4) a variety over a field k,
(5) a curve over a field k,
(6) a finite morphism of schemes,
(7) the cohomology of a sheaf of abelian groups $\mathcal{F}$ over a topological space X,
(8) a dualizing sheaf on a scheme X of dimension d proper over a field k, and
(9) a rational map from a variety X to a variety Y.

0G14 Exercise 111.63.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) cohomology of abelian sheaves on a Noetherian topological space X of dimension d,
(2) sheaf of differentials $\Omega^1_{X/k}$ of a smooth variety over a field k,
(3) dualizing sheaf ω_X of a smooth projective variety X over the field k,
(4) a smooth proper genus 0 curve over an algebraically closed field k, and
(5) the genus of a plane curve of degree d.

0G15 Exercise 111.63.3. Let k be a field. Let X be a scheme over k. Assume $X = X_1 \cup X_2$ is an open covering with X_1, X_2 both isomorphic to $\mathbf{P}^1_k$ and $X_1 \cap X_2$ isomorphic to $\mathbf{A}^1_k$. (Such a scheme exists, for example you can take $\mathbf{P}^1_k$ with ∞ doubled.) Show that $\dim_k H^1(X, \mathcal{O}_X)$ is infinite.

0G16 Exercise 111.63.4. Let k be an algebraically closed field. Let Y be a smooth projective curve of genus 10. Find a good lower bound for the genus of a smooth projective curve X such that there exists a nonconstant morphism $f : X \to Y$ which is not an isomorphism.

0G17 Exercise 111.63.5. Let k be an algebraically closed field of characteristic 0. Let

$$X : T_0^d + T_1^d - T_2^d = 0 \subset \mathbf{P}^2_k$$

be the Fermat curve of degree $d \geq 3$. Consider the closed points $p = [1 : 0 : 1]$ and $q = [0 : 1 : 1]$ on X. Set $D = [p] - [q]$.

(1) Show that D is nontrivial in the Weil divisor class group.
(2) Show that dD is trivial in the Weil divisor class group. (Hint: try to show that both $d[p]$ and $d[q]$ are the intersection of X with a line in the plane.)

0G18 Exercise 111.63.6. Let k be an algebraically closed field. Consider the 2-uple embedding

$$\varphi : \mathbf{P}^2 \longrightarrow \mathbf{P}^5$$

In terms of the material/notation in the lectures this is the morphism

$$\varphi = \varphi_{\mathcal{O}_{\mathbf{P}^2}(2)} : \mathbf{P}^2 \longrightarrow \mathbf{P}(\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)))$$

In terms of homogeneous coordinates it is given by

$$[a_0 : a_1 : a_2] \longmapsto [a_0^2 : a_0a_1 : a_0a_2 : a_1^2 : a_1a_2 : a_2^2]$$

It is a closed immersion (please just use this). Let $I \subset k[T_0, \ldots, T_5]$ be the homogeneous ideal of $\varphi(\mathbf{P}^2)$, i.e., the elements of the homogeneous part I_d are the homogeneous polynomials $F(T_0, \ldots, T_5)$ of degree d which restrict to zero on the closed subscheme $\varphi(\mathbf{P}^2)$. Compute $\dim_k I_d$ as a function of d.

0G19 Exercise 111.63.7. Let k be an algebraically closed field. Let X be a proper scheme of dimension d over k with dualizing module ω_X. You are given the following information:

(1) $\text{Ext}^i_X(\mathcal{F}, \omega_X) \times H^{d-i}(X, \mathcal{F}) \to H^d(X, \omega_X) \xrightarrow{t} k$ is nondegenerate for all i and for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$, and
(2) ω_X is finite locally free of some rank r.

Show that $r = 1$. (Hint: see what happens if you take $\mathcal{F}$ a suitable module supported at a closed point.)

111.64. Commutative Algebra, Final Exam, Fall 2021

0GRS These were the questions in the final exam of a course on commutative algebra, in the Fall of 2021 at Columbia University.

0GRT Exercise 111.64.1 (Definitions). Provide brief definitions of the italicized concepts.

(1) a multiplicative subset of a ring A,
(2) an Artinian ring A,
(3) the spectrum of a ring A as a topological space,
(4) a flat ring map $A \to B$,
(5) the height of a prime ideal $\mathfrak{p}$ in A, and
(6) the functors $\text{Tor}_i^A(-,-)$ over a ring A.

0GRU Exercise 111.64.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Artinian rings,
(2) flatness and prime ideals,
(3) lengths of $A/\mathfrak{m}^n$ for $(A, \mathfrak{m})$ Noetherian local,
(4) the dimension formula for universally catenary Noetherian rings,
(5) completion of a Noetherian local ring, and
(6) Matlis duality for Artinian local rings.

0GRV Exercise 111.64.3 (Units). What is the structure of the group of units of $\mathbf{Z}[x, 1/x]$ as an abelian group? No explanation necessary.

0GRW Exercise 111.64.4 (Ideals). Let $A = \mathbf{F}_2[x, y]/(x^2, xy, y^2)$ and denote $\overline{x}$ and $\overline{y}$ the images of x and y in A. List the ideals of A. No explanation necessary.

0GRX Exercise 111.64.5 (Tor and Ext). Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Set $\varphi(n) = \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

(1) Show that $\text{Tor}_1^A(A/\mathfrak{m}^n, \kappa)$ has dimension $\varphi(n)$ as a κ-vector space.
(2) Show that $\text{Ext}_A^1(A/\mathfrak{m}^n, \kappa)$ has dimension $\varphi(n)$ as a κ-vector space.

0GRY Exercise 111.64.6 (Two vectors). Let $A = \mathbf{Z}[a_1, a_2, a_3, b_1, b_2, b3]$. Set $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $A^{\oplus 3}$. Consider the set

$$Z = \{\mathfrak{p} \in \text{Spec}(A) \mid a, b \text{ map to linearly dependent vectors of } \kappa(\mathfrak{p})^{\oplus 3}\}$$

(1) Prove the Z is a closed subset of $\text{Spec}(A)$.
(2) What is the dimension $\dim(Z)$ of Z?
(3) What would happen to $\dim(Z)$ if we replaced $\mathbf{Z}$ by a field?

0GRZ Exercise 111.64.7 (Injectives). Let $(A, \mathfrak{m}, \kappa)$ be an Artinian local ring. Assume A is injective as an A-module. Show that $\text{Hom}_A(\kappa, A)$ has dimension 1 has a κ-vector space.

111.65. Algebraic Geometry, Final Exam, Spring 2022

0GY8 These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2022 at Columbia University.

0GY9 Exercise 111.65.1 (Definitions). Provide brief definitions of the italicized concepts.

(1) a scheme,
(2) a quasi-coherent module on a scheme X,
(3) a flat morphism of schemes $X \to Y$,
(4) a finite morphism of schemes $X \to Y$,
(5) a group scheme G over a base scheme S,
(6) a family of varieties over a base scheme S,
(7) the degree of a closed point x on a variety X over the field k,
(8) the usual logarithmic height of a point $p = (a_0 : \ldots : a_n)$ in $\mathbf{P}^n(\mathbf{Q})$, and
(9) a C_i field.

0GYA Exercise 111.65.2 (Theorems). Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) morphisms from a scheme X to the affine scheme $\text{Spec}(A)$,
(2) cohomology of a quasi-coherent module $\mathcal{F}$ on an affine scheme X,
(3) the Picard group of $\mathbf{P}^1_k$ where k is a field,
(4) the dimensions of fibres of a flat proper morphism $X \to S$ for S Noetherian,
(5) $\mathbf{G}_m$-equivariant modules on a scheme S, and
(6) Bezout's theorem on intersections (restrict to a special case if you like).

0GYB Exercise 111.65.3 (Cubic hypersurfaces). Let $F \in \mathbf{C}[T_0, \ldots, T_n]$ be homogeneous of degree 3. Given 3 vectors $x, y, z \in \mathbf{C}^{n+1}$ consider the condition

$$(*) \quad F(\lambda x + \mu y + \nu z) = 0 \text{ in } \mathbf{C}[\lambda, \mu, \nu]$$

(1) What is the dimension of the space of all choices of x, y, z?

(2) How many equations on the coordinates of x, y, and z is condition (*)?
(3) What is the expected dimension of the space of all triples x, y, z such that (*) is true?
(4) What is the dimension of the space of all triples such that x, y, z are linearly dependent?
(5) Conclude that on a hypersurface of degree 3 in $\mathbf{P}^n$ we expect to find a linear subspace of dimension 2 provided $n \geq a$ where it is up to you to find a.

0GYC Exercise 111.65.4 (Heights). Let K be a field. Let $h_n : \mathbf{P}^n(K) \to \mathbf{R}$, $n \geq 0$ be a collection of functions satisfying the 2 axioms we discussed in the lectures. Let X be a projective variety over K. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and recall that we have constructed in the lectures an associated height function $h_\mathcal{L} : X(K) \to \mathbf{R}$. Let $\alpha : X \to X$ be an automorphism of X over K.

(1) Prove that $P \mapsto h_\mathcal{L}(\alpha(P))$ differs from the function $h_{\alpha^*\mathcal{L}}$ by a bounded amount. (Hint: recall that if there is a morphism $\varphi : X \to \mathbf{P}^n$ with $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbf{P}^n}(1)$, then by construction $h_\mathcal{L}(P) = h_n(\varphi(P))$ and play around with that. In general write $\mathcal{L}$ as a difference of two of these.)
(2) Assume that $h_\mathcal{L}(P) - h_\mathcal{L}(\alpha(P))$ is unbounded on $X(K)$. Show that $h_\mathcal{N}$ with $\mathcal{N} = \mathcal{L} \otimes \alpha^*\mathcal{L}^{\otimes -1}$ is unbounded on $X(K)$.
(3) Assume X is an elliptic curve and that $\mathcal{L}$ is a symmetric ample invertible module on X such that $h_\mathcal{L}$ is unbounded on $X(K)$. Show that there exists an invertible module $\mathcal{N}$ of degree 0 such that $h_\mathcal{N}$ is unbounded. (Hints: Recall that X is an abelian variety of dimension 1. Thus $h_\mathcal{L}$ is quadratic up to a constant by results in the lectures. Choose a suitable point $P_0 \in X(K)$. Let $\alpha : X \to X$ be translation by P_0. Consider $P \mapsto h_\mathcal{L}(P) - h_\mathcal{L}(P + P_0)$. Apply the results you proved above.)

0GYD Exercise 111.65.5 (Monomorphisms). Let $f : X \to Y$ be a monomorphism in the category of schemes: for any pair of morphisms $a, b : T \to X$ of schemes if $f \circ a = f \circ b$, then $a = b$. Show that f is injective on points. Does you argument say anything else?

0GYE Exercise 111.65.6 (Fixed points). Let k be an algebraically closed field.

(1) If $G = \mathbf{G}_{m,k}$ show that if G acts on a projective variety X over k, then the action has a fixed point, i.e., prove there exists a point $x \in X(k)$ such that $a(g, x) = x$ for all $g \in G(k)$.
(2) Same with $G = (\mathbf{G}_{m,k})^n$ equal to the product of $n \geq 1$ copies of the multiplicative group.
(3) Give an example of an action of a connected group scheme G on a smooth projective variety X which does not have a fixed point.

111.66. Other chapters

Preliminaries

(96) Sheaves on Algebraic Stacks
(97) Criteria for Representability
(98) Artin's Axioms
(99) Quot and Hilbert Spaces
(100) Properties of Algebraic Stacks
(101) Morphisms of Algebraic Stacks
(102) Limits of Algebraic Stacks
(103) Cohomology of Algebraic Stacks
(104) Derived Categories of Stacks
(105) Introducing Algebraic Stacks
(106) More on Morphisms of Stacks
(107) The Geometry of Stacks

Topics in Moduli Theory

(108) Moduli Stacks
(109) Moduli of Curves

Miscellany

(110) Examples
(111) Exercises
(112) Guide to Literature
(113) Desirables
(114) Coding Style
(115) Obsolete
(116) GNU Free Documentation License
(117) Auto Generated Index

CHAPTER 112

A Guide to the Literature

03B0

112.1. Short introductory articles

03B1

- Barbara Fantechi: Stacks for Everybody [Fan01]
- Dan Edidin: What is a stack? [Edi03]
- Dan Edidin: Notes on the construction of the moduli space of curves [Edi00]
- Angelo Vistoli: Intersection theory on algebraic stacks and on their moduli spaces, and especially the appendix. [Vis89]

112.2. Classic references

03B2

- Mumford: Picard groups of moduli problems [Mum65]
  Mumford never uses the term "stack" here but the concept is implicit in the paper; he computes the picard group of the moduli stack of elliptic curves.
- Deligne, Mumford: The irreducibility of the space of curves of given genus [DM69]
  This influential paper introduces "algebraic stacks" in the sense which are now universally called Deligne-Mumford stacks (stacks with representable diagonal which admit étale presentations by schemes). There are many foundational results without proof. The paper uses stacks to give two proofs of the irreducibility of the moduli space of curves of genus g.
- Artin: Versal deformations and algebraic stacks [Art74]
  This paper introduces "algebraic stacks" which generalize Deligne-Mumford stacks and are now commonly referred to as Artin stacks, stacks with representable diagonal which admit smooth presentations by schemes. This paper gives deformation-theoretic criterion known as Artin's criterion which allows one to prove that a given moduli stack is an Artin stack without explicitly exhibiting a presentation.

112.3. Books and online notes

03B3

- Laumon, Moret-Bailly: Champs Algébriques [LMB00]
  This book is currently the most exhaustive reference on stacks containing many foundational results. It assumes the reader is

familiar with algebraic spaces and frequently references Knutson's book [Knu71]. There is an error in chapter 12 concerning the functoriality of the lisse-étale site of an algebraic stack. One doesn't need to worry about this as the error has been patched by Martin Olsson (see [Ols07b]) and the results in the remaining chapters (after perhaps slight modification) are correct.

- The Stacks Project Authors: Stacks Project [Aut].
  You are reading it!
- Anton Geraschenko: Lecture notes for Martin Olsson's class on stacks [Ols07a]
  This course systematically develops the theory of algebraic spaces before introducing algebraic stacks (first defined in Lecture 27!). In addition to basic properties, the course covers the equivalence between being Deligne-Mumford and having unramified diagonal, the lisse-étale site on an Artin stack, the theory of quasi-coherent sheaves, the Keel-Mori theorem, cohomological descent, and gerbes (and their relation to the Brauer group). There are also some exercises.
- Behrend, Conrad, Edidin, Fantechi, Fulton, Göttsche, and Kresch: Algebraic stacks, online notes for a book being currently written [BCE+07]
  The aim of this book is to give a friendly introduction to stacks without assuming a sophisticated background with a focus on examples and applications. Unlike [LMB00], it is not assumed that the reader has digested the theory of algebraic spaces. Instead, Deligne-Mumford stacks are introduced with algebraic spaces being a special case with part of the goal being to develop enough theory to prove the assertions in [DM69]. The general theory of Artin stacks is to be developed in the second part. Only a fraction of the book is now available on Kresch's website.
- Olsson, Martin: Algebraic spaces and stacks, [Ols16]
  Highly recommended introduction to algebraic spaces and algebraic stacks starting at the level of somebody who has mastered Hartshorne's book on algebraic geometry.

112.4. Related references on foundations of stacks

03B4

- Vistoli: Notes on Grothendieck topologies, fibered categories and descent theory [Vis05]
  Contains useful facts on fibered categories, stacks and descent theory in the fpqc topology as well as rigorous proofs.
- Knutson: Algebraic Spaces [Knu71]
  This book, which evolved from his PhD thesis under Michael Artin, contains the foundations of the theory of algebraic spaces. The book [LMB00] frequently references this text. See also Artin's papers on algebraic spaces: [Art69a], [Art69b], [Art69c], [Art70], [Art71b], [Art71a], [Art73], and [Art74]

- Grothendieck et al, Théorie des Topos et Cohomologie Étale des Schémas I, II, III also known as SGA4 [AGV71]
  Volume 1 contains many general facts on universes, sites and fibered categories. The word "champ" (French for "stack") appears in Deligne's Exposé XVIII.
- Jean Giraud: Cohomologie non abélienne [Gir65]
  The book discusses fibered categories, stacks, torsors and gerbes over general sites but does not discuss algebraic stacks. For instance, if G is a sheaf of abelian groups on X, then in the same way $H^1(X, G)$ can be identified with G-torsors, $H^2(X, G)$ can be identified with an appropriately defined set of G-gerbes. When G is not abelian, then $H^2(X, G)$ is defined as the set of G-gerbes.
- Kelly and Street: Review of the elements of 2-categories [KS74]
  The category of stacks form a 2-category although a simple type of 2-category where are 2-morphisms are invertible. This is a reference on general 2-categories. I have never used this so I cannot say how useful it is. Also note that [Aut] contains some basics on 2-categories.

112.5. Papers in the literature

03B6 Below is a list of research papers which contain fundamental results on stacks and algebraic spaces. The intention of the summaries is to indicate only the results of the paper which contribute toward stack theory; in many cases these results are subsidiary to the main goals of the paper. We divide the papers into categories with some papers falling into multiple categories.

04UW 112.5.1. Deformation theory and algebraic stacks. The first three papers by Artin do not contain anything on stacks but they contain powerful results with the first two papers being essential for [Art74].

- Artin: Algebraic approximation of structures over complete local rings [Art69a]
  It is proved that under mild hypotheses any effective formal deformation can be approximated: if $F : (Sch/S) \to (\text{Sets})$ is a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$, and $\hat{\xi} \in F(\hat{\mathcal{O}}_{S,s})$ is an effective formal deformation, then for any $n > 0$, there exists an residually trivial étale neighborhood $(S', s') \to (S, s)$ and $\xi' \in F(S')$ such that ξ' and $\hat{\xi}$ agree up to order n (ie. have the same restriction in $F(\mathcal{O}_{S,s}/\mathfrak{m}^n)$).
- Artin: Algebraization of formal moduli I [Art69b]
  It is proved that under mild hypotheses any effective formal versal deformation is algebraizable. Let $F : (Sch/S) \to (\text{Sets})$ be a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$ be a locally closed point, $\hat{A}$ be a complete Noetherian local $\mathcal{O}_S$-algebra with residue field k' a finite extension of $k(s)$, and $\hat{\xi} \in F(\hat{A})$ be an effective formal versal deformation of an element $\xi_0 \in F(k')$.

Then there is a scheme X finite type over S and a closed point $x \in X$ with residue field $k(x) = k'$ and an element $\xi \in F(X)$ such that there is an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{A}$ identifying the restrictions of ξ and $\hat{\xi}$ in each $F(\hat{A}/\mathfrak{m}^n)$. The algebraization is unique if $\hat{\xi}$ is a universal deformation. Applications are given to the representability of the Hilbert and Picard schemes.

- Artin: Algebraization of formal moduli. II [Art70]
  Vaguely, it is shown that if one can contract a closed subset $Y' \subset X'$ formally locally around Y', then exists a global morphism $X' \to X$ contracting Y with X an algebraic space.
- Artin: Versal deformations and algebraic stacks [Art74]
  This momentous paper builds on his work in [Art69a] and [Art69b]. This paper introduces Artin's criterion which allows one to prove algebraicity of a stack by verifying deformation-theoretic properties. More precisely (but not very precisely), Artin constructs a presentation of a limit preserving stack $\mathcal{X}$ locally around a point $x \in \mathcal{X}(k)$ as follows: assuming the stack $\mathcal{X}$ satisfies Schlessinger's criterion([Sch68]), there exists a formal versal deformation $\hat{\xi} \in \lim \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ of x. Assuming that formal deformations are effective (i.e., $\mathcal{X}(\hat{A}) \to \lim \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ is bijective), then one obtains an effective formal versal deformation $\xi \in \mathcal{X}(\hat{A})$. Using results in [Art69b], one produces a finite type scheme U and an element $\xi_U : U \to \mathcal{X}$ which is formally versal at a point $u \in U$ over x. Then if we assume $\mathcal{X}$ admits a deformation and obstruction theory satisfying certain conditions (ie. compatibility with étale localization and completion as well as constructibility condition), then it is shown in section 4 that formal versality is an open condition so that after shrinking U, $U \to \mathcal{X}$ is smooth. Artin also presents a proof that any stack admitting an fppf presentation by a scheme admits a smooth presentation by a scheme so that in particular one can form quotient stacks by flat, separated, finitely presented group schemes.
- Conrad, de Jong: Approximation of Versal Deformations [CdJ02]
  This paper offers an approach to Artin's algebraization result by applying Popescu's powerful result: if A is a Noetherian ring and B a Noetherian A-algebra, then the map $A \to B$ is a regular morphism if and only if B is a direct limit of smooth A-algebras. It is not hard to see that Popescu's result implies Artin's approximation over an arbitrary excellent scheme (the excellence hypothesis implies that for a local ring A, the map $A^{\mathrm{h}} \to \hat{A}$ from the henselization to the completion is regular). The paper uses Popescu's result to give a "groupoid" generalization of the main theorem in [Art69b] which is valid over arbitrary excellent base schemes and for arbitrary points $s \in S$. In particular, the results in [Art74] hold under an arbitrary excellent base. They discuss the étale-local uniqueness of the algebraization and whether the automorphism group of the object acts naturally on the henselization of the algebraization.

- Jason Starr: Artin's axioms, composition, and moduli spaces [Sta06]
  The paper establishes that Artin's axioms for algebraization are compatible with the composition of 1-morphisms.
- Martin Olsson: Deformation theory of representable morphism of algebraic stacks [Ols06a]
  This generalizes standard deformation theory results for morphisms of schemes to representable morphisms of algebraic stacks in terms of the cotangent complex. These results cannot be viewed as consequences of Illusie's general theory as the cotangent complex of a representable morphism $X \to \mathcal{X}$ is not defined in terms of cotangent complex of a morphism of ringed topoi (because the lisse-étale site is not functorial).

04UX 112.5.2. Coarse moduli spaces. Papers discussing coarse moduli spaces.

- Keel, Mori: Quotients in Groupoids [KM97]
  It had apparently long been "folklore" that separated Deligne-Mumford stacks admitted coarse moduli spaces. A rigorous (although terse) proof of the following theorem is presented here: if $\mathcal{X}$ is an Artin stack locally of finite type over a Noetherian base scheme such that the inertia stack $I_\mathcal{X} \to \mathcal{X}$ is finite, then there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated and Y an algebraic space locally of finite type over S. The hypothesis that the inertia is finite is precisely the right condition: there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated if and only if the inertia is finite.
- Conrad: The Keel-Mori Theorem via Stacks [Con05b]
  Keel and Mori's paper [KM97] is written in the groupoid language and some find it challenging to grasp. Brian Conrad presents a stack-theoretic version of the proof which is quite transparent although it uses the sophisticated language of stacks. Conrad also removes the Noetherian hypothesis.
- Rydh: Existence of quotients by finite groups and coarse moduli spaces [Ryd07a]
  Rydh removes the hypothesis from [KM97] and [Con05b] that $\mathcal{X}$ be finitely presented over some base.
- Abramovich, Olsson, Vistoli: Tame stacks in positive characteristic [AOV08]
  They define a tame Artin stack as an Artin stack with finite inertia such that if $\phi : \mathcal{X} \to Y$ is the coarse moduli space, ϕ_* is exact on quasi-coherent sheaves. They prove that for an Artin stack with finite inertia, the following are equivalent: $\mathcal{X}$ is tame if and only if the stabilizers of $\mathcal{X}$ are linearly reductive if and only if $\mathcal{X}$ is étale locally on the coarse moduli space a quotient of an affine scheme by a linearly reductive group scheme. For a tame Artin stack, the coarse moduli space is particularly nice. For instance, the coarse moduli space commutes with arbitrary base change while a general coarse moduli space for an Artin stack with finite inertia will only commute with flat base change.
- Alper: Good moduli spaces for Artin stacks [Alp08]

For general Artin stacks with infinite affine stabilizer groups (which are necessarily non-separated), coarse moduli spaces often do not exist. The simplest example is $[\mathbf{A}^1/\mathbf{G}_m]$. It is defined here that a quasi-compact morphism $\phi : \mathcal{X} \to Y$ is a good moduli space if $\mathcal{O}_Y \to \phi_*\mathcal{O}_\mathcal{X}$ is an isomorphism and ϕ_* is exact on quasi-coherent sheaves. This notion generalizes a tame Artin stack in [AOV08] as well as encapsulates Mumford's geometric invariant theory: if G is a reductive group acting linearly on $X \subset \mathbf{P}^n$, then the morphism from the quotient stack of the semi-stable locus to the GIT quotient $[X^{ss}/G] \to X//G$ is a good moduli space. The notion of a good moduli space has many nice geometric properties: (1) ϕ is surjective, universally closed, and universally submersive, (2) ϕ identifies points in Y with points in $\mathcal{X}$ up to closure equivalence, (3) ϕ is universal for maps to algebraic spaces, (4) good moduli spaces are stable under arbitrary base change, and (5) a vector bundle on an Artin stack descends to the good moduli space if and only if the representations are trivial at closed points.

04UY 112.5.3. Intersection theory. Papers discussing intersection theory on algebraic stacks.

- Vistoli: Intersection theory on algebraic stacks and on their moduli spaces [Vis89]
  This paper develops the foundations for intersection theory with rational coefficients for Deligne-Mumford stacks. If $\mathcal{X}$ is a separated Deligne-Mumford stack, the chow group $\mathrm{CH}_*(\mathcal{X})$ with rational coefficients is defined as the free abelian group of integral closed substacks of dimension k up to rational equivalence. There is a flat pullback, a proper push-forward and a generalized Gysin homomorphism for regular local embeddings. If $\phi : \mathcal{X} \to Y$ is a moduli space (ie. a proper morphism with is bijective on geometric points), there is an induced push-forward $\mathrm{CH}_*(\mathcal{X}) \to \mathrm{CH}_k(Y)$ which is an isomorphism.
- Edidin, Graham: Equivariant Intersection Theory [EG98]
  The purpose of this article is to develop intersection theory with integral coefficients for a quotient stack $[X/G]$ of an action of an algebraic group G on an algebraic space X or, in other words, to develop a G-equivariant intersection theory of X. Equivariant chow groups defined using only invariant cycles does not produce a theory with nice properties. Instead, generalizing Totaro's definition in the case of BG and motivated by the fact that if $V \to X$ is a vector bundle then $\mathrm{CH}_i(X) \cong \mathrm{CH}_i(V)$ naturally, the authors define $\mathrm{CH}_i^G(X)$ as follows: Let $\dim(X) = n$ and $\dim(G) = g$. For each i, choose a l-dimensional G-representation V where G acts freely on an open subset $U \subset V$ whose complement as codimension $d > n - i$. So $X_G = [X \times U/G]$ is an algebraic space (it can even be chosen to be a scheme). Then they define $\mathrm{CH}_i^G(X) = \mathrm{CH}_{i+l-g}(X_G)$. For the quotient stack, one defines $\mathrm{CH}_i([X/G]) = \mathrm{CH}^G_{i+g}(X) = \mathrm{CH}_{i+l}(X_G)$. In particular,

$\mathrm{CH}_i([X/G]) = 0$ for $i > \dim[X/G] = n - g$ but can be non-zero for $i \ll 0$. For example $\mathrm{CH}_i(B\mathbf{G}_m) = \mathbf{Z}$ for $i \leq 0$. They establish that these equivariant Chow groups enjoy the same functorial properties as ordinary Chow groups. Furthermore, they establish that if $[X/G] \cong [Y/H]$ that $\mathrm{CH}_i([X/G]) = \mathrm{CH}_i([Y/H])$ so that the definition is independent on how the stack is presented as a quotient stack.

- Kresch: Cycle Groups for Artin Stacks [Kre99]
  Kresch defines Chow groups for arbitrary Artin stacks agreeing with Edidin and Graham's definition in [EG98] in the case of quotient stack. For algebraic stacks with affine stabilizer groups, the theory satisfies the usual properties.
- Behrend and Fantechi: The intrinsic normal cone [BF97]
  Generalizing a construction due to Li and Tian, Behrend and Fantechi construct a virtual fundamental class for a Deligne-Mumford stack.

04UZ 112.5.4. Quotient stacks. Quotient stacks[1] form a very important subclass of Artin stacks which include almost all moduli stacks studied by algebraic geometers. The geometry of a quotient stack $[X/G]$ is the G-equivariant geometry of X. It is often easier to show properties are true for quotient stacks and some results are only known to be true for quotient stacks. The following papers address: When is an algebraic stack a global quotient stack? Is an algebraic stack "locally" a quotient stack?

- Laumon, Moret-Bailly: [LMB00, Chapter 6]
  Chapter 6 contains several facts about the local and global structure of algebraic stacks. It is proved that an algebraic stack $\mathcal{X}$ over S is a quotient stack $[Y/G]$ with Y an algebraic space (resp. scheme, resp. affine scheme) and G a finite group if and only if there exists an algebraic space (resp. scheme, resp. affine scheme) Y' and an finite étale morphism $Y' \to \mathcal{X}$. It is shown that any Deligne-Mumford stack over S and $x : \mathrm{Spec}(K) \to \mathcal{X}$ admits an representable, étale and separated morphism $\phi : [X/G] \to \mathcal{X}$ where G is a finite group acting on an affine scheme over S such that $\mathrm{Spec}(K) = [X/G] \times_{\mathcal{X}} \mathrm{Spec}(K)$. The existence of presentations with geometrically connected fibers is also discussed in detail.
- Edidin, Hassett, Kresch, Vistoli: Brauer Groups and Quotient stacks [EHKV01]
  First, they establish some fundamental (although not very difficult) facts concerning when a given algebraic stack (always assumed finite type over a Noetherian scheme in this paper) is a quotient stack. For an algebraic stack $\mathcal{X}$: $\mathcal{X}$ is a quotient stack if and only if there exists a vector bundle $V \to \mathcal{X}$ such that for every geometric point, the stabilizer acts faithfully on the fiber

[1] In the literature, quotient stack often means a stack of the form $[X/G]$ with X an algebraic space and G a subgroup scheme of GL_n rather than an arbitrary flat group scheme.

if and only if there exists a vector bundle $V \to \mathcal{X}$ and a locally closed substack $V^0 \subset V$ such that V^0 is representable and surjects onto $\mathcal{X}$. They establish that an algebraic stack is a quotient stack if there exists finite flat cover by an algebraic space. Any smooth Deligne-Mumford stack with generically trivial stabilizer is a quotient stack. They show that a $\mathbf{G}_m$-gerbe over a Noetherian scheme X corresponding to $\beta \in H^2(X, \mathbf{G}_m)$ is a quotient stack if and only if β is in the image of the Brauer map $\text{Br}(X) \to \text{Br}'(X)$. They use this to produce a non-separated Deligne-Mumford stack that is not a quotient stack.

- Totaro: The resolution property for schemes and stacks [Tot04]
  A stack has the resolution property if every coherent sheaf is the quotient of a vector bundle. The first main theorem is that if $\mathcal{X}$ is a normal Noetherian algebraic stack with affine stabilizer groups at closed points, then the following are equivalent: (1) $\mathcal{X}$ has the resolution property and (2) $\mathcal{X} = [Y/\text{GL}_n]$ with Y quasi-affine. In the case $\mathcal{X}$ is finite type over a field, then (1) and (2) are equivalent to: (3) $\mathcal{X} = [\text{Spec}(A)/G]$ with G an affine group scheme finite type over k. The implication that quotient stacks have the resolution property was proven by Thomason. The second main theorem is that if $\mathcal{X}$ is a smooth Deligne-Mumford stack over a field which has a finite and generically trivial stabilizer group $I_{\mathcal{X}} \to \mathcal{X}$ and whose coarse moduli space is a scheme with affine diagonal, then $\mathcal{X}$ has the resolution property. Another cool result states that if $\mathcal{X}$ is a Noetherian algebraic stack satisfying the resolution property, then $\mathcal{X}$ has affine diagonal if and only if the closed points have affine stabilizer.
- Kresch: On the Geometry of Deligne-Mumford Stacks [Kre09]
  This article summarizes general structure results of Deligne-Mumford stacks (of finite type over a field) and contains some interesting results concerning quotient stacks. It is shown that any smooth, separated, generically tame Deligne-Mumford stack with quasi-projective coarse moduli space is a quotient stack $[Y/G]$ with Y quasi-projective and G an algebraic group. If $\mathcal{X}$ is a Deligne-Mumford stack whose coarse moduli space is a scheme, then $\mathcal{X}$ is Zariski-locally a quotient stack if and only if it admits a Zariski-open covering by stack quotients of schemes by finite groups. If $\mathcal{X}$ is a Deligne-Mumford stack proper over a field of characteristic 0 with coarse moduli space Y, then: Y is projective and $\mathcal{X}$ is a quotient stack if and only if Y is projective and $\mathcal{X}$ possesses a generating sheaf if and only if $\mathcal{X}$ admits a closed embedding into a smooth proper DM stack with projective coarse moduli space. This motivates a definition that a Deligne-Mumford stack is projective if there exists a closed embedding into a smooth, proper Deligne-Mumford stack with projective coarse moduli space.
- Kresch, Vistoli On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map [KV04]

It is shown that in characteristic 0 and for a fixed n, the following two statements are equivalent: (1) every smooth Deligne-Mumford stack of dimension n is a quotient stack and (2) the Azumaya Brauer group coincides with the cohomological Brauer group for smooth schemes of dimension n.

- Kresch: Cycle Groups for Artin Stacks [Kre99]
  It is shown that a reduced Artin stack finite type over a field with affine stabilizer groups admits a stratification by quotient stacks.
- Abramovich-Vistoli: Compactifying the space of stable maps [AV02]
  Lemma 2.2.3 establishes that for any separated Deligne-Mumford stack is étale-locally on the coarse moduli space a quotient stack $[U/G]$ where U affine and G a finite group. [Ols06b, Theorem 2.12] shows in this argument G is even the stabilizer group.
- Abramovich, Olsson, Vistoli: Tame stacks in positive characteristic [AOV08]
  This paper shows that a tame Artin stack is étale locally on the coarse moduli space a quotient stack of an affine by the stabilizer group.
- Alper: On the local quotient structure of Artin stacks [Alp10]
  It is conjectured that for an Artin stack $\mathcal{X}$ and a closed point $x \in \mathcal{X}$ with linearly reductive stabilizer, then there is an étale morphism $[V/G_x] \to \mathcal{X}$ with V an algebraic space. Some evidence for this conjecture is given. A simple deformation theory argument (based on ideas in [AOV08]) shows that it is true formally locally. A stack-theoretic proof of Luna's étale slice theorem is presented proving that for stacks $\mathcal{X} = [\mathrm{Spec}(A)/G]$ with G linearly reductive, then étale locally on the GIT quotient $\mathrm{Spec}(A^G)$, $\mathcal{X}$ is a quotient stack by the stabilizer.

04V0 112.5.5. Cohomology. Papers discussing cohomology of sheaves on algebraic stacks.

- Olsson: Sheaves on Artin stacks [Ols07b]
  This paper develops the theory of quasi-coherent and constructible sheaves proving basic cohomological properties. This paper corrects a mistake in [LMB00] in the functoriality of the lisse-étale site. The cotangent complex is constructed. In addition, the following theorems are proved: Grothendieck's Fundamental Theorem for proper morphisms, Grothendieck's Existence Theorem, Zariski's Connectedness Theorem and finiteness theorem for proper pushforwards of coherent and constructible sheaves.
- Behrend: Derived l-adic categories for algebraic stacks [Beh03]
  Proves the Lefschetz trace formula for algebraic stacks.
- Behrend: Cohomology of stacks [Beh04]
  Defines the de Rham cohomology for differentiable stacks and singular cohomology for topological stacks.
- Faltings: Finiteness of coherent cohomology for proper fppf stacks [Fal03]
  Proves coherence for direct images of coherent sheaves for proper morphisms.
- Abramovich, Corti, Vistoli: Twisted bundles and admissible covers [ACV03]

The appendix contains the proper base change theorem for étale cohomology for tame Deligne-Mumford stacks.

04V1 112.5.6. Existence of finite covers by schemes. The existence of finite covers of Deligne-Mumford stacks by schemes is an important result. In intersection theory on Deligne-Mumford stacks, it is an essential ingredient in defining proper push-forward for non-representable morphisms. There are several results about $\overline{\mathcal{M}}_g$ relying on the existence of a finite cover by a smooth scheme which was proven by Looijenga. Perhaps the first result in this direction is [Ses72, Theorem 6.1] which treats the equivariant setting.

- Vistoli: Intersection theory on algebraic stacks and on their moduli spaces [Vis89]

  If $\mathcal{X}$ is a Deligne-Mumford stack with a moduli space (ie. a proper morphism which is bijective on geometric points), then there exists a finite morphism $X \to \mathcal{X}$ from a scheme X.
- Laumon, Moret-Bailly: [LMB00, Chapter 16]

  As an application of Zariski's main theorem, Theorem 16.6 establishes: if $\mathcal{X}$ is a Deligne-Mumford stack finite type over a Noetherian scheme, then there exists a finite, surjective, generically étale morphism $Z \to \mathcal{X}$ with Z a scheme. It is also shown in Corollary 16.6.2 that any Noetherian normal algebraic space is isomorphic to the algebraic space quotient X'/G for a finite group G acting a normal scheme X.
- Edidin, Hassett, Kresch, Vistoli: Brauer Groups and Quotient stacks [EHKV01]

  Theorem 2.7 states: if $\mathcal{X}$ is an algebraic stack of finite type over a Noetherian ground scheme S, then the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is quasi-finite if and only if there exists a finite surjective morphism $X \to F$ from a scheme X.
- Kresch, Vistoli: On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map [KV04]

  It is proved here that any smooth, separated Deligne-Mumford stack finite type over a field with quasi-projective coarse moduli space admits a finite, flat cover by a smooth quasi-projective scheme.
- Olsson: On proper coverings of Artin stacks [Ols05]

  Proves that if $\mathcal{X}$ is an Artin stack separated and finite type over S, then there exists a proper surjective morphism $X \to \mathcal{X}$ from a scheme X quasi-projective over S. As an application, Olsson proves coherence and constructibility of direct image sheaves under proper morphisms. As an application, he proves Grothendieck's existence theorem for proper Artin stacks.
- Rydh: Noetherian approximation of algebraic spaces and stacks [Ryd08]

  Theorem B of this paper is as follows. Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal (resp. a quasi-compact Deligne-Mumford stack with quasi-compact and separated diagonal). Then there exists a scheme Z and a finite, finitely presented and surjective morphism $Z \to X$ that is flat (resp. étale) over a dense quasi-compact open substack $U \subset X$.

04V2 112.5.7. Rigidification. Rigidification is a process for removing a flat subgroup from the inertia. For example, if X is a projective variety, the morphism from the Picard stack to the Picard scheme is a rigidification of the group of automorphism $\mathbf{G}_m$.

- Abramovich, Corti, Vistoli: Twisted bundles and admissible covers [ACV03]
  Let $\mathcal{X}$ be an algebraic stack over S and H be a flat, finitely presented separated group scheme over S. Assume that for every object $\xi \in \mathcal{X}(T)$ there is an embedding $H(T) \hookrightarrow \text{Aut}_{\mathcal{X}(T)}(\xi)$ which is compatible under pullbacks in the sense that for every arrow $\phi : \xi \to \xi'$ over $f : T \to T'$ and $g \in H(T')$, $g \circ \phi = \phi \circ f^*g$. Then there exists an algebraic stack $\mathcal{X}/H$ and a morphism $\rho : \mathcal{X} \to \mathcal{X}/H$ which is an fppf gerbe such that for every $\xi \in \mathcal{X}(T)$, the morphism $\text{Aut}_{\mathcal{X}(T)}(\xi) \to \text{Aut}_{\mathcal{X}/H(T)}(\xi)$ is surjective with kernel $H(T)$.
- Romagny: Group actions on stacks and applications [Rom05]
  Discusses how group actions behave with respect to rigidifications.
- Abramovich, Graber, Vistoli: Gromov-Witten theory for Deligne-Mumford stacks [AGV08]
  The appendix gives a summary of rigidification as in [ACV03] with two alternative interpretations. This paper also contains constructions for gluing algebraic stacks along closed substacks and for taking roots of line bundles.
- Abramovich, Olsson, Vistoli: Tame stacks in positive characteristic ([AOV08])
  The appendix handles the more complicated situation where the flat subgroup stack of the inertia $H \subset I_{\mathcal{X}}$ is normal but not necessarily central.

04V3 112.5.8. Stacky curves. Papers discussing stacky curves.

- Abramovich, Vistoli: Compactifying the space of stable maps [AV02]
  This paper introduces twisted curves. The moduli space of stable maps from stable curves into an algebraic stack is typically not compact. By using maps from twisted curves, the authors construct a moduli stack which is proper when the target is a tame Deligne-Mumford stack whose coarse moduli space is projective.
- Behrend, Noohi: Uniformization of Deligne-Mumford curves [BN06]
  Proves a uniformization theorem of Deligne-Mumford analytic curves.

04V4 112.5.9. Hilbert, Quot, Hom and branchvariety stacks. Papers discussing Hilbert schemes and the like.

- Vistoli: The Hilbert stack and the theory of moduli of families [Vis91]
  If $\mathcal{X}$ is a algebraic stack separated and locally of finite type over a locally Noetherian and locally separated algebraic space S, Vistoli defines the Hilbert stack $\mathcal{H}\text{ilb}(\mathcal{F}/S)$ parameterizing finite and unramified morphisms from proper schemes. It is claimed without proof that $\mathcal{H}\text{ilb}(\mathcal{F}/S)$ is an algebraic stack. As a consequence, it is proved that with $\mathcal{X}$ as above, the Hom stack $\mathcal{H}\text{om}_S(T, \mathcal{X})$ is an algebraic stack if T is proper and flat over S.

- Olsson, Starr: Quot functors for Deligne-Mumford stacks [OS03]
  If $\mathcal{X}$ is a Deligne-Mumford stack separated and locally of finite presentation over an algebraic space S and $\mathcal{F}$ is a locally finitely-presented $\mathcal{O}_\mathcal{X}$-module, the quot functor $\text{Quot}(\mathcal{F}/\mathcal{X}/S)$ is represented by an algebraic space separated and locally of finite presentation over S. This paper also defines generating sheaves and proves existence of a generating sheaf for tame, separated Deligne-Mumford stacks which are global quotient stacks of a scheme by a finite group.
- Olsson: Hom-stacks and Restrictions of Scalars [Ols06b]
  Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Artin stacks locally of finite presentation over an algebraic space S with finite diagonal with $\mathcal{X}$ proper and flat over S such that fppf-locally on S, $\mathcal{X}$ admits a finite finitely presented flat cover by an algebraic space (eg. $\mathcal{X}$ is Deligne-Mumford or a tame Artin stack). Then $\text{Hom}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack locally of finite presentation over S.
- Alexeev and Knutson: Complete moduli spaces of branchvarieties ([AK10])
  They define a branchvariety of $\mathbf{P}^n$ as a finite morphism $X \to \mathbf{P}^n$ from a reduced scheme X. They prove that the moduli stack of branchvarieties with fixed Hilbert polynomial and total degrees of i-dimensional components is a proper Artin stack with finite stabilizer. They compare the stack of branchvarieties with the Hilbert scheme, Chow scheme and moduli space of stable maps.
- Lieblich: Remarks on the stack of coherent algebras [Lie06b]
  This paper constructs a generalization of Alexeev and Knutson's stack of branch-varieties over a scheme Y by building the stack as a stack of algebras over the structure sheaf of Y. Existence proofs of Quot and Hom spaces are given.
- Starr: Artin's axioms, composition, and moduli spaces [Sta06]
  As an application of the main result, a common generalization of Vistoli's Hilbert stack [Vis91] and Alexeev and Knutson's stack of branchvarieties [AK10] is provided. If $\mathcal{X}$ is an algebraic stack locally of finite type over an excellent scheme S with finite diagonal, then the stack $\mathcal{H}$ parameterizing morphisms $g : T \to \mathcal{X}$ from a proper algebraic space T with a G-ample line bundle L is an Artin stack locally of finite type over S.
- Lundkvist and Skjelnes: Non-effective deformations of Grothendieck's Hilbert functor [LS08]
  Shows that the Hilbert functor of a non-separated scheme is not represented since there are non-effective deformations.
- Halpern-Leistner and Preygel: Mapping stacks and categorical notions of properness [HLP14]
  This paper gives a proof that the Hom stack is algebraic under some hypotheses on source and target which are more general than, or at least different from, the ones in Olsson's paper.

04V5 112.5.10. Toric stacks. Toric stacks provide a great class of examples and a natural testing ground for conjectures due to the dictionary between the geometry of a toric stack and the combinatorics of its stacky fan in a similar way that toric varieties provide examples and counterexamples in scheme theory.

- Borisov, Chen and Smith: The orbifold Chow ring of toric Deligne-Mumford stacks [BCS05]
  Inspired by Cox's construction for toric varieties, this paper defines smooth toric DM stacks as explicit quotient stacks associated to a combinatorial object called a stacky fan.
- Iwanari: The category of toric stacks [Iwa09]
  This paper defines a toric triple as a smooth Deligne-Mumford stack $\mathcal{X}$ with an open immersion $\mathbf{G}_m \hookrightarrow \mathcal{X}$ with dense image (and therefore $\mathcal{X}$ is an orbifold) and an action $\mathcal{X} \times \mathbf{G}_m \to \mathcal{X}$. It is shown that there is an equivalence between the 2-category of toric triples and the 1-category of stacky fans. The relationship between toric triples and the definition of smooth toric DM stacks in [BCS05] is discussed.
- Iwanari: Integral Chow rings for toric stacks [Iwa07]
  Generalizes Cox's Δ-collections for toric varieties to toric orbifolds.
- Perroni: A note on toric Deligne-Mumford stacks [Per08]
  Generalizes Cox's Δ-collections and Iwanari's paper [Iwa07] to general smooth toric DM stacks.
- Fantechi, Mann, and Nironi: Smooth toric DM stacks [FMN07]
  This paper defines a smooth toric DM stack as a smooth DM stack $\mathcal{X}$ with the action of a DM torus $\mathcal{T}$ (ie. a Picard stack isomorphic to $T \times BG$ with G finite) having an open dense orbit isomorphic to $\mathcal{T}$. They give a "bottom-up description" and prove an equivalence between smooth toric DM stacks and stacky fans.
- Geraschenko and Satriano: Toric Stacks I and II [GS11a] and [GS11b]
  These papers define a toric stack as the stack quotient of a toric variety by a subgroup of its torus. A generically stacky toric stack is defined as a torus invariant substack of a toric stack. This definition encompasses and extends previous definitions of toric stacks. The first paper develops a dictionary between the combinatorics of stacky fans and the geometry of the corresponding stacks. It also gives a moduli interpretation of smooth toric stacks, generalizing the one in [Per08]. The second paper proves an intrinsic characterization of toric stacks.

112.5.11. Theorem on formal functions and Grothendieck's Existence Theorem.
04V6 These papers give generalizations of the theorem on formal functions [DG67, III.4.1.5] (sometimes referred to Grothendieck's Fundamental Theorem for proper morphisms) and Grothendieck's Existence Theorem [DG67, III.5.1.4].

- Knutson: Algebraic spaces [Knu71, Chapter V]
  Generalizes these theorems to algebraic spaces.
- Abramovich-Vistoli: Compactifying the space of stable maps [AV02, A.1.1]
  Generalizes these theorems to tame Deligne-Mumford stacks
- Olsson and Starr: Quot functors for Deligne-Mumford stacks [OS03]
  Generalizes these theorems to separated Deligne-Mumford stacks.

- Olsson: On proper coverings of Artin stacks [Ols05]
  Provides a generalization to proper Artin stacks.
- Conrad: Formal GAGA on Artin stacks [Con05a]
  Provides a generalization to proper Artin stacks and proves a formal GAGA theorem.
- Olsson: Sheaves on Artin stacks [Ols07b]
  Provides another proof for the generalization to proper Artin stacks.

04V7 112.5.12. Group actions on stacks. Actions of groups on algebraic stacks naturally appear. For instance, symmetric group S_n acts on $\overline{\mathcal{M}}_{g,n}$ and for an action of a group G on a scheme X, the normalizer of G in $\text{Aut}(X)$ acts on $[X/G]$. Furthermore, torus actions on stacks often appear in Gromov-Witten theory.

- Romagny: Group actions on stacks and applications [Rom05]
  This paper makes precise what it means for a group to act on an algebraic stack and proves existence of fixed points as well as existence of quotients for actions of group schemes on algebraic stacks. See also Romagny's earlier note [Rom03].

04V8 112.5.13. Taking roots of line bundles. This useful construction was discovered independently by Cadman and by Abramovich, Graber and Vistoli. Given a scheme X with an effective Cartier divisor D, the rth root stack is an Artin stack branched over X at D with a μ_r stabilizer over D and scheme-like away from D.

- Charles Cadman Using Stacks to Impose Tangency Conditions on Curves [Cad07]
- Abramovich, Graber, Vistoli: Gromov-Witten theory for Deligne-Mumford stacks [AGV08]

04V9 112.5.14. Other papers. Potpourri of other papers.

- Lieblich: Moduli of twisted sheaves [Lie07]
  This paper contains a summary of gerbes and twisted sheaves. If $\mathcal{X} \to X$ is a μ_n-gerbe with X a projective relative surface with smooth connected geometric fibers, it is shown that the stack of semistable $\mathcal{X}$-twisted sheaves is an Artin stack locally of finite presentation over S. This paper also develops the theory of associated points and purity of sheaves on Artin stacks.
- Lieblich, Osserman: Functorial reconstruction theorem for stacks [LO08b]
  Proves some surprising and interesting results on when an algebraic stack can be reconstructed from its associated functor.
- David Rydh: Noetherian approximation of algebraic spaces and stacks [Ryd08]
  This paper shows that every quasi-compact algebraic stack with quasi-finite diagonal can be approximated by a Noetherian stack. There are applications to removing the Noetherian hypothesis in results of Chevalley, Serre, Zariski and Chow.

112.6. Stacks in other fields

03B5

- Behrend and Noohi: Uniformization of Deligne-Mumford curves [BN06]

Gives an overview and comparison of topological, analytic and algebraic stacks.

- Behrang Noohi: Foundations of topological stacks I [Noo05]
- David Metzler: Topological and smooth stacks [Met05]

112.7. Higher stacks

05BF

- Lurie: Higher topos theory [Lur09f]
- Lurie: Derived Algebraic Geometry I - V [Lur09a], [Lur09b], [Lur09c], [Lur09d], [Lur09e]
- Toën: Higher and derived stacks: a global overview [Toë09]
- Toën and Vezzosi: Homotopical algebraic geometry I, II [TV05], [TV08]

112.8. Other chapters

CHAPTER 113

Desirables

02B4 113.1. Introduction

02B5 This is basically just a list of things that we want to put in the stacks project. As we add material to the Stacks project continuously this is always somewhat behind the current state of the Stacks project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

Last updated: Thursday, August 31, 2017.

113.2. Conventions

02B7 We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 2.1, but a lot more could be added there. Especially useful would be to find "hidden" conventions and tacit assumptions and put those there.

113.3. Sites and Topoi

02BA We have a chapter on sites and sheaves, see Sites, Section 7.1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 18.1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 21.1. But a lot more could be added, especially in the chapter on cohomology.

113.4. Stacks

02BB We have a chapter on (abstract) stacks, see Stacks, Section 8.1. It would be nice if

(1) improve the discussion on "stackyfication",
(2) give examples of stackyfication,
(3) more examples in general,
(4) improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups $\mathcal{F}$ over $\mathcal{C}$ the set of equivalence classes of gerbes banded by $\mathcal{F}$ is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

113.5. Simplicial methods

03MZ We have a chapter on simplicial methods, see Simplicial, Section 14.1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. There is a chapter on simplicial spaces, see Simplicial Spaces, Section 85.1. This chapter briefly discusses simplicial topological spaces, simplicial sites, and simplicial topoi. We can further develop "simplicial algebraic geometry" to discuss simplicial schemes (or simplicial algebraic spaces, or simplicial algebraic stacks) and treat geometric questions, their cohomology, etc.

113.6. Cohomology of schemes

02BE There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 30.1. We have a chapter discussing the derived category of quasi-coherent sheaves on a scheme, see Derived Categories of Schemes, Section 36.1. We have a chapter discussing duality for Noetherian schemes and relative duality for morphisms of schemes, see Duality for Schemes, Section 48.1. We also have chapters on étale cohomology of schemes and on crystalline cohomology of schemes. But most of the material in these chapters is very basic and a lot more could/should be added there.

113.7. Deformation theory à la Schlessinger

02BF We have a chapter on this material, see Formal Deformation Theory, Section 90.1. We have a chapter discussing examples of the general theory, see Deformation Problems, Section 93.1. We have a chapter, see Deformation Theory, Section 91.1 which discusses deformations of rings (and modules), deformations of ringed spaces (and sheaves of modules), deformations of ringed topoi (and sheaves of modules). In this chapter we use the naive cotangent complex to describe obstructions, first order deformations, and infinitesimal automorphisms. This material has found some applications to algebraicity of moduli stacks in later chapters. There is also a chapter discussing the full cotangent complex, see Cotangent, Section 92.1.

113.8. Definition of algebraic stacks

02BK An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. See Algebraic Stacks, Section 94.12. A "Deligne-Mumford stack" is an algebraic stack for which there exists a scheme and a surjective étale morphism from that scheme to it as in the paper [DM69] of Deligne and Mumford, see Algebraic Stacks, Definition 94.12.2. We will reserve the term "Artin stack" for a stack such as in the papers by Artin, see [Art69b], [Art70], and [Art74]. A possible definition is that an Artin stack is an algebraic stack $\mathcal{X}$ over a locally Noetherian scheme S such that $\mathcal{X} \to S$ is locally of finite type[1].

113.9. Examples of schemes, algebraic spaces, algebraic stacks

02BL The Stacks project currently contains two chapters discussing moduli stacks and their properties, see Moduli Stacks, Section 108.1 and Moduli of Curves, Section 109.1. Over time we intend to add more, for example:

(1) $\mathcal{A}_g$, i.e., principally polarized abelian schemes of genus g,
(2) $\mathcal{A}_1 = \mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
(3) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g-curves with n pairwise distinct labeled points,
(4) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n-pointed nodal projective genus g-curves,
(5) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks $\mathcal{X}$, $\mathcal{Y}$ and the base scheme S),

[1] Namely, these are exactly the algebraic stacks over S satisfying Artin's axioms [-1], [0], [1], [2], [3], [4], [5] of Artin's Axioms, Section 98.14.

(6) $\mathrm{Bun}_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G-bundles of the geometric Langlands programme (with suitable conditions on the scheme X, the group scheme G, and the base scheme S),
(7) $\mathcal{P}ic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

More generally, the Stacks project is somewhat lacking in geometrically meaningful examples.

113.10. Properties of algebraic stacks

02BM This is perhaps one of the easier projects to work on, as most of the basic theory is there now. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

113.11. Lisse étale site of an algebraic stack

02BN This has been introduced in Cohomology of Stacks, Section 103.14. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 110.58. Of course a lot more could be said about this, but it turns out to be very useful to prove things using the "big" étale site as much as possible.

113.12. Things you always wanted to know but were afraid to ask

02BO There are going to be lots of lemmas that you use over and over again that are useful but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \Rightarrow U$ and $R' \Rightarrow U'$ what does it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes.

113.13. Quasi-coherent sheaves on stacks

02BP These are defined and discussed in the chapter Cohomology of Stacks, Section 103.1. Derived categories of modules are discussed in the chapter Derived Categories of Stacks, Section 104.1. A lot more could be added to these chapters.

113.14. Flat and smooth

02BR Artin's theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition. This is now available as Criteria for Representability, Theorem 97.16.1.

113.15. Artin's representability theorem

02BS This is discussed in the chapter Artin's Axioms, Section 98.1. We also have an application, see Quot, Theorem 99.5.12. There should be a lot more applications and the chapter itself has to be cleaned up as well.

113.16. DM stacks are finitely covered by schemes

02BT We already have the corresponding result for algebraic spaces, see Limits of Spaces, Section 70.16. What is missing is the result for DM and quasi-DM stacks.

113.17. Martin Olsson's paper on properness

02BU This proves two notions of proper are the same. The first part of this is now available in the form of Chow's lemma for algebraic stacks, see More on Morphisms of Stacks, Theorem 106.10.3. As a consequence we show that it suffices to use DVR's in checking the valuative criterion for properness for algebraic stacks in certain cases, see More on Morphisms of Stacks, Section 106.11.

113.18. Proper pushforward of coherent sheaves

02BV We can start working on this now that we have Chow's lemma for algebraic stacks, see previous section.

113.19. Keel and Mori

02BW See [KM97]. Their result has been added in More on Morphisms of Stacks, Section 106.13.

113.20. Add more here

02BX Actually, no we should never have started this list as part of the Stacks project itself! There is a todo list somewhere else which is much easier to update.

113.21. Other chapters

CHAPTER 114

Coding Style

02BY 114.1. List of style comments

02BZ These will be changed over time, but having some here now will hopefully encourage a consistent LaTeX style. We will call "code[1]" the contents of the source files.

(1) Keep all lines in all tex files to at most 80 characters.
(2) Do not use indentation in the tex file. Use syntax highlighting in your editor, instead of indentation, to visualize environments, etc.
(3) Use

```
\medskip\noindent
```

to start a new paragraph, and use

```
\noindent
```

to start a new paragraph just after an environment.
(4) Do not break the code for mathematical formulas across lines if possible. If the complete code complete with enclosing dollar signs does not fit on the line, then start with the first dollar sign on the first character of the next line. If it still does not fit, find a mathematically reasonable spot to break the code.
(5) Displayed math equations should be coded as follows

```
$$
...
...
$$
```

In other words, start with a double dollar sign on a line by itself and end similarly.
(6) Do not use any macros. Rationale: This makes it easier to read the tex file, and start editing an arbitrary part without having to learn innumerable macros. And it doesn't make it harder or more timeconsuming to write. Of course the disadvantage is that the same mathematical object may be TeXed differently in different places in the text, but this should be easy to spot.
(7) The theorem environments we use are: "theorem", "proposition", "lemma" (plain), "definition", "example", "exercise", "situation" (definition), "remark", "remarks" (remark). Of course there is also a "proof" environment.
(8) An environment "foo" should be coded as follows

```
\begin{foo}
...
```

[1] It is all Knuth's fault. See [Knu79].

```
...
\end{foo}
```

similarly to the way displayed equations are coded.

(9) Instead of a "corollary", just use "lemma" environment since likely the result will be used to prove the next bigger theorem anyway.

(10) Directly following each lemma, proposition, or theorem is the proof of said lemma, proposition, or theorem. No nested proofs please.

(11) The files preamble.tex, chapters.tex and fdl.tex are special tex files. Apart from these, each tex file has the following structure

```
\input{preamble}
\begin{document}
\title{Title}
\maketitle
\tableofcontents
...
...
\input{chapters}
\bibliography{my}
\bibliographystyle{amsalpha}
\end{document}
```

(12) Try to add labels to lemmas, propositions, theorems, and even remarks, exercise, and other environments. If labelling a lemma use something like

```
\begin{lemma}
\label{lemma-bar}
...
\end{lemma}
```

Similarly for all other environments. In other words, the label of a environment named "foo" starts with "foo-". In addition to this please make all labels consist only of lower case letters, digits, and the symbol "-".

(13) Never refer to "the lemma above" (or proposition, etc). Instead use:

```
Lemma \ref{lemma-bar} above
```

This means that later moving lemmas around is basically harmless.

(14) Cross-file referencing. To reference a lemma labeled "lemma-bar" in the file foo.tex which has title "Foo", please use the following code

```
Foo, Lemma \ref{foo-lemma-bar}
```

If this does not work, then take a look at the file preamble.tex to find the correct expression to use. This will produce the "Foo, Lemma <link>" in the output file so it will be clear that the link points out of the file.

(15) If at all possible avoid forward references in proof environments. (It should be possible to write an automated test for this.)

(16) Do not start any sentence with a mathematical symbol.

(17) Do not have a sentence of the type "This follows from the following" just before a lemma, proposition, or theorem. Every sentence ends with a period.

(18) State all hypotheses in each lemma, proposition, theorem. This makes it easier for readers to see if a given lemma, proposition, or theorem applies to their particular problem.

(19) Keep proofs short; less than 1 page in pdf or dvi. You can always achieve this by splitting out the proof in lemmas etc.
(20) In a defining property foobar use

```
{\it foobar}
```

in the code inside the definition environment. Similarly if the definition occurs in the text of the document. This will make it easier for the reader to see what it is that is being defined.
(21) Put any definition that will be used outside the section it is in, in its own definition environment. Temporary definitions may be made in the text. A tricky case is that of mathematical constructions (which are often definitions involving interrelated lemmas). Maybe a good solution is to have them in their own short section so users can refer to the section instead of a definition.
(22) Do not number equations unless they are actually being referenced somewhere in the text. We can always add labels later.
(23) In statements of lemmas, propositions and theorems and in proofs keep the sentences short. For example, instead of "Let R be a ring and let M be an R-module." write "Let R be a ring. Let M be an R-module.". Rationale: This makes it easier to parse the trickier parts of proofs and statements.
(24) Use the

```
\section
```

command to make sections, but try to avoid using subsections and subsubsections.
(25) Avoid using complicated latex constructions.

114.2. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
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CHAPTER 115

Obsolete

073T

115.1. Introduction

073U In this chapter we put some lemmas that have become "obsolete" (see [Mil17]).

115.2. Preliminaries

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0EGF Remark 115.2.1. The information which used to be contained in this remark is now subsumed in the combination of Categories, Lemmas 4.24.4 and 4.24.3.

115.3. Homological algebra

0BFJ

076K Remark 115.3.1. The following remarks are obsolete as they are subsumed in Homology, Lemmas 12.24.11 and 12.25.3. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Homology, Definition 12.10.1). Suppose that $K^{\bullet,\bullet}$ is a double complex to which Homology, Lemma 12.25.3 applies such that for some $r \geq 0$ all the objects $'E_r^{p,q}$ belong to $\mathcal{C}$. Then all the cohomology groups $H^n(sK^\bullet)$ belong to $\mathcal{C}$. Namely, the assumptions imply that the kernels and images of $'d_r^{p,q}$ are in $\mathcal{C}$. Whereupon we see that each $'E_{r+1}^{p,q}$ is in $\mathcal{C}$. By induction we see that each $'E_\infty^{p,q}$ is in $\mathcal{C}$. Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in $\mathcal{C}$. Using that $\mathcal{C}$ is closed under extensions we conclude that $H^n(sK^\bullet)$ is in $\mathcal{C}$ as claimed. The same result holds for the second spectral sequence associated to $K^{\bullet,\bullet}$. Similarly, if $(K^\bullet, F)$ is a filtered complex to which Homology, Lemma 12.24.11 applies and for some $r \geq 0$ all the objects $E_r^{p,q}$ belong to $\mathcal{C}$, then each $H^n(K^\bullet)$ is an object of $\mathcal{C}$.

115.4. Obsolete algebra lemmas

088X

055Z Lemma 115.4.1. Let M be an R-module of finite presentation. For any surjection $\alpha : R^{\oplus n} \to M$ the kernel of α is a finite R-module.

Proof. This is a special case of Algebra, Lemma 10.5.3. $\square$

00I5 Lemma 115.4.2. Let $\varphi : R \to S$ be a ring map. If

(1) for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and
(2) for any $x \in \mathrm{Ker}(\varphi)$ there exists $n > 0$ such that $x^n = 0$,

then φ induces a homeomorphism on spectra. Given a prime number p such that

(a) S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and
(b) the kernel of φ is generated by nilpotent elements,

then (1) and (2) hold, and for any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. This is a combination of Algebra, Lemmas 10.46.3 and 10.46.7. □

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

00SX Lemma 115.4.3. Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$. Assume $S_{\mathfrak{q}}$ is essentially of finite type over $R_{\mathfrak{p}}$. Assume given

(1) an integer $n \geq 0$,
(2) a prime $\mathfrak{a} \subset \kappa(\mathfrak{p})[x_1, \ldots, x_n]$,
(3) a surjective $\kappa(\mathfrak{p})$-homomorphism
$$\psi : (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_{\mathfrak{a}} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}},$$
and
(4) elements $\overline{f}_1, \ldots, \overline{f}_e$ in $\mathrm{Ker}(\psi)$.

Then there exist

(1) an integer $m \geq 0$,
(2) and element $g \in S$, $g \notin \mathfrak{q}$,
(3) a map
$$\Psi : R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \longrightarrow S_g,$$
and
(4) elements $f_1, \ldots, f_e, f_{e+1}, \ldots, f_{e+m}$ of $\mathrm{Ker}(\Psi)$

such that

(1) the following diagram commutes
$$\begin{tikzcd} R[x_1, \ldots, x_{n+m}] \arrow[r, "x_{n+j} \mapsto 0"'] \arrow[d, "\Psi"'] & (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_{\mathfrak{a}} \arrow[d, "\psi"] \\ S_g \arrow[r] & S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \end{tikzcd}$$
(2) the element f_i, $i \leq n$ maps to a unit times $\overline{f}_i$ in the local ring
$$(\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}])_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})},$$
(3) the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and
(4) the induced map $R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}}$ is surjective, where $\mathfrak{b} = \Psi^{-1}(\mathfrak{q}S_g)$.

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Namely, suppose we have constructed
$$\Psi' : R_{\mathfrak{p}}[x_1, \ldots, x_{n+m}] \longrightarrow S_{\mathfrak{q}}$$
and $f'_1, \ldots, f'_{e+m} \in R_{\mathfrak{p}}[x_1, \ldots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \notin \mathfrak{p}$ such that each ff'_k comes from an element $f_k \in R[x_1, \ldots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \notin \mathfrak{q}$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R-algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_{\mathfrak{q}}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Pick $y_1, \ldots, y_n \in S$ such that $y_i \bmod \mathfrak{p}S = \psi(x_i)$. Let $y_{n+1}, \ldots, y_{n+m} \in S$ be elements which generate an R-subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R. Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak{p}S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]_{\mathfrak{a}}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$ for some common denominator $d \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$, $d \notin \mathfrak{a}$. Choose lifts $G_j, D \in R[x_1, \ldots, x_n]$ of g_j and d. Set $y'_{n+j} = D(y_1, \ldots, y_n)y_{n+j} - G_j(y_1, \ldots, y_n)$. By construction $y'_{n+j} \in \mathfrak{p}S$. It is clear that $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ generate an R-subalgebra of S whose localization is S. We define

$$\Psi : R[x_1, \ldots, x_{n+m}] \to S$$

to be the map that sends x_i to y_i for $i = 1, \ldots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \ldots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal $\mathfrak{b}$ maps onto the ideal $(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]$.

Denote $J = \text{Ker}(\Psi)$. We have a short exact sequence

$$0 \to J_{\mathfrak{b}} \to R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}} \to 0.$$

The surjectivity comes from our choice of $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak{b}}/\mathfrak{p}J_{\mathfrak{b}} \to \kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})} \to S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \to 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. □

01DE Remark 115.4.4 (Projective resolutions). Let R be a ring. For any set S we let $F(S)$ denote the free R-module on S. Then any left R-module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \to F(M) \to M \to 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

02CA Lemma 115.4.5. Let S be a multiplicative set of A. Then the map

$$f : \text{Spec}(S^{-1}A) \longrightarrow \text{Spec}(A)$$

induced by the canonical ring map $A \to S^{-1}A$ is a homeomorphism onto its image and $\text{Im}(f) = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$.

Proof. This is a duplicate of Algebra, Lemma 10.17.5. □

05IP Lemma 115.4.6. Let $A \to B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A-algebra.

Proof. Special case of More on Flatness, Proposition 38.13.10. □

053F Lemma 115.4.7. Let R be a domain with fraction field K. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be a finite S-module. Assume that M is flat over R. If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S.

Proof. This lemma is true because M is finitely presented even without the assumption that $M \otimes_R R'$ is finitely presented for every R' as in the statement of the lemma. This follows from More on Flatness, Proposition 38.13.10. Originally this lemma had an erroneous proof (thanks to Ofer Gabber for finding the gap) and was used in an alternative proof of the proposition cited. To reinstate this lemma, we need a correct argument in case R is a local normal domain using only results from the chapters on commutative algebra; please email stacks.project@gmail.com if you have an argument. $\square$

02TQ **Lemma 115.4.8.** Let $A \to B$ be a ring map. Let $f \in B$. Assume that

(1) $A \to B$ is flat,
(2) f is a nonzerodivisor, and
(3) $A \to B/fB$ is flat.

Then for every ideal $I \subset A$ the map $f : B/IB \to B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A. In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB. Hence the result follows from the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A B \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "f"] & B/IB \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I \otimes_A B \arrow[r] & B \arrow[r] & B/IB \arrow[r] & 0 \end{tikzcd}$$

with exact rows. $\square$

051A **Lemma 115.4.9.** If $R \to S$ is a faithfully flat ring map then for every R-module M the map $M \to S \otimes_R M$, $x \mapsto 1 \otimes x$ is injective.

Proof. This lemma is a duplicate of Algebra, Lemma 10.82.11. $\square$

07C2 **Remark 115.4.10.** This reference/tag used to refer to a Section in the chapter Smoothing Ring Maps, but the material has since been subsumed in Algebra, Section 10.127.

07K3 **Lemma 115.4.11.** Let $(R, \mathfrak m)$ be a reduced Noetherian local ring of dimension 1 and let $x \in \mathfrak m$ be a nonzerodivisor. Let $\mathfrak q_1, \ldots, \mathfrak q_r$ be the minimal primes of R. Then

$$\text{length}_R(R/(x)) = \sum\nolimits_i \text{ord}_{R/\mathfrak q_i}(x)$$

Proof. Special (very easy) case of Chow Homology, Lemma 42.3.2. $\square$

0AXG **Lemma 115.4.12.** Let A be a Noetherian local normal domain of dimension 2. For $f \in \mathfrak m$ nonzero denote $\text{div}(f) = \sum n_i(\mathfrak p_i)$ the divisor associated to f on the punctured spectrum of A. We set $|f| = \sum n_i$. There exist integers N and M such that $|f + g| \leq M$ for all $g \in \mathfrak m^N$.

Proof. Pick $h \in \mathfrak m$ such that f, h is a regular sequence in A (this follows from Algebra, Lemmas 10.157.4 and 10.72.7). We will prove the lemma with $M = \text{length}_A(A/(f, h))$ and with N any integer such that $\mathfrak m^N \subset (f, h)$. Such an integer N exists because $\sqrt{(f, h)} = \mathfrak m$. Note that $M = \text{length}_A(A/(f+g, h))$ for all $g \in \mathfrak m^N$ because $(f, h) = (f+g, h)$. This moreover implies that $f+g, h$ is a regular sequence in A too, see Algebra, Lemma 10.104.2. Now suppose that $\text{div}(f+g) = \sum m_j(\mathfrak{q}_j)$. Then consider the map

$$c : A/(f+g) \longrightarrow \prod A/\mathfrak{q}_j^{(m_j)}$$

where $\mathfrak{q}_j^{(m_j)}$ is the symbolic power, see Algebra, Section 10.64. Since A is normal, we see that $A_{\mathfrak{q}_i}$ is a discrete valuation ring and hence

$$A_{\mathfrak{q}_i}/(f+g) = A_{\mathfrak{q}_i}/\mathfrak{q}_i^{m_i}A_{\mathfrak{q}_i} = (A/\mathfrak{q}_i^{(m_i)})_{\mathfrak{q}_i}$$

Since $V(f+g,h) = \{\mathfrak{m}\}$ this implies that c becomes an isomorphism on inverting h (small detail omitted). Since h is a nonzerodivisor on $A/(f+g)$ we see that the length of $A/(f+g,h)$ equals the Herbrand quotient $e_A(A/(f+g),0,h)$ as defined in Chow Homology, Section 42.2. Similarly the length of $A/(h,\mathfrak{q}_j^{(m_j)})$ equals $e_A(A/\mathfrak{q}_j^{(m_j)},0,h)$. Then we have

$$\begin{aligned} M &= \text{length}_A(A/(f+g,h) \\ &= e_A(A/(f+g),0,h) \\ &= \sum\nolimits_i e_A(A/\mathfrak{q}_j^{(m_j)},0,h) \\ &= \sum\nolimits_i \sum\nolimits_{m=0,\ldots,m_j-1} e_A(\mathfrak{q}_j^{(m)}/\mathfrak{q}_j^{(m+1)},0,h) \end{aligned}$$

The equalities follow from Chow Homology, Lemmas 42.2.3 and 42.2.4 using in particular that the cokernel of c has finite length as discussed above. It is straightforward to prove that $e_A(\mathfrak{q}^{(m)}/\mathfrak{q}^{(m+1)},0,h)$ is at least 1 by Nakayama's lemma. This finishes the proof of the lemma. □

0BJK Lemma 115.4.13. Let $A \to B$ be a flat local homomorphism of Noetherian local rings. If A and $B/\mathfrak{m}_AB$ are Gorenstein, then B is Gorenstein.

Proof. Follows immediately from Dualizing Complexes, Lemma 47.21.8. □

0DXL Lemma 115.4.14. Let $(A,\mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s be an integer. Assume

(1) A has a dualizing complex,
(2) if $\mathfrak{p} \not\in V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > s$.

Then there exists an $n > 0$ and an ideal $J \subset A$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$ such that JI^n annihilates $H^i_\mathfrak{m}(M)$ for $i \leq s$.

Proof. According to Local Cohomology, Lemma 51.9.4 we have to show this for the finite A-module $E^i = \text{Ext}^{-i}_A(M,\omega_A^\bullet)$ for $i \leq s$. The support Z of $E^0 \oplus \ldots \oplus E^s$ is closed in $\text{Spec}(A)$ and does not contain any prime as in (2). Hence it is contained in $V(JI^n)$ for some J as in the statement of the lemma. □

0EFS Lemma 115.4.15. Let $(A,\mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume

(a) A has a dualizing complex,
(b) $\text{cd}(A,I) \leq d$,
(c) if $\mathfrak{p} \not\in V(I)$ then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$ or $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d+s$.

Then the assumptions of Algebraic and Formal Geometry, Lemma 52.10.4 hold for $A, I, \mathfrak{m}, M$ and $H^i_\mathfrak{m}(M) \to \lim H^i_\mathfrak{m}(M/I^nM)$ is an isomorphism for $i \leq s$ and these modules are annihilated by a power of I.

Proof. The assumptions of Algebraic and Formal Geometry, Lemma 52.10.4 by the more general Algebraic and Formal Geometry, Lemma 52.10.5. Then the conclusion of Algebraic and Formal Geometry, Lemma 52.10.4 gives the second statement. □

0EFZ Lemma 115.4.16. In Algebraic and Formal Geometry, Situation 52.10.1 we have $H^s_\mathfrak{a}(M) = \lim H^s_\mathfrak{a}(M/I^nM)$.

Proof. This is immediate from Algebraic and Formal Geometry, Theorem 52.10.8. The original version of this lemma, which had additional assumptions, was superseded by the this theorem. □

0EKR Lemma 115.4.17. Let A be a Noetherian ring. Let $f \in \mathfrak{a}$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume

(1) A has a dualizing complex and is complete with respect to f,
(2) A_f is (S_2) and for every minimal prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) \geq 3$.

Then the completion functor

$$\mathrm{Coh}(\mathcal{O}_U) \longrightarrow \mathrm{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. This lemma is a special case of Algebraic and Formal Geometry, Lemma 52.15.6. □

115.5. Lemmas related to ZMT

073V The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 10.123.12.

00PU Lemma 115.5.1. Let R be a ring and let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. If t is integral over $R[x]$, then there exists an $\ell \geq 0$ such that for every $a \in R$ the element $\varphi(a)^\ell t$ is integral over $\varphi_a : R[y] \to S$, defined by $y \mapsto \varphi(ax)$ and $r \mapsto \varphi(r)$ for $r \in R$.

Proof. Say $t^d + \sum_{i<d} \varphi(f_i)t^i = 0$ with $f_i \in R[x]$. Let ℓ be the maximum degree in x of all the f_i. Multiply the equation by $\varphi(a)^\ell$ to get $\varphi(a)^\ell t^d + \sum_{i<d} \varphi(a^\ell f_i)t^i = 0$. Note that each $\varphi(a^\ell f_i)$ is in the image of φ_a. The result follows from Algebra, Lemma 10.123.1. □

00PR Lemma 115.5.2. Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Set $u_n = \varphi(a_n)$, $u_{n-1} = u_nt + \varphi(a_{n-1})$, and so on till $u_1 = u_2t + \varphi(a_1)$. Then all of $u_n, u_{n-1}, \ldots, u_1$ and $u_nt, u_{n-1}t, \ldots, u_1t$ are integral over R, and the ideals $(\varphi(a_0), \ldots, \varphi(a_n))$ and $(u_n, \ldots, u_1)$ of S are equal.

Proof. We prove this by induction on n. As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 10.123.1 that u_nt is integral over R. Of course $u_n = \varphi(a_n)$ is integral over R. Then $u_{n-1} = u_nt + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 10.36.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \to S$ where S' is the integral closure of R in S and the displayed equation we see that $u_{n-1}, \ldots, u_1$ and $u_{n-1}t, \ldots, u_1t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1t + \varphi(a_0) = 0$. □

00PS Lemma 115.5.3. Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R.

Proof. This is immediate from Lemma 115.5.2 since one of the elements u_i will not be in J. $\square$

The following two lemmas are a way of describing closed subschemes of $\mathbf{P}^1_R$ cut out by one (nondegenerate) equation.

00Q4 Lemma 115.5.4. Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Assume that for every prime $\mathfrak{p}$ of R at least one coefficient of F is not in $\mathfrak{p}$. Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R-module S_n is finite locally free of rank d.

Proof. The R-module S_n has a presentation

$$R[X, Y]_{n-d} \to R[X, Y]_n \to S_n \to 0.$$

Thus by Algebra, Lemma 10.79.4 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \to \kappa(\mathfrak{p})[X, Y]$ for all primes $\mathfrak{p}$. This is clear from the assumption that F does not map to the zero polynomial mod $\mathfrak{p}$. The assertion on ranks is clear from this as well. $\square$

00Q5 Lemma 115.5.5. Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e. Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.

Proof. This is one way to define "relatively prime". If you have another definition, then you can show it is equivalent to this one. $\square$

00Q6 Lemma 115.5.6. Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Let $S = R[X, Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in $\mathfrak{p}$. There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e, and a $G \in R[X, Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \to (S_{n+e})_f$ for all $n \geq d$.

Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R. In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 115.5.4. Pick any $G \in R[X, Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X, Y]$ is relatively prime to the image of $F(X, Y)$ (this is possible for some e). Apply Algebra, Lemma 10.79.4 to the map induced by multiplication by G from $S_d \to S_{d+e}$. By our choice of G and Lemma 115.5.5 we see $S_d \otimes \kappa(\mathfrak{p}) \to S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \to S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X, Y]$ is relatively prime to the image of F for all primes $\mathfrak{p}'$ of R. And then by Algebra, Lemma 10.79.4 again we see that all the maps $G : S_d \to S_{d+e}$, $n \geq d$ are isomorphisms. $\square$

00Q7 Remark 115.5.7. Let R be a ring. Suppose that we have $F \in R[X, Y]_d$ and $G \in R[X, Y]_e$ such that, setting $S = R[X, Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines isomorphisms

$S_n \to S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R-algebra A as follows:

(1) as an R-module $A = S_{ed}$, and
(2) multiplication $A \times A \to A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed}.

This makes sense because multiplication by G^d induces a bijective map $S_{de} \to S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A.

00Q3 Lemma 115.5.8. Let R be a ring, let $f \in R$. Suppose we have S, S' and the solid arrows forming the following commutative diagram of rings

$$\begin{tikzcd} & S'' \arrow[rd, dashed] \arrow[dd, dashed] & \\ R \arrow[ru, dashed] \arrow[rr] \arrow[d] & & S \arrow[d] \\ R_f \arrow[r] & S' \arrow[r] & S_f \end{tikzcd}$$

Assume that $R_f \to S'$ is finite. Then we can find a finite ring map $R \to S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f, $i = 1, \ldots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree $d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j<d_i} (a_{ij}/f^n) t^j$ for some uniform n. Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{nd_i} g_i^{d_i} + \sum_{j<d_i} f^{n(d_i-j)} a_{ij} g_i^j$ of S is killed by a power of f. Hence upon increasing n to n', which replaces g_i by $f^{n'-n} g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j<d_i} f^{n(d_i-j)} a_{ij} t^j$ with coefficients in R. Also, by construction $Q_i(f^n g_i) = 0$ in S. Thus we get a finite R-algebra $S'' = R[z_1, \ldots, z_w]/(Q_1(z_1), \ldots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \to S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \to S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N \beta(h) = 0$ for some N. Thus $f^N h$ is an element ot the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S''. OK, and it is easy to see that S''/J does the job. □

115.6. Formally smooth ring maps

07GD

00TO Lemma 115.6.1. Let R be a ring. Let S be a R-algebra. If S is of finite presentation and formally smooth over R then S is smooth over R.

Proof. See Algebra, Proposition 10.138.13. □

0AKC Remark 115.6.2. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AKD Remark 115.6.3. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AKE Remark 115.6.4. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AKF Remark 115.6.5. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AM9 Remark 115.6.6. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Lemma 88.9.1 which became unused because of a rearrangement of the material.

0AK9 Lemma 115.6.7. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let t be the minimal number of generators for I. Let C be a Noetherian I-adically complete A-algebra. There exists an integer $d \geq 0$ depending only on $I \subset A \to C$ with the following property: given

(1) $c \geq 0$ and B in Algebraization of Formal Spaces, Equation (88.2.0.2) such that for $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$,
(2) an integer $n > 2t\max(c, d)$,
(3) an A/I^n-algebra map $\psi_n : B/I^nB \to C/I^nC$,

there exists a map $\varphi : B \to C$ of A-algebras such that $\psi_n \bmod I^{m-c} = \varphi \bmod I^{m-c}$ with $m = \lfloor \frac{n}{t} \rfloor$.

Proof. This lemma has been obsoleted by the stronger Algebraization of Formal Spaces, Lemma 88.5.3. In fact, we will deduce the lemma from it.

Let $I \subset A \to C$ be given as in the statement above. Denote $d(\mathrm{Gr}_I(C))$ and $q(\mathrm{Gr}_I(C))$ the integers found in Local Cohomology, Section 51.22. Observe that t is an upper bound for the minimal number of generators of IC and hence we have $d(\mathrm{Gr}_I(C)) + 1 \leq t$, see discussion in Local Cohomology, Section 51.22. We may and do assume $t \geq 1$ since otherwise the lemma does not say anything. We claim that the lemma is true with

$$d = q(\mathrm{Gr}_I(C))$$

Namely, suppose that c, B, n, ψ_n are as in the statement above. Then we see that

$$n > 2t\max(c, d) \Rightarrow n \geq 2tc + 1 \Rightarrow n \geq 2(d(\mathrm{Gr}_I(C)) + 1)c + 1$$

On the other hand, we have

$$n > 2t\max(c, d) \Rightarrow n > t(c + d) \Rightarrow n \geq q(C) + tc \geq q(\mathrm{Gr}_I(C)) + (d(\mathrm{Gr}_I(C)) + 1)c$$

Hence the assumptions of Algebraization of Formal Spaces, Lemma 88.5.3 are satisfied and we obtain an A-algebra homomorphism $\varphi : B \to C$ which is congruent with ψ_n module $I^{n-(d(\mathrm{Gr}_I(C))+1)c}C$. Since

$$\begin{aligned} n-(d(\mathrm{Gr}_I(C))+1)c &= \frac{n}{t}+\frac{(t-1)n}{t}-(d(\mathrm{Gr}_I(C))+1)c \\ &\geq \frac{n}{t}+\frac{(d(\mathrm{Gr}_I(C))n}{t}-(d(\mathrm{Gr}_I(C))+1)c \\ &> \frac{n}{t}+\frac{d(\mathrm{Gr}_I(C))2tc}{t}-(d(\mathrm{Gr}_I(C))+1)c \\ &= \frac{n}{t}+2d(\mathrm{Gr}_I(C))c-(d(\mathrm{Gr}_I(C))+1)c \\ &= \frac{n}{t}+d(\mathrm{Gr}_I(C))c-c \\ &\geq m-c \end{aligned}$$

we see that we have the congruence of φ and ψ_n module $I^{m-c}C$ as desired. $\square$

115.7. Sites and sheaves

0EGM

0931 Remark 115.7.1 (No map from lower shriek to pushforward). Let U be an object of a site $\mathcal{C}$. For any abelian sheaf $\mathcal{G}$ on $\mathcal{C}/U$ one may wonder whether there is a canonical map

$$c : j_{U!}\mathcal{G} \longrightarrow j_{U*}\mathcal{G}$$

To construct such a thing is the same as constructing a map $j_U^{-1}j_{U!}\mathcal{G} \to \mathcal{G}$. Note that restriction commutes with sheafification. Thus we can use the presheaf of Modules on Sites, Lemma 18.19.2. Hence it suffices to define for V/U a map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{G}(V/U)$$

compatible with restrictions. It looks like we can take the which is zero on all summands except for the one where φ is the structure morphism $\varphi_0 : V \to U$ where we take 1. However, this isn't compatible with restriction mappings: namely, if $\alpha : V' \to V$ is a morphism of $\mathcal{C}$, then denote V'/U the object of $\mathcal{C}/U$ with structure morphism $\varphi_0' = \varphi_0 \circ \alpha$. We need to check that the diagram

$$\begin{tikzcd} \bigoplus_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \arrow[r] \arrow[d] & \mathcal{G}(V/U) \arrow[d] \\ \bigoplus_{\varphi' \in \mathrm{Mor}_\mathcal{C}(V',U)} \mathcal{G}(V' \xrightarrow{\varphi'} U) \arrow[r] & \mathcal{G}(V'/U) \end{tikzcd}$$

commutes. The problem here is that there may be a morphism $\varphi : V \to U$ different from φ_0 such that $\varphi \circ \alpha = \varphi_0'$. Thus the left vertical arrow will send the summand corresponding to φ into the summand on which the lower horizontal arrow is equal to 1 and almost surely the diagram doesn't commute.

115.8. Cohomology

0BM0 Obsolete lemmas about cohomology.

0EH4 Lemma 115.8.1. Let I be an ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^1(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. This is a special case of the more general Cohomology, Lemma 20.35.1. □

0EH5 Lemma 115.8.2. Let I be an ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Given n define

$$H^1_n = \bigcap\nolimits_{m\geq n} \mathrm{Im}\left(H^1(X, I^n\mathcal{F}_{m+1}) \to H^1(X, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus H^1_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. This is a special case of the more general Cohomology, Lemma 20.35.2. □

0EI7 Lemma 115.8.3. Let I be a finitely generated ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^0(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim \Gamma(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. This is a special case of the more general Cohomology, Lemma 20.35.3. □

06YW Lemma 115.8.4. Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. For any complex of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{G}^\bullet$ there exists a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ such that $f^*\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules for any morphism $f : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ of ringed topoi.

Proof. This follows from Cohomology on Sites, Lemmas 21.17.11 and 21.18.1. □

06YX Remark 115.8.5. This remark used to discuss what we know about pullbacks of K-flat complexes being K-flat or not, but is now obsoleted by Cohomology on Sites, Lemma 21.18.1.

The following lemma computes the cohomology sheaves of the derived limit in a special case.

0A08 Lemma 115.8.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume

(1) K_n is an object of $D^+(\mathcal{O})$ for all n,
(2) for $q \in \mathbf{Z}$ there exists $n(q)$ such that $H^q(K_{n+1}) \to H^q(K_n)$ is an isomorphism for $n \geq n(q)$,
(3) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(4) for every $U \in \mathcal{B}$ we have $H^p(U, H^q(K_n)) = 0$ for $p > d$ and all q.

Then we have $H^m(R\lim K_n) = \lim H^m(K_n)$ for all $m \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. Let $U \in \mathcal{B}$. For each n there is a spectral sequence

$$H^p(U, H^q(K_n)) \Rightarrow H^{p+q}(U, K_n)$$

which converges as K_n is bounded below, see Derived Categories, Lemma 13.21.3. If we fix $m \in \mathbf{Z}$, then we see from our assumption (4) that only $H^p(U, H^q(K_n))$ contribute to $H^m(U, K_n)$ for $0 \leq p \leq d$ and $m - d \leq q \leq m$. By assumption (2) this implies that $H^m(U, K_{n+1}) \to H^m(U, K_n)$ is an isomorphism as soon as $n \geq \max n(m), \ldots, n(m-d)$. The functor $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 19.13.6. Thus we have

$$H^m(U, K) = H^m(R\lim R\Gamma(U, K_n))$$

On the other hand we have just seen that the complexes $R\Gamma(U, K_n)$ have eventually constant cohomology groups. Thus by More on Algebra, Remark 15.86.10 we find that $H^m(U, K)$ is equal to $H^m(U, K_n)$ for all $n \gg 0$ for some bound independent of $U \in \mathcal{B}$. Pick such an n. Finally, recall that $H^m(K)$ is the sheafification of the presheaf $U \mapsto H^m(U, K)$ and $H^m(K_n)$ is the sheafification of the presheaf $U \mapsto H^m(U, K_n)$. On the elements of $\mathcal{B}$ these presheaves have the same values. Therefore assumption (3) guarantees that the sheafifications are the same too. The lemma follows. □

0D7P Lemma 115.8.7. In Simplicial Spaces, Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Simplicial Spaces, Remark 85.4.1. Suppose given strictly full weak Serre subcategories

$$\mathcal{A} \subset \text{Ab}(\mathcal{D}), \quad \mathcal{A}_n \subset \text{Ab}(\mathcal{C}_n)$$

Then

(1) the collection of abelian sheaves $\mathcal{F}$ on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_n$ is in $\mathcal{A}_n$ for all n is a strictly full weak Serre subcategory $\mathcal{A}_{total} \subset \text{Ab}(\mathcal{C}_{total})$.

If a_n^{-1} sends $\mathcal{A}$ into $\mathcal{A}_n$ for all n, then

(2) a^{-1} sends $\mathcal{A}$ into $\mathcal{A}_{total}$ and
(3) a^{-1} sends $D_{\mathcal{A}}(\mathcal{D})$ into $D_{\mathcal{A}_{total}}(\mathcal{C}_{total})$.

If $R^q a_{n,*}$ sends $\mathcal{A}_n$ into $\mathcal{A}$ for all n, q, then

(4) $R^q a_*$ sends $\mathcal{A}_{total}$ into $\mathcal{A}$ for all q, and
(5) Ra_* sends $D^+_{\mathcal{A}_{total}}(\mathcal{C}_{total})$ into $D^+_{\mathcal{A}}(\mathcal{D})$.

Proof. The only interesting assertions are (4) and (5). Part (4) follows from the spectral sequence in Simplicial Spaces, Lemma 85.9.3 and Homology, Lemma 12.24.11. Then part (5) follows by considering the spectral sequence associated to the canonical filtration on an object K of $D^+_{\mathcal{A}_{total}}(\mathcal{C}_{total})$ given by truncations. We omit the details. □

01DY Remark 115.8.8. This tag used to refer to a section of the chapter on cohomology listing topics to be treated.

01FS Remark 115.8.9. This tag used to refer to a section of the chapter on cohomology listing topics to be treated.

0DCV Remark 115.8.10. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Cohomology on Sites, Lemma 21.31.9.

0DCW Remark 115.8.11. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Cohomology on Sites, Lemma 21.31.9.

0DCX Remark 115.8.12. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Cohomology on Sites, Lemma 21.31.9.

0DDP Remark 115.8.13. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Étale Cohomology, Lemma 59.100.5.

0DDQ Remark 115.8.14. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Étale Cohomology, Lemma 59.100.5.

0DDR Remark 115.8.15. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Étale Cohomology, Lemma 59.100.5.

0DDZ Remark 115.8.16. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE0 Remark 115.8.17. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE1 Remark 115.8.18. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.5 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE2 Remark 115.8.19. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.6 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE3 Remark 115.8.20. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0EWB Remark 115.8.21. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

0EWC Remark 115.8.22. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

0EWD Remark 115.8.23. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.5 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

0EWE Remark 115.8.24. This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

03TU Remark 115.8.25. This tag used to be in the chapter on étale cohomology, but is no longer suitable there because of a reorganization. The content of the tag was the following: Étale Cohomology, Lemma 59.77.3 can be used to prove that if $f : X \to Y$ is a separated, finite type morphism of schemes and Y is Noetherian, then $Rf_!$ induces a functor $D_{ctf}(X_{\acute{e}tale}, \Lambda) \to D_{ctf}(Y_{\acute{e}tale}, \Lambda)$. An example of this argument, when Y is the spectrum of a field and X is a curve is given in The Trace Formula, Proposition 64.13.1.

0F5D Lemma 115.8.26. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. There exists a unique functor $f^! : \textit{Ab}(Y_{\acute{e}tale}) \to \textit{Ab}(X_{\acute{e}tale})$ such that

(1) for any open $j : U \to X$ with $f \circ j$ separated there is a canonical isomorphism $j^! \circ f^! = (f \circ j)^!$, and
(2) these isomorphisms for $U \subset U' \subset X$ are compatible with the isomorphisms in More Étale Cohomology, Lemma 63.6.3.

Proof. Immediate consequence of More Étale Cohomology, Lemmas 63.6.1 and 63.6.3. □

0F5E Proposition 115.8.27. Let $f : X \to Y$ be a locally quasi-finite morphism. There exist adjoint functors $f_! : \textit{Ab}(X_{\acute{e}tale}) \to \textit{Ab}(Y_{\acute{e}tale})$ and $f^! : \textit{Ab}(Y_{\acute{e}tale}) \to \textit{Ab}(X_{\acute{e}tale})$ with the following properties

(1) the functor $f^!$ is the one constructed in More Étale Cohomology, Lemma 63.6.1,
(2) for any open $j : U \to X$ with $f \circ j$ separated there is a canonical isomorphism $f_! \circ j_! = (f \circ j)_!$, and
(3) these isomorphisms for $U \subset U' \subset X$ are compatible with the isomorphisms in More Étale Cohomology, Lemma 63.3.13.

Proof. See More Étale Cohomology, Sections 63.4 and 63.6. □

0F5G Lemma 115.8.28. Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. For an abelian group A and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have $f^!(\overline{y}_*A) = \prod_{f(\overline{x})=\overline{y}} \overline{x}_*A$.

Proof. Follows from the corresponding statement in More Étale Cohomology, Lemma 63.6.1. □

0F5K Lemma 115.8.29. Let $f : X \to Y$ and $g : Y \to Z$ be composable locally quasi-finite morphisms of schemes. Then $g_! \circ f_! = (g \circ f)_!$ and $f^! \circ g^! = (g \circ f)^!$.

Proof. Combination of More Étale Cohomology, Lemmas 63.4.12 and 63.6.3. □

115.9. Differential graded algebra

0FU5
0BYY Lemma 115.9.1. Let (A, d) and (B, d) be differential graded algebras. Let N be a differential graded (A, B)-bimodule with property (P). Let M be a differential graded A-module with property (P). Then $Q = M \otimes_A N$ is a differential graded B-module which represents $M \otimes_A^{\mathbf{L}} N$ in $D(B)$ and which has a filtration

$$0 = F_{-1}Q \subset F_0Q \subset F_1Q \subset \ldots \subset Q$$

by differential graded submodules such that $Q = \bigcup F_p Q$, the inclusions $F_i Q \to F_{i+1} Q$ are admissible monomorphisms, the quotients $F_{i+1}Q/F_iQ$ are isomorphic as differential graded B-modules to a direct sum of $(A \otimes_R B)[k]$.

Proof. Choose filtrations $F_\bullet$ on M and N. Then consider the filtration on $Q = M \otimes_A N$ given by

$$F_n(Q) = \sum\nolimits_{i+j=n} F_i(M) \otimes_A F_j(N)$$

This is clearly a differential graded B-submodule. We see that

$$F_n(Q)/F_{n-1}(Q) = \bigoplus\nolimits_{i+j=n} F_i(M)/F_{i-1}(M) \otimes_A F_j(N)/F_{j-1}(N)$$

for example because the filtration of M is split in the category of graded A-modules. Since by assumption the quotients on the right hand side are isomorphic to direct sums of shifts of A and $A \otimes_R B$ and since $A \otimes_A (A \otimes_R B) = A \otimes_R B$, we conclude that the left hand side is a direct sum of shifts of $A \otimes_R B$ as a differential graded B-module. (Warning: Q does not have a structure of (A, B)-bimodule.) This proves the first statement of the lemma. The second statement is immediate from the definition of the functor in Differential Graded Algebra, Lemma 22.33.2. $\square$

115.10. Simplicial methods

08Q0

01AA Lemma 115.10.1. Assumptions and notation as in Simplicial, Lemma 14.32.1. There exists a section $g : U \to V$ to the morphism f and the composition $g \circ f$ is homotopy equivalent to the identity on V. In particular, the morphism f is a homotopy equivalence.

Proof. Immediate from Simplicial, Lemmas 14.32.1 and 14.30.8. $\square$

018W Lemma 115.10.2. Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map

$$\mathrm{Hom}(\Delta[k], X) \longrightarrow \mathrm{cosk}_1 \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$\begin{aligned} \mathrm{Mor}(V, \mathrm{cosk}_1 \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)) &= \mathrm{Mor}(\mathrm{sk}_1 V, \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)) \\ &= \mathrm{Mor}(i_{1!}\mathrm{sk}_1 V, \mathrm{Hom}(\Delta[k], X)) \\ &= \mathrm{Mor}(i_{1!}\mathrm{sk}_1 V \times \Delta[k], X) \end{aligned}$$

The first equality by the adjointness of sk and cosk, the second equality by the adjointness of $i_{1!}$ and sk_1, and the first equality by Simplicial, Definition 14.17.1 where the last X denotes the constant simplicial object with value X. By Simplicial, Lemma 14.20.2 an element in this set depends only on the terms of degree 0 and 1 of $i_{1!}\mathrm{sk}_1 V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see Simplicial, Lemma 14.21.3. Thus the set above is equal to $\mathrm{Mor}(V \times \Delta[k], X) = \mathrm{Mor}(V, \mathrm{Hom}(\Delta[k], X))$. $\square$

018X Lemma 115.10.3. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$ such that the self products $X \times \ldots \times X$ exist. Let $k \geq 0$ and let $C[k]$ be as in Simplicial, Example 14.5.6. With notation as in Simplicial, Lemma 14.15.2 the canonical map

$$\mathrm{Hom}(C[k], X)_1 \longrightarrow (\mathrm{cosk}_0 \mathrm{sk}_0 \mathrm{Hom}(C[k], X))_1$$

is identified with the map

$$\prod\nolimits_{\alpha : [k] \to [1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. This is shown in the proof of Hypercoverings, Lemma 25.7.3. $\square$

115.11. Results on schemes

07VA Lemmas that seem superfluous.

03H1 Lemma 115.11.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}^n_R$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.

Proof. This is a special case of More on Morphisms, Theorem 37.53.5. $\square$

01YJ Lemma 115.11.2. Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that

(1) For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.
(2) If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.
(3) For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.
(4) There exists some coherent sheaf $\mathcal{G}$ on X such that
   (a) $\text{Supp}(\mathcal{G}) = Z_0$,
   (b) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and
   (c) property $\mathcal{P}$ holds for $\mathcal{G}$.

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. The proof is a variant on the proof of Cohomology of Schemes, Lemma 30.12.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$.

Consider a coherent sheaf $\mathcal{G}$ as in (3). By Cohomology of Schemes, Lemma 30.12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z. Since $\mathcal{Q}$ has property $\mathcal{P}$ we conclude that also $((Z_0 \to X)_*\mathcal{I})^{\oplus r}$ has property $\mathcal{P}$. By (2) we deduce property $\mathcal{P}$ for $(Z_0 \to X)_*\mathcal{I}$. Slotting this into the proof of Cohomology of Schemes, Lemma 30.12.5 at the appropriate point gives the lemma. Some details omitted. $\square$

01YK Lemma 115.11.3. Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that

(1) For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.
(2) If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.
(3) For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that
   (a) $\text{Supp}(\mathcal{G}) = Z$,

(b) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and
(c) property $\mathcal{P}$ holds for $\mathcal{G}$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. This follows from Lemma 115.11.2 in exactly the same way that Cohomology of Schemes, Lemma 30.12.6 follows from Cohomology of Schemes, Lemma 30.12.5. $\square$

01XP Lemma 115.11.4. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-modules. Assume that

(1) X is quasi-compact,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.

Then there exists an $n \geq 0$ such that multiplication by s^n on $\mathcal{F}$ factors through $\mathcal{F}'$.

Proof. In other words we claim that $s^n\mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. In other words, we claim that the quotient map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ becomes zero after multiplying by a power of s. This follows from Properties, Lemma 28.17.3. $\square$

0CC3 Lemma 115.11.5. Let $f : X \to Y$ be a morphism schemes. Assume

(1) X and Y are integral schemes,
(2) f is locally of finite type and dominant,
(3) f is either quasi-compact or separated,
(4) f is generically finite, i.e., one of (1) – (5) of Morphisms, Lemma 29.51.7 holds.

Then there is a nonempty open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite locally free of degree $\deg(X/Y)$. In particular, the degrees of the fibres of $f^{-1}(V) \to V$ are bounded by $\deg(X/Y)$.

Proof. We may choose V such that $f^{-1}(V) \to V$ is finite. Then we may shrink V and assume that $f^{-1}(V) \to V$ is flat and of finite presentation by generic flatness (Morphisms, Proposition 29.27.1). Then the morphism is finite locally free by Morphisms, Lemma 29.48.2. Since V is irreducible the morphism has a fixed degree. The final statement follows from this and Morphisms, Lemma 29.57.3. $\square$

115.12. Derived categories of varieties

0GXZ Some lemma which were originally part of the chapter on derived categories of varieties but are no longer needed.

0G04 Lemma 115.12.1. Let k be a field. Let X be a separated scheme of finite type over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(\text{Supp}(\mathcal{F})) = 0$ there is an isomorphism of k-vector spaces

$$\text{Hom}_X(\mathcal{F}, M) = \text{Hom}_X(\mathcal{F}, F(M))$$

functorial in M in $D_{perf}(\mathcal{O}_X)$. Then there exists an automorphism $f : X \to X$ over k which induces the identity on the underlying topological space[1] and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that F and $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$ are siblings.

[1]This often forces f to be the identity, see Varieties, Lemma 33.32.1.

Proof. By Derived Categories of Varieties, Lemma 57.11.2 we conclude that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose support is a closed point there are isomorphisms

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = H^0(X, F(M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$$

functorial in M.

Let $x \in X$ be a closed point and apply the above with $\mathcal{F} = \mathcal{O}_x$ the skyscraper sheaf with value $\kappa(x)$ at x. We find

$$\dim_{\kappa(x)} \text{Tor}_p^{\mathcal{O}_{X,x}}(M_x, \kappa(x)) = \dim_{\kappa(x)} \text{Tor}_p^{\mathcal{O}_{X,x}}(F(M)_x, \kappa(x))$$

for all $p \in \mathbf{Z}$. In particular, if $H^i(M) = 0$ for $i > 0$, then $H^i(F(M)) = 0$ for $i > 0$ by Derived Categories of Varieties, Lemma 57.11.3.

If $\mathcal{E}$ is locally free of rank r, then $F(\mathcal{E})$ is locally free of rank r. This is true because a perfect complex K over $\mathcal{O}_{X,x}$ with

$$\dim_{\kappa(x)} \text{Tor}_i^{\mathcal{O}_{X,x}}(K, \kappa(x)) = \begin{cases} r & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0 \end{cases}$$

is equal to a free module of rank r placed in degree 0. See for example More on Algebra, Lemma 15.75.6.

If M is supported on a closed subscheme $Z \subset X$, then $F(M)$ is also supported on Z. This is clear because we will have $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x = 0$ for $x \notin Z$ and hence the same will be true for $F(M)$ and hence we get the conclusion from Derived Categories of Varieties, Lemma 57.11.3.

In particular $F(\mathcal{O}_x)$ is supported at $\{x\}$. Let $i \in \mathbf{Z}$ be the minimal integer such that $H^i(\mathcal{O}_x) \neq 0$. We know that $i \leq 0$. If $i < 0$, then there is a morphism $\mathcal{O}_x[-i] \to F(\mathcal{O}_x)$ which contradicts the fact that all morphisms $\mathcal{O}_x[-i] \to \mathcal{O}_x$ are zero. Thus $F(\mathcal{O}_x) = \mathcal{H}[0]$ where $\mathcal{H}$ is a skyscraper sheaf at x.

Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module with $\dim(\text{Supp}(\mathcal{G})) = 0$. Then there exists a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_n = \mathcal{G}$$

such that for $n \geq i \geq 1$ the quotient $\mathcal{G}_i/\mathcal{G}_{i-1}$ is isomorphic to $\mathcal{O}_{x_i}$ for some closed point $x_i \in X$. Then we get distinguished triangles

$$F(\mathcal{G}_{i-1}) \to F(\mathcal{G}_i) \to F(\mathcal{O}_{x_i})$$

and using induction we find that $F(\mathcal{G}_i)$ is a coherent sheaf placed in degree 0.

Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. We know that $H^i(F(\mathcal{G})) = 0$ for $i > 0$. To get a contradiction assume that $H^i(F(\mathcal{G}))$ is nonzero for some $i < 0$. We choose i minimal with this property so that we have a morphism $H^i(F(\mathcal{G}))[-i] \to F(\mathcal{G})$ in $D_{perf}(\mathcal{O}_X)$. Choose a closed point $x \in X$ in the support of $H^i(F(\mathcal{G}))$. By More on Algebra, Lemma 15.100.2 there exists an $n > 0$ such that

$$H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n \longrightarrow \text{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

is nonzero. Next, we take $m \geq 1$ and we consider the short exact sequence

$$0 \to \mathfrak{m}_x^m \mathcal{G} \to \mathcal{G} \to \mathcal{G}/\mathfrak{m}_x^m \mathcal{G} \to 0$$

By the above we know that $F(\mathcal{G}/\mathfrak{m}_x^m\mathcal{G})$ is a sheaf placed in degree 0. Hence $H^i(F(\mathfrak{m}_x^m\mathcal{G})) \to H^i(F(\mathcal{G}))$ is an isomorphism. Consider the commutative diagram

$$\begin{tikzcd}
H^i(F(\mathfrak{m}_x^m\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n \arrow[r] \arrow[d] & \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathfrak{m}_x^m\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \arrow[d] \\
H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n \arrow[r] & \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)
\end{tikzcd}$$

Since the left vertical arrow is an isomorphism and the bottom arrow is nonzero, we conclude that the right vertical arrow is nonzero for all $m \geq 1$. On the other hand, by the first paragraph of the proof, we know this arrow is isomorphic to the arrow

$$\mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathfrak{m}_x^m\mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \longrightarrow \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

However, this arrow is zero for $m \gg n$ by More on Algebra, Lemma 15.102.2 which is the contradiction we're looking for.

Thus we know that F preserves coherent modules. By Derived Categories of Varieties, Lemma 57.12.2 we find F is a sibling to the Fourier-Mukai functor F' given by a coherent $\mathcal{O}_{X \times X}$-module $\mathcal{K}$ flat over X via pr_1 and finite over X via pr_2. Since $F(\mathcal{O}_X)$ is an invertible $\mathcal{O}_X$-module $\mathcal{L}$ placed in degree 0 we see that

$$\mathcal{L} \cong F(\mathcal{O}_X) \cong F'(\mathcal{O}_X) \cong \mathrm{pr}_{2,*}\mathcal{K}$$

Thus by Functors and Morphisms, Lemma 56.7.6 there is a morphism $s : X \to X \times X$ with $\mathrm{pr}_2 \circ s = \mathrm{id}_X$ such that $\mathcal{K} = s_*\mathcal{L}$. Set $f = \mathrm{pr}_1 \circ s$. Then we have

$$\begin{aligned}
F'(M) &= R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*K \otimes \mathcal{K}) \\
&= R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*M \otimes s_*\mathcal{L}) \\
&= R\mathrm{pr}_{2,*}(Rs_*(Lf^*M \otimes \mathcal{L})) \\
&= Lf^*M \otimes \mathcal{L}
\end{aligned}$$

where we have used Derived Categories of Schemes, Lemma 36.22.1 in the third step. Since for all closed points $x \in X$ the module $F(\mathcal{O}_x)$ is supported at x, we see that f induces the identity on the underlying topological space of X. We still have to show that f is an isomorphism which we will do in the next paragraph.

Let $x \in X$ be a closed point. For $n \geq 1$ denote $\mathcal{O}_{x,n}$ the skyscaper sheaf at x with value $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$. We have

$$\mathrm{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n}) \cong \mathrm{Hom}_X(\mathcal{O}_{x,m}, F(\mathcal{O}_{x,n})) \cong \mathrm{Hom}_X(\mathcal{O}_{x,m}, f^*\mathcal{O}_{x,n} \otimes \mathcal{L})$$

functorially with respect to $\mathcal{O}_X$-module homomorphisms between the $\mathcal{O}_{x,n}$. (The first isomorphism exists by assumption and the second isomorphism because F and F' are siblings.) For $m \geq n$ we have $\mathcal{O}_{X,x}/\mathfrak{m}^n = \mathrm{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n})$ via the action on $\mathcal{O}_{x,n}$ we conclude that $f^\sharp : \mathcal{O}_{X,x}/\mathfrak{m}_x^n \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is bijective for all n. Thus f induces isomorphisms on complete local rings at closed points and hence is étale (Étale Morphisms, Lemma 41.11.3). Looking at closed points we see that $\Delta_f : X \to X \times_{f,X,f} X$ (which is an open immersion as f is étale) is bijective hence an isomorphism. Hence f is a monomorphism. Finally, we conclude f is an isomorphism as Descent, Lemma 35.25.1 tells us it is an open immersion. $\square$

115.13. Representability in the regular proper case

0FYI This section is obsolete because we improved Derived Categories of Varieties, Theorem 57.6.3 to apply to all proper schemes over a field (whereas before we only proved it for projective schemes over a field).

0FYJ **Lemma 115.13.1.** Let $f : X' \to X$ be a proper birational morphism of integral Noetherian schemes with X regular. The map $\mathcal{O}_X \to Rf_*\mathcal{O}_{X'}$ canonically splits in $D(\mathcal{O}_X)$.

The proof given here follows the argument given in [MS20, Remark 3.4]

Proof. Set $E = Rf_*\mathcal{O}_{X'}$ in $D(\mathcal{O}_X)$. Observe that E is in $D^b_{\text{Coh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.3. By Derived Categories of Schemes, Lemma 36.11.8 we find that E is a perfect object of $D(\mathcal{O}_X)$. Since $\mathcal{O}_{X'}$ is a sheaf of algebras, we have the relative cup product $\mu : E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to E$ by Cohomology, Remark 20.28.7. Let $\sigma : E \otimes E^\vee \to E^\vee \otimes E$ be the commutativity constraint on the symmetric monoidal category $D(\mathcal{O}_X)$ (Cohomology, Lemma 20.50.6). Denote $\eta : \mathcal{O}_X \to E \otimes E^\vee$ and $\epsilon : E^\vee \otimes E \to \mathcal{O}_X$ the maps constructed in Cohomology, Example 20.50.7. Then we can consider the map

$$E \xrightarrow{\eta \otimes 1} E \otimes E^\vee \otimes E \xrightarrow{\sigma \otimes 1} E^\vee \otimes E \otimes E \xrightarrow{1 \otimes \mu} E^\vee \otimes E \xrightarrow{\epsilon} \mathcal{O}_X$$

We claim that this map is a one sided inverse to the map in the statement of the lemma. To see this it suffices to show that the composition $\mathcal{O}_X \to \mathcal{O}_X$ is the identity map. This we may do in the generic point of X (or on an open subscheme of X over which f is an isomorphism). In this case $E = \mathcal{O}_X$ and μ is the usual multiplication map and the result is clear. $\square$

0FYK **Lemma 115.13.2.** Let X be a proper scheme over a field k which is regular. Let $K \in \text{Ob}(D_{QCoh}(\mathcal{O}_X))$. The following are equivalent

(1) $K \in D^b_{\text{Coh}}(\mathcal{O}_X) = D_{perf}(\mathcal{O}_X)$, and
(2) $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}^i_X(E, K) < \infty$ for all perfect E in $D(\mathcal{O}_X)$.

Proof. The equality in (1) holds by Derived Categories of Schemes, Lemma 36.11.8. The implication (1) $\Rightarrow$ (2) follows from Derived Categories of Varieties, Lemma 57.5.3. The implication (2) $\Rightarrow$ (1) follows from More on Morphisms, Lemma 37.69.6. $\square$

0FYL **Lemma 115.13.3.** Let X be a proper scheme over a field k which is regular.

(1) Let $F : D_{perf}(\mathcal{O}_X)^{opp} \to \text{Vect}_k$ be a k-linear cohomological functor such that
$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$
for all $E \in D_{perf}(\mathcal{O}_X)$. Then F is isomorphic to a functor of the form $E \mapsto \text{Hom}_X(E, K)$ for some $K \in D_{perf}(\mathcal{O}_X)$.
(2) Let $G : D_{perf}(\mathcal{O}_X) \to \text{Vect}_k$ be a k-linear homological functor such that
$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k G(E[n]) < \infty$$
for all $E \in D_{perf}(\mathcal{O}_X)$. Then G is isomorphic to a functor of the form $E \mapsto \text{Hom}_X(K, E)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

Proof. This follows from Derived Categories of Varieties, Theorem 57.6.3 and Lemma 57.6.4. We also give another proof below.

Proof of (1). The derived category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, is compactly generated, and $D_{perf}(\mathcal{O}_X)$ is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 36.3.1, Theorem 36.15.3, and Proposition 36.17.1. By Derived Categories of Varieties, Lemma 57.6.2 we may assume $F(E) = \mathrm{Hom}_X(E, K)$ for some $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. Then it follows that K is in $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ by Lemma 115.13.2.

Proof of (2). Consider the contravariant functor $E \mapsto E^\vee$ on $D_{perf}(\mathcal{O}_X)$, see Cohomology, Lemma 20.50.5. This functor is an exact anti-self-equivalence of $D_{perf}(\mathcal{O}_X)$. Hence we may apply part (1) to the functor $F(E) = G(E^\vee)$ to find $K \in D_{perf}(\mathcal{O}_X)$ such that $G(E^\vee) = \mathrm{Hom}_X(E, K)$. It follows that $G(E) = \mathrm{Hom}_X(E^\vee, K) = \mathrm{Hom}_X(K^\vee, E)$ and we conclude that taking $K^\vee$ works. $\square$

115.14. Functor of quotients

08J4

082R Lemma 115.14.1. Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $q_i : \mathcal{F} \to \mathcal{Q}_i$, $i = 1, 2$ be surjective maps of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{Q}_1$ flat over S. Let $T \to S$ be a quasi-compact morphism of schemes such that there exists a factorization

$$\begin{tikzcd} & \mathcal{F}_T \arrow[dl, "q_{1,T}"'] \arrow[dr, "q_{2,T}"] & \\ \mathcal{Q}_{1,T} & & \mathcal{Q}_{2,T} \arrow[ll, dotted] \end{tikzcd}$$

Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) $q_{1,Z}$ factors through $q_{2,Z}$. If $\mathrm{Ker}(q_2)$ is a finite type $\mathcal{O}_X$-module and X quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Apply Flatness on Spaces, Lemma 77.8.2 to the map $\mathrm{Ker}(q_2) \to \mathcal{Q}_1$. $\square$

115.15. Spaces and fpqc coverings

0ARG The material here was made obsolete by Gabber's argument showing that algebraic spaces satisfy the sheaf condition with respect to fpqc coverings. Please visit Properties of Spaces, Section 66.17.

03W9 Lemma 115.15.1. Let S be a scheme. Let X be an algebraic space over S. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Then the map

$$\mathrm{Mor}_S(T, X) \longrightarrow \prod\nolimits_{i \in I} \mathrm{Mor}_S(T_i, X)$$

is injective.

Proof. Immediate consequence of Properties of Spaces, Proposition 66.17.1. $\square$

03WA Lemma 115.15.2. Let S be a scheme. Let X be an algebraic space over S. Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 65.12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.

Proof. This is true because all algebraic spaces satisfy the sheaf property for the fpqc topology, see Properties of Spaces, Proposition 66.17.1. $\square$

03WB Lemma 115.15.3. Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated over S, then X satisfies the sheaf condition for the fpqc topology.

Proof. Immediate consequence of the general Properties of Spaces, Proposition 66.17.1. □

03WC Remark 115.15.4. This remark used to discuss to what extend the original proof of Lemma 115.15.3 (of December 18, 2009) generalizes.

115.16. Very reasonable algebraic spaces

07T6 Material that is somewhat obsolete.

03IN Lemma 115.16.1. Let S be a scheme. Let X be a reasonable algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.8.6).

Proof. Follows from the definitions and Decent Spaces, Lemma 68.12.3. □

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \to X$, then X is very reasonable, see Decent Spaces, Lemma 68.4.7.

03I9 Lemma 115.16.2. A scheme is very reasonable.

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. □

03IA Lemma 115.16.3. Let S be a scheme. Let X be an algebraic space over S. If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.

Proof. This is case (ϵ) of Decent Spaces, Lemma 68.5.2. □

03IB Lemma 115.16.4. An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.

Proof. This is one of the implications of Decent Spaces, Lemma 68.5.1. □

03JF Lemma 115.16.5. Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Y \to X$ be a representable morphism. If X is very reasonable, so is Y.

Proof. This is case (ϵ) of Decent Spaces, Lemma 68.5.3. □

03IC Remark 115.16.6. Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 65.14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U. Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact, hence X is very reasonable. On the other hand, the diagonal $U \times_X U \to U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 26.21.4. Let $0_1, 0_2$ be the two zeros of U. Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2. Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

03K7 Lemma 115.16.7. Let S be a scheme. Let X be a very reasonable algebraic space over S. There exists a set of schemes U_i and morphisms $U_i \to X$ such that

(1) each U_i is a quasi-compact scheme,
(2) each $U_i \to X$ is étale,
(3) both projections $U_i \times_X U_i \to U_i$ are quasi-compact, and
(4) the morphism $\coprod U_i \to X$ is surjective (and étale).

Proof. Decent Spaces, Definition 68.6.1 says that there exist $U_i \to X$ such that (2), (3) and (4) hold. Fix i, and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \to U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i-invariant open (see Groupoids, Lemma 39.19.2). Hence W' is a quasi-compact scheme, $W' \to X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \to W'$ are quasi-compact. This means the family of $W' \to X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. □

115.17. Obsolete lemmas on algebraic spaces

0D45 Lemmas that seem superfluous or are no longer used in the text.

07V2 Lemma 115.17.1. In Cohomology of Spaces, Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is surjective.

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 69.16.7 but now is a consequence of it. □

07V3 Lemma 115.17.2. In Cohomology of Spaces, Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally closed.

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 69.16.7 but now is a consequence of it. □

0AKU Remark 115.17.3. This tag used to refer to an equation in the proof of Formal Spaces, Lemma 87.20.4.

0AKV Remark 115.17.4. This tag used to refer to an equation in the proof of Formal Spaces, Lemma 87.20.4.

115.18. Obsolete lemmas on algebraic stacks

0G2T Lemmas that seem superfluous or are no longer used in the text.

0CXS Lemma 115.18.1. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u_n \in U$, $n \geq 1$ be pairwise distinct finite type points such that x is not versal at u_n for all n. After replacing u_n by a subsequence, there exist morphisms

$$x \to x_1 \to x_2 \to \ldots \quad \text{in } \mathcal{X} \text{ lying over} \quad U \to U_1 \to U_2 \to \ldots$$

over S such that

(1) for each n the morphism $U \to U_n$ is a first order thickening,

(2) for each n we have a short exact sequence
$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0$$
with $U_0 = U$ for $n = 1$,

(3) for each n there does not exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \to x$ such that the composition
$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to x_n} x_n|_W \xrightarrow{\alpha} x$$
is the canonical morphism $x|_{U \cap W} \to x$.

Proof. This lemma was originally used in the proof of a criterion for openness of versality (Artin's Axioms, Lemma 98.20.3) but it got replaced by Artin's Axioms, Lemma 98.20.1 from which it readily follows. Namely, after replacing u_n, $n \geq 1$ by a subsequence we may and do assume that there are no specializations among these points, see Properties, Lemma 28.5.11. Then we can apply Artin's Axioms, Lemma 98.20.1 to finish the proof. $\square$

115.19. Variants of cotangent complexes for schemes

08T5 This section gives an alternative construction of the cotangent complex of a morphism of schemes. This section is currently in the obsolete chapter as we can get by with the easier version discussed in Cotangent, Section 92.25 for applications.

Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{C}_{X/Y}$ be the category whose objects are commutative diagrams

08T6 (115.19.0.1)
$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "i"'] \arrow[d] & A \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

of schemes where

(1) U is an open subscheme of X,
(2) V is an open subscheme of Y, and
(3) there exists an isomorphism $A = V \times \text{Spec}(P)$ over V where P is a polynomial algebra over $\mathbf{Z}$ (on some set of variables).

In other words, A is an (infinite dimensional) affine space over V. Morphisms are given by commutative diagrams.

Notation. An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (115.19.0.1), is often denoted $U \to A$ where it is understood that (a) U is an open subscheme of X, (b) $U \to A$ is a morphism over Y, (c) the image of the structure morphism $A \to Y$ is an open $V \subset Y$, and (d) $A \to V$ is an affine space. We'll write $U \to A/V$ to indicate $V \subset Y$ is the image of $A \to Y$. Recall that X_{Zar} denotes the small Zariski site X. There are forgetful functors
$$\mathcal{C}_{X/Y} \to X_{Zar},\ (U \to A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \mapsto Y_{Zar},\ (U \to A/V) \mapsto V.$$

08T7 Lemma 115.19.1. Let $X \to Y$ be a morphism of schemes.

(1) The category $\mathcal{C}_{X/Y}$ is fibred over X_{Zar}.
(2) The category $\mathcal{C}_{X/Y}$ is fibred over Y_{Zar}.

(3) The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs (U, V) where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.

Proof. Ad (1). Given an object $U \to A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \to U$ of X_{Zar} consider the object $i' : U' \to A$ of $\mathcal{C}_{X/Y}$ where i' is the composition of i and $U' \to U$. The morphism $(U' \to A) \to (U \to A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over X_{Zar}.

Ad (2). Given an object $U \to A/V$ and $V' \to V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_V A$ to obtain a strongly cartesian morphism $(U' \to A') \to (U \to A)$ over $V' \to V$.

Ad (3). Denote $(X/Y)_{Zar}$ the category in (3). Given $U \to A/V$ and a morphism $(U', V') \to (U, V)$ in $(X/Y)_{Zar}$ we can consider $A' = V' \times_V A$. Then the morphism $(U' \to A'/V') \to (U \to A/V)$ is strongly cartesian in $\mathcal{C}_{X/Y}$ over $(X/Y)_{Zar}$. $\square$

We obtain a topology τ_X on $\mathcal{C}_{X/Y}$ by using the topology inherited from X_{Zar} (see Stacks, Section 8.10). If not otherwise stated this is the topology on $\mathcal{C}_{X/Y}$ we will consider. To be precise, a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering of $\mathcal{C}_{X/Y}$ if and only if

(1) $U = \bigcup U_i$, and
(2) $A_i = A$ for all i.

We obtain the same collection of sheaves if we allow $A_i \cong A$ in (2). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/Y}) \to Sh(X_{Zar})$.

The site $\mathcal{C}_{X/Y}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to A) \mapsto \mathcal{O}(A)$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to A) \mapsto \mathcal{O}(U)$.
(3) The sheaf $\underline{\mathcal{O}}_Y$ given by the rule $(U \to A/V) \mapsto \mathcal{O}(V)$.

We obtain morphisms of ringed topoi

08T8 (115.19.1.1)
$$\begin{tikzcd} (Sh(\mathcal{C}_{X/Y}), \underline{\mathcal{O}}_X) \arrow[r, "i"'] \arrow[d, "\pi"'] & (Sh(\mathcal{C}_{X/Y}), \mathcal{O}) \\ (Sh(X_{Zar}), \mathcal{O}_X) & \end{tikzcd}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$.

08TA Remark 115.19.2. We obtain a second topology τ_Y on $\mathcal{C}_{X/Y}$ by taking the topology inherited from Y_{Zar}. There is a third topology $\tau_{X \to Y}$ where a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering if and only if $U = \bigcup U_i$, $V = \bigcup V_i$ and $A_i \cong V_i \times_V A$. This is the topology inherited from the topology on the site $(X/Y)_{Zar}$ whose underlying category is the category of pairs (U, V) as in Lemma 115.19.1 part (3). The coverings of $(X/Y)_{Zar}$ are families $\{(U_i, V_i) \to (U, V)\}$ such that $U = \bigcup U_i$ and $V = \bigcup V_i$. There are morphisms of topoi

$$Sh(\mathcal{C}_{X/Y}) = Sh(\mathcal{C}_{X/Y}, \tau_X) \longleftarrow Sh(\mathcal{C}_{X/Y}, \tau_{X \to Y}) \longrightarrow Sh(\mathcal{C}_{X/Y}, \tau_Y)$$

(recall that τ_X is our "default" topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topoi

$$\begin{tikzcd} Sh(X_{Zar}) \arrow[d, "f"] & Sh(\mathcal{C}_{X/Y}) \arrow[l, "\pi"'] & Sh(\mathcal{C}_{X/Y}, \tau_{X \to Y}) \arrow[l] \arrow[d] \\ Sh(Y_{Zar}) & & Sh(\mathcal{C}_{X/Y}, \tau_Y) \arrow[ll] \end{tikzcd}$$

is not commutative. Namely, the pullback of a nonzero abelian sheaf on Y is a nonzero abelian sheaf on $(\mathcal{C}_{X/Y}, \tau_{X \to Y})$, but we can certainly find examples where such a sheaf pulls back to zero on X. Note that any presheaf $\mathcal{F}$ on Y_{Zar} gives a sheaf $\underline{\mathcal{F}}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \to A/V)$ the set $\mathcal{F}(V)$. Even if $\mathcal{F}$ happens to be a sheaf it isn't true in general that $\underline{\mathcal{F}} = \pi^{-1}f^{-1}\mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\underline{\mathcal{F}}$ as the pushforward of the pullback of $\mathcal{F}$ to $Sh(\mathcal{C}_{X/Y}, \tau_{X \to Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\underline{\mathcal{O}_Y}$. But what is true is that there is a map $\underline{\mathcal{F}} \to \pi^{-1}f^{-1}\mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi_!$.

115.20. Deformations and obstructions of flat modules

08VZ In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space X over a ring Λ. This material is obsolete due to the improved discussion in Quot, Section 99.6.

Our setup will be the following. We assume given

(1) a ring Λ,
(2) an algebraic space X over Λ,
(3) a Λ-algebra A, set $X_A = X \times_{\text{Spec}(\Lambda)} \text{Spec}(A)$, and
(4) a finitely presented $\mathcal{O}_{X_A}$-module $\mathcal{F}$ flat over A.

In this situation we will consider all possible surjections

$$0 \to I \to A' \to A \to 0$$

where A' is a Λ-algebra whose kernel I is an ideal of square zero in A'. Given A' we obtain a first order thickening $X_A \to X_{A'}$ of algebraic spaces over $\text{Spec}(\Lambda)$. For each of these we consider the problem of lifting $\mathcal{F}$ to a finitely presented module $\mathcal{F}'$ on $X_{A'}$ flat over A'. We would like to replicate the results of Deformation Theory, Lemma 91.12.1 in this setting.

To be more precise let $\text{Lift}(\mathcal{F}, A')$ denote the category of pairs $(\mathcal{F}', \alpha)$ where $\mathcal{F}'$ is a finitely presented module on $X_{A'}$ flat over A' and $\alpha : \mathcal{F}'|_{X_A} \to \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}'_1, \alpha_1) \to (\mathcal{F}'_2, \alpha_2)$ are isomorphisms $\mathcal{F}'_1 \to \mathcal{F}'_2$ which are compatible with α_1 and α_2. The set of isomorphism classes of $\text{Lift}(\mathcal{F}, A')$ is denoted $\text{Lift}(\mathcal{F}, A')$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{\acute{e}tale}$ flat over A. We introduce the category $\text{Lift}(\mathcal{G}, A')$ of pairs $(\mathcal{G}', \beta)$ where $\mathcal{G}'$ is a sheaf of $\mathcal{O}_X \otimes_\Lambda A'$-modules flat over A' and β is an isomorphism $\mathcal{G}' \otimes_{A'} A \to \mathcal{G}$.

08W0 **Lemma 115.20.1.** Notation and assumptions as above. Let $p : X_A \to X$ denote the projection. Given A' denote $p' : X_{A'} \to X$ the projection. The functor p'_* induces an equivalence of categories between

(1) the category $\text{Lift}(\mathcal{F}, A')$, and
(2) the category $\text{Lift}(p_*\mathcal{F}, A')$.

Proof. FIXME. □

Let $\mathcal{H}$ be a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules on $\mathcal{C}_{X/\Lambda}$ flat over A. We introduce the category $\text{Lift}_{\mathcal{O}}(\mathcal{H}, A')$ whose objects are pairs $(\mathcal{H}', \gamma)$ where $\mathcal{H}'$ is a sheaf of $\mathcal{O} \otimes_\Lambda A'$-modules flat over A' and $\gamma : \mathcal{H}' \otimes_A A' \to \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_\Lambda A$-modules.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{\acute{e}tale}$ flat over A. Consider the morphisms i and π of Cotangent, Equation (92.27.1.1). Denote $\underline{\mathcal{G}} = \pi^{-1}(\mathcal{G})$. It is simply given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{G}(U)$ hence it is a sheaf of $\underline{\mathcal{O}}_X \otimes_\Lambda A$-modules. Denote $i_*\underline{\mathcal{G}}$ the same sheaf but viewed as a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules.

08W1 **Lemma 115.20.2.** Notation and assumptions as above. The functor $\pi_!$ induces an equivalence of categories between

(1) the category $\text{Lift}_{\mathcal{O}}(i_*\underline{\mathcal{G}}, A')$, and
(2) the category $\text{Lift}(\mathcal{G}, A')$.

Proof. FIXME. □

08W2 **Lemma 115.20.3.** Notation and assumptions as in Lemma 115.20.2. Consider the object

$$L = L(\Lambda, X, A, \mathcal{G}) = L\pi_!(Li^*(i_*(\underline{\mathcal{G}})))$$

of $D(\mathcal{O}_X \otimes_\Lambda A)$. Given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have

(1) The category $\text{Lift}(\mathcal{G}, A')$ is nonempty if and only if a certain class $\xi \in \text{Ext}^2_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$ is zero.
(2) If $\text{Lift}(\mathcal{G}, A')$ is nonempty, then $\text{Lift}(\mathcal{G}, A')$ is principal homogeneous under $\text{Ext}^1_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.
(3) Given a lift $\mathcal{G}'$, the set of automorphisms of $\mathcal{G}'$ which pull back to $\text{id}_{\mathcal{G}}$ is canonically isomorphic to $\text{Ext}^0_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.

Proof. FIXME. □

Finally, we put everything together as follows.

08W3 **Proposition 115.20.4.** With Λ, X, A, $\mathcal{F}$ as above. There exists a canonical object $L = L(\Lambda, X, A, \mathcal{F})$ of $D(X_A)$ such that given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have

(1) The category $\text{Lift}(\mathcal{F}, A')$ is nonempty if and only if a certain class $\xi \in \text{Ext}^2_{X_A}(L, \mathcal{F} \otimes_A I)$ is zero.
(2) If $\text{Lift}(\mathcal{F}, A')$ is nonempty, then $\text{Lift}(\mathcal{F}, A')$ is principal homogeneous under $\text{Ext}^1_{X_A}(L, \mathcal{F} \otimes_A I)$.
(3) Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\text{id}_{\mathcal{F}}$ is canonically isomorphic to $\text{Ext}^0_{X_A}(L, \mathcal{F} \otimes_A I)$.

Proof. FIXME. □

08W4 **Lemma 115.20.5.** In the situation of Proposition 115.20.4, if $X \to \text{Spec}(\Lambda)$ is locally of finite type and Λ is Noetherian, then L is pseudo-coherent.

Proof. FIXME. □

115.21. The stack of coherent sheaves in the non-flat case

0CXY In Quot, Theorem 99.5.12 the assumption that $f : X \to B$ is flat is not necessary. In this section we modify the method of proof based on ideas from derived algebraic geometry to get around the flatness hypothesis. An entirely different method is used in Quot, Section 99.6 to get exactly the same result; this is why the method from this section is obsolete.

The only step in the proof of Quot, Theorem 99.5.12 which uses flatness is in the application of Quot, Lemma 99.5.11. The lemma is used to construct an obstruction theory as in Artin's Axioms, Section 98.24. The proof of the lemma relies on Deformation Theory, Lemmas 91.12.1 and 91.12.5 from Deformation Theory, Section 91.12. This is how the assumption that f is flat comes about. Before we go on, note that results (2) and (3) of Deformation Theory, Lemmas 91.12.1 do hold without the assumption that f is flat as they rely on Deformation Theory, Lemmas 91.11.7 and 91.11.4 which do not have any flatness assumptions.

Before we give the details we give some motivation for the construction from derived algebraic geometry, since we think it will clarify what follows. Let A be a finite type algebra over the locally Noetherian base S. Denote $X \otimes^{\mathbf{L}} A$ a "derived base change" of X to A and denote $i : X_A \to X \otimes^{\mathbf{L}} A$ the canonical inclusion morphism. The object $X \otimes^{\mathbf{L}} A$ does not (yet) have a definition in the Stacks project; we may think of it as the algebraic space X_A endowed with a simplicial sheaf of rings $\mathcal{O}_{X\otimes^{\mathbf{L}}A}$ whose homology sheaves are

$$H_i(\mathcal{O}_{X\otimes^{\mathbf{L}}A}) = \text{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, A).$$

The morphism $X \otimes^{\mathbf{L}} A \to \text{Spec}(A)$ is flat (the terms of the simplicial sheaf of rings being A-flat), so the usual material for deformations of flat modules applies to it. Thus we see that we get an obstruction theory using the groups

$$\text{Ext}^i_{X\otimes^{\mathbf{L}}A}(i_*\mathcal{F}, i_*\mathcal{F} \otimes_A M)$$

where $i = 0, 1, 2$ for inf auts, inf defs, obstructions. Note that a flat deformation of $i_*\mathcal{F}$ to $X \otimes^{\mathbf{L}} A'$ is automatically of the form $i'_*\mathcal{F}'$ where $\mathcal{F}'$ is a flat deformation of $\mathcal{F}$. By adjunction of the functors Li^* and $i_* = Ri_*$ these ext groups are equal to

$$\text{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$$

Thus we obtain obstruction groups of exactly the same form as in the proof of Quot, Lemma 99.5.11 with the only change being that one replaces the first occurrence of $\mathcal{F}$ by the complex $Li^*(i_*\mathcal{F})$.

Below we prove the non-flat version of the lemma by a "direct" construction of $E(\mathcal{F}) = Li^*(i_*\mathcal{F})$ and direct proof of its relationship to the deformation theory of $\mathcal{F}$. In fact, it suffices to construct $\tau_{\geq -2}E(\mathcal{F})$, as we are only interested in the ext groups $\text{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i = 0, 1, 2$. We can even identify the cohomology sheaves

$$H^i(E(\mathcal{F})) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{F} & \text{if } i = 0 \\ 0 & \text{if } i = -1 \\ \text{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_X, A) \otimes_{\mathcal{O}_X} \mathcal{F} & \text{if } i = -2 \end{cases}$$

This observation will guide our construction of $E(\mathcal{F})$ in the remarks below.

09DN Remark 115.21.1 (Direct construction). Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let U be another algebraic space over B. Denote $q : X \times_B U \to U$ the second projection. Consider the distinguished triangle

$$Lq^*L_{U/B} \to L_{X \times_B U/B} \to E \to Lq^*L_{U/B}[1]$$

of Cotangent, Section 92.28. For any sheaf $\mathcal{F}$ of $\mathcal{O}_{X \times_B U}$-modules we have the Atiyah class

$$\mathcal{F} \to L_{X \times_B U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

see Cotangent, Section 92.19. We can compose this with the map to E and choose a distinguished triangle

$$E(\mathcal{F}) \to \mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} E[1] \to E(\mathcal{F})[1]$$

in $D(\mathcal{O}_{X \times_B U})$. By construction the Atiyah class lifts to a map

$$e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow Lq^*L_{U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

fitting into a morphism of distinguished triangles

$$\begin{tikzcd}
\mathcal{F} \otimes^{\mathbf{L}} Lq^*L_{U/B}[1] \arrow[r] & \mathcal{F} \otimes^{\mathbf{L}} L_{X \times_B U/B}[1] \arrow[r] & \mathcal{F} \otimes^{\mathbf{L}} E[1] \\
E(\mathcal{F}) \arrow[r] \arrow[u, "e_{\mathcal{F}}"] & \mathcal{F} \arrow[r] \arrow[u, "Atiyah"] & \mathcal{F} \otimes^{\mathbf{L}} E[1] \arrow[u, "="]
\end{tikzcd}$$

Given $S, B, X, f, U, \mathcal{F}$ we fix a choice of $E(\mathcal{F})$ and $e_{\mathcal{F}}$.

09DP Remark 115.21.2 (Construction of obstruction class). With notation as in Remark 115.21.1 let $i : U \to U'$ be a first order thickening of U over B. Let $\mathcal{I} \subset \mathcal{O}_{U'}$ be the quasi-coherent sheaf of ideals cutting out B in B'. The fundamental triangle

$$Li^*L_{U'/B} \to L_{U/B} \to L_{U/U'} \to Li^*L_{U'/B}[1]$$

together with the map $L_{U/U'} \to \mathcal{I}[1]$ determine a map $e_{U'} : L_{U/B} \to \mathcal{I}[1]$. Combined with the map $e_{\mathcal{F}}$ of the previous remark we obtain

$$(\mathrm{id}_{\mathcal{F}} \otimes Lq^*e_{U'}) \cup e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I}[2]$$

(we have also composed with the map from the derived tensor product to the usual tensor product). In other words, we obtain an element

$$\xi_{U'} \in \mathrm{Ext}^2_{\mathcal{O}_{X \times_B U}}(E(\mathcal{F}), \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I})$$

09DQ Lemma 115.21.3. In the situation of Remark 115.21.2 assume that $\mathcal{F}$ is flat over U. Then the vanishing of the class $\xi_{U'}$ is a necessary and sufficient condition for the existence of a $\mathcal{O}_{X \times_B U'}$-module $\mathcal{F}'$ flat over U' with $i^*\mathcal{F}' \cong \mathcal{F}$.

Proof (sketch). We will use the criterion of Deformation Theory, Lemma 91.11.8. We will abbreviate $\mathcal{O} = \mathcal{O}_{X \times_B U}$ and $\mathcal{O}' = \mathcal{O}_{X \times_B U'}$. Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{U'} \to \mathcal{O}_U \to 0.$$

Let $\mathcal{J} \subset \mathcal{O}'$ be the quasi-coherent sheaf of ideals cutting out $X \times_B U$. By the above we obtain an exact sequence

$$\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to q^*\mathcal{I} \to \mathcal{J} \to 0$$

where the $\text{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U)$ is an abbreviation for

$$\text{Tor}_1^{h^{-1}\mathcal{O}_B}(p^{-1}\mathcal{O}_X, q^{-1}\mathcal{O}_U) \otimes_{(p^{-1}\mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_B} q^{-1}\mathcal{O}_U)} \mathcal{O}.$$

Tensoring with $\mathcal{F}$ we obtain the exact sequence

$$\mathcal{F} \otimes_{\mathcal{O}} \text{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to \mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J} \to 0$$

(Note that the roles of the letters $\mathcal{I}$ and $\mathcal{J}$ are reversed relative to the notation in Deformation Theory, Lemma 91.11.8.) Condition (1) of the lemma is that the last map above is an isomorphism, i.e., that the first map is zero. The vanishing of this map may be checked on stalks at geometric points $\overline{z} = (\overline{x}, \overline{u}) : \text{Spec}(k) \to X \times_B U$. Set $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{U,\overline{u}}$, and $C = \mathcal{O}_{\overline{z}}$. By Cotangent, Lemma 92.28.2 and the defining triangle for $E(\mathcal{F})$ we see that

$$H^{-2}(E(\mathcal{F}))_{\overline{z}} = \mathcal{F}_{\overline{z}} \otimes \text{Tor}_1^R(A, B)$$

The map $\xi_{U'}$ therefore induces a map

$$\mathcal{F}_{\overline{z}} \otimes \text{Tor}_1^R(A, B) \longrightarrow \mathcal{F}_{\overline{z}} \otimes_B \mathcal{I}_{\overline{u}}$$

We claim this map is the same as the stalk of the map described above (proof omitted; this is a purely ring theoretic statement). Thus we see that condition (1) of Deformation Theory, Lemma 91.11.8 is equivalent to the vanishing $H^{-2}(\xi_{U'}) : H^{-2}(E(\mathcal{F})) \to \mathcal{F} \otimes \mathcal{I}$.

To finish the proof we show that, assuming that condition (1) is satisfied, condition (2) is equivalent to the vanishing of $\xi_{U'}$. In the rest of the proof we write $\mathcal{F} \otimes \mathcal{I}$ to denote $\mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J}$. A consideration of the spectral sequence

$$\text{Ext}^i(H^{-j}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I}) \Rightarrow \text{Ext}^{i+j}(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I})$$

using that $H^0(E(\mathcal{F})) = \mathcal{F}$ and $H^{-1}(E(\mathcal{F})) = 0$ shows that there is an exact sequence

$$0 \to \text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) \to \text{Ext}^2(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I}) \to \text{Hom}(H^{-2}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I})$$

Thus our element $\xi_{U'}$ is an element of $\text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I})$. The proof is finished by showing this element agrees with the element of Deformation Theory, Lemma 91.11.8 a verification we omit. $\square$

09DR Lemma 115.21.4. In Quot, Situation 99.5.1 assume that S is a locally Noetherian scheme and $S = B$. Let $\mathcal{X} = \text{Coh}_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 98.13.1).

Proof (sketch). Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 29.16.1) and $U = \text{Spec}(A)$ with $U \to S$ mapping into an affine open $\text{Spec}(\Lambda)$ of S. We will use Artin's Axioms, Lemma 98.24.4 to prove the lemma. Let $\mathcal{F}$ be the coherent module on $X_A = \text{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

Choose $E(\mathcal{F})$ and $e_{\mathcal{F}}$ as in Remark 115.21.1. The description of the cohomology sheaves of $E(\mathcal{F})$ shows that

$$\text{Ext}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M) = \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for any A-module M. Using this and using Deformation Theory, Lemma 91.11.7 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

By Lemma 115.21.3 given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A I)$$

Apply Derived Categories of Spaces, Lemma 75.23.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We omit the verification that $E(\mathcal{F})$ is in D^-_{Coh}; hint: use Cotangent, Lemma 92.5.4. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \mathrm{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 98.24.2. Finally, condition (iv) of Artin's Axioms, Lemma 98.24.3 holds by a variant of Deformation Theory, Lemma 91.12.5 whose formulation and proof we omit. Thus Artin's Axioms, Lemma 98.24.4 applies and the lemma is proved. □

0CXZ Theorem 115.21.5. Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation and separated. Then $\mathrm{Coh}_{X/B}$ is an algebraic stack over S.

Proof. This theorem is a copy of Quot, Theorem 99.6.1. The reason we have this copy here is that with the material in this section we get a second proof (as discussed at the beginning of this section). Namely, we argue exactly as in the proof of Quot, Theorem 99.5.12 except that we substitute Lemma 115.21.4 for Quot, Lemma 99.5.11. □

115.22. Modifications

0AS3 Here is a obsolete result on the category of Algebraization of Formal Spaces, Equation (88.30.0.1). Please visit Algebraization of Formal Spaces, Section 88.30 for the current material.

0AE4 Lemma 115.22.1. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The category of Algebraization of Formal Spaces, Equation (88.30.0.1) for A is equivalent to the category Algebraization of Formal Spaces, Equation (88.30.0.1) for the henselization A^h of A.

Proof. This is a special case of Algebraization of Formal Spaces, Lemma 88.30.3. □

The following lemma on rational singularities is no longer needed in the chapter on resolving surface singularities.

0B50 Lemma 115.22.2. In Resolution of Surfaces, Situation 54.9.1. Let M be a finite reflexive A-module. Let $M \otimes_A \mathcal{O}_X$ denote the pullback of the associated $\mathcal{O}_S$-module. Then $M \otimes_A \mathcal{O}_X$ maps onto its double dual.

Proof. Let $\mathcal{F} = (M \otimes_A \mathcal{O}_X)^{**}$ be the double dual and let $\mathcal{F}' \subset \mathcal{F}$ be the image of the evaluation map $M \otimes_A \mathcal{O}_X \to \mathcal{F}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q} \to 0$$

Since X is normal, the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings for points of codimension 1 (see Properties, Lemma 28.12.5). Hence $\mathcal{Q}_x = 0$ for such points by More on Algebra, Lemma 15.23.3. Thus $\mathcal{Q}$ is supported in finitely many closed points and is globally generated by Cohomology of Schemes, Lemma 30.9.10. We obtain the exact sequence

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{Q}) \to 0$$

because $\mathcal{F}'$ is generated by global sections (Resolution of Surfaces, Lemma 54.9.2). Since $X \to \mathrm{Spec}(A)$ is an isomorphism over the complement of the closed point, and since M is reflexive, we see that the maps

$$M \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F})$$

induce isomorphisms after localization at any nonmaximal prime of A. Hence these maps are isomorphisms by More on Algebra, Lemma 15.23.13 and the fact that reflexive modules over normal rings have property (S_2) (More on Algebra, Lemma 15.23.18). Thus we conclude that $\mathcal{Q} = 0$ as desired. $\square$

115.23. Intersection theory

0AYK

0FIG Lemma 115.23.1. Let $b : X' \to X$ be the blowing up of a smooth projective scheme over a field k in a smooth closed subscheme $Z \subset X$. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Assume there exists an element of $K_0(X)$ whose restriction to Z is equal to the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$. Then $[Lb^*\mathcal{O}_Z] = [\mathcal{O}_E] \cdot \alpha''$ in $K_0(X')$ for some $\alpha'' \in K_0(X')$.

Proof. The schemes X, X', E, Z are smooth and projective over k and hence we have $K'_0(X) = K_0(X) = K_0(\mathrm{Vect}(X)) = K_0(D^b_{\mathrm{Coh}}(X)))$ and similarly for the other 3. See Derived Categories of Schemes, Lemmas 36.38.1, 36.38.4, and 36.38.5. We will switch between these versions at will in this proof. Consider the short exact sequence

$$0 \to \mathcal{F} \to \pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'} \to 0$$

of finite locally free $\mathcal{O}_E$-modules defining $\mathcal{F}$. Observe that $\mathcal{C}_{E/X'} = \mathcal{O}_{X'}(-E)|_E$ is the restriction of the invertible $\mathcal{O}_X$-module $\mathcal{O}_{X'}(-E)$. Let $\alpha \in K_0(X)$ be an element such that $i^*\alpha = [\mathcal{C}_{Z/X}]$ in $K_0(Z)$. Let $\alpha' = b^*\alpha - [\mathcal{O}_{X'}(-E)]$. Then $j^*\alpha' = [\mathcal{F}]$. We deduce that $j^*\lambda^i(\alpha') = [\wedge^i(\mathcal{F})]$ by Weil Cohomology Theories, Lemma 45.13.1. This means that $[\mathcal{O}_E]\cdot\alpha' = [\wedge^i\mathcal{F}]$ in $K_0(X)$, see Derived Categories of Schemes, Lemma 36.38.8. Let r be the maximum codimension of an irreducible component of Z in X. A computation which we omit shows that $H^{-i}(Lb^*\mathcal{O}_Z) =$

$\wedge^i \mathcal{F}$ for $i \geq 0, 1, \ldots, r-1$ and zero in other degrees. It follows that in $K_0(X)$ we have

$$\begin{aligned}[Lb^*\mathcal{O}_Z] &= \sum\nolimits_{i=0,\ldots,r-1} (-1)^i [\wedge^i \mathcal{F}] \\ &= \sum\nolimits_{i=0,\ldots,r-1} (-1)^i [\mathcal{O}_E] \lambda^i(\alpha') \\ &= [\mathcal{O}_E] \left(\sum\nolimits_{i=0,\ldots,r-1} (-1)^i \lambda^i(\alpha')\right)\end{aligned}$$

This proves the lemma with $\alpha'' = \sum_{i=0,\ldots,r-1} (-1)^i \lambda^i(\alpha')$. □

02TL Lemma 115.23.2. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \to X$ be the closed immersion of the zero scheme of a. Let $f \in R(X)^*$. In this case $i^*\text{div}_X(f) = 0$ in $A_{n-2}(D)$.

Proof. Special case of Chow Homology, Lemma 42.30.1. □

02SA Remark 115.23.3. This remark used to say that it wasn't clear whether the arrows of Chow Homology, Lemma 42.23.2 were isomorphisms in general. However, we've now found a proof of this fact.

02SY 115.23.4. Blowing up lemmas. In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blowups. These lemmas can be found in [Ful98, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism b of Lemma 115.23.11 is a composition of infinitely many blowups, but over any given quasi-compact open $W \subset X$ one needs only finitely many blowups (and this is the result of loc. cit.).

02SZ Lemma 115.23.5. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $D \subset Y$ be an effective Cartier divisor. Assume X, Y integral, $n = \dim_\delta(X) = \dim_\delta(Y)$ and f dominant. Then

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

In particular if f is birational then $f_*[f^{-1}(D)]_{n-1} = [D]_{n-1}$.

Proof. Immediate from Chow Homology, Lemma 42.26.3 and the fact that D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$. □

02T0 Lemma 115.23.6. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral with $\dim_\delta(X) = n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. There exists a projective morphism

$$\pi : X' \longrightarrow X$$

such that

(1) X' is integral,
(2) $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism,
(3) there exist effective Cartier divisors $D, E \subset X'$ such that

$$\pi^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

(4) the meromorphic section s corresponds, via the isomorphism above, to the meromorphic section $1_D \otimes (1_E)^{-1}$ (see Divisors, Definition 31.14.1),
(5) we have
$$\pi_*([D]_{n-1} - [E]_{n-1}) = \mathrm{div}_{\mathcal{L}}(s)$$
in $Z_{n-1}(X)$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of denominators of s, see Divisors, Definition 31.23.10. By Divisors, Lemma 31.34.6 we get (2), (3), and (4). By Divisors, Lemma 31.32.9 we get (1). By Divisors, Lemma 31.32.13 the morphism π is projective. We still have to prove (5). By Chow Homology, Lemma 42.26.3 we have
$$\pi_*(\mathrm{div}_{\mathcal{L}'}(s')) = \mathrm{div}_{\mathcal{L}}(s).$$
Hence it suffices to show that $\mathrm{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$. This follows from the equality $s' = 1_D \otimes 1_E^{-1}$ and additivity, see Divisors, Lemma 31.27.5. □

02T1 Definition 115.23.7. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = n - 1$. The ϵ-invariant of this situation is
$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$
where n_Z, resp. m_Z is the coefficient of Z in the $(n-1)$-cycle $[D_1]_{n-1}$, resp. $[D_2]_{n-1}$.

02T2 Lemma 115.23.8. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let Z be an open and closed subscheme of the scheme $D_1 \cap D_2$. Assume $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n-2$. Then there exists a morphism $b : X' \to X$, and Cartier divisors D_1', D_2', E on X' with the following properties
(1) X' is integral,
(2) b is projective,
(3) b is the blowup of X in the closed subscheme Z,
(4) $E = b^{-1}(Z)$,
(5) $b^{-1}(D_1) = D_1' + E$, and $b^{-1}D_2 = D_2' + E$,
(6) $\dim_\delta(D_1' \cap D_2') \leq n - 2$, and if $Z = D_1 \cap D_2$ then $D_1' \cap D_2' = \emptyset$,
(7) for every integral closed subscheme W' with $\dim_\delta(W') = n - 1$ we have
  (a) if $\epsilon_{W'}(D_1', E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n-1$ and
  $$\epsilon_{W'}(D_1', E) < \epsilon_W(D_1, D_2),$$
  (b) if $\epsilon_{W'}(D_2', E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n-1$ and
  $$\epsilon_{W'}(D_2', E) < \epsilon_W(D_1, D_2),$$

Proof. Note that the quasi-coherent ideal sheaf $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ defines the scheme theoretic intersection $D_1 \cap D_2 \subset X$. Since Z is a union of connected components of $D_1 \cap D_2$ we see that for every $z \in Z$ the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is equal to $\mathcal{I}_z$. Let $b : X' \to X$ be the blowup of X in Z. (So Zariski locally around Z it is the blowup of X in $\mathcal{I}$.) Denote $E = b^{-1}(Z)$ the corresponding effective Cartier divisor, see Divisors, Lemma 31.32.4. Since $Z \subset D_1$ we have $E \subset f^{-1}(D_1)$ and hence $D_1 = D_1' + E$ for some effective Cartier divisor $D_1' \subset X'$, see Divisors, Lemma 31.13.8. Similarly $D_2 = D_2' + E$. This takes care of assertions (1) – (5).

Note that if W' is as in (7) (a) or (7) (b), then the image W of W' is contained in $D_1 \cap D_2$. If W is not contained in Z, then b is an isomorphism at the generic point of W and we see that $\dim_\delta(W) = \dim_\delta(W') = n - 1$ which contradicts the assumption that $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$. Hence $W \subset Z$. This means that to prove (6) and (7) we may work locally around Z on X.

Thus we may assume that $X = \mathrm{Spec}(A)$ with A a Noetherian domain, and $D_1 = \mathrm{Spec}(A/a)$, $D_2 = \mathrm{Spec}(A/b)$ and $Z = D_1 \cap D_2$. Set $I = (a, b)$. Since A is a domain and $a, b \neq 0$ we can cover the blowup by two patches, namely $U = \mathrm{Spec}(A[s]/(as - b))$ and $V = \mathrm{Spec}(A[t]/(bt - a))$. These patches are glued using the isomorphism $A[s, s^{-1}]/(as - b) \cong A[t, t^{-1}]/(bt - a)$ which maps s to t^{-1}. The effective Cartier divisor E is described by $\mathrm{Spec}(A[s]/(as - b, a)) \subset U$ and $\mathrm{Spec}(A[t]/(bt - a, b)) \subset V$. The closed subscheme D_1' corresponds to $\mathrm{Spec}(A[t]/(bt - a, t)) \subset U$. The closed subscheme D_2' corresponds to $\mathrm{Spec}(A[s]/(as - b, s)) \subset V$. Since "$ts = 1$" we see that $D_1' \cap D_2' = \emptyset$.

Suppose we have a prime $\mathfrak{q} \subset A[s]/(as - b)$ of height one with $s, a \in \mathfrak{q}$. Let $\mathfrak{p} \subset A$ be the corresponding prime of A. Observe that $a, b \in \mathfrak{p}$. By the dimension formula we see that $\dim(A_\mathfrak{p}) = 1$ as well. The final assertion to be shown is that

$$\mathrm{ord}_{A_\mathfrak{p}}(a)\mathrm{ord}_{A_\mathfrak{p}}(b) > \mathrm{ord}_{B_\mathfrak{q}}(a)\mathrm{ord}_{B_\mathfrak{q}}(s)$$

where $B = A[s]/(as - b)$. By Algebra, Lemma 10.124.1 we have $\mathrm{ord}_{A_\mathfrak{p}}(x) \geq \mathrm{ord}_{B_\mathfrak{q}}(x)$ for $x = a, b$. Since $\mathrm{ord}_{B_\mathfrak{q}}(s) > 0$ we win by additivity of the ord function and the fact that $as = b$. □

02T3 Definition 115.23.9. Let X be a scheme. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given a function $I \to \mathbf{Z}_{\geq 0}$, $i \mapsto n_i$. The sum of the effective Cartier divisors $D = \sum n_i D_i$, is the unique effective Cartier divisor $D \subset X$ such that on any quasi-compact open $U \subset X$ we have $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ is the sum as in Divisors, Definition 31.13.6.

02T4 Lemma 115.23.10. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given $n_i \geq 0$ for $i \in I$. Then

$$[D]_{n-1} = \sum\nolimits_i n_i [D_i]_{n-1}$$

in $Z_{n-1}(X)$.

Proof. Since we are proving an equality of cycles we may work locally on X. Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors $D = D_1 + D_2$. By Chow Homology, Lemma 42.24.2 we see that $D_1 = \mathrm{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$ where 1_{D_1} denotes the canonical section of $\mathcal{O}_X(D_1)$. Of course we have the same statement for D_2 and D. Since $1_D = 1_{D_1} \otimes 1_{D_2}$ via the identification $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ we win by Divisors, Lemma 31.27.5. □

02T5 Lemma 115.23.11. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = d$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Assume that for all $\{i, j, k\} \subset I$, $\#\{i, j, k\} = 3$ we have $D_i \cap D_j \cap D_k = \emptyset$. Then there exist

(1) an open subscheme $U \subset X$ with $\dim_\delta(X \setminus U) \leq d - 3$,
(2) a morphism $b : U' \to U$, and

(3) effective Cartier divisors $\{D'_j\}_{j \in J}$ on U'

with the following properties:

(1) b is proper morphism $b : U' \to U$,
(2) U' is integral,
(3) b is an isomorphism over the complement of the union of the pairwise intersections of the $D_i|_U$,
(4) $\{D'_j\}_{j \in J}$ is a locally finite collection of effective Cartier divisors on U',
(5) $\dim_\delta(D'_j \cap D'_{j'}) \leq d - 2$ if $j \neq j'$, and
(6) $b^{-1}(D_i|_U) = \sum n_{ij} D'_j$ for certain $n_{ij} \geq 0$.

Moreover, if X is quasi-compact, then we may assume $U = X$ in the above.

Proof. Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \ldots$$

and finite sets of effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \amalg P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each $n \geq 0$ we have $(b_0 \circ \ldots \circ b_n)^{-1}(D_i)$ is a nonnegative integer combination of the divisors $D_{n+1,j}$, $j \in I_{n+1}$.

To start the induction we set $X_0 = X$ and $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\}_{i \in I_n})$ let X_{n+1} be the blowup of X_n in the closed subscheme $Z_n = \bigcup_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Note that the closed subschemes $D_{n,i} \cap D_{n,i'}$ are pairwise disjoint by our assumption on triple intersections. In other words we may write $Z_n = \coprod_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Moreover, in a Zariski neighbourhood of $D_{n,i} \cap D_{n,i'}$ the morphism b_n is equal to the blowup of the scheme X_n in the closed subscheme $D_{n,i} \cap D_{n,i'}$, and the results of Lemma 115.23.8 apply. Hence setting $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$ we get an effective Cartier divisor. The Cartier divisors $D_{n+1,\{i,i'\}}$ are pairwise disjoint. Clearly we have $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$ for every $i' \in I_n$, $i' \neq i$. Hence, applying Divisors, Lemma 31.13.8 we see that indeed $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$ for some effective Cartier divisor $D_{n+1,i}$ on X_{n+1}. In a neighbourhood of $D_{n+1,\{i,i'\}}$ these divisors $D_{n+1,i}$ play the role of the primed divisors of Lemma 115.23.8. In particular we conclude that $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$ if $i \neq i'$, $i, i' \in I_n$ by part (6) of Lemma 115.23.8. This already implies that triple intersections of the divisors $D_{n+1,i}$ are zero.

OK, and at this point we can use the quasi-compactness of X to conclude that the invariant

(115.23.11.1)

02T6
$$\epsilon(X, \{D_i\}_{i \in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d - 1, \{i, i'\} \in P(I)\}$$

is finite, since after all each D_i has at most finitely many irreducible components. We claim that for some n the invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ is zero. Namely, if not then by Lemma 115.23.8 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i \in I}) = \epsilon(X_0, \{D_{0,i}\}_{i \in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i \in I_1}) > \ldots$$

of positive integers which is a contradiction. Take n with invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ equal to zero. This means that there is no integral closed subscheme $Z \subset X_n$ and no pair of indices $i, i' \in I_n$ such that $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$. In other words, $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d - 2$ for all pairs $\{i, i'\} \in P(I_n)$ as desired.

Next, we come to the general case where we no longer assume that the scheme X is quasi-compact. The problem with the idea from the first part of the proof is that we may get and infinite sequence of blowups with centers dominating a fixed point of X. In order to avoid this we cut out suitable closed subsets of codimension ≥ 3 at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$
\begin{tikzcd}
X = X_0 & & & \\
U_0 \arrow[u, "j_0"] & X_1 \arrow[l, "b_0"'] & & \\
& U_1 \arrow[u, "j_1"] & X_2 \arrow[l, "b_1"'] & \\
& & U_2 \arrow[u, "j_2"] & X_3 \arrow[l, "b_2"']
\end{tikzcd}
$$

Each of the morphisms $j_n : U_n \to X_n$ will be an open immersion. Each of the morphisms $b_n : X_{n+1} \to U_n$ will be a proper birational morphism of integral schemes. As in the quasi-compact case we will have effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$ on X_n. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \amalg P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will arrange it so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We start the induction by setting $X_0 = X$, $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\})$ we construct the open subscheme U_n as follows. For each pair $\{i, i'\} \in P(I_n)$ consider the closed subscheme $D_{n,i} \cap D_{n,i'}$. This has "good" irreducible components which have δ-dimension $d-2$ and "bad" irreducible components which have δ-dimension $d - 1$. Let us set

$$\text{Bad}(i, i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-1} W$$

and similarly

$$\text{Good}(i, i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-2} W.$$

Then $D_{n,i} \cap D_{n,i'} = \text{Bad}(i, i') \cup \text{Good}(i, i')$ and moreover we have $\dim_\delta(\text{Bad}(i, i') \cap \text{Good}(i, i')) \leq d - 3$. Here is our choice of U_n:

$$U_n = X_n \setminus \bigcup\nolimits_{\{i,i'\} \in P(I_n)} \text{Bad}(i, i') \cap \text{Good}(i, i').$$

By our condition on triple intersections of the divisors $D_{n,i}$ we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \amalg G_{n,i,i'}$$

where $Z_{n,i,i'}$ is δ-equidimensional of dimension $d-1$ and $G_{n,i,i'}$ is δ-equidimensional of dimension $d-2$. (So topologically $Z_{n,i,i'}$ is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup\nolimits_{\{i,i'\}\in P(I_n)} Z_{n,i,i'} = \coprod\nolimits_{\{i,i'\}\in P(I_n)} Z_{n,i,i'},$$

and we let $b_n : X_{n+1} \to X_n$ be the blowup in Z_n. Note that Lemma 115.23.8 applies to the morphism $b_n : X_{n+1} \to X_n$ locally around each of the loci $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$. Hence, exactly as in the first part of the proof we obtain effective Cartier divisors $D_{n+1,\{i,i'\}}$ for $\{i,i'\} \in P(I_n)$ and effective Cartier divisors $D_{n+1,i}$ for $i \in I_n$ such that $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i'\in I_n, i'\neq i} D_{n+1,\{i,i'\}}$. For each n denote $\pi_n : X_n \to X$ the morphism obtained as the composition $j_0 \circ \ldots \circ j_{n-1} \circ b_{n-1}$.

Claim: given any quasi-compact open $V \subset X$ for all sufficiently large n the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \ldots$$

are all isomorphisms. Namely, if the map $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$ is not an isomorphism, then $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$ for some $\{i,i'\} \in P(I_n)$. Hence there exists an irreducible component $W \subset D_{n,i} \cap D_{n,i'}$ with $\dim_\delta(W) = d-1$. In particular we see that $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$. Applying Lemma 115.23.8 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with $\epsilon(V, \{D_i|_V\})$ as in (115.23.11.1). Since V is quasi-compact, we have $\epsilon(V, \{D_i|_V\}) < \infty$ and taking $n > \epsilon(V, \{D_i|_V\})$ we see the result.

Note that by construction the difference $X_n \setminus U_n$ has $\dim_\delta(X_n \setminus U_n) \leq d-3$. Let $T_n = \pi_n(X_n \setminus U_n)$ be its image in X. Traversing in the diagram of maps above using each b_n is closed it follows that $T_0 \cup \ldots \cup T_n$ is a closed subset of X for each n. Any $t \in T_n$ satisfies $\delta(t) \leq d-3$ by construction. Hence $\overline{T_n} \subset X$ is a closed subset with $\dim_\delta(T_n) \leq d-3$. By the claim above we see that for any quasi-compact open $V \subset X$ we have $T_n \cap V \neq \emptyset$ for at most finitely many n. Hence $\{\overline{T_n}\}_{n\geq 0}$ is a locally finite collection of closed subsets, and we may set $U = X \setminus \bigcup \overline{T_n}$. This will be U as in the lemma.

Note that $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$ by construction of U. Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of X. Hence we can define

$$U' = \lim\nolimits_n \pi_n^{-1}(U).$$

The induced morphism $b : U' \to U$ is proper since this is local on U, and over each compact open the limit stabilizes. Similarly we set $J = \bigcup_{n\geq 0} I_n$ using the inclusions $I_n \to I_{n+1}$ from the construction. For $j \in J$ choose an n_0 such that j corresponds to $i \in I_{n_0}$ and define $D'_j = \lim_{n\geq n_0} D_{n,i}$. Again this makes sense as locally over X the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact X. $\square$

115.24. Commutativity of intersecting divisors

0AYE The results of this section were originally used to provide an alternative proof of the lemmas of Chow Homology, Section 42.28 and a weak version of Chow Homology, Lemma 42.30.5.

02TC Lemma 115.24.1. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Let $\{i_j : D_j \to X\}_{j\in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j > 0$, $j \in J$. Set $D = \sum_{j\in J} n_j D_j$, and denote $i : D \to X$ the inclusion morphism. Let $\alpha \in Z_{k+1}(X)$. Then

$$p : \coprod\nolimits_{j\in J} D_j \longrightarrow D$$

is proper and

$$i^*\alpha = p_*\left(\sum n_j i_j^*\alpha\right)$$

in $\mathrm{CH}_k(D)$.

Proof. The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family D_j is finite and the result is altogether easy and a straightforward consequence of Chow Homology, Lemma 42.24.2 and Divisors, Lemma 31.27.5 and the definitions.

The morphism p is proper since the family $\{D_j\}_{j\in J}$ is locally finite. Write $\alpha = \sum_{a\in A} m_a[W_a]$ with $W_a \subset X$ an integral closed subscheme of δ-dimension $k + 1$. Denote $i_a : W_a \to X$ the closed immersion. We assume that $m_a \neq 0$ for all $a \in A$ such that $\{W_a\}_{a\in A}$ is locally finite on X.

Observe that by Chow Homology, Definition 42.29.1 the class $i^*\alpha$ is the class of a cycle $\sum m_a\beta_a$ for certain $\beta_a \in Z_k(W_a\cap D)$. Namely, if $W_a \not\subset D$ then $\beta_a = [D\cap W_a]_k$ and if $W_a \subset D$, then β_a is a cycle representing $c_1(\mathcal{O}_X(D)) \cap [W_a]$.

For each $a \in A$ write $J = J_{a,1} \amalg J_{a,2} \amalg J_{a,3}$ where

(1) $j \in J_{a,1}$ if and only if $W_a \cap D_j = \emptyset$,
(2) $j \in J_{a,2}$ if and only if $W_a \neq W_a \cap D_1 \neq \emptyset$, and
(3) $j \in J_{a,3}$ if and only if $W_a \subset D_j$.

Since the family $\{D_j\}$ is locally finite we see that $J_{a,3}$ is a finite set. For every $a \in A$ and $j \in J$ we choose a cycle $\beta_{a,j} \in Z_k(W_a \cap D_j)$ as follows

(1) if $j \in J_{a,1}$ we set $\beta_{a,j} = 0$,
(2) if $j \in J_{a,2}$ we set $\beta_{a,j} = [D_j \cap W_a]_k$, and
(3) if $j \in J_{a,3}$ we choose $\beta_{a,j} \in Z_k(W_a)$ representing $c_1(i_a^*\mathcal{O}_X(D_j)) \cap [W_j]$.

We claim that

$$\beta_a \sim_{rat} \sum\nolimits_{j\in J} n_j\beta_{a,j}$$

in $\mathrm{CH}_k(W_a \cap D)$.

Case I: $W_a \not\subset D$. In this case $J_{a,3} = \emptyset$. Thus it suffices to show that $[D \cap W_a]_k = \sum n_j[D_j \cap W_a]_k$ as cycles. This is Lemma 115.23.10.

Case II: $W_a \subset D$. In this case β_a is a cycle representing $c_1(i_a^*\mathcal{O}_X(D))\cap[W_a]$. Write $D = D_{a,1} + D_{a,2} + D_{a,3}$ with $D_{a,s} = \sum_{j\in J_{a,s}} n_jD_j$. By Divisors, Lemma 31.27.5

we have

$$\begin{aligned} c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^*\mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^*\mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^*\mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since $J_{a,3}$ is finite we see that the last term agrees with $\sum_{j \in J_{a,3}} n_j c_1(i_a^*\mathcal{L}_j) \cap [W_a]$, see Divisors, Lemma 31.27.5. This is represented by $\sum_{j \in J_{a,3}} n_j \beta_{a,j}$. Finally, by Case I we see that the middle term is represented by the cycle $\sum_{j \in J_{a,2}} n_j [D_j \cap W_a]_k = \sum_{j \in J_{a,2}} n_j \beta_{a,j}$. Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have $i^*D \sim_{rat} \sum m_a \beta_a$ by our choice of β_a. For each a we have $\beta_a \sim_{rat} \sum_j \beta_{a,j}$ with the rational equivalence taking place on $D \cap W_a$. Since the collection of closed subschemes $D \cap W_a$ is locally finite on D, we see that also $\sum m_a \beta_a \sim_{rat} \sum_{a,j} m_a \beta_{a,j}$ on D! (See Chow Homology, Remark 42.19.6.) Ok, and now it is clear that $\sum_a m_a \beta_{a,j}$ (viewed as a cycle on D_j) represents $i_j^*\alpha$ and hence $\sum_{a,j} m_a \beta_{a,j}$ represents $p_* \sum_j i_j^*\alpha$ and we win. □

02TD Lemma 115.24.2. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Assume $\dim_\delta(D \cap D') = n - 2$. Let $i : D \to X$, resp. $i' : D' \to X$ be the corresponding closed immersions. Then

(1) there exists a cycle $\alpha \in Z_{n-2}(D \cap D')$ whose pushforward to D represents $i^*[D']_{n-1} \in \mathrm{CH}_{n-2}(D)$ and whose pushforward to D' represents $(i')^*[D]_{n-1} \in \mathrm{CH}_{n-2}(D')$, and
(2) we have
$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$
in $\mathrm{CH}_{n-2}(X)$.

Proof. Part (2) is a trivial consequence of part (1). Let us write $[D]_{n-1} = \sum n_a [Z_a]$ and $[D']_{n-1} = \sum m_b [Z_b]$ with Z_a the irreducible components of D and $[Z_b]$ the irreducible components of D'. According to Chow Homology, Definition 42.29.1, we have $i^*D' = \sum m_b i^*[Z_b]$ and $(i')^*D = \sum n_a (i')^*[Z_a]$. By assumption, none of the irreducible components Z_b is contained in D, and hence $i^*[Z_b] = [Z_b \cap D]_{n-2}$ by definition. Similarly $(i')^*[Z_a] = [Z_a \cap D']_{n-2}$. Hence we are trying to prove the equality of cycles

$$\sum n_a [Z_a \cap D']_{n-2} = \sum m_b [Z_b \cap D]_{n-2}$$

which are indeed supported on $D \cap D'$. Let $W \subset X$ be an integral closed subscheme with $\dim_\delta(W) = n - 2$. Let $\xi \in W$ be its generic point. Set $R = \mathcal{O}_{X,\xi}$. It is a Noetherian local domain. Note that $\dim(R) = 2$. Let $f \in R$, resp. $f' \in R$ be an element defining the ideal of D, resp. D'. By assumption $\dim(R/(f, f')) = 0$. Let $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_t \subset R$ be the minimal primes over (f'), let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R$ be the minimal primes over (f). The equality above comes down to the equality

$$\sum_{i=1,\ldots,s} \text{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f)) \text{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1,\ldots,t} \text{length}_{R_{\mathfrak{q}'_j}}(R_{\mathfrak{q}'_j}/(f')) \text{ord}_{R/\mathfrak{q}'_j}(f).$$

By Chow Homology, Lemma 42.3.1 applied with $M = R/(f)$ the left hand side of this equation is equal to

$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \to R/(f)))$$

OK, and now we note that $\text{Ker}(f' : R/(f) \to R/(f))$ is canonically isomorphic to $((f) \cap (f'))/(ff')$ via the map $x \bmod (f) \mapsto f'x \bmod (ff')$. Hence the left hand side is

$$\text{length}_R(R/(f, f')) - \text{length}_R((f) \cap (f')/(ff'))$$

Since this is symmetric in f and f' we win. $\square$

02TE Lemma 115.24.3. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_j\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j, m_j \geq 0$ be collections of nonnegative integers. Set $D = \sum n_j D_j$ and $D' = \sum m_j D_j$. Assume that $\dim_\delta(D_j \cap D_{j'}) = n - 2$ for every $j \neq j'$. Then $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ in $\text{CH}_{n-2}(X)$.

Proof. This lemma is a trivial consequence of Lemmas 115.23.10 and 115.24.2 in case the sums are finite, e.g., if X is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let $i_j : D_j \to X$ be the closed immersions Let $p : \coprod D_j \to X$ denote coproduct of the morphisms i_j. Let $\{Z_a\}_{a \in A}$ be the collection of irreducible components of $\bigcup D_j$. For each j we write

$$[D_j]_{n-1} = \sum d_{j,a}[Z_a].$$

By Lemma 115.23.10 we have

$$[D]_{n-1} = \sum n_j d_{j,a}[Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a}[Z_a].$$

By Lemma 115.24.1 we have

$$D \cdot [D']_{n-1} = p_* \left(\sum n_j i_j^*[D']_{n-1}\right), \quad D' \cdot [D]_{n-1} = p_* \left(\sum m_{j'} i_{j'}^*[D]_{n-1}\right).$$

As in the definition of the Gysin homomorphisms (see Chow Homology, Definition 42.29.1) we choose cycles $\beta_{a,j}$ on $D_j \cap Z_a$ representing $i_j^*[Z_a]$. (Note that in fact $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$ if Z_a is not contained in D_j, i.e., there is no choice in that case.) Now since p is a closed immersion when restricted to each of the D_j we can (and we will) view $\beta_{a,j}$ as a cycle on X. Plugging in the formulas for $[D]_{n-1}$ and $[D']_{n-1}$ obtained above we see that

$$D \cdot [D']_{n-1} = \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 115.24.2 (see also its proof) says that for $j \neq j'$ the cycles $\sum d_{j',a}\beta_{a,j}$ and $\sum d_{j,a}\beta_{a,j'}$ are equal as cycles! Hence we see that

$$\begin{aligned} D \cdot [D']_{n-1} &= \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a}\beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j',a}\beta_{a,j}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a}\beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j,a}\beta_{a,j'}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a}\beta_{a,j} \\ &= \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a}\beta_{a,j'} \\ &= D' \cdot [D]_{n-1} \end{aligned}$$

and we win. □

02TF Lemma 115.24.4. Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Then

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $\mathrm{CH}_{n-2}(X)$.

First proof of Lemma 115.24.4. First, let us prove this in case X is quasi-compact. In this case, apply Lemma 115.23.11 to X and the two element set $\{D, D'\}$ of effective Cartier divisors. Thus we get a proper morphism $b : X' \to X$, a finite collection of effective Cartier divisors $D'_j \subset X'$ intersecting pairwise in codimension ≥ 2, with $b^{-1}(D) = \sum n_j D'_j$, and $b^{-1}(D') = \sum m_j D'_j$. Note that $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$ in $Z_{n-1}(X)$ and similarly for D', see Lemma 115.23.5. Hence, by Chow Homology, Lemma 42.26.4 we have

$$D \cdot [D']_{n-1} = b_*\left(b^{-1}(D) \cdot [b^{-1}(D')]_{n-1}\right)$$

in $\mathrm{CH}_{n-2}(X)$ and similarly for the other term. Hence the lemma follows from the equality $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$ in $\mathrm{CH}_{n-2}(X')$ of Lemma 115.24.3.

Note that in the proof above, each referenced lemma works also in the general case (when X is not assumed quasi-compact). The only minor change in the general case is that the morphism $b : U' \to U$ we get from applying Lemma 115.23.11 has as its target an open $U \subset X$ whose complement has codimension ≥ 3. Hence by Chow Homology, Lemma 42.19.3 we see that $\mathrm{CH}_{n-2}(U) = \mathrm{CH}_{n-2}(X)$ and after replacing X by U the rest of the proof goes through unchanged. □

Second proof of Lemma 115.24.4. Let $\mathcal{I} = \mathcal{O}_X(-D)$ and $\mathcal{I}' = \mathcal{O}_X(-D')$ be the invertible ideal sheaves of D and D'. We denote $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$ and $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$. We can restrict the inclusion map $\mathcal{I} \to \mathcal{O}_X$ to D' to get a map

$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$

and similarly

$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$

It is clear that

$$\mathrm{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \mathrm{Coker}(\psi)$$

and

$$\mathrm{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \mathrm{Ker}(\psi).$$

Hence we see that

$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$

in $K_0(\mathrm{Coh}_{\leq n-1}(X))$. On the other hand it is clear that

$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$

and that

$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Chow Homology, Lemma 42.69.7 (applied two times) this means that the element γ is an element of $B_{n-2}(X)$, and maps to both $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$ and to $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$ and we win (since the map $B_{n-2}(X) \to \mathrm{CH}_{n-2}(X)$ is well defined – which is the key to this proof). □

115.25. Dualizing modules on regular proper models

0C6D In Semistable Reduction, Situation 55.9.3 we let $\omega^\bullet_{X/R} = f^!\mathcal{O}_{\mathrm{Spec}(R)}$ be the relative dualizing complex of $f : X \to \mathrm{Spec}(R)$ as introduced in Duality for Schemes, Remark 48.12.5. Since f is Gorenstein of relative dimension 1 by Semistable Reduction, Lemma 55.9.2 we can use Duality for Schemes, Lemmas 48.25.10, 48.21.7, and 48.25.4 to see that

$$\omega^\bullet_{X/R} = \omega_X[1]$$

for some invertible $\mathcal{O}_X$-module ω_X. This invertible module is often called the relative dualizing module of X over R. Since R is regular (hence Gorenstein) of dimension 1 we see that $\omega_R^\bullet = R[1]$ is a normalized dualizing complex for R. Hence $\omega_X = H^{-2}(f^!\omega_R^\bullet)$ and we see that ω_X is not just a relative dualizing module but also a dualizing module, see Duality for Schemes, Example 48.22.1. Thus ω_X represents the functor

$$\mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Sets}, \quad \mathcal{F} \mapsto \mathrm{Hom}_R(H^1(X, \mathcal{F}), R)$$

by Duality for Schemes, Lemma 48.22.5. This gives an alternative definition of the relative dualizing module in Semistable Reduction, Situation 55.9.3. The formation of ω_X commutes with arbitrary base change (for any proper Gorenstein morphism of given relative dimension); this follows from the corresponding fact for the relative dualizing complex discussed in Duality for Schemes, Remark 48.12.5 which goes back to Duality for Schemes, Lemma 48.12.4. Thus ω_X pulls back to the dualizing module ω_C of C over K discussed in Algebraic Curves, Lemma 53.4.2. Note that ω_C is isomorphic to $\Omega_{C/K}$ by Algebraic Curves, Lemma 53.4.1. Similarly $\omega_X|_{X_k}$ is the dualizing module ω_{X_k} of X_k over k.

0C6E **Lemma 115.25.1.** *In Semistable Reduction, Situation 55.9.3 the dualizing module of C_i over k is*

$$\omega_{C_i} = \omega_X(C_i)|_{C_i}$$

where ω_X is as above.

Proof. Let $t : C_i \to X$ be the closed immersion. Since t is the inclusion of an effective Cartier divisor we conclude from Duality for Schemes, Lemmas 48.9.7 and

48.14.2 that we have $t^!(\mathcal{L}) = \mathcal{L}(C_i)|_{C_i}$ for every invertible $\mathcal{O}_X$-module $\mathcal{L}$. Consider the commutative diagram

$$\begin{tikzcd} C_i \arrow[r, "t"] \arrow[d, "g"] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & \mathrm{Spec}(R) \end{tikzcd}$$

Observe that C_i is a Gorenstein curve (Semistable Reduction, Lemma 55.9.2) with invertible dualizing module ω_{C_i} characterized by the property $\omega_{C_i}[0] = g^!\mathcal{O}_{\mathrm{Spec}(k)}$. See Algebraic Curves, Lemma 53.4.1, its proof, and Algebraic Curves, Lemmas 53.4.2 and 53.5.2. On the other hand, $s^!(R[1]) = k$ and hence

$$\omega_{C_i}[0] = g^!s^!(R[1]) = t^!f^!(R[1]) = t^!\omega_X$$

Combining the above we obtain the statement of the lemma. $\square$

115.26. Duplicate and split out references

09AQ This section is a place where we collect duplicates and references which used to say several things at the same time but are now split into their constituent parts.

05JR **Lemma 115.26.1.** Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.

Proof. This is a duplicate of Properties, Lemma 28.22.3. $\square$

03IF **Lemma 115.26.2.** Let S be a scheme. Let X be an algebraic space over S. The map $\{\mathrm{Spec}(k) \to X \text{ monomorphism}\} \to |X|$ is injective.

Proof. This is a duplicate of Properties of Spaces, Lemma 66.4.12. $\square$

03QZ **Theorem 115.26.3.** Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = \mathrm{Gal}_{\kappa(s)}$ denote the absolute Galois group. Then there is an equivalence of categories $Sh(S_{\acute{e}tale}) \to G\text{-Sets}$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$.

Proof. This is a duplicate of Étale Cohomology, Theorem 59.56.3. $\square$

06IF **Remark 115.26.4.** You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 90.12 for the actual content.

08E6 **Lemma 115.26.5.** Let X be a locally ringed space. A direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.

Proof. This is a duplicate of Modules, Lemma 17.14.6. $\square$

08XU **Lemma 115.26.6.** Let R be a ring. Let E be an R-module. The following are equivalent

(1) E is an injective R-module, and
(2) given an ideal $I \subset R$ and a module map $\varphi : I \to E$ there exists an extension of φ to an R-module map $R \to E$.

Proof. This is Baer's criterion, see Injectives, Lemma 19.2.6. $\square$

02PI **Lemma 115.26.7.** Let R be a local ring.

(1) If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = \text{length}_R(M) - \text{length}_R(N)$.
(2) If (M, φ, ψ) is a $(2, 1)$-periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.
(3) Suppose that we have a short exact sequence of 2-periodic complexes

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. This follows from Chow Homology, Lemmas 42.2.3 and 42.2.4. □

08S8 Lemma 115.26.8. Let A be a ring and let I be an A-module.
(1) The set of extensions of rings $0 \to I \to A' \to A \to 0$ where I is an ideal of square zero is canonically bijective to $\text{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$.
(2) Given a ring map $A \to B$, a B-module N, an A-module map $c : I \to N$, and given extensions of rings with square zero kernels:
   (a) $0 \to I \to A' \to A \to 0$ corresponding to $\alpha \in \text{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$, and
   (b) $0 \to N \to B' \to B \to 0$ corresponding to $\beta \in \text{Ext}^1_B(NL_{B/\mathbf{Z}}, N)$
   then there is a map $A' \to B'$ fitting into Deformation Theory, Equation (91.2.0.1) if and only if β and α map to the same element of $\text{Ext}^1_A(NL_{A/\mathbf{Z}}, N)$.

Proof. This follows from Deformation Theory, Lemmas 91.2.3 and 91.2.5. □

08UD Lemma 115.26.9. Let $(S, \mathcal{O}_S)$ be a ringed space and let $\mathcal{J}$ be an $\mathcal{O}_S$-module.
(1) The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $\text{Ext}^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$.
(2) Given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, an f-map $c : \mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:
   (a) $0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ corresponding to $\alpha \in \text{Ext}^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$,
   (b) $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ corresponding to $\beta \in \text{Ext}^1_{\mathcal{O}_X}(NL_{X/\mathbf{Z}}, \mathcal{G})$
   then there is a morphism $X' \to S'$ fitting into Deformation Theory, Equation (91.7.0.1) if and only if β and α map to the same element of $\text{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G})$.

Proof. This follows from Deformation Theory, Lemmas 91.7.4 and 91.7.6. □

08UL Lemma 115.26.10. Let $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a ringed topos and let $\mathcal{J}$ be an $\mathcal{O}_\mathcal{B}$-module.
(1) The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $\text{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$.
(2) Given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, an $\mathcal{O}$-module $\mathcal{G}$, an $f^{-1}\mathcal{O}_\mathcal{B}$-module map $c : f^{-1}\mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:
   (a) $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ corresponding to $\alpha \in \text{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$,
   (b) $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ corresponding to $\beta \in \text{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathbf{Z}}, \mathcal{G})$
   then there is a morphism $(Sh(\mathcal{C}), \mathcal{O}') \to (Sh(\mathcal{B}, \mathcal{O}_{\mathcal{B}'})$ fitting into Deformation Theory, Equation (91.13.0.1) if and only if β and α map to the same element of $\text{Ext}^1_{\mathcal{O}}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{G})$.

Proof. This follows from Deformation Theory, Lemmas 91.13.4 and 91.13.6. $\square$

0D3H Remark 115.26.11. This tag used to point to a section describing several examples of deformation problems. These now each have their own section. See Deformation Problems, Sections 93.4, 93.5, 93.6, and 93.7.

0D3M Lemma 115.26.12. Deformation Problems, Examples 93.4.1, 93.5.1, 93.6.1, and 93.7.1 satisfy the Rim-Schlessinger condition (RS).

Proof. This follows from Deformation Problems, Lemmas 93.4.2, 93.5.2, 93.6.2, and 93.7.2. $\square$

0D3N Lemma 115.26.13. We have the following canonical k-vector space identifications:

(1) In Deformation Problems, Example 93.4.1 if $x_0 = (k, V)$, then $T_{x_0}\mathcal{F} = (0)$ and $\text{Inf}_{x_0}(\mathcal{F}) = \text{End}_k(V)$ are finite dimensional.
(2) In Deformation Problems, Example 93.5.1 if $x_0 = (k, V, \rho_0)$, then $T_{x_0}\mathcal{F} = \text{Ext}^1_{k[\Gamma]}(V, V) = H^1(\Gamma, \text{End}_k(V))$ and $\text{Inf}_{x_0}(\mathcal{F}) = H^0(\Gamma, \text{End}_k(V))$ are finite dimensional if Γ is finitely generated.
(3) In Deformation Problems, Example 93.6.1 if $x_0 = (k, V, \rho_0)$, then $T_{x_0}\mathcal{F} = H^1_{cont}(\Gamma, \text{End}_k(V))$ and $\text{Inf}_{x_0}(\mathcal{F}) = H^0_{cont}(\Gamma, \text{End}_k(V))$ are finite dimensional if Γ is topologically finitely generated.
(4) In Deformation Problems, Example 93.7.1 if $x_0 = (k, P)$, then $T_{x_0}\mathcal{F}$ and $\text{Inf}_{x_0}(\mathcal{F}) = \text{Der}_k(P, P)$ are finite dimensional if P is finitely generated over k.

Proof. This follows from Deformation Problems, Lemmas 93.4.3, 93.5.3, 93.6.3, and 93.7.3. $\square$

115.27. Other chapters

CHAPTER 116

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Version 1.2, November 2002

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CHAPTER 117

Auto generated index

117.1. Alphabetized definitions

117.2. Definitions listed per chapter

In 4.30.1: $(2, 1)$-category
In 4.31.1: final object
In 4.31.2: 2-fibre product of f and g
In 4.32.1: 2-category of categories over $\mathcal{C}$
In 4.32.2: fibre category, lift, x lies over U, lift, ϕ lies over f
In 4.33.1: strongly cartesian morphism, strongly $\mathcal{C}$-cartesian morphism
In 4.33.5: fibred category over $\mathcal{C}$
In 4.33.6: choice of pullbacks, pullback functor
In 4.33.9: 2-category of fibred categories over $\mathcal{C}$
In 4.34.2: relative inertia of $\mathcal{S}$ over $\mathcal{S}'$, inertia fibred category $\mathcal{I}_\mathcal{S}$ of $\mathcal{S}$
In 4.35.1: fibred in groupoids
In 4.35.6: 2-category of categories fibred in groupoids over $\mathcal{C}$
In 4.36.2: split fibred category, $\mathcal{S}_F$
In 4.37.2: split category fibred in groupoids, $\mathcal{S}_F$
In 4.38.1: discrete
In 4.38.2: category fibred in sets, category fibred in discrete categories
In 4.38.3: 2-category of categories fibred in sets over $\mathcal{C}$
In 4.39.1: setoid
In 4.39.2: category fibred in setoids
In 4.39.3: 2-category of categories fibred in setoids over $\mathcal{C}$
In 4.40.1: representable
In 4.42.3: representable, $\mathcal{X}$ is relatively representable over $\mathcal{Y}$
In 4.43.1: monoidal category
In 4.43.2: functor of monoidal categories
In 4.43.4: invertible
In 4.43.5: left dual, right dual
In 4.43.9: symmetric monoidal category
In 4.43.11: functor of symmetric monoidal categories
In 4.44.1: morphism of dotted arrows

Topology

In 5.4.1: separated
In 5.5.1: base for the topology on X, basis for the topology on X
In 5.5.4: subbase for the topology on X, subbasis for the topology on X
In 5.6.3: strict map of topological spaces, submersive
In 5.7.1: connected, connected component
In 5.7.8: totally disconnected
In 5.7.10: locally connected
In 5.8.1: irreducible, irreducible component
In 5.8.6: generic point, Kolmogorov, quasi-sober, sober
In 5.9.1: Noetherian, locally Noetherian
In 5.10.1: chain of irreducible closed subsets, length, dimension, Krull dimension, Krull dimension of X at x
In 5.10.5: equidimensional
In 5.11.1: codimension
In 5.11.4: catenary
In 5.12.1: quasi-compact, quasi-compact, retrocompact
In 5.13.1: locally quasi-compact
In 5.15.1: constructible, locally constructible
In 5.17.2: closed, Bourbaki-proper, quasi-proper, universally closed, proper
In 5.18.1: Jacobson
In 5.19.1: specialization, generalization, stable under specialization, stable under generalization
In 5.19.4: specializations lift along f, specializing, generalizations lift along f, generalizing
In 5.20.1: immediate specialization, dimension function
In 5.21.1: interior, nowhere dense
In 5.22.1: profinite
In 5.23.1: spectral, spectral
In 5.26.1: extremally disconnected
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In 5.28.1: partition, parts, refines
In 5.28.2: good stratification
In 5.28.3: stratification, strata
In 5.28.4: locally finite
In 5.30.1: topological group, homomorphism of topological groups
In 5.30.5: profinite group
In 5.30.7: topological ring, homomorphism of topological rings

In 5.30.10: topological module, homomorphism of topological modules

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In 6.3.1: presheaf $\mathcal{F}$ of sets on X, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X
In 6.3.2: constant presheaf with value A
In 6.4.4: presheaf of abelian groups on X, abelian presheaf over X, morphism of abelian presheaves over X
In 6.5.1: presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$
In 6.5.2: underlying presheaf of sets of $\mathcal{F}$
In 6.6.1: presheaf of $\mathcal{O}$-modules, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules
In 6.7.1: sheaf $\mathcal{F}$ of sets on X, morphism of sheaves of sets
In 6.7.4: constant sheaf with value A
In 6.8.1: abelian sheaf on X, sheaf of abelian groups on X
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In 6.10.1: sheaf of $\mathcal{O}$-modules, morphism of sheaves of $\mathcal{O}$-modules
In 6.11.2: separated
In 6.15.1: type of algebraic structure
In 6.16.2: subpresheaf, subsheaf, injective, surjective, injective, surjective
In 6.21.7: f-map $\xi : \mathcal{G} \to \mathcal{F}$
In 6.21.9: composition of φ and ψ
In 6.25.1: ringed space, morphism of ringed spaces
In 6.25.3: composition of morphisms of ringed spaces
In 6.26.1: pushforward, pullback
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In 6.30.1: presheaf $\mathcal{F}$ of sets on $\mathcal{B}$, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$
In 6.30.2: sheaf $\mathcal{F}$ of sets on $\mathcal{B}$, morphism of sheaves of sets on $\mathcal{B}$
In 6.30.8: presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$, sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$
In 6.30.11: presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$, sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$
In 6.31.2: restriction of $\mathcal{G}$ to U, restriction of $\mathcal{G}$ to U, open subspace of $(X, \mathcal{O})$ associated to U, restriction of $\mathcal{G}$ to U
In 6.31.3: extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$, extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$
In 6.31.5: extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by 0, extension $j_!\mathcal{F}$ of $\mathcal{F}$ by 0, extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e, extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e, extension by 0, extension by 0

Sites and Sheaves

In 7.2.1: presheaf of sets, Morphisms of presheaves
In 7.2.2: presheaf, morphism
In 7.3.1: injective, surjective
In 7.3.3: subpresheaf
In 7.3.5: image of φ
In 7.6.1: family of morphisms with fixed target
In 7.6.2: site, coverings of $\mathcal{C}$
In 7.7.1: sheaf
In 7.7.5: $Sh(\mathcal{C})$
In 7.7.6: sheaf
In 7.8.1: morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$, morphism from $\mathcal{U}$ to $\mathcal{V}$, refinement
In 7.8.2: combinatorially equivalent, tautologically equivalent
In 7.10.9: separated
In 7.10.11: sheaf associated to $\mathcal{F}$
In 7.11.1: injective, surjective
In 7.12.1: effective epimorphism, universal effective epimorphism
In 7.12.2: weaker than the canonical topology, subcanonical
In 7.12.3: representable sheaves, $\underline{U}$
In 7.13.1: continuous
In 7.14.1: morphism of sites
In 7.14.5: composition
In 7.15.1: topos, morphism of topoi, composition $f \circ g$
In 7.17.1: quasi-compact
In 7.17.4: quasi-compact, quasi-compact
In 7.20.1: cocontinuous

In 7.25.1: localization of the site $\mathcal{C}$ at the object U, localization morphism, direct image functor, restriction of $\mathcal{F}$ to $\mathcal{C}/U$, extension of $\mathcal{G}$ by the empty set
In 7.29.2: special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$
In 7.30.4: localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$, localization morphism
In 7.32.1: point of the topos $Sh(\mathcal{C})$
In 7.32.2: point p of the site $\mathcal{C}$
In 7.32.6: skyscraper sheaf
In 7.36.1: 2-morphism from f to g
In 7.37.2: morphism $f : p \to p'$
In 7.38.1: conservative, has enough points
In 7.40.2: weakly contractible, enough weakly contractible objects, enough P objects
In 7.42.1: sheaf theoretically empty
In 7.42.3: almost cocontinuous
In 7.43.1: embedding
In 7.43.2: subtopos
In 7.43.4: open subtopos
In 7.43.6: closed subtopos
In 7.43.7: open immersion, closed immersion
In 7.44.1: pushforward
In 7.45.1: global sections
In 7.47.1: sieve S on U
In 7.47.3: sieve on U generated by the morphisms f_i
In 7.47.4: pullback of S by f
In 7.47.6: topology on $\mathcal{C}$
In 7.47.8: finer, stronger, coarser, weaker
In 7.47.10: sheaf
In 7.47.12: canonical topology
In 7.48.2: topology associated to $\mathcal{C}$
In 7.49.2: separated
In 7.49.4: sheaf associated to $\mathcal{F}$
In 7.52.1: point p

Stacks

In 8.2.2: presheaf of morphisms from x to y, presheaf of isomorphisms from x to y
In 8.3.1: descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$, cocycle condition, morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data
In 8.3.4: pullback functor
In 8.3.5: trivial descent datum, canonical descent datum, effective
In 8.4.1: stack
In 8.4.5: 2-category of stacks over $\mathcal{C}$
In 8.5.1: stack in groupoids
In 8.5.5: 2-category of stacks in groupoids over $\mathcal{C}$
In 8.6.1: stack in setoids, stack in sets, stack in discrete categories
In 8.6.5: 2-category of stacks in setoids over $\mathcal{C}$
In 8.10.2: structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$, $\mathcal{S}$ is endowed with the topology inherited from $\mathcal{C}$
In 8.11.1: gerbe
In 8.11.4: gerbe over
In 8.12.4: $f_*\mathcal{S}$, pushforward of $\mathcal{S}$ along f
In 8.12.9: $f^{-1}\mathcal{S}$, pullback of $\mathcal{S}$ along f

Fields

In 9.2.1: field, subfield
In 9.2.2: domain, integral domain
In 9.5.1: characteristic, prime subfield of F
In 9.6.2: field extension
In 9.6.3: tower
In 9.6.6: generates the field extension, finitely generated field extension
In 9.7.1: degree, finite
In 9.7.8: number field
In 9.8.1: algebraic, algebraic extension
In 9.9.1: minimal polynomial
In 9.10.1: algebraically closed
In 9.10.3: algebraic closure
In 9.11.1: relatively prime
In 9.12.2: separable, separable, separable
In 9.12.6: separable degree
In 9.14.1: purely inseparable, purely inseparable
In 9.14.7: separable degree, inseparable degree, degree of inseparability
In 9.15.1: normal
In 9.15.8: automorphisms of E over F, automorphisms of E/F
In 9.16.2: splitting field of P over F
In 9.16.4: normal closure E over F
In 9.20.1: trace, norm
In 9.20.6: trace pairing

Commutative Algebra

In 10.70.1: blowup algebra, Rees algebra, affine blowup algebra
In 10.71.2: resolution, resolution of M by free R-modules, resolution of M by finite free R-modules
In 10.72.1: I-depth, depth
In 10.77.1: projective
In 10.78.1: locally free, finite locally free, finite locally free of rank r
In 10.82.1: universally injective, universally exact
In 10.84.1: direct sum dévissage, Kaplansky dévissage
In 10.86.1: Mittag-Leffler
In 10.88.1: Mittag-Leffler directed system of modules
In 10.88.2: dominates
In 10.88.7: Mittag-Leffler
In 10.90.1: coherent module, coherent ring
In 10.96.2: I-adically complete, I-adically complete
In 10.102.5: rank
In 10.103.1: Cohen-Macaulay
In 10.103.8: maximal Cohen-Macaulay
In 10.103.12: Cohen-Macaulay
In 10.104.1: Cohen-Macaulay
In 10.104.6: Cohen-Macaulay
In 10.105.1: catenary
In 10.105.3: universally catenary
In 10.108.1: pure
In 10.109.2: finite projective dimension, projective dimension
In 10.109.10: finite global dimension, global dimension
In 10.110.7: regular
In 10.112.5: local ring of the fibre at $\mathfrak{q}$
In 10.119.8: uniformizer
In 10.120.1: associates, irreducible, prime
In 10.120.4: unique factorization domain, UFD
In 10.120.12: principal ideal domain, PID
In 10.120.14: Dedekind domain
In 10.121.2: order of vanishing along R
In 10.121.3: lattice in V
In 10.121.5: distance between M and M'
In 10.122.3: quasi-finite at $\mathfrak{q}$, quasi-finite
In 10.123.7: strongly transcendental over R
In 10.125.1: relative dimension of S/R at $\mathfrak{q}$, relative dimension of
In 10.131.1: derivation, R-derivation, Leibniz rule
In 10.131.2: module of Kähler differentials, module of differentials
In 10.133.1: differential operator $D : M \to N$ of order k
In 10.133.4: module of principal parts of order k
In 10.134.1: naive cotangent complex
In 10.135.1: global complete intersection over k, local complete intersection over k
In 10.135.5: complete intersection (over k)
In 10.136.1: syntomic, flat local complete intersection over R
In 10.136.5: relative global complete intersection
In 10.137.1: smooth
In 10.137.6: standard smooth algebra over R
In 10.137.11: smooth at $\mathfrak{q}$
In 10.138.1: formally smooth over R
In 10.141.1: small extension
In 10.143.1: étale, étale at $\mathfrak{q}$
In 10.144.1: standard étale
In 10.148.1: formally unramified over R
In 10.149.2: universal first order thickening, conormal module, $C_{S/R}$
In 10.150.1: formally étale over R
In 10.151.1: unramified, G-unramified, unramified at $\mathfrak{q}$, G-unramified at $\mathfrak{q}$
In 10.153.1: henselian, strictly henselian
In 10.155.3: henselization, strict henselization of R with respect to $\kappa \subset \kappa^{sep}$, strict henselization
In 10.157.1: (R_k), regular in codimension $\leq k$, (S_k)
In 10.160.1: complete local ring
In 10.160.4: coefficient ring
In 10.160.5: Cohen ring
In 10.161.1: N-1, N-2, Japanese

In 10.162.1: universally Japanese, Nagata ring
In 10.162.9: analytically unramified, analytically unramified
In 10.165.2: geometrically normal
In 10.166.2: geometrically regular

Brauer groups

In 11.2.1: finite
In 11.2.2: skew field
In 11.2.3: simple, simple
In 11.2.4: central
In 11.2.5: opposite algebra
In 11.5.2: Brauer group
In 11.8.1: splits, splitting field

Homological Algebra

In 12.3.1: preadditive, additive
In 12.3.3: zero object
In 12.3.5: direct sum
In 12.3.8: additive
In 12.3.9: kernel, cokernel, coimage of f, image of f
In 12.4.1: Karoubian
In 12.5.1: abelian
In 12.5.3: injective, surjective, subobject, quotient
In 12.5.7: complex, exact at y, exact at x_i, exact, exact sequence, exact complex, short exact sequence
In 12.5.9: split
In 12.6.1: extension E of B by A, morphism of extensions
In 12.6.2: Ext-group
In 12.9.1: simple
In 12.9.2: Artinian, Artinian
In 12.9.3: Noetherian, Noetherian
In 12.10.1: Serre subcategory, weak Serre subcategory
In 12.10.5: kernel of the functor F
In 12.11.1: zeroth K-group of $\mathcal{A}$
In 12.12.1: cohomological δ-functor, δ-functor
In 12.12.2: morphism of δ-functors from F to G
In 12.12.3: universal δ-functor
In 12.13.2: homotopy equivalence, homotopy equivalent
In 12.13.4: quasi-isomorphism, acyclic
In 12.13.8: homotopy equivalence, homotopy equivalent
In 12.13.10: quasi-isomorphism, acyclic
In 12.14.1: k-shifted chain complex $A[k]_\bullet$
In 12.14.2: $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$
In 12.14.7: k-shifted cochain complex $A[k]^\bullet$
In 12.14.8: $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$
In 12.16.1: category of graded objects of $\mathcal{A}$
In 12.16.4: shift
In 12.17.1: additive monoidal category
In 12.18.1: double complex
In 12.18.3: associated simple complex, associated total complex
In 12.19.1: decreasing filtration, filtered object of $\mathcal{A}$, morphism $(A, F) \to (B, F)$ of filtered objects, induced filtration, quotient filtration, finite, separated, exhaustive
In 12.19.3: strict
In 12.20.1: spectral sequence in $\mathcal{A}$, morphism of spectral sequences
In 12.20.2: limit, degenerates at E_r
In 12.21.1: exact couple, morphism of exact couples
In 12.21.3: spectral sequence associated to the exact couple
In 12.22.1: differential object, morphism of differential objects
In 12.22.3: homology
In 12.22.5: spectral sequence associated to (A, d, α)
In 12.23.1: filtered differential object
In 12.23.4: induced filtration
In 12.23.6: weakly converges to $H(K)$, abuts to $H(K)$
In 12.24.1: filtered complex $K^\bullet$ of $\mathcal{A}$
In 12.24.5: induced filtration
In 12.24.7: regular, coregular, bounded, bounded below, bounded above
In 12.24.9: weakly converges to $H^*(K^\bullet)$, abuts to $H^*(K^\bullet)$, converges to $H^*(K^\bullet)$
In 12.25.2: weakly converges to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, abuts to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, converges to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, weakly converges to $H^n(\text{Tot}(K^{\bullet,\bullet}))$,

abuts to $H^n(\mathrm{Tot}(K^{\bullet,\bullet}))$, converges to $H^n(\mathrm{Tot}(K^{\bullet,\bullet}))$
In 12.27.1: injective
In 12.27.4: enough injectives
In 12.27.5: functorial injective embeddings
In 12.28.1: projective
In 12.28.4: enough projectives
In 12.28.5: functorial projective surjections
In 12.31.2: Mittag-Leffler condition, ML

Derived Categories

In 13.3.1: triangle, morphism of triangles
In 13.3.2: triangulated category, distinguished triangles, pre-triangulated category
In 13.3.3: exact functor, triangulated functor
In 13.3.4: pre-triangulated subcategory, triangulated subcategory
In 13.3.5: homological, cohomological
In 13.3.6: δ-functor from $\mathcal{A}$ to $\mathcal{D}$, image of the short exact sequence under the given δ-functor
In 13.5.1: compatible with the triangulated structure
In 13.6.1: saturated
In 13.6.5: kernel of F, kernel of H
In 13.6.7: quotient category $\mathcal{D}/\mathcal{B}$, quotient functor
In 13.8.1: category of (cochain) complexes, bounded below, bounded above, bounded
In 13.9.1: cone
In 13.9.4: termwise split injection $\alpha : A^\bullet \to B^\bullet$, termwise split surjection $\beta : B^\bullet \to C^\bullet$
In 13.9.9: termwise split exact sequence of complexes of $\mathcal{A}$, triangle associated to the termwise split sequence of complexes
In 13.10.1: distinguished triangle of $K(\mathcal{A})$
In 13.11.3: derived category of $\mathcal{A}$, bounded derived category
In 13.13.1: category of finite filtered objects of $\mathcal{A}$
In 13.13.2: filtered quasi-isomorphism, filtered acyclic
In 13.13.5: filtered derived category of $\mathcal{A}$
In 13.13.7: bounded filtered derived category
In 13.14.2: right derived functor RF is defined at, value of RF at X, left derived functor LF is defined at, value of LF at X
In 13.14.9: right derivable, everywhere defined, left derivable, everywhere defined
In 13.14.10: computes, computes
In 13.15.3: right derived functors of F, left derived functors of F, right acyclic for F, acyclic for RF, left acyclic for F, acyclic for LF
In 13.16.2: ith right derived functor R^iF of F
In 13.18.1: injective resolution of A, injective resolution of $K^\bullet$
In 13.19.1: projective resolution of A, projective resolution of $K^\bullet$
In 13.21.1: Cartan-Eilenberg resolution
In 13.23.2: resolution functor
In 13.26.1: filtered injective
In 13.27.1: ith extension group
In 13.27.4: Yoneda extension, equivalent
In 13.28.1: zeroth K-group of $\mathcal{D}$
In 13.31.1: K-injective
In 13.33.1: derived colimit, homotopy colimit
In 13.34.1: derived limit, homotopy limit
In 13.36.3: classical generator, strong generator, weak generator, generator
In 13.37.1: compact object
In 13.37.5: compactly generated
In 13.40.1: right orthogonal, left orthogonal
In 13.40.9: right admissible, left admissible, two-sided admissible
In 13.41.1: Postnikov system, morphism of Postnikov systems

Simplicial Methods

In 14.2.1: $\delta^n_j : [n-1] \to [n]$, $\sigma^n_j : [n+1] \to [n]$
In 14.3.1: simplicial object U of $\mathcal{C}$, simplicial set, simplicial abelian group, morphism of simplicial objects $U \to U'$, category of simplicial objects of $\mathcal{C}$

In 14.5.1: cosimplicial object U of $\mathcal{C}$, cosimplicial set, cosimplicial abelian group, morphism of cosimplicial objects $U \to U'$, category of cosimplicial objects of $\mathcal{C}$
In 14.6.1: product of U and V
In 14.7.1: fibre product of V and W over U
In 14.8.1: pushout of V and W over U
In 14.9.1: product of U and V
In 14.10.1: fibre product of V and W over U
In 14.11.1: n-simplex of U, face of x, degeneracy of x, degenerate
In 14.12.1: n-truncated simplicial object of $\mathcal{C}$, morphism of n-truncated simplicial objects
In 14.13.1: product $U \times V$ of U and V, product $U \times V$ exists
In 14.14.1: $\text{Hom}(U, V)$
In 14.15.1: $\text{Hom}(U, V)$
In 14.17.1: $\text{Hom}(U, V)$
In 14.18.1: split, split
In 14.20.1: augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$
In 14.22.3: Eilenberg-Maclane object $K(A, k)$
In 14.26.1: homotopy from a to b, homotopic, in the same homotopy class
In 14.26.6: homotopy equivalence, homotopy equivalent
In 14.28.1: homotopy from a to b, homotopic, in the same homotopy class
In 14.30.1: trivial Kan fibration
In 14.31.1: Kan fibration, Kan complex

More on Algebra

In 15.3.1: stably isomorphic, stably free
In 15.8.3: kth Fitting ideal
In 15.10.1: Zariski pair
In 15.11.1: henselian pair
In 15.14.1: absolutely integrally closed
In 15.15.1: auto-associated
In 15.22.1: torsion, torsion free
In 15.23.1: reflexive
In 15.23.9: reflexive hull
In 15.24.1: content ideal of x
In 15.26.1: strict transform of M along $R \to R[\frac{I}{a}]$
In 15.28.1: Koszul complex
In 15.28.2: Koszul complex on $f_1, \ldots, f_r$
In 15.30.1: M-Koszul-regular, M-H_1-regular, Koszul-regular, H_1-regular
In 15.32.1: regular ideal, Koszul-regular ideal, H_1-regular ideal, quasi-regular ideal
In 15.33.2: local complete intersection
In 15.36.1: topological ring, topological module, homomorphism of topological modules, homomorphism of topological rings, linearly topologized, linearly topologized, ideal of definition, pre-admissible, admissible, pre-adic, adic
In 15.37.1: formally smooth over R
In 15.37.3: formally smooth for the $\mathfrak{n}$-adic topology
In 15.41.1: regular
In 15.46.1: p-independent over k, p-basis of K over k
In 15.47.1: J-0, J-1, J-2
In 15.50.1: G-ring
In 15.52.1: quasi-excellent, excellent
In 15.55.1: injective
In 15.55.5: $M \mapsto M^\vee$, free module
In 15.59.1: K-flat
In 15.59.13: derived tensor product
In 15.61.1: Tor independent over R
In 15.64.1: m-pseudo-coherent, pseudo-coherent, m-pseudo-coherent, pseudo-coherent
In 15.66.1: tor-amplitude in $[a, b]$, finite tor dimension, tor dimension $\leq d$, finite tor dimension
In 15.68.1: finite projective dimension, projective-amplitude in $[a, b]$
In 15.69.1: finite injective dimension, injective-amplitude in $[a, b]$
In 15.70.4: I-projective
In 15.74.1: perfect, perfect
In 15.80.2: finitely presented relative to R
In 15.81.4: m-pseudo-coherent relative to R, pseudo-coherent relative to R, m-pseudo-coherent relative to R, pseudo-coherent relative to R
In 15.82.1: pseudo-coherent ring map, perfect ring map

In 15.83.1: R-perfect, perfect relative to R
In 15.88.1: I-power torsion module, an f-power torsion module
In 15.91.4: derived complete with respect to I, derived complete with respect to I
In 15.104.1: absolutely flat, weakly étale, absolutely flat
In 15.104.3: weak dimension $\leq d$
In 15.106.1: unibranch, geometrically unibranch
In 15.106.6: number of branches of A, number of geometric branches of A
In 15.109.1: formally catenary
In 15.111.1: extension of discrete valuation rings, ramification index, weakly unramified, residual degree, residue degree
In 15.111.7: unramified with respect to A, tamely ramified with respect to A, totally ramified with respect to A
In 15.112.3: decomposition group of $\mathfrak{m}$, inertia group of $\mathfrak{m}$
In 15.112.6: wild inertia group of $\mathfrak{m}$, tame inertia group of $\mathfrak{m}$
In 15.113.3: mixed characteristic, absolute ramification index
In 15.115.1: weak solution for $A \subset B$, solution for $A \subset B$, separable solution
In 15.117.1: invertible, trivial
In 15.123.1: extension of valuation rings, weakly unramified, residual degree, residue degree
In 15.124.5: Bézout domain, elementary divisor domain

Smoothing Ring Maps

In 16.2.1: singular ideal of A over R
In 16.2.3: elementary standard in A over R, strictly standard in A over R

Sheaves of Modules

In 17.4.1: generated by global sections, generate
In 17.4.5: subsheaf generated by the s_i
In 17.5.1: support of $\mathcal{F}$, support of s
In 17.8.1: locally generated by sections
In 17.9.1: finite type
In 17.10.1: quasi-coherent sheaf of $\mathcal{O}_X$-modules
In 17.10.6: sheaf associated to the module M and the ring map α, sheaf associated to the module M
In 17.11.1: finite presentation
In 17.12.1: coherent $\mathcal{O}_X$-module
In 17.13.1: closed immersion of ringed spaces
In 17.14.1: locally free, finite locally free, finite locally free of rank r
In 17.17.1: flat
In 17.17.3: flat at x
In 17.20.1: flat at x, flat
In 17.20.3: flat over Y at a point $x \in X$, flat over Y
In 17.23.1: annihilator
In 17.24.1: Koszul complex
In 17.24.2: Koszul complex on $f_1, \ldots, f_r$
In 17.25.1: invertible $\mathcal{O}_X$-module, trivial
In 17.25.6: tensor power
In 17.25.7: associated graded ring
In 17.25.9: Picard group
In 17.28.1: $\mathcal{O}_1$-derivation, φ-derivation, Leibniz rule
In 17.28.3: module of differentials, universal φ-derivation
In 17.28.10: S-derivation, sheaf of differentials $\Omega_{X/S}$ of X over S
In 17.29.1: differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k
In 17.29.4: module of principal parts of order k
In 17.29.8: differential operator of order k on X/S
In 17.30.1: de Rham complex of $\mathcal{B}$ over $\mathcal{A}$
In 17.30.4: de Rham complex
In 17.31.1: naive cotangent complex
In 17.31.6: naive cotangent complex

Modules on Sites

In 18.4.1: free abelian presheaf
In 18.5.1: free abelian sheaf
In 18.6.1: ringed site, structure sheaf, morphism of ringed sites, composition of morphisms of ringed sites

In 18.7.1: ringed topos, structure sheaf, morphism of ringed topoi, composition of morphisms of ringed topoi
In 18.8.1: 2-morphism from f to g
In 18.9.1: presheaf of $\mathcal{O}$-modules, morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules
In 18.10.1: sheaf of $\mathcal{O}$-modules, morphism of sheaves of $\mathcal{O}$-modules
In 18.13.1: pushforward, pullback
In 18.16.1: $g_{p!}\mathcal{F}$, $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$
In 18.17.1: free $\mathcal{O}$-module, finite free, generated by global sections, generated by r global sections, generated by finitely many global sections, global presentation, global finite presentation
In 18.19.1: localization of the ringed site $(\mathcal{C}, \mathcal{O})$ at the object U, localization morphism, direct image functor, restriction of $\mathcal{F}$ to $\mathcal{C}/U$, extension by zero
In 18.21.2: localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$, localization morphism
In 18.23.1: locally free, finite locally free, locally generated by sections, locally generated by r sections, of finite type, quasi-coherent, of finite presentation, coherent
In 18.28.1: flat, flat, flat, flat
In 18.31.1: flat, flat
In 18.31.3: flat over $(Sh(\mathcal{D}), \mathcal{O}')$
In 18.32.1: rank r, invertible, $\mathcal{O}^*$
In 18.32.6: Picard group
In 18.33.1: $\mathcal{O}_1$-derivation, φ-derivation, Leibniz rule
In 18.33.3: module of differentials, universal φ-derivation
In 18.33.10: Y-derivation, sheaf of differentials $\Omega_{X/Y}$ of X over Y, universal Y-derivation
In 18.34.1: differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k
In 18.34.4: module of principal parts of order k
In 18.35.1: naive cotangent complex
In 18.35.4: naive cotangent complex
In 18.40.4: locally ringed site
In 18.40.6: locally ringed
In 18.40.9: morphism of locally ringed topoi, morphism of locally ringed sites
In 18.43.1: constant sheaf, locally constant, finite locally constant

Injectives

In 19.2.4: α-small with respect to I
In 19.10.1: generator, Grothendieck abelian category
In 19.11.2: size

Cohomology of Sheaves

In 20.4.1: torsor, $\mathcal{G}$-torsor, morphism of $\mathcal{G}$-torsors, trivial $\mathcal{G}$-torsor
In 20.9.1: Čech complex, Čech cohomology groups
In 20.12.1: flasque, flabby
In 20.23.1: alternating Čech complex
In 20.23.2: ordered Čech complex
In 20.24.2: locally finite
In 20.26.2: K-flat
In 20.26.14: derived tensor product
In 20.26.15: Tor
In 20.46.1: strictly perfect
In 20.47.1: m-pseudo-coherent, pseudo-coherent, m-pseudo-coherent, pseudo-coherent
In 20.48.1: tor-amplitude in $[a, b]$, finite tor dimension, locally has finite tor dimension, tor dimension $\leq d$
In 20.49.1: perfect, perfect

Cohomology on Sites

In 21.4.1: pseudo torsor, pseudo $\mathcal{G}$-torsor, morphism of pseudo $\mathcal{G}$-torsors, torsor, $\mathcal{G}$-torsor, morphism of $\mathcal{G}$-torsors, trivial $\mathcal{G}$-torsor
In 21.8.1: Čech complex, Čech cohomology groups
In 21.13.4: totally acyclic
In 21.17.2: K-flat
In 21.17.13: derived tensor product
In 21.17.14: Tor
In 21.31.2: qc covering
In 21.41.1: simplicial $\mathcal{A}_\bullet$-module, simplicial sheaf of $\mathcal{A}_\bullet$-modules
In 21.43.1: $QC(\mathcal{O})$
In 21.44.1: strictly perfect

In 21.45.1: m-pseudo-coherent, pseudo-coherent, m-pseudo-coherent, pseudo-coherent
In 21.46.1: tor-amplitude in $[a, b]$, finite tor dimension, locally has finite tor dimension, tor dimension $\leq d$
In 21.47.1: perfect, perfect

Differential Graded Algebra

In 22.3.1: differential graded algebra over R
In 22.3.2: homomorphism of differential graded algebras
In 22.3.3: commutative, strictly commutative
In 22.3.4: tensor product differential graded algebra
In 22.4.1: differential graded module, homomorphism of differential graded modules
In 22.4.3: k-shifted module
In 22.5.1: homotopy between f and g, homotopic
In 22.5.3: homotopy category
In 22.6.1: cone
In 22.7.1: admissible monomorphism, admissible epimorphism, admissible short exact sequence
In 22.8.2: triangle associated to $0 \to K \to L \to M \to 0$, distinguished triangle
In 22.11.1: opposite differential graded algebra
In 22.11.3: kth shifted A-module, kth shifted A-module
In 22.22.2: derived category of (A, d)
In 22.24.1: R-linear category $\mathcal{A}$
In 22.24.2: functor of R-linear categories, R-linear functor
In 22.25.1: graded category $\mathcal{A}$ over R
In 22.25.2: functor of graded categories over R, graded functor
In 22.25.3: $\mathcal{A}^0$
In 22.25.4: graded direct sum
In 22.26.1: differential graded category $\mathcal{A}$ over R
In 22.26.2: functor of differential graded categories over R
In 22.26.3: category of complexes of $\mathcal{A}$, homotopy category of $\mathcal{A}$
In 22.26.4: differential graded direct sum
In 22.28.1: (A, B)-bimodule, graded (A, B)-bimodule, differential graded (A, B)-bimodule

Divided Power Algebra

In 23.2.1: divided power structure
In 23.3.1: divided power ring, homomorphism of divided power rings
In 23.4.1: extends
In 23.6.1: divided power structure
In 23.6.5: compatible with the differential graded structure
In 23.8.5: complete intersection, local complete intersection

Differential Graded Sheaves

In 24.3.1: sheaf of graded $\mathcal{O}$-algebras, sheaf of graded algebras, homomorphism of graded $\mathcal{O}$-algebras
In 24.4.1: graded $\mathcal{A}$-module, graded module, homomorphism of graded $\mathcal{A}$-modules
In 24.8.1: graded $(\mathcal{A}, \mathcal{B})$-bimodule, homomorphism of graded $(\mathcal{A}, \mathcal{B})$-bimodules
In 24.12.1: sheaf of differential graded $\mathcal{O}$-algebras, sheaf of differential graded algebras, homomorphism of differential graded $\mathcal{O}$-algebras
In 24.13.1: differential graded $\mathcal{A}$-module, differential graded module, homomorphism of differential graded $\mathcal{A}$-modules
In 24.17.1: differential graded $(\mathcal{A}, \mathcal{B})$-bimodule, homomorphism of differential graded $(\mathcal{A}, \mathcal{B})$-bimodules
In 24.21.1: homotopy between f and g, homotopic
In 24.21.2: homotopy category
In 24.22.2: cone
In 24.25.2: graded injective
In 24.25.7: K-injective
In 24.26.4: derived category of $(\mathcal{A}, \mathrm{d})$
In 24.28.2: derived tensor product, derived pullback
In 24.29.2: derived internal hom, derived pushforward

In 24.33.1: $QC(\mathcal{A}, \text{d})$

Hypercoverings

In 25.2.1: semi-representable objects, semi-representable objects over X
In 25.2.2: which associates a presheaf to a semi-representable object
In 25.3.1: covering, covering
In 25.3.3: hypercovering of X
In 25.4.1: homology of K
In 25.6.1: hypercovering of $\mathcal{G}$, hypercovering

Schemes

In 26.2.1: locally ringed space $(X, \mathcal{O}_X)$, local ring of X at x, residue field of X at x, morphism of locally ringed spaces
In 26.3.1: open immersion
In 26.3.3: open subspace of X associated to U
In 26.4.1: closed immersion
In 26.4.4: closed subspace of X associated to the sheaf of ideals $\mathcal{I}$
In 26.5.2: standard open covering, standard open covering
In 26.5.3: structure sheaf $\mathcal{O}_{\text{Spec}(R)}$ of the spectrum of R, spectrum
In 26.5.5: affine scheme, morphism of affine schemes
In 26.9.1: scheme, morphism of schemes
In 26.10.2: open immersion, open subscheme, closed immersion, closed subscheme, immersion, locally closed immersion
In 26.12.1: reduced
In 26.12.5: scheme structure on Z, reduced induced scheme structure, reduction X_{red} of X
In 26.15.1: representable by a scheme, representable
In 26.15.3: satisfies the sheaf property for the Zariski topology, subfunctor $H \subset F$, representable by open immersions, covers F
In 26.17.1: fibre product
In 26.17.7: inverse image $f^{-1}(Z)$ of the closed subscheme Z
In 26.18.1: scheme over S, structure morphism, scheme over R, morphism $f : X \to Y$ of schemes over S, base change, base change, base change
In 26.18.3: preserved under arbitrary base change, preserved under base change, preserved under arbitrary base change, preserved under base change
In 26.18.4: scheme theoretic fibre X_s of f over s, fibre of f over s
In 26.19.1: quasi-compact
In 26.20.1: universally closed
In 26.20.3: satisfies the existence part of the valuative criterion, satisfies the uniqueness part of the valuative criterion
In 26.21.3: separated, quasi-separated, separated, quasi-separated
In 26.23.1: monomorphism

Constructions of Schemes

In 27.4.5: relative spectrum of $\mathcal{A}$ over S, spectrum of $\mathcal{A}$ over S
In 27.5.1: affine n-space over S, affine n-space over R
In 27.6.1: vector bundle associated to $\mathcal{E}$
In 27.6.2: vector bundle $\pi : V \to S$ over S, morphism of vector bundles over S
In 27.7.1: cone associated to $\mathcal{A}$, affine cone associated to $\mathcal{A}$
In 27.7.2: cone $\pi : C \to S$ over S, morphism of cones
In 27.8.2: standard open covering
In 27.8.3: structure sheaf $\mathcal{O}_{\text{Proj}(S)}$ of the homogeneous spectrum of S, homogeneous spectrum
In 27.10.1: twist of the structure sheaf of $\text{Proj}(S)$
In 27.13.2: projective n-space over $\mathbf{Z}$, projective n-space over S, projective n-space over R
In 27.16.7: relative homogeneous spectrum of $\mathcal{A}$ over S, homogeneous spectrum of $\mathcal{A}$ over S, relative Proj of $\mathcal{A}$ over S
In 27.21.1: projective bundle associated to $\mathcal{E}$, twist of the structure sheaf
In 27.22.2: Grassmannian over $\mathbf{Z}$, Grassmannian over S, Grassmannian over R

Properties of Schemes

In 28.3.1: integral

In 28.4.1: local
In 28.4.2: locally P
In 28.5.1: locally Noetherian, Noetherian
In 28.6.1: Jacobson
In 28.7.1: normal
In 28.8.1: Cohen-Macaulay
In 28.9.1: regular, nonsingular
In 28.10.1: dimension, dimension of X at x
In 28.11.1: catenary
In 28.12.1: regular in codimension k, (R_k), (S_k)
In 28.13.1: Japanese, universally Japanese, Nagata
In 28.14.1: regular locus, singular locus
In 28.15.1: unibranch at x, geometrically unibranch at x, unibranch, geometrically unibranch
In 28.15.4: number of branches of X at x, number of geometric branches of X at x
In 28.18.1: quasi-affine
In 28.21.1: locally projective
In 28.23.1: κ-generated
In 28.24.3: subsheaf of sections annihilated by $\mathcal{I}$
In 28.24.6: subsheaf of sections supported on T
In 28.26.1: ample

Morphisms of Schemes

In 29.4.4: scheme theoretic intersection, scheme theoretic union
In 29.5.5: scheme theoretic support of $\mathcal{F}$
In 29.6.2: scheme theoretic image
In 29.7.1: scheme theoretic closure of U in X, scheme theoretically dense in X
In 29.8.1: dominant
In 29.9.1: surjective
In 29.10.1: universally injective, radicial
In 29.11.1: affine
In 29.12.1: ample family of invertible modules on X
In 29.13.1: quasi-affine
In 29.14.1: local, stable under base change, stable under composition
In 29.14.2: locally of type P
In 29.15.1: finite type at $x \in X$, locally of finite type, finite type
In 29.16.3: finite type point
In 29.17.1: universally catenary
In 29.19.1: J-2
In 29.20.1: quasi-finite at a point $x \in X$, locally quasi-finite, quasi-finite
In 29.21.1: finite presentation at $x \in X$, locally of finite presentation, finite presentation
In 29.23.1: open, universally open
In 29.24.1: submersive, universally submersive
In 29.25.1: flat at a point $x \in X$, flat over S at a point $x \in X$, flat, flat over S
In 29.26.3: canonical scheme structure on T
In 29.29.1: relative dimension $\leq d$ at x, relative dimension $\leq d$, relative dimension d
In 29.30.1: syntomic at $x \in X$, syntomic, local complete intersection over k, standard syntomic
In 29.30.15: syntomic of relative dimension d
In 29.31.1: conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X, conormal sheaf of i
In 29.32.1: sheaf of differentials $\Omega_{X/S}$ of X over S, universal S-derivation
In 29.34.1: smooth at $x \in X$, smooth, standard smooth
In 29.34.13: smooth of relative dimension d
In 29.35.1: unramified at $x \in X$, G-unramified at $x \in X$, unramified, G-unramified
In 29.36.1: étale at $x \in X$, étale, standard étale
In 29.37.1: relatively ample, f-relatively ample, ample on X/S, f-ample
In 29.38.1: relatively very ample, f-relatively very ample, very ample on X/S, f-very ample
In 29.40.1: quasi-projective, H-quasi-projective, locally quasi-projective
In 29.41.1: proper
In 29.43.1: projective, H-projective, locally projective
In 29.44.1: integral, finite
In 29.45.1: universal homeomorphism

In 29.47.1: seminormal, absolutely weakly normal
In 29.47.3: seminormal, absolutely weakly normal
In 29.47.8: seminormalization, absolute weak normalization
In 29.48.1: finite locally free, rank, degree
In 29.49.1: equivalent, rational map from X to Y, S-rational map from X to Y
In 29.49.3: rational function on X
In 29.49.4: ring of rational functions on X
In 29.49.6: function field, field of rational functions
In 29.49.8: defined in a point $x \in X$, domain of definition
In 29.49.10: dominant
In 29.49.11: birational, S-birational
In 29.50.1: birational
In 29.51.8: degree of X over Y
In 29.51.11: modification of X
In 29.51.12: alteration of X
In 29.53.2: integral closure of $\mathcal{O}_X$ in $\mathcal{A}$
In 29.53.3: normalization of X in Y
In 29.54.1: normalization
In 29.55.6: seminormalization of X in Y, weak normalization of X in Y
In 29.55.8: weak normalization
In 29.55.9: weakly normal
In 29.57.1: bounds the degrees of the fibres of f, fibres of f are universally bounded

Cohomology of Schemes

In 30.11.1: depth k at a point, depth k at a point, (S_k), (S_k)
In 30.11.4: Cohen-Macaulay
In 30.26.2: Z is proper over S

Divisors

In 31.2.1: associated, associated points of X
In 31.4.1: embedded associated point, embedded point, embedded component
In 31.5.1: weakly associated, weakly associated points of X
In 31.7.1: relative assassin of $\mathcal{F}$ in X over S
In 31.8.1: relative weak assassin of $\mathcal{F}$ in X over S
In 31.11.2: torsion, torsion free
In 31.12.1: reflexive hull, reflexive
In 31.13.1: locally principal closed subscheme, effective Cartier divisor
In 31.13.6: sum of the effective Cartier divisors D_1 and D_2
In 31.13.12: pullback of D by f is defined, pullback of the effective Cartier divisor
In 31.14.1: invertible sheaf $\mathcal{O}_S(D)$ associated to D, canonical section
In 31.14.6: regular section
In 31.14.8: zero scheme
In 31.18.2: relative effective Cartier divisor
In 31.19.1: conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X, conormal algebra of f
In 31.19.5: normal cone C_ZX, normal bundle
In 31.20.2: regular, Koszul-regular, H_1-regular, quasi-regular
In 31.21.1: regular immersion, Koszul-regular immersion, H_1-regular immersion, quasi-regular immersion
In 31.22.2: relative quasi-regular immersion, relative H_1-regular immersion
In 31.23.1: sheaf of meromorphic functions on X, $\mathcal{K}_X$, meromorphic function
In 31.23.3: meromorphic section of $\mathcal{F}$
In 31.23.4: pullbacks of meromorphic functions are defined for f
In 31.23.7: regular
In 31.23.10: ideal sheaf of denominators of s
In 31.26.2: prime divisor, Weil divisor
In 31.26.3: order of vanishing of f along Z
In 31.26.5: principal Weil divisor associated to f
In 31.26.7: Weil divisor class group
In 31.27.1: order of vanishing of s along Z
In 31.27.4: Weil divisor associated to s, Weil divisor class associated to $\mathcal{L}$

In 31.32.1: blowing up of X along Z, blowing up of X in the ideal sheaf $\mathcal{I}$, exceptional divisor, center
In 31.33.1: strict transform, strict transform
In 31.34.1: U-admissible blowup

Limits of Schemes

Varieties

In 33.3.1: variety
In 33.6.1: geometrically reduced at x, geometrically reduced
In 33.7.1: geometrically connected
In 33.8.1: geometrically irreducible
In 33.9.1: geometrically pointwise integral at x, geometrically pointwise integral, geometrically integral
In 33.10.1: geometrically normal at x, geometrically normal
In 33.12.1: geometrically regular at x, geometrically regular over k
In 33.16.1: dual numbers
In 33.16.3: tangent space of X over S at x, tangent vector
In 33.20.1: algebraic k-scheme, locally algebraic k-scheme
In 33.26.1: affine variety, projective variety, quasi-projective variety, proper variety, smooth variety
In 33.33.1: Euler characteristic of $\mathcal{F}$
In 33.35.7: m-regular
In 33.35.15: Hilbert polynomial
In 33.36.1: absolute frobenius of X
In 33.36.4: relative Frobenius morphism of X/S
In 33.39.3: δ-invariant of A
In 33.39.7: δ-invariant of X at x
In 33.40.4: A is a wedge of $A_1, \ldots, A_n$
In 33.43.1: curve
In 33.44.1: degree, degree
In 33.45.3: intersection number
In 33.45.10: degree of Z with respect to $\mathcal{L}$
In 33.46.1: embedding dimension of X at x
In 33.46.2: embedding dimension of X/k at x

Topologies on Schemes

In 34.3.1: Zariski covering of T
In 34.3.4: standard Zariski covering
In 34.3.5: big Zariski site
In 34.3.7: big Zariski site of S, small Zariski site of S, big affine Zariski site of S, small affine Zariski site of S
In 34.3.15: restriction to the small Zariski site
In 34.4.1: étale covering of T
In 34.4.5: standard étale covering
In 34.4.6: big étale site
In 34.4.8: big étale site of S, small étale site of S, big affine étale site of S, small affine étale site of S
In 34.4.15: restriction to the small étale site
In 34.5.1: smooth covering of T
In 34.5.5: standard smooth covering
In 34.5.6: big smooth site
In 34.5.8: big smooth site of S, big affine smooth site of S
In 34.6.1: syntomic covering of T
In 34.6.5: standard syntomic covering
In 34.6.6: big syntomic site
In 34.6.8: big syntomic site of S, big affine syntomic site of S
In 34.7.1: fppf covering of T
In 34.7.5: standard fppf covering
In 34.7.6: big fppf site
In 34.7.8: big fppf site of S, big affine fppf site of S
In 34.8.1: standard ph covering
In 34.8.4: ph covering of T
In 34.8.9: big ph site
In 34.8.11: big ph site of S, big affine ph site of S
In 34.9.1: fpqc covering of T
In 34.9.9: standard fpqc covering
In 34.9.12: satisfies the sheaf property for the given family, satisfies the sheaf property for the fpqc topology
In 34.10.1: standard V covering
In 34.10.7: V covering of T
In 34.10.11: satisfies the sheaf property for the V topology

Descent

In 35.2.1: descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves, cocycle condition, morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data
In 35.2.3: trivial descent datum, canonical descent datum, effective
In 35.3.1: descent datum (N, φ) for modules with respect to $R \to A$, cocycle condition, morphism $(N, \varphi) \to (N', \varphi')$ of descent data
In 35.3.4: effective
In 35.4.2: split equalizer
In 35.4.5: universally injective
In 35.4.9: C
In 35.4.15: base extension along f, descent morphism for modules, effective descent morphism for modules
In 35.4.19: f_*
In 35.8.2: structure sheaf of the big site $(Sch/S)_\tau$, structure sheaf of the small site, sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$, sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$
In 35.12.1: parasitic, parasitic for the τ-topology
In 35.15.1: local in the τ-topology
In 35.20.1: germ of X at x, morphism of germs, composition of morphisms of germs
In 35.20.2: étale, smooth
In 35.21.1: étale local, smooth local
In 35.22.1: τ local on the base, τ local on the target, local on the base for the τ-topology
In 35.26.1: τ local on the source, local on the source for the τ-topology
In 35.32.3: étale local on source-and-target
In 35.33.1: étale local on the source-and-target
In 35.34.1: descent datum for $V/X/S$, cocycle condition, descent datum relative to $X \to S$, morphism $f : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$
In 35.34.3: descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to S\}$, morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data
In 35.34.7: pullback functor
In 35.34.9: pullback functor
In 35.34.10: trivial descent datum, canonical descent datum, effective
In 35.34.11: canonical descent datum, effective
In 35.36.1: morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings

Derived Categories of Schemes

In 36.6.1: supported on T
In 36.14.1: approximation holds for the triple
In 36.14.2: approximation by perfect complexes holds
In 36.22.2: Tor independent over S
In 36.35.1: perfect relative to S, S-perfect
In 36.36.1: resolution property
In 36.38.2: Grothendieck group of X, Grothendieck group of coherent sheaves on X

More on Morphisms

In 37.2.1: thickening, first order thickening, morphism of thickenings, thickenings over S, morphisms of thickenings over S
In 37.5.1: first order infinitesimal neighbourhood
In 37.6.1: formally unramified
In 37.7.2: universal first order thickening, conormal sheaf of Z over X
In 37.8.1: formally étale
In 37.11.1: formally smooth
In 37.13.1: naive cotangent complex of f
In 37.20.1: normal at x, normal morphism
In 37.21.1: regular at x, regular morphism
In 37.22.1: Cohen-Macaulay at x, Cohen-Macaulay morphism
In 37.35.1: étale neighbourhood of (S, s), morphism of étale neighbourhoods, elementary étale neighbourhood
In 37.58.1: finitely presented relative to S, of finite presentation relative to S

In 37.59.2: m-pseudo-coherent relative to S, pseudo-coherent relative to S, m-pseudo-coherent relative to S, pseudo-coherent relative to S
In 37.60.2: pseudo-coherent
In 37.61.2: perfect
In 37.62.2: Koszul at x, Koszul morphism, local complete intersection morphism
In 37.64.1: weakly étale, absolutely flat
In 37.66.1: ind-quasi-affine, ind-quasi-affine
In 37.73.1: affine stratification
In 37.73.4: affine stratification number
In 37.75.2: weighting, pondération
In 37.78.1: completely decomposed, completely decomposed

More on Flatness

In 38.4.1: one step dévissage of $\mathcal{F}/X/S$ over s
In 38.4.2: one step dévissage of $\mathcal{F}/X/S$ at x
In 38.4.6: standard shrinking
In 38.5.1: complete dévissage of $\mathcal{F}/X/S$ over s
In 38.5.2: complete dévissage of $\mathcal{F}/X/S$ at x
In 38.5.5: standard shrinking
In 38.6.1: elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$
In 38.6.2: complete dévissage of $N/S/R$ over $\mathfrak{r}$
In 38.6.4: complete dévissage of $N/S/R$ at $\mathfrak{q}$
In 38.15.2: impurity of $\mathcal{F}$ above s
In 38.16.1: pure along X_s, universally pure along X_s, pure along X_s, universally S-pure, universally pure relative to S, S-pure, pure relative to S, S-pure, pure relative to S
In 38.20.10: $\mathcal{F}$ is flat over S in dimensions $\geq n$
In 38.21.1: universal flattening of $\mathcal{F}$ exists, universal flattening of X exists
In 38.21.3: flattening stratification, flattening stratification
In 38.34.2: h covering of T
In 38.34.10: big h site
In 38.34.11: standard h covering
In 38.34.13: big h site of S, big affine h site of S

Groupoid Schemes

In 39.3.1: pre-relation, relation, pre-equivalence relation, equivalence relation on U over S
In 39.3.3: restriction, pullback
In 39.4.1: group scheme over S, morphism $\psi : (G, m) \to (G', m')$ of group schemes over S
In 39.4.3: closed subgroup scheme, open subgroup scheme
In 39.4.5: smooth group scheme, flat group scheme, separated group scheme
In 39.9.1: abelian variety
In 39.10.1: action of G on the scheme X/S, equivariant, G-equivariant
In 39.10.2: free
In 39.11.1: pseudo G-torsor, formally principally homogeneous under G, trivial
In 39.11.3: principal homogeneous space, G-torsor, G-torsor in the τ topology, τ G-torsor, τ torsor, quasi-isotrivial, locally trivial
In 39.12.1: G-equivariant quasi-coherent $\mathcal{O}_X$-module, equivariant quasi-coherent $\mathcal{O}_X$-module
In 39.13.1: groupoid scheme over S, groupoid over S, morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S
In 39.14.1: quasi-coherent module on (U, R, s, t, c)
In 39.17.2: stabilizer of the groupoid scheme (U, R, s, t, c)
In 39.18.2: restriction of (U, R, s, t, c) to U'
In 39.19.1: set-theoretically R-invariant, R-invariant, R-invariant, R-invariant
In 39.20.1: quotient sheaf U/R
In 39.20.2: representable quotient, representable quotient
In 39.21.1: cartesian, (U', R', s', t', c') is cartesian over (U, R, s, t, c), morphism of groupoid schemes cartesian over (U, R, s, t, c)

More on Groupoid Schemes

Étale Morphisms of Schemes

In 41.3.1: unramified homomorphism of local rings
In 41.3.5: unramified at x, unramified
In 41.9.1: flat, faithfully flat, flat (resp. faithfully flat)
In 41.9.3: flat over Y at $x \in X$, flat at $x \in X$, flat, faithfully flat
In 41.11.1: étale homomorphism of local rings
In 41.11.4: étale at $x \in X$, étale
In 41.21.1: strict normal crossings divisor
In 41.21.4: normal crossings divisor

Chow Homology and Chern Classes

In 42.2.1: 2-periodic complex, cohomology modules, exact, $(2, 1)$-periodic complex, cohomology modules
In 42.2.2: multiplicity, (additive) Herbrand quotient
In 42.7.6: δ-dimension of Z
In 42.8.1: cycle on X, k-cycle
In 42.8.3: support
In 42.8.4: effective
In 42.9.2: multiplicity of Z' in Z, k-cycle associated to Z
In 42.10.2: multiplicity of Z' in $\mathcal{F}$, k-cycle associated to $\mathcal{F}$
In 42.12.1: pushforward
In 42.14.1: flat pullback of α by f
In 42.17.1: principal divisor associated to f
In 42.19.1: rationally equivalent to zero, rationally equivalent, Chow group of k-cycles on X, Chow group of k-cycles modulo rational equivalence on X
In 42.22.1: envelope
In 42.24.1: Weil divisor associated to s, Weil divisor associated to $\mathcal{L}$
In 42.25.1: intersection with the first Chern class of $\mathcal{L}$
In 42.29.1: Gysin homomorphism
In 42.33.1: bivariant class c of degree p for f
In 42.34.1: Chow cohomology
In 42.34.4: first Chern class
In 42.37.1: Chern classes of $\mathcal{E}$ on X, total Chern class of $\mathcal{E}$ on X
In 42.38.1: intersection with the jth Chern class of $\mathcal{E}$
In 42.38.8: ith Chern class, total Chern class
In 42.41.1: degree of a zero cycle
In 42.46.3: Chern classes of E are defined
In 42.50.3: localized Chern character, localized pth Chern class
In 42.59.4: the gysin map for f exists, gysin map
In 42.68.2: admissible, symbol, admissible relation, determinant of the finite length R-module M
In 42.68.13: determinant of (M, φ, ψ)
In 42.68.29: symbol associated to M, a, b
In 42.68.31: tame symbol

Intersection Theory

In 43.13.5: intersect properly, intersect properly
In 43.15.1: multiplicity of M for the ideal of definition I

Picard Schemes of Curves

In 44.4.1: Picard functor
In 44.6.3: genus

Weil Cohomology Theories

In 45.5.1: ith Chow group of M
In 45.7.3: classical Weil cohomology theory
In 45.11.4: Weil cohomology theory

Adequate Modules

In 46.3.1: module-valued functor, morphism of module-valued functors
In 46.3.2: adequate, linearly adequate
In 46.5.1: adequate
In 46.5.7: Adeq$(\mathcal{O})$, Adeq$((Sch/S)_\tau, \mathcal{O})$, Adeq$(S)$
In 46.8.1: pure projective, pure injective
In 46.8.5: pure projective resolution, pure injective resolution
In 46.8.8: pure extension module

Dualizing Complexes

In 47.2.1: essential, essential extension of, essential

In 47.4.1: projective cover, projective envelope
In 47.5.1: injective hull
In 47.5.5: indecomposable
In 47.15.1: dualizing complex
In 47.21.1: Gorenstein, Gorenstein
In 47.27.1: relative dualizing complex

Duality for Schemes

In 48.2.2: dualizing complex
In 48.20.5: dualizing complex normalized relative to $\omega_S^\bullet$
In 48.24.1: Gorenstein
In 48.25.2: Gorenstein at x, Gorenstein morphism
In 48.28.1: relative dualizing complex

Discriminants and Differents

In 49.4.1: trace element
In 49.7.1: Kähler different
In 49.9.1: different

de Rham Cohomology

In 50.7.1: Hodge filtration
In 50.15.1: de Rham complex of log poles is defined for $Y \subset X$ over S
In 50.15.3: de Rham complex of log poles for $Y \subset X$ over S

Local Cohomology

In 51.4.2: cohomological dimension of I in A
In 51.13.1: I-depth, depth

Algebraic and Formal Geometry

In 52.6.4: derived complete with respect to $\mathcal{I}$
In 52.16.5: $(\mathcal{F}_n)$ extends to X
In 52.16.7: $(\mathcal{F}_n)$ canonically extends to X
In 52.19.1: $(\mathcal{F}_n)$ satisfies the (a, b)-inequalities, $(\mathcal{F}_n)$ satisfies the strict (a, b)-inequalities

Algebraic Curves

In 53.2.7: nonsingular projective model of X
In 53.3.1: linear series of degree d and dimension r, $\mathfrak{g}^r_d$
In 53.8.1: genus
In 53.11.1: geometric genus
In 53.16.2: multicross singularity, node, ordinary double point, defines a nodal singularity
In 53.19.1: node, ordinary double point, defines a nodal singularity, singularities of X are at-worst-nodal
In 53.19.10: split node
In 53.20.2: at-worst-nodal of relative dimension 1

Resolution of Surfaces

In 54.5.1: normalized blowup of X at x
In 54.8.3: defines a rational singularity, reduction to rational singularities is possible for A
In 54.14.1: resolution of singularities
In 54.14.2: resolution of singularities by normalized blowups

Semistable Reduction

In 55.3.1: numerical type
In 55.3.2: equivalent types
In 55.3.4: numerical type of genus g
In 55.3.8: (-1)-index
In 55.3.11: topological genus of T
In 55.3.12: minimal
In 55.3.16: (-2)-index
In 55.4.1: Picard group of T
In 55.8.4: minimal model
In 55.11.4: numerical type associated to X
In 55.14.6: semistable reduction
In 55.14.8: good reduction

Functors and Morphisms

Derived Categories of Varieties

In 57.3.2: a Serre functor exists, Serre functor
In 57.8.1: Fourier-Mukai functor, Fourier-Mukai kernel
In 57.10.1: siblings, sibling
In 57.12.1: siblings, sibling
In 57.15.1: the Fourier-Mukai kernel of a relative equivalence from X to Y over S
In 57.18.1: derived equivalent

Fundamental Groups of Schemes

In 58.2.1: G-set, discrete G-set, morphism of G-sets, G-Sets
In 58.3.6: Galois category
In 58.6.1: fundamental group, base point

Étale Cohomology

In 59.4.1: étale covering
In 59.9.1: presheaf of sets, abelian presheaf
In 59.10.1: family of morphisms with fixed target
In 59.10.2: site, coverings
In 59.11.1: separated presheaf, sheaf
In 59.11.4: category of sheaves of sets, abelian sheaves
In 59.13.1: zeroth Čech cohomology group
In 59.15.1: fpqc covering
In 59.15.5: satisfies the sheaf property for the fpqc topology
In 59.16.1: descent datum, cocycle condition, effective
In 59.16.5: descent datum
In 59.16.6: effective
In 59.17.2: ringed site, quasi-coherent
In 59.18.1: Čech complex, Čech cohomology groups
In 59.18.4: free abelian presheaf on $\mathcal{G}$
In 59.20.1: τ-covering
In 59.20.2: big τ-site of S, small τ-site of S
In 59.20.4: standard τ-covering
In 59.21.1: étale topos, small étale topos, Zariski topos, small Zariski topos, big τ-topos
In 59.23.1: constant sheaf
In 59.23.3: structure sheaf
In 59.26.1: étale
In 59.26.3: standard étale
In 59.27.1: étale covering
In 59.27.3: big étale site over S, small étale site over S, big, small Zariski sites
In 59.29.1: geometric point, lies over, étale neighborhood, morphism of étale neighborhoods
In 59.29.6: stalk
In 59.31.3: support of $\mathcal{F}$, support of σ
In 59.32.2: henselian
In 59.32.6: strictly henselian
In 59.33.2: étale local ring of S at $\overline{s}$, strict henselization of $\mathcal{O}_{S,s}$, henselization of $\mathcal{O}_{S,s}$, strict henselization of S at $\overline{s}$, henselization of S at s
In 59.35.1: direct image, pushforward
In 59.35.3: direct image, pushforward
In 59.35.4: higher direct images
In 59.36.1: inverse image, pullback
In 59.51.1: system $(\mathcal{F}_i, \varphi_{i'i})$ of sheaves on $(X_i, f_{i'i})$
In 59.56.1: absolute Galois group, algebraic
In 59.57.1: G-module, discrete G-module, morphism of G-modules, R-G-module, morphism of R-G-modules
In 59.57.2: continuous group cohomology groups, group cohomology groups, Galois cohomology groups, Galois cohomology groups of K with coefficients in M
In 59.61.3: similar, equivalent
In 59.61.4: Brauer group
In 59.64.1: constant sheaf with value E, constant sheaf, locally constant, finite locally constant, constant sheaf with value A, constant sheaf, locally constant, finite locally constant, constant sheaf with value M, constant sheaf, locally constant
In 59.66.1: trace
In 59.67.5: C_r, nontrivial solution
In 59.67.9: variety, curve
In 59.70.1: extension by zero, extension by zero
In 59.71.1: constructible, constructible, constructible
In 59.76.1: $D_c(X_{étale}, \Lambda)$
In 59.77.1: $D_{ctf}(X_{étale}, \Lambda)$
In 59.93.1: locally acyclic at $\overline{x}$ relative to K, locally acyclic relative to K, universally locally acyclic relative to K, locally acyclic, universally locally acyclic
In 59.95.1: cohomological dimension of X
In 59.96.1: cohomological dimension of f

Crystalline Cohomology

In 60.2.2: divided power envelope of J in B relative to (A, I, γ)
In 60.4.1: δ is compatible with γ

Pro-étale Cohomology

Relative Cycles

More Étale Cohomology

The Trace Formula

Algebraic Spaces

In 65.12.1: open immersion, open subspace, closed immersion, closed subspace, immersion, locally closed subspace
In 65.12.5: Zariski covering
In 65.12.6: small Zariski site F_{Zar}
In 65.13.2: separated over S, locally separated over S, quasi-separated over S, Zariski locally quasi-separated over S
In 65.14.4: acts freely, quotient of U by G
In 65.16.2: base change of F' to S, viewed as an algebraic space over S'

Properties of Algebraic Spaces

In 66.3.1: separated, locally separated, quasi-separated, Zariski locally quasi-separated, separated, locally separated, quasi-separated, Zariski locally quasi-separated
In 66.4.1: point
In 66.4.7: topological space
In 66.5.1: quasi-compact
In 66.7.2: has property $\mathcal{P}$
In 66.7.5: has property $\mathcal{P}$ at x
In 66.8.2: étale locally constructible
In 66.9.1: dimension of X at x
In 66.9.2: dimension
In 66.10.2: dimension of the local ring of X at x, x is a point of codimension d on X
In 66.12.5: algebraic space structure on Z, reduced induced algebraic space structure, reduction X_{red} of X
In 66.16.2: étale
In 66.18.1: small étale site $X_{étale}$
In 66.18.2: $X_{spaces,étale}$
In 66.18.5: $X_{affine,étale}$
In 66.18.7: étale topos, small étale topos
In 66.18.9: f-map $\varphi : \mathcal{G} \to \mathcal{F}$
In 66.19.1: geometric point, geometric point lying over x
In 66.19.2: étale neighborhood, morphism of étale neighborhoods
In 66.19.6: stalk
In 66.20.3: support of $\mathcal{F}$, support of σ
In 66.21.2: structure sheaf
In 66.22.2: étale local ring of X at $\overline{x}$, strict henselization of X at $\overline{x}$
In 66.23.2: geometrically unibranch at x, geometrically unibranch
In 66.23.4: number of geometric branches of X at x
In 66.24.1: Noetherian
In 66.25.2: X is regular at x
In 66.29.1: quasi-coherent
In 66.31.2: locally projective

Morphisms of Algebraic Spaces

In 67.4.2: separated, locally separated, quasi-separated
In 67.5.2: surjective
In 67.6.2: open, universally open
In 67.7.2: submersive, universally submersive
In 67.8.2: quasi-compact
In 67.9.2: closed, universally closed
In 67.10.1: monomorphism
In 67.13.2: inverse image $f^{-1}(Z)$ of the closed subspace Z
In 67.14.4: scheme theoretic intersection, scheme theoretic union
In 67.15.4: scheme theoretic support of $\mathcal{F}$
In 67.16.2: scheme theoretic image
In 67.17.3: scheme theoretic closure of U in X, scheme theoretically dense in X
In 67.18.1: dominant
In 67.19.3: universally injective
In 67.20.2: affine
In 67.20.8: relative spectrum of $\mathcal{A}$ over X, spectrum of $\mathcal{A}$ over X
In 67.21.2: quasi-affine
In 67.22.2: has property $\mathcal{P}$
In 67.22.6: has property $\mathcal{Q}$ at x
In 67.23.1: locally of finite type, finite type at x, of finite type
In 67.25.2: finite type point
In 67.27.1: locally quasi-finite, quasi-finite at x, quasi-finite
In 67.28.1: locally of finite presentation, finite presentation at x, of finite presentation
In 67.30.1: flat, flat at x
In 67.31.2: flat at x over Y, flat over Y
In 67.33.1: dimension of the local ring of the fibre of f at x, transcendence degree

Decent Algebraic Spaces

Cohomology of Algebraic Spaces

Limits of Algebraic Spaces

Divisors on Algebraic Spaces

Algebraic Spaces over Fields

Topologies on Algebraic Spaces

Descent and Algebraic Spaces

Derived Categories of Spaces

More on Morphisms of Spaces

In 76.5.1: conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X, conormal sheaf of i
In 76.6.1: conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X, conormal algebra of i
In 76.6.5: normal cone $C_Z X$, normal bundle
In 76.7.2: sheaf of differentials $\Omega_{X/Y}$ of X over Y, universal Y-derivation
In 76.9.1: thickening, first order thickening, morphism of thickenings, thickenings over B, morphisms of thickenings over B
In 76.12.1: first order infinitesimal neighbourhood
In 76.13.1: formally smooth, formally étale, formally unramified
In 76.14.1: formally unramified
In 76.15.5: universal first order thickening, conormal sheaf of Z over X
In 76.16.1: formally étale
In 76.19.1: formally smooth
In 76.21.1: naive cotangent complex of f
In 76.23.2: the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, the fibre of X over z is flat at x over the fibre of Y over z, the fibre of X over z is flat over the fibre of Y over z
In 76.26.2: Cohen-Macaulay at x, Cohen-Macaulay morphism
In 76.27.2: Gorenstein at x, Gorenstein morphism
In 76.29.2: the fibre of $f : X \to Y$ at y is geometrically reduced
In 76.44.2: Koszul-regular immersion, H_1-regular immersion, quasi-regular immersion
In 76.45.3: m-pseudo-coherent relative to Y, pseudo-coherent relative to Y, m-pseudo-coherent relative to Y, pseudo-coherent relative to Y
In 76.46.1: pseudo-coherent, pseudo-coherent at x
In 76.47.1: perfect, perfect at x
In 76.48.1: Koszul morphism, local complete intersection morphism, Koszul at x
In 76.52.1: perfect relative to Y, Y-perfect
In 76.55.1: at-worst-nodal of relative dimension 1

Flatness on Algebraic Spaces

In 77.2.2: impurity of $\mathcal{F}$ above y
In 77.3.1: pure above y, universally pure above y, pure above y, universally Y-pure, universally pure relative to Y, Y-pure, pure relative to Y, Y-pure, pure relative to Y
In 77.11.1: universal flattening of $\mathcal{F}$ exists, universal flattening of X exists
In 77.11.3: $\mathcal{F}$ is flat over Y in dimensions $\geq n$

Groupoids in Algebraic Spaces

In 78.4.1: pre-relation, relation, pre-equivalence relation, equivalence relation on U over B
In 78.4.3: restriction, pullback
In 78.5.1: group algebraic space over B, morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B
In 78.8.1: action of G on the algebraic space X/B, equivariant, G-equivariant
In 78.8.2: free
In 78.9.1: pseudo G-torsor, formally principally homogeneous under G, trivial
In 78.9.3: principal homogeneous space, principal homogeneous G-space over B, G-torsor in the τ topology, τ G-torsor, τ torsor, quasi-isotrivial, locally trivial
In 78.10.1: G-equivariant quasi-coherent $\mathcal{O}_X$-module, equivariant quasi-coherent $\mathcal{O}_X$-module
In 78.11.1: groupoid in algebraic spaces over B, morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B
In 78.12.1: quasi-coherent module on (U, R, s, t, c)
In 78.16.2: stabilizer of the groupoid in algebraic spaces (U, R, s, t, c)
In 78.17.2: restriction of (U, R, s, t, c) to U'
In 78.18.1: R-invariant, R-invariant, R-invariant
In 78.19.1: quotient sheaf U/R

In 78.19.3: quotient representable by an algebraic space, representable quotient, representable quotient, quotient representable by an algebraic space
In 78.20.1: quotient stack, quotient stack

More on Groupoids in Spaces

In 79.15.1: strongly split over u, strong splitting of R over u, split over u, splitting of R over u, quasi-split over u, quasi-splitting of R over u

Bootstrap

In 80.3.1: representable by algebraic spaces
In 80.4.1: property $\mathcal{P}$

Pushouts of Algebraic Spaces

Chow Groups of Spaces

In 82.2.5: δ-dimension of T
In 82.3.1: cycle on X, k-cycle
In 82.4.2: $\mathcal{F}$ has length d at x
In 82.5.2: multiplicity of Z in Y, k-cycle associated to Y
In 82.6.1: multiplicity of Z in $\mathcal{F}$, k-cycle associated to $\mathcal{F}$
In 82.8.1: pushforward
In 82.10.1: flat pullback of α by f
In 82.13.1: principal divisor associated to f
In 82.15.1: rationally equivalent to zero, rationally equivalent, Chow group of k-cycles on X, Chow group of k-cycles modulo rational equivalence on X
In 82.17.1: Weil divisor associated to s, Weil divisor associated to $\mathcal{L}$
In 82.18.1: intersection with the first Chern class of $\mathcal{L}$
In 82.22.1: Gysin homomorphism
In 82.26.1: bivariant class c of degree p for f
In 82.26.2: Chow cohomology
In 82.28.2: ith Chern class of $\mathcal{E}$, total Chern class of $\mathcal{E}$
In 82.32.1: degree of a zero cycle

Quotients of Groupoids

In 83.3.1: R-invariant, G-invariant
In 83.3.4: base change, flat base change
In 83.4.1: categorical quotient, categorical quotient in $\mathcal{C}$, categorical quotient in the category of schemes, categorical quotient in schemes
In 83.4.4: universal categorical quotient, uniform categorical quotient
In 83.5.1: orbit, R-orbit
In 83.5.4: weakly R-equivalent, R-equivalent, weak orbit, weak R-orbit, orbit, R-orbit
In 83.5.8: set-theoretically R-invariant, separates orbits, separates R-orbits
In 83.5.13: set-theoretic pre-equivalence relation, set-theoretic equivalence relation
In 83.5.18: orbit space for R
In 83.6.1: coarse quotient, coarse quotient in schemes
In 83.7.1: uniformly, universally
In 83.8.1: sheaf of R-invariant functions on X, the functions on X are the R-invariant functions on U
In 83.9.1: good quotient
In 83.10.1: geometric quotient

More on Cohomology of Spaces

Simplicial Spaces

In 85.12.1: cartesian, cartesian, cartesian, cartesian
In 85.13.1: simplicial system of the derived category, cartesian, morphism of simplicial systems of the derived category
In 85.14.1: simplicial system of the derived category of modules, cartesian, morphism of simplicial systems of the derived category of modules
In 85.27.1: cartesian, Y is cartesian over X
In 85.27.3: simplicial scheme associated to f

Duality for Spaces

In 86.2.2: dualizing complex
In 86.9.1: relative dualizing complex

Formal Algebraic Spaces

In 87.4.7: tensor product, completed tensor product

In 87.4.8: topologically nilpotent, weak ideal of definition, weakly pre-admissible, weakly admissible
In 87.5.1: taut
In 87.6.1: adic
In 87.7.1: weakly pre-adic, c-adic, weakly adic
In 87.9.1: affine formal algebraic space, morphism of affine formal algebraic spaces
In 87.9.7: McQuillan, classical, weakly adic, adic, adic*, Noetherian
In 87.9.9: formal spectrum
In 87.10.2: countably indexed
In 87.11.1: formal algebraic space, morphism of formal algebraic spaces
In 87.14.3: completion of X along T
In 87.16.3: quasi-separated, separated
In 87.17.2: quasi-compact
In 87.17.4: quasi-compact
In 87.20.7: locally countably indexed, locally countably indexed and classical, locally weakly adic, locally adic*, locally Noetherian
In 87.23.2: adic morphism
In 87.24.1: locally of finite type, finite type
In 87.25.1: surjective
In 87.26.1: monomorphism
In 87.27.1: closed immersion
In 87.29.1: topologically of finite type over
In 87.30.1: separated, quasi-separated
In 87.31.1: proper
In 87.34.1: small étale site
In 87.37.3: completion of X along T
In 87.38.1: completion of X along Z

Algebraization of Formal Spaces

In 88.4.1: rig-smooth over (A, I)
In 88.8.1: rig-étale over (A, I)
In 88.13.4: flat
In 88.14.2: rig-closed
In 88.14.7: completed principal localization
In 88.15.2: naively rig-flat
In 88.15.4: rig-flat
In 88.16.1: rig-flat
In 88.17.2: rig-smooth
In 88.18.1: rig-smooth
In 88.19.2: rig-etale
In 88.20.1: rig-étale
In 88.21.1: rig-surjective
In 88.24.1: formal modification

Resolution of Surfaces Revisited

In 89.4.1: blowing up $X' \to X$ of X at x
In 89.5.1: normalized blowup of X at x
In 89.8.1: resolution of singularities
In 89.8.2: resolution of singularities by normalized blowups

Formal Deformation Theory

In 90.3.1: $\mathcal{C}_\Lambda$, classical case
In 90.3.2: small extension
In 90.3.6: relative cotangent space
In 90.3.9: essential surjection
In 90.4.1: $\widehat{\mathcal{C}}_\Lambda$
In 90.5.1: category cofibered in groupoids over $\mathcal{C}$
In 90.6.1: prorepresentable
In 90.6.2: predeformation category, morphism of predeformation categories
In 90.7.1: category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$, formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$, morphism $a : \xi \to \eta$ of formal objects
In 90.7.3: completion of $\mathcal{F}$
In 90.8.1: smooth
In 90.8.9: versal
In 90.9.1: smooth, unobstructed
In 90.10.1: conditions (S1) and (S2)
In 90.11.1: R-linear
In 90.11.9: tangent space TF of F
In 90.12.1: tangent space $T\mathcal{F}$ of $\mathcal{F}$
In 90.12.3: differential $d\varphi : T\mathcal{F} \to T\mathcal{G}$ of φ
In 90.14.4: minimal, miniversal
In 90.16.1: condition (RS)
In 90.16.8: deformation category
In 90.17.1: lift of x along f, morphism of lifts
In 90.19.1: group of infinitesimal automorphisms of x' over x
In 90.19.2: group of infinitesimal automorphisms of x_0
In 90.19.5: automorphism functor of x

In 98.9.1: formal object, morphism of formal objects, lies over
In 98.9.4: effective
In 98.11.1: limit preserving
In 98.12.1: versal
In 98.12.2: versal
In 98.13.1: openness of versality, openness of versality
In 98.18.1: condition (RS*)
In 98.22.1: obstruction theory, obstruction modules, obstruction
In 98.23.5: naive obstruction theory

Quot and Hilbert Spaces

Properties of Algebraic Stacks

In 100.4.2: point
In 100.4.8: topological space
In 100.5.1: surjective
In 100.6.1: quasi-compact
In 100.7.2: has property $\mathcal{P}$
In 100.7.5: has property $\mathcal{P}$ at x
In 100.8.1: monomorphism
In 100.9.1: open immersion, closed immersion, immersion
In 100.9.9: open substack, closed substack, locally closed substack
In 100.10.4: algebraic stack structure on Z, reduced induced algebraic stack structure, reduction $\mathcal{X}_{red}$ of $\mathcal{X}$
In 100.11.8: residual gerbe of $\mathcal{X}$ at x exists, residual gerbe of $\mathcal{X}$ at x
In 100.12.2: dimension of $\mathcal{X}$ at x
In 100.12.3: dimension
In 100.13.1: number of geometric branches of $\mathcal{X}$ at x, geometrically unibranch at x

Morphisms of Algebraic Stacks

In 101.4.1: DM, quasi-DM, separated, quasi-separated
In 101.4.2: DM over S, quasi-DM over S, separated over S, quasi-separated over S, DM, quasi-DM, separated, quasi-separated
In 101.5.3: relative sheaf of automorphisms of x, relative sheaf of isomorphisms from x_1 to x_2, sheaf of automorphisms of x, sheaf of isomorphisms from x_1 to x_2
In 101.7.2: quasi-compact
In 101.8.1: Noetherian
In 101.9.1: affine
In 101.10.1: integral, finite
In 101.11.2: open, universally open
In 101.12.2: submersive, universally submersive
In 101.13.2: closed, universally closed
In 101.14.2: universally injective
In 101.15.2: universal homeomorphism
In 101.16.2: has property $\mathcal{P}$
In 101.17.1: locally of finite type, of finite type
In 101.18.2: finite type point
In 101.23.2: locally quasi-finite
In 101.24.1: quasi-finite
In 101.25.1: flat
In 101.26.2: flat at x
In 101.27.1: locally of finite presentation, of finite presentation
In 101.28.1: gerbe over, gerbe
In 101.33.1: smooth
In 101.34.2: has property $\mathcal{P}$
In 101.35.1: étale
In 101.36.1: unramified
In 101.37.1: proper
In 101.38.1: scheme theoretic image
In 101.39.1: dotted arrow, morphism of dotted arrows
In 101.39.6: uniqueness part of the valuative criterion
In 101.39.10: existence part of the valuative criterion
In 101.44.1: local complete intersection morphism, Koszul
In 101.46.3: normalization
In 101.48.1: decent
In 101.50.1: integral

Limits of Algebraic Stacks

In 102.3.1: limit preserving

Cohomology of Algebraic Stacks

In 103.7.1: flat base change property

In 103.9.1: parasitic
In 103.14.1: lisse-étale site, flat-fppf site
In 103.17.2: coherent

Derived Categories of Stacks

In 104.5.1: derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves

Introducing Algebraic Stacks

In 105.4.3: smooth
In 105.5.1: algebraic stack

More on Morphisms of Stacks

In 106.3.1: thickening, morphism of thickenings, thickenings over $\mathcal{Z}$, morphisms of thickenings over $\mathcal{Z}$
In 106.3.3: first order thickening
In 106.8.1: formally smooth
In 106.12.1: categorical moduli space, uniform categorical moduli space, categorical moduli space in $\mathcal{C}$, uniform categorical moduli space in $\mathcal{C}$
In 106.13.1: well-nigh affine

The Geometry of Algebraic Stacks

In 107.2.2: versal ring to $\mathcal{X}$ at x_0
In 107.3.4: multiplicity
In 107.4.1: formal branches of $\mathcal{X}$ through x_0
In 107.4.3: multiplicity of a formal branch of $\mathcal{X}$ through x_0
In 107.5.2: the relative dimension
In 107.5.7: relative dimension
In 107.5.14: pseudo-catenary
In 107.6.3: dimension of the local ring of $\mathcal{X}$ at x

Moduli Stacks

Moduli of Curves

In 109.16.4: moduli stack of smooth proper curves, moduli stack of smooth proper curves of genus g
In 109.19.2: relative dualizing sheaf
In 109.20.1: prestable family of curves
In 109.21.2: semistable family of curves
In 109.22.2: stable family of curves
In 109.22.4: moduli stack of stable curves, moduli stack of stable curves of genus g

Examples

Exercises

In 111.2.1: directed set, system of rings
In 111.2.3: colimit
In 111.2.8: finite presentation
In 111.6.4: quasi-compact
In 111.6.6: Hausdorff
In 111.6.9: irreducible, irreducible
In 111.6.12: generic point
In 111.6.16: Noetherian, Artinian
In 111.6.18: irreducible component
In 111.6.22: closed, specialization, generalization
In 111.6.26: connected, connected component
In 111.9.1: length
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In 111.22.3: class group of A, Picard group of A
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In 111.26.2: graded module, locally finite, Euler-Poincaré function, Hilbert function, Hilbert polynomial
In 111.26.3: graded A-algebra, graded module M over a graded A-algebra B, homomorphisms of graded modules/rings, graded submodules, graded ideals, exact sequences of graded modules
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117.3. Other chapters

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